PROPERTIES AND APPLICATIONS OF BESSEL FUNCTIONS OF IMAGINARY ORDER

Thesis by

Samuel P. Morgan, Jr.

In Partial Fulfilment of the Requirements
for the Degree of Doctor of Philosophy

California Institute of Technology
Pasadena, California
1947

ACKNOWLEDGMENTS

It is a pleasure to thank Professor W. R. Smythe for stimulating discussions of the problems involved in this thesis, for many helpful suggestions, and above all for his enthusiasm and his active interest in all stages of the work.

For several of the more important references of chapters II and III I am indebted to the late Professor Harry Bateman, whose untimely death prevented his seeing the full development of the research.

The use of the computing equipment at the Cooperative Wind Tunnel was arranged for me by Professor C. B. Millikan and Mr. F. H. Felberg. For instruction in the operation of the punched card machines I have to thank Dr. E. C. Bower of Douglas Aircraft Company and various members of the wind tunnel staff.

CONTENTS

Page

Chapter II
Physical Applications of Bessel Functions
of Imaginary Order and Imaginary Argument

Chapter III
Physical Applications of Bessel Functions
of Imaginary Order and Real Argument

Appendix
Tables of the Wedge Functions $F_\nu(e^x)$ and $G_\nu(e^x)$

Summary

0.1. Nature and Purpose of Thesis.

Although solutions of Bessel's differential equation,

$$z^2 \, d^2w/dz^2 + z \, dw/dz + (z^2 - \nu^2)w = 0, \qquad (1)$$

have been widely studied and extensively tabulated for real values of the index ν because of their applications to all fields of mathematical physics, much less attention has been given to Bessel functions for which the order ν is purely imaginary, the argument z being either imaginary or real. Inasmuch as the functions of imaginary order appear in various problems from different branches of mathematical physics, it seems worth while to give a connected discussion of their properties and to list the various physical applications which have come to the author's notice.

The purpose of this thesis will be to suggest canonical definitions for Bessel functions of imaginary order and either imaginary or real argument, and to develop the mathematical properties of these functions, including series and integral representations, location of zeros, orthogonality properties, methods of representation of arbitrary functions, and methods of numerical calculation. Physical applications of the functions with imaginary argument will then be exhibited. These functions provide solutions of Laplace's equation useful in certain types of potential and heat flow problems in cylindrical coordinates; they also occur in the investigation of the stability of flow of a layer of fluid whose density and velocity vary with height, and in the study of the propagation of Love waves over the surface of an inhomogeneous elastic medium. Bessel functions of imaginary order and real argument give solutions of the wave equation which can be used to calculate the propagation of sound waves

or electromagnetic waves around a circular bend in a rectangular wave guide. They also occur in the solution of Schrödinger's equation for a particle in a radial force field when the potential is approximated by an exponential function, and in the solution of the relativistic Schrödinger equation for a free particle in an expanding universe when the radius of the universe is a linear function of time.

The appendix of the thesis contains a table of numerical values of Bessel functions of imaginary order and imaginary argument, covering representative ranges in both order and argument. It is felt that this table, even though it is of limited accuracy, will be of interest because it represents the only numerical tabulation of Bessel functions of imaginary order at present in existence.

CHAPTER I

Mathematical Properties of Bessel Functions of Imaginary Order

1.0. General Theory of Sturm-Liouville Equations.

The differential equation for Bessel functions of imaginary order $i\nu$ and imaginary argument ikx is obtained from 0.1 (1), by writing ikx for z, $i\nu$ for ν, and y for w, and dividing through by x, in the form

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) - \left(k^2x - \frac{\nu^2}{x}\right)y = 0, \tag{1}$$

where unless otherwise specified ν and x will always be regarded as real. The equation for functions of imaginary order $i\nu$ and real argument kx is similarly obtained as

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(k^2x + \frac{\nu^2}{x}\right)y = 0. \tag{2}$$

Both (1) and (2) are special cases of the self-adjoint Sturm equation

$$\frac{d}{dx}\left\{K(x)\frac{dy}{dx}\right\} - G(x)y = 0; \tag{3}$$

and many of the properties of their solutions can be deduced from general theorems concerning the solutions of (3) under specified boundary conditions. The Sturmian theory has been elegantly presented by Ince;[1)] a number of pertinent theorems will be quoted here for convenient reference.

We shall consider solutions of (3) in the closed interval $a \leq x \leq b$, throughout which K and G are continuous real functions of the real variable x. K does not vanish and may therefore be assumed positive; also K has a continuous first derivative throughout the interval. The theorems which we shall need are concerned principally with the zeros in (a, b) of the solutions of (3), and with the behavior of these zeros when the functions K(x) and G(x) are varied.

---

1) Ince, E. L., Ordinary Differential Equations, chaps. X-XI.

Theorem 1.[2] Let $y_1(x)$ and $y_2(x)$ be any two real linearly independent solutions of (3), and assume that y_1 vanishes at least twice in (a, b). Then between any two consecutive zeros of y_1 there is one and only one zero of y_2.

If a continuous function of x has two or more zeros in a given interval it is said to be oscillatory in that interval; if it has not more than one zero it is said to be non-oscillatory in the interval.

Theorem 2.[3] If the solutions of (3) oscillate in (a, b), they will oscillate more rapidly when K or G or both are diminished. For example, the solutions of (1) and (2) oscillate more rapidly with increasing ν^2.

It is not difficult to set up sufficient conditions for the oscillatory or non-oscillatory character of the solutions of an equation in a given interval.

Theorem 3.[4] Let K(x) and G(x) be bounded as follows: $\mathcal{K} \geq K \geq k > 0$ and $\mathcal{G} \geq G \geq g$ throughout (a, b). Then the solutions of (3) are non-oscillatory in (a, b) if either $g \geq 0$ or $-(g/k) < \pi^2/(b-a)^2$. A sufficient condition that the solutions of (3) should have at least m zeros in (a, b) is that $-(\mathcal{G}/\mathcal{K}) \geq m^2\pi^2/(b-a)^2$.

Theorem 4.[5] Let y(x) be that solution of (3) which satisfies the one-point boundary conditions $y(a) = \alpha$; $y'(a) = \alpha'$. If the zeros of y(x) are marked in order on the segment (a, b), the effect of diminishing K and/or G, while leaving α and α' invariant, is to cause all the roots to move in the direction from b toward a. If K and/or G diminish con-

---

2) Ince, op. cit., 224.
3) Ibid., 225-6.
4) Ibid., 227.
5) Ibid., 229.

tinuously (a process which may most easily be effected by supposing K and G to depend upon an auxiliary parameter λ), from time to time a new zero may enter the segment at b and move to the left toward a.

In an important special case of the Sturm equation the function G has the form $G = \ell - \lambda g$, where ℓ and g are real continuous functions of x in $a \leq x \leq b$, and λ is an arbitrary parameter. Many problems of mathematical physics require the solution of such an equation subject to assigned boundary conditions at two points; i. e., one must simultaneously satisfy:

$$\frac{d}{dx}\left\{K\frac{dy}{dx}\right\} - (\ell - \lambda g)y = 0, \tag{4.1}$$

$$\alpha' y(a) - \alpha y'(a) = 0, \tag{4.2}$$

$$\beta' y(b) + \beta y'(b) = 0, \tag{4.3}$$

where α, α', β, β' are independent of λ. Eqs. (4.1)-(4.3) comprise what is known as a Sturm-Liouville system. For any value of λ, (4.1), together with the boundary condition (4.2), has one and only one distinct solution, say $y = Y(x, \lambda)$. This solution, taken together with the second boundary condition (4.3), furnishes the characteristic equation

$$\mathcal{I}(\lambda) \equiv \beta' Y(b, \lambda) + \beta Y'(b, \lambda) = 0, \tag{5}$$

whose roots in λ are the eigenvalues (characteristic numbers) of the system (4). The solutions of (4) corresponding to the various eigenvalues are called eigenfunctions (characteristic functions) of the system.

Theorem 5.[6] If in the system (4) K, g, and ℓ are real continuous functions of x when $a \leq x \leq b$, are independent of λ, and are such that $K > 0$, $g > 0$, and if α, α', β, and β' are also independent of λ, then there exists an infinite set of real characteristic numbers $\lambda_0, \lambda_1, \lambda_2, \ldots$ which have no limit-point except $\lambda = +\infty$; if the corresponding character-

6) Ince, op. cit., 235.

istic functions are $y_0, y_1, y_2, \ldots$, then y_m has exactly m zeros in the interval $a < x < b$. If the additional conditions $\ell \geq 0$, $\alpha\alpha' \geq 0$, $\beta\beta' \geq 0$ are satisfied, then the characteristic numbers are all positive.

Theorem 6.[7] The eigenfunctions of the Sturm-Liouville system (4) are orthogonal over the range (a, b) with respect to the weight-function g; i. e., if $i \neq j$,

$$\int_a^b g(x) y_i(x) y_j(x)\, dx = 0. \tag{6.1}$$

If $i = j$,

$$\beta' \int_a^b g(x) y_i^2(x)\, dx = K(b) y_i(b) \mathcal{F}'(\lambda_i). \tag{6.2}$$

The eigenfunctions of (4) may, in case $g > 0$, conveniently be normalized so that

$$\int_a^b g y_i y_j\, dx = \delta_{ij} = \begin{cases} 0 \text{ if } i \neq j, \\ 1 \text{ if } i = j. \end{cases} \tag{7}$$

Theorem 7.[8] If, in the system (4), $g > 0$, then the characteristic numbers are all real and occur as simple roots of the characteristic equation (5).

Much of the importance of the eigenfunctions of a Sturm-Liouville system lies in the possibility of representing an arbitrary function f(x) in the interval (a, b) by means of a series of such functions. If we assume that it is possible to write, for $a \leq x \leq b$,

$$f(x) = \sum_{n=0}^{\infty} A_n y_n(x), \tag{8.1}$$

the coefficients may be formally determined, using (6), as

$$A_n = \frac{\int_a^b g(t) y_n(t) f(t)\, dt}{\int_a^b g(t) y_n^2(t)\, dt}. \tag{8.2}$$

Mercer[9] and others have in fact shown that the general Sturm-Liouville

---

7) Ince, op. cit., 237-241.
8) Ibid., 238, 241.
9) Mercer, J., Phil. Trans. Roy. Soc., (A), 211, 111-198 (1912).

series (8) corresponding to f(x) behaves in the same way as the ordinary Fourier series corresponding to f(x). A typical result is the following:

Theorem 8.[10] Let the function f(x) possess a Lebesgue integral in (a, b), and let f(x) have limited total fluctuation in an arbitrarily small neighborhood of a point x = s belonging to the open interval (a, b). Then the Sturm-Liouville series (8) converges at the point s to the sum $\frac{1}{2}[f(s+0) + f(s-0)]$.

For the comprehensive extension of the principal theorems on Fourier series to the whole class of Sturm-Liouville expansions, reference may be made to the work of Mercer cited above.

1.1. Bessel Functions of Imaginary Order and Imaginary Argument. Wedge Functions.

We pass now to consideration of Bessel's equation 0.1 (1) written with imaginary variable ix and imaginary parameter $i\nu$, so that it becomes:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 - \nu^2)y = 0. \tag{1}$$

Our first task will be to obtain a fundamental pair of solutions of (1) in useful form.

A series solution of the ordinary Bessel equation 0.1 (1) is customarily obtained around the regular singular point z = 0 in the form:[11]

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-)^m (\frac{1}{2}z)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}. \tag{2}$$

$J_\nu(z)$ is called the ordinary Bessel function of the first kind of argument z and order ν. It is a solution of 0.1 (1) for unrestricted complex values of z and ν; it is an analytic function of z for all values of z

10) Mercer, op. cit., 196.
11) Watson, G. N., Theory of Bessel Functions, 2nd ed., chap. 3, 38-45.

($z = 0$ possibly excepted), and an analytic function of ν for all values of ν. The function $J_{-\nu}(z)$ also satisfies 0.1 (1) and is linearly independent of $J_\nu(z)$ if ν is not a real integer, so that if and only if ν is not an integer $J_\nu(z)$ and $J_{-\nu}(z)$ form a fundamental system of solutions of Bessel's equation.

It is frequently convenient to take as standard solutions of Bessel's equation linear combinations of $J_\nu(z)$ and $J_{-\nu}(z)$ which approach distinct limits as ν becomes an integer. Particularly important are the two Hankel functions:[12)]

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\nu\pi i} J_\nu(z)}{i \sin \nu\pi}, \tag{3.1}$$

$$H_\nu^{(2)}(z) = \frac{J_{-\nu}(z) - e^{\nu\pi i} J_\nu(z)}{-i \sin \nu\pi}; \tag{3.2}$$

these (or their limits as ν approaches a real integer) represent a fundamental pair of solutions for all values of z and ν.

Standard notations for a fundamental pair of solutions of 0.1 (1), when the argument z is purely imaginary and the order ν is unrestricted, have been adopted as follows:[13)]

$$I_\nu(x) = e^{-\frac{\nu\pi i}{2}} J_\nu(ix) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}x\right)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}, \tag{4.1}$$

$$K_\nu(x) = \frac{\pi}{2}\,\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi} = \frac{\pi i}{2}\, e^{\frac{\nu\pi i}{2}} H_\nu^{(1)}(ix). \tag{4.2}$$

$I_\nu(x)$ and $K_\nu(x)$ are called modified Bessel functions of the first and second kinds respectively; they have been widely tabulated for real values of ν.

Since no restrictions were laid upon z and ν in the derivation of the series (2) for $J_\nu(z)$, it is evident that $J_{i\nu}(ix)$ and $J_{-i\nu}(ix)$ both

---

12) Watson, *op. cit.*, 73.
13) *Ibid.*, 77-8.

furnish solutions of Bessel's equation (1) with imaginary order and imaginary argument; in fact, since (1) has purely real coefficients, it is satisfied by both the real and the imaginary parts of $J_{i\nu}(ix)$ separately. However it is not convenient in practice to define standard solutions of (1) directly in terms of $J_{i\nu}(ix)$. Rather we wish a fundamental real pair of solutions whose form is as well adapted as possible to numerical computation, and whose asymptotic behavior for large values of the argument is simple. Such a pair may be compactly defined by forming from the modified Bessel functions $I_{i\nu}(x)$ and $I_{-i\nu}(x)$ the following real linear combinations, which will henceforth be regarded as canonical solutions of (1):

$$F_\nu(x) \equiv \frac{\pi}{2}\,\frac{I_{i\nu}(x)+I_{-i\nu}(x)}{\operatorname{sh}\nu\pi} \equiv \frac{\pi}{\operatorname{sh}\nu\pi}\,\operatorname{Re} I_{i\nu}(x),$$

$$G_\nu(x) \equiv \frac{i\pi}{2}\,\frac{I_{i\nu}(x)-I_{-i\nu}(x)}{\operatorname{sh}\nu\pi} \equiv -\frac{\pi}{\operatorname{sh}\nu\pi}\,\operatorname{Im} I_{i\nu}(x) \tag{5.1}$$

$$\equiv K_{i\nu}(x) \equiv \frac{\pi i}{2}\,e^{-\frac{\nu\pi}{2}}\,H^{(1)}_{i\nu}(ix), \tag{5.2}$$

where ν is real and x is real and positive. The linear independence of $F_\nu(x)$ and $G_\nu(x)$ follows from the independence of $I_{i\nu}(x)$ and $I_{-i\nu}(x)$ (Watson, p. 78), $i\nu$ being not a real integer.

Any solution of (1) may be called a Bessel function of imaginary order and imaginary argument; the special solutions $F_\nu(x)$ and $G_\nu(x)$ will be referred to as "wedge functions"* of the first and second kinds respectively. In the following sections various properties of the wedge functions will be developed.

1.11. Series and Integral Representations of Wedge Functions.

Series representations of the wedge functions in the neighborhood of the origin are complicated by the fact that both functions have an oscil-

---

*This name was suggested by Prof. Smythe in view of the application of these functions to potential theory (Art. 2.1), where they show a certain analogy to the solutions of Legendre's equation called "cone functions".

latory discontinuity at $x = 0$, the nature of this singularity being due to the circumstance that the exponents $\pm i\nu$ of the differential equation at the origin are purely imaginary. We may however obtain expressions for the wedge functions in terms of series of modified Bessel functions $I_m(x)$, which indicate clearly the behavior of $F_\nu(x)$ and $G_\nu(x)$ near the origin and which can be used for numerical calculation when x is small.

We consider the following series[14] due to Lommel, which is valid for unrestricted z if $\mu \neq \nu$ and ν is not a negative integer:

$$J_\nu(z) = \frac{\Gamma(\mu+\nu)}{\Gamma(\nu-\mu)} \sum_{m=0}^{\infty} \frac{\Gamma(\nu-\mu+m)}{\Gamma(\nu+m+1)} \frac{\left(\frac{z}{2}\right)^{\nu-\mu+m}}{m!} J_{\mu+m}(z). \tag{1}$$

Replacing z by ix, ν by $i\nu$, μ by 0, multiplying through by $e^{\frac{1}{2}\nu\pi}$, and using 1.1 (4.1), we have, after some simplification,

$$\begin{aligned} I_{i\nu}(x) &= e^{\nu\pi/2} J_{i\nu}(ix) \\ &= \frac{e^{i\nu \log \frac{x}{2}}}{\Gamma(i\nu)} \sum_{m=0}^{\infty} \frac{(-)^m \left(\frac{x}{2}\right)^m I_m(x)}{m!\,(m+i\nu)} = \frac{e^{i\nu \log \frac{x}{2}}}{\Gamma(i\nu)} \left[A(\nu,x) - iB(\nu,x)\right] \end{aligned} \tag{2}$$

where
$$A(\nu,x) = \sum_{m=1}^{\infty} \frac{m(-)^m \left(\frac{x}{2}\right)^m I_m(x)}{m!\,(m^2+\nu^2)}. \tag{3.1}$$

and
$$B(\nu,x) = \sum_{m=0}^{\infty} \frac{\nu(-)^m \left(\frac{x}{2}\right)^m I_m(x)}{m!\,(m^2+\nu^2)}. \tag{3.2}$$

Setting
$$\Theta(\nu,x) = \nu \log \frac{x}{2} - \arg \Gamma(i\nu)$$
and using the known relation[15]
$$|\Gamma(i\nu)| = (\pi/\nu \sinh \nu\pi)^{\frac{1}{2}}, \tag{4}$$

we have from 1.1 (5.1) and (5.2) the results:

$$\begin{aligned} F_\nu(x) &= (\pi/\sinh \nu\pi)\, \mathrm{Re}\, I_{i\nu}(x) \\ &= \sqrt{\nu\pi/\sinh \nu\pi}\, \left[A(\nu,x) \cos \Theta(\nu,x) + B(\nu,x) \sin \Theta(\nu,x)\right], \end{aligned} \tag{5.1}$$

$$\begin{aligned} G_\nu(x) &= -(\pi/\sinh \nu\pi)\, \mathrm{Im}\, I_{i\nu}(x) \\ &= \sqrt{\nu\pi/\sinh \nu\pi}\, \left[B(\nu,x) \cos \Theta(\nu,x) - A(\nu,x) \sin \Theta(\nu,x)\right]. \end{aligned} \tag{5.2}$$

---

14) Watson, op. cit., 143.

15) Whittaker, E. T., and Watson, G. N., Modern Analysis, 4th ed., 259, ex. 7.

The function arg $\Gamma(i\nu)$ may be computed from power series[16)] for small values of ν and from Stirling's asymptotic series [16)] for log $\Gamma(i\nu)$ when ν is large. Successive terms of the series for $A(\nu, x)$ and $B(\nu, x)$ decrease sufficiently rapidly when x is moderately small to facilitate computation of these auxiliary functions. In Art. 1.13 $A(\nu, x)$ and $B(\nu, x)$ will be expressed as series in ascending powers of x.

Simple definite integral expressions for the wedge functions may be obtained from known integral representations for $I_\nu(z)$. We have[17)]

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos\nu\theta\, d\theta - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-z\,ch\,t-\nu t}\,dt$$

for unrestricted values of ν if $|\arg z| < \frac{1}{2}\pi$. Letting z be real (= x) and positive and replacing ν by $i\nu$, the formula becomes

$$I_{i\nu}(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\,ch\,\nu\theta\, d\theta - \frac{i\,sh\,\nu\pi}{\pi}\int_0^\infty e^{-x\,ch\,t-i\nu t}\,dt.$$

Separation of real and imaginary parts according to 1.1 (5.1) and (5.2) leads to the useful results:

$$F_\nu(x) = \frac{1}{sh\,\nu\pi}\int_0^\pi e^{x\cos\theta}\,ch\,\nu\theta\, d\theta - \int_0^\infty e^{-x\,ch\,t}\sin\nu t\, dt, \tag{6.1}$$

$$G_\nu(x) = \int_0^\infty e^{-x\,ch\,t}\cos\nu t\, dt. \tag{6.2}$$

The integral representing $G_\nu(x)$ is particularly simple and is easily evaluated by mechanical quadrature provided x is moderately large, so that the exponential factor in the integrand becomes negligible before the cosine term has undergone many oscillations. The first (finite) integral in $F_\nu(x)$ may be split into various parts which are not difficult to calculate separately; some details are given in connection with the numerical table in the appendix.

---

16) Davis, H. T., Tables of the Higher Mathematical Functions, vol. 1, 181-185.

17) Watson, op. cit., 181, eq. (4).

1.12. Asymptotic Behavior of Wedge Functions.

Asymptotic representations of the wedge functions for large argument and fixed order are easily obtained from the known asymptotic series[18] for $I_\nu(z)$ and $K_\nu(z)$. Using the notation

$$(\nu, m) \equiv (-\nu, m) \equiv \frac{\Gamma(\nu+m+\frac{1}{2})}{m!\,\Gamma(\nu-m+\frac{1}{2})}, \tag{1}$$

we have, for $|\arg z| < 3\pi/2$,

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2z)^m}, \tag{2.1}$$

and, for $-\pi/2 < \arg z < 3\pi/2$,

$$I_\nu(z) \sim \frac{e^z}{(2\pi z)^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{(-)^m(\nu, m)}{(2z)^m} + \frac{e^{-z+(\nu+\frac{1}{2})\pi i}}{(2\pi z)^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2z)^m}, \tag{2.2}$$

the second series on the right being negligible compared with the first if $|\arg z| < \frac{1}{2}\pi$. Putting $i\nu$ for ν and x for z in (2.2), substituting (2.2) into 1.1 (5.1), and expanding $(i\nu, m)$, we find that, when ν is fixed and x is large and positive,

$$E_\nu(x) \sim \frac{e^x}{sh\,\nu\pi}\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}\left[1 + \frac{(4\nu^2+1^2)}{1!(8x)} + \frac{(4\nu^2+1^2)(4\nu^2+3^2)}{2!(8x)^2} + \dots\right]. \tag{3.1}$$

Similarly from (2.1) and 1.1 (5.2) we have,* when ν is fixed and x is large and positive,

$$G_\nu(x) = K_{i\nu}(x) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}\left[1 - \frac{(4\nu^2+1^2)}{1!(8x)} + \frac{(4\nu^2+1^2)(4\nu^2+3^2)}{2!(8x)^2} - \dots\right]. \tag{3.2}$$

---

18) Watson, op. cit., 202-203.

*It is a little tricky to calculate the asymptotic expansion of $G_\nu(x)$ directly by substituting the "negligible" series of (2.2) into the first equation of 1.1 (5.2), since if we include this series in the expression for $I_\nu(z)$ we no longer find $I_{-i\nu}(x) = \overline{I_{+i\nu}(x)}$. Furthermore the "negligible" series exhibits Stokes' phenomenon in passing through the region $|\arg z| < \frac{1}{2}\pi$, since in the range $-3\pi/2 < \arg z < \pi/2$ the exponential factor is exp $[-z-(\nu+\frac{1}{2})\pi i]$.

It may be noted that the two wedge functions behave at infinity in a manner similar to that of the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ of real order; i. e., one tends exponentially to infinity, the other exponentially to zero. Since in applications to physical problems it is frequently necessary to find a solution of 1.1 (1) which vanishes for large positive values of the argument, the canonical definitions 1.1 (5.1) and (5.2) were chosen with this end in view; evidently no Bessel function of imaginary order and imaginary argument can vanish at infinity if it is linearly distinct from $G_\nu(x)$.

The asymptotic series (3.1) and (3.2) are useful for numerical calculation when x is moderately large, provided that ν is not of magnitude comparable to x. (The larger ν, the less rapidly do successive terms diminish.) In practice the number of significant figures obtainable from an asymptotic series may be greatly increased by the use of a "convergence factor." This technique has been developed by J. R. Airey[19] and is adapted for the calculation of Bessel functions of imaginary order, as Airey shows by an illustrative example.

In the neighborhood of $x = 0$ the wedge functions both oscillate infinitely rapidly, being essentially sinusoidal functions of $\nu \log x$ with phase constants depending on ν. Their limiting forms may be deduced from 1.11 (5). We substitute into 1.11 (3) the relation $I_m(0) = \delta_{om}$ and obtain $A(\nu,0) = 0$, $B(\nu,0) = 1/\nu$; then we find from 1.11 (5) that if ν is fixed as x tends to zero,

$$F_\nu(x) \longrightarrow \sqrt{\pi/\nu \sinh \nu\pi}\, \sin\left[\nu \log \tfrac{1}{2}x - \arg \Gamma(i\nu)\right], \tag{4.1}$$

$$G_\nu(x) \longrightarrow \sqrt{\pi/\nu \sinh \nu\pi}\, \cos\left[\nu \log \tfrac{1}{2}x - \arg \Gamma(i\nu)\right]. \tag{4.2}$$

---

19) Airey, J. R., Phil. Mag., (7), 24, 521-552 (1937).

It is sometimes of interest to know the behavior of the wedge functions for large values of the order ν, the argument remaining fixed. The dominant terms of the asymptotic expansions in ν of the functions may be obtained[20] from the defining series 1.1 (4.1) for $I_\nu(x)$. If we write $i\nu$ for ν, the series becomes:

$$I_{i\nu}(x) = \frac{(\frac{1}{2}x)^{i\nu}}{\Gamma(i\nu+1)}\left[1 + \frac{(\frac{x}{2})^2}{1!\,(i\nu+1)} + \cdots\right].$$

We substitute for the Γ-function Stirling's approximation,

$$\Gamma(i\nu+1) \sim (i\nu/e)^{i\nu}\sqrt{2\pi i\nu}\,\left[1 + O(\tfrac{1}{\nu})\right], \tag{5}$$

and obtain

$$I_{i\nu}(x) \sim \frac{1}{\sqrt{2\pi\nu}}\exp\left[i\nu\log\frac{x}{2} - i\nu(\log\nu + \frac{i\pi}{2} - 1) - \frac{i\pi}{4}\right]\left\{1 + O(\tfrac{1}{\nu})\right\}$$

$$= \frac{e^{\nu\pi/2}}{\sqrt{2\pi\nu}}\exp i\left[\nu(\log\frac{x}{2} - \log\nu + 1) - \frac{\pi}{4}\right]\left\{1 + O(\tfrac{1}{\nu})\right\} \tag{6}$$

If we recall that for large ν $\operatorname{sh}\nu\pi$ differs negligibly from $\frac{1}{2}e^{\nu\pi}$, equations 1.1 (5.1) and (5.2) yield the following asymptotic expressions for the wedge functions when ν is large and x is fixed:

$$F_\nu(x) \sim e^{-\nu\pi/2}\sqrt{\frac{2\pi}{\nu}}\cos\left[\nu(\log\nu - \log\frac{x}{2} - 1) + \frac{\pi}{4}\right]\left\{1 + O(\tfrac{1}{\nu})\right\}, \tag{7.1}$$

$$G_\nu(x) \sim e^{-\nu\pi/2}\sqrt{\frac{2\pi}{\nu}}\sin\left[\nu(\log\nu - \log\frac{x}{2} - 1) + \frac{\pi}{4}\right]\left\{1 + O(\tfrac{1}{\nu})\right\}. \tag{7.2}$$

From these expressions it is evident that both canonical solutions of Bessel's equation 1.1 (1) with imaginary order and imaginary argument, regarded as functions of their order ν, undergo an infinite number of oscillations of exponentially decreasing amplitude and slowly decreasing wavelength as ν increases without limit.

The limiting forms of the wedge functions when ν tends to zero, x remaining fixed, may be seen immediately from the defining equations

---

20) Cf. Watson, op. cit., 225.

1.1 (5.1) and (5.2). These forms are:

$$F_\nu(x) \underset{\nu\to 0}{\longrightarrow} \frac{I_0(x)}{\nu} \underset{\nu\to 0}{\longrightarrow} \infty, \tag{8.1}$$

$$G_\nu(x) \underset{\nu\to 0}{\longrightarrow} K_0(x). \tag{8.2}$$

Asymptotic expressions for the various solutions of Bessel's equation valid when the order ν and the argument z are simultaneously large and of comparable magnitude have been derived for general complex values of ν and z; but the analysis is lengthy and the results are complicated by the necessity for treating numerous subcases separately. We shall not take space here to apply these general results to the special case of our wedge functions; reference may be made if desired to the complete treatment given by Watson.[21)]

1.13. Alternative Definitions of Bessel Functions of Imaginary Order and Imaginary Argument.

In connection with the definitions of the wedge functions $F_\nu(x)$ and $G_\nu(x)$ which we have adopted in this work, we may naturally inquire whether any other fundamental set of solutions with more convenient properties has ever been suggested. A brief discussion of the real and imaginary parts of the function $J_{\nu+\mu i}(x)$ of complex order was given by Lommel[22)] many years ago; but the only attempt at anything like a systematic treatment of Bessel functions of purely imaginary order is that of M. Bocher.[23)] We shall summarize the relations between the functions defined by Bocher and our functions $F_\nu(x)$ and $G_\nu(x)$.

Bocher first defines a particular solution $\{J_n(z)\}$ of the ordinary Bessel equation 0.1 (1) by writing, for unrestricted complex values of

---

21) Watson, op. cit., chap. VIII.
22) Lommel, E., Math. Ann., 3, 481-486 (1871).
23) Bocher, M., Annals of Mathematics, 6, 137-160 (1892).

ν and z,

$$\{J_n(z)\} \equiv 2^n \Gamma(n+1)\, J_n(z), \tag{1}$$

where $J_n(z)$ is the ordinary Bessel function of the first kind. When z (= x) is real and positive and n ($= i\nu$) is purely imaginary, he defines two real independent solutions of the differential equation as:

$$H_{i\nu}(x) \equiv \tfrac{1}{2}\left[\{J_{i\nu}(x)\} + \{J_{-i\nu}(x)\}\right], \tag{2.1}$$

$$I_{i\nu}(x) \equiv \tfrac{1}{2i}\left[\{J_{i\nu}(x)\} - \{J_{-i\nu}(x)\}\right]. \tag{2.2}$$

(Bocher's $I_{i\nu}(x)$ is not to be confused with the modified Bessel function of the first kind, for which elsewhere in this thesis we use the customary modern notation $I_\nu(z)$.)

Bocher goes on to find that

$$H_{i\nu}(x) = \cos(\nu \log x)\, S_1(x) + \sin(\nu \log x)\, S_2(x), \tag{3.1}$$

$$I_{i\nu}(x) = -\cos(\nu \log x)\, S_2(x) + \sin(\nu \log x)\, S_1(x), \tag{3.2}$$

where $S_1(x)$ and $S_2(x)$ denote the following power series:

$$S_1(x) = 1 - \frac{1}{4(1^2+\nu^2)}x^2 + \frac{(2)_2 - (2)_0\nu^2}{4^2 2!\,(1^2+\nu^2)(2^2+\nu^2)}x^4 - \frac{(3)_3 - (3)_1\nu^2}{4^3 3!\,(1^2+\nu^2)(2^2+\nu^2)(3^2+\nu^2)}x^6 + \frac{(4)_4 - (4)_2\nu^2 + (4)_0\nu^4}{4^4 4!\,(1^2+\nu^2)\cdots(4^2+\nu^2)}x^8 - \frac{(5)_5 - (5)_3\nu^2 + (5)_1\nu^4}{4^5 5!\,(1^2+\nu^2)\cdots(5^2+\nu^2)}x^{10} + \frac{(6)_6 - (6)_4\nu^2 + (6)_2\nu^4 - (6)_0\nu^6}{4^6 6!\,(1^2+\nu^2)\cdots(6^2+\nu^2)}x^{12} - \cdots \tag{4.1}$$

$$S_2(x) = -\frac{\nu}{4(1^2+\nu^2)}x^2 + \frac{(2)_1\nu}{4^2 2!\,(1^2+\nu^2)(2^2+\nu^2)}x^4 - \frac{(3)_2\nu - (3)_0\nu^3}{4^3 3!\,(1^2+\nu^2)\cdots(3^2+\nu^2)}x^6 + \frac{(4)_3\nu - (4)_1\nu^3}{4^4 4!\,(1^2+\nu^2)\cdots(4^2+\nu^2)}x^8 - \frac{(5)_4\nu - (5)_2\nu^3 + (5)_0\nu^5}{4^5 5!\,(1^2+\nu^2)\cdots(5^2+\nu^2)}x^{10} + \frac{(6)_5\nu - (6)_3\nu^3 + (6)_1\nu^5}{4^6 6!\,(1^2+\nu^2)\cdots(6^2+\nu^2)}x^{12} - \cdots \tag{4.2}$$

The symbol $(p)_q$, where p and q are any positive integers such that $q \leq p$, denotes the sum of all of the different products which can be formed by multiplying together q of the p factors 1, 2,..., p.* By definition $(p)_0 = 1$ and $(p)_q = 0$ if $q > p$ or if $q < 0$.

As a fundamental real set of Bessel functions whose order $i\nu$ and argument ix are both purely imaginary, Bocher defines:

$$\begin{aligned}\overline{H}_{i\nu}(ix) &\equiv Re\left[e^{\frac{\nu\pi}{2}}\{J_{i\nu}(ix)\}\right] \equiv Re\, e^{\frac{\nu\pi}{2}}\left[H_{i\nu}(ix) + i\,I_{i\nu}(ix)\right] \\ &\equiv e^{\frac{\nu\pi}{2}} Re\left[e^{i\nu\log(ix)} S_1(ix) - i\,e^{i\nu\log(ix)} S_2(ix)\right] \\ &\equiv \cos(\nu\log x)\, S_1(ix) + \sin(\nu\log x)\, S_2(ix),\end{aligned} \qquad (5.1)$$

$$\begin{aligned}\overline{I}_{i\nu}(ix) &\equiv Im\left[e^{\frac{\nu\pi}{2}}\{J_{i\nu}(ix)\}\right] \\ &\equiv \sin(\nu\log x)\, S_1(ix) - \cos(\nu\log x)\, S_2(ix).\end{aligned} \qquad (5.2)$$

The series $S_1(ix)$ and $S_2(ix)$ are evidently real when x is real; they are simply related to the functions which were denoted by $A(\nu,x)$ and $B(\nu,x)$ in Art. 1.11. We may deduce this relation by substituting for $\{J_{i\nu}(ix)\}$ from (1) into (5.1) and then comparing (5.1) with 1.11 (2); thus:

$$\begin{aligned}e^{\frac{\nu\pi}{2}}\{J_{i\nu}(ix)\} &\equiv e^{i\nu\log x}\left[S_1(ix) - i\,S_2(ix)\right] \\ &\equiv 2^{i\nu}\Gamma(i\nu+1)\, e^{\frac{\nu\pi}{2}} J_{i\nu}(ix) \equiv \frac{\Gamma(i\nu+1)\, 2^{i\nu} e^{i\nu\log\frac{1}{2}x}}{\Gamma(i\nu)}\left[A(\nu,x) - i\,B(\nu,x)\right] \\ &\equiv e^{i\nu\log x}\left[i\nu A(\nu,x) + \nu B(\nu,x)\right].\end{aligned}$$

If we cancel the exponential factor from the second and fifth members of this equation and equate separately the real and imaginary parts, we have at once

$$A(\nu,x) = -\frac{1}{\nu} S_2(ix); \quad B(\nu,x) = \frac{1}{\nu} S_1(ix). \qquad (6)$$

Bocher's solutions $\overline{H}_{i\nu}(ix)$ and $\overline{I}_{i\nu}(ix)$ must of course be expressible in terms of any other fundamental set of solutions of the differential equation; it is an elementary exercise to write them as linear combinations, with coefficients depending on ν, of $F_\nu(x)$ and $G_\nu(x)$. Since clearly both

---

*For example, $(p)_p = p!$, $(p)_{p-1} = p!(1 + 1/2 + 1/3 + \dots + 1/p)$, and $(p)_1 = 1 + 2 + \dots + p$; Bocher presents a short table of values of $(p)_q$ calculated from the recursion formula $(p)_q = (p-1)_q + p(p-1)_{q-1}$.

$\overline{H}_{i\nu}(ix)$ and $\overline{I}_{i\nu}(ix)$ depend linearly upon $F_\nu(x)$, which becomes exponentially infinite for large positive values of the argument while $G_\nu(x)$ tends to zero, <u>both</u> functions tend to infinity for large x. But in the physical problems where Bessel functions occur, e. g. in electromagnetic theory, a frequent boundary condition is the requirement that the quantities involved shall vanish at infinity. It is therefore of considerable importance to choose one of the canonical solutions of our differential equation so that it does vanish at infinity. For this reason, in spite of the relatively simple limiting forms of Bocher's functions near x = 0, we shall not employ these functions in our work.

The requirement that one of the wedge functions vanish for large values of the argument still leaves at our disposal in fixing the canonical definition of the function an arbitrary multiplicative factor which may depend upon ν. The definition actually chosen for $G_\nu(x)$ in Art. 1.1 was suggested by the observation that the familiar modified Bessel function $K_\nu(x)$ of real positive argument, defined for general values of ν by 1.1 (4.2), is a real function when the order is purely imaginary, and that this function has the simple definite integral representation 1.11 (6.2). We accordingly defined $G_\nu(x) \equiv K_{i\nu}(x)$, and then chose the definition of the other canonical solution $F_\nu(x)$ to exhibit as much formal symmetry as possible with $G_\nu(x)$.

The fact that the amplitudes of both $F_\nu(x)$ and $G_\nu(x)$ decrease exponentially with increasing order for any fixed value of x (cf. 1.12 (7)) necessitates the use in numerical tables of negative powers of 10 to take account of the wide variation of the wedge functions in absolute magnitude. It is likely that if more extensive tables than ours are ever undertaken, the functions tabulated will be the more convenient ones $e^{\frac{\nu\pi}{2}} F_\nu(x)$ and

$e^{\frac{\nu\pi}{2}} G_\nu(x)$, with a short auxiliary table of $e^{\frac{\nu\pi}{2}}$. A similar device has already been used with the modified Bessel functions;[24] namely, for large values of the argument one tabulates not the functions themselves but the combinations $e^{-x}I_n(x)$ and $e^{x}K_n(x)$. These latter functions vary slowly over a wide range of values of x and are smooth enough to permit accurate interpolation.

One of the considerations involved in fixing the standard definitions of the various kinds of Bessel functions is the desirability of giving as simple a form as possible to the recurrence relations which exist between the functions of different orders. These recurrence relations, which connect for example the function $K_n(x)$ with the functions $K_{n\pm 1}(x)$ and their derivatives, are a consequence of the fact that Bessel's equation is a confluent form of the hypergeometric equation;[25] they are quite useful in simplifying the results of analysis and especially in the calculation of numerical tables. However the recurrence formulas are of little practical value if the orders of the functions concerned are not all real; for example the relations involving $K_{i\nu}(x)$ connect this function with the functions $K_{i\nu\pm 1}(x)$ of complex order, or in our notation they connect $G_\nu(x)$ with $G_{\nu\mp i}(x)$. The existence of a linear relation connecting $G_\nu(x)$ with $G_{\nu\pm 1}(x)$ is not guaranteed by the form of the differential equation; and it does not appear likely that any such recurrence formula can be secured by adjusting the definitions of the wedge functions.*

---

24) British Association for the Advancement of Science, Mathematical Tables, vol. VI, part 1, Cambridge, 1937. Table VIII.

25) Whittaker and Watson, op. cit., 359-360 et seq.

*Professor Bateman expressed in conversation with the author the opinion that the chances of finding such a relation were very remote.

1.2. Zeros of Bessel Functions of Imaginary Order and Imaginary or Complex Argument.

In the first part of the present section we are concerned with the zeros of the solutions of the equation

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) - \left(x - \frac{\nu^2}{x}\right)y = 0 \tag{1}$$

for Bessel functions of imaginary order and imaginary argument, when the solutions are regarded as functions of the real variables x and ν. Later we shall prove certain theorems involving the zeros of Bessel functions of imaginary order and complex argument, which will be of use in the hydrodynamical investigations of Art. 2.2.

With the notation of Art. 1.0, where ν and x are real, (1) is a Sturm equation in which $K(x) = x$, $G(x) = x - \nu^2/x$.* We shall be interested in the solutions of (1) in the closed interval $0 < a \leq x \leq b < \infty$, throughout which K(x) and G(x) are bounded by $\mathcal{K} \geq K \geq \mathcal{k} > 0$ and $\mathcal{G} \geq G \geq \mathcal{g}$, where $\mathcal{K} = b$, $\mathcal{k} = a$, $\mathcal{G} = b - \nu^2/b$, and $\mathcal{g} = a - \nu^2/a$.

Theorem 1. (i). Any real solution of (1), considered as a function of x, has an infinite number of real zeros in the interval between x = 0 and $x = \nu$.

(ii). No solution of (1) has more than one real zero to the right of $x = \nu$.

(iii). If (a, b) is any preassigned finite interval of the positive x-axis and m is any given positive integer, then for sufficiently large values of ν every real solution of (1) will have at least m zeros in (a, b).

Part (i) of the theorem follows most readily by observing from 1.12

---

*No confusion will be caused by this notation, since when K and G are used to denote Bessel functions they will always carry appropriate subscripts.

(4.1) and (4.2) that for small values of x every real solution of (1) has the limiting form

$$y_\nu(x) \to A(\nu)\ \sin[\nu \log x + \delta(\nu)]\ , \qquad (2)$$

where the amplitude $A(\nu)$ and the phase constant $\delta(\nu)$ are independent of x. The argument of the sine passes through all negative integral multiples of π as $x \to +0$; so the origin is a limit-point of zeros of all real solutions of (1). Parts (ii) and (iii) follow directly from theorem 3 of Art. 1.0. If $a \geqq \nu$, then $g = a(1 - \nu^2/a^2) \geqq 0$; so the solutions of (1) cannot oscillate for $x \geqq \nu$. If a, b, and m are fixed, a sufficient condition for the solutions to have at least m zeros in (a, b) is

$$-g/K = \nu^2/b^2 - 1 \geqq m^2\pi^2/(b - a)^2;$$

and the inequality certainly holds for all sufficiently large values of ν.

Since $G(x) = x - \nu^2/x$ is decreased by increasing ν^2, theorem 2 of Art. 1.0 shows that the higher the order ν, the more rapidly will the solutions of (1) oscillate in the neighborhood of a given point; the increased rate of oscillation is of course obvious in the limiting form (2).

It is qualitatively apparent from theorem 1 above that, as the order ν of the wedge functions $F_\nu(x)$ and $G_\nu(x)$ is continuously increased, the real zeros of these functions move steadily to the right into intervals previously zero-free. The sudden appearance of a new zero between two old zeros of either function is precluded; since $F_\nu(x)$ and $G_\nu(x)$ are continuous functions varying continuously with ν, any such new zero would have to appear as a double zero,[26] and no solution of (1) can possess a double zero at an ordinary point unless it vanish identically.

Theorem 2. If A and B are real constants independent of ν and if x has any fixed value, the linear combination of wedge functions

---

26) Cf. Ince, op. cit., 229, n. 2.

$$y_\nu(x) = AF_\nu(x) + BG_\nu(x), \tag{3}$$

considered as a function of ν, has an infinite number of zeros for increasing values of ν with a limit-point at $+\infty$.

From 1.12 (7.1) and (7.2) we have the asymptotic form of y_ν when ν is large and x is fixed; namely,

$$y_\nu(x) \sim C e^{-\frac{\nu\pi}{2}} \sqrt{\frac{2\pi}{\nu}} \sin\left[\nu\left(\log\nu - \log\frac{x}{2} - 1\right) + \delta\right]\left\{1 + O\left(\frac{1}{\nu}\right)\right\}, \tag{4}$$

from which the theorem is evident.

The complex zeros of the solutions of the Sturm equation,

$$\frac{d}{dz}\left\{K(z)\frac{dw}{dz}\right\} - G(z)w = 0, \tag{5}$$

may be investigated by the use of a certain integral equality known as the Green's transform.[27] It is supposed that K(z) and G(z) are analytic in a domain D throughout which K(z) does not vanish; and (5) is replaced by the pair of equations

$$dw_1/dz = w_2/K(z), \qquad dw_2/dz = G(z)\,w_1, \tag{6}$$

$$\text{where } w_1 = w,\ w_2 = K(z)\,dw/dz. \tag{7}$$

On combining the complex conjugate of the first member of (6) with the second member, we get

$$w_2\,\overline{dw_1} + \overline{w_1}\,dw_2 = |w_2|^2\,\overline{dz/K(z)} + |w_1|^2\,G(z)\,dz,$$

which, being integrated between limits z_1 and z_2 along a path of integration lying wholly within D, yields the Green's transform of (5), namely:

$$\left[\overline{w_1}\,w_2\right]_{z_1}^{z_2} - \int_{z_1}^{z_2}\frac{|w_2|^2\,\overline{dz}}{\overline{K(z)}} - \int_{z_1}^{z_2}|w_1|^2 G(z)\,dz = 0. \tag{8}$$

Let $dz/K(z) \equiv dK \equiv dK_1 + i\,dK_2, \quad G(z)\,dz \equiv dG \equiv dG_1 + i\,dG_2,$ (9)

and split the Green's transform into real and imaginary parts:

$$\mathcal{R}e\left[\overline{w_1}\,w_2\right]_{z_1}^{z_2} = \int_{z_1}^{z_2}|w_2|^2\,dK_1 + \int_{z_1}^{z_2}|w_1|^2\,dG_1, \tag{10.1}$$

---

27) Ince, op. cit., chap. XXI. We define G(z) with opposite sign to that used by Ince, in order to keep our notation consistent with Art. 1.0.

$$\mathcal{I}m\left[\bar{w}_1 w_2\right]_{z_1}^{z_2} = -\int_{z_1}^{z_2} |w_2|^2 d\mathcal{K}_2 + \int_{z_1}^{z_2} |w_1|^2 d\mathcal{G}_2. \tag{10.2}$$

Recalling (7), we see that if the solution $w(z)$ of (5) vanishes at z_1, it cannot also vanish at z_2 unless the right sides of (10.1) and (10.2) both vanish. In particular it does not vanish at z_2 if we can find a path of integration in D connecting z_1 and z_2 throughout which a definite one of the following four pairs of inequalities is satisfied:

$$(a)\ \begin{matrix} d\mathcal{K}_1 \geq 0, \\ d\mathcal{G}_1 \geq 0; \end{matrix} \quad (b)\ \begin{matrix} d\mathcal{K}_1 \leq 0, \\ d\mathcal{G}_1 \leq 0; \end{matrix} \quad (c)\ \begin{matrix} d\mathcal{K}_2 \leq 0, \\ d\mathcal{G}_2 \geq 0; \end{matrix} \quad (d)\ \begin{matrix} d\mathcal{K}_2 \geq 0, \\ d\mathcal{G}_2 \leq 0. \end{matrix} \tag{11}$$

Our first application of this theory will be to the modified Bessel equation

$$\frac{d}{dz}\left(z\,\frac{dw}{dz}\right) - \left(z + \frac{\nu^2}{z}\right)w = 0, \tag{12}$$

obtained by writing iz for z in Bessel's equation 0.1 (1). Any solution of (12) will be called a modified Bessel function of order ν (here assumed real) and argument z. In this case we have $K(z) = z$, $G(z) = z + \nu^2/z$; the domain D includes the whole complex plane, cut along the negative half of the real axis, except for a small circle excluding the origin. An elementary calculation gives, for the quantities defined in (9),

$$d\mathcal{K} \equiv d\mathcal{K}_1 + i\,d\mathcal{K}_2 = dr/r + i\,d\theta, \tag{13.1}$$

$$\begin{aligned} d\mathcal{G} &\equiv d\mathcal{G}_1 + i\,d\mathcal{G}_2 \\ &= \left\{\left(r\cos 2\theta + \frac{\nu^2}{r}\right)dr - r^2\sin 2\theta\, d\theta\right\} + i\left\{r\sin 2\theta\, dr + (r^2\cos 2\theta + \nu^2)\,d\theta\right\} \end{aligned} \tag{13.2}$$

where $z = re^{i\theta}$, $-\pi < \theta < \pi$.

Useful in the statement of the results which we shall prove are the two curves whose equations in polar coordinates are

$$\mathcal{G}_2(r, \theta) = \tfrac{1}{2}r^2\sin 2\theta + \nu^2\theta = \pm\tfrac{1}{2}\pi\nu^2. \tag{14}$$

The equation $\mathcal{G}_2(r, \theta) = +\frac{1}{2}\pi\nu^2$ represents the positive imaginary axis $\theta = \frac{1}{2}\pi$ plus the locus of points satisfying the relation

$$r^2 = \nu^2\,(\pi - 2\theta)\csc 2\theta \qquad (14.1)$$

for $0 < |\pi - 2\theta| < \pi$. The latter locus is a bell-shaped or witch-shaped curve symmetrical about the imaginary axis $\theta = \frac{1}{2}\pi$, having a flat maximum $y = \nu$ at $x = 0$, and asymptotic to the real axis for large values of x ($\theta \to 0 + 0$ or $\theta \to \pi - 0$). The equation $\mathcal{J}_1(r, \theta) = -\frac{1}{2}\pi\nu^2$ represents the reflection in the real axis of $\mathcal{J}_1(r, \theta) = +\frac{1}{2}\pi\nu^2$.

Theorem 3. (i). No modified Bessel function of real order can have two complex roots whose imaginary parts are equal and whose real parts have the same sign.

(ii). No such function can have two complex roots with equal imaginary parts whose representative points lie outside the open region between the two curves $r^2 = \pm\nu^2\,(\pi - 2\theta)\csc 2\theta$.

For part (i) assume that the modified Bessel function $R_\nu(z)$ which vanishes at $z_1 = x_1 + ib$ also vanishes at $z_2 = x_2 + ib$, where for convenience we take $x_2 > x_1$. Assume at first that both roots are in the first quadrant, so that $x_2 > x_1 \geq 0$ and $b > 0$. We carry out the integration of (8) over the straight line $y = b$ from z_1 to z_2. Along this segment $x \geq 0$, $dx > 0$, and $dy = 0$; so on writing out in rectangular coordinates the quantities defined in (9) we find that

$$dK_1 = \operatorname{Re}\,(dz/z) = (x\,dx + y\,dy)/(x^2 + y^2) = (x\,dx)/(x^2 + b^2) \geq 0,$$

$$d\mathcal{J}_1 = \operatorname{Re}\,(z + \nu^2/z)dz = x[1 + \nu^2/(x^2 + y^2)]\,dx + y[-1 + \nu^2/(x^2 + y^2)]\,dy$$

$$= x[1 + \nu^2/(x^2 + b^2)]\,dx \geq 0.$$

Hence the inequalities (11a) are satisfied throughout the path of integration, and $R_\nu(z)$ cannot vanish both at z_1 and at z_2. The occurrence of a pair of complex roots with equal imaginary parts in any other quadrant is ruled out in an exactly similar way.

For part (ii) assume that $R_\nu(z)$ vanishes both at $z_1 = x_1 + ib = r_1 e^{i\theta_1}$

and at $z_2 = x_2 + ib = r_2 e^{i\theta_2}$. In view of the result just proved, it suffices to take x_1 and x_2 of opposite sign, say $x_1 < 0 < x_2$; and for convenience we consider first the case $b > 0$, so that $\pi > \theta_1 > \frac{1}{2}\pi > \theta_2 > 0$. By hypothesis the representative points of z_1 and z_2 lie on or above the curve $r^2 = \nu^2 (\pi - 2\theta) \csc 2\theta$; let the radii vectores to z_1 and z_2 intersect this curve in the points $\zeta_1 = \rho_1 e^{i\theta_1}$ and $\zeta_2 = \rho_2 e^{i\theta_2}$ respectively. We carry out the integration of (8) along a path consisting of the following parts: (1) the radial segment from z_1 to ζ_1; (2) that portion of the curve $r^2 = \nu^2 (\pi - 2\theta) \csc 2\theta$ from ζ_1 to ζ_2; (3) the radial segment from ζ_2 to z_2. Along (1) we have $d\theta = 0$, $dr < 0$, and $\sin 2\theta < 0$; so from (13.1) and (13.2), $dK_2 = 0$ and $d\mathcal{G}_2 > 0$. On (2) $d\mathcal{G}_2 \quad 0$ by definition, and $dK_2 = d\theta < 0$. Along (3) $dK_2 = d\theta = 0$ and $d\mathcal{G}_2 = r \sin 2\theta \, dr > 0$. Hence the pair of inequalities (11c) are satisfied, and $R_\nu(z)$ cannot vanish both at z_1 and at z_2. The case $b < 0$, in which both imaginary parts are negative, is treated in a similar way to complete the proof.

It may be noted here that our methods do not permit us to dispose of the exceptional possibility that a solution of the modified Bessel equation (12) of real order may have two complex roots of equal imaginary part, lying on opposite sides of the imaginary axis and within the open region* between the curves $r^2 = \pm\nu^2 (\pi - 2\theta) \csc 2\theta$.

Analysis similar to the preceding may be applied to the solutions of the equation

$$\frac{d}{dz}\left(z\frac{dw}{dz}\right) - \left(z - \frac{\nu^2}{z}\right)w = 0 \tag{15}$$

obtained by writing $-\nu^2$ for ν^2 in (12). Any solution of (15) will be called a modified Bessel function of purely imaginary order $i\nu$ and complex argument z.** The functions $K(z) = z$ and $G(z) = z - \nu^2/z$ are analytic

*This region is somewhat less extensive than the strip $|y| < \nu$.

**The wedge functions defined in 1.1 are of course particular solutions when the independent variable of the equation is regarded as real.

in the whole complex plane (cut along the negative real axis) excluding the origin. The quantities $d\mathcal{K}$ and $d\mathcal{J}$ may be obtained by replacing ν^2 by $-\nu^2$ in (13.1) and (13.2).

Theorem 4. No modified Bessel function of purely imaginary order can have two complex roots with equal imaginary parts.

To prove the theorem, assume that the modified Bessel function $R_{i\nu}(z)$ vanishes both at $z_1 = x_1 + ib$ and at $z_2 = x_2 + ib$, where for definiteness $x_1 < x_2$, and we assume at first for convenience $b > 0$. We carry out the integration of (8) along the straight line $y = b$ from z_1 to z_2. On this segment $dx > 0$, $dy = 0$; so on writing out the expressions for $d\mathcal{K}_2$ and $d\mathcal{J}_2$ we find that

$$d\mathcal{K}_2 = \operatorname{Im}\,(dz/z) = (-y\,dx + x\,dy)/(x^2 + y^2) = -(b\,dx)/(x^2 + b^2) < 0, \tag{16.1}$$

$$d\mathcal{J}_2 = \operatorname{Im}\,(z - \nu^2/z)dz = \left[y + \nu^2 y/(x^2 + y^2)\right] dx + \left[x - \nu^2 x/(x^2 + y^2)\right] dy$$
$$= b\left[1 + \nu^2/(x^2 + b^2)\right] dx > 0. \tag{16.2}$$

Thus the pair of inequalities (11c) are satisfied, and $R_{i\nu}(z)$ cannot vanish both at z_1 and at z_2. If we assume $b < 0$, we merely reverse both inequalities and obtain (11d); thus the theorem is completely established.

In the following theorem use will be made of the curve

$$r^2 = 2\nu^2\theta \csc 2\theta,\ 0 < |\theta| < \tfrac{1}{2}\pi; \qquad r(0) = \nu, \tag{17}$$

which is just the symmetrical bell-sahped curve of (14.1) rotated through an angle of $-\frac{1}{2}\pi$, so that it now lies on the right side of the imaginary axis, passes through the point $(\nu, 0)$, and is asymptotic to the imaginary axis at $\pm i\infty$. Writing (17) in rectangular coordinates, $xy - \nu^2\tan^{-1}(y/x) = 0$, and comparing with (16.2), we see that the differential equation of this curve is just $d\mathcal{J}_2 = 0$.

Theorem 5. The modified Bessel function $K_{i\nu}(z)$ of imaginary order has no complex zeros on or to the left of the curve $r^2 = 2\nu^2\theta \csc 2\theta$.

Let $z_1 = x_1 > 0$ be one of the real positive zeros which $K_{i\nu}(x) \equiv G_\nu(x)$ has by theorem 1, (i). Assume $K_{i\nu}(z_2)$ vanishes, where $z_2 = x_2 + ib$ is a complex number on or to the left of the curve (17); let the line $y = b$ intersect this curve in the point $\zeta = \xi + ib$. Assume for the moment $b > 0$. Carry out the integration of (8) along a path consisting of the following parts: (1) the x-axis from $(x, 0)$ to $(\nu, 0)$; (2) the curve (17) from $(\nu, 0)$ to (ξ, b); (3) the line $y = b$ from (ξ, b) to (x_2, b). On (1) $y = 0$, $dy = 0$, so from (16.1) and (16.2) $dK_2 = d\mathcal{G}_2 = 0$. On (2) $dK_2 = \text{Im}\,(dz/z) = \text{Im}\, d(\log z) = d\theta > 0$; $d\mathcal{G}_2 = 0$ by definition of the curve (17). On (3) $y = b > 0$, $dy = 0$, and $dx < 0$; so from (16.1) and (16.2) $dK_2 = -b\, dx/(x^2 + b^2) > 0$; $\qquad d\mathcal{G}_2 = b\, dx\, [1 + \nu^2/(x^2 + b^2)] < 0$. Hence the pair of inequalities (11d) are satisfied throughout the path of integration, so $K_{i\nu}(z)$ cannot vanish at z_2. In a similar way it is established that $K_{i\nu}(z)$ cannot have a complex zero to the left of the curve (17) with negative imaginary part.

The possibility that $K_{i\nu}(z)$ may have complex zeros in the extensive region of the right half-plane to the right of the curve $r^2 = 2\nu^2\theta \csc 2\theta$ cannot be excluded by our methods.

1.31. Expansion of an Arbitrary Function in a Series of Wedge Functions.

The possibility of representing an arbitrary function over a finite interval of the positive x-axis by means of a series of wedge functions follows directly from the general theory of Art. 1.0; we summarize here the results.

Consider the Sturm-Liouville system:

$$\frac{d}{dx}\left(x \frac{dy}{dx}\right) - \left(x - \frac{\nu^2}{x}\right) y = 0, \tag{1.1}$$

$$\alpha' y(a) - \alpha y'(a) = 0, \tag{1.2}$$
$$\beta' y(b) + \beta y'(b) = 0, \tag{1.3}$$

where $0 < a < b < \infty$ and, with the notation of 1.0 (4.1), $K(x) = x$, $\ell(x) = x$, $g(x) = 1/x$, and $\lambda = \nu^2$. Let $Y(x, \nu)$ be the solution of (1.1) satisfying the first boundary condition; then the second boundary condition yields the characteristic equation (cf. 1.0 (5)) which must be satisfied by the eigenvalues ν_i. In the simple case $\alpha = \beta = 0$ and $\alpha' = \beta' = 1$ the boundary conditions are $y(a) = y(b) = 0$, so $Y(x, \nu)$ may be taken as the linear combination of wedge functions

$$Y(x, \nu) = F_\nu(a) G_\nu(x) - G_\nu(a) F_\nu(x);$$

the characteristic equation then becomes

$$\mathcal{F}(\nu^2) = F_\nu(a) G_\nu(b) - G_\nu(a) F_\nu(b) = 0. \tag{2}$$

Evidently the system (1) satisfies the conditions of theorem 5 of Art. 1.0, so there will be an infinite set of real, all positive* eigenvalues $\nu_0^2, \nu_1^2, \nu_2^2, \ldots$, which have no limit-point but $+\infty$; and the eigenfunction corresponding to ν_m will have exactly m zeros between a and b. Methods for actually calculating the roots of the characteristic equation numerically will be briefly discussed in Art. 1.4.

An arbitrary function f(x) may be represented in (a, b) by the series of wedge functions

$$f(x) = \sum_{n=0}^{\infty} A_n y_{\nu_n}(x), \tag{3}$$

where ν_n is the <u>nth</u> eigenvalue of the system (1), $y_{\nu_n}(x)$ is the corresponding eigenfunction, and the coefficient A_n is determined by

$$A_n = \frac{\int_a^b f(t)\, y_{\nu_n}(t)\, \frac{dt}{t}}{\int_a^b y_{\nu_n}^2(t)\, \frac{dt}{t}}. \tag{4}$$

---

*Provided, of course, that $\alpha\alpha' \geq 0$ and $\beta\beta' \geq 0$, as is almost always the case in practice.

By theorem 8 of 1.0, if f(x) possesses a Lebesgue integral in (a, b) and is of limited total fluctuation in the neighborhood of an interior point s of (a, b), the series (3) converges at s to the (mean) value of f(s). It can also be shown to converge to f(x) at the end-points of the interval unless the functions $y_{\nu_m}(x)$ are constrained to vanish at the end-points. In the latter case the series vanishes at the end-points, no matter whether f(a) = f(b) = 0 or not.

The integral in the denominator of (4) may be calculated from 1.0 (5) and (6.2); recalling that $\lambda = \nu^2$, we obtain

$$\int_a^b \frac{y_{\nu_m}^2(t)\,dt}{t} = \frac{b}{2\nu_m}\left[\frac{\partial y_{\nu_m}(x)}{\partial x}\right]_{x=b}\left[\frac{\partial y_\nu(x)}{\partial \nu} + \frac{\beta}{\beta'}\frac{\partial^2 y_\nu(x)}{\partial\nu\,\partial x}\right]_{\substack{x=b\\ \nu=\nu_m}}. \quad (5)$$

Formulas equivalent to this have been given by Dougall[28] and Bocher[29].
The right side of (5) cannot be simplified, as can the coefficients in an ordinary Fourier-Bessel expansion, because as noted in 1.13 we have no recurrence relations involving derivatives of the wedge functions.

If we attempt to represent a function f(x) over the interval (0, b) or over the infinite interval $(0, \infty)$ by means of wedge functions, we find that our boundary conditions no longer select discrete values of ν. We have now to use all values of ν in the representation of f(x), and the infinite series (2) passes over into an infinite integral in a way similar to the well-known transition of an ordinary Fourier series into a Fourier integral as the fundamental interval is extended to infinity. In the case at hand we obtain what may be called a Fourier-Bessel integral, though of a form not previously discussed. Sufficient conditions for representing a function in the interval (0, b) by such an integral will be given in the following article; but since the rigorous demonstration is long and

---

28) Dougall, J., Proc. Edinburgh Math. Soc., 18, 40 (1900).
29) Bocher, op. cit., 149. Bocher treats only the case $\beta = 0$ and writes $i\nu$ for our ν; this accounts for his negative sign.

involved, we shall first give a heuristic development which, while making no pretense of rigor, will indicate formally the result which we may expect.

We assume that a suitably behaved function f(x) may be expanded in the interval (a, ∞) in a series of wedge functions of the second kind, which vanish at a and at infinity; we shall eventually find the limiting form of this series as $a \to +0$.* We have from (3), (4), and (5), on setting $\beta = 0$,

$$f(x) = \sum_{n=0}^{\infty} \frac{\int_a^\infty f(t) G_{\nu_n}(t) \frac{dt}{t} G_{\nu_n}(x)}{-\frac{a}{2\nu_n}\left[\frac{\partial G_\nu(x)}{\partial x} \frac{\partial G_\nu(x)}{\partial \nu}\right]_{\substack{x=a \\ \nu=\nu_n}}}, \tag{6}$$

where the negative sign arises from evaluating (5) at the lower limit of the interval. Since we are interested in the limiting case $a \to +0$, we calculate the denominator of (6) approximately from 1.12 (4.2); and we also assume that we are considering only those terms of the series for which ν is so large that $\Gamma(i\nu)$ may be represented by Stirling's asymptotic formula. Then

$$\arg \Gamma(i\nu) = \mathrm{Im} \log \Gamma(i\nu) \sim \mathrm{Im}\left[(i\nu - \tfrac{1}{2})\log(i\nu) - i\nu + \log\sqrt{2\pi}\right]$$
$$= \nu(\log \nu - 1) - \frac{\pi}{4},$$

and 1.12 (4.2) becomes, for ν large and x small,

$$G_\nu(x) \sim \sqrt{\pi/\nu \sinh \nu\pi} \cos\left[\nu(\log \tfrac{1}{2}x - \log \nu + 1) + \frac{\pi}{4}\right]. \tag{7}$$

Equation (7), together with the boundary condition $G_{\nu_k}(a) = 0$, yields the equation for the eigenvalues:

$$\nu_k(\log \tfrac{1}{2}a - \log \nu_k + 1) + \frac{\pi}{4} = -(k + \tfrac{1}{2})\pi. \tag{8}$$

If we subtract this equation from the similar equation satisfied by ν_{k+1} and write $\nu_{k+1} = \nu_k + \delta\nu_k$, then neglecting squares of $\delta\nu_k$ we have

---

*The theorem of the next section actually permits the representation of a function in the finite interval (0, b), where b is any preassigned number, however large. This is not quite the same as representing the function in the infinite interval $(0, \infty)$.

$$\delta\nu_k\left(\log\tfrac{1}{2}a - \log\nu_k + 1\right) - \delta\nu_k = -\pi;\quad \delta\nu_k = \frac{\pi}{\log\frac{2\nu_k}{a}}. \tag{9}$$

Differentiating (7) with respect to x and ν in turn and then employing (8), we see that

$$\left.\frac{\partial G_{\nu_k}(x)}{\partial x}\right|_{x=a} \sim (-)^k \frac{\nu_k}{a}\sqrt{\frac{\pi}{\nu_k \operatorname{sh}\nu_k\pi}}; \tag{10.1}$$

$$\left.\frac{\partial G_\nu(a)}{\partial\nu}\right|_{\nu=\nu_k} \sim (-)^k \log\frac{a}{2\nu_k}\sqrt{\frac{\pi}{\nu_k \operatorname{sh}\nu_k\pi}}. \tag{10.2}$$

Substituting (10.1) and (10.2) into the denominator of (6), replacing $\log(2\nu_k/a)$ by $(\pi/\delta\nu_k)$ in accordance with (9), and letting the series pass into an integral as $a \to +0$, we are led to the formula

$$f(x) = \frac{2}{\pi^2}\int_0^\infty \nu \operatorname{sh}\nu\pi\, G_\nu(x)\left\{\int_0^\infty f(t)G_\nu(t)\frac{dt}{t}\right\}d\nu, \tag{11}$$

which presumably represents f(x) in the open interval $(0,\infty)$.

The rigorous proof of a formula similar to (11), valid when f(x) satisfies certain sufficient conditions, will be given in the next article.

1.32. A Fourier-Bessel Integral Involving Wedge Functions.

The main result which we shall prove in this section is contained in the following

Theorem. Let f(t) be a function of the real variable t in the range $0 \leqslant t \leqslant T$, and let x be a fixed point of the open interval (0, T). If

(i) f(t) is continuous except at a finite number of discontinuities in (0, T),

(ii) f(t) has limited total fluctuation in an interval surrounding x, and

(iii) $\int_0^T \frac{|f(t)|\,dt}{t}$ exists, then

$$\frac{2}{\pi^2}\int_0^\infty \nu \sinh \nu\pi\, G_\nu(x)\left\{\int_0^\infty f(t) G_\nu(t)\frac{dt}{t}\right\}d\nu = \tfrac{1}{2}\left[f(x+0)+f(x-0)\right]. \tag{1}$$

We require certain preliminary lemmas, which will for convenience be expressed in terms of the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ defined for unrestricted complex values of the order by 1.1 (4.1) and (4.2).

Lemma 1. If $0 \leq t \leq T$ and if $|\zeta| \geqslant N$, where $\mathrm{Re}\,\zeta \geqslant 0$, then

$$I_\zeta(t) = \frac{(\frac{1}{2}t)^\zeta}{\zeta\Gamma(\zeta)}\left[1 + O\left(\frac{1}{N}\right)\right]. \tag{2}$$

Proof: From 1.1 (4.1) we have

$$I_\zeta(t) = \frac{(\frac{1}{2}t)^\zeta}{\zeta\Gamma(\zeta)}\left[1 + \sum_{m=1}^\infty \frac{(\frac{1}{2}t)^{2m}}{m!(\zeta+1)(\zeta+2)\cdots(\zeta+m)}\right].$$

If $|\zeta| \geqslant N$ and $\mathrm{Re}\,\zeta \geqslant 0$, then $|\zeta + n| > N$ for $n = 1, 2, \ldots$; so

$$\left|\sum_{m=1}^\infty \frac{(\frac{1}{2}t)^{2m}}{m!(\zeta+1)\cdots(\zeta+m)}\right| < \sum_{m=1}^\infty \frac{(\frac{1}{2}T)^{2m}}{m!\,N^m} = \exp\frac{T^2}{4N} - 1 = O\left(\frac{1}{N}\right).$$

Q.E.D.

Let $\zeta = \xi + i\eta = \rho e^{i\varphi}$, where $\rho \geqslant N$ and $|\varphi| \leqslant \frac{1}{2}\pi$; then the following asymptotic expression may be obtained[30] from Stirling's formula:

$$\begin{aligned} |\Gamma(\zeta)|^2 &= 2\pi e^{-2\xi}(\xi^2+\eta^2)^{\xi-\frac{1}{2}} e^{-2\eta\varphi}\left[1 + O\left(\frac{1}{\zeta}\right)\right] \\ &> 2\pi e^{-2\xi - \pi|\eta|} N^{2\xi}\rho^{-1}\left[1 + O\left(\frac{1}{N}\right)\right]. \end{aligned} \tag{3}$$

Lemma 2. Let $0 < t \leq T$ and let N be a positive integer greater than unity.

(i). If $\zeta = (N + \frac{1}{2}) + i\eta$, where $-(N + \frac{1}{2}) \leq \eta \leq (N + \frac{1}{2})$, then

$$K_\zeta(t) = \tfrac{1}{2}\Gamma(\zeta)\left(\tfrac{1}{2}t\right)^{-\zeta}\left[1 + O\left(\frac{1}{N}\right)\right]. \tag{4}$$

---

30) Copson, E. T., Theory of Functions of a Complex Variable, 224.

