Gromov-Witten theory, non-archimedean geometry, and mirror symmetry

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ABSTRACT

This thesis consists of three projects related to enumerative geometry and mirror symmetry, with an eye towards birational geometry.

The first project studies how certain non-archimedean Gromov-Witten invariants of log Calabi-Yau surfaces, called infinitesimal cylinder counts, behave under blowup. We discuss the case of primitive cylinders, and establish a formula that expresses cylinder counts on a blow up of a toric surface in terms of counts in a simpler surface. The proof of the formula uses non-archimedean geometry techniques in an essential way to produce suitable degenerations of the geometric objects enumerated by the counts.

The next two projects introduce and study the notion of F-bundle, a structure which can be used to formulate mirror symmetry type results using the language of differential geometry. Our spectral decomposition theorem provides a canonical decomposition for F-bundles satisfying a condition called maximality. We develop the theory of framing, and use it to obtain reconstruction theorems for isomorphisms between maximal F-bundles. As an application of this theory, we prove the uniqueness of certain decompositions of quantum cohomology related to birational geometry, complementing the existence results found in the literature. We also extend the framework of F-bundles to the setting of equivariant mirror symmetry, and prove an unfolding result which can be used to strengthen mirror symmetry statements from the small quantum cohomology to the big quantum cohomology. We apply this unfolding theorem to the equivariant mirror symmetry of general flag varieties, for which only the small quantum cohomology mirror symmetry was known until now.

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Chapter 1

INTRODUCTION

1.1 Motivations

Enumerative geometry is concerned with counting geometric objects subject to certain constraints. For example, in the complex plane there is a unique line passing through two distinct points, and a unique conic passing through five points in general position. Another, more challenging, computation shows that there are $609,250$ rational conics in a generic quintinc threefold ([Kat86]). There are many examples of such computations using classical algebraic geometry techniques, but for a long time no general theory was available to enumerate curves subject to certain incidence conditions inside a given manifold.

Developed at the end of the 20th century, Gromov-Witten theory provides a general mathematical framework which produces curve counting invariants. Although it originated from the mathematics of quantum field theory, Gromov-Witten theory has grown into a major field of mathematics and has led to decades of groundbreaking research in symplectic and algebraic geometry. Early works in Gromov-Witten theory revolved around two central themes: producing mathematically rigorous foundations to the theory ([KM94; Kon95a; BM96; Beh97; FP97]), and studying the mirror symmetry phenomenon observed by physicists in string theory ([Can+91; Giv95; SYZ96; HV00]).

Mirror symmetry was originally described as a duality between physical theories, and realized mathematically as a duality between the complex and symplectic geometry of pairs of Calabi-Yau manifolds. Physicists' computations suggested that certain numerical quantities associated to each side of the mirror correspondence should be equal, and led to conjectural formulas for certain Gromov-Witten invariants which were unkown to mathematicians at the time ([Can+91]). The development of Gromov-Witten theory led to a rigorous proof of these identities ([Giv96; LLY97; LLY99]), and initiated the mathematical study of mirror symmetry.

Since those early days, mirror symmetry has grown into a vast field of geometry. Efforts were made to provide a conceptual understanding of the numerical coincidence observed on concrete examples, leading to two main formulations of mirror symmetry. The Homological Mirror Symmetry conjecture is a categorical framework proposed by Kontsevich ([Kon95b]), which expresses mirror symmetry as an equivalence of categories associated to each side of the mirror correspondence. Another formulation is the Hodge-theoritic mirror symmetry, also known as D-module mirror symmetry, which associates to each side of the mirror correspondence a system of differential equations and conjectures that those systems are equivalent ([Dub96; Bar00]). In this formulation, the equality of numbers is obtained by comparing coefficients of solutions. Those two approaches are related, and in certain cases homological mirror symmetry implies Hodge-theoritic mirror symmetry ([KKP08; GPS15]).

While the above formulations of mirror symmetry were obtained by studying concrete examples, such as Calabi-Yau geometries and toric varieties, they are not precise and consequently out of reach. Before proving any kind of mirror symmetry statement, one needs to produce mirror pairs. So far, the only general mechanism to produce mirror pairs is the Strominger-Yau-Zaslow (SYZ) conjecture ([SYZ96; Gro13]). It conjectures that every Calabi-Yau manifold admits a fibration into special Lagrangian tori, and that the mirror Calabi-Yau manifold can be constructed by dualizing this fibration in a certain sense. We discuss the SYZ conjecture in more details in Section 2.2. We note that in its modern formulation, mirror symmetry has been generalized beyond the Calabi-Yau case, and that no construction exists in that generality.

In recent years, methods used in Hodge-theoritic mirror symmetry were fruitfully applied to problems at the interaction between enumerative geometry and birational geometry ([Iri23; IK23]). Some of the results in this thesis contribute to these developments. In particular, the main results of Chapter 4 are foundational for the definition of new birational invariants obtained from enumerative geometry in [Kat+24].

In this thesis, we use non-archimedean geometry and formal geometry to study questions related to mirror symmetry. Non-archimedean geometry is a generalization of analytic geometry, where the notion of convergence is made a lot weaker by the use of non-archimedean absolute values. This allows us to use analytic techniques without worrying too much about convergence, and also introduces features that are unique to this setting (see Section 2.2.2). The relevance of non-archimedan geometry for mirror symmetry was outlined in [KS06]. In particular, a non-archimedean version of the SYZ conjecture was proved in [NXY19], eventually leading to a construction of mirror to a log Calabi-Yau affine manifold ([KY23; KY24]), a non-compact analogue of a Calabi-Yau manifold. The use of formal geometry in mirror symmetry is well-established, and provides a geometric language to talk about properties of generating series associated to Gromov-Witten invariants.

1.2 Overview of the main results

We now give an overview of the results of the thesis. General background used throughout the thesis is given in Chapter 2.

Non-archimedean cylinder counts

In Chapter 3, we study certain non-archimedean Gromov-Witten invariants involved in the construction of a mirror to a log Calabi-Yau manifold as outlined by the SYZ conjecture ([KY23]). The definition of those invariants, called infinitesimal cylinder counts, involves the theory of skeleta for non-archimedean analytic spaces which we review in Section 2.2. The counts encode the mirror manifold through a combinatorial object called a scattering diagram. It is desirable to obtain closed-form formulae for those invariants in order to study the mirror manifold, because the scattering diagram can ultimately be used to produce equations for the mirror (see [Arg23]).

The main result of the chapter provides a formula which computes infinitesimal cylinder counts for a primitive curve class in the surface case. Specifically, the project deals with an affine log Calabi-Yau surface U with a fixed compactification $U \subset Y$, obtained as a blowup of a toric variety $\pi\colon Y \to Y_t$. For surfaces this setup is in fact non-restrictive, because every log Calabi-Yau surface is a blowup of a toric variety up to a single blowup.

We denote by $N(V, \beta)$ the count of *infinitesimal cylinders of type V and curve class β*. In this notation, V is a combinatorial object which encodes conditions on the skeleta of the analytic stable maps we are counting, and $\beta \in \mathrm{NE}(Y, \mathbb{Z})$ is an effective curve class. The count is a refinement of a 3-pointed relative Gromov-Witten invariant to (Y, D), where $D = Y \setminus U$, with two boundary points and a generic interior point constraint. In the toric case, those counts can be computed and are always 0 or 1. But in the non toric case, interactions of analytic stable maps with the exceptional locus E of the blowup morphism $\pi\colon Y \to Y_t$ produce non trivial counts. The complexity of the interaction with the exceptional locus is measured by the *twig type* of V, which is a tuple $\mathbf{w} = (\mathbf{w}_s)_{1 \leq s \leq t}$ where each $\mathbf{w}_s$ corresponds to a point of the domain curve mapped to the exceptional locus. For example, in the toric case if $N(V, \beta) \neq 0$ then the twig type of V is empty as there is no exceptional locus. The next simplest case corresponds to cylinder types V whose twig type consists of a single element: the associated stable maps meet the exceptional locus of π at a single point. Consequently, those cylinder counts can be expressed in terms of counts on a variety which is obtained by blowing up Y_t at a single boundary point.

Our result deals with *primitive* cylinder types, which geometrically correspond to stable maps intersecting each component of the exceptional locus E at most at a single point, with intersection number 1. It expresses a general primitive cylinder count in terms of counts of primitive cylinders with a single twig, which are the simplest kind of non toric counts.

Theorem 1.2.1 (Theorem 3.1.1)**.** *Let V be a primitive infinitesimal tropical cylinder type with twig type $\mathbf{w} = (\mathbf{w}_s)_{1 \leq t \leq s}$, let $\beta \in \mathrm{NE}(Y, \mathbb{Z})$. Then*

$$N(V, \beta) = \sum_{\beta_1 + \cdots + \beta_t = \beta} \prod_{s=1}^{t} N(V_s, \beta_s),$$

where V_s is an infinitesimal cylinder of twig type $\mathbf{w}_s$.

The result is established using non-archimedean techniques. Geometrically, we isolate intersections with the exceptional locus by swapping them for components which avoid it via an inductive procedure. This is done by a gluing and degeneration procedure at the level of the domain curve, reminiscent of the classical argument by Kontsevich and Manin which computes the Gromov-Witten invariants of $\mathbb{P}^2$ ([KM94]). However, in this case the deformation is first encoded tropically, and lifting it to an analytic deformation uses non-archimedean geometry in an essential way. We then relate counts at each side of the degeneration using deformation invariance, and eventually obtain our formula.

Our approach contrasts with the typical methods used to compute Gromov-Witten invariants, which usually rely on degeneration of the target, or take advantage of a torus action on the target to apply localization techniques. In our setup, Y admits a so-called *degeneration to the normal cone*, whose central fiber is the union of Y_t and projective bundles over components of the toric boundary. In principle, applying the degeneration formula in this setting should provide a way to compute cylinder counts. However, there is no clear analogue of the degeneration formula for non-archimedean Gromov-Witten invariants. We note that such a formula is available in logarithmic Gromov-Witten theory ([Abr+20]), and that conjecturally non-archimedean cylinder counts should correspond to certain logarithmic Gromov-Witten invariants. We plan to address the question of the comparison of non-archimedean and logarithmic Gromov-Witten invariants in future works.

Decomposition of F-bundles

In Chapter 4, we introduce the notion of formal and non-archimedean F-bundle and establish essential theorems about them, which we call the *spectral decomposition theorem* and the *extension of framing theorem*. We apply the extension of framing to the study of the decomposition of quantum cohomology for a blowup and for a projective bundle.

The notion of F-bundle emerged as the de Rham part of a non-commutative Hodge structure. The latter appeared in [KKP08] as an attempt to provide a common framework for homological mirror symmetry and Hodge-theoritic mirror symmetry. We review the differential-geometric data associated to the enumerative geometry in a variety in Section 2.3, which motivates the notion of F-bundle studied in this thesis.

Geometrically, an F-bundle $(\mathcal{H}, \nabla)$ parametrized by a base B is a vector bundle $\mathcal{H} \to B \times \mathbb{D}_u$, where $\mathbb{D}_u$ is a neighborhood of $u = 0$ in an affine line, and a meromorphic flat connection ∇ on $\mathcal{H}$ with poles at $u = 0$, such that $\nabla_{u^2\partial_u}$ and $\nabla_{u\xi}$ are regular for any tangent vector field ξ to B. We consider the cases when B is a formal scheme, and when B is a smooth non-archimedean analytic space.

We prove the following decomposition theorems in the formal and non-archimedean settings, they provide a canonical decomposition for *maximal* F-bundles with respect to the eigenvalues of the operator $K_b \coloneqq \nabla_{u^2\partial_u}|_{b,0}$, where $b \in B$ is a closed point. We refer to Chapter 4 for the notion of maximal F-bundle, which can be thought of as a weaker version of Frobenius manifolds.

Theorem 1.2.2 (Formal spectral decomposition, Theorem 4.1.1)**.** *Let B be a formal neighborhood of a rational point b in a smooth $\Bbbk$-variety, and $(\mathcal{H}, \nabla)$ an F-bundle over B maximal at b. Assume that we have a decomposition $\mathcal{H}_{b,0} \simeq \bigoplus_{i\in I} H_i$ stable under K_b, and that for any $i \neq j \in I$, the spectra of $K_b|_{H_i}$ and $K_b|_{H_j}$ are disjoint. Then $(\mathcal{H}, \nabla)/B$ decomposes into a product of maximal F-bundles $(\mathcal{H}_i, \nabla_i)/B_i$ extending the decomposition of $\mathcal{H}|_{b,0}$.*

Theorem 1.2.3 (Non-archimedean spectral decomposition, Theorem 4.1.2)**.** *Let B be an admissible open neighborhood of a rational point b in a smooth $\Bbbk$-analytic space, and $(\mathcal{H}, \nabla)$ an F-bundle over B maximal at b. Assume that we have a decomposition $\mathcal{H}_{b,0} \simeq \bigoplus_{i\in I} H_i$ stable under K_b, and that for any $i \neq j \in I$, the spectra of $K_b|_{H_i}$ and $K_b|_{H_j}$ are disjoint. Then there exists an admissible open neighborhood U of b such that $(\mathcal{H}|_U, \nabla|_U)/U$ decomposes into a product of maximal F-bundles $(\mathcal{H}_i, \nabla_i)/U_i$ extending the decomposition of $\mathcal{H}_{b,0}$.*

The next main result concerns the existence and uniqueness of framings for F-bundles. Roughly, a framing is a trivialization of $\mathcal{H}$ in which the connection ∇ has no non-negative powers of u. We prove the following non-archimedean version (see Theorem 4.1.3 for the formal version).

Theorem 1.2.4 (Theorem 4.1.4)**.** *Let B be an admissible open neighborhood of a rational point b in a smooth $\Bbbk$-analytic space. Let $(\mathcal{H}, \nabla)$ be a non-archimedean F-bundle over B. Then every framing at b extends uniquely and explicitly to a framing over an admissible open neighborhood U of b in B.*

Ultimately, the extension of framing provides reconstruction results for morphisms of F-bundles (Proposition 4.1.5). The main applications we present in this chapter concern the A-model F-bundle associated to a blowup or a projective bundle. It was proved in [Iri23; IK23] that in those two cases, the quantum D-module (an incarnation of the A-model F-bundle) splits in a way that extends the classical splitting of cohomology. In Section 4.5, we obtain uniqueness results for these decompositions, as well as partial existence results (see Theorems 4.1.8 and 4.1.10 for the case of projective bundles).

The results obtained in Chapter 4 pave the way for the development of the theory of atoms in [Kat+24]. The theory produces new birational invariants from enumerative geometry, providing new ways to study birationality problems. The theory of F-bundles also provides a clean geometric framework which can be used to express Hodge-theoritic mirror symmetry.

Mirror symmetry of flag varieties

In Chapter 5, we use the theory of F-bundles to prove Hodge-theoritic mirror symmetry for general flag varieties. On the A-side, we consider the enumerative geometry of a flag variety $X = G/P$ of general type. On the B-side, we consider a Landau-Ginzburg model, which encodes the singularity theory of a superpotential $\mathcal{W} \colon X^\vee \to \mathbb{A}^1$ defined on the mirror $X^\vee = G^\vee/P^\vee$, where $G^\vee$ and $P^\vee$ are Langlands dual to G and P respectively.

In the recent work [Cho23], a restricted version of mirror symmetry for the pair $(X, X^\vee)$ was proved after restricting the quantum cohomology to the small locus (i.e. allowing only divisor insertions in the quantum product). Our goal is to extend this correspondence to the big quantum cohomology, by constructing an appropriate unfolding of the superpotential and extending the mirror isomorphism.

While some reconstruction results are available in the literature, none of them applies to the situation we consider. Those results either require the small quantum cohomology to be H^2-generated, or to be semisimple. The former result is known as the Hertling-Manin unfolding theorem ([HM04]), and generalizes the Kontsevich-Manin reconstruction theorem for Gromov-Witten invariants ([KM94]). We observe that a flag variety X has a natural torus action T, and that when working T-equivariantly the H^2-generation condition holds for *equivariant quantum cohomology*.

The structure of F-bundle studied in Chapter 4 does not exactly fit the setup of equivariant quantum cohomology. Consequently, we introduce the notion of *equivariant F-bundle* and prove the following analogue of the Hertling-Manin unfolding theorem.

Theorem 1.2.5 (Unfolding of equivariant F-bundles, Theorem 5.1.1)**.** *Let* $\mathcal{F} = \{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}$ *be an equivariant F-bundle over* $\Bbbk[\![\mathbf{t}_I]\!]$*, and fix* $v \in \mathcal{H}_R|_{u=t_I=0}$*.*

1. If v satisfies (IC), (GC) and $\operatorname{coker} \mu_v$ *is free, then $\mathcal{F}$ admits a maximal unfolding with a cyclic vector induced from v.*

2. If v satisfies (GC'), then any two maximal unfoldings of $\mathcal{F}$ with cyclic vectors induced from v are isomorphic under a unique isomorphism.

Furthermore, any framing for $\mathcal{F}$ induces a unique framing on a maximal unfolding.

This new unfolding theorem successfully generalizes the existing reconstruction results for quantum cohomology to the equivariant setting. Applying our unfolding theorem, we are able to obtain equivariant mirror symmetry for general flag varieties (Theorem 5.1.4).

Hodge-theoritic mirror symmetry typically involves other structures, such as a pairing and an integral structure. Some of those structures are also included in the existing reconstruction results. It is natural to ask whether this reconstruction can be generalized to the equivariant setting, and we plan to address this question in the future.

Chapter 2

BACKGROUND

In this chapter, we present background material used throughout the thesis. Gromov-Witten theory is used in each chapter and in Section 2.1 we recall the definition of the moduli spaces and of Gromov-Witten invariants. The next sections discuss various aspects of mirror symmetry which are at the heart of the next chapters. In Section 2.2 we discuss mirror symmetry in the context of Calabi-Yau varieties through the lense of the SYZ conjecture, explain how certain features of non-archimedean geometry are used to implement the SYZ conjecture in the non-archimedean setting, and define the non-archimedean Gromov-Witten invariants which are studied in Chapter 3. In Section 2.3, we discuss Hodge-theoritic mirror symmetry in a more general context and relations to birational geometry, motivating Chapters 4 and 5.

2.1 Gromov-Witten theory

In this section, we present the basics of Gromov-Witten theory, which is used throughout the thesis. The standard references are [Kon95a; FP97].

Let X be a smooth proper complex variety of dimension d. One is interested in counting curves in X subject to various incidence conditions, and there are many ways of doing this ([PT14]). Gromov-Witten theory provides a way to count parametrized curves inside of X, with marked points at which incidence conditions are imposed. The basic objects one wants to count are maps $f\colon (C, p_1, \dots, p_n) \to X$, where C is a smooth proper curve with marked points $(p_1, \dots, p_n)$ and f is a smooth proper map.

2.1.1 Stable maps

To extract enumerative invariants from this situation, one defines an intersection theory on the moduli stack classifying such morphisms. There are two issues to address.

1. Transversality: a good understanding of the deformation theory of maps $f\colon (C, p_1, \dots, p_n) \to X$ is required in order to define an intrinsic normal cone to deal with non-transverse intersections, in the style of [Ful98].

2. Properness: the moduli stack needs to be proper in order to define numbers. This

requires to include degenerations of maps $(C, p_1, \dots, p_n) \to X$ in the definition of the moduli stack.

Those observations motivate Kontsevich's definition of stable maps, which allows nodal singularities in the domain curve and kills infinitesimal automorphisms by imposing a stability condition. As usual in moduli theory, one needs to study families of such objects. We give the definition over a field, but note that the theory was developed for X a scheme locally of finite presentation over a locally noetherian scheme S.

Definition 2.1.1 (Stable map, [Kon95a, §1.1],[Yu18, §7])**.** Let $\Bbbk$ be a field, X a scheme over $\Bbbk$ and T a $\Bbbk$-scheme. An *n-pointed, genus g stable map* $(C \to T, (s_i), f)$ *into X over T* consists of a morphism $C \to T$, a morphism $f\colon C \to X$ and n morphisms $s_i\colon T \to C$ such that:

1. $C \to T$ is a proper flat family of curves,
2. the geometric fibers of $C \to T$ are reduced with at worst nodal singularities, and have arithmetic genus g,
3. the n morphisms $s_i\colon T \to C$ are disjoint sections of $C \to T$ whose image is in the smooth locus of $C \to T$, and
4. (stability condition) for any geometric fiber C_t of $C \to T$, every irreducible component of C_t of genus 0 (resp. 1) has at least 3 (resp. 1) special points on its normalization, where special points are the marked points and the points coming from nodes.

We say that $(C \to T, (s_i), f)$ is an *n-pointed, genus g pre-stable map into X over T* if it satisfies conditions 1-3.

Definition 2.1.2 (Stable curve)**.** Let $\Bbbk$ be a field, T a $\Bbbk$-scheme. A (pre)-stable curve over T is a (pre)-stable map $(C \to T, (s_i), f)$ to a point, with $f\colon C \to \operatorname{Spec}\Bbbk$ being the structure morphism.

When X is projective over $\Bbbk$, the class of a stable map $(C \to \operatorname{Spec}\Bbbk, (s_i), f)$ into X is $f_*[C] \in \mathrm{NE}(X, \mathbb{Z})$. For $\beta \in \mathrm{NE}(X, \mathbb{Z})$, we denote by $\overline{\mathcal{M}}_{g,n}(X, \beta)$ the moduli stack parametrizing n-pointed, genus g stable maps to X of class β. We also denote by $\overline{\mathcal{M}}_{0,n}$ (resp. $\overline{\mathcal{M}}^{\mathrm{pre}}_{g,n}$) the moduli stack of stable (resp. pre-stable) curves.

Theorem 2.1.3 ([Kon95a],[FP97])**.** *Let $\Bbbk$ be a field of characteristic 0, X a smooth projective $\Bbbk$-variety. Fix $g, n \geq 0$ and $\beta \in \mathrm{NE}(X, \mathbb{Z})$. The moduli stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a proper Deligne-Mumford stack.*

While the moduli stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ may be singular and have components of varying dimensions, it has an expected dimension called *virtual dimension* which is given by ([Beh97])

$$\operatorname{vdim} \overline{\mathcal{M}}_{g,n}(X, \beta) = n + (1 - g)(\dim X - 3) + \beta \cdot T_X.$$

The formula is obtained by counting the difference between the number of deformation parameters of a stable map and the number of constraints a deformation must satisfy. The standard reference for deformation theory in algebraic geometry is [Har10], we refer to [Yu18, Lemma 7.14] for the description in the case of stable maps.

An intersection theory for Deligne-Mumford stacks was developed in [Vis89], generalizing [Ful98]. In order to obtain invariants, a *virtual fundamental class* is required. This is a cycle in $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \in A_{\operatorname{vdim} \overline{\mathcal{M}}_{g,n}(X,\beta)}(\overline{\mathcal{M}}_{g,n}(X, \beta))$. The theory of virtual fundamental classes was originally developed in [BF97] using the notion of perfect obstruction theory, which can be seen as a precursor of the cotangent complex for algebraic stacks. The perfect obstruction theory for Gromov-Witten theory was then constructed in [BM96; Beh97]. We refer to [Kha19] for a modern treatment.

Let $(C \to T, (s_i), f)$ be a stable map into X over T. Remembering only the data $(C \to T, (s_i))$ induces a morphism to the moduli stack $\overline{\mathcal{M}}^{\mathrm{pre}}_{g,n}$ of n-pointed, genus g pre-stable curves, called the *domain morphism*

$$\mathrm{dom}\colon \overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow \overline{\mathcal{M}}^{\mathrm{pre}}_{g,n}.$$

Composing the section s_i with f provides a tuple of *evaluation morphisms*

$$\mathrm{ev} = (\mathrm{ev}_1, \dots, \mathrm{ev}_n)\colon \overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow X^n.$$

Using those morphisms and the virtual fundamental class, we obtain *Gromov-Witten invariants* associated to any classes $\alpha_1, \dots, \alpha_n \in H^*(X, \Bbbk)$ and $\gamma \in H^*(\overline{\mathcal{M}}^{\mathrm{pre}}_{g,n}, \Bbbk)$:

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}^*(\alpha_1 \otimes \cdots \otimes \alpha_n) \cup \mathrm{dom}^* \gamma \in H^*(\operatorname{Spec} \Bbbk, \Bbbk) \simeq \Bbbk.$$

Two other types of operations on stable maps are used in Chapter 3. If $\overline{\mathcal{M}}^{\mathrm{pre}}_{g,n}(X, \beta)$ denotes the moduli stack of pre-stable maps into X of class β, contracting unstable components defines a *stabilization map*

$$\overline{\mathcal{M}}^{\mathrm{pre}}_{g,n}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta).$$

Forgetting a marked point and applying the stabilization morphism defines a *forgetful morphism*

$$\overline{\mathcal{M}}_{g,n+1}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta).$$

2.1.2 Relative Gromov-Witten theory

In Chapter 3, we are interested in counting stable maps into an open geometry U. This is done by fixing an snc compactification $U \subset X$, and using relative Gromov-Witten theory for the pair $(X, D \coloneqq X \setminus U)$.

Relative Gromov-Witten theory for a pair (X, D), where $D \subset X$ is an snc divisor, is a way to count stable maps into X with higher tangency conditions along the divisor D at the marked points. The theory was initially developed in [Li01; Li02] when D is smooth, with the goal of obtaining a degeneration formula in Gromov-Witten theory. Given a degeneration of X to a singular variety $X_1 \coprod_D X_2$ with two irreducible components meeting along a divisor D, the degeneration formula expresses the Gromov-Witten invariants of X in terms of relative Gromov-Witten invariants of (X_1, D) and (X_2, D).

Let us describe relative stable maps more precisely. Denote by $D = D_1 + \cdots + D_N$ the irreducible components of D. A *contact order* along D is a N-tuple $(p_1, \dots, p_N) \in \mathbb{N}^N$, with the i-th component specifying the tangency order with D_i. Fix $g, n \geq 0$ and contact orders $\mathbf{P} = (\mathbf{P}_i)_{1 \leq i \leq n}$. We want to count n-pointed, genus g stable maps $(C, (p_i), f\colon C \to X)$ of class β, such that the order of f at p_i along D is specified by $\mathbf{P}_i$. The moduli space of such maps is not proper, because contact orders can jump in families, and components of C can even degenerate to components mapped into D. It is possible to define a compactification $\overline{\mathcal{M}}_{g,n}(X, \mathbf{P}, \beta)$ of this moduli space. The virtual dimension of this moduli stack is

$$\begin{aligned} \operatorname{vdim} \overline{\mathcal{M}}_{g,n}(X, \mathbf{P}, \beta) &= \operatorname{vdim} \overline{\mathcal{M}}_{g,n}(X,\beta) - \beta \cdot D \\ &= n + (1-g)(\dim X - 3) + \beta \cdot T_X(-\log D). \end{aligned}$$

Constructing a suitable compactification of the moduli space and defining a virtual fundamental class is a major difficulty of the theory. Various substacks of this moduli space are considered in Section 3.2.2.2.

Associated to a relative stable map $(C, (p_i), f) \in \overline{\mathcal{M}}_{g,n}(X, \mathbf{P}, \beta)$ is a number of combinatorial data that can be encoded into a decorated graph (see [Li02; ACP15;

Yu15]). Specifically, one considers the *dual graph* of the domain curve C: to each irreducible components of C one associates a vertex, to each node an edge, and to each marked point a leg at the corresponding vertex. Each vertex is decorated with a genus and a curve class, and each edge and leg is assigned an integral vector encoding contact orders with D. Finally each vertex, edge and leg is assigned a stratum of the divisor D, to which the corresponding irreducible component, node or marked point is mapped. We loosely refer to this as the tropical data underlying a relative stable map.

2.1.3 Variants: non-archimedean, logarithmic, derived

Gromov-Witten theory has been developed beyond the classical algebro-geometric context. Specifically, in Chapter 3 we use *non-archimedean Gromov-Witten theory* and a derived enhancement of the theory.

The tropical data associated to a relative stable map arises naturally in the non-archimedean setting, as we explain in Section 2.2.3. This is a general feature of non-archimedean analytic geometry, and it can be seen as a motivation for the development of non-archimedean Gromov-Witten theory. Non-archimedean analytic stable maps were introduced in [Yu15; Yu18], where the moduli space is constructed and the properness of the moduli space is proved. The series of papers [PY24; PY22] further develops the theory in the setting of derived analytic geometry and constructs a virtual fundamental class, allowing the definition of non-archimedean Gromov-Witten invariants in full generality.

While we do not use it in this thesis we briefly mention *logarithmic Gromov-Witten theory*, which is a far-reaching generalization of relative Gromov-Witten theory. The theory enhances stable maps with a log-structure which records contact orders. The log-structure also encodes the tropical data underlying a relative stable map, and allows one to analyze combinatorics of the geometric situation using piecewise linear geometry. We refer to [GS13; Abr+25] for the theory of logarithmic stable maps, and to [Abr+20] for a discussion of the degeneration formula in the logarithmic setting.

2.2 SYZ conjecture and non-archimedean mirror construction

Mirror symmetry, in a broad sense, is a conjectured duality between geometries, and it underpins much of the work in this thesis. Various aspects of mirror symmetry are discussed in Section 2.3. In this section, we discuss the SYZ conjecture and its implementation in the non-archimedean framework. In particular, in Section 2.2.3 we define the non-archimedean cylinder counts which are studied in Chapter 3.

2.2.1 SYZ Conjecture

The SYZ conjecture, originally formulated in [SYZ96], is a conjectural construction that associates to a symplectic Calabi-Yau variety X a mirror partner $\check{X}$. It originates from the observation that given a torus T, the Jacobian variety $\check{T}$ is again a torus that realizes various predictions of mirror symmetry, and that the correspondence $T \mapsto \check{T}$ is an involution. The general conjecture is unprecise, but can be losely formulated as follows.

Conjecture 2.2.1 (SYZ mirror symmetry)**.** *Let X be a symplectic Calabi-Yau manifold. There exists continuous surjections $f\colon X \to B$, $\check{f}\colon \check{X} \to B$ and a codimension 2 submanifold $\Delta \subset B$ such that:*

1. *on $B \setminus \Delta$, the maps f and $\check{f}$ are fibrations into nonsingular special Lagrangian tori, and for each $b \in B \setminus \Delta$ the fibers $f^{-1}(b)$ and $\check{f}^{-1}(b)$ are dual tori, and*
2. *for $b \in \Delta$, the fibers $f^{-1}(b)$ and $\check{f}^{-1}(b)$ are singular special Lagrangian tori.*

Furthermore, the dual fibration $\check{f}\colon \check{X} \to B$ is obtained by counting holomorphic disks connecting singular fibers of f.

Let us comment on some aspects on this conjecture.

• It is expected that the correct statement should involve a family version. Instead of considering a Calabi-Yau variety X, one should consider a *maximally unipotent degeneration* $\mathcal{X} \to \mathbb{D}^*$ of Calabi-Yau varieties parametrized by a punctured disk. The maximally unipotent condition states that the monodromy should have a Jordan block of maximal rank. The total space $\mathcal{X}$ of the family is a non-compact Calabi-Yau manifold, and conjecturally under the maximally degenerate assumption there exists an snc compactification $\overline{\mathcal{X}}$ of $\mathcal{X}$ such that the volume form extends to a volume form with at worst logarithmic singularities along the divisor $\overline{\mathcal{X}} \setminus \mathcal{X}$. Such geometries are called *log Calabi-Yau* and are expected to be the correct setting for the SYZ conjecture.

• The last part of the conjecture is a *reconstruction proposal*, and says that the mirror should be determined by enumerating special kinds of open holomorphic curves in X which we call *holomorphc cylinders*. It motivates the introduction of the cylinder counts we study in Chapter 3. Roughly, locally on the smooth locus $B \setminus \Delta$ one can construct a dual fibration. The conjecture says that the gluing and extension data required to extend the dual fibration to the singular locus Δ can be extracted from holomorphic cylinder counts. In the family version of the SYZ conjecture, those counts should in some way be relative to the compactification divisor.

Aspects of the reconstruction problem and its relation to the Homological Mirror Symmetry conjecture were studied in [KS01; KS06]. Gross and Siebert formulated an algebro-geometric version of the reconstruction problem in [GS03; GS06], leading to what is now known as the *Gross-Siebert program*, whose goal is to understand mirror symmetry through logarithmic geometry; see [Gro13] for a survey. The program prompted the development of logarithmic Gromov-Witten theory, and culminated in the mirror constructions in [GS19; GS22].

The idea to reconstruct the dual fibration $\check{f}\colon \check{X} \to B$ from the enumerative information of X is as follows. The base B should be equipped with an *affine structure* away from the singular locus $\Delta \subset B$. This affine structure encodes piecewise linear geometry, also known as tropical geometry. The dual fibration can be constructed on affine charts, but the gluing along those charts might not be consistent. To remedy this, one encodes corrections to the gluing maps into a combinatorial object defined on B known as a *scattering diagram*. It consists of a collection of codimension 1 affine subspaces, called walls, with data attached to them called *wall-crossing functions*. Those wall-crossing functions are automorphisms of the coordinate charts, whose coefficients are defined by counting holomorphic cylinders. The consistency of the gluing can be expressed as a series of identities satisfied by wall-crossing functions. Formulated in this way, the reconstruction problem boils down to defining a suitable affine structure on B and wall-crossing functions in such a way that the resulting scattering diagram satisfies the consistency condition. We refer to [KS14; GPS10; GHS22; AB23] for a treatment of scattering diagrams in the context of mirror symmetry.

While the SYZ conjecture is compelling, in general the existence of special tori fibrations remains conjectural in the symplectic category. Recent works in this direction include [Li23; Yua22]. On the other hand, in the non-archimedean category the retraction to the skeleton provides an analogue of the SYZ fibration, motivating the development of a non-archimedean approach to mirror symmetry [KS06; KY23].

2.2.2 Formal models and skeleta of non-archimedean analytic spaces

The most important feature of non-archimedean geometry used in this thesis is the theory of formal models and their skeleta. We refer to [BGR84; Bos14; Ber90] for a general introduction to non-archimedean analytic geometry.

2.2.2.1 Formal models

We start by recalling Raynaud's theory of formal models, which provides a link between formal and non-archimedean analytic geometry. Fix a non-archimedean field $\Bbbk$ of characteristic 0, let R denote the ring of integers. For simplicity, we assume that R is a complete valuation ring of height 1, and denote by I an ideal of definition for the topology on R. The typical case is $R = \mathbb{C}[\![T]\!]$ equipped with the T-adic topology.

The R-algebra $R\langle x_1, \dots, x_n\rangle$ of *restricted power series* in $(x_1, \dots, x_n)$ is the completion of the polynomial ring $R[x_1, \dots, x_n]$ for the I-adic topology

$$R\langle x_1, \dots, x_n\rangle \coloneqq \lim_k R[x_1, \dots, x_n]/I^k.$$

It is the prototypical example of a formal model: applying $\otimes_R \Bbbk$, we obtain the Tate algebra in n-variables corresponding to the $\Bbbk$-analytic closed unit disk of dimension n. More generally, gluing together closed formal subschemes of $\operatorname{Spf} R\langle x_1, \dots, x_n\rangle$ provides formal models of $\Bbbk$-analytic spaces. Such formal schemes are called *admissible*; we give the precise definition below.

Definition 2.2.2 (Admissible formal scheme)**.**

1. A topological R-algebra A is *admissible* if it is isomorphic to a quotient $R\langle x_1, \dots, x_n\rangle/\mathfrak{a}$ with the I-adic topology, $\mathfrak{a}$ a finitely generated ideal, and if it has no I-torsion.
2. A formal R-scheme $\mathfrak{X}$ is *admissible* if there exists an open affine covering $(\mathfrak{U}_i)_{i\in I}$ of $\mathfrak{X}$ with $\mathfrak{U}_i = \operatorname{Spf} A_i$, where A_i is an admissible R-algebra.

We denote by fSch_R the category of admissible formal schemes over R, and by $\mathrm{fAn}_\Bbbk$ the category of $\Bbbk$-analytic spaces. The functor $\otimes_R \Bbbk$ from admissible R-algebras to affinoid $\Bbbk$-algebras induces Raynaud's *rigidification functor*

$$(\cdot)^{\mathrm{rig}} \colon \mathrm{fSch}_R \longrightarrow \mathrm{An}_\Bbbk.$$

To state Raynaud's theorem, we denote by S the class of admissible formal blowups in fSch_R, see [Bos14, §8.2].

Theorem 2.2.3 (Raynaud's theorem, [Bos14, Theorem 8.4.3])**.** *The rigidification functor* $(\cdot)^{\mathrm{rig}} \colon \mathrm{fSch}_R \to \mathrm{An}_\Bbbk$ *factors through the localization* $\mathrm{fSch}_R[S^{-1}]$ *of* fSch_R *at admissible formal blowups. Furthermore, this induces an equivalence of categories* *between* $\mathrm{fSch}_R[S^{-1}]$ *and the category* $\mathrm{An}_{\Bbbk}^{\mathrm{qcqs}}$ *of quasi-paracompact quasi-separated* $\Bbbk$*-analytic spaces.*

We mention the work [Ant18], which generalizes the theory of formal models to derived $\Bbbk$-analytic spaces.

2.2.2.2 Skeleta of non-archimedean analytic spaces

The study of formal models leads to the notion of *skeleton* of $\Bbbk$-analytic spaces, first introduced in [Ber99; Ber04]. Kontsevich and Soibelman in [KS01; KS06] introduce the notion of *essential skeleton* associated to a maximal degeneration of Calabi-Yau varieties, and conjecture that one should be able to construct a non-archimedean analogue of the SYZ fibration over the essential skeleton.

Roughly, associated to a good enough formal model $\mathfrak{X}$ of a $\Bbbk$-analytic space X is a polyhedral complex $\mathrm{Sk}(\mathfrak{X}) \subset X$ which is a strong deformation retract of X, and which is equipped with an affine structure. The combinatorial structure of $\mathrm{Sk}(\mathfrak{X})$ reflects the geometry of the special fiber $\mathfrak{X}_s$. It is used in [NXY19, §6] as the base parametrizing the non-archimedean SYZ fibration, under some assumption on the formal model.

As mentioned in Section 2.2.1, any implementation of the SYZ conjecture should really be about pairs (X, D) where $D \subset X$ is an snc divisor compactifying the open Calabi-Yau variety $X \setminus D$. The theory of formal models and skeleta is generalized to strictly semistable pairs in [GRW16].

Definition 2.2.4. A *formal strictly semistable pair* $(\mathfrak{X}, H)$ consists of a connected quasi-compact admissible formal R-scheme $\mathfrak{X}$ and a sum $H = H_1 + \cdots + H_S$ of distinguished effective Cartier divisors on $\mathfrak{X}$ such that $\mathfrak{X}$ is covered by formal open subset $\mathfrak{U}$ which admit an étale morphism

$$\psi \colon \mathfrak{U} \longrightarrow \mathrm{Spf}\, R\langle x_0, \dots, x_d\rangle/(x_0 \cdots x_r - \pi),$$

for $r \leq d$ and $\pi \in \Bbbk^\times$ with $|\pi| < 1$. Furthermore, the generic fiber of each H_i has irreducible support, and $H_i|_{\mathfrak{U}}$ is defined by $\psi^*(x_j)$ for some $j > r$, unless it is trivial.

Strictly semistable pairs provide the correct setting to produce formal models for an snc compactification X, while keeping track of the boundary divisor D.

Theorem 2.2.5 ([GRW16, Theorem 4.13])**.** *Let $(\mathfrak{X}, H)$ be a strictly semistable pair and let X be the generic fiber of $\mathfrak{X}$. Then there is a canonical retraction map τ from $X \setminus H$ onto the skeleton $\mathrm{Sk}(\mathfrak{X}, H)$ which extends to a proper strong deformation retraction $\widehat{\tau}$ from X^{an} onto the compactified skeleton $\widehat{\mathrm{Sk}}(\mathfrak{X}, H)$.*

It is also proved that away from a codimension 1 locus, the fibers of the retraction are $\Bbbk$-affinoid tori.

While the notion of skeleton depends on a choice of formal model, a subset called the *essential skeleton* is present in every skeleton. Its construction was outlined in [KS01] for a projective Calabi-Yau variety, and made precise in full generality in [MN15; MMS22]. If X is a smooth $\Bbbk$-algebraic variety, not necessarily proper, and $\omega \in H^0(X, K_X^{\otimes \ell})$, one defines a piecewise linear subset $\mathrm{Sk}(\omega) \subset X^{\mathrm{an}}$ as the maximum locus of $\|\omega\| \colon X^{\mathrm{an}} \to \mathbb{R}_{\geq 0}$, where $\|\omega\|$ is defined via Temkin's theory of Kähler seminorms ([Tem16, §8]).

Definition 2.2.6 ([KY23, Definition 8.13])**.** Let X be a smooth $\Bbbk$-algebraic variety, the *essential skeleton of* X is

$$\mathrm{Sk}^{\mathrm{ess}}(X) \coloneqq \bigcup_{\substack{\omega \in H^0(Y, K_Y(D)^{\otimes \ell}) \setminus 0 \\ \ell \in \mathbb{N}_{>0}}} \mathrm{Sk}(\omega) \subset X^{\mathrm{an}},$$

for an snc compactification $X \subset Y$, $D \coloneqq Y \setminus X$. This definition is independent of the compactification Y.

The essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X) \subset X^{\mathrm{an}}$ is a birational invariant, and in the log Calabi-Yau case it only depends on the canonical volume form. However, one only obtains a retraction map $X^{\mathrm{an}} \to \mathrm{Sk}^{\mathrm{ess}}(X)$ and an affine structure on $\mathrm{Sk}^{\mathrm{ess}}(X)$ after fixing a formal model. In general it is not known if $\mathrm{Sk}^{\mathrm{ess}}(X)$ arises as the skeleton associated to a formal model, and the question is related to the existence of a minimal model for pairs [NX16; BM19]. When it does, the retraction and the affine structure are canonical, and the retraction $X^{\mathrm{an}} \to \mathrm{Sk}^{\mathrm{ess}}(X)$ is a non-archimedean analogue of the SYZ fibration ([NXY19]).

2.2.3 Non-archimedean spine counts and mirror construction

In Chapter 3, we study non-archimedean analogues of the cylinder counts that appear in the SYZ conjecture. Those counts were originally defined in the surface case in [Yue16; Yu21]. Their construction was generalized to higher dimension for affine log Calabi-Yau varieties containing a dense torus in [KY23], and that is the setup we consider. We note however that in the recent work [KY24] the construction is generalized to any smooth affine log Calabi-Yau variety with maximal boundary.

We now outline the construction of these counts. Fix a smooth affine log Calabi-Yau variety U and an snc compactification (X, D) with maximal boundary, i.e. such that the boundary divisor D contains a 0-dimensional stratum. We state the toric model assumption of [KY23].

Assumption 2.2.7 (Existence of a toric model)**.** There exists a dense torus $T \subset U$ and a birational morphism $\pi\colon X \dashrightarrow X_t$ to a toric variety, which restricts to an isomorphism on the tori.

Up to blowing up a subvariety of D, we assume that the map π is a blowup map, in particular it is defined everywhere. The toric model assumption is used in two essential ways.

• The cylinder counts considered in the SYZ conjecture are open curve counts. The torus action is used to transform those open curves into closed curves by capping them, through the *toric tail condition* ([KY23, Construction 9.3]).

• The torus has a canonical tropicalization map $\tau\colon T^{\mathrm{an}} \to M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, where M is the cocharacter lattice of T, and the toric variety X_t produces an affine structure on $M \otimes_{\mathbb{Z}} \mathbb{R}$. Furthermore, under π the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(U)$ is identified with $M_{\mathbb{R}}$. In the absence of a minimal model for (X, D), the composition $\tau := \tau_t \circ \pi\colon U^{\mathrm{an}} \to M_{\mathbb{R}} \simeq \mathrm{Sk}^{\mathrm{ess}}(U)$ plays the role of a retraction to the essential skeleton ([KY23, §2]). Furthermore, the proper toric variety X_t induces a compactification $\overline{M}_{\mathbb{R}}$ of $M_{\mathbb{R}}$, and the retraction map τ extends to a map $\overline{\tau}\colon X^{\mathrm{an}} \to \overline{M}_{\mathbb{R}}$.

Consider the moduli space $\overline{\mathcal{M}}_{0,n}(X, \mathbb{P}, \beta)$ of n-pointed, genus 0 relative stable maps to (X, D) of class $\beta \in \mathrm{NE}(X, \mathbb{Z})$ with contact data along D specified by $\mathbf{P} = (\mathbf{P}_i)_{1 \leq i \leq n}$. We call marked points p_i associated to a 0 contact order *interior points*, other points are called *boundary points*. Consider the open substack $\mathcal{M}^{\mathrm{sm}}(U, \mathbb{P}, \beta) \subset \overline{\mathcal{M}}_{0,n}(X, \mathbb{P}, \beta)$ parametrizing stable maps $(C, (p_i), f)$ such that:

1. for every boundary point p_i, the image $f(p_i)$ lies in an open codimension 1 stratum of D,
2. $(C, (p_i))$ is a stable curve,

3. $f^*(T_X(-\log D))$ is a trivial vector bundle, i.e. $(C,(p_i),f)$ has unobstructed deformations, and

4. the preimage of the exceptional locus of π by f is a finite set of points without multiplicity.

Fix $1 \leq i \leq n$ corresponding to an interior point, consider the map

$$\Phi_i := (\mathrm{dom}, \mathrm{ev}_i)\colon \mathcal{M}^{\mathrm{sm}}(U, \mathbf{P}, \beta) \to \overline{\mathcal{M}}_{0,n} \times X.$$

A key observation is the following.

Proposition 2.2.8 ([KY23, Lemma 3.6])**.** *Under conditions 1-3, the map Φ_i is étale.*

Condition 4 is used to control the tropicalization of stable maps in $M_{\mathbb{R}}$.

Given an analytic stable map $(C,(p_i),f) \in \mathcal{M}^{\mathrm{sm}}(U,\mathbf{P},\beta)^{\mathrm{an}}$, its image through $\overline{\tau}$ factors through a piecewise linear map $h\colon \Gamma \to \overline{M}_{\mathbb{R}}$, where Γ is a metric graph with marked points and semi-infinite edges called legs, producing a *tropical curve* in $\overline{M}_{\mathbb{R}}$. The counts relevant for the construction of the mirror are a combinatorial refinement of Gromov-Witten invariants, parametrized by a part of the tropical curve intrinsic to f called the *spine*.

Definition 2.2.9 (Spine of a stable map)**.** Let $(C,(p_i),f) \in \mathcal{M}^{\mathrm{sm}}(U,\mathbf{P},\beta)$. The *spine* $\mathrm{Sp}(f)$ *of* f is the restriction of the map $h\colon \Gamma \to \overline{M}_{\mathbb{R}}$ to the convex hull $\Gamma^s \subset \Gamma$ of the marked points.

The spine of $(C,(p_i),f)$ is intrinsic in the following sense: for any choice of formal model for (X,D) producing a retraction to $\mathrm{Sk}^{\mathrm{ess}}(U)$, the induced tropical curve in $\mathrm{Sk}^{\mathrm{ess}}(U)$ contains $\mathrm{Sp}(f)$. The space of spines is denoted by $\mathsf{SP}(M_{\mathbb{R}},\mathbf{P})$. Denoting by $\overline{\mathsf{M}}_{0,n}$ the moduli space of stable n-pointed tropical curves ([ACP15]), and fixing $1 \leq i \leq n$ corresponding to an interior point, we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{sm}}(U,\mathbf{P},\beta)^{\mathrm{an}} & \xrightarrow{\Phi_i^{\mathrm{an}}} & \overline{\mathcal{M}}_{0,n}^{\mathrm{an}} \times U^{\mathrm{an}} \\ \Big\downarrow{\scriptstyle \mathrm{Sp}} & & \Big\downarrow \\ \mathsf{SP}(M_{\mathbb{R}},\mathbf{P}) & \xrightarrow{\Phi_i^{\mathrm{trop}}} & \overline{\mathsf{M}}_{0,n} \times M_{\mathbb{R}}. \end{array}$$

In short, the count associated to a spine $S \in \mathsf{SP}(M_{\mathbb{R}},\mathbf{P})$ is obtained by intersecting $\mathrm{Sp}^{-1}(S)$ with a fiber of Φ_i^{an} at a point lying above $\Phi_i^{\mathrm{trop}}(S)$. It is a naive count, as opposed to a virtual count, in the sense that it is an integer counting unobstructed stable maps and carrying a direct enumerative meaning. We omit many details, and refer to [KY23, §9-10] for the precise construction of the counts.

We can now define the non-archimedean cylinder counts studied in Chapter 3. They are counts of 3-pointed stable maps with two boundary points and a single interior point associated to a *cylinder spine* S and satisfying the toric tail condition. A cylinder spine is the simplest kind of spine: the underlying graph has two semi-infinite edges connected at a single vertex, and a contracted leg corresponding to the interior marked point. We refer to [KY23, §20] for the construction of the scattering diagram using cylinder counts.

2.3 Mirror symmetry

In this section, we present some aspects of mirror symmetry that motivate Chapters 4 and 5. While in Section 2.2 we discussed mirror symmetry in the Calabi-Yau case, here we introduce the general A-model and B-model. We focus on Hodge-theoritic mirror symmetry and connections to birational geometry, rather than attempting an exhaustive discussion of mirror symmetry.

2.3.1 Hodge-theoritic mirror symmetry

The notion of F-bundle studied in Chapters 4 and 5 emerges from the Hodge-theoritic approach to mirror symmetry. This approach seeks to express mirror symmetry as a duality which exchanges differential-geometric data associated to each side of the correspondence. In this thesis we only work within the algebro-geometric context. Below, we present each side of the correspondence called the A-model and B-model, and review Hodge-theoritic mirror symmetry.

The A-model encodes the enumerative geometry of a connected compact symplectic variety (X, ω), which in this thesis we will assume to be algebraic. The enumerative geometry of X is captured by genus 0 Gromov-Witten theory. Given a curve class $\beta \in \mathrm{NE}(X, \mathbb{Z})$ and cohomology classes $\gamma_1, \dots, \gamma_n \in H^*(X, \mathbb{C})$ we obtain n-pointed genus 0 Gromov-Witten invariants

$$\langle \gamma_1, \dots, \gamma_n \rangle_{0,n}^{\beta} := \int_{[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^* \gamma_1 \cup \dots \cup \mathrm{ev}_n^* \gamma_n.$$

After fixing a homogeneous basis $(T_0, T_1, \dots, T_k, \dots, T_N)$ of $H^*(X, \mathbb{C})$, with $(T_1, \dots, T_k)$ a basis of $H^2(X, \mathbb{C})$, we can define the *genus zero Gromov-Witten* *potential*

$$\Phi = \sum_{n\geq 0, \beta\in \mathrm{NE}(X,\mathbb{Z})} \frac{q^\beta}{n!} \sum_{i_1,\dots,i_n} \langle T_{i_1},\dots,T_{i_n}\rangle^\beta_{0,n} t_{i_1}\dots t_{i_n},$$

where $\{t_0,\dots,t_N\}$ are formal variables and q is a Novikov variable recording the curve class. Denoting by $(\cdot,\cdot)$ the Poincaré pairing on $H^*(X,\mathbb{C})$, we define the *(big) quantum product* by

$$(T_i \star T_j, T_r) = \frac{\partial^3\Phi}{\partial t_i \partial t_j \partial t_r} \in \mathbb{C}[\![\mathrm{NE}(X,\mathbb{Z})]\!][\![t_0,\dots,t_N]\!],$$

where $\mathbb{C}[\![\mathrm{NE}(X,\mathbb{Z})]\!]$ denotes the Novikov ring obtained as the completion of the semigroup algebra $\mathbb{C}[\mathrm{NE}(X,\mathbb{Z})]$ with respect to the ideal $(q^\beta, \beta\neq 0)$. The following fact is well-known, and proved using geometric properties of spaces of stable maps (see [KM94]).

Proposition 2.3.1. *The big quantum product defines a commutative and associative ring structure on $H^*(X,\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\![\mathrm{NE}(X,\mathbb{Z})]\!][\![t_0,\dots,t_N]\!]$ that deforms the classical cup-product around $t=0$.*

Dubrovin observed that the information of the quantum product can be conveniently stored into a differential-geometric language by introducing a meromorphic connection on $H^*(X,\mathbb{C}) \otimes \mathbb{C}[\![\mathrm{NE}(X,\mathbb{Z})]\!][\![t_0,\dots,t_N]\!]$ defined by

$$\nabla_{\partial t_i} = \partial_{t_i} + u^{-1} T_i \star \cdot,$$

where u is a formal parameter. Commutativity of the quantum product implies that this connection is flat. One can further introduce a connection operator in the u-direction as

$$\nabla_{\partial_u} = \partial_u - u^{-2}\mathbf{K} + u^{-1}\mathbf{G},$$

with

$$\mathbf{K} = \left[c_1(T_X) + \sum_{i:\ \deg T_i \neq 2} \frac{\deg T_i - 2}{2} t_i T_i\right]\star,$$
$$\mathbf{G} = \frac{1}{2}(\deg_X - \dim X),$$

where $\deg_X$ is the grading on $H^*(X,\mathbb{C})$. The u-extended connection is still a flat meromorphic connection, with a pole of order 2 at $u=0$ in the u-direction, and logarithmic singularities in the t-directions at $u=0$. In this thesis we refer to the base-changed cohomology together with the connection ∇ as the *A-model F-bundle*, but note that it is also called the quantum D-module in the literature ([Iri20; Iri23]).

The B-model is attached to a pair $(Y, W\colon Y \to \mathbb{C})$, where Y is a smooth quasiprojective complex variety and W is a holomorphic function with compact critical locus. It encodes information about the singularities of W. We refrain from giving a general definition of the B-model connection, but mention that it is defined through the *twisted de Rham complex*

$$(\Omega_Y^\bullet, d_W \coloneqq ud + dW\wedge),$$

where approriate formal variables need to be introduced. Much like the usual Gauss-Manin connection, the differental d_W induces a connection on the top-degree hypercohomology which is reffered to as the *twisted Gauss-Manin connection*. A concrete example is considered in Chapter 5. The whole data of the B-model together with its connection is referred to as the *Gauss-Manin system*. Other names found in the literature are Landau-Ginzburg model, Brieskorn lattice ([KKP17; Coa+15]).

In its simplest form, the Hodge-theoritic approach to mirror symmetry can be stated as follows.

Conjecture 2.3.2. *Let (X, ω) be a compact symplectic variety. There exists a pair $(Y, W\colon Y \to \mathbb{C})$ such that the quantum D-module of X is isomorphic to the twisted Gauss-Manin system of (Y, W).*

Using the differential-geometric language, an isomorphism of D-module consists of a change of coordinates and a gauge-equivalence between the connections. Additionally, the choice of function W on Y is expected to correspond to a choice of divisor in X. Below we give the example of $X = \mathbb{P}^1$, equipped with the toric boundary $\{0, \infty\} \subset \mathbb{P}^1$.

Example 2.3.3 (Mirror symmetry for $\mathbb{P}^1$)**.** The classical cohomology ring is $H^*(\mathbb{P}^1, \mathbb{C}) = \mathbb{C}[h]/(h^2)$, where h is the hyperplane class. The Novikov ring is isomorphic to $\mathbb{C}[\![q]\!]$, and the big quantum cohomology ring is isomorphic to

$$\mathrm{QH}(X) \simeq \mathbb{C}[\![q, t_0, t_1, u]\!][h]/(h^2 - qe^{t_1}).$$

The A-model connection is given on a cohomology class s by

$$\begin{aligned}
\nabla_{\partial_{t_0}}(s) &= \partial_{t_0} s + u^{-1} s,\\
\nabla_{\partial_{t_1}}(s) &= \partial_{t_1} s + u^{-1} h \star s,\\
\nabla_{u\partial_u}(s) &= u\partial_u s + u^{-1}(2h - t_0) \star s + \frac{1}{2}(\deg_X(s) - 2).
\end{aligned}$$

The corresponding B-model is the torus $Y = \mathbb{G}_m$ equipped with the function $W\colon x \mapsto x + \frac{qe^{t_1}}{x} + t_0$. The twisted de Rham complex on Y is the complex of $\mathbb{C}[\![q, y_0, y_1, u]\!]$-modules

$$\mathbb{C}[x, x^{-1}][\![q, t_0, t_1, u]\!] \xrightarrow{d_W} \mathbb{C}[x, x^{-1}][\![q, t_0, t_1, u]\!]dx,$$

with

$$d_W(\eta) = ud\eta + \left(1 - \frac{qe^{t_1}}{x^2}\right)\eta dx.$$

The cokernel is a free $\mathbb{C}[\![q, t_0, t_1, u]\!]$-module generated by $\{1, x\}$. The differential d_W induces a connection on $\operatorname{coker}(d_W)$ with regular singularities at $u = 0$ along ∂_{y_0} and ∂_{y_1}, given by

$$\begin{aligned}
\nabla'_{\partial_{t_0}}[\eta] &= [\partial_{t_0}\eta] + u^{-1}[\eta],\\
\nabla'_{\partial_{t_1}}[\eta] &= [\partial_{t_1}\eta] + u^{-1}\Big[\frac{qe^{t_1}}{x}\eta\Big] = [\partial_{t_1}\eta] + u^{-1}\Big[x\eta\Big],\\
\nabla'_{u\partial_u}[\eta] &= [u\partial_u\eta] + u^{-1}\Big[(2\frac{qe^{t_1}}{x} + t_0)\eta\Big] + \frac{1}{2}[\deg(\eta) - 2],
\end{aligned}$$

where $\deg(1) = 0$ and $\deg(x) = 2$. Identifying the basis $\{1, x\}$ with the basis of cohomology $\{1, h\}$ produces an isomorphism betwen the A-model and the B-model which is compatible with the connections. To see it, we use the relation $\frac{qe^{t_1}}{x} = x$ in $\operatorname{coker}(d_W)$, obtained from $[d_W(1)] = 0$.

Additional structures are expected to be reflected under mirror symmetry. For instance, the Poincaré pairing on the A-side has a counterpart on the B-side in known examples. More interesting, there is a natural integral structure on the A-side induced by $H^*(X, \mathbb{Z}) \subset H^*(X, \mathbb{C})$. It is expected to correspond to an integral structure on the B-side. In [KKP08], this additional data is axiomatized in the notion of *nc-Hodge structure* and general conjectures are made regarding the existence of nc-Hodge structures on the A-side and B-side. The notion of F-bundle that we study in this thesis is part of the data forming a nc-Hodge structure, known as the de Rham data. It was previously studied under the name of (TE)-structure in [HM99; Her02].

Let us mention that known examples of Hodge-theoritic mirror symmetry include the case of toric varieties ([Bat93; Coa+15]), as well as hypersurfaces and complete intersections inside toric varities ([Giv98]). Partial results when restricting to the small locus were recently proved for general flag varieties [MR20; Li+24; Cho23], and the main result of Chapter 5 extends mirror symmetry to the big locus.

2.3.2 Connections to birational geometry

Interactions between Gromov-Witten theory and birational geometry have been studied since the beginning of Gromov-Witten theory [Gat01; Hu00; Bay04]. There are two typical situations one wants to consider.

Question 1. *Let X be a smooth projective algebraic variety.*

1. Let $\widetilde{X} \to X$ be the blowup of X along a smooth closed subvariety $Z \subset X$. What is the relation between the Gromov-Witten theory of $\widetilde{X}$ and that of X and Z?

2. Let $V \to X$ be a vector bundle, let $\mathbb{P}(V) \to X$ be its projectivization. What is the relation between the Gromov-Witten theory of $\mathbb{P}(V)$ and that of X?

The most direct approach to this question consists in trying to express individual Gromov-Witten invariants of the top space in terms of Gromov-Witten invariants of the base. It is hard to obtain explicit formulas, but reconstruction results such as the following theorem can be obtained through recursive procedures

Theorem 2.3.4 ([Fan21; FL20])**.** *Let X be a smooth projective variety, let $V \to X$ be a vector bundle. The Gromov-Witten theory of $\mathbb{P}(V)$ is uniquely determined by the Gromov-Witten theory of X and the total Chern class $c(V)$.*

A fruitful approach is to try to compare the quantum D-modules directly, in the style of Hodge-theoritic mirror symmetry. For example in the case of a projective bundle $\mathbb{P}(V) \to X$ with V of rank $m \geq 2$, the Leray-Hirsch theorem produces the additive graded decomposition

$$H^*(\mathbb{P}(V), \mathbb{Q}) \simeq \bigoplus_{i=0}^{m-1} H^*(X, \mathbb{Q})[-2i],$$

where $[\cdot]$ indicates a degree shift. The main result of [IK23] extends this to an isomorphism between the quantum D-modules.

Theorem 2.3.5 ([IK23])**.** *The classical isomorphism deforms into an isomorphism of quantum D-modules. In particular, the quantum D-module of $\mathbb{P}(V)$ splits into m copies of the quantum D-module of X.*

A similar result for blowups is proved in [Iri23], producing a decomposition that deforms the classical isomorphism

$$H^*(\widetilde{X}, \mathbb{Q}) \simeq H^*(X, \mathbb{Q}) \oplus \bigoplus_{i=0}^{\operatorname{codim} Z-1} H^*(Z, \mathbb{Q})[-2i].$$

The main application of the general theory of F-bundles developped in Chapter 4 proves a uniqueness result for the above decompositions.

Those decompositions of the quantum D-modules are a key ingredient in the theory of atoms developped in [Kat+24]. The theory produces new birational invariants of algebraic varieties extracted from the quantum D-module, and has found applications in the proof of irrationality results. Our spectral decomposition theorem (Theorem 4.3.42) is key to establishing the general framework of the theory, by applying it to the A-model maximal F-bundle (Definition 4.2.17). The theory of atoms investigates how the decomposition of the F-bundle varies as we move the base point, hence requires some analytic theory of F-bundles. While the convergence of the quantum product in the complex analytic setting is conjectural, the convergence in the non-archimedean sense follows directly from geometric constraints on Gromov-Witten invariants. This motivates our study of F-bundles in the *non-archimedean* setting.

We conclude with a brief discussion of relations to the Homological Mirror Symmetry conjecture, from which the idea of extracting birational invariants from enumerative geometry stems. The conjecture associates to each side a derived category, and expresses mirror symmetry as a derived equivalence [Kon95b; KS01; KKP08]. Various versions of homological mirror symmetry have been obtained in the toric case ([HV00; Fan+14; HH22]). In [DKK16; DKK13], the mirror of toric varieties is studied and semi-orthogonal decompositions of the Landau-Ginzburg model category are constructed. It is shown that under homological symmetry, those decompositions produce decompositions of the A-model category which are related to the birational geometry of the A-model. Producing semiorthogonal decompositions is hard, and homological mirror symmetry is still conjectural in general. However, since homological mirror symmetry conjecturally implies Hodge-theoritic mirror symmetry (see [KKP08; GPS15]), semiorthogonal decompositions are expected to produce decompositions of the quantum D-module that contain meaningful information about the birational geometry of the variety.

Chapter 3

CYLINDER COUNTS IN BLOWUPS OF TORIC SURFACES

3.1 Introduction

3.1.1 Main result

Some algebro-geometric implementations of the SYZ picture of mirror symmetry consist in constructing explicitly a mirror algebra reflecting the enumerative geometry of the initial variety. The way this enumerative data enters in the definition of the mirror algebra is to be thought of as the instanton correction, in a broad sense.

To this day, in the log Calabi-Yau case we have at our disposal essentially two constructions: one using punctured log-Gromov-Witten invariants [Abr+25; AG22; GS19; GS22], and the other relying on non-archimedean enumerative geometry [KY23; PY24; PY22; Yue16; Yu21; Yu18; Yu15]. These constructions are known to be equivalent in restricted cases [Joh24], although a general comparison result between punctured Gromov-Witten theory and non-archimedean Gromov-Witten theory has not been achieved yet. In both of these constructions, instanton corrections enter as structure constants of the mirror algebra. Structure constants can be expressed in terms of counts of analytic cylinders in the initial variety [Gro+18; KY23]. These analytic cylinders in the non-archimedean setting correspond to the broken lines of [Gro+18].

In this paper, we are interested in computing the non-archimedean cylinder counts for a log Calabi-Yau surface $(\mathcal{Y}, \mathcal{D})$. We assume that $(\mathcal{Y}, \mathcal{D})$ is the blow up of a toric surface: this is not a restrictive assumption, as every log Calabi-Yau surface admits a toric model [GHK15, Proposition 1.3]. The idea is then to relate counts in the blown up variety to counts after we blow down a (-1)-curve in the exceptional locus of the toric model. We do it using a deformation procedure parametrized by tropical data, and use analytic geometry to cut out appropriate connected components in the moduli space of non-archimedean stable maps. Deformation invariant counts are then defined using the powerful formalism of virtual fundamental class applied to derived analytic stacks of stable maps [PY22]. The upshot is that the geometry of the derived moduli spaces reflects perfectly the axioms of Gromov-Witten theory, so we can handle degenerations of the domain in an easy way. We relate these virtual counts to the naive counts of the mirror construction [KY23] using a smoothness argument, and obtain in this way a formula involving only cylinder counts.

The advantage of this approach is that it leads to explicit, closed-form formula for the counts defining structure constants of the mirror. To the best of our knowledge, this is a new result which contrasts with the scattering diagram approach [GPS10], that gives an algorithmic way to compute the structure constants to a finite order. Closed form formulas are of interest as they allow one to compute the mirror algebra, give nontrivial relations between various invariants, and enter in the coefficients of wall-crossing functions whose expressions are in most cases conjectural [GP10].

To state the main results of the paper, let $(\mathcal{Y}, \mathcal{D})$ be a log Calabi-Yau surface with a toric model $\pi\colon (\mathcal{Y}, \mathcal{D}) \to (\mathcal{Y}_t, \mathcal{D}_t)$. The map π is a blowup of non-torus fixed points x_{ij} in the toric boundary, where the i index refers to the toric boundary component, and the j index enumerates the points in the i component. We denote by $\mathcal{E}_{ij}$ the irreducible divisor above p_{ij}, by $\mathcal{E}_i$ the union of the exceptional divisors lying above a fixed toric boundary divisor, and by $\mathcal{E}$ the full exceptional locus.

The formulas involve counts of *primitive (infinitesimal) cylinders* (Section 3.3.2). A cylinder is a tropical curve that parametrizes non-archimedean stable maps that (i) meet only two prescribed components of the boundary $\mathcal{D}$ at a single unspecified point with a given multiplicity, and (ii) have prescribed intersection numbers with each $\mathcal{E}_i$. We work with primitive cylinders, by which we mean cylinders parametrizing stable maps that have intersection number at most 1 with each $\mathcal{E}_i$. To a cylinder V and a curve class β, we associate a cylinder count $N(V, \beta)$. Our cylinder counts refine the *cylinder spine counts* defined in [KY23], in the sense that summing our counts over all possible prescriptions in condition (ii) above gives the cylinder spine count. Given a primitive cylinder V, we call the part of the tropical curve parametrizing the intersections with the exceptional divisor the *twig*, and encode the intersection numbers in a tuple of weight vectors called the *twig type*. The length of this tuple is the number of irreducible components of $\mathcal{E}$ that the associated stable maps meet. In particular twig types of length 1, as those of the cylinders that appear on the right-hand side of our formula, parametrize stable maps meeting $\mathcal{E}$ at a single point.

Theorem 3.1.1 (Theorem 3.4.1)**.** *Let V be a primitive infinitesimal tropical cylinder* *with twig type* $\mathbf{w} = (\mathbf{w}_s)_{1\leq s\leq t}$*, let* $\beta \in \mathrm{NE}(Y)$*. Then*

$$N(V,\beta) = \sum_{\beta_1+\cdots+\beta_t=\beta} \prod_{s=1}^{t} N(V_s,\beta_s),$$

where V_s *is an infinitesimal cylinder of twig type* $\mathbf{w}_s$ *(see Construction 3.4.4).*

For each component of the toric boundary, let ℓ_i denote the number of irreducible components of $\mathcal{E}_i$. An easy inspection of the curve classes leads to the following refined result.

Corollary 3.1.2 (Corollary 3.4.3)**.** *Let* V *be a primitive infinitesimal cylinder of twig type* $\mathbf{w} = (\mathbf{w}_s)_{1\leq s\leq t}$*. For each* s*, let* $E^{\mathrm{trop}}_{i(s)}$ *be the direction of the corresponding twig.*

Then there are at most $\prod_{1\leq s\leq t} \ell_{i(s)}$ *curve classes such that* $N(V,\beta) \neq 0$*. Such a curve class* β *is determined by the choice of an irreducible component* $E_{i(s)j}$ *for all* s*, and we then have*

$$N(V,\beta) = \prod_{s=1}^{t} N(V_s,\beta_s),$$

where β_s *is the curve class whose intersection number with each irreducible component of* E *is* 0 *except for* $E_{i(s)j}$*.*

This result is a first step towards expressing general cylinder counts (meeting multiple irreducible components of the exceptional divisor) in terms of counts of cylinders touching only one irreducible component of the exceptional divisor. In other words, it expresses primitive cylinder counts of an arbitrary log Calabi-Yau surface in terms of cylinder counts on a toric surface blown up at one point of the toric boundary.

Cylinder counts really depend on the interior of the log Calabi-Yau $(\mathcal{Y},\mathcal{D})$. In the case of a single blowup the interior is $\mathbb{G}_m^2 \cup (\mathcal{E}\setminus\mathcal{D})$, where $\mathcal{E}$ is a (-1)-curve and $\mathcal{D}$ is an snc anticanonical divisor, strict transform of the toric boundary upon taking a toric model. In practice, we can choose any snc compactification of the interior arising from a toric model to compute these counts.

The methods of this paper only work for primitive cylinders, that parametrize stable maps that have simple intersections with the exceptional divisor. This is because we cannot control the virtual contributions induced by higher intersection numbers solely by tropical means. Concretely, for higher multiplicities the domain curves can have "bubbles" mapped to the exceptional divisor, and these are not seen by the tropical picture. Because of this phenomenon, our argument to select connected components in the moduli spaces, which is key to proving deformation invariance of the counts, does not hold anymore.

3.1.2 Organization of the paper

The paper is organized as follows: in Section 3.2 we introduce the geometric set-up, and review non-archimedean Gromov-Witten theory. In Section 3.3 we set up conventions for tropical curves, and define cylinder counts. Section 3.4 is the main body of this work: first we define a tropical deformation, then lift it to a deformation of analytic stable maps, and finally we look at the degeneration of this deformation. We apply the deformation procedure inductively, reducing the number of blowups by one at each step.

3.1.3 Statements and Declarations

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I wish to thank my advisor, Tony Yue Yu, for his help and support throughout this project.

3.2 Notations and conventions

Let $(\mathcal{Y}, \mathcal{D})$ be a log Calabi-Yau surface over a non-archimedean field k of characteristic 0. In this section, we fix a toric model and define the associated tropicalization map. After that, we define the relevant moduli spaces of non-archimedean analytic stable maps. Even though we make use of the powerful derived formalism in Section 3.4, the important Lemma 3.2.5 allows to identify the virtual counts with the naive counts defined in [KY23].

3.2.1 Geometric setup.

3.2.1.1 Blow-up of toric surfaces.

Up to applying a toric blowup, we assume that $(\mathcal{Y}, \mathcal{D})$ admits a toric model [GHK15, Proposition 1.3]. In the counting of stable maps we consider later, applying a toric blowup does not change the counts.

By a toric model, we mean that $(\mathcal{Y}, \mathcal{D})$ is obtained as a sequence of non-toric blow-ups of toric boundary points of a complete smooth toric surface $\mathcal{Y}_t$. We choose a cyclic ordering of the irreducible components of the toric boundary $\mathcal{D}_t = \sum_{i\in I_{\mathcal{D}_t}} \mathcal{D}_{t,i}$, and denote by $\pi\colon (\mathcal{Y},\mathcal{D}) \to (\mathcal{Y}_t,\mathcal{D}_t)$ the toric model. We denote by $T_M = \mathcal{Y}_t \setminus \mathcal{D}_t$ the big torus with cocharacter lattice M.