(ii). If $\zeta = \xi \pm i(N + \frac{1}{2})$, where $0 \leq \xi \leq (N + \frac{1}{2})$, then

$$K_\zeta(t) = \tfrac{1}{2}\Gamma(\zeta)(\tfrac{1}{2}t)^{-\zeta}\left[1 + O(\tfrac{1}{N}) + O(\tfrac{1}{N^{2\xi}})\right]. \qquad (5)$$

Proof: From 1.1 (4.2) and (4.1) we have

$$K_\zeta(t) = \frac{\pi}{2\sin\zeta\pi}\left[\frac{(\frac{1}{2}t)^{-\zeta}}{-\zeta\Gamma(-\zeta)}\left\{1 + \sum_{m=1}^{\infty}\frac{(\frac{1}{2}t)^{2m}}{m!(1-\zeta)(2-\zeta)\cdots(m-\zeta)}\right\} - \frac{(\frac{1}{2}t)^{\zeta}}{\Gamma(\zeta)}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}t)^{2m}}{m!\,\zeta(\zeta+1)\cdots(\zeta+m)}\right]. \qquad (6)$$

We employ the identity[31)]

$$\Gamma(\zeta)\Gamma(-\zeta) = -\pi/(\zeta\sin\zeta\pi)$$

and obtain, after factoring $\frac{1}{2}\Gamma(\zeta)(\frac{1}{2}t)^{-\zeta}$ out of the right side of (6),

$$K_\zeta(t) = \tfrac{1}{2}\Gamma(\zeta)(\tfrac{1}{2}t)^{-\zeta}\left[1 + \sum_{m=1}^{\infty}\frac{(\frac{1}{2}t)^{2m}}{m!\,(1-\zeta)(2-\zeta)\cdots(m-\zeta)} - \frac{\pi(\frac{1}{2}t)^{2\zeta}}{\sin\zeta\pi\,\Gamma^2(\zeta)}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}t)^{2m}}{m!\,\zeta(\zeta+1)\cdots(\zeta+m)}\right]. \qquad (7)$$

In part (i), where $\zeta = (N + \frac{1}{2}) + i\eta$, it is easily seen that for $N > 1$ we have $\rho = |\zeta| \leq \sqrt{2}\,(N + \frac{1}{2}) < 2N$, so (3) implies that

$$|\Gamma(\zeta)|^2 > \pi e^{-(2N+1)\pi} N^{2N}\left[1 + O(\tfrac{1}{N})\right].$$

If k is any non-negative integer we have the following evident inequalities:

$$|1-\zeta| > \frac{N}{2},\quad |k-\zeta| \geq \frac{1}{2},\quad \text{and } |\zeta + k| > N.$$

We also have $|\sin\zeta\pi| = \operatorname{ch}\eta\pi \geq 1$ and $|(\frac{1}{2}t)^{2\zeta}| = (\frac{1}{2}t)^{2N+1}$.

Hence we may dominate the remainder terms on the right side of (7) as follows:

$$\left|\sum_{m=1}^{\infty}\frac{(\frac{1}{2}t)^{2m}}{m!(1-\zeta)\cdots(m-\zeta)} - \frac{\pi(\frac{1}{2}t)^{2\zeta}}{\sin\zeta\pi\,\Gamma^2(\zeta)}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}t)^{2m}}{m!\,\zeta(\zeta+1)\cdots(\zeta+m)}\right|$$

$$< \sum_{m=1}^{\infty}\frac{(\frac{1}{2}T)^{2m}}{m!\,N(\frac{1}{2})^m} + \frac{\pi(\frac{1}{2}T)^{2N+1}}{\pi e^{-(2N+1)\pi}N^{2N}}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}T)^{2m}}{m!\,N^{m+1}}\left[1+O(\tfrac{1}{N})\right]$$

$$= \frac{1}{N}\left[\exp\tfrac{1}{2}T^2 - 1\right] + \left(\frac{Te^{\pi}}{2N}\right)^{2N+1}\exp\frac{T^2}{4N}\left[1+O(\tfrac{1}{N})\right] = O(\tfrac{1}{N}).$$

Q.E.D.

For part (ii), in which $\zeta = \xi \pm i(N + \frac{1}{2})$, we have again $\rho < 2N$,

---

31) Whittaker and Watson, op. cit., 239.

and (3) becomes

$$|\Gamma(\zeta)|^2 > \pi e^{-2\xi-(N+\frac{1}{2})\pi} N^{2\xi-1}[1+O(\tfrac{1}{N})].$$

We have also, if $N > 1$,

$$|\sin\zeta\pi| = [\sin^2\xi\pi + sh^2(N+\tfrac{1}{2})\pi]^{\frac{1}{2}} \geqslant sh(N+\tfrac{1}{2})\pi > \tfrac{1}{4}e^{(N+\frac{1}{2})\pi},$$

as well as $|(\frac{1}{2}t)^{2\zeta}| = (\frac{1}{2}t)^{2\xi}$, and $|k \pm \zeta| > N$,

where k is any integer. Accordingly:

$$\left|\sum_{m=1}^{\infty} \frac{(\frac{1}{2}t)^{2m}}{m!(1-\zeta)\cdots(m-\zeta)} - \frac{\pi(\frac{1}{2}t)^{2\zeta}}{\sin\zeta\pi\,\Gamma^2(\zeta)}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}t)^{2m}}{m!\zeta(\zeta+1)\cdots(\zeta+m)}\right|$$

$$< \sum_{m=1}^{\infty}\frac{(\frac{1}{2}T)^{2m}}{m!N^m} + \frac{4(\frac{1}{2}T)^{2\xi}}{e^{-2\xi}N^{2\xi-1}}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}T)^{2m}}{m!N^{m+1}}[1+O(\tfrac{1}{N})]$$

$$= \left(\exp\frac{T^2}{4N}-1\right) + 4\left(\frac{Te}{2N}\right)^{2\xi}\exp\frac{T^2}{4N}[1+O(\tfrac{1}{N})] = O(\tfrac{1}{N}) + O(\tfrac{1}{N^{2\xi}}).$$

Q.E.D.

The term $O(1/N^{2\xi})$ may evidently be disregarded if $\xi \geqslant \frac{1}{2}$.

<u>Lemma 3.</u> Let x be a fixed positive number and let f(t) be a function defined in the range $(0,\infty)$ such that

(i) f(t) is continuous except at a finite number of discontinuities in $(0,\infty)$, and

(ii) $\int_0^\infty \frac{|f(t)|}{t}\,dt$ exists; then

$$\lim_{R\to\infty}\int_0^\infty \frac{f(t)}{t}\left\{\int_0^R \nu\,sh\,\nu\pi\,K_{i\nu}(x)K_{i\nu}(t)\,d\nu\right\}dt$$
$$= \int_0^\infty \nu\,sh\,\nu\pi\,K_{i\nu}(x)\left\{\int_0^\infty f(t)K_{i\nu}(t)\frac{dt}{t}\right\}d\nu, \qquad (8)$$

provided the limit on the left exists.

<u>Proof:</u> The function $\sqrt{\nu\,sh\,\nu\pi}\,K_{i\nu}(t)$ is a continuous function of ν and t provided $0 \leqslant \nu \leqslant R$ and $t \geqslant t_0 > 0$; furthermore it tends to zero by 1.12 (2.2) as $t \to \infty$. It is therefore a bounded function when the variables are in the stated ranges. On replacing ζ by $i\nu$ in (7) and making use of the relation (1.11 (4)) $\sqrt{\nu\,sh\,\nu\pi}\,|\Gamma(i\nu)| = \sqrt{\pi}$, we have

after some elementary manipulations the inequality

$$\left|\sqrt{\nu \operatorname{sh} \nu\pi}\; K_{i\nu}(t)\right| \leq \sqrt{\pi} \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}t\right)^{2m}}{(m!)^2}; \qquad (9)$$

and the right side of (9) is certainly bounded for all real ν as $t \to +0$. Hence if we define

$$\phi(\nu; x, t) \equiv \nu \operatorname{sh} \nu\pi K_{i\nu}(x)\, K_{i\nu}(t),$$

where x and t are any positive real numbers, there exists a constant A such that $|\phi(\nu; x, t)| \leq A$ so long as $0 \leq \nu \leq R$.

For any preassigned positive values of R and ε we may by hypothesis (ii) choose β so that

$$\int_\beta^\infty \frac{|f(t)|}{t}\, dt < \frac{\varepsilon}{2RA}.$$

We then have

$$\left|\int_0^\infty \frac{f(t)}{t} \int_0^R \phi(\nu; x, t)\, d\nu\, dt - \int_0^R \int_0^\infty \frac{f(t)}{t} \phi(\nu; x, t)\, dt\, d\nu\right|$$

$$= \left|\int_0^\beta \frac{f(t)}{t} \int_0^R \phi(\nu; x, t)\, d\nu\, dt + \int_\beta^\infty \frac{f(t)}{t} \int_0^R \phi(\nu; x, t)\, d\nu\, dt\right.$$

$$\left. - \int_0^R \int_0^\beta \frac{f(t)}{t} \phi(\nu; x, t)\, dt\, d\nu - \int_0^R \int_\beta^\infty \frac{f(t)}{t} \phi(\nu; x, t)\, dt\, d\nu\right|$$

$$= \left|\int_\beta^\infty \frac{f(t)}{t} \int_0^R \phi(\nu; x, t)\, d\nu\, dt - \int_0^R \int_\beta^\infty \frac{f(t)}{t} \phi(\nu; x, t)\, dt\, d\nu\right|$$

$$\leq \int_\beta^\infty \frac{|f(t)|}{t} \int_0^R |\phi(\nu; x, t)|\, d\nu\, dt + \int_0^R \int_\beta^\infty \frac{|f(t)|}{t} |\phi(\nu; x, t)|\, dt\, d\nu$$

$$< \frac{\varepsilon}{2RA} \cdot RA + \frac{\varepsilon A}{2RA} \cdot R = \varepsilon,$$

since we may evidently justify $\int_0^\beta \int_0^R = \int_0^R \int_0^\beta$ by considering the ranges within which f(t) is continuous.* Since the preceding inequality is true for an arbitrarily small ε, the repeated integrals $\int_0^\infty \int_0^R$ and $\int_0^R \int_0^\infty$ are equal for all finite values of R; the desired result follows by passage

---

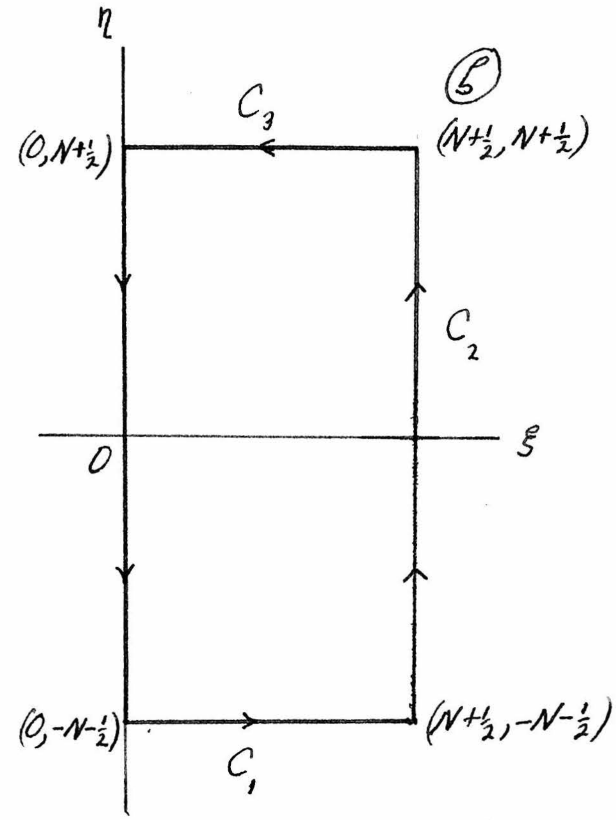
*In particular, if the integrand $|f(t)|/t$ has a singularity at $t = 0$, we break the range of integration into two parts and treat the lower limit in the same way that we have just treated the upper limit.

to the limit as $R \to \infty$.

If it happens that $f(t) \equiv 0$ for $t > T$, we may evidently write T for the upper limits of the integrals over t. We shall use this special case of (8) in what follows.

We proceed now to the proof of the main theorem.

Consider the contour integral $\oint \zeta K_\zeta(x) I_\zeta(t)\, d\zeta$, where $0 < t \leq x$, around the rectangular contour in the ζ-plane having corners at $(0, \pm(N+\frac{1}{2}))$ and at $(N+\frac{1}{2}, \pm(N+\frac{1}{2}))$. Since the integrand is everywhere an analytic function of ζ, by Cauchy's theorem the contour integral vanishes; i. e.,



$$\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} i\eta K_{i\eta}(x) I_{i\eta}(t)\, i\, d\eta = \int_{C_1+C_2+C_3} \zeta K_\zeta(x) I_\zeta(t)\, d\zeta, \tag{10}$$

where the right-hand integral is evaluated over the bottom, right side, and top of the rectangular contour.

Writing ν for η in the left side of (10) and noting from the definition 1.1 (5.2) that $K_{i\nu}(x)$ is an even function of ν, we may transform the integral over the imaginary axis and obtain the following relation:

$$\int_{C_1+C_2+C_3} \zeta K_\zeta(x) I_\zeta(t)\, d\zeta = -\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \nu K_{i\nu}(x) I_{i\nu}(t)\, d\nu$$

$$= -\int_{-N-\frac{1}{2}}^{0} \nu K_{i\nu}(x) I_{i\nu}(t)\, d\nu - \int_{0}^{N+\frac{1}{2}} \nu K_{i\nu}(x) I_{i\nu}(t)\, d\nu$$

$$= \int_{0}^{N+\frac{1}{2}} \nu K_{i\nu}(x) I_{-i\nu}(t)\, d\nu - \int_{0}^{N+\frac{1}{2}} \nu K_{i\nu}(x) I_{i\nu}(t)\, d\nu$$

$$= \frac{2i}{\pi}\int_0^{N+\frac{1}{2}} \nu \, sh\, \nu\pi \, K_{i\nu}(x) K_{i\nu}(t)\, d\nu. \tag{11.1}$$

Similarly if $0 < x \leq t$, we have

$$\frac{2i}{\pi}\int_0^{N+\frac{1}{2}} \nu \, sh\, \nu\pi \, K_{i\nu}(x) K_{i\nu}(t)\, d\nu = \int_{C_1+C_2+C_3} \zeta K_\zeta(t) I_\zeta(x)\, d\zeta. \tag{11.2}$$

Let u be the greater of the two quantities x and t and let v be the other, so that $0 < v \leq u \leq T$. From (2), (4), and (5) we have

$$\int_{C_1+C_2+C_3} \zeta K_\zeta(u) I_\zeta(v)\, d\zeta = \frac{1}{2}\int_{C_1+C_2+C_3} \left(\frac{v}{u}\right)^\zeta [1 + R(\zeta; u, v)]\, d\zeta, \tag{12}$$

where on the segments C_1 and C_3 $|R(\zeta;\ u,\ v)| \leq A/N + B/N^{2\xi}$, and on the segment C_2 $|R(\zeta;\ u,\ v)| \leq \frac{1}{2}C/N$, A, B, and C being constants independent of N. Now

$$\frac{1}{2}\int_{C_1+C_2+C_3} \left(\frac{v}{u}\right)^\zeta d\zeta = \frac{1}{2}\int_{-i(N+\frac{1}{2})}^{i(N+\frac{1}{2})} \exp\left(\zeta \log \frac{v}{u}\right) d\zeta = \frac{i \sin[(N+\frac{1}{2}) \log \frac{v}{u}]}{\log \frac{v}{u}}; \tag{13}$$

and setting $\log(v/u) = -\lambda < 0$ and $\zeta = \xi + i\eta$, where on C_1 and C_3 $|\eta| = N + \frac{1}{2}$ and on C_2 $\xi = N + \frac{1}{2}$, we can dominate the remainder term of (12) as follows:

$$\begin{aligned} R_N &= \left|\frac{1}{2}\int_{C_1+C_2+C_3} e^{-\lambda\zeta} R(\zeta; u, v)\, d\zeta\right| \\ &\leq \int_0^{N+\frac{1}{2}} e^{-\lambda\xi}\left[\frac{A}{N} + \frac{B}{N^{2\xi}}\right] d\xi + \int_0^{N+\frac{1}{2}} e^{-\lambda(N+\frac{1}{2})} \frac{C}{2N}\, d\eta \\ &= \frac{A}{N\lambda}\left[1 - e^{-\lambda(N+\frac{1}{2})}\right] + \frac{B[1 - e^{-(\lambda + 2\log N)}]}{\lambda + 2\log N} + \frac{(N+\frac{1}{2})C}{2N} e^{-\lambda(N+\frac{1}{2})}. \end{aligned} \tag{14}$$

We consider separately the case $\lambda \leq N^{-\frac{1}{2}}$, where t is inside the interval $(xe^{-\frac{1}{\sqrt{N}}}, xe^{\frac{1}{\sqrt{N}}})$, and the case $\lambda > N^{-\frac{1}{2}}$, where t is outside the interval. If $\lambda > N^{-\frac{1}{2}}$, then from (14)

$$R_N < \frac{A}{\sqrt{N}} + \frac{B}{2\log N} + Ce^{-\sqrt{N}} = O\left(\frac{1}{\log N}\right); \tag{15.1}$$

while if $\lambda \leq N^{-\frac{1}{2}}$, then by sutstituting $\lambda = 0$ in the second line of (14) we have

$$R_N < \frac{(A+\frac{1}{2}C)(N+\frac{1}{2})}{N} + \frac{B}{2\log N} = O(1). \tag{15.2}$$

Collecting the results of (11), (12), (13), and (15), we have

$$\int_0^T \frac{f(t)}{t} \int_0^{N+\frac{1}{2}} \nu \operatorname{sh} \nu\pi \, K_{i\nu}(x) \, K_{i\nu}(t) \, d\nu \, dt$$

$$= \frac{\pi}{2} \int_0^T \frac{f(t)}{t} \frac{\sin[(N+\frac{1}{2})\log\frac{x}{t}]}{\log\frac{x}{t}} dt + \int_0^T \frac{f(t)}{t} O\left(\frac{1}{\log N}\right) dt + \int_{x\exp(-N^{-\frac{1}{2}})}^{x\exp(N^{-\frac{1}{2}})} \frac{f(t)}{t} O(1) \, dt. \tag{16}$$

Since by hypothesis $\int_0^T \frac{|f(t)|}{t} dt$ exists and $x(e^{\frac{1}{\sqrt{N}}} - e^{-\frac{1}{\sqrt{N}}}) = O(N^{-\frac{1}{2}})$, both remainder terms in (16) are o(1) and vanish as $N \to \infty$. On setting $\log x = \sigma$, $\log t = \tau$ in the first integral on the right side of (16), inverting the order of integration on the left side by lemma 3, and letting $N \to \infty$, we have

$$\int_0^\infty \nu \operatorname{sh} \nu\pi \, K_{i\nu}(x) \int_0^T f(t) K_{i\nu}(t) \frac{dt}{t} d\nu$$

$$= \lim_{N\to\infty} \frac{\pi}{2} \int_{-\infty}^{\log T} f(e^\tau) \sin \frac{[(N+\frac{1}{2})(\sigma-\tau)]}{(\sigma-\tau)} d\tau.$$

If $f(e^\tau) \equiv g(\tau)$, then by hypothesis $\int_0^{\log T} |g(\tau)| d\tau$ exists and $g(\tau)$ is of limited total fluctuation in an interval surrounding σ. Hence by Fourier's single integral formula[32] we have

$$\int_0^\infty \nu \operatorname{sh} \nu\pi \, K_{i\nu}(x) \int_0^T f(t) K_{i\nu}(t) \frac{dt}{t} d\nu$$

$$= \frac{\pi^2}{2} \cdot \frac{1}{2}\left[f(e^{\sigma+0}) + f(e^{\sigma-0})\right] = \frac{\pi^2}{2} \cdot \frac{1}{2}\left[f(x+0) + f(x-0)\right].$$

---

32) Titchmarsh, E. C., Theory of Fourier Integrals, Art. 1.14. The formula

$$\lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-\infty}^{a} g(\tau) \frac{\sin\lambda(\sigma-\tau)}{\sigma-\tau} d\tau = \frac{1}{2}\left[g(\sigma+0) + g(\sigma-0)\right], \quad \sigma < a,$$

is valid even under less stringent conditions than we have imposed on $g(\tau)$.

On writing the wedge function G_ν for $K_{i\nu}$ we obtain the result (1) stated at the beginning of this section.

The theorem just proved appears to be of a somewhat different type from the ordinary Fourier-Bessel integral theorem,[33] since it involves integration over the order as well as the argument of the functions concerned.[34] It would be of interest to know whether an integral of the form (1) can represent a function over the entire range $(0,\infty)$, as the considerations at the end of the preceding section might lead one to believe; but the question does not seem easy to decide by our methods.

A number of formulas involving integration of $G_\nu(x)$ with respect to ν follow from 1.11 (6.2), which defines $G_\nu(x)$ as the Fourier cosine transform of $\sqrt{\tfrac{1}{2}\pi}\,\exp(-x \operatorname{ch} t)$. In general if $f(t)$ is a continuous function in $(0,\infty)$ such that $\int_0^\infty |f(t)|\,dt$ exists, and if $f(t)$ has limited total fluctuation in the neighborhood of the point $t = s$, there exist the following reciprocal relations between $f(t)$ and its Fourier cosine transform $\mathcal{F}_c(\nu)$:[35]

$$\mathcal{F}_c(\nu) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos \nu t\,dt; \tag{17.1}$$

$$f(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}_c(\nu)\cos s\nu\,d\nu. \tag{17.2}$$

If in 1.11 (6.2) we make the following identifications:

---

33) Watson, *op. cit.*, Arts. 14.3 *et seq.*

34) But see T. M. MacRobert, *Proc. Roy. Soc. Edinburgh*, 51, 116-126 (1931), where several Fourier-type integrals are obtained by contour integration. One of MacRobert's results is the following:

If $f(\lambda) = \int_p^q \phi(\rho) J_\rho(\rho\lambda)\rho\,d\rho, \quad (0 \le p < q)$

then $\int_0^\infty \left(\lambda - \frac{1}{\lambda}\right) f(\lambda) J_m(m\lambda)\,d\lambda = \begin{cases} \frac{1}{2}[\phi(m+0) + \phi(m-0)], & p < m < q; \\ 0, & 0 < m < p \text{ or } m > q. \end{cases}$

35) Titchmarsh, *op. cit.*, 1-4, 13.

$$\mathcal{F}_c(\nu) = G_\nu(x), \quad f(t) = \sqrt{\tfrac{1}{2}\pi}\, e^{-x \operatorname{ch} t}, \tag{18}$$

then we have the pair of relations

$$G_\nu(x) = \int_0^\infty e^{-x \operatorname{ch} t} \cos \nu t \, dt, \tag{19.1}$$

$$\tfrac{1}{2}\pi e^{-x \operatorname{ch} s} = \int_0^\infty G_\nu(x) \cos \nu s \, d\nu. \tag{19.2}$$

In the special case $s = 0$, (19.2) becomes

$$\int_0^\infty G_\nu(x)\, d\nu = \tfrac{1}{2}\pi e^{-x}. \tag{20}$$

If $\mathcal{F}_c(\nu)$ and $\mathcal{G}_c(\nu)$ are the Fourier cosine transforms of $f(t)$ and $g(t)$ respectively, then we have the formula[36)]

$$2\int_0^\infty \mathcal{F}_c(\nu)\mathcal{G}_c(\nu) \cos \nu t \, d\nu = \int_{-\infty}^\infty g(u) f(t-u)\, du. \tag{21}$$

Applied to the wedge functions $G_\nu(x)$ and $G_\nu(y)$, this gives

$$\int_0^\infty G_\nu(x) G_\nu(y) \cos \nu t \, d\nu = \tfrac{1}{4}\pi \int_{-\infty}^\infty e^{-x \operatorname{ch}(t-u) - y \operatorname{ch} u} du, \tag{22}$$

or, in the special case $t = 0$,

$$\int_0^\infty G_\nu(x) G_\nu(y) d\nu = \tfrac{1}{2}\pi \int_0^\infty e^{-(x+y)\operatorname{ch} u} du = \tfrac{1}{2}\pi G_0(x+y). \tag{23}$$

The theory of Fourier integrals could doubtless be made to yield other such results involving $G_\nu(x)$; but since the applications which we have in view do not require the use of these formulas we shall not carry the investigation further here.

1.4. Transformation of the Differential Equation for the Wedge Functions. Calculation of the Eigenvalues.

The wedge functions $F_\nu(x)$ and $G_\nu(x)$ are difficult to tabulate and to employ in numerical calculations for small values of the argument because of their oscillatory discontinuity at $x = 0$. It is possible to facilitate their use in practical problems, as well as formally to simplify some theoretical developments, by transforming the independent variable of the defining differential equation so as to remove the singularity at

---

36) Titchmarsh, *op. cit.*, 51.

the origin from the finite part of the plane. For this purpose let

$$u = \log x, \text{ or } x = e^u. \tag{1}$$

The transformation (1) takes the triad of points $(0, 1, \infty)$ of the x-axis into the triad $(-\infty, 0, \infty)$ of the u-axis; and, since $d/dx = e^{-u}d/du$, it transforms the equation 1.1 (1) into

$$d^2y/du^2 + (\nu^2 - e^{2u})\, y = 0, \tag{2}$$

which has no singularities for finite values of u, and of which the general solution is evidently

$$y = c_1 F_\nu(e^u) + c_2 G_\nu(e^u). \tag{3}$$

The quantities $F_\nu(e^u)$ and $G_\nu(e^u)$ are tabulated as functions of u and ν in the appendix of this thesis. It is evident from (1) and 1.12 (4.1) and (4.2) that for large negative values of u, $F_\nu(e^u)$ and $G_\nu(e^u)$ are approximately sinusoidal functions of νu; this fact is plausible since $\sin \nu u$ and $\cos \nu u$ both satisfy (2) when u is negative and large enough to make e^{2u} negligible compared with ν^2. By theorem 3 of Art. 1.0 the solutions of (2) are non-oscillatory when $u \geqslant \log \nu$; their asymptotic form for u large and positive may easily be obtained from (1) and 1.12 (3.1) and (3.2).

One may write a series expansion such as 1.31 (3) directly in terms of $F_\nu(e^u)$ and $G_\nu(e^u)$ by making the simple transformation of variable (1) in the integrals of 1.31 (4) and (5); but the eigenvalues ν_n must be computed as the roots of a transcendental equation. For example, if the boundary conditions are $y = 0$ at $u = c$ and at $u = d$, then the eigenvalues are the roots in ν of the equation (cf. 1.31 (2))

$$\mathcal{F}(\nu^2) = F_\nu(e^c)G_\nu(e^d) - G_\nu(e^c)F_\nu(e^d) = 0. \tag{4}$$

The only practicable way to obtain the first few roots of (4) for given values of c and d appears to be interpolation in a table of wedge functions. One evaluates $\mathcal{F}(\nu^2)$ for several adjacent tabular values of ν around the expected root ν_n and interpolates to find the value of ν for which the function vanishes. Then it is possible to calculate by double interpolation the value of the eigenfunction corresponding to ν_n for any desired value of u.

If it is necessary to calculate the roots of (4) beyond the range of the available tables, recourse may be had to the asymptotic developments, first given by Horn,[37] of the large eigenvalues of (2) and their corresponding eigenfunctions. Horn's results will be briefly quoted here and applied to the case at hand.

We consider the equation

$$\frac{d}{du}\left(A\frac{dy}{du}\right) + (\nu^2 B + C)y = 0, \qquad (5)$$

where A, B, and C are real continuous functions of the real variable u and possess continuous derivatives of all orders in $c \leq u \leq d$, A and B being positive in the given interval; and ν^2 is an arbitrary parameter. Horn shows that the solution of (5) which satisfies the boundary conditions $y = \alpha$, $dy/du = \alpha'$ at $u = c$ is represented asymptotically for large values of ν^2 by the series

$$y = \cos\nu\omega\left(\varphi_0 + \frac{\varphi_2}{\nu^2} + \cdots\right) + \sin\nu\omega\left(\frac{\varphi_1}{\nu} + \frac{\varphi_3}{\nu^3} + \cdots\right), \qquad (6)$$

where ω and the ϕ's are functions of u defined by:

$$\omega(u) = \int_c^u \sqrt[4]{\frac{B}{A}}\,du; \qquad (7.1)$$

$$\varphi_0(u) = \frac{\alpha\sqrt[4]{A(c)B(c)}}{\sqrt[4]{AB}}; \qquad (7.2)$$

$$\varphi_1(u) = -\frac{1}{2\sqrt[4]{AB}}\int_c^u \frac{A\varphi_0'' + A'\varphi_0' + C\varphi_0}{\sqrt[4]{AB}}\,du + \frac{[\alpha' - \varphi_0'(c)]\sqrt[4]{A(c)^3}}{\sqrt[4]{AB}\,\sqrt[4]{B(c)}}; \qquad (7.3)$$

$$\varphi_{2n}(u) = \frac{1}{2\sqrt[4]{AB}}\int_c^u \frac{A\varphi_{2n-1}'' + A'\varphi_{2n-1}' + C\varphi_{2n-1}}{\sqrt[4]{AB}}\,du \quad (n = 1, 2, \cdots) \qquad (7.4)$$

$$\varphi_{2n+1}(u) = -\frac{1}{2\sqrt[4]{AB}}\int_c^u \frac{A\varphi_{2n}'' + A'\varphi_{2n}' + C\varphi_{2n}}{\sqrt[4]{AB}}\,du - \frac{\varphi_{2n}'(c)\sqrt[4]{A(c)^3}}{\sqrt[4]{AB}\,\sqrt[4]{B(c)}} \qquad (7.5)$$

$(n = 1, 2, \cdots).$

---

37) Horn, J., Math. Ann., 52, 271-292 (1899).

If we impose the boundary conditions

$$y(c) = y(d) = 0, \tag{8}$$

we get from (6), on setting $\alpha = 0$ in (7) and introducing the notations

$$\tilde{\omega} \equiv \omega(d) \equiv \int_c^{d_4} \sqrt{\frac{B}{A}}\, dw, \quad \gamma_n \equiv \varphi_{n+2}(d) \text{ for } n \geq -1, \tag{9}$$

the characteristic equation

$$\nu\tilde{\omega} = \tan^{-1}\left\{ - \frac{\gamma_0 + \frac{\gamma_2}{\nu^2} + \cdots}{\nu\gamma_{-1} + \frac{\gamma_1}{\nu} + \cdots} \right\} = k\pi + \frac{\delta_1}{\nu} + \frac{\delta_3}{\nu^3} + \cdots \tag{10}$$

where k is an integer (assumed positive) and an elementary calculation gives

$$\delta_1 = -\frac{\gamma_0}{\gamma_{-1}}, \quad \delta_3 = -\left(\frac{\gamma_2}{\gamma_{-1}} - \frac{\gamma_0\gamma_1}{\gamma_{-1}^2} - \frac{\gamma_0^3}{3\gamma_{-1}^3} \right). \tag{11}$$

On setting

$$\nu_k = \frac{k\pi}{\tilde{\omega}} + \frac{\varepsilon_1}{k} + \frac{\varepsilon_3}{k^3} + \cdots \tag{12}$$

in (10) and equating to zero coefficients of successive powers of 1/k, we find that

$$\varepsilon_1 = \frac{\delta_1}{\pi}, \quad \varepsilon_3 = \frac{1}{\pi^3}\left(\tilde{\omega}^2 \delta_3 - \tilde{\omega}\delta_1^2 \right). \tag{13}$$

Hence from (9), (11), (12), and (13) the eigenvalues of (5) with the boundary conditions (8) are given by

$$\nu_k = \frac{k\pi}{\tilde{\omega}} - \frac{\varphi_2(d)}{k\pi\,\varphi_1(d)} + O\left(\frac{1}{(k\pi)^3}\right), \tag{14}$$

and the eigenfunctions are given to the same degree of approximation by (6) if we keep terms in $1/\nu^2$.

The case in which the boundary conditions are

$$y'(c) - hy(c) = 0, \; y'(d) + Hy(d) = 0 \tag{15}$$

is treated in Horn's paper. One sets $\alpha = 1$ and $\alpha' = h$ in (7) and obtains formally the same results as in (10) - (13) above, except that the quantities γ_n are now defined by

$$\gamma_{-1} = -\omega'\varphi_0, \qquad \gamma_0 = \varphi_0' + H\varphi_0 + \omega'\varphi_1,$$

$$\gamma_1 = \varphi_1' + H\varphi_1 - \omega'\varphi_2, \quad \gamma_2 = \varphi_2' + H\varphi_2 + \omega'\varphi_3, \ \cdots, \tag{16}$$

the functions all being evaluated at $u = d$.

If we consider specifically equation (2) under the boundary conditions (8), we have $A = 1$, $B = 1$, $C = -e^{2u}$, and $\alpha = 0$, so that from (7),

$$\omega(u) = \int_c^u du = u - c;$$
$$\varphi_0(u) = 0; \quad \varphi_1(u) = \alpha';$$
$$\varphi_2(u) = -\frac{\alpha'}{2}\int_c^u e^{2u}du = -\frac{\alpha'}{4}\left[e^{2u} - e^{2c}\right].$$

Hence the eigenvalues are given approximately by

$$\nu_k = \frac{k\pi}{(d-c)} + \frac{e^{2d} - e^{2c}}{4k\pi}, \tag{17.1}$$

and the corresponding eigenfunctions by

$$y_k = \alpha'\cos\nu_k(u-c)\left[\frac{(e^{2u}-e^{2c})}{4\nu_k^2} + \cdots\right] + \alpha'\sin\nu_k(u-c)\left[\frac{1}{\nu_k} + \cdots\right]. \tag{17.2}$$

If we transform back to the original variable x by means of (1) and let $a = e^c$, $b = e^d$, the eigenvalues are given by

$$\nu_k = \frac{k\pi}{\log b/a} + \frac{b^2 - a^2}{4k\pi} + \cdots; \tag{18.1}$$

and the eigenfunctions by

$$\alpha'\cos\left(\nu_k \log\frac{x}{a}\right)\left[\frac{x^2-a^2}{4\nu_k^2} + \cdots\right] + \alpha'\sin\left(\nu_k\log\frac{x}{a}\right)\left[\frac{1}{\nu_k} + \cdots\right], \tag{18.2}$$

the multiplicative constant α' being arbitrary. It would of course be possible to improve the approximations by computing more terms, but the quantities ϕ_n, δ_n, and $\mathcal{E}_n$ increase rapidly in complexity for larger values of n.

A great many theoretical results involving the eigenfunctions and eigenvalues of a Sturm-Liouville system, as well as some actual numerical information, may be obtained by adopting the viewpoint of the calculus of variations.[38] In connection with the system

$$\frac{d}{dx}\left\{K\frac{dy}{dx}\right\} - (\ell - \lambda g)y = 0 \qquad (K > 0,\ g > 0), \tag{19.1}$$

$$y'(a) - hy(a) = 0, \tag{19.2}$$

$$y'(b) + Hy(b) = 0, \tag{19.3}$$

one considers the functional expressions

$$\mathcal{D}[\varphi] = \int_a^b (K\varphi'^2 + \ell\varphi^2)dx + hK(a)\varphi(a)^2 + HK(b)\varphi(b)^2, \tag{20.1}$$

$$\mathcal{H}[\varphi, \psi] = \int_a^b g\varphi\psi\,dx; \tag{20.2}$$

in the case of the differential equation 1.2 (1) satisfied by the wedge functions, $K = x$, $\ell = x$, and $g = 1/x$, so that

$$\mathcal{D}[\varphi] = \int_a^b x(\varphi^2 + \varphi'^2)dx + ha\,\varphi(a)^2 + H(b)\varphi(b)^2, \tag{21.1}$$

$$\mathcal{H}[\varphi, \psi] = \int_a^b \varphi\psi\,\frac{dx}{x}. \tag{21.2}$$

Now it is known that if $y_0, y_1, \ldots, y_{n-1}$ are the first n eigenfunctions of the system (19), then the (n+1)st eigenfunction of (19) is that function y_n which minimizes the quotient $Q[y_n] = \mathcal{D}[y_n]/\mathcal{H}[y_n, y_n]$ under the n subsidiary conditions $\mathcal{H}[y_i, y_n] = 0$, $i = 0, 1, \ldots, n-1$; and the actual minimum value of Q is the (n+1)st eigenvalue λ_n. In particular, if λ_0 is the least eigenvalue of (19) and ϕ is any continuous function with a piecewise continuous first derivative, then

$$Q[\varphi] = \frac{\mathcal{D}[\varphi]}{\mathcal{H}[\varphi, \varphi]} \geq \lambda_0; \tag{22}$$

the more exactly ϕ approximates to the true eigenfunction, the more closely does the value of the quotient approach λ_0. One may improve the approximation by following the procedure of Ritz[39] and assuming for ϕ a series $c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$ with adjustable coefficients, then minimizing $Q[\phi]$ qua function of the coefficients.

The ideas just developed evidently apply also to Bessel functions of real argument and either real or purely imaginary order. Application

---

38) Courant, R., and Hilbert, D., Methoden der Mathematischen Physik, vol. 1, 2nd ed., chap. 6, 345-348.

39) Ibid, 149-151.

to a numerical example will be made in Art. 3.11.

1.5. Bessel Functions of Imaginary Order and Real Argument. Definitions of $U_\nu(x)$ and $V_\nu(x)$.

The remainder of this chapter will be devoted to a development of the properties of Bessel functions of purely imaginary order and real argument. The treatment will be similar to that just given the functions of imaginary order and imaginary argument, but somewhat less detailed.

Bessel's differential equation 0.1 (1) becomes, when $i\nu$ is written for ν, x for z, and y for w,

$$x^2 d^2y/dx^2 + x\, dy/dx + (x^2 + \nu^2)\, y = 0 \tag{1}$$

Two linearly independent solutions of (1), namely $J_{i\nu}(x)$ and $J_{-i\nu}(x)$, are given immediately by the power series 1.1 (2). Since when ν is real and x is real and positive $J_{i\nu}(x)$ and $J_{-i\nu}(x)$ are evidently complex conjugate quantities, under these conditions we shall regard as our fundamental pair of solutions of (1) the following real combinations:

$$U_\nu(x) \equiv \tfrac{1}{2}[J_{i\nu}(x) + J_{-i\nu}(x)] \equiv \mathrm{Re}\, J_{i\nu}(x); \tag{2.1}$$

$$V_\nu(x) \equiv \tfrac{1}{2i}[J_{i\nu}(x) - J_{-i\nu}(x)] \equiv \mathrm{Im}\, J_{i\nu}(x). \tag{2.2}$$

We observe that $U_\nu(x)$ is an even function of ν and $V_\nu(x)$ is an odd function of ν.

It may be noted that while the definitions of $F_\nu(x)$ and $G_\nu(x)$ can be so chosen that the two wedge functions exhibit very different behavior at infinity, no such marked difference in asymptotic behavior exists among the various real solutions of (1) with imaginary order and real argument to dictate the form which we shall adopt for the definitions of $U_\nu(x)$ and $V_\nu(x)$. It might be well, before any extensive numerical calculations of these functions are undertaken, to consider more carefully whether they are indeed the most convenient pair of solutions of equation (1).* We shall mention briefly some alternative solutions of (1) in Art. 1.53; meantime we proceed to develop the properties of $U_\nu(x)$ and $V_\nu(x)$.

1.51. Series and Integral Representations of $U_\nu(x)$ and $V_\nu(x)$.

Like the wedge functions, the functions $U_\nu(x)$ and $V_\nu(x)$ possess an oscillatory discontinuity at the origin. They may however be conveniently represented for small values of x in terms of series of ordinary Bessel functions or power series.

From Lommel's series 1.11 (1), on replacing ν by $i\nu$, z by x, and μ by 0, we obtain without difficulty

$$J_{i\nu}(x) = \frac{e^{i\nu\log\frac{1}{2}x}}{\Gamma(i\nu)}\left[C(\nu,x) - iD(\nu,x)\right], \tag{1}$$

where

$$C(\nu,x) = \sum_{m=1}^{\infty} \frac{m(\frac{1}{2}x)^m J_m(x)}{m!\,(m^2+\nu^2)}; \tag{2.1}$$

$$D(\nu,x) = \sum_{m=0}^{\infty} \frac{\nu(\frac{1}{2}x)^m J_m(x)}{m!\,(m^2+\nu^2)}. \tag{2.2}$$

As in 1.11 we set $\Theta(\nu, x) = \nu\log\frac{1}{2}x - \arg\Gamma(i\nu)$ and employ 1.11 (4); then on separating real and imaginary parts of (1) by 1.5 (2.1) and (2.2) we get

$$U_\nu(x) = \sqrt{\frac{\nu\sinh\nu\pi}{\pi}}\left[C(\nu,x)\cos\Theta(\nu,x) + D(\nu,x)\sin\Theta(\nu,x)\right]; \tag{3.1}$$

$$V_\nu(x) = \sqrt{\frac{\nu\sinh\nu\pi}{\pi}}\left[C(\nu,x)\sin\Theta(\nu,x) - D(\nu,x)\cos\Theta(\nu,x)\right]. \tag{3.2}$$

Comparing (2.1) and (2.2) with 1.11 (3.1) and (3.2) and using 1.13 (6.1) and (6.2), we see that $C(\nu, ix) = A(\nu, x) = -S_2(ix)/\nu$ and $D(\nu, ix) = B(\nu, x) = S_1(ix)/\nu$, so that

$$C(\nu, x) = -S_2(x)/\nu, \quad D(\nu, x) = S_1(x)/\nu, \tag{4}$$

*The matter of notation is also open to discussion.

where $S_1(x)$ and $S_2(x)$ are the power series defined by 1.13 (4.1) and (4.2).

A large number of contour integrals representing $J_\nu(z)$, most of which remain valid when ν is purely imaginary, are given by Watson.[40] Of theoretical interest is Poisson's integral, valid for Re $(\nu) > -\frac{1}{2}$,

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \cos(z\cos\theta)\sin^{2\nu}\theta\, d\theta, \tag{5}$$

which was used by Lommel in the work previously cited[22] to define Bessel functions of complex order. However if ν is complex, say $\nu = \sigma + i\tau$, then separation of real and imaginary parts of (5) leads to oscillatory factors under the integral sign of the form $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(2\tau \log \sin\theta)$ which, while they do not impair the theoretical usefulness of (5), render it practically worthless for purposes of numerical computation. The same criticism applies to the various transformations of this integral given by Watson.

A much more useful representation of $U_\nu(x)$ and $V_\nu(x)$ is furnished by Schläfli's generalization of Bessel's integral.[41] If Re $(z) > 0$, then for unrestricted values of ν,

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - z\sin\theta)\,d\theta - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-\nu t - z\,\mathrm{sh}\,t}\,dt. \tag{6}$$

If we replace ν by $i\nu$ and z by x and separate real and imaginary parts, we get by 1.5 (2.1) and (2.2)

$$U_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\,\mathrm{ch}\,\nu\theta\,d\theta - \frac{\mathrm{sh}\,\nu\pi}{\pi}\int_0^\infty e^{-x\,\mathrm{sh}\,t}\sin\nu t\,dt, \tag{7.1}$$

$$V_\nu(x) = \frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\,\mathrm{sh}\,\nu\theta\,d\theta - \frac{\mathrm{ch}\,\nu\pi}{\pi}\int_0^\infty e^{-x\,\mathrm{sh}\,t}\cos\nu t\,dt. \tag{7.2}$$

Another integral representation of $J_\nu(x)$, valid for $|\mathrm{Re}\,(\nu)| < 1$

---

40) Watson, op. cit., chap. VI.
41) Ibid., 176.

and x > 0, is[42)]

$$J_\nu(x) = \frac{2}{\pi}\int_0^\infty \sin\left(x\,\mathrm{ch}\,t - \frac{\nu\pi}{2}\right)\mathrm{ch}\,\nu t\,dt. \tag{8}$$

This yields, on setting $i\nu$ for ν,

$$U_\nu(x) = \frac{2}{\pi}\,\mathrm{ch}\,\frac{\nu\pi}{2}\int_0^\infty \sin(x\,\mathrm{ch}\,t)\cos\nu t\,dt, \tag{9.1}$$

$$V_\nu(x) = -\frac{2}{\pi}\,\mathrm{sh}\,\frac{\nu\pi}{2}\int_0^\infty \cos(x\,\mathrm{ch}\,t)\cos\nu t\,dt. \tag{9.2}$$

Since the convergence of the last two integrals is obtained only by the rapidity of oscillation of the integrands, they are probably not so well adapted to evaluation by mechanical quadrature as the infinite integrals of (7.1) and (7.2), whose convergence is secured by the factor exp (-x sh t).

1.52. Asymptotic Behavior of $U_\nu(x)$ and $V_\nu(x)$.

Using the notation of 1.12 (1), we have if ν is fixed and $|z|$ is large and positive with $|\arg z| < \pi$, the following asymptotic expansion of $J_\nu(z)$:[43)]

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left[\cos\left(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right)\sum_{m=0}^{\infty}\frac{(-)^m(\nu, 2m)}{(2z)^{2m}} - \sin\left(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right)\sum_{m=0}^{\infty}\frac{(-)^m(\nu, 2m+1)}{(2z)^{2m+1}}\right]. \tag{1}$$

The coefficient (ν, m) may be written if $m \geq 1$ in the form

$$(\nu, m) = \frac{\{4\nu^2 - 1^2\}\{4\nu^2 - 3^2\}\cdots\{4\nu^2 - (2m-1)^2\}}{2^{2m}\,m!}, \tag{2}$$

while $(\nu, 0) = 1$. Replacing ν by $i\nu$ and z by x and separating real and imaginary parts of (1) we have if ν is fixed and x is large and positive:

$$U_\nu(x) \sim \mathrm{ch}\frac{\nu\pi}{2}\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[\cos\left(x - \frac{\pi}{4}\right)\sum_{m=0}^{\infty}\frac{(-)^m(i\nu, 2m)}{(2x)^{2m}} - \sin\left(x - \frac{\pi}{4}\right)\sum_{m=0}^{\infty}\frac{(-)^m(i\nu, 2m+1)}{(2x)^{2m+1}}\right] \tag{3.1}$$

---

42) Watson, op. cit., 180.

43) Ibid., 199. The convergence factor for the series (1) is given explicitly by Airey in reference 19.

$$V_\nu(x) \sim \operatorname{sh}\frac{\nu\pi}{2}\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[\sin\left(x-\frac{\pi}{4}\right)\sum_{m=0}^{\infty}\frac{(-)^m(i\nu,2m)}{(2x)^{2m}} + \cos\left(x-\frac{\pi}{4}\right)\sum_{m=0}^{\infty}\frac{(-)^m(i\nu,2m+1)}{(2x)^{2m+1}}\right]. \quad (3.2)$$

From (3.1) and (3.2) we see that any real solution of 1.5 (1) has for sufficiently large values of x the asymptotic form

$$y \sim Ax^{-\frac{1}{2}} \sin (x + \delta), \quad (4)$$

thus confirming the remark made at the end of Art. 1.5 that all real Bessel functions of imaginary order and real argument exhibit the same (oscillatory) asymptotic behavior for large values of the argument. This result is to be contrasted with the non-oscillatory character of the functions of imaginary order and <u>imaginary</u> argument for $x \gg \nu$.

The limiting forms of $U_\nu(x)$ and $V_\nu(x)$ as $x \to +0$, ν being fixed, may be obtained from 1.51 (2) and (3) if we recall that $J_m(0) = \delta_{om}$; these forms are

$$U_\nu(x) \to \sqrt{\frac{\operatorname{sh}\nu\pi}{\nu\pi}} \sin\left[\nu \log \tfrac{1}{2}x - \arg \Gamma(i\nu)\right], \quad (5.1)$$

$$V_\nu(x) \to -\sqrt{\frac{\operatorname{sh}\nu\pi}{\nu\pi}} \cos\left[\nu \log \tfrac{1}{2}x - \arg \Gamma(i\nu)\right]. \quad (5.2)$$

Both functions evidently undergo an infinite number of oscillations in the neighborhood of the origin.

To find asymptotic expressions for $U_\nu(x)$ and $V_\nu(x)$ when ν is large and x is fixed, we substitute Stirling's approximation 1.12 (5) for the Γ-function into the first term of the series (cf. 1.1 (2)) for $J_{i\nu}(x)$ and obtain

$$J_{i\nu}(x) \sim \frac{1}{\sqrt{2\pi\nu}} \exp\left[i\nu\left(\log\frac{x}{2} - \log\nu + 1 - \frac{i\pi}{2}\right) - \frac{i\pi}{4}\right]\left\{1 + O\left(\frac{1}{\nu}\right)\right\}$$

$$= \frac{e^{\nu\pi/2}}{\sqrt{2\pi\nu}} \exp i\left[\nu\left(\log\frac{x}{2} - \log\nu + 1\right) - \frac{\pi}{4}\right]\left\{1 + O\left(\frac{1}{\nu}\right)\right\}. \quad (6)$$

Hence we have, for ν large and x fixed,

$$U_\nu(x) \sim \frac{e^{\nu\pi/2}}{\sqrt{2\pi\nu}} \cos\left[\nu\left(\log\tfrac{x}{2} - \log\nu + 1\right) - \tfrac{\pi}{4}\right]\left\{1 + O\left(\tfrac{1}{\nu}\right)\right\}, \quad (7.1)$$

$$V_\nu(x) \sim \frac{e^{\nu\pi/2}}{\sqrt{2\pi\nu}} \sin\left[\nu\left(\log\tfrac{x}{2} - \log\nu + 1\right) - \tfrac{\pi}{4}\right]\left\{1 + O\left(\tfrac{1}{\nu}\right)\right\}. \quad (7.2)$$

Both canonical solutions of Bessel's equation with imaginary order and real argument, regarded as functions of their order ν, undergo an infinite number of oscillations of exponentially increasing amplitude and slowly decreasing wavelength as ν increases without limit.

Since $J_\nu(z)$ is a continuous function of ν, we see from the definitions 1.5 (2.1) and (2.2) that as $\nu \to 0$, x remaining fixed,

$$U_\nu(x) \xrightarrow[\nu\to 0]{} J_0(x) \quad (8.1); \qquad V_\nu(x) \xrightarrow[\nu\to 0]{} 0. \quad (8.2)$$

Furthermore

$$\lim_{\nu\to 0} \frac{V_\nu(x)}{\nu} = \lim_{\nu\to 0} \frac{J_{i\nu}(x) - J_{-i\nu}(x)}{2i\nu} = \frac{1}{2} Y_0(x) \quad (8.3)$$

by definition, where $Y_0(x)$ is the Bessel function of the second kind of Hankel's type.[44]

Asymptotic expressions for $U_\nu(x)$ and $V_\nu(x)$ when ν and x are simultaneously large and of comparable magnitude may be obtained if necessary by specializing the formulas for Bessel functions of large order contained in the reference[23] mentioned at the end of Art. 1.12.

1.53. Alternative Definitions of Bessel Functions of Imaginary Order and Real Argument.

The equation

$$x^2 d^2y/dx^2 + x\, dy/dx + (x^2 + \nu^2)\, y = 0 \quad (1)$$

appears to have been first solved by Boole,[45] who obtained by the methods of operational calculus the general solution

---

44) Watson, op. cit., Arts. 3.5, 3.6.
45) Boole, G., Phil. Trans. Roy. Soc. (1844), 239.

$$y = \cos(\nu \log x)\sum_{n=0}^{\infty} a_{2n}x^{2n} + \sin(\nu \log x)\sum_{n=0}^{\infty} b_{2n}x^{2n}, \tag{2}$$

where a_0 and b_0 are arbitrary and for $n \geqslant 1$

$$a_{2n} = -\frac{n a_{2n-2} - \nu b_{2n-2}}{4n(n^2+\nu^2)}, \quad b_{2n} = -\frac{n b_{2n-2} + \nu a_{2n-2}}{4n(n^2+\nu^2)}. \tag{3}$$

Bocher's functions, denoted in 1.13 by $H_{i\nu}(x)$ and $I_{i\nu}(x)$ and defined by 1.13 (3.1) and (3.2), may evidently be obtained from Boole's solution by taking a_0 and b_0 to be 1 and 0 or 0 and 1 respectively.

If canonical solutions of (1) be defined by assigning simple values to the constants a_0 and b_0 in the general solution (2), the resultant series give precise information about the behavior of the functions which they represent in the neighborhood of the origin; but they do not convey a good idea of the nature of these functions for large values of the argument. On the other hand the functions $U_\nu(x)$ and $V_\nu(x)$, despite the fact that to represent them in the form (2) would require choosing a_0 and b_0 to have a complicated dependence on ν, are defined as simple combinations of the functions $J_{\pm i\nu}(x)$, whose behavior for all values of order and argument is already well known.* In view of the present state of development of the theory of Bessel functions, it seems convenient to choose the canonical functions of imaginary order and real argument to be related as directly as possible to $J_{i\nu}(x)$; whether or not we insert a multiplicative factor depending on ν, as in the case of the wedge functions $F_\nu(x)$ and $G_\nu(x)$, does not appear to be particularly significant. If $U_\nu(x)$ and $V_\nu(x)$ are to be tabulated over a considerable range of values of ν, it is evident from 1.52 (7.1) and (7.2) that the functions will

---

*So many of the known properties of the solutions of Bessel's equation have been expressed in terms of the function $J_\nu(z)$ that now the easiest way to investigate the series (2) would probably be to express it by 1.13 (1) and (2) as a linear combination of $J_{i\nu}(x)$ and $J_{-i\nu}(x)$, from which its properties could be quickly deduced.

show a wide variation in absolute magnitude; this complication may be avoided by tabulating the combinations exp $(-\frac{1}{2}\nu\pi)U_\nu(x)$ and exp $(-\frac{1}{2}\nu\pi)V_\nu(x)$.

For the reasons discussed at the end of Art. 1.13, it is not to be expected that recurrence relations will exist among Bessel functions of real argument and purely imaginary order.

1.6. Zeros of the Functions $U_\nu(x)$ and $V_\nu(x)$.

With the notation of Art. 1.0, the equation

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(x + \frac{\nu^2}{x}\right)y = 0 \qquad (1)$$

is a Sturm equation in which $K(x) = x$, $G(x) = -x - \nu^2/x$. The function $G(x)$ attains its maximum value -2ν when $x = \nu$ and tends to $-\infty$ as $x \to +0$ or as $x \to +\infty$. The following theorem summarizes various results concerning the distribution of the positive real zeros of the solutions of (1).

Theorem 1. (i). Every real solution of (1) has an infinite number of positive real zeros, with limit points at $x = 0$ and at $x = +\infty$.

(ii). If (a, b) is any preassigned finite interval of the positive real axis and m is any given positive integer, then for sufficiently large values of ν every real solution of (1) will have at least m zeros in (a, b).

(iii). If $y_\nu(x)$ is any real solution of (1) which vanishes at $x = c$ (>0), then the next smaller zero of $y_\nu(x)$ exceeds $c - \pi c/(c^2+\nu^2+\frac{1}{4})^{\frac{1}{2}}$.[46)]

Part (i) is an evident consequence of the limiting forms 1.52 (4) and (5) of the fundamental pair of solutions of (1) for large and small values of x.

For part (ii), if $\nu \geq b$ we have throughout (a, b) the inequalities $G(x) \leq \mathcal{G} = -b\,(1 + \nu^2/b^2)$ and $K(x) \leq \mathcal{K} = b$. The desired result follows

46) Cf. Watson, op. cit., Art. 15.82.

from theorem 3 of Art. 1.0, since

$$-\mathcal{G}/K = 1 + \nu^2/b^2 \geqslant m^2\pi^2/(b - a)^2$$

for all sufficiently large values of ν. The larger the order ν, the more rapidly will the solutions of (1) oscillate in the neighborhood of a given point.

To prove part (iii) we write $u = \sqrt{x}y$ in (1); u obviously has the same positive zeros as y. This substitution transforms (1) into

$$\frac{d^2u}{dx^2} + \left(1 + \frac{\nu^2 + \frac{1}{4}}{x^2}\right)u = 0, \tag{2}$$

which is to be compared with the equation

$$\frac{d^2v}{dx^2} + \left(1 + \frac{\nu^2 + \frac{1}{4}}{c^2}\right)v = 0. \tag{3}$$

The solutions of the latter equation are sinusoids in x with an interval $\pi c/(c^2 + \nu^2 + \frac{1}{4})^{\frac{1}{2}}$ between successive zeros. Now if $x \leqq c$, we have $(\nu^2 + \frac{1}{4})/x^2 \geqslant (\nu^2 + \frac{1}{4})/c^2$, so that by theorem 2 of Art. 1.0 the solutions of (2) oscillate more rapidly than the solutions of (3). This implies that if $y_\nu(x)$ vanishes at $x = c$, it must have vanished previously to the right of $c - \pi c/(c^2 + \nu^2 + \frac{1}{4})^{\frac{1}{2}}$. Since $c/(c^2 + \nu^2 + \frac{1}{4})^{\frac{1}{2}} < 1$, we see that every real solution of (1) vanishes at least once in any interval of length π of the positive real axis.

The following theorem is concerned with the zeros of the solutions of (1) regarded as functions of their order.

<u>Theorem 2.</u> If A and B are real constants independent of ν and if x has any fixed value, the linear combination

$$y_\nu(x) = AU_\nu(x) + BV_\nu(x), \tag{4}$$

regarded as a function of ν, has an infinite number of zeros for increasing values of ν with a limit-point at $+\infty$.

The theorem follows from a consideration of the asymptotic forms 1.52 (7.1) and (7.2) for $U_\nu(x)$ and $V_\nu(x)$ when ν is large and x is fixed.

in a manner analogous to the proof of theorem 2, Art. 1.2, which involves the wedge functions.

1.7. Expansion of an Arbitrary Function in a Series of Bessel Functions of Imaginary Order and Real Argument.

If we attempt to represent an arbitrary function by a series of Bessel functions of imaginary order and real argument over a finite interval of the positive x-axis, we arrive at results somewhat different from those encountered in the similar problem involving wedge functions, for we find that the representation usually requires a finite number of ordinary Bessel functions of real order and real argument in addition to an infinite series of functions of imaginary order and real argument.

Consider the Sturm-Liouville system:

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) - \left(-x - \frac{\nu^2}{x}\right)y = 0, \qquad (1.1)$$

$$\alpha' y(a) - \alpha y'(a) = 0, \qquad (1.2)$$

$$\beta' y(b) + \beta y'(b) = 0, \qquad (1.3)$$

where $0 < a < b < \infty$ and, with the notation of 1.0 (4.1), $K(x) = x$, $\ell(x) = -x$, $g(x) = 1/x$, and $\lambda = \nu^2$. The two boundary conditions together furnish the characteristic equation (cf. 1.0 (5)) which must be satisfied by the eigenvalues; e. g., in the simple case where $\alpha' = \beta' = 0$ so that the boundary conditions are $y'(a) = y'(b) = 0$, the characteristic equation is

$$\mathcal{F}(\nu^2) = U_\nu'(a)V_\nu'(b) - U_\nu'(b)V_\nu'(a) = 0. \qquad (2)$$

In any case the system (1) satisfies the conditions of the first sentence of theorem 5, Art. 1.0, so there will be an infinite set of real eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$, which have no limit-point but $+\infty$. Since $\ell(x)$ is negative, in general a finite number, say k, of these eigenvalues will be negative. If the eigenvalues be arranged in order of increasing algebraic magnitude and denoted by $-\mu_0^2 = \lambda_0$, $-\mu_1^2 = \lambda_1$, ... , $-\mu_{k-1}^2 = \lambda_{k-1}$, $\nu_k^2 = \lambda_k$, $\nu_{k+1}^2 = \lambda_{k+1}$, ... , we see that the eigenfunctions y_{μ_i} corresponding to the first k eigenvalues will be ordinary Bessel functions of real order μ_i and real argument, while the remaining eigenfunctions y_{ν_j} will be Bessel functions of purely imaginary order and real argument. It may be noted that the functions $y_{\mu_i}(x)$ and $y_{\nu_j}(x)$ exhibit no qualitative differences in behavior within the interval (a, b) except for the regular increase in number of zeros required by the fundamental theorem 5; outside the given interval in the neighborhood of the origin there is of course a marked difference in the behavior of the functions of real order and those of imaginary order.

An arbitrary function f(x) may be represented in (a, b) by means of the eigenfunctions of the system (1) in the form

$$f(x) = \sum_{m=0}^{k-1} A_m y_{\mu_m}(x) + \sum_{n=k}^{\infty} A_n y_{\nu_n}(x), \tag{3}$$

where the coefficients A_n of the second series are given formally by 1.31 (4) and (5),* and the coefficients A_m of the first series are given, since $\mu_m = i\nu_m$, by

$$A_m = \frac{\int_a^b f(t)\, y_{\mu_m}(t)\, \frac{dt}{t}}{\int_a^b y_{\mu_m}^2(t)\, \frac{dt}{t}}, \tag{4}$$

$$\int_a^b y_{\mu_m}^2(t)\, \frac{dt}{t} = -\frac{b}{2\mu_m}\left[\frac{\partial y_{\mu_m}(x)}{\partial x}\right]_{x=b}\left[\frac{\partial y_\mu(x)}{\partial \mu} + \frac{\beta}{\beta'}\frac{\partial^2 y_\mu(x)}{\partial \mu\, \partial x}\right]_{\substack{x=b\\ \mu=\mu_m}}. \tag{5}$$

The convergence properties of the series (3) depend upon such results as theorem 8 of Art. 1.0, which apply to Sturm-Liouville series in general.

We now consider briefly what form the series (3) must take if we try to represent f(x) in the infinite interval (a, ∞) or in the interval

---

*Of course y_{ν_n} is not the same function in the present section that it was in 1.31.

(0, b) which includes the origin.[47] From 1.52 (1) and (3) we see that every solution of Bessel's equation with real argument and either real or purely imaginary order vanishes, together with its derivatives, as $x \to \infty$, so that if a is positive and b is infinite, the boundary conditions (1.2) and (1.3) no longer select discrete values of ν. We have instead to use all values of ν, both real and purely imaginary, in the representation of f(x); and we are led to expect that if a representation of f(x) analogous to (3) is possible in the interval (a, ∞), it will be of the form

$$f(x) = \int_0^\infty C(\mu)\, y_\mu(x)\, d\mu + \int_0^\infty D(\nu)\, y_\nu(x)\, d\nu, \tag{6}$$

where $y_\mu(x)$ and $y_\nu(x)$ are those solutions of Bessel's equation, of real and purely imaginary order respectively, which satisfy the boundary condition (1.2). The coefficients $C(\mu)$ and $D(\nu)$, which will themselves involve definite integrals, may be expressed, as in the derivation at the end of 1.31, by

$$C(\mu) = \lim_{b\to\infty} \frac{A_m}{\mu_{m+1} - \mu_m}\, ; \quad D(\nu) = \lim_{b\to\infty} \frac{A_n}{\nu_{n+1} - \nu_n}. \tag{7}$$

If the left-hand end-point of the fundamental interval (a, b) is taken to be the origin, the boundary conditions (1.2) and (1.3) still give us a finite number of discrete negative eigenvalues $\lambda_i = -\mu_i^2$ corresponding to eigenfunctions $J_{\mu_i}(x)$. Since we have automatically $J_\mu(0) = 0$ if $\mu > 0$ and $J_\mu'(0) = 0$ if $\mu > 1$, the μ_i's are the roots in μ of the equation $\beta' J_\mu(b) + \beta J_\mu'(b) = 0$. On the other hand if $\lambda = \nu^2 > 0$, we cannot satisfy the boundary condition (1.2), since in this case from 1.52 (5.1) and (5.2) all solutions of (1.1) oscillate infinitely rapidly

---

47) Bocher, op. cit., 159-160, discusses both of these possibilities in the case where the boundary condition at the end-points is $y = 0$.

in the neighborhood of the origin and their derivatives are unbounded. However in some applications, such as those arising in potential theory, it is sufficient to require, not that (1.2) be satisfied, but merely that the functions $y_\nu(x)$ remain bounded as $x \to +0$. This latter condition is satisfied by the solutions of (1.1) for all positive values of ν^2. Hence the representation analogous to (3) of f(x) in the open interval (0, b) must consist of a finite series of ordinary Bessel functions of real order plus an infinite integral over the functions of imaginary order; viz.,

$$f(x) = \sum_{m=0}^{k-1} A_m y_{\mu_m}(x) + \int_0^\infty B(\nu) y_\nu(x)\, d\nu, \tag{8}$$

where $y_{\mu_m}(x)$ and $y_\nu(x)$ are bounded at the origin and satisfy the boundary condition (1.3). The coefficient A_m is given by (4) and the coefficient $B(\nu)$ by

$$B(\nu) = \lim_{a\to 0} \frac{A_n}{\nu_{n+1} - \nu_n}. \tag{9}$$

The representation of f(x) over the whole range $(0, \infty)$ will evidently require two infinite integrals of the form (6) with a = 0.

The actual existence of formulas such as (6) and (8) appears very plausible in view of the known validity of several similar integral formulas of Fourier's type; however we shall not here investigate the explicit form of the coefficients in (6) and (8) or the conditions under which these formulas may be rigorously valid.

1.8. Transformation of the Differential Equation for the Functions $U_\nu(x)$ and $V_\nu(x)$. Calculation of the Eigenvalues.

For purposes of numerical calculation it is convenient to subject the differential equation 1.5 (1) for Bessel functions of imaginary order and real argument to the transformation of Art. 1.4, namely

$$u = \log x, \text{ or } x = e^u. \qquad (1)$$

The equation then becomes

$$d^2y/dx^2 + (\nu^2 + e^{2u})\, y = 0, \qquad (2)$$

which has no singularities for finite values of u, and of which the general solution is evidently

$$y = c_1 U_\nu(e^u) + c_2 V_\nu(e^u). \qquad (3)$$

From (1) and 1.52 (5.1) and (5.2), or by inspection of the transformed equation (2), the solutions are approximately sinusoidal functions of νu when u is negative and so large that $e^{2u} << \nu^2$.

To find the small eigenvalues of the Sturm-Liouville system 1.7 (1), it is necessary to obtain the first few roots of some such characteristic equation as 1.7 (2). As in the case of the wedge functions, the only practicable way of doing this appears to be interpolation in a table of the functions U_ν and V_ν, or preferably in a table of their derivatives; until such tables are available the practical value of Bessel functions of imaginary order and real argument will be limited. If the characteristic equation happens to have negative roots, as discussed in the preceding article, recourse must be had to a table of ordinary Bessel functions which includes non-integral as well as integral values of the order.

The large positive eigenvalues of the system 1.7 (1) are given asymptotically by Horn's method, outlined in Art. 1.4. If for example the boundary conditions are dy/du = 0 at u = c and at u = d, corresponding in virtue of (1) to dy/dx = 0 at $x = e^c$ and at $x = e^d$, then we have h = H = 0 in 1.4 (15). Setting A = B = 1, $C = e^{2u}$, and $\alpha' = 0$ in 1.4 (7), we get

$$\omega(u) = u - c, \quad \varphi_0(u) = \alpha, \quad \varphi_1(u) = -\frac{\alpha}{4}(e^{2u} - e^{2c}).$$

Hence from 1.4 (16), (11), (12), and (13) the positive eigenvalues are given approximately by

$$\nu_k = k\pi/(d - c) - (e^{2d} - e^{2c})/4k\pi,$$

or transforming back to the variable x by (1) and setting $a = e^c$, $b = e^d$,

$$\nu_k = k\pi/\log (b/a) - (b^2 - a^2)/4k\pi. \tag{4}$$

The corresponding eigenfunctions are, by 1.4 (6),

$$\alpha\left\{\cos\left(\nu_k \log \frac{x}{a}\right)\left[1 + O\left(\frac{1}{\nu_k^2}\right)\right] - \sin\left(\nu_k \log \frac{x}{a}\right)\left[\frac{x^2 - a^2}{4\nu_k} + O\left(\frac{1}{\nu_k^3}\right)\right]\right\}. \tag{5}$$

As in Art. 1.4, we may employ the calculus of variations to estimate the lowest eigenvalue λ_0 of the system 1.7 (1). In the case where the boundary conditions are $y'(a) = y'(b) = 0$, we find on comparing 1.7 (1) with 1.4 (19), (20), and (22), that

$$\lambda_0 \leq Q[\varphi] = \frac{D[\varphi]}{H[\varphi, \varphi]}, \tag{6.1}$$

where $D[\varphi] = \int_a^b x(\varphi'^2 - \varphi^2)\,dx$ and $H[\varphi, \varphi] = \int_a^b \varphi^2 \frac{dx}{x}$, (6.2)

ϕ being any continuous function with a piecewise continuous derivative in (a, b) such that

$$\phi'(a) = \phi'(b) = 0. \tag{7}$$

If we take $\phi(x) \equiv 1$, we get at once from (6)

$$\lambda_0 \leq Q[1] = -\tfrac{1}{2}(b^2 - a^2)/\log (b/a). \tag{8}$$

It is also of some interest to take $\phi = \cos[n\pi(x - a)/(b - a)]$, where n is a positive integer. This function evidently satisfies (7), and an elementary calculation gives for the expressions defined by (6.2)

$$D_n = \int_a^b x\left[\frac{n^2\pi^2}{(b-a)^2}\sin^2\frac{n\pi(x-a)}{(b-a)} - \cos^2\frac{n\pi(x-a)}{(b-a)}\right]dx$$
$$= \frac{(b^2 - a^2)}{4}\left[\frac{n^2\pi^2}{(b-a)^2} - 1\right]; \tag{9.1}$$

$$H_n = \int_a^b \cos^2\frac{n\pi(x-a)}{(b-a)}\,\frac{dx}{x}$$
$$= \frac{1}{2}\log\frac{b}{a} + \frac{1}{2}\cos\frac{2n\pi a}{b-a}\left[Ci\,\frac{2n\pi b}{b-a} - Ci\,\frac{2n\pi a}{b-a}\right]$$
$$+ \frac{1}{2}\sin\frac{2n\pi a}{b-a}\left[Si\,\frac{2n\pi b}{b-a} - Si\,\frac{2n\pi a}{b-a}\right], \tag{9.2}$$

where $\mathrm{Si}\ u = \int_0^u \frac{\sin t}{t}\,dt$ and $\mathrm{Ci}\ u = -\int_u^\infty \frac{\cos t}{t}\,dt$. It is not necessarily true that the ratio of the last given expressions for $\mathcal{D}_n$ and $\mathcal{H}_n$ dominates the (n+1)st eigenvalue of the system 1.7 (1), because the function $\phi = \cos[n\pi(x - a)/(b - a)]$ is not strictly orthogonal to the preceding n eigenfunctions; nevertheless the ratio $\mathcal{D}_n/\mathcal{H}_n$ should give a rough approximation to the (n+1)st eigenvalue for small values of n, and in the absence of tables this value may be improved by using such definite integrals as 1.51 (7.1) and (7.2) actually to compute the functions U_ν' and V_ν' occurring in the characteristic equation for a pair of adjacent values of ν.

In Art. 3.11 we shall compare for a particular numerical case the estimates of the eigenvalues given by (4) and by (9).

CHAPTER II

Physical Applications of Bessel Functions of Imaginary Order and Imaginary Argument

2.1. A General Potential Problem in Cylindrical Coordinates.

Bessel functions of imaginary order were introduced into mathematical physics by M. Bocher[1] in the investigation of a certain problem of potential theory. The essential features of Bocher's discussion will be given in the present article.

The potential problem in question is the following: Given a space S bounded externally by two coaxial cylinders of revolution, two planes through the axis of these cylinders, and two planes perpendicular to this axis. It is required to find a potential function V which 1) everywhere within S satisfies Laplace's equation $\nabla^2 V = 0$, and is finite, continuous, and single-valued, together with its first space derivatives, and 2) assumes on the surface of S arbitrarily assigned values.

The space S may be defined in a conveniently chosen system of cylindrical coordinates (ρ, ϕ, z) by the inequalities $0 < a \leq \rho \leq b$, $0 \leq \phi \leq \alpha < 2\pi$, and $0 \leq z \leq c$. We may solve the general potential problem by superposing six simpler potential functions, each of which takes on assigned values on a single (different) face of S and vanishes on the other five faces. The face of S on which a given potential function does not vanish will be called the exceptional face; it turns out that we shall get essentially three different types of solution, corresponding to the following cases:

---

1) Bocher, M., Annals of Mathematics, 6, 137-160 (1892).

(i) The exceptional face is one of the planes perpendicular to the axis, say $z = c$;

(ii) the exceptional face is one of the cylindrical surfaces of S, say $\rho = b$;

(iii) the exceptional face is one of the azimuthal planes, say $\phi = \alpha$.

Our first task is to find suitable solutions of Laplace's equation, which in cylindrical coordinates takes the form

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{1}$$

If we assume that V may be expressed as a product of cylindrical harmonics,

$$V = R(\rho)\, \Phi(\phi)\, Z(z), \tag{2}$$

we find that V will be a solution of (1) provided that R, Φ, and Z satisfy the following equations:[2)]

$$d^2\Phi/d\phi^2 = -\nu^2\Phi, \tag{3}$$

$$d^2Z/dz^2 = k^2 Z, \tag{4}$$

$$\rho^2 d^2R/d\rho^2 + \rho\, dR/d\rho + (k^2\rho^2 - \nu^2)\, R = 0, \tag{5}$$

where k^2 and ν^2 are arbitrary separation constants. It is seen that Φ and Z will be exponential or trigonometric functions of their arguments, and, by comparing (5) with 0.1 (1), that R will be a Bessel function of order ν and argument $k\rho$.* The behavior of all three functions depends largely upon the nature of the separation constants.

If u_1, u_2, u_3 represent the three cylindrical coordinates in any order, the exceptional face of S being given by $u_3 =$ constant, then the product of harmonics (2) must vanish on a pair of faces $u_1 =$ constant and on a pair of faces $u_2 =$ constant; therefore k^2 and ν^2 must be so chosen that $U_1(u_1)$ and $U_2(u_2)$ are oscillatory functions of their arguments in

---

2) Smythe, W. R., Static and Dynamic Electricity, 1st ed., Arts. 5.29, 5.291.

*In the special cases $k = 0$ and $k = \nu = 0$, the solutions of (3), (4), and (5) are all elementary functions; these are not of interest in our present development.

the relevant intervals. This requirement fixes the nature of the separation constants in the three different cases listed above.

In case (i) the condition that R and Φ must be oscillatory functions is secured by taking ν and k both real, so that the solutions of the Φ-equation will be sinusoids in $\nu\phi$ and the solutions of the R-equation will be ordinary Bessel functions of order ν and argument $k\rho$. The solutions of the Z-equation will be real exponential (hyperbolic) functions of kz. It is easy to show from the general theory of Sturm-Liouville systems that the boundary conditions on Φ and R determine an infinite number of admissible values of the constants ν and k.

In case (ii) we make both Φ and Z sinusoidal functions by taking ν real and k purely imaginary; the boundary conditions are satisfied by an infinite number of values of each. The solutions of the R-equation are now Bessel functions of real order and imaginary argument, i. e., modified Bessel functions I_ν and K_ν.

In case (iii) we take ν and k both imaginary, so that Z is a sinusoidal function of $|k|z$ and R is a Bessel function of imaginary order and imaginary argument; Φ will be a sum of real exponential (hyperbolic) functions of $|\nu|\phi$. If we now write k for ik and ν for iν and employ the notation introduced in 1.1 for Bessel functions of imaginary order and imaginary argument, we get for the typical product of harmonics

$$V = [AF_\nu(k\rho) + BG_\nu(k\rho)][C \operatorname{sh} \nu\phi + D \operatorname{ch} \nu\phi][E \sin kz + F \cos kz]. \quad (6)$$

The separate solutions of our three partial problems all proceed now in much the same way; but since problems of types (i) and (ii) involve only well-known functions and are treated in standard works on potential theory,[3)] we turn our attention immediately to (iii). In this case the

---

3) See for example Smythe, op. cit., chap. V.

boundary condition on the surfaces $\rho = a$, $\rho = b$, $z = 0$, $z = c$, and $\phi = 0$ is $V = 0$; on the remaining surface $\phi = \alpha$ the condition is

$$V(\rho, \alpha, z) = f(\rho, z). \tag{7}$$

We secure the vanishing of V on the first five surfaces by choosing from the set of all products of the type (6) every one which has the form

$$V_{mn} = C_{mn}\left[G_{\nu_{nm}}(n\pi a/c)F_{\nu_{nm}}(n\pi\rho/c) - F_{\nu_{nm}}(n\pi a/c)G_{\nu_{nm}}(n\pi\rho/c)\right]\mathrm{sh}\,\nu_{nm}\phi \,\sin(n\pi z/c), \tag{8}$$

where we have taken $k = n\pi/c$, n a positive integer, and ν_{nm} is the m<u>th</u> positive root of the equation

$$G_\nu(n\pi a/c)F_\nu(n\pi b/c) - F_\nu(n\pi a/c)G_\nu(n\pi b/c) = 0. \tag{9}$$

The last equation is equivalent to 1.31 (2); it has an infinite number of real positive roots in ν. We shall denote the Bessel function enclosed in square brackets in (8) by $R_{\nu_{nm}}(n\pi\rho/c)$.

We now build from the set of products (8) the double series

$$V(\rho, \phi, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}R_{\nu_{nm}}(n\pi\rho/c)\ \mathrm{sh}\,\nu_{nm}\phi\ \sin n\pi z/c, \tag{10}$$

which vanishes on the non-exceptional faces of S and which we shall assume satisfies Laplace's equation, as it certainly would if it consisted of only a finite number of terms.* Inasmuch as the functions $\sin n\pi z/c$ and $R_{\nu_{nm}}(n\pi\rho/c)$ form complete orthogonal sets over the intervals $0 \leq z \leq c$ and $a \leq \rho \leq b$, we may formally determine the coefficients to satisfy the boundary condition on the exceptional face. Substitution of (10) into (7) gives

---

*In the present section and the two following we shall assume without further investigation that the infinite series and infinite integrals with which we deal are convergent and that the necessary interchanges of limit-operations are justified; such formal procedure is often fruitful, even though from the point of view of pure mathematics a more rigorous treatment would be desirable. As Bocher[4)] points out, the <u>practical utility</u> of such a series as (10) depends not so much on its <u>ultimate</u> convergence as on the numerical accuracy with which the first few terms approximate to the desired function.

4) Bocher, M., <u>Über die Reihenentwickelungen der Potentialtheorie</u>, 157-158.

$$f(\rho, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} R_{\nu_{nm}}(n\pi\rho/c)\, \mathrm{sh}\,\nu_{nm}\alpha \,\sin n\pi z/c. \tag{11}$$

We multiply both sides of (11) by $\rho^{-1} R_{\nu_{pq}}(p\pi\rho/c) \sin p\pi z/c$ and integrate first over z from 0 to c, then over ρ from a to b. The orthogonality properties of the sine and Bessel functions cause all terms to drop out except the one for which n = p, m = q; and we get, using 1.31 (5) with $\beta = 0$,

$$\int_a^b \int_0^c \frac{1}{\rho} f(\rho, z)\, R_{\nu_{pq}}(p\pi\rho/c) \sin(p\pi z/c)\, dz\, d\rho$$
$$= C_{qp}\, \mathrm{sh}\,\nu_{pq}\alpha \cdot \frac{p\pi b}{4\nu_{pq}} \left[\frac{\partial R_\nu(x)}{\partial x} \frac{\partial R_\nu(x)}{\partial \nu} \right]_{\substack{x = p\pi b/c \\ \nu = \nu_{pq}}} \tag{12}$$

The last equation expresses the value of C_{qp} in terms of the given function $f(\rho, z)$ on the exceptional face and completes the formal determination of the potential function corresponding to subcase (iii) of our general potential problem. Cases (i) and (ii) may evidently be treated in a similar manner; the latter requires the expansion of an arbitrary function in a double Fourier series in ϕ and z, and the former involves a mixed Fourier and Fourier-Bessel expansion in ϕ and ρ.

It is apparent from what we have just done that the complete solution, by the method of development in series, of the general potential problem stated at the beginning of this section requires the use not only of ordinary and modified Bessel functions of real order but also of Bessel functions whose order and argument are both purely imaginary, the latter functions being necessary to secure assigned boundary values on portions of the wedge surfaces ϕ = constant. The analogous potential problem in spherical polar coordinates involves the space S bounded externally by the concentric spheres r = a and r = b, the coaxial cones $\theta = \alpha$ and $\theta = \beta$, and the azimuthal planes $\phi = 0$ and $\phi = \phi_0$. The subcase (iii) above corresponds to a potential in the spherical polar system which takes assigned values on the cone $\theta = \beta$ and vanishes on the remaining faces of S. To satisfy these boundary conditions we require associated Legendre functions $P_{-\frac{1}{2}+ip}^{m}(\cos\theta)$ of complex degree;[5] these have been called "cone functions" on account of the manner of their introduction into mathematical physics. Because the functions $F_{\nu}(x)$ and $G_{\nu}(x)$ appear in the analogous problem involving wedges, we refer to them in this work as "wedge functions".

Although potential problems as general as ours are not often solved explicitly in textbooks, it is easy to see that many practical problems of potential theory are merely special cases of the one discussed here, in which one or more of the six surfaces of the space S have disappeared. These degenerate cases lead to changes, such as the replacement of an infinite series by an infinite integral, in the various formal expressions for the potential. We note particularly the various alternative forms which the solution involving wedge functions may assume.[6]

If we take $a = 0$, so that the inner cylindrical surface of S shrinks to the axis, then the condition that $R_{\nu}(n\pi\rho/c)$ vanish at $\rho = a$ can no longer be satisfied, because of the oscillatory behavior of the wedge functions at the origin; but it may be replaced by the weaker requirement that $R_{\nu}(n\pi\rho/c)$ remain finite as $\rho \to 0$, which is met by every real value of ν. Hence the summation over m in (10) is replaced by an integration over all positive values of ν, and the solution retains this form whether or not the outer cylindrical boundary of S is let move away to infinity. (Compare in this connection the latter part of Art. 1.31.) Similarly if one or both of the bounding surfaces z = constant is removed to infinity, the boundary condition which restricts k ($= n\pi/c$) to discrete values in (10) is abolished, and the summation over n becomes a Fourier integral

5) Hobson, E. W., Spherical and Ellipsoidal Harmonics, 444-448.
6) Bocher, M., Annals of Mathematics, 6, 152-154 (1892).

over all real values of k. An example of a potential distribution involving wedge boundaries where the fields extend to infinity will be given in Art. 2.12.

It may be noted finally that if the angle α between the inclined surfaces of S is allowed to increase to 2π and the azimuthal planes are removed, leaving a ring-shaped region with only four faces, the potential problem corresponding to case (iii) vanishes and we need to use only ordinary and modified Bessel functions of real order.

The reader is doubtless aware that the usefulness of solutions of Laplace's equation is not confined to electrostatics. This important equation is satisfied by such quantities as the magnetic scalar potential, the velocity potential of irrotational flow of a perfect fluid, and the temperature in the steady state of diffusion of heat, all of which may occur in problems involving wedge boundaries. Since the mathematical treatment of all of these functions is very similar, we shall confine ourselves in the next two articles to examples from the field of electrostatics; in Art. 2.13 we shall make a few remarks concerning the equation for the conduction of heat.

2.11. Potential Distribution Due to a Point Charge inside a Cylindrical Conducting Ring with Two Dielectrics.

In this article we shall consider the problem of finding the potential distribution within a hollow ring bounded by the earthed conducting surfaces $\rho = a$, $\rho = b$, $z = 0$, and $z = c$, under the influence of an interior point charge q at (ρ_0, ϕ_0, z_0), the region $0 < \phi < \alpha < \phi_0$ within the ring being filled with a dielectric of capacitivity ε_1, and the region $\alpha < \phi < 2\pi$ containing the charge being filled with a dielectric of capacitivity ε_2.

Like most potential theory problems involving point charges, this problem reduces essentially to the determination of a function (called the Green's function) which satisfies Laplace's equation in a given region, vanishes on the boundaries of the region, and has a simple pole at an interior point of the region, such that the difference between the Green's function and the reciprocal of the distance from the pole tends to a definite limit as the variable point approaches the pole. The Green's function is just proportional to the potential which would be produced by a point charge situated at the pole, the boundaries of the region being held at potential zero. Since the problem at hand requires us to satisfy boundary conditions on the wedge surfaces ϕ = constant, obviously we shall need in the determination of the Green's function the harmonics of 2.1 (6) which involve the wedge functions.

Now it turns out that a systematic and mathematically rigorous study of the various forms of Green's function for spaces bounded by surfaces of the cylindrical coordinate system was made several decades ago by J. Dougall,[7] his results being reproduced in the textbook of Gray, Matthews, and MacRobert.[8] It would therefore be possible, using the methods of Dougall, to develop rigorously the Green's function for the cylindrical ring-shaped region $a \leq \rho \leq b$, $0 \leq z \leq c$ (a result which he does not write down explicitly), for the case where the region is filled with a single homogeneous isotropic dielectric, and then to solve the problem at hand by superposing series of the form 2.1 (10) which represent the effect of the second dielectric, the coefficients being determined so as to intro-

---

7) Dougall, J., Proc. Edinburgh Math. Soc., 18, 33-83 (1900).

8) Gray, A., Matthews, G. B., and MacRobert, T. M., Bessel Functions, 2nd ed., 101-110. Note that these authors express Dougall's results in terms of the function K_{is} employed in the present work; the function which Dougall calls $G_{is}(ix)$ (cf. G. M. M., p. 23) is equal to $\exp(\frac{1}{2}s\pi)K_{is}(x)$, or, in terms of our wedge functions, to $\exp(\frac{1}{2}s\pi)G_s(x)$.

duce no additional singularities and to satisfy the continuity conditions at the dielectric boundaries. Instead of proceeding in this manner, however, we shall solve the problem ab initio, taking account of the singularity due to the point charge by a method often employed by Smythe, which, if it lacks anything in mathematical rigor, has at least the advantage of physical clarity.

Using the notation of 2.1, where

$$R_\nu(n\pi\rho/c) = G_\nu(n\pi a/c)\, F_\nu(n\pi\rho/c) - F_\nu(n\pi a/c)\, G_\nu(n\pi\rho/c) \qquad (1)$$

and ν_{nm} is the mth positive root of $R_\nu(n\pi b/c) = 0$, we assume potentials in the different regions of interest as follows: For $0 < \phi < \alpha$,

$$V_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{nm}e^{\nu_{nm}\phi} + B_{nm}e^{-\nu_{nm}\phi}\right]R_{\nu_{nm}}(n\pi\rho/c)\sin n\pi z/c; \qquad (2)$$

for $\alpha < \phi < \phi_0$,

$$V_2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[C_{nm}e^{\nu_{nm}\phi} + D_{nm}e^{-\nu_{nm}\phi}\right]R_{\nu_{nm}}(n\pi\rho/c)\sin n\pi z/c; \qquad (3)$$

and for $\phi_0 < \phi < 2\pi$,

$$V_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[E_{nm}e^{\nu_{nm}\phi} + F_{nm}e^{-\nu_{nm}\phi}\right]R_{\nu_{nm}}(n\pi\rho/c)\sin n\pi z/c. \qquad (4)$$

These potentials have been chosen so as to vanish on all conducting surfaces. The remaining arbitrary constants will be determined from the conditions that V and $\varepsilon\frac{\partial V}{\partial n}$ must be continuous across dielectric boundaries (where $\frac{\partial}{\partial n} = \frac{1}{\rho}\frac{\partial}{\partial\phi}$ represents the normal derivative) and that V must be continuous except at the charge itself across the surface $\phi = \phi_0$, while for any closed surface S within the ring and surrounding the point charge,

$$-\int_S \varepsilon\frac{\partial V}{\partial n}\,dS = q, \qquad (5)$$

n being the outward normal to S. Equation (5) is the statement of Gauss's electric flux theorem in rationalized MKS units.

Applying the continuity conditions to V_1 at $\phi = 0$ and to V_3 at $\phi = 2\pi$ and noting that because the sine and Bessel functions form orthogonal sets, corresponding terms of the resultant series must be equal separately, we get, on cancelling common factors,

$$A_{nm} + B_{nm} = E_{nm}e^{2\pi\nu_{nm}} + F_{nm}e^{-2\pi\nu_{nm}}, \tag{6}$$

$$\varepsilon_1(A_{nm} - B_{nm}) = \varepsilon_2(E_{nm}e^{2\pi\nu_{nm}} - F_{nm}e^{-2\pi\nu_{nm}}). \tag{7}$$

Similarly at $\phi = \alpha$,

$$A_{nm}e^{\nu_{nm}\alpha} + B_{nm}e^{-\nu_{nm}\alpha} = C_{nm}e^{\nu_{nm}\alpha} + D_{nm}e^{-\nu_{nm}\alpha}, \tag{8}$$

$$\varepsilon_1(A_{nm}e^{\nu_{nm}\alpha} - B_{nm}e^{-\nu_{nm}\alpha}) = \varepsilon_2(C_{nm}e^{\nu_{nm}\alpha} - D_{nm}e^{-\nu_{nm}\alpha}); \tag{9}$$

and at $\phi = \phi_0$,

$$C_{nm}e^{\nu_{nm}\phi_0} + D_{nm}e^{-\nu_{nm}\phi_0} = E_{nm}e^{\nu_{nm}\phi_0} + F_{nm}e^{-\nu_{nm}\phi_0}. \tag{10}$$

We shall apply Gauss's electric flux theorem (5) to the pair of planes $\phi = \phi_0 - 0$ and $\phi = \phi_0 + 0$ which fit snugly around the point charge at (ρ_0, ϕ_0, z_0); for this purpose we write

$$\frac{\varepsilon_2}{\rho}\left[\frac{\partial V_2}{\partial \varphi} - \frac{\partial V_3}{\partial \varphi}\right]_{\varphi=\varphi_0}$$

$$= \frac{\varepsilon_2}{\rho}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\nu_{nm}\left[(C_{nm}-E_{nm})e^{\nu_{nm}\varphi_0} - (D_{nm}-F_{nm})e^{-\nu_{nm}\varphi_0}\right]R_{\nu_{nm}}\left(\frac{n\pi\rho}{c}\right)\sin\frac{n\pi z}{c}, \tag{11}$$

multiply both sides of (11) by $R_{\nu_{kl}}(k\pi\rho/c)\sin k\pi z/c$, and integrate over z from 0 to c and over ρ from a to b. On the left side we get the integral

$$\int_a^b\int_0^c \frac{\varepsilon_2}{\rho}\left[\frac{\partial V_2}{\partial \varphi} - \frac{\partial V_3}{\partial \varphi}\right]_{\varphi=\varphi_0} R_{\nu_{kl}}\left(\frac{k\pi\rho}{c}\right)\sin\left(\frac{k\pi z}{c}\right)dz\,d\rho.$$

Since the electric field is continuous across the plane $\phi = \phi_0$ except at the point where the charge is situated, the factor in square brackets vanishes except in an area around $\rho = \rho_0$, $z = z_0$, which will be taken to be so small that in it the sine and Bessel functions may be regarded as constant and taken out from under the integral sign.[9] The remaining integral is just the left-hand side of (5) and is equal to q. Hence the left side of (11) becomes, after integration, $qR_{\nu_{kl}}(k\pi\rho_0/c)\sin k\pi z_0/c$, while on the right side all terms drop out except the one for which n = k,

---

9) Cf. Smythe, op. cit., Art. 5.297. The factor in brackets is assumed to possess the characteristic property of the δ-function employed by Dirac and others in quantum mechanics.

$m = \ell$, which we evaluate by 1.31 (5).

On introducing the notation

$$I_{nm} = \frac{4 R_{\nu_{nm}}(n\pi \rho_0/c) \sin(n\pi z_0/c)}{n\pi b \left[\frac{\partial R_\nu(x)}{\partial x} \frac{\partial R_\nu(x)}{\partial \nu}\right]_{x = n\pi b/c,\ \nu = \nu_{nm}}}, \tag{12}$$

we get

$$\varepsilon_2 \left[(C_{nm} - E_{nm}) e^{\nu_{nm}\phi_0} - (D_{nm} - F_{nm}) e^{-\nu_{nm}\phi_0}\right] = q I_{nm}. \tag{13}$$

Simultaneous solution of the six linear equations (6) - (10) and (13) is a tedious but elementary exercise; on carrying out the algebra and combining terms we get for the factor depending on ϕ in (2) the following expression:

$$A_{nm} e^{\nu_{nm}\phi} + B_{nm} e^{-\nu_{nm}\phi}$$
$$= \frac{(1-\beta) q I_{nm}}{2\varepsilon_2} \frac{\operatorname{sh}\nu_{nm}\pi \operatorname{ch}\nu_{nm}(\phi - \phi_0 + \pi) + \beta \operatorname{sh}\nu_{nm}(\pi - \alpha) \operatorname{ch}\nu_{nm}(\phi + \phi_0 - \pi - \alpha)}{\operatorname{sh}^2 \nu_{nm}\pi - \beta^2 \operatorname{sh}^2 \nu_{nm}(\pi - \alpha)}, \tag{14}$$

where $\beta = (\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2)$. Eqs. (2) and (14) provide an explicit expression for the potential in the region of capacitivity ε_1; similar expressions may easily be written down for the region of capacitivity ε_2.

2.12. Potential Distribution Due to a Point Charge in the Neighborhood of a Dielectric Wedge.

We shall next determine by the use of wedge functions the potential distribution produced by a point charge near an infinite dielectric wedge. Let the charge q be located at (ρ_0, ϕ_0, z_0), the region $0 < \phi < \alpha < \phi_0$ being filled with a dielectric of capacitivity ε_1, and the region $\alpha < \phi < 2\pi$ containing the charge being filled with a dielectric of capacitivity ε_2. Since in this case the fields are not limited to a finite region by conducting boundaries, we shall expect the potentials to be expressed by integrals rather than by series of discrete terms, as noted at the end of Art. 2.1.

We require first an expression in terms of wedge functions for the inverse distance from the pole (ρ_0, ϕ_0, z_0) to the variable point (ρ, ϕ, z); such an expression has been obtained by Dougall[10] by the method of contour integration. One considers the function of ζ

$$f(\zeta) = \csc\zeta\pi \cos\zeta(\pi - \phi + \phi_0)\, K_\zeta(k\rho_0)\, I_\zeta(k\rho), \qquad (1)$$

where $0 < \rho < \rho_0$ and $0 < \phi - \phi_0 < 2\pi$; this function is analytic except for simple poles corresponding to all real integral values of ζ. Let $f(\zeta)$ be integrated around a contour in the ζ-plane consisting of a large semicircle of radius half an odd integer in the right half-plane, and the imaginary axis indented at the origin. It is easy to show that the integral around the infinite semicircle vanishes. The integral over the imaginary axis, which may be transformed as in 1.32 (11.1), is then equal, by the theorem of residues, to an infinite series of products of modified Bessel functions. Using the addition theorem for modified Bessel functions,[11]

$$K_0(kR) = \sum_{m=0}^{\infty}(2 - \delta_{0m})\, I_m(k\rho)\, K_m(k\rho_0)\cos m(\phi - \phi_0), \qquad (2)$$

where $0 < \rho < \rho_0$, $0 < \phi - \phi_0 < 2\pi$, and $R = \left[\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)\right]^{\frac{1}{2}}$, we find that

$$\frac{2}{\pi}\int_0^\infty \operatorname{ch}\nu(\pi - \phi + \phi_0)\, K_{i\nu}(k\rho)\, K_{i\nu}(k\rho_0)\, d\nu = K_0(kR), \qquad (3)$$

and by symmetry this equality evidently holds whatever be the relative magnitudes of ρ and ρ_0. From the known result[12]

$$\frac{2}{\pi}\int_0^\infty \cos k(z - z_0)\, K_0(kR)\, dk = \frac{1}{r}, \qquad (4)$$

where $r = \left[R^2 + (z - z_0)^2\right]^{\frac{1}{2}} = \left[\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + (z - z_0)^2\right]^{\frac{1}{2}}$ is the distance between the points (ρ, ϕ, z) and (ρ_0, ϕ_0, z_0), we get finally the desired expression for inverse distance, namely

---

10) See Gray, Matthews, and MacRobert, op. cit., 101-103.
11) Ibid., 74.
12) Ibid., 101.

$$\frac{1}{r} = \frac{4}{\pi^2}\int_0^\infty \cos k(z - z_0)\int_0^\infty \operatorname{ch}\nu(\pi - \phi + \phi_0)\, G_\nu(k\rho)\, G_\nu(k\rho_0)\, d\nu dk, \qquad (5)$$

where $0 < \phi - \phi_0 < 2\pi$ and we have introduced the wedge function notation G_ν for $K_{i\nu}$.

The potential problem at hand is now to be solved by writing the total potential in the wedge in the form $V_1 = V_0 + V_{11}$, and the total potential outside the wedge in the form $V_2 = V_0 + V_{12}$. Here

$$V_0 = \frac{q}{4\pi\varepsilon_2 r} = \frac{q}{\pi^3\varepsilon_2}\int_0^\infty \cos k(z - z_0)\int_0^\infty \operatorname{ch}\nu(\pi - \phi + \phi_0) G_\nu(k\rho) G_\nu(k\rho_0) d\nu dk, \qquad (6)$$

where $0 < \phi - \phi_0 < 2\pi$, is the potential which would be produced by the charge q in the absence of the wedge; the presence of the wedge introduces the additional term

$$V_{11} = \frac{q}{\pi^3\varepsilon_2}\int_0^\infty \cos k(z - z_0)\int_0^\infty \left[A(\nu, k)e^{\nu\phi} + B(\nu, k)e^{-\nu\phi}\right] G_\nu(k\rho) G_\nu(k\rho_0) d\nu dk \qquad (7)$$

for $0 < \phi < \alpha$, and the additional term

$$V_{12} = \frac{q}{\pi^3\varepsilon_2}\int_0^\infty \cos k(z - z_0)\int_0^\infty \left[C(\nu, k)e^{\nu\phi} + D(\nu, k)e^{-\nu\phi}\right] G_\nu(k\rho) G_\nu(k\rho_0) d\nu dk \qquad (8)$$

for $\alpha < \phi < 2\pi$, where the functions A, B, C, and D must be chosen so as to insure the continuity of the total potential V and the normal component $\frac{\varepsilon}{\rho}\frac{\partial V}{\partial \phi}$ of the total displacement at dielectric boundaries. (The integrals (7) and (8) are assumed to converge and, since the integrands are cylindrical harmonics of the form 2.1 (6), to satisfy Laplace's equation.) We insure that the integrals will satisfy the continuity conditions by requiring that the integrands themselves do so (assuming, of course, the legitimacy of differentiation under the sign of integration). Recalling that if $0 < \phi < \phi_0$ we must write $\phi + 2\pi$ for ϕ in (6), we find that the boundary conditions lead to the following four simultaneous equations:

At $\phi = 0$,

$$A + B = Ce^{2\pi\nu} + De^{-2\pi\nu}, \qquad (9)$$

$$\varepsilon_1\left[A - B + \operatorname{sh}\nu(\pi - \phi_0)\right] = \varepsilon_2\left[Ce^{2\pi\nu} - De^{-2\pi\nu} + \operatorname{sh}\nu(\pi - \phi_0)\right]; \qquad (10)$$

and at $\phi = \alpha$,

$$Ae^{\nu\alpha} + Be^{-\nu\alpha} = Ce^{\nu\alpha} + De^{-\nu\alpha}, \qquad (11)$$

$$\varepsilon_1\left[Ae^{\nu\alpha} - Be^{-\nu\alpha} + \operatorname{sh}\nu(\pi + \alpha - \phi_0)\right] = \varepsilon_2\left[Ce^{\nu\alpha} - De^{-\nu\alpha} + \operatorname{sh}\nu(\pi + \alpha - \phi_0)\right]. \quad (12)$$

If we solve equations (9) - (12) simultaneously and then combine terms, we get at length the following expression for the factor depending on ϕ in (8):

$$C(\nu,k)e^{\nu\varphi} + D(\nu,k)e^{-\nu\varphi}$$
$$= \frac{-\beta \operatorname{sh}\nu\alpha\left[\operatorname{sh}\nu\pi \operatorname{ch}\nu(\varphi+\varphi_0-\alpha-2\pi) + \beta \operatorname{ch}\nu(\varphi-\varphi_0)\operatorname{sh}\nu(\pi-\alpha)\right]}{\operatorname{sh}^2\nu\pi - \beta^2\operatorname{sh}^2\nu(\pi-\alpha)}, \quad (13)$$

where $\beta = (\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2)$. Hence we have a formal representation of the potential function in the region outside the wedge; the solution inside the wedge may obviously be worked out in the same way.*

As is well known,[13] in the case of steady flow of electric current in an extended conducting medium the potential function satisfies Laplace's equation and the conductivity of the medium plays exactly the same role as the capacitivity in electrostatics, so that all the mathematical technique used in electrostatics also applies here. This fact is sometimes used by geophysicists to investigate the structure below the earth's surface by observing the distribution of potential on the surface when current is passed through the soil between two or more surface electrodes. It is evident that with slight changes in notation the problem just solved will provide expressions for the potential distribution in the conducting half-space $0 \leq \phi \leq \pi$ when current enters the surface through a single point electrode, if the wedge-shaped region $0 \leq \phi < \gamma$ has uniform conductivity σ_1 and the region $\gamma < \phi \leq \pi$ has uniform conductivity σ_2; the results may be generalized to the case of several electrodes if desired.

A solution of the problem treated in this section has been given

---

*It would probably be possible to convert the integrals over ν in (7) and (8) into infinite series by the method of contour integration used to derive (3); but the results would be complicated and there seems to be no reason for attempting the transformation.

13) Smythe, op. cit., Art. 6.10.

in an entirely different form by S. O. Rice[14] in terms of a single infinite integral of the Legendre function $Q_{i\lambda-\frac{1}{2}}$ of complex order with respect to the parameter λ. Rice's development is mathematically rigorous, though his result is not well adapted to numerical calculation in the absence of tables of the function $Q_{i\lambda-\frac{1}{2}}$.

It is worth noting in conclusion that the analogous two-dimensional problem of a line charge parallel to the vertex of a dielectric wedge is solved in the second edition of Smythe's textbook[15] by the use of the circular harmonics

$$V(\rho,\varphi) = {\sin \atop \cos}(\nu \log \rho) e^{\pm\nu\varphi}. \tag{14}$$

2.13. The Equation of Conduction of Heat.

The equation of conduction of heat in a homogeneous isotropic solid may be written in the form[16]

$$\nabla^2 v = \frac{1}{K}\frac{\partial v}{\partial t}, \tag{1}$$

where v(x, y, z, t) represents the temperature, t the time, and K is a constant of the material called the diffusivity. We wish to consider briefly whether useful solutions of (1) may be found involving Bessel functions of imaginary order.

In the special case where the flow of heat has reached a steady state, the right side of (1) vanishes and the distribution of temperature satisfies Laplace's equation, so that all the methods of potential theory are available to determine it. Thus if we want to find the steady-state temperature in a general solid bounded by surfaces of the cylindrical

---

14) Rice, S. O., Phil. Mag., (7), 29, 36-46 (1940).

15) Smythe, Static and Dynamic Electricity, 2nd ed., in press. Art. 4.07.

16) Carslaw, H. S., Mathematical Theory of the Conduction of Heat, 2nd ed., 8.

coordinate system, when the surface temperature is specified on two axial planes ϕ = constant, we shall require harmonics of the form 2.1 (6) involving the wedge functions, and the solution will be mathematically identical with case (iii) of Art. 2.1.

In the more general case where the temperature varies with time, we may seek a solution of (1) which is the product of four functions each depending on a single variable; in cylindrical coordinates such a solution is readily obtained in the form[17)]

$$v = e^{-\kappa(\alpha^2+\mu^2)t} R_\nu(\mu\rho) \frac{\cos}{\sin} \nu\varphi \frac{\cos}{\sin} \alpha z, \tag{2}$$

where $R_\nu(\mu\rho)$ is a Bessel function of order ν and argument $\mu\rho$, μ, ν, and α being completely arbitrary separation constants. If these constants all be taken as real, we see that all three of the space-dependent factors on the right side of (2) are oscillatory, so that by giving special values to the separation constants a triple series can be built from products of the form (2) which vanishes for all values of t on all six faces of the general solid bounded by surfaces of the cylindrical coordinate system, and assumes for t = 0 arbitrary values in the interior of the solid. Now any problem in heat conduction (or radiation) with surface conditions independent of time can be reduced to two simpler problems, one of which is a case of steady temperature, while the other is a case of variable temperature with the surface (or the surrounding medium, in the case of radiation) held at zero temperature; and finally any conduction or radiation problem where surface conditions vary with time can be reduced by a method due to Duhamel to a simpler problem with surface conditions independent of time.[18)] We thus get from the general heat conduction problem in cylindrical coordinates no new applications of Bessel functions

---

17) Carslaw, op. cit., 123.
18) Ibid., 16-19.

of imaginary order beyond those treated in Art. 2.1.*

In spherical polar coordinates (r, θ, ϕ) a particular time dependent solution of (1) is given by[20)]

$$v = e^{-\kappa\alpha^2 t}(\alpha r)^{-\frac{1}{2}} R_{\nu+\frac{1}{2}}(\alpha r)\, \Theta_{\nu}^{-\mu}(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} \mu\phi, \tag{3}$$

where $R_{\nu+\frac{1}{2}}$ is a Bessel function and $\Theta_{\nu}^{-\mu}$ is an associated Legendre function of degree ν and order $-\mu$, the separation constants α, μ, and ν being completely arbitrary. If we write $i\nu$ for $\nu+\frac{1}{2}$ in (3) we get the solution:

$$v = e^{-\kappa\alpha^2 t}(\alpha r)^{-\frac{1}{2}} R_{i\nu}(\alpha r)\, \Theta_{-\frac{1}{2}+i\nu}^{-\mu}(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} \mu\phi, \tag{4}$$

which involves Bessel functions of imaginary order and cone functions (see Art. 2.1) if μ and ν are real. The last expression for v certainly satisfies the conduction equation (1), but it does not seem to be adapted to the solution of any problems which cannot be treated by the use of harmonics of the form (3); in any case the usefulness of (4) is limited because of the singularity of the radial factor at $r = 0$.

A differential equation analogous to the equation of conduction of heat occurs in the treatment of induced electric currents (eddy currents) in extended conductors. In the case where the inducing magnetic field is axially symmetric and varies sinusoidally with time, the vector potential of the eddy currents may be expressed in terms of modified Bessel functions of complex argument $xi^{\frac{1}{2}}$, which lead to the ber and bei functions of Lord Kelvin.[21)] It is formally possible, by choosing the separation constants properly in the differential equation describing the eddy currents, to

---

*Carslaw and Jaeger[19)] in deriving the Green's function for the conduction equation in cylindrical coordinates make use of Dougall's contour integrals of Bessel functions with respect to their order (compare the derivation of 2.12 (5)), but the functions of imaginary order do not come into the expressions for the final results.

19) Carslaw, H. S., and Jaeger, J. C., J. London Math. Soc., 15, 278 (1940).

20) Carslaw, op. cit., 144.

21) Smythe, op. cit., Arts. 11.02 - 11.04.

obtain solutions involving the complex functions $F_\nu(xi^{\pm\frac{3}{2}})$ and $G_\nu(xi^{\pm\frac{3}{2}})$; but no problems have been found whose solutions would be facilitated by the use of these functions.

2.2. An Application to Hydrodynamics. Stability of Superposed Streams of Fluids of Different Densities.

We turn now to quite a different application of Bessel functions of imaginary order, which occurs in some hydrodynamical investigations of G. I. Taylor[22)] and S. Goldstein.[23)] The problem which occasions the use of these functions may be introduced as follows:

It is well known that when the wind near the ground drops at night with the cooling of the ground, the wind at a higher level frequently remains unchanged, so that the effect of a decrease in density with height is to suppress turbulence and to enable a large velocity gradient to be maintained. This at once presents to the mathematician the problem of the stability of a fluid in which the density and velocity vary with height above the ground, regarded as a horizontal plane. It turns out that if the velocity is assumed to vary linearly with height and the density exponentially, the stability investigations involve Bessel functions, and the results are simple enough to admit physical interpretation.

Taylor's analysis proceeds in the following manner: We assume an undisturbed flow in the direction of the axis of x with a velocity $u_0(z)$ depending in a manner later to be specified on z, the height; the density of the undisturbed fluid at height z is taken as $\rho_0 e^{-\beta z}$. We now superimpose a small sinusoidal disturbance on the original flow, so that the total vector velocity $\vec{q}$ is given by

---

22) Taylor, G. I., Proc. Roy. Soc. London, (A), 132, 499-507 (1931).
23) Goldstein, S., ibid., 524-548.

$$\vec{q} = \vec{i}\left[u_0(z) + u_1(z)\varepsilon\right] + \vec{j}v_1(z)\varepsilon + \vec{k}w_1(z)\varepsilon, \tag{1}$$

where $\varepsilon = \exp i(kx - \sigma t)$.* Here k is regarded as a real number (evidently $k = 2\pi/\lambda$, where λ is the wavelength of the disturbance), and the nature of σ is to be determined from the equations of motion together with the boundary conditions. Real values of σ correspond to stable progressive waves, while complex values of σ correspond either to exponentially amplified waves and instability or to exponentially attenuated waves; the criterion for stability of the original flow against small disturbances of the form (1) is thus $\mathrm{Im}\,\sigma \leq 0$. The total density ρ and pressure p also fluctuate about their undisturbed values in the same manner as the velocity; they may accordingly be written as

$$\rho = \rho_0 e^{-\beta z} + \rho_1(z)\varepsilon, \tag{2}$$

$$p = (\rho_0 g/\beta)e^{-\beta z} + p_1(z)\varepsilon, \tag{3}$$

since the undisturbed pressure is given by $\int_z^\infty \rho_0 g e^{-\beta z} dz$.

It is assumed that the variation of the undisturbed density with altitude is due to the changing physical characteristics of the fluid, any small element of fluid being regarded as incompressible. Hence for points that move with the flow the particle derivative[24)] $D\rho/Dt = \partial\rho/\partial t + q\cdot\vec{\nabla}\rho$ vanishes; on taking account of (1) and (2) we get, to the first order of small quantities,

$$-i\sigma\rho_1(z) + iku_0(z)\rho_1(z) - \beta w_1(z)\rho_0 e^{-\beta z} = 0. \tag{4}$$

The continuity equation $\vec{\nabla}\cdot\vec{q} = 0$ gives

$$iku_1(z) + dw_1/dz = 0. \tag{5}$$

On substituting (1) - (3) into Euler's dynamical equation

$$\frac{\partial \vec{q}}{\partial t} + \vec{q}\cdot\vec{\nabla}\vec{q} = \vec{F} - \frac{\vec{\nabla} p}{\rho}, \tag{6}$$

---

*The more general assumption $\varepsilon = \exp i(kx + \ell y - \sigma t)$ would lead to no essential change in the form of our results.

24) Webster, A. G., Dynamics, 2nd ed., 496-499, develops the hydrodynamical equations used in this paragraph.

writing $-\vec{k}g$ for the body force $\vec{F}$ per unit mass, and dropping products of small quantities, we get the equations of motion to first order:

$$-ikp_1 = ik\rho(u_o - \sigma/k)u_1 + \rho w_1 du_o/dz, \tag{7}$$

$$0 = ik\rho(u_o - \sigma/k)v_1, \tag{8}$$

$$-dp_1/dz = ik\rho(u_o - \sigma/k)w_1 + g\rho_1; \tag{9}$$

and in these three equations ρ may be replaced wherever it occurs by $\rho_o e^{-\beta z}$ to the same order of approximation. If we eliminate the quantities p_1, u_1, and ρ_1 among the equations (4), (5), (7), and (9), we obtain the equation for the vertical component of velocity $w_1(z)$:

$$d^2w_1/dz^2 - \beta dw_1/dz + w_1\left[(u_o - \sigma/k)^{-1}(\beta du_o/dz - d^2u_o/dz^2) - k^2 + g\beta(u_o - \sigma/k)^{-2}\right] = 0. \tag{10}$$

We now consider the case of a uniform velocity gradient $u_o(z) = \alpha z$ and write

$$w_1(z) = f(z)\exp\left(\tfrac{1}{2}\beta z\right), \tag{11}$$

so that (10) becomes

$$d^2f/dz^2 - f\left[k^2 + \tfrac{1}{4}\beta^2 - g\beta/(\alpha z - \sigma/k)^2 - \alpha\beta/(\alpha z - \sigma/k)\right] = 0. \tag{12}$$

In order to reduce (12) to a tractable form, we assume with Taylor that the density of the fluid does not change appreciably in a distance equal to the wavelength of the disturbance; i. e., $\lambda = 2\pi/k \ll 1/\beta$, or $k \gg \beta$, so that $\frac{1}{4}\beta^2$ is negligible compared with k^2. We also assume that the wave-length is small compared with the characteristic length g/α^2 (the velocity gradient is not too high); i. e., $1/k \ll g/\alpha^2$. If $g\beta/\alpha^2$ is of order of magnitude unity and if z is comparable with a wavelength ($z \approx 1/k$), then

$$g\beta/(\alpha z - \sigma/k)^2 \approx k^2 \gg \alpha\beta/(\alpha z - \sigma/k) \approx k\beta$$

provided σ/k is not comparable with αz; while if $\alpha z - \sigma/k$ is very small the term with the squared denominator certainly dominates the other. Hence the last term in (12) may be neglected and the equation for f becomes

$$d^2f/dz^2 - \left[k^2 - g\beta/(\alpha z - \sigma/k)^2\right]f = 0. \tag{13}$$

On writing

$$kz - \sigma/\alpha, \qquad f = \zeta^{\frac{1}{2}}h(\zeta), \tag{14}$$

(13) becomes

$$\zeta^2 d^2h/d\zeta^2 + \zeta dh/d\zeta - [\zeta^2 + (\tfrac{1}{4} - g\beta/\alpha^2)]\, h = 0, \tag{15}$$

which, by comparison with 1.2 (12), is the equation for modified Bessel functions of argument ζ and order

$$\nu = (\tfrac{1}{4} - g\beta/\alpha^2)^{\frac{1}{2}}. \tag{16}$$

Clearly if $\alpha^2 > 4g\beta$, corresponding to a large velocity gradient, ν will be a real number between $-\frac{1}{2}$ and $+\frac{1}{2}$, while if $\alpha^2 < 4g\beta$, corresponding to a small velocity gradient, ν will be purely imaginary. Returning via (14) and (11) to the original variables, we see that the vertical component of velocity is given by

$$w_1(z)\mathcal{E} = (z - \sigma/\alpha k)^{\frac{1}{2}} e^{\frac{1}{2}\beta z} R_\nu(kz - \sigma/\alpha) e^{i(kx - \sigma t)}, \tag{17}$$

where $R_\nu(z)$ represents any solution of the modified Bessel equation 1.2 (12). We shall now investigate some special cases.

<u>Case of a Fluid of Variable Density Contained between Two Horizontal Planes.</u> If the moving fluid is bounded by the rigid horizontal planes $z = z_1$ and $z = z_2$, the boundary conditions are $w_1(z_1) = w_1(z_2) = 0$; since k, z, and α are real, the conditions can be satisfied only if there exist two zeros of the function $\zeta^{\frac{1}{2}}R(\zeta)$ with the same imaginary part. Suppose for the moment that we have two such roots, say $\zeta_1 = \xi_1 + ib$ and $\zeta_2 = \xi_2 + ib$; then k and σ are determined by the pair of equations

$$kz_1 - \sigma/\alpha = \xi_1 + ib, \tag{18.1}$$

$$kz_2 - \sigma/\alpha = \xi_2 + ib. \tag{18.2}$$

Hence $k = 2\pi/\lambda = (\xi_2 - \xi_1)/(z_2 - z_1)$ (19.1)

and $\sigma = \alpha(z_1\xi_2 - z_2\xi_1)/(z_2 - z_1) - i\alpha b,$ (19.2)

and the phase velocity $(\mathrm{Re}\,\sigma)/k$ is given by

$$(\mathrm{Re}\,\sigma)/k = \alpha(z_1\xi_2 - z_2\xi_1)/(\xi_2 - \xi_1). \tag{20}$$

In the case $\alpha^2 > 4g\beta$, where ν is real and $-\frac{1}{2} < \nu < \frac{1}{2}$, stable waves of all wavelengths can propagate, since the function $\zeta^{\frac{1}{2}} R_\nu(\zeta)$ vanishes at $\zeta = 0$ because of the first factor and the modified Bessel function can certainly be chosen to vanish for $\zeta = \xi_2$, where ξ_2 is any desired real number. From (20), the phase velocity of these waves is just αz_1, the velocity of the fluid at the lower boundary z_1, so that they are all moving backward with respect to the upper layers of fluid. The possibility of unstable waves in the case $\alpha^2 > 4g\beta$ cannot be decided with our present knowledge of the complex zeros of modified Bessel functions of real order, which is summarized in theorem 3 of Art. 1.2. The most we can say is that if unstable waves do exist, they correspond to values of σ for which $|\mathrm{Im}\,\sigma| < \nu\alpha < \frac{1}{2}\alpha$.

In the case $\alpha^2 < 4g\beta$, we may write for clarity $(\frac{1}{4} - g\beta/\alpha^2)^{\frac{1}{2}} = i\nu$, where $i\nu$ is purely imaginary; then the function $R_{i\nu}(\zeta)$ occurring in (17) is a linear combination of wedge functions. Stable waves of all lengths can be propagated, since both wedge functions have an infinite number of real roots with limit-points at the origin, and if $R_{i\nu}(\xi)$ is the linear combination of these functions vanishing at ξ_2, it is evidently possible by continuous variation of the coefficients in $R_{i\nu}$ to vary ξ_2 continuously and to make the difference between ξ_2 and the next smaller real root ξ_1 assume a value corresponding, by (19.1), to any desired wavelength. The velocity of this wave is then determined by (20). On the other hand, since by theorem 4 of Art. 1.2 $R_{i\nu}$ cannot have two complex roots with equal imaginary parts, <u>no unstable waves</u> can be propagated. (Taylor could not show the absence of unstable waves.)

<u>Case of a Fluid of Variable Density Bounded by a Horizontal Plane and Extending to Infinity.</u> If the fluid is bounded by the horizontal plane $z = z_1$ and extends to $+\infty$, the Bessel function in (17) must vanish

as its argument tends to infinity in the right half-plane, so that from 1.12 (2) it must be a constant multiple of the function K_ν. If $\alpha^2 > 4g\beta$ so that the order ν is real, no unstable waves can exist in the semi-infinite fluid, since it is known that, if ν is real and $-\frac{1}{2} < \nu < \frac{1}{2}$, $K_\nu(\zeta)$ has no zeros in the region $|\arg\zeta| \leq \pi$.[25] The only stable waves are those given by $\zeta = 0$. (Taylor seems to have overlooked this possibility when he states that no waves, either stable or unstable, can exist for $\alpha^2 > 4g\beta$.) From (18.1) we see that these waves can have any wavelength, but that they all move with the same velocity $\sigma/k = \alpha z_1$, which is the velocity of the fluid at the boundary plane $z = z_1$.

In the case $\alpha^2 < 4g\beta$ we have to deal with the function $K_{i\nu}(\zeta)$ of imaginary order, which by theorem 1 of 1.2 has an infinite number of real zeros in ζ between the origin and the point $\zeta = +\nu$.* Corresponding to any particular real root ξ_1 of $K_{i\nu}(\zeta) = 0$, stable waves of all wavelengths can propagate, the dependence of velocity on wavelength being given by σ/k from (18.1). The possibility of unstable waves must remain open, since theorem 5 of 1.2 does not preclude the existence of complex zeros of $K_{i\nu}(\zeta)$ in the right half-plane.

The stability problem treated by Goldstein in the second paper[23] cited above is somewhat different from Taylor's; it may be stated as follows: We consider an infinite expanse of perfect fluid with a layer of constant velocity and density and infinite depth on top, a layer of

---

25) Watson, G. N., Theory of Bessel Functions, 2nd ed. The region $|\arg\zeta| < \pi$ is proved zero-free in Art. 15.7, while the absence of zeros on the negative real axis follows from the formula [Art. 3.71, eq. (18)]

$$K_\nu(xe^{\pm\pi i}) = e^{\mp\nu\pi i}K_\nu(x) \mp \pi i I_\nu(x).$$

*Taylor asserts that the Hankel function $H^{(1)}_{i\nu}(i\xi)$ $[= (-2i/\pi)\exp(\frac{1}{2}\nu\pi)K_{i\nu}(\xi)]$ is purely imaginary whenever ξ is real, and that it has an infinite number of real positive and negative zeros in ξ in the neighborhood of the origin. His argument is actually valid only for real positive values of ξ; since $(-1)^{i\nu}$ is not the complex conjugate of $(-1)^{-i\nu}$, the functions $H^{(1)}_{i\nu}(i\xi)$ and $K_{i\nu}(\xi)$ are complex valued when ξ is real and negative.

different constant velocity and slightly larger constant density and infinite depth at the bottom, and a finite transition layer in between, where the velocity varies linearly and the density varies exponentially from one boundary to the other. We wish to investigate the behavior of a small sinusoidal disturbance progressing in the direction of the steady flow.

We assume the following steady-state distribution of velocity, density, and pressure gradient, the notation being chosen to agree as closely as possible with the first part of this article:

For $z < 0$, $\quad u_0 = 0, \quad \rho = \rho_0, \quad dp_0/dz = -\rho_0 g;$ (21.1)

for $0 < z < h$, $\quad u_0 = Uz/h = \alpha z, \quad \rho = \rho_0 e^{-\beta z}, \quad dp_0/dz = -\rho_0 g e^{-\beta z};$ (21.2)

for $z > h$, $\quad u_0 = U = \alpha h, \quad \rho = \rho_0 e^{-\beta h}, \quad dp_0/dz = -\rho_0 g e^{-\beta h}.$ (21.3)

On these steady-state quantities we superimpose fluctuations of the form given by eqs. (1) - (3) and obtain as before the hydrodynamical equations (4), (5), (7), (8), and (9), noting that in the regions of constant density $\rho_1(z) = 0$ so that (4) is nugatory.

The expression for the vertical component of velocity in the transition layer is derived in the form (17) by exactly the same arguments as before, but with the added simplification that if we assume the change in density to be only a small fraction of the mean density,* we may consider the factor $\exp(\frac{1}{2}\beta z)$ to be essentially equal to unity, so that

$$w_1(z)\mathcal{E} = (z - \sigma/\alpha k)^{\frac{1}{2}} R_\nu(kz - \sigma/\alpha) e^{i(kx - \sigma t)}, \tag{22}$$

where ν is defined by (16) and R_ν is a modified Bessel function. The equation for $w_1(z)$ in the top and bottom layers is easily derived from (5), (7), and (9); on setting u_0 = constant, ρ = constant, and $\rho_1 = 0$, we get directly

---

*Actually Goldstein does not make explicit use of the approximations introduced by Taylor to simplify eq. (12) above, but bases all the approximations necessary to obtain (22) on the single assumption that the total change in density is small compared with the mean density.

$$d^2w_1/dz^2 = k^2w_1, \tag{23}$$

so that for $z < 0$ or $z > h$, $w_1(z)$ is proportional to $\exp(\pm kz)$.

The boundary conditions in our problem are that the normal component of velocity and the pressure must be continuous across surfaces where the velocity gradient is discontinuous. Let $z = z_o$ be the equation of such a surface in the undisturbed flow, and let $z = z_o + \eta$ be the equation of this surface in the disturbed motion. Then to first order $w_1(z)$ must be continuous at $z = z_o$. Also, to first order, the value of w_1 at $z = z_o$ is connected with η by the equation

$$w_1 = \partial\eta/\partial t + u_o\,\partial\eta/\partial x = i(ku_o - \sigma)\eta. \tag{24}$$

The pressure must be continuous at $z = z_o + \eta$, so that to first order $p_o + p_1 + \eta\, dp_o/dz$ must be continuous at z_o; i. e., $p_1 - \eta\rho_o g$ must be continuous at z_o. Substituting for η from (24) and for p_1 from (7) and (5), and dropping terms like ρ_o and $w_1(z)$ which are already assumed to be continuous at $z = z_o$, we find that the expression

$$kw_1\, du_o/dz - (ku_o - \sigma)\, dw_1/dz \tag{25}$$

must be continuous at $z = z_o$.

Solutions of the equations of motion which vanish at $z = \pm\infty$ are, from (22) and (23):

For $z < 0$, $\quad w_1(z) = Ae^{kz}$, (26.1)

for $0 < z < h$, $\quad w_1(z) = \zeta^{\frac{1}{2}}[BI_\nu(\zeta) + CI_{-\nu}(\zeta)]$, (26.2)

and for $z > h$, $\quad w_1(z) = De^{-kz}$, (26.3)

where $\zeta = kz - \sigma/\alpha$, and if $\nu = 0$ the term $I_{-\nu}(\zeta)$ in (26.2) is to be replaced by $K_o(\zeta)$. If we set

$$\zeta_1 = -\sigma/\alpha, \qquad \zeta_2 = kh - \sigma/\alpha, \tag{27}$$

the continuity of $w_1(z)$ at $z = 0$ and at $z = h$ leads to the conditions

$$A = \zeta_1^{\frac{1}{2}}[BI_\nu(\zeta_1) + CI_{-\nu}(\zeta_1)], \tag{28.1}$$

$$D = \zeta_2^{\frac{1}{2}}[BI_\nu(\zeta_2) + CI_{-\nu}(\zeta_2)]. \tag{28.2}$$

The continuity of the expression (25) leads, with use of (28) and some rearrangement, to the pair of conditions:

$$B\left[(1+\tfrac{1}{2}\zeta_1^{-1})I_\nu(\zeta_1) - I_\nu'(\zeta_1)\right] + C\left[(1+\tfrac{1}{2}\zeta_1^{-1})I_{-\nu}(\zeta_1) - I_{-\nu}'(\zeta_1)\right] = 0, \quad (29.1)$$

$$B\left[(1-\tfrac{1}{2}\zeta_2^{-1})I_\nu(\zeta_2) + I_\nu'(\zeta_2)\right] + C\left[(1-\tfrac{1}{2}\zeta_2^{-1})I_{-\nu}(\zeta_2) + I_{-\nu}'(\zeta_2)\right] = 0, \quad (29.2)$$

which have a non-zero solution in B and C provided that

$$\left[(1+\tfrac{1}{2}\zeta_1^{-1})I_\nu(\zeta_1) - I_\nu'(\zeta_1)\right]\left[(1-\tfrac{1}{2}\zeta_2^{-1})I_{-\nu}(\zeta_2) + I_{-\nu}'(\zeta_2)\right]$$
$$- \left[(1+\tfrac{1}{2}\zeta_1^{-1})I_{-\nu}(\zeta_1) - I_{-\nu}'(\zeta_1)\right]\left[(1-\tfrac{1}{2}\zeta_2^{-1})I_\nu(\zeta_2) + I_\nu'(\zeta_2)\right] = 0. \quad (30)$$

The real roots of equation (30), regarded in virtue of (27) as an equation in σ when k is a given real number, correspond to stable progressive waves, while the complex roots with $\operatorname{Im}\sigma > 0$ correspond to amplified unstable waves.

A rigorous theoretical treatment of the roots of the period equation (30) apparently being infeasible, Goldstein attacks the problem indirectly. By the use of asymptotic formulas for the modified Bessel functions when the order and the argument are simultaneously large, he obtains the limiting form of (30) as $\alpha \to 0$ ($|\zeta_1| \to \infty$, $|\zeta_2| \to \infty$, $\nu \to i\infty$), which corresponds to the case of no steady motion. The system is then completely stable, and there are an infinite number of principal periods of oscillation, which are shown to vary continuously and to remain real and distinct as α^2 increases from zero to just less than $4g\beta$, hk being supposed small for this part of the work. Then when α^2 is just less than $4g\beta$, the periods are shown to vary continuously and to remain real and distinct when the wavelength is varied over all possible values. It is deduced that the motion is stable for $\alpha^2 < 4g\beta$. When $\alpha^2 = 4g\beta$ there is one real principal period if kh is less than about 0.4 and none otherwise; and when $\alpha^2 > 4g\beta$ there is, for kh small, one real principal period and an infinite number of imaginary ones, which correspond to unstable modes of oscillation. It is deduced that the motion is unstable for $\alpha^2 > 4g\beta$, and it appears that this is true for all wavelengths.

Since the rather lengthy mathematical calculations involved in carrying out the argument which has just been sketched yield no outstanding new results for the theory of Bessel functions of imaginary order, the reader is referred to Goldstein's original paper for details of the work.

2.3. Propagation of Love Waves over the Surface of an Elastically Inhomogeneous Medium.

Bessel functions of imaginary order and imaginary argument occur in the solution of the problem of propagation of transverse elastic waves over the surface of a semi-infinite body whose modulus of rigidity increases as a quadratic function of the depth. This problem is of some practical interest in seismology for the following reasons:

When an earthquake disturbance is transmitted to a great distance from its point of origin, the main shock reaches the distant stations at times corresponding to the passage of waves over the surface of the earth with nearly constant velocity; and the oscillations are largely in a horizontal plane and transverse to the direction of propagation of the shock. Observations indicate the existence of dispersion, i. e., some variation of velocity with wavelength, in these surface waves.

Such transverse surface waves oscillating in a horizontal plane are called Love waves in honor of A. E. H. Love,[26] who showed that waves of the type described may be transmitted if we have a homogeneous surface layer of rigidity μ, density ρ, and finite thickness overlying a semi-infinite homogeneous solid of different rigidity μ' and density ρ', such

---

26) Love, A. E. H., Some Problems of Geodynamics, 160-165.

that $(\mu'/\rho')^{\frac{1}{2}} > (\mu/\rho)^{\frac{1}{2}}$. It has been shown by E. Meissner[27] and others that Love waves may propagate over the surface of an elastic solid whose modulus of rigidity and/or density vary continuously with the depth. It is known from seismological data[28] that the velocities of both the dilatational and the distortional waves through the body of the earth increase with depth according to a law which is nearly linear for the first 1200 km. This fact is not sufficient to determine completely the variation of the density or of the elastic moduli in the interior of the earth; but in order to give some sort of theoretical treatment of seismic waves we may make mathematically simple assumptions which are not too widely at variance with our present incomplete knowledge.

We first recall the general dynamical equations for an isotropic elastic solid.[29] Let $\vec{s} = \vec{i}u + \vec{j}v + \vec{k}w$ be the vector displacement of any point in the body from its equilibrium position. The strain tensor is then

$$\underset{\sim}{\Phi} = \begin{pmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{pmatrix}, \qquad (1)$$

where $\varepsilon_x = \partial u/\partial x$, $\gamma_{xy} = \gamma_{yx} = \frac{1}{2}(\partial u/\partial y + \partial v/\partial x)$, etc. If the stress tensor is

$$\underset{\sim}{T} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}, \qquad (2)$$

---

27) Meissner, E., Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, 66, 181-195 (1921).

28) Ibid., 182.

29) Page, L., Introduction to Theoretical Physics, 2nd ed., chap. III. Observe changes in notation.

where τ_{xy} is the y-component of stress across the plane x = constant, and the other components have similar significance, then the total force per unit volume, including the body force $\vec{G}$, is

$$\vec{F} = \vec{G} + \vec{\nabla}\cdot\vec{T}. \tag{3}$$

The relation between stress and strain for an isotropic elastic medium is given by the tensor equation[30)]

$$\underset{\sim}{T} = \left[\left(K - \tfrac{2}{3}\mu\right)\vec{\nabla}\cdot\vec{s}\right]\underset{\sim}{I} + 2\mu\underset{\sim}{\Phi}, \tag{4}$$

where K is the bulk modulus, μ the modulus of rigidity, and $\underset{\sim}{I}$ a unit tensor. Substituting for $\vec{F}$ and $\underset{\sim}{T}$ in Newton's equation $\vec{F} = \rho\partial^2\vec{s}/\partial t^2$, we have the general equation of motion for an isotropic elastic medium:

$$\rho\frac{\partial^2\vec{s}}{\partial t^2} = \vec{G} + \vec{\nabla}\cdot\left\{\left[\left(K - \tfrac{2}{3}\mu\right)\vec{\nabla}\cdot\vec{s}\right]\underset{\sim}{I}\right\} + \vec{\nabla}\cdot\left(2\mu\underset{\sim}{\Phi}\right). \tag{5}$$

We now consider the following problem: A semi-infinite elastic solid is given by $z \geq 0$, its density $\rho(z)$ and rigidity $\mu(z)$ being functions of the depth z only. A distortional wave propagating in the positive x-direction and vibrating in the y-direction is given by

$$\vec{s} = \vec{j}v = \vec{j}Z(z)e^{i(kx-\sigma t)}. \tag{6}$$

For such a wave the only non-vanishing components of the strain tensor are

$$\gamma_{xy} = \gamma_{yx} = \frac{1}{2}\frac{\partial v}{\partial x}; \quad \gamma_{yz} = \gamma_{zy} = \frac{1}{2}\frac{\partial v}{\partial z}. \tag{7}$$

If we neglect the body force due to gravity and note that the dilatation $\vec{\nabla}\cdot\vec{s} = 0$, the equation of motion (5) becomes:

$$\rho(z)\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(2\mu\gamma_{xy}\right) + \frac{\partial}{\partial z}\left(2\mu\gamma_{zy}\right) = \mu(z)\frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial z}\left[\mu(z)\frac{\partial v}{\partial z}\right]. \tag{8}$$

If v has the form (6), substitution into (8) leads to

$$\frac{d}{dz}\left[\mu(z)\frac{dZ}{dz}\right] + \left[\sigma^2\rho(z) - k^2\mu(z)\right]Z(z) = 0. \tag{9}$$

One boundary condition is that the stress must vanish at the free surface $z = 0$; from (4), (6), and (7), since $\vec{\nabla}\cdot\vec{s} = 0$, we see that this implies

30) Page, op. cit., 162, eq. (47-16).

that at z = 0

$$\tau_{zy} = 2\mu\,\gamma_{zy} = \mu\frac{\partial v}{\partial z} = 0, \quad \text{or} \quad Z'(0) = 0. \tag{10.1}$$

The other boundary condition is that the wave must be essentially confined to the surface of the medium; i. e.,

$$\lim_{z \to +\infty} Z(z) = 0. \tag{10.2}$$

We now further assume that the modulus of rigidity and the density are given by[31]

$$\mu = \mu_0(1 + z/\ell)^2; \quad \rho = \rho_0 = \text{constant.} \tag{11}$$

Then the local velocity c of distortional waves is a linear function of the depth, namely

$$c = \sqrt{\mu/\rho} = c_0(1 + z/\ell), \text{ where } c_0 = \sqrt{\mu_0/\rho_0}. \tag{12}$$

The fact that μ is infinite at an infinite depth makes very little difference in the results, since the waves with which we shall be concerned are of relatively short wavelength and are confined largely to the surface of the medium, so that its elastic properties at great depths do not come into account.[32]

If we substitute μ and ρ from (11) into (9) and introduce the dimensionless variable $\zeta \;\; 1 = z/\ell$, we get

$$\zeta^2 d^2Z/d\zeta^2 + 2\zeta\, dZ/d\zeta + \ell^2[\sigma^2/c_0^2 - k^2\zeta^2]Z = 0, \tag{13}$$

the boundary conditions (10) becoming

$$dZ/d\zeta = 0 \text{ at } \zeta = 1; \quad \lim_{\zeta \to +\infty} Z(\zeta) = 0. \tag{14}$$

Introducing $V = \sigma/k$ for the phase velocity of the waves represented by (6) and $\lambda = 2\pi/k$ for the wavelength, and making the substitution

$$Z(\zeta) = \zeta^{-\frac{1}{2}} f(\zeta), \tag{15}$$

we find that (13) becomes

---

31) Sakuraba, S., Geophysical Mag., Tokyo, 9, 211-214 (1935), has given a very brief treatment of this case. I am indebted to Prof. Bateman for calling Sakuraba's paper to my attention.

32) Cf. Meissner, op. cit., 195.

$$\zeta^2 d^2f/d\zeta^2 + \zeta\, df/d\zeta - \left[(2\pi\ell/\lambda)^2\zeta^2 + \tfrac{1}{4} - (2\pi\ell V/\lambda c_0)^2\right] f = 0, \qquad (16)$$

which is just the modified Bessel equation 1.2 (12) for functions of argument $2\pi\ell\zeta/\lambda$ and order ν, where

$$\nu^2 = \tfrac{1}{4} - (2\pi\ell V/\lambda c_0)^2. \qquad (17)$$

Before proceeding further with the analysis, we shall try to get an idea of the order of magnitude of the numbers involved in our work. From representative observational data given by Gutenberg,[33)] we see that the velocity of distortional waves through the interior of the earth increases uniformly from the surface value of 4.4 km/sec to a value 50% greater at a depth of 1200 km. Hence we have in (12) the approximate values $c_0 = 4.4$ km/sec, $\ell = 2400$ km. Transverse surface waves with a representative period T = 20 sec all have observed velocities near V = 3.3 km/sec,[34)] corresponding to the wavelength $\lambda = VT = 66$ km. For such waves, $2\pi\ell/\lambda = 230$ and $2\pi\ell V/\lambda c_0 = 170$; hence we see from (17) that ν is purely imaginary and very nearly equal to $2\pi\ell Vi/\lambda c_0$. Since we are interested in values of ζ slightly greater than unity and in values of V/c_0 slightly less than unity, we shall be dealing with Bessel functions whose argument is roughly equal to 250, the magnitude of the ratio order/argument being somewhat less than unity.

The modified Bessel function which vanishes (exponentially) for large positive values of the argument, thus satisfying the second boundary condition (14), is known from 1.12 (2) to be the function $K_\nu(2\pi\ell\zeta/\lambda)$; hence from (15) we have, on writing $i\nu$ instead of ν for the order since we shall henceforth be concerned only with functions of purely imaginary order,

$$Z(\zeta) = A\zeta^{-\frac{1}{2}}K_{i\nu}(2\pi\ell\zeta/\lambda) = A\zeta^{-\frac{1}{2}}G_\nu(2\pi\ell\zeta/\lambda). \qquad (18)$$

---

33) Gutenberg, B., <u>Der Aufbau der Erde</u>, 31, table 5.
34) <u>Ibid.</u>, 109, table 49a.

The first boundary condition (14) reduces to

$$(2\pi\ell/\lambda)G_\nu'(2\pi\ell/\lambda) - \tfrac{1}{2}G_\nu(2\pi\ell/\lambda) = 0. \qquad (19)$$

Eqs. (13) and (14) represent a standard two-point Sturm-Liouville boundary value problem, so we know from theorem 5 of Art. 1.0 or directly from the properties of the wedge function G_ν that for any fixed value of $2\pi\ell/\lambda$, eq. (19) will determine a series of increasing real positive values of $\nu (\approx 2\pi\ell V/\lambda c_0)$ corresponding to waves of a given wavelength traveling with a discrete series of velocities. The slowest wave of a given wavelength will have no nodal planes below the surface of the medium; the faster waves will have 1, 2, 3, ... nodal planes, corresponding to the higher values of ν. It turns out that only the slowest wave corresponding to a given wavelength, i. e., the wave without nodal planes, is of seismological interest.[35)]

From eq. (19) we may in principle obtain the dispersion curve of phase velocity $V \approx \nu\lambda c_0/2\pi\ell$ against wavelength λ. The group velocity (velocity of propagation of energy) is then obtainable as $V - \lambda dV/d\lambda$. Since the values of order and argument under consideration are far outside the range covered by our table of wedge functions, it is necessary to work from the asymptotic representations of Bessel functions whose order and argument are simultaneously large and of comparable magnitude (see reference 21 of chapter I), the results in the case at hand being conveniently expressible in terms of the ratio $|\text{order}/\text{argument}| \approx V/c_0\zeta$. This problem has been treated numerically by H. Jeffreys[36)] in an attempt

---

35) Meissner, op. cit., 186. We may note that Sakuraba (reference 31) is guilty of the incorrect statement that "the Love wave exists, which is characterized by an infinite large number of nodal planes." The waves of finite frequency and finite wavelength certainly have only a finite number of nodal planes. He also remarks that the solution involving Bessel functions of purely imaginary order is "only of theoretical interest," whereas we have seen above that for all physically occurring values of the quantities involved, the order is purely imaginary.

36) Jeffreys, H., Monthly Notices of the Royal Astronomical Society, Geophysical Supplements, 2, 101-111 (1928-31).

to get quantitative agreement with observational data. Jeffreys considers the slightly more general problem of a homogeneous surface layer of finite thickness overlying a deep layer wherein the velocity of distortional waves increases linearly with the depth, and plots phase and group velocities vs. wavelength of the Love waves for various values of the parameters involved. He obtains asymptotic expressions for the necessary modified Bessel functions of imaginary order directly from the differential equation. For the various curves obtained reference may be had to Jeffreys's paper. His analytical expressions might easily be derived in the standard Bessel function notation by the methods of this thesis, though it is unlikely that any significant extension of the results would be suggested by so doing, particularly in view of the relatively meagre seismological data at present available for comparison with any detailed theory.

Note added June 23, 1947: The flow of electric current between coaxial cylindrical electrodes, taking account of both convection and diffusion, has been investigated by F. Borgnis* using Bessel functions of imaginary order and imaginary argument, in a paper of which the author was unaware when the preceding chapter was written. Subsequently F. Emde** has discussed in some detail asymptotic representations of Bessel functions of large purely imaginary order.

*Borgnis, F., Ann. d. Phys. (5), 31, 745-754 (1938).
**Emde, F., Z. f. Angew. Math. u. Mech., 19, 101-118 (1939).

CHAPTER III

Physical Applications of Bessel Functions of Imaginary Order and Real Argument

3.1. Solutions of the Wave Equation Involving Bessel Functions of Imaginary Order.

The most important practical use of Bessel functions of purely imaginary order and real argument is for the construction of solutions of the wave equation,

$$v^2 \nabla^2 \Omega = \partial^2 \Omega / \partial t^2, \tag{1}$$

in cylindrical coordinates. In the present article we shall write out some useful scalar solutions of (1) which may be applied to the acoustic and electromagnetic problems of the next two sections.

Restricting ourselves from the start to functions which are harmonic in time, we assume that a solution of (1) in cylindrical coordinates may be written in the form

$$\Omega(\rho, \phi, z, t) = R(\rho)\Phi(\phi)Z(z)e^{-i\omega t}, \tag{2}$$

where ω is real. We find that Ω will be a solution of the wave equation provided that

$$d^2\Phi/d\phi^2 = -\nu^2\Phi, \tag{3}$$

$$d^2Z/dz^2 = -k_z^2 Z, \tag{4}$$

$$\rho^2 d^2R/d\rho^2 + \rho dR/d\rho + [\rho^2(\omega^2/v^2 - k_z^2) - \nu^2]R = 0, \tag{5}$$

where ν and k_z are arbitrary separation constants. If we introduce the notation

$$k^2 = \omega^2/v^2, \; k_c^2 = k^2 - k_z^2, \tag{6}$$

and take ν^2 and k_z^2 to be real, we see that Φ will consist of trigonometric or exponential functions depending on the sign of ν^2 and that Z will also be trigonometric or exponential in form depending on the sign of k_z^2, while R will be a Bessel function of (real or imaginary) argument $k_c\rho$ and (real or imaginary) order ν.*

The boundary conditions usually imposed upon Ω are that $\Omega = 0$ or that $\partial\Omega/\partial n = 0$ on two pairs of level surfaces of the cylindrical coordinate system; we therefore choose the separation constants so that two of the three space factors on the right side of (2) are oscillatory functions of their respective arguments over the desired ranges. In the applications of the next two articles the boundary conditions will be that Ω or its normal derivative must vanish on a pair of planes z = constant and on a pair of cylinders ρ = constant. Hence we must choose k_z real to make Z a sinusoidal function of z, and we must choose ν purely imaginary to make R oscillatory, since the Bessel functions of real order are not oscillatory if the argument $k_c\rho$ happens to be imaginary, that is, if $\omega^2/v^2 - k_z^2 = k_c^2 < 0$. On writing $i\nu$ for ν and denoting by $R_{i\nu}(k_c\rho)$ any Bessel function of order $i\nu$ and argument $k_c\rho$, we have for a typical solution of (1),

$$\Omega = R_{i\nu}(k_c\rho)\left[Ce^{\nu\phi} + De^{-\nu\phi}\right]\left[E \sin k_z z + F \cos k_z z\right]e^{-i\omega t}. \qquad (7)$$

To fix our ideas, let us consider the case where the boundary con-

---

*If we take $k_z^2 = 0$ and $Z(z)$ = constant, we get solutions of the two-dimensional wave equation,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Omega}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Omega}{\partial\phi^2} = \frac{1}{v^2}\frac{\partial^2\Omega}{\partial t^2},$$

which may involve Bessel functions of imaginary order. Bocher[1)] has noted the application of these functions to the problem of the transverse vibrations of a thin uniform membrane bounded by two concentric circular arcs and two radii of these circles; the functions of imaginary order and real argument occur when an arbitrary harmonic displacement of the membrane is specified along the bounding radii ϕ = constant.