More precisely, $(\mathcal{Y},\mathcal{D})$ is obtained from $(\mathcal{Y}_t,\mathcal{D}_t)$ as follows: fix a tuple $(\ell_1,\dots,\ell_m)$ of integers, and over each divisor $\mathcal{D}_{t,i}$ choose ℓ_i distinct points x_{ij}, not lying in any 0-strata of $\mathcal{D}_t$. Let $\pi\colon \mathcal{Y}\to\mathcal{Y}_t$ be the blowup at these points. We denote by $\mathcal{D}_i$ the strict transform of $\mathcal{D}_{i,t}$, and by $\mathcal{E}_{ij}$ the exceptional divisor lying above x_{ij}. We also set $\mathcal{D} = \sum_i \mathcal{D}_i$ and $\mathcal{E} = \sum_{i,j}\mathcal{E}_{ij}$. We thus have the relation

$$c_1(T_{\mathcal{Y}}) = \pi^* c_1(T_{\mathcal{Y}_t}) - \mathcal{E} = \mathcal{D} - \mathcal{E}.$$

We denote by Σ the fan associated to $\mathcal{Y}_t$.

3.2.1.2 Tropicalization through toric model.

From now on, we work with Berkovich analytifications which we denote with straight letters: $Y = \mathcal{Y}^{\mathrm{an}}$, $D = \mathcal{D}^{\mathrm{an}}$, $Y_t = \mathcal{Y}_t^{\mathrm{an}}$, $D_t = \mathcal{D}_t^{\mathrm{an}}$, and so on.

Since $U = Y \setminus D$ is log Calabi-Yau, we can consider its essential skeleton $\mathrm{Sk}(U)$, which comes with an integral affine structure. It is constructed from the log Calabi-Yau volume form on U using Temkin's Kähler seminorm. Similarly, we consider the skeleton $\mathrm{Sk}(T_M)$ with its canonical integral affine structure. There is an isomorphism $\mathrm{Sk}(T_M) \simeq M_{\mathbb{R}} = M \otimes \mathbb{R}$, compatible with integral affine structures meaning that $\mathrm{Sk}(T_M,\mathbb{Z}) \simeq M$. Up to applying this isomorphism, we assume $\mathrm{Sk}(T_M) = M_{\mathbb{R}}$.

Remark 3.2.1. The skeleton $\mathrm{Sk}(U)$ is naturally included inside the Clemens polytope of (Y,D), inducing an embedding $\mathrm{Sk}(U) \hookrightarrow \mathbb{R}^m_{\geq 0}$ where m is the number of components of D. Integral points of the skeleton have coordinates in $\mathbb{Z}^m_{\geq 0}$ under this inclusion, and using the identification $M \simeq \mathrm{Sk}(U,\mathbb{Z})$ the *norm* of a vector $v \in M$ is defined as the sum of the absolute value of its coordinates under the embedding $M \hookrightarrow \mathbb{Z}^m$.

We denote by $\overline{M}_{\mathbb{R}}$ the natural compactification of the essential skeleton induced by the $\mathbb{G}_m^2 \hookrightarrow Y_t$ (*cf.* Fig. 3.1 for $\mathbb{G}_m^2 \hookrightarrow \mathbb{P}^2$). We consider the tropicalization of U through the toric model π, and extend it to the compactification Y

$$\tau\colon Y \xrightarrow{\pi} Y_t \xrightarrow{\tau_t} \overline{M}_{\mathbb{R}}.$$

We refer to τ as the tropicalization map.

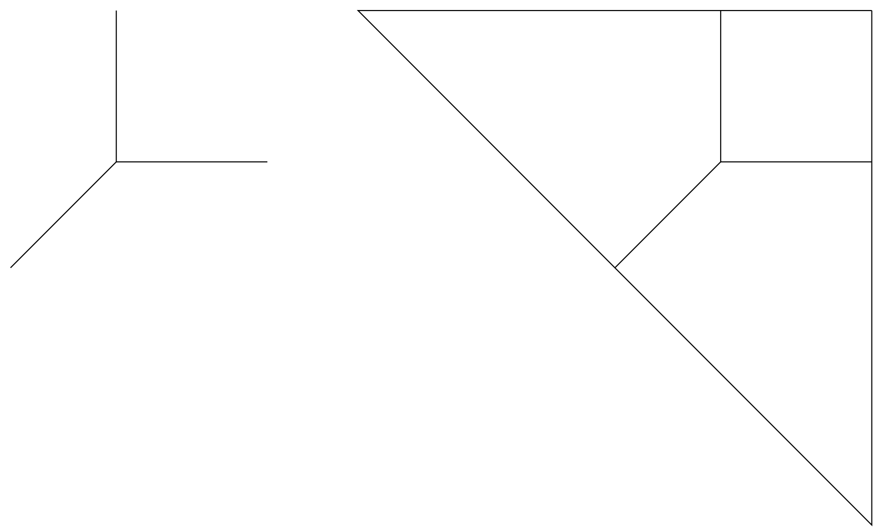


Figure 3.1: The fan of $\mathbb{P}^2$ drawed in $\mathrm{Sk}(\mathbb{G}_m^2)$, and the natural compactification $\overline{\mathrm{Sk}(\mathbb{G}_m^2)}$.

3.2.2 Non-archimedean stable maps

In this paper, we work with non-archimedean stable maps, and non-archimedean Gromov-Witten invariants. We can always recover algebraic statements using GAGA theorems for non-archimedean analytic stacks, given that Y is proper [PY16].

We refer to [PY24; PY22] for the general theory of stable maps in non-archimedean geometry. Below we recall the main definitions and results.

3.2.2.1 Derived non-archimedean stable maps.

Definition 3.2.2. Let S be a rigid k-analytic space, let $X \to S$ be a smooth rigid k-analytic space over S. We denote by $\overline{M}(X/S, \tau, \beta)$ the moduli space of (τ, β)-stable maps.

In the special case when τ is an n-valent vertex and β has genus 0, we denote this moduli space by $\overline{M}_{0,n}(X/S, \beta)$.

Theorem 3.2.3 ([PY22, Theorems 1.1, 1.2])**.** *Let S be a rigid k-analytic space and let $X \to S$ be a smooth rigid k-analytic space over S. Let (τ, β) be an A-graph. Then:*

1. *The moduli stack $\overline{M}(X, \tau, \beta)$ of (τ, β)-stable maps admits a derived enhancement $\mathbb{R}\overline{M}(X/S, \tau, \beta)$ that is a derived k-analytic stack, locally of finite presentation and derived lci over S.*

2. *If S is an algebraic variety and X is an algebraic variety over S, then*

$$\mathbb{R}\overline{M}(X/S,\tau,\beta)^{\mathrm{an}} \xrightarrow{\sim} \mathbb{R}\overline{M}(X^{\mathrm{an}}/S^{\mathrm{an}},\tau,\beta).$$

3. *The derived moduli stacks $\mathbb{R}\overline{M}(X/S,\tau,\beta)$ satisfy a list of geometric relations reflecting the Behrend-Manin axioms of Gromov-Witten theory.*

We denote by t_0 the truncation functor, so that $\overline{M}\ (X,\tau,\beta) = \mathrm{t}_0\,\mathbb{R}\overline{M}(X/S,\tau,\beta)$.

3.2.2.2 Relative derived non-archimedean stable maps.

Higher tangency conditions at the i-th marked point can be considered in the derived theory, using infinitesimal thickenings of the domain curve along the sections [PY24, §9; PY22, §8]. Given a contact order $m_i \in \mathbb{N}_{>0}$, evaluation maps with multiplicity are constructed

$$\mathrm{ev}_i^{m_i} \colon \mathbb{R}\overline{M}(X,\tau,\beta) \to X_{i,\tau}^{m_i}.$$

Using this map, one can parametrize stable maps with contact order m_i along a lci closed analytic subspace $\iota\colon Z_i \hookrightarrow Y$ at the i-th marked point by considering the substack given by the fiber product

$$\mathbb{R}\overline{M}(Y,\tau,\beta) \times_{X_{i,\tau}^{m_i}} Z_{i,\tau}^{m_i},$$

where $Z_{i,\tau}^{m_i}$ is obtained from Z_i using the same procedure of thickening along the i-th section.

We use these evaluation maps with multiplicities to construct derived versions of the moduli spaces considered in [KY23]. Let J be a finite set of cardinality n, and let $\mathbf{P} = (\mathbf{P}_j)_{j\in J}$ be a tuple of points in $\mathrm{Sk}(U,\mathbb{Z})$. Recall that points in $\mathrm{Sk}(U,\mathbb{Z})$ are valuations on $k(U)$ with integral values on $k^0(U^0)$. Define

$$B = \{j \in J \mid \mathbf{P}_j \neq 0\} \text{ and } I = \{j \in J \mid \mathbf{P}_j = 0\}.$$

For $j \in B$, we write $\mathbf{P}_j = m_j\nu_j$ where ν_j is a divisorial valuation with divisorial center $D_j \subset D$ and $m_j \in \mathbb{N}_{>0}$. Given $\beta \in \mathrm{NE}(\mathcal{Y})$, we define a sequence of derived moduli spaces:

$$\mathbb{R}M^{\mathrm{sm}}(U,\mathbf{P},\beta) \subset \mathbb{R}M^{\mathrm{sd}}(U,\mathbf{P},\beta) \subset \mathbb{R}M(U,\mathbf{P},\beta) \subset \mathbb{R}\overline{M}(Y,\mathbf{P},\beta) \subset \mathbb{R}\overline{M}(Y,\beta).$$

These moduli spaces are defined as follows:

- $\mathbb{R}\overline{M}(Y, \mathbf{P}, \beta)$ corresponds to stable maps $[C, (p_j)_{j\in J}, f\colon C \to Y]$ such that for every $j \in B$, p_j is mapped to D_j with multiplicity at least m_j.

- $\mathbb{R}M(U, \mathbf{P}, \beta)$ corresponds to the substack of stable maps $[C, (p_j)_{j\in J}, f\colon C \to Y]$ such that p_j is mapped to the open stratum D_j° for all $j \in B$, and $f^{-1}(D) = \sum_{j\in B} m_j p_j$.

- $\mathbb{R}M^{\mathrm{sd}}(U, \mathbf{P}, \beta)$ corresponds to the substack of stable maps with stable domain curve.

- $\mathbb{R}M^{\mathrm{sm}}(U, \mathbf{P}, \beta)$ corresponds to the substack of stable maps $[C, (p_j)_{j\in J}, f\colon C \to Y]$ such that:

  1. $f^*(T_Y(-\log D))$ is a trivial vector bundle on C.
  2. $f(C) \cap (D \cap E) = \emptyset$.
  3. $f^{-1}(E)$ is a finite set of points without multiplicities, disjoint from the nodes and the marked points of C.

We also consider the underived moduli spaces

$$\overline{M}(Y, \mathbf{P}, \beta) = \mathrm{t}_0\, \mathbb{R}\overline{M}(Y, \mathbf{P}, \beta),$$

and so on, which agree with the moduli spaces defined in [KY23, §3] Note that all of these stacks are analytification of the corresponding algebraic versions, that the three leftmost inclusions are (Zariski) open, and that $M^{\mathrm{sd}}(U, \mathbf{P}, \beta)$ and $M^{\mathrm{sm}}(U, \mathbf{P}, \beta)$ are varieties since we only consider rational curves.

3.2.2.3 Non-archimedean Gromov-Witten invariants.

Two conditions are needed to define numerical Gromov-Witten invariants: properness of the moduli space, to get a pushforward to a point, and a virtual fundamental class to cap cycles on the moduli space with. In the non-archimedean theory, rigid motivic Borel-Moore homology is used and a virtual fundamental class $[X/S] \in \mathrm{H}_d^{\mathrm{BM}}(X/S, \mathbb{Q}_S(2d))$ is associated to any derived lci morphism of derived analytic stack $\varphi\colon X \to S$ of virtual dimension d [PY24, Definition 4.4].

Theorem 3.2.4 ([PY24, Theorem 1.1])**.** *Let S be a rigid k-anaytic space and let $X \to S$ be a rigid k-analytic space smooth over S. Let (τ, β) be an A-graph.*

1. *There exists a virtual fundamental class*

$$[\mathbb{R}\overline{M}(X/S,\tau,\beta)] \in \mathrm{H}^{\mathrm{BM}}_d(\mathbb{R}\overline{M}(X/S,\tau,\beta)/S, \mathbb{Q}_S(2d)),$$

where d *is the virtual dimension.*

2. *The system of virtual fundamental classes satisfies the Behrend-Manin axioms of Gromov-Witten theory.*

Take $S = \operatorname{Spf} k$ and assume k has characteristic 0. Given an A-graph (τ, β), the virtual fundamental class, lci closed subvarieties $Z_i \subset X$ with contact order $m_i \in \mathbb{N}_{>0}$ and the associated diagrams

$$\begin{tikzcd}
\mathbb{R}\overline{M}(X,\tau,\beta) \arrow[r, "\mathrm{ev}_i^{m_i}"] \arrow[d, "\mathrm{st}"] & X_{i,\tau}^{m_i} & Z_{i,\tau}^{m_i} \arrow[l] \\
\overline{M}_\tau & &
\end{tikzcd}$$

one can define numerical Gromov-Witten classes and associated numerical invariants using the usual procedures [PY24, Definition 8.1]. That the subvarieties Z_i be lci is crucial to be able to define the virtual fundamental class. These classes satisfy the Behrend-Manin axioms of Gromov-Witten theory as a consequence of these same axioms for the derived moduli spaces and the functoriality properties of the virtual fundamental class.

We need the following lemma to define the relevant numerical Gromov-Witten invariants later.

Lemma 3.2.5. *The derived moduli stack* $\mathbb{R}M^{\mathrm{sm}}(U, \mathbf{P}, \beta)$ *is underived, meaning we have a canonical equivalence*

$$M^{\mathrm{sm}}(U, \mathbf{P}, \beta) \xrightarrow{\sim} \mathbb{R}M^{\mathrm{sm}}(U, \mathbf{P}, \beta).$$

In particular, it is smooth over $\overline{M}_{0,n}$.

Proof. By [KY23, Lemma 3.6], the moduli space $M^{\mathrm{sm}}(U, \mathbf{P}, \beta)$ is smooth over $\overline{M}_{0,n}$, thus its dimension equals the virtual dimension of $\mathbb{R}M^{\mathrm{sm}}(U, \mathbf{P}, \beta)$. By [PY24, Proposition 2.14], we deduce that the canonical closed immersion

$$M^{\mathrm{sm}}(U, \mathbf{P}, \beta) \hookrightarrow \mathbb{R}M^{\mathrm{sm}}(U, \mathbf{P}, \beta)$$

is an equivalence. □

3.3 Tropical curves

In this section, we review the notion of spines, tropical curves and twigs in the affine manifold $M_{\mathbb{R}}$. In Construction 3.3.7, we define an explicit topology on the space of tropical curves. Then we define the *tropical cylinders*, which are tropicalizations of the analytic cylinders we want to count, and the associated counts. These refine the spine counts defined in [KY23]. In order to keep track of the combinatorics, we define the notion of twig type associated to a cylinder.

3.3.1 Space of tropical curves

We will consider tropical curves in the $\mathbb{Z}$-affine manifold $M_{\mathbb{R}}$, which we refer to as the tropical base. These tropical curve will be used to parametrize analytic stable maps in $M^{\mathrm{sm}}(U, \mathbf{P}, \beta)$, so we require more than the usual balancing condition in their definition. Rather than giving precise definitions, which are spelled out in [KY23, §4], we illustrate the relevant notions in the 2-dimensional case through concrete examples.

Example 3.3.1. We choose as our running example the log Calabi-Yau surface (Y, D) obtained from $\mathbb{P}^2$ by blowing up the three toric divisors at 2 points. We denote by D_1, D_2, and D_3 the irreducible components of D, and by E_{11}, E_{12}, E_{21}, E_{22}, E_{31}, and E_{32} the exceptional divisors.

3.3.1.1 The canonical wall structure.

The tropical base carries a canonical wall structure denoted by $\mathrm{Wall} = \bigcup_{n\geq 0} \mathrm{Wall}^n$, which is essentially a collection of codimension 1 integral cones with an attached wall-crossing function constructed inductively in a combinatorial way. Concretely, in the 2-dimensional case, walls are rays starting from the origin of $M_{\mathbb{R}}$. The wall-crossing functions will not be considered in this article, so we omit them in the following outline of the construction (see [KY23, Construction 4.16]):

- Initial walls Wall^0: let $E_{ij}^{\mathrm{trop}} := \tau(E_{ij})$ be the tropicalization of the exceptional divisor E_{ij}. It is a point in $\partial\overline{M}_{\mathbb{R}}$ at the end of the ray corresponding to D_i. We also denote by E^{trop} the union of E_{ij}^{trop}, so Wall^0 corresponds to rays of the fan of Y_t that contain points of E^{trop}.
- Wall^{n+1} from Wall^n: add to Wall^n the rays generated by sums of two vectors in walls of Wall^n.

Example 3.3.2. For our running example (Y, D), Fig. 3.2 shows Wall^3.

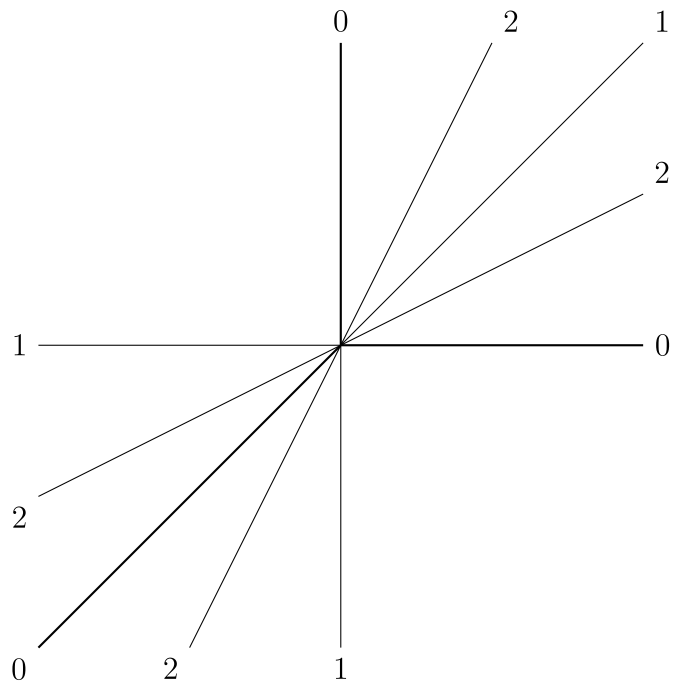


Figure 3.2: The Wall^3 part of the wall structure of Example 3.3.1 (obtained in three steps). The numbers on the rays indicate at which step of the construction a ray was added; thick rays are initial walls.

The wall structure is a necessary ingredient in our notion of tropical curves. More importantly, it carries a lot of geometric information as the structure constants of the mirror algebra can be obtained by analyzing the interaction between tropical cylinders and walls. We note that by construction, in the toric case there are no walls.

3.3.1.2 Nodal metric trees

A metric tree is a finite abstract tree Γ together with an identification of every edge e with a closed interval in $[0;\ell]$ where $\ell \in (0;+\infty]$. We refer to [KY23] for the notions of infinite and finite vertices, leg, node, irreducibility, and stability of metric trees.

Let J be a finite set of cardinality n, a metric tree with n legs $[\Gamma,(v_j)_{j\in J}]$ is a nodal metric tree Γ with 1-valent vertices $(v_j)_{j\in J}$ and no other 1-valent vertices. It is called *extended* if every v_j is infinite. It is called *simple* if there are no finite 2-valent vertices. We denote by $F\subset J$ the subset of indices corresponding to finite legs. Given a pointed tree $[\Gamma,(v_j)_{j\in J}]$, we will frequently denote by P_{ij} the path in Γ connecting the marked points v_i and v_j for $i,j\in J$.

Nodal metric trees will be used as domains of tropical curves, spines, or twigs to

$M_{\mathbb{R}}$, notions which we now define.

3.3.1.3 Tropical curves, spines, and twigs

We refer the reader to Fig. 3.3 for illustrations of the following notions.

Let J be an indexing set as above. We will always fix a partition $I \coprod B = J$ and a subset $F \subset J$. In particular, we allow $F = \emptyset$ and systematically omit it from the notations in this case. In the following definitions, when $F = \emptyset$ we call the objects *extended*.

Given a $\mathbb{Z}$-affine immersion $h\colon \Gamma \to \overline{M}_{\mathbb{R}}$ from a nodal metric tree to $\overline{M}_{\mathbb{R}}$, the slope of h at a vertex v of Γ along an edge $e \in \Gamma$ is an integral vector which we call the *weight vector* of h at v in the direction e, and denote by $w_{(v,e)}$. The *degree* of h at v along the edge e is the norm of $w_{(v,e)}$, in the sense of Remark 3.2.1.

Remark 3.3.3. If v is a vector parallel to the direction of a ray of the fan Σ_t, then the norm of v is the index of v in M (that is, the least common multiple of its coordinates).

A *tropical curve in* $M_{\mathbb{R}}$ is a $\mathbb{Z}$-affine immersion $T = [\Gamma, (v_j)_{j\in J}, h\colon \Gamma \to \overline{M}_{\mathbb{R}}]$ from an extended stable simple nodal metric tree to $\overline{M}_{\mathbb{R}}$ that is balanced at every vertex of valency greater than 1, constant on the v_i-leg for every $i \in I$, has weight vector on each v_j-leg in the direction of a ray of the fan if $j \in B$, and such that every infinite leg not labeled by a marked point is mapped to E^{trop}.
We denote by $\mathbf{P} = (\mathbf{P}_j)_{j\in J}$ the tuple of vectors given by the slope of h along the marked legs, and say that T is a tropical curve *of type* $\mathbf{P}$. Note that by definition $\mathbf{P}_j$ is nonzero if and only if $j \in B$. These tropical curves parametrize stable maps in $M^{\mathrm{sm}}(U, \mathbf{P}, \beta)$, and their space is denoted by $\mathsf{TC}(M_{\mathbb{R}},\ \mathbf{P})$.

A *spine in* $M_{\mathbb{R}}$ is a $\mathbb{Z}$-affine immersion $S = [\Gamma, (v_j)_{j\in J}, h\colon \Gamma \to \overline{M}_{\mathbb{R}}]$ from a stable nodal metric tree to $\overline{M}_{\mathbb{R}}$ with legs indexed by J, whose image meets $\partial\overline{M}_{\mathbb{R}}$ precisely at the marked points indexed by B and such that the sum of the weight vectors at each vertex v of valency greater than 1 is either 0 (h is balanced at v) or is contained in a wall (v is a bending vertex). Furthermore, marked points indexed by F correspond to finite legs.
If we denote by $\mathbf{P} = (\mathbf{P}_j)_{j\in J}$ the tuple of weight vectors of h along the legs, we say that S is a spine of type $\mathbf{P}^F$. Spines parametrize restriction of analytic stable maps

to the convex hull of the marked points in the domain curve. In particular, fixing a spine does not specify how the associated analytic stable maps meet the exceptional divisor. The space of spines of type $\mathbf{P}^F$ is denoted by $\mathsf{SP}(M_{\mathbb{R}}, \mathbf{P}^F)$.

A *twig in* $M_{\mathbb{R}}$ is a $\mathbb{Z}$-affine immersion $[\Gamma, (r, u_1, \ldots, u_t), h \colon \Gamma \to \overline{M}_{\mathbb{R}}]$ where Γ is a nodal metric tree, all the legs are marked and only the r-leg is finite, the image of Γ is contained in Wall, each u_i is mapped to E^{trop}, and h is balanced at every vertex of valency greater than 1. We refer to r as the *root* of the twig, and to the u_i as the *leaves*.
The weight vector of h at r is the *direction* of the twig, and the (ordered) tuple of weights of h at each leaves is called the *combinatorial type* of the twig.

Remark 3.3.4. These notions play well together, in the sense that given a tropical curve $[\Gamma, (v_j)_{j\in J}, h \colon \Gamma \to \overline{M}_{\mathbb{R}}]$ of type $\mathbf{P}$, if we denote by Γ^s the convex hull of the marked points in Γ then:

1. $[\Gamma^s, (v_j)_{j\in J}, h_{|\Gamma^s}]$ is a spine of type $\mathbf{P}$ [KY23, Lemma 4.23].
2. The restriction of h to the closure of connected components of $\Gamma \setminus \Gamma^s$ are twigs.

In particular, we have *spine map*

$$\mathrm{Sp} \colon \mathsf{TC}(M_{\mathbb{R}}, \mathbf{P}) \to \mathsf{SP}(M_{\mathbb{R}}, \mathbf{P}).$$

Example 3.3.5. In Fig. 3.3 several tropical curves are drawn in the skeleton. They illustrate the general fact that many twigs are compatible with a given spine, and that different twigs associated to a spine can have varying number of leaves. Furthermore, we can very often vary the degree of the leaves in such a way that the "shape" of the twig is invariant, but the degrees of the leaves become very large. For example, the shape of the twig in the bottom right corner is realized by the twig types $\{(2+n, 0), (1+n, 0), -(n, n)\}$ for every $n \in \mathbb{N}$, where the degrees are $2+n$, $1+n$ and n.

In practice to define a spine, it is enough to specify its behaviour around vertices. We can then recover an extended spine by extending the map using the $\mathbb{Z}$-affine structure of $M_{\mathbb{R}}$. This is made precise in the following construction, which we state mostly to set up notations about curve classes.

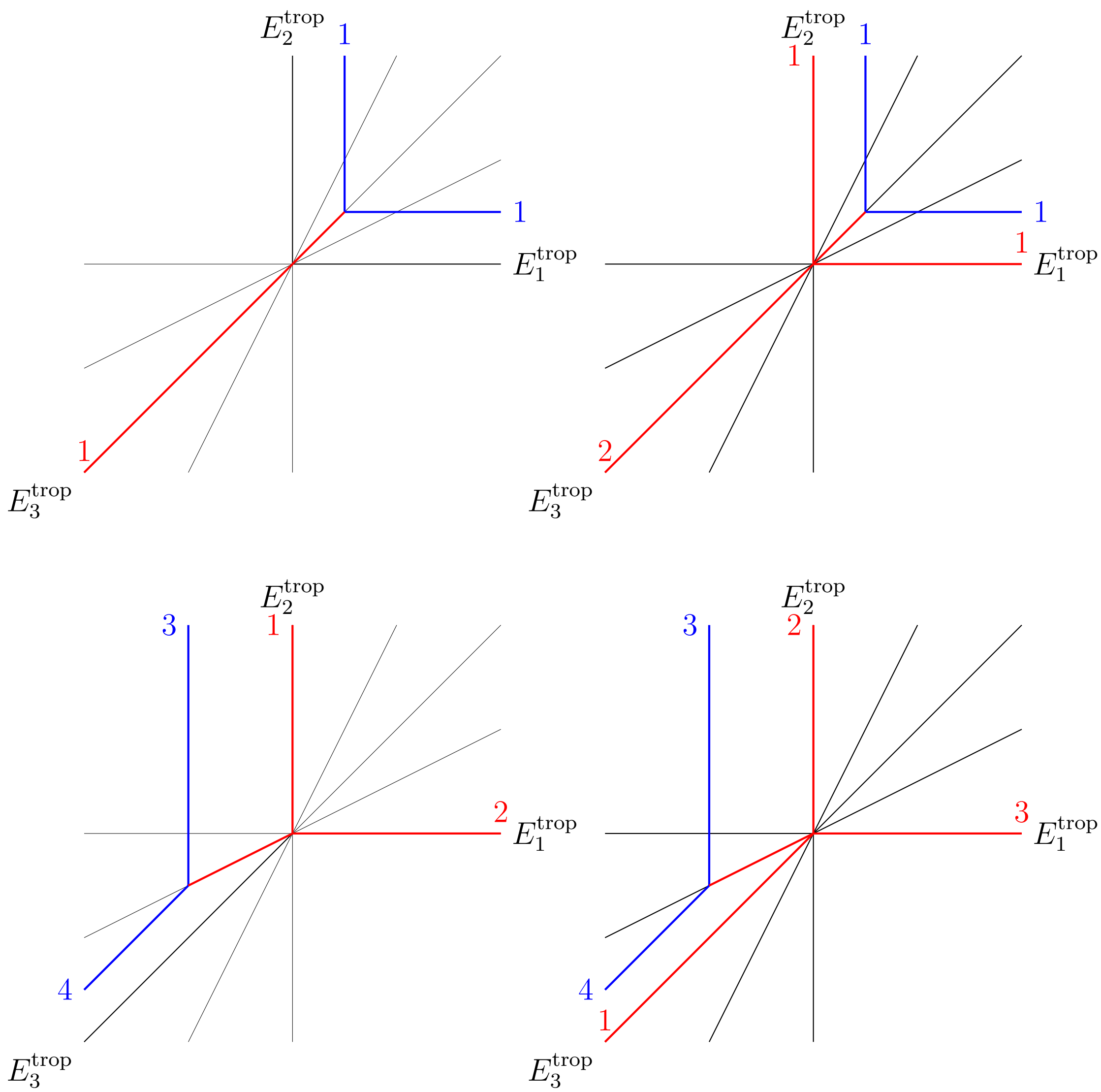


Figure 3.3: Tropical curves in $M_{\mathbb{R}}$ for Example 3.3.1. The spines are drawn in blue, and the twigs in red. The numbers correspond to the degree of the $\mathbb{Z}$-affine immersion along the legs.

Construction 3.3.6. Given an unextended spine $S = [\Gamma, (v_j)_{j\in J}, h]$ of type $\mathbf{P}^F$, the associated extended spine $\widehat{S} = [\widehat{\Gamma}, (\widehat{v}_j)_{j\in J}, \widehat{h}]$ is the spine obtained by applying the following procedure for each $j \in F$:

1. Glue a copy of $\ell_j := [0; +\infty]$ at v_j, and replace the marked point v_j by $\widehat{v}_j = \infty$ the infinite endpoint of ℓ_j.
2. Extend h affinely on ℓ_j with slope $\mathbf{P}_j$.

To each new leg, we can associate a curve class $\delta_j \in \mathrm{NE}(Y)$ using a piecewise-linear function $\varphi\colon M_{\mathbb{R}} \to N_1(Y_t, \mathbb{R})$. The curve class corresponding to the extension from S to $\widehat{S}$ is then defined as $\widehat{\delta} = \sum_{j\in F} \delta_j$. Given $\beta \in \mathrm{NE}(Y)$, the associated extended curve class is $\widehat{\beta} := \beta + \widehat{\delta}$.

3.3.1.4 Topology on $\mathsf{TC}(M_{\mathbb{R}}, \mathbf{P})$

The last ingredient we will need to parametrize the deformation procedure at the tropical level is a topology on $\mathsf{TC}(M_{\mathbb{R}}, \mathbf{P})$. We define it explicitly by giving a basis of open neighbourhoods, that we will use in Section 3.4.3 to prove that we select connected components in the relevant spaces of tropical curves. This topology was considered in the first version of [KY23], where it is proved that it is Hausdorff and that the natural tropicalization maps $M^{\mathrm{sm}}(U, \mathbf{P}, \beta) \to \mathsf{TC}(M_{\mathbb{R}}, \mathbf{P})$ are continuous. The topology is essentially given by deformations of the domain and of the image of tropical curves.

Construction 3.3.7. Let $T = [\Gamma, (v_j)_{j\in J}, h] \in \mathsf{TC}(M_{\mathbb{R}}, \mathbf{P})$, let $\varepsilon > 0$ and let $\mathcal{W} = (W_k)_{k\in K}$ be an open covering of $\overline{M}_{\mathbb{R}}$. Define a basic open neighbourhood $U(T, \varepsilon, \mathcal{W})$ of T as the set of tropical curves $[\Gamma', (v'_j)_{j\in J}, h']$ that satisfy:

1. There is a continuous map $c\colon \Gamma' \to \Gamma$ contracting a subset of topological edges of Γ', sending each v'_j to v_j and each node of Γ' to a node of Γ.
2. The sum of the length of all edges contracted by c is less then ε.
3. For each edge e of Γ, let e' be the edge of Γ' such that $c(e') = e$. If e has finite length, then the difference between the lengths of e and e' is less than ε. If e has finite length, then the length of e' is greater than $1/\varepsilon$.
4. For each vertex v' of Γ', if $h(c(v')) \in W_k$ then $h'(v') \in W_k$.

5. For each edge e' of Γ' not contracted by c, the derivative of h on $c(e')$ is equal to the derivative of h' on e'.

3.3.2 Tropical cylinders

In this subsection, we refine the infinitesimal cylinder counts of [KY23] based on the combinatorial structure of twigs.

Definition 3.3.8. Let $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2)$ be a vector of weights, such that $\mathbf{P}_1 + \mathbf{P}_2$ is parallel to the direction of a wall.

1. An *(infinitesimal) cylinder spine* of type $\mathbf{P}$ is a spine $S = [[-\varepsilon; \varepsilon], (v_1, v_2), h]$ of type $\mathbf{P}$, where 0 is the only bending vertex, and where $\varepsilon > 0$ is chosen so that the image of S intersects a single wall.
2. An *(infinitesimal) tropical cylinder* of type $\mathbf{P}$ is a balanced pointed tree in $M_{\mathbb{R}}$ obtained from an infinitesimal cylinder spine of type $\mathbf{P}$ by adding a single twig, and gluing an infinite constant leg to a point of the spine distinct from the bending vertex.

By construction, infinitesimal tropical cylinders have a single twig. We refer to the type of this twig as the *twig type of the cylinder*. We call a tropical cylinder *primitive* if the degree of every leg of the twig is equal to 1 and all the legs have a different direction.

Given an infinitesimal cylinder V, we define the associated extended tropical cylinder $\widehat{V}$ to be the tropical curve obtained after applying the extension procedure of Construction 3.3.6 to $\mathrm{Sp}(V)$.

Example 3.3.9. All the tropical curves in Fig. 3.3 are extended tropical cylinders. If we restrict the spine to a region of the domain such that the image only meets Wall at the bending vertex, then we obtain infinitesimal cylinders. Only the top right tropical cylinder is primitive.

Definition 3.3.10. Given $\beta \in \mathrm{NE}(Y)$ and a spine S of type $\mathbf{P}^F$, the count $N(S, \beta)$ is the length of a certain subset $F_w(S_w, \beta) \subset M_{0,n}(U, \mathbf{P}, \widehat{\beta})$, where S_w is obtained from S by gluing an interior leg to a point of S distinct from the bending vertex. The subset is constructed in three steps:

1. Extend the spine to $\widehat{S}_w$.

2. Look at $\mathrm{ev}_w^{-1}(h(w))$ in $M_{0,n}^{\mathrm{sm}}(U, \mathbf{P}, \widehat{\beta}) \cap \mathrm{Sp}^{-1}(\widehat{S}_w)$.

3. Consider the subset of stable maps that satisfy the toric tail condition [KY23, Construction 9.3].

It is proved in [KY23, Proposition 9.5] that this count is independent of the choice of w, by showing that it is the degree of $(\mathrm{st}, \mathrm{ev}_w)$ restricted to some closed subset of the target. In particular, for cylinder spines, the count is just the degree of some restriction of the evaluation map at w.

We now define the counts associated to an infinitesimal tropical cylinder, they refine the cylinder spine count.

Definition 3.3.11. Let V be a tropical cylinder (infinitesimal or extended), and let $\beta \in \mathrm{NE}(Y)$. Let $S = \mathrm{Sp}(V)$ be the associated cylinder spine, and let S_w be as in Definition 3.3.10. The *count of tropical cylinders* associated to V is

$$N(V, \beta) := \mathrm{length}\left(F_w(S_w, \beta) \cap \mathrm{Trop}^{-1}(\widehat{V})\right).$$

Remark 3.3.12. If V is extended, then $\widehat{V} = V$ and $\widehat{\beta} = \beta$. If V is not extended, then $N(V, \beta) = N(\widehat{V}, \widehat{\beta})$.

Remark 3.3.13. Given a cylinder spine S and a mark w, we have

$$N(S, \beta) = \sum_{V \in \mathrm{Sp}^{-1}(S_w)} N(V, \beta).$$

Alternatively, since cylinder spines only have one twig, this sum can be indexed by twig types. Given a curve class β, only finitely many twig types are realized by stable maps of class β.

3.4 Primitive holomorphic cylinders

In this section we prove the main Theorem 3.4.1 together with its Corollary 3.4.3. The theorem is proved by using a deformation procedure parametrized by tropical curves. Using ideas similar to Kontsevich's formula for plane rational curves, we define a subspace in a moduli space of analytic stable maps, and obtain an equality between counts by looking at different degenerations of the domain curve in this subspace. The subspace is defined using tropical data. To prove that the counts are deformation invariants, we express them as the degree of a map which is proper and flat over the analytic deformation.

In Section 3.4.2 we set up the notations for the tropical curves that will parametrize our deformation, and in Section 3.4.3 we define the tropical deformation. In Section 3.4.4 we pull back the tropical deformation to the analytic moduli space. The key results are Proposition 3.4.11, which shows that the tropical deformation cuts out connected components in the space of tropical curves, and Proposition 3.4.15, which shows that these connected components pull back to connected components of the smooth locus of the analytic moduli space up to restricting the domain curve. This properness argument relies on Proposition 3.4.13 which is a small generalization of [KY23, Proposition 10.1].

Finally, in Section 3.4.5 we look at different degenerations of the domain curve, and express the counts in terms of cylinder counts with one less twig together with some toric counts, which we evaluate explicitly. The key result is Proposition 3.4.19, relating counts before and after removing one twig to the tropical cylinder. Applying this formula inductively, we easily deduce Theorem 3.4.1.

3.4.1 Main results

Theorem 3.4.1. *Let V be a primitive infinitesimal tropical cylinder with twig type $\mathbf{w} = (\mathbf{w}_s)_{1 \le s \le t}$, let $\beta \in \mathrm{NE}(Y)$. Then*

$$N(V, \beta) = \sum_{\beta_1 + \cdots + \beta_t = \beta} \prod_{s=1}^{t} N(V_s, \beta_s),$$

where V_s is an infinitesimal cylinder of twig type $\mathbf{w}_s$ (see Construction 3.4.4).

The next lemma allows to simplify the sum, by identifying the curve classes contributing to non zero invariants.

Lemma 3.4.2. *Let V be a primitive infinitesimal cylinder of type $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2)$, whose twig has a single leaf of type $\mathbf{w} = \mathbf{P}_1 + \mathbf{P}_2$ in the direction of E_i^{trop}. Then, then are at most ℓ_i curve classes β such that $N(V_s, \beta) \neq 0$.*

Proof. Since the extension curve class is uniquely determined, the question is equivalent to determining β such that $M_{0,3}^{\mathrm{sm}}(U, \mathbf{P}, \beta) \cap \mathrm{Trop}^{-1}(\widehat{V}_w)$ is non empty.

A necessary condition for $M^{\mathrm{sm}}(U, \mathbf{P}, \beta)$ to be non empty is that the curve class β be compatible with $\mathbf{P}$ [KY23, Remark 3.5]. This determines the intersection numbers $\beta \cdot D_i$, so the only freedom in choosing β lies in the intersection numbers $\beta \cdot E_{ij}$.

By construction, if a stable map η in $M^{\mathrm{sm}}(U, \mathbf{P}, \beta)$ meets the exceptional divisor E_{kj} with multiplicty m, then $\mathrm{Trop}(\eta)$ has a twig in the direction E_k^{trop} of degree m. Thus, elements of $M^{\mathrm{sm}}(U, \mathbf{P}, \beta) \cap \mathrm{Trop}^{-1}(\widehat{V}_w)$ only meet a single irreducible component of E_i with multiplicty 1. There are ℓ_i such components, each giving rise to a different curve class since the intersection numbers induce an isomorphism $N_1(Y) \simeq N_1(Y_t) \oplus \mathbb{Z}^E$. □

As a direct consequence, we get:

Corollary 3.4.3. *Let V be a primitive infinitesimal cylinder of twig type $\mathbf{w} = (\mathbf{w}_s)_{1 \le s \le t}$. For each s, let $E_{i(s)}^{\mathrm{trop}}$ be the direction of the corresponding twig.*

Then there are at most $\prod_{1 \le s \le t} \ell_{i(s)}$ curve classes such that $N(V, \beta) \neq 0$. Such a curve class β is determined by the choice of an irreducible component $E_{i(s)j}$ for all s, and we then have

$$N(V, \beta) = \prod_{s=1}^{t} N(V_s, \beta_s),$$

where β_s is the curve class whose intersection number with each irreducible component of E is 0 except for $E_{i(s)j}$.

3.4.2 Initial data and notations

We fix once and for all a tropical cylinder $V = [\Gamma^0, (v_1^0, v_2^0, v_w^0), h^0]$ associated to a cylinder spine. Let $\widehat{V}$ denote the extended tropical cylinder associated to V. We assume that V is primitive of type $\mathbf{w} = (\mathbf{w}_s)_{1 \le s \le t}$. This means that each $\mathbf{w}_s$ is a primitive integral vector, and that V has a single twig with t leaves.
Let r and $(u_1, \ldots, u_t)$ denote respectively the root and the leaves of the twig. We denote by $\mathbf{w}_0 = \sum_s \mathbf{w}_s$, and let σ be the wall with direction $-\mathbf{w}_0$.

Our assumptions imply that the twig is a tree with a single $(t+1)$-vertex mapped to the origin. The k-th leaf is an interval $[0, +\infty]$, with 0 mapped to the origin and h^0 being a bijection to the ray $\mathbb{R}_{\ge 0}\mathbf{w}_k$. For each k, we fix $x_{t^k} \neq O$ and $x_{g^k} \in (O; x_{t^k})$ in $\mathbb{R}_{\ge 0}\mathbf{w}_k$. We also fix points x_w and $x_{w'}$ not lying inside walls.

Construction 3.4.4. For each $1 \le k \le t$, we define an infinitesimal cylinder V_k with twig type $\mathbf{w}_k$, *i.e.* whose twig has a single leaf of degree 1 in the $\mathbf{w}_k$ direction. We let the bending vertex be mapped to a point in $\left(0; x_g^k\right)$, and choose the contact orders of the two boundary legs such that V_k is balanced.

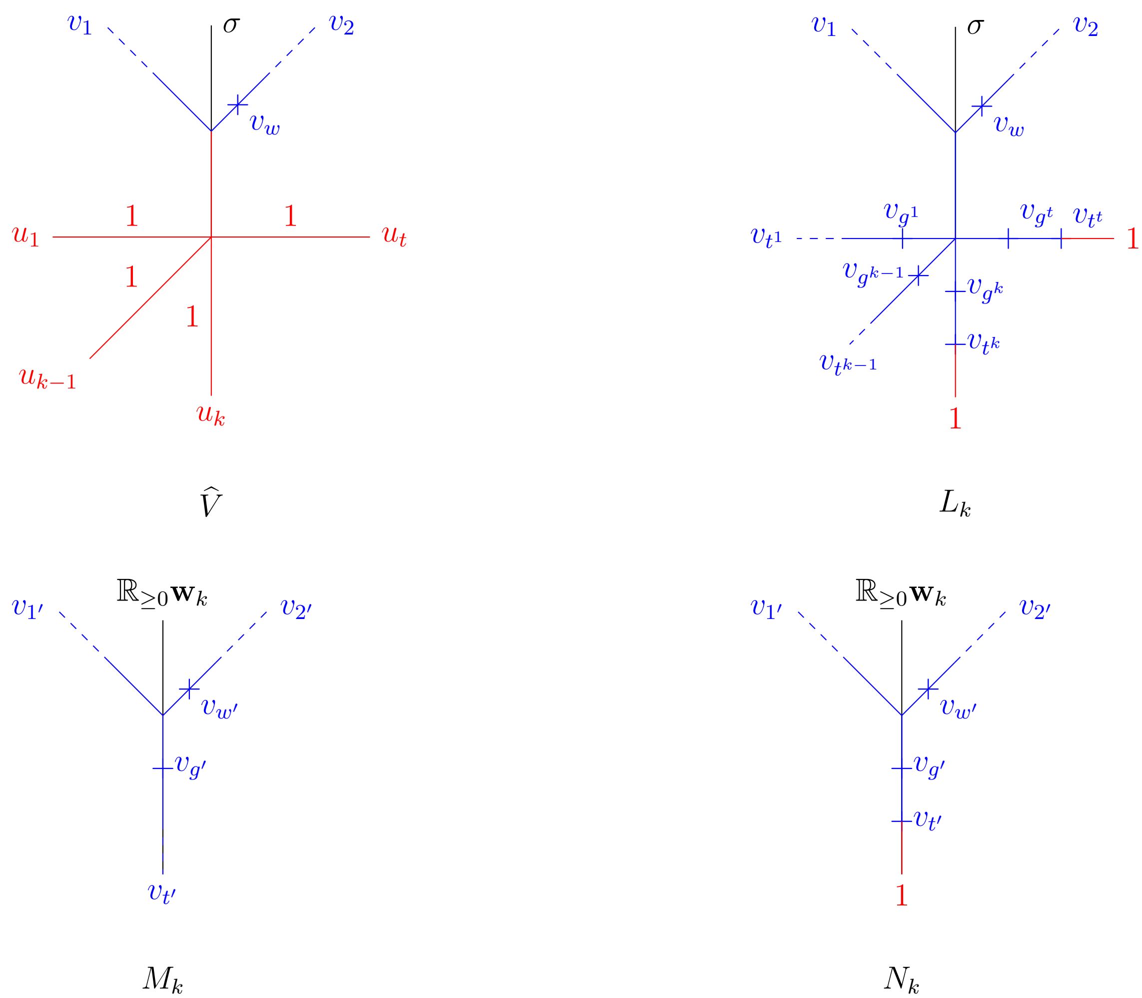


Figure 3.4: The initial tropical cylinder $\widehat{V}$, the and the families of tropical curves L_k, M_k and N_k.

We now define the following tropical curves (Fig. 3.4):

1. L_1 is obtained from $\widehat{V}$ by adding two interior points attached to every direction of the twig. More precisely for the i-th leaf, we denote by $\overline{v}_{g^i}$ (resp. $\overline{v}_{t^i}$) the unique point being mapped to x_{g^i} (resp. x_{t^i}), and glue to it the constant leg $[0, +\infty = v_{g^i}]$ (resp. $[0, +\infty = v_{t^i}]$).

2. For $2 \leq k \leq t+1$, L_k is obtained from L_{k-1} by forgetting the k-th leaf of the twig, and assigning to the map the weight w_k along the t^k-leg.

3. M_k is the balanced spine obtained from the extended tropical cylinder $\widehat{V}_k$ by substituting the leaf of the twig with a boundary marked point $v^M_{t'}$, and adding

an interior leg $v^M_{g'}$ mapped to x_{g^k} to this new leg.

4. N_k is obtained from the extended tropical cylinder $\widehat{V}_k$ by adding two interior legs attached to the twig, the new marked point $v^N_{t'}$ (resp. $v^N_{g'}$) being mapped to x_{t^k} (resp. x_{g^k}).

The curve L_k has $2t+3$ marked points, indexed by the set

$$J^L = \left\{1, 2, w, g^1, t^1, \ldots, g^t, t^t\right\}.$$

Let $B^L_0 = \{1, 2\}$ and $I^L_0 = J^L \setminus B^L_0$ denote the boundary and interior marked points of L_0. For $1 \leq k \leq t+1$, we set $B^L_k = B^L_{k-1} \cup \left\{t^{k-1}\right\}$ and $I^L_k = I^L_{k-1} \setminus \left\{t^{k-1}\right\}$. These sets index the boundary and interior marked points of L_k.

The tropical curves M_k and N_k have 5 marked points, indexed by the set $J^M = J^N = \{1', 2', w', g', t'\}$. For the M-sequence the interior points are indexed by $I^M = \{w', g'\}$ and the boundary points by $B^M = \{1', 2', t'\}$, while for the N-sequence the interior points are indexed by $I^N = \{w', g', t'\}$ and the boundary points by $B^N = \{1', 2'\}$.

Finally, we introduce

$$J^g = \left\{1, 2, 1', 2', w, w', g^1, t^1 \ldots, g^t, t^t, t'\right\} = J^L \cup J^M \setminus \{g'\}.$$

The set J^g has cardinality $2t+7$. For $0 \leq k \leq t$ we define the partition given by boundary indices

$$B^g_k = \left\{1, 2, 1', 2', t^1, \ldots, t^{k-1}, t'\right\} = B^L_k \cup B^M,$$

and interior indices

$$I^g_k = \left\{w, w', g^1, \ldots, g^{k-1}, g^k, t^k, \ldots, g^t, t^t\right\} = I^L_k \cup I^M_k \setminus \{g'\}.$$

Given a set J indexing marked points and a subset of interior indices $\tilde{I} \subset J$, we denote by $\mathrm{ev}_{\tilde{I}}$ the map given by simultaneous evaluation at marked points of $\widetilde{I}$.

For $i \in \{L, M, N, g\}$ and $1 \leq k \leq t$, denote by $\mathbf{P}^i_k$ a tuple of weights of length $n_i := |J^i|$. At the level of analytic moduli spaces, we define the following maps:

1. $\Phi^L_k = (\mathrm{st}, \mathrm{ev}_{I^L_k})\colon \overline{M}(Y^{\mathrm{an}}, \mathbf{P}^L_k, \beta) \to \overline{M}^{\mathrm{an}}_{0,2t+3} \times (Y^{\mathrm{an}})^{2t-k+2}$.
2. $\Phi^M_k = (\mathrm{st}, \mathrm{ev}_{I^M})\colon \overline{M}(Y^{\mathrm{an}}, \mathbf{P}^M_k, \beta) \to \overline{M}^{\mathrm{an}}_{0,5} \times (Y^{\mathrm{an}})^2$.

3. $\Phi_k^N = (\mathrm{st}, \mathrm{ev}_{I^N})\colon \overline{M}(Y^{\mathrm{an}}, \mathbf{P}_k^N, \beta) \to \overline{M}_{0,5}^{\mathrm{an}} \times (Y^{\mathrm{an}})^3$.

4. $\Phi_k^g = (\mathrm{st}, \mathrm{ev}_{I_k^g})\colon \overline{M}(Y^{\mathrm{an}}, \mathbf{P}_k^g, \beta) \to \overline{M}_{0,2t+7}^{\mathrm{an}} \times (Y^{\mathrm{an}})^{2t-k+3}$.

These maps tropicalize to maps between the tropicalization of spaces, which we denote by

$$\mathsf{\Phi}_k^i\colon \mathsf{TC}(M_{\mathbb{R}}, \mathbf{P}_k^i) \to \overline{\mathsf{M}}_{0,n_i} \times M_{\mathbb{R}}^{I_k^i},$$

where $\overline{\mathsf{M}}_{0,n_i}$ denotes the moduli space of pointed tropical curves [ACP15].

3.4.3 Tropical deformation

Construction 3.4.5. For $1 \leq k \leq t$, fix a primitive vector $\mathbf{w}_k'$ such that the mixed volume of $(\mathbf{w}_k, \mathbf{w}_k')$ is equal to 1. We consider the line H_k^g with direction $\mathbf{w}_k'$ going through x^{g^k}, and the line H_k^t with direction $\mathbf{w}_k'$ going through x_k^t. Their equations are given by integral affine functions on $M_{\mathbb{R}}$, that we pull back to a equations defining Cartier divisors H_k^g and H_k^t on Y. By the tropical intersection formula [Kat12], we have $H_k^g \cdot D_k = H_k^t \cdot D_k = 1$.

Construction 3.4.6. For $1 \leq k \leq t+1$, we define:

1. $\mathsf{V}_k^L = \mathsf{st}(L_k) \times x_w \times x_{g^1} \times \cdots x_{g^{k-1}} \times \mathsf{H}_k^g \times \mathsf{H}_k^t \times \cdots \times \mathsf{H}_k^t \times \mathsf{H}_k^g$,

2. $\mathsf{V}_k^M = \mathsf{st}(M_k) \times x_{w'} \times x_{g^k}$,

3. $\mathsf{V}_k^N = \mathsf{st}(N_k) \times x_{w'} \times \mathsf{H}_k^g \times \mathsf{H}_k^t$,

and set $\mathsf{TC}_k^i = (\mathsf{\Phi}_k^i)^{-1}(\mathsf{V}_k^i)$ for $i \in \{L, M, N\}$.

Proposition 3.4.7. *The point $L_k \in \mathsf{TC}_k^L$ is isolated. Similarly, $M_k \in \mathsf{TC}_k^M$ and $N_k \in \mathsf{TC}_k^N$ are isolated.*

Proof. We need to prove that L_k does not deform in TC_k^L. Let $U = U(L_k, \varepsilon, \mathcal{W})$ be an elementary open neighbourhood of L_k in $\mathsf{TC}(M_{\mathbb{R}}, \mathbf{P}_k^L)$.

We first note that the only possible deformations of the domain Γ_k^L of L_k in $\mathsf{TC}(M_{\mathbb{R}}, \mathbf{P}_k^L)$ consist in changing the length of the finite edges (in particular, moving around the roots of the twigs), and deforming the unique $(t+1)$-vertex into lower valency vertices. Up to choosing the cover $\mathcal{W}$ such that twigs of L_k are mapped to disjoint regions of $\overline{M}_{\mathbb{R}}$, we can assume that twigs remain intervals throughout every deformation in U, and that their images do not contain the origin.

Now let $K = [\Gamma, (v_j)_{j\in J_k^L}, h] \in U \cap \mathsf{TC}_k^L$, by definition we have a contraction $c\colon \Gamma \to \Gamma_k^L$. First we claim that the condition on the domain of K ensures that the $(t+1)$-valent vertex does not break into several vertices of lower valency. Otherwise, since every edge incident to this vertex lies in the spine, these multiple vertices would show in $\mathsf{st}(K)$, which would then not equal $\mathsf{st}(L_k)$. Hence the domains of K and L_k coincide as combinatorial trees (without metric structure). The weight of h on every edge is fixed, so K is completely determined by the length of finite edges and the image of a single point. Let v_0 the $(t+1)$-valent vertex in K.

For $i \geq k$, the root of the i-th twig in L_k is a 3-valent vertex. Thus it does not deform, and comes from a 3-valent vertex r_i' in Γ. Since intervals with a one-valent infinite vertex do not deform either, two of the edges incident to r_i' are fixed: one is the i-th twig, the other is the t^i-leg. Then r_i' lies in $\mathsf{H}_i^t \cap \mathbb{R}_{\geq 0} w_i = \{x_{t^i}\}$, and hence the root of the twig does not deform. The condition on v_{g^i} fixes the length of the two finite edges making up the path from v_0 to r_i'.

For $i < k$, the length of the edge connecting v_0 to the g^i-leg is fixed by the condition that v_{g^i} maps to x_{g^i}. Similarly, the condition on the image of v_w fixes the lengths of the edges in the path connecting v_w to v_0. Finally, we note that v_0 is mapped to the origin, since there are multiple edges incident to v_0 that are mapped to distinct walls, so the length of ever finite edge in Γ equals the length of the corresponding vertices in Γ_k^L.

All these observations put together show that $K = L_k$, proving the claim. The proof is similar, but simpler, for M_k and N_k. $\square$

We let $\mathsf{T}_k^L = \{L_k\}$, $\mathsf{T}_k^M = \{M_k\}$ and $\mathsf{T}_k^N = \{N_k\}$. The proposition is saying that $\mathsf{T}_k^i \subset \mathsf{TC}_k^i$ is a connected component.

Construction 3.4.8. Let $r \in [0; +\infty]$. For $1 \leq k \leq t$, consider the element $\Gamma_{k,r}$ (resp. $\Gamma_{k,r'}$) in $\overline{\mathsf{M}}_{0,2t+7}$ as in Fig. 3.5, obtained by gluing the stabilization of domains of L_k and M_k (resp. L_{k+1} and N_k) along the vertices v_{g^k} and $v_{g'}$, and varying the length of the horizontal edge, equal to the parameter r. When $r = 0$ the two abstract graphs are the same, so we obtain a path $\Delta \subset \overline{\mathsf{M}}_{0,n+7}$ parametrized by $r \in [-\infty; +\infty]$, whose marked points indexed by J^g and are partitioned into interior and boundary marked points as $J^g = I_k^g \cup B_k^g$.

Construction 3.4.9. For $1 \leq k \leq t$, we set:

$$\mathsf{V}_k^g = \Delta \times x_w \times x_{w'} \times x_{g^1} \times \cdots \times x_{g^{k-1}} \times \mathsf{H}_k^g \times \mathsf{H}_k^t \times \cdots \times \mathsf{H}_t^g \times \mathsf{H}_t^t,$$

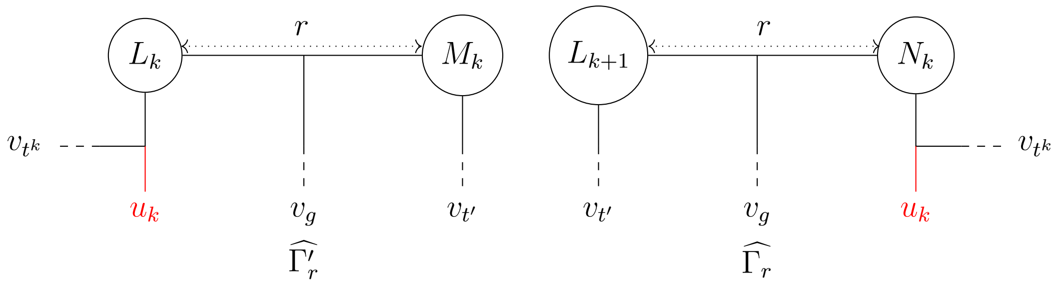


Figure 3.5: The path of tropical curves in $\overline{\mathsf{M}}_{0,2t+7}$, we only swap the t^k-leg together with the k-th twig with the t'-leg. The other twigs ($k+1$ through t) of L_k are not modified.

and define $\mathsf{TC}_k^g := (\Phi_k^g)^{-1}(\mathsf{V}_k^g)$.

We will work with tropical curves in TC_k^g. However in TC_k^g there are many tropical curves which are irrelevant to us, since we only impose conditions on the domain tropical curve and on interior points, but not on the twigs. In the next construction, we select the connected components in TC_k^g containing the relevant tropical curves for our count.

Construction 3.4.10. For $K = [\Gamma, (v_j), h] \in \mathsf{TC}_k^g$, we denote by P_{ij} the path in Γ from that vertex v_i to the vertex v_j in the spine of K.
Let T_k^g be the subset of TC^g consisting of tropical curves $K = [\Gamma, (v_j), h] \in \mathsf{TC}^g$ such that:

1. To $P_{wt^k} \cap P_{w't^k}$ is attached a single twig of degree 1, with direction $\mathbf{w}_k$.
2. For $k+1 \leq i \leq t$, to $P_{wt^i} \cap P_{g^it^i}$ is attached a single twig of degree 1, with direction $\mathbf{w}_i$.

Proposition 3.4.11. *For $1 \leq k \leq t$, the subset $\mathsf{T}_k^g \subset \mathsf{TC}_k^g$ is a union of connected components.*

Proof. We first prove that T_k^g is open. Let $K = [\Gamma, (v_j), h] \in \mathsf{T}_k^g$. Let $\varepsilon > 0$, let $\mathcal{W}$ be a finite open cover of $\overline{M}_{\mathbb{R}}$ and consider $U = U(K, \varepsilon, \mathcal{W}) \cap \mathsf{TC}_k^g$. By definition, if $K' = [\Gamma', (v'_j), h'] \in U$ then we have a continuous map $c\colon \Gamma' \to \Gamma$ contracting a subset of topological edges, sending v'_j to v_j and nodes to nodes.
Let $P_k = P_{wt^k} \cap P_{w't^k}$ in the domain of Γ', and for $i \geq k+1$ let $P_i = P_{wt^i} \cap P_{g^it^i}$ in Γ'. For $i \geq k$, the root r_i of the i-th twig does not deform since it is a 3-valent vertex.

Thus $r_i' = c^{-1}(r_i)$ is still a 3-valent vertex in Γ'. Furthermore, as intervals with an infinite 1-valent vertex do not deform either, two of the edges incident to r_i' are fixed. One of them is the t^i-leg and the other is the i-th twig. In particular r_i' is the root of the i-th twig in K', and is also the endpoint of the t^i-leg. Thus $r_i' \in P_i$, proving conditions (1) and (2) of Construction 3.4.10.

We now prove T_k^g is closed. Let $(K_\lambda)_{\lambda\in\Lambda}$ be a net in T_k^g converging to $K_\infty = [\Gamma_\infty, (v_j^\infty), h_\infty]$ in TC_k^g. Let U be a basic neighbourhood of K_∞ in TC_k^g as in the proof of openness. By definition, there exists $\lambda_0 \in \Lambda$ such that $K_{\lambda_0} = [\Gamma_{\lambda_0}, (v_j^{\lambda_0}), h_{\lambda_0}] \in U$. Thus we have a continuous map $c\colon \Gamma_{\lambda_0} \to \Gamma$ contracting a subset of edges. Conditions (1) and (2) of Construction 3.4.10 are still satisfied after contraction of some edges, thus $K_\infty \in \mathsf{T}_k^g$. $\square$

3.4.4 Analytic deformation

We proceed to defining the relevant spaces of analytic curves lying above the tropical spaces. To do this, recall the commutative diagram with tropicalization maps

$$\begin{tikzcd}
M^{\mathrm{sm}}(U, \mathbf{P}_k^i, \beta) \arrow[r, hook] \arrow[d, "\mathrm{Trop}^i"] & \overline{M}(Y, \mathbf{P}_k^i, \beta) \arrow[r, "\Phi_k^i"] & \overline{M}_{0,n_i} \times Y^{I_k^i} \arrow[d, "\mathrm{Trop}^i"] \\
\mathsf{TC}(M_{\mathbb{R}}, \mathbf{P}_k^i) \arrow[rr, "\Phi_k^i"] & & \overline{\mathsf{M}}_{0,n_i} \times \overline{M}_{\mathbb{R}}^{I_k^i}
\end{tikzcd}$$

At the level of domain curves, the tropicalization map corresponds to taking the convex hull of the marked points. For stable maps, the tropicalization map gives a well-defined map to $\mathsf{TC}(M_{\mathbb{R}}, \mathbf{P}_k^i)$ on the smooth locus only due to our notion of tropical curves.

Construction 3.4.12. Given a substack $M \subset \overline{M}(Y, \mathbf{P}_k^i, \beta)$ we denote by M^{sd} (resp. M^{sm}) its restriction to $M^{\mathrm{sd}}(U, \mathbf{P}_k^i, \beta)$ (resp. to $M^{\mathrm{sm}}(U, \mathbf{P}_k^i, \beta)$). For $i \in \{L, M, N, g\}$ and $1 \le k \le t+1$, define the following substacks:

1. $V_k^i \coloneqq \left(\mathrm{Trop}^i\right)^{-1}(\mathsf{V}_k^i)$ in $\overline{M}_{0,n_i} \times Y^{I_k^i}$,
2. $\overline{M}(V_k^i, \beta) = (\Phi_k^i)^{-1}(V_k^i)$ in $\overline{M}(Y, \mathbf{P}_k^i, \beta)$, and
3. $M(T_k^i, \beta) = \left(\mathrm{Trop}^i\right)^{-1}(\mathsf{T}_k^i) \cap M^{\mathrm{sm}}(U, \mathbf{P}_k^i, \beta)$.

By construction, we have $M(T_k^i, \beta) \subset \overline{M}(V_k^i, \beta)^{\mathrm{sm}}$. We continue to denote by $\Phi_k^i\colon M(T_k^i, \beta) \to V_k^i$ the restriction.

We have natural maps induced by forgetting marked points:

$$V_k^g \longrightarrow V_k^i,\ i \in \{L, M, N\}\,, \text{ and } V_k^g \longrightarrow V_{k+1}^L.$$

These maps are proper and flat, in particular open, since they are given by forgetting marked points and projections $Y^{I_k^g} \to Y^{I_k^i}$.

The following proposition expresses that $M(T_k^i, \beta)$ is not too far from being compact.

Proposition 3.4.13. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a net in $M(T_k^i, \beta)$, such that $(\mathrm{Trop}^i(f_\lambda))_{\lambda\in\Lambda}$ converges in TC_k^i. Then a subnet of $(f_\lambda)_{\lambda\in\Lambda}$ converges in $\overline{M}(V_k^i, \beta)^{\mathrm{sd}}$.*

Proof. By properness of $\overline{M}(Y, \mathbf{P}_k^i, \beta)$, up to passing to a subnet we may assume that (f_λ) converges to some $f_\infty = [C_\infty, (p_{j,\infty})_{j\in J}, C_\infty \to Y] \in \overline{M}(Y, \mathbf{P}_k^i, \beta)$.

We proceed as in [KY23, Proposition 10.1] by cutting the domain curves into body and caps. This decomposition is obtained by choosing a trivial family of closed disks centered at each boundary point. For $\lambda \in \Lambda \cup \{\infty\}$, denote by $\mathbb{D}_{i,\lambda}$ the (closed) cap associated to the i-th boundary marked point of f_λ and by $\mathbb{B}_\lambda$ the corresponding body.

The proof goes in three steps:

1. The boundaries of caps are mapped to a compact subset inside the torus [KY23, Claim 10.3].
2. $f_\infty(\mathbb{B}_\infty) \cap D = \emptyset$ [KY23, Claim 10.4].
3. The limit caps $\mathbb{D}_{\infty,i}$ do not have bubbles [KY23, Claim 10.5].

The first two claims carry on to our situation, but the proof of the third claim fails because of the non-transverse at infinity part of the spines we are considering. In the surface case, we can still prove that $\mathbb{D}_{i,\lambda}$ has no bubbles. Let $i \in B$ correspond to a non-transverse at infinity boundary point. For $\lambda \in \Lambda \cup \{\infty\}$, let $v_{i,\lambda}$ be the image of the i-th marked point on the domain curve of the tropicalization, and let $b_{i,\lambda}$ denote the image of $\partial\mathbb{D}_{i,\lambda}$. Let V be a compact polyhedral subset containing $h_\lambda([b_{i,\lambda}; v_{i,\lambda}])$ for all $\lambda \in \Lambda$, so that it also contains $h_\infty([b_{i,\infty}; v_{i,\infty}])$. Let $\widetilde{V} = (\pi \circ \tau_t)^{-1}(V)$. We can shrink V so that $\overline{\widetilde{V} \setminus E}$ is affinoid and only meets the irreducible component D_i of D. In addition to this affinoid domain, $\widetilde{V}$ contains a union of irreducible components of the exceptional divisor E. As $f_\lambda(\mathbb{D}_{i,\lambda}) \subset \widetilde{V}$ for all $\lambda \neq \infty$, this inclusion also holds for $\lambda = \infty$ by compactness of $\widetilde{V}$ and continuity of the universal stable map.

Let C_i denote the unique irreducible component in C_∞ intersecting both $\mathbb{B}_\infty$ and $\mathbb{D}_{i,\infty}$, which satisfies $f_{\infty,*}[C_i] \cdot D_i > 0$. Let C_b be a connected component of $\overline{\mathbb{D}_{i,\infty} \setminus C_i}$, it is a tree of $\mathbb{P}^1$. We note that each irreducible component $C' \subset C_b$ not contracted by f_∞ has image equal to an irreducible component of E, and thus contributes by $f_{\infty,*}[C'] \cdot D = \deg f_{\infty|C'} > 0$ to the intersection number $\beta \cdot D_i$. If $C' \subset C_b$ is such a component, then we have

$$1 = \beta \cdot D_i \geq f_{\infty,*}[C_i] \cdot D_i + \deg f_{\infty|C'} \geq 2.$$

This is a contradiction, so every component in C_b is contracted to a point. In turn, this contradicts the stability of f_∞ so we must have $C_b = \emptyset$. Then f_∞ has stable domain.

$\square$

We will use the next lemma to reduce enumerative computations to the smooth part of the moduli space. Defining the invariants in this way allows to interprete simple invariants as naive counts. The idea is that if families in a subspace $M^{\mathrm{sm}}(U, \mathbf{P}, \beta)$ degenerate to at worst stable maps with stable domains, then we can obtain a closed subspace in $M^{\mathrm{sm}}(U, \mathbf{P}, \beta)$ by removing stable maps with domain curves arising as degenerations.

Lemma 3.4.14. *Let $V \subset \overline{M}_{0,n} \times Y^I$, let $M_V = \Phi^{-1}(V) \subset \overline{M}(Y, \mathbf{P}, \beta)$, where $\Phi = (\mathrm{st}, \mathrm{ev}_I)$. Let $M' \subset M_V$, and assume that:*

- *M' is a union of connected components in M_V^{sm}.*
- *M' has Zariski closure in M_V contained in M_V^{sd}.*

Then there exists a Zariski open $W \subset V$, such that $M_W := \Phi^{-1}(W)$ satisfies:

1. *$M'_W := M' \cap M_W$ is a union of connected components in M_W. In particular $M'_W \subset M_V^{\mathrm{sm}}$.*
2. *The restriction $\Phi\colon M'_W \to W$ is proper.*
3. *W intersects every fiber of the first projection.*

Proof. Let $W = V \setminus \Phi(\overline{M'} \setminus M_V^{\mathrm{sm}})$, it is Zariski open by properness of Φ. Let $M_W := \Phi^{-1}(W)$.

By construction $\overline{M'} \cap M_W \subset M_V^{\mathrm{sm}}$, so the Zariski closure of M'_W in M_W lies in M_W^{sm}. Since $M' \subset M_V^{\mathrm{sm}}$ is a union of connected components, M'_W is a union of connected components in M_W^{sm}. In particular it equals its Zariski closure in M_W, thus $M'_W \subset M_W$ is a union of connected components proving (1).

Since $\Phi\colon M_V \to V$ is proper, we deduce the properness of the restriction of Φ to M'_W by base change and restriction to a union of connected components, proving (2).

(3) follows from [KY23, Lemma 3.12], the lemma states that fibers over a fixed domain curve have dense images under evaluation maps. □

We apply the previous lemma to the subspaces $M(T_k^i, \beta)$. For the degeneration argument to work, we impose a further compatibility condition on our choice of Zariski open susets.

Proposition 3.4.15. *There exist Zariski opens $W_k^i \subset V_k^i$ such that:*

1. *$M(T_k^i, \beta)_{W_k^i}$ is contained in $\overline{M}(V_k^i, \beta)^{\mathrm{sm}}$.*
2. *The restriction $\Phi_k^i\colon M(T_k^i, \beta)_{W_k^i} \to W_k^i$ is proper.*
3. *W_k^i intersects every fiber of the first projection map.*
4. *The forgetful maps $W_k^g \to W_k^i$ and $W_k^g \to W_{k+1}^L$ are surjective.*

Proof. By Proposition 3.4.11 and Proposition 3.4.13, we can apply Lemma 3.4.14 with $V = V_k^i$ and $M' = M(T_k^i, \beta)$. This gives Zariski opens $\widetilde{W}_k^g$, $\widetilde{W}_k^i$ and $\widetilde{W}_{k+1}^L$ that satisfy (1), (2), and (3).

We then let

$$W_k^g = \widetilde{W}_k^g \cap \mathrm{Fgt}^{-1}\widetilde{W}_k^L \cap \mathrm{Fgt}^{-1}\widetilde{W}_k^M \cap \mathrm{Fgt}^{-1}\widetilde{W}_k^N \cap \mathrm{Fgt}^{-1}\widetilde{W}_{k+1}^L.$$

This is an open subset, and noting that the forgetful maps are proper and flat thus open, we define the open subsets

$$W_k^i = \mathrm{Fgt}(W_k^g) \subset \widetilde{W}_k^i \text{ and } W_{k+1}^L = \mathrm{Fgt}(W_k^g) \subset \widetilde{W}_{k+1}^L.$$

Conditions (1) and (2) hold by pullback, condition (3) holds by [KY23, Lemma 3.12], and (4) holds by construction. □

We denote by $M(T^i_k,\beta)_W$ the fiber product $M(T^i_k,\beta)\times_{V^i_k} W^i_k$. The following corollary is precisely what we need to define deformation invariant counts in the next subsection.