1) Bocher, M., Annals of Mathematics, 6, 155-160 (1892).

ditions are $\partial\Omega/\partial z = 0$ at $z = 0$ and at $z = b$, and $\partial\Omega/\partial\rho = 0$ at $\rho = \rho_1$ and at $\rho = \rho_2$. Then the admissible solutions of (1) are of the form

$$\Omega_{mn} = R_{i\nu_{nm}}(k_{cn}\rho)\left[C_{mn}e^{\nu_{nm}\phi} + D_{mn}e^{-\nu_{nm}\phi}\right]\cos(n\pi z/b)e^{-i\omega t}, \qquad (8)$$

where we have taken $k_z = n\pi/b$, n an integer, and

$$k_{cn}^2 = \omega^2/v^2 - (n\pi/b)^2. \qquad (9)$$

$R_{i\nu}(k_{cn}\rho)$ is the particular Bessel function of order $i\nu$ and argument $k_{cn}\rho$ which satisfies the initial condition

$$R_{i\nu}'(k_{cn}\rho_1) = 0, \qquad (10.1)$$

and ν_{nm} is the mth root* in ν of the equation

$$R_{i\nu}'(k_{cn}\rho_2) = 0. \qquad (10.2)$$

If k_{cn} is real we know from Art. 1.7 that there will be in general a finite number of ordinary Bessel functions of real order satisfying the boundary conditions (10.1) and (10.2), in addition to an infinite number of functions of imaginary order, while if k_{cn} is imaginary we know from Art. 1.31 that the boundary conditions will determine merely an infinite number of functions of imaginary order. The same conclusions follow, of course, in case the boundary conditions are that $\Omega = 0$ at ρ_1 and ρ_2.

Solutions of the wave equation in rectangular coordinates (x, y, z) may also easily be obtained as products of harmonics. Since the results are well known, we shall merely note here for future reference the form

$$\Omega = [A \sin k_x x + B \cos k_x x][C \sin k_y y + D \cos k_y y][E \sin k_z z + F \cos k_z z]e^{-i\omega t}. \qquad (11)$$

This expression, as well as the equivalent form in terms of imaginary exponentials, obviously satisfies (1) provided that

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2/v^2. \qquad (12)$$

---

*The roots are ordered so that $\nu_{n1}^2 < \nu_{n2}^2 < \nu_{n3}^2 < \ldots$; evidently we need take only one root corresponding to each different value of ν^2.

3.11. Propagation of Sound Waves around a Circular Bend in a Rectangular Pipe.

We are now ready to consider the following problem: Given two similar semi-infinite straight pipes of rectangular cross section, whose upper and lower surfaces are respectively coplanar and whose axes intersect at a specified angle. The ends of the pipes are connected by a circular elbow of the same rectangular cross section, whose lateral surfaces are cylindrical. An infinite harmonic wave train of given frequency and amplitude is sent through one pipe and impinges upon the bent section. It is desired to calculate the form and amplitude of the wave train which is transmitted into the second pipe, and also of the reflected wave train. The practical interest of this problem lies in the calculation of the transmission of high-frequency electromagnetic waves through conducting wave guides; but since the electrical problem is complicated by the vectorial nature of the electromagnetic field, it seems worth while to discuss first the same problem as applied to sound waves, which may be handled in terms of scalar quantities.

In treating the irrotational motion of a compressible non-viscous fluid it is convenient to introduce the scalar velocity potential Ω, whose gradient is the velocity $\vec{q}$. For small oscillations Ω satisfies the wave equation

$$v^2 \nabla^2 \Omega = \partial^2 \Omega / \partial t^2, \qquad (1)$$

where $v^2 = dp/d\rho$ is the square of the velocity of sound in the given fluid.[2] The boundary condition on Ω at a rigid boundary is that the normal component of the velocity shall vanish, i. e.,

$$\partial \Omega / \partial n = 0. \qquad (2)$$

---

2) Rayleigh, Theory of Sound, 2nd ed., vol. II, Art. 244.

At the interface between two media of different densities δ_1 and δ_2 we must have the normal component of velocity continuous, so that

$$\partial\Omega_1/\partial n = \partial\Omega_2/\partial n. \tag{3.1}$$

The requirement that the pressure must be continuous across the boundary implies that[3]

$$\delta_2\, \partial\Omega_1/\partial t = \delta_2\, \partial\Omega_2/\partial t.$$

If the two media are of equal density, the last condition may be satisfied by taking

$$\Omega_1 = \Omega_2 \tag{3.2}$$

at the boundary.

We now consider the propagation of an infinite wave train of constant frequency in the positive x-direction through a rectangular pipe bounded by the planes $y = 0$, $y = a$, and $z = 0$, $z = b$.[4] From 3.1 (11) and (12) we see that the most general wave train satisfying the boundary condition (2) is given by

$$\Omega = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn} \cos(m\pi y/a)\cos(n\pi z/b)\exp i(h_{mn}x - \omega t), \tag{4}$$

where

$$h_{mn}{}^2 = \omega^2/v^2 - (m\pi/a)^2 - (n\pi/b)^2. \tag{5}$$

The coefficients A_{mn} in (4) may evidently be determined by Fourier's method so as to represent arbitrarily prescribed values of $\partial\Omega/\partial x$ over any desired section $x = x_0$ of the pipe.

It will be noted that the individual terms of (4), such as

$$\Omega_{mn} = A_{mn}\cos(m\pi y/a)\cos(n\pi z/b)\exp i(h_{mn}x - \omega t), \tag{6}$$

correspond to wave types in which the velocity components $\partial\Omega/\partial y$ and $\partial\Omega/\partial z$ perpendicular to the axis of the pipe have m-1 and n-1 nodes respectively between the bounding planes. Furthermore we see from (5) that for any

---

3) Rayleigh, *op. cit.*, 79.
4) *Ibid.*, Art. 268.

fixed value of the frequency and sufficiently large values of m and n, h_{mn}^2 is negative, so that $h_{mn} = i\gamma_{mn}$, say, and the factor depending on x in (4) becomes a real negative exponential $e^{-\gamma_{mn}x}$. Thus at a given frequency only a finite number of the lower modes can be propagated along the pipe without attenuation, the higher modes becoming rapidly insensible as we leave the neighborhood of the source. The "cut-off" of the higher modes is a phenomenon well known to workers with ultra-high-frequency electromagnetic waves. Of course at sufficiently high frequencies any given mode can be propagated through the pipe; the cut-off frequency, by (5), corresponds to

$$\omega_{mn}^2 = v^2[(m\pi/a)^2 + (n\pi/b)^2], \qquad (7)$$

and thus depends on the dimensions a and b of the pipe as well as on the integers m and n. We note that the mode for which m = n = 0, which is a purely longitudinal plane wave with all particles vibrating in the direction of propagation, is passed by the pipe without attenuation at <u>all</u> frequencies.

In the problem at hand, we shall assume for negative values of x an infinite train of plane waves traveling in the positive x-direction in a rectangular pipe bounded by y = 0, y = a, z = 0, and z = b, and given by the velocity potential

$$\Omega_0 = Ae^{ikx}, \; x < 0, \qquad (8)$$

where $k = \omega/v$, and it is understood throughout the rest of this section that all potentials vary with time according to the factor $e^{-i\omega t}$. Incident plane waves of the form (8) may be obtained either by choosing the dimensions of the pipe so that the given frequency is below cut-off (see (7)) for all modes except the one with m = n = 0, or by arranging a source which does not excite the higher modes; the case in which the incident wave train is a mixture of several modes merely leads to greater complication in the form of the solution.

At $x = 0$ the incident wave train enters the circular elbow bounded by $\rho = \rho_1$, $\rho = \rho_2 = \rho_1 + a$, $z = 0$, and $z = b$, and extending from $\phi = +\alpha$ to $\phi = -\alpha$. Observing that no modes with z-components of velocity will be excited in the bent pipe because no such components are present in the incident wave, we may set $n = 0$ in 3.1 (8) and assume for the steady-state velocity potential in the elbow

$$\Omega_2 = \sum_{m=0}^{\infty} R_{i\nu_m}(k\rho)\left[B_m e^{\nu_m \phi} + C_m e^{-\nu_m \phi}\right], \quad +\alpha > \phi > -\alpha, \tag{9}$$

where $k(= \omega/v)$ is always real, so that the Bessel function satisfying 3.1 (10.1) may be written in terms of the functions U_ν and V_ν of Art. 1.5 as

$$R_{i\nu}(k\rho) = V_\nu'(k\rho_1)U_\nu(k\rho) - U_\nu'(k\rho_1)V_\nu(k\rho), \tag{10}$$

and ν_m is the (m+1)st root of

$$R_{i\nu}'(k\rho_2) = 0. \tag{11}$$

As pointed out in 3.1, there will in general be, in addition to the infinite number of functions of imaginary order, a finite number of functions of real order which satisfy the boundary conditions; these latter may conveniently be expressed in a form similar to (10) by any fundamental pair of ordinary Bessel functions of real order.

On the other side of the elbow we shall assume for the wave train which is transmitted into the second pipe the velocity potential

$$\Omega_3 = \sum_{m=0}^{\infty} D_m \cos(m\pi y/a)e^{ih_m x}, \quad x > 0, \tag{12}$$

and for the waves reflected back into the first pipe,

$$\Omega_1 = \sum_{m=0}^{\infty} A_m \cos(m\pi y/a)e^{-ih_m x}, \quad x < 0, \tag{13}$$

where

$$h_m^2 = \omega^2/v^2 - (m\pi/a)^2, \tag{14}$$

and in case h_m ($= i\gamma_m$, say) is imaginary the sign is so chosen that the corresponding waves in the first pipe vanish as $x \to -\infty$ and in the second pipe as $x \to +\infty$. Depending on the dimensions of the pipe and the frequency, only a finite number of the lower modes (possibly only the mode for which $m = n = 0$) are propagated, with determinate phases and amplitudes, to any great distance from the bend; but in order to satisfy the boundary conditions at the bend it is necessary to take into account also the modes which are attenuated within a short distance.

The boundary conditions (3.2) and (3.1) are to be applied to match $\Omega_0 + \Omega_1$ at $x = 0$ with Ω_2 at $\phi = +\alpha$. From (3.2) we get

$$A + \sum_{m=0}^{\infty} A_m \cos(m\pi y/a) = \sum_{m=0}^{\infty} R_{i\nu_m}(k\rho)\left[B_m e^{\nu_m \alpha} + C_m e^{-\nu_m \alpha}\right], \tag{15}$$

and from (3.1) $\partial\Omega_0/\partial x + \partial\Omega_1/\partial x = -\rho^{-1}\partial\Omega_2/\partial\phi$, or

$$i\left[kA - \sum_{m=0}^{\infty} h_m A_m \cos(m\pi y/a)\right] = -\frac{1}{\rho}\sum_{m=0}^{\infty} \nu_m R_{i\nu_m}(k\rho)\left[B_m e^{\nu_m \alpha} - C_m e^{-\nu_m \alpha}\right]. \tag{16}$$

Similarly on matching Ω_2 at $\phi = -\alpha$ with Ω_3 at $x = 0$, we get

$$\sum_{m=0}^{\infty} R_{i\nu_m}(k\rho)\left[B_m e^{-\nu_m \alpha} + C_m e^{\nu_m \alpha}\right] = \sum_{m=0}^{\infty} D_m \cos(m\pi y/a), \tag{17}$$

$$-\frac{1}{\rho}\sum_{m=0}^{\infty} \nu_m R_{i\nu_m}(k\rho)\left[B_m e^{-\nu_m \alpha} - C_m e^{\nu_m \alpha}\right] = i\sum_{m=0}^{\infty} h_m D_m \cos(m\pi y/a). \tag{18}$$

On replacing ρ by $\rho_1 + y$ in eqs. (15) - (18), multiplying through by $\cos(n\pi y/a)$, and integrating from $y = 0$ to $y = a$, we get the set of equations:

$$\tfrac{1}{2}a\left[2A\delta_{on} + A_n(1 + \delta_{on})\right] = \sum_{m=0}^{\infty} M_{mn}\left[B_m e^{\nu_m \alpha} + C_m e^{-\nu_m \alpha}\right], \tag{19}$$

$$\tfrac{1}{2}ia\left[2kA\delta_{on} - h_n A_n(1 + \delta_{on})\right] = -\sum_{m=0}^{\infty} \nu_m N_{mn}\left[B_m e^{\nu_m \alpha} - C_m e^{-\nu_m \alpha}\right], \tag{20}$$

$$\sum_{m=0}^{\infty} M_{mn}\left[B_m e^{-\nu_m \alpha} + C_m e^{\nu_m \alpha}\right] = \tfrac{1}{2}aD_n(1 + \delta_{on}), \tag{21}$$

$$-\sum_{m=0}^{\infty} \nu_m N_{mn}\left[B_m e^{-\nu_m \alpha} - C_m e^{\nu_m \alpha}\right] = \tfrac{1}{2}aih_n D_n(1 + \delta_{on}), \tag{22}$$

where $$M_{mn} = \int_0^a R_{i\nu_m}(\rho_1 + y)\cos(n\pi y/a)\,dy \tag{23}$$

and $$N_{mn} = \int_0^a \frac{R_{i\nu_m}(\rho_1 + y)}{(\rho_1 + y)}\cos(n\pi y/a)\,dy. \tag{24}$$

If we could solve the infinite set of equations (19) - (22) for the infinite set of ratios A_m/A, B_m/A, C_m/A, and D_m/A, we should presumably have the rigorous solution of our original problem. Although the exact solution is not feasible, similar sets of equations have been used by

W. C. Hahn[5] and others to obtain approximate solutions of various electromagnetic problems involving cavity resonators and wave guides which it is not practicable to treat in any other way. The procedure is to take only a finite number of values of m, say three or four, and to solve the resultant equations for the coefficients of the first few terms in the expansions for the potential. The convergence of the process is sufficiently demonstrated, from an engineer's point of view, if the amplitudes of the higher order waves diminish rapidly compared with the amplitude of the original wave; this will be seen only by carrying out a numerical calculation in a particular case. It is worth noting that since the functions $R_{i\nu}(k\rho)$ and $R_{i\nu}(k\rho)/\rho$ undergo an integral number of oscillations in the interval $(\rho_1, \rho_1 + a)$, they will be in a manner of speaking approximately orthogonal to the cosine functions, so that when $m \neq n$ the quantities M_{mn} and N_{mn} defined by (23) and (24) may be expected to be much smaller than the quantities M_{nn} and N_{nn} with equal subscripts. Thus the largest coefficients in the set of equations (19) - (22) will be those of the diagonal terms, a circumstance which greatly facilitates the solution of a finite number of these equations by the method of successive approximations. In the cases published by Hahn (which involved only trigonometric functions), the rapidity of convergence of the solutions was increased by the introduction of certain auxiliary functions which could be separately calculated; possibly further investigation might disclose the usefulness of similar auxiliary functions in the present problem.

We shall not here undertake any extensive numerical calculations for the problem which we have been discussing; but it may nevertheless be of interest to see what would be the magnitude of order and argument of the

---

5) Hahn, W. C., Journal of Applied Physics, 12, 62-68 (1941).

Bessel functions involved in a typical case. Consider a square pipe for which a = b = 10 cm, transmitting a plane wave of wavelength 10π cm, which corresponds to a frequency at 0°C of $(331.5 \times 10^2)/10\pi = 1055$ cps. For such a wave $k = \omega/v = 2\pi/\lambda = 1/5$ cm^{-1}, so that from (7) all modes except the one for which m = n = 0 are below the cut-off frequency. If we take 15 cm for the radius of the center line of the bend, then $\rho_1 = 10$ cm, $\rho_2 = 20$ cm, and the equations 3.1 (10.1) and (10.2) for the admissible values of ν become

$$R_{i\nu}'(2) = R_{i\nu}'(4) = 0. \qquad (25)$$

If the squares of the successive roots of (25) are $\nu_0^2 < \nu_1^2 < \nu_2^2 <$ ..., we get from the calculus of variations, on putting a = 2, b = 4 in 1.8 (8), the inequality

$$\nu_0^2 < -6/\log 2 = -8.656 = -(2.942)^2,$$

so that the first root of (25) corresponds to an ordinary Bessel function of real order somewhat greater than 2.94. A rough approximation to ν_1^2 may be obtained from 1.8 (9.1) and (9.2); with the aid of a table of cosine integrals we get

$$\nu_1^2 \approx \mathcal{D}_1/\mathcal{H}_1 = 12.41 = (3.52)^2,$$

corresponding to a Bessel function of imaginary order in the neighborhood of 3.5i. On setting k = 1 in Horn's approximation 1.8 (4) to ν_k, we find $\nu_1 \approx 3.58$, with no a priori way of knowing which approximation is closer to the true value of ν_1. It appears, however, that with the chosen values of the various parameters we should need to obtain by trial and error from a table only the first two eigenvalues ν_0 and ν_1, the others being given with sufficient accuracy for all practical purposes by Horn's asymptotic formula, which improves rapidly for the higher eigenvalues. Likewise the eigenfunctions $R_{i\nu_m}(k\rho)$ for $m \geq 2$ would be represented quite simply by the asymptotic formula 1.8 (5).

3.12. Propagation of Electromagnetic Waves around a Circular Bend in a Rectangular Wave Guide.

In free space or in a perfect homogeneous isotropic dielectric of capacitivity ε and permeability μ the electric field intensity $\vec{E}$ satisfies the vector wave equation

$$\nabla^2 \vec{\mathcal{E}} = \mu\varepsilon \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2}, \qquad (1)$$

and all the other field vectors and potentials satisfy equations of the same form. The rectangular components of the field vectors individually satisfy scalar wave equations of the form (1), but the components of these vectors with respect to a general curvilinear coordinate system do not individually satisfy the scalar wave equation, because the unit vectors in a curvilinear system are not in general constant. The difficulty may be avoided by deriving the fields from potentials which satisfy (1) and which can be obtained by various methods; or, if by any means we have expressions for one component each of the electric and magnetic field vectors $\vec{E}$ and $\vec{B}$, the other components may be derived from the interrelations expressed by Maxwell's equations. For our purpose the latter procedure will be sufficient.

If we assume that the time variation of all field quantities is given by the harmonic factor $e^{-i\omega t}$, so that differentiation with respect to time is equivalent to multiplication by $-i\omega$, the curl equations of Maxwell become, for a homogeneous isotropic dielectric,

$$\vec{\nabla} \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t} = i\omega\vec{\mathcal{B}}\,; \quad \vec{\nabla} \times \left(\frac{\vec{\mathcal{B}}}{\mu}\right) = \frac{\partial}{\partial t}\left(\varepsilon\vec{\mathcal{E}}\right) = -i\omega\varepsilon\vec{\mathcal{E}}. \qquad (2)$$

If we write out the six component equations in cylindrical coordinates and further assume that all components vary as $\sin k_z z$ or $\cos k_z z$, so that the operation of $\partial^2/\partial z^2$ is equivalent to multiplication by $-k_z^2$, it is possible to solve for E_ρ, E_ϕ, B_ρ, and B_ϕ in terms of E_z and B_z; we get[6])

$$\mathcal{E}_\rho = \frac{1}{k_c^2}\left[\frac{\partial^2 \mathcal{E}_z}{\partial\rho\partial z} + \frac{i\omega}{\rho}\frac{\partial \mathcal{B}_z}{\partial\varphi}\right], \quad \mathcal{E}_\varphi = \frac{1}{k_c^2}\left[\frac{1}{\rho}\frac{\partial^2 \mathcal{E}_z}{\partial\varphi\partial z} - i\omega\frac{\partial \mathcal{B}_z}{\partial\rho}\right]; \tag{3}$$

$$\mathcal{B}_\rho = \frac{1}{k_c^2}\left[\frac{\partial^2 \mathcal{B}_z}{\partial\rho\partial z} - \frac{ik^2}{\omega\rho}\frac{\partial \mathcal{E}_z}{\partial\varphi}\right], \quad \mathcal{B}_\varphi = \frac{1}{k_c^2}\left[\frac{1}{\rho}\frac{\partial^2 \mathcal{B}_z}{\partial\varphi\partial z} + \frac{ik^2}{\omega}\frac{\partial \mathcal{E}_z}{\partial\rho}\right]; \tag{4}$$

where $k^2 = \omega^2\mu\varepsilon$, $k_c^2 = k^2 - k_z^2$. Similarly in rectangular coordinates, if the components E_x and B_x are supposed known and the fields are assumed to be propagating in the positive x-direction so that their dependence on x and t is given by $e^{i(hx-\omega t)}$, we get from the curl equations (2)

$$\mathcal{E}_y = \frac{1}{k^2-h^2}\left[ih\frac{\partial \mathcal{E}_x}{\partial y} + i\omega\frac{\partial \mathcal{B}_x}{\partial z}\right], \quad \mathcal{E}_z = \frac{1}{k^2-h^2}\left[ih\frac{\partial \mathcal{E}_x}{\partial z} - i\omega\frac{\partial \mathcal{B}_x}{\partial y}\right]; \tag{5}$$

$$\mathcal{B}_y = \frac{1}{k^2-h^2}\left[ih\frac{\partial \mathcal{B}_x}{\partial y} - \frac{ik^2}{\omega}\frac{\partial \mathcal{E}_x}{\partial z}\right], \quad \mathcal{B}_z = \frac{1}{k^2-h^2}\left[ih\frac{\partial \mathcal{B}_x}{\partial z} + \frac{ik^2}{\omega}\frac{\partial \mathcal{E}_x}{\partial y}\right]. \tag{6}$$

Since it happens that in the cylindrical coordinate system the unit vector in the z-direction is constant, the z-components of the field vectors do satisfy the scalar wave equations

$$\nabla^2 \mathcal{E}_z = \mu\varepsilon\frac{\partial^2 \mathcal{E}_z}{\partial t^2}, \qquad \nabla^2 \mathcal{B}_z = \mu\varepsilon\frac{\partial^2 \mathcal{B}_z}{\partial t^2}, \tag{7}$$

of which solutions are given by 3.1 (7). It is therefore easy to write down from 3.1 (7) and from (3) and (4) above various types of fields which satisfy the boundary conditions that the tangential component of $\overline{E}$ and the normal component of $\overline{B}$ shall vanish on the perfectly conducting surfaces $\rho = \rho_1$, $\rho = \rho_2 = \rho_1 + a$, $z = 0$, and $z = b$.

We consider first the case in which $B_z = 0$ and E_z is a suitably

---

6) Compare Ramo, S., and Whinnery, J. R., <u>Fields and Waves in Modern Radio</u>, 299-300 and 326-327. Note that in Ramo and Whinnery's notation the time dependence of the field quantities is given by $e^{+j\omega t}$.

specialized function of the form given by 3.1 (7). On making use of eqs. (3) and (4), we find that a field satisfying the specified boundary conditions is given by the following set of components (the time dependence $e^{-i\omega t}$ being understood):

$$E_z = R_{i\nu_{nm}}(k_{cn}\rho)\left[A_{mn}e^{\nu_{nm}\phi} + B_{mn}e^{-\nu_{nm}\phi}\right]\cos(n\pi z/b), \qquad (8.1)$$

$$E_\rho = -(n\pi/bk_{cn})R_{i\nu_{nm}}'(k_{cn}\rho)\left[A_{mn}e^{\nu_{nm}\phi} + B_{mn}e^{-\nu_{nm}\phi}\right]\sin(n\pi z/b), \qquad (8.2)$$

$$E_\phi = -(n\pi\nu_{nm}/bk_{cn}\rho)\, R_{i\nu_{nm}}(k_{cn}\rho)\left[A_{mn}e^{\nu_{nm}\phi} - B_{mn}e^{-\nu_{nm}\phi}\right]\sin(n\pi z/b), \qquad (8.3)$$

$$B_z = 0, \qquad (8.4)$$

$$B_\rho = -(ik^2\nu_{nm}/k_{cn}^2\omega\rho)R_{i\nu_{nm}}(k_{cn}\rho)\left[A_{mn}e^{\nu_{nm}\phi} - B_{mn}e^{-\nu_{nm}\phi}\right]\cos(n\pi z/b), \qquad (8.5)$$

$$B_\phi = (ik^2/k_{cn}\omega)\, R_{i\nu_{nm}}'(k_{cn}\rho)\left[A_{mn}e^{\nu_{nm}\phi} + B_{mn}e^{-\nu_{nm}\phi}\right]\cos(n\pi z/b), \qquad (8.6)$$

where n is any non-negative integer, $k_{cn}^2 = \omega^2\mu\varepsilon - (n\pi/b)^2$, $R_{i\nu}(k_{cn}\rho)$ is a Bessel function of order $i\nu$ vanishing at $\rho = \rho_1$, and ν_{nm} is the m<u>th</u> root of the equation $R_{i\nu}(k_{cn}\rho_2) = 0$. A field in which $B_z = 0$ will be designated as "transverse magnetic", or TM;* and the particular oscillation specified, as in (8), by the integers m and n will be called the TM_{mn} mode.

Similarly we may write down the components of a "transverse electric" field for which $E_z = 0$ and B_z is given by 3.1 (7); these are, for the TE_{mn} mode,

$$B_z = R_{i\nu_{nm}}(k_{cn}\rho)\left[C_{mn}e^{\nu_{nm}\phi} + D_{mn}e^{-\nu_{nm}\phi}\right]\sin(n\pi z/b), \qquad (9.1)$$

$$B_\rho = (n\pi/k_{cn}b)\, R_{i\nu_{nm}}'(k_{cn}\rho)\left[C_{mn}e^{\nu_{nm}\phi} + D_{mn}e^{-\nu_{nm}\phi}\right]\cos(n\pi z/b), \qquad (9.2)$$

$$B_\phi = (n\pi\nu_{nm}/k_{cn}^2 b\rho)\, R_{i\nu_{nm}}(k_{cn}\rho)\left[C_{mn}e^{\nu_{nm}\phi} - D_{mn}e^{-\nu_{nm}\phi}\right]\cos(n\pi z/b), \qquad (9.3)$$

$$E_z = 0, \qquad (9.4)$$

$$E_\rho = (i\omega\nu_{nm}/k_{cn}^2\rho)R_{i\nu_{nm}}(k_{cn}\rho)\left[C_{mn}e^{\nu_{nm}\phi} - D_{mn}e^{-\nu_{nm}\phi}\right]\sin(n\pi z/b), \qquad (9.5)$$

---

*The designation TM has in this case no particular advantage except brevity. It was originally introduced to describe fields which were propagating in the z-direction, and for which therefore the magnetic field was transverse to the direction of propagation.

$$E_\phi = -(i\omega/k_{cn})\, R_{i\nu_{nm}}{}'(k_{cn}\rho)\left[C_{mn}e^{\nu_{nm}\phi} + D_{mn}e^{-\nu_{nm}\phi}\right] \sin(n\pi z/b), \qquad (9.6)$$

where to satisfy the boundary conditions in this case the Bessel function $R_{i\nu}(k_{cn}\rho)$ must be so chosen that $R_{i\nu}'(k_{cn}\rho_1) = 0$, and ν_{nm} is the m<u>th</u> root of the equation $R_{i\nu}'(k_{cn}\rho_2) = 0$.

The types of waves which may propagate in a conducting guide of rectangular cross section have been widely discussed in the literature.[7] For the TE_{mn} mode propagating in the positive x-direction in a guide bounded by the conducting planes $y = 0$, $y = a$, $z = 0$, and $z = b$, the fields are given by (5) and (6) in connection with 3.1 (11) and (12) as follows:

$$B_x = A_{mn} \cos(m\pi y/a) \cos(n\pi z/b)e^{ih_{mn}x}, \qquad (10.1)$$

$$B_y = -ih_{mn}(m\pi/a)(k^2 - h_{mn}^2)^{-1}A_{mn} \sin(m\pi y/a) \cos(n\pi z/b)e^{ih_{mn}x}, \qquad (10.2)$$

$$B_z = -ih_{mn}(n\pi/b)(k^2 - h_{mn}^2)^{-1}A_{mn} \cos(m\pi y/a) \sin(n\pi z/b)e^{ih_{mn}x}, \qquad (10.3)$$

$$E_x = 0, \qquad (10.4)$$

$$E_y = -i\omega(n\pi/b)(k^2 - h_{mn}^2)^{-1}A_{mn} \cos(m\pi y/a) \sin(n\pi z/b)e^{ih_{mn}x}, \qquad (10.5)$$

$$E_z = i\omega(m\pi/a)(k^2 - h_{mn}^2)^{-1}A_{mn} \sin(m\pi y/a) \cos(n\pi z/b)e^{ih_{mn}x}, \qquad (10.6)$$

where

$$k^2 = \omega^2\mu\varepsilon = (m\pi/a)^2 + (n\pi/b)^2 + h_{mn}^2. \qquad (11)$$

The corresponding components in the transverse magnetic modes may be written down in a similar way, or they may be found in the work of Ramo and Whinnery.

One of the simplest wave types which may exist in a hollow rectangular pipe is the TE_{10} mode; this mode is also of great engineering importance.[8] We suppose that we have a TE_{10} wave traveling in the positive x-direction through a guide bounded by $y = 0$, $y = a$, $z = 0$, and $z = b$, which is con-

---

7) Ramo and Whinnery, *op. cit.*, Arts. 9.04-9.05.
8) *Ibid.*, Art. 9.05.

nected at $x = 0$ to a similar guide at a different angle through the circular elbow bounded by $\rho = \rho_1$, $\rho = \rho_2 = \rho_1 + a$, $z = 0$, $z = b$, and extending from $\phi = +\alpha$ to $\phi = -\alpha$; and we proceed to write down the equations which determine the amount and form of the transmitted and reflected waves at the bend.

In the first pipe, where $x < 0$, the non-vanishing field components of the incident TE_{10} wave are given by (10) and (11) as

$$B_{ox} = A\cos(\pi y/a)e^{ih_1 x}, \tag{12.1}$$

$$B_{oy} = -(ih_1 a/\pi)A\sin(\pi y/a)e^{ih_1 x}, \tag{12.2}$$

$$E_{oz} = (i\omega a/\pi)A\sin(\pi y/a)e^{ih_1 x}, \tag{12.3}$$

where

$$h_m^{\ 2} = \omega^2\mu\varepsilon - (m\pi/a)^2. \tag{13}$$

Since there is no y-component of electric field in the incident wave there will be no radial component of $\vec{E}$ in the bend; from (8), the only modes that will be excited are transverse magnetic modes with $n = 0$. Accordingly we assume a sum of such modes to represent the fields in the bend, the non-vanishing components being, for $\alpha > \phi > -\alpha$,

$$E_{2z} = \sum_{m=1}^{\infty} R_{i\nu_m}(k\rho)\left[B_m e^{\nu_m\phi} + C_m e^{-\nu_m\phi}\right], \tag{14.1}$$

$$B_{2\rho} = \sum_{m=1}^{\infty} -(i\nu_m/\omega\rho)R_{i\nu_m}(k\rho)\left[B_m e^{\nu_m\phi} - C_m e^{-\nu_m\phi}\right], \tag{14.2}$$

$$B_{2\phi} = \sum_{m=1}^{\infty} (ik/\omega)R_{i\nu_m}'(k\rho)\left[B_m e^{\nu_m\phi} + C_m e^{-\nu_m\phi}\right], \tag{14.3}$$

where $R_{i\nu}(k\rho_1) = 0$ and ν_m is the m<u>th</u> root of $R_{i\nu}(k\rho_2) = 0$.* In the transmitted and reflected waves we shall find only those transverse electric modes for which $E_y = 0$; hence for the reflected waves, $x < 0$, we assume

$$B_{1x} = \sum_{m=1}^{\infty} A_m\cos(m\pi y/a)e^{-ih_m x}, \tag{15.1}$$

$$B_{1y} = \sum_{m=1}^{\infty} (+ih_m a/m\pi)A_m\sin(m\pi y/a)e^{-ih_m x}, \tag{15.2}$$

---

*As in Arts. 3.1 and 3.11, the boundary conditions will in general be satisfied by a finite number of Bessel functions of real order as well as an infinite number of functions of imaginary order; these latter functions may be written if desired in the form

$$R_{i\nu}(k\rho) = V_\nu(k\rho_1)\,U_\nu(k\rho) - U_\nu(k\rho_1)\,V_\nu(k\rho).$$

$$E_{1z} = \sum_{m=1}^{\infty} (+i\omega a/m\pi)\, A_m \sin(m\pi y/a) e^{-ih_m x}, \tag{15.3}$$

and for the transmitted waves, $x > 0$,

$$B_{3x} = \sum_{m=1}^{\infty} D_m \cos(m\pi y/a) e^{ih_m x}, \tag{16.1}$$

$$B_{3y} = \sum_{m=1}^{\infty} -(ih_m a/m\pi) D_m \sin(m\pi y/a) e^{ih_m x}, \tag{16.2}$$

$$E_{3z} = \sum_{m=1}^{\infty} (i\omega a/m\pi) D_m \sin(m\pi y/a) e^{ih_m x}, \tag{16.3}$$

It is evident from (13) that if the dimensions of the guide are properly chosen h_m may be imaginary for $m > 1$, in which case all the modes except TE_{10} will be rapidly attenuated.

The boundary conditions in this problem require the continuity of the fields at all points; hence we must have $E_{0z} + E_{1z} = E_{2z}$ and $B_{oy} + B_{1y} = B_{2\rho}$ over the plane $x = -0$, $\phi = \alpha$, as well as $E_{2z} = E_{3z}$ and $B_{2\rho} = B_{3y}$ over the plane $\phi = -\alpha$, $x = +0$. (If these four conditions are satisfied, the curl equations (2) imply that the remaining component of the magnetic field is also continuous.) Evidently these conditions, applied to the expressions which we have written down for the field components, will lead to four sets of equations for the four sets of ratios A_m/A, B_m/A, C_m/A, and D_m/A, precisely similar to eqs. (19) - (22) of Art. 3.11.

Space limitations due to the original plan of this work, which was to exhibit as many <u>different</u> occurrences of Bessel functions of imaginary order as possible rather than to discuss any single application exhaustively, prevent us from continuing here the treatment of the wave guide problem which we have thus briefly introduced. It can scarcely be doubted, however, that this general problem currently represents the most important practical application of Bessel functions of imaginary order, and that it merits a much more extensive treatment than we have been able to give. Probably the equivalent circuit concepts which have already proved so fruitful in analyzing the transmission of microwaves[9] can be applied

9) For example, Whinnery, J. R., and Jamieson, H. W., "Equivalent Circuits for Discontinuities in Transmission Lines," <u>Proc. I. R. E.</u>, <u>32</u>, 98-114 (1944).

here, the reflection and transmission coefficients of the bent portion of the guide being represented by an equivalent impedance network at the junction between two sections of uniform line. For numerical analysis it would be highly desirable to have a table of the Bessel functions $U_\nu(x)$ and $V_\nu(x)$ of imaginary order and real argument comparable in range with the table of functions of imaginary order and imaginary argument contained in the appendix of the present work. If such a table can be made available, we feel that Bessel functions of imaginary order will find a very practical use in electromagnetic theory.

3.2. Schrödinger Wave Functions for a Particle in an Exponential Field of Force.

Among the more important physical applications of Bessel functions are those which occur in the quantum theory. Most of the elementary quantum mechanical problems which require the use of Bessel functions lead only to functions of real order; but within the past three or four years several investigations have been published which involve the functions of purely imaginary order. We shall now formulate the basic problem which gives rise to these latter functions.

The quantum mechanical behavior of a particle of mass m and total energy E in a field of force given by the potential function V is determined by the time independent Schrödinger equation

$$\nabla^2 \psi + \frac{2m}{\hbar^2}(E-V)\psi = 0, \tag{1}$$

where ψ is the wave function of the particle and $2\pi\hbar$ is Planck's quantum of action h. In a central force field, where V $[= V(r)]$ is a function of the radial distance only, it is well known[10)] that the wave function

10) See for example Pauling, L., and Wilson, E. B., Jr., *Introduction to Quantum Mechanics*, 113-121.

may be written in spherical coordinates as

$$\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi), \tag{2}$$

where Φ and Θ are respectively trigonometric and associated Legendre functions depending on two integral quantum numbers. The differential equation satisfied by the radial function is then

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{2m}{\hbar^2}\{E - V(r)\} - \frac{l(l+1)}{r^2}\right]R = 0, \tag{3}$$

where the non-negative integer l measures the total angular momentum of the particle in units of $\hbar$. If we consider the spherically symmetric state of zero total angular momentum (the so-called s-state) and write

$$R(r) = u(r)/r, \tag{4}$$

eq. (3) becomes

$$\frac{d^2u}{dr} + \frac{2m}{\hbar^2}[E - V(r)]u = 0. \tag{5}$$

We now specialize the problem under consideration by assuming for the potential $V(r)$ the exponential form

$$V(r) = -V_0 \exp(-r/a), \tag{6}$$

which represents an attractive force field if the constant V_0 is positive. The exponential field given by (6) evidently vanishes at large distances much more rapidly than the Coulomb potential $-V_0a/r$; it has an effective range given essentially by the characteristic length a. Such a potential has often been used as a convenient and mathematically tractable approximation to the short-range non-Coulomb fields of nuclear particles.

If we substitute (6) into (5) and introduce the notations

$$x = \exp(-r/2a),\ E = \hbar^2k^2/2m,\ V_0 = \hbar^2p^2/2m, \tag{7}$$

we find that the equation for u becomes

$$x^2d^2u/dx^2 + x\,du/dx + \left[(2ak)^2 + (2ap)^2x^2\right]u = 0, \tag{8}$$

which by comparison with 1.5 (1) is seen to have the solutions $u = J_{\pm 2aki}(2apx)$

for unrestricted values of k and p.* Various cases may arise, according to whether k and p are real or imaginary.

For a bound particle, i. e., one with negative total energy E, in the neighborhood of an attractive center of force, $p = (2mV_0/\hbar^2)^{\frac{1}{2}}$ is real but $k = (2mE/\hbar^2)^{\frac{1}{2}}$ is purely imaginary, so that u is proportional to the ordinary Bessel function $J_{2a|k|}(2apx)$ of real order and real argument. (The second solution of Bessel's equation is infinite at $x = 0$, which corresponds to $r = \infty$.) The admissible values, if any, of the total energy are determined by the boundary condition that u must vanish at $r = 0$, so each root in $|k|$ of the equation

$$J_{2a|k|}(2ap) \quad 0 \tag{9}$$

corresponds to a stationary state of the bound particle defined by a particular value of the total energy E. This problem has been discussed by Bethe and Bacher[11)] in their treatment of the ground state of the deuteron. If on the other hand E is positive, corresponding to a net kinetic energy of the particle at infinity, then k is real and we have to do with Bessel functions of imaginary order and real argument; we shall discuss this case briefly in the following paragraphs. If E is positive but V_0 is negative, so that (6) represents a repulsive field of force, then p is imaginary and we are led to functions of imaginary order and imaginary argument; these functions would arise in the problem of scattering of a stream of particles by a repulsive center of force.

Application of the functions of imaginary order and real argument

---

*If 2aki is a real integer, then in the general solution of (8) we must replace J_{-2aki}, which is no longer distinct from J_{2aki}, by any one of the various so-called functions of the second kind which are linearly independent of J_{2aki}; or the general solution may be expressed in terms of the pair of Hankel functions defined by 1.5 (3).

11) Bethe, H. A., and Bacher, R. F., Rev. Mod. Phys., 8, 110-111 (1936).

to nuclear physics has been made by Dube and Jha[12] in a paper on the emission of alpha-particles from radioactive nuclei. As is well known, the first successful theory of alpha-radioactivity was given by Condon and Gurney and by Gamow.[13] In the original form of the theory, the nucleus was represented by a rectangular potential hole of constant depth V_0 and of radius a equal by definition to the nuclear radius. Outside the nucleus the potential function was taken to be the ordinary Coulomb one between alpha-particle and product nucleus. It was then possible to compute the quantum mechanical probability that an alpha-particle of energy E would "leak through" the potential barrier and escape from the nucleus; and the result was found to agree with the empirical Geiger-Nuttall relation between half-life and disintegration energy for alpha-radioactive nuclei.

Since the model of the nucleus described above is admittedly very crude, Dube and Jha set out to try the effect of replacing the rectangular potential function by an exponential function. They accordingly assume that for $r < a$ the potential is given by the exponential law (6), and for $r > a$ by the Coulomb law

$$V = zZq^2/r, \tag{10}$$

where q is the electronic charge, Z is the atomic number of the product nucleus, and z ($= 2$) is the atomic number of the alpha-particle.

Now the wave function of an alpha-particle of (positive) energy E and zero total angular momentum is spherically symmetric and may be written in the form $\psi(r) = u(r)/r$, where u(r) satisfies (5). Inside the nucleus,

---

12) Dube, G. P., and Jha, S. N., Indian Journal of Physics, 17, 344-356 (1943). This paper was called to my attention by Prof. Bateman.

13) Bethe, H. A., Rev. Mod. Phys., 9, 161-163 (1937), gives a simple derivation of the result.

for $r < a$, $V(r)$ is the exponential function (6), so using the notation of (7),

$$u(r) = D[J_{-2aki}(2ap)J_{2aki}(2ape^{-r/2a}) - J_{2aki}(2ap)J_{-2aki}(2ape^{-r/2a})], \quad (11)$$

which vanishes at $r = 0$, D being an arbitrary constant. For $r > a$, $V(r)$ is the Coulomb potential (10), and the corresponding solutions of (5) are of different types in the regions $a < r < r_E$ and $r > r_E$, where $r_E = zZq^2/E$ is the classical turning point of an alpha-particle of energy E falling on the nucleus from outside. In the region $a < r < r_E$ of the potential barrier, where $E - V$ is negative, $u(r)$ is of exponential type, while in the outer region $r > r_E$, where $E - V$ is positive, $u(r)$ is of wave type. At large distances from the nucleus $u(r)$ must represent an outgoing spherical wave:

$$u(r) \sim Ae^{ikr}. \quad (12)$$

To obtain the relation between the amplitude A of the outgoing wave and the coefficient D of the wave function inside the nucleus it is simplest to use the well-known Wentzel-Kramers-Brillouin (WKB) approximation, which connects the asymptotic form (12) with the exponential function in the potential barrier, and so finally with the inside function (11) at $r = a$. From the value of A/D we may compute the decay constant λ of the given nucleus, which is defined as the ratio of the number of particles emitted per second to the total number of particles inside the nucleus.

The details of the calculation of the decay constant have been carried out by Dube and Jha for the exponential well in a form entirely similar to Bethe's calculation[13)] for the rectangular well, and the values of λ are expressed in terms of E, a, and Z for the two limiting cases $V_0 \to 0$ and $V_0 \to \infty$, which correspond respectively to $p \to 0$ and to $p \to \infty$. It is found[14] that for reasonable values of the parameters the ratio λ_0/λ_∞ $= 1/6$, approximately, from which the authors conclude that the decay constant does not depend critically on the exact depth of the potential well inside the nucleus. They also give a more complicated expression for λ when E and V_0 are of the same order of magnitude, derived from the known asymptotic representation of $J_\nu(z)$ when ν and z are simultaneously large.* The nuclear radii computed from observed values of the decay constant agree closely with the values obtained by earlier workers with the simple rectangular potential well, thus confirming the expectation that the results calculated from the one-body theory of alpha-decay are not sensitive to changes in the form of the assumed potential function. However, as Dube and Jha point out, in view of the present more correct many-body model of the nucleus, calculations such as theirs based on any one-body model must now be regarded as rough approximations and are therefore mainly of theoretical rather than of practical interest.

In recent months various writers[15, 16] have discussed the problem of scattering by an exponential field of the form (6) as it is formulated in Heisenberg's recent theory of the characteristic matrix. Without entering into details here, it may be stated that Heisenberg's new theory centers around a certain unitary matrix S, which vanishes

---

14) Dube and Jha, op. cit., 353.

*For a numerical estimate of the quantities involved we employ the values $\hbar = 1.054 \times 10^{-27}$ erg sec, $m = 4.003 \times 1.660 \times 10^{-24}$ gm, and take for an average radioactive nucleus (Bethe, loc. cit.) $a = 9 \times 10^{-13}$ cm, $E = 6$ Mev $= 6 \times 1.6 \times 10^{-6}$ ergs. The order of the Bessel functions is $2aki = (2a/\hbar)(2mE)^{\frac{1}{2}}i = 19i$, approximately, and the argument is of the same magnitude if V_0 and E are comparable.

15) Ter Haar, D., Physica, 12, 501-508 (1946).

16) Ma, S. T., Phys. Rev., 71, 195-200 (1947). See also an exchange of letters between Ma and W. Opechowski, Phys. Rev., 69, 668 (1946); 70, 772 (1946); 71, 210 (1947).

for those values of the energy which correspond to stationary states of the system. It was originally surmised that all values of the energy for which S vanishes correspond to closed stationary states; but ter Haar and Ma have shown by the example of the attractive exponential field, solved as above in terms of Bessel functions of imaginary order and real argument, that there may be redundant zeros of S which do not correspond to stationary states of the energy. The proper method for excluding these redundant zeros does not yet appear to have been convincingly settled; and though for the sake of completeness we have called attention to this newest occurrence of Bessel functions of imaginary order in the literature, a detailed discussion of the problem which occasioned their use or of the ultimate significance of the characteristic matrix in quantum mechanics falls outside the scope of this thesis.

3.2. Relativistic Wave Functions for a Free Particle in an Expanding Universe.

The last application of Bessel functions of imaginary order which we shall discuss occurs in a paper by E. Schrödinger[17] on the proper vibrations of an expanding universe. In order to present Schrödinger's results we must make a brief excursion into the field of relativistic quantum theory.

The simplest Lorentz-invariant wave equation is the scalar Klein-Gordon equation[18]

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - K^2\psi = 0,$$

17) Schrödinger, E., Physica, 6, 899-912 (1939). This paper was called to my attention by Prof. Bateman.

18) Pauli, W., Rev. Mod. Phys., 13, 208-210 (1941), derives the results stated in this paragraph.

where $K = mc/\hbar = 2\pi/\lambda_0$, c being the velocity of light, $2\pi\hbar$ the quantum of action, m the rest mass of the particle, and λ_0 its Compton wavelength. It is known that the most general solution of (1) can be decomposed into a sum of proper vibrations of the form

$$\psi(\vec{x}, t) = A(\vec{k}) \exp\left[i(-\vec{k}\cdot\vec{x} + \omega t)\right] + B(\vec{k}) \exp\left[i(\vec{k}\cdot\vec{x} - \omega t)\right], \tag{2}$$

where $\vec{x}$ is the ordinary three-dimensional position vector and $\omega^2/c^2 = k^2 + K^2$. The solution (2) evidently represents a plane wave with propagation vector $\vec{k}$ and angular frequency ω. It turns out that if the particles described by (1) are charged and if we are to define an energy-momentum tensor and a charge-current vector which satisfies the equation of continuity (cf. Pauli, loc. cit.), then we must regard the proper vibrations of negative frequency, such as the second term on the right side of (2), as representing particles of opposite charge from the proper vibrations of positive frequency. This convention is necessary because, as Pauli shows, if we interchange the factors $\exp\left[i(-\vec{k}\cdot\vec{x} + \omega t)\right]$ and $\exp\left[i(\vec{k}\cdot\vec{x} - \omega t)\right]$ in (2) we change the sign of the charge-current vector while the energy-momentum tensor remains unaltered.

In the general metric space defined by the line element*

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta \tag{3}$$

the Klein-Gordon equation is to be regarded as the covariant equation

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha}\left(g^{\alpha\beta}\sqrt{-g}\,\frac{\partial \psi}{\partial x^\beta}\right) + K^2\psi = 0, \tag{4}$$

where g is the determinant $|g_{\alpha\beta}|$ of the components of the metric tensor; evidently (4) reduces to (1) if ds^2 is the special relativity line element $-dx^2 - dy^2 - dz^2 + c^2dt^2$. We now wish to extend the investigation to the case of the non-static homogeneous universe whose line element is given

---

*Greek indices assume the values 1, 2, 3, 4; and the usual summation convention applies to repeated indices.

by

$$ds^2 = -R^2(t)\left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\right] + c^2dt^2, \qquad (5)$$

where χ, θ, and ϕ are the well-known co-moving angular coordinates and $R(t)$, the radius of curvature of space, is a function as yet unspecified of the time t.

The general equation (4) may be expressed in terms of the coordinates (χ, θ, ϕ, t) with the aid of the line element (5) and a solution obtained by the standard method of separation of variables in the form

$$\psi(\chi, \theta, \phi, t) = \Omega(\chi, \theta, \phi)\, f(t). \qquad (6)$$

The details of the transformation of variables and the calculation of the angle dependent function Ω have been given elsewhere by Schrödinger; our present interest is only in the resultant equation for the time dependent factor $f(t)$, which he finds to be

$$\frac{1}{R^3}\frac{d}{dt}\left[R^3\frac{df}{dt}\right] + c^2\left[\frac{n(n+2)}{R^2} + K^2\right]f = 0. \qquad (7)$$

The integer n is related to the wavelength λ of the proper vibration by the formula

$$\lambda = 2\pi R/n, \text{ or } n = 2\pi R/\lambda; \qquad (8)$$

in the cases of practical interest n is thus an enormously large number.

For any given value of n we take for the time dependent factor in (6) a linear combination $f_n(t)$ of two independent solutions of (7) with arbitrary coefficients. The general solution of (4) for this universe may then be written in the form of an infinite series of products of the type (6) including all non-negative integral values of n, and this series can in the familiar manner be adapted to an arbitrary initial state. The different members of the series are all independent of one another; if at the outset only one is present, no others will turn up in the course of time. We thus have a genuine decomposition into proper vibrations, although the time factors are in general not trigonometric functions.

They will be trigonometric functions whenever R(t) ceases to vary and remains constant for a time, since during any period when R is constant the general solution of (7) is

$$f_n(t) = A_n e^{i\omega_n t} + B_n e^{-i\omega_n t}, \text{ where } \omega_n = c\left[n(n+2)/R^2 + \kappa^2\right]^{\frac{1}{2}}. \quad (9)$$

Suppose that initially R(t) is constant for a time and that we fix our attention on the particular proper vibration $f_n(t) = Ae^{i\omega_n t}$, which corresponds, in virtue of the remarks following (2), to a particle of positive charge. Now suppose that R(t) undergoes a period of arbitrary variation, during which time of course the particular solution $f_n(t)$ loses its trigonometric character, and then returns to constancy. As soon as R(t) ceases to vary $f_n(t)$ will assume the form $A'e^{i\omega_n' t} + B'e^{-i\omega_n' t}$, but now - and this is the essential point - we have no guarantee that the coefficient B' of the negative frequency term will be zero. In other words there will be a mutual adulteration of positive and negative frequency terms in the course of time. This means with particles the production of oppositely charged pairs merely by the expansion, while with light it implies a production of light traveling in the opposite direction, thus a sort of reflection of light in homogeneous space. Alarmed by these prospects, Schrödinger has investigated the question in more detail in the case in which R is a linear function of the time. This case is soluble in terms of Bessel functions; we proceed to outline Schrödinger's analysis.

We assume that the radius of our universe is given by

$$R = a + bt, \quad (10)$$

and we introduce into (7) the new variables

$$z = \kappa cR/b = \kappa c(a/b + t), \; w(z) = zf, \quad (11)$$

so that after an elementary calculation (7) becomes

$$z^2 d^2w/dz^2 + z\,dw/dz + (\nu^2 + z^2)w = 0, \quad (12)$$

where $\nu^2 + 1 = n(n+2)c^2/b^2$. (13)

We see on comparing (12) with 1.5 (1) that w is a Bessel function of real argument z and imaginary order $i\nu$.

The solution of (12) corresponding to a proper vibration of positive frequency is the Hankel function of the first kind $H_{i\nu}^{(1)}(z)$ defined, if we write $i\nu$ for ν, by 1.1 (3.1). Recalling that $K = 2\pi/\lambda_0$, we see from (8), (13), and (11) that both ν and z are enormously large numbers, while the <u>ratio</u> z/ν is of the comparatively moderate order of magnitude λ/λ_0, which is the ratio of the actual wavelength to the Compton wavelength of the particle. An asymptotic representation of $H_{i\nu}^{(1)}(z)$ may therefore be obtained by Debye's method of saddle-point integration[19] in terms of the ratio

$$\nu/z \equiv \operatorname{sh}\alpha. \tag{14}$$

The result is

$$H_{i\nu}^{(1)}(z) \sim \frac{\sqrt{2}\, e^{\nu\pi/2 - i\pi/4}\, e^{i\nu(\operatorname{coth}\alpha - \alpha)}}{\sqrt{\pi z \operatorname{ch}\alpha}} \tag{15}$$

to a <u>very</u> high degree of approximation, because of the enormous magnitude of ν and z. Hence from (11), on dropping an irrelevant constant multiplier,

$$f(t) = z^{-\frac{3}{2}}(\operatorname{ch}\alpha)^{-\frac{1}{2}} e^{i\nu(\operatorname{coth}\alpha - \alpha)}. \tag{16}$$

In order to find the angular frequency ω, we differentiate the phase of (16) with respect to t and obtain, on making use of (11), (14), and (13) to simplify the result,

$$\begin{aligned} \omega &= \frac{d}{dt}\left[\nu(\operatorname{coth}\alpha - \alpha)\right] = -\nu(\operatorname{csch}^2\alpha + 1)\frac{d\alpha}{dt} \\ &= Kc\operatorname{ch}\alpha = c\left[\frac{n(n+2) - b^2/c^2}{R^2(t)} + K^2\right]^{\frac{1}{2}}. \end{aligned} \tag{17}$$

---

19) Asymptotic expressions for all kinds of Bessel functions of large complex order are derived by G. N. Watson, <u>Theory of Bessel Functions</u>, 2nd ed., Arts. 8.6-8.61. In connection with the functions of purely imaginary order see particularly the last paragraph of Art. 8.61.

The phase velocity is, from (8) and (17),

$$v_{ph} = \omega\lambda/2\pi = (\kappa cR \operatorname{ch}\alpha)/n,$$

while the group velocity is

$$v_g = R\frac{\partial\omega}{\partial n} = R\kappa c\frac{\partial \operatorname{ch}\alpha}{\partial n} = \frac{c(n+1)}{\kappa R \operatorname{ch}\alpha},$$

so that

$$v_{ph}v_g = c^2(n+1)/n. \tag{18}$$

Since n is very large, (18) is equivalent to the usual relation between phase and group velocities for both de Broglie waves and light waves.

The Hankel function $H_{i\nu}^{(2)}(z)$ of the second kind can be worked out in the same way and gives the exponential of negative frequency, corresponding to a particle of opposite charge from that represented by $H_{i\nu}^{(1)}(z)$. Since $H_{i\nu}^{(1)}(z)$ and $H_{i\nu}^{(2)}(z)$ are linearly independent solutions of (12), we see that the positive and negative frequency solutions of (7) keep clear of each other indefinitely so long as R(t) increases or decreases uniformly with time, so that under these circumstances we do not get the pair production anticipated above.* This latter phenomenon is evidently not caused by the velocity of expansion, but would probably be caused by accelerated expansion. It might play an important role in the critical periods of cosmology, when expansion changes to contraction or vice versa.

Solutions of the Dirac equation for a free electron in various cosmological spaces have been obtained by Taub;[20] it happens that the time dependence of the solutions in a De Sitter universe is given by Bessel functions $J_{\pm i\nu\pm\frac{1}{2}}$ of complex order, where ν is a very large number of

---

*Similarly the positive and negative frequency solutions of D'Alembert's equation for light, which is obtained by setting $\kappa = 0$ in (4), may be rigorously separated for all time if R(t) has the form a + bt; in this case D'Alembert's equation may be solved in terms of elementary functions, and there is nothing in the solutions which would correspond to a reflection of light in free space.

20) Taub, A. H., Phys. Rev., 51, 512-525 (1937).

the order of 10^{37}, so that for all practical purposes the behavior of the functions is completely described by their asymptotic representations.

APPENDIX

Tables of the Wedge Functions $F_\nu(e^x)$ and $G_\nu(e^x)$

We have shown in the preceding chapters that Bessel functions of imaginary order find application to several fields of mathematical physics, but before our formal results can be of much practical use in calculation we need adequate numerical tables of the functions of imaginary order. We shall present with this thesis a table of the wedge functions studied in chapters I and II. Although the scope of our table has been limited by requirements of time and the lack of elaborate facilities for computation, we feel that it will be of interest because no other such table is at present in existence.

The quantities tabulated are the wedge functions $F_\nu(e^x)$ and $G_\nu(e^x)$ defined in Art. 1.1, the argument being taken as e^x for the reasons discussed in Art. 1.4. Since, as we have seen, in the physical applications where these functions occur the order ν is not restricted to integral values but must be regarded as a continuous variable, we have essentially to tabulate them as functions of two continuous variables x and ν. Obviously the calculation of a function of two variables over representative ranges in both variables is a much more laborious task than the calculation of a function of a single variable, and the resultant table is correspondingly bulkier.

In the following sections we shall set forth the method used for computing the main body of the table. This work was done on the automatic punched card machines at the Southern California Cooperative Wind Tunnel in Pasadena, which is directed by the California Institute. We shall then describe the actual table, indicating the method of checking and the estimated accuracy of the published figures.

A.1. General Method of Numerical Integration by Means of Punched Card Machines.

The following method for the automatic numerical integration of the differential equation

$$d^2y/dx^2 = b(x)y \tag{1}$$

by means of the punched card machines manufactured by International Business Machines Corporation has been published by L. Feinstein and M. Schwarzchild.[1] One expands y in the Taylor series

$$y(x+h) = \sum_{n=0}^{\infty} (h^n/n!)y^{(n)}(x) \tag{2}$$

and obtains, on making use of (1) to eliminate the second derivative,

$$y(x+h) + y(x-h) = [2 + h^2b(x)]y(x) + 2\sum_{n=2}^{\infty}[h^{2n}/(2n)!]y^{(2n)}(x). \tag{3}$$

Similarly,

$$y''(x+h) + y''(x-h) - 2y''(x) = 2\sum_{n=1}^{\infty}[h^{2n}/(2n)!]y^{(2n+2)}(x). \tag{4}$$

Solving (4) for $h^2y^{(4)}(x)$ and using (1) to eliminate the second derivatives at x and at $x \pm h$, we get

$$[1 - (h^2/12)b(x+h)]y(x+h) + [1 - (h^2/12)b(x-h)]y(x-h)$$
$$- [2 + (5h^2/6)b(x)]y(x) - (h^6/240)y^{(6)}(x) + 0(h^8) = 0. \tag{5}$$

If we let

$$x_n = x_0 + nh,\ y_n = y(x_n),\ Z_n = (1/12)[1 - (h^2/12)b(x_n)]y(x_n),$$
$$B_n = \frac{2 + (5h^2/6)b(x_n)}{1 - (h^2/12)b(x_n)} = 2 + \frac{h^2b(x_n)}{1 - (h^2/12)b(x_n)}, \tag{6}$$

and neglect the sixth and higher powers of h, (5) becomes

$$Z_{n+1} = B_nZ_n - Z_{n-1}. \tag{7}$$

---

1) Feinstein, L., and Schwarzchild, M., Rev. Sci. Inst., 12, 405-408 (1941). These authors treat the general linear differential equation of the second order, but we shall be concerned only with an equation of the special form (1).

Solving the third member of (6) for y_n in terms of Z_n, we obtain

$$y_n = (B_n + 10)Z_n. \tag{8}$$

The extrapolation formula (7) permits us to calculate from any two adjacent values of Z the next succeeding value using only the operations of multiplication and addition, which can be performed by the IBM automatic multiplying punch. The quantities B_1, B_2, ... may be computed by (6) from the coefficient b(x) of the differential equation and punched into a deck of IBM cards; then if we punch into the first card the starting values Z_0 and Z_1 obtained from the initial conditions of the given problem, the multiplier punch will compute Z_2 and record it in the same card. We then transfer Z_1 and Z_2 to the card containing B_2* and repeat the process. When we are finished we obtain y_n from Z_n via (8); since B_n and Z_n are already punched in the same card, this step is easily carried out.

The punched card method of integration is particularly useful when the coefficient b(x) of the differential equation (1) depends linearly on a parameter and we wish to obtain solutions for several different values of the parameter. Suppose for instance that

$$b(x) = {}_0b(x) + \nu^2 {}_1b(x); \tag{9}$$

then from (6) we have, on dropping the sixth and higher powers of h,

$$B_{\nu n} = {}_0B_n + \nu^2 {}_1B_n + \nu^4 {}_2B_n, \tag{10}$$

where

$${}_0B_n = 2 + h^2 {}_0b_n + (h^4/12) {}_0b_n^{\,2}, \tag{10.1}$$

$${}_1B_n = h^2 {}_1b_n + (h^4/6) {}_0b_n \, {}_1b_n, \tag{10.2}$$

$${}_2B_n = (h^4/12) {}_1b_n^{\,2}. \tag{10.3}$$

---

*The multiplying punch described by Feinstein and Schwarzchild contained a mechanism for storing Z_n and Z_{n+1} in the machine between steps. Our machine was not thus equipped, so the quantities Z_n and Z_{n+1} had to be transferred from one card to the next with the IBM reproducing punch.

Only the quantities ${}_0B_n$, ${}_1B_n$, and ${}_2B_n$, which are all independent of ν^2, have to be computed beforehand; $B_{\nu h}$ can then be obtained with the help of the punched card machines for any value of ν^2.

Punched card methods are most efficient in calculations where the same numerical data are used over and over in different combinations; thus in the problem at hand their relative efficiency, as compared with other methods, increases with the number of different values of the parameter ν which we desire to consider. The method of numerical integration just described has, however, the disadvantage that it is not self-checking or self-correcting; we have no way of knowing how the inherent errors in the extrapolation formula (7) are mounting up during the course of an extended calculation. (The estimates given by Feinstein and Schwarzchild of the obtainable accuracy proved of little use in the calculation which we performed.) In cases where it is possible, the safest procedure would be to check the last value y_N of the sequence by some independent method of calculation.

A.2. Method of Calculation of the Wedge Functions.

The equation satisfied by the wedge functions $F_\nu(e^x)$ and $G_\nu(e^x)$ is just 1.4 (2), namely

$$d^2y/dx^2 = (e^{2x} - \nu^2)y. \qquad (1)$$

With the notation of A.1 (9) we have $b(x) = e^{2x} - \nu^2$, ${}_0b(x) = e^{2x}$, ${}_1b(x) = -1$, so that the quantities ${}_0B_n$, ${}_1B_n$, and ${}_2B_n$ may easily be written down from A.1 (10.1) - (10.3) and evaluated from tables of the exponential function.

It was originally planned to tabulate both $F_\nu(e^x)$ and $G_\nu(e^x)$ for 50 values of ν extending from $\nu = 0.2$ to $\nu = 10.0$, and for 300 values of x extending from $x = -0.49$ ($e^x = 0.613$) to $x = 2.50$ ($e^x = 12.18$) with a step interval $h = 0.01$. The starting values $Z_{\nu 0}$ and $Z_{\nu 1}$, corresponding to $x_0 = -0.50$ and $x_1 = -0.49$, were computed for both functions from the series representations 1.11 (5.1) and (5.2), the quantities A and B being expressed by 1.13 (6) in terms of the power series S_1 and S_2. This work was done on a 10 x 10 x 20 Friden automatic calculating machine, the results being recorded to ten figures and checked by repeating the entire computation at another time. Because of the necessity for computing the auxiliary functions A, B, $\sin\theta$, and $\cos\theta$ to a high degree of accuracy before undertaking the actual evaluation of $F_\nu(e^x)$ and $G_\nu(e^x)$, this calculation of the starting values was the most laborious and time-consuming part of the whole project. It would be of considerable value to have simpler representations of the canonical functions which are adapted to easy numerical evaluation when the argument is small.

The punched card machines were used to calculate and record on cards the coefficients $B_{\nu n}$ given by A.1 (10) for 50 values of ν and 300 values of n, in preparation for the step-by-step process of evaluating the Z_n's from the recurrence formula A.1 (7). Each step of the actual numerical integration involved the processing of 100 cards (50 values of ν for each function), and in order to guard against mechanical errors the entire calculation was carried out with two identical decks and two multiplying punches, the IBM reproducer being used to compare results at the end of each step. All numbers appearing on the cards were expressed to eight significant figures.

When the integration had been completed, it was clear that the inherent errors in the approximate extrapolation formula A.1 (7) had accumulated, in some cases to an intolerable degree, in the latter part of the range of integration. It would have been infeasible at this stage to repeat the integration with a smaller step interval or with additional starting values at intermediate points of the original range; so we compromised by checking the results at various points, correcting the errors where possible, and discarding the relatively few values which were too much in error to be easily corrected.

The table of $G_\nu(e^x)$ was checked by means of the definite integral representation 1.11 (6.2):

$$G_\nu(e^x) = \int_0^\infty \exp(-e^x \operatorname{ch} t) \cos \nu t \, dt. \qquad (2)$$

This integral converges quite rapidly when x is greater than $\log \nu$; for example, when x = 2.50 it was found possible to evaluate $G_\nu(e^x)$ for $0.2 \leq \nu \leq 10$ to one more significant figure than was desired in the table by breaking off the range of integration at t = 1.5 and applying Tschebyscheff's mechanical quadrature formula[2)] with fifteen subdivisions. The integral (2) was therefore used with a Monroe automatic calculating machine to evaluate $G_\nu(e^x)$ for x = 0.00, 0.50, 1.00, 1.50, 2.00, and 2.50. In the portion of the table where $G_\nu(e^x)$ was oscillatory (i. e., $x < \log \nu$; cf. Art. 1.2), the values given by the punched card integration were found to be accurate to one or two units in the fifth significant place; presumably the errors in the integration formula A.1 (7) cancelled out on the average in this region. On the other hand, where x was appreciably larger than

---

2) Encyklopädie der Mathematischen Wissenschaften, Bd. II, 3.1, 72-74. The formula in question is:

$$\int_a^b f(x)dx = (b - a)/n \sum_{r=1}^{n} f[a + (2r - 1)(b - a)/2n] + R_n.$$

This is not the most accurate quadrature formula available for a given number of subdivisions, but it is very easy to use because the coefficients are simple. In practice the remainder term R_n may be controlled by investigating whether an increase in the number of subdivisions n gives a significantly different value for the integral.

log ν, the punched card integration led to rapidly accumulating errors, the resulting functions tending to go off to $\pm\infty$ rather than to approach $+0$ as they should. Since the errors seemed to be varying continuously, it was possible to approximate to the correction terms by means of Lagrange's interpolating polynomial fitted to the known values of the corrections at $x = 0.00$, $x = 0.50$, etc. Corrections were applied in general where they did not exceed 1% of the uncorrected value, and the remainder of the table, where x was considerably greater than $\log \nu$, was discarded.

In order to check the table of $F_\nu(e^x)$ it was necessary to use the integral representation 1.11 (6.1),

$$F_\nu(e^x) = \operatorname{csch}\nu\pi \int_0^\pi \exp(e^x \cos\theta) \operatorname{ch}\nu\theta \, d\theta - \int_0^\infty \exp(-e^x \operatorname{ch} t) \sin\nu t \, dt, \quad (3)$$

since for most of the values of ν and x in the table neither power series nor asymptotic series converge rapidly enough to be useful. The second integral in (3) can be evaluated without difficulty by mechanical quadrature; but depending on the relative magnitudes of x and ν the first integrand may have sharp peaks at either end of the range of integration, which must be subtracted off and integrated separately. If $v = e^x$, it is easy to show that the first term on the right side of (3) is equal to

$$\operatorname{csch}\nu\pi \int_0^\pi \left\{ e^{v\cos\theta} - e^{-v}\left[1 + \tfrac{1}{2}x(\pi - \theta)^2\right]\right\} (\operatorname{ch}\nu\theta - 1)\, d\theta + \pi \operatorname{csch}\nu\pi\, I_0(v)$$
$$+ e^{-v}(1/\nu - \pi\operatorname{csch}\nu\pi) + ve^{-v}\left[1/\nu^3 - \pi\operatorname{csch}\nu\pi(1/\nu^2 + \pi^2/6)\right]. \quad (4)$$

Even with this transformation the remaining integral in (4) is surprisingly intractable; it apparently cannot be calculated by mechanical quadrature with a reasonable number of subdivisions to the accuracy desired in our table if ν is greater than about $0.6\, e^x$. Consequently it was not possible to check the table of $F_\nu(e^x)$ completely in the time at our disposal. We did find however that at $x = 2.50$, for $\nu \leq 8.0$ only ten values were in error by more than 0.05% (5 parts in 10,000); all of these erroneous values occurred for $\nu < 3.0$.* In the range $3.0 \leq \nu \leq 8.0$ most of the values were in error by not more than a few units in the fifth significant figure. Though it was not possible to compute accurate check values from (3) for $\nu > 8.0$, we know from the behavior of $G_\nu(e^x)$ that for large values of ν, when the functions are oscillatory over most of the range, the integration formula A.1 (7) is unlikely to accumulate errors of large absolute magnitude. The ten erroneous values mentioned above were adjusted by fitting a continuous correction curve to cancel the known relative error at $x = 0.00, 0.50, \ldots, 2.50$, and with these alterations the entire table of $F_\nu(e^x)$ is printed.

It is worth noting that since the errors involved in the integration formula A.1 (7) seem to compensate on the average when the solutions of the differential equation are oscillatory, the punched card method might be used with considerable success to calculate the Bessel functions $U_\nu(e^x)$ and $V_\nu(e^x)$ of imaginary order and real argument, since by 1.52 (3) and (5) these functions are oscillatory for all values of x. The results of such a calculation would of course have to be checked before publication.

A.3. Description of the Tables.

Since the tables of the wedge functions were printed directly from punched cards on an IBM tabulator, some changes, such as the placing of the negative sign on the right of the entry to which it applies, have had to be made in the usual format of such tables. The position of the decimal point is determined by multiplying the tabular entry by 10^p, where the

---

*The probable reason for the absence of large relative errors in the functions $F_\nu(e^x)$ is that these functions tend rapidly to infinity when $x \gg \log \nu$, and so increase in absolute magnitude as fast as the errors accumulate.

(negative) integer p is tabulated to the right of each row of the table. Thus $G_{5.0}(e^{1.00}) = 22452 \times 10^{-8} = 2.2452 \times 10^{-4}$, etc. In case the value of p increases in algebraic magnitude in the middle of a row, the entries marked with an asterisk should be used with the value of p on the <u>preceding</u> page.

The range and accuracy of the tables may be summarized as follows:

The function $F_\nu(e^x)$ is tabulated over the complete ranges $0.2 \leq \nu \leq 10.0$, $-0.49 \leq x \leq 2.50$. In the region where $F_\nu(e^x)$ is oscillatory the error in the last figure given should never exceed 5 units. In the region where $F_\nu(e^x)$ is non-oscillatory the error in the tabulated values should not exceed 5 parts in 10,000.

The function $G_\nu(e^x)$ is tabulated over the following ranges in ν and x:

$0.2 \leq \nu \leq 1.0$, $-0.49 \leq x \leq 0.50$;
$1.2 \leq \nu \leq 2.0$, $-0.49 \leq x \leq 1.00$;
$2.2 \leq \nu \leq 4.0$, $-0.49 \leq x \leq 1.50$;
$4.2 \leq \nu \leq 7.0$, $-0.49 \leq x \leq 2.00$;
$7.2 \leq \nu \leq 10.0$, $-0.49 \leq x \leq 2.50$.

The error in the last figure of any tabulated value does not exceed 5 units.

As a matter of interest the values of $G_\nu(e^x)$ computed from the definite integral A.2 (2) and correct to the last printed figure are given for x = 1.00, 1.50, 2.00, and 2.50 for those values of ν not included in the main table.

BIBLIOGRAPHY

Books

Bocher, M., Über die Reihenentwickelungen der Potentialtheorie. Leipzig, Teubner, 1894.

British Association for the Advancement of Science, Mathematical Tables, vol. VI, part 1. Cambridge, 1937.

Carslaw, H. S., Mathematical Theory of the Conduction of Heat, 2nd ed. New York, Dover, 1945.

Copson, E. T., Theory of Functions of a Complex Variable. Oxford, 1935.

Courant, R., and Hilbert, D., Methoden der Mathematischen Physik, vol. 1, 2nd ed. Berlin, Springer, 1931.

Davis, H. T., Tables of the Higher Mathematical Functions, vol. 1. Bloomington, Ind., Principia Press, 1933.

Encyklopädie der Mathematischen Wissenschaften, vol. II, 3.1. Leipzig, Teubner, 1909-21.

Grey A., Matthews, G. B., and MacRobert, T. M., Bessel Functions, 2nd ed. London, Macmillan, 1931.

Gutenberg, B., Der Aufbau der Erde. Berlin, Borntraeger, 1925.

Hobson, E. W., Spherical and Ellipsoidal Harmonics. Cambridge, 1931.

Ince, E. L., Ordinary Differential Equations. New York, Dover, 1944.

Love, A. E. H., Some Problems of Geodynamics. Cambridge, 1911.

Page, L., Introduction to Theoretical Physics, 2nd ed. New York, Van Nostrand, 1935.

Pauling, L., and Wilson, E. B., Jr., Introduction to Quantum Mechanics. New York, McGraw-Hill, 1935.

Ramo, S., and Whinnery, J. R., Fields and Waves in Modern Radio. New York, Wiley, 1944.

Rayleigh, Theory of Sound, 2nd ed., vol. II. New York, Dover, 1945.

Smythe, W. R., Static and Dynamic Electricity. New York, McGraw-Hill, 1939.

Titchmarsh, E. C., Theory of Fourier Integrals. Oxford, 1937.

Watson, G. N., Theory of Bessel Functions. New York, Macmillan, 1944.

Webster, A. G., Dynamics, 2nd ed. Leipzig, Teubner, 1912.

Whittaker, E. T., and Watson, G. N., Modern Analysis, 4th ed. Cambridge, 1940.

Papers

Airey, J. R., "Convergence Factor in Asymptotic Series," Phil. Mag. (7), 24, 521-552 (1937).

Bethe, H. A., "Nuclear Physics: Nuclear Dynamics, Theoretical," Rev. Mod. Phys., 9, 161-163 (1937).

Bethe, H. A., and Bacher, R. F., "Nuclear Physics: Stationary States of Nuclei," Rev. Mod. Phys., 8, 110-111 (1936).

Bocher, M., "On Some Applications of Bessel's Functions with Pure Imaginary Index," Annals of Mathematics, 6, 137-160 (1892).

Boole, G., "On a General Method in Analysis," Phil. Trans. Roy. Soc. (1844), 239.

Carslaw, H. S., and Jaeger, J. C., "The Determination of Green's Function for the Equation of Conduction of Heat in Cylindrical Coordinates by the Laplace Transformation," J. London Math. Soc., 15, 278 (1940).

Dougall, J., "The Determination of Green's Function by Means of Cylindrical or Spherical Harmonics," Proc. Edinburgh Math. Soc., 18, 33-83 (1900).

Dube, G. P., and Jha, S. N., "On the Theory of the Emission of Alpha-Particles from Radioactive Nuclei," Indian Journal of Physics, 17, 344-356 (1943).

Feinstein, L., and Schwarzchild, M., "Automatic Integration of Linear Second-Order Differential Equations by Means of Punched Card Machines," Rev. Sci. Inst., 12, 405-408 (1941).

Goldstein, S., "On the Stability of Superposed Streams of Fluids of Different Densities," Proc. Roy. Soc. London, (A), 132, 524-548 (1931).

Hahn, W. C., "A New Method for the Calculation of Cavity Resonators," Journal of Applied Physics, 12, 62-68 (1941).

Horn, J., "Über eine Lineare Differentialgleichung Zweiter Ordnung mit einem Willkürlichen Parameter," Math. Ann., 52, 271-292 (1899).

Jeffreys, H., "The Effect on Love Waves of Heterogeneity in the Lower Layer," Monthly Notices of the Royal Astronomical Society, Geophysical Supplements, 2, 101-111 (1928-31).

Lommel, E., "Zur Theorie der Bessel'schen Functionen," Math. Ann., 3, 481-486 (1871).

Ma, S. T., "On a General Condition of Heisenberg for the S Matrix," Phys. Rev., 71, 195-200 (1947).

MacRobert, T. M., "Fourier Integrals," Proc. Roy. Soc. Edinburgh, 51, 116-126 (1931).

Meissner, E., "Elastiche Oberflächenwellen mit Dispersion in einem Inhomogenen Medium," Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, 66, 181-195 (1921).

Mercer, J., "Sturm-Liouville Series of Normal Functions in the Theory of Integral Equations," Phil. Trans. Roy. Soc., (A), 211, 111-198 (1912).

Pauli, W., "Relativistic Field Theories of Elementary Particles," Rev. Mod. Phys., 13, 208-210 (1941).

Rice, S. O., "The Electric Field Produced by a Point-Charge Located outside a Dielectric Wedge," Phil. Mag. (7), 29, 36-46 (1940).

Sakuraba, S., "A Contribution to the Theory of the Love Waves Propagating over a Semi-Infinite Solid Body of Varying Elasticity," Geophysical Magazine, Tokyo, 9, 211-214 (1935).

Schrödinger, E., "The Proper Vibrations of the Expanding Universe," Physica, 6, 899-912 (1939).

Taub, A. H., "Quantum Equations in Cosmological Spaces," Phys. Rev., 51, 512-525 (1937).

Taylor, G. I., "Effect of Variation in Density on the Stability of Superposed Streams of Fluid," Proc. Roy. Soc. London, (A), 132, 499-507 (1931).

Ter Haar, D., "On the Redundant Zeros in the Theory of the Heisenberg Matrix," Physica, 12, 501-508 (1946).

Whinnery, J. R., and Jamieson, H. W., "Equivalent Circuits for Discontinuities in Transmission Lines," Proc. I. R. E., 32, 98-114 (1944).

Borgnis, F., "Stromleitung durch Konvektion und Diffusion in zylindrischen Anordnungen," Ann. d. Phys. (5), 31, 745-754 (1938).

Emde, F., "Passintegrale für Zylinderfunktionen von komplexem Index," Z. f. Angew. Math. u. Mech., 19, 101-118 (1939).

TABLES OF BESSEL FUNCTIONS OF IMAGINARY ORDER AND IMAGINARY ARGUMENT

BY

SAMUEL P. MORGAN

CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA
1947

TABLES OF BESSEL FUNCTIONS OF IMAGINARY ORDER AND IMAGINARY ARGUMENT

INTRODUCTION

1. Summary of Mathematical Formulas.

Various problems from different branches of mathematical physics give rise to the differential equation

$$v^2 d^2w/dv^2 + v dw/dv - (v^2 - \nu^2)w = 0, \tag{1}$$

in which v and ν are real quantities. Eq. (1) is a special case of Bessel's equation,

$$z^2 d^2w/dz^2 + z dw/dz + (z^2 - \rho^2)w = 0, \tag{2}$$

in which $z = iv$, $\rho = i\nu$; and its solutions are therefore Bessel functions whose order and argument are both purely imaginary. The accompanying table gives numerical values of a fundamental real set of solutions of (1) over representative ranges in both order and argument; it is the only numerical tabulation of Bessel functions of imaginary order at present in existence.

A fundamental real pair of solutions of (1) may be defined as follows:

$$F_\nu(v) \equiv \frac{\pi}{2}\,\frac{I_{i\nu}(v) + I_{-i\nu}(v)}{\operatorname{sh}\nu\pi} \equiv \frac{\pi}{\operatorname{sh}\nu\pi}\operatorname{Re} I_{i\nu}(v), \tag{3}$$

$$G_\nu(v) \equiv \frac{i\pi}{2}\,\frac{I_{i\nu}(v) - I_{-i\nu}(v)}{\operatorname{sh}\nu\pi} \equiv -\frac{\pi}{\operatorname{sh}\nu\pi}\operatorname{Im} I_{i\nu}(v)$$

$$\equiv K_{i\nu}(v) \equiv \tfrac{1}{2}\pi i e^{-\frac{1}{2}\nu\pi} H_{i\nu}^{(1)}(iv), \tag{4}$$

where ν is real and v is real and positive. In these definitions $I_{i\nu}(v)$ is the modified Bessel function of the first kind of purely imaginary order, being related to the ordinary Bessel function $J_{i\nu}(iv)$ of imaginary order and imaginary argument by

$$I_{i\nu}(v) \equiv e^{\frac{1}{2}\nu\pi} J_{i\nu}(iv) \equiv \sum_{m=0}^{\infty} \frac{(\frac{1}{2}v)^{i\nu + 2m}}{m!\,\Gamma(i\nu + m + 1)} \tag{5}$$

$K_{i\nu}(v)$ is the modified Bessel function of the second kind of purely imaginary order, and $H_{i\nu}^{(1)}(iv)$ is the Hankel function of the first kind with imaginary order and imaginary argument. For brevity the functions $F_\nu(v)$ and $G_\nu(v)$ may be called "wedge functions" of the first and second kinds respectively, since in potential theory they show a certain analogy to the solutions of Legendre's equation called "cone functions".

Representations of $F_\nu(v)$ and $G_\nu(v)$ in terms of series of modified Bessel functions of positive integral order are given by

$$F_\nu(v) = (\nu\pi/\mathrm{sh}\,\nu\pi)^{\frac{1}{2}}[A(\nu,v)\cos\theta(\nu,v) + B(\nu,v)\sin\theta(\nu,v)], \qquad (6)$$

$$G_\nu(v) = (\nu\pi/\mathrm{sh}\,\nu\pi)^{\frac{1}{2}}[B(\nu,v)\cos\theta(\nu,v) - A(\nu,v)\sin\theta(\nu,v)], \qquad (7)$$

where

$$\theta(\nu,v) = \nu\log\tfrac{1}{2}v - \arg\Gamma(i\nu), \qquad (8)$$

$$A(\nu,v) = \sum_{m=1}^{\infty} \frac{m(-)^m(\frac{1}{2}v)^m}{m!\,(m^2+\nu^2)} I_m(v), \qquad (9)$$

$$B(\nu,v) = \sum_{m=0}^{\infty} \frac{\nu(-)^m(\frac{1}{2}v)^m}{m!\,(m^2+\nu^2)} I_m(v). \qquad (10)$$

$A(\nu,v)$ and $B(\nu,v)$ may also be expressed as power series in v.

$$A(\nu,v) = -\sum_{n=1}^{\infty}\sum_{k=0}^{[\frac{1}{2}n-\frac{1}{2}]} \frac{(-)^k\,(n)_{n-1-2k}\,\nu^{2k}\,v^{2n}}{4^n\,n\,(1^2+\nu^2)\cdots(n^2+\nu^2)} \qquad (11)$$

$$B(\nu,v) = \frac{1}{\nu} + \frac{1}{\nu}\sum_{n=1}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-)^k\,(n)_{n-2k}\,\nu^{2k}\,v^{2n}}{4^n\,n!\,(1^2+\nu^2)(2^2+\nu^2)\cdots(v^2+\nu^2)}, \qquad (12)$$

where $[s]$ represents the greatest integer contained in s and the symbol $(p)_q$, where p and q are any positive integers such that $q \leq p$, denotes the sum of all the different products which can be formed by multiplying together q of the p factors $1,2,\ldots,p$, $(p)_0$ being equal to 1 by definition. A short table of values of $(p)_q$ has been given by Bocher.[1]

Definite integral representations of $F_\nu(v)$ and $G_\nu(v)$ are the following:

$$F_\nu(v) = \frac{1}{\mathrm{sh}\nu\pi}\int_0^\pi e^{v\cos\theta}\,\mathrm{ch}\nu\theta\,d\theta - \int_0^\infty e^{-v\,\mathrm{ch}t}\sin\nu t\,dt, \tag{13}$$

$$G_\nu(v) = \int_0^\infty e^{-v\,\mathrm{ch}t}\cos\nu t\,dt. \tag{14}$$

When ν is fixed and v is large and positive we have the asymptotic series:

$$F_\nu(v) \sim \frac{e^v}{\mathrm{sh}\nu\pi}\left(\frac{\pi}{2v}\right)^{\frac{1}{2}}\left[1 + \frac{(4\nu^2+1^2)}{1!(8v)} + \frac{(4\nu^2+1^2)(4\nu^2+3^2)}{2!\ (8v)^2} + \cdots\right], \tag{15}$$

$$G_\nu(v) \sim e^{-v}\left(\frac{\pi}{2v}\right)^{\frac{1}{2}}\left[1 - \frac{(4\nu^2+1^2)}{1!(8v)} + \frac{(4\nu^2+1^2)(4\nu^2+3^2)}{2!\ (8v)^2} - \cdots\right], \tag{16}$$

while if ν is fixed as v tends to zero through positive values,

$$F_\nu(v) \underset{v\to+0}{\to} (\pi/\nu\mathrm{sh}\nu\pi)^{\frac{1}{2}}\ \sin[\nu\log\tfrac{1}{2}v - \arg\Gamma(i\nu)], \tag{17}$$

$$G_\nu(v) \underset{v\to+0}{\to} (\pi/\nu\mathrm{sh}\nu\pi)^{\frac{1}{2}}\ \cos[\nu\log\tfrac{1}{2}v - \arg\Gamma(i\nu)]. \tag{18}$$

When ν is large and v is fixed,

$$F_\nu(v) \sim e^{-\frac{1}{2}\nu\pi}(2\pi/\nu)^{\frac{1}{2}}\cos[\nu(\log\nu - \log\tfrac{1}{2}v - 1) + \tfrac{1}{4}\pi](1+O(1/\nu)), \tag{19}$$

$$G_\nu(v) \sim e^{-\frac{1}{2}\nu\pi}(2\pi/\nu)^{\frac{1}{2}}\sin[\nu(\log\nu - \log\tfrac{1}{2}v - 1) + \tfrac{1}{4}\pi](1+O(1/\nu)), \tag{20}$$

while if ν tends to zero, v being fixed,

$$F_\nu(v) \underset{\nu\to 0}{\to} \frac{I_0(v)}{\nu} \underset{\nu\to 0}{\to} \infty, \tag{21}$$

$$G_\nu(v) \underset{\nu\to 0}{\to} K_0(v) \tag{22}$$

2. <u>Method</u> <u>of</u> <u>Computation</u> <u>of</u> <u>the</u> <u>Tables</u>.

Since the functions $F_\nu(v)$ and $G_\nu(v)$ have an oscillatory singularity at $v = 0$ (cf. (17) and (18)), it is more convenient to tabulate the the related quantities $F_\nu(e^x)$ and $G_\nu(e^x)$ as functions of x. These latter functions satisfy the differential equation

$$d^2w/dx^2 + (\nu^2 - e^{2x})\, w = 0, \tag{23}$$

obtained from (1) by the transformation of variable

$$v = e^x, \qquad x = \log v, \tag{24}$$

which takes the triad of points (0, 1, ∞) of the v-axis into the triad (−∞, 0, ∞) of the x-axis. The functions $F_\nu(e^x)$ and $G_\nu(e^x)$ have no singularities on the finite part of the x-axis, and they approach sinusoids in νx as $x \to -\infty$ $(v \to +0)$.

The accompanying table of the functions $F_\nu(e^x)$ and $G_\nu(e^x)$ was computed by step-by-step numerical integration of the differential equation (23) on the punched card machines at the Southern California Cooperative Wind Tunnel in Pasadena, using a method described by Feinstein and Schwarzchild.[2] The starting values for the numerical integration were obtained from (6) and (7) using the series (11) and (12), this preliminary work being carried out on a 10 x 10 x 20 Friden automatic calculating machine and the computations checked by repetition at another time. Mechanical errors in the punched card machines were avoided by performing the entire calculation with two identical sets of cards on two multiplying punches, the results being compared at the end of each step.

The numerical integration was carried from x = −0.49 (v=0.613) to x = 2.50 (v=12.18), and the accuracy of the results checked by evaluating $F_\nu(e^x)$ and $G_\nu(e^x)$ at the right-hand end-point and various intermediate points of the interval from the definite integrals (13) and (14). In the portion of the table where the functions are oscillatory (essentially $x < \log \nu$), the results of the punched card integration were found to be accurate to one or two units in the fifth significant figure. Over most parts of the non-oscillatory region the errors were small enough to be approximated by a continuous correction curve fitted to the known values of the corrections at certain check points; a small part of the table of $G_\nu(e^x)$ had however to be entirely discarded.*

---

*It is not to be expected that any step-by-step numerical integration formula will follow the solution $G_\nu(e^x)$ accurately in the region $x \gg \log \nu$ where this function is very small, because any inherent errors in the formula quickly introduce a small amount of the other solution $F_\nu(e^x)$ which in this region tends rapidly to infinity.

3. Description of the Tables.

Since the tables of the wedge functions were printed directly from punched cards on an International Business Machines tabulator, some changes, such as the placing of the negative sign on the right of the right of the entry to which it applies, have had to be made in the usual format of such tables. The position of the decimal point is determined by multiplying the tabular entry by 10^p, where the (negative) integer p is tabulated to the right of each row of the table. Thus $G_{5.0}(e^{1.00}) = 22452 \times 10^{-8} = 2.2452 \times 10^{-4}$, etc. In case the value of p increases in algebraic magnitude in the middle of a row, the entries marked with an asterisk should be used with the value of p on the preceding page.

The tabular interval is 0.01 in x and 0.2 in ν. The range and accuracy of the tables may be summarized as follows:

The function $F_\nu(e^x)$ is tabulated over the complete ranges $0.2 \leq \nu \leq 10.0$, $-0.49 \leq x \leq 2.50$. In the region where $F_\nu(e^x)$ is oscillatory the error in the last figure given should not exceed 5 units. In the region where $F_\nu(e^x)$ is non-oscillatory the error in the tabulated values should not exceed 5 parts in 10,000.

The function $G_\nu(e^x)$ is tabulated over the following ranges in ν and x:

$0.2 \leq \nu \leq 1.0$, $-0.49 \leq x \leq 0.50$;
$1.2 \leq \nu \leq 2.0$, $-0.49 \leq x \leq 1.00$;
$2.2 \leq \nu \leq 4.0$, $-0.49 \leq x \leq 1.50$;
$4.2 \leq \nu \leq 7.0$, $0.49 \leq x \leq 2.00$;
$7.2 \leq \nu \leq 10.0$, $0.49 \leq x \leq 2.50$.

The error in the last figure of any tabulated value does not exceed 5 units.

As a matter of interest the value of $G_\nu(e^x)$ computed from the definite integral (14) and correct to the last printed figure are given for X = 1.00, 1.50, 2.00, and 2.50 for those values of ν not included in the main table.

The use of the computing equipment at the Cooperative Wind Tunnel was arranged by Professor C. B. Millikan and Mr. F. H. Felberg. For instruction in the operation of the punched card machines the author thanks Dr. E. C. Bower of Douglas Aircraft Company and various members of the wind tunnel staff.

References:

[1]Bocher, M., "On Some Applications of Bessel's Functions with Pure Imaginary Index," Annals of Mathematics, 6, 144 (1892).

[2]Feinstein, L., and Schwarzchild, M., "Automatic Integration of Linear Second-Order Differential Equations by Means of Punched Card Machines," Rev. Sci. Inst., 12, 405-408 (1941).

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^{x})$

ν \ x	.49-	.48-	.47-	.46-	.45-	.44-	.43-	.42-	.41-	.40-	p
0.2	52349	52453	52559	52667	52775	52887	53001	53115	53233	53353	4-
0.4	22842	22903	22964	23024	23086	23148	23211	23274	23339	23403	4-
0.6	12138	12188	12237	12286	12337	12386	12436	12485	12535	12585	4-
0.8	65533	65992	66448	66903	67356	67808	68258	68706	69154	69599	5-
1.0	32800	33229	33656	34080	34502	34922	35340	35757	36171	36584	5-
1.2	13161	13544	13926	14307	14686	15064	15441	15814	16188	16558	5-
1.4	1811	2130	2449	2767	3085	3402	3719	4035	4351	4666	5-
1.6	4059-	3818-	3576-	3334-	3090-	2847-	2602-	2357-	2112-	1866-	5-
1.8	6338-	6179-	6018-	5855-	5691-	5525-	5358-	5188-	5018-	4846-	5-
2.0	64071-	63225-	62357-	61466-	60552-	59617-	58661-	57683-	56685-	55667-	6-
2.2	52895-	52647-	52376-	52081-	51764-	51423-	51060-	50674-	50265-	49835-	6-
2.4	37201-	37362-	37503-	37625-	37725-	37806-	37867-	37907-	37927-	37927-	6-
2.6	21818-	22198-	22564-	22916-	23253-	23575-	23883-	24175-	24452-	24713-	6-
2.8	9437-	9876-	10307-	10730-	11146-	11553-	11952-	12342-	12723-	13094-	6-
3.0	1067-	1452-	1836-	2218-	2598-	2976-	3351-	3723-	4093-	4458-	6-
3.2	3455	3183	2908	2630	2350	2067	1782	1495	1208	918	6-
3.4	49636	48177	46665	45100	43485	41822	40112	38358	36560	34722	7-
3.6	45375	44982	44533	44028	43467	42852	42183	41461	40688	39863	7-
3.8	31860	32175	32446	32670	32849	32982	33069	33109	33103	33050	7-
4.0	16712	17356	17973	18562	19133	19653	20153	20621	21058	21461	7-
4.2	4484	5152	5812	6461	7099	7725	8338	8936	9519	10085	7-
4.4	3048-	2541-	2030-	1514-	996-	476-	45	566	1086	1604	7-
4.6	60976-	58191-	55286-	52265-	49137-	45906-	42580-	39165-	35670-	32101-	8-
4.8	59447-	58723-	57866-	56878-	55762-	54519-	53153-	51667-	50064-	48347-	8-
5.0	41391-	42035-	42575-	43011-	43341-	43565-	43681-	43690-	43592-	43386-	8-
5.2	19747-	20974-	22146-	23258-	24308-	25293-	26212-	27060-	27837-	28539-	8-
5.4	2533-	3724-	4904-	6070-	7218-	8346-	9450-	10527-	11573-	12586-	8-
5.6	7196	6386	5556	4708	3847	2973	2090	1201	307	587-	8-
5.8	9997	9649	9270	8859	8420	7952	7458	6939	6397	5834	8-
6.0	82591	82793	82700	82313	81632	80661	79404	77864	76047	73959	9-
6.2	46693	48878	50878	52684	54290	55689	56877	57848	58600	59129	9-
6.4	12554	15131	17647	20091	22454	24726	26898	28960	30906	32726	9-
6.6	9465-	7524-	5550-	3553-	1540-	480	2497	4504	6491	8451	9-
6.8	17733-	16798-	15786-	14701-	13550-	12336-	11066-	9745-	8379-	6975-	9-
7.0	15921-	15863-	15728-	15517-	15230-	14869-	14436-	13933-	13362-	12726-	9-
7.2	9394-	9842-	10239-	10583-	10873-	11107-	11284-	11402-	11463-	11464-	9-
7.4	26450-	32145-	37666-	42982-	48064-	52886-	57420-	61642-	65529-	69061-	10-
7.6	18368	14030	9612	5138	635	3871-	8356-	12792-	17155-	21420-	10-
7.8	35121	33100	30878	28471	25891	23155	20279	17280	14177	10989	10-
8.0	30929	30924	30723	30327	29737	28959	27997	26857	25546	24073	10-
8.2	17182	18290	19275	20132	20854	21437	21877	22171	22316	22313	10-
8.4	3489	4793	6064	7292	8468	9586	10636	11612	12506	13313	10-
8.6	5020-	4103-	3156-	2185-	1199-	203-	794	1785	2763	3720	10-
8.8	75418-	71893-	67815-	63214-	58126-	52591-	46651-	40352-	33743-	26874-	11-
9.0	59198-	60141-	60599-	60569-	60050-	59049-	57571-	55631-	53242-	50425-	11-
9.2	27337-	30361-	33128-	35617-	37806-	39677-	41214-	42404-	43237-	43707-	11-
9.4	276	2690-	5632-	8525-	11343-	14061-	16655-	19104-	21384-	23476-	11-
9.6	14660	12913	11048	9081	7031	4917	2758	573	1617-	3792-	11-
9.8	16257	15884	15359	14687	13875	12931	11862	10681	9397	8024	11-
10.0	10430	10961	11382	11691	11883	11957	11912	11749	11468	11074	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^{x})$

$\nu \backslash x$	.39-	.38-	.37-	.36-	.35-	.34-	.33-	.32-	.31-	.30-	p
0.2	53475	53599	53725	53854	53985	54120	54255	54394	54536	54679	4-
0.4	23469	23534	23601	23668	23737	23807	23876	23947	24018	24091	4-
0.6	12635	12685	12736	12786	12837	12887	12938	12989	13040	13092	4-
0.8	70044	70487	70929	71370	71810	72248	72686	73123	73559	73994	5-
1.0	36993	37401	37807	38212	38614	39014	39412	39808	40203	40596	5-
1.2	16929	17296	17663	18027	18390	18751	19109	19467	19823	20177	5-
1.4	4980	5293	5606	5918	6229	6539	6848	7156	7463	7769	5-
1.6	1619-	1373-	1126-	878-	631-	383-	136-	112	359	607	5-
1.8	4673-	4498-	4322-	4145-	3967-	3788-	3608-	3427-	3244-	3061-	5-
2.0	54629-	53572-	52496-	51401-	50288-	49158-	48010-	46846-	45666-	44469-	6-
2.2	49383-	48909-	48414-	47897-	47360-	46802-	46224-	45626-	45008-	44371-	6-
2.4	37907-	37867-	37807-	37726-	37626-	37506-	37367-	37207-	37029-	36831-	6-
2.6	24959-	25190-	25404-	25602-	25785-	25951-	26101-	26234-	26352-	26452-	6-
2.8	13456-	13807-	14149-	14480-	14800-	15110-	15409-	15696-	15972-	16236-	6-
3.0	4820-	5178-	5532-	5880-	6224-	6562-	6895-	7222-	7543-	7857-	6-
3.2	628	338	47	244-	535-	826-	1115-	1403-	1690-	1976-	6-
3.4	32846	30933	28986	27006	24997	22960	20897	18812	16706	14581	7-
3.6	38988	38065	37094	36077	35015	33909	32761	31573	30345	29079	7-
3.8	32952	32807	32616	32380	32099	31773	31402	30988	30531	30032	7-
4.0	21831	22167	22469	22736	22968	23164	23324	23448	23536	23587	7-
4.2	10635	11165	11677	12169	12640	13089	13516	13919	14299	14655	7-
4.4	2118	2629	3135	3635	4128	4613	5089	5556	6013	6458	7-
4.6	28465-	24770-	21024-	17234-	13409-	9557-	5684-	1800-	2088	5972	8-
4.8	46522-	44591-	42560-	40433-	38215-	35911-	33525-	31064-	28534-	25939-	8-
5.0	43075-	42657-	42135-	41510-	40782-	39955-	39030-	38010-	36896-	35693-	8-
5.2	29165-	29715-	30185-	30575-	30883-	31110-	31254-	31316-	31294-	31190-	8-
5.4	13563-	14502-	15398-	16251-	17057-	17814-	18519-	19172-	19770-	20312-	8-
5.6	1479-	2367-	3247-	4117-	4975-	5817-	6641-	7445-	8226-	8982-	8-
5.8	5251	4651	4036	3407	2767	2118	1461	800	137	527-	8-
6.0	71609	69004	66155	63070	59761	56241	52520	48614	44535	40298	9-
6.2	59434	59513	59365	58993	58397	57579	56543	55293	53833	52169	9-
6.4	34413	35962	37364	38616	39711	40646	41416	42019	42452	42713	9-
6.6	10374	12252	14078	15842	17539	19160	20699	22149	23503	24756	9-
6.8	5539-	4078-	2598-	1106-	391	1886	3372	4843	6292	7712	9-
7.0	12029-	11273-	10463-	9602-	8694-	7744-	6756-	5736-	4688-	3618-	9-
7.2	11407-	11290-	11116-	10885-	10598-	10257-	9863-	9418-	8925-	8386-	9-
7.4	72218-	74983-	77341-	79279-	80787-	81857-	82483-	82662-	82393-	81677-	10-
7.6	25563-	29558-	33385-	37020-	40444-	43636-	46578-	49254-	51648-	53747-	10-
7.8	7734	4433	1105	2230-	5552-	8839-	12074-	15236-	18306-	21265-	10-
8.0	22447	20678	18779	16760	14634	12416	10119	7757	5347	2902	10-
8.2	22161	21861	21415	20826	20098	19236	18245	17134	15907	14575	10-
8.4	14026	14642	15154	15561	15858	16044	16119	16080	15928	15666	10-
8.6	4651	5547	6402	7210	7966	8663	9296	9861	10354	10771	10-
8.8	19798-	12570-	5245-	2120	9469	16745	23893	30856	37583	44021	11-
9.0	47201-	43598-	39644-	35371-	30814-	26008-	20993-	15810-	10499-	5104-	11-
9.2	43808-	43542-	42909-	41915-	40568-	38881-	36866-	34542-	31928-	29045-	11-
9.4	25363-	27026-	28452-	29628-	30544-	31192-	31566-	31663-	31482-	31025-	11-
9.6	5933-	8019-	10031-	11952-	13763-	15448-	16992-	18380-	19599-	20640-	11-
9.8	6574	5061	3500	1905	293	1323-	2926-	4501-	6033-	7507-	11-
10.0	10569	9959	9250	8449	7565	6605	5579	4498	3372	2213	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	.29-	.28-	.27-	.26-	.25-	.24-	.23-	.22-	.21-	.20-	p
0.2	54826	54976	55129	55285	55444	55606	55770	55939	56112	56286	4-
0.4	24164	24239	24315	24391	24469	24547	24628	24709	24791	24874	4-
0.6	13144	13196	13249	13301	13354	13407	13461	13516	13570	13625	4-
0.8	74429	74863	75296	75729	76162	76595	77027	77459	77891	78324	5-
1.0	40986	41375	41763	42148	42531	42913	43294	43673	44050	44425	5-
1.2	20529	20879	21228	21574	21918	22261	22603	22942	23279	23614	5-
1.4	8073	8377	8679	8981	9280	9580	9877	10173	10467	10761	5-
1.6	853	1101	1348	1594	1840	2086	2332	2577	2821	3065	5-
1.8	2877-	2693-	2507-	2321-	2135-	1948-	1760-	1572-	1383-	1194-	5-
2.0	43258-	42031-	40790-	39535-	38267-	36985-	35691-	34386-	33068-	31740-	6-
2.2	43715-	43040-	42347-	41635-	40906-	40160-	39397-	38617-	37821-	37009-	6-
2.4	36613-	36377-	36122-	35848-	35556-	35245-	34916-	34570-	34205-	33823-	6-
2.6	26537-	26605-	26657-	26692-	26710-	26712-	26698-	26668-	26621-	26558-	6-
2.8	16488-	16728-	16956-	17172-	17376-	17566-	17744-	17910-	18062-	18202-	6-
3.0	8165-	8466-	8760-	9046-	9325-	9596-	9859-	10114-	10360-	10598-	6-
3.2	2259-	2541-	2820-	3096-	3369-	3639-	3905-	4168-	4426-	4681-	6-
3.4	12441	10286	8121	5946	3765	1580	607-	2793-	4976-	7154-	7-
3.6	27777	26441	25072	23672	22243	20786	19304	17797	16269	14721	7-
3.8	29490	28908	28286	27624	26925	26188	25414	24606	23764	22889	7-
4.0	23602	23581	23523	23429	23299	23133	22931	22694	22422	22116	7-
4.2	14985	15290	15568	15820	16045	16243	16413	16556	16670	16755	7-
4.4	6891	7311	7717	8109	8486	8846	9191	9517	9827	10117	7-
4.6	9843	13694	17517	21304	25047	28739	32372	35938	39430	42842	8-
4.8	23285-	20580-	17828-	15036-	12211-	9358-	6484-	3595-	699-	2200	8-
5.0	34402-	33027-	31571-	30038-	28432-	26757-	25016-	23215-	21357-	19447-	8-
5.2	31003-	30733-	30383-	29952-	29442-	28854-	28190-	27451-	26640-	25759-	8-
5.4	20795-	21219-	21582-	21884-	22123-	22299-	22411-	22460-	22444-	22364-	8-
5.6	9710-	10408-	11074-	11706-	12301-	12859-	13378-	13855-	14290-	14681-	8-
5.8	1190-	1848-	2501-	3145-	3779-	4400-	5007-	5597-	6169-	6720-	8-
6.0	35918	31412	26793	22080	17289	12437	7541	2618	2315-	7239-	9-
6.2	50307	48255	46021	43612	41038	38309	35435	32427	29296	26055	9-
6.4	42802	42718	42462	42034	41436	40672	39743	38654	37410	36014	9-
6.6	25903	26938	27858	28658	29335	29886	30309	30602	30763	30792	9-
6.8	9097	10441	11736	12979	14162	15280	16329	17303	18198	19011	9-
7.0	2529-	1429-	322-	787	1893	2989	4070	5132	6169	7176	9-
7.2	7805-	7183-	6525-	5833-	5111-	4363-	3592-	2804-	2001-	1187-	9-
7.4	80519-	78924-	76902-	74464-	71622-	68393-	64794-	60844-	56565-	51980-	10-
7.6	55538-	57012-	58160-	58975-	59454-	59593-	59391-	58851-	57974-	56767-	10-
7.8	24097-	26783-	29308-	31657-	33814-	35769-	37508-	39021-	40299-	41335-	10-
8.0	440	2026-	4479-	6903-	9284-	11606-	13854-	16015-	18074-	20019-	10-
8.2	13146	11629	10034	8373	6656	4895	3101	1286	537-	2357-	10-
8.4	15293	14814	14231	13549	12771	11905	10955	9929	8833	7676	10-
8.6	11109	11365	11538	11626	11629	11547	11380	11130	10798	10388	10-
8.8	50120	55834	61120	65936	70246	74016	77218	79827	81824	83193	11-
9.0	332	5766	11153	16450	21616	26607	31385	35911	40149	44064	11-
9.2	25918-	22573-	19038-	15344-	11521-	7601-	3617-	397	4408	8382	11-
9.4	30296-	29301-	28048-	26550-	24819-	22870-	20721-	18390-	15898-	13266-	11-
9.6	21491-	22146-	22598-	22843-	22879-	22706-	22325-	21740-	20957-	19981-	11-
9.8	8910-	10228-	11448-	12559-	13550-	14413-	15137-	15718-	16148-	16425-	11-
10.0	1031	160-	1350-	2527-	3679-	4793-	5861-	6870-	7811-	8674-	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	.19-	.18-	.17-	.16-	.15-	.14-	.13-	.12-	.11-	.10-	p
0.2	56465	56648	56834	57024	57218	57416	57618	57825	58035	58250	4-
0.4	24958	25045	25132	25221	25311	25403	25496	25591	25687	25785	4-
0.6	13680	13736	13792	13848	13906	13964	14022	14080	14140	14200	4-
0.8	78757	79189	79623	80057	80492	80927	81364	81801	82240	82680	5-
1.0	44799	45172	45543	45914	46282	46650	47016	47381	47746	48109	5-
1.2	23948	24280	24610	24938	25264	25589	25911	26234	26553	26870	5-
1.4	11053	11344	11632	11921	12206	12491	12774	13056	13335	13614	5-
1.6	3308	3550	3793	4033	4274	4514	4753	4991	5228	5464	5-
1.8	1005-	815-	626-	436-	246-	56-	134	324	514	704	5-
2.0	30400-	29051-	27693-	26325-	24948-	23564-	22171-	20772-	19366-	17954-	6-
2.2	36182-	35340-	34483-	33612-	32727-	31828-	30917-	29993-	29057-	28109-	6-
2.4	33424-	33008-	32575-	32126-	31661-	31179-	30683-	30170-	29643-	29101-	6-
2.6	26478-	26383-	26271-	26144-	26001-	25843-	25668-	25479-	25274-	25054-	6-
2.8	18328-	18441-	18541-	18628-	18702-	18762-	18810-	18843-	18864-	18871-	6-
3.0	10826-	11046-	11257-	11458-	11650-	11832-	12004-	12167-	12320-	12462-	6-
3.2	4931-	5176-	5416-	5651-	5881-	6105-	6323-	6536-	6742-	6942-	6-
3.4	9324-	11484-	13631-	15764-	17879-	19976-	22050-	24101-	26126-	28123-	7-
3.6	13155	11573	9976	8367	6748	5121	3488	1850	210	1430-	7-
3.8	21982	21046	20080	19087	18068	17024	15956	14867	13757	12629	7-
4.0	21776	21403	20997	20559	20089	19589	19059	18499	17912	17298	7-
4.2	16813	16842	16842	16814	16757	16672	16559	16418	16250	16054	7-
4.4	10389	10642	10874	11086	11278	11449	11598	11726	11832	11916	7-
4.6	46165	49395	52523	55544	58451	61240	63903	66436	68833	71091	8-
4.8	5093	7975	10839	13678	16488	19260	21990	24670	27296	29861	8-
5.0	17489-	15489-	13452-	11382-	9284-	7164-	5026-	2876-	719-	1439	8-
5.2	24810-	23795-	22718-	21581-	20387-	19140-	17842-	16498-	15110-	13683-	8-
5.4	22221-	22015-	21745-	21414-	21022-	20571-	20061-	19494-	18871-	18196-	8-
5.6	15027-	15327-	15580-	15785-	15942-	16050-	16109-	16119-	16079-	15990-	8-
5.8	7249-	7755-	8234-	8687-	9111-	9505-	9868-	10199-	10496-	10758-	8-
6.0	12137-	16993-	21788-	26507-	31133-	35648-	40038-	44287-	48380-	52303-	9-
6.2	22716	19291	15793	12235	8631	4995	1340	2320-	5971-	9600-	9-
6.4	34474	32795	30983	29048	26995	24834	22573	20221	17788	15284	9-
6.6	30690	30456	30091	29598	28977	28233	27368	26386	25291	24088	9-
6.8	19736	20372	20916	21364	21715	21967	22119	22170	22121	21972	9-
7.0	8149	9082	9971	10812	11601	12335	13008	13619	14164	14641	9-
7.2	368-	453	1272	2085	2887	3674	4442	5188	5907	6596	9-
7.4	47114-	41993-	36645-	31100-	25386-	19536-	13580-	7550-	1480-	4598	10-
7.6	55235-	53389-	51238-	48795-	46073-	43090-	39860-	36404-	32740-	28890-	10-
7.8	42123-	42657-	42935-	42955-	42716-	42221-	41473-	40475-	39235-	37759-	10-
8.0	21837-	23516-	25047-	26420-	27626-	28656-	29506-	30169-	30642-	30921-	10-
8.2	4161-	5937-	7674-	9359-	10983-	12533-	14001-	15375-	16647-	17809-	10-
8.4	6465	5208	3916	2596	1258	89-	1435-	2771-	4088-	5376-	10-
8.6	9901	9341	8713	8022	7271	6468	5617	4725	3798	2844	10-
8.8	83923	84010	83452	82255	80427	77983	74940	71324	67162	62485	11-
9.0	47625	50804	53576	55918	57811	59240	60194	60666	60652	60151	11-
9.2	12285	16085	19751	23251	26556	29638	32472	35033	37301	39257	11-
9.4	10518-	7679-	4772-	1823-	1142	4097	7016	9874	12645	15305	11-
9.6	18823-	17493-	16003-	14367-	12599-	10717-	8736-	6676-	4555-	2392-	11-
9.8	16545-	16507-	16312-	15962-	15460-	14811-	14020-	13096-	12048-	10885-	11-
10.0	9452-	10135-	10718-	11195-	11560-	11811-	11945-	11961-	11857-	11637-	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^{x})$

ν \ x	.09-	.08-	.07-	.06-	.05-	.04-	.03-	.02-	.01-	.00	p
0.2	58471	58695	58924	59158	59397	59641	59892	60146	60407	60674	4-
0.4	25885	25986	26089	26195	26302	26410	26522	26635	26749	26867	4-
0.6	14261	14322	14384	14447	14510	14575	14640	14707	14773	14841	4-
0.8	83122	83565	84010	84457	84906	85357	85810	86267	86726	87188	5-
1.0	48471	48833	49194	49554	49914	50273	50633	50990	51349	51707	5-
1.2	27186	27501	27813	28124	28434	28742	29049	29354	29657	29960	5-
1.4	13891	14166	14440	14712	14983	15251	15519	15785	16049	16311	5-
1.6	5699	5934	6166	6399	6630	6860	7088	7316	7543	7768	5-
1.8	894	1083	1272	1461	1650	1838	2026	2213	2400	2585	5-
2.0	16536-	15113-	13685-	12252-	10816-	9377-	7935-	6490-	5043-	3595-	6-
2.2	27150-	26180-	25199-	24209-	23208-	22199-	21181-	20155-	19121-	18080-	6-
2.4	28545-	27975-	27391-	26793-	26183-	25560-	24924-	24277-	23617-	22947-	6-
2.6	24819-	24570-	24306-	24028-	23735-	23429-	23109-	22775-	22429-	22069-	6-
2.8	18865-	18846-	18814-	18768-	18710-	18638-	18554-	18456-	18346-	18224-	6-
3.0	12594-	12716-	12828-	12929-	13020-	13100-	13170-	13229-	13277-	13315-	6-
3.2	7135-	7322-	7501-	7674-	7840-	7998-	8148-	8292-	8427-	8555-	6-
3.4	30089-	32023-	33923-	35787-	37613-	39398-	41142-	42842-	44496-	46104-	7-
3.6	3069-	4704-	6333-	7954-	9566-	11166-	12753-	14325-	15879-	17415-	7-
3.8	11484	10323	9148	7960	6762	5554	4339	3118	1894	666	7-
4.0	16657	15991	15301	14587	13852	13096	12319	11525	10713	9885	7-
4.2	15831	15581	15305	15004	14677	14326	13951	13553	13132	12689	7-
4.4	11978	12018	12035	12031	12004	11954	11883	11790	11676	11540	7-
4.6	73204	75168	76979	78635	80131	81464	82633	83635	84468	85131	8-
4.8	32360	34787	37137	39404	41584	43672	45664	47554	49340	51017	8-
5.0	3594	5741	7873	9987	12076	14137	16163	18151	20094	21990	8-
5.2	12219-	10724-	9200-	7653-	6085-	4502-	2907-	1304-	302	1908	8-
5.4	17468-	16691-	15867-	14998-	14087-	13136-	12148-	11125-	10071-	8989-	8-
5.6	15853-	15667-	15433-	15153-	14826-	14454-	14038-	13579-	13079-	12539-	8-
5.8	10986-	11177-	11331-	11449-	11529-	11571-	11575-	11542-	11471-	11362-	8-
6.0	56042-	59584-	62917-	66028-	68908-	71546-	73933-	76062-	77923-	79512-	9-
6.2	13193-	16736-	20216-	23620-	26936-	30151-	33252-	36229-	39070-	41765-	9-
6.4	12718	10101	7444	4757	2051	663-	3375-	6073-	8747-	11385-	9-
6.6	22782	21379	19884	18305	16647	14919	13127	11279	9383	7447	9-
6.8	21723	21375	20930	20390	19758	19036	18228	17338	16369	15326	9-
7.0	15048	15382	15642	15827	15935	15967	15923	15802	15604	15333	9-
7.2	7251	7869	8448	8983	9472	9914	10305	10643	10927	11156	9-
7.4	10651	16647	22553	28338	33970	39419	44656	49653	54383	58821	10-
7.6	24875-	20719-	16445-	12078-	7641-	3162-	1335	5825	10282	14681	10-
7.8	36057-	34138-	32015-	29699-	27206-	24550-	21747-	18813-	15768-	12627-	10-
8.0	31005-	30893-	30586-	30086-	29397-	28522-	27467-	26239-	24846-	23296-	10-
8.2	18852-	19770-	20557-	21208-	21718-	22084-	22304-	22376-	22300-	22076-	10-
8.4	6627-	7832-	8982-	10069-	11087-	12027-	12883-	13650-	14322-	14895-	10-
8.6	1869	880	115-	1109-	2095-	3066-	4015-	4934-	5817-	6658-	10-
8.8	57329	51735	45744	39404	32762	25869	18779	11545	4223	3131-	11-
9.0	59168	57712	55793	53427	50634	47435	43857	39928	35680	31146	11-
9.2	40884	42168	43099	43670	43875	43713	43185	42296	41053	39467	11-
9.4	17832	20203	22397	24395	26180	27736	29050	30110	30907	31435	11-
9.6	207-	1979	4147	6278	8351	10349	12251	14043	15706	17226	11-
9.8	9618-	8260-	6823-	5321-	3769-	2181-	572-	1042	2646	4226	11-
10.0	11301-	10853-	10297-	9640-	8887-	8046-	7125-	6134-	5082-	3980-	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	p
0.2	60944	61221	61504	61794	62089	62392	62701	63017	63341	63671	4-
0.4	26986	27108	27232	27359	27488	27620	27754	27891	28031	28173	4-
0.6	14910	14979	15050	15122	15195	15269	15345	15421	15499	15577	4-
0.8	87653	88121	88593	89069	89548	90032	90520	91013	91511	92014	5-
1.0	52065	52424	52783	53143	53502	53863	54223	54586	54948	55313	5-
1.2	30261	30561	30860	31158	31454	31750	32045	32339	32631	32924	5-
1.4	16573	16833	17091	17347	17603	17857	18109	18360	18609	18857	5-
1.6	7992	8216	8437	8658	8877	9095	9312	9527	9741	9954	5-
1.8	2771	2956	3140	3324	3507	3689	3870	4051	4231	4410	5-
2.0	2146-	696-	754	2204	3654	5102	6549	7993	9436	10876	6-
2.2	17032-	15977-	14916-	13849-	12778-	11701-	10620-	9536-	8447-	7356-	6-
2.4	22265-	21573-	20871-	20159-	19438-	18707-	17968-	17221-	16466-	15703-	6-
2.6	21697-	21312-	20915-	20506-	20086-	19654-	19211-	18757-	18293-	17819-	6-
2.8	18089-	17942-	17782-	17611-	17427-	17232-	17025-	16807-	16578-	16338-	6-
3.0	13342-	13358-	13364-	13359-	13344-	13318-	13281-	13234-	13177-	13110-	6-
3.2	8674-	8786-	8890-	8986-	9073-	9152-	9223-	9285-	9339-	9384-	6-
3.4	47662-	49171-	50627-	52031-	53380-	54673-	55910-	57088-	58207-	59265-	7-
3.6	18929-	20421-	21888-	23330-	24744-	26128-	27482-	28803-	30090-	31342-	7-
3.8	562-	1790-	3016-	4237-	5452-	6661-	7860-	9049-	10226-	11390-	7-
4.0	9042	8186	7317	6437	5548	4651	3747	2837	1922	1005	7-
4.2	12225	11741	11237	10715	10175	9618	9045	8457	7856	7241	7-
4.4	11382	11204	11005	10787	10548	10290	10014	9719	9406	9077	7-
4.6	85622	85940	86086	86058	85858	85486	84943	84229	83348	82300	8-
4.8	52581	54030	55360	56568	57652	58610	59439	60138	60705	61140	8-
5.0	23833	25618	27343	29002	30591	32108	33548	34907	36184	37375	8-
5.2	3508	5099	6677	8238	9777	11291	12776	14227	15642	17017	8-
5.4	7882-	6752-	5603-	4439-	3262-	2076-	884-	310	1504	2693	8-
5.6	11961-	11347-	10699-	10018-	9307-	8567-	7802-	7013-	6203-	5375-	8-
5.8	11216-	11034-	10816-	10563-	10275-	9953-	9600-	9215-	8800-	8357-	8-
6.0	80823-	81851-	82593-	83046-	83210-	83083-	82667-	81962-	80973-	79702-	9-
6.2	44304-	46676-	48875-	50890-	52716-	54345-	55771-	56989-	57995-	58785-	9-
6.4	13979-	16516-	18988-	21384-	23695-	25911-	28024-	30026-	31908-	33663-	9-
6.6	5479	3488	1483	529-	2539-	4538-	6518-	8470-	10386-	12258-	9-
6.8	14213	13037	11801	10512	9176	7798	6386	4944	3480	2000	9-
7.0	14987	14569	14082	13528	12908	12227	11487	10692	9846	8953	9-
7.2	11328	11442	11499	11497	11436	11318	11142	10910	10623	10282	9-
7.4	62942	66725	70150	73199	75855	78104	79934	81336	82302	82828	10-
7.6	18996	23204	27279	31201	34946	38492	41821	44913	47751	50320	10-
7.8	9411-	6139-	2830-	495	3818	7118	10375	13571	16686	19701	10-
8.0	21599-	19767-	17810-	15741-	13573-	11320-	8996-	6615-	4193-	1744-	10-
8.2	21706-	21192-	20538-	19748-	18827-	17783-	16620-	15348-	13975-	12509-	10-
8.4	15363-	15725-	15978-	16120-	16150-	16068-	15874-	15570-	15158-	14641-	10-
8.6	7450-	8188-	8867-	9480-	10025-	10497-	10892-	11208-	11442-	11594-	10-
8.8	10461-	17712-	24827-	31753-	38437-	44827-	50875-	56536-	61766-	66525-	11-
9.0	26363	21370	16205	10912	5531	106	5320-	10703-	16001-	21172-	11-
9.2	37552	35322	32798	29999	26950	23676	20204	16564	12786	8901	11-
9.4	31688	31664	31364	30790	29948	28845	27490	25896	24076	22047	11-
9.6	18589	19783	20797	21621	22249	22674	22893	22904	22707	22303	11-
9.8	5765	7249	8665	9998	11236	12368	13383	14271	15023	15633	11-
10.0	2839-	1669-	483-	708	1892	3057	4192	5285	6327	7305	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20	p
0.2	64010	64356	64710	65073	65443	65822	66209	66606	67013	67428	4-
0.4	28320	28468	28621	28776	28935	29096	29263	29431	29605	29781	4-
0.6	15658	15739	15822	15907	15993	16081	16171	16262	16354	16449	4-
0.8	92522	93035	93555	94081	94613	95152	95698	96251	96812	97381	5-
1.0	55677	56043	56412	56781	57153	57526	57901	58279	58660	59043	5-
1.2	33215	33505	33795	34085	34374	34663	34952	35240	35528	35817	5-
1.4	19104	19349	19593	19835	20076	20317	20555	20793	21029	21265	5-
1.6	10165	10375	10584	10791	10998	11202	11406	11608	11809	12008	5-
1.8	4588	4765	4941	5116	5290	5464	5635	5806	5976	6145	5-
2.0	12313	13747	15177	16602	18023	19440	20851	22257	23658	25052	6-
2.2	6262-	5166-	4068-	2968-	1868-	766-	335	1436	2537	3637	6-
2.4	14933-	14156-	13373-	12584-	11789-	10990-	10185-	9376-	8563-	7746-	6-
2.6	17335-	16841-	16338-	15826-	15305-	14776-	14240-	13695-	13143-	12585-	6-
2.8	16087-	15825-	15553-	15270-	14978-	14676-	14365-	14044-	13714-	13376-	6-
3.0	13032-	12944-	12846-	12739-	12621-	12494-	12358-	12212-	12056-	11892-	6-
3.2	9421-	9450-	9470-	9481-	9484-	9479-	9465-	9443-	9413-	9374-	6-
3.4	60262-	61198-	62070-	62878-	63622-	64301-	64915-	65463-	65944-	66359-	7-
3.6	32557-	33734-	34871-	35968-	37023-	38035-	39003-	39926-	40803-	41633-	7-
3.8	12538-	13670-	14784-	15879-	16952-	18004-	19032-	20035-	21012-	21962-	7-
4.0	87	831-	1749-	2664-	3574-	4480-	5379-	6270-	7153-	8024-	7-
4.2	6615	5977	5330	4675	4011	3341	2666	1986	1303	618	7-
4.4	8731	8369	7992	7601	7195	6777	6347	5906	5453	4991	7-
4.6	81088	79714	78182	76495	74656	72669	70538	68268	65864	63329	8-
4.8	61442	61610	61644	61543	61309	60942	60443	59814	59055	58168	8-
5.0	38476	39487	40403	41224	41948	42572	43095	43517	43837	44053	8-
5.2	18347	19631	20863	22042	23164	24227	25228	26164	27033	27833	8-
5.4	3875	5046	6202	7342	8461	9557	10626	11665	12673	13645	8-
5.6	4530-	3671-	2802-	1924-	1040-	153-	734	1619	2499	3372	8-
5.8	7887-	7391-	6871-	6329-	5767-	5186-	4588-	3975-	3350-	2714-	8-
6.0	78153-	76333-	74248-	71905-	69313-	66481-	63419-	60138-	56649-	52964-	9-
6.2	59356-	59707-	59836-	59742-	59427-	58891-	58138-	57169-	55988-	54601-	9-
6.4	35285-	36767-	38102-	39287-	40316-	41185-	41891-	42432-	42805-	43009-	9-
6.6	14079-	15840-	17534-	19153-	20692-	22144-	23502-	24761-	25916-	26962-	9-
6.8	512	979-	2465-	3941-	5398-	6832-	8235-	9600-	10923-	12197-	9-
7.0	8018	7044	6036	5000	3939	2861	1768	667	437-	1539-	9-
7.2	9889	9445	8954	8418	7839	7221	6566	5879	5161	4418	9-
7.4	82911	82549	81747	80508	78839	76748	74249	71353	68077	64438	10-
7.6	52603	54590	56269	57630	58665	59370	59741	59775	59472	58835	10-
7.8	22599	25362	27974	30420	32685	34755	36619	38265	39684	40868	10-
8.0	715	3171	5606	8006	10356	12641	14847	16960	18967	20856	10-
8.2	10961-	9340-	7658-	5925-	4154-	2355-	540-	1278	3087	4877	10-
8.4	14023-	13307-	12500-	11606-	10631-	9583-	8469-	7296-	6073-	4807-	10-
8.6	11661-	11643-	11540-	11354-	11086-	10737-	10310-	9808-	9236-	8596-	10-
8.8	70777-	74490-	77637-	80193-	82138-	83459-	84146-	84193-	83601-	82374-	11-
9.0	26173-	30967-	35513-	39776-	43723-	47322-	50544-	53364-	55760-	57712-	11-
9.2	4942	941	3067-	7049-	10973-	14806-	18515-	22071-	25442-	28603-	11-
9.4	19825	17431	14885	12210	9428	6565	3644	692	2266-	5204-	11-
9.6	21697	20893	19900	18726	17382	15880	14234	12459	10572	8588	11-
9.8	16095	16404	16559	16556	16397	16082	15616	15001	14245	13355	11-
10.0	8212	9038	9775	10415	10953	11382	11700	11902	11987	11953	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^{x})$

ν \ x	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	p
0.2	67854	68290	68736	69192	69660	70139	70628	71131	71645	72172	4-
0.4	29963	30147	30337	30530	30728	30930	31138	31350	31567	31789	4-
0.6	16545	16643	16744	16847	16951	17058	17167	17278	17392	17508	4-
0.8	97958	98543	99138	99742	100356	100980	101615	102260	102917	103585	5-
1.0	59428	59818	60210	60607	61006	61410	61817	62230	62647	63069	5-
1.2	36106	36395	36684	36975	37265	37557	37849	38143	38437	38733	5-
1.4	21500	21733	21966	22199	22429	22660	22890	23119	23348	23576	5-
1.6	12206	12403	12599	12793	12987	13179	13370	13559	13748	13936	5-
1.8	6313	6480	6646	6811	6974	7137	7298	7459	7618	7776	5-
2.0	26440	27821	29196	30564	31924	33277	34622	35959	37288	38609	6-
2.2	4736	5833	6929	8022	9112	10200	11284	12365	13442	14515	6-
2.4	6926-	6103-	5277-	4450-	3620-	2789-	1957-	1124-	290-	543	6-
2.6	12019-	11448-	10870-	10287-	9699-	9105-	8507-	7905-	7298-	6688-	6-
2.8	13029-	12674-	12311-	11940-	11562-	11176-	10784-	10385-	9980-	9568-	6-
3.0	11719-	11537-	11346-	11147-	10940-	10724-	10501-	10270-	10032-	9787-	6-
3.2	9327-	9272-	9209-	9138-	9059-	8972-	8878-	8776-	8667-	8550-	6-
3.4	66707-	66988-	67202-	67349-	67429-	67443-	67389-	67270-	67084-	66833-	7-
3.6	42415-	43148-	43833-	44467-	45052-	45585-	46067-	46497-	46874-	47200-	7-
3.8	22883-	23775-	24636-	25466-	26263-	27026-	27755-	28448-	29106-	29726-	7-
4.0	8884-	9732-	10565-	11383-	12184-	12968-	13734-	14480-	15205-	15909-	7-
4.2	68-	754-	1438-	2121-	2800-	3474-	4143-	4805-	5460-	6106-	7-
4.4	4520	4041	3555	3063	2565	2062	1556	1047	537	25	7-
4.6	60671	57893	55001	52002	48901	45705	42420	39053	35609	32097	8-
4.8	57157	56022	54767	53395	51908	50310	48605	46796	44888	42884	8-
5.0	44166	44175	44080	43883	43582	43180	42677	42075	41376	40580	8-
5.2	28561	29217	29799	30304	30733	31083	31355	31547	31660	31693	8-
5.4	14579	15473	16325	17131	17891	18601	19260	19866	20417	20913	8-
5.6	4235	5085	5920	6737	7535	8310	9060	9783	10478	11141	8-
5.8	2069-	1417-	761-	103-	556	1213	1866	2513	3153	3782	8-
6.0	49096-	45059-	40867-	36535-	32076-	27508-	22845-	18104-	13301-	8453-	9-
6.2	53012-	51227-	49254-	47098-	44770-	42277-	39628-	36834-	33905-	30852-	9-
6.4	43043-	42908-	42604-	42131-	41496-	40693-	39732-	38615-	37348-	35933-	9-
6.6	27895-	28710-	29405-	29976-	30422-	30740-	30929-	30989-	30920-	30721-	9-
6.8	13417-	14576-	15671-	16695-	17645-	18516-	19305-	20008-	20622-	21144-	9-
7.0	2634-	3716-	4781-	5823-	6837-	7819-	8764-	9668-	10526-	11334-	9-
7.2	3652	2868	2070	1261	446	372-	1187-	1997-	2796-	3582-	9-
7.4	60457	56153	51551	46675	41551	36206	30669	24970	19138	13205	10-
7.6	57867	56574	54962	53043	50825	48322	45549	42521	39254	35768	10-
7.8	41809	42502	42944	43131	43063	42740	42164	41339	40270	38964	10-
8.0	22614	24231	25697	27002	28139	29101	29881	30476	30881	31094	10-
8.2	6634	8348	10007	11600	13117	14548	15884	17116	18236	19236	10-
8.4	3509-	2186-	849-	495	1835	3162	4468	5743	6978	8166	10-
8.6	7894-	7135-	6324-	5468-	4572-	3643-	2688-	1713-	726-	266	10-
8.8	80521-	78058-	75003-	71379-	67215-	62541-	57394-	51812-	45839-	39519-	11-
9.0	59206-	60230-	60776-	60839-	60420-	59521-	58151-	56320-	54043-	51338-	11-
9.2	31526-	34187-	36564-	38637-	40390-	41808-	42880-	43596-	43951-	43942-	11-
9.4	8098-	10921-	13649-	16259-	18729-	21035-	23160-	25084-	26791-	28266-	11-
9.6	6527	4406	2246	65	2116-	4278-	6402-	8468-	10457-	12351-	11-
9.8	12338	11205	9965	8632	7218	5735	4198	2622	1021	590-	11-
10.0	11803	11536	11156	10665	10071	9377	8591	7720	6774	5762	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

$\nu \backslash x$	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	p
0.2	72712	73264	73831	74411	75007	75617	76242	76882	77539	78213	4-
0.4	32016	32249	32488	32732	32983	33239	33502	33771	34047	34330	4-
0.6	17626	17749	17873	17999	18129	18262	18398	18537	18680	18826	4-
0.8	104266	104960	105667	106387	107122	107871	108635	109416	110213	111026	5-
1.0	63496	63927	64366	64810	65262	65719	66183	66654	67132	67619	5-
1.2	39031	39329	39630	39933	40238	40545	40855	41167	41483	41801	5-
1.4	23804	24032	24259	24487	24714	24943	25171	25399	25628	25857	5-
1.6	14123	14309	14494	14677	14860	15042	15224	15403	15584	15763	5-
1.8	7932	8088	8242	8396	8549	8700	8851	9001	9149	9297	5-
2.0	39921	41225	42521	43807	45085	46353	47613	48863	50105	51337	6-
2.2	15583	16647	17706	18760	19809	20851	21888	22919	23944	24963	6-
2.4	1377	2210	3042	3873	4702	5530	6355	7179	7999	8817	6-
2.6	6075-	5459-	4840-	4219-	3596-	2971-	2344-	1717-	1088-	460-	6-
2.8	9151-	8729-	8301-	7869-	7431-	6990-	6544-	6095-	5642-	5186-	6-
3.0	9534-	9275-	9009-	8737-	8458-	8174-	7884-	7588-	7288-	6982-	6-
3.2	8426-	8295-	8157-	8013-	7861-	7704-	7540-	7369-	7193-	7011-	6-
3.4	66516-	66135-	65691-	65183-	64612-	63980-	63287-	62535-	61723-	60853-	7-
3.6	47473-	47693-	47861-	47975-	48037-	48047-	48004-	47909-	47762-	47564-	7-
3.8	30309-	30854-	31361-	31828-	32255-	32643-	32989-	33296-	33561-	33785-	7-
4.0	16590-	17247-	17881-	18489-	19071-	19627-	20155-	20655-	21127-	21569-	7-
4.2	6743-	7368-	7983-	8584-	9173-	9746-	10305-	10848-	11374-	11882-	7-
4.4	486-	997-	1506-	2013-	2516-	3014-	3508-	3995-	4476-	4948-	7-
4.6	28523	24893	21216	17498	13746	9968	6171	2362	1451-	5262-	8-
4.8	40789	38608	36345	34006	31594	29117	26578	23984	21339	18650	8-
5.0	39690	38708	37637	36479	35237	33914	32513	31038	29492	27878	8-
5.2	31646	31519	31312	31028	30665	30226	29711	29123	28461	27729	8-
5.4	21352	21732	22053	22314	22515	22654	22732	22749	22703	22597	8-
5.6	11772	12368	12927	13448	13930	14371	14770	15126	15437	15704	8-
5.8	4399	5002	5590	6159	6709	7238	7744	8226	8681	9110	8-
6.0	3575-	1314	6199	11063	15890	20662	25364	29981	34496	38894	9-
6.2	27685-	24418-	21061-	17627-	14129-	10580-	6992-	3379-	246	3871	9-
6.4	34379-	32689-	30873-	28935-	26885-	24731-	22480-	20142-	17725-	15240-	9-
6.6	30395-	29941-	29363-	28663-	27844-	26908-	25862-	24708-	23452-	22098-	9-
6.8	21573-	21905-	22141-	22279-	22318-	22258-	22100-	21844-	21493-	21046-	9-
7.0	12089-	12787-	13425-	13999-	14508-	14948-	15319-	15617-	15843-	15994-	9-
7.2	4350-	5096-	5816-	6508-	7167-	7790-	8375-	8918-	9416-	9868-	9-
7.4	7202	1161	4886-	10907-	16871-	22745-	28500-	34105-	39531-	44749-	10-
7.6	32081	28216	24193	20035	15765	11408	6987	2527	1947-	6410-	10-
7.8	37427	35670	33703	31537	29186	26663	23983	21163	18218	15167	10-
8.0	31113	30940	30574	30019	29277	28354	27256	25989	24561	22982	10-
8.2	20111	20855	21462	21929	22253	22432	22465	22351	22092	21690	10-
8.4	9297	10365	11361	12280	13114	13859	14508	15058	15506	15847	10-
8.6	1257	2238	3203	4146	5058	5934	6767	7552	8282	8954	10-
8.8	32900-	26033-	18969-	11762-	4467-	2862	10170	17401	24501	31417	11-
9.0	48227-	44734-	40888-	36719-	32260-	27546-	22615-	17506-	12258-	6914-	11-
9.2	43570-	42837-	41749-	40317-	38552-	36468-	34084-	31418-	28494-	25334-	11-
9.4	29497-	30473-	31185-	31629-	31799-	31695-	31317-	30670-	29758-	28590-	11-
9.6	14135-	15790-	17304-	18661-	19850-	20860-	21683-	22310-	22737-	22959-	11-
9.8	2195-	3779-	5328-	6827	8262-	9619-	10886-	12050-	13102-	14030-	11-
10.0	4693	3577	2427	1253	67	1120-	2296-	3449-	4569-	5644-	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50	p
0.2	78903	79612	80338	81083	81847	82631	83436	84262	85110	85978	4-
0.4	34620	34916	35221	35534	35854	36183	36520	36867	37221	37585	4-
0.6	18975	19128	19284	19446	19610	19779	19951	20129	20311	20498	4-
0.8	11186	11271	11358	11446	11537	11630	11725	11823	11922	12024	4-
1.0	68114	68619	69130	69653	70185	70728	71281	71845	72420	73008	5-
1.2	42122	42447	42776	43109	43445	43787	44133	44484	44840	45202	5-
1.4	26087	26318	26550	26783	27016	27252	27490	27728	27968	28210	5-
1.6	15942	16121	16298	16476	16654	16831	17009	17186	17364	17540	5-
1.8	9443	9588	9733	9877	10020	10162	10304	10445	10585	10725	5-
2.0	52560	53774	54980	56176	57363	58541	59711	60872	62024	63169	6-
2.2	25975	26980	27979	28970	29955	30932	31902	32865	33821	34769	6-
2.4	9632	10443	11251	12055	12854	13650	14441	15228	16010	16786	6-
2.6	169	798	1426	2054	2681	3307	3932	4554	5175	5794	6-
2.8	4727-	4266-	3802-	3336-	2868-	2399-	1928-	1456-	984-	511-	6-
3.0	6672-	6357-	6038-	5715-	5388-	5058-	4724-	4387-	4048-	3706-	6-
3.2	6824-	6631-	6432-	6229-	6021-	5808-	5591-	5369-	5143-	4914-	6-
3.4	59927-	58945-	57908-	56819-	55677-	54484-	53243-	51953-	50617-	49236-	7-
3.6	47314-	47014-	46664-	46264-	45816-	45320-	44776-	44186-	43550-	42869-	7-
3.8	33968-	34109-	34210-	34268-	34286-	34263-	34198-	34093-	33948-	33763-	7-
4.0	21982-	22364-	22716-	23037-	23327-	23585-	23811-	24005-	24167-	24297-	7-
4.2	12372-	12843-	13294-	13725-	14135-	14524-	14891-	15235-	15556-	15854-	7-
4.4	5413-	5868-	6313-	6747-	7170-	7581-	7979-	8363-	8734-	9090-	7-
4.6	9063-	12846-	16606-	20334-	24024-	27669-	31262-	34798-	38268-	41669-	8-
4.8	15922	13162	10373	7564	4739	1904	935-	3772-	6601-	9417-	8-
5.0	26201	24465	22673	20830	18939	17006	15035	13030	10996	8937	8-
5.2	26928	26061	25129	24135	23081	21971	20807	19592	18330	17022	8-
5.4	22430	22203	21916	21570	21167	20707	20192	19623	19002	18331	8-
5.6	15925	16099	16227	16307	16341	16327	16266	16158	16004	15804	8-
5.8	9510	9879	10218	10526	10800	11040	11246	11418	11553	11654	8-
6.0	43161	47282	51244	55034	58639	62048	65248	68231	70986	73504	9-
6.2	7482	11065	14609	18100	21525	24874	28133	31291	34337	37260	9-
6.4	12696-	10103-	7471-	4810-	2130-	558	3243	5917	8567	11185	9-
6.6	20654-	19124-	17516-	15835-	14089-	12285-	10431-	8534-	6602-	4643-	9-
6.8	20507-	19878-	19161-	18361-	17480-	16522-	15493-	14395-	13235-	12017-	9-
7.0	16071-	16072-	15999-	15851-	15628-	15334-	14968-	14533-	14030-	13462-	9-
7.2	10271-	10623-	10923-	11168-	11359-	11493-	11570-	11591-	11555-	11461-	9-
7.4	49732-	54454-	58890-	63018-	66817-	70266-	73348-	76047-	78350-	80245-	10-
7.6	10838-	15206-	19489-	23664-	27709-	31600-	35317-	38839-	42146-	45222-	10-
7.8	12027	8816	5554	2259	1049-	4350-	7627-	10859-	14028-	17115-	10-
8.0	21260	19408	17436	15356	13182	10927	8604	6229	3816	1379	10-
8.2	21146	20466	19652	18711	17649	16472	15189	13808	12338	10788	10-
8.4	16080	16204	16217	16119	15912	15596	15174	14649	14025	13305	10-
8.6	9560	10099	10565	10956	11268	11499	11649	11715	11698	11598	10-
8.8	38097	44490	50550	56230	61489	66287	70588	74362	77579	80217	11-
9.0	1516-	3894	9274	14581	19773	24811	29653	34263	38605	42644	11-
9.2	21966-	18417-	14717-	10895-	6984-	3016-	977	4962	8907	12778	11-
9.4	27176-	25528-	23661-	21590-	19334-	16912-	14345-	11655-	8865-	5999-	11-
9.6	22975-	22784-	22389-	21793	21002-	20023-	18864-	17536-	16051-	14423-	11-
9.8	14827-	15485-	15998-	16362-	16572-	16627-	16527-	16273-	15867-	15312-	11-
10.0	6664-	7619-	8500-	9297	10004-	10614-	11120-	11518-	11803-	11974-	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	.51	.52	.53	.54	.55	.56	.57	.58	.59	.60	p
0.2	86871	87788	88728	89695	90688	91708	92754	93830	94935	96071	4-
0.4	37959	38343	38737	39142	39559	39985	40424	40874	41337	41813	4-
0.6	20689	20885	21086	21294	21506	21724	21948	22179	22415	22657	4-
0.8	12129	12236	12346	12459	12574	12693	12814	12939	13067	13199	4-
1.0	73608	74221	74848	75489	76145	76815	77503	78205	78926	79663	5-
1.2	45569	45943	46323	46709	47103	47504	47913	48329	48755	49188	5-
1.4	28455	28702	28951	29203	29458	29716	29977	30242	30510	30783	5-
1.6	17718	17897	18075	18255	18435	18615	18797	18980	19164	19349	5-
1.8	10864	11002	11140	11278	11415	11552	11689	11826	11963	12100	5-
2.0	64305	65433	66554	67667	68773	69872	70965	72051	73132	74207	6-
2.2	35710	36643	37570	38488	39400	40304	41201	42091	42974	43851	6-
2.4	17558	18325	19086	19841	20591	21336	22074	22807	23534	24255	6-
2.6	6411	7024	7635	8244	8849	9450	10049	10643	11234	11821	6-
2.8	38-	435	908	1381	1852	2323	2793	3262	3729	4194	6-
3.0	3361-	3015-	2666-	2316-	1965-	1612-	1259-	905-	550-	195-	6-
3.2	4680-	4443-	4203-	3960-	3714-	3465-	3214-	2960-	2704-	2447-	6-
3.4	47811-	46344-	44837-	43291-	41708-	40089-	38436-	36750-	35034-	33289-	7-
3.6	42144-	41376-	40567-	39716-	38826-	37897-	36931-	35928-	34890-	33818-	7-
3.8	33538-	33274-	32972-	32631-	32253-	31838-	31387-	30900-	30379-	29823-	7-
4.0	24395-	24460-	24494-	24494-	24464-	24401-	24307-	24181-	24024-	23837-	7-
4.2	16129-	16379-	16605-	16807-	16984-	17136-	17263-	17365-	17442-	17494-	7-
4.4	9431-	9756-	10065-	10358-	10633-	10892-	11132-	11354-	11558-	11744-	7-
4.6	44992-	48232-	51384-	54443-	57402-	60257-	63003-	65635-	68149-	70542-	8-
4.8	12214-	14985-	17727-	20433-	23098-	25716-	28284-	30795-	33244-	35629-	8-
5.0	6859	4765	2661	550	1561-	3669-	5769-	7857-	9927-	11976-	8-
5.2	15673	14286	12865	11412	9932	8429	6905	5364	3810	2248	8-
5.4	17611	16845	16035	15182	14290	13360	12395	11399	10372	9319	8-
5.6	15559	15269	14936	14560	14143	13685	13189	12656	12087	11484	8-
5.8	11718	11745	11737	11693	11612	11497	11346	11160	10940	10687	8-
6.0	75778	77800	79564	81064	82297	83258	83946	84357	84491	84349	9-
6.2	40050	42697	45193	47527	49693	51683	53491	55109	56533	57759	9-
6.4	13760	16282	18743	21132	23440	25660	27783	29800	31705	33491	9-
6.6	2665-	677-	1315	3301	5274	7225	9147	11032	12873	14662	9-
6.8	10747-	9430-	8073-	6680-	5258-	3814-	2353-	882-	592	2064	9-
7.0	12832-	12143-	11397-	10599-	9753-	8861-	7929-	6960-	5960-	4932-	9-
7.2	11312-	11107-	10847-	10535-	10171-	9757-	9296-	8789-	8240-	7651-	9-
7.4	81723-	82775-	83399-	83589-	83347-	82674-	81573-	80052-	78118-	75781-	10-
7.6	48049-	50612-	52897-	54892-	56586-	57970-	59038-	59783-	60202-	60293-	10-
7.8	20103-	22974-	25712-	28301-	30726-	32973-	35031-	36886-	38528-	39949-	10-
8.0	1066-	3505-	5922-	8303-	10634-	12900-	15087-	17182-	19173-	21047-	10-
8.2	9168	7489	5762	3998	2209	405	1402-	3199-	4976-	6721-	10-
8.4	12496	11601	10628	9583	8474	7307	6091	4833	3544	2230	10-
8.6	11415	11151	10807	10387	9893	9329	8698	8007	7258	6458	10-
8.8	82255	83680	84480	84652	84193	83108	81405	79097	76203	72743	11-
9.0	46350	49694	52649	55193	57307	58974	60182	60922	61188	60978	11-
9.2	16545	20176	23643	26917	29970	32779	35321	37576	39524	41151	11-
9.4	3081-	138-	2807	5728	8600	11398	14100	16681	19120	21397	11-
9.6	12665-	10795-	8828-	6783-	4677-	2529-	359-	1814	3971	6093	11-
9.8	14615-	13781-	12820-	11739-	10548-	9260-	7886-	6438-	4931-	3378-	11-
10.0	12029-	11967-	11788-	11495-	11091-	10579-	9965-	9254-	8453-	7571-	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^{x})$