Corollary 3.4.16. *Let:*

1. $\mathrm{Fgt}^L_k\colon \overline{M}_{0,2t+3}\times Y^{2t-k+2}\to Y^{2t-k+2}$ *denote the map forgetting every marked point except* w, g^k, t^k.

2. $\mathrm{Fgt}^M_k\colon \overline{M}_{0,5}\times Y^2\to Y^2$ *denote the map forgetting every marked point except* w', g', t'.

3. $\mathrm{Fgt}^N_k\colon \overline{M}_{0,5}\times Y^3\to Y^3$ *denote the map forgetting every marked point except* w', g', t'.

4. $\mathrm{Fgt}^g_k\colon \overline{M}_{0,2t+7}\times Y^{2t-k+3}\to \overline{\mathcal{M}}_{0,5}\times Y^{2t-k+3}$ *denote the map forgetting every marked point except* w, w', g^k, t^k, t'.

The composition $\Psi^i_k := \mathrm{Fgt}^i_k\circ\Phi^i_k$ *restricted to* $M(T^i_k,\beta)_{W^i_k}$ *is proper and flat.*

Proof. By Proposition 3.4.15, the map Φ^i_k is proper. It is also smooth by [KY23, Lemma 3.6]. Forgetful maps between moduli space of stable curves correspond to universal families. Thus they are proper and flat, so Fgt^i_k is proper and flat. We deduce that the composition Ψ^i_k is proper and flat. □

3.4.5 Enumerative invariants and degeneration

We now define enumerative invariants associated to the various spaces constucted. The definitions circumvents fundamental class because we managed to restrict to the smooth locus of the moduli spaces.

Recall that the dimension of the moduli space $M^{\mathrm{sm}}(U,\mathbf{P},\beta)$ for full-tangency n-pointed stable maps is $n-3+\dim Y = n-1$ for the surface case. We denote by $q^i\colon M(T^i_k,\beta)\to \mathrm{pt}$ the structure morphism, which is proper.

Given a (derived) k-analytic space over a point $q\colon X\to\mathrm{pt}$, the *motivic cohomology groups* $H^{2r}(X,\mathbb{Q}(r))$ are defined as the Borel-Moore homology of the identity morphism $\mathrm{id}_X\colon X\to X$. Throughout, when q is proper derived lci and $\gamma\in H^*(X,\mathbb{Q}(*))$ we use the virtual fundamental class $[X]$ to define

$$\int_X \gamma := q_*(\gamma\cap[X])\in\mathbb{Q}.$$

Denote by $\mathrm{pt} \in H^4(Y, \mathbb{Q}(2))$ and by $\mathrm{pt}_\mu \in H^{2(n-3)}(\overline{M}_{0,n}, \mathbb{Q}(n-3))$ point classes, and let $\delta_s^\ell = [H_s^\ell] \in H^2(Y, \mathbb{Q}(1))$ for $1 \leq s \leq t$ and $\ell \in \{g, t\}$.

Definition 3.4.17. Let $\beta \in \mathrm{NE}(Y)$, we define:

1. $N(T_k^g, \beta) \coloneqq \int_{M(T_k^g,\beta)_W} (\Psi_k^g)^* \left(\mathrm{pt}_\mu \boxtimes_{I_k^g \setminus \{g^s,t^s\}_{s\geq k}} \mathrm{pt} \boxtimes_{k\leq s\leq t} (\delta_s^g \boxtimes \delta_s^t)\right)$.
2. $N(L_k, \beta) = \int_{M(T_k^L,\beta)_W} \left(\Psi_k^L\right)^* \left(\boxtimes_{i\in I_k^l \setminus\, \{g^s,t^s\}_{s\geq k}} \mathrm{pt} \boxtimes_{k\leq s\leq t} (\delta_s^g \boxtimes \delta_s^t)\right)$.
3. $N(M_k, \beta) = \int_{M(T_k^M,\beta)_W} \left(\Psi_k^M\right)^* (\mathrm{pt} \boxtimes \mathrm{pt})$.
4. $N(N_k, \beta) = \int_{M(T_k^N,\beta)_W} \left(\Psi_k^N\right)^* (\mathrm{pt} \boxtimes \delta_k^g \boxtimes \delta_k^t)$.

We relate these invariants by computing $N(T_k^g, \beta)$ at different degenerations of the domain curve (Fig. 3.6). To characterize these degenerations, we only need to remember the shape of the domain curve. This is why we introduced the forgetful maps and defined our invariants through the maps Ψ_k^i.

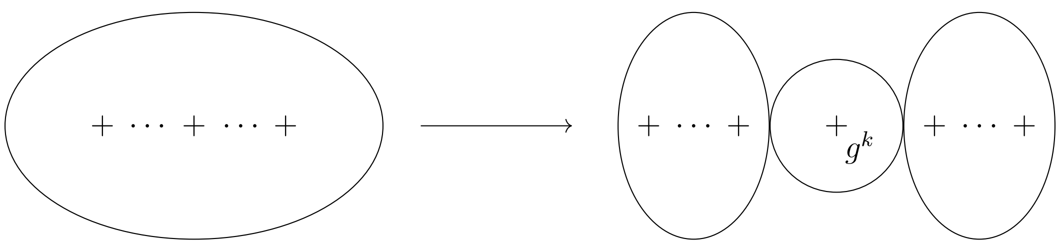


Figure 3.6: Degeneration of the domain curve. The middle component is contracted to a point in H_k^g.

We will need to keep track of the extension curve classes, so we introduce the following notations:

- $\widehat{\delta}_V$ is the extension curve class corresponding to the extension from V to $\widehat{V}$.
- Let V_k be the infinitesimal cylinder of twig type w_k of Construction 3.4.4, whose extension is N_k. We denote by $\widehat{\delta}_k$ the extension curve class.
  By construction, if we truncate the $1'$ and $2'$ legs of M_k to finite legs whose image intersect at most one wall, then the associated extension curve class is precisely $\widehat{\delta}_k$.
- Consider the infinitesimal spine obtained by truncating every boundary leg of L_{t+1} to a finite leg whose image intersects at most one wall. The corresponding extension curve class is $\widehat{\delta}_V + \sum_{s=1}^t \delta_s$, where δ_s is associated to the t^s-leg.

The next lemma computes the toric counts that appear in the inductive formula of Proposition 3.4.19.

Lemma 3.4.18. *For* $1 \leq k \leq t$ *and* $\beta \in \mathrm{NE}(Y)$*, we have*

$$N(M_k, \beta) = \begin{cases} 1 & \text{if} \quad \beta = \widehat{\delta}_k, \\ 0 & \text{else}. \end{cases}$$

Proof. By construction of W_k^M, the moduli space $M(T_k^M, \beta)$ is contained in the smooth locus $M^{\mathrm{sm}}(U, \mathbf{P}_k^M, \beta)$. Thus the evaluation maps $\mathrm{ev}_{w'}$ and $\mathrm{ev}_{g'}$ are étale [KY23, Lemma 3.6], and the invariant is the cardinality of the intersection of two fibers of these maps over arbitrary points.

Let F denote the fiber of $(\mathrm{ev}_{w'}, \mathrm{ev}_{g'})$ over the points $(x_{w'}, x_{g^k}) \in M_{\mathbb{R}} \subset U$ (recall that the essential skeleton is naturally included in Y). Any $f \in F$ is skeletal, meaning it has image contained in $\overline{M}_{\mathbb{R}}$, by [KY23, Theorem 8.18].

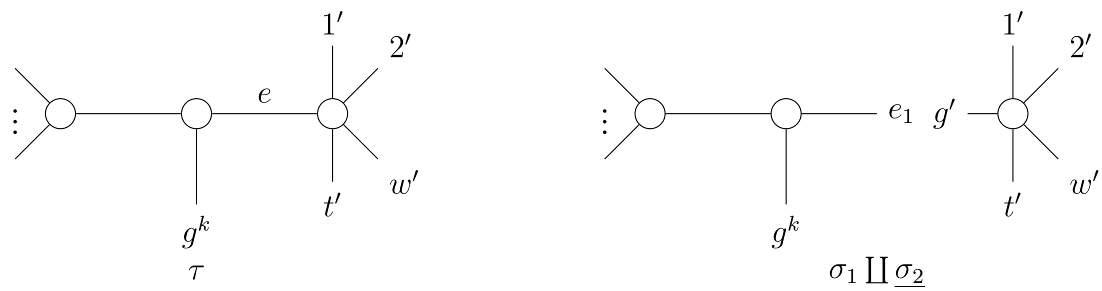
Let Γ_k^M denote $\mathrm{st}(M_k)$, we claim that $F \subset \mathrm{st}^{-1}(\Gamma_k^M)$. Indeed, for $f \in F$ the stablization $\mathrm{st}(f)$ is obtained by taking the convex hull of the marked points. Here, the image of f is fixed, equal to the image of M_k, and the domain of f does not have any nodes. This fixes the combinatorial type and the slope of f on every edge of the domain of f. Then $\mathrm{st}(f)$ is completely determined by the choice of the length of the two finite edges. These lengths can be uniquely recovered from the image of the two interior marked points and the slopes of f.

The previous observation implies that F is contained in $(\mathrm{st}, \mathrm{ev}_{w'})^{-1}(\Gamma_k^M, x_{w'}) \cap \mathrm{Sp}^{-1}(M_k)$. Stable maps in this set do not meet the exceptional locus E, so they correspond uniquely to stable map in $M^{\mathrm{sm}}(U_t, \mathbf{P}_k^M, \pi_*\beta)$. In the toric case, the map $(\mathrm{st}, \mathrm{ev}_{w'})$ is an open immersion with image $M_{0,5} \times U$ [KY23, Proposition 6.2]. Thus F has cardinality at most 1, and it is non empty if and only if $\beta = \widehat{\delta}_k$, proving the lemma. □

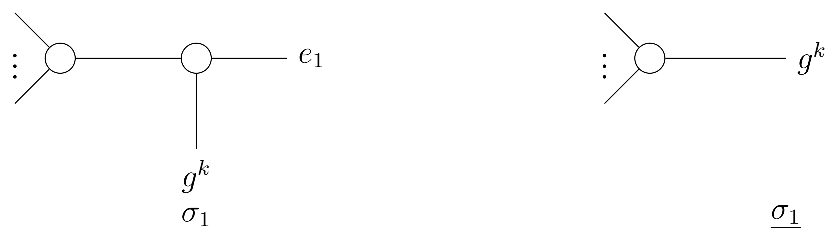
Proposition 3.4.19. *Same notations as in Lemma 3.4.18. For* $\beta \in \mathrm{NE}(Y)$ *and* $1 \leq k \leq t$*, we have*

$$N(L_k, \beta - \widehat{\delta}_k) = N(T_k^g, \beta) = \sum_{\beta = \beta_1 + \beta_2} N(L_{k+1}, \beta_1) N(N_k, \beta_2).$$

Proof. Let us prove the first equality. We fix $\mu \in \overline{M}_{0,5}$ given by the partition $(w, t^k | g^k | w', t')$, and let $M(T_k^g, \beta)_{W,\mu}$ denote the substack of stable maps over μ. For



(a) Cutting the edge e of τ to obtain σ.



(b) Forgetting the tail e_1 to obtain σ'.

Figure 3.7: Degeneration of the combinatorial type.

$i \in I_k^g$, we denote by $\gamma_i \in H^*(X, \mathbb{Q}(*))$ a cycle represented by a smooth subvariety Z_i of Y. We assume $Z_{g^k} = H_k^g$.

We shall use (τ, β)-marked stable maps to keep track of the combinatorics of the degeneration [BM96; PY24; PY22]. By construction, stable maps in $M(T_k^g, \beta)_{W,\mu}$ have a fixed graph type τ. Fix a decomposition of β into effective curve classes $\beta = \beta_1 + \beta_2$. Denote by $\underline{\tau} = (\tau, (\beta_1, 0, \beta_2))$ the associated A-graph.

Consider the moduli space

$$M^g(\underline{\tau}) := M(T_k^g, \beta)_{W,\mu} \bigcap_{i \in I_k^g} \mathrm{ev}_i^{-1}(Z_i) \cap \overline{M}(Y, \underline{\tau}).$$

Denote by $q\colon M^g(\underline{\tau}) \to \mathrm{pt}$ its structure morphism. Let $\underline{\sigma} = \underline{\sigma_1} \coprod \underline{\sigma_2}$ denote the new A-graph obtained by cutting the edge connecting the middle vertex to the rightmost vertex and forgetting the new tail (Fig. 3.7). Consider the moduli spaces

$$M^1(\underline{\sigma_1}) := M(T_k^L, \beta_1)_W \bigcap_{i \in I_k^L} \mathrm{ev}_i^{-1}(Z_i) \cap \overline{M}(Y, \underline{\sigma_1}),$$

$$M^2(\underline{\sigma_2}) := M(T_k^M, \beta_2)_W \bigcap_{i \in I_k^M} \mathrm{ev}_i^{-1}(Z_i) \cap \overline{M}(Y, \underline{\sigma_2}).$$

Let $M^{\mathrm{cut}}(\underline{\sigma})$ be the moduli space defined by the derived pullback diagram:

$$\begin{tikzcd}
M^{\mathrm{cut}}(\underline{\sigma}) \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, "q'"] & M^2(\underline{\sigma_2}) \arrow[d, "\mathrm{ev}_{g'}"] \arrow[ddr, bend left, "q_2"] & \\
M^1(\underline{\sigma_1}) \arrow[r, "\mathrm{ev}_{g^k}"] \arrow[drr, bend right, "q_1"] & Z_{g^k} \arrow[dr] & \\
& & \mathrm{pt}
\end{tikzcd}$$

Let $c\colon M^g(\underline{\tau}) \to M^1(\underline{\sigma_1}) \times M^2(\underline{\sigma_2})$ denote the composition of the morphism cutting the edge e and forgetting the new tail e_1. Note that in this case, at the level of domain curves the map forgetting the tail e_1 is an isomorphism. From Proposition 3.4.15.(4) we deduce that the map c induces an isomorphism $M^g(\underline{\tau}) \xrightarrow{\sim} M^{\mathrm{cut}}(\underline{\sigma})$.

The map c is the composition of a cutting-an-edge morphism and a forgetting-a-tail morphism, so it is compatible with virtual fundamental classes by [PY22, Theorem 1.1]. Together with Lemma 3.4.22 this gives

$$q_*[M^g(\underline{\tau})] = q'_*[M^{\mathrm{cut}}(\underline{\sigma})] = q_{1*}[M^1(\underline{\sigma_1})] \cdot q_{2*}[M^2(\underline{\sigma_2})/Z_{g^k}].$$

Now we specialize to Z_i a smooth point for $i \in \left\{w, w', g^1, \cdots, g^{k-1}\right\}$, $Z_{g^\ell} = H^g_\ell$ and $Z_{t^\ell} = H^t_\ell$ for $k \le \ell \le t$.

By Lemma 3.4.21 and Lemma 3.4.20

$$\begin{aligned} q_{2*}[M^2(\underline{\sigma_2})/H^g_k] &= q^M_* \left((\mathrm{ev}^*_{w'}[\mathrm{pt}] \cup \mathrm{ev}^*_{g'}[\mathrm{pt}] \cup \mathrm{ev}^*_{t'}[H^t_k]) \cap [M(T^M_k, \beta_2)_W] \right) \\ &= N(M_k, \beta_2). \end{aligned}$$

Similarly, Lemma 3.4.20 gives

$$q_{1*}[M^1(\underline{\sigma_1})] = N(L_k, \beta_1).$$

Given that the union over all A-graph $\underline{\tau}$ associated to a splitting $\beta = \beta_1 + 0 + \beta_2$ into effective classes equals the moduli space responsible for the count $N(T^g_k, \beta)$, we deduce the first eqality

$$N(T^g_k, \beta) = \sum_{\beta_1+\beta_2=\beta} N(L_k, \beta_1) N(M_k, \beta_2) = N(L_k, \beta - \widehat{\delta}_k).$$

The last equality being obtained by Lemma 3.4.18. A similar reasoning based on the choice of domain $\mu' \in \overline{M}_{0,5}$ given by the partition $(w, t'|g^k|w', t^k)$ proves the second equality. $\square$

The previous proof relies on the following lemmas, which are direct computations making use of the very good functoriality properties of the analytic Borel-Moore motivic cohomology of derived analytic spaces. Lemma 3.4.20 expresses the compatibility of the restriction of the virtual fundamental class to a derived lci subspace. Lemma 3.4.21 relate the relative virtual fundamental class to an absolute virtual fundamental class, and Lemma 3.4.22 is a splitting formula of the virtual fundamental class of a derived fiber product.

Lemma 3.4.20. *Consider a pullback square of derived k-analytic spaces over a point*

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "j"] & Z \end{tikzcd}$$

Assume j is derived lci and Z is smooth. Then $\left(f^* \operatorname{PD}^{-1} j_*[Y]\right) \cap [X] = i_*[W]$, where PD denotes the Poincaré duality.

Proof. First, we note that since Z is smooth, we have $\gamma \cap [Z] = \operatorname{PD}(\gamma)$ for all cycle γ [Kha19, Theorem 2.26]. In particular, we get $\operatorname{PD}^{-1} j_*[Y] \cap [Z] = j_*[Y]$. Then

$$\begin{aligned} (f^* \operatorname{PD}^{-1} j_*[Y]) \cap [X] &= (f^* \operatorname{PD}^{-1} j_*[Y]) \cap f^![Z] \\ &= f^!(\operatorname{PD}^{-1} j_*[Y] \cap [Z]) && \text{by [PY22, Proposition 4.10.(1)]} \\ &= f^! j_*[Y] \\ &= i_* g^![Y] && \text{by [PY22, Proposition 4.11]} \\ &= i_*[W]. \end{aligned}$$

□

Lemma 3.4.21. *Consider a pullback square of derived k-analytic spaces over a point*

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "g"] & X \arrow[d, "f"] \arrow[ddr, bend left, "q"] & \\ \mathrm{pt} \arrow[r, "j"] \arrow[drr, equal, bend right] & Z \arrow[dr, "p"] & \\ & & \mathrm{pt} \end{tikzcd}$$

Assume j is a smooth point of Z, and f and p are proper. Let $\gamma \in H^*(X, \mathcal{F})$, then

$$g_*(i^*\gamma \cap [W]) = q_*(\gamma \cap [X/Z]).$$

Proof. This is a consequence of the projection formula

$$\begin{aligned} g_*(i^*\gamma \cap [W]) &= q_* i_*(i^*\gamma \cap [W]) && \\ &= q_*(\gamma \cap i_*[W]) && \text{by the projection formula} \\ &= q_*(\gamma \cap i_* i^*[X/Z]) && \text{by base change [PY22, Proposition 4.7]} \\ &= q_*(\gamma \cap [X/Z]) && \text{because } i \text{ is an immersion.} \end{aligned}$$

□

Lemma 3.4.22. *Consider a pullback square of derived k-analytic spaces over a point*

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "g"] \arrow[dr, "e"] & X \arrow[d, "f"] \arrow[ddr, bend left, "b"] & \\ Y \arrow[r, "h"] \arrow[drr, bend right, "c"] & Z \arrow[dr, "d"] & \\ & & \mathrm{pt} \end{tikzcd}$$

Let $a = d \circ e$. Assume d, f and h are proper and derived lci. Then in $\mathrm{H}_0^{\mathrm{BM}}(\mathrm{pt}, \mathcal{F})$

$$a_*[W] = b_*[X] c_*[Y/Z].$$

Proof. Recall the compatibility between pushforward and composition product [Kha19, §2.3.4] for a proper map $f\colon X \to Y$ of derived k-analytic spaces over S, for all $\alpha \in \mathrm{H}_s^{\mathrm{BM}}(X/Y, \mathcal{F}(r))$ and $\beta \in \mathrm{H}_{s'}^{\mathrm{BM}}(Y/S, \mathcal{F}(r'))$ we have $f_*(\alpha \circ \beta) = f_*(\alpha) \circ \beta$.

We also recall that for a derived k-analytic space X, in $\mathrm{H}_*^{\mathrm{BM}}(X/X, \mathcal{F}(*))$ the right composition product, the left composition product and the external product $\boxtimes$ coincide. Furthermore, under the identification with motivic cohomology $\mathrm{H}_s^{\mathrm{BM}}(X/X, \mathcal{F}(r)) = \mathrm{H}^{-s}(X, \mathcal{F}(-r))$ these coincide with the cup-product of motivic cohomology on Borel-Moore homology, and with the cap-product of motivic cohomology. In particular, the composition product becomes commutative in this case.

We now compute

$$
\begin{aligned}
a_*[W] &= a_*\left([W/Z] \circ [Z]\right) && \text{by [PY22, Proposition 4.6]} \\
&= d_* e_*\left(([X/Z] \boxtimes [Y/Z]) \circ [Z]\right) && \text{by [PY22, Propositions 4.6, 4.7 and 4.10.(4)]} \\
&= d_*\left(e_*\left([X/Z] \boxtimes [Y/Z]\right) \circ [Z]\right) && \text{since } e_*(\alpha \circ \beta) = e_*(\alpha) \circ \beta \\
&= d_*\left((f_*[X/Z] \boxtimes h_*[Y/Z]) \circ [Z]\right) && \text{by [PY22, Proposition 4.10.(6)]} \\
&= d_*\left(f_*[X/Z] \circ h_*[Y/Z] \circ [Z]\right) && \text{since } \boxtimes = \circ \\
&= d_*\left(f_*[X/Z] \circ [Z] \circ h_*[Y/Z]\right) && \text{by commutativity} \\
&= d_*\left(f_*\left([X/Z] \circ [Z]\right) \circ h_*[Y/Z]\right) && \text{since } f_*(\alpha \circ \beta) = f_*(\alpha) \circ \beta \\
&= d_*\left(f_*[X] \circ h_*[Y/Z]\right) && \text{by [PY22, Proposition 4.6]} \\
&= b_*[X] c_*[Y/Z] && \text{by [PY22, Proposition 4.10.(6)].}
\end{aligned}
$$

□

The next proposition expresses the counts that appear in the initial step and the final steps of the inductive twig-removal procedure.

Proposition 3.4.23. *Given $\beta \in \mathrm{NE}(Y)$, we have:*

1. $N(L_1, \beta) = N(\widehat{V}, \beta)$*, the count of the extended initial cylinder.*
2. $N(L_{t+1}, \beta) = 0$ *for* $\beta \neq \widehat{\delta}_V + \sum_{s=1}^{t} \delta_s$*, and* $N(L_{t+1}, \widehat{\delta}_V + \sum_{s=1}^{t} \delta_s) = 1$.

Proof. For (1), we note that the count $N(L_1, \beta)$ is given by imposing divisorial conditions at the marked points corresponding to indices in $I_1^L \setminus \{w\}$. By construction, the divisors H_k^g and H_k^t have intersection number 1 with β. Thus, applying repeatedly the divisor axiom we see that $N(L_1, \beta)$ is given by a count of 3-pointed curves with two boundary marked points and one interior marked point lying above the initial tropical cylinder. As we evaluate at the interior marked point, we get the count of the initial cylinder by definition.

For (2), note that $N(L_{t+1}, \beta)$ counts curves without any twigs – *i.e.* curves that do not meet the exceptional divisor E. We can argue as in the proof of Lemma 3.4.18: evaluate $N(L_{t+1}, \beta)$ as the cardinality of $F = \mathrm{ev}_{I_{t+1}^g}^{-1}((x_i)_{i \in I_{t+1}^g})$ which consists of skeletal curves, prove that the domain of an element in F is completely determined by the image of the interior points, and use the result on toric spine counts. □

We can now prove our main result.

Proof of Theorem 3.4.1. Applying Proposition 3.4.19 inductively, and using Proposition 3.4.23, for $\beta \in \mathrm{NE}(Y)$ we compute

$$\begin{aligned}
N(V,\beta) &= N(\widehat{V}, \beta + \widehat{\delta}_V) \\
&= N(L_1, \beta + \widehat{\delta}_V) \\
&= \sum_{\gamma_1+\cdots+\gamma_t+\gamma_{t+1}=\beta+\widehat{\delta}_V+\sum_{s=1}^t \delta_s} N\left(L_{t+1}, \gamma_{t+1} - \sum_{s=1}^t \widehat{\delta}_s\right) \prod_{s=1}^t N(N_s, \gamma_s) \\
&= \sum_{\gamma_1+\cdots+\gamma_t=\beta+\sum_{s=1}^t \widehat{\delta}_s} \prod_{s=1}^t N(N_s, \gamma_s) \\
&= \sum_{\gamma_1+\cdots+\gamma_t=\beta+\sum_{s=1}^t \widehat{\delta}_s} \prod_{s=1}^t N(V_s, \gamma_s - \widehat{\delta}_s) \\
&= \sum_{\beta_1+\cdots+\beta_t=\beta} \prod_{s=1}^t N(V_s, \beta_s).
\end{aligned}$$

□

Chapter 4

DECOMPOSITION AND FRAMING OF F-BUNDLES AND APPLICATIONS TO QUANTUM COHOMOLOGY

This chapter is based on [Hin+24], joint work with Tony Yue Yu, Chi Zhang and Shaowu Zhang.

4.1 Introduction

4.1.1 Motivations

Let X be a smooth projective complex variety. The enumeration of curves in X is a classical subject in algebraic geometry, enjoying a variety of approaches (see [PT14]). Gromov-Witten theory is one of the most widely known and the most general (e.g. no restriction on the dimension of X) (see [KM94; LT98; Beh97]). The Gromov-Witten invariants of X are rational numbers depending on the genus g, number of marked points n and cohomology classes $\phi_1, \dots, \phi_n$ of X. They satisfy a notable relation called the WDVV equation, which allows them to be packaged into differential geometric data, such as Frobenius manifolds by Dubrovin ([Dub96]) or semi-infinite Hodge structures by Barannikov ([Bar00]). The differential geometric framework facilitates intuitions from geometry and mirror symmetry and contributes tremendously to the development of the subject. The framework was further extended to incorporate the integral/rational structure via the notion of nc-Hodge structure by Katzarkov-Kontsevich-Pantev [KKP08]. They established a profound gluing/decomposition theorem using the Fourier-Laplace transform of the associated D-modules (see §2.4.2 in loc. cit.). This motivated the development of the theory of atoms for applications to birational geometry (see [Kat+24]). The idea is to apply the decomposition to the A-model nc-Hodge structure (defined using Gromov-Witten invariants) associated to a smooth projective variety at a generic point of the base, and view the resulting pieces as elementary pieces of the variety called atoms. The collection of atoms (modulo an equivalence relation induced by blowups) is expected to serve as a powerful birational invariant.

While the notions of nc-Hodge structure and atom are natural and beautiful, it is still conjectural that Gromov-Witten invariants actually give rise to an nc-Hodge structure satisfying all the axioms in [KKP08, §2.1.5]. The difficulties include the convergence of the Gromov-Witten potential ([Iri07]), the Gamma conjecture ([GGI16]) and the opposedness axiom ([RS17a]). This means that the theory of nc-Hodge structure cannot yet be unconditionally employed for the study of Gromov-Witten invariants and their applications in general. In this paper, we consider a formal/non-archimedean distilled version of variation of nc-Hodge structures, which we call F-bundles (see Section 4.2, and see Section 4.1.3 for related notions). We establish the spectral decomposition theorem for F-bundles, according to the generalized eigenspaces of the Euler vector field action, motivated by the gluing theorem for nc-Hodge structures via Fourier-Laplace transform, see Section 4.3. The comparison of the F-bundle decomposition and the nc-Hodge structure decompositions is studied in [YZ24, §8].

Furthermore, we study the notion of framing of F-bundles, analogous to the decoration on variations of nc-Hodge structures, and prove the existence and uniqueness of the extension of framing, see Section 4.4. This allows us to identify F-bundles via maps on the base (analogous to a mirror map) together with a gauge transformation on the bundle. As an application, we prove the uniqueness of the decomposition map for the A-model F-bundle (hence quantum D-module and quantum cohomology) associated to a projective bundle, as well as to a blowup of an algebraic variety. This complements the existence result by Iritani-Koto [IK23] and Iritani [Iri23].

4.1.2 Main results

Below we give a more detailed description of our results.

Throughout the paper, we fix a field $\Bbbk$ of characteristic 0. In the non-archimedean setting, we assume that $\Bbbk$ has a nontrivial valuation whose restriction to $\mathbb{Q}$ is trivial. Let B be a smooth $\Bbbk$-analytic space, and $\mathbb{D}_u$ the germ at 0 in a $\Bbbk$-analytic closed unit disk with coordinate u.

An *F-bundle* $(\mathcal{H}, \nabla)$ over B consists of a vector bundle $\mathcal{H}$ over $B \times \mathbb{D}_u$ and a meromorphic flat connection ∇ with poles at $u = 0$, such that $\nabla_{u^2\partial_u}$ and $\nabla_{u\xi}$ are regular for any tangent vector field ξ on B. We refer to Definition 4.2.2 for the definition of logarithmic F-bundle in the formal case.

For any $b \in B$, we have a natural action

$$\begin{aligned} \mu_b \colon T_b B &\longrightarrow \operatorname{End}(\mathcal{H}_{b,0}) \\ \xi &\longmapsto \nabla_{u\xi}|_{\mathcal{H}_{b,0}}. \end{aligned}$$

The F-bundle is called *maximal* at b if the action induces an isomorphism between $T_b B$ and $\mathcal{H}_{b,0}$ via a cyclic vector, see Definition 4.2.6. This gives a commutative product on $T_b B$ by the flatness of ∇.

4.1.2.1 Spectral decomposition theorems

Let $K_b \coloneqq \nabla_{u^2\partial_u}|_{b,0}$, it is the action of the Euler vector field on the fiber $\mathcal{H}_{b,0}$. We show that the generalized eigenspace decomposition of K_b extends locally to a product decomposition of the F-bundle. Here are the precise statements.

Theorem 4.1.1 (Formal spectral decomposition, Theorem 4.3.32)**.** *Let B be a formal neighborhood of a rational point b in a smooth $\Bbbk$-variety, and $(\mathcal{H}, \nabla)$ an F-bundle over B maximal at b. Assume that we have a decomposition $\mathcal{H}_{b,0} \simeq \bigoplus_{i\in I} H_i$ stable under K_b, and that for any $i \neq j \in I$, the spectra of $K_b|_{H_i}$ and $K_b|_{H_j}$ are disjoint. Then $(\mathcal{H}, \nabla)/B$ decomposes into a product of maximal F-bundles $(\mathcal{H}_i, \nabla_i)/B_i$ extending the decomposition of $\mathcal{H}|_{b,0}$.*

Theorem 4.1.2 (Non-archimedean spectral decomposition, Theorem 4.3.42)**.** *Let B be an admissible open neighborhood of a rational point b in a smooth $\Bbbk$-analytic space, and $(\mathcal{H}, \nabla)$ an F-bundle over B maximal at b. Assume that we have a decomposition $\mathcal{H}_{b,0} \simeq \bigoplus_{i\in I} H_i$ stable under K_b, and that for any $i \neq j \in I$, the spectra of $K_b|_{H_i}$ and $K_b|_{H_j}$ are disjoint. Then there exists an admissible open neighborhood U of b such that $(\mathcal{H}|_U, \nabla|_U)/U$ decomposes into a product of maximal F-bundles $(\mathcal{H}_i, \nabla_i)/U_i$ extending the decomposition of $\mathcal{H}_{b,0}$.*

For proving the spectral decomposition, first we establish a formal and a non-archimedean version of the Frobenius theorem (see Theorems 4.3.7 and 4.3.10), by solving recursively a system of partial differential equations (see Proposition 4.3.4). By the maximality assumption, we obtain an F-manifold structure on the base B of the F-bundle (see Definition 4.3.11 and Lemmas 4.3.24 and 4.3.35). The eigenspaces of K_b induce a decomposition of the tangent space T_bB as a $\Bbbk$-algebra, and we show that this decomposition extends locally around b (Theorems 4.3.13 and 4.3.20). To do so, we first prove that the algebra structure on the tangent spaces decomposes via deformations of $\Bbbk$-algebras (Lemmas 4.3.15 and 4.3.22). Then, using the F-identity (4.3.12) of the F-manifold, we prove that the induced decomposition of the tangent bundle is a decomposition into commuting integrable distributions (Proposition 4.3.19). Finally, using the formal and non-archimedean versions of the Frobenius theorem, we integrate those distributions and produce a decomposition of the F-manifold $B \simeq \prod_{i\in I} B_i$.

Having decomposed the base B, we use maximality again to obtain a splitting of $\mathcal{H}|_{u=0}$. The link between the connection ∇ and the F-manifold structure implies that this decomposition is stable under the residue endomorphisms $\nabla_{u^2\partial_u}|_{u=0}$ and

$\nabla_{u\xi}|_{u=0}$ for any $\xi \in TB$. Using the disjoint spectra assumption, we extend this decomposition to a decomposition of $\mathcal{H}$ stable under $\nabla_{u^2\partial_u}$ by a recursive procedure, and obtain the decomposition in the formal case, see Proposition 4.3.26. In the non-archimedean case, through a careful analysis of the recursion and the norms of the coefficients, we show that the decomposition is convergent over an admissible open neighborhood of b; see Proposition 4.3.36. Finally, using flatness, we prove that the connection also decomposes according to the splitting of $\mathcal{H}$ (Proposition 4.3.29), and that each piece is the pullback of a maximal F-bundle on B_i from the decomposition of the base B.

4.1.2.2 Extension of framing

A *framing* for an F-bundle $(\mathcal{H}, \nabla)/B$ is roughly a local trivialization of $\mathcal{H}$ in which the connection involves no positive powers of u (Definition 4.2.9). It is analogous to the notion of decoration on variations of nc-Hodge structures in [KKP08, §4.1.3]. Framings do not exist in general. We prove in the following that if a framing exists at a point $b \in B$ and is strong in the logarithmic case, then it extends uniquely and explicitly to a formal or non-archimedean analytic neighborhood.

Theorem 4.1.3 (Theorem 4.4.2)**.** *Let $(\mathcal{H}, \nabla)/(B, D)$ be a logarithmic F-bundle, where B is a formal neighborhood of a rational point b in a smooth $\Bbbk$-variety. A framing at b extends to a framing over B if and only if it is strong with respect to D (see Definition 4.4.1). In this case, the extension is unique and explicitly determined.*

Theorem 4.1.4 (Theorem 4.4.26)**.** *Let B be an admissible open neighborhood of a rational point b in a smooth $\Bbbk$-analytic space. Let $(\mathcal{H}, \nabla)$ be a non-archimedean F-bundle over B. Then every framing at b extends uniquely and explicitly to a framing over an admissible open neighborhood U of b in B.*

The proofs are carried out by reformulating the problem into a system of partial differential equations ((4.4.4)-(4.4.7)), which is then solved inductively on the number of variables. If there are no logarithmic directions and $(t_1, \ldots, t_n)$ are coordinates on B centered at b, we first solve (4.4.6) in the t_1-direction at $t_2 = \cdots = t_n = 0$ order by order in t_1, by observing that the equation provides a recursive relation. We use this solution as an initial condition, and then solve (4.4.6) in the t_2-direction at $t_3 = \cdots = t_n = 0$. Using flatness of the connection, we prove that the solution obtained solves the equation in the t_1-direction as well, for all t_2. In this way, we solve (4.4.6) for all directions, and we show that the solution also solves (4.4.4) using flatness again. In the non-archimedean case, we prove that the solution converges by bounding its coefficients using (4.4.6).

The extension in the formal setting works also for logarithmic F-bundles, under the assumption that the framing at b is strong with respect to D (see Definition 4.4.1). This condition implies that the residues $\mu_b(v)$ at b along $u = 0$ have nilpotent adjoint endomorphism for $v \in T_bD$, a property we call the nilpotency condition (see Definition 4.4.11). This nilpotency condition allows us to extract a recursive relation from (4.4.5), so we can reconstruct a solution to the equation order by order. We proceed as in the non-logarithmic case and solve the system of PDEs inductively on the number of variables, this time starting from the logarithmic directions.

Based on the extension of framing theorem, we give a reconstruction result for isomorphisms of logarithmic F-bundles with framing in Section 4.4.3. We can always reconstruct the bundle isomorphism from its restriction to a point and the framing. In the maximal case, we can also reconstruct the map on the base from its restriction to a point, up to some multiplicative constants in the logarithmic directions.

Proposition 4.1.5 (Proposition 4.4.31)**.** *For $i = 1, 2$, let $(\mathcal{H}_i, \nabla_i)/(B_i, D_i)$ be a logarithmic F-bundle where B_i is the formal neighborhood of a rational point in a smooth $\Bbbk$-variety. Let $(f, \Phi)\colon (\mathcal{H}_1, \nabla_1)/(B_1, D_1) \to (\mathcal{H}_2, \nabla_2)/(B_2, D_2)$ be an isomorphism between logarithmic F-bundles with $f(b_1) = b_2$. Assume $(\mathcal{H}_1, \nabla_1)/(B_1, D_1)$ has a framing ∇_1^{fr}.*

1. The bundle map Φ is uniquely and explicitly determined by its restriction to $\mathcal{H}_1|_{b_1 \times \operatorname{Spf} \Bbbk[\![u]\!]}$.

2. If $(\mathcal{H}_1, \nabla_1)$ and $(\mathcal{H}_2, \nabla_2)$ are maximal, then the base map f is also uniquely determined by its restriction to b_1, up to some multiplicative constants in the logarithmic directions. The reconstruction is explicit after fixing compatible cyclic vectors at b_1 and b_2.

Motivated by the extension of framing theorem and our applications in Section 4.5, we prove the following classification result for framed F-bundles over a point.

Proposition 4.1.6 (Corollary 4.4.35)**.** *Let $\mathcal{H} \simeq H \times \Bbbk[\![u]\!]$ be a trivialized rank m vector bundle over $\Bbbk[\![u]\!]$. Let $(\mathcal{H}, \nabla)$ and $(\mathcal{H}, \nabla')$ be two F-bundles framed in the* *given trivialization, and write*

$$\nabla_{u\partial_u} = u\partial_u + u^{-1}\mathbf{K} + \mathbf{G},$$
$$\nabla'_{u\partial_u} = u\partial_u + u^{-1}\mathbf{K}' + \mathbf{G}'.$$

Assume $\mathbf{K}$ *has simple eigenvalues. Then* $(\mathcal{H}, \nabla)$ *is isomorphic to* $(\mathcal{H}, \nabla')$ *if and only if there exists* $\phi \in \mathrm{GL}(H)$ *such that*

1. $\mathbf{K} = \phi^{-1} \circ \mathbf{K}' \circ \phi$*, and*
2. *in an eigenbasis of* $\mathbf{K}$*, we have* $(\mathbf{G})_{ii} = (\phi^{-1} \circ \mathbf{G}' \circ \phi)_{ii}$ *for* $1 \leq i \leq m$*.*

Furthermore, the gauge equivalence is uniquely and explicitly determined by the initial condition ϕ *at* $u = 0$*.*

Proceeding order by order in u, we reformulate the gauge equivalence of the two connections as a system of equations (4.4.38)-(4.4.39) involving the adjoint map $[\mathbf{K}, \cdot]$. When the eigenvalues are not simple, the equations are hard to solve because the map $[\mathbf{K}, \cdot]$ does not have an easy description. We provide a partial classification in Theorem 4.4.34, under the assumption that all the generalized eigenspaces of $\mathbf{K}$ have the same dimension, and by restricting the type of coefficients we allow in the connections. The assumption on the coefficients allows us to work relative to a universal algebra. Relative to this algebra, the endomorphism $\mathbf{K}$ has simple eigenvalues, and we are able to solve the system.

We illustrate an application of these results in the next paragraph. The reconstruction of isomorphisms also has applications in the reconstruction of mirror maps in Hodge-theoretic mirror symmetry, which we plan to explore in a subsequent work.

4.1.2.3 Application to the decomposition of quantum cohomology

Let $V \to X$ be a rank m vector bundle on a smooth complex projective variety X, $P := \mathbb{P}(V) \xrightarrow{\pi} X$ the associated projective bundle, and $h := c_1(\mathcal{O}_P(1))$. We have a natural splitting

$$iso\colon \bigoplus_{i=0}^{m-1} H^*(X, \mathbb{Q})[-2i] \xrightarrow{\sum h^i \cup \pi^*} H^*(P, \mathbb{Q}). \tag{4.1.7}$$

Fix an ample class $\omega_X \in H^2(X, \mathbb{Z})$, and a homogeneous basis $\{T_j\}_{0 \leq j \leq N}$ of $H^*(X, \mathbb{Q})$ extending $\{\mathbf{1}, \omega_X\}$. We obtain the A-model maximal F-bundle $(\mathcal{H}, \nabla)$

for P over a formal base B with closed point b given by $0 \in H^*(X, \mathbb{Q})$ (see Example 4.2.25). Let $X' := \coprod_{0 \leq i \leq m-1} X$, and $(\mathcal{H}', \nabla')$ the A-model maximal F-bundle over a formal base B' with closed point b' given by $\Delta(a) \in H^*(X', \mathbb{C})$. We denote by $(a_{i,j})$ the coordinates of $\Delta(a)$ in the basis of $H^*(X', \mathbb{C})$ induced from $\{T_j\}$ using (4.1.7).

Our first result shows the existence and uniqueness of a gauge equivalence over the base points.

Theorem 4.1.8 (Theorem 4.5.16)**.** *There exists an F-bundle isomorphism*

$$\Phi(u) \colon (\mathcal{H}, \nabla)|_b \to (\mathcal{H}', \nabla')|_{b'},$$

whose components Φ_{ij} (as power series in u) are given by the cup-product with elements in $H^*(X, \mathbb{C})$ if and only if the coordinates of the base point $\Delta(a)$ satisfy

$$\sum_{j \colon \deg T_j \neq 2} \frac{\deg T_j - 2}{2} a_{i,j} T_j = c_1 V + m\lambda_i, \tag{4.1.9}$$

where λ_i was defined in Lemma 4.5.8.

Furthermore, in this case Φ is uniquely determined by the H^0-components of $\Phi_{ij}|_{u=0}$, and $\Delta(a)$ is uniquely determined by (4.1.9), up to a shift in $\bigoplus_{i=1}^m H^2(X, \mathbb{C})$.

Next, we extend the uniqueness result over the bases B and B'. The existence is shown by Iritani-Koto [IK23].

Theorem 4.1.10 (Theorem 4.5.20)**.** *Let $(f, \Phi) \colon (\mathcal{H}, \nabla)/B \to (\mathcal{H}', \nabla')/B'$ be an isomorphism of F-bundles. Then*

1. *The bundle map Φ is uniquely and explicitly determined by its restriction to $b \in B$.*

2. *The base map f is uniquely and explicitly determined by its restriction to $b \in B$, up to a multiplicative constant in the q direction.*

In Theorems 4.5.22 and 4.5.24, we state the analogous results in the case of blowups of smooth complex projective varieties.

4.1.3 Related works

Various related but slightly different concepts of F-bundles have been studied in the literature. We refer to [Dub96; Man99] for Frobenius manifolds, [Sai83; Sab07] for Saito structures, [Her03; DH21] for (TE)-structures and variations, [Bar00] for semi-infinite variations of Hodge structures, [HM99; Her02] for F-manifolds, [KKP08; KKP17] for nc-Hodge structures, and [CV91; Sim97; Moc06] for other related works. Logarithmic variants of Frobenius manifolds and (TE)-structures were introduced in [Rei09; RS15].

Works related to our decomposition theorems for F-bundles include [Dub96] for the decomposition of semisimple Frobenius manifolds, [Sab07] for the decomposition of meromorphic connections, [Her02] for the decomposition of F-manifolds, and [YZ24] for the comparison of the spectral decomposition and the vanishing cycle decomposition of nc-Hodge structures. Analogs of our extension of framing theorem were studied in [Iri06; Coa+20] for the q-direction, in [DH21] for the t-direction, and in [Iri08] under different assumptions. We refer to [Iri23; IK23] for the decomposition of quantum D-modules for projective bundles and blowups.

4.1.4 Acknowledgements

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4.2 Basic definitions and examples

In this section, we give the basic definitions regarding F-bundles and give the example of the A-model F-bundle.

4.2.1 Notion of F-bundle

Let $\mathbb{D}_u$ denote the germ at 0 in a $\Bbbk$-analytic closed unit disk with coordinate u.

Definition 4.2.1 (F-bundle)**.** Let B be a smooth $\Bbbk$-analytic space (resp. a smooth formal scheme over $\Bbbk$). An *F-bundle* $(\mathcal{H}, \nabla)$ over B consists of a vector bundle $\mathcal{H}$ over $B \times \mathbb{D}_u$ (resp. over $B \times \operatorname{Spf} \Bbbk[\![u]\!]$), and a meromorphic flat connection ∇ on $\mathcal{H}$ with poles along $u = 0$, such that $\nabla_{u^2\partial_u}$ and $\nabla_{u\xi}$ are regular for any tangent vector field ξ on B.

For applications to Gromov-Witten theory (see Section 4.2.2), the base B, the vector bundle $\mathcal{H}$ and the connection ∇ should all be understood in the context of supergeometry (see [Man97, §4]).

Given a map $f\colon B' \to B$, the pullback $f^*(\mathcal{H}, \nabla) := ((f \times \mathrm{id}_u)^*\mathcal{H}, (f \times \mathrm{id}_u)^*\nabla)$ is an F-bundle on B'.

In the formal case, we introduce the notion of logarithmic F-bundle.

Definition 4.2.2 (Logarithmic F-bundle)**.** Let B be a smooth formal scheme over $\Bbbk$ together with a normal crossing divisor $D \subset B$. A *logarithmic F-bundle* over (B, D) consists of a vector bundle $\mathcal{H}$ over $B \times \operatorname{Spf} \Bbbk[\![u]\!]$ and a meromorphic flat connection ∇ on $\mathcal{H}$ with poles along $u = 0$, such that $\nabla_{u^2\partial_u}$ and $\nabla_{u\xi}$ are regular for any log tangent vector field ξ on B.

Below we formulate several definitions for logarithmic F-bundles, which also apply to non-archimedean F-bundles, up to replacing logarithmic tangent vectors by analytic tangent vectors.

Remark 4.2.3 (Restriction to $u = 0$)**.** Let $(\mathcal{H}, \nabla)/(B, D)$ be a logarithmic F-bundle and ξ a logarithmic vector field on B. The failure of $\mathcal{O}_B$-linearity of the operator $\nabla_{u\xi}$ is given by the symbol

$$\begin{aligned} \sigma(\nabla_{u\xi})\colon T^*B \otimes_{\mathcal{O}_{B\times \operatorname{Spf}\Bbbk[\![u]\!]}} \mathcal{H} &\longrightarrow \mathcal{H} \\ df \otimes h &\longmapsto [\nabla_{u\xi}, f]h. \end{aligned}$$

We have $\sigma(\nabla_{u\xi})(df \otimes h) = df(u\xi)h$, which vanishes at $u = 0$. We thus obtain a map

$$\begin{aligned} \mu\colon TB(-\log D) &\longrightarrow \operatorname{End}_{\mathcal{O}_B}(\mathcal{H}|_{u=0}) \\ \xi &\longmapsto \nabla_{u\xi}|_{u=0}. \end{aligned} \tag{4.2.4}$$

In a similar way, the restriction of $\nabla_{u^2\partial_u}$ to $\mathcal{H}|_{u=0}$ is $\mathcal{O}_B$-linear.

Let $b = \operatorname{Spec}\Bbbk \to B$ be a closed point. The map (4.2.4) induces a map

$$\mu_b\colon T_bB(-\log D) \longrightarrow \operatorname{End}(\mathcal{H}_{b,0}). \tag{4.2.5}$$

Let K_b denote the action of $\nabla_{u^2\partial_u}$ on $\mathcal{H}_{b,0}$. The flatness of ∇ implies that the image of μ_b consists of commuting operators, which also commute with K_b.

Definition 4.2.6. A logarithmic F-bundle $(\mathcal{H}, \nabla)/(B, D)$ is called *maximal* (resp. *over-maximal*) at a closed point $b = \operatorname{Spec} \Bbbk \to B$ if there exists a *cyclic vector* for the action μ_b, i.e. a vector $h \in \mathcal{H}_{b,0}$ such that the map

$$T_b B(-\log D) \longrightarrow \mathcal{H}_{b,0}, \quad v \longmapsto \mu_b(v)(h)$$

is an isomorphism (resp. epimorphism). It is called maximal (resp. over-maximal) if it is maximal (resp. over-maximal) everywhere.

In the maximal case, the dimension of $T_b B(-\log D)$ is equal to the rank of $\mathcal{H}$, and μ_b induces an inclusion from $T_b B(-\log D)$ to $\operatorname{End}(\mathcal{H}_{b,0})$. We obtain a commutative associative product structure on $T_b B$, given by

$$\mu_b(v_1 \star v_2)(h) = \mu_b(v_2) \circ \mu_b(v_1)(h). \tag{4.2.7}$$

Definition 4.2.8. Let $(\mathcal{H}, \nabla)/(B, D)$ be a maximal logarithmic F-bundle. The unique logarithmic vector field Eu on B with $\mu(\mathrm{Eu}) = K \coloneqq \nabla_{u^2\partial_u}|_{u=0}$ is called the *Euler vector field*.

Definition 4.2.9. For a logarithmic F-bundle $(\mathcal{H}, \nabla)$ over (B, D), a *framing* is another flat connection ∇^{fr} on $\mathcal{H}$ without poles, such that in the local trivializations of $\mathcal{H}$ given by ∇^{fr}, if we denote by H the vector space of local flat sections, the original connection ∇ has the form

$$\nabla_{\partial_u} = \partial_u + \frac{1}{u^2}\mathbf{K} + \frac{1}{u}\mathbf{G}, \qquad \nabla_\xi = \xi + \frac{1}{u}\mathbf{A}(\xi) \tag{4.2.10}$$

for any logarithmic vector field ξ on B, where $\mathbf{K}, \mathbf{G}$ are $\operatorname{End}(H)$-valued functions on B, and $\mathbf{A}$ is an $\operatorname{End}(H)$-valued 1-form on B.

We give the definition of product of logarithmic F-bundles. The definition is analogous in the non-archimedean case.

Definition 4.2.11 (Product of F-bundles)**.** The product of two logarithmic F-bundles $(\mathcal{H}_1, \nabla_1)/(B_1, D_1)$ and $(\mathcal{H}_2, \nabla_2)/(B_2, D_2)$ is the F-bundle $(\mathcal{H}, \nabla)/(B, D)$ defined over $B = B_1 \times B_2$, with divisor $D = (D_1 \times B_2) \cup (B_1 \times D_2)$, by

$$\mathcal{H} = \mathrm{pr}_1^*\mathcal{H}_1 \oplus \mathrm{pr}_2^*\mathcal{H}_2,$$
$$\nabla = \mathrm{pr}_1^*\nabla_1 \oplus \mathrm{pr}_2^*\nabla_2,$$

where $\mathrm{pr}_i \colon B_1 \times B_2 \times \operatorname{Spf} \Bbbk[\![u]\!] \to B_i \times \operatorname{Spf} \Bbbk[\![u]\!]$ denotes the projection for $i = 1, 2$.

4.2.2 Example of A-model F-bundle

Let X be a smooth projective variety over $\mathbb{C}$. The rational Gromov-Witten invariants of X can be encoded in an F-bundle, called the A-model F-bundle associated to X, also known as the quantum D-module (see [Giv95]). Here we will give a logarithmic version and a non-archimedean version of the A-model F-bundle.

4.2.2.1 Gromov-Witten potential and quantum product

Fix a homogeneous basis $(T_i)_{0\le i\le N}$ of $H^*(X,\mathbb{Q})$, such that $T_0 = \mathbf{1}$ is the unit, and $(T_1,\dots,T_k)$ is a basis of $H^2(X,\mathbb{Q})$. Let $(T^i)_{0\le i\le N}$ denote the dual basis with respect to the cup product pairing.

Let $\mathbb{Q}[\![\mathrm{NE}(X,\mathbb{Z})]\!]$ denote the completion of $\mathbb{Q}[\mathrm{NE}(X,\mathbb{Z})] = \mathbb{Q}[q^\beta \mid \beta \in \mathrm{NE}(X,\mathbb{Z})]$ with respect to the maximal ideal $(q^\beta,\ \beta \neq 0)$. We write $\Bbbk[\![\mathrm{NE}(X,\mathbb{Z})]\!] \coloneqq \mathbb{Q}[\![\mathrm{NE}(X,\mathbb{Z})]\!] \otimes_{\mathbb{Q}} \Bbbk$.

The *genus* 0 *Gromov-Witten potential* is

$$\Phi = \sum_{n\ge 0,\beta} \frac{q^\beta}{n!} \sum_{i_1,\dots,i_n} \langle T_{i_1}\cdots T_{i_n}\rangle_{0,n}^{\beta}\, t_{i_1}\cdots t_{i_n} \in \mathbb{Q}[\![\mathrm{NE}(X,\mathbb{Z})]\!][\![t_0,\dots,t_N]\!], \tag{4.2.12}$$

where $\langle\cdots\rangle_{0,n}^{\beta}$ denotes the Gromov-Witten invariants of X of genus 0, class β and observables $T_{i_1},\dots,T_{i_n}$.

The *(big) quantum product* is given by

$$\begin{aligned} \star\colon H^*(X,\mathbb{Q})\otimes H^*(X,\mathbb{Q}) &\longrightarrow H^*(X,\mathbb{Q})\otimes \mathbb{Q}[\![\mathrm{NE}(X,\mathbb{Z})]\!][\![t_0,\dots,t_N]\!] \\ T_i\star T_j &\longmapsto \sum_r \frac{\partial^3\Phi}{\partial t_i\partial t_j\partial t_r} T^r, \end{aligned} \tag{4.2.13}$$

where

$$\frac{\partial^3\Phi}{\partial t_i\partial t_j\partial t_r} = \sum_{n\ge 0,\beta} \frac{q^\beta}{n!} \sum_{i_1,\dots,i_n} \langle T_iT_jT_rT_{i_1}\cdots T_{i_n}\rangle_{0,n+3}^{\beta}\, t_{i_1}\cdots t_{i_n}. \tag{4.2.14}$$

In Section 4.5, we will use a quantum product at a shifted origin, which we explain in the following lemma.

Lemma 4.2.15. *Let $\Delta(a) = \sum_{0\le i\le N} a_iT_i \in H^*(X,\Bbbk)$. Assume $\Delta(a)$ has no terms of degree 1 or 2. Then applying the shift $t = (t_0,\dots,t_N) \mapsto t+a = (t_0+a_0,\dots,t_N+a_N)$ to Φ produces a well-defined element $\Phi(t+a) \in \Bbbk[\![\mathrm{NE}(X,\mathbb{Z})]\!][\![t_0,\dots,t_N]\!]$.*

Proof. Before the shift, for $\alpha = (\alpha_0, \dots, \alpha_N) \in \mathbb{N}^{N+1}$, the coefficient of the monomial $q^\beta t_0^{\alpha_0} \cdots t_N^{\alpha_N}$ in Φ is $\frac{1}{\alpha_0! \cdots \alpha_N!} \langle T_0^{\alpha_0} \cdots T_N^{\alpha_N} \rangle_{0,|\alpha|}^\beta$. The coefficient of the monomial $t_0^{\alpha_0} \cdots t_N^{\alpha_N}$ of $\Phi(t+a)$ is given by evaluating

$$\frac{1}{\alpha_0! \cdots \alpha_N!} \frac{\partial^{|\alpha|} \Phi(t+a)}{\partial^{\alpha_0} t_0 \cdots \partial^{\alpha_N} t_N}$$

at $t = 0$. By the chain rule, this is the same as evaluating the derivative of the unshifted potential Φ at $t = a$. So, it is enough to check that this evaluation makes sense, i.e. that it gives an element in $\Bbbk[\![\mathrm{NE}(X, \mathbb{Z})]\!]$. The coefficient of q^β in the derivative of Φ is

$$\sum_{\gamma \in \mathbb{N}^{N+1}} \frac{1}{\gamma_0! \cdots \gamma_N!} \langle T_0^{\alpha_0+\gamma_0} \cdots T_N^{\alpha_N+\gamma_N} \rangle_{0,|\alpha|+|\gamma|}^\beta t_0^{\gamma_0} \cdots t_N^{\gamma_N}. \tag{4.2.16}$$

We claim that the above sum is finite when evaluated at a. By the unit axiom of Gromov-Witten invariants, the part of the sum with $\gamma_0 > 0$ is finite: if T_0 appears in a nonzero n-pointed Gromov-Witten invariant, then $n = 3$ and $\beta = 0$. We now prove that there are finitely many terms with $\gamma_0 = 0$. If a nonzero Gromov-Witten invariant $\langle T_0^{\alpha_0} T_1^{\alpha_1+\gamma_1} \cdots T_N^{\alpha_N+\gamma_N} \rangle_{0,|\alpha|+|\gamma|}^\beta$ contributes to the sum, the formula for the virtual dimension gives

$$\sum_{0 \le i \le N} \alpha_i \operatorname{codim} T_i + \sum_{1 \le i \le N} \gamma_i \operatorname{codim} T_i = 2(\dim X - 3 + |\alpha| + |\gamma| + \beta \cdot c_1 T_X).$$

Since we assume that there is no shift in the H^1 and H^2-directions, the monomial $t_0^{\gamma_0} \cdots t_N^{\gamma_N}$ evaluated at $t = a$ is 0, unless $\gamma_i = 0$ for $\operatorname{codim} T_i \in \{1, 2\}$. Then, when evaluating the $\gamma_0 = 0$ part of (4.2.16) at $t = a$, nonzero terms satisfy $\operatorname{codim} T_i \ge 3$ for $\gamma_i \neq 0$. We deduce

$$3|\gamma| \le \sum_{1 \le i \le N} \gamma_i \operatorname{codim} T_i = 2|\gamma| + \text{constant}.$$

It follows that the sum (4.2.16) is finite at $t = a$, completing the proof. ☐

By Lemma 4.2.15, the quantum product is also well-defined on a formal neighborhood of the shifted point $\Delta(a) \in H^*(X, \Bbbk)$.

4.2.2.2 Logarithmic A-model F-bundle

Let U be the formal neighborhood of a cohomology class $\Delta(a) \in H^*(X, \Bbbk)$ at which the quantum potential is well-defined. Using the basis $(T_i)_{0 \le i \le N}$, we write $U = \operatorname{Spf} \Bbbk[\![t_0, \dots, t_N]\!]$.

For $\xi \in H^2(X, \Bbbk)$, we define a derivation $\xi q \partial_q$ of $\Bbbk[\![\mathrm{NE}(X, \mathbb{Z})]\!]$ by

$$\xi q \partial_q (q^\beta) \coloneqq (\beta \cdot \xi) q^\beta.$$

Definition 4.2.17. The *logarithmic A-model F-bundle of X at base point* $\Delta(a)$ is the logarithmic F-bundle $(\mathcal{H}, \nabla)$ over $\operatorname{Spf} \Bbbk[\![\mathrm{NE}(X, \mathbb{Z})]\!] \times U$ defined as follows:

1. The bundle $\mathcal{H}$ is trivial with fiber $H^*(X, \Bbbk)$.

2. Let

$$\begin{aligned}
\mathbf{K} &\coloneqq \left[c_1(T_X) + \sum_{i:\deg T_i \neq 2} \frac{\deg T_i - 2}{2} t_i T_i \right] \star, \\
\mathbf{G} &\coloneqq \frac{1}{2}(\deg_X - \dim X), \\
\mathbf{A}(\tau) &\coloneqq \tau \star, \quad \tau \in H^*(X, \Bbbk), \\
\mathbf{A}(\xi) &\coloneqq \xi \star, \quad \xi \in H^2(X, \Bbbk),
\end{aligned}$$

where $\deg_X(\alpha) = i\alpha$ for $\alpha \in H^i(X, \Bbbk)$, and $\star$ is the quantum product shifted at $\Delta(a)$. The connection ∇ is given by

$$\begin{aligned}
\nabla_{\partial_u} &= \partial_u - \frac{1}{u^2}\mathbf{K} + \frac{1}{u}\mathbf{G}, \\
\nabla_{\partial_\tau} &= \partial_\tau + \frac{1}{u}\mathbf{A}(\tau), \\
\nabla_{\xi q \partial_q} &= \xi q \partial_q + \frac{1}{u}\mathbf{A}(\xi).
\end{aligned}$$

4.2.2.3 Non-archimedean A-model F-bundle

In the non-archimedean setting, $\Bbbk$ is a complete non-archimedean field of characteristic 0 with a nontrivial valuation whose restriction to $\mathbb{Q}$ is trivial.

Let $N^1(X)/\operatorname{Tor}$ denote the Néron-Severi group of X modulo torsion. The valuation of $\Bbbk$ induces a map

$$v \colon (N^1(X)/\operatorname{Tor}) \otimes_{\mathbb{Z}} \mathbb{G}_{\mathrm{m}/\Bbbk} \to (N^1(X)/\operatorname{Tor}) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{4.2.18}$$

Since the ample cone $\operatorname{Amp}(X)$ is open in $N^1(X)_{\mathbb{R}}$, its preimage $B_q \coloneqq v^{-1}(\operatorname{Amp}(X))$ is a $\Bbbk$-analytic space. Let B_t^{even} be the product of a $\Bbbk$-analytic affine line and an open polyunit disk, where the affine line has coordinate t_0 and the polyunit disk has coordinates t_i for $\deg T_i \in \{2, 4, 6, \dots\}$. Let B_t^{odd} be the purely odd vector space with coordinates t_i for $\deg T_i \in \{1, 3, 5, \dots\}$. Let $B \coloneqq B_q \times B_t^{\mathrm{even}} \times B_t^{\mathrm{odd}}$.

Lemma 4.2.19. *The genus* 0 *Gromov-Witten potential*

$$\Phi = \sum_{n\geq 0,\beta} \frac{q^\beta}{n!} \sum_{i_1,\cdots,i_n} \langle T_{i_1}\cdots T_{i_n}\rangle_\beta t_{i_1}\cdots t_{i_n} \in \mathbb{Q}[\![\mathrm{NE}(X,\mathbb{Z})]\!][\![\{t_i\}]\!],$$

defines an analytic function over B.

Proof. Let $\sigma \subset N^1(X)_\mathbb{R}$ be any simplicial cone generated by ample classes $\omega_1,\cdots,\omega_m$. Let B'_q be the preimage of σ under the valuation map (4.2.18), and $B' = B'_q \times B_t^{\mathrm{even}} \times B_t^{\mathrm{odd}}$. Then Φ is analytic over B', since the restriction of Φ to B' is given by the power series with rational coefficients

$$\Phi = \sum_{n\geq 0,\beta} \frac{1}{n!} q_1^{\beta\cdot\omega_1}\cdots q_m^{\beta\cdot\omega_m} \sum_{i_1,\cdots,i_n} \langle T_{i_1}\cdots T_{i_n}\rangle_\beta t_{i_1}\cdots t_{i_n} \in \mathbb{Q}[\![\{q_j\},\{t_i\}]\!],$$

which is polynomial in t_0 by the unit axiom. Since the union of all such σ covers the ample cone, the proof is complete. □

Lemma 4.2.19 implies that the quantum product is convergent over the non-archimedean base space B.

Definition 4.2.20. The *non-archimedean A-model F-bundle of* X is the F-bundle $(\mathcal{H},\nabla)$ over B defined by the same formulas as in Definition 4.2.17.

4.2.2.4 Maximal logarithmic F-bundle

The F-bundles defined above are not maximal because the base has larger dimension than the fibers. We can cut down the base dimension by choosing one q-variable and eliminating one t-variable as follows.

Fix a nef class $\omega \in N^1(X)$. It induces a projection

$$\Bbbk[\mathrm{NE}(X,\mathbb{Z})] \to \Bbbk[q], \quad q^\beta \mapsto q^{\beta\cdot\omega}. \tag{4.2.21}$$

Assumption 4.2.22. Assume that for any $i_1,\dots,i_n$ and d, there are finitely many β such that $\beta\cdot\omega = d$ and $\langle T_{i_1}\cdots T_{i_n}\rangle^\beta_{0,n} \neq 0$.

Lemma 4.2.23. *Assumption 4.2.22 holds if there exists* $\epsilon \in \mathbb{Q}$ *such that* $\omega + \epsilon c_1(T_X)$ *is ample. In particular, it holds if* ω *is ample, or if* X *is Fano.*

Proof. Recall that the virtual dimension of $\mathcal{M}_{0,n}(X,\beta)$ is equal to $\dim X - 3 + \beta\cdot c_1(T_X) + n$. If $\langle T_{i_1}\cdots T_{i_n}\rangle^\beta_{0,n} \neq 0$, we have $\dim_{\mathrm{vir}} \mathcal{M}_{0,n}(X,\beta) = \sum_{j=1}^n \operatorname{codim} T_{i_j}$.

So $\beta \cdot c_1(T_X)$ is fixed given $T_{i_1}, \dots, T_{i_n}$. If $\beta \cdot \omega$ is also given, then $\beta \cdot \Big(\omega + \epsilon\, c_1(T_X)\Big)$ is fixed too. This is only possible for finitely many β, since $\omega + \epsilon c_1(T_X)$ is assumed ample. □

Lemma 4.2.24. *Under Assumption 4.2.22, the Gromov-Witten potential* $\Phi \in \mathbb{Q}[\![\mathrm{NE}(X,\mathbb{Z})]\!][\![t_0,\dots,t_N]\!]$ *as in* (4.2.12) *induces an element* $\Phi^\omega \in \mathbb{Q}[\![q]\!][\![t_0,\dots,t_N]\!]$*, via the projection* (4.2.21)*. Conversely,* Φ *is uniquely determined by* Φ^ω*.*

Proof. Assumption 4.2.22 guarantees that Φ^ω is well-defined. Let us prove the other direction. Fix $i_1,\dots,i_n$ and d. Knowing Φ^ω, we can form the following series:

$$\Psi = \sum_{r_1,\dots,r_k} \frac{1}{r_1!\cdots r_k!} \sum_{\beta\cdot\omega=d} \langle T_{i_1}\cdots T_{i_n} T_1^{r_1}\cdots T_k^{r_k}\rangle^\beta_{0,n+r_1+\cdots+r_k}\, s_1^{r_1}\cdots s_k^{r_k} \in \mathbb{Q}[\![s_1,\dots,s_k]\!],$$

where $T_1,\dots,T_k$ constitute a basis of $H^2(X,\mathbb{Q})$. By the divisor axiom, we have

$$\Psi = \sum_{r_1,\dots,r_k} \frac{1}{r_1!\cdots r_k!} \sum_{\beta\cdot\omega=d} \langle T_{i_1}\cdots T_{i_n}\rangle^\beta_{0,n}\, (\beta\cdot T_1)^{r_1} s_1^{r_1}\cdots(\beta\cdot T_k)^{r_k} s_k^{r_k} \in \mathbb{Q}[\![s_1,\dots,s_k]\!].$$

Comparing the coefficients, we conclude that every $\langle T_{i_1}\cdots T_{i_n}\rangle^\beta_{0,n}$ is uniquely determined by Ψ, and therefore by Φ^ω. □

Example 4.2.25 (Maximal A-model F-bundle)**.** Assume $\omega = T_1$ satisfies Assumption 4.2.22. Let $\Delta(a) \in H^*(X,\Bbbk)$ be a cohomology class at which the quantum potential is well-defined. Let $U = \operatorname{Spf}\Bbbk[\![t_0,\cdots,t_N]\!]$ be the formal neighborhood of $\Delta(a)$ in $H^*(X,\Bbbk)$, $U' \subset U$ the closed subspace given by $t_1 = 0$, and $B' = \operatorname{Spf}\Bbbk[\![q]\!]\times U'$. Then the potential Φ^ω in Lemma 4.2.24 produces a maximal logarithmic F-bundle over B' by the same formulas as in Definition 4.2.17. Indeed, the multiplicative unit $\mathbf{1}$ is a cyclic vector at 0 by the unit axiom.

4.3 Spectral decomposition of maximal F-bundles

In this section, we establish the spectral decomposition theorem for maximal F-bundles in the formal and non-archimedean settings, see Theorems 4.3.32 and 4.3.42. We first prove in §4.3.1 formal and non-archimedean analogs of the Frobenius theorem in differential geometry using an argument that we call "generalized flatness". We study the decomposition of the base as F-manifolds in Section 4.3.2. The spectral decomposition theorems are presented and proved in Section 4.3.3.

Recall that $\Bbbk$ is a field of characteristic 0. In the non-archimedean setting, we equip $\Bbbk$ with a complete nontrivial valuation whose restriction to $\mathbb{Q}$ is trivial.

4.3.1 Frobenius theorem

4.3.1.1 Generalized flatness for systems of PDEs

We prove a criterion ensuring the existence of a unique formal solution to some systems of quasi-linear PDEs in Proposition 4.3.4. We also prove a non-archimedean version in a special case in Lemma 4.3.6. Throughout, we set $M_0 \coloneqq \Bbbk^m$. We denote by $\mathfrak{m}$ the maximal ideal $(t_1, \dots, t_n)$ in $\Bbbk[\![t_1, \dots, t_n]\!]$.

Notation 4.3.1. We use the following notations for tuples of integers:

1. Let $\preceq$ denote the partial order on $\mathbb{N}^n$ defined by
$$(r_i)_{1\leq i\leq n} \preceq (s_i)_{1\leq i\leq n} \iff \forall 1 \leq i \leq n,\ r_i \leq s_i.$$
2. For $r = (r_i) \in \mathbb{N}^n$, let $|r| \coloneqq \sum_{1\leq i\leq n} r_i$.
3. For $r = (r_i)_{1\leq i\leq n} \in \mathbb{N}^n$ and $1 \leq j \leq n$, we set
$$\tau_j(r) \coloneqq (r_1, \dots, r_{j-1}, r_j + 1, r_{j+1}, \dots, r_n) \in \mathbb{N}^n.$$

Definition 4.3.2. Let $\mathcal{D} = (D_i \colon M_0 \otimes_\Bbbk \mathfrak{m} \to M_0 \otimes_\Bbbk \Bbbk[\![t_1, \dots, t_n]\!])_{1\leq i\leq n}$ be a system of differential operators of the form $D_i = \partial_{t_i} - f_i$, with $f_i \colon M_0 \otimes_\Bbbk \mathfrak{m} \to M_0 \otimes_\Bbbk \Bbbk[\![t_1, \dots, t_n]\!]$ an arbitrary map. We say that the system $\mathcal{D}$ is *generalized flat* if the two following conditions are satisfied:

1. For every $d \in \mathbb{N}$ and every $1 \leq i \leq n$, the composition
$$M_0 \otimes_\Bbbk \mathfrak{m} \xrightarrow{f_i} M_0 \otimes_\Bbbk \Bbbk[\![t_1, \dots, t_n]\!] \longrightarrow M_0 \otimes_\Bbbk \left(\Bbbk[\![t_1, \dots, t_n]\!]/\mathfrak{m}^d\right)$$
factors through $M_0 \otimes_\Bbbk \left(\mathfrak{m}/\mathfrak{m}^d\right)$.
2. If $\varphi \in M_0 \otimes_\Bbbk \mathfrak{m}$ satisfies $D_i(\varphi) = 0 \mod \mathfrak{m}^d$ for all $1 \leq i \leq n$, then $\partial_{t_i}(f_j(\varphi)) = \partial_{t_j}(f_i(\varphi)) \mod \mathfrak{m}^d$ for all $1 \leq i, j \leq n$.

Remark 4.3.3. Condition (1) means that for $\varphi \in M_0 \otimes_\Bbbk \mathfrak{m}$, the total t-degree d terms of $f_i(\varphi)$ depend on terms in φ of total t-degree at most d. This assumption allows to solve the associated system of PDEs recursively. It is automatically satisfied if the components of $f(\varphi)$ are power series in the components of φ.

Our notion of generalized flat systems of PDEs allows us to prove the following existence and uniqueness result.

Proposition 4.3.4. *Let $(D_i\colon M_0\otimes_{\Bbbk}\mathfrak{m}\to M_0\otimes_{\Bbbk}\Bbbk[\![t_1,\dots,t_n]\!])_{1\le i\le n}$ be a generalized flat system of differential operators. Then there exists a unique $\varphi\in M_0\otimes_{\Bbbk}\mathfrak{m}$ satisfying $D_i(\varphi)=0$ for all $1\le i\le n$.*

Proof. In this proof, for $\varphi\in M_0\otimes_{\Bbbk}\mathfrak{m}$ and $\ell\in\mathbb{N}^n$, we denote by $[f_i(\varphi)]_\ell$ the coefficient of t^ℓ in $f_i(\varphi)$.

For the uniqueness, if $\varphi=\sum_{\ell\in\mathbb{N}^n}\varphi_\ell t^\ell$ is a solution of the differential system, then φ satisfies the recursive relations with respect to t-monomials

$$(\ell_i+1)\varphi_{\tau_i(\ell)}=[f_i(\varphi)]_\ell. \tag{4.3.5}$$

This uniquely determines the coefficients of φ from the initial condition $\varphi_0=0$.

For the existence, we construct inductively on $d\in\mathbb{N}$ an element $\varphi^{(d)}\in M_0\otimes_{\Bbbk}\mathfrak{m}$ such that

1. $\varphi^{(d)}$ has terms of degree at most $d+1$,
2. if $d\ge 1$, then $\varphi^{(d)}=\varphi^{(d-1)}\mod\mathfrak{m}^{d+1}$,
3. $D_i(\varphi^{(d)})=0\mod\mathfrak{m}^{d+1}$ for all $1\le i\le n$.

Set $\varphi^{(0)}:=\sum_{i=1}^n[f_i(0)]_0t_i$, it satisfies (1), (2), and (3) for $d=0$.

For the inductive step, fix $d\in\mathbb{N}$ and assume $\varphi^{(d)}$ is constructed. Given $\ell\in\mathbb{N}^n$ with $|\ell|=d+2$, there exists a minimal index i_0 and a unique $\ell'\in\mathbb{N}^n$ such that $\ell=\tau_{i_0}(\ell')$. The index i_0 corresponds to the first nonzero component of ℓ. We set $\varphi_\ell:=\frac{1}{\ell_{i_0}}[f_{i_0}(\varphi^{(d)})]_{\ell'}$, and define

$$\varphi^{(d+1)}:=\varphi^{(d)}+\sum_{\substack{\ell\in\mathbb{N}^n\\|\ell|=d+2}}\varphi_\ell t^\ell.$$

By construction $\varphi^{(d+1)}$ satisfies (1) and (2), it remains to check (3). By the inductive assumption (2) and Condition (1) of generalized flatness, we have $[f_i(\varphi^{(d+1)})]_\ell=[f_i(\varphi^{(d)})]_\ell$ for all $\ell\in\mathbb{N}^n$ such that $|\ell|\le d+1$. Thus we only need to check that the added coefficients φ_ℓ with $|\ell|=d+2$ satisfy the recursive relations (4.3.5) for all $1\le i\le n$.

Fix $\ell\in\mathbb{N}^n$ with $|\ell|=d+2$, and an index i. Let i_0 be as in the definition of φ_ℓ, then there exists a unique $\ell_0\in\mathbb{N}^n$ with $|\ell_0|=d$ such that $\ell=\tau_i\tau_{i_0}(\ell_0)=\tau_{i_0}\tau_i(\ell_0)$. By the construction of φ_ℓ, the recursive relation (4.3.5) in the t_i-direction is equivalent to

$$\ell_i[f_{i_0}(\varphi^{(d+1)})]_{\tau_i(\ell_0)}=\ell_{i_0}[f_i(\varphi^{(d+1)})]_{\tau_{i_0}(\ell_0)}.$$

Since $|\tau_i(\ell_0)| = |\tau_{i_0}(\ell_0)| = d+1$, the induction hypothesis (2) and Condition (1) of generalized flatness imply $[f_{i_0}(\varphi^{(d+1)})]_{\tau_i(\ell_0)} = [f_{i_0}(\varphi^{(d)})]_{\tau_i(\ell_0)}$, and similarly for the right hand side. Then the recursion relation for ℓ is equivalent to

$$[\partial_{t_{i_0}} f_i(\varphi^{(d)})]_{\ell_0} = [\partial_{t_i} f_{i_0}(\varphi^{(d)})]_{\ell_0},$$

which follows from Condition (2) of generalized flatness. We conclude that $\varphi^{(d+1)}$ satisfies (3), proving the inductive step.

Condition (2) of the construction implies that $\{\varphi^{(d)} \mod \mathfrak{m}^d\}_{d\geq 0}$ is an inductive system producing a well-defined element $\widetilde{\varphi} \in M_0 \otimes_{\Bbbk} \Bbbk[\![t_1, \ldots, t_n]\!]$ such that $\widetilde{\varphi} = \varphi^{(d)} \mod \mathfrak{m}^{d+2}$ for all $d \geq 0$. Condition (3) of the construction implies that $\widetilde{\varphi}$ satisfies the recursive relations (4.3.5) for all $\ell \in \mathbb{N}^n$, hence it is a solution of $D_i(\varphi) = 0$. Thus $\widetilde{\varphi}$ satisfies $D_i(\widetilde{\varphi}) = 0$ for $1 \leq i \leq n$, completing the proof. □

We denote by T_n the Tate $\Bbbk$-algebra in n variables. For $\rho \in \sqrt{|\Bbbk^\times|}$, we denote by $T_n(\rho)$ the $\Bbbk$-affinoid algebra associated to the closed polydisk of radius ρ and dimension n ([BGR84, §6.1.5]), consider the norm

$$\left| \sum_{\alpha\in\mathbb{N}^n} a_\alpha t^\alpha \right|_\rho := \max_\alpha |a_\alpha| \rho^{|\alpha|}.$$

Lemma 4.3.6. *For $1 \leq i \leq n$ and $1 \leq k \leq m$, let $Y_i^k \in T_m = \Bbbk\langle x_1, \ldots, x_m\rangle$. Let $|Y| := \max_{1\leq i,k\leq n} |Y_i^k|$, assume $|Y| > 0$. Let $f = (f_k)_{1\leq k\leq m} \in M_0 \otimes_{\Bbbk} \mathfrak{m}$ satisfying ($1 \leq i \leq n$, $1 \leq k \leq m$)*

$$\partial_{t_i} f_k = Y_i^k(f_1, \ldots, f_m).$$

Then the components of f are convergent on the open polydisk of radius $|Y|^{-1}$ and have norms bounded by 1. Equivalently, f induces a map $\operatorname{Sp} T_n(\rho) \to \operatorname{Sp} T_n$ for all $\rho \in \sqrt{|\Bbbk^\times|}$ with $0 < \rho < |Y|^{-1}$.

Proof. Write $f_i = \sum_{\alpha\in\mathbb{N}^n} f_{i,\alpha} t^\alpha$ and $Y_i^k = \sum_{r\in\mathbb{N}^m} y_r^{(i,k)} x^r$. We have $|Y| = \sup |y_r^{(i,k)}|$. By assumption we have $f_{i,0} = 0$, which ensures that the composition $Y_i^k(f_1, \ldots, f_m)$ is well-defined.

For $d \in \mathbb{N}$, we set $v_d := \max_{1\leq i\leq m, |\alpha|=d} |f_{i,\alpha}|$. We will prove $v_d \leq |Y|^d$ by induction on d. We have $v_0 = 0 \leq 1 = |Y|^0$. Next, fix $d > 0$ and assume we have proved $v_e \leq |Y|^e$ for all $e < d$. Let $\alpha \in \mathbb{N}^n$ with $|\alpha| = d-1$. Then for $1 \leq k \leq n$, as in (4.3.5), we have the recursion

$$f_{i,\tau_k(\alpha)} = \frac{1}{\alpha_k + 1} [Y_i^k(f_1, \ldots, f_m)]_\alpha,$$

where the right hand side is the coefficient of t^α in $Y_i^k(f_1, \dots, f_m)$. We now express this coefficient. For $r \in \mathbb{N}^m$, let $\mathcal{P}(r, \alpha)$ denote the set of partitions of α into $|r|$-tuples. We write an element of $\mathcal{P}(r, \alpha)$ as $\{\alpha_1^{(1)}, \dots, \alpha_{r_1}^{(1)}, \alpha_1^{(2)}, \dots, \alpha_{r_m}^{(m)}\}$, where $\alpha_p^{(q)} \in \mathbb{N}^n$ for each p, q. The coefficient can then be expressed as the finite sum

$$[Y_i^k(f_1, \dots, f_m)]_\alpha = \sum_{r \in \mathbb{N}^m} y_r^{(i,k)} \sum_{\{\alpha_p^{(q)}\} \in \mathcal{P}(r,\alpha)} \prod_{1 \le q \le m} \prod_{1 \le p \le r_q} f_{q, \alpha_p^{(q)}}.$$

We deduce

$$\begin{aligned} |f_{i, \tau_k(\alpha)}| &\le |Y| \max_{\{\alpha_p^{(q)}\} \in \mathcal{P}(r,\alpha)} \prod_{1 \le q \le m} \prod_{1 \le p \le r_q} |f_{q, \alpha_p^{(q)}}| \\ &\le |Y| \max_{\{\alpha_p^{(q)}\} \in \mathcal{P}(r,\alpha)} \prod_{1 \le q \le m} \prod_{1 \le p \le r_q} |Y|^{|\alpha_p^{(q)}|} \\ &\le |Y| \times |Y|^{|\alpha|} = |Y|^d. \end{aligned}$$

Let $0 < \rho < |Y|^{-1}$ in $\sqrt{|\mathbb{k}^\times|}$, we then have $|f_{i,\alpha}|\rho^{|\alpha|} \le (\rho|Y|)^{|\alpha|} \le 1$. This implies that $f_i \in T_n(\rho)$, since $\rho|Y| < 1$, and that $|f_i|_\rho \le 1$, and the lemma follows. □

4.3.1.2 Frobenius theorem

We prove the formal and non-archimedean analogs of the Frobenius theorem in differential geometry, which states that a local basis of commuting vector fields comes from coordinates.

Theorem 4.3.7. *Let $B = \operatorname{Spf} \mathbb{k}[\![t_1, \dots, t_n]\!]$ and let $(Y_i)_{1 \le i \le n}$ be a commuting basis of vector fields on B. Then there exists a unique automorphism $\varphi \colon B \to B$ such that $d\varphi(\partial_{t_i}) = \varphi^* Y_i$ for all $1 \le i \le n$.*

Proof. Let b be the closed point of B, given by $t_1 = \cdots = t_n = 0$. Let $\mathfrak{m} = (t_1, \dots, t_n)$ denote the maximal ideal of $\mathbb{k}[\![t_1, \dots, t_n]\!]$. We write $Y_i = \sum_k Y_i^k \partial_{t_k}$, with $Y_i^k \in \mathbb{k}[\![t_1, \dots, t_n]\!]$. Working in coordinates, giving $\varphi \colon B \to B$ is equivalent to giving $\varphi_1, \dots, \varphi_n \in \mathbb{k}[\![t_1, \dots, t_n]\!]$ such that $\varphi_i(0) = 0$. Furthemore, φ is invertible if and only if the differential at b is invertible, i.e. if and only if the matrix $\left(\frac{\partial \varphi_i}{\partial t_j}\right)_{1 \le i,j \le n}$ is invertible at $t = 0$. The condition $d\varphi(\partial_{t_i}) = \varphi^* Y_i$ is equivalent to

$$\sum_{1 \le k \le n} \frac{\partial \varphi_i}{\partial t_k}(t) \partial_{t_k} = \sum_{1 \le k \le n} Y_i^k(\varphi_1(t), \dots, \varphi_n(t)) \partial_{t_k}. \tag{4.3.8}$$

Since $\varphi_i(0) = 0$, the composition on the right hand side is well-defined.

For $1 \leq i \leq n$, consider the first-order quasi-linear differential operator

$$\begin{aligned} D_i \colon \mathbb{k}^n \otimes_{\mathbb{k}} \mathfrak{m} &\longrightarrow \mathbb{k}^n \otimes_{\mathbb{k}} \mathbb{k}[\![t_1, \dots, t_n]\!] \\ (\varphi_1, \dots, \varphi_n) &\longmapsto \left(\frac{\partial \varphi_k}{\partial t_i} - Y_i^k(\varphi_1, \dots, \varphi_n) \right)_{1 \leq k \leq n}. \end{aligned}$$

Equation (4.3.8) is equivalent to $D_i(\varphi_1, \dots, \varphi_n) = 0$ for $1 \leq i \leq n$. We will prove that the system $\{D_i = 0\}$ is generalized flat, and apply Proposition 4.3.4. Condition (1) of Definition 4.3.2 is satisfied because the components of Y_i are power series in the argument. We now check Condition (2). Assume $(\varphi_1, \dots, \varphi_n) \in \mathbb{k}^n \otimes_{\mathbb{k}} \mathfrak{m}$ satisfies $D_i(\varphi_1, \dots, \varphi_n) = 0 \mod \mathfrak{m}^d$ for all $1 \leq i \leq n$. Then, since $[Y_i, Y_j] = 0$, we have $(1 \leq i, k \leq n)$

$$\begin{aligned} \frac{\partial (Y_j^k(\varphi_1, \dots, \varphi_n))}{\partial t_i} &= \sum_s \frac{\partial \varphi_s}{\partial t_i} \frac{\partial Y_j^k}{\partial t_s}(\varphi_1, \dots, \varphi_n) \mod \mathfrak{m}^d \\ &= \sum_s Y_i^s(\varphi_1, \dots, \varphi_n) \frac{\partial Y_j^k}{\partial t_s}(\varphi_1, \dots, \varphi_n) \mod \mathfrak{m}^d \\ &= \sum_s Y_j^s(\varphi_1, \dots, \varphi_n) \frac{\partial Y_i^k}{\partial t_s}(\varphi_1, \dots, \varphi_n) \mod \mathfrak{m}^d \\ &= \frac{\partial (Y_i^k(\varphi_1, \dots, \varphi_n))}{\partial t_j} \mod \mathfrak{m}^d. \end{aligned}$$

We deduce from Proposition 4.3.4 that the components $(\varphi_1, \dots, \varphi_n)$ of φ are uniquely determined and that they can be constructed inductively. The associated morphism $\varphi \colon B \to B$ is automatically an automorphism, because its differential at b is given by the matrix $(Y_j^i(0))_{1 \leq i,j \leq n}$, which is invertible by assumption. □

Lemma 4.3.9. *Let X be a $\mathbb{k}$-analytic space, and $x \in X$ a smooth $\mathbb{k}$-rational point. There exists an admissible open neighborhood $U \subset X$ of x and an open immersion $U \hookrightarrow \operatorname{Sp} T_n$, where $n = \dim_x X$.*

Proof. Since x is a smooth rigid point, there exists an admissible affinoid neighborhood $U \subset X$ of x and an étale map $U \to Y := \operatorname{Sp} T_n$, with $n = \dim_x X$. Up to shrinking U, we may assume that $f^{-1}(f(x)) = \{x\}$. We will show that $f_x^* \colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism, then f restricts to an open immersion on an affinoid open neighborhood of x by [BGR84, 7.3.3/Corollary 6]. By [BGR84, 7.3.3/Proposition 5], it is enough to check that the induced morphism $\hat{f}_x^* \colon \widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ on the completed local rings is an isomorphism.

Since f is étale, we have $f_x^*(\mathfrak{m}_{f(x)}) = \mathfrak{m}_x$, in particular $\widehat{\mathcal{O}}_{X,x}$ is a complete $\widehat{\mathcal{O}}_{Y,f(x)}$-module. Since x is a $\mathbb{k}$-rational, the map $\hat{f}_x^*$ is an isomorphism modulo $\mathfrak{m}_{f(x)}$, and hence $\hat{f}_x^*$ is surjective by [Stacks, Tag 0315]. The Krull dimension of noetherian local rings is invariant under completion. Since $\dim_x X = \dim_{f(x)} Y$, necessarily $\hat{f}_x^*$ is injective using [Stacks, Tag 00KW]. This concludes the proof. $\square$

Theorem 4.3.10. *Let B be a smooth $\Bbbk$-analytic space, and $(Y_1, \ldots, Y_n)$ be a commuting basis of local vector fields around a rational point $b \in B$. Then, there exists admissible open neighborhoods $V \subset B$ of b and $U \subset \operatorname{Sp} T_n$ of 0 and an isomorphism $\varphi \colon U \to V$ such that $\varphi(b) = 0$ and $d\varphi(\partial_{t_i}|_U) = \varphi^*(Y_i|_V)$.*

Proof. By Lemma 4.3.9, we may assume that $B \simeq \operatorname{Sp} T_n$. We start by applying Theorem 4.3.7 to the restriction of the vector fields $(Y_i)_{1 \leq i \leq n}$ to a formal neighborhood $\widehat{B} = \operatorname{Spf} \Bbbk[\![t_1, \ldots, t_n]\!]$ of $0 \in \operatorname{Sp} T_n$. This produces a unique formal automorphism $\widehat{\varphi} = (\varphi_1, \ldots, \varphi_n) \colon \widehat{B} \to \widehat{B}$ satisfying the relations (4.3.8). We will prove that $\widehat{\varphi}$ extends to admissible open neighborhoods of 0.

Let $|Y| := \max_i |Y_i|$, and let $\rho \in \sqrt{|\Bbbk^\times|}$ such that $\rho < \min(1, |Y|^{-1})$. By Lemma 4.3.6, $\widehat{\varphi}$ extends to a map $\varphi \colon \operatorname{Sp} T_n(\rho) \to \operatorname{Sp} T_n$. The truncations of φ coincide with the truncations of $\widehat{\varphi}$. In particular, they induce isomorphisms $T_n/\mathfrak{m}^d \xrightarrow{\sim} T_n(\rho)/\mathfrak{m}^d$ for all $d \geq 0$. We conclude the proof using [Bos14, §3.3 Lemma 18(ii)]. $\square$

4.3.2 Decomposition theorems for F-manifolds

In this subsection, we prove the decomposition theorems for formal and non-archimedean versions of F-manifolds; see Theorems 4.3.13 and 4.3.20.

4.3.2.1 Decomposition theorem for formal F-manifolds

The notion of F-manifold was introduced by Hertling and Manin as a weaker version of Frobenius manifolds; see [HM99] and the monograph [Man99, I.§5].

Definition 4.3.11 (F-manifold)**.** Let B be a smooth formal scheme or a smooth $\Bbbk$-analytic space. An F-manifold structure on B is a $\mathcal{O}_B$-bilinear commutative associative product $\star$ on the tangent bundle TB, satisfying the *F-identity*: for any (local) vector fields X, Y, Z, W we have

$$P_{X \star Y}(Z, W) = X \star P_Y(Z, W) + (-1)^{|X||Y|} Y \star P_X(Z, W), \tag{4.3.12}$$

where

$$P_X(Z, W) := [X, Z \star W] - [X, Z] \star W - (-1)^{|X||Z|} Z \star [X, W].$$

We prove the following decomposition result for formal F-manifolds.

Theorem 4.3.13. *Let B be a formal neighborhood of a rational point b in a smooth $\Bbbk$-variety. Let $\star$ denote an F-manifold structure with unit on B. Assume that there exists a splitting as $\Bbbk$-algebras*

$$T_bB = \bigoplus_{i \in I} A_i. \tag{4.3.14}$$

Then there exists formal F-manifolds $(B_i, \star_i)$ such that

1. *$(B, \star)$ is isomorphic to $\prod_{i \in I}(B_i, \star_i)$ as F-manifolds with unit,*
2. *and the induced decomposition of $(TB, \star)$ restricts to (4.3.14) at b.*

The idea of the proof is the following. We obtain a decomposition of TB into sheaves of subalgebras in Lemma 4.3.15, induced from that of T_bB. Proposition 4.3.19 will show that the direct summands of TB define commuting foliations (in the sense of [AD13, Definition 2.1]). We can then integrate them using the Frobenius theorem (Theorem 4.3.7).

Lemma 4.3.15. *Let A be a unital associative commutative $\Bbbk$-algebra and I a finite set. Assume A admits a splitting $A \simeq \bigoplus_{i \in I} A_i$ as $\Bbbk$-algebras. Then the splitting extends over any deformation of A over $\Bbbk[\![t_1, \dots, t_n]\!]$.*

Proof. Let $\widetilde{R} := \Bbbk[\![t_1, \dots, t_n]\!]$ and let $\widetilde{A}$ be an $\widetilde{R}$-algebra which is a deformation of A. Let $\mathfrak{m} = (t_1, \dots, t_n)$, and for $k \in \mathbb{N}$, $A_k := \widetilde{A}/\mathfrak{m}^{k+1}\widetilde{A}$ and $B_k := (\widetilde{R}/\mathfrak{m}^{k+1})^{\oplus I}$.

We will prove by induction on $\ell \geq 0$ that for any $0 \leq k \leq \ell$, there are $\widetilde{R}$-algebra maps $B_k \to A_k$ that fit into a commutative diagram

$$\begin{tikzcd} A_\ell \arrow[r] & A_{\ell-1} \arrow[r] & \cdots \arrow[r] & A_1 \arrow[r] & A_0 \\ B_\ell \arrow[r] \arrow[u] & B_{\ell-1} \arrow[r] \arrow[u] & \cdots \arrow[r] & B_1 \arrow[r] \arrow[u] & B_0. \arrow[u] \end{tikzcd} \tag{4.3.16}$$

For $\ell = 0$, the $\widetilde{R}$-algebra structures on $A_0 \simeq A$ and $B_0 \simeq \Bbbk^{\oplus I}$ are induced by the compositions of the quotient map $\widetilde{R} \to \Bbbk$ with the structural maps $\Bbbk \to A$ and $\Bbbk \to \Bbbk^{\oplus I}$. In particular, the map $B_0 \to A_0$ provided by the splitting $A \simeq \bigoplus_{i \in I} A_i$ is a map of $\widetilde{R}$-algebras.

Now assume that the maps $B_k \to A_k$ are constructed for $k \leq \ell$. Let us prove that the dashed arrow exists in the commutative diagram of $\widetilde{R}$-algebras

$$\begin{tikzcd} A_{\ell+1} \arrow[r] & A_\ell \\ B_{\ell+1} \arrow[u, dashed] \arrow[r] & B_\ell. \arrow[u] \end{tikzcd}$$

In other words, we are looking for a lift of the composite map $B_{\ell+1} \to A_\ell$ to $A_{\ell+1}$. Since $\ker(A_{\ell+1} \to A_\ell) = \mathfrak{m}^{\ell+1} A_{\ell+1}$, the algebra $A_{\ell+1}$ is a square-zero extension of A_ℓ. Then, the obstruction to the existence of this lift is a class in $\mathrm{Ext}^1_{\widetilde{R}}(\mathbb{L}_{B_{\ell+1}/\widetilde{R}} \otimes_{B_{\ell+1}} A_\ell, \mathfrak{m}^{\ell+1} A_{\ell+1})$. Since

$$\mathbb{L}_{B_{\ell+1}/\widetilde{R}} \simeq \mathbb{L}_{B_{\ell+1}/(\widetilde{R}/\mathfrak{m}^{\ell+1})} = 0,$$

the obstruction vanishes, and the lift always exists, concluding the induction.

By functoriality of limits in the category of $\widetilde{R}$-algebras, we obtain a map of $\widetilde{R}$-algebras

$$\widetilde{R}^{\oplus I} \simeq \lim_k B_k \longrightarrow \lim_k A_k \simeq \widetilde{A},$$

concluding the proof. □

We now state two lemmas needed to prove Proposition 4.3.19.

Lemma 4.3.17. *Let R be a local domain. Let $f\colon M \to N$ be a surjective morphism of finite free R-modules, and $D \subset M$ a free submodule. Assume (1) $D \cap \ker f = 0$, (2) $\operatorname{rk} D = \operatorname{rk} N$ and (3) M/D is torsion-free. Then f restricts to an isomorphism $D \xrightarrow{\sim} N$.*

Proof. Let $S \coloneqq \mathrm{Frac}(R)/R$. We have $N/f(D) \simeq M/(\ker f + D)$. We prove that this module is torsion-free. Since $\ker f \cap D = 0$, we have a short exact sequence:

$$0 \longrightarrow \ker f \longrightarrow M/D \longrightarrow M/(\ker f + D) \longrightarrow 0.$$

Applying $\otimes_R S$ gives the exact sequence

$$0 = \mathrm{Tor}_1^R(M/D, S) \longrightarrow \mathrm{Tor}_1^R(M/(\ker f + D), S) \longrightarrow \ker f \otimes_R S \xrightarrow{\varphi} M/D \otimes_R S,$$

and we see that $M/(\ker f + D)$ is torsion-free if and only if φ is injective. Since M/D is torsion-free, the module $D \otimes_S R$ is identified with a submodule of $M \otimes_R S$ and we have $M/D \otimes_R S \simeq (M \otimes_R S)/(D \otimes_R S)$. Since $M/\ker f \simeq N$ is torsion-free, the module $\ker f \otimes_R S$ is identified with a submodule of $M \otimes_R S$, and φ corresponds to the composition

$$\ker f \otimes_R S \longrightarrow M \otimes_R S \longrightarrow (M \otimes_R S)/(D \otimes_R S),$$

where the first map is the canonical inclusion and the second map is the canonical projection. Then, since $\ker f + D$ is torsion-free, we have

$$\ker(\varphi) \simeq (\ker f \otimes_R S) \cap (D \otimes_R S) \simeq (\ker f \cap D) \otimes_R S = 0.$$

We deduce that $N/f(D)$ is torsion-free. But since $\operatorname{rk} N = \operatorname{rk} f(D)$, the quotient $N/f(D)$ is a torsion module. We conclude that $N/f(D) = 0$, and the lemma follows. $\square$

Lemma 4.3.18. *Let $B = \operatorname{Spf} \Bbbk[\![t_1, \ldots, t_n]\!]$. Let $\mathcal{D}$ be a free $\mathcal{O}_B$-subsheaf of TB stable under the Lie bracket and such that $TB/\mathcal{D}$ is torsion-free. Then $\mathcal{D}$ admits an $\mathcal{O}_B$-basis of commuting vector fields.*

Proof. We denote by ∂_i the vector field associated to t_i. The coordinates $(t_i)_{1 \leq i \leq n}$ provide a trivialization $TB = \bigoplus_{1 \leq i \leq n} \mathcal{O}_B \partial_i$.

Let m denote the rank of $\mathcal{D}$, then up to reordering the coordinates we may assume $\mathcal{D} \cap \bigoplus_{m+1 \leq i \leq n} \mathcal{O}_B \partial_i = 0$. If $m = n$ there is nothing to show. Assume $m < n$, then there exists i_1 such that $\mathcal{O}_B \partial_{i_1} \cap \mathcal{D} = 0$. Then $\mathcal{D}^{(1)} \coloneqq \mathcal{D} \oplus \mathcal{O}_B \partial_{i_1}$ is a free $\mathcal{O}_B$-module of rank $m+1$. We can thus apply the same argument inductively until we obtain a free $\mathcal{O}_B$-module of rank n, and obtain in this way vector fields $(\partial_{i_1}, \ldots, \partial_{i_{n-m+1}})$ such that $\mathcal{D} \cap \bigoplus_{m+1 \leq j \leq n} \mathcal{O}_B \partial_{i_j} = 0$.

Let $B' = \operatorname{Spf} \Bbbk[\![t_1, \cdots, t_m]\!]$ and $\pi \colon B \to B'$ denote the canonical projection. Let $\psi \colon \mathcal{D} \to \pi^* TB'$ denote the restriction of $d\pi \colon TB \to \pi^* TB'$. The kernel of $d\pi$ is $\bigoplus_{m+1 \leq i \leq n} \mathcal{O}_B \partial_i$, so ψ is injective. By Lemma 4.3.17, ψ is an isomorphism. Let ∂'_i denote the vector field of B' associated to t_i. We define $X_i \coloneqq \psi^{-1}(\pi^* \partial'_i)$. By construction $(X_i)_{1 \leq i \leq m}$ is an $\mathcal{O}_B$-basis of $\mathcal{D}$.

We now check that $[X_i, X_j] = 0$. The $\mathcal{O}_B$-linearity of $d\pi$ and π^* implies

$$d\pi [X_i, X_j] = \pi^* [\partial'_i, \partial'_j] = 0.$$

Since $[X_i, X_j]$ is a section of $\mathcal{D}$ and $d\pi$ restricted to $\mathcal{D}$ is an isomorphism, we deduce that X_i and X_j commute. $\square$

Proposition 4.3.19. *Let $B = \operatorname{Spf} \Bbbk[\![t_1, \dots, t_n]\!]$ and $\star$ an F-manifold structure with unit on B. Assume that we have a decomposition into sheaves of subalgebras $(TB, \star) = \bigoplus_{i \in I} (\mathcal{D}_i, \star|_{\mathcal{D}_i})$. Then:*

1. *For all i we have $[\mathcal{D}_i, \mathcal{D}_i] \subset \mathcal{D}_i$.*
2. *For $i \neq j$ we have $[\mathcal{D}_i, \mathcal{D}_j] = 0$.*
3. *There exists an automorphism $\varphi \colon B \to B$ and a partition $\{1, \cdots, n\} = \coprod_{i \in I} J_i$ such that, for each $i \in I$, the pullback $\varphi^* \mathcal{D}_i$ is generated by $\{d\varphi(\partial_{t_j})\}_{j \in J_i}$.*

Proof. For $i \in I$, let $p_i \colon TB \to \mathcal{D}_i$ denote the projection, corresponding to the multiplication by the identity section e_i of $\mathcal{D}_i$. We have $p_i^2 = p_i$, $p_i \circ p_j = \delta_{ij}$ and $\bigoplus_{i \in I} p_i = \mathrm{id}$, thus $\ker p_i = \bigoplus_{j \neq i} \mathcal{D}_j$.

Let $i \in I$, we prove that $\mathcal{D}_i$ is stable under Lie bracket. Let X, Y be sections of $\mathcal{D}_i$. Since $e_i \star X = X$, the F-identity gives

$$P_X(e_i, Y) = e_i \star P_X(e_i, Y) + X \star P_{e_i}(e_i, Y).$$

The left-hand side equals

$$P_X(e_i, Y) = [X, Y] - [X, e_i] \star Y - e_i \star [X, Y],$$

and the terms on the right-hand side are

$$\begin{aligned} e_i \star P_X(e_i, Y) &= e_i \star ([X, Y] - [X, e_i] \star Y - e_i \star [X, Y]) \\ &= -e_i \star [X, e_i] \star Y \\ &= -Y \star [X, e_i], \end{aligned}$$

and

$$\begin{aligned} X \star P_{e_i}(e_i, Y) &= X \star ([e_i, Y] - e_i \star [e_i, Y]) \\ &= X \star [e_i, Y] - X \star e_i \star [e_i, Y] \\ &= 0, \end{aligned}$$

where we used $e_i \star X = X$, $e_i \star Y = Y$, $e_i \star e_i = e_i$ and the commutativity of the product. Thus, the F-identity above reduces to $[X, Y] = e_i \star [X, Y]$. Equivalently, $[X, Y]$ is a section of $\mathcal{D}_i$, proving (1).

Fix $i, j \in I$ with $i \neq j$. Let X and Y be sections of $\mathcal{D}_i$ and $\mathcal{D}_j$ respectively, in particular $e_i \star X = X$ and $e_j \star Y = Y$. We need to show $[X, Y] = 0$. We have

$$\begin{aligned}
[X, Y] &= [e_i \star X, e_j \star Y] \\
&= P_{e_i \star X}(e_j, Y) + [e_i \star X, Y] \star Y + [e_i \star X, Y] \star e_j \\
&= P_{e_i \star X}(e_j, Y) + \Big(P_{e_j}(e_j, X) + [e_j, e_i] \star X + [e_j, X] \star e_i\Big) \star Y \\
&\quad + (P_Y(e_i, X) + [Y, e_i] \star X + [Y, X] \star e_i) \star e_j \\
&= e_i \star P_X(e_j, Y) + X \star P_{e_i}(e_j, Y) + Y \star P_{e_j}(e_i, X) + e_j \star P_Y(e_i, X) \\
&= (e_i - e_j) \star [X, Y] + X \star [e_i, Y] + Y \star [e_j, X].
\end{aligned}$$

Multiplication by e_i shows that $X \star [e_i, Y] = 0$. By symmetry, we also have $Y \star [e_j, X] = 0$, so the equation reduces to

$$[X, Y] = (e_i - e_j) \star [X, Y].$$

Multiplication by e_k for k different from i and j gives $e_k \star [X, Y] = 0$, so $[X, Y]$ is a section of $\mathcal{D}_i \oplus \mathcal{D}_j$. We then have

$$(e_i + e_j) \star [X, Y] = [X, Y] = (e_i - e_j) \star [X, Y].$$

We deduce $e_j \star [X, Y] = 0$, and by symmetry $e_i \star [X, Y] = -e_i \star [Y, X] = 0$. Thus $[X, Y] = 0$, and (2) is proved.

By (1) and (2), the decomposition $TB = \bigoplus_{i \in I} \mathcal{D}_i$ is a decomposition into commuting subsheaves of Lie algebras. For each $i \in I$, we have $TB/\mathcal{D}_i \simeq \bigoplus_{j \neq i} \mathcal{D}_j$, which is torsion-free. By Lemma 4.3.18, $\mathcal{D}_i$ admits an $\mathcal{O}_B$-basis of commuting vector fields. By (2), these bases assemble into a basis of commuting vector fields for sections of TB. Then (3) follows by applying Theorem 4.3.7 to the union of these bases. $\square$

Proof of Theorem 4.3.13. By [Stacks, Tag 0C0S(2)], we may assume that the base B has the form $\operatorname{Spf} \Bbbk[\![t_1, \dots, t_n]\!]$. The sheaf of algebras $(TB, \star)$ corresponds to a formal deformation of $(T_b B, \star|_b)$ over $\Bbbk[\![t_1, \dots, t_n]\!]$. By Lemma 4.3.15, we obtain a decomposition into sheaves of subalgebras $(TB, \star) = \bigoplus_{i \in I}(\mathcal{D}_i, \star|_{\mathcal{D}_i})$ extending the decomposition of the fiber at b. Let $\varphi\colon B \to B$ be the change of coordinates provided by Proposition 4.3.19(3) and let $\{1, \dots, n\} = \coprod_{i \in I} J_i$ be the associated partition. Let $\mathcal{E}_i \coloneqq \bigoplus_{j \in J_i} \mathcal{O}_B \partial_{t_j} \subset TB$, its image under $d\varphi$ generates $\varphi^* \mathcal{D}_i$.

Since φ is an automorphism of the formal neighborhood of a point, the differential $d\varphi\colon TB \to \varphi^* TB$ is an isomorphism. Then, we can produce another F-manifold structure on B, which we denote by $\varphi^*(\star)$, such that $\varphi\colon (B, \varphi^*(\star)) \to (B, \star)$ is an isomorphism of F-manifolds. Let $B_i \coloneqq \operatorname{Spf} \Bbbk[\![t_j,\ j \in J_i]\!]$, let $\iota_i\colon B_i \to B$ be the canonical closed immersion. By construction the subsheaves $\mathcal{E}_i$ are stable under $\varphi^*(\star)$. Thus the restriction $\varphi^*(\star)|_{\mathcal{E}_i}$ is well-defined, and induces an F-manifold structure $\star_i$ on B_i, such that $\iota_i\colon (B_i, \star_i) \to (B, \varphi^*(\star))$ is a closed immersion of F-manifolds. Since $(B, \varphi^*(\star)) \simeq \prod_{i\in I}(B_i, \star_i)$, we obtain (1), and (2) holds by construction. □

4.3.2.2 Decomposition theorem for non-archimedean F-manifolds

Theorem 4.3.20. *Let B be a smooth $\Bbbk$-analytic space endowed with an F-manifold structure $\star$ with unit, and $b \in B$ a $\Bbbk$-rational point. Assume there exists a splitting as $\Bbbk$-algebras*

$$T_bB = \bigoplus_{i\in I} A_i. \tag{4.3.21}$$

Then there exist an admissible open neighborhood U of b and non-archimedean F-manifolds with unit $(U_i, \star_i)$ such that

1. *$(U, \star|_U)$ is isomorphic to $\prod_{i\in I}(U_i, \star_i)$ as F-manifolds with unit,*
2. *and the induced decomposition of $(TU, \star|_U)$ restricts to (4.3.21) at b.*

Lemma 4.3.22. *Let $(B, \star)$ and b be as in Theorem 4.3.20. Assume there exists a splitting as $\Bbbk$-algebras*

$$T_bB = \bigoplus_{i\in I} A_i.$$

Then there exists an admissible open neighborhood U of b, and a decomposition into sheaves of subalgebras $(TU, \star|_U) = \bigoplus_{i\in I}(\mathcal{D}_i, \star|_{\mathcal{D}_i})$ extending the decomposition of T_bB.

Proof. In this proof, we view the rigid $\Bbbk$-analytic spaces as Berkovich spaces. Then the base B is Hausdorff. Let $X \coloneqq \operatorname{Spec}^{\mathrm{an}} TB$ be the relative analytic spectrum. Since TB is a finite free $\mathcal{O}_B$-module, the structural map $f\colon \operatorname{Spec}^{\mathrm{an}} TB \to B$ is proper as Berkovich spaces, in particular proper as topological spaces.

The splitting of T_bB produces a surjection $X_b = \operatorname{Spec}^{\mathrm{an}} T_bB \to \coprod_{i\in I} \operatorname{Sp} \Bbbk$. This implies that $X_b = \coprod_{i\in I} X_{b,i}$, where $X_{b,i}$ is the preimage of the i-th copy of $\operatorname{Sp} \Bbbk$. Let $U \subset B$ be the open neighborhood of b given by Lemma 4.3.23, with $f^{-1}(U) = \coprod_{i\in I} W_i$. We obtain a map $X \times_B U \to \coprod_{i\in I} U$ extending $X_b \to \coprod_{i\in I} \operatorname{Sp} \Bbbk$ by mapping W_i to the i-th copy of U under f. This is equivalent to a map of sheaves of $\mathcal{O}_U$-algebras $\mathcal{O}_U^{\oplus I} \to TU$, producing the desired splitting. $\square$

Lemma 4.3.23. *Let $f\colon X \to B$ be a proper map between Haussdorff topological spaces. Let $b \in B$, assume that $f^{-1}(b) = \coprod_{i\in I} X_{b,i}$ for a finite set I. Then, there exists an open neighborhood $U \subset B$ of b such that $f^{-1}(U)$ is a disjoint union $\coprod_{i\in I} W_i$, and $W_i \cap f^{-1}(b) = X_{b,i}$.*

Proof. Since f is proper, the fiber $f^{-1}(b)$ is compact. Hence, each $X_{b,i}$ is compact. Since X is Hausdorff, there exists open subsets $V_i \subset X$ containing $X_{b,i}$ with $V_i \cap V_j = \emptyset$ for $i \neq j$. Since f is proper, it is closed, so $U := f\Big(\big(\bigcup_i V_i\big)^{\complement}\Big)^{\complement}$ is open in B. Let $W_i := V_i \cap f^{-1}(U)$. Since $f^{-1}(U) \cap (\bigcup_{i\in I} V_i)^{\complement} = \emptyset$, we have $f^{-1}(U) = \coprod_{i\in I} W_i$. By construction of V_i, we have $W_i \cap f^{-1}(b) = X_{b,i}$, completing the proof. $\square$

Proof of Theorem 4.3.20. By Lemma 4.3.22, there exists an admissible open neighborhood U_1 of b and a decomposition into sheaves of subalgebras

$$(TU_1, \star|_{U_1}) = \bigoplus_{i\in I} (\mathcal{D}_i, \star|_{\mathcal{D}_i}),$$

extending the decomposition of T_bB. As in the proof of Proposition 4.3.19, the F-identity implies that $\{\mathcal{D}_i\}_{i\in I}$ define commuting integrable distributions on TU_1.

Up to shrinking U_1, we can choose a local basis of commuting vector fields $(Y_j)_{j\in J_i}$ of $\mathcal{D}_i$ at b, and assemble them into a local commuting basis of TU_1 at b. By Theorem 4.3.10, there exists admissible opens $U_2 \subset U_1$ and $V \subset \operatorname{Sp} T_n$ and an isomorphism $\varphi\colon V \to U_2$ such that $d\varphi(\partial_{t_j}) = \varphi^*(Y_j)$, where $\{t_j\}$ are the analytic coordinates on V centered at 0. We conclude as in the formal case (see Theorem 4.3.13).

$\square$

4.3.3 Decomposition theorems for maximal F-bundles

In this subsection, we establish the spectral decomposition theorem for maximal F-bundles (see Theorems 4.3.32 and 4.3.42).

We consider a maximal F-bundle $(\mathcal{H}, \nabla)$ over a formal (resp. admissible open) neighborhood of a rational point b in a smooth $\Bbbk$-variety (resp. $\Bbbk$-analytic space). Let $K_b := \nabla_{u^2\partial_u}|_{b,0}$. Consider a decomposition of the fiber $\mathcal{H}_{b,0} \simeq \bigoplus_{i\in I} H_i$ stable under

K_b, such that the induced endomorphisms $K_b|_{H_i}$ and $K_b|_{H_j}$ have disjoint spectra for each $i \neq j$. Our spectral theorem asserts that this produces a decomposition of $(\mathcal{H}, \nabla)$ into a product of maximal F-bundles $(\mathcal{H}_i, \nabla_i)/B_i$ extending the decomposition of $\mathcal{H}_{b,0}$. We refer to Section 4.1 for an outline of the proof.

4.3.3.1 The formal case

Lemma 4.3.24. *Let B be a formal neighborhood of a rational point b in a smooth $\Bbbk$-variety. Let $(\mathcal{H}, \nabla)/B$ be an F-bundle maximal at b, and let $h\colon B \to \mathcal{H}|_{u=0}$ be a section of cyclic vectors (see Definition 4.2.6). The data $\{(\mathcal{H}, \nabla), h\}$ induce a formal F-manifold structure on B with identity.*

Proof. Evaluation on the section of cyclic vectors h provides an isomorphism $\eta := \mu(\cdot)(h)\colon TB \to \mathcal{H}|_{u=0}$, and a commutative and associative product on TB as in (4.2.7). Furthermore $e := \eta^{-1}(h)$ is an identity for this product since for a vector field X we have

$$\eta(X \star e) = \mu(X) \circ \eta(\eta^{-1}(h)) = \mu(X)(h) = \eta(X).$$

We refer to [DH20, Lemma 10] for the proof of the F-identity, which is given there for (TE)-structures. □

Lemma 4.3.25. *Let H be a $\Bbbk$-vector space of finite dimension, and $U \in \operatorname{End}_{\Bbbk}(H)$. Assume we have a decomposition $H = \bigoplus_{i\in I} H_i$ stable under U, such that the induced endomorphisms $U|_{H_i}$ and $U|_{H_j}$ have disjoint spectra for $i \neq j$. Then*

1. $\ker[\cdot, U] \subset \bigoplus_{i\in I} \operatorname{End}_{\Bbbk}(H_i)$, *and*
2. $[\cdot, U]$ *restricts to an isomorphism of $\bigoplus_{i\neq j} \operatorname{Hom}_{\Bbbk}(H_j, H_i)$ onto itself.*

Proof. Let $\Bbbk^a$ denote an algebraic closure of $\Bbbk$. The disjoint spectra assumption implies that $H_i \otimes_{\Bbbk} \Bbbk^a$ is a direct sum of generalized eigenspaces for U. In particular, any endomorphism that commutes with U preserves this decomposition, proving (1). It follows that the restriction $[\cdot, U] : \bigoplus_{i\neq j} \operatorname{Hom}_{\Bbbk}(H_j, H_i) \to \bigoplus_{i\neq j} \operatorname{Hom}_{\Bbbk}(H_j, H_i)$ is injective, hence an isomorphism by comparing dimensions, proving (2). □

Proposition 4.3.26. *Let $(\mathcal{H}, \nabla)$ be an F-bundle over a formal neighborhood $B = \operatorname{Spf} \Bbbk[\![t_1, \ldots, t_n]\!]$ of $b = 0$ in an affine space. Let $K = \nabla_{u^2\partial_u}|_{u=0}$ and $\mathcal{H}_{b,0} = \bigoplus_{i\in I} H_i$ a decomposition stable under K_b such that the induced endomorphisms on H_i have disjoint spectra.*

Let $\mathcal{H}|_{u=0} = \bigoplus_{i\in I}\mathcal{H}_{i,0}$ be a decomposition extending the decomposition of $\mathcal{H}_{b,0}$, and stable under K. Then it extends to a decomposition $\mathcal{H} = \bigoplus_{i\in I}\mathcal{H}_i$ such that $u^2\nabla_{\partial_u}(\mathcal{H}_i) \subset \mathcal{H}_i$.

Proof. Write $t = (t_1, \dots, t_n)$ and $H = \mathcal{H}_{b,0}$. Choose a trivialization $\Phi\colon \mathcal{H} \simeq H \times \operatorname{Spf} \Bbbk[\![t, u]\!]$ such that $\mathcal{H}_{i,0} \simeq H_i \times \operatorname{Spf} \Bbbk[\![t]\!]$. Write the connection in the u direction as

$$\nabla_{\partial_u} = \frac{\partial}{\partial u} + \frac{U(t)}{u^2},$$

where $U(t,u) = \sum_{k\geq 0} U_k(t)u^k$ for $U_k(t) \in \operatorname{End}(H)[\![t]\!]$. By assumption, $U_0(t) \in \bigoplus_{i\in I}\operatorname{End}(H_i)[\![t]\!]$.

We will construct an automorphism $P(t,u) \in \operatorname{Aut}(H \times \operatorname{Spf}\Bbbk[\![t,u]\!])$ with $P(t,0) = \mathrm{id}$, such that $P(t,0) = \mathrm{id}$ and $P^{-1}UP + P^{-1}\frac{\partial P}{\partial u} \in \bigoplus_{i\in I}\operatorname{End}(H_i)[\![t,u]\!]$. Given such a $P(t,u)$, defining $\mathcal{H}_i$ to be the constant extension of H_i in the trivialization $P^{-1}\circ\Phi$ provides the desired splitting of $\mathcal{H}$.

For $m \geq 1$, $T_m(t) \in \operatorname{End}(H)[\![t]\!]$ and $P(t,u) = \mathrm{id} + u^m T_m(t) \in \mathrm{GL}(H)[\![t,u]\!]$, write $(P^*\nabla)_{\partial_u} = \frac{\partial}{\partial u} + u^{-2}\widetilde{U}(t,u)$. We have

$$\widetilde{U}(t,u) - U(t,u) = \sum_{k\geq 0}(-1)^{k+1}u^{m(k+1)}T_m(t)^k[T_m(t), U] + \sum_{k\geq 0}(-1)^k m u^{m(k+1)+1}T_m(t)^{k+1}. \tag{4.3.27}$$

Note that the right-hand side of (4.3.27) has degree at least degree m in u, and the coefficient of u^m is $-[T_m(t), U_0(t)]$.

Let $<$ denote the degree lexicographic order on $\mathbb{N}^n$. For $v = (v_1, \cdots, v_n)$, we write $t^v = t_1^{v_1}\cdots t_n^{v_n}$. Now for $T_m(t) = t^v T_{m,v}$ with $T_{m,v} \in \operatorname{End}(H)$, we have $-[T_m(t), U_0(t)] = -[T_{m,v}, U_0(0)]t^v + T't^{v'}$ where $T' \in \operatorname{End}(H)[\![t]\!]$ and $v < v'$. Write $U_k(t) = \sum_{w\in\mathbb{N}^n} U_{k,w}t^w$. By Lemma 4.3.25, we can choose $T_{m,v}$ such that $U_{m,v} - [T_{m,v}, U_0(0)] \in \bigoplus_{i\in I}\operatorname{End}(H_i)$. By induction on $v \in \mathbb{N}^n$ using the lexicographic order on $\mathbb{N}^n$, we can assume $U_m(t) \in \bigoplus_{i\in I}\operatorname{End}(H_i)[\![t]\!]$. By induction on $m \geq 1$, we can further make $\widetilde{U}(t,u) \in \bigoplus_{i\in I}\operatorname{End}(H_i)[\![t,u]\!]$, completing the proof. □

Lemma 4.3.28. *Write $t = (t_1, \dots, t_n)$. Let $\widetilde{H}$ be a finite free $\Bbbk[\![t]\!]$-module, and $U(t) \in \operatorname{End}(\widetilde{H})$. Let $\widetilde{H} = \bigoplus_{i\in I}\widetilde{H}_i$ be a decomposition of $\widetilde{H}$ stable under $U(t)$. Assume that for $i \neq j$, the induced endomorphisms $U(t)|_{\widetilde{H}_i}$ and $U(t)|_{\widetilde{H}_j}$ have disjoint spectra. Let $X(t) \in \operatorname{End}(\widetilde{H})$ such that $[X(t), U(t)] \in \bigoplus_{i\in I}\operatorname{End}(\widetilde{H}_i)$, then $X(t) \in \bigoplus_{i\in I}\operatorname{End}(\widetilde{H}_i)$.*

Proof. Let $R := \Bbbk[\![t_1, \dots, t_n]\!]$ and $K := \mathrm{Frac}(R)$ its fraction field. Working over K, Lemma 4.3.25 implies that $\ker[\cdot, U] \subset \bigoplus_{i\in I} \mathrm{End}_R(\widetilde{H}_i)$.

We have a decomposition $\mathrm{End}_R(\widetilde{H}) = \bigoplus_{i,j\in I} \mathrm{Hom}_R(\widetilde{H}_j, \widetilde{H}_i)$. Let $X_{i,j}$ denote the components of X with respect to this splitting. Let $Y := \sum_{i\neq j} X_{i,j}$ denote the off-diagonal part of X. We will prove that $Y = 0$. Since $U \in \bigoplus_{i\in I} \mathrm{End}(\widetilde{H}_i)$, the commutator $[Y, U]$ has vanishing diagonal, i.e. it lies in $\bigoplus_{i\neq j} \mathrm{Hom}_R(\widetilde{H}_j, \widetilde{H}_i)$. Furthermore, using the assumption, we see that $[Y, U] = [X, U] - \sum_{i\in I}[X_{i,i}, U]$ is block diagonal. It follows that $[Y, U] = 0$, hence $Y \in \bigoplus_{i\in I} \mathrm{End}_R(\widetilde{H}_i)$. By definition, Y is off-diagonal, so $Y = 0$, proving the lemma. □

The following proposition implies that the decomposition in Proposition 4.3.26 induces a decomposition of F-bundle $(\mathcal{H}, \nabla) \simeq \bigoplus_{i\in I}(\mathcal{H}_i, \nabla_i)$ over B, where ∇_i is the restriction of ∇ to $\mathcal{H}_i$.

Proposition 4.3.29. *In the setting of Proposition 4.3.26, we have $u\nabla_\xi(\mathcal{H}_i) \subset \mathcal{H}_i$ for any vector field ξ on B.*

Proof. Write $t = (t_1, \cdots, t_n)$. Let $H := \mathcal{H}|_{b,0}$, and $H = \bigoplus_{i\in I} H_i$ the splitting induced by the decomposition of $\mathcal{H}$. Fix a trivialization $\mathcal{H} \simeq H \times \mathrm{Spf}\, \Bbbk - [\![t, u]\!]$ such that $\mathcal{H}_i \simeq H_i \times \mathrm{Spf}\, \Bbbk[\![t, u]\!]$, and write

$$\nabla = d + u^{-1} \sum_{1\leq i\leq n} T_i(t, u)dt_i + u^{-2}U(t, u)du,$$

with $U(t, u) = \sum_{k\geq 0} U_k(t)u^k$ and $T_i(t, u) = \sum_{k\geq 0} T_{i,k}(t)u^k$. By assumption, we have $U(t, u) \in \bigoplus_{i\in I} \mathrm{End}(H_i)[\![t, u]\!]$. In particular, $U_0(t)$ induces endomorphisms in $\mathrm{End}(H_i)[\![t]\!]$ for all $i \in I$, and the assumption on the decomposition at $t = u = 0$ implies that those have disjoint spectra.

Fix $i \in \{1, \dots, n\}$. The flatness equation $[\nabla_{\partial_u}, \nabla_{\partial_{t_i}}] = 0$ reads

$$\frac{\partial(u^{-1}T_i)}{\partial u} - \frac{\partial(u^{-2}U)}{\partial t_i} = u^{-3}[T_i, U].$$

Splitting this equation according to powers of u gives $[T_{i,0}, U_0] = 0$, and for $k \geq 1$:

$$[T_{i,k}, U_0] = (k-2)T_{i,k-1} - \frac{\partial U_{k-1}}{\partial t_i} - \sum_{\substack{k_1+k_2=k \\ k_1<k}} [T_{k_1}, U_{k_2}]. \tag{4.3.30}$$

We prove by induction on $k \geq 0$ that $T_{i,k}$ is block diagonal, i.e. $T_{i,k} \in \bigoplus_{i\in I} \mathrm{End}(H_i)[\![t]\!]$. The base case $k = 0$ follows from Lemma 4.3.28, because $T_{i,0}$ commutes with

$U_0(t)$. Now, let $k \geq 1$ and assume $T_{i,\ell}(t)$ is block diagonal for $\ell < k$. Since each $U_\ell(t)$ is assumed block diagonal, the right-hand side of (4.3.30) is block diagonal. Applying Lemma 4.3.28, we obtain that $T_{i,k}(t)$ is also block diagonal, completing the proof. $\square$

It remains to show that the above decomposition $(\mathcal{H}, \nabla) \simeq \bigoplus_{i\in I}(\mathcal{H}_i, \nabla_i)$ is compatible with the decomposition of the base.

Lemma 4.3.31. *Let $B \simeq B_1 \times B_2$ be a formal neighborhood of $b = 0$ in a product of affine spaces, and $(\mathcal{H}, \nabla)/B$ be an F-bundle over B. Assume that $\nabla_{u\xi}|_{u=0} = 0$ for all vector fields ξ in the directions of B_2. Then there exists an F-bundle $(\mathcal{H}_1, \nabla_1)/B_1$ such that $\mathrm{pr}_1^*(\mathcal{H}_1, \nabla_1) \simeq (\mathcal{H}, \nabla)$, where pr_1 is the projection $B \simeq B_1 \times B_2 \to B_1$.*

Proof. For $i = 1, 2$, let $t_i = (t_{i,j},\ 1 \leq j \leq n_i)$ denote coordinates on B_i. Let $\mathcal{H}_1 \coloneqq \mathcal{H}|_{B_1\times\{0\}\times \operatorname{Spf} \Bbbk[\![u]\!]}$. By assumption, ∇ has no pole at $u = 0$ in the directions of B_2. Since ∇ is flat, given any trivialization of $\mathcal{H}_1$ we can extend it uniquely by ∇ to a trivialization of $\mathcal{H}$ over $B_1 \times B_2 \times \operatorname{Spf} \Bbbk[\![u]\!]$. This defines an isomorphism $\mathrm{pr}_1^*\mathcal{H}_1 \simeq \mathcal{H}$, and in this trivialization we have

$$\nabla = d + u^{-1} \sum_{1\leq j\leq n_1} T_{1,j}(t_1, t_2, u)dt_{1,j} + u^{-2}U(t_1, t_2, u)du.$$

Since ∇ is flat, we have for all $1 \leq j \leq n_1$ and $1 \leq k \leq n_2$

$$\frac{\partial(u^{-1}T_{1,j})}{\partial t_{2,k}} = 0, \qquad \frac{\partial(u^{-2}U)}{\partial t_{2,k}} = 0.$$

Hence, the connection matrices in the directions of B_1 and the u-direction are independent of t_2. This means that the connection is equal to the pullback of a connection on $B_1 \times \operatorname{Spf} \Bbbk[\![u]\!]$, completing the proof. $\square$

Theorem 4.3.32 (Spectral decomposition theorem)**.** *Let B be a formal neighborhood of a rational point b in a smooth $\Bbbk$-variety, and $(\mathcal{H}, \nabla)$ an F-bundle over B maximal at b. Write $K_b = \nabla_{u^2\partial_u}|_{b,0}$. Assume that we have a decomposition $\mathcal{H}_{b,0} \simeq \bigoplus_{i\in I} H_i$ stable under K_b, and that for any $i \neq j \in I$, the spectra of $K_b|_{H_i}$ and $K_b|_{H_j}$ are disjoint. Then $(\mathcal{H}, \nabla)/B$ decomposes into a product of maximal F-bundles $(\mathcal{H}_i, \nabla_i)/B_i$ extending the decomposition of $\mathcal{H}|_{b,0}$.*

Proof. As in the proof of Theorem 4.3.13, we may assume the base B has the form $\operatorname{Spf} \Bbbk[\![t_1, \cdots, t_n]\!]$. Let $h : B \to \mathcal{H}|_{u=0}$ be a section of cyclic vectors, providing an isomorphism

$$\eta \coloneqq u\nabla|_{u=0}(h) : TB \xrightarrow{\sim} \mathcal{H}|_{u=0}.$$

This induces an F-manifold structure $(B, \star)$ on B by Lemma 4.3.24. In particular, we have a decomposition $T_bB = \bigoplus_{i\in I} E_i$ with $E_i = \eta_b^{-1}(H_i)$. Since the spectra of $K_b|_{H_i}$ and $K_b|_{H_j}$ are disjoint, up to extending the base field, each H_i is a direct sum of generalized eigenspaces for K_b. Since ∇ is flat, it follows that $T_bB = \bigoplus_{i\in I} E_i$ is a splitting of $\Bbbk$-algebra. By Theorem 4.3.13, we obtain a decomposition of F-manifold $B \simeq \prod_{i\in I}(B_i, \star_i)$, extending the decomposition at T_bB. This induces a decomposition of $TB = \bigoplus_{i\in I} \mathcal{E}_i$ as $\mathcal{O}_B$-algebras. We refer to sections of $\mathcal{E}_i$ as being in the directions of B_i.

Under η, we obtain a decomposition $\mathcal{H}|_{u=0} \simeq \bigoplus_{i\in I} \mathcal{H}_{i,0}$. Since the action of K corresponds to multiplication by the Euler vector field, this decomposition is stable under K, and extends the decomposition of $\mathcal{H}_{b,0} \simeq \bigoplus_{i\in I} H_i$. By Propositions 4.3.26 and 4.3.29, this further extends to a decomposition $(\mathcal{H}, \nabla) \simeq \bigoplus_{i\in I}(\mathcal{H}_i, \nabla_i)$.

For each $i \in I$ and ξ not in the directions of B_i, the action of $(\nabla_i)_{u\xi}|_{u=0}$ on $\mathcal{H}_{i,0}$ under η is the restriction of $\xi\star$ to the subalgebra $\mathcal{E}_i$, hence it vanishes. Then by Lemma 4.3.31, $(\mathcal{H}_i, \nabla_i)/B$ isomorphic to a pullback of F-bundle from B_i, which we also denote as $(\mathcal{H}_i, \nabla_i)/B_i$. We thus have a decomposition of F-bundle

$$(\mathcal{H}, \nabla) \simeq \bigoplus_{i\in I} \mathrm{pr}_i^*(\mathcal{H}_i, \nabla_i),$$

where $\mathrm{pr}_i\colon B \simeq \prod_{j\in I} B_j \to B_i$ is the projection to the i-th component.

It remains to check that each F-bundle in the decomposition is maximal. Let $j_i\colon B_i \hookrightarrow B$ be the canonical closed immersion, and $h_i \coloneqq j_i^*h$. We claim that h_i is a section of cyclic vectors for $(\mathcal{H}_i, \nabla_i)/B_i$, i.e. the map $\eta_i\colon \xi \mapsto (\nabla_i)_{u\xi}|_{u=0}(h_i)$ is an isomorphism $TB_i \xrightarrow{\sim} \mathcal{H}_i|_{u=0}$. Since B_i is the formal neighborhood of a point in an affine space, it is enough to check that the stalk of η_i at the closed point b_i of B_i is an isomorphism. This stalk is the composition of the isomorphisms

$$T_{b_i}B_i \longrightarrow E_i \xrightarrow{\eta_b|_{E_i}} H_i,$$

and hence it is an isomorphism, completing the proof. □

Example 4.3.33 (rank 1 maximal F-bundle)**.** Let $B = \operatorname{Spf} \Bbbk[\![t]\!]$ and $b = 0 \in B$. Let $(\mathcal{H}, \nabla)/B$ be an F-bundle, maximal at b. Fixing a trivialization of $\mathcal{H}$, we write the connection as $\nabla = d + u^{-2}p(t,u)du + u^{-1}q(t,u)dt$. Flatness of ∇ reduces to the equation $\frac{\partial(u^{-2}p)}{\partial t} = \frac{\partial(u^{-1}q)}{\partial u}$. Solutions are parameterized by pairs $(\psi(t,u), c) \in \Bbbk[\![t,u]\!] \times \Bbbk$ by the rule

$$p = u\frac{\partial\psi}{\partial u} - \psi + uc, \quad q = \frac{\partial\psi}{\partial t}.$$

The F-bundle is maximal at $t = 0$ if and only if $q(0,0) \neq 0$ or, in terms of ψ, $\frac{\partial \psi}{\partial t}(0,0) \neq 0$.

Example 4.3.34 (simple eigenvalues)**.** Let B be the formal neighborhood of $b = 0$ in an n-dimensional affine space. Let $(\mathcal{H}, \nabla)/B$ be an F-bundle, maximal at b. Assume that $K_b = u^2 \nabla_{\partial_u}|_{b,u=0}$ has simple eigenvalues. Then $(\mathcal{H}, \nabla)/B$ is isomorphic to a product of rank 1 maximal F-bundles.

Concretely, there exists a change of coordinates $f\colon \prod_{1\leq i\leq n} \operatorname{Spf} \Bbbk[\![t_i]\!] \xrightarrow{\sim} B$, and a trivialization of $f^*(\mathcal{H}, \nabla)$ in which the connection takes the form

$$f^*\nabla = d + u^{-1} \begin{pmatrix} \frac{\partial \psi_1}{\partial t_1} dt_1 & & 0 \\ & \ddots & \\ 0 & & \frac{\partial \psi_n}{\partial t_n} dt_n \end{pmatrix} + u^{-2} \begin{pmatrix} u\frac{\partial \psi_1}{\partial u} - \psi_1 + uc_1 & & 0 \\ & \ddots & \\ 0 & & u\frac{\partial \psi_n}{\partial u} - \psi_n + uc_n \end{pmatrix} du,$$

with $(\psi_i, c_i) \in \Bbbk[\![t_{,i}\,, u]\!] \times \Bbbk$ such that $-\psi_i(0,0)$ is an eigenvalue of K_b, and $\frac{\partial \psi_i}{\partial t_i}(0,0) \neq 0$ (see Example 4.3.33).

When K_b has simple eigenvalues, the change of coordinates is obtained by integrating a basis of sections of eigenvectors for the connection in the u-direction.

4.3.3.2 The non-archimedean case

Next, we prove the spectral decomposition theorem in the non-archimedean case. The proof builds on the formal case, but an additional challenge lies in bounding the norms of the coefficients of the gauge transform and establishing non-archimedean convergence. We achieve these bounds through a detailed analysis of the recursive relations of the coefficients; see Proposition 4.3.36.

Lemma 4.3.35. *Let B be an admissible open neighborhood of a rational point b in a smooth $\Bbbk$-analytic space. Let $(\mathcal{H}, \nabla)/B$ be a non-archimedean F-bundle maximal at b. Then there exists an admissible open neighborhood $U \subset B$ of b such that $(\mathcal{H}, \nabla)$ admits a section of cyclic vectors, and the data $\{(\mathcal{H}, \nabla), h\}$ induces a non-archimedean F-manifold structure on U with identity.*

Proof. Being maximal is an open condition, so there exists an admissible open neighborhood $U \subset B$ of b over which a section of cyclic vector h exists. The proof is then identical to the formal case, and relies on explicit computations in local analytic coordinates centered at b. □

Proposition 4.3.36. *Let $(\mathcal{H}, \nabla)$ be an F-bundle over $B = \operatorname{Sp} \Bbbk\langle t_1, \dots, t_n\rangle$, and let $b = 0 \in B$. Let $K = \nabla_{u^2\partial_u}|_{u=0}$ and $\mathcal{H}_{b,0} = \bigoplus_{i\in I} H_i$ a decomposition stable under K_b such that the induced endomorphisms on H_i have disjoint spectra.*

Let $\mathcal{H}|_{u=0} = \bigoplus_{i\in I} \mathcal{H}_{i,0}$ be a decomposition extending the decomposition of $\mathcal{H}_{b,0}$, and stable under K. Then, there exists an admissible open neighborhood $U \subset B$ of b and a decomposition $\mathcal{H}|_U = \bigoplus_{i\in I} \mathcal{H}_i$ such that $\mathcal{H}_i|_{u=0} = \mathcal{H}_{i,0}|_U$ and $u^2\nabla_{\partial_u}(\mathcal{H}_i) \subset \mathcal{H}_i$.

Proof. We keep the setting and notations of Proposition 4.3.26, in particular $H \coloneqq \mathcal{H}_{b,0}$. Let $\leq$ denote the degree lexicographic order on $\mathbb{N}^n$. We denote by $\tau(v)$ the direct successor of $v \in \mathbb{N}^n$ for this order. The gauge transformation P constructed in the formal case is an ordered product

$$P = \prod_{m\geq 1} P_m, \quad P_m = \prod_{v\in\mathbb{N}^n} P_{m,v},$$

where $P_{m,v} = \mathrm{id} + u^m t^v T_{m,v}$ and $T_{m,v} \in \operatorname{End}(H)$. Let ϕ denote the inverse of the restriction of $[\cdot, U_0(0)]$ to $\bigoplus_{i\neq j} \operatorname{Hom}(H_j, H_i)$. The gauge transformations $P_{m,v}$ are constructed inductively, and characterized by the following relations:

$$T_{m,v} = \phi(\text{off-diagonal part of the term } u^m t^v \text{ in } \widetilde{U}_{m,v}), \tag{4.3.37}$$

$$\widetilde{U}_{m,\tau(v)} = P_{m,v}^{-1}\widetilde{U}_{m,v}P_{m,v} + u^2 P_{m,v}^{-1}\frac{\partial P_{m,v}}{\partial u}, \tag{4.3.38}$$

$$\widetilde{U}_{m+1,0} = P_m^{-1}\widetilde{U}_{m,0}P_m + u^2 P_m^{-1}\frac{\partial P_m}{\partial u}, \tag{4.3.39}$$

and $\widetilde{U}_{1,0} = U(t,u)$ is the initial connection matrix. For an element $M(t,u) = \sum_{m,v} M_{m,v}u^m t^v \in \operatorname{End}(H)[\![t,u]\!]$ and $\delta, \varepsilon > 0$, we let

$$|M(t,u)|_{\delta,\varepsilon} \coloneqq \sup_{m\in\mathbb{N}, v\in\mathbb{N}^n} |M_{m,v}|\delta^m\varepsilon^{|v|}.$$

We denote by $\mathbb{D}(\delta,\varepsilon)$ the polydisk $\{|u| \leq \delta, |t| \leq \varepsilon\}$.

Since the gauge transformations restrict to id at $u = 0$, all the matrices $\widetilde{U}_{m,v}(t,u)$ have the same constant term. We denote this common value by U_0, and set $\widetilde{V}_{m,v}(t,u) \coloneqq \widetilde{U}_{m,v}(t,u) - U_0$. Fix $\delta \leq 1$ and $\varepsilon \leq 1$ such that $\delta|\phi| \leq 1$ and $|\phi||\widetilde{V}_{1,0}|_{\delta,\varepsilon} < 1$. This is possible, since $\widetilde{V}_{1,0}(0,0) = 0$.