ν \ x	.61	.62	.63	.64	.65	.66	.67	.68	.69	.70	p
0.2	97239	98439	99672	100941	102245	103586	104966	106387	107847	109349	4-
0.4	42302	42805	43321	43853	44399	44961	45539	46134	46746	47376	4-
0.6	22906	23164	23427	23698	23976	24264	24558	24862	25174	25495	4-
0.8	13334	13472	13615	13761	13912	14066	14225	14389	14557	14730	4-
1.0	80420	81196	81992	82809	83646	84507	85390	86298	87230	88188	5-
1.2	49632	50085	50548	51021	51506	52002	52510	53031	53564	54111	5-
1.4	31059	31340	31625	31917	32212	32514	32821	33134	33454	33781	5-
1.6	19536	19725	19914	20107	20301	20498	20697	20898	21103	21309	5-
1.8	12237	12373	12510	12648	12786	12925	13064	13204	13345	13487	5-
2.0	75277	76343	77404	78462	79517	80569	81619	82667	83715	84762	6-
2.2	44720	45583	46440	47291	48136	48975	49809	50638	51462	52281	6-
2.4	24970	25680	26363	27080	27772	28457	29137	29812	30481	31144	6-
2.6	12403	12982	13556	14126	14692	15253	15809	16361	16907	17450	6-
2.8	4658	5119	5578	6035	6489	6940	7389	7834	8277	8716	6-
3.0	160	515	870	1225	1578	1931	2283	2634	2983	3331	6-
3.2	2187-	1927-	1665-	1402-	1137-	873-	607-	341-	75-	191	6-
3.4	31516-	29717-	27895-	26050-	24184-	22299-	20397-	18479-	16547-	14602-	7-
3.6	32714-	31578-	30413-	29218-	27997-	26749-	25477-	24181-	22864-	21525-	7-
3.8	29235-	28614-	27961-	27279-	26566-	25825-	25056-	24261-	23440-	22595-	7-
4.0	23620-	23373-	23096-	22791-	22457-	22096-	21708-	21293-	20852-	20387-	7-
4.2	17521-	17523-	17500-	17452-	17380-	17284-	17163-	17019-	16851-	16661-	7-
4.4	11910-	12058-	12186-	12295-	12385-	12455-	12506-	12537-	12549-	12542-	7-
4.6	72808-	74945-	76950-	78819-	80549-	82139-	83585-	84887-	86042-	87050-	8-
4.8	37942-	40181-	42342-	44420-	46412-	48313-	50122-	51834-	53447-	54958-	8-
5.0	13998-	15991-	17949-	19869-	21746-	23576-	25357-	27084-	28753-	30363-	8-
5.2	680	889-	2456-	4018-	5570-	7109-	8632-	10134-	11613-	13065-	8-
5.4	8241	7143	6026	4893	3748	2594	1433	268	897-	2060-	8-
5.6	10849	10184	9490	8770	8025	7259	6472	5667	4847	4013	8-
5.8	10402	10086	9738	9362	8957	8526	8069	7588	7085	6560	8-
6.0	83932	83241	82278	81049	79558	77809	75809	73565	71085	68378	9-
6.2	58781	59598	60206	60604	60791	60766	60531	60086	59434	58577	9-
6.4	35150	36678	38068	39315	40416	41366	42162	42801	43282	43603	9-
6.6	16391	18055	19647	21160	22588	23927	25170	26313	27352	28282	9-
6.8	3528	4976	6403	7803	9169	10496	11779	13012	14189	15307	9-
7.0	3882-	2814-	1733-	644-	447	1537	2619	3690	4744	5777	9-
7.2	7024-	6364-	5673-	4954-	4212-	3449-	2670-	1878-	1076-	270-	9-
7.4	73055-	69954-	66494-	62694-	58573-	54152-	49456-	44508-	39333-	33959-	10-
7.6	60056-	59492-	58606-	57402-	55887-	54070-	51961-	49572-	46916-	44008-	10-
7.8	41140-	42095-	42809-	43277-	43497-	43469-	43194-	42671-	41906-	40903-	10-
8.0	22794-	24403-	25864-	27169-	28309-	29279-	30073-	30686-	31114-	31356-	10-
8.2	8423-	10072-	11656-	13166-	14593-	15926-	17159-	18282-	19291-	20176-	10-
8.4	902	433-	1764-	3084-	4384-	5654-	6886-	8072-	9205-	10276-	10-
8.6	5613	4728	3810	2865	1900	921	64-	1048-	2026-	2989-	10-
8.8	68745	64238	59256	53836	48019	41848	35369	28629	21679	14570	11-
9.0	60295	59144	57535	55481	52997	50104	46824	43183	39209	34933	11-
9.2	42443	43391	43987	44227	44108	43632	42804	41629	40119	38286	11-
9.4	23492	25388	27068	28518	29727	30685	31383	31816	31980	31875	11-
9.6	8161	10157	12062	13861	15536	17075	18463	19687	20739	21607	11-
9.8	1793-	193-	1410	3000	4562	6082	7545	8940	10252	11469	11-
10.0	6616-	5597-	4524-	3408	2258-	1087-	95	1275	2444	3589	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

$\nu \backslash x$	.71	.72	.73	.74	.75	.76	.77	.78	.79	.80	p
0.2	11090	11249	11413	11581	11756	11935	12119	12309	12506	12707	3-
0.4	48024	48692	49379	50086	50816	51567	52340	53138	53959	54806	4-
0.6	25825	26166	26517	26878	27250	27632	28028	28434	28854	29286	4-
0.8	14909	15092	15281	15475	15675	15881	16094	16313	16538	16771	4-
1.0	89174	90187	91230	92302	93407	94542	95713	96918	98159	99439	5-
1.2	54673	55248	55840	56448	57072	57713	58373	59050	59749	60466	5-
1.4	34114	34455	34804	35160	35525	35900	36284	36677	37081	37496	5-
1.6	21520	21734	21952	22173	22398	22627	22862	23101	23345	23594	5-
1.8	13629	13773	13919	14066	14215	14365	14518	14672	14828	14987	5-
2.0	85810	86859	87910	88963	90020	91082	92149	93221	94301	95389	6-
2.2	53096	53908	54717	55522	56325	57126	57925	58724	59522	60320	6-
2.4	31802	32455	33103	33746	34384	35018	35648	36274	36896	37516	6-
2.6	17987	18520	19048	19572	20090	20605	21114	21620	22121	22617	6-
2.8	9152	9585	10014	10440	10862	11280	11695	12106	12513	12917	6-
3.0	3677	4021	4364	4704	5042	5378	5712	6043	6372	6698	6-
3.2	457	722	988	1252	1516	1779	2041	2302	2562	2820	6-
3.4	12646-	10681-	8708-	6728-	4744-	2757-	767-	1222	3212	5198	7-
3.6	20168-	18793-	17402-	15995-	14575-	13142-	11699-	10245-	8784-	7315-	7-
3.8	21726-	20835-	19922-	18990-	18038-	17068-	16082-	15079-	14062-	13032-	7-
4.0	19897-	19383-	18847-	18288-	17709-	17109-	16489-	15851-	15195-	14522-	7-
4.2	16448-	16212-	15955-	15677-	15377-	15058-	14719-	14361-	13984-	13589-	7-
4.4	12515-	12470-	12405-	12322-	12221-	12101-	11963-	11808-	11636-	11446-	7-
4.6	87908-	88617-	89176-	89584-	89842-	89951-	89910-	89721-	89385-	88903-	8-
4.8	56365-	57665-	58857-	59938-	60908-	61765-	62508-	63135-	63648-	64045-	8-
5.0	31908-	33387-	34797-	36135-	37398-	38585-	39693-	40720-	41664-	42525-	8-
5.2	14488-	15877-	17230-	18543-	19815-	21042-	22222-	23352-	24430-	25454-	8-
5.4	3218-	4368-	5507-	6632-	7741-	8831-	9899-	10943-	11960-	12949-	8-
5.6	3169	2316	1456	593	272-	1136-	1997-	2853-	3701-	4539-	8-
5.8	6016	5454	4877	4285	3680	3065	2441	1810	1173	533	8-
6.0	65452	62318	58986	55466	51772	47914	43906	39761	35491	31111	9-
6.2	57519	56263	54815	53180	51364	49373	47216	44899	42430	39820	9-
6.4	43763	43762	43601	43279	42799	42164	41374	40435	39349	38122	9-
6.6	29101	29805	30392	30860	31207	31432	31534	31514	31372	31108	9-
6.8	16360	17345	18256	19091	19846	20519	21105	21605	22014	22333	9-
7.0	6784	7760	8702	9605	10465	11278	12041	12751	13405	13999	9-
7.2	537	1342	2141	2929	3704	4461	5197	5909	6592	7245	9-
7.4	28413-	22723-	16918-	11028-	5082-	889	6856	12788	18657	24432	10-
7.6	40864-	37500-	33936-	30191-	26284-	22238-	18073-	13812-	9478-	5094-	10-
7.8	39668-	38207-	36531-	34648-	32570-	30308-	27875-	25287-	22556-	19699-	10-
8.0	31410-	31276-	30955-	30449-	29762-	28898-	27863-	26662-	25303-	23794-	10-
8.2	20935-	21562-	22053-	22405-	22616-	22686-	22613-	22399-	22045-	21554-	10-
8.4	11279-	12207-	13054-	13815-	14486-	15058-	15532-	15904-	16171-	16333-	10-
8.6	3931-	4846-	5727-	6568-	7363-	8108-	8796-	9423-	9985-	10478-	10-
8.8	7355	85	7186-	14404-	21517-	28473-	35221-	41714-	47903-	53745-	11-
9.0	30389	25611	20637	15504	10253	4924	444-	5807-	11127-	16362-	11-
9.2	36144	33711	31008	28056	24878	21501	17953	14260	10454	6565	11-
9.4	31501	30862	29964	28813	27422	25800	23962	21924	19702	17317	11-
9.6	22285	22767	23049	23129	23006	22681	22158	21441	20536	19453	11-
9.8	12581	13578	14450	15190	15791	16247	16554	16711	16715	16567	11-
10.0	4700	5765	6776	7721	8593	9383	10084	10688	11191	11587	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^{x})$

ν \ x	.81	.82	.83	.84	.85	.86	.87	.88	.89	.90	p
0.2	12915	13130	13352	13581	13815	14059	14310	14569	14837	15113	3-
0.4	55680	56580	57510	58468	59457	60477	61530	62618	63741	64900	4-
0.6	29731	30191	30666	31155	31660	32181	32719	33274	33847	34440	4-
0.8	17011	17258	17513	17777	18048	18328	18618	18917	19225	19544	4-
1.0	00757	102116	103519	104965	106456	107996	109585	111226	112920	114670	5-
1.2	61207	61968	62753	63561	64394	65255	66141	67056	68000	68974	5-
1.4	37921	38360	38809	39272	39747	40237	40743	41261	41797	42350	5-
1.6	23849	24111	24378	24653	24934	25221	25517	25821	26133	26454	5-
1.8	15149	15313	15480	15651	15824	16001	16182	16367	1655[illegible]	16750	5-
2.0	96486	97593	98712	99843	100987	102147	103323	104516	105728	106960	6-
2.2	61118	61918	62720	63525	64333	65145	65962	66784	67613	68450	6-
2.4	38132	38745	39356	39965	40572	41179	41784	42389	42995	43601	6-
2.6	23110	23599	24084	24565	25043	25518	25989	26458	26925	27389	6-
2.8	13316	13713	14105	14494	14879	15261	15639	16014	16385	16754	6-
3.0	7021	7341	7659	7974	8286	8594	8900	9204	9504	9801	6-
3.2	3077	3332	3586	3837	4087	4335	4581	4825	5066	5306	6-
3.4	7182	9161	11134	13100	15058	17007	18945	20873	22788	24691	7-
3.6	5840-	4361-	2878-	1393-	93	1579	3064	4546	6026	7501	7-
3.8	11989-	10935-	9871-	8798-	7717-	6628-	5534-	4436-	3333-	2228-	7-
4.0	13832-	13128-	12410-	11678-	10934-	10178-	9412-	8636-	7851-	7058-	7-
4.2	13178-	12749-	12305-	11845-	11371-	10883-	10382-	9869-	9344-	8808-	7-
4.4	11240-	11019-	10781-	10528-	10261-	9979-	9684-	9375-	9054-	8720-	7-
4.6	88277-	87509-	86601-	85555-	84374-	83061-	81619-	80050-	78358-	76547-	8-
4.8	64326-	64491-	64541-	64476-	64297-	64005-	63602-	63088-	62466-	61737-	8-
5.0	43301-	43990-	44592-	45106-	45531-	45868-	46115-	46274-	46343-	46325-	8-
5.2	26422-	27331-	28181-	28969-	29695-	30357-	30953-	31484-	31947-	32343-	8-
5.4	13905-	14829-	15716-	16566-	17377-	18147-	18873-	19556-	20193-	20783-	8-
5.6	5366-	6178-	6974-	7752-	8510-	9246-	9958-	10644-	11303-	11934-	8-
5.8	108-	749-	1388-	2023-	2652-	3274-	3887-	4489-	5078-	5653-	8-
6.0	26634	22074	17447	12766	8046	3301	1454-	6204-	10935-	15634-	9-
6.2	37076	34208	31226	28141	24963	21702	18370	14978	11537	8059	9-
6.4	36757	35260	33637	31894	30037	28074	26011	23857	21619	19306	9-
6.6	30724	30222	29604	28872	28031	27082	26031	24881	23638	22305	9-
6.8	22559	22692	22732	22679	22534	22296	21969	21551	21047	20458	9-
7.0	14532	15001	15404	15740	16007	16205	16332	16389	16374	16290	9-
7.2	7864	8446	8988	9489	9945	10356	10719	11032	11294	11505	9-
7.4	30086	35591	40919	46044	50942	55590	59964	64044	67810	71246	10-
7.6	683-	3731	8126	12478	16765	20964	25054	29014	32823	36462	10-
7.8	16732-	13672-	10536-	7341-	4106-	848-	2415	5665	8883	12053	10-
8.0	22146-	20366-	18467-	16460-	14356-	12168-	9909-	7592-	5231-	2840-	10-
8.2	20928-	20172-	19292-	18291-	17178-	15959-	14641-	13233-	11744-	10183-	10-
8.4	16387-	16334-	16174-	15909-	15539-	15069-	14501-	13839-	13088-	12251-	10-
8.6	10899-	11245-	11513-	11702-	11811-	11840-	11787-	11654-	11442-	11152-	10-
8.8	59198-	64222-	68783-	72847-	76387-	79377-	81797-	83631-	84866-	85495-	11-
9.0	21473-	26420-	31168-	35679-	39921-	43861-	47471-	50724-	53596-	56065-	11-
9.2	2623	1339-	5291-	9201-	13038-	16772-	20374-	23815-	27068-	30108-	11-
9.4	14787	12133	9379	6547	3661	744	2178-	5083-	7946-	10743-	11-
9.6	18200	16789	15232	13542	11735	9826	7833	5771	3660	1518	11-
9.8	16267	15820	15230	14501	13641	12657	11559	10357	9061	7684	11-
10.0	11873	12047	12106	12051	11882	11601	11210	10713	10116	9424	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^x)$

$\nu \backslash x$	.91	.92	.93	.94	.95	.96	.97	.98	.99	1.00	p
0.2	15398	15693	15997	16312	16638	16975	17323	17682	18055	18441	3-
0.4	66098	67336	68614	69936	71303	72716	74178	75690	77255	78874	4-
0.6	35051	35684	36337	37013	37711	38434	39181	39954	40754	41582	4-
0.8	19874	20214	20566	20930	21306	21695	22097	22513	22944	23390	4-
1.0	11647	11834	12028	12226	12433	12647	12868	13096	13333	13577	5-
1.2	69981	71019	72095	73204	74351	75537	76764	78033	79347	80707	5-
1.4	42920	43507	44113	44739	45386	46054	46745	47458	48197	48961	5-
1.6	26784	27124	27475	27835	28207	28590	28985	29394	29815	30251	5-
1.8	16949	17151	17360	17574	17794	18021	18252	18492	18738	18992	5-
2.0	10821	10949	11080	11213	11348	11487	11629	11775	11924	12078	6-
2.2	69294	70148	71012	71888	72776	73677	74593	75524	76473	77440	6-
2.4	44209	44818	45430	46045	46663	47286	47913	48547	49187	49834	6-
2.6	27851	28312	28771	29229	29687	30145	30602	31061	31520	31981	6-
2.8	17120	17482	17842	18200	18555	18908	19259	19608	19956	20303	6-
3.0	10095	10387	10675	10961	11245	11525	11803	12079	12352	12624	6-
3.2	5543	5778	6011	6241	6469	6695	6919	7140	7359	7576	6-
3.4	26580	28455	30314	32159	33987	35798	37593	39370	41130	42872	7-
3.6	8971	10435	11891	13341	14781	16212	17634	19045	20444	21832	7-
3.8	1120-	12-	1096	2204	3310	4413	5512	6608	7699	8784	7-
4.0	6258-	5452-	4641-	3825-	3006-	2184-	1360-	535-	291	1117	7-
4.2	8262-	7706-	7141-	6569-	5989-	5403-	4811-	4213-	3612-	3006-	7-
4.4	8375-	8019-	7652-	7276-	6890-	6495-	6092-	5682-	5265-	4841-	7-
4.6	74621-	72582-	70436-	68186-	65836-	63391-	60855-	58233-	55529-	52748-	8-
4.8	60903-	59966-	58929-	57793-	56563-	55240-	53828-	52329-	50746-	49084-	8-
5.0	46218-	46025-	45745-	45381-	44932-	44402-	43791-	43100-	42333-	41491-	8-
5.2	32672-	32932-	33124-	33247-	33303-	33291-	33211-	33065-	32852-	32575-	8-
5.4	21325-	21818-	22261-	22653-	22995-	23284-	23522-	23707-	23840-	23921-	8-
5.6	12534-	13103-	13639-	14140-	14607-	15038-	15432-	15788-	16106-	16385-	8-
5.8	6213-	6756-	7280-	7784-	8267-	8728-	9166-	9579-	9966-	10328-	8-
6.0	20286-	24878-	29395-	33826-	38157-	42376-	46472-	50433-	54248-	57907-	9-
6.2	4555	1036	2487-	6001-	9497-	12962-	16386-	19759-	23070-	26309-	9-
6.4	16925	14485	11995	9463	6899	4311	1709	900-	3505-	6098-	9-
6.6	20889	19394	17828	16195	14503	12757	10963	9130	7264	5371	9-
6.8	19787	19037	18210	17311	16344	15311	14219	13070	11871	10625	9-
7.0	16135	15911	15619	15261	14838	14353	13806	13202	12543	11831	9-
7.2	11663	11768	11819	11816	11760	11651	11490	11276	11013	10700	9-
7.4	74335	77062	79417	81387	82966	84146	84922	85293	85258	84817	10-
7.6	39912	43157	46179	48965	51500	53772	55771	57486	58910	60036	10-
7.8	15157	18178	21100	23907	26584	29117	31493	33700	35726	37560	10-
8.0	432-	1978	4377	6751	9086	11368	13586	15727	17778	19728	10-
8.2	8559-	6884-	5166-	3417-	1648-	132	1911	3678	5423	7136	10-
8.4	11336-	10348-	9294-	8180-	7013-	5802-	4554-	3276-	1978-	668-	10-
8.6	10786-	10347-	9838-	9263-	8625-	7929-	7180-	6383-	5544-	4667-	10-
8.8	85513-	84923-	83729-	81940-	79571-	76639-	73167-	69179-	64705-	59776-	11-
9.0	58115-	59730-	60899-	61614-	61871-	61668-	61008-	59897-	58343-	56360-	11-
9.2	32912-	35458-	37726-	39699-	41362-	42704-	43713-	44383-	44709-	44689-	11-
9.4	13452-	16051-	18518-	20833-	22978-	24936-	26690-	28227-	29535-	30604-	11-
9.6	638-	2788-	4915-	6999-	9023-	10971-	12825-	14569-	16190-	17673-	11-
9.8	6237	4735	3190	1617	29	1559-	3134-	4680-	6185-	7635-	11-
10.0	8643	7781	6846	5847	4794	3696	2564	1407	238	934-	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^x)$

$\nu \backslash x$	1.01	1.02	1.03	1.04	1.05	1.06	1.07	1.08	1.09	1.10	p
0.2	18839	19253	19681	20123	20583	21058	21551	22063	22594	23144	3-
0.4	80550	82285	84082	85944	87874	89873	91947	94096	96326	98640	4-
0.6	42439	43327	44246	45198	46186	47209	48270	49370	50511	51695	4-
0.8	23852	24330	24826	25339	25871	26423	26994	27587	28203	28841	4-
1.0	13831	14094	14365	14647	14940	15242	15556	15881	16219	16569	5-
1.2	82115	83573	85083	86648	88270	89950	91694	93502	95377	97323	5-
1.4	49752	50570	51417	52295	53204	54146	55123	56135	57186	58276	5-
1.6	30702	31167	31649	32147	32662	33197	33749	34322	34917	35533	5-
1.8	19253	19524	19803	20091	20388	20696	21013	21343	21684	22037	5-
2.0	12235	12396	12562	12733	12909	13091	13278	13471	13670	13876	6-
2.2	78427	79435	80465	81520	82601	83709	84846	86014	87216	88452	6-
2.4	50490	51154	51828	52513	53210	53920	54643	55382	56137	56909	6-
2.6	32444	32909	33378	33850	34326	34807	35294	35787	36286	36793	6-
2.8	20649	20994	21339	21684	22029	22375	22722	23070	23421	23774	6-
3.0	12893	13160	13426	13690	13953	14214	14475	14735	14994	15253	6-
3.2	7791	8003	8214	8422	8629	8834	9037	9239	9439	9638	6-
3.4	44596	46302	47991	49661	51314	52949	54568	56169	57755	59324	7-
3.6	23208	24571	25921	27258	28582	29892	31188	32470	33738	34992	7-
3.8	9863	10936	12000	13057	14106	15146	16176	17197	18208	19209	7-
4.0	1941	2764	3585	4402	5217	6027	6832	7633	8427	9216	7-
4.2	2398-	1787-	1174-	560-	55	669	1283	1895	2506	3115	7-
4.4	4412-	3977-	3538-	3094-	2647-	2197-	1745-	1291-	835-	379-	7-
4.6	49894-	46972-	43987-	40943-	37845-	34698-	31507-	28276-	25009-	21712-	8-
4.8	47344-	45531-	43649-	41699-	39688-	37617-	35491-	33313-	31088-	28818-	8-
5.0	40575-	39589-	38534-	37413-	36228-	34982-	33679-	32319-	30907-	29445-	8-
5.2	32234-	31830-	31364-	30838-	30254-	29612-	28915-	28165-	27362-	26510-	8-
5.4	23950-	23927-	23853-	23728-	23552-	23328-	23054-	22733-	22366-	21953-	8-
5.6	16625-	16825-	16986-	17106-	17187-	17227-	17228-	17190-	17113-	16997-	8-
5.8	10662-	10969-	11247-	11496-	11715-	11905-	12064-	12194-	12292-	12361-	8-
6.0	61401-	64719-	67855-	70800-	73546-	76088-	78419-	80535-	82431-	84103-	9-
6.2	29466-	32532-	35498-	38356-	41097-	43714-	46199-	48545-	50747-	52800-	9-
6.4	8671-	11215-	13722-	16183-	18591-	20938-	23217-	25421-	27542-	29575-	9-
6.6	3459	1534	396-	2324-	4245-	6150-	8034-	9890-	11711-	13491-	9-
6.8	9338	8015	6661	5281	3881	2466	1042	385-	1812-	3232-	9-
7.0	11070	10263	9413	8525	7602	6648	5667	4663	3640	2603	9-
7.2	10340	9934	9484	8993	8462	7894	7292	6658	5996	5307	9-
7.4	83975	82737	81109	79101	76723	73988	70909	67503	63785	59775	10-
7.6	60861	61380	61591	61496	61094	60390	59387	58092	56512	54655	10-
7.8	39193	40618	41827	42815	43576	44108	44408	44476	44312	43918	10-
8.0	21566	23283	24869	26315	27614	28759	29744	30564	31214	31693	10-
8.2	8806	10424	11979	13463	14868	16185	17406	18526	19537	20434	10-
8.4	647	1957	3256	4534	5783	6997	8167	9286	10348	11346	10-
8.6	3759-	2827-	1875-	911-	58	1027	1990	2940	3870	4775	10-
8.8	54430-	48702-	42636-	36273-	29658-	22838-	15860-	8773-	1625-	5533	11-
9.0	53961-	51167-	47997-	44477-	40632-	36492-	32086-	27448-	22612-	17612-	11-
9.2	44324-	43617-	42576-	41207-	39523-	37537-	35264-	32723-	29934-	26918-	11-
9.4	31425-	31992-	32301-	32350-	32140-	31672-	30951-	29982-	28775-	27340-	11-
9.6	19007-	20180-	21183-	22007	22646-	23095-	23350-	23411-	23276-	22947-	11-
9.8	9017-	10320-	11531-	12641-	13640-	14519-	15270-	15888-	16368-	16704-	11-
10.0	2096-	3240-	4354-	5427-	6451-	7415-	8312-	9133-	9870-	10517-	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	1.11	1.12	1.13	1.14	1.15	1.16	1.17	1.18	1.19	1.20	p
0.2	23716	24309	24925	25564	26228	26920	27637	28384	29160	29968	3-
0.4	*101041	*103533	*106121	*108809	*111602	11450	11753	12066	12392	12732	3-
0.6	52924	54199	55523	56899	58329	59814	61358	62964	64634	66372	4-
0.8	29504	30191	30906	31647	32418	33219	34052	34918	35819	36756	4-
1.0	16934	17311	17704	18111	18535	18974	19432	19908	20402	20917	4-
1.2	*99343	*101440	*103617	*105881	*108231	11067	11321	11585	11860	12146	4-
1.4	59406	60580	61799	63066	64381	65749	67170	68647	70185	71783	5-
1.6	36172	36836	37524	38239	38981	39752	40553	41386	42253	43154	5-
1.8	22403	22781	23174	23582	24005	24444	24901	25375	25867	26380	5-
2.0	14089	14310	14538	14774	15019	15272	15535	15808	16092	16386	5-
2.2	89725	91038	92392	93790	95234	96727	98272	99871	101528	103245	6-
2.4	57699	58510	59341	60196	61074	61979	62910	63871	64863	65887	6-
2.6	37309	37834	38368	38914	39471	40041	40624	41223	41837	42468	6-
2.8	24129	24488	24851	25218	25590	25968	26351	26742	27139	27545	6-
3.0	15512	15772	16032	16293	16556	16820	17086	17354	17625	17899	6-
3.2	9835	10032	10228	10423	10618	10812	11007	11201	11396	11592	6-
3.4	60879	62420	63946	65461	66963	68455	69938	71412	72879	74341	7-
3.6	36233	37460	38673	39874	41062	42237	43401	44553	45696	46828	7-
3.8	20199	21179	22148	23107	24054	24991	25917	26833	27739	28634	7-
4.0	9998	10774	11542	12303	13057	13802	14540	15269	15990	16703	7-
4.2	3721	4324	4923	5519	6110	6696	7278	7854	8424	8989	7-
4.4	78	534	991	1446	1900	2352	2802	3249	3694	4135	7-
4.6	18388-	15042-	11679-	8302-	4916-	1524-	1870	5261	8648	12025	8-
4.8	26508-	24161-	21782-	19373-	16938-	14481-	12005-	9515-	7012-	4501-	8-
5.0	27936-	26382-	24788-	23155-	21487-	19787-	18058-	16302-	14522-	12722-	8-
5.2	25610-	24665-	23676-	22645-	21576-	20469-	19329-	18156-	16953-	15722-	8-
5.4	21496-	20995-	20454-	19872-	19252-	18595-	17903-	17177-	16419-	15631-	8-
5.6	16843-	16652-	16424-	16161-	15863-	15530-	15165-	14768-	14341-	13884-	8-
5.8	12399-	12406-	12384-	12332-	12250-	12139-	12000-	11833-	11639-	11418-	8-
6.0	85548-	86764-	87750-	88503-	89025-	89315-	89374-	89204-	88808-	88188-	9-
6.2	54696-	56433-	58007-	59412-	60648-	61711-	62599-	63312-	63849-	64209-	9-
6.4	31513-	33352-	35085-	36708-	38217-	39608-	40876-	42020-	43036-	43922-	9-
6.6	15225-	16907-	18531-	20092-	21585-	23006-	24350-	25613-	26792-	27883-	9-
6.8	4639-	6030-	7398-	8739-	10049-	11322-	12554-	13741-	14879-	15965-	9-
7.0	1555	501	555-	1609-	2656-	3693-	4716-	5721-	6704-	7661-	9-
7.2	4596	3866	3119	2359	1588	812	32	749-	1526-	2297-	9-
7.4	55492	50956	46188	41213	36051	30729	25269	19698	14039	8319	10-
7.6	52533	50155	47535	44685	41622	38360	34915	31305	27547	23660	10-
7.8	43296	42451	41388	40112	38632	36956	35092	33051	30844	28483	10-
8.0	31998	32127	32081	31860	31467	30904	30175	29283	28236	27038	10-
8.2	21212	21867	22395	22795	23063	23200	23204	23077	22818	22431	10-
8.4	12274	13127	13900	14588	15188	15695	16108	16424	16641	16759	10-
8.6	5650	6487	7283	8032	8730	9371	9953	10472	10924	11308	10-
8.8	12654	19689	26590	33310	39805	46032	51949	57519	62704	67471	11-
9.0	12486-	7270-	2002-	3280	8539	13737	18838	23806	28607	33207	11-
9.2	23699-	20301-	16750-	13073-	9299-	5456-	1571-	2324	6203	10036	11-
9.4	25687-	23831-	21788-	19572	17203-	14700-	12081-	9368-	6583-	3746-	11-
9.6	22427-	21722-	20837-	19780-	18561-	17188-	15675-	14034-	12279-	10423-	11-
9.8	16896-	16941-	16839-	16592-	16202-	15672-	15008-	14215-	13301-	12274-	11-
10.0	11069-	11520-	11867-	12107-	12238-	12258-	12169-	11970-	11665-	11256-	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^{x})$

ν \ x	1.21	1.22	1.23	1.24	1.25	1.26	1.27	1.28	1.29	1.30	p
0.2	30808	31684	32595	33544	34534	35565	36641	37763	38933	40155	3-
0.4	13085	13453	13836	14235	14651	15085	15537	16008	16500	17013	3-
0.6	68180	70063	72024	74066	76194	78412	80725	83137	85654	88281	4-
0.8	37731	38746	39804	40905	42053	43249	44496	45796	47154	48570	4-
1.0	21453	22010	22592	23197	23827	24484	25168	25883	26628	27406	4-
1.2	12444	12753	13075	13411	13761	14126	14506	14902	15317	15748	4-
1.4	73448	75180	76984	78863	80821	82861	84988	87206	89521	91935	5-
1.6	44092	45068	46085	47144	48246	49395	50594	51842	53146	54505	5-
1.8	26913	27467	28044	28645	29270	29922	30602	31310	32049	32819	5-
2.0	16692	17010	17341	17685	18043	18416	18804	19209	19631	20071	5-
2.2	*105026	*106874	*108794	*110788	*112860	11502	11726	11959	12202	12456	5-
2.4	66946	68042	69176	70352	71571	72837	74150	75516	76935	78412	6-
2.6	43117	43785	44474	45185	45920	46679	47465	48279	49123	49998	6-
2.8	27959	28383	28817	29263	29720	30191	30675	31174	31689	32221	6-
3.0	18177	18459	18746	19037	19334	19638	19948	20265	20591	20925	6-
3.2	11788	11986	12185	12386	12589	12794	13002	13214	13429	13648	6-
3.4	75799	77255	78709	80165	81624	83088	84560	86041	87533	89040	7-
3.6	47951	49066	50174	51275	52371	53463	54552	55640	56728	57817	7-
3.8	29520	30396	31263	32121	32972	33815	34651	35482	36306	37127	7-
4.0	17407	18104	18792	19472	20144	20808	21465	22115	22758	23394	7-
4.2	9548	10101	10647	11188	11721	12249	12770	13285	13793	14296	7-
4.4	4573	5008	5438	5864	6286	6703	7116	7524	7927	8325	7-
4.6	15390	18739	22071	25381	28669	31930	35164	38368	41540	44680	8-
4.8	1985-	534	3053	5567	8076	10576	13065	15540	17999	20441	8-
5.0	10904-	9071-	7226-	5372-	3510-	1644-	225	2093	3958	5819	8-
5.2	14467-	13188-	11890-	10572-	9239-	7892-	6534-	5166-	3791-	2411-	8-
5.4	14815-	13972-	13104-	12214-	11303-	10372-	9424-	8460-	7483-	6494-	8-
5.6	13398-	12886-	12348-	11786-	11201-	10594-	9968-	9323-	8660-	7982-	8-
5.8	11171-	10900-	10604-	10285-	9943-	9581-	9198-	8795-	8375-	7937-	8-
6.0	87348-	86292-	85024-	83549-	81874-	80004-	77945-	75704-	73289-	70707-	9-
6.2	64394-	64403-	64239-	63903-	63397-	62725-	61891-	60896-	59746-	58446-	9-
6.4	44677-	45299-	45787-	46142-	46363-	46450-	46405-	46229-	45923-	45490-	9-
6.6	28883-	29790-	30601-	31315-	31930-	32445-	32859-	33172-	33383-	33492-	9-
6.8	16994-	17964-	18871-	19713-	20488-	21193-	21826-	22386-	22872-	23282-	9-
7.0	8589-	9484-	10344-	11166-	11946-	12682-	13372-	14013-	14604-	15142-	9-
7.2	3058-	3807-	4541-	5257-	5951-	6622-	7267-	7884-	8470-	9023-	9-
7.4	2562	3206-	8960-	14676-	20328-	25895-	31351-	36676-	41848-	46845-	10-
7.6	19663	15574	11413	7199	2952	1308-	5562-	9791-	13976-	18099-	10-
7.8	25980	23349	20602	17754	14819	11812	8748	5641	2508	637-	10-
8.0	25698	24221	22618	20896	19066	17137	15119	13024	10862	8646	10-
8.2	21918	21283	20528	19660	18683	17603	16427	15161	13813	12390	10-
8.4	16778	16696	16517	16240	15868	15403	14849	14209	13487	12688	10-
8.6	11620	11859	12025	12115	12131	12071	11938	11731	11452	11104	10-
8.8	71791	75635	78981	81808	84100	85843	87028	87650	87707	87201	11-
9.0	37575	41680	45497	48998	52161	54965	57392	59428	61060	62277	11-
9.2	13795	17453	20984	24361	27561	30562	33343	35883	38167	40177	11-
9.4	881-	1992	4849	7669	10430	13112	15695	18159	20485	22659	11-
9.6	8483-	6475-	4414-	2318-	203-	1913	4014	6084	8105	10062	11-
9.8	11143-	9917-	8607-	7225-	5782-	4290-	2763-	1213-	348	1905	11-
10.0	10746-	10142-	9447-	8669	7815-	6892-	5909-	4875-	3798-	2688-	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

$\nu \backslash x$	1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.40	p
0.2	41430	42763	44155	45610	47131	48723	50388	52131	53956	55868	3-
0.4	17549	18109	18693	19304	19944	20612	21313	22044	22811	23614	3-
0.6	°91024	°93889	°96881	°100009	°103279	10670	11028	11402	11794	12205	3-
0.8	50048	51592	53206	54891	56654	58497	60425	62443	64555	66768	4-
1.0	28218	29066	29952	30878	31845	32856	33915	35022	36182	37396	4-
1.2	16199	16669	17161	17675	18212	18773	19360	19975	20617	21290	4-
1.4	°94456	°97087	°99836	°102707	°105710	10884	11213	11556	11916	12292	4-
1.6	55924	57405	58952	60569	62258	64024	65871	67803	69825	71942	5-
1.8	33623	34462	35339	36255	37212	38212	39257	40351	41496	42693	5-
2.0	20530	21009	21509	22031	22577	23147	23743	24366	25018	25700	5-
2.2	12720	12995	13283	13583	13896	14223	14565	14922	15296	15687	5-
2.4	79949	81549	83218	84957	86771	88664	90641	92706	94864	97121	6-
2.6	50907	51852	52834	53856	54920	56028	57184	58390	59649	60963	6-
2.8	32771	33340	33930	34542	35177	35837	36524	37238	37982	38757	6-
3.0	21268	21622	21986	22362	22751	23153	23569	24000	24448	24913	6-
3.2	13871	14099	14333	14572	14818	15070	15330	15599	15875	16161	6-
3.4	90564	92108	93675	95267	96887	98540	100229	101957	103728	105547	7-
3.6	58908	60005	61108	62219	63340	64473	65620	66785	67968	69172	7-
3.8	37944	38758	39570	40382	41195	42009	42826	43648	44476	45312	7-
4.0	24025	24650	25271	25887	26499	27109	27716	28322	28927	29533	7-
4.2	14792	15283	15768	16248	16722	17193	17659	18121	18580	19036	7-
4.4	8718	9107	9490	9869	10243	10612	10977	11338	11695	12048	7-
4.6	47785	50854	53887	56884	59842	62764	65647	68494	71304	74078	8-
4.8	22863	25264	27643	29997	32325	34627	36903	39150	41369	43559	8-
5.0	7673	9519	11354	13177	14986	16780	18558	20318	22059	23781	8-
5.2	1027-	358	1743	3125	4503	5876	7242	8600	9948	11285	8-
5.4	5494-	4487-	3472-	2452-	1429-	403-	623	1648	2671	3690	8-
5.6	7290-	6584-	5868-	5141-	4406-	3663-	2915-	2162-	1405-	646-	8-
5.8	7483-	7015-	6532-	6037-	5531-	5014-	4488-	3954-	3412-	2865-	8-
6.0	67965-	65072-	62036-	58865-	55569-	52155-	48632-	45010-	41297-	37501-	9-
6.2	57000-	55413-	53690-	51838-	49862-	47769-	45565-	43256-	40849-	38351-	9-
6.4	44931-	44251-	43451-	42535-	41507-	40371-	39131-	37791-	36357-	34831-	9-
6.6	33501-	33410-	33220-	32933-	32551-	32075-	31508-	30853-	30113-	29289-	9-
6.8	23615-	23872-	24052-	24156-	24182-	24133-	24009-	23812-	23541-	23200-	9-
7.0	15627-	16056-	16430-	16746-	17005-	17206-	17348-	17433-	17460-	17429-	9-
7.2	9541-	10023-	10468-	10873-	11237-	11560-	11841-	12078-	12272-	12422-	9-
7.4	51650-	56242-	60605-	64724-	68582-	72167-	75467-	78470-	81168-	83552-	10-
7.6	22142-	26088-	29919-	33620-	37177-	40574-	43798-	46838-	49681-	52317-	10-
7.8	3779-	6904-	9996-	13042-	16027-	18939-	21764-	24491-	27107-	29603-	10-
8.0	6385	4093	1780	542-	2861-	5166-	7446-	9690-	11887-	14028-	10-
8.2	10901	9353	7756	6117	4447	2753	1045	669-	2379-	4077-	10-
8.4	11816	10877	9877	8821	7716	6568	5384	4169	2933	1680	10-
8.6	10689	10210	9669	9071	8419	7717	6970	6182	5358	4504	10-
8.8	86138	84525	82377	79710	76541	72894	68793	64266	59343	54057	11-
9.0	63075	63448	63397	62922	62031	60729	59028	56940	54482	51670	11-
9.2	41903	43331	44453	45263	45756	45930	45785	45324	44551	43473	11-
9.4	24662	26482	28104	29519	30717	31688	32428	32932	33197	33222	11-
9.6	11940	13725	15402	16960	18386	19670	20803	21776	22583	23219	11-
9.8	3447	4961	6433	7854	9210	10492	11689	12791	13792	14682	11-
10.0	1555-	408-	742	1886	3013	4115	5183	6206	7177	8088	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	1.41	1.42	1.43	1.44	1.45	1.46	1.47	1.48	1.49	1.50	p
0.2	57872	59972	62176	64487	66913	69461	72137	74949	77906	81015	3-
0.4	24456	25338	26263	27234	28253	29322	30446	31627	32868	34174	3-
0.6	12635	13086	13560	14055	14577	15123	15698	16300	16935	17602	3-
0.8	69086	71516	74063	76735	79539	82483	85574	88821	92235	95824	4-
1.0	38668	40001	41398	42863	44401	46015	47710	49490	51360	53328	4-
1.2	21996	22735	23509	24322	25174	26068	27007	27994	29030	30119	4-
1.4	12686	13099	13531	13985	14462	14961	15485	16035	16613	17221	4-
1.6	74159	76480	78913	81464	84139	86946	89892	92984	96233	99646	5-
1.8	43948	45262	46638	48081	49594	51182	52847	54596	56432	58361	5-
2.0	26414	27162	27946	28767	29628	30532	31479	32474	33518	34615	5-
2.2	16096	16524	16973	17443	17936	18452	18994	19563	20160	20787	5-
2.4	*99481	*101951	*104536	*107244	*110082	11306	11617	11945	12288	12648	5-
2.6	62337	63773	65275	66846	68492	70216	72023	73917	75904	77990	6-
2.8	39566	40410	41292	42213	43176	44184	45240	46345	47504	48719	6-
3.0	25397	25901	26425	26972	27543	28139	28762	29413	30095	30809	6-
3.2	16457	16764	17083	17413	17757	18115	18488	18878	19284	19708	6-
3.4	*107417	*109343	*111330	*113383	*115507	11771	11999	12236	12482	12739	6-
3.6	70401	71657	72942	74260	75614	77008	78445	79928	81462	83051	7-
3.8	46156	47011	47879	48762	49660	50578	51516	52477	53464	54478	7-
4.0	30140	30749	31362	31979	32602	33232	33871	34519	35179	35852	7-
4.2	19490	19942	20393	20844	21295	21747	22202	22659	23119	23585	7-
4.4	12397	12743	13087	13427	13766	14103	14439	14775	15110	15446	7-
4.6	76817	79522	82196	84838	87453	90041	92606	95150	97677	100189	8-
4.8	45721	47854	49958	52035	54085	56108	58106	60080	62032	63963	8-
5.0	25482	27163	28822	30459	32075	33668	35240	36790	38320	39829	8-
5.2	12610	13923	15221	16505	17774	19027	20263	21484	22687	23874	8-
5.4	4705	5714	6716	7711	8696	9672	10638	11593	12536	13468	8-
5.6	113	873	1631	2387	3139	3888	4632	5369	6101	6825	8-
5.8	2313-	1757-	1198-	637-	75-	487	1049	1609	2166	2721	8-
6.0	33633-	29699-	25710-	21673-	17596-	13488-	9357-	5209-	1053-	3105	9-
6.2	35769-	33109-	30379-	27585-	24734-	21833-	18888-	15907-	12895-	9859-	9-
6.4	33221-	31530-	29764-	27928-	26027-	24068-	22054-	19991-	17885-	15742-	9-
6.6	28386-	27407-	26356-	25236-	24051-	22805-	21502-	20146-	18741-	17291-	9-
6.8	22790-	22313-	21770-	21165-	20500-	19778-	19000-	18171-	17293-	16368-	9-
7.0	17341-	17198-	17000-	16748-	16444-	16090-	15686-	15235-	14739-	14200-	9-
7.2	12529-	12591-	12610-	12585-	12518-	12408-	12257-	12066-	11835-	11567-	9-
7.4	85616-	87355-	88765-	89845-	90592-	91009-	91095-	90856-	90294-	89416-	10-
7.6	54737-	56932-	58897-	60624-	62109-	63349-	64340-	65081-	65572-	65814-	10-
7.8	31967-	34190-	36264-	38180-	39932-	41514-	42921-	44148-	45191-	46048-	10-
8.0	16101-	18099-	20012-	21831-	23550-	25161-	26658-	28035-	29286-	30408-	10-
8.2	5755-	7403-	9015-	10581-	12095-	13550-	14939-	16256-	17495-	18650-	10-
8.4	418	846-	2106-	3355-	4586-	5793-	6969-	8110-	9209-	10262-	10-
8.6	3623	2722	1805	878	54-	986-	1912-	2828-	3728-	4608-	10-
8.8	48441	42531	36364	29980	23417	16715	9915	3057	3819-	10673-	11-
9.0	48524	45067	41323	37315	33073	28622	23993	19215	14318	9334	11-
9.2	42098	40438	38505	36314	33880	31221	28355	25304	22087	18727	11-
9.4	33009	32559	31877	30970	29844	28509	26975	25254	23359	21303	11-
9.6	23679	23960	24062	23984	23728	23297	22694	21925	20996	19916	11-
9.8	15455	16106	16630	17023	17283	17409	17400	17257	16981	16576	11-
10.0	8932	9701	10390	10993	11506	11925	12246	12469	12591	12612	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^{x})$

ν \ x	1.51	1.52	1.53	1.54	1.55	1.56	1.57	1.58	1.59	1.60	p
0.2	•84287	•87732	•91359	•95182	•99211	•103459	•107943	•112675	11767	12295	2-
0.4	35547	36993	38516	40120	41811	43594	45475	47460	49557	51772	3-
0.6	18304	19043	19820	20639	21503	22413	23373	24387	25457	26588	3-
0.8	•99599	•103573	•107756	•112162	•116805	12170	12686	13231	13806	14414	3-
1.0	55396	57572	59864	62277	64819	67498	70324	73306	76452	79776	4-
1.2	31264	32469	33737	35073	36479	37961	39524	41172	42912	44749	4-
1.4	17860	18532	19239	19984	20768	21594	22465	23383	24352	25374	4-
1.6	•103234	•107007	•110976	11516	11955	12419	12907	13422	13965	14538	4-
1.8	60388	62519	64761	67121	69604	72220	74975	77880	80943	84174	5-
2.0	35768	36978	38253	39595	41005	42492	44057	45706	47445	49279	5-
2.2	21445	22138	22866	23631	24436	25284	26177	27118	28110	29155	5-
2.4	13027	13425	13843	14282	14745	15232	15744	16284	16853	17452	5-
2.6	80179	82478	84894	87434	90104	92914	95872	98986	102267	105725	6-
2.8	49993	51330	52734	54209	55759	57390	59106	60911	62812	64816	6-
3.0	31557	32341	33164	34027	34935	35888	36890	37944	39054	40222	6-
3.2	20152	20617	21103	21613	22148	22709	23299	23918	24569	25255	6-
3.4	13007	13286	13577	13882	14201	14535	14886	15253	15639	16044	6-
3.6	84700	86412	88192	90047	91980	93999	96109	98316	100628	103051	7-
3.8	55524	56603	57719	58874	60073	61318	62614	63964	65372	66843	7-
4.0	36540	37244	37966	38709	39473	40263	41079	41924	42801	43712	7-
4.2	24056	24535	25021	25516	26022	26540	27071	27617	28180	28760	7-
4.4	15783	16121	16463	16807	17156	17509	17868	18234	18607	18990	7-
4.6	•102690	•105185	•107677	•110170	•112671	11518	11771	12026	12284	12546	7-
4.8	65875	67770	69651	71520	73379	75233	77084	78935	80790	82654	8-
5.0	41318	42789	44243	45680	47102	48512	49909	51298	52679	54055	8-
5.2	25045	26199	27337	28459	29567	30660	31739	32806	33862	34908	8-
5.4	14387	15294	16188	17070	17939	18795	19640	20472	21293	22103	8-
5.6	7542	8250	8950	9641	10322	10994	11657	12310	12953	13587	8-
5.8	3272	3818	4360	4896	5427	5951	6469	6980	7484	7980	8-
6.0	7258	11400	15524	19626	23700	27742	31746	35709	39628	43500	9-
6.2	6804-	3738-	665-	2409	5480	8541	11589	14619	17627	20610	9-
6.4	13565-	11360-	9133-	6888-	4629-	2362-	91-	2181	4449	6709	9-
6.6	15801-	14274-	12715-	11128-	9516-	7884-	6236-	4575-	2904-	1228-	9-
6.8	15401-	14395-	13352-	12275-	11169-	10036-	8879-	7701-	6506-	5297-	9-
7.0	13619-	13000-	12345-	11655-	10933-	10182-	9405-	8603-	7779-	6935-	9-
7.2	11261-	10921-	10546-	10140-	9703-	9237-	8744-	8226-	7685-	7123-	9-
7.4	88227-	86737-	84953-	82885-	80543-	77939-	75085-	71994-	68678-	65151-	10-
7.6	65807-	65556-	65063-	64333-	63371-	62184-	60779-	59163-	57346-	55334-	10-
7.8	46718-	47199-	47492-	47598-	47517-	47253-	46809-	46187-	45394-	44434-	10-
8.0	31396-	32248-	32960-	33533-	33963-	34252-	34399-	34405-	34272-	34003-	10-
8.2	19718-	20694-	21574-	22355-	23034-	23610-	24080-	24444-	24701-	24852-	10-
8.4	11262-	12207-	13090-	13910-	14661-	15342-	15949-	16480-	16934-	17310-	10-
8.6	5466-	6290-	7082-	7838-	8552-	9223-	9846-	10419-	10939-	11405-	10-
8.8	17466-	24160-	30717-	37102-	43280-	49220-	54890-	60261-	65309-	70008-	11-
9.0	4293	774-	5837-	10865-	15828-	20699-	25449-	30051-	34481-	38715-	11-
9.2	15245	11666	8013	4309	578	3156-	6872-	10545-	14153-	17676-	11-
9.4	19101	16771	14327	11787	9169	6490	3768	1022	1731-	4473-	11-
9.6	18691	17333	15852	14258	12564	10782	8925	7007	5040	3040	11-
9.8	16045	15393	14625	13749	12770	11696	10537	9302	7998	6638	11-
10.0	12532	12353	12076	11704	11240	10689	10056	9345	8563	7716	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^{x})$

$\nu \backslash x$	1.61	1.62	1.63	1.64	1.65	1.66	1.67	1.68	1.69	1.70	p
0.2	12853	13443	14068	14729	15429	16171	16958	17792	18678	19619	2-
0.4	54113	56590	59209	61982	64919	68031	71329	74829	78543	82485	3-
0.6	27783	29046	30382	31798	33295	34882	36565	38349	40243	42253	3-
0.8	15056	15734	16452	17212	18016	18868	19771	20729	21745	22823	3-
1.0	*83287	*86998	*90923	*95075	*99472	*104126	*109060	*114290	11983	12573	3-
1.2	46689	48739	50906	53199	55626	58196	60918	63804	66864	70111	4-
1.4	26454	27595	28802	30076	31426	32855	34368	35971	37671	39474	4-
1.6	15144	15782	16457	17171	17926	18726	19572	20469	21419	22427	4-
1.8	*87585	*91187	*94993	*99015	*103267	*107767	*112530	11757	12292	12858	4-
2.0	51215	53258	55416	57696	60107	62657	65355	68212	71237	74444	5-
2.2	30258	31423	32652	33950	35323	36774	38308	39933	41653	43476	5-
2.4	18085	18752	19456	20200	20986	21817	22696	23625	24609	25651	5-
2.6	*109371	*113216	11727	12156	12608	13087	13592	14127	14693	15292	5-
2.8	66927	69152	71500	73980	76598	79362	82284	85372	88641	92099	6-
3.0	41453	42750	44119	45562	47085	48694	50394	52190	54090	56100	6-
3.2	25976	26736	27537	28381	29272	30212	31205	32254	33362	34535	6-
3.4	16470	16919	17391	17888	18412	18964	19548	20163	20814	21502	6-
3.6	*105594	*108264	*111070	11402	11713	12040	12385	12749	13133	13539	6-
3.8	68382	69993	71681	73453	75314	77271	79330	81498	83783	86193	7-
4.0	44661	45650	46683	47762	48893	50077	51320	52625	53998	55444	7-
4.2	29360	29982	30628	31299	31999	32728	33491	34289	35125	36002	7-
4.4	19382	19785	20200	20629	21072	21532	22010	22508	23026	23568	7-
4.6	12812	13083	13359	13643	13933	14232	14540	14858	15188	15530	7-
4.8	84529	86422	88336	90277	92250	94260	96314	98418	100579	102803	8-
5.0	55428	56802	58179	59562	60954	62359	63781	65223	66690	68186	8-
5.2	35945	36974	37998	39019	40037	41056	42077	43104	44138	45182	8-
5.4	22903	23693	24475	25249	26017	26779	27538	28294	29048	29804	8-
5.6	14211	14827	15434	16033	16624	17208	17786	18358	18925	19489	8-
5.8	8470	8952	9427	9895	10356	10810	11257	11699	12135	12565	8-
6.0	47322	51092	54809	58471	62078	65630	69127	72571	75962	79303	9-
6.2	23565	26488	29376	32229	35043	37817	40551	43244	45894	48503	9-
6.4	8958	11194	13411	15609	17785	19936	22060	24157	26223	28260	9-
6.6	450	2128	3801	5468	7126	8772	10405	12021	13619	15199	9-
6.8	4076-	2846-	1611-	372-	867	2105	3339	4567	5787	6997	9-
7.0	6075-	5200-	4313-	3416-	2512-	1601-	688-	227	1142	2054	9-
7.2	6541-	5941-	5326-	4697-	4057-	3406-	2747-	2081-	1411-	738-	9-
7.4	61427-	57520-	53446-	49218-	44851-	40360-	35760-	31065-	26289-	21447-	10-
7.6	53139-	50770-	48238-	45552-	42725-	39766-	36688-	33502-	30218-	26849-	10-
7.8	43313-	42036-	40611-	39045-	37345-	35519-	33576-	31523-	29368-	27122-	10-
8.0	33599-	33065-	32403-	31618-	30715-	29699-	28575-	27349-	26026-	24613-	10-
8.2	24897-	24836-	24672-	24407-	24042-	23581-	23026-	22382-	21652-	20839-	10-
8.4	17605-	17821-	17956-	18012-	17988-	17886-	17708-	17454-	17128-	16731-	10-
8.6	11815-	12167-	12460-	12693-	12866-	12979-	13032-	13025-	12960-	12836-	10-
8.8	74337-	78276-	81809-	84923-	87604-	89845-	91638-	92981-	93872-	94312-	11-
9.0	42731-	46507-	50026-	53270-	56224-	58875-	61213-	63227-	64912-	66262-	11-
9.2	21093-	24384-	27531-	30517-	33326-	35943-	38356-	40552-	42522-	44257-	11-
9.4	7186-	9854-	12460-	14988-	17424-	19753-	21963-	24040-	25974-	27755-	11-
9.6	1018	1010-	3031-	5033-	7001-	8924-	10790-	12587-	14303-	15931-	11-
9.8	5230	3784	2312	824	670-	2159-	3634-	5084-	6499-	7872-	11-
10.0	6810	5854	4854	3818	2754	1670	574	526-	1622-	2707-	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	1.71	1.72	1.73	1.74	1.75	1.76	1.77	1.78	1.79	1.80	p
0.2	20618	21680	22810	24012	25293	26657	28110	29661	31316	33083	2-
0.4	*86674	*91127	*95862	*100901	*106267	*111983	11808	12457	13151	13891	2-
0.6	44389	46658	49072	51639	54372	57285	60387	63697	67228	70999	3-
0.8	23969	25186	26480	27857	29323	30883	32547	34320	36212	38231	3-
1.0	13198	13862	14568	15319	16119	16970	17877	18843	19874	20975	3-
1.2	*73558	*77219	*81110	*85248	*89651	*94336	*99329	*104649	*110321	11637	3-
1.4	41388	43421	45579	47876	50317	52916	55682	58630	61772	65125	4-
1.6	23496	24632	25837	27118	28480	29930	31473	33116	34868	36736	4-
1.8	13460	14098	14774	15494	16258	17072	17937	18859	19841	20887	4-
2.0	*77844	*81450	*85278	*89345	*93665	*98257	*103144	*108344	*113882	11978	4-
2.2	45408	47456	49630	51938	54389	56994	59764	62711	65848	69190	5-
2.4	26755	27926	29168	30485	31885	33371	34951	36631	38419	40323	5-
2.6	15927	16599	17312	18069	18872	19725	20631	21594	22619	23709	5-
2.8	*95762	*99643	*103755	*108117	*112744	11766	12288	12842	13432	14059	5-
3.0	58228	60481	62869	65400	68085	70933	73958	77172	80587	84219	6-
3.2	35776	37089	38480	39954	41517	43176	44935	46804	48789	50900	6-
3.4	22229	22998	23813	24676	25590	26560	27589	28681	29841	31073	6-
3.6	13967	14421	14900	15408	15946	16516	17120	17761	18442	19164	6-
3.8	*88738	*91425	*94266	*97270	*100450	*103817	*107386	*111169	11518	11944	6-
4.0	56966	58572	60267	62056	63949	65950	68069	70314	72694	75218	7-
4.2	36924	37894	38915	39991	41126	42326	43593	44934	46354	47858	7-
4.4	24135	24728	25352	26006	26695	27420	28185	28993	29846	30749	7-
4.6	15886	16257	16644	17049	17473	17918	18386	18878	19396	19943	7-
4.8	*105098	*107473	*109935	*112494	11516	11794	12085	12389	12709	13045	7-
5.0	69717	71286	72899	74561	76280	78060	79909	81834	83842	85941	8-
5.2	46240	47313	48407	49523	50667	51841	53050	54299	55592	56934	8-
5.4	30561	31324	32092	32869	33657	34458	35275	36111	36969	37851	8-
5.6	20050	20609	21168	21728	22290	22857	23429	24008	24596	25195	8-
5.8	12991	13413	13832	14249	14663	15077	15491	15906	16324	16746	8-
6.0	82596	85844	89051	92221	95358	98468	101556	104628	107692	110755	9-
6.2	51071	53599	56088	58540	60957	63341	65696	68025	70332	72621	9-
6.4	30266	32240	34182	36094	37975	39826	41648	43444	45214	46961	9-
6.6	16757	18294	19808	21299	22767	24210	25630	27027	28401	29753	9-
6.8	8196	9382	10554	11711	12851	13975	15081	16170	17241	18294	9-
7.0	2962	3865	4762	5649	6528	7396	8253	9098	9931	10750	9-
7.2	63-	612	1286	1958	2625	3288	3945	4595	5237	5871	9-
7.4	16551-	11615-	6652-	1673-	3309	8284	13242	18173	23069	27921	10-
7.6	23405-	19897-	16337-	12734-	9099-	5441-	1770-	1904	5575	9233	10-
7.8	24791-	22386-	19914-	17385-	14806-	12186-	9533-	6855-	4159-	1453-	10-
8.0	23117-	21543-	19899-	18190-	16424-	14608-	12747-	10849-	8919-	6964-	10-
8.2	19949-	18986-	17955-	16859-	15706-	14498-	13242-	11943-	10606-	9235-	10-
8.4	16266-	15736-	15144-	14494-	13788-	13031-	12226-	11376-	10487-	9561-	10-
8.6	12656-	12422-	12134-	11795-	11407-	10973-	10494-	9974-	9416-	8821-	10-
8.8	94304-	93854-	92970-	91663-	89943-	87825-	85323-	82455-	79239-	75693-	11-
9.0	67273-	67946-	68280-	68279-	67946-	67288-	66313-	65029-	63447-	61579-	11-
9.2	45750-	46996-	47991-	48733-	49220-	49454-	49436-	49171-	48662-	47916-	11-
9.4	29375-	30824-	32097-	33189-	34096-	34814-	35343-	35681-	35829-	35790-	11-
9.6	17459-	18879-	20185-	21369-	22425-	23349-	24137-	24786-	25293-	25659-	11-
9.8	9192-	10453-	11646-	12763-	13800-	14750-	15609-	16371-	17033-	17594-	11-
10.0	3773-	4813-	5820-	6787-	7709-	8579-	9392-	10143-	10829-	11445-	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^{x})$

ν \ x	1.81	1.82	1.83	1.84	1.85	1.86	1.87	1.88	1.89	1.90	p
0.2	34971	36990	39149	41461	43936	46590	49436	52490	55769	59292	2-
0.4	14682	15528	16432	17400	18438	19549	20740	22019	23392	24867	2-
0.6	*75026	*79332	*83936	*88864	*94142	*99797	*105860	*112366	11935	12685	2-
0.8	40388	42694	45159	47797	50622	53648	56892	60372	64108	68121	3-
1.0	22150	23406	24748	26184	27721	29368	31133	33027	35058	37241	3-
1.2	12284	12974	13712	14501	15346	16250	17219	18260	19375	20572	3-
1.4	68703	72524	76607	80973	85645	90647	96004	101748	107908	114521	4-
1.6	38728	40856	43128	45558	48157	50937	53916	57107	60529	64202	4-
1.8	22004	23195	24467	25826	27280	28835	30500	32283	34194	36245	4-
2.0	12607	13278	13994	14760	15578	16452	17388	18390	19464	20615	4-
2.2	*72752	*76550	*80603	*84931	*89554	*94496	*99783	*105441	*111502	11800	4-
2.4	42351	44514	46820	49281	51909	54717	57720	60933	64372	68056	5-
2.6	24870	26108	27427	28834	30336	31941	33656	35489	37452	39553	5-
2.8	14727	15438	16196	17004	17867	18787	19771	20822	21947	23150	5-
3.0	*88083	*92197	*96580	*101251	*106232	*111548	11722	12329	12977	13671	5-
3.2	53145	55534	58077	60787	63676	66757	70045	73557	77309	81322	6-
3.4	32383	33777	35260	36840	38523	40317	42231	44275	46457	48790	6-
3.6	19933	20749	21618	22543	23529	24578	25698	26892	28168	29530	6-
3.8	12397	12878	13389	13934	14513	15130	15788	16489	17237	18036	6-
4.0	77897	80742	83766	86982	90403	94046	97925	102061	106472	111179	7-
4.2	49453	51145	52942	54852	56882	59042	61342	63792	66403	69188	7-
4.4	31705	32718	33792	34932	36143	37430	38799	40257	41809	43464	7-
4.6	20521	21132	21779	22465	23192	23964	24784	25656	26584	27572	7-
4.8	13398	13771	14165	14581	15021	15487	15982	16507	17065	17658	7-
5.0	88140	90448	92874	95429	98123	100970	103981	107169	110550	114139	8-
5.2	58330	59787	61309	62905	64579	66340	68196	70154	72224	74415	8-
5.4	38762	39704	40682	41699	42760	43869	45032	46253	47537	48892	8-
5.6	25808	26436	27082	27748	28437	29152	29896	30672	31484	32335	8-
5.8	17172	17605	18045	18495	18957	19431	19919	20425	20950	21496	8-
6.0	*113824	11691	12002	12316	12635	12959	13291	13630	13979	14339	8-
6.2	74896	77163	79428	81696	83973	86268	88587	90939	93332	95774	9-
6.4	48688	50397	52092	53776	55453	57127	58803	60486	62180	63893	9-
6.6	31085	32397	33691	34970	36235	37489	38734	39974	41211	42449	9-
6.8	19329	20347	21348	22334	23305	24263	25208	26143	27070	27990	9-
7.0	11556	12349	13128	13894	14647	15387	16116	16833	17540	18239	9-
7.2	6496	7112	7717	8313	8899	9474	10040	10596	11142	11680	9-
7.4	32722	37467	42149	46764	51309	55780	60176	64496	68740	72909	10-
7.6	12872	16485	20065	23608	27108	30561	33963	37313	40607	43845	10-
7.8	1257	3964	6662	9345	12008	14647	17258	19836	22379	24883	10-
8.0	4990-	3003-	1008-	990	2985	4973	6950	8912	10855	12776	10-
8.2	7835-	6413-	4972-	3516-	2051-	580-	892	2362	3826	5281	10-
8.4	8602-	7615-	6604-	5571-	4521-	3458-	2384-	1303-	219-	865	10-
8.6	8194-	7537-	6852-	6143-	5414-	4666-	3903-	3128-	2343-	1550-	10-
8.8	71838-	67695-	63286-	58632-	53757-	48683-	43433-	38030-	32495-	26851-	11-
9.0	59437-	57036-	54391-	51516-	48428-	45144-	41681-	38055-	34285-	30387-	11-
9.2	46941-	45743-	44332-	42719-	40913-	38926-	36771-	34459-	32004-	29418-	11-
9.4	35565-	35159-	34576-	33822-	32903-	31827-	30601-	29233-	27733-	26109-	11-
9.6	25881-	25962-	25903-	25705-	25373-	24910-	24319-	23608-	22779-	21841-	11-
9.8	18049-	18399-	18642-	18779-	18810-	18736-	18560-	18284-	17910-	17444-	11-
10.0	11989-	12458-	12850-	13163-	13397-	13551-	13626-	13622-	13540-	13382-	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99	2.00	p
0.2	63081	67157	71545	76273	81369	86869	92806	99220	106155	113657	2-
0.4	26453	28159	29996	31975	34108	36409	38895	41578	44479	47619	2-
0.6	13492	14359	15293	16299	17384	18554	19817	21180	22655	24250	2-
0.8	*72434	*77073	*82066	*87444	*93239	*99490	*106236	*113522	12140	12991	2-
1.0	39585	42107	44820	47742	50890	54284	57948	61903	66178	70800	3-
1.2	21858	23241	24729	26331	28057	29916	31924	34090	36430	38961	3-
1.4	12162	12925	13747	14630	15582	16607	17713	18908	20196	21590	3-
1.6	*68145	*72381	*76936	*81837	*87113	*92797	*98926	*105538	11268	12039	3-
1.8	38446	40809	43350	46082	49024	52191	55604	59285	63259	67550	4-
2.0	21849	23175	24600	26131	27777	29550	31461	33519	35740	38137	4-
2.2	12496	13244	14047	14909	15837	16836	17911	19069	20319	21667	4-
2.4	*72006	*76243	*80790	*85675	*90926	*96574	*102653	*109201	11626	12387	4-
2.6	41804	44218	46807	49588	52575	55786	59241	62961	66968	71289	5-
2.8	24439	25820	27302	28892	30599	32435	34407	36532	38819	41283	5-
3.0	14413	15208	16060	16974	17955	19009	20141	21359	22670	24082	5-
3.2	*85614	*90209	*95133	*100410	*106071	*112147	11867	12569	13324	14136	5-
3.4	51284	53952	56810	59872	63154	66676	70456	74518	78885	83583	6-
3.6	30985	32542	34209	35993	37905	39956	42156	44518	47057	49786	6-
3.8	18890	19803	20779	21824	22943	24142	25428	26808	28291	29884	6-
4.0	11621	12158	12732	13346	14004	14708	15463	16273	17142	18076	6-
4.2	72161	75335	78727	82354	86235	90390	94841	99613	104732	110226	7-
4.4	45229	47113	49125	51275	53574	56034	58668	61491	64517	67763	7-
4.6	28625	29748	30946	32226	33594	35057	36623	38299	40095	42021	7-
4.8	18290	18963	19680	20446	21263	22137	23071	24071	25141	26288	7-
5.0	11795	12201	12633	13093	13584	14109	14669	15268	15909	16595	7-
5.2	76738	79203	81822	84609	87577	90741	94116	97722	101576	105699	8-
5.4	50322	51835	53438	55139	56946	58869	60917	63100	65431	67921	8-
5.6	33229	34170	35164	36213	37325	38504	39757	41089	42508	44021	8-
5.8	22066	22662	23288	23946	24639	25371	26145	26966	27838	28766	8-
6.0	14711	15097	15500	15920	16360	16822	17308	17821	18363	18937	8-
6.2	*98277	*100851	*103505	*106254	*109109	*112083	11519	11845	12187	12548	8-
6.4	65629	67396	69201	71050	72953	74917	76953	79068	81275	83584	9-
6.6	43692	44944	46210	47493	48800	50135	51504	52915	54372	55883	9-
6.8	28906	29819	30734	31653	32578	33514	34464	35431	36421	37437	9-
7.0	18929	19613	20292	20968	21643	22318	22997	23682	24374	25078	9-
7.2	12209	12730	13245	13754	14258	14759	15258	15756	16255	16757	9-
7.4	77004	81029	84987	88884	92724	96515	100264	103979	107670	111349	10-
7.6	47025	50149	53216	56228	59188	62099	64964	67788	70576	73334	10-
7.8	27348	29770	32151	34488	36783	39035	41247	43420	45557	47660	10-
8.0	14672	16542	18383	20193	21972	23719	25433	27116	28767	30389	10-
8.2	6725	8154	9566	10959	12331	13682	15010	16314	17593	18849	10-
8.4	1948	3025	4096	5157	6206	7243	8265	9271	10260	11232	10-
8.6	754-	45	844	1641	2433	3219	3998	4768	5528	6276	10-
8.8	21120-	15320-	9474-	3599-	2287	8166	14024	19845	25616	31327	11-
9.0	26379-	22277-	18099-	13860-	9575-	5261-	930-	3404	7728	12031	11-
9.2	26715-	23907-	21008-	18031-	14989-	11894-	8758-	5593-	2410-	781	11-
9.4	24371-	22530-	20593-	18573-	16478-	14320-	12107-	9849-	7555-	5236-	11-
9.6	20800-	19661-	18433-	17122-	15737-	14284-	12771-	11207-	9598-	7952-	11-
9.8	16887-	16246-	15524-	14726-	13859-	12927-	11936-	10892-	9800-	8667-	11-
10.0	13150-	12846-	12474-	12036-	11537-	10979-	10366-	9703-	8994-	8243-	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	2.01	2.02	2.03	2.04	2.05	2.06	2.07	2.08	2.09	2.10	p
0.2	12177	13058	14013	15048	16172	17395	18724	20171	21747	23465	1-
0.4	51017	54700	58691	63023	67727	72839	78398	84449	91040	98225	2-
0.6	25977	27847	29875	32075	34464	37059	39882	42953	46299	49946	2-
0.8	13913	14912	15994	17168	18443	19827	21333	22972	24756	26700	2-
1.0	*75801	*81218	*87089	*93455	*100366	*107874	11604	12491	13457	14511	2-
1.2	41698	44663	47875	51357	55136	59240	63700	68552	73833	79587	3-
1.4	23098	24729	26496	28412	30490	32746	35198	37865	40766	43926	3-
1.6	12873	13776	14753	15813	16960	18208	19562	21035	22636	24380	3-
1.8	*72190	*77208	*82639	*88524	*94904	*101826	*109342	11751	12639	13607	3-
2.0	40727	43528	46558	49837	53391	57247	61429	65972	70910	76282	4-
2.2	23124	24698	26400	28243	30240	32404	34752	37302	40073	43087	4-
2.4	13209	14097	15057	16096	17220	18439	19761	21195	22753	24447	4-
2.6	*75951	*80986	*86426	*92309	*98675	*105571	*113046	12115	12996	13952	4-
2.8	43942	46811	49911	53262	56887	60812	65065	69677	74682	80119	5-
3.0	25603	27244	29016	30930	32999	35237	37661	40288	43137	46229	5-
3.2	15011	15955	16973	18072	19259	20543	21932	23437	25068	26837	5-
3.4	*88641	*94090	*99966	*106306	11315	12055	12855	13721	14659	15677	5-
3.6	52722	55885	59292	62967	66932	71215	75844	80852	86273	92147	6-
3.8	31596	33440	35425	37565	39872	42363	45053	47962	51109	54516	6-
4.0	19079	20158	21320	22571	23919	25374	26944	28640	30475	32460	6-
4.2	11613	12247	12930	13664	14456	15309	16230	17223	18297	19459	6-
4.4	*71249	*74993	*79019	*83350	*88013	*93036	*98453	*104298	*110609	11743	6-
4.6	44088	46307	48691	51255	54013	56984	60184	63636	67360	71383	7-
4.8	27519	28839	30256	31780	33418	35181	37080	39126	41332	43714	7-
5.0	17330	18119	18965	19874	20851	21901	23032	24250	25562	26978	7-
5.2	*110113	11484	11992	12536	13121	13749	14425	15152	15936	16781	7-
5.4	70584	73434	76487	79761	83275	87048	91104	95466	100161	105220	8-
5.6	45637	47363	49211	51189	53310	55586	58029	60655	63479	66520	8-
5.8	29753	30806	31931	33133	34420	35799	37278	38866	40572	42607	8-
6.0	19547	20195	20885	21621	22407	23248	24148	25113	26149	27261	8-
6.2	12929	13333	13761	14215	14700	15216	15768	16358	16990	17668	8-
6.4	*86006	*88556	*91246	*94092	*97109	*100314	*103726	*107365	*111252	11541	8-
6.6	57456	59099	60820	62630	64536	66551	68686	70954	73367	75940	9-
6.8	38485	39569	40695	41868	43096	44385	45741	47174	48690	50300	9-
7.0	25795	26530	27286	28067	28875	29717	30596	31517	32486	33507	9-
7.2	17263	17776	18298	18831	19378	19942	20525	21130	21761	22422	9-
7.4	11502	11871	12242	12618	12998	13386	13783	14191	14613	15050	9-
7.6	76069	78788	81500	84214	86940	89688	92470	95300	98189	101153	10-
7.8	49734	51782	53810	55822	57825	59825	61829	63845	65882	67950	10-
8.0	31982	33548	35090	36611	38115	39604	41084	42559	44034	45516	10-
8.2	20081	21290	22476	23642	24789	25918	27033	28136	29230	30319	10-
8.4	12186	13121	14039	14939	15823	16690	17542	18380	19207	20023	10-
8.6	7012	7736	8446	9143	9826	10496	11153	11797	12430	13052	10-
8.8	36966	42524	47996	53374	58654	63834	68913	73891	78771	83556	11-
9.0	16302	20532	24712	28836	32897	36890	40811	44658	48430	52126	11-
9.2	3969	7146	10303	13433	16529	19584	22594	25555	28464	31318	11-
9.4	2898-	551-	1797	4140	6471	8785	11074	13336	15565	17758	11-
9.6	6276-	4577-	2862-	1137-	592	2319	4038	5746	7438	9110	11-
9.8	7497-	6298-	5073-	3830-	2572-	1305-	33-	1238	2506	3766	11-
10.0	7455-	6634-	5784-	4910-	4016-	3105-	2183-	1253-	318-	618	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^{x})$

ν \ x	2.11	2.12	2.13	2.14	2.15	2.16	2.17	2.18	2.19	2.20	p
0.2	25341	27388	29626	32073	34752	37687	40905	44439	48319	52585	1-
0.4	*106064	11462	12398	13421	14540	15768	17113	18590	20211	21994	1-
0.6	53925	58269	63015	68205	73887	80111	86935	94423	102647	111688	2-
0.8	28821	31137	33667	36433	39460	42776	46411	50400	54780	59594	2-
1.0	15660	16914	18283	19781	21419	23214	25180	27337	29706	32310	2-
1.2	*85860	*92706	*100182	*108354	11730	12708	13781	14957	16249	17668	2-
1.4	47371	51130	55232	59716	64620	69988	75869	82317	89395	97168	3-
1.6	26281	28353	30616	33087	35790	38746	41986	45536	49431	53709	3-
1.8	14660	15809	17062	18430	19925	21562	23353	25317	27471	29834	3-
2.0	*82129	*88500	*95450	*103033	11510	12417	13411	14503	15702	17020	3-
2.2	46367	49941	53837	58089	62733	67809	73363	79444	86109	93421	4-
2.4	26290	28297	30484	32870	35474	38320	41432	44837	48569	52660	4-
2.6	14993	16125	17358	18703	20170	21772	23524	25440	27538	29838	4-
2.8	*86032	*92464	*99468	*107102	11164	12072	13062	14143	15324	16616	4-
3.0	49588	53240	57214	61542	66259	71405	77025	83166	89884	97239	5-
3.2	28758	30846	33116	35586	38278	41213	44415	47913	51737	55922	5-
3.4	16780	17979	19281	20698	22241	23921	25755	27756	29942	32334	5-
3.6	*98516	*105428	11293	12110	12998	13965	15019	16168	17424	18796	5-
3.8	58208	62213	66560	71282	76416	82004	88091	94726	101967	109874	6-
4.0	34609	36939	39466	42209	45191	48433	51962	55807	59999	64575	6-
4.2	20715	22077	23552	25153	26891	28781	30836	33073	35511	38170	6-
4.4	12481	13279	14144	15082	16099	17205	18406	19713	21136	22687	6-
4.6	*75731	*80434	*85526	*91042	*97024	*103516	*110568	11823	12658	13566	6-
4.8	46287	49068	52077	55335	58865	62694	66850	71365	76274	81617	7-
5.0	28506	30158	31943	33874	35966	38233	40692	43362	46262	49417	7-
5.2	17692	18676	19739	20888	22133	23480	24940	26525	28245	30115	7-
5.4	*110674	11656	12291	12978	13720	14524	15395	16338	17363	18475	7-
5.6	*69795	*73327	*77138	*81253	*85701	*90512	*95720	*101362	*107480	11412	7-
5.8	44382	46510	48804	51280	53954	56844	59971	63356	67024	71002	8-
6.0	28457	29744	31131	32626	34240	35982	37866	39904	42111	44503	8-
6.2	18396	19178	20020	20927	21904	22959	24098	25329	26661	28104	8-
6.4	11987	12465	12978	13530	14125	14766	15457	16204	17012	17885	8-
6.6	*78688	*81630	*84782	*88166	*91803	*95717	*99933	*104481	*109392	11470	8-
6.8	52013	53839	55790	57878	60117	62521	65106	67889	70891	74130	9-
7.0	34589	35736	36956	38257	39646	41134	42730	44443	46288	48275	9-
7.2	23116	23847	24621	25441	26313	27243	28236	29300	30441	31667	9-
7.4	15505	15980	16479	17004	17559	18148	18773	19440	20153	20916	9-
7.6	*104208	*107369	*110656	11409	11768	12147	12546	12970	13420	13900	9-
7.8	70056	72214	74433	76727	79108	81590	84191	86924	89810	92866	10-
8.0	47009	48522	50061	51634	53250	54917	56646	58447	60332	62312	10-
8.2	31406	32496	33592	34701	35827	36976	38155	39370	40628	41937	10-
8.4	20832	21636	22437	23239	24044	24857	25682	26523	27384	28270	10-
8.6	13664	14269	14867	15460	16051	16642	17234	17831	18436	19052	10-
8.8	*88251	*92863	*97400	*101874	*106294	*110675	*115031	11938	12374	12812	10-
9.0	55747	59297	62777	66194	69553	72862	76128	79361	82573	85775	11-
9.2	34115	36857	39542	42172	44750	47280	49764	52210	54622	57008	11-
9.4	19913	22027	24100	26131	28119	30066	31974	33845	35681	37487	11-
9.6	10759	12382	13977	15541	17076	18578	20049	21490	22900	24282	11-
9.8	5015	6250	7468	8668	9847	11004	12138	13249	14336	15400	11-
10.0	1551	2480	3402	4313	5213	6099	6969	7824	8662	9482	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	2.21	2.22	2.23	2.24	2.25	2.26	2.27	2.28	2.29	2.30	p
0.2	*57280	*62452	*68153	*74444	*81392	*89073	*97573	*106988	11742	12901	0
0.4	23956	26117	28499	31127	34030	37239	40789	44723	49082	53921	1-
0.6	12164	13259	14467	15799	17270	18896	20696	22689	24898	27349	1-
0.8	*64891	*70723	*77150	*84241	*92070	*100724	*110296	12090	13264	14568	1-
1.0	35173	38326	41801	45632	49862	54537	59707	65432	71776	78813	2-
1.2	19228	20946	22838	24925	27228	29773	32587	35702	39155	42982	2-
1.4	*105716	11512	12548	13690	14951	16343	17882	19585	21472	23565	2-
1.6	*58411	*63584	*69280	*75558	*82484	*90132	*98586	*107940	11829	12978	2-
1.8	32432	35289	38433	41899	45721	49939	54600	59757	65466	71792	3-
2.0	18468	20062	21816	23750	25881	28235	30835	33709	36890	40413	3-
2.2	*101449	*110271	11998	13066	14244	15543	16977	18563	20317	22259	3-
2.4	*57151	*62084	*67508	*73477	*80053	*87305	*95308	*104151	11393	12475	3-
2.6	32360	35130	38175	41524	45211	49276	53760	58712	64186	70243	4-
2.8	18030	19582	21287	23159	25220	27489	29993	32755	35809	39186	4-
3.0	*105299	11414	12385	13452	14625	15917	17341	18912	20647	22565	4-
3.2	*60505	*65530	*71043	*77099	*83756	*91082	*99153	*108051	11787	12872	4-
3.4	34951	37818	40963	44415	48208	52379	56971	62032	67615	73779	5-
3.6	20297	21940	23742	25717	27887	30272	32896	35785	38971	42486	5-
3.8	11852	12798	13834	14970	16216	17585	19090	20747	22572	24585	5-
4.0	*69574	*75040	*81023	*87577	*94765	*102655	*111324	12086	13135	14292	5-
4.2	41073	44245	47714	51512	55674	60240	65253	70762	76824	83500	6-
4.4	24379	26227	28246	30455	32874	35526	38435	41630	45143	49009	6-
4.6	14557	15637	16817	18108	19519	21066	22761	24622	26665	28913	6-
4.8	*87437	*93783	*100708	*108272	11654	12560	13551	14639	15833	17145	6-
5.0	52851	56592	60672	65125	69990	75311	81136	87518	94519	102206	7-
5.2	32149	34363	36775	39407	42280	45419	48853	52613	56733	61254	7-
5.4	19684	20999	22431	23992	25694	27553	29585	31808	34243	36912	7-
5.6	12133	12917	13770	14699	15712	16817	18024	19343	20786	22367	7-
5.8	*75320	*80010	*85110	*90660	*96705	*103296	*110489	11835	12694	13634	7-
6.0	47097	49912	52972	56299	59921	63866	68169	72865	77996	83606	8-
6.2	29668	31364	33206	35207	37384	39754	42336	45153	48227	51587	8-
6.4	18831	19856	20969	22177	23490	24918	26473	28168	30017	32035	8-
6.6	12044	12666	13340	14072	14866	15730	16669	17693	18808	20025	8-
6.8	*77631	*81417	*85518	*89964	*94787	*100026	*105721	*111919	11867	12603	8-
7.0	50418	52734	55238	57950	60890	64079	67543	71309	75408	79872	9-
7.2	32987	34411	35947	37609	39407	41356	43471	45767	48264	50982	9-
7.4	21734	22615	23563	24587	25692	26889	28185	29591	31117	32777	9-
7.6	14412	14961	15551	16185	16869	17607	18405	19270	20207	21225	9-
7.8	*96113	*99575	*103275	*107241	*111500	11608	12103	12637	13215	13842	9-
8.0	64401	66613	68963	71469	74148	77020	80106	83431	87020	90900	10-
8.2	43307	44744	46261	47866	49572	51392	53337	55424	57668	60087	10-
8.4	29187	30141	31138	32185	33288	34456	35698	37022	38439	39959	10-
8.6	19682	20330	21000	21696	22424	23187	23992	24844	25750	26717	10-
8.8	13256	13708	14169	14643	15134	15643	16174	16733	17321	17944	10-
9.0	*88982	*92207	*95467	*98780	*102164	*105640	*109231	*112961	11686	12094	10-
9.2	59376	61734	64094	66466	68861	71294	73778	76329	78964	81701	11-
9.4	39268	41027	42772	44509	46246	47990	49751	51539	53364	55238	11-
9.6	25638	26971	28284	29582	30867	32146	33424	34708	36003	37317	11-
9.8	16441	17462	18462	19446	20413	21369	22316	23257	24198	25142	11-
10.0	10284	11069	11838	12590	13328	14052	14765	15469	16166	16858	11-

TABLE OF THE WEDGE FUNCTION $F_\nu(e^x)$

ν \ x	2.31	2.32	2.33	2.34	2.35	2.36	2.37	2.38	2.39	2.40	p
0.2	14187	15619	17211	18986	20966	23176	25647	28411	31507	34980	0
0.4	*59295	*65271	*71923	*79334	*87600	*96829	*107144	11869	13161	14611	0
0.6	30071	33098	36468	40221	44407	49080	54302	60145	66689	74026	1-
0.8	16016	17625	19416	21412	23636	26120	28895	32000	35476	39373	1-
1.0	*86628	*95315	*104979	11574	12774	14115	15611	17285	19159	21260	1-
1.2	*47233	*51955	*57210	*63061	*69584	*76864	*84995	*94087	*104265	11567	1-
1.4	25888	28468	31338	34534	38096	42068	46506	51467	57020	63241	2-
1.6	14253	15668	17241	18993	20945	23122	25554	28270	31310	34716	2-
1.8	*78810	*86603	*95267	*104906	11564	12761	14099	15592	17263	19134	2-
2.0	44318	48652	53467	58820	64777	71414	78816	87080	96315	106648	3-
2.2	24413	26803	29457	32409	35695	39357	43441	48002	53101	58808	3-
2.4	13675	15006	16483	18126	19953	21989	24259	26793	29625	32793	3-
2.6	*76952	*84391	*92648	*101822	11203	12339	13605	15018	16596	18361	3-
2.8	42927	47074	51676	56790	62478	68813	75875	83755	92560	102408	4-
3.0	24688	27039	29646	32539	35755	39332	43315	47755	52711	58248	4-
3.2	14073	15401	16874	18508	20322	22340	24585	27087	29877	32994	4-
3.4	*80593	*88132	*96483	*105743	11602	12744	14015	15430	17007	18767	4-
3.6	46370	50664	55418	60687	66531	73021	80237	88267	97215	107197	5-
3.8	26808	29264	31981	34990	38326	42028	46141	50717	55812	61492	5-
4.0	15569	16978	18537	20262	22173	24292	26646	29262	32173	35416	5-
4.2	*90860	*98983	*107957	11788	12887	14105	15456	16958	18627	20486	5-
4.4	*53269	*57966	*63152	*68883	*75224	*82247	*90034	*98678	*108285	11897	5-
4.6	31388	34115	37123	40445	44118	48183	52687	57684	63232	69400	6-
4.8	18588	20177	21928	23861	25997	28358	30973	33871	37087	40659	6-
5.0	*110655	11995	13019	14148	15394	16772	18295	19982	21853	23930	6-
5.2	*66220	*71678	*77686	*84305	*91605	*99666	*108576	11843	12936	14147	6-
5.4	39840	43057	46595	50490	54781	59516	64746	70528	76928	84022	7-
5.6	24101	26004	28095	30394	32927	35718	38798	42201	45964	50132	7-
5.8	14664	15794	17034	18397	19896	21548	23368	25378	27599	30057	7-
6.0	*89749	*96479	*103862	*111969	12088	13069	14149	15340	16656	18110	7-
6.2	55262	59286	63696	68535	73849	79692	86123	93209	101027	109659	8-
6.4	34242	36656	39299	42197	45377	48871	52713	56942	61604	66747	8-
6.6	21354	22807	24396	26138	28047	30143	32446	34978	37768	40842	8-
6.8	13406	14283	15242	16292	17442	18703	20088	21610	23284	25129	8-
7.0	*84741	*90054	*95859	*102208	*109158	11677	12513	13430	14439	15548	8-
7.2	53942	57171	60696	64547	68760	73373	78428	83976	90068	96768	9-
7.4	34584	36552	38699	41043	43605	46407	49477	52842	56534	60591	9-
7.6	22331	23536	24848	26279	27841	29549	31418	33465	35710	38174	9-
7.8	14523	15262	16066	16943	17899	18944	20085	21334	22703	24204	9-
8.0	*95103	*99663	*104617	*110008	11588	12228	12928	13693	14530	15447	9-
8.2	62699	65527	68593	71922	75543	79487	83789	88486	93622	99243	10-
8.4	41596	43361	45269	47336	49580	52018	54674	57570	60731	64188	10-
8.6	27752	28864	30061	31353	32752	34268	35915	37708	39663	41796	10-
8.8	18607	19315	20073	20888	21766	22715	23743	24859	26072	27395	10-
9.0	12525	12982	13469	13988	14545	15144	15790	16489	17247	18071	10-
9.2	*84559	*87559	*90726	*94083	*97658	*101480	*105582	*109998	11477	11993	10-
9.4	57173	59184	61285	63491	65821	68294	70929	73749	76778	80044	11-
9.6	38659	40037	41461	42941	44487	46113	47832	49657	51604	53690	11-
9.8	26095	27062	28049	29064	30112	31203	32344	33544	34814	36165	11-
10.0	17550	18245	18946	19658	20385	21132	21906	22711	23554	24442	11-

TABLE OF THE WEDGE FUNCTION $F_{\nu}(e^{x})$

$\nu \backslash x$	2.41	2.42	2.43	2.44	2.45	2.46	2.47	2.48	2.49	2.50	p
0.2	38876	43256	48184	53735	59992	67058	75041	84076	94312	105922	0
0.4	16237	18066	20122	22439	25051	28000	31332	35102	39373	44218	0
0.6	*82262	*91515	*101924	*113646	12686	14178	15864	17771	19931	22381	0
0.8	*43747	*48662	*54189	*60413	*67430	*75349	*84297	*94420	*105886	11889	0
1.0	23617	26266	29244	32598	36377	40642	45462	50913	57085	64084	1-
1.2	12847	14284	15901	17720	19771	22085	24697	27653	31000	34794	1-
1.4	*70220	*78056	*86867	*96782	*107954	12056	13479	15089	16910	18975	1-
1.6	38536	42823	47642	53065	59173	66062	73841	82634	92586	103864	2-
1.8	21231	23585	26230	29206	32557	36336	40602	45422	50876	57056	2-
2.0	11822	13119	14576	16212	18052	20125	22462	25099	28081	31453	2-
2.2	*65202	*72373	*80426	*89480	*99669	*111151	12410	13873	15528	17401	2
2.4	36341	40319	44785	49803	55449	61809	68982	77080	86234	96596	3-
2.6	20338	22552	25038	27829	30969	34504	38489	42988	48070	53821	3-
2.8	*113439	12581	13969	15529	17284	19262	21492	24011	26859	30083	3-
3.0	*64441	*71376	*79150	*87876	*97681	*108712	12114	13515	15097	16885	3-
3.2	36478	40377	44747	49648	55153	61344	68313	76168	85033	95050	4-
3.4	20734	22935	25399	28163	31264	34750	38673	43092	48077	53706	4-
3.6	11834	13080	14475	16038	17792	19762	21977	24472	27284	30458	4-
3.8	*67832	*74915	*82838	*91712	*101663	*112833	12539	13952	15544	17340	4-
4.0	39034	43073	47589	52643	58308	64662	71801	79829	88869	99062	5-
4.2	22558	24870	27453	30342	33578	37206	41278	45856	51007	56811	5-
4.4	13088	14415	15897	17554	19408	21486	23816	26434	29378	32693	5-
4.6	*76265	*83916	*92452	*101986	*112648	12459	13797	15299	16988	18887	5-
4.8	44633	49057	53990	59495	65648	72531	80243	88892	98606	109529	6-
5.0	26238	28806	31667	34857	38420	42403	46862	51860	57468	63769	6-
5.2	15493	16989	18654	20509	22579	24892	27479	30376	33624	37271	6-
5.4	*91892	*100635	*110358	12118	13325	14673	16178	17863	19751	21869	6-
5.6	*54752	*59879	*65577	*71916	*78977	*86852	*95646	*105478	11648	12882	6-
5.8	32778	35796	39147	42871	47017	51636	56789	62546	68986	76198	7-
6.0	19718	21501	23478	25675	28116	30835	33866	37248	41028	45258	7-
6.2	11920	12977	14148	15447	16890	18496	20284	22278	24504	26993	7-
6.4	*72429	*78713	*85670	*93383	*101944	*111457	12204	13383	14699	16168	7-
6.6	44236	47986	52134	56728	61822	67478	73765	80763	88562	97264	8-
6.8	27162	29407	31888	34634	37676	41049	44796	48962	53601	58772	8-
7.0	16771	18119	19608	21255	23077	25096	27336	29824	32593	35676	8-
7.2	*104142	11227	12123	13113	14208	15421	16764	18255	19913	21757	8-
7.4	*65053	*69965	*75380	*81355	*87956	*95258	*103344	11231	12226	13333	8-
7.6	40882	43861	47141	50758	54750	59161	64042	69449	75445	82105	9-
7.8	25852	27663	29657	31852	34274	36947	39902	43173	46796	50817	9-
8.0	16453	17558	18772	20109	21582	23207	25002	26986	29182	31617	9-
8.2	*105404	11216	11959	12775	13674	14664	15757	16965	18300	19779	9-
8.4	*67972	*72120	*76671	*81671	*87170	*93225	*99901	*107268	11541	12442	9-
8.6	44129	46683	49481	52553	55928	59641	63730	68239	73217	78719	10-
8.8	28838	30416	32142	34035	36113	38395	40907	43674	46726	50097	10-
9.0	18968	19947	21016	22187	23470	24878	26426	28129	30005	32076	10-
9.2	12554	13165	13830	14557	15352	16224	17180	18232	19389	20665	10-
9.4	*83576	*87405	*91568	*96104	*101055	*106472	*112407	11892	12608	13396	10-
9.6	55934	58356	60979	63827	66926	70309	74006	78057	82501	[illegible]7386	11-
9.8	37607	39155	40822	42624	44577	46701	49017	51548	54318	57358	11-
10.0	25382	26384	27455	28606	29848	31192	32652	34242	35977	37877	11-

TABLE OF THE WEDGE FUNCTION $G_{\nu}(e^{x})$

ν \ x	.49-	.48-	.47-	.46-	.45-	.44-	.43-	.42-	.41-	.40-	p
0.2	7454	7378	7304	7230	7156	7081	7008	6935	6863	6789	4-
0.4	6992	6924	6858	6791	6725	6659	6592	6527	6460	6395	4-
0.6	6274	6219	6164	6110	6055	5999	5945	5890	5836	5781	4-
0.8	53740	53334	52926	52516	52107	51695	51281	50868	50453	50037	5-
1.0	43755	43497	43236	42974	42707	42439	42168	41894	41619	41341	5-
1.2	33652	33530	33406	33277	33145	33011	32872	32730	32584	32436	5-
1.4	24197	24187	24173	24156	24134	24109	24080	24048	24012	23972	5-
1.6	15981	16049	16115	16175	16233	16288	16339	16387	16431	16471	5-
1.8	9362	9474	9582	9689	9793	9894	9992	10088	10180	10270	5-
2.0	44606	45848	47074	48283	49475	50649	51803	52941	54059	55158	6-
2.2	11910	13039	14160	15275	16384	17485	18580	19665	20741	21809	6-
2.4	6880-	6011-	5140-	4266-	3390-	2511-	1632-	751-	126	1005	6-
2.6	15041-	14486-	13922-	13349-	12768-	12179-	11582-	10977-	10366-	9748-	6-
2.8	16055-	15794-	15521-	15237-	14941-	14634-	14317-	13988-	13650-	13301-	6-
3.0	13081-	13047-	13001-	12944-	12876-	12797-	12707-	12606-	12494-	12372-	6-
3.2	8605-	8711-	8808-	8897-	8977-	9048-	9110-	9163-	9208-	9243-	6-
3.4	42958-	44603-	46198-	47741-	49231-	50667-	52046-	53367-	54628-	55829-	7-
3.6	10437-	12043-	13634-	15208-	16763-	18296-	19807-	21293-	22753-	24183-	7-
3.8	8897	7696	6485	5264	4036	2802	1564	324	916-	2156-	7-
4.0	16572	15900	15204	14483	13740	12975	12191	11387	10565	9728	7-
4.2	16157	15959	15732	15479	15199	14893	14561	14204	13823	13418	7-
4.4	11566	11689	11790	11868	11924	11957	11968	11956	11921	11864	7-
4.6	59803	62527	65120	67579	69897	72070	74093	75963	77676	79227	8-
4.8	13891	16706	19484	22218	24901	27528	30093	32589	35012	37356	8-
5.0	13943-	11873-	9774-	7651-	5509-	3353-	1189-	977	3142	5298	8-
5.2	24250-	23200-	22088-	20917-	19690-	18411-	17083-	15710-	14295-	12841-	8-
5.4	22266-	22100-	21870-	21577-	21222-	20806-	20330-	19796-	19205-	18559-	8-
5.6	14369-	14748-	15081-	15368-	15607-	15798-	15940-	16032-	16075-	16069-	8-
5.8	5748-	6315-	6861-	7384-	7883-	8355-	8800-	9216-	9601-	9954-	8-
6.0	5774	836	4104-	9030-	13923-	18768-	23545-	28238-	32832-	37308-	9-
6.2	36845	33897	30821	27627	24328	20937	17466	13928	10338	6709	9-
6.4	40869	39988	38946	37746	36392	34891	33249	31471	29567	27542	9-
6.6	29253	29813	30244	30545	30714	30750	30654	30425	30065	9576	9-
6.8	13226	14396	15501	16534	17492	18369	19162	19868	20482	1003	9-
7.0	286	1394	2496	3586	4658	5708	6730	7720	8671	9581	9-
7.2	6575-	5885-	5165-	4418-	3649-	2860-	2057-	1243-	423-	399	9-
7.4	78258-	76099-	73526-	70554-	67199-	63478-	59413-	55025-	50338-	45378-	10-
7.6	56627-	57857-	58755-	59316-	59537-	59416-	58955-	58157-	57024-	55566-	10-
7.8	24667-	27321-	29811-	32120-	34236-	36144-	37835-	39296-	40521-	41501-	10-
8.0	1166	1302-	3762-	6198-	8595-	10937-	13209-	15398-	17488-	19468-	10-
8.2	14258	12807	11271	9659	7983	6254	4483	2681	862	962-	10-
8.4	15729	15383	14928	14369	13709	12953	12106	11174	10164	9082	10-
8.6	10490	10882	11194	11423	11569	11630	11605	11494	11300	1022	10-
8.8	36915	43386	49523	55279	60609	65473	69832	73654	76910	79573	11-
9.0	13153-	7792-	2368-	3075	8493	13843	19082	24166	29057	33713	11-
9.2	34230-	31581-	28666-	25509-	22138-	18580-	14866-	11027-	7095-	3104-	11-
9.4	31652-	31539-	31149-	30486-	29554-	28363-	26922-	25245-	23346-	21242-	11-
9.6	17562-	18884-	20033-	20998-	21771-	22344-	22713-	22873-	22823-	22565-	11-
9.8	3029-	4602-	6131-	7602-	9000-	10311-	11525-	12628-	13611-	14464-	11-
10.0	5847	4778	3663	2510	1333	142	1049-	2231-	3390-	4516-	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	.39-	.38-	.37-	.36-	.35-	.34-	.33-	.32-	.31-	.30-	p
0.2	6717	6645	6574	6502	6430	6360	6289	6219	6149	6079	4-
0.4	6330	6265	6200	6136	6071	6007	5943	5878	5815	5752	4-
0.6	5726	5672	5617	5563	5508	5453	5400	5345	5292	5237	4-
0.8	49620	49202	48784	48364	47944	47523	47102	46680	46258	45834	5-
1.0	41060	40778	40493	40206	39918	39626	39333	39038	38742	38444	5-
1.2	32284	32130	31971	31811	31646	31479	31309	31136	30960	30782	5-
1.4	23928	23881	23831	23777	23719	23658	23594	23527	23455	23381	5-
1.6	16509	16541	16572	16598	16622	16642	16658	16671	16681	16687	5-
1.8	10357	10441	10522	10600	10675	10747	10817	10883	10947	11008	5-
2.0	56237	57297	58336	59355	60353	61329	62285	63218	64130	65019	6-
2.2	22867	23915	24953	25979	26995	27998	28989	29969	30935	31888	6-
2.4	1886	2765	3643	4517	5391	6261	7128	7991	8851	9705	6-
2.6	9124-	8494-	7859-	7219-	6574-	5926-	5273-	4618-	3959-	3298-	6-
2.8	12942-	12574-	12197-	11810-	11415-	11012-	10600-	10181-	9754-	9320-	6-
3.0	12239-	12095-	11941-	11777-	11603-	11419-	11226-	11023-	10810-	10589-	6-
3.2	9269-	9286-	9294-	9293-	9283-	9264-	9236-	9199-	9153-	9098-	6-
3.4	56968-	58043-	59054-	60000-	60879-	61691-	62435-	63110-	63715-	64250-	7-
3.6	25584-	26953-	28288-	29587-	30850-	32074-	33259-	34402-	35502-	36558-	7-
3.8	3392-	4623-	5848-	7065-	8272-	9467-	10649-	11816-	12967-	14100-	7-
4.0	8874	8008	7128	6238	5338	4430	3515	2594	1669	742	7-
4.2	12990	12539	12067	11575	11062	10531	9981	9414	8832	8234	7-
4.4	11785	11683	11559	11413	11246	11057	10848	10618	10368	10099	7-
4.6	80615	81835	82887	83767	84474	85006	85363	85543	85548	85375	8-
4.8	39615	41786	43861	45838	47711	49477	51132	52671	54092	55391	8-
5.0	7442	9567	11669	13742	15782	17783	19740	21649	23505	25304	8-
5.2	11354-	9837-	8293-	6727-	5143-	3546-	1940-	328-	1285	2894	8-
5.4	17859-	17109-	16309-	15462-	14571-	13639-	12667-	11659-	10618-	9546-	8-
5.6	16012-	15907-	15752-	15548-	15297-	14998-	14653-	14263-	13829-	13352-	8-
5.8	10274-	10560-	10811-	11026-	11205-	11346-	11450-	11516-	11544-	11534-	8-
6.0	41652-	45848-	49881-	53737-	57402-	60863-	64109-	67127-	69907-	72439-	9-
6.2	3054	613-	4277-	7925-	11543-	15118-	18635-	22081-	25443-	28709-	9-
6.4	25406	23167	20835	18418	15926	13370	10761	8107	5421	2713	9-
6.6	28960	28218	27355	26374	25280	24077	22770	21365	19868	18286	9-
6.8	21428	21754	21981	22108	22133	22057	21881	21604	21229	20756	9-
7.0	10444	11257	12015	12715	13353	13926	14432	14868	15232	15522	9-
7.2	1219	2033	2836	3625	4396	5143	5865	6556	7214	7834	9-
7.4	40172-	34747-	29134-	23363-	17465-	11473-	5418-	666	6747	12791	10-
7.6	53789-	51706-	49322-	46659-	43728-	40547-	37134-	33508-	29691-	25704-	10-
7.8	42230-	42705-	42922-	42880-	42579-	42022-	41211-	40151-	38850-	37314-	10-
8.0	21324-	23045-	24619-	26037-	27289-	28369-	29268-	29982-	30505-	30835-	10-
8.2	2781-	4581-	6350-	8077-	9750-	11358-	12890-	14336-	15687-	16933-	10-
8.4	7937	6737	5489	4203	2887	1551	205	1144-	2484-	3807-	10-
8.6	10663	10226	9714	9130	8480	7767	6997	6176	5310	4404	10-
8.8	81625	83048	83832	83972	83466	82318	80538	78138	75137	71559	11-
9.0	38098	42176	45915	49284	52257	54809	56921	58575	59757	60460	11-
9.2	914	4924	8893	12786	16573	20220	23697	26974	30026	32824	11-
9.4	18951-	16494-	13892-	11168-	8346-	5450-	2507-	458	3419	6351	11-
9.6	22099-	21431-	20567-	19514-	18283-	16884-	15330-	13636-	11818-	9891-	11-
9.8	15178-	15748-	16167-	16432-	16540-	16490-	16282-	15920-	15405-	14743-	11-
10.0	5597-	6622-	7581-	8465-	9264-	9972-	10580-	11083-	11476-	11755-	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

$\nu \backslash x$	.29-	.28-	.27-	.26-	.25-	.24-	.23-	.22-	.21-	.20-	p
0.2	6010	5941	5872	5804	5736	5667	5600	5533	5466	5400	4-
0.4	5688	5625	5563	5500	5438	5376	5314	5252	5191	5130	4-
0.6	5184	5130	5076	5023	4969	4916	4862	4810	4757	4704	4-
0.8	45410	44987	44562	44139	43714	43289	42865	42441	42015	41591	5-
1.0	38144	37842	37538	37233	36927	36619	36309	35996	35686	35374	5-
1.2	30600	30417	30230	30040	29849	29654	29458	29258	29057	28853	5-
1.4	23303	23221	23137	23049	22959	22865	22768	22668	22565	22459	5-
1.6	16689	16689	16685	16678	16668	16655	16638	16618	16595	16568	5-
1.8	11065	11120	11171	11220	11265	11308	11348	11385	11419	11450	5-
2.0	65886	66731	67552	68350	69125	69877	70606	71310	71990	72646	6-
2.2	32827	33752	34663	35558	36439	37304	38154	38987	39805	40605	6-
2.4	10554	11398	12237	13068	13893	14711	15521	16323	17117	17902	6-
2.6	2635-	1970-	1304-	637-	27	694	1361	2027	2691	3353	6-
2.8	8880-	8432-	7979-	7520-	7056-	6586-	6112-	5633-	5150-	4664-	6-
3.0	10358-	10119-	9871-	9615-	9351-	9078-	8799-	8512-	8218-	7917-	6-
3.2	9034-	8962-	8881-	8791-	8693-	8586-	8472-	8349-	8218-	8079-	6-
3.4	64714-	65107-	65429-	65678-	65856-	65962-	65995-	65956-	65846-	65663-	7-
3.6	37569-	38533-	39449-	40317-	41135-	41902-	42617-	43280-	43889-	44445-	7-
3.8	15214-	16306-	17375-	18421-	19441-	20434-	21399-	22334-	23239-	24111-	7-
4.0	186-	1114-	2040-	2964-	3882-	4795-	5700-	6597-	7483-	8358-	7-
4.2	7622	6997	6360	5712	5055	4389	3715	3035	2350	1661	7-
4.4	9811	9504	9180	8838	8479	8105	7716	7312	6894	6464	7-
4.6	85027	84504	83807	82937	81897	80688	79314	77777	76081	74228	8-
4.8	56565	57613	58531	59317	59970	60489	60873	61119	61229	61203	8-
5.0	27040	28711	30311	31838	33287	34654	35938	37133	38238	39251	8-
5.2	4496	6085	7659	9212	10741	12242	13710	15142	16534	17883	8-
5.4	8447-	7324-	6180-	5018-	3842-	2655-	1460-	262-	938	2135	8-
5.6	12834-	12277-	11682-	11051-	10386-	9689-	8962-	8208-	7428-	6626-	8-
5.8	11485-	11399-	11275-	11113-	10915-	10681-	10412-	10108-	9771-	9401-	8-
6.0	74715-	76725-	78464-	79925-	81104-	81995-	82596-	82906-	82922-	82645-	9-
6.2	31867-	34904-	37808-	40570-	43178-	45622-	47895-	49986-	51888-	53595-	9-
6.4	6-	2725-	5434-	8120-	10773-	13383-	15939-	18430-	20848-	23181-	9-
6.6	16625	14893	13097	11244	9344	7403	5430	3434	1423	594-	9-
6.8	20189	19530	18781	17947	17031	16037	14970	13835	12637	11381	9-
7.0	15737	15875	15937	15922	15830	15661	15416	15097	14705	14242	9-
7.2	8415	8952	9444	9887	10279	10619	10904	11134	11307	11421	9-
7.4	18766	24639	30378	35953	41334	46490	51396	56023	60347	64344	10-
7.6	21570-	17313-	12957-	8527-	4047-	455	4954	9425	13843	18181	10-
7.8	35554-	33579-	31402-	29036-	26495-	23795-	20951-	17982-	14904-	11736-	10-
8.0	30970-	30908-	30650-	30198-	29555-	28724-	27711-	26523-	25167-	23651-	10-
8.2	18066-	19079-	19965-	20718-	21332-	21805-	22132-	22313-	22344-	22227-	10-
8.4	5103-	6363-	7579-	8742-	9844-	10877-	11834-	12708-	13493-	14185-	10-
8.6	3466	2503	1522	529	468-	1461-	2443-	3408-	4347-	5255-	10-
8.8	67432	62786	57658	52087	46116	39791	33160	26275	19188	11954	11-
9.0	60676	60404	59647	58411	56705	54544	51944	48927	45518	41743	11-
9.2	35347	37573	39483	41062	42296	43174	43690	43840	43621	43036	11-
9.4	9226	12021	14710	17270	19678	21914	23958	25792	27400	28768	11-
9.6	7873-	5784-	3641-	1465-	724	2907	5063	7172	9217	11176	11-
9.8	13941-	13005-	11946-	10773-	9496-	8130-	6686-	5178-	3620-	2029-	11-
10.0	11918-	11961-	11886-	11693-	11384-	10962-	10430-	9796-	9064-	8242-	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	.19-	.18-	.17-	.16-	.15-	.14-	.13-	.12-	.11-	.10-	p
0.2	5334	5269	5203	5138	5074	5009	4945	4881	4818	4755	4-
0.4	5069	5009	4948	4889	4829	4770	4710	4652	4594	4535	4-
0.6	4651	4599	4547	4494	4443	4391	4340	4288	4237	4185	4-
0.8	41166	40742	40318	39894	39470	39048	38624	38203	37781	37360	5-
1.0	35059	34744	34427	34109	33792	33472	33152	32831	32509	32187	5-
1.2	28646	28439	28228	28016	27802	27585	27368	27147	26925	26702	5-
1.4	22350	22239	22124	22007	21887	21764	21639	21511	21381	21248	5-
1.6	16539	16506	16470	16431	16390	16345	16297	16247	16193	16137	5-
1.8	11477	11502	11525	11544	11560	11574	11585	11592	11597	11599	5-
2.0	73278	73887	74470	75029	75563	76073	76558	77018	77455	77865	6-
2.2	41389	42155	42905	43635	44349	45044	45721	46379	47018	47638	6-
2.4	18679	19445	20202	20948	21685	22410	23124	23825	24516	25195	6-
2.6	4014	4673	5328	5981	6630	7274	7915	8551	9181	9806	6-
2.8	4174-	3681-	3185-	2688-	2188-	1687-	1184-	681-	177-	327	6-
3.0	7609-	7295-	6975-	6649-	6318-	5982-	5640-	5294-	4944-	4589-	6-
3.2	7932-	7778-	7617-	7448-	7272-	7089-	6899-	6703-	6501-	6292-	6-
3.4	65409-	65084-	64688-	64222-	63687-	63082-	62410-	61670-	60863-	59992-	7-
3.6	44945-	45391-	45781-	46115-	46392-	46613-	46777-	46884-	46934-	46927-	7-
3.8	24950-	25755-	26525-	27258-	27954-	28612-	29230-	29808-	30346-	30842-	7-
4.0	9220-	10068-	10900-	11716-	12514-	13293-	14052-	14789-	15504-	16195-	7-
4.2	969	276	418-	1111-	1802-	2491-	3175-	3854-	4526-	5190-	7-
4.4	6021	5567	5103	4629	4147	3657	3160	2657	2150	1638	7-
4.6	72224	70071	67775	65341	62773	60077	57259	54324	51278	48128	8-
4.8	61039	60738	60302	59732	59028	58192	57227	56135	54917	53578	8-
5.0	40167	40986	41706	42324	42839	43251	43557	43759	43854	43843	8-
5.2	19184	20435	21632	22772	23852	24869	25821	26706	27520	28262	8-
5.4	3325	4507	5675	6828	7961	9071	10156	11211	12235	13224	8-
5.6	5803-	4963-	4107-	3239-	2361-	1475-	585-	306	1197	2084	8-
5.8	9001-	8571-	8113-	7628-	7118-	6584-	6029-	5454-	4861-	4253-	8-
6.0	82076-	81218-	80073-	78646-	76941-	74965-	72725-	70228-	67485-	64504-	9-
6.2	55099-	56395-	57478-	58344-	58991-	59415-	59615-	59591-	59343-	58871-	9-
6.4	25421-	27559-	29586-	31494-	33275-	34922-	36429-	37790-	38998-	40051-	9-
6.6	2608-	4611-	6594-	8549-	10468-	12342-	14163-	15923-	17615-	19232-	9-
6.8	10073	8720	7326	5900	4446	2973	1486	8-	1502-	2989-	9-
7.0	13710	13111	12450	11728	10950	10119	9239	8315	7350	6350	9-
7.2	11478	11475	11414	11295	11118	10884	10595	10251	9856	9410	9-
7.4	67994	71276	74173	76670	78752	80409	81632	82415	82752	82644	10-
7.6	22416	26523	30478	34260	37847	41217	44353	47236	49851	52181	10-
7.8	8498-	5209-	1888-	1443	4767	8061	11307	14486	17577	20563	10-
8.0	21986-	20182-	18250-	16202-	14052-	11814-	9500-	7127-	4708-	2260-	10-
8.2	21962-	21551-	20997-	20303-	19474-	18515-	17434-	16237-	14932-	13528-	10-
8.4	14777-	15266-	15648-	15921-	16083-	16133-	16070-	15895-	15609-	15215-	10-
8.6	6124-	6949-	7722-	8439-	9094-	9683-	10201-	10644-	11009-	11294-	10-
8.8	4628	2733-	10073-	17336-	24466-	31409-	38111-	44520-	50589-	56271-	11-
9.0	37633	33221	28542	23634	18537	13291	7938	2522	2914-	8327-	11-
9.2	42090	40791	39150	37180	34899	32325	29480	26388	23075	19569	11-
9.4	29883	30737	31321	31632	31665	31421	30902	30112	29059	27752	11-
9.6	13034	14773	16376	17830	19121	20237	21169	21907	22445	22778	11-
9.8	417-	1197	2801	4378	5913	7392	8801	10126	11354	12475	11-
10.0	7338-	6361-	5322-	4229	3095-	1930-	746-	446	1633	2804	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	.09-	.08-	.07	.06-	.05-	.04-	.03-	.02-	.01-	.00	p
0.2	4693	4631	4569	4508	4447	4387	4327	4267	4207	4149	4-
0.4	4477	4420	4363	4305	4249	4193	4137	4080	4025	3970	4-
0.6	4135	4084	4034	3983	3934	3884	3834	3785	3736	3687	4-
0.8	36940	36520	36102	35684	35266	34851	34436	34022	33609	33197	5-
1.0	31864	31541	31217	30893	30569	30244	29918	29594	29268	28943	5-
1.2	26477	26250	26022	25792	25560	25328	25094	24859	24623	24385	5-
1.4	21114	20976	20837	20694	20550	20404	20255	20104	19951	19797	5-
1.6	16078	16016	15952	15884	15814	15741	15666	15588	15508	15426	5-
1.8	11599	11595	11588	11579	11567	11553	11536	11516	11494	11469	5-
2.0	78251	78613	78949	79261	79549	79811	80049	80262	80452	80617	6-
2.2	48239	48822	49386	49927	50450	50953	51436	51900	52342	52765	6-
2.4	25861	26514	27155	27782	28394	28994	29579	30151	30708	31250	6-
2.6	10426	11039	11646	12245	12838	13423	14000	14569	15130	15682	6-
2.8	831	1334	1836	2337	2836	3333	3828	4321	4810	5296	6-
3.0	4231-	3869-	3505-	3137-	2767-	2394-	2020-	1644-	1267-	888-	6-
3.2	6078-	5858-	5632-	5401-	5165-	4924-	4679-	4429-	4175-	3918-	6-
3.4	59055-	58056-	56994-	55871-	54689-	53448-	52151-	50798-	49392-	47933-	7-
3.6	46862-	46742-	46564-	46330-	46040-	45695-	45295-	44840-	44331-	43770-	7-
3.8	31297-	31709-	32077-	32403-	32684-	32921-	33113-	33261-	33364-	33422-	7-
4.0	16862-	17503-	18117-	18704-	19263-	19793-	20293-	20762-	21201-	21607-	7-
4.2	5846-	6492-	7128-	7751-	8361-	8957-	9539-	10104-	10652-	11183-	7-
4.4	1123	606	89	429-	947-	1462-	1975-	2484-	2989-	3488-	7-
4.6	44879	41540	38116	34615	31044	27410	23721	19983	16206	12395	8-
4.8	52119	50545	48858	47064	45165	43166	41072	38886	36615	34264	8-
5.0	43726	43504	43176	42745	42210	41574	40837	40002	39071	38047	8-
5.2	28930	29522	30036	30473	30829	31105	31300	31413	31444	31393	8-
5.4	14176	15088	15957	16781	17557	18284	18959	19581	20147	20657	8-
5.6	2965	3837	4697	5542	6371	7180	7968	8731	9468	10176	8-
5.8	3630-	2995-	2351-	1699-	1041-	380-	282	944	1602	2255	8-
6.0	61296-	57873-	54246-	50429-	46435-	42278-	37972-	33534-	28978-	24321-	9-
6.2	58178-	57266-	56139-	54801-	53257-	51514-	49577-	47455-	45155-	42685-	9-
6.4	40942-	41670-	42230-	42621-	42841-	42890-	42767-	42473-	42009-	41378-	9-
6.6	20767-	22213-	23564-	24815-	25960-	26994-	27913-	28713-	29391-	29944-	9-
6.8	4463-	5916-	7342-	8735-	10089-	11397-	12653-	13852-	14988-	16056-	9-
7.0	5320	4264	3187	2095	993	113-	1219-	2320-	3409-	4482-	9-
7.2	8915	8376	7794	7172	6513	5822	5100	4353	3584	2796	9-
7.4	82090	81094	79660	77797	75516	72828	69748	66293	62481	58334	10-
7.6	54215	55940	57347	58427	59176	59588	59662	59397	58795	57859	10-
7.8	23425	26147	28713	31106	33312	35320	37115	38689	40031	41133	10-
8.0	203	2664	5108	7520	9885	12188	14413	16548	18578	20492	10-
8.2	12034-	10460-	8816-	7115-	5366-	3581-	1773-	47	1867	3674	10-
8.4	14714-	14111-	13410-	12615-	11732-	10768-	9728-	8621-	7454-	6236-	10-
8.6	11497-	11615-	11648-	11597-	11460-	11240-	10938-	10556-	10098-	9565-	10-
8.8	61521-	66301-	70573-	74305-	77468-	80039-	81999-	83331-	84026-	84079-	11-
9.0	13674-	18911-	23996-	28889-	33551-	37944-	42034-	45788-	49175-	52169-	11-
9.2	15899	12095	8191	4217	209	3802-	7780-	11694-	15509-	19195-	11-
9.4	26202	24422	22430	20241	17875	15353	12697	9929	7076	4160	11-
9.6	22904	22820	22528	22031	21333	20440	19361	18105	16684	15112	11-
9.8	13476	14350	15087	15681	16125	16416	16551	16529	16350	16016	11-
10.0	3948	5052	6106	7099	8023	8867	9623	10284	10843	11295	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	p
0.2	4090	4032	3974	3917	3860	3803	3747	3691	3636	3581	4-
0.4	3916	3861	3807	3754	3700	3648	3594	3542	3491	3439	4-
0.6	3638	3590	3542	3493	3446	3398	3352	3304	3258	3211	4-
0.8	32787	32377	31969	31563	31158	30753	30351	29950	29551	29154	5-
1.0	28617	28292	27967	27642	27317	26992	26669	26345	26021	25698	5-
1.2	24146	23907	23666	23425	23183	22940	22696	22452	22207	21961	5-
1.4	19640	19482	19321	19159	18996	18830	18664	18495	18325	18155	5-
1.6	15341	15253	15164	15072	14978	14881	14783	14681	14579	14474	5-
1.8	11441	11411	11378	11343	11306	11266	11223	11179	11131	11082	5-
2.0	80758	80874	80966	81035	81081	81102	81101	81076	81029	80958	6-
2.2	53168	53550	53912	54254	54574	54875	55156	55417	55656	55875	6-
2.4	31778	32290	32787	33269	33735	34185	34620	35038	35440	35826	6-
2.6	16225	16758	17282	17796	18300	18794	19277	19749	20210	20660	6-
2.8	5778	6257	6731	7201	7665	8125	8579	9028	9470	9906	6-
3.0	509-	130-	249	629	1008	1386	1763	2138	2512	2884	6-
3.2	3656-	3391-	3124-	2853-	2580-	2304-	2026-	1746-	1465-	1183-	6-
3.4	46423-	44865-	43259-	41608-	39914-	38177-	36401-	34587-	32737-	30853-	7-
3.6	43156-	42490-	41774-	41008-	40194-	39332-	38423-	37469-	36471-	35430-	7-
3.8	33436-	33404-	33328-	33207-	33041-	32832-	32579-	32283-	31943-	31562-	7-
4.0	21981-	22322-	22630-	22903-	23143-	23348-	23519-	23654-	23755-	23820-	7-
4.2	11695-	12188-	12661-	13112-	13542-	13949-	14334-	14695-	15031-	15343-	7-
4.4	3981-	4466-	4943-	5412-	5870-	6318-	6754-	7177-	7588-	7985-	7-
4.6	8560	4707	845	3019-	6877-	10721-	14543-	18337-	22094-	25807-	8-
4.8	31837	29339	26777	24157	21483	18762	16000	13203	10378	7529	8-
5.0	36931	35726	34436	33064	31612	30085	28486	26819	25089	23298	8-
5.2	31261	31047	30752	30378	29925	29394	28787	28106	27352	26527	8-
5.4	21109	21501	21832	22103	22311	22457	22540	22560	22516	22410	8-
5.6	10853	11497	12107	12679	13214	13708	14161	14571	14937	15258	8-
5.8	2901	3537	4162	4773	5369	5947	6506	7044	7559	8049	8-
6.0	19578-	14767-	9905-	5008-	93-	4822	9720	14584	19397	24143	9-
6.2	40056-	37277-	34359-	31312-	28148-	24879-	21518-	18076-	14567-	11004-	9-
6.4	40580-	39621-	38504-	37234-	35814-	34253-	32554-	30727-	28777-	26712-	9-
6.6	30369-	30666-	30831-	30866-	30770-	30544-	30187-	29703-	29093-	28360-	9-
6.8	17052-	17970-	18808-	19560-	20225-	20798-	21277-	21660-	21946-	22133-	9-
7.0	5533-	6558-	7551-	8509-	9425-	10296-	11118-	11887-	12599-	13251-	9-
7.2	1995	1183	365	455-	1272-	2083-	2884-	3669-	4437-	5182-	9-
7.4	53873	49122	44108	38857	33398	27760	21972	16067	10076	4031	10-
7.6	56595	55009	53112	50915	48428	45667	42648	39388	35904	32218	10-
7.8	41990	42595	42946	43040	42877	42458	41786	40864	39699	38297	10-
8.0	22276	23920	25414	26748	27914	28904	29712	30334	30766	31004	10-
8.2	5457	7204	8903	10543	12114	13605	15005	16307	17501	18579	10-
8.4	4973-	3677-	2354-	1016-	330	1673	3005	4316	5597	6839	10-
8.6	8963-	8295-	7567-	6784-	5951-	5075-	4163-	3219-	2253-	1270-	10-
8.8	83490-	82263-	80408-	77940-	74877-	71242-	67064-	63375-	57211-	51610-	11-
9.0	54746-	56885-	58570-	59787-	60526-	60783-	60554-	59842-	58652-	56995-	11-
9.2	22721-	26057-	29175-	32049-	34656-	36973-	38982-	40666-	42011-	43005-	11-
9.4	1208	1754-	4701-	7608-	10447-	13196-	15830-	18325-	20661-	22818-	11-
9.6	13402	11569	9632	7606	5512	3367	1192	995-	3172-	5320-	11-
9.8	15529	14895	14120	13210	12176	11025	9771	8423	6996	5502	11-
10.0	11635	11860	11967	11957	11828	11582	11222	10752	10175	9497	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	.11	.12	.13	.14	.15	.16	.17	.18	.19	.20	p
0.2	3526	3473	3420	3366	3314	3261	3209	3158	3107	3056	4-
0.4	3388	3338	3286	3236	3187	3138	3089	3040	2992	2944	4-
0.6	3166	3119	3074	3028	2984	2939	2895	2850	2807	2763	4-
0.8	28758	28364	27972	27581	27193	26806	26421	26038	25658	25279	5-
1.0	25376	25055	24733	24414	24094	23776	23458	23142	22826	22511	5-
1.2	21715	21469	21221	20975	20727	20479	20233	19984	19736	19489	5-
1.4	17983	17809	17634	17458	17280	17102	16922	16742	16561	16379	5-
1.6	14368	14259	14149	14037	13923	13808	13690	13572	13452	13330	5-
1.8	11030	10976	10920	10862	10802	10740	10676	10610	10542	10472	5-
2.0	80866	80750	80613	80455	80274	80073	79850	79606	79343	79059	6-
2.2	56075	56254	56413	56552	56671	56770	56850	56911	56952	56973	6-
2.4	36196	36550	36887	37208	37512	37800	38071	38326	38564	38785	6-
2.6	21099	21526	21941	22344	22735	23113	23479	23833	24174	24502	6-
2.8	10336	10759	11174	11583	11983	12377	12762	13138	13507	13867	6-
3.0	3255	3622	3987	4348	4706	5060	5411	5757	6099	6437	6-
3.2	899-	615-	330-	44-	241	526	810	1094	1377	1659	6-
3.4	28938-	26992-	25018-	23019-	20997-	18953-	16889-	14809-	12713-	10605-	7-
3.6	34347-	33224-	32062-	30863-	29628-	28358-	27056-	25722-	24359-	22968-	7-
3.8	31139-	30674-	30170-	29625-	29042-	28421-	27763-	27068-	26339-	25575-	7-
4.0	23850-	23845-	23805-	23730-	23620-	23475-	23296-	23084-	22837-	22557-	7-
4.2	15630-	15891-	16126-	16334-	16517-	16672-	16800-	16901-	16974-	17020-	7-
4.4	8368-	8735-	9086-	9422-	9740-	10040-	10323-	10587-	10832-	11057-	7-
4.6	29468-	33070-	36607-	40071-	43456-	46755-	49961-	53068-	56071-	58964-	8-
4.8	4664	1789	1090-	3966-	6834-	9688-	12520-	15325-	18097-	20830-	8-
5.0	21452	19556	17613	15628	13606	11552	9470	7367	5246	3112	8-
5.2	25633	24674	23651	22567	21425	20229	18980	17682	16340	14955	8-
5.4	22241	22010	21718	21365	20952	20482	19954	19372	18735	18047	8-
5.6	15534	15762	15943	16076	16161	16197	16185	16124	16015	15858	8-
5.8	8514	8950	9358	9735	10082	10395	10675	10921	11131	11306	8-
6.0	28805	33367	37814	42128	46297	50306	54140	57787	61235	64471	9-
6.2	7399-	3768-	122-	3524	7158	10764	14331	17845	21292	24661	9-
6.4	24542-	22274-	19918-	17483-	14978-	12414-	9801-	7149-	4469-	1772-	9-
6.6	27507-	26537-	25455-	24266-	22974-	21585-	20105-	18541-	16898-	15185-	9-
6.8	22220-	22208-	22095-	21883-	21573-	21167-	20665-	20071-	19386-	18615-	9-
7.0	13840-	14362-	14816-	15200-	15510-	15747-	15909-	15996-	16006-	15940-	9-
7.2	5900-	6589-	7244-	7863-	8442-	8979-	9470-	9913-	10307-	10649-	9-
7.4	2035-	8091-	14103-	20041-	25871-	31563-	37086-	42412-	47512-	52358-	10-
7.6	28350	24321	20156	15876	11507	7073	2599	1889-	6367-	10809-	10-
7.8	36667	34818	32762	30511	28079	25480	22729	19843	16840	13736	10-
8.0	31048	30897	30552	30016	29292	28384	27299	26043	24624	23051	10-
8.2	19535	20362	21054	21608	22020	22286	22406	22379	22204	21883	10-
8.4	8034	9173	10249	11253	12180	13023	13775	14433	14990	15444	10-
8.6	278-	716	1705	2682	3639	4570	5468	6326	7138	7898	10-
8.8	45617-	39276-	32636-	25748-	18664-	11438-	4125-	3220	10539	17779	11-
9.0	54883-	52334-	49368-	46009-	42283-	38231-	33855-	29219-	24351-	19290-	11-
9.2	43640-	43913-	43819-	43360-	42540-	41366-	39848-	37999-	35834-	33370-	11-
9.4	24775-	26517-	28028-	29295-	30307-	31056-	31535-	31740-	31669-	31323-	11-
9.6	7420-	9453-	11399-	13243-	14966-	16553-	17990-	19264-	20363-	21278-	11-
9.8	3957	2374	768	845-	2450-	4032-	5575-	7066-	8490-	9834-	11-
10.0	8726	7869	6934	5930	4868	3759	2612	1439	253	937-	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	.21	.22	.23	.24	.25	.26	.27	.28	.29	.30	p
0.2	3007	2957	2908	2859	2810	2762	2715	2668	2622	2576	4-
0.4	2897	2850	2803	2757	2710	2665	2620	2576	2532	2488	4-
0.6	2720	2678	2635	2593	2551	2510	2468	2428	2388	2347	4-
0.8	24903	24529	24157	23788	23419	23055	22692	22332	21974	21619	5-
1.0	22197	21885	21574	21263	20955	20648	20342	20037	19735	19433	5-
1.2	19241	18994	18747	18500	18253	18007	17761	17515	17271	17027	5-
1.4	16196	16013	15828	15644	15458	15274	15087	14901	14714	14527	5-
1.6	13207	13082	12956	12828	12700	12571	12440	12309	12176	12042	5-
1.8	10400	10327	10252	10175	10096	10016	9934	9851	9766	9680	5-
2.0	78755	78432	78089	77728	77347	76950	76534	76100	75649	75182	6-
2.2	56975	56959	56924	56870	56798	56708	56600	56474	56331	56171	6-
2.4	38990	39179	39351	39507	39646	39769	39875	39965	40040	40098	6-
2.6	24817	25120	25409	25685	25948	26198	26434	26656	26866	27062	6-
2.8	14218	14560	14893	15216	15530	15835	16130	16415	16689	16954	6-
3.0	6769	7097	7419	7736	8047	8352	8651	8944	9230	9510	6-
3.2	1939	2218	2495	2769	3041	3311	3577	3841	4101	4358	6-
3.4	8486-	6359-	4225-	2086-	54	2194	4332	6465	8593	10712	7-
3.6	21550-	20108-	18643-	17156-	15651-	14127-	12588-	11034-	9468-	7891-	7-
3.8	24778-	23949-	23089-	22199-	21281-	20336-	19365-	18369-	17350-	16309-	7-
4.0	22245-	21900-	21524-	21117-	20679-	20212-	19715-	19191-	18639-	18061-	7-
4.2	17038-	17029-	16993-	16929-	16838-	16721-	16576-	16406-	16209-	15986-	7-
4.4	11263-	11449-	11615-	11759-	11883-	11986-	12068-	12128-	12167-	12185-	7-
4.6	61740-	64395-	66924-	69322-	71585-	73708-	75687-	77519-	79201-	80730-	8-
4.8	23518-	26155-	28737-	31256-	33709-	36090-	38393-	40615-	42750-	44795-	8-
5.0	972	1171-	3312-	5444-	7564-	9666-	11745-	13798-	15818-	17801-	8-
5.2	13533	12075	10587	9072	7534	5977	4405	2822	1231	362-	8-
5.4	17308	16522	15690	14815	13900	12946	11956	10934	9881	8802	8-
5.6	15654	15403	15106	14765	14379	13951	13481	12971	12423	11838	8-
5.8	11444	11545	11610	11637	11627	11580	11496	11375	11218	11025	8-
6.0	67484	70266	72804	75093	77124	78889	80384	81603	82543	83201	9-
6.2	27939	31114	34174	37108	39906	42556	45051	47380	49535	51509	9-
6.4	933	3634	6320	8982	11608	14189	16714	19174	21558	23858	9-
6.6	13407-	11573-	9690-	7767-	5812-	3831-	1835-	168	2171	4165	9-
6.8	17761-	16828-	15819-	14739-	13594-	12388-	11127-	9817-	8463-	7071-	9-
7.0	15799-	15582-	15292-	14929-	14496-	13993-	13425-	12793-	12101-	11352-	9-
7.2	10937-	11170-	11347-	11466-	11529-	11533-	11480-	11369-	11201-	10977-	9-
7.4	56926-	61191-	65130-	68724-	71952-	74798-	77248-	79287-	80907-	82098-	10-
7.6	15190-	19486-	23672-	27726-	31624-	35344-	38867-	42171-	45240-	48056-	10-
7.8	10551	7304	4013	698	2621-	5924-	9192-	12406-	15547-	18595-	10-
8.0	21334	19484	17512	15431	13254	10995	8666	6284	3863	1418	10-
8.2	21419	20814	20072	19198	18199	17080	15849	14515	13085	11570	10-
8.4	15792	16030	16158	16175	16080	15874	15560	15138	14612	13985	10-
8.6	8601	9242	9816	10319	10748	11098	11369	11558	11663	11684	10-
8.8	24884	31800	38475	44858	50902	56560	61789	66551	70810	74533	11-
9.0	14075-	8748-	3352-	2070	7476	12823	18069	23171	28089	32786	11-
9.2	30630-	27635-	24410-	20983-	17382-	13637-	9779-	5839-	1852-	2151	11-
9.4	30706-	29822-	28679-	27287-	25659-	23809-	21752-	19507-	17093-	14532-	11-
9.6	22000-	22522-	22841-	22953-	22858-	22555-	22049-	21344-	20445-	19362-	11-
9.8	11085-	12231-	13262-	14168-	14940-	15571-	16055-	16388-	16566-	16589-	11-
10.0	2117-	3276-	4403-	5486-	6516-	7482-	8374-	9184-	9904-	10526-	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	.31	.32	.33	.34	.35	.36	.37	.38	.39	.40	p
0.2	2530	2484	2440	2396	2352	2308	2265	2223	2181	2139	4-
0.4	2445	2401	2358	2316	2275	2233	2193	2152	2112	2073	4-
0.6	2307	2267	2228	2189	2151	2113	2075	2038	2000	1964	4-
0.8	21266	20916	20569	20224	19882	19542	19206	18872	18540	18212	5-
1.0	19133	18835	18539	18244	17951	17660	17370	17083	16798	16514	5-
1.2	16784	16541	16298	16058	15817	15578	15339	15101	14865	14630	5-
1.4	14339	14152	13964	13777	13589	13402	13214	13027	12840	12653	5-
1.6	11908	11773	11637	11500	11362	11224	11086	10947	10807	10667	5-
1.8	9593	9504	9414	9322	9229	9135	9040	8944	8848	8750	5-
2.0	74699	74199	73683	73153	72608	72048	71475	70886	70286	69673	6-
2.2	55994	55800	55589	55363	55120	54862	54589	54300	53997	53680	6-
2.4	40141	40168	40179	40175	40156	40122	40073	40010	39932	39840	6-
2.6	27245	27414	27570	27713	27842	27959	28062	28152	28229	28293	6-
2.8	17209	17453	17687	17910	18123	18326	18518	18699	18869	19028	6-
3.0	9783	10049	10307	10559	10803	11039	11268	11489	11702	11907	6-
3.2	4612	4861	5106	5347	5584	5816	6043	6266	6483	6695	6-
3.4	12820	14916	16998	19064	21111	23138	25142	27124	29079	31008	7-
3.6	6306-	4714-	3116-	1515-	86	1688	3289	4886	6478	8063	7-
3.8	15248-	14167-	13068-	11953-	10824-	9680-	8525-	7359-	6185-	5002-	7-
4.0	17457-	16829-	16176-	15501-	14804-	14087-	13350-	12594-	11821-	11032-	7-
4.2	15738-	15466-	15169-	14848-	14504-	14137-	13748-	13338-	12907-	12457-	7-
4.4	12181-	12156-	12110-	12043-	11954-	11845-	11716-	11566-	11396-	11207-	7-
4.6	82102-	83316-	84369-	85261-	85988-	86551-	86949-	87181-	87247-	87148-	8-
4.8	46744-	48594-	50342-	51984-	53515-	54935-	56239-	57425-	58492-	59436-	8-
5.0	19744-	21640-	23487-	25279-	27013-	28685-	30291-	31828-	33293-	34681-	8-
5.2	1955-	3543-	5122-	6688-	8237-	9766-	11270-	12747-	14191-	15600-	8-
5.4	7698	6574	5431	4274	3105	1928	746	439-	1622-	2801-	8-
5.6	11218	10565	9881	9168	8428	7663	6876	6069	5244	4403	8-
5.8	10797	10535	10239	9911	9552	9162	8743	8297	7825	7328	8-
6.0	83574	83663	83466	82984	82221	81178	79859	78270	76416	74304	9-
6.2	53294	54884	56274	57458	58433	59195	59742	60072	60183	60077	9-
6.4	26065	28169	30164	32041	33793	35414	36897	38236	39428	40466	9-
6.6	6142	8093	10011	11886	13713	15482	17187	18821	20377	21848	9-
6.8	5648-	4200-	2733-	1254-	230	1714	3190	4652	6093	7507	9-
7.0	10549-	9696-	8798-	7858-	6882-	5873-	4837-	3777-	2701-	1611-	9-
7.2	10699-	10366-	9983-	9549-	9068-	8541-	7972-	7363-	6718-	6040-	9-
7.4	82855-	83174-	83053-	82494-	81500-	80076-	78230-	75972-	73314-	70271-	10-
7.6	50603-	52867-	54835-	56498-	57846-	58872-	59570-	59936-	59969-	59669-	10-
7.8	21535-	24347-	27016-	29525-	31861-	34010-	35959-	37697-	39213-	40500-	10-
8.0	1036-	3484-	5910-	8299-	10637-	12909-	15101-	17200-	19192-	21066-	10-
8.2	9979	8323	6613	4860	3074	1269	544-	2354-	4149-	5916-	10-
8.4	13262	12449	11550	10571	9521	8405	7231	6008	4744	3448	10-
8.6	11620	11473	11244	10933	10544	10079	9542	8936	8265	7536	10-
8.8	77693	80266	82234	83580	84297	84378	83823	82637	80829	78414	11-
9.0	37222	41365	45180	48639	51713	54380	56617	58407	59738	60598	11-
9.2	6137	10071	13922	17658	21249	24664	27875	30856	33583	36033	11-
9.4	11844-	9055-	6187-	3265-	316-	2636	5566	8447	11256	13968	11-
9.6	18105-	16683-	15112-	13404-	11575-	9642-	7623-	5534-	3396-	1228-	11-
9.8	16455-	16167-	15726-	15138-	14407-	13541-	12547-	11436-	10218-	8904-	11-
10.0	11046-	11457-	11755-	11939-	12006-	11955-	11787-	11503-	11108-	10603-	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	.41	.42	.43	.44	.45	.46	.47	.48	.49	.50	p
0.2	2098	2058	2018	1978	1939	1900	1861	1823	1786	1749	4-
0.4	2033	1995	1955	1917	1880	1843	1806	1770	1734	1698	4-
0.6	1928	1892	1856	1821	1786	1752	1717	1684	1650	1617	4-
0.8	17887	17564	17244	16927	16613	16304	15996	15691	15389	15090	5-
1.0	16232	15953	15676	15400	15127	14857	14588	14321	14058	13795	5-
1.2	14396	14163	13931	13701	13472	13244	13018	12793	12570	12348	5-
1.4	12467	12281	12097	11912	11727	11543	11360	11177	10995	10814	5-
1.6	10527	10385	10244	10103	9962	9820	9679	9537	9396	9255	5-
1.8	8651	8552	8452	8351	8249	8147	8044	7940	7836	7732	5-
2.0	69047	68409	67760	67100	66429	65747	65056	64356	63647	62929	6-
2.2	53349	53003	52645	52273	51889	51493	51085	50665	50233	49791	6-
2.4	39733	39613	39478	39331	39170	38997	38811	38613	38402	38179	6-
2.6	28344	28384	28410	28423	28424	28413	28390	28355	28308	28249	6-
2.8	19178	19316	19444	19561	19667	19763	19849	19924	19988	20043	6-
3.0	12104	12293	12474	12646	12810	12966	13114	13252	13383	13505	6-
3.2	6902	7103	7299	7488	7672	7850	8022	8188	8348	8501	6-
3.4	32907	34776	36612	38415	40183	41915	43608	45262	46876	48448	7-
3.6	9639	11205	12759	14299	15824	17333	18823	20295	21745	23173	7-
3.8	3814-	2621-	1424-	226-	974	2171	3366	4556	5741	6920	7-
4.0	10227-	9408-	8577-	7733-	6880-	6016-	5145-	4267-	3383-	2495-	7-
4.2	11987-	11498-	10992-	10470-	9931-	9377-	8809-	8228-	7634-	7029-	7-
4.4	10999-	10772-	10526-	10263-	9982-	9684-	9370-	9040-	8695-	8336-	7-
4.6	86883-	86455-	85864-	85111-	84199-	83129-	81905-	80528-	79001-	77329-	8-
4.8	60257-	60952-	61522-	61964-	62278-	62464-	62522-	62452-	62254-	61930-	8-
5.0	35990-	37217-	38360-	39417-	40384-	41260-	42044-	42733-	43326-	43823-	8-
5.2	16971-	18299-	19582-	20817-	22001-	23131-	24203-	25217-	26169-	27057-	8-
5.4	3972-	5133-	6280-	7410-	8520-	9608-	10670-	11703-	12706-	13675-	8-
5.6	3550	2687	1815	939	59	820-	1697-	2569-	3434-	4289-	8-
5.8	6808	6266	5705	5127	4532	3923	3302	2671	2031	1385	8-
6.0	71940	69335	66495	63432	60156	56679	53011	49166	45158	40998	9-
6.2	59753	59213	58459	57495	56323	54949	53377	51614	49666	47540	9-
6.4	41347	42069	42628	43023	43252	43314	43210	42940	42506	41909	9-
6.6	23230	24515	25700	26778	27747	28601	29338	29955	30449	30819	9-
6.8	8889	10231	11529	12776	13967	15096	16160	17153	18072	18912	9-
7.0	51	585	1682	2771	3847	4905	5940	6947	7923	8862	9-
7.2	5331-	4596-	3839-	3062-	2270-	1467-	657-	157	969	1777	9-
7.4	66860-	63097-	59004-	54601-	49913-	44964-	39781-	34389-	28818-	23098-	10-
7.6	59038-	58079-	56799-	55204-	53303-	51108-	48631-	45885-	42887-	39652-	10-
7.8	41549-	42356-	42914-	43221-	43277-	43079-	42631-	41934-	40993-	39814-	10-
8.0	22810-	24413-	25865-	27158-	28284-	29236-	30008-	30596-	30997-	31207-	10-
8.2	7646-	9325-	10944-	12492-	13959-	15336-	16614-	17784-	18839-	19773-	10-
8.4	2128	793	547-	1883-	3207-	4508-	5779-	7011-	8195-	9323-	10-
8.6	6753	5921	5047	4136	3196	2234	1255	267	722-	1707-	10-
8.8	75409	71837	67726	63107	58015	52488	46568	40299	33730	26908	11-
9.0	60981	60884	60308	59259	57744	55776	53370	50546	47326	43736	11-
9.2	38187	40026	41536	42704	43522	43982	44081	43819	43197	42222	11-
9.4	16560	19009	21295	23397	25299	26984	28437	29646	30602	31295	11-
9.6	952	3123	5266	7362	9392	11338	13182	14908	16500	17946	11-
9.8	7506-	6038-	4514-	2947-	1352-	254	1859	3446	5001	6510	11-
10.0	9996-	9290-	8494-	7615-	6662-	5643-	4570-	3453-	2301-	1128-	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	.51	.52	.53	.54	.55	.56	.57	.58	.59	.60	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2	12128	11909	11692	11477	11264	11051	10842	10633	10427	10223	5-
1.4	10634	10455	10276	10098	9922	9746	9572	9400	9227	9056	5-
1.6	9114	8973	8832	8692	8551	8412	8273	8134	7994	7857	5-
1.8	7627	7522	7416	7311	7205	7099	6992	6886	6780	6673	5-
2.0	62203	61470	60729	59982	59228	58468	57703	56932	56157	55378	6-
2.2	49338	48874	48401	47919	47428	46928	46419	45902	45379	44847	6-
2.4	37944	37699	37442	37175	36897	36609	36311	36003	35686	35362	6-
2.6	28179	28098	28006	27903	27789	27665	27530	27387	27233	27069	6-
2.8	20087	20120	20144	20158	20162	20156	20140	20115	20080	20036	6-
3.0	13618	13723	13820	13908	13987	14059	14122	14176	14223	14261	6-
3.2	8648	8788	8922	9049	9170	9284	9391	9492	9586	9673	6-
3.4	49977	51462	52901	54296	55643	56943	58194	59397	60550	61652	7-
3.6	24576	25955	27308	28633	29930	31196	32431	33635	34806	35943	7-
3.8	8090	9251	10401	11539	12664	13775	14869	15946	17006	18047	7-
4.0	1603-	709-	185	1080	1973	2863	3750	4632	5508	6377	7-
4.2	6414-	5788-	5155-	4514-	3866-	3212-	2554-	1892-	1227-	561-	7-
4.4	7963-	7576-	7177-	6766-	6344-	5912-	5470-	5019-	4560-	4094-	7-
4.6	75514-	73560-	71471-	69252-	66907-	64440-	61856-	59161-	56360-	53458-	8-
4.8	61480-	60905-	60207-	59387-	58449-	57393-	56223-	54940-	53549-	52052-	8-
5.0	44222-	44523-	44725-	44828-	44832-	44738-	44546-	44256-	43870-	43388-	8-
5.2	27880-	28635-	29320-	29935-	30477-	30947-	31342-	31662-	31907-	32076-	8-
5.4	14607-	15502-	16355-	17165-	17931-	18649-	19319-	19939-	20507-	21021-	8-
5.6	5132-	5960-	6771-	7563-	8333-	9080-	9801-	10494-	11158-	11790-	8-
5.8	735	82	570-	1221-	1869-	2510-	3144-	3768-	4381-	4981-	8-
6.0	36703	32285	27760	23144	18451	13697	8898	4070	772-	5611-	9-
6.2	45245	42788	40179	37427	34542	31535	28417	25198	21890	18505	9-
6.4	41151	40236	39168	37951	36590	35090	33457	31697	29818	27826	9-
6.6	31063	31180	31170	31034	30772	30384	29874	29243	28494	27630	9-
6.8	19669	20341	20925	21418	21818	22124	22335	22449	22467	22388	9-
7.0	9759	10612	11416	12167	12862	13498	14072	14581	15024	15398	9-
7.2	2576	3363	4133	4883	5609	6308	6976	7609	8206	8763	9-
7.4	17257-	11326-	5337-	680	6694	12673	18586	24404	30095	35632	10-
7.6	36199-	32547-	28717-	24729-	20605-	16369-	12043-	7651-	3218-	1233	10-
7.8	38403-	36770-	34923-	32874-	30635-	28219-	25639-	22912-	20053-	17079-	10-
8.0	31226-	31054-	30692-	30143-	29410-	28497-	27411-	26158-	24746-	23184-	10-
8.2	20579-	21253-	21789-	22186-	22440-	22550-	22516-	22337-	22015-	21552-	10-
8.4	10388-	11383-	12301-	13136-	13882-	14534-	15088-	15540-	15888-	16129-	10-
8.6	2679-	3632-	4559-	5454-	6311-	7122-	7883-	8589-	9233-	9812-	10-
8.8	19885	12714	5447	1859-	9152-	16377-	23480-	30408-	37111-	43539-	11-
9.0	39803	35559	31038	26274	21305	16170	10909	5564	175	5216-	11-
9.2	40900	39245	37268	34987	32420	29589	26517	23228	19751	16113	11-
9.4	31720	31874	31756	31367	30709	29790	28617	27200	25553	23686	11-
9.6	19230	20343	21275	22016	22561	22906	23046	22982	22713	22242	11-
9.8	7958	9331	10618	11806	12884	13842	14672	15365	15916	16319	11-
10.0	57	1241	2413	3562	4676	5744	6758	7705	8578	9369	11-