We prove by a double induction on m and v the inequalities

$$|u^m t^v T_{m,v}|_{\delta,\varepsilon} \leq |\phi||\widetilde{V}_{m,v}|_{\delta,\varepsilon} \leq |\phi||\widetilde{V}_{m,0}|_{\delta,\varepsilon} \leq |\phi||\widetilde{V}_{1,0}|_{\delta,\varepsilon} < 1. \tag{4.3.40}$$

We use the lexicographic order on the product $\mathbb{N}_{>0} \times \mathbb{N}^n$, i.e. $(m, v) < (m', v')$ if and only if $m < m'$ or $m = m'$ and $v < v'$. For $m = 1, v = 0$, the inequalities follow from (4.3.37) and the choice of (δ, ε). Now fix $(m, v) \in \mathbb{N}_{>0} \times \mathbb{N}^n$ with $(m, v) > (1, 0)$, and assume all the inequalities proved for $(m', v') < (m, v)$. Equation (4.3.37) gives

$$|u^m t^v T_{m,v}|_{\delta,\varepsilon} \leq |\phi||\widetilde{V}_{m,v}|_{\delta,\varepsilon}.$$

We now bound $|\widetilde{V}_{m,v}|_{\delta,\varepsilon}$. If $v > 0$, then we can write $v = \tau(w)$ for some $w \geq 0$. The difference between $\widetilde{V}_{m,\tau(w)}$ and $\widetilde{V}_{m,w}$ is given by (4.3.27):

$$\begin{aligned}
\widetilde{V}_{m,\tau(w)} - \widetilde{V}_{m,w} = \widetilde{U}_{m,\tau(w)} - \widetilde{U}_{m,w}& \\
&= \sum_{k\geq 0}(-1)^{k+1}(u^m t^w)^{k+1} T_{m,w}^k [T_{m,w}, \widetilde{V}_{m,w}] \\
&+ \sum_{k\geq 0}(-1)^{k+1}(u^m t^w)^{k+1} T_{m,w}^k [T_{m,w}, U_0] \\
&+ \sum_{k\geq 0}(-1)^{k} u (u^m t^w)^{k+1} T_{m,w}^{k+1}.
\end{aligned}$$

Let us bound each term on the right hand side. Since $|u^m t^w T_{m,w}|_{\delta,\varepsilon} < 1$, we have for all $k \geq 0$

$$|(u^m t^w)^{k+1} T_{m,w}^k [T_{m,w}, \widetilde{V}_{m,w}]|_{\delta,\varepsilon} \leq |u^m t^w T_{m,w}|_{\delta,\varepsilon}^{k+1} |\widetilde{V}_{m,w}|_{\delta,\varepsilon} < |\widetilde{V}_{m,w}|_{\delta,\varepsilon}.$$

By the definition of ϕ and (4.3.37), we have

$$|u^m t^w [T_{m,w}, U_0]|_{\delta,\varepsilon} = |u^m t^w \phi^{-1}(T_{m,w})|_{\delta,\varepsilon} \leq |\widetilde{V}_{m,w}|_{\delta,\varepsilon}. \tag{4.3.41}$$

We can then bound the second term for all $k \geq 0$

$$|(u^m t^w)^{k+1} T_{m,w}^k [T_{m,w}, U_0]|_{\delta,\varepsilon} \leq |u^m t^w T_{m,w}|_{\delta,\varepsilon}^k |[T_{m,w}, U_0]|_{\delta,\varepsilon} < |\widetilde{V}_{m,w}|_{\delta,\varepsilon}.$$

For the third term, using the induction hypothesis and $\delta|\phi| \leq 1$, we obtain for all $k \geq 0$

$$|u^{m(k+1)+1} t^{w(k+1)} T_{m,w}^{k+1}|_{\delta,\varepsilon} \leq \delta(|\phi||\widetilde{V}_{m,w}|_{\delta,\varepsilon})^{k+1} \leq \delta|\phi||\widetilde{V}_{m,w}|_{\delta,\varepsilon} \leq |\widetilde{V}_{m,w}|_{\delta,\varepsilon},$$

where we used $|\phi||\widetilde{V}_{m,w}|_{\delta,\varepsilon} \leq 1$ in the second inequality. Using those bounds, we obtain the inequalities

$$|\widetilde{V}_{m,\tau(w)}|_{\delta,\varepsilon} \leq \max(|\widetilde{V}_{m,w}|_{\delta,\varepsilon}, |\widetilde{V}_{m,\tau(w)} - \widetilde{V}_{m,w}|_{\delta,\varepsilon}) \leq |\widetilde{V}_{m,w}|_{\delta,\varepsilon} \leq |\widetilde{V}_{m,0}|_{\delta,\varepsilon},$$

proving the inductive step when $v > 0$. If $v = 0$, then necessarily $m > 1$ and we can write $m = m' + 1$. We compare $\widetilde{V}_{m'+1,0}$ to $\widetilde{V}_{m',0}$. To do so, write $P_{m'} = \mathrm{id} + u^{m'} R_{m'}(t,u)$. Similarly to the previous case, using (4.3.27) we obtain

$$\begin{aligned}\widetilde{V}_{m'+1,0} - \widetilde{V}_{m',0} &= \widetilde{U}_{m'+1,0} - \widetilde{U}_{m',0} \\ &= \sum_{k\geq 0} (-1)^{k+1} u^{m'(k+1)} R_{m'}^k [R_{m'}, \widetilde{V}_{m,w}] \\ &+ \sum_{k\geq 0} (-1)^{k+1} u^{m'(k+1)} R_{m'}^k [R_{m'}, U_0] \\ &+ \sum_{k\geq 0} (-1)^{k} u^{m'(k+1)+1} R_{m'}^{k+1},\end{aligned}$$

and we will use the induction hypothesis to bound each term. Since $|u^{m'} t^v T_{m',v}|_{\delta,\varepsilon} < 1$ for all $v \in \mathbb{N}^n$, we have $|u^{m'} R_{m'}|_{\delta,\varepsilon} < 1$. In particular, similarly to the case $v > 0$, the first term is bounded by $|\widetilde{V}_{m',0}|_{\delta,\varepsilon}$. To handle the other terms, we use the explicit formula

$$u^{m'} R_{m'} = \sum_{\substack{k\geq 1\\ w\in\mathbb{N}^n}} u^{km'} t^w \sum_{\substack{w_1+\cdots+w_k=w\\ w_1>\cdots>w_k}} T_{m',w_1} \cdots T_{m',w_k}.$$

Using this formula, we obtain

$$\begin{aligned}|[u^{m'} R_{m'}, U_0]|_{\delta,\varepsilon} &\leq \max_{\substack{k\geq 1, w\in\mathbb{N}^n\\ w_1+\cdots+w_k=w}} |u^{km'} t^w [T_{m',w_1}\cdots T_{m',w_k}, U_0]|_{\delta,\varepsilon} \\ &\leq \max_{\substack{k\geq 1, w\in\mathbb{N}^n\\ w_1+\cdots+w_k=w}} \max_{1\leq i\leq k} \left(\prod_{\substack{1\leq j\leq k\\ j\neq i}} |u^{m'} t^{w_j} T_{m',w_j}|_{\delta,\varepsilon} \times |u^{m'} t^{w_i} [T_{m',w_i}, U_0]|_{\delta,\varepsilon} \right) \\ &\leq \max_{\substack{k\geq 1, w\in\mathbb{N}^n\\ w_1+\cdots+w_k=w}} \max_{1\leq i\leq k} \left(\prod_{\substack{1\leq j\leq k\\ j\neq i}} |\phi|||\widetilde{V}_{m',w_j}|_{\delta,\varepsilon} \times |\widetilde{V}_{m',w_i}|_{\delta,\varepsilon} \right) \\ &\leq \max_{\substack{k\geq 1, w\in\mathbb{N}^n\\ w_1+\cdots+w_k=w}} |\widetilde{V}_{m',w_i}|_{\delta,\varepsilon} \leq |\widetilde{V}_{m',0}|_{\delta,\varepsilon}.\end{aligned}$$

For the second inequality, we used the formula for the commutator of a product. The third inequality follows from the induction hypothesis at step (m', w_j), and the inequality (4.3.41) applied to T_{m',w_i}. The fourth and fifth inequalities follow from the induction hypothesis. Then, similarly to the case $v > 0$, we obtain that the second term is bounded by $|\widetilde{V}_{m',0}|_{\delta,\varepsilon}$. We now consider the third term. For $k \geq 1$ and $w_1, \cdots, w_k \in \mathbb{N}^n$, since $|\phi||\widetilde{V}_{m',0}|_{\delta,\varepsilon} \leq 1$ by the induction hypothesis, we have

$$|u^{km'} t^{w_1+\cdots+w_k} T_{m',w_1} \cdots T_{m',w_k}|_{\delta,\varepsilon} \leq (|\phi||\widetilde{V}_{m',0}|_{\delta,\varepsilon})^k \leq |\phi||\widetilde{V}_{m',0}|_{\delta,\varepsilon}.$$

In particular, we have the better bound $|u^{m'} R_{m'}|_{\delta,\varepsilon} \leq |\phi||\widetilde{V}_{m',0}|_{\delta,\varepsilon}$. Since $|\phi||\widetilde{V}_{m',0}|_{\delta,\varepsilon} \leq 1$, we obtain the bound on the third term for all $k \geq 0$

$$|u^{m'(k+1)+1} R_{m'}^{k+1}|_{\delta,\varepsilon} \leq \delta(|\phi||\widetilde{V}_{m',0}|_{\delta,\varepsilon})^{k+1} \leq \delta|\phi||\widetilde{V}_{m',0}|_{\delta,\varepsilon} \leq |\widetilde{V}_{m',0}|_{\delta,\varepsilon}.$$

Similarly to the case $v > 0$, we deduce

$$|\widetilde{V}_{m'+1,0}|_{\delta,\varepsilon} \leq \max(|\widetilde{V}_{m'+1,0}|_{\delta,\varepsilon}, |\widetilde{V}_{m'+1,0} - \widetilde{V}_{m',0}|_{\delta,\varepsilon}) \leq |\widetilde{V}_{m',0}|_{\delta,\varepsilon} \leq |\widetilde{V}_{1,0}|_{\delta,\varepsilon},$$

concluding the induction.

Now, (4.3.40) implies that the product defining P is convergent on the polydisk $\mathbb{D}(\delta,\varepsilon)$, that P^{-1} is also convergent on $\mathbb{D}(\delta,\varepsilon)$, and that $|P|_{\delta,\varepsilon} = |P^{-1}|_{\delta,\varepsilon} = 1$. In particular, the decomposition constructed in the formal case extends to an admissible open neighborhood of $(b, 0)$, completing the proof. □

Theorem 4.3.42 (Non-archimedean spectral decomposition theorem)**.** *Let B be a $\Bbbk$-analytic space, $b \in B$ a smooth $\Bbbk$-rational point, and $(\mathcal{H}, \nabla)$ an F-bundle over B maximal at b. Write $K_b = \nabla_{u^2\partial_u}|_{b,0}$. Assume that we have a decomposition $\mathcal{H}_{b,0} \simeq \bigoplus_{i\in I} H_i$ stable under K_b, and that for any $i \neq j \in I$, the spectra of $K_b|_{H_i}$ and $K_b|_{H_j}$ are disjoint. Then there exists an admissible open neighborhood U of b such that the restriction $(\mathcal{H}|_U, \nabla|_U)/U$ decomposes into a product of maximal F-bundles $(\mathcal{H}_i, \nabla_i)/U_i$ extending the decomposition of $\mathcal{H}_{b,0}$.*

Proof. By Lemma 4.3.9, we can find an admissible neighborhood U of b isomorphic to an admissible open neighborhood of 0 in a $\Bbbk$-analytic affine space. Hence, we may assume that $B = \operatorname{Sp} T_n$ and $b = 0$. By Lemma 4.3.35, up to shrinking B we can find a section of cyclic vectors $h\colon B \to \mathcal{H}|_{u=0}$, providing an isomorphism

$$\eta \coloneqq (u\nabla)|_{u=0}(h)\colon TB \longrightarrow \mathcal{H}|_{u=0},$$

and an F-manifold structure $\star$ on B. The splitting of $\mathcal{H}_{b,0}$ induces a splitting of T_bB as a $\Bbbk$-algebra. By Theorem 4.3.20, there exists an admissible neighborhood U of b such that $(U, \star|_U)$ is isomorphic to a product of F-manifolds $\prod_{i\in I}(U_i, \star_i)$, and the induced decomposition of TU extends the decomposition of T_bB.

We keep denoting by $(\mathcal{H}, \nabla)$ the restriction of the F-bundle to U. The decomposition of TU induces a decomposition $\mathcal{H}|_{u=0} \simeq \bigoplus_{i\in I} \mathcal{H}_{i,0}$ satisfying the assumptions of Proposition 4.3.36. As in the formal case, this implies that there exists F-bundles $(\mathcal{H}_i, \nabla_i)/U_i$ such that

$$(\mathcal{H}, \nabla) \simeq \bigoplus_{i\in I} \operatorname{pr}_i^*(\mathcal{H}_i, \nabla_i),$$

where $\mathrm{pr}_i \colon U \simeq \prod_{j\in I} U_j \to U_i$ is the projection.

Let b_i denote the image of b under the projection $U \to U_i$, let $j_i \colon U_i \hookrightarrow U$ denote the canonical closed immersion and $h_i := j_i^* h$. As in the formal case, the stalk at b_i of the map $\eta_i := (u\nabla_i)|_{u=0}(h_i) \colon TU_i \to \mathcal{H}_i|_{u=0}$ is an isomorphism. Hence $(\mathcal{H}_i, \nabla_i)/U_i$ is maximal at b_i. Up to shrinking U_i, this implies that $(\mathcal{H}_i, \nabla_i)/U_i$ is maximal, completing the proof. $\square$

4.4 Framing of F-bundles

In this section, we prove the extension of framing theorems (Theorems 4.4.2 and 4.4.26). In Section 4.4.3, we apply the extension of framing to obtain a uniqueness result for isomorphisms between maximal F-bundles admitting a framing (Proposition 4.4.31). In Section 4.4.4, we provide a partial classification of framed F-bundles over a point, up to gauge equivalence, under some assumptions on the coefficients of the connection (Theorem 4.4.34). When the K-operator of the F-bundle has simple eigenvalues, we obtain a full classification in Corollary 4.4.35. We will apply those results to the A-model F-bundles in Section 4.5.

4.4.1 Extension of framing for logarithmic formal F-bundles

4.4.1.1 Main result

Here we state the theorem of extension of framing, and fix the notations for the proof.

Definition 4.4.1. Let $(\mathcal{H}, \nabla)/(B, D)$ be a logarithmic F-bundle and $b \in B$ a rational point. We say that a framing ∇_b^{fr} for the restricted F-bundle $(\mathcal{H}, \nabla)|_b$ is *strong* with respect to D if for any function q vanishing on D, the endomorphism $\nabla_{uq\partial_q}|_{b\times \operatorname{Spf} \Bbbk[\![u]\!]}$ is independent of u in a ∇_b^{fr}-flat trivialization of $\mathcal{H}|_{b\times \operatorname{Spf} \Bbbk[\![u]\!]}$.

Theorem 4.4.2 (Extension of framing)**.** *Let $(\mathcal{H}, \nabla)/(B, D)$ be a logarithmic F-bundle, where B is a formal neighborhood of a rational point b in a smooth $\Bbbk$-variety. A framing ∇_b^{fr} for the restricted F-bundle $(\mathcal{H}, \nabla)|_b$ extends to a framing for $(\mathcal{H}, \nabla)$ if and only if ∇_b^{fr} is strong with respect to D. In this case, the extension is uniquely and explicitly determined from ∇_b^{fr} and $(\mathcal{H}, \nabla)$.*

We refer to Example 4.4.25 for a counter-example to the existence part of Theorem 4.4.2 without assuming the framing is strong with respect to D.

Write $B = \operatorname{Spf} \Bbbk[\![q_1, \dots, q_s, t_1, \dots, t_n]\!]$, where $\prod_{1\le i\le s} q_i = 0$ is a local equation for D at b. Let m be the rank of $\mathcal{H}$ and $H := \mathcal{H}_{b,0}$ the fiber of $\mathcal{H}$. We start with any trivialization $iso\colon \mathcal{H} \simeq H \times B \times \operatorname{Spf} \Bbbk[\![u]\!]$ extending a ∇^{fr}_b-flat trivialization of $\mathcal{H}|_{b\times \operatorname{Spf}\Bbbk[\![u]\!]}$. Let Ω denote the connection form of ∇ in the trivialization iso. Fix a basis of H, and write

$$\Omega = \sum_{1\leq i\leq s} u^{-1}q_i^{-1}Q^i(q,t,u)dq_i + \sum_{1\leq j\leq n} u^{-1}T^j(q,t,u)dt_j + u^{-2}U(q,t,u)du, \tag{4.4.3}$$

where $U, Q^i, T^j \in \operatorname{Mat}(m\times m, \Bbbk[\![q_i,t_j,u]\!])$. The framing assumption at b allows us to assume that $U(0,0,u)$ is linear in u. The assumption that the endomorphism $\nabla_{uq_i\partial_{q_i}}|_{q=t=0}$ is ∇^{fr}_b-flat means that $Q^i(0,0,u)$ is independent of u.

We want to modify the trivialization iso by an automorphism of $H \times B \times \operatorname{Spf}\Bbbk[\![u]\!]$, to produce a new trivialization extending $iso|_{b\times\operatorname{Spf}\Bbbk[\![u]\!]}$ and in which ∇ is framed. Equivalently, we seek a gauge transformation $P(q,t,u) \in \mathrm{GL}(m, \Bbbk[\![q_i,t_j,u]\!])$ and matrices $K(q,t), G(q,t), \widetilde{Q}^i(q,t), \widetilde{T}^j(q,t)$ in $\operatorname{Mat}(m\times m, \Bbbk[\![q_i,t_j]\!])$ such that

$$P^{-1}\partial_u P + u^{-2}P^{-1}UP = u^{-2}K + u^{-1}G, \tag{4.4.4}$$

$$P^{-1}\partial_{q_i} P + u^{-1}q_i^{-1}P^{-1}Q^iP = u^{-1}q_i^{-1}\widetilde{Q}^i, \tag{4.4.5}$$

$$P^{-1}\partial_{t_j} P + u^{-1}P^{-1}T^jP = u^{-1}\widetilde{T}^j, \tag{4.4.6}$$

and satisfying $P(0,0,u) = \mathrm{id}$. By identifying the polar part at $u = 0$, we get an expression for the matrices $K, G, \widetilde{Q}^i, \widetilde{T}^j$. In particular, setting $P_0 \coloneqq P(q,t,0)$, we have the following expressions

$$\widetilde{Q}^i = P_0^{-1}Q^i_{-1}P_0 \quad\text{and}\quad \widetilde{T}^j = P_0^{-1}T^j_{-1}P_0, \tag{4.4.7}$$

with $Q^i_{-1} = \nabla_{uq_i\partial_{q_i}}|_{u=0}$ and $T^j_{-1} = \nabla_{u\partial_{t_j}}|_{u=0}$. We will construct P in Section 4.4.1.3 order by order in each variable, starting with the logarithmic directions.

4.4.1.2 Two matrix lemmas

We now state two matrix lemmas that we will use for the proof of Theorem 4.4.2.

Lemma 4.4.8. *Let R be a ring.*

1. Let $T \in \operatorname{Mat}(m\times m, R[\![t]\!])$. Let $(X_k(t))_{k\in\mathbb{N}}$ be a sequence of matrices in $\operatorname{Mat}(m\times m, R[\![t]\!])$ satisfying

$$\partial_t X_k = -[T, X_{k+1}].$$

Then $(X_k(t))_{k\in\mathbb{N}}$ is uniquely determined by $(X_k(0))_{k\in\mathbb{N}}$. In particular, if $X_k(0) = 0$ for all $k \geq 0$, then $X_k(t) = 0$ for all $k \geq 0$.

2. *Let* $n \in \mathbb{N}$*, and* $T_1, \dots, T_n \in \mathrm{Mat}(m \times m, R[\![t_1, \dots, t_n]\!])$*. Let* $(X_k(t))_{k\in\mathbb{N}}$ *be a sequence of matrices in* $\mathrm{Mat}(m \times m, R[\![t_1, \dots, t_n]\!])$ *satisfying for all* $1 \le i \le n$

$$\partial_{t_i} X_k = -[T_i, X_{k+1}].$$

Then $(X_k(t))_{k\in\mathbb{N}}$ *is uniquely determined by* $(X_k(0))_{k\in\mathbb{N}}$*. In particular, if* $X_k(0) = 0$ *for all* $k \ge 0$*, then* $X_k(t) = 0$ *for all* $k \ge 0$*.*

Proof. For (1), we write $X_k(t) = \sum_{\ell\in\mathbb{N}} X_{\ell,k} t^\ell$. For $d \ge 0$, we have

$$(d+1)! X_{d+1,k} = \left.\frac{\partial^{d+1} X_k}{\partial^{d+1} t}\right|_{t=0} = -\left.\frac{\partial^d}{\partial^d t}[T, X_{k+1}]\right|_{t=0} = -\sum_{s=0}^{d} \binom{d}{s} \left.\left[\frac{\partial^{d-s} T}{\partial^{d-s} t}, \frac{\partial^s X_{k+1}}{\partial^s t}\right]\right|_{t=0}.$$

This provides a recursive relation for $\{X_{d+1,k}\}_{k\in\mathbb{N}}$ in terms of $\{X_{r,k},\ r \le d\}_{k\in\mathbb{N}}$. Thus, $(X_k)_{k\ge 0}$ is uniquely determined by $(X_k(0))_{k\ge 0}$.

For (2), we apply inductively on $1 \le i \le n$ the single variable case with the ring $R[\![t_1, \dots, t_{i-1}]\!]$. In this way, we prove that for $1 \le i \le n$, the sequence $(X_k|_{t_{i+1}=\dots=t_n=0})_{k\in\mathbb{N}}$ is uniquely determined by the sequence $(X_k|_{t_i=\dots=t_n=0})_{k\in\mathbb{N}}$. Thus $(X_k)_{k\in\mathbb{N}}$ is uniquely determined by the initial condition $(X_k|_{t_1=\dots=t_n=0})_{k\in\mathbb{N}}$.

For both (1) and (2), choosing $X_k(t) = 0$ for all $k \ge 0$ provides a sequence that satisfies the assumptions of the lemma, with the initial condition $X_k(0) = 0$. It follows from the uniqueness that this is the only solution to the equations such that $X_k(0) = 0$ for all $k \ge 0$. $\square$

Lemma 4.4.9. *Let* R *be a ring. For* $1 \le i \le s$*, let* $Q_i \in \mathrm{Mat}(m \times m, R[\![q_1, \dots, q_s]\!])$ *such that* $\phi_i := \mathrm{ad}(Q_i)|_{q=0}$ *is nilpotent. Let* $(X_k(q))_{k\in\mathbb{N}}$ *be a sequence of matrices in* $\mathrm{Mat}(m \times m, R[\![q_1, \dots, q_s]\!])$ *satisfying for all* $1 \le i \le s$

$$q_i \partial_{q_i} X_k = [Q_i, X_{k+1}].$$

Then, for any initial condition $(X_k(0))_{k\in\mathbb{N}}$*, there exists at most one solution* $(X_k(q))_{k\in\mathbb{N}}$*. In particular, if* $X_k(0) = 0$ *for all* $k \ge 0$*, then* $X_k(q) = 0$ *for all* $k \ge 0$*.*

Proof. We use Notation 4.3.1. In particular, given tuples of integers $\ell = (\ell_i)_{1\le i\le n}$ and $r = (r_i)_{1\le i\le n}$, the length of ℓ is $|\ell| = \ell_1 + \dots + \ell_n$, and we write $r \preceq \ell$ if $r_i \le \ell_i$ for all $1 \le i \le n$. We denote the linear differential operator $q_i \partial_{q_i}$ by D_i, so the equations are $D_i X_k = [Q_i, X_{k+1}]$.

First, a direct induction shows that for all $n \in \mathbb{N}$ we can express $D_i^{n+1} X_k$ as a linear combination of terms of the form

$$[D_i^{a_1} Q_i, [\cdots, [D_i^{a_u} Q_i, X_{k+u}] \cdots]], \tag{4.4.10}$$

with $1 \leq u \leq n+1$ and $(a_v)_{1 \leq v \leq u} \in \mathbb{N}^u$ satisfying $a_1 + \cdots + a_u + u = n+1$. If we denote the coefficient of such a term by $\alpha_n(a_1, \ldots, a_u)$, it is elementary to see that the sequence $(\alpha_n)_{n \in \mathbb{N}}$ is fully determined by the initial condition $\alpha_0(0) = 1$ and the recursion relation

$$\alpha_{n+1}(a_1, \ldots, a_u) = \sum_{a_v \neq 0} \alpha_n(a_1, \ldots, a_v - 1, \ldots, a_u) + \delta_{a_u, 0} \alpha_n(a_1, \ldots, a_{u-1}).$$

Write $X_k(q) = \sum_{r \in \mathbb{N}^s} X_{r,k} q_1^{r_1} \cdots q_s^{r_s}$. We will show that for $d \geq 1$, the terms $\{X_{\ell,k},\ |\ell| = d\}_{k \in \mathbb{N}}$ are determined by $\{X_{r,k},\ |r| < d\}_{k \in \mathbb{N}}$. It will follow directly that $(X_k(q))_{k \in \mathbb{N}}$ is uniquely determined by the initial term $(X_k(0))_{k \in \mathbb{N}}$. Fix $\ell \in \mathbb{N}^s$ with $|\ell| = d$ and $k \in \mathbb{N}$. We express $X_{\ell,k}$ in terms of $\{X_{r,k+s},\ |r| < d,\ s \geq 1\}$. Fix i such that $\ell_i \neq 0$, and let $n \in \mathbb{N}$. We note that the coefficient of q^ℓ in $D_i^{n+1} X_k$ is $\ell_i^{s+1} X_{\ell,k}$. On the other hand, by the previous paragraph $D_i^{n+1} X_k$ is a linear combination of terms of the form (4.4.10). The coefficient of q^ℓ in (4.4.10) is expressed in terms of derivatives of Q_i and coefficients $X_{r,k+u}$ with $r \preceq \ell$ and $u \geq 1$. If $X_{\ell,k+u}$ appears in a term, then only the constant term of the terms involving Q_i contribute. If $a > 0$, then $D_i^a Q_i$ has no constant term, so $X_{\ell,k+u}$ appears in the relation if and only if $a_1 = \cdots = a_u = 0$. Given the condition $a_1 + \cdots + a_u + u = n+1$, this implies $u = n+1$ and we conclude that

$$\ell_i^{n+1} X_{\ell,k} = \phi_i^{n+1}(X_{\ell,k+n+1}) + \{\text{terms involving derivatives of } Q_i \text{ and } X_{r,k+u} \text{ with } |r| < d\}.$$

Since ϕ_i is nilpotent, for n large enough the right hand side does not depend on $\{X_{\ell,k}\}_{k \in \mathbb{N}}$, and we obtain a recursive relation determining uniquely $X_{\ell,k}$ as a function of terms already known. This completes the proof. □

4.4.1.3 Proof of Theorem 4.4.2

We formulate a condition under which we are able to solve the system of PDEs (4.4.4)-(4.4.6) recursively.

Definition 4.4.11 (Nilpotency condition)**.** Let $(\mathcal{H}, \nabla)/(B, D)$ be a logarithmic F-bundle, where B is a formal neighborhood of a rational point b in a smooth $\Bbbk$-variety. We say that $(\mathcal{H}, \nabla)/(B, D)$ satisfies *the nilpotency condition at* b if for all vector $v \in T_b D$, the adjoint $\operatorname{ad} \mu_b(v)$ is nilpotent (see (4.2.5) for μ_b).

Lemma 4.4.12. *Let $(\mathcal{H},\nabla)/(B,D)$ be a logarithmic F-bundle, where B is a formal neighborhood of a rational point b in a smooth $\Bbbk$-variety. If there exists a framing for $(\mathcal{H},\nabla)$ at b that is strong with respect to D, then $\mu_b(v)$ is nilpotent for every $v \in T_bB$. In particular, $(\mathcal{H},\nabla)$ satisfies the nilpotency condition at b.*

Proof. Write $B = \operatorname{Spf} \Bbbk[\![q_1,\dots,q_s,t_1,\dots,t_n]\!]$, with q_i the logarithmic directions. Let ∇_b^{fr} be a framing at b that is strong with respect to D, fix a trivialization of $\mathcal{H}$ extending a ∇_b^{fr}-flat trivialization. Fix $1 \le i \le s$ and write

$$\nabla_{q_i\partial_{q_i}} = q_i\partial_{q_i} + u^{-1}Q(q,t,u).$$

By the assumption, $Q_0 \coloneqq Q(0,0,u)$ is independent of u. Since ∇_b^{fr} is a framing, we have $\nabla_{\partial_u}|_{b\times \operatorname{Spf}\Bbbk[\![u]\!]} = \partial_u + u^{-2}K + u^{-1}G$, with K and G constant endomorphisms of $\mathcal{H}_{b,0}$. In this trivialization, the flatness equation $[\nabla_{\partial_u}, \nabla_{q_i\partial_{q_i}}] = 0$ restricted to $b \times \operatorname{Spf}\Bbbk[\![u]\!]$ reads

$$-Q_0 = u^{-1}[Q_0,K] + [Q_0,G].$$

In particular $[Q_0,G] = -Q_0$. It follows that $[Q_0,[Q_0,-G]] = [Q_0,Q_0] = 0$. Jacobson's lemma ([Jac62, Lemma 4, p. 44]) implies that $[Q_0,-G] = Q_0$ is nilpotent, proving the first part of the lemma. Since the adjoint of a nilpotent endomorphism is nilpotent, the second part follows. □

The next series of lemmas will enable us to prove Theorem 4.4.2 by framing the connection inductively in each direction. Given a logarithmic F-bundle $(\mathcal{H},\nabla)/(B,D)$ over $B = \operatorname{Spf}\Bbbk[\![q_1,\dots,q_s,t_1,\dots,t_n]\!]$, a closed subscheme $B' \subset B$ and a subsheaf $\mathcal{F} \subset T_B(-\log D)$, we will say that $(\mathcal{H},\nabla)$ is *framed in the directions of $\mathcal{F}$ at B'* if there exists a trivialization of $\mathcal{H}$ such that $\nabla_\xi|_{B'}$ takes the form (4.2.10) for any section ξ of $\mathcal{F}$, i.e. the restriction of the connection matrix in the direction ξ to B' has no positive powers of u. If we formulate multiple conditions involving several subsheaves and closed subschemes, we mean that there exists a trivialization in which the connection form satisfies all the formulated conditions.

Lemma 4.4.13. *Let $(\mathcal{H},\nabla)$ be a logarithmic F-bundle over $\operatorname{Spf}\Bbbk[\![q_1,\dots,q_s]\!]$ (without t-variables) satisfying the nilpotency condition (Definition 4.4.11), fix $1 \le i \le s$. Assume it is framed in all q-directions at $\{q_j = 0, i \le j \le s\}$. Then there exists a gauge transformation $P(q_1,\dots,q_s,u)$ such that $P|_{q_i=\dots=q_s=0} = \mathrm{id}$ and $P^*\nabla$ is framed in the q_i-direction at $\{q_j = 0, i+1 \le j \le s\}$. In particular, $P^*\nabla$ is still framed in all q-directions at $\{q_j = 0, i \le j \le s\}$.*

Proof. We let $q := \{1, \dots, s\}$, $q^{\leq i} := \{q_1, \dots, q_i\}$, $q^{\geq i} := \{q_i, \dots, q_s\}$ and $q^{>i} := \{q_{i+1}, \dots, q_s\}$. Let $u^{-1}q_i^{-1}Q(q,u)$ denote the connection matrix in the q_i-direction in a trivialization of $\mathcal{H}$ provided by the partial framing assumption. Write $Q(q,u) = \sum_{\ell,k\geq 0} Q_{\ell,k-1}q_i^\ell u^k$, by the framing assumption we have $Q|_{q^{\geq i}=0} = Q_{0,-1}|_{q^{\geq i}=0}$.

We seek a gauge transformation $P(q,u)$ such that

$$\begin{aligned} \partial_{q_i} P|_{q^{>i}=0} &= u^{-1}q_i^{-1}\left(-QP + PP_0^{-1}Q_{-1}P_0\right)|_{q^{>i}=0}, \\ P|_{q^{\geq i}=0} &= \mathrm{id}, \end{aligned}$$

where $P_0 := P(q,0)$ and $Q_{-1} := Q(q,0)$. We look for P of the form $P(q,u) = \sum_{\ell,k\geq 0} P_{\ell,k}q_i^\ell u^k$, where $P_{\ell,k}$ depends on $\{q_1, \dots, q_{i-1}\}$, We construct the solution P order by order in powers of q_i, by expressing $\{P_{\ell+1,k}\}_{k\in\mathbb{N}}$ in terms of $\{P_{\ell',k},\ \ell' \leq \ell\}_{k\in\mathbb{N}}$ for $\ell \in \mathbb{N}$.

The initial condition gives $P_{0,0} = \mathrm{id}$ and $P_{0,k} = 0$ for $k > 0$. Let $\ell \in \mathbb{N}$ and $k \in \mathbb{N}$. We isolate a monomial $q_i^\ell u^k$ in the differential equation and obtain

$$(\ell+1)P_{\ell+1,k} = -\sum_{\substack{\ell_1+\ell_2=\ell+1\\ k_1+k_2=k+1}} Q_{\ell_1,k_1-1}|_{q^{>i}=0}P_{\ell_2,k_2} + \sum_{\ell_1+\ell_2+\ell_3+\ell_4=\ell+1} P_{\ell_1,k+1}(P_0^{-1})_{\ell_2}Q_{\ell_3,-1}|_{q^{>i}=0}P_{\ell_4,0},$$

where $(P_0^{-1})_{\ell_2}$ is the coefficient of $q_i^{\ell_2}$ in P_0^{-1}. Using the framing assumption at $q^{\geq i} = 0$ and the initial condition for P, we isolate terms involving $\{P_{\ell+1,k'}\}_{k'\in\mathbb{N}}$ and obtain the relation for all $k \geq 0$

$$P_{\ell+1,k} = \psi_{\ell,k}(P) - \frac{1}{\ell+1}[Q|_{q^{\geq i}=0}, P_{\ell+1,k+1}], \tag{4.4.14}$$

where

$$\begin{aligned} \psi_{\ell,k}(P) := \frac{1}{\ell+1}\Bigg(&-\sum_{\substack{k_1+k_2=k+1\\ \ell_1+\ell_2=\ell+1\\ \ell_2<\ell+1}} Q_{\ell_1,k_1-1}|_{q^{>i}=0}P_{\ell_2,k_2} \\ &+ \sum_{\substack{\ell_1+\ell_2+\ell_3+\ell_4=\ell+1\\ 0<\ell_1<\ell+1}} P_{\ell_1,k+1}(P_0^{-1})_{\ell_2}Q_{\ell_3,-1}|_{q^{>i}=0}P_{\ell_4,0}\Bigg). \end{aligned}$$

Note that $\psi_{\ell,k}(P)$ only depends on $\{P_{\ell',k'},\ \ell' < \ell+1,\ k' \leq k+1\}$.

Let $E := \mathrm{Mat}(m \times m, \Bbbk[\![q_1, \dots, q_{i-1}]\!])^{\mathbb{N}}$. Consider the linear maps $\tau\colon E \to E$ given by the shift $\{M_k\}_{k\in\mathbb{N}} \mapsto \{M_{k+1}\}_{k\in\mathbb{N}}$ and $\Phi\colon E \to E$ given by $\{M_k\}_{k\in\mathbb{N}} \mapsto \{[Q|_{q^{\geq i}=0}, M_k]\}_{k\in\mathbb{N}}$. The relations (4.4.14) give

$$\left(\mathrm{id}_E + \frac{1}{\ell+1}\Phi\circ\tau\right)\{P_{\ell+1,k}\}_{k\in\mathbb{N}} = \{\psi_{\ell,k}(P)\}_{k\in\mathbb{N}}. \tag{4.4.15}$$

We prove that $\mathrm{id}_E + \frac{1}{\ell+1}\Phi \circ \tau$ is invertible. To do so, it is enough to prove that it is invertible at $q_1 = \cdots = q_{i-1} = 0$. The map $\Phi|_{q_1=\cdots=q_{i-1}=0}$ is nilpotent, since $\mathrm{ad}(Q|_{q=u=0})$ is. The maps τ and Φ commute, so the composition $\Phi \circ \tau \colon E \to E$ is also nilpotent at $q_1 = \cdots = q_{i-1} = 0$. Hence $\mathrm{id}_E + \frac{1}{\ell+1}\Phi \circ \tau$ is invertible at $q_1 = \cdots = q_{i-1} = 0$. It follows that $\mathrm{id}_E + \frac{1}{\ell+1}\Phi \circ \tau$ is invertible, and composing (4.4.15) with its inverse provides a recursive relation determining the coefficient of $q_i^{\ell+1}$ from lower order terms. Hence the differential equation admits a solution $P(q,u)$ such that $P|_{q^{\geq i}=0} = \mathrm{id}$. The initial condition implies that the connection $P^*\nabla$ is still framed in all q-directions at $q^{\geq i} = 0$. This completes the proof. □

Lemma 4.4.16. *Let $(\mathcal{H}, \nabla)$ be a logarithmic F-bundle over $\operatorname{Spf} \Bbbk[\![q_1, \dots, q_s]\!]$ (with no t-variables) satisfying the nilpotency condition (Definition 4.4.11), fix $1 \leq i \leq s$. Assume it is framed in all q-directions at $\{q_j = 0, i \leq j \leq s\}$, and framed in the q_i-direction at $\{q_j = 0, i+1 \leq j \leq s\}$. Then $(\mathcal{H}, \nabla)$ is framed in all the q-directions at $\{q_j = 0, i+1 \leq j \leq s\}$.*

Proof. Let $q^{\leq i} := \{q_1, \dots, q_i\}$. The partial framing assumption provides a trivialization of $\mathcal{H}$. For $1 \leq i' \leq s$, let $u^{-1}q_{i'}^{-1}Q^{i'}(q^{\leq i}, u) = q_{i'}^{-1}\sum_{k\geq 0} Q_{k-1}^{i'}(q^{\leq i})u^{k-1}$ denote the restriction of the connection matrix in the $q_{i'}$-direction to $q_{i+1} = \cdots = q_s = 0$. The framing assumption means that $Q_k^{i'}|_{q_i=0} = 0$ and $Q_k^i = 0$ for all $k \geq 0$ and $1 \leq i' \leq s$.

Fix $1 \leq i' \leq s$, with $i' \neq i$. For $k \geq 0$, the u^k term of the flatness equation $[\nabla_{q_{i'}\partial_{q_{i'}}}, \nabla_{q_i\partial_{q_i}}] = 0$ provides the equation

$$q_i\partial_{q_i}Q_k^{i'} = -[Q_{-1}^i, Q_{k+1}^{i'}].$$

Since $\mathrm{ad}(Q_{-1}^s(0))$ is nilpotent, we can apply Lemma 4.4.9 with $R = \Bbbk[\![q_1, \dots, q_{i-1}]\!]$ and $X_k = Q_k^{i'}$. We deduce that $Q_k^{i'} = 0$ for all $k \geq 0$, proving that the connection is also framed in the $q_{i'}$-direction at $q_{i+1} = \cdots = q_s = 0$. □

Lemma 4.4.17. *Let $(\mathcal{H}, \nabla)$ be a logarithmic F-bundle over $\operatorname{Spf} \Bbbk[\![q_1, \dots, q_s, t_1, \dots, t_n]\!]$ framed in the q-directions at $t = 0$. Then there exists a gauge transformation P such that $P|_{t=0} = \mathrm{id}$ and $P^*\nabla$ is framed in all the q-directions and t-directions at $t = 0$.*

Proof. We work in a trivialization of $\mathcal{H}$ provided by the partial framing assumption. For $1 \leq i \leq s$, let $u^{-1}q_i^{-1}Q^i(q,t,u)$ denote the connection matrix in the q_i-direction in this trivialization. For $1 \leq j \leq n$, let $u^{-1}T^j(q,t,u)$ denote the connection matrix in the t_j-direction in this trivialization. Let

$$P(q,t,u) := \prod_{j=1}^{n}\left(\mathrm{id} - t_j \frac{T^j(q,0,u) - T^j(q,0,0)}{u}\right).$$

Note that $P(q,t,u)$ only has non-negative powers of u, because $T^j(q,0,u) - T^j(q,0,0)$ has no constant term in u. We have $P|_{t=0} = P^{-1}|_{t=0} = \mathrm{id}$, and we compute $\frac{\partial P}{\partial q_i}\Big|_{t=0} = 0$ and $\frac{\partial P}{\partial t_j}\Big|_{t=0} = -u^{-1}(T^j(q,0,u) - T^j(q,0,0))$. The connection matrix of $P^*\nabla$ in the t_j-direction at $t=0$ is

$$\left[P^{-1}\frac{\partial P}{\partial t_j} + u^{-1}P^{-1}T^jP\right]\Bigg|_{t=0} = u^{-1}(-T^j(q,0,u) + T^j(q,0,0) + T^j(q,0,u)) = u^{-1}T^j(q,0,0),$$

which is framed. The connection matrix of $P^*\nabla$ in the q_i-direction at $t=0$ is

$$\left[P^{-1}\frac{\partial P}{\partial q_i} + u^{-1}q_i^{-1}P^{-1}Q^iP\right]\Bigg|_{t=0} = u^{-1}q_i^{-1}Q^i(q,0,u),$$

which is also framed. The lemma is proved. □

Lemma 4.4.18. *Let $(\mathcal{H},\nabla)$ be a logarithmic F-bundle over $\mathrm{Spf}\,\Bbbk[\![q_1,\dots,q_s,t_1,\dots,t_n]\!]$ framed in the q-directions at $t=0$, fix $1 \le j \le n$. Assume it is framed in all t-directions at $\{t_i = 0, j \le i \le n\}$. Then there exists a gauge transformation $P(q,t,u)$ such that $P|_{t_j=\dots=t_n=0} = \mathrm{id}$ and $P^*\nabla$ is framed in the t_j-direction at $\{t_i = 0, j+1 \le i \le n\}$, framed in all the q-directions at $t=0$, and in all the t-directions at $\{t_i = 0, j \le i \le n\}$.*

Proof. Let $t^{\le j} := \{t_1,\dots,t_j\}$, $t^{\ge j} := \{t_j,\dots,t_n\}$ and $t^{>j} := \{t_{j+1},\dots,t_n\}$. Let $u^{-1}T(q,t,u)$ denote the connection matrix in the t_j-direction in a trivialization of $\mathcal{H}$ provided by the partial framing assumption. Write $T(q,t,u) = \sum_{\ell,k\in\mathbb{N}} T_{\ell,k-1}t_j^\ell u^k$, by the framing assumption we have $T|_{t^{\ge j}=0} = T_{0,-1}|_{t^{\ge j}=0}$.

We seek a gauge transformation $P(q,t,u)$ such that

$$\begin{aligned}\partial_{t_j}P|_{t^{>j}=0} &= u^{-1}\left(-TP + PP_0^{-1}T_{-1}P_0\right)|_{t^{>j}=0},\\ P|_{t^{\ge j}=0} &= \mathrm{id},\end{aligned}$$

where $P_0 := P(q,t,0)$ and $T_{-1} := T(q,t,0)$. We look for P of the form $P(q,t,u) = \sum_{\ell,k\ge 0} P_{\ell,k}t_j^\ell u^k$, where $P_{\ell,k}$ depends on the variables $\{q_1,\dots q_s,t_1,\dots,t_{j-1}\}$. The differential equation provides a recursive relation for $\{P_{\ell,k}\}_{k\in\mathbb{N}}$. By isolating the coefficient of $t_j^\ell u^k$ we obtain

$$(\ell+1)P_{\ell+1,k} = -\sum_{\substack{\ell_1+\ell_2=\ell\\ k_1+k_2=k+1}} T_{\ell_1,k_1-1}|_{t^{>j}=0}P_{\ell_2,k_2} + \sum_{\ell_1+\ell_2+\ell_3+\ell_4=\ell} P_{\ell_1,k+1}(P_0^{-1})_{\ell_2}T_{\ell_3,-1}|_{t^{>j}=0}P_{\ell_4,0}, \tag{4.4.19}$$

where $(P_0^{-1})_{\ell_2}$ denotes the coefficient of $t_j^{\ell_2}$ in P_0^{-1}. This determines P from the initial data $\{P_{0,k}\}_{k\in\mathbb{N}}$, i.e. from $P|_{t^{\geq j}=0} = \mathrm{id}$. Hence the differential equation admits a solution $P(q,t,u)$ such that $P|_{t^{\geq j}} = \mathrm{id}$. By construction, $P^*\nabla$ is framed in the t_j-direction at $t^{>j} = 0$.

We now check that the other t-directions are still framed at $t^{\geq j} = 0$, and that the q-directions are still framed at $t = 0$. Since $P|_{t^{\geq j}=0} = \mathrm{id}$, the connection matrices at $t^{\geq j} = 0$ are modified by the first derivatives of $\sum_{k\geq 0} P_{1,k}u^k$. From the recursion (4.4.19), the initial condition for P and the framing assumption for T we obtain that $P_{1,k} = -T_{0,k}|_{t^{>j}=0} = 0$ for all $k \geq 0$. We conclude that $P^*\nabla$ remains framed in all the t-directions at $t^{\geq j} = 0$ and in all the q-directions at $t = 0$, concluding the proof. □

Lemma 4.4.20. *Let $(\mathcal{H},\nabla)$ be a logarithmic F-bundle over $\operatorname{Spf}\Bbbk[\![q_1,\dots,q_s,t_1,\dots,t_n]\!]$, fix $1 \leq j \leq n$. Assume it is framed in all the t-directions at $\{t_i = 0, j \leq i \leq n\}$, and framed in the t_j-direction at $t_{j+1} = \cdots = t_n = 0$. Then $(\mathcal{H},\nabla)$ is framed in all the t-directions at $\{t_i = 0, j+1 \leq i \leq n\}$.*

Proof. Let $t^{\leq j} := \{t_1,\dots,t_j\}$. The partial framing assumption provides a trivialization of $\mathcal{H}$. For $1 \leq j' \leq n$, let $u^{-1}T^{j'}(q,t^{\leq j},u) = \sum_{k\geq 0} T^{j'}_{k-1}(q,t^{\leq j})u^{k-1}$ denote the restriction of the connection matrix in the $t_{j'}$-direction to $t_{j+1} = \cdots = t_n = 0$. The framing assumption means that $T^{j'}_k|_{t_j=0} = 0$ and $T^j_k = 0$ for all $k \geq 0$ and $1 \leq j' \leq n$.

Fix $1 \leq j' \leq n$, with $j' \neq j$. For $k \geq 0$, the u^k term of the flatness equation $[\nabla_{\partial_{t_{j'}}}, \nabla_{\partial_{t_j}}] = 0$ provides the equation

$$\partial_{t_j} T^{j'}_k = -[T^j_{-1}, T^{j'}_{k+1}].$$

We apply Lemma 4.4.8(1) with $R = \Bbbk[\![q_1,\dots,q_s,t_1,\dots,t_{j-1}]\!]$, $X_k = T^{j'}_k$ and the initial condition $T^{j'}_k|_{t_j=0} = 0$, and deduce that $T^{j'}_k(q,t^{\leq j}) = 0$ for all $k \geq 0$. Thus, the connection is also framed in the $t_{j'}$-direction at $t_{j+1} = \cdots = t_n = 0$. □

Lemma 4.4.21. *Let $(\mathcal{H},\nabla)$ be a logarithmic F-bundle over $\operatorname{Spf}\Bbbk[\![q_1,\dots,q_s,t_1,\dots,t_n]\!]$. Assume it is framed in the t-directions and framed in the q-directions at $t = 0$. Then $(\mathcal{H},\nabla)$ is also framed in the q-directions.*

Proof. In a trivialization provided by the framing assumption, denote by $u^{-1}T^j_{-1}(q,t)$ the connection matrix in the t_j-direction ($1 \leq j \leq n$) and by $u^{-1}q_i^{-1}Q^i(q,t,u)$ the connection matrix in the q_i-direction ($1 \leq i \leq s$). Write $Q^i = \sum_{k\geq 0} Q^i_{k-1}(q,t)u^k$. The framing assumption means that $Q^i_k|_{t=0} = 0$ for $1 \leq i \leq s$ and $k \geq 0$.

Fix $1 \leq i \leq s$. For $k \geq 0$, the u^k term of the flatness equation $[\nabla_{\partial_{t_j}}, \nabla_{q_i\partial_{q_i}}] = 0$ is

$$\partial_{t_j} Q^i_k = -[T^j_{-1}, Q^i_{k+1}].$$

We apply Lemma 4.4.8(2) with $R = \Bbbk[\![q_1,\dots,q_s]\!]$, $X_k = Q^i_k$ and the initial condition $Q^i_k|_{t=0} = 0$, and deduce that $Q^i_k(q,t) = 0$ for all $k \geq 0$. Thus, the connection is also framed in the q_i-direction. □

Lemma 4.4.22. *Let $(\mathcal{H},\nabla)$ be a logarithmic F-bundle over* $\operatorname{Spf}\Bbbk[\![q_1,\dots,q_s,t_1,\dots,t_n]\!]$ *satisfying the nilpotency condition (Definition 4.4.11). Assume it is framed in the q-directions and t-directions, and framed in the u-direction at $q = t = 0$. Then $(\mathcal{H},\nabla)$ is also framed in the u-direction.*

Proof. In a trivialization provided by the framing assumption, let $u^{-1}q_i^{-1}Q^i(q,t)$ (resp. $u^{-1}T^j(q,t)$) denote the connection matrix in the q_i-direction (resp. t_j-direction). Let $u^{-2}U(q,t,u)$ denote the connection matrix in the u-direction. Write $U(q,t,u) = \sum_{k\geq 0} U_{k-2}(q,t)u^k$. The framing assumption means that for $k \geq 0$, we have $U_k(0,0) = 0$.

For $k \geq 0$, and $1 \leq i \leq s$, the u^k term of the flatness equation $[\nabla_{\partial_u}, \nabla_{q_i\partial_{q_i}}] = 0$ provides the equation

$$q_i\partial_{q_i}(U_k) = -[Q^i, U_{k+1}].$$

We restrict this equation to $t = 0$. Since $\operatorname{ad}(Q^i(0,0))$ is nilpotent, we can apply Lemma 4.4.9 with $R = \Bbbk$ and $X_k = U_k(q,0)$ to deduce that $U_k(q,0) = 0$ for all $k \geq 0$.

Next, for $k \geq 0$, the u^k term of the flatness equation $[\nabla_{\partial_u}, \nabla_{\partial_{t_j}}] = 0$ provides the equation

$$\partial_{t_j}(U_k) = -[T^j, U_{k+1}].$$

We apply Lemma 4.4.8(2) with $R = \Bbbk[\![q_1,\dots,q_s]\!]$, $X_k = U_k(q,t)$ and the initial condition $U_k(q,0) = 0$, and deduce that $U_k(q,t) = 0$ for all $k \geq 0$. Thus, the connection is also framed in the u-direction. □

We can now finish the proof of Theorem 4.4.2.

Proof of Theorem 4.4.2. Fix a trivialization $\mathcal{H} \simeq H \times (B \times \operatorname{Spf} \Bbbk[\![u]\!])$ extending the trivialization of $\mathcal{H}|_{b \times \operatorname{Spf} \Bbbk[\![u]\!]}$ induced by ∇^{fr}_b. As explained after Theorem 4.4.2, the content of the theorem reduces to proving existence and uniqueness of a solution $P(q,t,u)$ to the overdetermined nonlinear system of PDEs (4.4.4)-(4.4.6) with initial condition $P(0,0,u) = \mathrm{id}$.

We prove the existence part of the statement. If there exists a framing ∇^{fr} extending ∇^{fr}_b, then we see that ∇^{fr}_b is strong with respect to D by working in a ∇^{fr}-flat trivialization. Conversely assume that ∇^{fr}_b is strong with respect to D, in particular the nilpotency condition is satisfied by Lemma 4.4.12. We first frame the restricted F-bundle $(\mathcal{H}', \nabla') \coloneqq (\mathcal{H}, \nabla)|_{t=0}$, defined over the base $B' \coloneqq \operatorname{Spf} \Bbbk[\![q_1, \dots, q_s]\!]$. Applying inductively Lemmas 4.4.13 and 4.4.16 on $i \in \{1, \dots, s\}$, we obtain a gauge transformation $P(q,u)$ such that $P(0,u) = \mathrm{id}$ and $P^*\nabla'$ is framed in all the q-directions. Note that to apply the lemmas for the base case $i = 1$, we use that ∇^{fr}_b is strong with respect to D. Extending this gauge transformation constantly in the t-directions, we obtain a gauge transformation $P_1(q,u) \in \operatorname{Aut}(\mathcal{H})$ with $P_1(0,u) = \mathrm{id}$ such that $\nabla_1 \coloneqq P_1^*\nabla$ is framed in all the q-directions at $t = 0$. By Lemma 4.4.17, we obtain a gauge transformation $P_2(q,t,u) \in \operatorname{Aut}(\mathcal{H})$ with $P_2(q,0,u) = \mathrm{id}$ such that $\nabla_2 \coloneqq P_2^*\nabla_1$ is framed in all the q-directions and t-directions at $t = 0$. Applying inductively Lemmas 4.4.18 and 4.4.20 on $j \in \{1, \dots, n\}$, we obtain a gauge transformation $P_3(q,t,u) \in \operatorname{Aut}(\mathcal{H})$ with $P(q,0,u) = \mathrm{id}$ such that $\nabla_3 \coloneqq P_3^*\nabla_2$ is framed in all the q-directions at $t = 0$, and in all the t-directions along B. By Lemma 4.4.21, the connection ∇_3 is also framed in all the q-directions along B. Since $\nabla_{3,\partial_u}|_{q=t=0} = \nabla_{\partial_u}|_{q=t=0}$, the connection ∇_3 is framed in the u-directions at $q = t = 0$. We conclude by Lemma 4.4.22 that ∇_3 is framed in the u-direction as well. Thus the gauge transformation $\widetilde{P} \coloneqq P_3 P_2 P_1$ solves the system (4.4.4)-(4.4.6) with the initial condition $\widetilde{P}(0,0,u) = \mathrm{id}$, concluding the proof of existence.

We now prove uniqueness. Assume the system of PDEs is written in a trivialization in which the connection is framed. In particular, the nilpotency condition is satisfied by Lemma 4.4.12. From the equations in the directions of B we obtain recursive relations as in (4.4.15) and (4.4.19). Hence, any solution is uniquely determined by the condition $P(0,0,u) = \mathrm{id}$. □

4.4.1.4 Framings on rank 1 F-bundles

F-bundles do not admit framings in general (see [Sab07, §IV.5.b] for a sufficient condition), even though we established the extension of framing in Theorem 4.4.2.

Here we discuss the existence of framing on rank 1 F-bundles.

Proposition 4.4.23. *Let B be a formal neighborhood of a rational point b in a smooth $\Bbbk$-variety. Let $(\mathcal{H}, \nabla)/B$ be a (non-logarithmic) formal F-bundle of rank 1. Then it admits a framing.*

Proof. We keep the notations of the proof of Theorem 4.4.2. In the non-logarithmic case there are no q-variables, and in the rank 1 case the matrices are elements of $\Bbbk[\![t, u]\!]$, so they commute. Then $K = U_{-2}$, $G = U_{-1}$ and $\widetilde{T}^i = T^i_{-1}$ for $1 \leq i \leq n$. The system of PDEs (4.4.4)-(4.4.6) is then

$$\partial_u P(t, u) + P(t, u)U_{\geq 0}(t, u) = 0,$$
$$\partial_{t_i} P(t, u) + P(q, t, u)T^i_{\geq 0}(t, u) = 0,$$

where $U_{\geq 0} = \sum_{k \geq 0} U_k u^k$ and $T^i_{\geq 0} = \sum_{k\geq 0} T^i_k u^k$. We furthermore need $P(0, 0) \neq 0$ in order for $P(t, u)$ to be invertible.

It is readily checked, using flatness, that the ansatz

$$P(t, u) = \exp\left(-\sum_{i=1}^{n} \int_0^{t_i} \left(T^i_{\geq 0}(t_1, \ldots, t_{i-1}, s_i, 0, \ldots, 0, u) + T^i_{\geq 0}(0, u)\right) ds_i - \int_0^u U_{\geq 0}(0, v)dv\right) \tag{4.4.24}$$

solves the system of PDEs, and is invertible since $P(0, 0) = 1$. □

In the following example, we discuss the case of rank 1 logarithmic F-bundle, and provide a counter-example to the existence part of Theorem 4.4.2 without assuming the framing is strong with respect to D.

Example 4.4.25. Let $\mathcal{H}$ be the trivial rank 1 bundle over $\operatorname{Spf} \Bbbk[\![q, u]\!]$. Let $\nabla = d + \Omega$ be the connection on $\mathcal{H}$ with $\Omega = \alpha\frac{dq}{q}$, where $\alpha \in \Bbbk$. Then $(\mathcal{H}, \nabla)$ is a F-bundle and $\nabla^{\mathrm{fr}}_0 = d$ is a framing for $(\mathcal{H}, \nabla)|_{q=0}$. It is strong with respect to D if and only if $\alpha = 0$. The differential system to solve in order to extend the framing is

$$\frac{\partial P}{\partial u} = 0,$$
$$q\frac{\partial P}{\partial q} + \alpha P = 0.$$

If $\alpha \neq 0$, all solutions to this system are scalar multiples of αq^{-1}. In particular, they are not well-defined at $q = 0$.

4.4.2 Extension of framing for non-archimedean F-bundles

In this subsection, we establish the theorem of extension of framing for non-archimedean F-bundles, building on the results of the previous subsection.

Theorem 4.4.26. *Let B be a smooth $\Bbbk$-analytic space, and $b \in B$ a $\Bbbk$-rational point. Let $(\mathcal{H}, \nabla)$ be a non-archimedean F-bundle over B. Then every framing of $(\mathcal{H}, \nabla)$ at b extends uniquely and explicitly to a framing over an admissible open neighborhood U of b in B.*

We need to show that the gauge transformation $P(t, u)$ constructed in the formal case is convergent on an admissible open neighborhood of $t = 0$, $u = 0$. This gauge transformation is characterized by $P(0, u) = 0$ and the equations (4.4.6) for $1 \le j \le n$. We use these equations to obtain estimates on the coefficients of $P(t, u)$.

Lemma 4.4.27. *Let $(R, |\cdot|)$ be a Banach $\Bbbk$-algebra. Let $Q = \mathrm{id} + \sum_{r\ge 1} Q_r t^r \in \mathrm{Mat}(m \times m, R)[\![t]\!]$, and write $Q^{-1} = \mathrm{id} + \sum_{r\ge 1}(Q^{-1})_r t^r$. For $\ell \ge 1$ we have*

$$|(Q^{-1})_\ell| \le \max_{\substack{\ell\ge k\ge 1,\ r_i\ge 1\\ r_1+\dots+r_k=\ell}} \prod_{i=1}^{k} |Q_{r_i}|.$$

Proof. We have $Q^{-1} = \sum_{k\ge 0}(-1)^k \left(\sum_{r\ge 1} Q_r t^r\right)^k$. Isolating the coefficient of t^ℓ ($\ell \ge 1$) we obtain

$$(Q^{-1})_\ell = \sum_{k\ge 0} \sum_{\substack{r_1+\dots+r_k=\ell\\ r_i\ge 1}} \prod_{1\le i\le k} Q_{r_i},$$

and we see that only the range $1 \le k \le \ell$ contributes. This completes the proof. □

Proposition 4.4.28. *Let $(R, |\cdot|)$ be a Banach $\Bbbk$-algebra and let $T \in \mathrm{Mat}(m \times m, R)\langle t, u\rangle$. Let $P(t, u) \in \mathrm{Mat}(m \times m, R)[\![t, u]\!]$ be the unique solution of the system*

$$\begin{aligned} \partial_t P &= u^{-1}\left(-TP + PP_0^{-1}T_{-1}P_0\right), \\ P(0, u) &= \mathrm{id}, \end{aligned}$$

where $P_0 \coloneqq P(t, 0)$. Then P is convergent on the open disk of radius $\min\left(1, \frac{1}{|T|}\right)$, meaning that for all $0 < \rho < \min\left(1, \frac{1}{|T|}\right)$ in $\sqrt{|\Bbbk^\times|}$ we have $P \in \mathrm{Mat}(m \times m, R)\langle \rho^{-1}t, \rho^{-1}u\rangle$.

Proof. We write $T = \sum_{\substack{\ell\ge 0\\ k\ge -1}} T_{\ell,k} t^\ell u^{k+1}$. Since we assume T is convergent on the closed unit disk, we have for all $\ell \ge 0$, $k \ge -1$

$$|T_{\ell,k}| \le |T|. \tag{4.4.29}$$

Let $P := \mathrm{id} + \sum_{\substack{\ell\geq 1\\ k\geq 0}} P_{\ell,k} t^\ell u^k$ and $v_{\ell,k} := |P_{\ell,k}|$. If we show $v_{\ell,k} \leq \alpha^{\ell+k}$ for $\alpha > 0$, then $P(t,u)$ converges on the open polydisk of radius $\frac{1}{\alpha}$.

We have seen in Lemma 4.4.18 that P is uniquely determined by the recursion

$$(\ell+1)P_{\ell+1,k} = -\sum_{\substack{\ell_1+\ell_2=\ell\\ k_1+k_2=k+1}} T_{\ell_1,k_1-1}P_{\ell_2,k_2} + \sum_{\ell_1+\ell_2+\ell_3+\ell_4=\ell} P_{\ell_1,k+1}(P_0^{-1})_{\ell_2}T_{\ell_3,-1}P_{\ell_4,0}.$$

Applying the norm, we obtain

$$\begin{aligned}(\ell+1)v_{\ell+1,k} &\leq \max\left(\max_{\substack{\ell_1+\ell_2=\ell\\ k_1+k_2=k+1}} |T_{\ell_1,k_1-1}||P_{\ell_2,k_2}|, \max_{\substack{\ell_1+\ell_2+\ell_3+\ell_4=\ell\\ \ell_1\neq 0}} |P_{\ell_1,k+1}||(P_0^{-1})_{\ell_2}||(T_{-1})_{\ell_3}||(P_0)_{\ell_4}|\right)\\ &\leq |T|\cdot\max\left(\max_{\substack{\ell_1+\ell_2=\ell\\ k_1+k_2=k+1}} v_{\ell_2,k_2}, \max_{\substack{\ell_1+\ell_2+\ell_3+\ell_4=\ell\\ \ell_1\neq 0}} v_{\ell_1,k+1}v_{\ell_4,0}|(P_0^{-1})_{\ell_2}|\right),\end{aligned}$$

where on the second inequality we use (4.4.29).

Let $\alpha := \max(1, |T|)$. We use the above inequality to prove by induction on $\ell \geq 0$ that

$$\forall k \geq 0, \quad v_{\ell,k} \leq \alpha^\ell.$$

For $\ell = 0$, we have $v_{0,k} = \delta_{0,k}$ so the inequality is obvious. Now assume $v_{r,k} \leq \alpha^r$ for all $r \leq \ell$. By Lemma 4.4.27, we then have $|(P_0^{-1})_r| \leq \max_{1\leq i\leq r} \alpha^i = \alpha^r$ for all $r \leq \ell$. Since $\alpha \geq |T|$, we deduce that

$$\begin{aligned}(\ell+1)v_{\ell+1,k} &\leq |T|\max\left(\max_{\substack{s\leq\ell\\ k_2\leq k+1}} \alpha^s, \max_{\substack{\ell_1+\ell_2+\ell_4\leq\ell\\ \ell_1\neq 0}} \alpha^{\ell_1+\ell_2+\ell_4}\right)\\ &= |T|\alpha^\ell \leq \alpha^{\ell+1} \leq (\ell+1)\alpha^{\ell+1}.\end{aligned}$$

This concludes the inductive step.

Since $\alpha \geq 1$, we have $v_{\ell,k} \leq \alpha^\ell \leq \alpha^{\ell+k}$ for all $\ell, k \geq 0$. We deduce that P converges on the open disk of radius $\frac{1}{\alpha}$, completing the proof. □

We can now finish the proof of Theorem 4.4.26.

Proof of Theorem 4.4.26. Up to restricting to an open neighborhood of b, we may assume that $B = \mathrm{Sp}\, T_n$ by Lemma 4.3.9. Let $(t_1, \ldots, t_n)$ be local analytic coordinates centered at b. After rescaling we can assume that the connection matrices converge on $\mathrm{Sp}\, \Bbbk\langle t_1, \ldots, t_n, u\rangle$.

As in the formal case, we can reformulate the extension of framing problem into a system of PDEs (4.4.4) and (4.4.6). We can solve the equations (4.4.6) inductively on the number of t-variables, and by Lemma 4.4.22 the equation (4.4.4) will be automatically satisfied. Using Proposition 4.4.28 inductively, we obtain that at each step the solution, i.e the gauge transformation, converges on an admissible open neighborhood of b. $\square$

4.4.3 Reconstruction of isomorphism of framed maximal F-bundles

In this subsection, we explain how to use the extension of framing for logarithmic F-bundles (Theorem 4.4.2) to reconstruct an isomorphism of framed maximal F-bundles compatible with the framings. This is useful for establishing the uniqueness of mirror maps in applications to enumerative geometry.

Definition 4.4.30 (Compatibility of framings)**.** For $i = 1, 2$ let $(\mathcal{H}_i, \nabla_i, \nabla_i^{\mathrm{fr}})/(B_i, D_i)$ be two framed logarithmic F-bundles. A morphism $(f, \Phi)\colon (\mathcal{H}_1, \nabla_1)/(B_1, D_1) \to (\mathcal{H}_2, \nabla_2)/(B_2, D_2)$ of logarithmic F-bundles is said to be *compatible with the framings* if $\Phi \circ \nabla_1^{\mathrm{fr}} = (f \times \mathrm{id}_u)^* \nabla_2^{\mathrm{fr}} \circ \Phi$.

Proposition 4.4.31. *For $i = 1, 2$, let $(\mathcal{H}_i, \nabla_i)/(B_i, D_i)$ be a logarithmic F-bundle where B_i is the formal neighborhood of a rational point in a smooth $\Bbbk$-variety. Let $(f, \Phi)\colon (\mathcal{H}_1, \nabla_1)/(B_1, D_1) \to (\mathcal{H}_2, \nabla_2)/(B_2, D_2)$ be an isomorphism of logarithmic F-bundles with $f(b_1) = b_2$. Assume $(\mathcal{H}_1, \nabla_1)/(B_1, D_1)$ has a framing ∇_1^{fr}.*

1. *The bundle map Φ is uniquely determined by its restriction to $\mathcal{H}_1|_{b_1 \times \operatorname{Spf} \Bbbk[\![u]\!]}$.*

2. *If $(\mathcal{H}_1, \nabla_1)$ and $(\mathcal{H}_2, \nabla_2)$ are maximal, then the map on the bases f is also uniquely determined by its restriction to b_1, up to some multiplicative constants in the logarithmic directions. The reconstruction is explicit after fixing compatible cyclic vectors at b_1 and b_2.*

Proof. For (1), let H_i denote the fiber of $\mathcal{H}_i$ over b_i, and $\phi \in \operatorname{Hom}(H_1, H_2)$ the restriction of Φ at b_1. Fix a ∇_1^{fr}-flat trivialization Ψ_1 of $\mathcal{H}_1$ and an arbitrary trivialization Ψ_2 of $(f \times \mathrm{id}_u)^* \mathcal{H}_2$, producing the commutative diagram

$$\begin{tikzcd}
\mathcal{H}_1 \arrow[r, "\Psi_1"] \arrow[d, "\Phi"] & H_1 \times B_1 \times \operatorname{Spf} \Bbbk[\![u]\!] \arrow[d, "\widetilde{\Phi}"] \\
(f \times \mathrm{id}_u)^* \mathcal{H}_2 \arrow[r, "\Psi_2"] & H_2 \times B_1 \times \operatorname{Spf} \Bbbk[\![u]\!].
\end{tikzcd}$$

Denote by $\varphi\colon \mathcal{H}_1 \to (f\times \mathrm{id}_u)^*\mathcal{H}_2$ the map obtained from ϕ by taking its constant extension with respect to the trivializations Ψ_1 and Ψ_2. If $\tilde{\phi} = \widetilde{\Phi}|_{(b_1,0)}$, then $\varphi = \Psi_2^{-1}\circ(\tilde{\phi}\times \mathrm{id}_{B_1\times \operatorname{Spf}\Bbbk[\![u]\!]})\circ\Psi_1$. Define two connections on $(f\times\mathrm{id}_u)^*\mathcal{H}_2$

$$
\begin{aligned}
\nabla_1' &\coloneqq (f\times \mathrm{id}_u)^*\nabla_2 = \Phi\circ\nabla_1\circ\Phi^{-1},\\
\nabla_2' &\coloneqq \varphi\circ\nabla_1\circ\varphi^{-1}.
\end{aligned}
$$

In the trivialization Ψ_2 we see that ∇_1' is framed over all B_1, and ∇_2' is framed only at b_1. Furthermore ∇_1' and ∇_2' are gauge equivalent under $\Phi\circ\varphi^{-1}$, and $\Phi\circ\varphi^{-1}|_{b_1} = \mathrm{id}$. We conclude from Theorem 4.4.2 that $\Phi\circ\varphi^{-1}$ is unique, and then so is Φ provided that we know $\Phi|_{b_1\times\operatorname{Spf}\Bbbk[\![u]\!]}$. This proves (1).

Next we prove (2), and assume that the F-bundles are maximal. The framing ∇_1^{fr} induces unique framings ∇_2^{fr} (resp. $\nabla_2^{\mathrm{fr}\prime}$) on $(\mathcal{H}_2,\nabla_2)$ (resp. $f^*(\mathcal{H}_2,\nabla_2)$) such that in the diagram

$$
(\mathcal{H}_1,\nabla_1) \xrightarrow{(\mathrm{id}_{B_1},\Phi)} f^*(\mathcal{H}_2,\nabla_2) \xrightarrow{(f,\mathrm{id})} (\mathcal{H}_2,\nabla_2),
$$

all the morphisms are compatible with the framings. Furthermore, the framing $\nabla_2^{\mathrm{fr}\prime}$ is determined by ∇_1^{fr} and Φ, and hence is already known by (1).

Let h_1 be a ∇_1^{fr}-flat section of cyclic vectors for $(\mathcal{H}_1,\nabla_1)$. Because of the compatibility of the framings, h_1 induces a $\nabla_2^{\mathrm{fr}\prime}$-flat section of cyclic vectors $h_2' \coloneqq \Phi(h_1)$ for $f^*(\mathcal{H}_2,\nabla_2)$, and a ∇_2^{fr}-flat section of cyclic vectors $h_2 \coloneqq (f^{-1}\times\mathrm{id}_u)^*(h_2')$ for $(\mathcal{H}_2,\nabla_2)$. We obtain isomorphisms $\eta_i \coloneqq \mu_{\mathcal{H}_i}(\cdot)(h_i)\colon TB_i(-\log D_i)\to \mathcal{H}_i|_{u=0}$ that fit into a commutative diagram

$$
\begin{array}{ccc}
TB_1(-\log D_1) & \xrightarrow{df} & f^*TB_2(-\log D_2)\\
\downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle f^*(\eta_2)}\\
\mathcal{H}_1|_{u=0} & \xrightarrow{\Phi|_{u=0}} & (f\times\mathrm{id}_u)^*\mathcal{H}_2|_{u=0},
\end{array}
$$

where all arrows are isomorphisms. The maps η_1 and $\Phi|_{u=0}$ are already known. We have $f^*(\eta_2) = \mu_{(f\times\mathrm{id}_u)^*\mathcal{H}_2}(\cdot)(h_2')$ by construction and compatibility of (f,id) with the framings. So $f^*(\eta_2)$ is determined by h_1, $\nabla_2^{\mathrm{fr}\prime}$ and Φ, hence is known. We deduce that df is determined by h_1, ∇_1^{fr} and Φ.

Since B_i are formal neighborhoods of points, the differential df determines f uniquely, up to some multiplicative constants in the logarithmic directions. To see this, choose coordinates $(q,t) = (q_1,\dots,q_r,t_1,\dots,t_n)$ for (B_1,D_1), centered at b_1, where $\prod_{1\le i\le r} q_i = 0$ is a local equation for D_1. Similarly, choose coordinates

$(p, s) = (p_1, \dots, p_r, s_1, \dots, s_n)$ for (B_2, D_2) centered at $f(b_1)$. In coordinates, the restriction of f to B_1 is given by $f = (f_1, \dots, f_{r+n})$ where $q_i = f_i(p, s)$ and $t_j = f_{r+j}(p, s)$. The differential df corresponds to a map of $\Bbbk[\![q, t]\!]$-modules

$$\Psi \colon \Gamma(B_1, f^*\Omega^1_{B_2}(\log D_2)) \to \Gamma(B_1, \Omega^1_{B_1}(\log D_1)),$$

given by the pullback of differential forms, i.e.

$$\Psi(d \log p_i) = d \log f_i = \frac{df_i}{f_i}, \quad \Psi(ds_j) = df_{r+j}.$$

We conclude the proof by integrating the differential forms. $\square$

4.4.4 Equivalence of F-bundles over a point

For applications in Section 4.5, we present some results here for the classification of framed F-bundles over a point up to gauge equivalence; see Theorem 4.4.34 and Corollary 4.4.35.

Let $(\mathcal{H}, \nabla, \nabla^{\mathrm{fr}})$ be a framed F-bundle over a point. Fix a ∇^{fr}-flat trivialization $\mathcal{H} \simeq H \otimes_{\Bbbk} \Bbbk[\![u]\!]$ and write

$$\nabla_{u\partial_u} = u\partial_u + u^{-1}\mathbf{K} + \mathbf{G},$$

with $\mathbf{K}, \mathbf{G} \in \mathrm{End}_{\Bbbk}(H)$.

We assume that the endomorphism $\mathbf{K}$ induces a $\Bbbk$-vector space decomposition $H = \bigoplus_{1 \le k \le m} H_k$ into generalized eigenspaces, and all H_k have same dimensions. Then we have a $\Bbbk$-vector space H_0 and a splitting of the fiber

$$iso \colon H_0^{\oplus m} \xrightarrow{\sim} H. \tag{4.4.32}$$

So we can represent endomorphisms on H as $m \times m$ matrices with coefficients in $\mathrm{End}_{\Bbbk}(H_0)$. In particular we write $\mathbf{K} = (\mathbf{K}_{ij})_{1 \le i,j \le m}$ and $\mathbf{G} = (\mathbf{G}_{ij})_{1 \le i,j \le m}$. By construction $\mathbf{K}_{ij} = 0$ if $i \neq j$ and $\mathbf{K}_{ii} = \xi_i \mathrm{id}_{H_0} + \mathbf{N}_i$ with $\xi_i \in \Bbbk$ and $\mathbf{N}_i$ a nilpotent endomorphism.

Fix $\mathbf{c}_1, \dots, \mathbf{c}_r, \mathbf{d} \in \mathrm{End}_{\Bbbk}(H_0)$ such that $\mathbf{c}_i$ are nilpotent endomorphisms, $[\mathbf{c}_i, \mathbf{c}_j] = 0$ and $[\mathbf{d}, \mathbf{c}_i] = d_i \mathbf{c}_i$ for $d_i \in \mathbb{N}_{>0}$.

Definition 4.4.33. We denote by $\mathcal{F}(H, iso, \mathbf{d}, (\mathbf{c}_i)_{1 \le i \le r})$ the space of connections ∇' on $\mathcal{H}$ which, in the fixed ∇^{fr}-flat trivialization, are of the form

$$\nabla'_{u\partial_u} = u\partial_u + u^{-1}\mathbf{K}' + (\mu'\mathbf{D} + \mathbf{H}'),$$

where

1. $\mu' \notin \mathbb{Q}_{<0} \subset \Bbbk$,
2. $\mathbf{K}', \mathbf{D}, \mathbf{H}' \in \mathrm{End}_{\Bbbk}(H)$,
3. $\mathbf{K}'_{ij}, \mathbf{H}'_{ij} \in \Bbbk[\mathbf{c}_1, \ldots, \mathbf{c}_r]$, and
4. $\mathbf{D}_{ii} = \mathbf{d}$ and $\mathbf{D}_{ij} = 0$ for $i \neq j$.

Theorem 4.4.34. *Let $(\mathcal{H}, \nabla, \nabla^{\mathrm{fr}})$ be as above. Assume $\nabla \in \mathcal{F}(H, iso, \mathbf{d}, (\mathbf{c}_i)_{1 \le i \le r})$ and let $\nabla' \in \mathcal{F}(H, iso, \mathbf{d}, (\mathbf{c}_i)_{1 \le i \le r})$. Write*

$$\begin{aligned} \nabla_{u\partial_u} &= u\partial_u + u^{-1}\mathbf{K} + (\mu\mathbf{D} + \mathbf{H}), \\ \nabla'_{u\partial_u} &= u\partial_u + u^{-1}\mathbf{K}' + (\mu'\mathbf{D} + \mathbf{H}'). \end{aligned}$$

Then ∇ is gauge-equivalent to ∇' under $\mathbf{\Phi}(u) \in \mathrm{GL}(H[\![u]\!])$ with $\mathbf{\Phi}_{ij}(u) \in \Bbbk[\mathbf{c}_1, \ldots, \mathbf{c}_r][\![u]\!]$ if and only if the following three conditions are satisfied:

1. *there exists $\phi \in \mathrm{GL}(H)$ with $\phi_{ij} \in \Bbbk[\mathbf{c}_1, \ldots, \mathbf{c}_r]$ such that $\mathbf{K} = \phi^{-1} \circ \mathbf{K}' \circ \phi$,*
2. *$\mu = \mu'$, and*
3. *for all $1 \le i \le m$, $\mathbf{H}_{ii} = (\phi^{-1} \circ \mathbf{H}' \circ \phi)_{ii} \mod (\mathbf{c}_1, \ldots, \mathbf{c}_r)$.*

Furthermore, $\mathbf{\Phi}$ is then uniquely determined by the initial condition $\mathbf{\Phi}|_{u=0} = \phi \mod (\mathbf{c}_1, \ldots, \mathbf{c}_r)$.

The assumptions on the form of the operators allow us to work in the non-commutative subalgebra $\Bbbk[\mathbf{d}, \mathbf{c}_1, \ldots, \mathbf{c}_r] \subset \mathrm{End}_{\Bbbk}(H_0)$. We then reduce to the case of simple eigenvalues by treating the operators $\mathbf{d}$, $(\mathbf{c}_i)_{1 \le i \le r}$ as formal variables. In the simple eigenvalues case, the gauge equivalence can be constructed inductively.

As a corollary, in the simple eigenvalue case we obtain a classification of F-bundles over a point with a fixed framing.

Corollary 4.4.35. *Let $\mathcal{H} \simeq H \times \Bbbk[\![u]\!]$ be a trivialized rank m vector bundle over $\Bbbk[\![u]\!]$. Let $(\mathcal{H}, \nabla)$ and $(\mathcal{H}, \nabla')$ be two F-bundle structures framed in the given trivialization, and write*

$$\begin{aligned} \nabla_{u\partial_u} &= u\partial_u + u^{-1}\mathbf{K} + \mathbf{G}, \\ \nabla'_{u\partial_u} &= u\partial_u + u^{-1}\mathbf{K}' + \mathbf{G}'. \end{aligned}$$

Assume $\mathbf{K}$ has simple eigenvalues. Then $(\mathcal{H}, \nabla)$ is isomorphic to $(\mathcal{H}, \nabla')$ if and only if there exists $\phi \in \mathrm{GL}(H)$ such that

1. $\mathbf{K} = \phi^{-1} \circ \mathbf{K}' \circ \phi$, *and*

2. *in an eigenbasis of* $\mathbf{K}$, *we have* $(\mathbf{G})_{ii} = (\phi^{-1} \circ \mathbf{G}' \circ \phi)_{ii}$ *for* $1 \leq i \leq m$.

Furthermore, the gauge equivalence is uniquely and explicitly determined by the initial condition ϕ *at* $u = 0$.

Proof. The choice of an eigenbasis for $\mathbf{K}$ produces a splitting $iso\colon \Bbbk^{\oplus m} \xrightarrow{\sim} H$ as in (4.4.32). Since there are no nilpotent operators in $\mathrm{End}_{\Bbbk}(\Bbbk) \simeq \Bbbk$, and this algebra is commutative, the content of Definition 4.4.33 becomes empty, and the corollary is just a reformulation of Theorem 4.4.34 in this special case. □

Proof of Theorem 4.4.34. Let $R_0 = \Bbbk[\![c_1, \ldots, c_r]\!]$ and $R = \Bbbk[\deg][\![c_1, \ldots, c_r]\!]$ where $\{(c_i)_{1 \leq i \leq r}, \deg\}$ are formal variables satisfying the commutation relations $[c_i, c_j] = 0$ and $[\deg, c_i] = d_i c_i$. There is a specialization map $R \to \Bbbk[\mathbf{d}, \mathbf{c}_1, \ldots, \mathbf{c}_r]$. Using iso we also have a specialization map

$$\mathrm{Mat}(m \times m, R) \to \mathrm{End}_{\Bbbk}(H).$$

By the definition of $\mathcal{F}(H, iso, \mathbf{d}, (\mathbf{c}_i)_{1 \leq i \leq r})$, the connections $\nabla_{u\partial_u}$ and $\nabla'_{u\partial_u}$ lift to differential operators of the form

$$u\partial_u + u^{-1}K + \mu D + H,$$
$$u\partial_u + u^{-1}K' + \mu' D + H',$$

with $K, K', H, H' \in \mathrm{Mat}(m \times m, R_0)$ and $D = \deg \cdot \mathrm{Id}_m$. A gauge equivalence $\mathbf{\Phi}$ as in the theorem also lifts along the specialization map, so we have reduced the problem to finding $\Phi(u) \in \mathrm{GL}(m, R_0[\![u]\!])$ such that

$$\Phi^{-1}(u\partial_u + u^{-1}K + \mu D + H)\Phi = u\partial_u + u^{-1}K' + \mu' D + H'. \tag{4.4.36}$$

The conditions (1)-(3) also lift under the specialization map, so we are left to prove the following lemma. □

Lemma 4.4.37. *There exists a gauge equivalence* $\Phi(u) \in \mathrm{GL}(m, R_0[\![u]\!])$ *solving* (4.4.36) *if and only if there exists* $Q \in \mathrm{GL}(m, R_0)$ *such that*

(a) $K = Q^{-1} K' Q$,

(b) $\mu = \mu'$, *and*

(c) $H_{ii} = (Q^{-1}H'Q)_{ii} \mod (c_1, \dots, c_r)$.

In this case, $\Phi(u)$ *is uniquely determined by the initial condition* $\Phi|_{u=0} = Q \mod (c_1, \dots, c_r)$.

Proof. By construction of the splitting (4.4.32) and Definition 4.4.33(2), the matrix K is diagonal, and $K_{ii} = \xi_i \mod (c_1, \dots, c_r)$, where $\{\xi_1, \dots, \xi_m\}$ are the distinct eigenvalues of $\mathbf{K}$. In particular K has simple eigenvalues, so $\mathrm{ad}_K = [K, \cdot]$ has kernel given by diagonal matrices, and image given by matrices with vanishing diagonal.

Let us first prove that the conditions (a)-(c) are sufficient. Fix $Q \in \mathrm{GL}(m, R_0)$ satisfying (a) and (c). We are looking for $\Phi(u)$ such that $\Phi|_{u=0} = Q \mod (c_1, \dots, c_r)$ solving (4.4.36). Write $\Phi(u) = QP(u)$ with $P(u) = \sum_{k\geq 0} P_k u^k$ satisfying $P_0 = \mathrm{Id}_m \mod (c_1, \dots, c_r)$. Equation (4.4.36) then reduces to the system

$$[K, P_0] = 0, \tag{4.4.38}$$
$$[K, P_{k+1}] = \varphi(P_k) - kP_k, \tag{4.4.39}$$

where

$$\varphi \colon M \mapsto M(\mu' Q^{-1}DQ + Q^{-1}H'Q) - (\mu D + H)M.$$

Before analyzing the existence of solutions, let us rewrite (4.4.39) in order to isolate the terms involving the non-commutative variable $\deg$. Define the $\Bbbk$-linear operator $\mathrm{Eu}(\cdot) := [\deg, \cdot]$ on R. The commutations relations in R give $\mathrm{Eu}(\cdot) = \sum_{1\leq i\leq r} d_i c_i \partial_{c_i}$. For $M \in \mathrm{Mat}(m \times m, R)$, we write $\mathrm{Eu}(M) := (\mathrm{Eu}(M_{ij}))_{1\leq i,j\leq m}$. We have $\mathrm{Eu}(M) = [D, M]$, so

$$\begin{aligned}\varphi(M) &= M(\mu' D + Q^{-1}\,\mathrm{Eu}(Q) + Q^{-1}H'Q) - (\mu D + H)M \\ &= \mu' MD - \mu DM + M(Q^{-1}\,\mathrm{Eu}(Q) + Q^{-1}H'Q) - HM \\ &= (\mu' - \mu)MD - \mu\,\mathrm{Eu}(M) + M(Q^{-1}\,\mathrm{Eu}(Q) + Q^{-1}H'Q) - HM.\end{aligned}$$

Since $\mu = \mu'$, the term involving D vanishes.

We now prove by induction on k the following: there exists a unique sequence of matrices $(P_0, \dots, P_k)$ such that (i) $P_0 = \mathrm{Id}_m \mod (c_1, \dots, c_r)$ and $[K, P_0] = 0$, (ii) $(P_\ell, P_{\ell+1})$ solves (4.4.39) for $0 \leq \ell \leq k-1$, and (iii) $\varphi(P_k) - kP_k \in \mathrm{im}\,\mathrm{ad}_K$.

We construct P_0 satisfying (i), (ii) and (iii). The condition $[K, P_0] = 0$ implies that P_0 is a diagonal matrix, $P_0 = \mathrm{Diag}(\delta_1, \dots, \delta_n)$. The initial condition $P_0 = \mathrm{Id}_m$

$\bmod (c_1, \ldots, c_r)$ gives $\delta_i = 1 \mod (c_1, \ldots, c_r)$. To ensure that we can solve the recursion for P_1, we need $\varphi(P_0)_{ii} = 0$ for all i. This provides the relation

$$\mu \operatorname{Eu}(\delta_i) = \alpha_i \delta_i, \tag{4.4.40}$$

for all i, where $\alpha_i = (Q^{-1} \operatorname{Eu}(Q) + Q^{-1} H' Q - H)_{ii}$. For any $x \in R_0$, we have $\operatorname{Eu}(x) \in (c_1, \ldots, c_r) R_0$. Together with Condition (c), this implies that $\alpha_i = 0 \mod (c_1, \ldots, c_r)$. We can then solve for δ_i order by order in $(c_1, \ldots, c_r)$ and determine P_0 uniquely from the initial condition $P_0 = \mathrm{Id}_m \mod (c_1, \ldots, c_r)$. Note that the condition on μ in Definition 4.4.33 ensures that we obtain a recursion that we can solve.

Let $k \geq 1$, and assume $(P_0, \ldots, P_{k-1})$ are constructed. The existence of a matrix P such that $[K, P] = \varphi(P_{k-1}) - (k-1) P_{k-1}$ is guaranteed by Condition (iii) of the induction hypothesis. The matrix P is determined up to a diagonal matrix. We first prove that for any choice of P, there exists a unique diagonal matrix Δ such that $\varphi(P + \Delta) - k(P + \Delta) \in \operatorname{im} \operatorname{ad}_K$, i.e. has vanishing diagonal. Let $\Delta = \operatorname{Diag}(\delta_1, \ldots, \delta_n)$ be a diagonal matrix, the vanishing of the i-th diagonal term of $\varphi(P + \Delta) - k(P + \Delta)$ is equivalent to an equation of the form

$$\mu \operatorname{Eu}(\delta_i) + k \delta_i = \alpha \delta_i + \beta, \tag{4.4.41}$$

with

$$\begin{aligned} \alpha &= (Q^{-1} \operatorname{Eu}(Q))_{ii} + (Q^{-1} H' Q)_{ii} - H_{ii}, \\ \beta &= (\varphi(P) - kP)_{ii}. \end{aligned}$$

As in the initial step ($k = 0$), we have $\alpha = 0 \mod (c_1, \ldots, c_r)$. Since δ_i is a power series in $(c_1, \ldots, c_r)$, (4.4.41) provides a recursion relation on the coefficients of δ_i. Since $k \geq 1$ the constant term of δ_i is uniquely determined by looking at the equation modulo $(c_1, \ldots, c_r)$, where it gives $k\delta_i = \beta \mod (c_1, \ldots, c_r)$. The other coefficients are then uniquely determined inductively. The condition on μ in Definition 4.4.33 ensures that we obtain a recursion that we can solve, thus δ_i is uniquely determined from P. We have proved the existence of a matrix P_k satisfying Conditions (ii) and (iii) of the induction. Now we prove uniqueness. Let P_k and $\widetilde{P_k}$ be two matrices satisfying (ii) and (iii). In particular, they are solutions of the equation $[K, P] = \varphi(P_{k-1}) - (k-1) P_{k-1}$, so there exists a diagonal matrix Δ such that $P_k = \widetilde{P_k} + \Delta$. Condition (iii) gives $\varphi(\widetilde{P_k} + \Delta) - k(\widetilde{P_k} + \Delta) \in \operatorname{im} \operatorname{ad}_K$. Since $\widetilde{P_k}$ already satisfies (iii) and Δ is diagonal, we deduce from the uniqueness in the previous paragraph that $\Delta = 0$. Hence $P_k = \widetilde{P_k}$, concluding the induction.

Now we prove Conditions (a)-(c) assuming that there exists $\Phi(u) = \sum_{k\geq 0} P_k u^k \in \mathrm{GL}(m, R_0[\![u]\!])$ solving (4.4.36). In particular $P_0 \in \mathrm{GL}(m, R_0)$. Multiplying (4.4.36) on the left by Φ and isolating the u^k term, we obtain for $k = -1$ and $k \geq 0$ respectively:

$$KP_0 = P_0 K',$$
$$kP_k + KP_{k+1} + (\mu D)P_k + HP_k = P_{k+1}K' + P_k(\mu' D) + P_k H'.$$

Let $Q = P_0^{-1}$, it satisfies Condition (a). For any $1 \leq i \leq r$ we have $\deg \cdot c_i = c_i \cdot \deg + d_i c_i$. By comparing the coefficient of the formal variable $\deg$ we obtain $\mu = \mu'$, verifying Condition (b). Looking at the u^0 term, using $K = P_0 K' P_0^{-1}$ and modding out $(c_i)_{1\leq i\leq r}$, we obtain

$$K(P_1 P_0^{-1}) - P_1 P_0^{-1} K + H = P_0 H' P_0^{-1} \mod (c_1, \ldots, c_r).$$

Since $[K, P_1(P_0)^{-1}]$ has vanishing diagonal, Condition (c) follows. □

4.5 Application: quantum cohomology of projective bundle

In this section, we study the decomposition of the maximal A-model F-bundle associated to a projective bundle. We prove the existence of the decomposition when restricting the F-bundle to a point, as well as the uniqueness of the decomposition (Theorems 4.5.16 and 4.5.20). In Section 4.5.4, we state the analogous results in the case of a blowup of algebraic varieties (Theorems 4.5.22 and 4.5.24).

Let X be a smooth complex projective variety of dimension d, $V \to X$ a vector bundle of rank m on X, $P := \mathbb{P}(V)$ the associated projective bundle of lines in V, and write $\pi\colon P \to X$. We fix an ample divisor class $\omega_X \in H^2(X, \mathbb{Z})$, and a homogeneous basis $\{T_i\}_{0\leq i\leq N}$ of $H^*(X, \mathbb{Q})$ extending $\{\mathbf{1}, \omega_X\}$.

4.5.1 A-model F-bundle of P at the limiting point

We have the following classical decomposition of the cohomology of P, as a special case of Leray-Hirsch theorem (see [Hat02, Theorem 4D-1]).

Proposition 4.5.1. *Let $h := c_1(\mathcal{O}_P(1))$. We have the splitting isomorphism of cohomology groups*

$$iso\colon H_{\text{split}} := \bigoplus_{i=0}^{m-1} H^*(X, \mathbb{Q})[-2i] \xrightarrow{\sum h^i \cup \pi^*} H^*(P, \mathbb{Q}). \tag{4.5.2}$$

Lemma 4.5.3. *We have*

$$K_P = \pi^* K_X - mh - \pi^* c_1 V.$$

Proof. It follows from the relative Euler sequence

$$0 \to \Omega_{P/X} \to \mathcal{O}_P(-1) \otimes \pi^* V^\vee \to \mathcal{O}_P \to 0$$

that

$$K_{P/X} = -mh - \pi^* c_1 V.$$

Hence

$$K_P = \pi^* K_X + K_{P/X} = \pi^* K_X - mh - \pi^* c_1 V. \qquad \square$$

Recall that we fixed an ample class ω_X on X. Let $\omega_P \coloneqq \pi^*\omega_X$.

Lemma 4.5.4. *The class ω_P is nef and satisfies Assumption 4.2.22.*

Proof. Since ω_X is ample, its pullback ω_P is nef. Furthermore, there exists $\varepsilon > 0$ such that $\omega_X + \varepsilon(c_1 T_X + c_1 V)$ is ample. Then, by Lemma 4.5.3, we have $\omega_P + \varepsilon c_1 P = \pi^*(\omega_X + \varepsilon(c_1 T_X + c_1 V)) + \varepsilon mh$. It is ample, since it is the sum of a nef class and an ample class ([Laz04, Corollary 1.4.10]). We conclude by Lemma 4.2.23. $\square$

Using the homogeneous basis $\{T_i\}_{0 \le i \le N}$ of $H^*(X, \mathbb{Q})$, we produce a homogeneous basis

$$\{\pi^*(T_i)h^j,\ 0 \le i \le N,\ 0 \le j \le m-1\}$$

of $H^*(P, \mathbb{C})$ extending ω_P. We denote by $\{t_{i,j}\}$ the induced linear coordinates on $H^*(P, \mathbb{C})$.

Let $(\mathcal{H}, \nabla)/B$ denote the maximal A-model F-bundle of P constructed from ω_P, with base point $0 \in H^*(P, \mathbb{C})$ (see Example 4.2.25). Write $(q, t = \{t_{i,j},\ (i,j) \neq (1,0)\})$ for the coordinates on B. Let b denote the closed point of B, given by $q = 0, t = 0$, which we refer to as the limiting point in this section. Let $\mathbf{K}_{\mathrm{lim}}$ and $\mathbf{G}_{\mathrm{lim}}$ denote the restrictions of the operators $\mathbf{K}$ and $\mathbf{G}$ at the limiting point (see Definition 4.2.17).

Let us compute the matrices of $\mathbf{K}_{\mathrm{lim}}$ and $\mathbf{G}_{\mathrm{lim}}$ under the splitting iso in (4.5.2).

We have

$$\mathbf{G}_{\mathrm{lim}} = \begin{pmatrix} \mathbf{G}_X - \frac{m-1}{2} & & & \\ & \mathbf{G}_X - \frac{m-3}{2} & & \\ & & \ddots & \\ & & & \mathbf{G}_X + \frac{m-1}{2} \end{pmatrix},$$

and $\mathbf{K}_{\mathrm{lim}}$ is computed in the following proposition.

Proposition 4.5.5. *The operator $\mathbf{K}_{\lim}$ on $H^*(P,\mathbb{C})$ has the following matrix with respect to the splitting in (4.5.2):*

$$\mathbf{K}_{\lim} = \begin{pmatrix} c_1T_X + c_1V & & & & m(1-c_mV) \\ m & c_1T_X + c_1V & & & -mc_{m-1}V \\ & m & \ddots & & \vdots \\ & & \ddots & c_1T_X + c_1V & -mc_2V \\ & & & m & c_1T_X + c_1V - mc_1V \end{pmatrix}.$$

Proof. Consider four operators $K_1, \dots, K_4$ on H_{split} such that for $\gamma \in H^*(P,\mathbb{C}) \simeq H_{\text{split}}$, we have

1. $K_1(\gamma) = \pi^*(c_1T_X) \cup \gamma$,
2. $K_2(\gamma) = h \cup \gamma$,
3. $K_3(\gamma) = \pi^*c_1V \cup \gamma$, and
4. $K_4(\gamma) = p_*q^*\gamma$, where $p, q\colon P \times_X P \to P$ are the projections.

By Lemma 4.5.3, the classical multiplication by c_1T_P has matrix $K_1 + mK_2 + K_3$.

The non-classical part of $\mathbf{K}_{\lim}$ is expressed in terms of 3-pointed Gromov-Witten invariants of the form $\langle c_1P, \gamma_1, \gamma_2\rangle_{0,3}^{\beta}$ for an effective curve class $\beta \neq 0$ such that $\beta \cdot \omega_P = 0$ and cohomology classes $\gamma_1, \gamma_2 \in H^*(P,\mathbb{C})$. Fix such a β, by the projection formula, we have $\beta \cdot \omega_P = (\pi_*\beta) \cdot \omega_X$. Since ω_X is ample, this implies that $\pi_*\beta = 0$, i.e. $\beta = \delta[L]$ for $[L]$ the class of a line in a fiber of π and $\delta \in \mathbb{N}_{>0}$ ($\delta = 0$ gives the classical contribution). By the divisor axiom and Lemma 4.5.3, we have

$$\langle c_1P, \gamma_1, \gamma_2\rangle_{0,3}^{\beta} = (\beta \cdot c_1P)\,\langle \gamma_1, \gamma_2\rangle_{0,2}^{\beta} = \delta m \langle \gamma_1, \gamma_2\rangle_{0,2}^{\beta}.$$

Let $M \coloneqq \overline{\mathcal{M}}_{0,2}(P, \delta[L])$ denote the moduli stack of 2-pointed rational stable maps of class β. By the Riemann-Roch formula, the virtual dimension $\dim_{\text{vir}} M$ of M is equal to $\dim P - 3 + \int_\beta c_1(P) + 2 = d - 2 + m(\delta + 1)$. Since β is a fiber class, the evaluation map

$$\mathrm{ev}_1 \times \mathrm{ev}_2\colon M \to P \times P$$

factors through

$$P \times_X P \subset P \times P.$$

In order to have nonzero counts, we need $\dim_{\mathrm{vir}} M \leq \dim P \times_X P$ which implies that $\delta = 1$, i.e. the curve class can only be $[L]$. We then have an isomorphism

$$\mathrm{ev}_1 \times \mathrm{ev}_2 \colon M \xrightarrow{\sim} P \times_X P \subset P \times P.$$

In particular, M is smooth, so $[M]^{\mathrm{vir}} = [M]$. Under this isomorphism, the operator

$$\gamma \mapsto \mathrm{ev}_{1,*}\left(\mathrm{ev}_2^* \gamma \cup [M]^{\mathrm{vir}}\right) = \mathrm{ev}_{1,*} \mathrm{ev}_2^* \gamma$$

is equal to mK_4. Therefore, the non-classical contribution to $\mathbf{K}_{\mathrm{lim}}$ is mK_4.

We obtain

$$\mathbf{K}_{\mathrm{lim}} = iso \circ (K_1 + mK_2 + K_3 + mK_4) \circ iso^{-1}. \tag{4.5.6}$$

Now let us calculate the four matrices $K_1, \ldots, K_4$. For any $\alpha_i \in H^*(X, \mathbb{C})[-2i]$, we have

$$\pi^*(c_1 T_X) \cup (h^i \cup \pi^* \alpha_i) = h^i \cup \pi^*(c_1 T_X \cup \alpha_i),$$

hence $K_1 = (c_1 T_X \cup) \cdot \mathrm{id}_{H_{\mathrm{split}}}$. Similarly, we have that $K_3 = (c_1 V \cup) \cdot \mathrm{id}_{H_{\mathrm{split}}}$. For $i = 0, \ldots, m-1$, we have

$$h \cup (h^i \cup \pi^* \alpha_i) = h^{i+1} \cup \pi^* \alpha_i.$$

When $i = m-1$, by [BT82, Eq. (20.6)] we have

$$h \cup (h^{m-1} \cup \pi^* \alpha_i) = h^m \cup \pi^* \alpha_{m-1} = -\sum_{j=0}^{m-1} h^j \cup \pi^*(c_{m-j} V \cup \alpha_{m-1}).$$

So

$$K_2 = \begin{pmatrix} & & & & -c_m V \\ 1 & & & & -c_{m-1} V \\ & \ddots & & & \vdots \\ & & 1 & & -c_2 V \\ & & & 1 & -c_1 V \end{pmatrix}.$$

For any $\alpha_i \in H^*(X, \mathbb{C})[-2i]$, $i = 0, \ldots, m-1$, since $\pi \circ p = \pi \circ q$, by the projection formula we have

$$p_* q^*(h^i \cup \pi^* \alpha_i) = p_*(q^*(h^i) \cup q^* \pi^* \alpha_i) = p_*(q^*(h^i) \cup p^* \pi^* \alpha_i) = p_* q^*(h^i) \cup \pi^* \alpha_i.$$

Since $p_* q^*(h^i) \in H^{2(i-(m-1))}(P, \mathbb{C})$, it vanishes unless $i = m-1$, in which case it is equal to the identity. We deduce that the matrix of K_4 has only one nonzero block: the top-right corner, which is $\mathrm{id}_{H^*(X,\mathbb{C})}$.

Substituting the above computations into (4.5.6), we conclude the proof. $\square$

4.5.2 Decomposition of $\mathbf{K}_{\mathrm{lim}}$

In this subsection, we study the generalized eigenspaces of $\mathbf{K}_{\mathrm{lim}}$. We will consider the commutative subalgebra $\mathbb{C}[\mathbf{t}, \mathbf{c}_1, \ldots, \mathbf{c}_m]$ of $\mathrm{End}_{\mathbb{C}}(H^*(X, \mathbb{C}))$ generated by the commuting nilpotent operators

$$\mathbf{t} \coloneqq c_1 T_X \cup \quad \text{and} \quad \mathbf{c}_i \coloneqq c_i V \cup \; (1 \le i \le m).$$

Let $\mathbf{d} \coloneqq \mathbf{G}_X = \frac{1}{2}(\deg_X - \dim X)$, where $\deg_X(\alpha) = i\alpha$ for $\alpha \in H^i(X, \mathbb{C})$. We have the commutation relations

$$[\mathbf{d}, \mathbf{t}] = \mathbf{t}, \quad [\mathbf{d}, \mathbf{c}_i] = i\mathbf{c}_i. \tag{4.5.7}$$

Lemma 4.5.8. *1. There exists $\phi = (\phi_{ij}) \in \mathrm{GL}(H_{\mathrm{split}})$ with entries $\phi_{ij} \in \mathbb{C}[\mathbf{c}_1, \ldots, \mathbf{c}_m]$, and $\boldsymbol{\lambda}_i = \lambda_i \cup \in \mathbb{C}[\mathbf{c}_1, \ldots, \mathbf{c}_m] \subset \mathrm{End}_{\mathbb{C}}(H^*(X, \mathbb{C}))$ such that*

$$\mathbf{K}_{\mathrm{split}} \coloneqq \phi^{-1}\mathbf{K}_{\mathrm{lim}}\phi = \begin{pmatrix} \mathbf{t}+\mathbf{c}_1 & & \\ & \ddots & \\ & & \mathbf{t}+\mathbf{c}_1 \end{pmatrix} + m \begin{pmatrix} \boldsymbol{\lambda}_1 & & \\ & \ddots & \\ & & \boldsymbol{\lambda}_m \end{pmatrix}.$$

2. Up to reordering the blocks, for $1 \le i \le m$ we have

$$\boldsymbol{\lambda}_i = \xi^{i-1} - \frac{\mathbf{c}_1}{m} \mod (\mathbf{c}_1^2, \mathbf{c}_2, \ldots, \mathbf{c}_m), \quad \xi = e^{\frac{2\pi \mathrm{i}}{m}}.$$

In particular the i-th diagonal block of $\mathbf{K}_{\mathrm{split}}$ is the cup-product with an element in $H^*(X, \mathbb{C})$ whose H^2-component is $c_1 T_X$.

Proof. As an element of $\mathrm{Mat}(m \times m, \mathbb{C}[\mathbf{c}_1, \ldots, \mathbf{c}_m])$, we have $\mathbf{K}_{\mathrm{lim}} = (\mathbf{t}+\mathbf{c}_1)\mathrm{Id}_m + mM$, where M is the companion matrix

$$M = \begin{pmatrix} 0 & & & & 1-\mathbf{c}_m \\ 1 & & & & -\mathbf{c}_{m-1} \\ & 1 & & & -\mathbf{c}_{m-2} \\ & & \ddots & & \vdots \\ & & & 1 & -\mathbf{c}_1 \end{pmatrix}. \tag{4.5.9}$$

The characteristic polynomial of M is $\lambda^m + \sum_{i=1}^{m-1} \mathbf{c}_{m-i}\lambda^i + (\mathbf{c}_m - 1)$. Modulo $(\mathbf{c}_1, \ldots, \mathbf{c}_m)$ this polynomial has simple roots given by m-th roots of unity. Since it is monic, we can lift these roots to $\mathbb{C}[\mathbf{c}_1, \ldots, \mathbf{c}_m]$ by solving the equation order by order. (1) follows.

For (2), the characteristic polynomial of M modulo $(\mathbf{c}_1^2, \mathbf{c}_2, \ldots, \mathbf{c}_m)$ is

$$\lambda^m + \mathbf{c}_1\lambda^{m-1} - 1 = \left(\lambda + \frac{\mathbf{c}_1}{m}\right)^m - 1.$$

We deduce that $m\boldsymbol{\lambda}_i = me^{\frac{2\pi \mathrm{i}(i-1)}{m}} - \mathbf{c}_1$ modulo those classes, proving (2). $\square$

Lemma 4.5.8 implies that $\mathbf{K}_{\text{lim}}$ has m generalized eigenspaces, all isomorphic to $H^*(X, \mathbb{C})$, matching the setup of Section 4.4.4. The splitting considered in (4.4.32) is given by the modified isomorphism

$$iso \circ \phi^{-1} \colon H_{\text{split}} \xrightarrow{\sim} H^*(P, \mathbb{C}). \tag{4.5.10}$$

We will use the following lemma to check Condition (c) of Theorem 4.4.34.

Lemma 4.5.11. *Let $H = \mathrm{Diag}(\mu_1, \cdots, \mu_m) \in \mathrm{GL}(H_{\text{split}})$ be a block diagonal matrix with scalar entries. Let $\phi = (\phi_{ij}) \in \mathrm{GL}(H_{\text{split}})$ be as in Lemma 4.5.8. Assume that $\sum_{1\leq j\leq m} \mu_j = 0$. Then $(\phi^{-1} \circ H \circ \phi)_{ii} = 0$ for all $1 \leq i \leq m$.*

Proof. As in the previous lemma, we view ϕ and H as elements in $\mathrm{Mat}(m \times m, \mathbb{C}[\![\mathbf{c}_1, \ldots, \mathbf{c}_m]\!])$. By construction, ϕ diagonalizes the companion matrix $M \in \mathrm{Mat}(m \times m, \mathbb{C}[\mathbf{c}_1, \ldots, \mathbf{c}_m])$ from (4.5.9). Let $\Lambda = \mathrm{Diag}(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m)$. By construction we have $M\phi = \phi\Lambda$. For every $1 \leq i \leq m$, we deduce

$$\phi_{mi} = \boldsymbol{\lambda}_i\phi_{1i},\ \phi_{1i} = \boldsymbol{\lambda}_i\phi_{2i},\ \phi_{2i} = \boldsymbol{\lambda}_i\phi_{3i}, \cdots, \phi_{m-1,i} = \boldsymbol{\lambda}_i\phi_{mi}.$$

Similarly, for $\psi := \phi^{-1}$ we have that $\Lambda\psi = \psi M$, and we obtain for all $1 \leq i \leq m$

$$\boldsymbol{\lambda}_i\psi_{i1} = \psi_{i2},\ \boldsymbol{\lambda}_i\psi_{i2} = \psi_{i3}, \ldots,\ \boldsymbol{\lambda}_i\psi_{i,m-1} = \psi_{im},\ \boldsymbol{\lambda}_i\psi_{im} = \psi_{i1}.$$

In particular for $1 \leq i \leq m$, we have

$$\psi_{i1}\phi_{1i} = \psi_{i2}\phi_{2i} = \cdots = \psi_{im}\phi_{mi}.$$

We deduce

$$(\phi^{-1} \circ H \circ \phi)_{ii} = \sum_{1\leq j\leq m} \psi_{ij}(H)_{jj}\phi_{ji} = \psi_{i1}\phi_{1i} \sum_{1\leq j\leq m} \mu_j = 0. \qquad \square$$

Remark 4.5.12. The automorphism $\phi \mod (\mathbf{c}_1, \ldots, \mathbf{c}_m)$ gives the initial condition for the gauge equivalence in Theorem 4.5.16. Since it diagonalizes the (block) circulant matrix $M \mod (\mathbf{c}_1, \ldots, \mathbf{c}_m)$ it can be chosen to be the matrix

$$Q = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & \xi^{-1} & \cdots & \xi^{-(m-1)} \\ 1 & \xi^{-2} & \cdots & (\xi^{-2})^{m-1} \\ \vdots & & & \vdots \\ 1 & \xi^{-(m-1)} & \cdots & (\xi^{-(m-1)})^{m-1} \end{pmatrix}.$$

Example 4.5.13 (Trivial bundle case)**.** If $V = \mathcal{O}_X^{\oplus m}$ is a trivial vector bundle, then $\mathbf{c}_i = 0$ for $1 \leq i \leq m$. In particular, we have $\lambda_i = \xi^{i-1}$, where $\xi = e^{\frac{2\pi i}{m}}$.

Example 4.5.14 ($\mathbb{P}^1$-bundle case)**.** Let V be a rank 2 bundle over X of dimension d. Then the classes (λ_1, λ_2) are obtained by solving the quadratic equation

$$\lambda^2 + \mathbf{c}_1\lambda + \mathbf{c}_2 - 1 = 0,$$

where $\mathbf{c}_i$ is the cup product with c_iV. Since $(\mathbf{c}_1^2)^{\frac{d}{2}} = (\mathbf{c}_2)^{\frac{d}{2}} = 0$, the discriminant $\Delta = \mathbf{c}_1^2 - 4\mathbf{c}_2 + 4$ admits a square-root in $\mathbb{C}[\mathbf{c}_1, \mathbf{c}_2]$ given by

$$\sqrt{\Delta} = 2\sqrt{1 + \frac{\mathbf{c}_1^2}{4} - \mathbf{c}_2} = 2\left(1 + \sum_{1 \leq n \leq \frac{d}{2}} \binom{1/2}{n}\left(\frac{\mathbf{c}_1^2}{4} - \mathbf{c}_2\right)^n\right).$$

Using the quadratic formula, we obtain the roots ($i = 1, 2$)

$$\lambda_i = (-1)^{i-1} - \frac{c_1V}{2} + (-1)^{i-1} \sum_{1 \leq n \leq \frac{d}{2}} \binom{1/2}{n}\left(\frac{(c_1V)^2}{4} - c_2V\right)^n.$$

4.5.3 Uniqueness of the decomposition

In this subsection, we prove the uniqueness of the decomposition of the maximal A-model associated to a projective bundle, as well as its existence at the limiting point (Theorems 4.5.16 and 4.5.20). We will consider a maximal A-model F-bundle $(\mathcal{H}', \nabla')$ of $X' := \coprod_{i=1}^m X$ with a shifted base point, and use Theorem 4.4.34 to construct a gauge equivalence between the F-bundle $(\mathcal{H}, \nabla)$ of P and $(\mathcal{H}', \nabla')$ over the base points. The uniqueness results will follow from Theorem 4.4.34 and the extension of framing theorem.

We have

$$H^*(X', \mathbb{Q}) \xrightarrow{\sim} \bigoplus_{i=1}^m H^*(X, \mathbb{Q}). \tag{4.5.15}$$

Let $\omega' \in H^2(X', \mathbb{Q})$ denote the class corresponding to $(\omega_X, \ldots, \omega_X)$ under (4.5.15), it is ample so Assumption 4.2.22 is satisfied.

Fix a homogeneous basis of $H^2(X', \mathbb{Q})$ extending ω'. Complete it to a homogeneous basis of $H^*(X', \mathbb{Q})$ by adding the elements $\{T_i,\ \deg T_i \neq 2\}$ in each copy of $H^*(X, \mathbb{Q})$.

Let $\Delta(a) \in H^*(X', \mathbb{C})$ be a cohomology class at which the quantum product is well-defined. We produce $(\mathcal{H}', \nabla')/B'$, the maximal A-model F-bundle of X' associated to ω' with base point $\Delta(a)$ as in Example 4.2.25. Let (q, t) denote the coordinates on B', and let b' denote the closed point of B', given by $t = 0, q = 0$, which we refer to as the limiting point for X'.

Using the last observation of Lemma 4.5.8, we will interpret $\mathbf{K}_{\text{split}}$ as the $\mathbf{K}$-operator of $(\mathcal{H}', \nabla')$ for certain values of $\Delta(a)$.

For $i \in \{1, \dots, m\}$ and j such that $\deg T_j \neq 2$, we denote by $a_{i,j}$ the coordinate of $\Delta(a)$ along the basis element T_j in the i-th copy of $H^*(X, \mathbb{C})$ in $H^*(X', \mathbb{C})$.

Theorem 4.5.16. *There exists an F-bundle isomorphism*

$$\Phi(u) \colon (\mathcal{H}, \nabla)|_b \to (\mathcal{H}', \nabla')|_{b'},$$

whose components Φ_{ij} (as power series in u) are given by the cup-product with elements in $H^*(X, \mathbb{C})$ if and only if the coordinates of the base point $\Delta(a)$ satisfy

$$\sum_{j \colon \deg T_j \neq 2} \frac{\deg T_j - 2}{2} a_{i,j} T_j = c_1 V + m\lambda_i, \tag{4.5.17}$$

where λ_i was defined in Lemma 4.5.8.

Furthermore, in this case Φ is uniquely and explicitly determined by the H^0-components of $\Phi_{ij}|_{u=0}$, and $\Delta(a)$ is uniquely determined by (4.5.17), up to a shift in $\bigoplus_{i=1}^m H^2(X, \mathbb{C})$.

Proof. The bundles $\mathcal{H}|_b$ and $\mathcal{H}'|_{b'}$ are trivial by definition, their fibers are identified with H_{split} through (4.5.2) and (4.5.15), and the the connections ∇ and ∇' are framed. We use Theorem 4.4.34 to prove the proposition.

The matrices of $\mathbf{K}_{\text{lim}}$, $\mathbf{G}_{\text{lim}}$ were computed in Section 4.5.1. Write $\nabla_{u\partial_u}|_b = u\partial_u - u^{-1}\mathbf{K}_{\text{split}} + \mathbf{G}_{\text{split}}$. We have

$$\mathbf{G}_{\text{split}} = \begin{pmatrix} \mathbf{G}_X & & \\ & \ddots & \\ & & \mathbf{G}_X \end{pmatrix}.$$

To compute $\mathbf{K}_{\text{split}}$, note that the class ω' is ample. In particular, the restriction to $q = t = 0$ of the quantum product associated to $\Phi^{\omega'}$ is the classical cup-product. Then, $\mathbf{K}_{\text{split}}$ is block diagonal, and its i-th block is given by

$$(\mathbf{K}_{\text{split}})_{ii} = \left(c_1 T_X + \sum_{j \colon \deg T_j \neq 2} \frac{\deg T_j - 2}{2} a_{i,j} T_j \right) \cup . \tag{4.5.18}$$

Thus, after identifying the fibers with H_{split}, the connections $\nabla|_b$ and $\nabla'|_{b'}$ lie in $\mathcal{F}(H_{\text{split}}, \mathrm{id}, \mathbf{G}_X, (T_j\cup)_{0\le j\le N})$, see Definition 4.4.33. We apply Theorem 4.4.34 with $\mathbf{K} = -\mathbf{K}_{\text{split}}$, $\mathbf{D} = \mathbf{G}_{\text{split}}$, $\mathbf{H} = 0$, $\mathbf{K}' = -\mathbf{K}_{\text{lim}}$ and

$$\mathbf{H}' = \mathbf{G}_{\text{lim}} - \mathbf{G}_{\text{split}} = \begin{pmatrix} -\frac{m-1}{2} & & & \\ & -\frac{m-3}{2} & & \\ & & \ddots & \\ & & & \frac{m-1}{2} \end{pmatrix}.$$

Assume first that the coordinates of $\Delta(a)$ satisfy (4.5.17). Let $\phi = (\phi_{ij}) \in \mathrm{GL}(H_{\text{split}})$ denote the automorphism from Lemma 4.5.8. Equations (4.5.17) and (4.5.18) imply that $\phi^{-1}\mathbf{K}_{\text{lim}}\phi = \mathbf{K}_{\text{split}}$, which is Condition (1) of the theorem. Condition (2) is satisfied with $\mu = \mu' = 1$. Condition (3) follows from Lemma 4.5.11 and our choice of $\mathbf{H}$. We conclude that the connections $\nabla|_b$ and $\nabla'|_{b'}$ are gauge equivalent through a bundle isomorphism $\Phi(u)$ satisfying the conditions of the theorem.

Now, assume that there exists a bundle isomorphism $\Phi(u)$ as in the theorem, in particular each component ϕ_{ij} of $\Phi|_{u=0}$ is given by the cup-product with a cohomology class. Let $\phi \coloneqq (\phi_{ij}) \in \mathrm{GL}(H_{\text{split}})$. Since $\Phi(u)$ is a gauge equivalence, we have in particular $\phi^{-1}\mathbf{K}_{\text{lim}}\phi = \mathbf{K}_{\text{split}}$. Recall from (4.5.18) that $\mathbf{K}_{\text{split}}$ is block diagonal, and that its coefficients are given by the cup-product with cohomology classes in $H^*(X, \mathbb{C})$. The assumption on the components of $\Phi|_{u=0}$ implies that ϕ diagonalizes $\mathbf{K}_{\text{lim}}$ viewed as an element of $\mathrm{Mat}(m \times m, R)$, where $R = \{\alpha \mapsto x \cup \alpha \mid x \in H^*(X, \mathbb{C})\}$. The eigenvalues of $\mathbf{K}_{\text{lim}}$ as an R-linear map were computed in Lemma 4.5.8: they are $(c_1 T_X + c_1 V + m\lambda_i)\cup$ with $1 \le i \le m$. In particular, $\Delta(a)$ satisfies (4.5.17).

The uniqueness part of the theorem follows from the uniqueness of Theorem 4.4.34, and the non-degeneracy of the Poincaré pairing. □

Remark 4.5.19. If the H^2-component of the base point $\Delta(a)$ is 0, then the quantum product converges at $\Delta(a)$ by Lemma 4.2.15.

Theorem 4.5.20. *Let $(f, \Phi)\colon (\mathcal{H}, \nabla)/B \to (\mathcal{H}', \nabla')/B'$ be an isomorphism of F-bundles. Then*

1. *The bundle map Φ is uniquely and explicitly determined by its restriction to $b \in B$.*

2. *The base map f is uniquely and explicitly determined by its restriction to $b \in B$, up to a multiplicative constant in the q direction.*

Proof. The F-bundle $(\mathcal{H}', \nabla')/B'$ is framed by definition. Since ω' is ample, at the point b' the quantum product reduces to the classical cup-product. In particular $(\mathbf{1}, \ldots, \mathbf{1}) \in H^*(X', \mathbb{C})$ is a cyclic vector. The theorem thus follows from a direct application of Proposition 4.4.31. □

We refer to [IK23] regarding the existence of the isomorphism.

4.5.4 Case of blowups of algebraic varieties

In this subsection, we state the analogs of the results in Section 4.5.3 in the case of blowups of algebraic varieties.

Let X be a smooth project complex algebraic variety, and $\sigma\colon Z \hookrightarrow X$ a smooth closed subvariety of codimension $m \geq 2$. Let $\pi\colon \widetilde{X} \to X$ be the blowup of X along Z. Similar to the projective bundle case, we have a classical decomposition

$$iso\colon H^*(X, \mathbb{Q}) \oplus \bigoplus_{i=1}^{m-1} H^*(Z, \mathbb{Q})[-2i] \xrightarrow{\sim} H^*(\widetilde{X}, \mathbb{Q}). \tag{4.5.21}$$

Let $X' := X \sqcup \coprod_{i=1}^{m-1} Z$. Fix an ample class $\omega_X \in H^2(X, \mathbb{Q})$.

Let $(\mathcal{H}, \nabla)/B$ denote the maximal A-model F-bundle of X associated to the nef class $\pi^*\omega_X$, with base point $b = 0 \in H^*(\widetilde{X}, \mathbb{Q})$ and coordinates (q, t). Fix a class $\Delta(a) \in H^*(X', \mathbb{C}) \simeq H^*(X, \mathbb{C}) \oplus \bigoplus_{1 \leq i \leq m-1} H^*(Z, \mathbb{C})$ at which the quantum product is well-defined. Let $(\mathcal{H}', \nabla')/B'$ denote the maximal A-model F-bundle associated to the class $(\omega_X, \sigma^*\omega_X, \cdots, \sigma^*\omega_X)$, with base point $b' = \Delta(a)$ and coordinates (q, t) such that $q = t = 0$ at b'. Since X' is a disjoint union, $(\mathcal{H}', \nabla')$ is the product of a maximal A-model F-bundle associated to X and ω_X, and $m - 1$ copies of maximal F-bundles associated to Z and $\sigma^*\omega_X$.

We can prove a result analogous to Theorem 4.5.16. For $1 \leq i \leq m$, let $\mathbf{c}_i$ denote the cup-product with $c_i(N_{Z/X})$. The polynomial $\lambda^m + \sum_{i=0} \mathbf{c}_{m-i}\lambda^i + \lambda$ has m distinct roots $\boldsymbol{\lambda}_i = \lambda_i \cup \in \mathbb{C}[\mathbf{c}_1, \ldots, \mathbf{c}_m]$, with

$$\lambda_1 = 0, \quad \lambda_i = \xi^{2(i-1)-1} - \frac{c_1 N_{Z/X}}{m-1} \mod H^{\geq 3}(X, \mathbb{C}),$$

where $\xi = e^{\frac{\pi i}{m-1}}$ and $2 \leq i \leq m$, up to a permutation of the indices $\{2, \ldots, m\}$. Those are the analogs of the eigenvalues computed for $\mathbf{K}_{\lim}$ in the projective bundle case.

Let $\{S_j\}_{1 \leq j \leq \dim H^*(Z, \mathbb{C})}$ be a basis of $H^*(Z, \mathbb{C})$ extending $\sigma^*\omega_X$. For $1 \leq i \leq m$, let $\Delta_i(a)$ denote the component of $\Delta(a)$ in the i-th summand of (4.5.21), and for

$2 \leq i \leq m$ decompose it as

$$\Delta_i(a) = \sum_j a_{i,j} S_j.$$

Using the splitting (4.5.21), we can view an element $\Phi \in \mathrm{End}_{\mathbb{C}}(H^*(\widetilde{X}, \mathbb{C}))$ as a matrix $(\Phi_{i,j})_{1\leq i,j\leq m}$, with $\Phi_{1,1} \in \mathrm{End}_{\mathbb{C}}(H^*(X, \mathbb{C}))$, and $\Phi_{i,i} \in \mathrm{End}_{\mathbb{C}}(H^*(Z, \mathbb{C}))$ for $2 \leq i \leq m$. The following result is analogous to Theorem 4.5.16.

Theorem 4.5.22. *Let $\Delta(a) \in H^*(X', \mathbb{C})$ be a cohomology class at which the quantum product converges, such that $\Delta_1(a) \in H^2(X, \mathbb{C})$, and for $2 \leq i \leq m$, we have*

$$\sum_{j\colon\ \deg S_j \neq 2} \frac{\deg_Z S_j - 2}{2} a_{i,j} S_j = c_1 N_{Z/X} + (m-1)\lambda_i. \tag{4.5.23}$$

Then, there exists an F-bundle isomorphism $\Phi\colon (\mathcal{H}, \nabla)|_b \to (\mathcal{H}', \nabla')|_{b'}$.

Furthermore, if we restrict the coefficients of Φ to lie in a universal algebra as in the projective bundle case, then Φ is uniquely determined by its restriction to $u = 0$, and the base point $\Delta(a)$ is uniquely determined up to a shift in $H^2(X, \mathbb{C}) \oplus \bigoplus_{i=1}^{m-1} H^2(Z, \mathbb{C})$.

A direct consequence of Proposition 4.4.31 is the following, which is analogous to Theorem 4.5.20.

Theorem 4.5.24. *Let $(f, \Phi)\colon (\mathcal{H}, \nabla)/B \to (\mathcal{H}', \nabla')/B'$ be an isomorphism of F-bundles. Then*

1. *The bundle map Φ is uniquely and explicitly determined by its restriction to $b \in B$.*

2. *The base map f is uniquely and explicitly determined by its restriction to $b \in B$, up to a multiplicative constant in the q direction.*

We refer to [IK23] regarding the existence of the isomorphism.

Chapter 5

UNFOLDING OF EQUIVARIANT F-BUNDLES AND APPLICATION TO THE MIRROR SYMMETRY OF FLAG VARIETIES

This chapter is based on [Hin+25], joint work with Li Changzheng, Tony Yue Yu, Chi Zhang and Shaowu Zhang.

5.1 Introduction

5.1.1 Motivations

For a smooth complex projective variety X, the Gromov-Witten invariants of X are roughly counts of algebraic curves in X with given genus, class, and constraints (see [Gro85; Wit91; KM94; BF97]). We can organize the rational (i.e. genus zero) Gromov-Witten invariants into a generating series as follows.

Fix a homogeneous basis $(T_i)_{0\le i\le N}$ of $H^*(X,\mathbb{Q})$, and let $(T_i^*)_{0\le i\le N}$ denote the dual basis with respect to the Poincaré pairing. Let $\mathbb{Q}[\![\mathrm{NE}(X,\mathbb{Z})]\!]$ denote the completion of $\mathbb{Q}[\mathrm{NE}(X,\mathbb{Z})] = \mathbb{Q}[q^\beta \mid \beta \in \mathrm{NE}(X,\mathbb{Z})]$ with respect to the maximal ideal $(q^\beta,\ \beta \neq 0)$.

The *genus* 0 *Gromov-Witten potential* is a formal power series

$$\Phi = \sum_{n\ge 0,\beta} \frac{q^\beta}{n!} \sum_{i_1,\dots,i_n} \langle T_{i_1}\cdots T_{i_n}\rangle_{0,n}^{\beta}\, t_{i_1}\cdots t_{i_n} \in \mathbb{Q}[\![\mathrm{NE}(X,\mathbb{Z})]\!][\![t_0,\dots,t_N]\!],$$

where $\langle\cdots\rangle_{0,n}^{\beta}$ denotes the Gromov-Witten invariants of X of genus 0, class β and cohomological constraints $T_{i_1},\dots,T_{i_n}$. It gives rise to the *big quantum cohomology* of X, i.e. a deformation of the classical cup product on $H^*(X,\mathbb{Q})$:

$$\begin{aligned} \star\colon H^*(X,\mathbb{Q}) \otimes H^*(X,\mathbb{Q}) &\longrightarrow H^*(X,\mathbb{Q}) \otimes \mathbb{Q}[\![\mathrm{NE}(X,\mathbb{Z})]\!][\![t_0,\dots,t_N]\!] \\ T_i \star T_j &\longmapsto \sum_r \frac{\partial^3\Phi}{\partial t_i \partial t_j \partial t_r} T_r^*. \end{aligned}$$

A simpler version called *small quantum cohomology* is the restriction of the big quantum cohomology to $t_i = 0$, for all $i = 0,\dots,N$ (or equivalently, by the divisor axiom of Gromov-Witten invariants, for all i with $\deg T_i \neq 2$). The idea of small quantum cohomology appeared before the big version, first in [Can+91], where the small quantum cohomology of a quintic Calabi-Yau threefold was computed using the mirror manifold's periods. This computation led to the curve counting invariants of the quintic that were previously unknown, and sparked decades of research of enumerative geometry and mirror symmetry from the mathematical viewpoint.

The small quantum cohomology mirror symmetry was proved in various cases, such as complete intersections in projective spaces in [Giv96; LLY97] and toric complete intersections in [Giv98; RS17b; LLY99]. Given that the small quantum cohomology is the restriction of the big quantum cohomology, a natural question is whether mirror symmetry still holds for the big quantum cohomology. The big quantum cohomology mirror symmetry was proved for projective spaces in [Bar01], for quadric hypersurfaces in [Hu22], for $\mathbb{P}^2$ via tropical geometry in [Gro11], for toric varieties in [Iri17b; Iri17a; RS15], and for toric Deligne-Mumford stacks in [Coa+20].

One tool for such an extension is the reconstruction theorem for Gromov-Witten invariants by Kontsevich-Manin [KM94], which is the prototype of the universal unfolding of Frobenius manifolds by Hertling-Manin [HM04] and that of logarithmic Frobenius manifolds by Reichelt [Rei09]. This is the essential ingredient in the proof of big quantum cohomology mirror symmetry for projective spaces in [Bar00; Bar01]. It is shown that the big quantum cohomology can be reconstructed from the small under the condition that the small quantum cohomology (or the classical cohomology) is H^2-generated. The Hertling-Manin unfolding theorem applies more generally to so called (TE)-structures, or F-bundles $(\mathcal{H}, \nabla)/B$, where $\mathcal{H}$ is a vector bundle over $B \times \operatorname{Spf} \Bbbk[\![u]\!]$ and ∇ is a flat connection on $\mathcal{H}$ with poles at $u = 0$, such that $\nabla_{u^2\partial_u}$ and $\nabla_{u\xi}$ are regular for any tangent vector field ξ on B. The H^2-generation condition is then replaced by two conditions called (IC) and (GC). For $b \in B$, the residues $\nabla_{u\xi}|_{(b,0)}$ and $\nabla_{u^2\partial_u}|_{(b,0)}$ are endomorphisms of the fiber $\mathcal{H}_{b,0}$. An element $v \in \mathcal{H}_{b,0}$ satisfies the (GC) condition if the iterated action of these endomorphisms on v generate $\mathcal{H}_{b,0}$. It satisfies the (IC) condition if the map $\xi \in T_bB \mapsto \nabla_{u\xi}|_{(b,0)}(v)$ is injective. Under those two conditions, the F-bundle admits a universal unfolding into a maximal F-bundle.

Another tool for such an extension from small to big quantum cohomology is the reconstruction from a semisimple point. In the context of Frobenius manifolds, the structure around a semisimple point was studied in [Dub96; CG17], and a reconstruction result was proved in [BM04; MT08]. Teleman also studied semisimplicity in the context of topological field theories in [Tel12].

In this paper, we aim to establish the big quantum cohomology mirror symmetry for flag varieties, in the sense of isomorphism of big quantum D-modules. The small quantum cohomology mirror symmetry for general flag varieties was recently established in [Cho23], as an isomorphism of small quantum D-modules. In general the small quantum cohomology of flag varieties is neither H^2-generated, nor semisimple, so neither of the above reconstruction methods can be applied here.

The *main discovery* of this paper is that an analogous H^2-generation condition can be recovered if we work equivariantly with respect to a torus action.

We first extend the definition of F-bundle (from [Kat+24; Hin+24]) to equivariant F-bundle (see Definition 5.2.10). Since the connection ∇_{∂_u} is not linear with respect to the equivariant variables, we need to work with infinite rank F-bundles over an infinite dimensional base. Nevertheless, most of the data can still be reduced to a finite rank (T)-structure relative to $H_T^*(\mathrm{pt}, \mathbb{Q})$.

Next we extend the (IC) and (GC) conditions to the equivariant setting, and establish an unfolding theorem for equivariant F-bundles under these conditions (see Theorem 5.3.36).

For application to the mirror symmetry of flag varieties, we produce an unfolding of the B-model by constructing an appropriate unfolding of the Landau-Ginzburg superpotential. We further check the various conditions on the big quantum D-module of flag varieties, and apply our equivariant unfolding theorem to obtain the mirror symmetry theorem for the big quantum cohomology of flag varieties.

5.1.2 Main results

An *F-bundle* $(\mathcal{H}, \nabla)$ over some base B consists of a vector bundle $\mathcal{H}$ over $B \times \operatorname{Spf} \Bbbk[\![u]\!]$ and a meromorphic flat connection ∇ with poles at $u = 0$, such that $\nabla_{u^2\partial_u}$ and $\nabla_{u\xi}$ are regular for any tangent vector field ξ on B. If the connection ∇ is only defined in the directions of B, we call $(\mathcal{H}, \nabla)$ a $\Bbbk$-linear *(T)-structure*. In order not to create confusion in the infinite rank/dimension setting, we formulate F-bundles and (T)-structures in purely algebraic terms in Section 5.2, replacing the vector bundle by a free module, and the connection by derivations.

Let us explain the various conditions involved in our equivariant unfolding theorem. Let $\Bbbk$ be a field of characteristic zero, R a $\Bbbk$-algebra and $(\mathcal{H}, \nabla)$ an F-bundle (resp. a (T)-structure) over $R[\![t_i, i \in I]\!]$ for a countable set I, with fiber H at $t = 0$, $u = 0$.

Residues of ∇ induce $\mathrm{K} \coloneqq \nabla_{u^2\partial_u}|_{u=t=0} \in \mathrm{End}_R(H)$,

$$\mu \colon \bigoplus_{i \in I} R\partial_{t_i} \longrightarrow \mathrm{End}_R(H), \quad \partial_{t_i} \longmapsto \nabla_{u\partial_{t_i}}|_{u=t=0},$$

and for any $v \in H$,

$$\mu_v \colon \bigoplus_{i \in I} R\partial_{t_i} \longrightarrow H, \quad \partial_{t_i} \longmapsto \nabla_{u\partial_{t_i}}|_{u=t=0}(v).$$

The F-bundle $(\mathcal{H}, \nabla)$ is called *maximal* if there exists $v \in H$ such that μ_v is an isomorphism, and v is called a cyclic vector. We further define the following conditions on v (see Definition 5.3.15):

(IC) The map μ_v is injective.

(GC) The orbit of v under the action of the subalgebra $R[\mathrm{im}\,\mu, \mathrm{K}] \subset \mathrm{End}_R(H)$ (resp. $R[\mathrm{im}\,\mu] \subset \mathrm{End}_R(H)$ in the case of a (T)-structure) is H.

(GC') The condition (GC) is satisfied after base change to $\mathrm{Frac}(R)$.

The conditions (IC) and (GC) were originally formulated in [HM04] as necessary conditions to obtain the existence and uniqueness of a maximal unfolding. We find that when working relative to a ring, condition (GC') is enough for uniqueness, while conditions (IC) and (GC) need to be complemented by the assumption that $\mathrm{coker}\,\mu_v$ is free in order to construct a maximal unfolding (see Theorem 5.1.3 for a precise statement).

5.1.2.1 Equivariant unfolding theorem

For our application to the mirror symmetry of a flag variety $X = G/P$, the F-bundle associated to the quantum cohomology of X does not satisfy conditions (GC) or (GC'). Our new idea is to consider the equivariant quantum cohomology of X induced by the natural torus action. Note that while the associated (T)-structure is linear over $R \coloneqq H_T^*(\mathrm{pt}, \Bbbk)$ and of finite rank, the connection ∇_{∂_u} in the u-direction connection is *not* R-linear due to the nontrivial grading on R. Therefore, the associated F-bundle can only be defined over the base field $\Bbbk$, and hence has infinite rank and depends on infinitely many variables, indexed by a $\Bbbk$-basis of $H_T^*(X, \Bbbk)$.

We introduce the notion of *equivariant F-bundle* in Definition 5.2.10. Let I be a finite set and choose a $\Bbbk$-linear basis of a $\Bbbk$-algebra R indexed by a countable set K. Let

$\mathbf{t}_I = \{t_{i,k} \,|\, (i,k) \in I \times K\}$ denote formal parameters over $\Bbbk$, and $t_I = \{t_i \,|\, i \in I\}$ formal parameters over R. An equivariant F-bundle consists of the data $\{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}$, where $(\mathcal{H}, \nabla)$ is a $\Bbbk$-linear F-bundle over $\Bbbk[\![\mathbf{t}_I]\!]$ and $\{(\mathcal{H}_R, \nabla_R), \alpha\}$ is an R-linear lift over $R[\![t_I]\!]$ of the $\Bbbk$-linear (T)-structure underlying $(\mathcal{H}, \nabla)$. An unfolding of an equivariant F-bundle is an extension over a bigger formal base (see Definition 5.3.33). We also generalize the notion of framing (from [Hin+24, Definition 2.9]) to equivariant F-bundles (Definition 5.2.13), which consists of framings for the $\Bbbk$-linear F-bundle and the R-linear (T)-structure that are compatible under the lift.

We extend the (IC), (GC), (GC') and maximality conditions to equivariant F-bundles by requiring that the R-linear (T)-structure satisfy those conditions. Our main theorem is the following unfolding theorem for equivariant F-bundles.

Theorem 5.1.1 (Unfolding of equivariant F-bundles, Theorem 5.3.36)**.** *Let $\mathcal{F} = \{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}$ be an equivariant F-bundle over $\Bbbk[\![\mathbf{t}_I]\!]$, and fix $v \in \mathcal{H}_R|_{u=t_I=0}$.*

1. If v satisfies (IC), (GC) and $\operatorname{coker} \mu_v$ is free, then $\mathcal{F}$ admits a maximal unfolding with a cyclic vector induced from v.

2. If v satisfies (GC'), then any two maximal unfoldings of $\mathcal{F}$ with cyclic vectors induced from v are isomorphic under a unique isomorphism.

Furthermore, any framing for $\mathcal{F}$ induces a unique framing on a maximal unfolding.

The fist step in our proof is to establish a formal version of the Hertling-Manin unfolding theorem in the finite rank case (see Theorem 5.1.3). Then we use it to unfold the R-linear (T)-structure. Finally we conclude by unfolding the $\Bbbk$-linear F-bundle in the u-direction. The key observation for the last step is the very useful Lemma 5.3.1. It states that an equivariant F-bundle is uniquely determined by the underlying (T)-structure and the value of the u-direction connection at one point, under the assumption that the (T)-structure admits a framing. This assumption always holds for the $\Bbbk$-linear (T)-structure associated to an equivariant F-bundle by Proposition 5.3.4 and Lemma 5.2.6.

Proposition 5.1.2 (Lemma 5.3.1)**.** *For $k = 1, 2$, let I_k be a countable set and $(\mathcal{H}_k, \nabla_k)/R[\![t_j, j \in I_k]\!]$ an F-bundle. Let $(f, \Phi)\colon (\mathcal{H}_1, \nabla_1)_0 \to (\mathcal{H}_2, \nabla_2)_0$ be a morphism of (T)-structures. Assume the (T)-structure $(\mathcal{H}_1, \nabla_1)_0$ admits a framing. Then*

1. ∇_1 *is uniquely determined by the underlying (T)-structure and* $\nabla_{1,\partial_u}|_{t_{I_1}=0}$*, and any such data determine a unique F-bundle connection extending the (T)-structure.*

2. (f, Φ) *is an isomorphism of F-bundles if and only if* $(f, \Phi)|_{t_{I_1}=0}$ *is an isomorphism of F-bundles.*

Here is our formal version of the Hertling-Manin unfolding theorem we mentioned above. We also deduce a version for (T)-structures in Corollary 5.3.30.

Theorem 5.1.3 (Formal Hertling-Manin unfolding, Theorem 5.3.28)**.** *Let* R *be an integral domain containing* $\mathbb{Q}$*. Let* $(\mathcal{H}, \nabla)/R[\![t_1, \ldots, t_d]\!]$ *be a finite rank F-bundle. Let* $v \in \mathcal{H}/(t_1, \ldots, t_d, u)\mathcal{H}$*.*

1. If v *satisfies (IC), (GC) and* $\operatorname{coker} \mu_v$ *is free, then there exists a maximal unfolding with a cyclic vector induced from* v*.*

2. If v *satisfies (GC'), then any two maximal unfoldings of* $(\mathcal{H}, \nabla)$ *with cyclic vectors induced from* v *are isomorphic under a unique isomorphism.*

Furthermore, any framing for $(\mathcal{H}, \nabla)$ *induces a unique framing on a maximal unfolding.*

Our proof follows mostly the original proof of Hertling and Manin, which was carried out in the complex analytic setting. In particular, we produce unfoldings using the (GC) condition in Lemma 5.3.16, which is the formal analogue of [HM04, Lemma 2.9]. While the original proof uses analytic methods to construct a framing of the (T)-structure in which the u-direction has a logarithmic pole at $u = 1$, we show that the proof actually works in any framing trivialization.

Since we are working over an integral domain R, the (IC) and (GC) conditions are not sufficient to prove existence, and we have to require that $\operatorname{coker} \mu_v$ is free in order to construct a maximal unfolding. This ensures that we can extend a basis of $\operatorname{im} \mu_v$ to a basis of $\mathcal{H}/(t_1, \ldots, t_d, u)\mathcal{H}$. We prove the uniqueness by observing that under (GC'), an unfolding $(\mathcal{H}', \nabla')$ is characterized by a choice of framing before unfolding and the action of ∇' on a section that extends v. This allows us to compare unfoldings through their action on a section extending v, and to establish the isomorphism. A priori, the isomorphism we produce is only defined over $\operatorname{Frac}(R)$, but we note that it is in fact defined over R if the unfoldings are. A key result is the canonical extension of framing for (T)-structures (Proposition 5.3.4), which was essentially proved in [Hin+24].

5.1.2.2 Application to mirror symmetry of flag varieties

We apply Theorem 5.1.1 to the mirror symmetry of flag varieties G/P, where G is a simply-connected complex simple Lie group and P is a parabolic subgroup of G. We begin by reviewing some relevant progress on the mirror symmetry of flag varieties.

On the A-side, there was a remarkable presentation of the small quantum cohomology ring $\mathrm{QH}^*(G/P)$ in terms of Peterson variety given in the unpublished lecture notes [Pet97] by Peterson. This was partially verified in [Rie03; LS10; Che09], and was recently proved in [Cho22] in full generality. On the B-side, Rietsch constructed a mirror Landau-Ginzburg model $(X_P^\vee, \mathcal{W})$ for G/P in [Rie08], and showed the coincidence between the critical loci of W and the Peterson variety strata. As a consequence, we obtain a first level of small quantum cohomology mirror symmetry in the sense of a ring isomorphism $\mathrm{QH}^*(G/P) \cong \mathrm{Jac}(\mathcal{W})$. We refer to [Bat+00; Li+24] and the references therein for more relevant studies in the special case $G = \mathrm{SL}(n+1, \mathbb{C})$.

Furthermore, on the A-side, we can consider the Dubrovin connection on the trivial $\mathrm{QH}^*(G/P)$-bundle over $\mathbb{C}^*$, which endows the vector bundle with a quantum D-module structure. The flag variety G/P admits a natural torus action by the maximal torus T of G, so that we can consider the equivariant quantum D-module structure as well. On the B-side, we consider the Brieskorn lattice $G_0(X_P^\vee, \mathcal{W}, p)$ assoicated to Rietsch's equivariant superpotential mirror to G/P (see Section 5.4.2 for more details). The small quantum cohomology mirror symmetry in the sense of isomorphism of small quantum D-modules has been studied for certain Grassmannians in [MR20; PRW16; PR18; LT24], and was recently established in [Cho23]. In the present paper, we first reformulate this in terms of an isomorphism $\mathcal{F}^A \cong \mathcal{F}^B$ of equivariant F-bundles. Then, as an application of Theorem 5.1.1, we obtain the following.

Theorem 5.1.4 (Big quantum D-module mirror symmetry, Theorem 5.4.35)**.** *The A-model big equivariant F-bundle $\mathcal{F}^{A,\mathrm{big}}$ is isomorphic to the B-model big equivariant F-bundle $\mathcal{F}^{B,\mathrm{big}}$. The isomorphism is uniquely determined by the small equivariant quantum D-module mirror symmetry.*

By taking the non-equivariant limit of the isomorphism in Theorem 5.1.4, we obtain a non-equivariant version of the big quantum cohomology mirror symmetry for flag varieties; see Theorem 5.4.38.

Note that the small quantum cohomology $\mathrm{QH}^*(G/P)$ can be neither semisimple nor H^2-generated, such as is the case when $G/P = \mathrm{SG}(2, 2n)$ is the Grassmannian of isotropic planes in Lie type C_n (see [CP11]). Therefore, the application of our unfolding theorem is essential in such cases, for which neither the unfolding in [HM04] nor the semisimple reconstruction in [Tel12] can be applied.

In addition to the mirror statement above, we further anticipate the complex analytic convergence of the mirror map, as well as the compatibility with the pairings on the F-bundles. These aspects present promising directions for future research.

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5.2 (T)-structures and equivariant F-bundles

We fix a field $\Bbbk$ of characteristic zero, a $\Bbbk$-algebra R and a $\Bbbk$-linear basis $(\lambda_k)_{k\in K}$ of R, with K a countable set.

5.2.1 Completions

We set the conventions for completions of rings of polynomials in infinitely many variables, following [Iri17a, §2.1]. Our reference for topological algebra is [FK18, 0§7].

Let I be a countable set indexing indeterminates $t_I = \{t_i, i \in I\}$. We denote by $\mathbb{N}^{(I)}$ the set of almost zero integer sequences indexed by I. Let M be a module or ring, we denote by $M[\![t_I]\!]$ the module consisting of formal power series $\sum_{\alpha\in\mathbb{N}^{(I)}} a_\alpha t_I^\alpha$, where $t_I^\alpha := \prod_{i\in I} t_i^{\alpha_i}$ and $a_\alpha \in M$. It is the projective limit of the modules $M[\![t_i, i \in I']\!]$, where $I' \subset I$ runs through finite subsets. For two countable sets I and I'', we have $M[\![t_i, i \in I]\!][\![t_i, i \in I'']\!] \simeq M[\![t_i, i \in I \cup I'']\!]$.

If M is linearly topologized by the descending chain of submodules $\{M_\lambda\}_{\lambda\in\Lambda}$, we equip $M[\![t_I]\!]$ with the linear topology induced by the submodules

$$M[\![t_I]\!]_{\lambda,\mathcal{I}} := \left\{ \sum_{\alpha\in\mathbb{N}^{(I)}} a_\alpha t_I^\alpha,\ a_\alpha \in M_\lambda \text{ for all } \alpha \in \mathcal{I} \right\}, \tag{5.2.1}$$

where $\lambda \in \Lambda$ and $\mathcal{I} \subset \mathbb{N}^{(I)}$ is a finite set of exponents. The convergence of a sequence for this topology means that the sequence of coefficients of each monomial converges in M. Hence, if M is complete, so is $M[\![t_I]\!]$. If R is a topological ring and M is a topological R-module, then $R[\![t_I]\!]$ is a topological ring and $M[\![t_I]\!]$ is a topological $R[\![t_I]\!]$-module. If R is a topological $\Bbbk$-algebra, then $R[\![t_I]\!]$ is a topological $\Bbbk[\![t_I]\!]$-algebra.

Let R be a discrete ring, let M be an R-module. The closure of the monomial ideal $(t_i, i \in I) \subset R[\![t_I]\!]$ is the ideal $\mathcal{J} := \{f \in R[\![t_I]\!],\ f|_{t_I=0} = 0\}$. If I is finite, those two ideals coincide and the topology on $M[\![t_I]\!]$ is equivalent to the usual $(t_i, i \in I)$-adic topology. When I is infinite, the $\mathcal{J}$-adic topology is finer, which means that for any finite subset $\mathcal{I} \subset \mathbb{N}^{(I)}$ there exists $n \in \mathbb{N}$ such that $\mathcal{J}^n M[\![t_I]\!] \subset M[\![t_I]\!]_{\mathcal{I}}$.

Remark 5.2.2. Let I be a countable set, $t_I = \{t_i, i \in I\}$ a set of indeterminates. Let R be a topological ring. Here are a few facts we will use about modules over $R[\![t_I]\!]$.

1. If M is a free R-module, then $M[\![t_I]\!]$ is free, and we have a canonical isomorphism $M \otimes_R R[\![t_I]\!] \simeq M[\![t_I]\!]$ given by $m \otimes 1 \mapsto m$.

2. If M and M' are free R-modules, there is a canonical isomorphism of $R[\![t_I]\!]$-modules
$$\mathrm{Hom}_{R[\![t_I]\!]}(M[\![t_I]\!], M'[\![t_I]\!]) \simeq \mathrm{Hom}_R(M, M')[\![t_I]\!].$$

3. If R is discrete and M is a free R-module, an element $\Phi \in \mathrm{End}_{R[\![t_I]\!]}(M[\![t_I]\!])$ is an isomorphism if and only if $\Phi|_{t_I=0} \in \mathrm{End}_R(M)$ is an isomorphism.

For (3), we may reduce to the case $\Phi = 1 + f$ with $f \in \mathcal{J}\,\mathrm{End}_R(M)[\![t_I]\!]$. Then it suffices to prove that the sequence $\Psi_n := \sum_{k=0}^{n}(-1)^k f^k$ converges in $\mathrm{End}_R(M)[\![t_I]\!]$. For $m \geq n$ we have $\Psi_m - \Psi_n = \sum_{k=n+1}^{m}(-1)^k f^k \in \mathcal{J}^{n+1}\,\mathrm{End}_R(M)[\![t_I]\!]$. Since the $\mathcal{J}$-adic topology is finer than the topology (5.2.1), the sequence $(\Psi_n)_n$ is a Cauchy sequence. Since $\mathrm{End}_R(M)$ is a discrete space, it is complete. We conclude that $(\Psi_n)_n$ converges to an element Ψ such that $\Phi \circ \Psi = \Psi \circ \Phi = 1$.

5.2.2 F-bundles and (T)-structures

We equip R with the discrete topology. Given a countable set I, the derivations $\partial_{t_j} \colon R[\![t_i, i \in I]\!] \to R[\![t_i, i \in I]\!]$ are continuous and linearly independent. Hence, it makes sense to define a (partial) connection in the t-directions on a $R[\![t_i, i \in I]\!]$-module $\mathcal{H}$ by specifying its action on ∂_{t_j} for all $j \in I$.

Definition 5.2.3 (F-bundle, (T)-structure)**.** Let I be a countable set and $t_I := \{t_i, i \in I\}$.

1. An (R-linear) ***F-bundle*** $(\mathcal{H}, \nabla)$ over $R[\![t_I]\!]$ is a free $R[\![t_I, u]\!]$-module $\mathcal{H}$ together with a (R-linear) connection

$$\nabla_{\partial_{t_i}} : \mathcal{H} \to u^{-1}\mathcal{H},$$
$$\nabla_{\partial_u} : \mathcal{H} \to u^{-2}\mathcal{H}$$

satisfying the flatness condition.

2. An (R-linear) ***(T)-structure*** $(\mathcal{H}, \nabla)$ over $R[\![t_I]\!]$ is a free $R[\![t_I, u]\!]$-module $\mathcal{H}$ together with a (R-linear) connection in the t-directions

$$\nabla_{\partial_{t_i}} : \mathcal{H} \to u^{-1}\mathcal{H}$$

satisfying the flatness condition.

A *morphism of F-bundles (resp. (T)-structures)* $(f, \phi)\colon (\mathcal{H}, \nabla)/R[\![t_I]\!] \to (\mathcal{H}', \nabla')/R[\![t_J]\!]$ consists of a continuous map of R-algebras $f\colon R[\![t_J]\!] \to R[\![t_I]\!]$, and a continuous map of $R[\![t_I, u]\!]$-modules $\phi\colon \mathcal{H} \to f^*\mathcal{H}' := \mathcal{H}' \otimes_{R[\![t_J,u]\!]} R[\![t_I, u]\!]$ such that $\phi \circ \nabla = f^*\nabla' \circ \phi$.

Underlying an F-bundle $(\mathcal{H}, \nabla)$ over $R[\![t_I]\!]$ is an R-linear (T)-structure $(\mathcal{H}, \nabla)_0$ over $R[\![t_I]\!]$ obtained by forgetting ∇_{∂_u}. This defines a functor $(\cdot)_0$ from R-linear F-bundles to R-linear (T)-structures.

Let $(\mathcal{H}, \nabla)$ be an R-linear (T)-structure over $R[\![t_I]\!]$. A *trivialization* of the (T)-structure is a choice of isomorphism $\mathcal{H} \simeq H \otimes_R R[\![t_I, u]\!]$, where H is a free R-module (necessarily isomorphic to $\mathcal{H}/J\mathcal{H}$, where J is the closure of the ideal (t_I, u)). Under such an isomorphism, the connection ∇ decomposes as $\nabla_{\partial_{t_i}} = \partial_{t_i} + u^{-1}\mathbf{A}_i(t_I, u)$, with $\mathbf{A}_i \in \mathrm{End}_R(H)[\![t_I, u]\!]$. We refer to $\mathbf{A}_i$ as the *connection matrix in the direction* t_i. Different choices of trivialization produce connection matrices related by the usual gauge-transformation formula.

We introduce special trivializations called framings.

Definition 5.2.4 (Framing)**.** 1. A *framing* for an R-linear F-bundle (resp. an R-linear (T)-structure) $(\mathcal{H}, \nabla)/R[\![t_I]\!]$ is a trivialization in which the connection matrices only have negative powers of u.

2. A morphism of framed F-bundles (resp. (T)-structures)

$$(f, \phi)\colon (\mathcal{H}, \nabla)/R[\![t_I]\!] \to (\mathcal{H}', \nabla')/R[\![t_J]\!]$$

is *compatible with the framings* if it is constant when read in framing trivializations. More precisely, the framings $\mathcal{H} \simeq H \otimes_R R[\![t_I, u]\!]$ and $\mathcal{H}' \simeq H' \otimes_R R[\![t_J, u]\!]$ induce an isomorphism

$$\mathrm{Hom}_{R[\![t_I,u]\!]}(\mathcal{H}, f^*\mathcal{H}') \simeq \mathrm{Hom}_R(H, H')[\![t_I, u]\!].$$

The condition is that the image of ϕ is independent of t_I and u.

5.2.3 Lift of (T)-structures

Recall that we have fixed R a $\Bbbk$-algebra and a $\Bbbk$-basis $\boldsymbol{\lambda} = (\lambda_k, k \in K)$ of R. Let I be a countable set, we introduce two sets of formal variables:

$$t_I := \{t_i, i \in I\}, \quad \mathbf{t}_I := \{t_{i,k}, (i,k) \in I \times K\}.$$

There is a continuous morphism of R-algebras:

$$\psi_{\boldsymbol{\lambda}} \colon R[\![t_I]\!] \longrightarrow R[\![\mathbf{t}_I]\!], \quad t_i \mapsto \sum_{k \in K} \lambda_k t_{i,k}. \tag{5.2.5}$$

This induces a functor $(\mathcal{H}, \nabla)/R[\![t_I]\!] \mapsto (\widetilde{\mathcal{H}}, \widetilde{\nabla})/\Bbbk[\![\mathbf{t}_I]\!]$ from R-linear (T)-structures to $\Bbbk$-linear (T)-structures.

Lemma 5.2.6. *Let R be a $\Bbbk$-algebra with a fixed $\Bbbk$-basis $\boldsymbol{\lambda} = (\lambda_k, k \in K)$.*

1. *There exists a functor $(\mathcal{H}, \nabla)/R[\![t_I]\!] \mapsto (\widetilde{\mathcal{H}}, \widetilde{\nabla})/\Bbbk[\![\mathbf{t}_I]\!]$ from R-linear (T)-structures to $\Bbbk$-linear (T)-structures. It is obtained by applying the change of variable* (5.2.5) *and forgetting the R-linear structure.*

2. *Any framing for $(\mathcal{H}, \nabla)/R[\![t_I]\!]$ induces a framing for $(\widetilde{\mathcal{H}}, \widetilde{\nabla})/\Bbbk[\![\mathbf{t}_I]\!]$.*

Proof. Let $(\mathcal{H}, \nabla)$ be an R-linear (T)-structure over $R[\![t_I]\!]$. We define $\widetilde{\mathcal{H}}$ to be the $\Bbbk[\![\mathbf{t}_I]\!]$-module obtained by forgetting the R-linear structure on $\mathcal{H} \otimes_{R[\![t_I]\!]} R[\![\mathbf{t}_I]\!]$.

To define the (T)-structure connection $\widetilde{\nabla}$ we fix a trivialization $\mathcal{H} \simeq H \otimes_R R[\![t_I]\!]$. This induces an isomorphism $\widetilde{\mathcal{H}} \simeq \widetilde{H} \otimes_{\Bbbk} \Bbbk[\![\mathbf{t}_I]\!]$, where $\widetilde{H}$ denotes the $\Bbbk$-module obtained from H by forgetting the R-linear structure. We have a map of $\Bbbk[\![t_I, u]\!]$-algebras

$$\Psi_{\boldsymbol{\lambda}} \colon \mathrm{End}_R(H)[\![t_I, u]\!] \longrightarrow \mathrm{End}_{\Bbbk}(\widetilde{H})[\![\mathbf{t}_I, u]\!], \tag{5.2.7}$$

given by applying the change of variable $\psi_{\boldsymbol{\lambda}}$ and forgetting the R-linear structure. Fix $(i,k) \in I \times K$, and write $\nabla_{\partial_{t_i}} = \partial_{t_i} + u^{-1}\mathbf{A}_i(t_I, u)$, with $\mathbf{A}_i(t_I, u) \in \mathrm{End}_R(H)[\![t_I, u]\!]$. We then set

$$\widetilde{\nabla}_{\partial_{t_{i,k}}} := \partial_{t_{i,k}} + u^{-1}\lambda_k \widetilde{\mathbf{A}}_i(\mathbf{t}_I, u),$$

where $\widetilde{\mathbf{A}}_i := \Psi_{\boldsymbol{\lambda}}(\mathbf{A}_i)$. The chain rule and the flatness of ∇ imply that $\widetilde{\nabla}$ is flat, producing a $\Bbbk$-linear (T)-structure $(\widetilde{\mathcal{H}}, \widetilde{\nabla})$ over $\Bbbk[\![\mathbf{t}_I]\!]$. It is easily checked that this (T)-structure is independent of the choice of trivialization for $(\mathcal{H}, \nabla)$.

We now check functoriality. Let $(f, \phi)\colon (\mathcal{H}, \nabla)/R[\![t_I]\!] \to (\mathcal{H}', \nabla')/R[\![s_J]\!]$ be a morphism of (T)-structures. Let $(\widetilde{\mathcal{H}}, \widetilde{\nabla})/\Bbbk[\![\mathbf{t}_I]\!]$ and $(\widetilde{\mathcal{H}'}, \widetilde{\nabla}')/\Bbbk[\![\mathbf{s}_J]\!]$ denote the induced $\Bbbk$-linear (T)-structures. There exists a unique morphism of $\Bbbk$-algebras $\widetilde{f}\colon \Bbbk[\![\mathbf{s}_J]\!] \to \Bbbk[\![\mathbf{t}_I]\!]$ making the following diagram of R-algebras commutative:

$$\begin{array}{ccc}
R[\![s_J]\!] & \xrightarrow{\ f\ } & R[\![t_I]\!] \\
\Big\downarrow{\scriptstyle \psi'_{\boldsymbol{\lambda}}} & & \Big\downarrow{\scriptstyle \psi_{\boldsymbol{\lambda}}} \\
R[\![\mathbf{s}_J]\!] & \xrightarrow{\widetilde{f}\otimes_{\Bbbk} 1} & R[\![\mathbf{t}_I]\!].
\end{array}$$

It is characterized by the relations $\psi_{\boldsymbol{\lambda}} \circ f(s_j) = \sum_{k\in K} \lambda_k \widetilde{f}(s_{j,k})$ for all $j \in J$, and is automatically continuous. The morphism of $R[\![t_I, u]\!]$-modules $\phi\colon \mathcal{H} \to \mathcal{H}' \otimes_{R[\![s_J,u]\!]} R[\![t_I, u]\!]$ induces a morphism of $\Bbbk[\![\mathbf{t}_I, u]\!]$-modules $\tilde{\phi}\colon \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}'} \otimes_{\Bbbk[\![\mathbf{s}_J,u]\!]} \Bbbk[\![\mathbf{t}_I, u]\!]$ obtained by forgetting the R-linear structure of the map of $R[\![\mathbf{t}_I, u]\!]$-modules

$$\begin{aligned}
\mathcal{H} \otimes_{R[\![t_I,u]\!]} R[\![\mathbf{t}_I, u]\!] &\xrightarrow{\phi\otimes 1} \mathcal{H}' \otimes_{R[\![s_J,u]\!]} R[\![t_I, u]\!] \otimes_{R[\![t_I,u]\!]} R[\![\mathbf{t}_I, u]\!] \\
&\simeq \Big(\mathcal{H}' \otimes_{R[\![s_J,u]\!]} R[\![\mathbf{s}_J, u]\!]\Big) \otimes_{R[\![\mathbf{s}_J,u]\!]} R[\![\mathbf{t}_I, u]\!].
\end{aligned}$$

Forgetting the R-linear structure, the right-hand side is naturally isomorphic to $\widetilde{\mathcal{H}'} \otimes_{\Bbbk[\![\mathbf{s}_J,u]\!]} \Bbbk[\![\mathbf{t}_I, u]\!]$. Fixing trivializations of the (T)-structures, we directly check that the pair $(\widetilde{f}, \tilde{\phi})$ is compatible with the connections. We omit the check that this is compatible with composition of morphisms. By construction, a framing trivialization for $(\mathcal{H}, \nabla)$ induces a framing trivialization for $(\widetilde{\mathcal{H}}, \widetilde{\nabla})$, concluding the proof. $\square$

Remark 5.2.8. The functor $(\mathcal{H}, \nabla)/R[\![t_I]\!] \mapsto (\widetilde{\mathcal{H}}, \widetilde{\nabla})/\Bbbk[\![\mathbf{t}_I]\!]$ defined above for (T)-structures is analogous to the composition of inverse image functor $\psi^*_{\boldsymbol{\lambda}}$ and the restriction of scalars from R to $\Bbbk$ in the theory of D-modules.

Definition 5.2.9. An *R-linear lift* of a $\Bbbk$-linear (T)-structure $(\mathcal{H}, \nabla)/\Bbbk[\![\mathbf{t}_I]\!]$ is the data of an R-linear (T)-structure $(\mathcal{H}_R, \nabla_R)/R[\![t_I]\!]$ and an isomorphism of $\Bbbk$-linear (T)-structures $\alpha\colon (\mathcal{H}, \nabla)_0 \xrightarrow{\sim} (\widetilde{\mathcal{H}}_R, \widetilde{\nabla}_R)$.

5.2.4 Equivariant F-bundles

Definition 5.2.10 (Equivariant F-bundle)**.** Let I and J be countable sets. An *R-equivariant F-bundle* over $\Bbbk[\![\mathbf{t}_I]\!]$ consists of the following data $\{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}$.

1. $(\mathcal{H}, \nabla)$ is a $\Bbbk$-linear F-bundle over $\Bbbk[\![\mathbf{t}_I]\!]$, and

2. $\alpha\colon (\mathcal{H},\nabla)_0 \xrightarrow{\sim} (\widetilde{\mathcal{H}}_R, \widetilde{\nabla}_R)$ is an R-linear lift of the underlying (T)-structure $(\mathcal{H},\nabla)_0$, where $\mathcal{H}_R$ has finite rank as a $R[\![t_I, u]\!]$-module.

A *morphism* of equivariant F-bundles

$$\{(\mathcal{H},\nabla),(\mathcal{H}_R,\nabla_R),\alpha\}/\mathbb{k}[\![\mathbf{t}_I]\!] \xrightarrow{(\beta,\beta_R)} \{(\mathcal{H}',\nabla'),(\mathcal{H}'_R,\nabla'_R),\alpha'\}/\mathbb{k}[\![\mathbf{t}_J]\!]$$

consists of

1. a morphism of $\mathbb{k}$-linear F-bundles $\beta\colon (\mathcal{H},\nabla)\to(\mathcal{H}',\nabla')$, and
2. a morphism of R-linear (T)-structures $\beta_R\colon (\mathcal{H}_R,\nabla_R)\to(\mathcal{H}'_R,\nabla'_R)$,

such that the following diagram of $\mathbb{k}$-linear (T)-structures commutes:

$$\begin{tikzcd}
(\mathcal{H},\nabla)_0 \arrow[r, "\beta_0"] \arrow[d, "\alpha"] & (\mathcal{H}',\nabla')_0 \arrow[d, "\alpha'"] \\
(\widetilde{\mathcal{H}}_R,\widetilde{\nabla}_R) \arrow[r, "\widetilde{\beta}_R"] & (\widetilde{\mathcal{H}}'_R,\widetilde{\nabla}'_R).
\end{tikzcd} \tag{5.2.11}$$

Remark 5.2.12. 1. We identify a $\mathbb{k}$-linear F-bundle $(\mathcal{H},\nabla)$ with the $\mathbb{k}$-equivariant F-bundle $\{(\mathcal{H},\nabla),(\mathcal{H},\nabla)_0,\mathrm{id}\}$, where we choose $1\in\mathbb{k}$ as a $\mathbb{k}$-basis of $\mathbb{k}$. This defines a fully faithful functor.

2. When $\dim_{\mathbb{k}} R = 1$, equivariant F-bundles correspond to $\mathbb{k}$-linear F-bundles of finite dimension and parametrized by finitely many variables, up to isomorphism. Indeed, after choosing the basis given by $1\in R$ the change of coordinate (5.2.5) is the identity and the formal variables t_I and $\mathbf{t}_I$ agree. Given an equivariant F-bundle $\mathcal{F}=\{(\mathcal{H},\nabla),(\mathcal{H}_R,\nabla_R),\alpha\}/\mathbb{k}[\![t_I]\!]$, using α we see that $\mathcal{H}$ has finite rank over $\mathbb{k}[\![t_I]\!]$ because $\mathcal{H}_R$ does, and we can define a u-direction connection on $\mathcal{H}_R$ compatible with the (T)-structure, making α an isomorphism of F-bundles.

Definition 5.2.13. 1. A *framing* for an equivariant F-bundle $\{(\mathcal{H},\nabla),(\mathcal{H}_R,\nabla_R),\alpha\}$ is the data of framings for $(\mathcal{H},\nabla)$ and $(\mathcal{H}_R,\nabla_R)$, such that $\alpha\colon (\mathcal{H},\nabla)_0\to(\widetilde{\mathcal{H}}_R,\widetilde{\nabla}_R)$ is compatible with the induced framings.

2. A morphism (β,β_R) of framed equivariant F-bundles is *compatible with the framings* if both β and β_R are compatible with the framings.

Remark 5.2.14. A morphism of equivariant F-bundles (β,β_R) is uniquely determined by β_R and the R-linear lifts through (5.2.11). Similarly, a framing of equivariant F-bundle is uniquely determined by the framing on the R-linear lift.

5.3 Unfolding of equivariant F-bundles

Recall the setting of Section 5.2, $\Bbbk$ is a field of characteristic 0 and R is a $\Bbbk$-algebra of countable dimension.

5.3.1 Framing of (T)-structures

In this subsection, we prove that an F-bundle $(\mathcal{H}, \nabla)$ is characterized by the underlying (T)-structure and the restriction of the F-bundle to a point using framing of (T)-structures (see Lemma 5.3.1). We deduce a criterion for lifting a morphism of (T)-structures to a morphism of F-bundles. We also prove the existence of framing and extension of framing results for (T)-structures over a noetherian base.

Lemma 5.3.1. *For $k = 1, 2$, let I_k be a countable set and $(\mathcal{H}_k, \nabla_k)/R[\![t_j, j \in I_k]\!]$ be an F-bundle. Let $(f, \Phi)\colon (\mathcal{H}_1, \nabla_1)_0 \to (\mathcal{H}_2, \nabla_2)_0$ be a morphism of (T)-structures. Assume that the (T)-structure $(\mathcal{H}_1, \nabla_1)_0$ admits a framing.*

1. ∇_1 is uniquely determined by the underlying (T)-structure and $\nabla_{1,\partial_u}|_{t_{I_1}=0}$, and any such data determine a unique F-bundle connection extending the (T)-structure.

2. (f, Φ) is an isomorphism of F-bundles if and only if $(f, \Phi)|_{t_{I_1}=0}$ is an isomorphism of F-bundles.

Proof. For (1), fix a framing trivialization $\mathcal{H} \simeq H \otimes R[\![t_i, i \in I_1, u]\!]$ of the underlying (T)-structure. In this trivialization, write $\nabla_{1,\partial_{t_i}} = \partial_{t_i} + u^{-1}T^i$ and $\nabla_{1,\partial_u} = \partial_u + u^{-2}U$. By assumption, the endomorphism T^i is independent of u. The flatness equations for the u-direction and t_i-direction give for all $i \in I_1$

$$\frac{\partial U}{\partial t_i} = -T^i + u\frac{\partial T^i}{\partial u} + u^{-1}[U, T^i] = -T^i + u^{-1}[U, T^i]. \tag{5.3.2}$$

Any U solving this system of equations gives rise to an F-bundle structure extending the (T)-structure. Then (1) reduces to proving that for any initial condition $U_0(u) \in \mathrm{End}_{R[\![u]\!]}(H[\![u]\!])$, there exists a unique $U(t, u)$ solving (5.3.2) with $U(0, u) = U_0(u)$. Introduce the differential operators $D_i\colon X \mapsto \frac{\partial X}{\partial t_i} + u^{-1}\,\mathrm{ad}_{T^i}(X)$, where $\mathrm{ad}_{T^i} = [T^i, \cdot]$. Then (5.3.2) can be written as $D_i(U) = -T^i$, and we need to prove that the system is compatible for any initial condition.

Since ∇_1 is flat, by comparing degrees in u, we have for all $i, j \in I_1$

$$[T^j, T^i] = u\left(\frac{\partial T^j}{\partial t_i} - \frac{\partial T^i}{\partial t_j}\right) = 0. \tag{5.3.3}$$

It follows that

$$\begin{aligned}[D_i, D_j] &= [\partial_{t_i}, \partial_{t_j}] + u^{-1}\left(\left[\frac{\partial}{\partial t_i}, \mathrm{ad}_{T^j}\right] + \left[\mathrm{ad}_{T^i}, \frac{\partial}{\partial t_j}\right]\right) + u^{-2}[\mathrm{ad}_{T^i}, \mathrm{ad}_{T^j}] \\ &= u^{-1}\Big(\mathrm{ad}_{\partial_{t_i} T^j} - \mathrm{ad}_{\partial_{t_j} T^i}\Big) + u^{-2}\,\mathrm{ad}_{[T^i, T^j]} = 0.\end{aligned}$$

Hence, by the usual theory of linear system of ODEs, the system is compatible if and only if for all $i, j \in I_1$, we have $D_i(T^j) = D_j(T^i)$. This follows from the flatness equations (5.3.3). We can thus construct a unique solution inductively on the number of variables from any initial condition. If I_1 is finite, we obtain a solution in finitely many steps. If $I_1 \simeq \mathbb{N}$ is infinite, we construct a solution in the projective limit $\varprojlim \mathrm{End}_R(H)[\![t_1, \ldots, t_n, u]\!] = \mathrm{End}_R(H)[\![t_i, i \in I_1, u]\!] \simeq \mathrm{End}_{R[\![t_i, i\in I_1, u]\!]}(H \otimes R[\![t_i, i \in I_1, u]\!])$. (1) is proved.

For (2), the first direction is obvious. For the converse, if $\Phi|_{t_{I_1}=0}$ is an isomorphism, then the $R[\![t_{I_1}]\!]$-module map Φ is an isomorphism (see Remark 5.2.2). The connection $\nabla_2' := \Phi^{-1} \circ f^*\nabla_2 \circ \Phi$ defines an F-bundle structure on $\mathcal{H}$. By assumption, the underlying (T)-structure agrees with $(\mathcal{H}, \nabla)_0$ and $\nabla_{1,\partial_u}|_{t_{I_1}=0} = \nabla'_{2,\partial_u}|_{t_{I_1}=0}$. It follows from the uniqueness in (1) that $\nabla_2' = \nabla_1$, hence (f, Φ) is a morphism of F-bundles. □

For (T)-structures defined over a Noetherian base $R[\![t_1, \ldots, t_n]\!]$, results from [Hin+24, §4.1] imply the existence of framing trivializations.

Proposition 5.3.4. *Let $(\mathcal{H}, \nabla)/R[\![t_1, \ldots, t_n]\!]$ be an R-linear (T)-structure. Any trivialization of $\mathcal{H}|_{t=0}$ extends uniquely to a framing of $(\mathcal{H}, \nabla)$.*

Proof. Fix a trivialization $\mathcal{H} \simeq H \otimes R[\![t_1, \ldots, t_n, u]\!]$ lifting the trivialization of $\mathcal{H}/(t_1, \ldots, t_n)\mathcal{H}$. Write the connection as $\nabla_{\partial_{t_i}} = \partial_{t_i} + u^{-1}T^i(t, u)$. We want to show that there exists a unique gauge transformation $P(t, u) \in \mathrm{GL}(H[\![t_1, \ldots, t_n, u]\!])$ with $P(0, u) = \mathrm{id}$ such that $uP^{-1}\frac{\partial P}{\partial t_i} + P^{-1}T^iP$ is independent of u for all $1 \leq i \leq n$. This amounts to solving the system of PDEs ($1 \leq i \leq n$)

$$\frac{\partial P}{\partial t_i} = u^{-1}(-T^iP + PP_0^{-1}T^i_{-1}P_0),$$

where $P_0 = P(t, 0)$ and $T^i_{-1} = T^i(t, 0)$, with the initial condition $P(0, u) = \mathrm{id}$. Uniqueness is clear, as the system provides recursive relations for the coefficients of P, and existence follows from [Hin+24, Lemmas 4.17, 4.18, 4.20]. The arguments there still apply, because we assume that R contains $\mathbb{Q}$. □

Fix I a finite set, let $(\mathcal{H}, \nabla)/R[\![t_I]\!]$ be a (T)-structure of finite rank $n \in \mathbb{N}$. Let $v_1 \in \mathcal{H}/(t_I, u)\mathcal{H}$. Any choice $(h_1, \ldots, h_n)$ of $R[\![t_I, u]\!]$-basis for $\mathcal{H}$ provides a trivialization through the isomorphisms

$$\mathcal{H} \simeq \bigoplus_{1 \le i \le n} R[\![t_I, u]\!] h_i \simeq R^{\oplus n} \otimes_R R[\![t_I, u]\!].$$

We call a basis $(h_1, \ldots, h_n)$ *good for* $(\mathcal{H}, \nabla)$ if it induces a framing trivialization. We say that it *extends* v_1 if h_1 is a lift of v_1. Proposition 5.3.4 implies that any basis of $\mathcal{H}/(t_I, u)\mathcal{H}$ lifts uniquely to a good basis of $(\mathcal{H}, \nabla)$. More generally, we have the following.

Lemma 5.3.5. *Let I and J be finite sets. Let $(f, \Phi)\colon (\mathcal{H}, \nabla)/R[\![t_I]\!] \to (\mathcal{H}', \nabla')/R[\![t_J]\!]$ be a morphism of finite rank (T)-structures. Assume that $\Phi|_{t_I=0}$ is an isomorphism.*

1. *Any good basis $(h_1, \ldots, h_n)$ of $(\mathcal{H}, \nabla)$ induces a unique good basis $(h'_1, \ldots, h'_n)$ of $(\mathcal{H}', \nabla')$ such that $\Phi(h_k) = f^*(h'_k)$ for all $1 \le k \le n$.*

2. *Φ is uniquely determined by its restriction to $\mathcal{H}|_{t_I=0}$.*

Proof. The assumptions imply that Φ is an isomorphism of $R[\![t_I, u]\!]$-modules. In particular, we have isomorphisms of $R[\![u]\!]$-modules

$$\mathcal{H}/(t_I)\mathcal{H} \simeq f^*\mathcal{H}'/(t_I) f^*\mathcal{H}' \simeq \mathcal{H}'/(t_J)\mathcal{H}'. \tag{5.3.6}$$

A good basis $(h'_1, \ldots, h'_n)$ for $(\mathcal{H}', \nabla')$ is uniquely characterized by its projection to $\mathcal{H}'/(t_J)\mathcal{H}'$. This value is uniquely specified by the condition $\Phi(h_k) = f^*(h'_k)$ using the isomorphism (5.3.6), which proves (1).

For (2), we note that Φ is uniquely determined by the image of a good basis $(h_1, \ldots, h_n)$ of $(\mathcal{H}, \nabla)$. By (1), the image $(\Phi(h_1), \ldots, \Phi(h_n))$ is a good basis for $f^*(\mathcal{H}', \nabla')$. In particular, it is uniquely determined by its restriction to $t_I = 0$, which only depends on $\Phi|_{t_I=0}$. The proof is complete. ☐

5.3.2 Formal Hertling-Manin unfolding theorem

In this subsection, we prove an analogue of the Hertling-Manin unfolding theorem for (TE)-structures (see [HM04, Theorem 2.5]) for formal R-linear F-bundles and (T)-structures.

Definition 5.3.7 (Unfolding of (T)-structure, F-bundle)**.** Let R be a $\Bbbk$-algebra, I and J countable sets. Let $(\mathcal{H}, \nabla)/R[\![t_I]\!]$ be an R-linear (T)-structure (resp. F-bundle). An *unfolding* of $(\mathcal{H}, \nabla)$ is a morphism of (T)-structures (resp. F-bundles) $(i, \phi)\colon (\mathcal{H}, \nabla)/R[\![t_I]\!] \to (\mathcal{H}', \nabla')/R[\![t_J]\!]$, where

1. $I \subset J$ and $i\colon R[\![t_J]\!] \to R[\![t_I]\!]$ is the quotient by the closure of the ideal $(t_j, j \in J \setminus I)$, and
2. $\phi\colon \mathcal{H} \to i^*\mathcal{H}'$ is an isomorphism of $R[\![t_I, u]\!]$-modules.

A *morphism between two unfoldings* $\iota_k\colon (\mathcal{H}, \nabla) \to (\mathcal{H}_k, \nabla_k)$ for $k = 1, 2$, is a morphism of (T)-structures (resp. F-bundles) $(f, \psi)\colon (\mathcal{H}_2, \nabla_2) \to (\mathcal{H}_1, \nabla_1)$ such that ψ is an isomorphism and the following diagram commutes

$$\begin{tikzcd} & (\mathcal{H}, \nabla) \arrow[dl, "\iota_2"'] \arrow[dr, "\iota_1"] & \\ (\mathcal{H}_2, \nabla_2) \arrow[rr, "(f,\psi)"] & & (\mathcal{H}_1, \nabla_1). \end{tikzcd}$$

Remark 5.3.8. In the above commutative diagram, assume that $(\mathcal{H}_k, \nabla_k)$ depends on finitely many variables indexed by a finite set J_k for $k = 1, 2$, and write

$$\iota_k = (i_k, \phi_k) : (\mathcal{H}, \nabla)/R[\![t_I]\!] \to (\mathcal{H}_2, \nabla_2)/R[\![t_{J_k}]\!].$$

Then for any two morphisms (f, ψ_k), $k = 1, 2$, between the unfoldings ι_2 and ι_1, we have $\psi_1 = \psi_2$. In other words, the morphism on the base f determines the bundle map. Indeed, the commutativity of the diagram implies that $i_2^*\psi_k \circ \phi_2 = \phi_1$. This determines $\psi_k|_{t_{J_2}=0} = \phi_1 \circ \phi_2^{-1}|_{t_{J_2}=0}$. By Lemma 5.3.5, ψ_k is uniquely determined by $\psi_k|_{t_{J_2}=0}$, and thus $\psi_1 = \psi_2$.

Remark 5.3.9. When I and J are finite, given an unfolding of R-linear (T)-structures

$$(i, \phi)\colon (\mathcal{H}, \nabla)/R[\![t_I]\!] \longrightarrow (\mathcal{H}', \nabla')/R[\![t_J]\!],$$

any framing for $(\mathcal{H}, \nabla)/R[\![t_I]\!]$ induces a unique framing for $(\mathcal{H}', \nabla')/R[\![t_J]\!]$, and vice versa. Indeed, ϕ takes the framing trivialization for $(\mathcal{H}, \nabla)$ to a framing trivialization for $i^*(\mathcal{H}', \nabla')$, which is uniquely determined by its restriction to the fiber $i^*\mathcal{H}'|_{t_I=0} = \mathcal{H}'|_{t_J=0}$. We can extend this to a framing trivialization for $(\mathcal{H}', \nabla')$ by Proposition 5.3.4.

Lemma 5.3.10. *For $k = 1, 2$, let I_k be countable sets, and let*

$$(f, \Phi)\colon (\mathcal{H}_1, \nabla_1)/R[\![t_1]\!] \to (\mathcal{H}_2, \nabla_2)/R[\![t_2]\!]$$

be an unfolding of R-linear (T)-structures. Assume the (T)-structure $(\mathcal{H}_2, \nabla_2)$ admits a framing. Given an F-bundle structure $(\mathcal{H}_1, \nabla_1^F)$ on $(\mathcal{H}_1, \nabla_1)$, there exists a unique F-bundle structure $(\mathcal{H}_2, \nabla_2^F)$ on $(\mathcal{H}_2, \nabla_2)$ such that (f, Φ) is an unfolding of F-bundles.

Proof. Since (f, Φ) is an unfolding of (T)-structures, we have isomorphisms of $R[\![u]\!]$-modules:

$$\mathcal{H}_1|_{t_1=0} \simeq f^*\mathcal{H}_2|_{t_1=0} \simeq \mathcal{H}_2|_{t_2=0}. \tag{5.3.11}$$

Under this isomorphism, the restriction $\nabla_1^F|_{t_1=0}$ produces a F-bundle connection on $\mathcal{H}_2|_{t_2=0}$. Since the latter admits a framing, applying Lemma 5.3.1(1) we obtain a unique F-bundle $(\mathcal{H}_2, \nabla_2^F)$ extending the (T)-structure $(\mathcal{H}_2, \nabla_2)$. We now check that (f, Φ) is a morphism of F-bundles. By construction, the connections $f^*\nabla_2^F$ and $\Phi \circ \nabla_1^F \circ \Phi^{-1}$ are F-bundle connections on $f^*\nabla_2$ which coincide at $t_1 = 0$, and with the same underlying (T)-structures. The framing for $(\mathcal{H}_2, \nabla_2)$ induces a framing on $f^*(\mathcal{H}_2, \nabla_2)$, as can be seen by fixing a framing trivialization of $(\mathcal{H}_2, \nabla_2)$ and pulling it back under f. Then, it follows from Lemma 5.3.1(1) that those two F-bundle structures agree. Hence, (f, Φ) is a morphism of F-bundles.

For uniqueness, note that the F-bundle connection ∇_2^F is uniquely determined by its restriction to $t_2 = 0$ since $(\mathcal{H}_2, \nabla_2)$ admits a framing, and that $\nabla_2^F|_{t_2=0}$ is uniquely specified by $\nabla_1^F|_{t_1=0}$ through the isomorphisms (5.3.11). □

For an R-linear (T)-structure $(\mathcal{H}, \nabla)/R[\![t_I]\!]$, there is a morphism of R-modules [Hin+24, Remark 2.3]

$$\begin{aligned} \mu\colon \bigoplus_{i\in I} R\partial_{t_i} &\longrightarrow \operatorname{End}_R(H), \\ \partial_{t_i} &\longmapsto \nabla_{u\partial_{t_i}}\Big|_{u=0,t_I=0}, \end{aligned} \tag{5.3.12}$$

where $H \coloneqq \mathcal{H}/J\mathcal{H}$ with J the closure of the ideal (t_I, u). For each $v \in H$ we obtain an evaluation map of R-modules:

$$\begin{aligned} \mu_v\colon \bigoplus_{i\in I} R\partial_{t_i} &\longrightarrow H, \\ \xi &\longmapsto \mu(\xi)(v). \end{aligned} \tag{5.3.13}$$

Furthermore, if $(\mathcal{H}, \nabla)$ is an F-bundle, we also have a residue endomorphism in the u-direction $\mathrm{K} \coloneqq [u^2\nabla_{\partial_u}]|_{u=t=0} \in \operatorname{End}_R(H)$. We introduce the notion of maximal (T)-structure and maximal F-bundle, analogous to [Hin+24, Definition 2.6].

Definition 5.3.14 (Maximal (T)-structure, maximal F-bundle)**.** Let R be a $\Bbbk$-algebra, I a countable set, and $J \subset R[\![t_I, u]\!]$ the closure of the ideal (t_I, u). An R-linear (T)-structure, or F-bundle, $(\mathcal{H}, \nabla)/R[\![t_I]\!]$ is *maximal* if there exists $v \in \mathcal{H}/J\mathcal{H}$ such that the map μ_v is an isomorphism. We call such a v a *cyclic vector*.

The Hertling-Manin unfolding theorem guarantees the existence and uniqueness of a maximal unfolding under certain conditions, which we introduce in the next definition.

Definition 5.3.15. Let I be a countable set, $(\mathcal{H}, \nabla)/R[\![t_I]\!]$ an R-linear (T)-structure (resp. F-bundle), and $J \subset R[\![t_I, u]\!]$ the closure of the ideal (t_I, u). We define the following conditions on an element $v \in H \coloneqq \mathcal{H}/J\mathcal{H}$:

(IC) The map μ_v in (5.3.13) is injective.

(GC) The orbit of v under the action of the subalgebra $R[\operatorname{im}\mu] \subset \operatorname{End}_R(H)$ (resp. $R[\operatorname{im}\mu, \mathrm{K}] \subset \operatorname{End}_R(H)$) defined by evaluation on v is H.

(GC') The condition (GC) is satisfied after base change to $\operatorname{Frac}(R)$.

If v satisfies (GC), we say that v is a *generating vector* for $(\mathcal{H}, \nabla)$.

The following lemma provides a construction of unfoldings under the (GC) condition. It is analogous to [HM04, Lemma 2.9], except that we use framings of (T)-structures to avoid the analytic argument used there.

Lemma 5.3.16. *Let $(\mathcal{H}^{(0)}, \nabla^{(0)})/R[\![t_1, \ldots, t_d]\!]$ be an F-bundle of rank n satisfying the (GC) condition, let $v_1 \in \mathcal{H}^{(0)}/(t_I, u)\mathcal{H}^{(0)}$ be a generating vector. Let $(h_1^{(0)}, \ldots, h_n^{(0)})$ be a good basis of $(\mathcal{H}^{(0)}, \nabla^{(0)})$ extending v_1. Fix $\ell \geq 1$ and let $f_1, \ldots, f_n \in R[\![t_1, \ldots, t_d, s_1, \ldots, s_\ell]\!]$ whose restrictions to $s = 0$ are 0.*

Then there exists an unfolding $\iota\colon (\mathcal{H}^{(0)}, \nabla^{(0)})/R[\![t_1, \ldots, t_d]\!] \to (\mathcal{H}, \nabla)/R[\![t_1, \ldots, t_d, s_1, \ldots, s_\ell]\!]$ such that, if $(h_1, \ldots, h_n)$ denotes the good basis of $(\mathcal{H}, \nabla)$ induced from $(h_1^{(0)}, \ldots, h_n^{(0)})$ (see Lemma 5.3.5), we have for $1 \leq j \leq \ell$

$$[u\nabla_{\partial_{s_j}}]|_{u=0}(h_1|_{u=0}) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial s_j} h_i|_{u=0}. \tag{5.3.17}$$

Any two unfoldings satisfying (5.3.17) are isomorphic under a morphism (id, ψ), where ψ identifies the canonical extensions of the good basis $(h_i^{(0)})_{1 \leq i \leq n}$.

Proof. Set $t \coloneqq \{t_1, \ldots, t_d\}$. We consider the case $\ell = 1$, as we can always decompose an unfolding as a sequence of 1-dimensional unfoldings.

Let $H \coloneqq R^{\oplus n}$. The good basis $(h_i^{(0)})_{1 \leq i \leq n}$ provides an isomorphism $\phi\colon \mathcal{H}^{(0)} \xrightarrow{\sim} H \otimes_R R[\![t, u]\!]$. Let $\mathcal{H} \coloneqq H \otimes_R R[\![t, s, u]\!]$. We first prove that there exists a unique connection ∇ on $\mathcal{H}$ such that $\iota = (i, \phi)\colon (\mathcal{H}^{(0)}, \nabla^{(0)}) \to (\mathcal{H}, \nabla)$ is an unfolding satisfying (5.3.17). This is equivalent to constructing unique matrices $T^i(t,s), S(t,s), U_k(t,s) \in \mathrm{Mat}(n \times n, R[\![t,s]\!])$ such that the connection form

$$\Omega \coloneqq \frac{1}{u}\sum_{i=1}^{n} T^i(t,s)dt_i + \frac{1}{u}S(t,s)ds + \frac{1}{u^2}\sum_{k\geq 0} U_{k-2}(t,s)u^k du$$

satisfies:

(a) the flatness equation $d\Omega + \Omega \wedge \Omega = 0$,

(b) $T^i(t,0)$ and $U_k(t,0)$ coincide with the connection matrix of $\nabla^{(0)}$ in $(h_i^{(0)})_{1\leq i\leq n}$, and

(c) $S(t,s)e_1 = \sum_{i=1}^{n} \frac{\partial f_i}{\partial s} e_i$, where $(e_i)_{1\leq i\leq n}$ is the canonical basis of $\mathcal{H}/(u)\mathcal{H} = R^{\oplus n} \otimes_R R[\![t,s]\!]$.

We further decompose the matrices into powers of s, and write $T^i_w(t)$ (resp. $S_w(t)$, $U_{k,w}(t)$) for the coefficient of s^w in $T^i(t,s)$ (resp. $S(t,s)$, $U_k(t,s)$). We will construct the matrices order by order in s.

Condition (a) is equivalent to the following system of equations:

$$[S, T^i] = 0 \tag{5.3.18}$$

$$[S, U_{-2}] = 0 \tag{5.3.19}$$

$$\partial_s T^i = \partial_{t_i} S \tag{5.3.20}$$

$$\partial_s U_{-2} = [U_{-1}, S] - S \tag{5.3.21}$$

$$\partial_s U_k = [U_{k+1}, S] \qquad (k \geq -1) \tag{5.3.22}$$

$$[T^i, T^j] = 0 \tag{5.3.23}$$

$$[U_{-2}, T^i] = 0 \tag{5.3.24}$$

$$\partial_{t_i} T^j = \partial_{t_j} T^j \tag{5.3.25}$$

$$\partial_{t_i} U_{-2} = [U_{-1}, T^i] - T^i \tag{5.3.26}$$

$$\partial_{t_i} U_k = [U_{k+1}, T^i] \qquad (k \geq -1). \tag{5.3.27}$$

We prove by induction on $m \in \mathbb{N}$ that there exists unique matrices $T^i_w(t)$ and $U_{k,w}(t)$ for $0 \leq w \leq m$ and $S_w(t)$ for $0 \leq w \leq m-1$ such that the equations (5.3.18) through (5.3.22) are satisfied modulo s^m, the equations (5.3.23) through (5.3.27) are satisfied modulo s^{m+1}, condition (b) is satisfied and condition (c) is satisfied modulo s^m.

For $m = 0$, condition (b) provides the matrices $T_0^i(t)$ and $U_{k,0}(t)$, and the equations (5.3.23)-(5.3.27) are satisfied modulo s by flatness of $\nabla^{(0)}$.

Now assume the induction carried out until step m, we prove step $m+1$. We only need to construct the matrices T_{m+1}^i, $U_{k,m+1}$ and S_m so that the various conditions of the induction are satisfied. The construction of a unique matrix S_m such that (5.3.18), (5.3.19) and condition (c) are satisfied modulo s^{m+1} is as in (i) of the proof of [HM04, Lemma 2.9]. The matrices T_{m+1}^i and $U_{k,m+1}$ are uniquely determined by imposing equations (5.3.20)-(5.3.22) modulo s^{m+1}.
It remains to check that equations (5.3.23)-(5.3.27) hold modulo s^{m+2}, assuming that equations (5.3.18)-(5.3.27) hold modulo s^{m+1}. Since they hold at $s = 0$, we simply check that the s-derivative of these equations is zero modulo s^{m+1}. For (5.3.23) we have modulo s^{m+1}

$$
\begin{aligned}
\partial_s[T^i, T^j] &= [\partial_s T^i, T^j] + [T^i, \partial_s T^j] \\
&= [\partial_{t_i} S, T^j] + [T^i, \partial_{t_j} S] \\
&= -[S, \partial_{t_i} T^j] - [\partial_{t_j} T^i, S] \\
&= 0.
\end{aligned}
$$

For (5.3.24) we have modulo s^{m+1}

$$
\begin{aligned}
\partial_s[U_{-2}, T^i] &= [\partial_s U^{-2}, T^i] + [U_{-2}, \partial_s T^i] \\
&= [[U_{-1}, S], T^i] + [U_{-2}, \partial_{t_i} S] \\
&= [[U_{-1}, S], T^i] - [\partial_{t_i} U_{-2}, S] \\
&= [[U_{-1}, S], T^i] - [[U_{-1}, T^i], S] \\
&= 0.
\end{aligned}
$$

For (5.3.25) we have modulo s^{m+1}

$$
\partial_s(\partial_{t_i} T^j - \partial_{t_j} T^i) = \partial_{t_i}\partial_s T^j - \partial_{t_j}\partial_s T^i = \partial_{t_i}\partial_{t_j} S - \partial_{t_j}\partial_{t_i} S = 0.
$$

For (5.3.26) we have modulo s^{m+1}

$$
\begin{aligned}
\partial_s\left(\partial_{t_i} U_{-2} + T^i + [T^i, U_{-1}]\right) &= \partial_{t_i}[U_{-1}, S] - \partial_{t_i} S + \partial_{t_i} S + [\partial_s T^i, U_{-1}] + [T^i \partial_s U_{-1}] \\
&= [\partial_{t_i} U_{-1}, S] + [U_{-1}, \partial_{t_i} S] + [\partial_{t_i} S, U_{-1}] + [T^i, [U_0, S]] \\
&= [[U_0, T^i], S] + [T^i, [U_0, S]] \\
&= 0,
\end{aligned}
$$

where on the first line we used (5.3.20) and (5.3.21), on the second line we used (5.3.20) and (5.3.22), on the third line we used (5.3.27), and on the last line we used the Jacobi identity and (5.3.18). For (5.3.27) we have modulo s^{m+1}

$$\begin{aligned}\partial_s\left(\partial_{t_i}U_k + [T^i, U_{k+1}]\right) &= \partial_{t_i}[U_{k+1}, S] + [\partial_s T^i, U_{k+1}] + [T^i, \partial_s U_{k+1}]\\ &= [\partial_{t_i}U_{k+1}, S] + [T^i, \partial_s U_{k+1}]\\ &= [[U_{k+2}, T^i], S] + [T^i, [U_{k+2}, S]]\\ &= 0.\end{aligned}$$

This finishes the induction step, and proves the existence.

For uniqueness up to isomorphism, assume that $\iota'\colon (\mathcal{H}^{(0)}, \nabla^{(0)}) \to (\mathcal{H}', \nabla')$ is another unfolding satisfying (5.3.17). We prove that it is isomorphic to the unfolding $(\mathcal{H}, \nabla)$ constructed above. Let $\psi\colon \mathcal{H} \to \mathcal{H}'$ denote the $R[\![t, s, u]\!]$-module isomorphism obtained by identifying the good bases obtained from $(h_i^{(0)})_{1\leq i\leq n}$. Then the connection form of $\psi^{-1}\circ\nabla'\circ\psi$ in the trivialization of $\mathcal{H}$ given by $(e_1, \ldots, e_n)$ satisfies conditions (a), (b), and (c) above. Thus $\psi^{-1}\circ\nabla'\circ\psi = \nabla$, and we conclude that $(\mathrm{id}\psi)\colon (\mathcal{H}, \nabla) \to (\mathcal{H}', \nabla')$ is an isomorphism of unfoldings. □

Lemma 5.3.16 says that under the (GC) assumption, an unfolding $\iota\colon (\mathcal{H}^{(0)}, \nabla^{(0)}) \to (\mathcal{H}, \nabla)$ is uniquely determined up to isomorphism by the choice of a good basis $(h_1, \ldots, h_n)$ extending a cyclic vector, and the action of the connection on h_1.

Theorem 5.3.28 (Hertling-Manin for F-bundles)**.** *Let R be an integral domain containing $\mathbb{Q}$. Let $(\mathcal{H}, \nabla)/R[\![t_1, \ldots, t_d]\!]$ be a finite rank F-bundle. Let $v \in \mathcal{H}/(t_1, \ldots, t_d, u)\mathcal{H}$.*

1. If v satisfies (IC), (GC) and $\operatorname{coker}\mu_v$ *is free, then there exists a maximal unfolding with cyclic vector induced from v.*

2. If v satisfies (GC'), then any two maximal unfoldings of $(\mathcal{H}, \nabla)$ with cyclic vector induced from v are isomorphic under a unique isomorphism.

Furthermore, any framing for $(\mathcal{H}, \nabla)$ induces a unique framing on a maximal unfolding.

Proof. Let n denote the rank of $\mathcal{H}$, and $\ell := n - d$. We assume $\ell \geq 0$, as otherwise the evaluation map μ_v cannot be injective and a maximal unfolding of $(\mathcal{H}, \nabla)$ does not exist. Write $t = \{t_1, \dots, t_d\}$ and $s = \{s_1, \dots, s_\ell\}$. Fix a good basis $(h_1, \dots, h_n)$ for $(\mathcal{H}, \nabla)$ extending v, i.e. with $h_1|_{t=u=0} = v$.

For (1), let $N \in \mathrm{Mat}(n \times d, R)$ denote the matrix of the evaluation map μ_v. Let $f_1, \dots, f_n \in R[\![t, s]\!]$ with $f_i(t, 0) = 0$. Applying Lemma 5.3.16 we obtain an unfolding $\iota\colon (\mathcal{H}, \nabla)/R[\![t]\!] \to (\mathcal{H}', \nabla')/R[\![t, s]\!]$. Let $v' \in \mathcal{H}'/(t, s, u)\mathcal{H}'$ corresponding to v, the matrix of the evaluation map $\mu_{v'}$ in the good basis obtained from $(h_i)_{1 \le i \le n}$ is

$$\left(N \quad (\tfrac{\partial f_i}{\partial s_j}\Big|_{t=s=0})_{1 \le i \le n, 1 \le j \le \ell}\right) \in \mathrm{Mat}(n \times n, R). \tag{5.3.29}$$

Since v satisfies (IC), the columns of N form a basis of $\operatorname{im} \mu_v \subset \mathcal{H}/(t_1, \dots, t_d, u)\mathcal{H}$. Since $\operatorname{coker} \mu_v$ is free, by the basis extension theorem, we can extend this basis to a basis of $\mathcal{H}/(t_1, \dots, t_d, u)\mathcal{H}$ by adding elements $\{v_1, \dots, v_\ell\}$. Any choice $(f_1, \dots, f_n)$ such that the vector $(\frac{\partial f_i}{\partial s_k}|_{t=s=0})_{1 \le i \le n}$ corresponds to v_k for all $1 \le k \le \ell$ gives rise to a maximal unfolding, since the columns of (5.3.29) then form a basis of H. This proves (1).

We now prove (2). For $k = 1, 2$, let $\iota_k = (i_k, \phi_k)\colon (\mathcal{H}, \nabla) \to (\mathcal{H}'_k, \nabla'_k)$ be a maximal unfolding. In the good bases induced from $(h_i)_{1 \le i \le n}$ the 1-forms defining the (T)-structures are closed by (5.3.25), and hence can be written as $u^{-1} dA_k$ for a unique $A_k \in \mathrm{Mat}(n \times n, R)[\![t, s]\!]$ satisfying $A_k(0, 0) = 0$. The first column of A_k provides n elements of $R[\![t, s]\!]$ that define a map of R-algebras $\psi_k\colon R[\![t, s]\!] \to R[\![t, s]\!]$. Since the unfoldings are assumed to be maximal, $d\psi_k|_{t=s=0}$ is an isomorphism. This follows from the fact that, by construction, its matrix in the basis $(dt_1, \dots, dt_d, ds_1, \dots, ds_\ell)$ coincides with the matrix of the evaluation map for $(\mathcal{H}', \nabla')$. We deduce that $\psi_k \in \mathrm{Aut}_R(R[\![t, s]\!])$. If $(f, j)\colon (\mathcal{H}_1, \nabla_1) \to (\mathcal{H}_2, \nabla_2)$ is an isomorphism of unfoldings, then $f^* dA_2 = dA_1$ which implies $A_2 \circ f = A_1$. In particular $\psi_2 \circ f = \psi_1$, and this determines f uniquely, since ψ_2 is an isomorphism. In turn, this determines j uniquely by Remark 5.3.8. Conversely, let $f = \psi_2^{-1} \circ \psi_1$ and define $j\colon \mathcal{H}'_1 \to f^*\mathcal{H}'_2$ by identifying the good bases induced from $(h_i)_{1 \le i \le n}$. In particular, we have $d\psi_1 = d\psi_2 \circ df$. Therefore $f^*(\mathcal{H}'_2, \nabla'_2)$ is a maximal unfolding whose action on the cyclic section that extends h_1 agrees with that of $(\mathcal{H}'_1, \nabla'_1)$. After base changing to $\mathrm{Frac}(R)$, the (GC) condition is satisfied. It follows from Lemma 5.3.16 that (f, j) is compatible with the connections and is an isomorphism of unfoldings after base changing to $\mathrm{Frac}(R)$. But f (resp. j) is invertible over R (resp. $R[\![t, s, u]\!]$) by construction, so the unfoldings are isomorphic over R.

The last claim follows from the extension of framing result in [Hin+24, Theorem 1.3]. The proof is complete. $\square$

Corollary 5.3.30 (Hertling-Manin for (T)-structures)**.** *Let R be an integral domain containing $\mathbb{Q}$. Let $(\mathcal{H}, \nabla)/R[\![t_1, \ldots, t_d]\!]$ be a finite rank (T)-structure. Let $v \in \mathcal{H}/(t_1, \ldots, t_d, u)\mathcal{H}$.*

1. If v satisfies (IC), (GC) and $\operatorname{coker} \mu_v$ *is free, then there exists a maximal unfolding with cyclic vector induced from v.*

2. If v satisfies (GC'), then any two maximal unfoldings of $(\mathcal{H}, \nabla)$ with cyclic vector induced from v are isomorphic under a unique isomorphism.

Proof. Let n denote the rank of $\mathcal{H}$. Write $t = \{t_1, \ldots, t_d\}$ and $s = \{s_1, \ldots, s_{n-d}\}$. We choose an F-bundle structure $(\mathcal{H}, \nabla^F)/R[\![t]\!]$ lifting the (T)-structure $(\mathcal{H}, \nabla)$. Then $(\mathcal{H}, \nabla^F)$ satisfies the conditions of Theorem 5.3.28(1), producing a maximal unfolding of F-bundle. Since being maximal is a property of the (T)-structure, the unfolding of the underlying (T)-structure is maximal, proving (1).

For (2), let $\iota_1 \colon (\mathcal{H}, \nabla)/R[\![t]\!] \to (\mathcal{H}_1, \nabla_1)/R[\![t, s]\!]$ and $\iota_2 \colon (\mathcal{H}, \nabla)/R[\![t]\!] \to (\mathcal{H}_2, \nabla_2)/R[\![t, s]\!]$ be two maximal unfoldings of (T)-structures, with cyclic vector induced from v. Since the base of $(\mathcal{H}, \nabla)$ has finitely many variables, it follows from Proposition 5.3.4 that it admits a framing. This induces a framing on any unfolding by Remark 5.3.9. Thus, we can apply Lemma 5.3.10 and extend the two unfoldings ι_1 and ι_2 uniquely to maximal unfoldings of the F-bundle $(\mathcal{H}, \nabla^F)$. We conclude from Theorem 5.3.28 that they are isomorphic under a unique isomorphism, and hence the same holds for the underlying unfoldings of (T)-structures. This concludes the proof. $\square$

Remark 5.3.31 (Existence when R is not a field)**.** Let R be an integral domain, $(\mathcal{H}, \nabla)/R[\![t_1, \ldots, t_d]\!]$ be a finite rank F-bundle, and $v \in H := \mathcal{H}/(t_1, \ldots, t_d, u)\mathcal{H}$.

1. If v only satisfies (IC) and (GC'), we know that a maximal unfolding exists after base change to $\operatorname{Frac}(R)$. In fact, the maximal unfolding is defined over any localization R' of R such that $\operatorname{coker} \mu_v \otimes_R R'$ is a free module, by Theorem 5.3.28(1).

2. Let $(\mathcal{H}, \nabla) \to (\mathcal{H}', \nabla')$ be an unfolding. We obtain maps μ and μ' as in (5.3.12). Let $\mathcal{A} := R[\operatorname{im} \mu]$ and $\mathcal{A}' := R[\operatorname{im} \mu']$ denote the associated commuting subalgebras of $\operatorname{End}_R(H)$. We have $\mathcal{A} \subset \mathcal{A}' \subset \mathcal{C}(\mathcal{A}') \subset \mathcal{C}(\mathcal{A})$, where $\mathcal{C}(\cdot)$ denotes the commutant algebra. Let $\widetilde{\mu}_v \colon \mathcal{A} \to H$ and $\widetilde{\mu}'_v \colon \mathcal{A}' \to H$ denote the evaluation on v. From the

commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d, "\widetilde{\mu}_v"] & \mathcal{A}' \arrow[r] \arrow[d, "\widetilde{\mu}'_v"] & \mathcal{A}'/\mathcal{A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r, "\mathrm{id}"] & H \arrow[r] & 0, &
\end{tikzcd}$$

we obtain the long exact sequence

$$0 \longrightarrow \ker \widetilde{\mu}_v \longrightarrow \ker \widetilde{\mu}'_v \longrightarrow \mathcal{A}'/\mathcal{A} \longrightarrow \operatorname{coker} \widetilde{\mu}_v \longrightarrow \operatorname{coker} \widetilde{\mu}'_v \longrightarrow 0.$$

If the unfolding is maximal, we have $\mathcal{A}' = \operatorname{im} \mu'$ and $\widetilde{\mu}'_v$ is an isomorphism. We deduce that $\ker \widetilde{\mu}_v = 0$ and $\operatorname{coker} \widetilde{\mu}_v \simeq \mathcal{A}'/\mathcal{A}$. Then, v satisfies the (IC) condition but not necessarily the (GC) condition. In the special case when $\mathcal{A} = \mathcal{C}(\mathcal{A})$, a maximal unfolding exists if and only if v satisfies (IC) and (GC).

This is illustrated in Example 5.3.32.

Example 5.3.32. Let $R = \Bbbk[\![\lambda_1, \lambda_2]\!]$, $H = R^{\oplus 3}$ and $\mathcal{H} = H \otimes_R R[\![t_1, t_2]\!]$. Let (e_1, e_2, e_3) denote the canonical basis of H. We consider the matrices

$$A = \mathrm{Id}_3,\ B = \begin{pmatrix} 0 & 0 & 1 \\ \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \end{pmatrix},\ C = B^2 = \begin{pmatrix} 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \\ \lambda_1\lambda_2 & 0 & 0 \end{pmatrix}.$$

Assume ∇ is an F-bundle connection on $\mathcal{H}$ such that $\mu(\partial_{t_1}) = A$ and $\mu(\partial_{t_2}) = B$. We have $R[\operatorname{im} \mu] = RA \oplus RB \oplus RC$ and $R[\operatorname{im} \mu] = \mathcal{C}(R[\operatorname{im} \mu])$. It follows from Remark 5.3.31(2) that there exists a maximal unfolding with cyclic vector $v = \alpha e_1 + \beta e_2 + \gamma e_3$ if and only if v satisfies (IC) and (GC). The matrix of the evaluation map $\widetilde{\mu}_v \colon R[\operatorname{im} \mu] \to H$ with respect to the bases (A, B, C) and (e_1, e_2, e_3) is

$$\widetilde{\mu}_v = \begin{pmatrix} \alpha & \gamma & \lambda_2\beta \\ \beta & \lambda_1\alpha & \lambda_1\gamma \\ \gamma & \lambda_2\beta & \lambda_1\lambda_2\alpha \end{pmatrix},$$

whose determinant is $\lambda_1^2\lambda_2\alpha^3 + \lambda_2^2\beta^3 + \lambda_1\gamma^3 - 3\lambda_1\lambda_2\alpha\beta\gamma$. The vector v satisfies (IC) and (GC) if and only if this determinant is invertible. For $v = e_3$, this determinant is λ_1 and we conclude that the associated maximal unfolding is defined over $\Bbbk[\![\lambda_1, \lambda_2]\!][\lambda_1^{-1}]$. For $v = e_2$, this determinant is λ_2^2 and the associated maximal unfolding is defined over $\Bbbk[\![\lambda_1, \lambda_2]\!][(\lambda_2^2)^{-1}]$.

5.3.3 Unfolding theorem for equivariant F-bundles

In this subsection, we prove the unfolding theorem for equivariant F-bundles. The strategy is to unfold the R-linear (T)-structure using Corollary 5.3.30, and then extend it in the u-direction using Lemma 5.3.1.

Definition 5.3.33. Let R be a $\mathbb{k}$-algebra, and let I be a countable set.

1. An *unfolding of* $\mathbb{k}$*-linear equivariant F-bundle* $\{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}/\mathbb{k}[\![\mathbf{t}_I]\!]$ is a morphism of equivariant F-bundles (ι, ι_R) such that ι is an unfolding of $\mathbb{k}$-linear F-bundles and ι_R is an unfolding of R-linear (T)-structure. In particular, ι and ι_R are compatible with the R-linear lifts as in (5.2.11).

2. A *morphism of unfoldings* is a morphism (β, β_R) of equivariant F-bundles such that both β and β_R are morphisms of unfoldings. In particular, (β, β_R) commutes with the unfolding maps.

3. An equivariant F-bundle is *maximal* if the underlying R-linear (T)-structure is maximal.

Lemma 5.3.34. *Let I be a countable set. Let $(\mathcal{H}, \nabla)/R[\![t_I]\!]$ be an F-bundle. A framing for the (T)-structure $(\mathcal{H}, \nabla)_0$ is a framing for the F-bundle if and only if it restricts to a framing of F-bundles at $t_I = 0$.*

Proof. The framing provides a trivialization $\mathcal{H} \simeq H \otimes_R R[\![t_I, u]\!]$. Write $\nabla_{\partial_{t_i}} = \partial_{t_i} + u^{-1}T_i(t)$ and $\nabla_{\partial_u} = \partial_u + u^{-2}U(t, u)$. By Lemma 5.3.1(1), $U(t, u)$ is uniquely determined by the system of differential equations (5.3.2) and the initial condition $U(0, u)$. Write $U(t, u) = \sum_{k \geq 0} U_{k-2}(t)u^k$. The differential equation implies for all $k \geq 0$

$$\frac{\partial U_k}{\partial t_i} = -[T_i, U_{k+1}].$$

Since we have the initial condition $U_k(0) = 0$, we deduce that $U_k(t) = 0$ for all $k \geq 0$ by applying [Hin+24, Lemma 4.8(1)] inductively on the number of variables. The reverse direction is obvious. □

Proposition 5.3.35. *Let I and J be finite sets, and R be a $\mathbb{k}$-algebra without zero divisors equipped with a fixed basis. Let $\mathcal{F} \to \mathcal{F}'$ be an unfolding of $\mathbb{k}$-linear equivariant F-bundles. Then any framing on $\mathcal{F}$ extends uniquely to a framing on $\mathcal{F}'$.*

Proof. Uniqueness follows from the uniqueness of extension of framing for $(\mathcal{H}_R, \nabla_R)$, together with Remark 5.2.14.

We now prove the existence part. Assume $\mathcal{F}$ admits a framing and

$$(\beta, \beta_R)\colon \mathcal{F} = \{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}/\Bbbk[\![t_I]\!] \longrightarrow \mathcal{F}' = \{(\mathcal{H}', \nabla'), (\mathcal{H}'_R, \nabla'_R), \alpha'\}/\Bbbk[\![\mathbf{t}_J]\!]$$

is an unfolding. By Remark 5.3.9, the framing for $(\mathcal{H}_R, \nabla_R)$ produces a unique framing on $(\mathcal{H}'_R, \nabla'_R)$. By Lemma 5.2.6(2), this framing induces a framing on $(\widetilde{\mathcal{H}}'_R, \widetilde{\nabla}'_R)$, thus a framing on the (T)-structure $(\mathcal{H}', \nabla')_0$ under α'. By construction, under $\beta|_{\mathbf{t}_I=0}$, the framing constructed on $(\mathcal{H}', \nabla')$ coincides with the initial framing of $(\mathcal{H}, \nabla)$. We conclude from Lemma 5.3.34 that it is a framing of F-bundle. This concludes the proof. □

Theorem 5.3.36 (Unfolding of equivariant F-bundles)**.** *Let $\mathcal{F} = \{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\}$ be an equivariant F-bundle over $\Bbbk[\![\mathbf{t}_I]\!]$, and fix $v \in \mathcal{H}_R/(t_I, u)\mathcal{H}_R$.*

1. *If v satisfies (IC), (GC) and* coker μ_v *is free, then $\mathcal{F}$ admits a maximal unfolding with cyclic vector induced from v.*

2. *If v satisfies (GC'), then any two maximal unfoldings of $\mathcal{F}$ with cyclic vector induced from v are isomorphic under a unique isomorphism.*

Furthermore, any framing for $\mathcal{F}$ induces a unique framing on a maximal unfolding.

Proof. We prove (1). For the R-linear (T)-structures, there exists a maximal unfolding by Corollary 5.3.30:

$$\beta_R\colon (\mathcal{H}_R, \nabla_R)/R[\![t_I]\!] \longrightarrow (\mathcal{H}'_R, \nabla'_R)/R[\![t_J]\!].$$

By functoriality, we obtain an unfolding of $\Bbbk$-linear (T)-structures:

$$\widetilde{\beta}_R \circ \alpha\colon (\mathcal{H}, \nabla)_0/\Bbbk[\![\mathbf{t}_I]\!] \longrightarrow (\widetilde{\mathcal{H}}'_R, \widetilde{\nabla}'_R)/\Bbbk[\![\mathbf{t}_J]\!].$$

By Proposition 5.3.4, the R-linear (T)-structures admit framings. Those framings induce framings on the $\Bbbk$-linear (T)-structures by Lemma 5.2.6(2). Hence, we can apply Lemma 5.3.10 to define an F-bundle structure $(\widetilde{\mathcal{H}}'_R, \widetilde{\nabla}'^F_R)$ extending the $\Bbbk$-linear (T)-structure $(\widetilde{\mathcal{H}}'_R, \widetilde{\nabla}'_R)$, such that $\widetilde{\beta}_R \circ \alpha$ becomes an unfolding of F-bundles. Then $\{(\widetilde{\mathcal{H}}'_R, \widetilde{\nabla}'^F_R), (\mathcal{H}'_R, \nabla'_R), \mathrm{id}\}$ is an equivariant F-bundle and $(\widetilde{\beta}_R \circ \alpha, \beta_R)$ is a maximal unfolding of $\mathcal{F}$ with cyclic vector v.

We now prove (2). For $k = 1, 2$, let

$$(\beta_k, \beta_{R,k})\colon \{(\mathcal{H}, \nabla), (\mathcal{H}_R, \nabla_R), \alpha\} \to \{(\mathcal{H}_k, \nabla_k), (\mathcal{H}_{R,k}, \nabla_{R,k}), \alpha_k\}$$

be two maximal unfoldings of equivariant F-bundles, with cyclic vectors $v_k \in \mathcal{H}_{R,k}/(t_J, u)\mathcal{H}_{R,k}$ induced from v. By Corollary 5.3.30, there exists a unique isomorphism of R-linear (T)-structures

$$iso_R\colon (\mathcal{H}_{R,1}, \nabla_{R,1}) \to (\mathcal{H}_{R,2}, \nabla_{R,2})$$

such that $\beta_{R,2} = iso_R \circ \beta_{R,1}$. This induces an isomorphism for the underlying $\mathbb{k}$-linear (T)-structures

$$iso := \alpha_2^{-1} \circ \widetilde{iso_R} \circ \alpha_1\colon (\mathcal{H}_1, \nabla_1) \to (\mathcal{H}_2, \nabla_2),$$

and it satisfies $\beta_2 = \beta_1 \circ iso$. It suffices to show that iso is compatible with the u-direction. Since β_k are unfoldings of F-bundles, they restrict to isomorphisms of F-bundles at $\mathbf{t}_J = 0$. Hence, iso is compatible with the u-direction at $\mathbf{t}_J = 0$. Since the $\mathbb{k}$-linear (T)-structures come from finite R-linear (T)-structures, they admit framings. Then Lemma 5.3.1(2) implies that iso is an isomorphism of F-bundles. We conclude that (iso, iso_R) is an isomorphism of equivariant F-bundles compatible with the unfoldings. This isomorphism is unique, since (iso, iso_R) is uniquely determined by iso_R. (2) is proved.

The last statement is a special case of Proposition 5.3.35. The theorem is proved. $\square$

5.4 Application to mirror symmetry of flag varieties

In this section, we apply our equivariant unfolding theorem to obtain the big D-module mirror symmetry for flag varieties G/P of general Lie type (Theorem 5.4.35).

We start with the $\mathbb{k}$-linear F-bundles given by the equivariant small quantum D-module for G/P on the A-side, and another one by the equivariant Gauss-Manin system with respect to Rietsch's superpotential on the B-side (see [Rie08]). Note that both F-bundles are of infinite rank, as the equivariant parameters are not yet included in the base ring. Moreover, their R-linear (T)-structure lifts coincide with the D-module structures defined in [Cho23], and are thus isomorphic to each other as shown therein. We will construct a suitable unfolding on the B-side, and apply our equivariant unfolding theorem to deduce the isomorphism between the unfoldings on both sides. We remark that in general, the classical cohomology of G/P is not generated by the divisor classes and the small quantum cohomology is not semisimple, so that neither the unfolding in [HM04] nor the semisimple reconstruction in [Tel12] is directly applicable.

5.4.1 Equivariant F-bundles for G/P

5.4.1.1 Equivariant quantum cohomology ring of G/P

Let G be a simply-connected complex simple Lie group, and P be a parabolic subgroup of G containing a Borel subgroup $B \subset G$. Let B_- denote the opposite Borel subgroup, and then $T := B \cap B_-$ is a maximal torus of G. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a basis of simple roots, and $\{\omega_1, \cdots, \omega_n\}$ be the fundamental weights. The Weyl group $W := N_G(T)/T$ is generated by simple reflections $s_i := s_{\alpha_i}$. The Weyl subgroup W_P of P is generated by the simple reflections s_α with $\alpha \in \Delta_P := \{\alpha_i \in \Delta \mid s_i P \subset P\}$. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ denote the standard length function, and w_0 (resp. w_P) denote the longest element in W (resp. W_P). Denote by $W^P \subset W$ the subset of minimal length representative of the cosets W/W_P.

The flag variety $X := G/P$ is a Fano manifold. It parametrizes partial flags (resp. isotropic partial flags) in a complex vector space when G is of type A (resp. B, C, D). For each $w \in W^P$, there are Schubert varieties $X^w := \overline{BwP/P}$ (resp. $X_w := \overline{B_- wP/P}$) of (co)dimension $\ell(w)$ inside X. We have

$$H^*(X, \mathbb{Z}) = \bigoplus_{w \in W^P} \mathbb{Z}\,\mathrm{PD}([X_w]),$$

where $\mathrm{PD}(\cdot)$ denotes the Poincaré dual, and

$$H_2(X, \mathbb{Z}) = \bigoplus_{\alpha \in \Delta \setminus \Delta_P} \mathbb{Z}[X^{s_\alpha}].$$

For each $w \in W^P$, the Schubert variety X_w (resp. X^w) is invariant under the natural T-action on X, so that it defines a fundamental class in the T-equivariant Borel-Moore homology. This class is identified with a T-equivariant cohomology class in $H_T^{2\ell(w)}(X, \mathbb{C})$ (resp. $H_T^{2(\dim X - \ell(w))}(X, \mathbb{C})$) denoted as σ_w (resp. σ^w). The fundamental weights produce equivariant parameters for the T-action which we denote by $\lambda = (\lambda_1, \ldots, \lambda_n)$. We have identifications

$$H_T^*(\mathrm{pt}, \mathbb{C}) = \mathbb{C}[\lambda_1, \ldots, \lambda_n] =: \mathbb{C}[\lambda], \tag{5.4.1}$$

$$H_T^*(X, \mathbb{C}) = \bigoplus_{w \in W^P} \mathbb{C}[\lambda]\sigma_w. \tag{5.4.2}$$

To be more precise, we view ω_i as a character in $\mathrm{Hom}(T, \mathbb{C}^*)$, and denote by $\mathbb{C}_{-\omega_i}$ the one-dimensional representation of T viewed a vector bundle over a point. Then we take $\lambda_i := c_1^T(\mathbb{C}_{-\omega_i})$ and consequently we have $\lambda_i = -\omega_i$. We denote by $(\cdot, \cdot)$ the equivariant Poincaré pairing on $H_T^*(X, \mathbb{C})$. The $\mathbb{C}[\lambda]$-bases $\{\sigma^w\}_w$ and $\{\sigma_w\}_w$ are dual with respect to the Poincaré pairing, i.e. $(\sigma_u, \sigma^v) = \delta_{u,v}$. In the following, we denote by $\mathbb{C}(\lambda)$ the fraction field of $\mathbb{C}[\lambda] = \mathbb{C}[\lambda_1, \ldots, \lambda_n]$.

Lemma 5.4.3 ([Buc+18, Lemma 5.11])**.** *The localized equivariant cohomology of X, $H_T^*(X) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}(\lambda)$ is generated by the element $\sum_{\alpha \in \Delta \setminus \Delta_P} \sigma_{s_\alpha}$ as a $\mathbb{C}(\lambda)$-algebra.*

Remark 5.4.4. The above lemma shows that $H_T^*(X, \mathbb{C})$ is generated by $H_T^2(X, \mathbb{C})$ after localization. This also follows from [CKS08, Lemma 4.1.3], and can be generalized to any smooth projective variety admitting a torus action with finitely many attractive torus-fixed points by [ACT22, Lemma 1].

Let $\overline{\mathcal{M}}_{0,m}(X, d)$ denote the moduli space of m-pointed stable maps to X of genus zero and degree $d \in H_2(X, \mathbb{Z})$, and $\mathrm{ev}_i \colon \overline{\mathcal{M}}_{0,m}(X, d) \to X$ denote the i-th T-equivariant evaluation map. The moduli space $\overline{\mathcal{M}}_{0,m}(X, d)$ carries a T-action, and has a T-equivariant virtual fundamental class $[\overline{\mathcal{M}}_{0,m}(X, d)]^{\mathrm{vir}}$. For $\gamma_1, \ldots, \gamma_m \in H_T^*(X, \mathbb{C})$, we have the genus-zero, m-point equivariant Gromov-Witten invariant

$$\langle \gamma_1, \ldots, \gamma_m \rangle_d := \int_{[\overline{\mathcal{M}}_{0,m}(X,d)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_m^*(\gamma_m) \in \mathbb{C}[\lambda]. \tag{5.4.5}$$

We introduce the necessary choices of bases, and associated coordinates, in order to define the equivariant big quantum cohomology ring of X. Write $\Delta \setminus \Delta_P = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$ and $W^P = \{v_1, \cdots, v_N\}$ with $v_j = s_{i_j}$ for $1 \le j \le r$. We introduce Novikov variables $q = (q_1, \ldots, q_r)$ corresponding to the basis $\{[X^{s_\alpha}] \mid \alpha \in \Delta \setminus \Delta_P\}$ of $H_2(X, \mathbb{Z})$. For $d \in H_2(X, \mathbb{Z})$, we have $d = \sum_j d_j [X^{s_{i_j}}]$ and denote $q^d := \prod_{j=1}^r q_j^{d_j}$. We use $\{\tau_i\}$ for the $\mathbb{C}[\lambda]$-linear coordinates of $H_T^*(X)$, whose elements are of the form $\alpha = \sum_{i=1}^N \tau_i \sigma_{v_i}$.

As a module, the equivariant big quantum cohomology ring is

$$\mathrm{QH}_T^{*,\mathrm{big}}(X) := H_T^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[q][\![\tau]\!].$$

It encodes all genus zero Gromov-Witten invariants in the quantum product $\star_\tau^{\mathrm{big}}$, defined by

$$\sigma_v \star_\tau^{\mathrm{big}} \sigma_w = \sum_{\eta \in W^P} \sum_{m \ge 0} \sum_{i_1, \ldots, i_m} \sum_{d \in H_2(X, \mathbb{Z})} \frac{\tau_{i_1} \cdots \tau_{i_m}}{m!} \left\langle \sigma_v, \sigma_w, \sigma^\eta, \sigma_{v_{i_1}}, \cdots, \sigma_{v_{i_m}} \right\rangle_d q^d \sigma_\eta.$$

Here the coefficient of $\tau_{i_1} \cdots \tau_{i_m}$ is indeed a polynomial in q since X is Fano.

Denote $\tilde{q}_j := q_j e^{\tau_j}$ and $\tilde{q}^d := \prod_j \tilde{q}_j^{d_j}$. Letting $\tau_i = 0$ for all $i > r$ and using the divisor axiom for Gromov-Witten invariants, we obtain the equivariant small quantum cohomology ring

$$\mathrm{QH}_T^*(X) = H_T^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\tilde{q}] \quad \text{with} \quad \sigma_v \star \sigma_w = \sum\nolimits_{\eta, d} \langle \sigma_v, \sigma_w, \sigma^\eta \rangle_d \, \tilde{q}^d \sigma_\eta.$$

The next lemma follows directly from Lemma 5.4.3 and [ST97, Lemma 2.1].

Lemma 5.4.6. *The localized equivariant small quantum cohomology of X, $\mathrm{QH}_T^*(X) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}(\lambda)$ is generated by $\{\sigma_{s_\alpha} \mid \alpha \in \Delta \setminus \Delta_P\}$ as a $\mathbb{C}(\lambda)[\tilde{q}]$-algebra.*

Remark 5.4.7. By further taking the nonequivariant limit $\lambda = 0$, we obtain the small quantum cohomology $\mathrm{QH}^*(X)$, which could be non-semisimple. For instance for G of type C_n and $\Delta_P = \Delta \setminus \{\alpha_2\}$, we obtain the isotropic Grassmannian $\mathrm{SG}(2, 2n) = \{V \leq \mathbb{C}^{2n} \mid \dim V = 2, \Omega(V, V) = 0\}$, where Ω is a symplectic form on $\mathbb{C}^{2n}$. It is shown in [CP11] that $\mathrm{QH}^*(\mathrm{SG}(2, 2n))$ is not semisimple. It is easy to see that $\mathrm{QH}^*(\mathrm{SG}(2, 2n))$ is not generated by $H^2(\mathrm{SG}(2, 2n), \mathbb{C})$ either.

5.4.1.2 Equivariant F-bundle structures for G/P

We recall that $\tau = (\tau_1, \cdots, \tau_N)$ are the $\mathbb{C}[\lambda]$ coordinates of $H_T^*(X)$ dual to the standard basis we chose, $q = (q_1, \cdots, q_r)$ are the Novikov variables and $\lambda = (\lambda_1, \cdots, \lambda_n)$ are the equivariant variables. For $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, we set $\lambda^k := \prod_{i=1}^n \lambda_i^{k_i}$ and $|k| := \sum_{i=1}^n k_i$. It is expected but remains unsolved in general that the big quantum cohomology is convergent around $\tau = 0$. Therefore we work on the formal neighborhood of $\tau = 0$.

Let $\Bbbk := \mathbb{C}(q)$ be the fraction field of $\mathbb{C}[q]$, and let $R := \Bbbk[\lambda]$. We fix the $\Bbbk$-basis $\boldsymbol{\lambda} = (\lambda^k, k \in \mathbb{N}^n)$ of R. We obtain $\Bbbk$-linear coordinates $\boldsymbol{\tau} = \{\tau_{i,k}, 1 \leq i \leq N, k \in \mathbb{N}^n\}$ on $H_T^*(X, \Bbbk)$ associated to the $\Bbbk$-basis $(\sigma_{v_i}\lambda^k, 1 \leq i \leq N, k \in \mathbb{N}^n)$. There is a continuous morphism of R-algebras:

$$\psi_{\boldsymbol{\lambda}} : R[\![\tau]\!] \to R[\![\boldsymbol{\tau}]\!], \tau_i \mapsto \sum_{k \in \mathbb{N}^n} \lambda^k \tau_{i,k}.$$

We define a $\Bbbk$-linear equivariant F-bundle equivariant F-bundle

$$\mathcal{F}^{A,\mathrm{big}} := \{(\mathcal{H}^{A,\mathrm{big}}, \nabla^{A,\mathrm{big}}), (\mathcal{H}_R^{A,\mathrm{big}}, \nabla_R^{A,\mathrm{big}}), \alpha\} / \Bbbk[\![\boldsymbol{\tau}]\!]$$

associated to the equivariant big quantum cohomology as follows. The R-linear (T)-structure $(\mathcal{H}_R^{A,\mathrm{big}}, \nabla_R^{A,\mathrm{big}})$ is given by the $R[\![\tau, u]\!]$-module

$$\begin{aligned} \mathcal{H}_R^{A,\mathrm{big}} &= H_T^*(X, \Bbbk) \otimes_R R[\![\tau, u]\!], \\ \nabla_{R,\partial_{\tau_j}}^{A,\mathrm{big}} &= \partial_{\tau_j} + u^{-1}\Big((\sigma_{v_j} + \lambda_{i_j}) \star_\tau^{\mathrm{big}}\Big), \end{aligned}$$

where $1 \leq j \leq N$ and we set $\lambda_{i_j} = 0$ for $j > r$. Here, i_j are the indices of $\Delta \setminus \Delta_P = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$. The $\Bbbk$-linear F-bundle $(\mathcal{H}^{A,\mathrm{big}}, \nabla^{A,\mathrm{big}})$ has underlying $\Bbbk[\![\boldsymbol{\tau}, u]\!]$-module

$$\mathcal{H}^{A,\mathrm{big}} = H_T^*(X, \Bbbk) \otimes_\Bbbk \Bbbk[\![\boldsymbol{\tau}, u]\!],$$

and the connection $\nabla^{A,\mathrm{big}}$ is specified by:

$$\nabla^{A,\mathrm{big}}_{\partial_{\tau_{j,k}}} = \partial_{\tau_{j,k}} + u^{-1}\Big(\lambda^k(\sigma_{v_j} + \lambda_{i_j}) \star^{\mathrm{big}}_{\tau}\Big),$$
$$\nabla^{A,\mathrm{big}}_{u\partial_u} = \mathrm{Gr}^{A,\mathrm{big}} - \nabla^{A,\mathrm{big}}_{E^{\mathrm{big}}_A}.$$

Here,

$$\mathrm{Gr}^{A,\mathrm{big}} = u\partial_u + E^{\mathrm{big}}_A + \mu_A,$$

where μ_A is the $\Bbbk[\![\boldsymbol{\tau}, y]\!]$-linear grading operator on the fiber $H^*_T(X, \Bbbk)$ linear defined by

$$\mu_A(\lambda^k \sigma_{v_i}) = \Big(\ell(v_i) + |k|\Big)\lambda^k \sigma_{v_i},$$

and E^{big}_A is the Euler vector field measuring degree on the base $\Bbbk[\![\boldsymbol{\tau}]\!]$, given by

$$E^{\mathrm{big}}_A = \sum_{1\leq j\leq r} \frac{\deg(q_j)}{2}\partial_{\tau_{j,0}} + \sum_{\substack{1\leq j\leq N\\ k\in\mathbb{N}^n}} (1 - \ell(v_j) - |k|)\tau_{j,k}\partial_{\tau_{j,k}},$$

where $\ell(v_j) = 1$ for $1 \leq j \leq r$ and the degree $\deg(q_j)$ is defined as:

$$\deg(q_j) := 2\int_{[X^{s_{i_j}}]} c_1(T_{G/P}).$$

Under the change of variables $R[\![\tau]\!] \to R[\![\boldsymbol{\tau}]\!]$, $\tau_i \mapsto \sum_{k\in\mathbb{N}^n} \lambda^k \tau_{i,k}$, the data $\{(\mathcal{H}^{A,\mathrm{big}}_R, \nabla^{A,\mathrm{big}}_R), \mathrm{id}\}$ provides an R-linear lift of the underlying (T)-structure $(\mathcal{H}^{A,\mathrm{big}}, \nabla^{A,\mathrm{big}})_0$. We obtain the A-model big equivariant F-bundle:

$$\mathcal{F}^{A,\mathrm{big}} = \{(\mathcal{H}^{A,\mathrm{big}}, \nabla^{A,\mathrm{big}}), (\mathcal{H}^{A,\mathrm{big}}_R, \nabla^{A,\mathrm{big}}_R), \mathrm{id}\}/\Bbbk[\![\boldsymbol{\tau}]\!].$$

Remark 5.4.8. For $1 \leq j \leq r$, consider the line bundle $L_j = G \times_P \mathbb{C}_{-\omega_{i_j}}$ over G/P. Since $c^T_1(L_j) = \sigma_{s_{i_j}} - \omega_{i_j} = \sigma_{v_j} + \lambda_{i_j}$, we can write $\nabla^{A,\mathrm{big}}_{R,\partial_{\tau_j}} = \partial_{\tau_j} + u^{-1}c^T_1(L_j)\star^{\mathrm{big}}_{\tau}$. For $j > r$, $\nabla^{A,\mathrm{big}}_{R,\partial_{\tau_j}}$ are not weighted.

Remark 5.4.9. The flatness of $\nabla^{A,\mathrm{big}}$ in the u-direction follows from a similar argument to that in [Coa+20, Section 3.2].

By restricting to the small locus $\tau_j = 0$ for $(\mathcal{H}^{A,\mathrm{big}}, \nabla^{A,\mathrm{big}})$, and $\tau_{j,k} = 0$ for $(\mathcal{H}^{A,\mathrm{big}}_R, \nabla^{A,\mathrm{big}}_R)$ when $j > r$, we obtain the A-model small equivariant F-bundle

$$\mathcal{F}^A := \{(\mathcal{H}^A, \nabla^A), (\mathcal{H}^A_R, \nabla^A_R), \mathrm{id}\}/\Bbbk[\![\boldsymbol{\tau}_{\leq r}]\!],$$

where $\boldsymbol{\tau}_{\leq r} = \{\tau_{i,k}, 1 \leq i \leq r, k \in \mathbb{N}^n\}$ parametrizes the $\Bbbk$-linear F-bundle, and $\tau_{\leq r} = \{\tau_i, 1 \leq i \leq r\}$ parametrizes the R-linear (T)-structure.

The quotient maps $\Bbbk[\![\boldsymbol{\tau}]\!] \to \Bbbk[\![\boldsymbol{\tau}_{\leq r}]\!]$ and $R[\![\tau]\!] \to R[\![\tau_{\leq r}]\!]$ together with the natural identification of the fibers produce an unfolding of the equivariant F-bundle $\iota\colon \mathcal{F}^A \to \mathcal{F}^{A,\mathrm{big}}$.

Proposition 5.4.10. *The morphism $\iota\colon \mathcal{F}^A \to \mathcal{F}^{A,\mathrm{big}}$ is a maximal unfolding of $\Bbbk$-linear equivariant F-bundles, with cyclic vector given by $1 \in H_T^*(X,\Bbbk)$.*

Proof. We have already proven the morphism $\iota : \mathcal{H}^A \to \mathcal{F}^{A,\mathrm{big}}$ is an unfolding, and it remains to check that $(\mathcal{H}_R^A, \nabla_R^A)$ is maximal. We take the cyclic vector $v = 1 \in \mathcal{H}_R^{A,\mathrm{big}}|_{\tau=u=0} = H_T^*(X,\Bbbk)$. The R-linear evaluation map is:

$$\mu_{v=1}\colon \bigoplus_{j=1}^N R\partial_{\tau_j} \longrightarrow H_T^*(X,\Bbbk)$$
$$\partial_{\tau_j} \longmapsto \nabla_{u\partial_{\tau_j}}|_{\tau=u=0}(1) = \sigma_{v_j} + \lambda_{i_j}.$$

Since $\{\sigma_{v_i}\}_i$ is an $R = \Bbbk[\lambda]$ basis of $H_T^*(X,\Bbbk)$ by equation (5.4.2), $\mu_{v=1}$ is an R-isomorphism and we conclude that ι is maximal unfolding. □

5.4.2 The small D-module mirror symmetry for G/P

In this section, we review the B-side of mirror symmetry for for G/P as in [Rie08], construct a R-linear (T)-structure $(\mathcal{H}_R^B, \nabla_R^B)$ from the Gauss-Manin connection, and state the small mirror symmetry as in [Cho23].

5.4.2.1 Small D-module mirror symmetry

Let $G^\vee$ be the Langlands dual group of G, and $T^\vee$, $B^\vee$, $P^\vee$ be the Langlands dual of T, B, P respectively. Rietsch's equivariant mirror superpotential is a triple $(X_P^\vee, \mathcal{W}, p)$. Here $X_P^\vee$ is a subvariety of $G^\vee \times Z$ isomorphic to

$$\Big((G^\vee/P^\vee) \setminus -K_{G^\vee/P^\vee}\Big) \times \text{Spec } \mathbb{C}[\tilde{q}_i^{\pm 1} | \alpha_i \in \Delta\backslash\Delta_P],$$

where Z is the center of the Levi subgroup of $P^\vee$ and $-K_{G^\vee/P^\vee}$ is the anti-canonical divisor of the dual partial flag variety $G^\vee/P^\vee$ given in [KLS14]. The holomorphic function $\mathcal{W} : X_P^\vee \to \mathbb{C}$ is the non-equivariant mirror superpotential of G/P, and $p : X_P^\vee \to T^\vee$ is a morphism which gives information on the equivariant part of Rietsch's original mirror superpotential $\mathcal{W} + \ln \phi(;h)$ (see [Rie08]).

Denote by $\Omega^i(X_P^\vee/Z)$ the space of holomorphic i-forms over $X_P^\vee$ with respect to

$$Z \cong \text{Spec } \mathbb{C}\Big[\tilde{q}_i^{\pm 1} \,\Big|\, \alpha_i \in \Delta\backslash\Delta_P\Big]$$

via the aforementioned isomorphism. Identify the Lie algebra $\mathfrak{t}^\vee = \mathrm{Lie}(T^\vee)$ with $\mathfrak{t}^*$. Let $\{(\lambda_i)^*\} \subset (\mathfrak{t}^\vee)^*$ be the dual base of $\{\lambda_i\} \subset \mathfrak{t}^\vee$, and $\mathrm{mc}_{T^\vee} \in \Omega^1(T^\vee; \mathfrak{t}^\vee)$ denote the Maurer-Cartan form of $T^\vee$.

In [Cho23], the B-model D-module $\Big(G_0(X_P^\vee, \mathcal{W}, p), \nabla\Big)$ consists of a $\mathbb{C}[\lambda, u][\tilde{q}_i^{\pm 1}]$-module defined by:

$$G_0(X_P^\vee, \mathcal{W}, p) = \operatorname{coker}\Big(\mathbb{C}[\lambda, u] \otimes_{\mathbb{C}} \Omega^{top-1}(X_P^\vee/Z) \xrightarrow{\partial} \mathbb{C}[\lambda, u] \otimes_{\mathbb{C}} \Omega^{\mathrm{top}}(X_P^\vee/Z)\Big),$$
$$\partial = 1 \otimes \left(ud + d\mathcal{W} \wedge - \sum_{j=1}^{n} \lambda_j (p^*\langle(\lambda_j)^*, \mathrm{mc}_{T^\vee}\rangle) \wedge \right).$$

It is equipped with a meromorphic connection having a logarithmic pole in the $\tilde{q}_i$-direction:

$$\nabla_{\partial_{\tilde{q}_i}}([\omega]) = \Big[\mathcal{L}_{\partial_{\tilde{q}_i}}(\omega) + u^{-1}\frac{\partial \mathcal{W}}{\partial \tilde{q}_i}\omega - u^{-1}\sum_{j=1}^{n} \lambda_j\Big(\iota_{\partial_{\tilde{q}_i}} p^*\Big\langle(\lambda_j)^*, \mathrm{mc}_{T^\vee}\Big\rangle\Big)\omega\Big].$$

Here, $p^*\Big\langle(\lambda_j)^*, \mathrm{mc}_{T^\vee}\Big\rangle \in \Omega^1(X_P^\vee/Z)$ and ∂, ∇ are linear on $\mathbb{C}[\lambda, u]$.

Remark 5.4.11. For any $\omega \in \mathbb{C}[\lambda, u] \otimes \Omega^{\mathrm{top}}(X_P^\vee/Z)$, $\omega = g\omega_0$ for some $g \in \mathcal{O}(X_P^\vee)$ and $\omega_0 \in \Omega^{\mathrm{top}}((G^\vee/P^\vee) \setminus -K_{G^\vee/P^\vee})$. We have the Lie derivative $\mathcal{L}_{\partial_{\tilde{q}_i}}(\omega) = \frac{\partial g}{\partial \tilde{q}_i}\omega_0$.

The small quantum D-module mirror symmetry holds for G/P in the following sense.

Proposition 5.4.12 ([Cho23, Theorem 1.2])**.** *There exists a unique $\mathbb{C}[\lambda, u][\tilde{q}_i^{\pm 1}]$-linear map*

$$\Phi_{\mathrm{mir}} : G_0(X_P^\vee, \mathcal{W}, p) \longrightarrow \mathrm{QH}_T^*(G/P)[u, \tilde{q}_1^{-1}, \cdots, \tilde{q}_r^{-1}] \tag{5.4.13}$$

satisfying the following:

1. *Φ_{mir} is bijective, and preserves the connection,*

2. *$\Phi_{\mathrm{mir}}([\Omega]) = 1$, where Ω is the unique (up to sign) volumn form in $\Omega^{\mathrm{top}}(X_P^\vee/Z)$, whose restrictions to every torus chart of $(G^\vee/P^\vee) \setminus -K_{G^\vee/P^\vee}$ is equal to the standard volumn form $\pm dz_1 \wedge \cdots \wedge dz_K/z_1 \dots z_K$,*

3. *at the semi-classical limit, we have a ring isomorphism*

$$\Phi_{\mathrm{mir}}^{u=0} : \mathrm{Jac}(X_P^\vee, \mathcal{W}, p) \xrightarrow{\cong} \mathrm{QH}_T^*(G/P)[\tilde{q}_1^{-1}, \cdots, \tilde{q}_r^{-1}], \tag{5.4.14}$$

4. *Φ_{mir} intertwines the shift operators (see [Cho23, Sections 3.3 and 4.5]), and*

5. *Φ_{mir} preserves the $\mathbb{Z}$-grading.*

In (5.4.14), $\mathrm{Jac}(X_P^\vee, \mathcal{W}, p)$ denotes the Jacobi ring, which is the coordinate ring of the scheme-theoretic zero locus of certain relative 1-forms in $\Omega^1(X_P^\vee \times \mathfrak{t}/Z \times \mathfrak{t})$ (see [Cho23, Definition 4.9] for more details). It corresponds to setting $u = 0$ in the B-model D-module.

We remark that the above isomorphism is a bit implicit. Below we provide an example with explicit isomorphism of small quantum D-modules from [MR20].

Example 5.4.15. For $G/P = \mathrm{Gr}(3,5) = \{V \leq \mathbb{C}^5 \mid \dim V = 3\}$, the Langlands dual flag variety $G^\vee/P^\vee$ is the Grassmannian $\mathrm{Gr}(2,5) \hookrightarrow \mathbb{P}^9$, whose image is defined by the Plücker relations $p_{a_1a_2}p_{a_3a_4} - p_{a_1a_3}p_{a_2a_4} + p_{a_1a_4}p_{a_2a_3} = 0$ for $1 \leq a_1 < a_2 < a_3 < a_4 \leq 5$. In this case, $-K_{G^\vee/P^\vee} = \{p_{12}p_{23}p_{34}p_{45}p_{15} = 0\}$. The $\mathcal{W}$-part of Rietsch's equivariant superpotential is given by:

$$\mathcal{W} = \frac{p_{13}}{p_{12}} + \frac{p_{24}}{p_{23}} + \frac{p_{35}}{p_{34}} + \tilde{q}\frac{p_{14}}{p_{45}} + \frac{p_{25}}{p_{15}}.$$

The degree of the inhomogeneous coordinate $\theta_{ij} = \frac{p_{ij}}{p_{12}}$ is equal to $2(i+j-3)$. The volume form $\Omega = \frac{d\theta_{13}d\theta_{14}d\theta_{15}d\theta_{23}d\theta_{24}d\theta_{25}}{\theta_{23}\theta_{34}\theta_{45}\theta_{15}}$ is of degree 0. For $1 \leq i < j \leq 5$, $\Phi_{\mathrm{mir}}([\theta_{ij}\Omega]) = \sigma_w$ with $w \in S_5$ the unique permutation satisfying $w(4) = 6-j$, $w(5) = 6-i$, and $w(1) < w(2) < w(3)$.

5.4.2.2 Equivariant F-bundles formulation

In our setting of (T)-structures, we need to replace the logarithmic $\tilde{q}_i$-directions with a regular meromorphic connection in y_i-directions. This is achieved by the D-module inverse image under

$$\begin{aligned} \psi_1 \colon \mathbb{C}[\lambda, u][\tilde{q}_i^{\pm 1}] &\longrightarrow \mathbb{C}[\lambda, u][q_i^{\pm 1}][\![y_{\leq r}]\!] \\ \tilde{q}_i &\longmapsto q_i e^{y_i}, \end{aligned}$$

where $y_{\leq r} = \{y_i, 1 \leq i \leq r\}$. For the purpose of applying our reconstruction theorem to obtain big mirror symmetry, we need to further take the fraction field of q and formalize u. So we compose ψ_1 with the following base change:

$$\psi_2 \colon \mathbb{C}[\lambda, u][q_i^{\pm 1}][\![y_{\leq r}]\!] \longrightarrow \mathbb{C}[\lambda, u][q_i^{\pm 1}][\![y_{\leq r}]\!] \otimes_{\mathbb{C}[q_i^{\pm 1}, u]} \mathbb{C}(q)[\![u]\!].$$

Namely, we have the following, where we recall $\Bbbk = \mathbb{C}(q)$ and $R = \Bbbk[\lambda]$:

$$\psi := \psi_2 \circ \psi_1 \colon \mathbb{C}[\lambda, u][\tilde{q}_i^{\pm 1}] \longrightarrow R[\![y_{\leq r}, u]\!]. \tag{5.4.16}$$

The B-model R-linear (T)-structure $(\mathcal{H}_R^B, \nabla_R^B)$ is the D-module inverse image $\psi^*\Big(\big(G_0(X_P^\vee, \mathcal{W}, p), \nabla\big)\Big)$, whose underlying $R[\![y_{\le r}, u]\!]$-module is

$$\mathcal{H}_R^B = G_0(X_P^\vee, \mathcal{W}, p) \otimes R[\![y_{\le r}, u]\!],$$

and the connection is given by

$$\nabla^B_{R,\partial_{y_i}}([\omega]) = \left[\mathcal{L}_{\partial_{y_i}}(\omega) + u^{-1}\frac{\partial \mathcal{W}}{\partial y_i}\omega - u^{-1}\sum_{j=1}^n \lambda_j\Big(\iota_{\partial_{y_i}} p^*\Big\langle(\lambda_j)^*, \mathrm{mc}_{T^\vee}\Big\rangle\Big)\omega\right]. \tag{5.4.17}$$

Next, we define the B-model $\Bbbk$-linear F-bundle. Fix the $\Bbbk$-basis of R given by $\boldsymbol{\lambda} = (\lambda^k, k \in \mathbb{N}^n)$ and let $\mathbf{y}_{\le r} \coloneqq \{y_{i,k}, 1 \le i \le r, k \in \mathbb{N}^n\}$. Consider the following change of variables:

$$\begin{aligned} \psi_{\boldsymbol{\lambda}} : R[\![y_{\le r}, u]\!] &\longrightarrow R[\![\mathbf{y}_{\le r}, u]\!] \\ y_i &\longmapsto \sum_{k\in\mathbb{N}^n} \lambda^k y_{i,k}. \end{aligned} \tag{5.4.18}$$

The underlying (T)-structure $(\mathcal{H}^B, \nabla^B)$ of the B-model $\Bbbk$-linear F-bundle is defined as the D-module inverse image $\psi^*_{\boldsymbol{\lambda}}(\mathcal{H}_R^B, \nabla_R^B)$ and restrict scalars from R to $\Bbbk$, as in Lemma 5.2.6. Explicitly, the underlying $\Bbbk[\![\mathbf{y}_{\le r}, u]\!]$-module is

$$\mathcal{H}^B = G_0(X_P^\vee, \mathcal{W}, p) \otimes R[\![\mathbf{y}_{\le r}, u]\!],$$

equipped with a regular meromorphic connection

$$\nabla^B_{\partial_{y_{i,k}}}([\omega]) = \Big[\mathcal{L}_{\partial_{y_{i,k}}}(\omega) + u^{-1}\frac{\partial \mathcal{W}}{\partial y_{i,k}}\omega - u^{-1}\sum_{j=1}^n \lambda_j\Big(\iota_{\partial_{y_{i,k}}} p^*\Big\langle(\lambda_j)^*, \mathrm{mc}_{T^\vee}\Big\rangle\Big)\omega\Big],$$

where $\mathcal{W}$ is in variables $y_{i,k}$. The u-direction is defined in Eq. (5.4.21), its definition uses the grading operator on the B-model, which we now define. We first construct a grading on $\Omega^{\mathrm{top}}(X_P^\vee/Z) \otimes \Bbbk[\![\mathbf{y}_{\le r}, u]\!]$.

Construction 5.4.19 (Grading on differential forms)**.** We construct a $\Bbbk[\![\mathbf{y}_{\le r}, u]\!]$-linear operator μ_B on $\Omega^{\mathrm{top}}(X_P^\vee/Z) \otimes \Bbbk[\![\mathbf{y}_{\le r}, u]\!]$, which defines a grading on differential forms.

Recall that $\Omega^{\mathrm{top}}(X_P^\vee/Z) \otimes \Bbbk[\![\mathbf{y}_{\le r}, u]\!]$ is a rank 1 free module over $\mathcal{O}\Big(X_P^\vee\Big)[\![\mathbf{y}_{\le r}, u]\!]$. The choice of Ω in Proposition 5.4.12(2) produces a basis of this module. For any differential form $\omega = h\Omega$ with $h \in \mathcal{O}\Big(X_P^\vee\Big)[\![\mathbf{y}_{\le r}, u]\!]$, we define

$$\mu_B(h\Omega) \coloneqq \frac{\deg_B(h)}{2}\Omega,$$

where the degree operator $\deg_B$ is defined on functions in $\mathcal{O}(X_P^\vee)$ using the $\mathbb{G}_m$-action on $X_P^\vee$ constructed in [Cho23, Lemma 4.6], and extended by $\Bbbk[\![\mathbf{y}_{\leq r}, u]\!]$-linearity. Using the Jacobian isomorphism (5.4.14) and the grading operator μ_A, we can describe $\deg_B$ as follows. For any local chart, we take a coordinate system $(z_i)_i$ so that z_i are homogeneous. For a monomial function $h \in \mathcal{O}\Big((G^\vee/P^\vee) \setminus -K_{G^\vee/P^\vee}\Big)$, we have $\deg_B(h) = 2dh$ where $d \in \mathbb{Z}$ is given by:

$$\left(\sum_{i=1}^{r} \frac{\deg(q_i)}{2} q_i \frac{\partial}{\partial q_i} + \mu_A\right) \Phi_{\mathrm{mir}}^{u=0,y=0}(\overline{h}) = d\Phi_{\mathrm{mir}}^{u=0,y=0}(\overline{h}),$$

where $\overline{h}$ denotes the image of h in $\mathrm{Jac}(X_P^\vee, \mathcal{W}, p)$ and $\Phi_{\mathrm{mir}}^{u=0,y=0}$ is the Jacobi isomorphism (5.4.14). It is then extended to a $\Bbbk[\![\mathbf{y}_{\leq r}, u]\!]$-linear operator on $\mathcal{O}(X_P^\vee)[\![\mathbf{y}_{\leq r}, u]\!]$.

We use the simple example of $X = \mathbb{CP}^1$ to illustrate the definition of $\deg_B$.

Example 5.4.20 (μ_B for $\mathbb{CP}^1$)**.** For $X = \mathbb{CP}^1$, we have $\Bbbk = \mathbb{C}(q)$. The equivariant (small) quantum cohomology $\mathrm{QH}_T^*(X)$ is isomorphic to $\Bbbk[H, \lambda]/(H^2 - H\lambda - q)$, where q has degree 4. The mirror $X_P^\vee$ is the family $\mathbb{G}_m \times \operatorname{Spec} \mathbb{C}[q^{\pm 1}] \to \operatorname{Spec} \mathbb{C}[q^{\pm 1}]$, the superpotential is $\mathcal{W} = z + \frac{q}{z}$, and the Jacobi ring $\mathrm{Jac}(X_P^\vee, \mathcal{W}, p)$ is isomorphic to $\Bbbk[z, z^{-1}, \lambda]/\big(1 - \frac{\lambda}{z} - \frac{q}{z^2}\big)$. The mirror isomorphism $\Phi_{\mathrm{mir}}^{u=0,y=0}$ at $u = 0, y = 0$ is given by the morphism of $\Bbbk[\lambda]$-modules defined by $z \mapsto H$.

We have $\deg_B(qz^3) = q \deg_B(z^3) = 6qz^3$, where the last equality follows from the computation:

$$\begin{aligned}\left(\frac{\deg(q)}{2} q \frac{\partial}{\partial q} + \mu_A\right) \Phi_{\mathrm{mir}}^{u=0,y=0}(\overline{z^3}) &= \left(\frac{\deg(q)}{2} q \frac{\partial}{\partial q} + \mu_A\right)(\lambda^2 H + qH + q\lambda) \\ &= 3(\lambda^2 H + qH + q\lambda).\end{aligned}$$

The u-direction connection of the B-model $\Bbbk$-linear F-bundle $(\mathcal{H}^B, \nabla^B)$ is defined as

$$\begin{aligned}\nabla^B_{u\partial_u} &\coloneqq \mathrm{Gr}^B - \nabla_{E_B}, \\ \mathrm{Gr}^B &\coloneqq u\partial_u + E_B + \mu_B,\end{aligned} \tag{5.4.21}$$

where $u\partial_u$ measures the degree in u, μ_B is the grading operator on differential forms (see Construction 5.4.19), and E_B is the Euler vector field measuring the degree of the $\mathbf{y}$-variables and accounting for the degree of q:

$$E_B \coloneqq \sum_{1 \leq j \leq r} \frac{\deg(q_j)}{2} \partial_{y_{j,0}} - \sum_{1 \leq j \leq r, k \in \mathbb{N}^n} |k| y_{j,k} \partial_{y_{j,k}}.$$

In the following proposition, we note that even though μ_B is only defined on differential forms, the total grading operator Gr^B is well-defined on equivalences classes.

Proposition 5.4.22. *The grading operator* Gr^B *produces a well-defined operator on* $\mathcal{H}^B$.

Proof. Let K denote the rank of $\Omega^1(X_P^\vee/Z)$. We only need to prove that for any $(K-1)$-form η, there exists a $(K-1)$-form η' such that

$$\mathrm{Gr}^B\Big((ud + d\mathcal{W}\wedge)\eta\Big) = (ud + d\mathcal{W}\wedge)\eta'. \tag{5.4.23}$$

Fix a torus-invariant local chart with coordinates $(z_i)_i$ and write

$$\eta = \sum_i g_i(z, q, \mathbf{y}_{\leq r}, u)\iota_{\partial_{z_i}}\left(\bigwedge_{j=1}^{K} dz_j\right).$$

Let

$$\eta' := \sum_i \Big((u\partial_u + E_B + \deg_B)(g_i) + \left(1 - \frac{\deg_B(z_i)}{2z_i}\right)g_i\iota_{\partial_{z_i}}\left(\bigwedge_{j=1}^{K} dz_j\right).$$

A direct computation shows that this choice of η' satisfies (5.4.23). We conclude that Gr^B descends to an operator on $\mathcal{H}^B$, completing the proof. □

We note the following property of μ_B, which we will use in Proposition 5.4.31.

Lemma 5.4.24 (Leibniz rule for μ_B)**.** *The grading operator μ_B on differential forms μ_B (see Construction 5.4.19) satisfies the Leibniz rule:*

$$\mu_B(g_1g_2\Omega) = g_1\mu_B(g_2\Omega) + g_2\mu_B(g_1\Omega),$$

for any $g_1, g_2 \in \mathcal{O}(X_P^\vee) \otimes \Bbbk[\![\mathbf{y}_{\leq r}, u]\!]$.

Proof. The statement follows from the facts that the Jacobian isomorphism $\Phi_{\mathrm{mir}}^{u=0,y=0}$ is a ring isomorphism, and that $(\sum_{j=1}^r \frac{\deg(q_j)}{2} q_j \frac{\partial}{\partial q_j} + \mu_A)$ satisfies the Leibniz rule. □

By construction, the data $\{(\mathcal{H}_R^B, \nabla_R^B), \mathrm{id}\}$ is an R-linear lift of the $\Bbbk$-linear (T)-structure $(\mathcal{H}^B, \nabla^B)_0$, and we obtain the B-model small equivariant F-bundle:

$$\mathcal{F}^B = \{(\mathcal{H}^B, \nabla^B), (\mathcal{H}_R^B, \nabla_R^B), \mathrm{id}\}/\Bbbk[\![\mathbf{y}_{\leq r}]\!].$$

Lemma 5.4.25 (Proposition 5.4.12(5))**.** *For $h \in \mathcal{O}((G^\vee/P^\vee)\backslash - K_{G^\vee\backslash P^\vee})$, we have*

$$\left(u\partial_u + \sum_{i=1}^{r} \frac{\deg(q_j)}{2} q_j \frac{\partial}{\partial q_j} + \mu_A\right) \Phi_{\mathrm{mir}}^{y=0}(h\Omega) = \Phi_{\mathrm{mir}}^{y=0}\Big((u\partial_u + \mu_B)(h\Omega)\Big).$$

The following proposition essentially follows from Proposition 5.4.12. Here we add the explanation in detail for completeness.

Proposition 5.4.26. *The A-model small equivariant F-bundle $\mathcal{F}^A$ over $\Bbbk[\![\boldsymbol{\tau}_{\leq r}]\!]$ is isomorphic to the B-model small equivariant F-bundle $\mathcal{F}^B$ over $\Bbbk[\![\mathbf{y}_{\leq r}]\!]$ under an isomorphism $\Big((\mathrm{mir}_\Bbbk, \Phi_{\mathrm{mir},\Bbbk}), (\mathrm{mir}, \Phi_{\mathrm{mir}})\Big)$ where Φ_{mir} is as in Proposition 5.4.12, $\Phi_{\mathrm{mir},\Bbbk}$ is the induced $\Bbbk$-linear map, and $\mathrm{mir}_\Bbbk \colon y_{i,k} \mapsto \tau_{i,k}$, $\mathrm{mir}\colon y_i \mapsto \tau_i$ identify the variables of the equivariant F-bundles.*

Proof. Our construction of B-model small equivariant F-bundle

$$\mathcal{F}^B = \{(\mathcal{H}^B, \nabla^B), (\mathcal{H}^B_R, \nabla^B_R), \mathrm{id}\}/\Bbbk[\![\mathbf{y}_{\leq r}]\!]$$

consists of the (T)-structure obtained as D-module inverse image:

$$(\mathcal{H}^B_R, \nabla^B_R) = \psi^*\Big(G_0(X_P^\vee, \mathcal{W}, p), \nabla\Big),$$
$$(\mathcal{H}^B, \nabla^B) = (\psi_{\boldsymbol{\lambda}} \circ \psi)^*\Big(G_0(X_P^\vee, \mathcal{W}, p), \nabla\Big),$$

where ψ and $\psi_{\boldsymbol{\lambda}}$ are base change maps defined in equations (5.4.16) and (5.4.18), together with a u-direction on $(\mathcal{H}^B, \nabla^B)$.

Our construction of A-model small equivariant F-bundle $\mathcal{F}^A = \{(\mathcal{H}^A, \nabla^A), (\mathcal{H}^A_R, \nabla^A_R), \mathrm{id}\}/\Bbbk[\![\boldsymbol{\tau}_{\leq r}]\!]$ in Section 5.4.1.2 can also be written as (T)-structures $(\mathcal{H}^A, \nabla^A)$ and $(\mathcal{H}^A_R, \nabla^A_R)$ obtained as D-module inverse image under the same maps, together with a u-direction on $(\mathcal{H}^A, \nabla^A)$.

By the functoriality of pullback, the isomorphism in Proposition 5.4.12 induces an isomorphism

$$((\mathrm{mir}_\Bbbk, \Phi_{\mathrm{mir},\Bbbk}), (\mathrm{mir}, \Phi_{\mathrm{mir}}))\colon \mathcal{F}^B \longrightarrow \mathcal{F}^A$$

of the underlying (T)-structures. It suffices to prove that in our setting Φ_{mir} is also graded, i.e. that $(\mathrm{mir}_\Bbbk)_* E_B = E_A$ and

$$\Phi_{\mathrm{mir},\Bbbk} \circ \mathrm{Gr}^B = \mathrm{mir}_\Bbbk^* \mathrm{Gr}^A \circ \Phi_{\mathrm{mir},\Bbbk}, \tag{5.4.27}$$

where $\mathrm{mir}_\Bbbk^*\mathrm{Gr}^A$ is the grading operator on $\mathrm{mir}_\Bbbk^*\mathcal{H}_A$ induced from Gr^A. The compatibility of the Euler vector fields is clear from their definition and the fact that

$\mathrm{mir}_{\Bbbk}$ identifies the variables $\mathbf{y}_{\leq r}$ and $\boldsymbol{\tau}_{\leq r}$. For (5.4.27), we need to check that for all $f \in O(X_P^\vee)[\![\mathbf{y}_{\leq r}, u]\!]$, we have

$$\Phi_{\mathrm{mir},\Bbbk}([(u\partial_u + E_B + \mu_B)(f\Omega)]) = (u\partial_u + E_B + \mu_A)(\Phi_{\mathrm{mir},\Bbbk}([f\Omega])).$$

Since both sides are linear in q and satisfy the Leibniz rule for y and u, it suffices to check the equation for $f \in \mathcal{O}\Big((G^\vee/P^\vee)\backslash - K_{G^\vee/P^\vee}\Big)$.

In this case, the left-hand side reduces to $\Phi_{\mathrm{mir},\Bbbk}([\mu_B(f\Omega)])$. For the right-hand side, observe that $\Phi_{\mathrm{mir},\Bbbk}([f\Omega])$ is obtained by first applying the mirror map (5.4.13) from [Cho23] to $[f\Omega]$, and then pulling back under the change of variables $\tilde{q}_i \mapsto q_i e^{\sum_k \lambda^k y_{i,k}}$. Hence, $\Phi_{\mathrm{mir},\Bbbk}([f\Omega])$ is a power series in the variables $\{q_i e^{\sum_k \lambda^k y_{i,k}}\}_{1\leq i\leq r}$ with coefficients in $H_T^*(X, \mathbb{C})$. For an equivariant cohomology class $\alpha \in H_T^*(X, \mathbb{C})$, we have:

$$q_i \frac{\partial}{\partial q_i}\Big(q_i e^{\sum_k \lambda^k y_{i,k}} \alpha\Big) = \frac{\partial}{\partial y_{i,0}}\Big(q_i e^{\sum_k \lambda^k y_{i,k}} \alpha\Big).$$

It follows that at $y = 0$, the right-hand side is

$$\begin{aligned}(u\partial_u + E_B + \mu_A)(\Phi_{\mathrm{mir},\Bbbk}([f\Omega])) &= \left(u\partial_u + \sum_{i=1}^{r} \frac{\deg(q_i)}{2} q_i \frac{\partial}{\partial q_i} + \mu_A\right)(\Phi_{\mathrm{mir},\Bbbk}([f\Omega])) \\ &= \Phi_{\mathrm{mir},\Bbbk}([\mu_B(f\Omega)]),\end{aligned}$$

where the last equality follows from Lemma 5.4.25. This implies that the bundle map is compatible with u-direction at $y = 0$, and by Lemma 5.3.5 that Eq. (5.4.27) holds for any y. We deduce that $(\mathrm{mir}_{\Bbbk}, \Phi_{\mathrm{mir},\Bbbk})$ is an isomorphism of F-bundles, concluding the proof. □

5.4.3 The big D-module mirror symmetry for G/P

In this subsection, we prove the big D-module mirror symmetry for G/P in the framework of equivariant F-bundles. We first construct a maximal unfolding of the small B-model equivariant F-bundle by unfolding the superpotential $\mathcal{W}$ (Construction 5.4.28). In particular, we discuss freeness in Lemma 5.4.30 and Proposition 5.4.29, and flatness in Propositions 5.4.29 and 5.4.31. We obtain the equivariant big mirror symmetry for flag varieties in Theorem 5.4.35 and deduce a non-equivariant version in Theorem 5.4.38.

5.4.3.1 Unfolding of the B-model

We fix formal variables $y = \{y_1, \ldots, y_N\}$ and $\mathbf{y} = \{y_{i,k}, 1 \leq i \leq N, k \in \mathbb{N}^n\}$.

Construction 5.4.28 (Unfolded superpotential)**.** For any $r < j \leq N$, let $\bar{f}_j := (\Phi_{\text{mir}}^{u=0,y=0})^{-1}(\sigma_{v_j}) \in \text{Jac}(X_P^\vee, \mathcal{W}, p)$. Let $f_j \in \mathcal{O}(X_P^\vee)$ be a lift of $\bar{f}_j$ such that the function f_j is independent of $y_1, ..., y_N$. The unfolded superpotential $\widetilde{\mathcal{W}}$ is:

$$\widetilde{\mathcal{W}} := \mathcal{W} + \sum_{j=r+1}^{N} y_j f_j.$$

Similar to the previous section on the small B-model equivariant F-bundle, we now associate to $\widetilde{\mathcal{W}}$ the big B-model equivariant F-bundle:

$$\mathcal{F}^{B,\text{big}} := \{(\mathcal{H}^{B,\text{big}}, \nabla^{B,\text{big}}), (\mathcal{H}_R^{B,\text{big}}, \nabla_R^{B,\text{big}}), \text{id}\}/\Bbbk[\![\mathbf{y}]\!].$$

We first construct an R-linear (T)-structure $(\mathcal{H}_R^{B,\text{big}}, \nabla_R^{B,\text{big}})$, consisting of a $R[\![y,u]\!]$-module defined by

$$\mathcal{H}_R^{B,\text{big}} := \text{coker}\Big(R[\![y,u]\!] \otimes_{\mathbb{C}[\tilde{q}_i^{\pm 1}]} \Omega^{top-1}(X_P^\vee/Z) \xrightarrow{\partial} R[\![y,u]\!] \otimes_{\mathbb{C}[\tilde{q}_i^{\pm 1}]} \Omega^{top}(X_P^\vee/Z)\Big),$$
$$\partial := 1 \otimes \left(ud + d\widetilde{\mathcal{W}} \wedge - \sum_{j=1}^{n} \lambda_j (p^*\langle(\lambda_j)^*, \text{mc}_{T^\vee}\rangle) \wedge\right),$$

equipped with a connection defined by:

$$\nabla_{R,\partial_{y_i}}^{B,\text{big}}([\omega]) := \left[\mathcal{L}_{\partial_{y_i}}(\omega) + u^{-1}\frac{\partial\widetilde{\mathcal{W}}}{\partial y_i}\omega - u^{-1}\sum_{j=1}^{n} \lambda_j\Big(\iota_{\partial_{y_i}} p^*\Big\langle(\lambda_j)^*, \text{mc}_{T^\vee}\Big\rangle\Big)\omega\right].$$

Proposition 5.4.29. *The data $(\mathcal{H}_R^{B,\text{big}}, \nabla_R^{B,\text{big}})/R[\![y]\!]$ defines an R-linear (T)-structure.*

Proof. We first check that $\mathcal{H}_R^{B,\text{big}}$ is a free $R[\![y,u]\!]$-module. By construction, we have $\mathcal{H}_R^{B,\text{big}}|_{y_{\geq r+1}=0} = \mathcal{H}_R^B$, which is a finite free $R[\![y_{\leq r}, u]\!]$-module (see [Cho23, p. 52]). Since $y_j\omega \in \text{im}(\partial)$ if and only if $\omega \in \text{im}(\partial)$, the element y_j is torsion-free. Hence, the freeness of $\mathcal{H}_R^{B,\text{big}}$ follows from Lemma 5.4.30 below. The flatness of $\nabla_R^{B,\text{big}}$ follows from the facts that $\frac{\partial\widetilde{\mathcal{W}}}{\partial y_i} = f_i$ is independent of $y_1, \ldots, y_N$, that $[\mathcal{L}_{\partial_{y_s}}, \mathcal{L}_{\partial_{y_t}}] = 0$, and that $[\mathcal{L}_{\partial_{y_s}}, \iota_{y_t}] = 0$. $\square$

Lemma 5.4.30. *Let R_0 be a commutative unital ring and M be an $R_0[\![z_1, \cdots, z_m]\!]$-module such that z_i is torsion-free for all $1 \leq i \leq m$. Let $\Omega_1, \ldots, \Omega_N \in M$. If $\{\bar{\Omega}_1, \cdots, \bar{\Omega}_N\}$ is an R_0-basis of $M/(z_1, ..., z_m)M$, then $\{\Omega_1, \cdots, \Omega_N\} \subset M$ is an $R_0[\![z_1, \cdots, z_m]\!]$-basis of $M[\![z_1, \cdots, z_m]\!]$.*

Proof. By Nakayama lemma [ZS60, §VIII.3, Corollary 2], $\{\Omega_1, \cdots, \Omega_N\}$ generates M as an $R_0[\![z_1, \cdots, z_m]\!]$-module. It remains to prove the freeness.

We first treat the case $m = 1$. Let $g_i(z_1) \in R_0[\![z_1]\!]$ be coefficients such that $\sum_i g_i(z_1)\Omega_i = 0$. Since $\{\Omega_i\}$ induces a basis of M/z_1M, we have $g_i(0) = 0$. Assume g_i has no terms of degree less than or equal to $b-1$ in z_1, i.e. we can write $g_i = z_1^b h_i$. Then we have:

$$\sum_i z_1^b h_i \Omega_i = 0.$$

Since z_1^b is torsion-free in M, we deduce that $\sum_i h_i\Omega_i = 0$, which implies $h_i(0) = 0$. Hence, g_i has no terms of degree less than or equal to b. By induction on b, this implies $g_i = 0$.

The case $m \geq 2$ follows from the case $m = 1$ by a direct induction on the number of variables. The proof is complete. $\square$

The $\Bbbk$-linear F-bundle $(\mathcal{H}^{B,\mathrm{big}}, \nabla^{B,\mathrm{big}})/\Bbbk[\![\mathbf{y}]\!]$ is given by the D-module inverse image of $(\mathcal{H}_R^{B,\mathrm{big}}, \nabla_R^{B,\mathrm{big}})$ under the map of $R[\![u]\!]$-algebras $R[\![y,u]\!] \to R[\![\mathbf{y},u]\!]$ given by $y_i \mapsto \sum_{k\in\mathbb{N}^n} \lambda^k y_{i,k}$. More explicitly:

$$\mathcal{H}^{B,\mathrm{big}} := \mathcal{H}_R^{B,\mathrm{big}} \otimes_{\Bbbk[\![y,u]\!]} \Bbbk[\![\mathbf{y},u]\!],$$
$$\nabla^{B,\mathrm{big}}_{\partial_{y_{i,k}}}([\omega]) := \left[\mathcal{L}_{\partial_{y_{i,k}}}(\omega) + u^{-1}\frac{\partial\widetilde{\mathcal{W}}}{\partial y_{i,k}}\omega - u^{-1}\sum_{j=1}^n \lambda_j\Big(\iota_{\partial_{y_{i,k}}} p^*\big\langle(\lambda_j)^*, \mathrm{m}c_{T^\vee}\big\rangle\Big)\omega\right],$$

equipped with the u-direction connection

$$\nabla^{B,\mathrm{big}}_{u\partial_u} := \mathrm{Gr}^{B,\mathrm{big}} - \nabla_{E_B^{\mathrm{big}}},$$
$$\mathrm{Gr}^{B,\mathrm{big}} := u\partial_u + E_B^{\mathrm{big}} + \mu_B,$$

where the Euler vector field is given by

$$E_B^{\mathrm{big}} := \sum_{1\leq j\leq r} \frac{\deg(q_j)}{2}\partial_{y_{j,0}} + \sum_{1\leq j\leq N, k\in\mathbb{N}^n} (1-\ell(v_j)-|k|)y_{j,k}\partial_{y_{j,k}},$$

and where the grading on differential forms μ_B (see Construction 5.4.19) is extended to a $\Bbbk[\![\mathbf{y},u]\!]$-linear operator. Similar to Proposition 5.4.22, the total grading operator $\mathrm{Gr}^{B,\mathrm{big}}$ lifts to $\mathcal{H}^{B,\mathrm{big}}$.

Proposition 5.4.31. *The big Gauss-Manin connection $\nabla^{B,\mathrm{big}}$ is flat.*

Proof. The underlying (T)-structure of $(\mathcal{H}^{B,\mathrm{big}}, \nabla^{B,\mathrm{big}})$ is flat, as it is obtained from the R-linear (T)-structure $(\mathcal{H}_R^{B,\mathrm{big}}, \nabla_R^{B,\mathrm{big}})$, whose flatness was established in Proposition 5.4.29. We check the flatness in the u-direction as follows. We have:

$$\left[\nabla^{B,\mathrm{big}}_{u\partial_u}, \nabla^{B,\mathrm{big}}_{\partial_{y_{j,k}}}\right] = \left[\mathrm{Gr}^{B,\mathrm{big}}, \nabla^{B,\mathrm{big}}_{\partial_{y_{j,k}}}\right] - \left[\nabla^{B,\mathrm{big}}_{E_B^{\mathrm{big}}}, \nabla^{B,\mathrm{big}}_{\partial_{y_{j,k}}}\right].$$

Since the underlying (T)-structure of $(\mathcal{H}^{B,\mathrm{big}}, \nabla^{B,\mathrm{big}})$ is flat, we have:

$$\left[\nabla^{B,\mathrm{big}}_{E_B^{\mathrm{big}}}, \nabla^{B,\mathrm{big}}_{\partial_{y_{j,k}}}\right] = \nabla^{B,\mathrm{big}}_{[E_B^{\mathrm{big}},\partial_{y_{j,k}}]} = -(1-\ell(v_j)-|k|)\nabla^{B,\mathrm{big}}_{\partial_{y_{j,k}}}. \tag{5.4.32}$$

We claim that the grading structure is compatible with the connection, in the sense that:

$$\left[\mathrm{Gr}^{B,\mathrm{big}}, \nabla^{B,\mathrm{big}}_{\partial_{y_{j,k}}}\right] = -(1-\ell(v_j)-|k|)\nabla^{B,\mathrm{big}}_{\partial_{y_{j,k}}}. \tag{5.4.33}$$

When $1 \le j \le r$, the equality holds because ∇^B is flat. When $r+1 \le j \le N$, the equality holds because for any $[\omega] \in \mathcal{H}^{B,big}$ we have:

$$\begin{aligned}
\mathrm{Gr}^{B,\mathrm{big}}([f_j\omega]) - f_j\mathrm{Gr}^{B,\mathrm{big}}([\omega]) &= [\ell(v_j)f_j\omega],\\
\mathrm{Gr}^{B,\mathrm{big}}([u^{-1}\omega]) - u^{-1}\mathrm{Gr}^{B,\mathrm{big}}([\omega]) &= -[u^{-1}\omega],\\
\mathrm{Gr}^{B,\mathrm{big}}([\lambda^k\omega]) - \lambda^k\mathrm{Gr}^{B,\mathrm{big}}([\omega]) &= [|k|\lambda^k\omega],
\end{aligned}$$

where we use Lemma 5.4.24 and the homogeneity of the elements f_j, u^{-1}, and λ^k. Note that f_j is homogeneous of degree $\ell(v_j)$ by our choice of lift in Construction 5.4.28. Now, flatness follows from equations (5.4.32) and (5.4.33), concluding the proof. □

Similar to the A-model, we note that the quotient maps $\Bbbk[\![\mathbf{y}]\!] \to \Bbbk[\![\mathbf{y}_{\le r}]\!]$ and $R[\![y]\!] \to R[\![y_{\le r}]\!]$ induce an unfolding of equivariant F-bundles $\iota_B \colon \mathcal{F}^B \to \mathcal{F}^{B,\mathrm{big}}$.

Proposition 5.4.34. *The morphism $\iota_B \colon \mathcal{F}^B \to \mathcal{F}^{B,\mathrm{big}}$ is a maximal unfolding of $\Bbbk$-linear equivariant F-bundles, with cyclic vector given by $[\Omega] := \Phi_{\mathrm{mir}}^{-1}(1)$, where $1 \in H_T^*(X,\Bbbk)$ is the cyclic vector on the A-side.*

Proof. We only need to check that $\mathcal{F}^{B,\mathrm{big}}$ is a maximal unfolding of $\mathcal{F}^B$ with cyclic vector induced from $[\Omega]$. By definition of $\mathcal{F}^{B,\mathrm{big}}$, for $r < j \le N$ we have:

$$\Phi_{\mathrm{mir}}\left(\left[u\nabla^{B,\mathrm{big}}_{R,\partial_{y_j}}\right]\Big|_{y=u=0}([\Omega])\right) = \Phi_{\mathrm{mir}}^{u=0,y=0}(\bar{f}_j) = \sigma_{v_j}.$$

For $1 \le j \le r$, by Proposition 5.4.26 we have:

$$\Phi_{\mathrm{mir}}\left(\left[u\nabla^{B,\mathrm{big}}_{R,\partial_{y_j}}\right]\Big|_{y=u=0}([\Omega])\right) = \left[u\nabla^{A}_{\partial t_j}\right]\Big|_{t=u=0}(1) = \sigma_{v_j}.$$

Since Φ_{mir} is an isomorphism and $\{\sigma_{v_j}, 1 \le j \le N\}$ is a basis of $H_T^*(X,\Bbbk)$, we conclude that the unfolding is maximal. □

5.4.3.2 Mirror symmetry

Now we can show the big D-module mirror symmetry for $X = G/P$ in the following sense.

Theorem 5.4.35 (Equivariant big mirror symmetry)**.** *There exists a unique isomorphism of equivariant F-bundles*

$$\left(\left(\mathrm{mir}_{\Bbbk}^{\mathrm{big}}, \Phi_{\mathrm{mir},\Bbbk}^{\mathrm{big}}\right), \left(\mathrm{mir}^{\mathrm{big}}, \Phi_{\mathrm{mir}}^{\mathrm{big}}\right)\right) : \mathcal{F}^{B,\mathrm{big}} \longrightarrow \mathcal{F}^{A,\mathrm{big}}$$

extending the small mirror map $\big((\mathrm{mir}_{\Bbbk}, \Phi_{\mathrm{mir},\Bbbk}), (\mathrm{mir}, \Phi_{\mathrm{mir}})\big)$ *in Proposition 5.4.26.*

Proof. By Proposition 5.4.10, we have a maximal unfolding of the small A-model equivariant F-bundle, given by $\mathcal{F}^A \to \mathcal{F}^{A,\mathrm{big}}$. Composing the small mirror isomorphism $\big((\mathrm{mir}_{\Bbbk}, \Phi_{\mathrm{mir},\Bbbk}), (\mathrm{mir}, \Phi_{\mathrm{mir}})\big) : \mathcal{F}^A \to \mathcal{F}^B$ of Proposition 5.4.26, with the B-model maximal unfolding $\iota : \mathcal{F}^B \to \mathcal{F}^{B,\mathrm{big}}$, we obtain another maximal unfolding $\mathcal{F}^A \to \mathcal{F}^{B,\mathrm{big}}$.

To apply Theorem 5.3.36 and obtain a unique isomorphism $\mathcal{F}^{A,\mathrm{big}} \to \mathcal{F}^{B,\mathrm{big}}$, we need to check the small A-model equivariant F-bundle satisfies the (GC') condition as in Definition 5.3.15. We take $v = 1 \in H_T^*(X, \Bbbk) = \mathcal{H}^A|_{\tau=u=0}$. After base change to $\mathrm{Frac}(R) = \Bbbk(\lambda)$, we have the evaluation map of $\Bbbk(\lambda)$-modules:

$$\begin{aligned} \mu_{v=1} \colon \bigoplus_{1 \le j \le r} \Bbbk(\lambda)\partial_{\tau_j} &\longrightarrow H_T^*(X, \Bbbk) \otimes \Bbbk(\lambda), \\ \partial_{\tau_j} &\longmapsto \mu(\partial_{\tau_j})(1) = \sigma_{v_j} + \lambda_{i_j}. \end{aligned} \tag{5.4.36}$$

By Lemma 5.4.6, $H_T^*(X, \Bbbk) \otimes \Bbbk(\lambda)$ is generated as a $\Bbbk(\lambda)$-algebra by σ_{v_j} for $1 \le j \le r$, so the orbit of $v = 1$ under the action of $R[\mathrm{im}\,\mu]$ is the fiber $\mathcal{H}_R^A/(\tau_1, \dots, \tau_N, u)\mathcal{H}_R^A \otimes_R \mathrm{Frac}(R)$, and (GC') is verified. The proof is complete. $\square$

We deduce a non-equivariant limit of the theorem by applying the base change associated to the quotient map $R \to R/(\lambda)$. This corresponds to setting $\lambda = 0$ in all the previous formulas. We use the superscript λ_0 to indicate that the non-equivariant limit is taken. We note that since $R/(\lambda) \simeq \Bbbk$, in the non-equivariant limit the equivariant F-bundles can be reduced to $\Bbbk$-linear F-bundles (see Remark 5.2.12).

On the A-side, we have $\tau_i = \sum \lambda^k \tau_{i,k} = \tau_{i,0}$. The big quantum D-module $(\mathcal{H}^{A,\mathrm{big},\lambda_0}, \nabla^{A,\mathrm{big},\lambda_0})$ is an F-bundle over $\Bbbk[\![\tau]\!] = \Bbbk[\![\tau_1, \ldots, \tau_N]\!]$ defined by

$$\begin{aligned}
\mathcal{H}^{A,\mathrm{big},\lambda_0} &= H^*(X, \mathbb{C}) \otimes \Bbbk[\![\tau, u]\!], \\
\nabla^{A,\mathrm{big},\lambda_0}_{\partial_{\tau_j}} &= \partial_{\tau_j} + u^{-1}\sigma_{v_j} \star^{\mathrm{big},\lambda_0}, \\
\nabla^{A,\mathrm{big},\lambda_0}_{u\partial_u} &= \mathrm{Gr}^{A,\mathrm{big},\lambda_0} - \nabla^{A,\mathrm{big},\lambda_0}_{E_A^{\mathrm{big},\lambda_0}},
\end{aligned}$$

where $\mathrm{Gr}^{A,\mathrm{big},\lambda_0} = u\partial_u + E_A^{\mathrm{big},\lambda_0} + \mu_A$, with the Euler vector field

$$E_A^{\mathrm{big},\lambda_0} := \sum_{1 \leq j \leq r} \frac{\deg(q_j)}{2} \partial_{\tau_j} + \sum_{1 \leq j \leq N} (1 - \ell(v_j)) \tau_j \partial_{\tau_j}.$$

On the B-side, we have $y_i = \sum \lambda^k y_{i,k} = y_{i,0}$. The non-equivariant big Gauss-Manin system is an F-bundle $(\mathcal{H}^{B,\mathrm{big},\lambda_0}, \nabla^{B,\mathrm{big},\lambda_0})$ over $\Bbbk[\![y]\!] = \Bbbk[\![y_1, \ldots, y_N]\!]$ defined by

$$\begin{aligned}
\mathcal{H}^{B,\mathrm{big},\lambda_0} &= \mathrm{coker}\left(\Bbbk[\![y, u]\!] \otimes \Omega^{\mathrm{top}-1}(X_P^\vee/Z) \xrightarrow{1 \otimes (ud + d\widetilde{\mathcal{W}} \wedge)} \Bbbk[\![y, u]\!] \otimes \Omega^{\mathrm{top}}(X_P^\vee/Z)\right), \\
\nabla^{B,\mathrm{big},\lambda_0}_{\partial_{y_j}} &= \mathcal{L}_{\partial_{y_j}} + \frac{\partial \widetilde{\mathcal{W}}}{\partial y_j}, \\
\nabla^{B,\mathrm{big},\lambda_0}_{u\partial_u} &= u\partial_u - u^{-1}\widetilde{\mathcal{W}} \qquad \text{(by Proposition 5.4.37 below)}.
\end{aligned}$$

Proposition 5.4.37. *At the non-equivariant limit, we have* $\nabla^{B,\mathrm{big},\lambda_0}_{u\partial_u} = u\partial_u - u^{-1}\widetilde{\mathcal{W}}$.

Proof. In this proof, to simplify notations we drop the superscript λ_0. For any $[\omega] \in \mathcal{H}^{B,\mathrm{big}}$, we can write $\omega = g\Omega$ where $g \in \mathcal{O}(X_P^\vee)[\![y, u]\!]$ and Ω is given in Proposition 5.4.12(2). We have:

$$\begin{aligned}
\nabla^{B,\mathrm{big}}_{u\partial_u}([g\Omega]) &= \Big[u\partial_u(g\Omega) + E_B^{\mathrm{big}}(g)\Omega + \mu_B(g\Omega)\Big] - \nabla^{B,\mathrm{big}}_{E_B^{\mathrm{big}}}([g\Omega]) \\
&= \Big[u\partial_u(g\Omega) + \mu_B(g\Omega) - u^{-1}E_B^{\mathrm{big}}(\widetilde{\mathcal{W}})g\Omega\Big] \\
&= \Big[u\partial_u(g\Omega) + \mu_B(g\Omega) - u^{-1}g\mathrm{Gr}^{B,\mathrm{big}}(\widetilde{\mathcal{W}}\Omega) + g\partial_u(\widetilde{\mathcal{W}})\Omega + g\mu_B(\widetilde{\mathcal{W}}\Omega)\Big].
\end{aligned}$$

We claim that $\mathrm{Gr}^{B,\mathrm{big}}(\widetilde{\mathcal{W}}\Omega) = \widetilde{\mathcal{W}}\Omega$, or equivalently that $\widetilde{\mathcal{W}}$ is homogeneous of degree 1 as an element of $\mathcal{O}(X_P^\vee)[\![y, u]\!]$. By [Cho25, Lemma 4.3], we have $\Phi_{\mathrm{mir}}([\mathcal{W}\Omega]) = c_1(G/P)$, and hence $\mathcal{W}$ is homogeneous of degree 1 since Φ_{mir} is graded (Proposition 5.4.12(5)). For $r < j \leq N$, our choice of lift in Construction 5.4.28 shows that $\mathrm{Gr}^{B,\mathrm{big}}(y_j f_j \Omega) = E_B^{\mathrm{big}}(y_j) f_j \Omega + y_j \mu_B(f_j\Omega) = y_j f_j \Omega$. Since $\widetilde{\mathcal{W}}$ is independent of u, we have:

$$\nabla^{B,\mathrm{big}}_{u\partial_u}([g\Omega]) = \Big[u\partial_u(g\Omega) - u^{-1}\widetilde{\mathcal{W}}g\Omega\Big] + \Big[u\mu_B(g\Omega) + g\mu_B(\widetilde{\mathcal{W}}\Omega)\Big].$$

By [Cho23, Remark 4.13] and the references therein, after fixing local coordinates $(z_i)_i$ we can write $\Omega = \frac{\bigwedge_{i=1}^{K} dz_i}{z_1 \cdots z_K}$. By Lemma 5.4.24, we have:

$$\mu_B(g\Omega) = \sum_{i=1}^{K} \frac{\partial g}{\partial z_i} \mu_B(z_i \Omega).$$

Consider the $K-1$ form $\eta \coloneqq \sum_{i=1}^{K} g \iota_{\partial_{z_i}} \mu_B(z_i \Omega)$. We have:

$$\begin{aligned} ud\eta &= u \sum_{i,j=1}^{K} \frac{\partial g}{\partial z_j} dz_j \wedge \iota_{\partial_{z_i}} \mu_B(z_i\Omega) + ug \sum_{i=1}^{K} d\Big(\iota_{\partial_{z_i}} \mu_B(z_i \Omega)\Big) \\ &= u \sum_{i=1}^{K} \frac{\partial g}{\partial z_i} \mu_B(z_i \Omega) = u\mu_B(g\Omega), \end{aligned}$$

and

$$\begin{aligned} d\widetilde{\mathcal{W}} \wedge \eta &= \sum_{i=1}^{K} \frac{\partial \widetilde{\mathcal{W}}}{\partial z_i} dz_i \wedge \Big(g \iota_{\partial_{z_i}} \mu_B(z_i \Omega)\Big) \\ &= g\mu_B\Big(\widetilde{\mathcal{W}}\Omega\Big). \end{aligned}$$

We deduce that

$$\Big[u\mu_B(g\Omega) + g\mu_B(\widetilde{\mathcal{W}}\Omega)\Big] = \Big[(ud + d\widetilde{\mathcal{W}} \wedge)\eta\Big] = 0,$$

concluding the proof. □

As a direct consequence of Theorem 5.4.35, we obtain the following non-equivariant big D-module mirror symmetry.

Theorem 5.4.38 (Non-equivariant big mirror symmetry)**.** *The non-equivariant limit of* $(\mathrm{mir}_{\Bbbk}^{\mathrm{big}}, \Phi_{\mathrm{mir},\Bbbk}^{\mathrm{big}})$ *in Theorem 5.4.35 gives an isomorphism of* $\Bbbk$*-linear F-bundles:*

$$\Big(\mathcal{H}^{A,\mathrm{big},\lambda_0}, \nabla^{B,\mathrm{big},\lambda_0}\Big) \xrightarrow{\sim} \Big(\mathcal{H}^{B,\mathrm{big},\lambda_0}, \nabla^{B,\mathrm{big},\lambda_0}\Big).$$

This isomorphism is uniquely determined by the non-equivariant small mirror isomorphism.

Proof. The existence is clear. The uniqueness follows from [Hin+24, Proposition 4.27], which applies because the F-bundles are maximal and admit framings. □

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