ν \ x	.61	.62	.63	.64	.65	.66	.67	.68	.69	.70	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2	10020	9820	9621	9425	9231	9038	8848	8660	8473	8289	5-
1.4	8887	8718	8551	8385	8221	8058	7896	7736	7577	7420	5-
1.6	7720	7583	7447	7312	7178	7044	6911	6780	6649	6519	5-
1.8	6567	6461	6355	6248	6142	6037	5932	5827	5722	5618	5-
2.0	54595	53809	53020	52229	51435	50640	49843	49045	48247	47449	6-
2.2	44309	43764	43214	42658	42096	41530	40958	40384	39805	39222	6-
2.4	35027	34684	34334	33975	33609	33238	32858	32472	32080	31683	6-
2.6	26896	26714	26523	26324	26116	25900	25676	25445	25206	24961	6-
2.8	19984	19922	19852	19773	19686	19591	19487	19376	19257	19131	6-
3.0	14290	14312	14326	14333	14331	14322	14306	14282	14250	14212	6-
3.2	9754	9828	9895	9956	10010	10057	10097	10132	10160	10181	6-
3.4	62704	63704	64653	65549	66394	67186	67925	68613	69246	69829	7-
3.6	37046	38113	39144	40138	41094	42013	42892	43733	44533	45294	7-
3.8	19068	20068	21046	22001	22931	23838	24719	25573	26400	27200	7-
4.0	7238	8090	8931	9762	10580	11386	12177	12954	13715	14459	7-
4.2	106	774	1440	2104	2765	3422	4075	4721	5362	5995	7-
4.4	3621-	3143-	2659-	2171-	1680-	1186-	690-	193-	304	800	7-
4.6	50461-	47373-	44202-	40953-	37633-	34246-	30799-	27300-	23753-	20165-	8-
4.8	50452-	48753-	46958-	45072-	43098-	41041-	38905-	36694-	34412-	32065-	8-
5.0	42812-	42144-	41385-	40537-	39603-	38584-	37483-	36302-	35045-	33714-	8-
5.2	32169-	32185-	32126-	31992-	31782-	31498-	31141-	30711-	30211-	29640-	8-
5.4	21482-	21887-	22235-	22527-	22762-	22938-	23056-	23115-	23117-	23059-	8-
5.6	12390-	12954-	13483-	13974-	14426-	14838-	15210-	15539-	15826-	16070-	8-
5.8	5565-	6133-	6682-	7211-	7718-	8202-	8662-	9095-	9502-	9881-	8-
6.0	10432-	15218-	19956-	24628-	29221-	33719-	38109-	42376-	46507-	50489-	9-
6.2	15056	11553	8010	4440	853	2736-	6316-	9874-	13398-	16875-	9-
6.4	25730	23537	21256	18895	16464	13971	11426	8839	6219	3576	9-
6.6	26656	25574	24389	23107	21733	20271	18730	17114	15429	13684	9-
6.8	22213	21943	21579	21123	20577	19943	19225	18425	17547	16595	9-
7.0	15701	15933	16093	16179	16192	16132	15998	15793	15516	15169	9-
7.2	9278	9748	10170	10543	10866	11136	11353	11515	11622	11673	9-
7.4	40986	46129	51035	55680	60041	64095	67822	71203	74223	76866	10-
7.6	5677	10090	14449	18729	22908	26963	30872	34615	38172	41523	10-
7.8	14006-	10853-	7637-	4378-	1094-	2197	5475	8721	11919	15048	10-
8.0	21481-	19648-	17697-	15638-	13485-	11250-	8948-	6592-	4196-	1775-	10-
8.2	20952-	20218-	19355-	18368-	17265-	16053-	14738-	13330-	11838-	10270-	10-
8.4	16261-	16285-	16199-	16004-	15703-	15296-	14788-	14180-	13478-	12687-	10-
8.6	10322-	10760-	11121-	11404-	11607-	11728-	11768-	11724-	11599-	11392-	10-
8.8	49645-	55383-	60711-	65591-	69987-	73867-	77203-	79971-	82151-	83728-	11-
9.0	10566-	15833-	20979-	25961-	30743-	35288-	39560-	43527-	47158-	50427-	11-
9.2	12343	8474	4536	561	3418-	7370-	11262-	15063-	18743-	22271-	11-
9.4	21619	19369	16954	14395	11715	8935	6080	3174	240	2695-	11-
9.6	21574	20714	19671	18454	17073	15541	13872	12081	10183	8195	11-
9.8	16571	16670	16615	16405	16045	15536	14884	14095	13176	12136	11-
10.0	10068	10671	11171	11563	11843	12010	12061	11996	11816	11522	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	.71	.72	.73	.74	.75	.76	.77	.78	.79	.80	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2	8108	7929	7750	7576	7403	7232	7063	6897	6734	6572	5-
1.4	7264	7110	6958	6808	6659	6512	6366	6222	6080	5939	5-
1.6	6390	6262	6134	6008	5883	5759	5637	5516	5396	5277	5-
1.8	5515	5412	5309	5208	5106	5005	4906	4806	4708	4610	5-
2.0	46651	45854	45057	44263	43469	42678	41890	41104	40321	39542	6-
2.2	38636	38049	37458	36865	36271	35676	35079	34483	33885	33287	6-
2.4	31280	30872	30459	30042	29621	29196	28768	28336	27901	27464	6-
2.6	24709	24451	24186	23915	23638	23356	23069	22777	22481	22180	6-
2.8	18996	18855	18708	18554	18393	18227	18054	17876	17692	17503	6-
3.0	14166	14114	14054	13988	13916	13838	13753	13663	13566	13464	6-
3.2	10197	10206	10209	10206	10197	10182	10161	10135	10103	10065	6-
3.4	70358	70836	71260	71634	71957	72228	72448	72620	72741	72813	7-
3.6	46014	46694	47332	47930	48487	49002	49477	49911	50303	50655	7-
3.8	27971	28714	29428	30111	30765	31388	31980	32540	33069	33566	7-
4.0	15186	15895	16586	17257	17908	18538	19147	19735	20300	20843	7-
4.2	6619	7235	7842	8437	9022	9594	10155	10702	11235	11754	7-
4.4	1296	1789	2280	2767	3251	3729	4202	4669	5128	5581	7-
4.6	16543-	12893-	9220-	5533-	1836-	1867	5564	9252	12925	16576	8-
4.8	29657-	27193-	24678-	22117-	19514-	16876-	14206-	11509-	8792-	6059-	8-
5.0	32313-	30844-	29312-	27718-	26068-	24364-	22610-	20810-	18968-	17088-	8-
5.2	29002-	28297-	27528-	26696-	25803-	24853-	23846-	22786-	21676-	20517-	8-
5.4	22944-	22772-	22543-	22258-	21918-	21523-	21076-	20577-	20028-	19431-	8-
5.6	16269-	16425-	16535-	16601-	16623-	16599-	16531-	16419-	16264-	16065-	8-
5.8	10230-	10549-	10838-	11094-	11318-	11509-	11666-	11790-	11880-	11935-	8-
6.0	54309-	57957-	61421-	64690-	67755-	70606-	73235-	75635-	77799-	79720-	9-
6.2	20295-	23645-	26914-	30091-	33166-	36129-	38969-	41677-	44245-	46665-	9-
6.4	920	1740-	4393-	7030-	9641-	12217-	14749-	17227-	19643-	21988-	9-
6.6	11885	10039	8154	6236	4294	2336	368	1601-	3564-	5513-	9-
6.8	15573	14485	13337	12133	10878	9577	8237	6862	5459	4034	9-
7.0	14754	14273	13728	13122	12457	11737	10965	10144	9278	8371	9-
7.2	11669	11609	11493	11324	11100	10824	10497	10120	9695	9225	9-
7.4	79119	80972	82416	83445	84053	84238	84002	83344	82270	80786	10-
7.6	44652	47541	50176	52542	54629	56424	57920	59108	59983	60541	10-
7.8	18092	21033	23856	26544	29081	31455	33653	35661	37469	39068	10-
8.0	656	3084	5493	7869	10199	12467	14662	16770	18778	20676	10-
8.2	8638-	6952-	5221-	3458-	1673-	123	1917	3700	5460	7185	10-
8.4	11811-	10857-	9831-	8739-	7590-	6391-	5150-	3874-	2574-	1256-	10-
8.6	11106-	10742-	10303-	9793-	9215-	8572-	7870-	7114-	6308-	5459-	10-
8.8	84691-	85034-	84754-	83855-	82344-	80232-	77537-	74277-	70477-	66167-	11-
9.0	53307-	55779-	57822-	59422-	60567-	61249-	61463-	61209-	60488-	59306-	11-
9.2	25620-	28763-	31675-	34333-	36715-	38804-	40581-	42035-	43153-	43927-	11-
9.4	5608-	8473-	11268-	13967-	16550-	18993-	21278-	23385-	25296-	26996-	11-
9.6	6135	4021	1872	293-	2456-	4598-	6699-	8742-	10708-	12581-	11-
9.8	10984	9732	8390	6971	5488	3955	2387	796	802-	2393-	11-
10.0	11118	10608	9996	9289	8492	7615	6665	5652	4585	3474	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	.81	.82	.83	.84	.85	.86	.87	.88	.89	.90	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2	6413	6256	6102	5948	5798	5651	5505	5362	5222	5084	5-
1.4	5800	5663	5528	5395	5263	5134	5006	4880	4756	4634	5-
1.6	5159	5043	4928	4814	4701	4590	4480	4372	4266	4160	5-
1.8	4513	4417	4320	4226	4132	4040	3948	3857	3768	3679	5-
2.0	38766	37995	37228	36466	35709	34957	34211	33470	32735	32007	6-
2.2	32690	32094	31499	30905	30312	29722	29134	28548	27965	27386	6-
2.4	27025	26584	26141	25697	25251	24806	24360	23913	23467	23022	6-
2.6	21875	21567	21255	20941	20623	20302	19979	19653	19326	18998	6-
2.8	17308	17109	16905	16697	16485	16269	16050	15827	15600	15370	6-
3.0	13357	13244	13126	13003	12876	12744	12608	12467	12323	12174	6-
3.2	10023	9975	9922	9864	9801	9734	9662	9586	9505	9420	6-
3.4	72838	72816	72746	72630	72470	72266	72018	71728	71398	71027	7-
3.6	50966	51238	51469	51660	51812	51926	52001	52038	52038	52002	7-
3.8	34032	34465	34867	35237	35574	35880	36154	36396	36607	36786	7-
4.0	21363	21859	22332	22780	23205	23604	23980	24331	24656	24958	7-
4.2	12257	12746	13219	13675	14114	14537	14941	15328	15697	16047	7-
4.4	6025	6461	6888	7304	7711	8107	8492	8865	9226	9575	7-
4.6	20201	23793	27347	30858	34320	37728	41078	44362	47580	50726	8-
4.8	3315-	565-	2183	4930	7668	10393	13100	15784	18441	21067	8-
5.0	15174-	13229-	11258-	9265-	7254-	5229-	3193-	1151-	893	2935	8-
5.2	19313-	18067-	16781-	15459-	14103-	12716-	11303-	9865-	8406-	6930-	8-
5.4	18786-	18096-	17363-	16588-	15773-	14922-	14035-	13115-	12164-	11186-	8-
5.6	15824-	15541-	15218-	14853-	14453-	14013-	13538-	13028-	12485-	11910-	8-
5.8	11956-	11944-	11897-	11816-	11702-	11555-	11375-	11164-	10922-	10649-	8-
6.0	81395-	82817-	83984-	84892-	85541-	85929-	86056-	85921-	85528-	84877-	9-
6.2	48928-	51028-	52959-	54713-	56287-	57675-	58874-	59880-	60691-	61304-	9-
6.4	24254-	26433-	28517-	30499-	32373-	34132-	35771-	37283-	38664-	39910-	9-
6.6	7441-	9340-	11204-	13024-	14795-	16510-	18162-	19745-	21253-	22682-	9-
6.8	2591	1138	319-	1775-	3225-	4660-	6077-	7470-	8832-	10159-	9-
7.0	7428	6451	5447	4418	3371	2309	1236	159	919-	1994-	9-
7.2	8712	8158	7566	6939	6279	5591	4876	4139	3383	2612	9-
7.4	78899	76620	73962	70937	67562	63855	59833	55518	50932	46096	10-
7.6	60779	60697	60296	59579	58549	57214	55580	53656	51455	48986	10-
7.8	40448	41603	42527	43214	43661	43867	43830	43553	43035	42281	10-
8.0	22452	24095	25596	26947	28141	29169	30028	30711	31216	31540	10-
8.2	8866	10492	12052	13538	14941	16251	17461	18563	19552	20421	10-
8.4	70	1395	2712	4010	5283	6521	7717	8863	9951	10975	10-
8.6	4573-	3654-	2711-	1749-	775-	204	1182	2152	3107	4041	10-
8.8	61377-	56143-	50503-	44499-	38175-	31576-	24750-	17747-	10617-	3410-	11-
9.0	57674-	55605-	53113-	50220-	46947-	43320-	39366-	35116-	30603-	25859-	11-
9.2	44351-	44422-	44140-	43509-	42532-	41219-	39581-	37630-	35383-	32858-	11-
9.4	28472-	29710-	30701-	31438-	31913-	32124-	32070-	31751-	31169-	30331-	11-
9.6	14345-	15983-	17483-	18830-	20015-	21026-	21855-	22495-	22941-	23190-	11-
9.8	3962-	5495-	6978-	8397-	9741-	10997-	12153-	13199-	14127-	14927-	11-
10.0	2330	1164	13-	1189-	2355-	3498-	4609-	5675-	6689-	7639-	11-

TABLE OF THE WEDGE FUNCTION $G_{\nu}(e^{x})$

ν \ x	.91	.92	.93	.94	.95	.96	.97	.98	.99	1.00	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2	4948	4814	4682	4553	4426	4302	4181	4060	3943	3827	5-
1.4	4514	4396	4280	4165	4052	3942	3833	3726	3621	3518	5-
1.6	4057	3954	3853	3754	3656	3559	3464	3371	3279	3189	5-
1.8	3591	3505	3419	3334	3250	3168	3087	3007	2928	2851	5-
2.0	31285	30571	29863	29162	28469	27783	27106	26436	25775	25122	6-
2.2	26809	26236	25667	25102	24541	23985	23433	22887	22346	21809	6-
2.4	22577	22132	21689	21248	20808	20369	19934	19501	19070	18641	6-
2.6	18668	18336	18004	17672	17340	17007	16674	16341	16009	15678	6-
2.8	15138	14904	14667	14428	14187	13945	13700	13455	13209	12962	6-
3.0	12022	11867	11708	11547	11383	11216	11046	10874	10700	10525	6-
3.2	9331	9239	9143	9043	8940	8834	8725	8613	8499	8381	6-
3.4	70616	70168	69682	69162	68605	68016	67395	66743	66060	65349	7-
3.6	51929	51822	51679	51504	51295	51053	50781	50478	50146	49784	7-
3.8	36936	37054	37142	37201	37230	37229	37201	37146	37062	36953	7-
4.0	25234	25485	25712	25914	26092	26245	26374	26480	26561	26620	7-
4.2	16379	16692	16986	17262	17517	17754	17972	18170	18349	18510	7-
4.4	9911	10234	10544	10840	11122	11390	11643	11883	12107	12318	7-
4.6	53795	56783	59687	62502	65227	67857	70389	72822	75153	77379	8-
4.8	23656	26206	28712	31170	33577	35928	38222	40454	42622	44722	8-
5.0	4972	6999	9013	11011	12988	14941	16868	18764	20626	22452	8-
5.2	5439-	3936-	2426-	910-	607	2123	3635	5140	6634	8115	8-
5.4	10181-	9153-	8104-	7037-	5954-	4857-	3750-	2634-	1512-	388-	8-
5.6	11305-	10671-	10010-	9325-	8617-	7887-	7138-	6371-	5589-	4794-	8-
5.8	10347-	10017-	9659-	9275-	8866-	8433-	7978-	7501-	7004-	6488-	8-
6.0	83972-	82817-	81416-	79774-	77898-	75793-	73466-	70927-	68183-	65242-	9-
6.2	61719-	61935-	61952-	61771-	61392-	60819-	60054-	59099-	57959-	56639-	9-
6.4	41017-	41981-	42799-	43470-	43991-	44361-	44580-	44648-	44564-	44330-	9-
6.6	24025-	25279-	26439-	27500-	28460-	29315-	30062-	30699-	31224-	31636-	9-
6.8	11444-	12684-	13874-	15008-	16082-	17093-	18037-	18910-	19708-	20430-	9-
7.0	3060-	4112-	5147-	6161-	7148-	8105-	9027-	9912-	10756-	11554-	9-
7.2	1828	1037	240	558-	1353-	2142-	2921-	3687-	4437-	5167-	9-
7.4	41037	35778	30345	24767	19068	13278	7425	1536	4361-	10236-	10-
7.6	46265	43305	40122	36733	33156	29410	25515	21489	17355	13133	10-
7.8	41296	40085	38655	37016	35175	33144	30934	28558	26028	23358	10-
8.0	31681	31639	31414	31008	30425	29667	28740	27649	26400	25002	10-
8.2	21164	21779	22261	22608	22817	22889	22822	22618	22278	21804	10-
8.4	11929	12805	13600	14307	14923	15443	15865	16186	16405	16520	10-
8.6	4948	5821	6655	7444	8182	8866	9490	10051	10544	10968	10-
8.8	3820	11023	18148	25144	31963	38555	44875	50878	56525	61774	11-
9.0	20922-	15828-	10617-	5326-	5	5335	10626	15839	20934	25875	11-
9.2	30074-	27055-	23823-	20406	16829-	13121-	9312-	5429-	1505-	2431	11-
9.4	29244-	27916-	26359-	24587-	22613-	20455-	18131-	15658-	13059-	10354-	11-
9.6	23239-	23088-	22739-	22196-	21462-	20545-	19453-	18196-	16783-	15229-	11-
9.8	15593-	16119-	16500-	16734-	16818-	16752-	16536-	16173-	15667-	15022-	11-
10.0	8518-	9317-	10029-	10647-	11166-	11580-	11887-	12083-	12167-	12138-	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	1.01	1.02	1.03	1.04	1.05	1.06	1.07	1.08	1.09	1.10	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2											5-
1.4											5-
1.6											5-
1.8											5-
2.0											6-
2.2	21279	20754	20236	19723	19216	18716	18222	17736	17256	16783	6-
2.4	18216	17795	17377	16962	16550	16144	15741	15344	14949	14560	6-
2.6	15347	15018	14690	14364	14041	13718	13398	13080	12765	12452	6-
2.8	12715	12467	12219	11970	11722	11475	11228	10982	10737	10492	6-
3.0	10348	10169	9989	9807	9625	9442	9258	9074	8890	8704	6-
3.2	8262	8140	8016	7890	7763	7634	7503	7371	7238	7103	6-
3.4	64612	63848	63058	62247	61412	60558	59682	58790	57880	56955	7-
3.6	49395	48980	48538	48072	47581	47069	46534	45978	45403	44809	7-
3.8	36817	36656	36470	36260	36025	35769	35491	35192	34873	34535	7-
4.0	26656	26669	26660	26629	26577	26504	26411	26297	26165	26014	7-
4.2	18651	18773	18877	18962	19029	19078	19109	19123	19119	19098	7-
4.4	12513	12693	12859	13010	13146	13268	13374	13467	13545	13608	7-
4.6	79498	81510	83410	85201	86879	88444	89897	91234	92458	93569	8-
4.8	46752	48710	50593	52400	54127	55774	57339	58821	60218	61529	8-
5.0	24238	25982	27681	29332	30933	32483	33978	35418	36799	38121	8-
5.2	9580	11027	12452	13854	15229	16575	17890	19172	20419	21629	8-
5.4	738	1862	2982	4096	5201	6296	7377	8442	9491	10520	8-
5.6	3987-	3170-	2346-	1517-	684-	151	985	1817	2645	3467	8-
5.8	5956-	5408-	4846-	4272-	3687-	3092-	2489-	1880-	1267-	650-	8-
6.0	62115-	58812-	55341-	51715-	47945-	44041-	40014-	35878-	31644-	27324-	9-
6.2	55143-	53476-	51645-	49656-	47516-	45232-	42812-	40264-	37596-	34816-	9-
6.4	43948-	43418-	42744-	41929-	40976-	39888-	38670-	37326-	35862-	34283-	9-
6.6	31934-	32116-	32184-	32136-	31974-	31699-	31313-	30817-	30213-	29504-	9-
6.8	21073-	21634-	22112-	22505-	22811-	23032-	23164-	23210-	23168-	23039-	9-
7.0	12304-	13004-	13649-	14239-	14770-	15240-	15649-	15995-	16276-	16492-	9-
7.2	5874-	6555-	7207-	7828-	8414-	8963-	9474-	9943-	10369-	10751-	9-
7.4	16064-	21815-	27464-	32984-	38350-	43537-	48522-	53283-	57799-	62049-	10-
7.6	8845	4513	158	4198-	8533-	12825-	17055-	21200-	25241-	29159-	10-
7.8	20564	17660	14663	11587	8451	5270	2061	1158-	4372-	7562-	10-
8.0	23462	21790	19995	18089	16080	13982	11806	9565	7271	4937	10-
8.2	21200	20469	19616	18647	17568	16385	15105	13737	12288	10768	10-
8.4	16530	16436	16239	15941	15542	15047	14458	13779	13015	12171	10-
8.6	11318	11593	11791	11911	11953	11917	11802	11610	11348	11000	10-
8.8	66592	70944	74801	78138	80933	83168	84828	85904	86390	86283	11-
9.0	30626	35152	39420	43400	47064	50385	53340	55909	58074	59819	11-
9.2	6348	10216	14006	17688	21235	24619	27816	30801	33552	36049	11-
9.4	7565-	4714-	1826-	1076	3971	6833	9641	12372	15004	17517	11-
9.6	13545-	11747-	9849-	7869-	5822-	3727-	1600-	540	2676	4789	11-
9.8	14243-	13339-	12316-	11186-	9956-	8640-	7248-	5792-	4285-	2742-	11-
10.0	11997-	11745-	11385-	10920-	10354-	9694-	8944-	8113-	7208-	6237-	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	1.11	1.12	1.13	1.14	1.15	1.16	1.17	1.18	1.19	1.20	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2											5-
1.4											5-
1.6											5-
1.8											5-
2.0											6-
2.2	16317	15857	15405	14961	14524	14095	13673	13258	12851	12452	6-
2.4	14175	13795	13420	13051	12687	12327	11974	11626	11284	10947	6-
2.6	12142	11835	11531	11231	10934	10640	10350	10064	9781	9503	6-
2.8	10249	10008	9768	9530	9293	9058	8826	8596	8368	8143	6-
3.0	8520	8335	8151	7968	7785	7602	7421	7240	7061	6884	6-
3.2	6968	6833	6696	6560	6423	6285	6148	6011	5874	5738	6-
3.4	56014	55061	54096	53120	52135	51142	50140	49134	48123	47108	7-
3.6	44197	43570	42926	42268	41597	40913	40217	39511	38796	38072	7-
3.8	34177	33802	33409	33000	32576	32137	31685	31220	30743	30255	7-
4.0	25845	25658	25455	25235	24999	24748	24483	24204	23912	23608	7-
4.2	19061	19008	18940	18856	18757	18645	18518	18377	18224	18058	7-
4.4	13658	13693	13715	13723	13719	13701	13671	13629	13575	13509	7-
4.6	94566	95449	96221	96881	97430	97871	98205	98432	98555	98577	8-
4.8	62755	63893	64945	65909	66785	67575	68278	68894	69424	69870	8-
5.0	39383	40581	41717	42789	43795	44735	45610	46417	47158	47831	8-
5.2	22800	23930	25018	26062	27062	28015	28921	29778	30587	31346	8-
5.4	11529	12514	13474	14408	15314	16191	17037	17851	18632	19379	8-
5.6	4281	5086	5879	6660	7426	8177	8910	9624	10319	10992	8-
5.8	32-	586	1203	1817	2427	3031	3628	4216	4795	5363	8-
6.0	22931-	18476-	13971-	9430-	4864-	286-	4294	8862	13408	17920	9-
6.2	31935-	28960-	25901-	22767-	19568-	16313-	13012-	9675-	6310-	2928-	9-
6.4	32595-	30803-	28914-	26934-	24870-	22729-	20518-	18244-	15915-	13538-	9-
6.6	28694-	27785-	26781-	25686-	24504-	23240-	21899-	20485-	19003-	17459-	9-
6.8	22825-	22527-	22145-	21682-	21140-	20522-	19830-	19066-	18235-	17339-	9-
7.0	16641-	16725-	16742-	16694-	16580-	16401-	16159-	15854-	15488-	15064-	9-
7.2	11087-	11376-	11616-	11807-	11949-	12041-	12082-	12074-	12015-	11908-	9-
7.4	66015-	69681-	73031-	76051-	78730-	81055-	83020-	84616-	85839-	86684-	10-
7.6	32935-	36551-	39991-	43237-	46277-	49095-	51680-	54020-	56106-	57928-	10-
7.8	10714-	13810-	16835-	19774-	22613-	25336-	27930-	30384-	32686-	34824-	10-
8.0	2575	200	2177-	4541-	6882-	9185-	11438-	13630-	15750-	17786-	10-
8.2	9184	7548	5868	4154	2416	665	1090-	2839-	4573-	6280-	10-
8.4	11252	10264	9213	8106	6950	5752	4519	3259	1979	688	10-
8.6	10587	10106	9559	8950	8285	7566	6799	5989	5141	4260	10-
8.8	85586	84306	82452	80039	77085	73611	69642	65206	60335	55062	11-
9.0	61135	62012	62445	62432	61975	61078	59749	57998	55839	53289	11-
9.2	38274	40210	41844	43164	44161	44829	45162	45161	44826	44160	11-
9.4	19891	22108	24151	26003	27651	29083	30287	31255	31981	32458	11-
9.6	6862	8879	10822	12676	14426	16057	17557	18913	20115	21153	11-
9.8	1174-	403	1977	3535	5061	6544	7971	9330	10609	11798	11-
10.0	5210-	4135-	3023-	1883-	727-	436	1595	2740	3861	4947	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	1.21	1.22	1.23	1.24	1.25	1.26	1.27	1.28	1.29	1.30	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2											5-
1.4											5-
1.6											5-
1.8											5-
2.0											6-
2.2	12060	11676	11300	10932	10571	10218	9872	9535	9206	8884	6-
2.4	10617	10291	9972	9658	9351	9050	8754	8465	8182	7905	6-
2.6	9229	8959	8693	8431	8174	7920	7672	7429	7189	6955	6-
2.8	7920	7700	7483	7269	7057	6848	6643	6442	6243	6048	6-
3.0	6706	6532	6358	6187	6017	5849	5682	5519	5357	5198	6-
3.2	5602	5466	5331	5197	5064	4932	4799	4670	4541	4413	6-
3.4	46091	45072	44052	43033	42016	41002	39991	38985	37984	36990	7-
3.6	37341	36604	35862	35115	34364	33610	32855	32099	31342	30588	7-
3.8	29756	29248	28731	28205	27674	27136	26592	26044	25492	24937	7-
4.0	23291	22964	22626	22279	21923	21558	21185	20806	20420	20028	7-
4.2	17880	17691	17491	17281	17060	16831	16592	16345	16091	15830	7-
4.4	13432	13344	13246	13137	13019	12891	12754	12608	12455	12294	7-
4.6	98498	98322	98050	97686	97232	96690	96065	95357	94571	93710	8-
4.8	70231	70510	70707	70824	70862	70823	70709	70522	70264	69936	8-
5.0	48438	48978	49453	49861	50204	50483	50698	50851	50943	50974	8-
5.2	32055	32713	33321	33877	34382	34836	35240	35592	35895	36147	8-
5.4	20090	20766	21405	22006	22570	23095	23582	24030	24439	24809	8-
5.6	11643	12271	12874	13452	14004	14528	15025	15494	15935	16347	8-
5.8	5918	6461	6989	7502	7998	8477	8939	9381	9805	10208	8-
6.0	22387	26799	31146	35416	39602	43693	47681	51558	55317	58949	9-
6.2	461	3850	7228	10587	13918	17212	20462	23659	26796	29865	9-
6.4	11121-	8670-	6194-	3700-	1195-	1314	3818	6312	8789	11241	9-
6.6	15858-	14206-	12509-	10772-	9000-	7201-	5379-	3540-	1690-	164	9-
6.8	16381-	15367-	14299-	13182-	12020-	10816-	9576-	8304-	7005-	5682-	9-
7.0	14582-	14044-	13455-	12815-	12128-	11396-	10622-	9810-	8962-	8083-	9-
7.2	11751-	11547-	11297-	11001-	10661-	10279-	9857-	9396-	8898-	8366-	9-
7.4	87151-	87238-	86948-	86284-	85251-	83854-	82103-	80006-	77575-	74822-	10-
7.6	59479-	60754-	61748-	62458-	62881-	63018-	62870-	62438-	61726-	60740-	10-
7.8	36788-	38570-	40162-	41557-	42748-	43732-	44504-	45061-	45403-	45529-	10-
8.0	19728-	21565-	23290-	24892-	26366-	27702-	28896-	29942-	30834-	31570-	10-
8.2	7952-	9579-	11154-	12666-	14108-	15473-	16753-	17941-	19032-	20020-	10-
8.4	608-	1900-	3181-	4443-	5679-	6882-	8045-	9161-	10225-	11230-	10-
8.6	3353	2425	1481	528	427-	1381-	2326-	3256-	4167-	5053-	10-
8.8	49424	43458	37206	30709	24012	17157	10191	3160	3892-	10919-	11-
9.0	50365	47090	43489	39586	35409	30990	26358	21545	16587	11516	11-
9.2	43169	41861	40247	38340	36154	33707	31016	28102	24987	21693	11-
9.4	32685	32660	32384	31859	31092	30088	28855	27404	25747	23896	11-
9.6	22020	22709	23214	23534	23664	23605	23359	22926	22313	21523	11-
9.8	12886	13866	14728	15466	16074	16547	16883	17077	17130	17041	11-
10.0	5989	6978	7905	8762	9542	10239	10846	11358	11772	12083	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.40	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2											5-
1.4											5-
1.6											5-
1.8											5-
2.0											6-
2.2	8570	8263	7964	7673	7388	7111	6842	6580	6325	6077	6-
2.4	7634	7369	7110	6857	6610	6370	6135	5906	5684	5467	6-
2.6	6724	6499	6278	6063	5852	5645	5444	5246	5055	4868	6-
2.8	5856	5667	5483	5302	5124	4949	4779	4612	4449	4289	6-
3.0	5041	4887	4733	4584	4437	4292	4151	4012	3875	3741	6-
3.2	4287	4163	4040	3919	3799	3682	3566	3452	3339	3229	6-
3.4	36002	35022	34050	33088	32136	31195	30265	29347	28442	27549	7-
3.6	29834	29082	28333	27588	26848	26114	25384	24661	23945	23236	7-
3.8	24380	23820	23260	22699	22139	21579	21021	20465	19912	19363	7-
4.0	19632	19230	18825	18416	18006	17593	17178	16763	16348	15933	7-
4.2	15562	15288	15007	14722	14433	14140	13844	13543	13241	12937	7-
4.4	12126	11951	11770	11582	11390	11192	10989	10781	10570	10356	7-
4.6	92777	91774	90707	89576	88388	87143	85847	84501	83110	81678	8-
4.8	69543	69084	68562	67981	67342	66648	65902	65106	64263	63376	8-
5.0	50947	50862	50722	50527	50279	49980	49633	49238	48798	48315	8-
5.2	36351	36506	36614	36675	36691	36662	36589	36474	36318	36123	8-
5.4	25140	25432	25684	25900	26078	26218	26322	26390	26422	26421	8-
5.6	16730	17083	17406	17700	17964	18199	18404	18580	18727	18845	8-
5.8	10590	10952	11292	11610	11906	12180	12431	12660	12866	13049	8-
6.0	62448	65809	69024	72090	75002	77755	80347	82773	85032	87122	9-
6.2	32859	35772	38598	41331	43965	46496	48919	51229	53424	55499	9-
6.4	13662	16045	18386	20678	22915	25092	27205	29248	31218	33111	9-
6.6	2018	3867	5703	7523	9322	11093	12833	14537	16201	17821	9-
6.8	4340-	2985-	1620-	249-	1122	2489	3849	5197	6530	7842	9-
7.0	7174-	6241-	5285-	4311-	3322-	2322-	1315-	302-	711	1722	9-
7.2	7802-	7208-	6587-	5941-	5273-	4586-	3881-	3163-	2433-	1695-	9-
7.4	71759-	68403-	64767-	60870-	56728-	52360-	47785-	43023-	38092-	33015-	10-
7.6	59484-	57967-	56196-	54182-	51933-	49461-	46779-	43900-	40836-	37602-	10-
7.8	45439-	45135-	44620-	43897-	42972-	41849-	40534-	39037-	37363-	35522-	10-
8.0	32147-	32562-	32815-	32904-	32831-	32597-	32204-	31654-	30952-	30102-	10-
8.2	20901-	21670-	22324-	22859-	23274-	23568-	23738-	23787-	23712-	23517-	10-
8.4	12171-	13043-	13841-	14561-	15200-	15754-	16221-	16599-	16886-	17081-	10-
8.6	5908-	6727-	7506-	8240-	8926-	9558-	10135-	10652-	11108-	11499-	10-
8.8	17877-	24720-	31408-	37896-	44147-	50122-	55785-	61102-	66043-	70579-	11-
9.0	6367	1175	4025-	9197-	14309-	19326-	24217-	28948-	33491-	37817-	11-
9.2	18245	14668	10987	7229	3420	413-	4242-	8043-	11788-	15452-	11-
9.4	21866	19674	17335	14868	12291	9624	6886	4098	1281	1546-	11-
9.6	20564	19443	18171	16758	15214	13553	11788	9932	8001	6009	11-
9.8	16811	16443	15940	15307	14549	13674	12687	11599	10418	9154	11-
10.0	12290	12391	12386	12274	12058	11739	11321	10807	10202	9511	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	1.41	1.42	1.43	1.44	1.45	1.46	1.47	1.48	1.49	1.50	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2											5-
1.4											5-
1.6											5-
1.8											5-
2.0											6-
2.2	5836	5603	5376	5156	4943	4735	4535	4341	4154	3972	6-
2.4	5255	5050	4850	4656	4468	4285	4108	3935	3769	3608	6-
2.6	4685	4506	4333	4165	4001	3841	3686	3536	3389	3248	6-
2.8	4133	3982	3833	3688	3547	3410	3277	3146	3020	2897	6-
3.0	3610	3482	3357	3235	3116	2999	2884	2774	2666	2561	6-
3.2	3121	3015	2910	2808	2708	2610	2515	2421	2330	2242	6-
3.4	26670	25805	24954	24118	23298	22492	21702	20929	20171	19430	7-
3.6	22537	21845	21162	20489	19827	19174	18532	17900	17282	16673	7-
3.8	18817	18276	17740	17209	16684	16165	15653	15147	14649	14159	7-
4.0	15518	15105	14693	14283	13877	13473	13072	12675	12283	11895	7-
4.2	12631	12324	12016	11709	11401	11094	10788	10484	10180	9880	7-
4.4	10139	9919	9697	9473	9248	9021	8793	8566	8338	8111	7-
4.6	80207	78701	77162	75595	74004	72390	70756	69108	67446	65774	8-
4.8	62447	61478	60473	59434	58365	57266	56143	54996	53829	52643	8-
5.0	47790	47226	46625	45989	45320	44619	43890	43135	42354	41551	8-
5.2	35889	35618	35312	34972	34599	34196	33765	33305	32820	32311	8-
5.4	26385	26317	26217	26087	25927	25737	25521	25279	25013	24722	8-
5.6	18935	18998	19034	19043	19026	18985	18919	18829	18717	18583	8-
5.8	13210	13349	13466	13561	13635	13687	13719	13731	13722	13695	8-
6.0	89042	90791	92369	93775	95011	96077	96976	97710	98279	98689	9-
6.2	57452	59281	60983	62557	64003	65319	66505	67561	68488	69286	9-
6.4	34922	36649	38288	39837	41295	42658	43926	45097	46170	47145	9-
6.6	19392	20911	22375	23780	25124	26405	27619	28766	29842	30848	9-
6.8	9132	10394	11626	12825	13987	15110	16191	17228	18219	19161	9-
7.0	2727	3724	4709	5678	6631	7562	8471	9354	10210	11035	9-
7.2	950-	203-	546	1292	2035	2770	3496	4211	4912	5597	9-
7.4	27811-	22501-	17107-	11649-	6148-	624-	4902	10410	15882	21298	10-
7.6	34213-	30684-	27030-	23267-	19412-	15479-	11486-	7448-	3381-	698	10-
7.8	33524-	31378-	29095-	26685-	24161-	21534-	18816-	16019-	13155-	10237-	10-
8.0	29109-	27978-	26716-	25330-	23827-	22215-	20502-	18696-	16807-	14844-	10-
8.2	23202-	22770-	22224-	21567-	20804-	19939-	18977-	17923-	16783-	15563-	10-
8.4	17184-	17194-	17112-	16940-	16678-	16329-	15895-	15379-	14784-	14114-	10-
8.6	11824-	12082-	12270-	12390-	12440-	12421-	12333-	12177-	11954-	11667-	10-
8.8	74686-	78339-	81520-	84212-	86403-	88081-	89241-	89880-	89995-	89592-	11-
9.0	41899-	45712-	49233-	52442-	55319-	57849-	60018-	61814-	63230-	64257-	11-
9.2	19012-	22442-	25721-	28827-	31740-	34443-	36918-	39151-	41128-	42839-	11-
9.4	4361-	7145-	9879-	12543-	15118-	17587-	19933-	22141-	24197-	26086-	11-
9.6	3971	1903	178-	2259-	4323-	6354-	8340-	10264-	12113-	13875-	11-
9.8	7816	6417	4967	3478	1962	431	1104-	2630-	4136-	5610-	11-
10.0	8742	7899	6991	6025	5010	3954	2866	1754	628	502-	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	1.51	1.52	1.53	1.54	1.55	1.56	1.57	1.58	1.59	1.60	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2											5-
1.4											5-
1.6											5-
1.8											5-
2.0											6-
2.2											6-
2.4											6-
2.6											6-
2.8											6-
3.0											6-
3.2											6-
3.4											7-
3.6											7-
3.8											7-
4.0											7-
4.2	9582	9285	8993	8704	8417	8136	7858	7584	7315	7050	7-
4.4	7884	7658	7433	7209	6987	6768	6551	6336	6124	5914	7-
4.6	64094	62412	60727	59043	57362	55688	54022	52367	50724	49095	8-
4.8	51442	50227	49002	47770	46530	45286	44041	42796	41554	40315	8-
5.0	40728	39886	39028	38156	37271	36376	35472	34562	33647	32728	8-
5.2	31780	31227	30656	30067	29461	28841	28209	27566	26914	26253	8-
5.4	24409	24075	23721	23348	22957	22551	22130	21697	21250	20793	8-
5.6	18428	18253	18059	17846	17617	17371	17111	16836	16548	16248	8-
5.8	13649	13586	13505	13407	13294	13165	13021	12865	12695	12513	8-
6.0	98941	99040	98990	98795	98459	97986	97383	96655	95806	94843	9-
6.2	69957	70503	70925	71226	71408	71474	71427	71270	71008	70644	9-
6.4	48021	48800	49480	50064	50551	50944	51243	51450	51569	51600	9-
6.6	31781	32641	33427	34138	34775	35338	35827	36243	36586	36858	9-
6.8	20053	20894	21681	22415	23094	23717	24284	24795	25250	25649	9-
7.0	11829	12589	13314	14002	14651	15262	15832	16361	16849	17294	9-
7.2	6264	6912	7539	8142	8721	9274	9800	10298	10767	11206	9-
7.4	26640	31891	37034	42053	46933	51661	56223	60607	64803	68799	10-
7.6	4775	8834	12861	16840	20760	24605	28364	32025	35576	39008	10-
7.8	7278-	4289-	1283-	1728	4733	7719	10675	13592	16458	19263	10-
8.0	12815-	10731-	8600-	6432-	4238-	2026-	195	2415	4625	6817	10-
8.2	14270-	12911-	11491-	10019-	8501-	6945-	5358-	3747-	2120-	484-	10-
8.4	13373-	12565-	11694-	10766-	9785-	8757-	7688-	6581-	5444-	4281-	10-
8.6	11317-	10906-	10437-	9914-	9339-	8715-	8047-	7338-	6592-	5813-	10-
8.8	88675-	87253-	85337-	82944-	80088-	76791-	73073-	68958-	64471-	59641-	11-
9.0	64894-	65138-	64991-	64456-	63540-	62250-	60597-	58594-	56254-	53594-	11-
9.2	44272-	45422-	46283-	46851-	47124-	47103-	46791-	46190-	45308-	44151-	11-
9.4	27797-	29320-	30645-	31765-	32673-	33365-	33838-	34091-	34123-	33935-	11-
9.6	15536-	17087-	18516-	19813-	20971-	21983-	22842-	23544-	24083-	24459-	11-
9.8	7042-	8421-	9736-	10979-	12141-	13212-	14187-	15059-	15821-	16470-	11-
10.0	1629-	2743-	3835-	4897-	5921-	6898-	7823-	8687-	9485-	10211-	11-

TABLE OF THE WEDGE FUNCTION $G_{\nu}(e^{x})$

ν \ x	1.61	1.62	1.63	1.64	1.65	1.66	1.67	1.68	1.69	1.70	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2											5-
1.4											5-
1.6											5-
1.8											5-
2.0											6-
2.2											6-
2.4											6-
2.6											6-
2.8											6-
3.0											6-
3.2											6-
3.4											7-
3.6											7-
3.8											7-
4.0											7-
4.2	6791	6537	6287	6043	5804	5571	5343	5122	4906	4695	7-
4.4	5707	5504	5305	5108	4916	4726	4542	4361	4185	4013	7-
4.6	47483	45890	44316	42764	41235	39731	38253	36802	35379	33985	8-
4.8	39083	37858	36642	35439	34247	33069	31906	30760	29633	28523	8-
5.0	31807	30887	29968	29053	28143	27238	26340	25451	24572	23703	8-
5.2	25586	24914	24236	23558	22878	22199	21521	20845	20173	19505	8-
5.4	20327	19851	19368	18880	18387	17890	17390	16889	16386	15885	8-
5.6	15937	15615	15285	14947	14601	14249	13892	13531	13166	12798	8-
5.8	12319	12115	11901	11678	11447	11208	10963	10711	10455	10194	8-
6.0	93771	92596	91324	89962	88514	86988	85389	83724	81999	80219	9-
6.2	70181	69624	68977	68245	67432	66543	65581	64553	63463	62315	9-
6.4	51546	51410	51195	50904	50539	50105	49604	49039	48415	47735	9-
6.6	37060	37193	37259	37260	37198	37074	36892	36653	36360	36015	9-
6.8	25992	26280	26514	26694	26822	26899	26925	26903	26834	26720	9-
7.0	17698	18059	18377	18653	18888	19081	19234	19346	19420	19455	9-
7.2	11614	11991	12337	12651	12933	13183	13401	13587	13742	13866	9-
7.4	72587	76160	79510	82632	85521	88174	90588	92762	94695	96387	10-
7.6	42310	45474	48491	51354	54058	56595	58963	61156	63172	65010	10-
7.8	21998	24655	27225	29700	32073	34338	36490	38522	40431	42212	10-
8.0	8982	11112	13199	15236	17216	19133	20980	22752	24444	26052	10-
8.2	1154	2787	4409	6013	7592	9141	10654	12125	13549	14923	10-
8.4	3099-	1903-	698-	510	1716	2915	4101	5269	6416	7536	10-
8.6	5006-	4173-	3321-	2453-	1573-	686-	205	1094	1979	2854	10-
8.8	54495-	49064-	43378-	37469-	31369-	25112-	18730-	12255-	5721-	841	11-
9.0	50631-	47385-	43877-	40127-	36159-	31996-	27663-	23183-	18583-	13887-	11-
9.2	42729-	41053-	39133-	36985-	34622-	32060-	29316-	26405-	23348-	20162-	11-
9.4	33531-	32915-	32092-	31069-	29854-	28457-	26886-	25155-	23273-	21254-	11-
9.6	24670-	24715-	24596-	24314-	23873-	23276-	22530-	21639-	20612-	19455-	11-
9.8	17001-	17411-	17698-	17862-	17902-	17818-	17613-	17288-	16847-	16294-	11-
10.0	10860-	11427-	11910-	12304-	12609-	12821-	12941-	12968-	12902-	12746-	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	1.71	1.72	1.73	1.74	1.75	1.76	1.77	1.78	1.79	1.80	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2											5-
1.4											5-
1.6											5-
1.8											5-
2.0											6-
2.2											6-
2.4											6-
2.6											6-
2.8											6-
3.0											6-
3.2											6-
3.4											7-
3.6											7-
3.8											7-
4.0											7-
4.2	4491	4292	4099	3911	3730	3554	3384	3221	3062	2909	7-
4.4	3845	3680	3520	3366	3215	3069	2926	2789	2656	2528	7-
4.6	32621	31287	29985	28715	27478	26273	25102	23964	22859	21788	8-
4.8	27433	26363	25315	24291	23288	22309	21353	20423	19517	18636	8-
5.0	22846	22002	21172	20355	19555	18768	18000	17248	16514	15798	8-
5.2	18843	18189	17541	16902	16272	15651	15042	14444	13857	13283	8-
5.4	15385	14887	14391	13900	13413	12931	12456	11987	11525	11071	8-
5.6	12429	12059	11688	11318	10949	10582	10217	9856	9499	9146	8-
5.8	9929	9661	9391	9119	8846	8573	8300	8027	7756	7486	8-
6.0	78391	76520	74612	72673	70709	68724	66723	64713	62696	60680	9-
6.2	61114	59865	58572	57241	55876	54480	53060	51618	50160	48688	9-
6.4	47002	46221	45394	44526	43620	42680	41709	40711	39690	38648	9-
6.6	35621	35181	34698	34173	33611	33014	32384	31726	31040	30330	9-
6.8	26562	26363	26124	25847	25535	25188	24811	24405	23971	23513	9-
7.0	19453	19416	19344	19239	19101	18934	18738	18514	18265	17991	9-
7.2	13960	14024	14060	14067	14047	14000	13929	13833	13714	13573	9-
7.4	97840	99056	100040	100794	101323	101634	101731	101620	101312	100812	10-
7.6	66665	68140	69434	70549	71484	72244	72830	73246	73498	73588	10-
7.8	43863	45382	46766	48015	49127	50104	50944	51651	52224	52667	10-
8.0	27571	28997	30329	31563	32698	33731	34663	35492	36219	36845	10-
8.2	16240	17499	18694	19822	20882	21871	22787	23628	24392	25081	10-
8.4	8626	9681	10698	11674	12605	13489	14323	15106	15836	16510	10-
8.6	3718	4565	5392	6197	6977	7727	8447	9133	9784	10398	10-
8.8	7398	13922	20380	26746	32992	39092	45021	50757	56278	61565	11-
9.0	9119-	4306-	529	5361	10167	14925	19612	24208	28693	33049	11-
9.2	16865-	13477-	10017-	6505-	2958-	604	4163	7701	11201	14647	11-
9.4	19111-	16858-	14508-	12077-	9578-	7026-	4435-	1821-	803	3423	11-
9.6	18178-	16788-	15296-	13711-	12045-	10308-	8511-	6664-	4779-	2868-	11-
9.8	15633-	14870-	14011-	13062-	12030-	10922-	9746-	8510-	7222-	5891-	11-
10.0	12500-	12168-	11751-	11255-	10683-	10039-	9329-	8558-	7731-	6855-	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	1.81	1.82	1.83	1.84	1.85	1.86	1.87	1.88	1.89	1.90	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2											5-
1.4											5-
1.6											5-
1.8											5-
2.0											6-
2.2											6-
2.4											6-
2.6											6-
2.8											6-
3.0											6-
3.2											6-
3.4											7-
3.6											7-
3.8											7-
4.0											7-
4.2	2762	2620	2484	2352	2227	2106	1989	1879	1773	1671	7-
4.4	2404	2283	2168	2057	1950	1847	1747	1652	1562	1475	7-
4.6	20751	19748	18778	17841	16937	16067	15228	14421	13647	12902	8-
4.8	17779	16950	16145	15365	14612	13883	13180	12502	11849	11220	8-
5.0	15100	14421	13761	13120	12498	11895	11311	10747	10202	9677	8-
5.2	12721	12172	11637	11116	10609	10115	9636	9172	8723	8287	8-
5.4	10625	10188	9760	9342	8932	8534	8146	7768	7402	7045	8-
5.6	8798	8455	8117	7786	7461	7143	6832	6529	6233	5944	8-
5.8	7219	6954	6692	6434	6180	5929	5684	5443	5207	4977	8-
6.0	58666	56661	54667	52690	50731	48794	46884	45001	43151	41334	9-
6.2	47208	45722	44235	42748	41268	39796	38335	36888	35457	34045	9-
6.4	37588	36516	35434	34343	33248	32151	31056	29963	28877	27799	9-
6.6	29599	28850	28085	27307	26517	25720	24917	24111	23302	22494	9-
6.8	23032	22531	22011	21474	20924	20361	19789	19208	18620	18029	9-
7.0	17695	17379	17044	16692	16324	15942	15548	15143	14729	14308	9-
7.2	13411	13230	13031	12815	12583	12337	12078	11807	11526	11235	9-
7.4	100129	99271	98248	97069	95743	94280	92690	90982	89167	87254	10-
7.6	73523	73307	72947	72450	71822	71068	70198	69217	68135	66956	10-
7.8	52983	53174	53244	53198	53039	52770	52399	51931	51369	50718	10-
8.0	37369	37794	38120	38351	38488	38535	38495	38370	38164	37882	10-
8.2	25693	26228	26687	27070	27378	27613	27776	27870	27896	27856	10-
8.4	17128	17690	18193	18639	19028	19360	19635	19854	20018	20129	10-
8.6	10973	11508	12001	12452	12861	13227	13549	13828	14065	14259	10-
8.8	66602	71371	75860	80056	83950	87534	90801	93746	96369	98668	11-
9.0	37258	41305	45174	48854	52331	55596	58641	61457	64040	66384	11-
9.2	18022	21312	24503	27582	30537	33357	36032	38555	40916	43111	11-
9.4	6025	8597	11125	13598	16005	18336	20580	22729	24775	26711	11-
9.6	941-	994	2921	4833	6719	8571	10379	12135	13831	15461	11-
9.8	4524-	3131-	1721-	300-	1125	2541	3944	5325	6678	7996	11-
10.0	5936-	4979-	3991-	2979-	1949-	907-	140	1186	2225	3252	11-

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

$\nu \backslash x$	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99	2.00	p
0.2											4-
0.4											4-
0.6											4-
0.8											5-
1.0											5-
1.2											5-
1.4											5-
1.6											5-
1.8											5-
2.0											6-
2.2											6-
2.4											6-
2.6											6-
2.8											6-
3.0											6-
3.2											6-
3.4											7-
3.6											7-
3.8											7-
4.0											7-
4.2	1575	1482	1394	1311	1231	1156	1083	1016	952	891	7-
4.4	1390	1311	1235	1162	1092	1027	964	905	848	794	7-
4.6	12189	11506	10852	10227	9629	9059	8515	7998	7507	7039	8-
4.8	10616	10036	9479	8945	8434	7946	7478	7032	6607	6202	8-
5.0	9170	8683	8214	7764	7331	6916	6520	6140	5777	5431	8-
5.2	7867	7461	7070	6694	6331	5983	5649	5328	5021	4727	8-
5.4	6700	6366	6043	5730	5429	5139	4861	4592	4334	4088	8-
5.6	5664	5392	5128	4872	4624	4385	4155	3932	3718	3512	8-
5.8	4752	4533	4319	4112	3910	3715	3526	3344	3167	2997	8-
6.0	39553	37810	36106	34445	32826	31251	29721	28236	26799	25408	9-
6.2	32656	31288	29947	28632	27346	26090	24865	23673	22513	21387	9-
6.4	26731	25676	24635	23609	22602	21614	20645	19698	18774	17874	9-
6.6	21689	20888	20093	19307	18529	17762	17007	16266	15540	14828	9-
6.8	17433	16838	16243	15650	15060	14475	13895	13324	12760	12206	9-
7.0	13879	13446	13010	12572	12134	11695	11259	10824	10394	9969	9-
7.2	10937	10631	10320	10004	9685	9364	9041	8718	8395	8074	9-
7.4	85254	83175	81028	78822	76566	74268	71938	69585	67215	64837	10-
7.6	65692	64346	62929	61446	59907	58317	56684	55016	53317	51596	10-
7.8	49987	49180	48302	47359	46358	45304	44202	43059	41879	40669	10-
8.0	37526	37101	36612	36062	35455	34796	34091	33343	32556	31735	10-
8.2	27753	27591	27371	27098	26773	26401	25984	25525	25029	24500	10-
8.4	20187	20196	20155	20068	19937	19764	19551	19300	19014	18697	10-
8.6	14411	14522	14593	14626	14621	14581	14506	14399	14261	14093	10-
8.8	100645	102300	103639	104669	105395	105824	105968	105836	105440	104792	11-
9.0	68488	70350	71969	73347	74486	75390	76063	76511	76741	76760	11-
9.2	45134	46980	48648	50134	51440	52564	53507	54274	54866	55287	11-
9.4	28530	30227	31798	33239	34547	35721	36759	37662	38429	39063	11-
9.6	17017	18495	19888	21192	22403	23519	24536	25454	26271	26986	11-
9.8	9273	10503	11682	12803	13864	14860	15789	16647	17432	18144	11-
10.0	4261	5247	6205	7130	8019	8867	9671	10428	11135	11791	11-

TABLE OF THE WEDGE FUNCTION $G_{\nu}(e^{x})$

ν \ x	2.01	2.02	2.03	2.04	2.05	2.06	2.07	2.08	2.09	2.10	p
0.2											4 -
0.4											4 -
0.6											4 -
0.8											5 -
1.0											5 -
1.2											5 -
1.4											5 -
1.6											5 -
1.8											5 -
2.0											6 -
2.2											6 -
2.4											6 -
2.6											6 -
2.8											6 -
3.0											6 -
3.2											6 -
3.4											7 -
3.6											7 -
3.8											7 -
4.0											7 -
4.2											7 -
4.4											7 -
4.6											8 -
4.8											8 -
5.0											8 -
5.2											8 -
5.4											8 -
5.6											8 -
5.8											8 -
6.0											9 -
6.2											9 -
6.4											9 -
6.6											9 -
6.8											9 -
7.0											9 -
7.2	7755	7439	7127	6819	6516	6219	5927	5642	5364	5094	9 -
7.4	62457	60083	57722	55379	53060	50771	48517	46300	44128	42002	10 -
7.6	49859	48112	46360	44608	42864	41130	39411	37712	36037	34390	10 -
7.8	39434	38177	36906	35624	34335	33046	31758	30478	29206	27947	10 -
8.0	30885	30008	29110	28194	27265	26326	25380	24432	23482	22537	10 -
8.2	23939	23350	22736	22102	21450	20784	20104	19417	18723	18025	10 -
8.4	18349	17974	17574	17152	16710	16252	15779	15293	14798	14295	10 -
8.6	13899	13678	13435	13170	12885	12582	12265	11933	11590	11237	10 -
8.8	103905	102791	101466	99943	98237	96363	94337	92172	89884	87487	11 -
9.0	76577	76199	75637	74902	74003	72951	71756	70432	68988	67436	11 -
9.2	55542	55637	55576	55367	55018	54535	53926	53199	52363	51426	11 -
9.4	39564	39937	40183	40308	40315	40209	39994	39678	39265	38763	11 -
9.6	27600	28113	28527	28844	29065	29194	29234	29187	29059	28853	11 -
9.8	18780	19341	19826	20235	20570	20831	21021	21140	21193	21180	11 -
10.0	12394	12941	13433	13868	14246	14568	14835	15046	15202	15307	11 -

TABLE OF THE WEDGE FUNCTION $G_{\nu}(e^x)$

ν \ x	2.11	2.12	2.13	2.14	2.15	2.16	2.17	2.18	2.19	2.20	p
0.2											4 -
0.4											4 -
0.6											4 -
0.8											5 -
1.0											5 -
1.2											5 -
1.4											5 -
1.6											5 -
1.8											5 -
2.0											6 -
2.2											6 -
2.4											6 -
2.6											6 -
2.8											6 -
3.0											6 -
3.2											6 -
3.4											7 -
3.6											7 -
3.8											7 -
4.0											7 -
4.2											7 -
4.4											7 -
4.6											8 -
4.8											8 -
5.0											8 -
5.2											8 -
5.4											8 -
5.6											8 -
5.8											8 -
6.0											9 -
6.2											9 -
6.4											9 -
6.6											9 -
6.8											9 -
7.0											9 -
7.2	4830	4575	4328	4089	3858	3636	3422	3216	3019	2831	9 -
7.4	39926	37903	35935	34026	32175	30385	28658	26993	25392	23854	10 -
7.6	32773	31190	29642	28135	26667	25241	23860	22523	21233	19990	10 -
7.8	26706	25483	24282	23105	21954	20833	19740	18679	17650	16655	10 -
8.0	21598	20667	19748	18841	17951	17078	16223	15390	14578	13790	10 -
8.2	17327	16630	15936	15249	14569	13898	13238	12590	11957	11338	10 -
8.4	13786	13274	12760	12247	11736	11227	10725	10229	9741	9262	10 -
8.6	10876	10508	10134	9759	9381	9003	8627	8252	7881	7515	10 -
8.8	84996	82424	79788	77098	74367	71609	68834	66054	63280	60521	11 -
9.0	65788	64054	62247	60376	58451	56484	54484	52461	50423	48379	11 -
9.2	50395	49282	48094	46839	45526	44163	42758	41319	39855	38370	11 -
9.4	38176	37512	36777	35976	35118	34209	33254	32261	31234	30182	11 -
9.6	28574	28225	27812	27340	26814	26238	25617	24957	24263	23538	11 -
9.8	21105	20971	20781	20539	20247	19910	19532	19117	18667	18187	11 -
10.0	15360	15364	15321	15234	15104	14935	14728	14487	14215	13914	11 -

TABLE OF THE WEDGE FUNCTION $G_{\nu}(e^{x})$

ν \ x	2.21	2.22	2.23	2.24	2.25	2.26	2.27	2.28	2.29	2.30	p
0.2											4 -
0.4											4 -
0.6											4 -
0.8											5 -
1.0											5 -
1.2											5 -
1.4											5 -
1.6											5 -
1.8											5 -
2.0											6 -
2.2											6 -
2.4											6 -
2.6											6 -
2.8											6 -
3.0											6 -
3.2											6 -
3.4											7 -
3.6											7 -
3.8											7 -
4.0											7 -
4.2											7 -
4.4											7 -
4.6											8 -
4.8											8 -
5.0											8 -
5.2											8 -
5.4											8 -
5.6											8 -
5.8											8 -
6.0											9 -
6.2											9 -
6.4											9 -
6.6											9 -
6.8											9 -
7.0											9 -
7.2	2651	2479	2316	2160	2012	1873	1741	1616	1497	1386	9 -
7.4	22380	20969	19622	18336	17113	15949	14845	13798	12809	11874	10 -
7.6	18795	17646	16546	15492	14487	13528	12615	11748	10925	10146	10 -
7.8	15695	14768	13877	13023	12204	11419	10671	9959	9280	8636	10 -
8.0	13025	12285	11571	10883	10221	9586	8978	8395	7840	7311	10 -
8.2	10737	10151	9584	9034	8505	7994	7502	7032	6580	6148	10 -
8.4	8793	8335	7889	7456	7036	6629	6237	5859	5495	5147	10 -
8.6	7155	6800	6453	6115	5785	5463	5153	4851	4561	4282	10 -
8.8	57786	55084	52422	49807	47246	44744	42307	39939	37642	35422	11 -
9.0	46338	44305	42290	40298	38335	36406	34516	32671	30874	29127	11 -
9.2	36874	35371	33870	32374	30890	29423	27976	26555	25162	23803	11 -
9.4	29109	28020	26920	25817	24712	23612	22520	21440	20375	19329	11 -
9.6	22787	22017	21230	20431	19624	18812	18001	17192	16388	15593	11 -
9.8	17681	17151	16603	16038	15461	14874	14281	13685	13089	12494	11 -
10.0	13587	13238	12868	12480	12079	11664	11242	10811	10376	9938	11 -

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

ν \ x	2.31	2.32	2.33	2.34	2.35	2.36	2.37	2.38	2.39	2.40	p
0.2											4 -
0.4											4 -
0.6											4 -
0.8											5 -
1.0											5 -
1.2											5 -
1.4											5 -
1.6											5 -
1.8											5 -
2.0											6 -
2.2											6 -
2.4											6 -
2.6											6 -
2.8											6 -
3.0											6 -
3.2											6 -
3.4											7 -
3.6											7 -
3.8											7 -
4.0											7 -
4.2											7 -
4.4											7 -
4.6											8 -
4.8											8 -
5.0											8 -
5.2											8 -
5.4											8 -
5.6											8 -
5.8											8 -
6.0											9 -
6.2											9 -
6.4											9 -
6.6											9 -
6.8											9 -
7.0											9 -
7.2	1281	1183	1091	1004	924	848	778	712	652	595	9 -
7.4	10992	10162	9381	8649	7963	7321	6722	6162	5642	5158	10 -
7.6	9408	8713	8058	7442	6863	6320	5812	5336	4894	4481	10 -
7.8	8024	7446	6900	6385	5899	5444	5015	4614	4239	3889	10 -
8.0	6808	6330	5878	5450	5046	4665	4306	3970	3654	3358	10 -
8.2	5737	5346	4973	4620	4285	3969	3670	3389	3125	2878	10 -
8.4	4813	4494	4190	3901	3626	3366	3119	2886	2667	2460	10 -
8.6	4014	3756	3510	3274	3050	2836	2634	2442	2261	2089	10 -
8.8	33278	31215	29234	27336	25520	23787	22137	20569	19081	17673	11 -
9.0	27434	25797	24218	22699	21240	19843	18508	17235	16023	14872	11 -
9.2	22479	21192	19945	18740	17579	16462	15389	14364	13384	12451	11 -
9.4	18304	17303	16329	15382	14466	13581	12729	11908	11122	10371	11 -
9.6	14810	14041	13288	12552	11836	11140	10468	9819	9193	8593	11 -
9.8	11903	11319	10743	10178	9625	9084	8559	8049	7555	7080	11 -
10.0	9499	9062	8628	8199	7776	7361	6955	6559	6174	5800	11 -

TABLE OF THE WEDGE FUNCTION $G_\nu(e^x)$

$\nu \backslash x$	2.41	2.42	2.43	2.44	2.45	2.46	2.47	2.48	2.49	2.50	p
0.2											4 -
0.4											4 -
0.6											4 -
0.8											5 -
1.0											5 -
1.2											5 -
1.4											5 -
1.6											5 -
1.8											5 -
2.0											6 -
2.2											6 -
2.4											6 -
2.6											6 -
2.8											6 -
3.0											6 -
3.2											6 -
3.4											7 -
3.6											7 -
3.8											7 -
4.0											7 -
4.2											7 -
4.4											7 -
4.6											8 -
4.8											8 -
5.0											8 -
5.2											8 -
5.4											8 -
5.6											8 -
5.8											8 -
6.0											9 -
6.2											9 -
6.4											9 -
6.6											9 -
6.8											9 -
7.0											9 -
7.2	543	494	449	408	371	336	304	275	248	223	9 -
7.4	4710	4295	3911	3557	3231	2932	2658	2407	2178	1971	10 -
7.6	4097	3741	3412	3108	2827	2570	2333	2116	1918	1739	10 -
7.8	3563	3259	2978	2717	2475	2252	2047	1858	1685	1527	10 -
8.0	3083	2825	2585	2362	2157	1965	1788	1624	1474	1336	10 -
8.2	2646	2429	2227	2038	1863	1700	1550	1411	1282	1165	10 -
8.4	2267	2085	1915	1756	1607	1470	1341	1222	1112	1011	10 -
8.6	1929	1777	1635	1502	1377	1262	1154	1054	961	875	10 -
8.8	16344	15091	13911	12804	11766	10795	9889	9044	8260	7532	11 -
9.0	13783	12751	11779	10862	10001	9194	8437	7731	7072	6460	11 -
9.2	11564	10721	9924	9170	8461	7793	7165	6579	6030	5517	11 -
9.4	9653	8970	8322	7706	7124	6575	6059	5574	5118	4692	11 -
9.6	8017	7468	6943	6445	5971	5524	5100	4700	4326	3973	11 -
9.8	6623	6183	5763	5362	4979	4616	4271	3945	3639	3349	11 -
10.0	5440	5092	4757	4437	4131	3839	3560	3296	3046	2810	11 -

ν \ x	1.00	1.50	2.00	2.50	p
0.2	479.06	65.044	2.7972	0.018173	4-
0.4	470.00	64.257	2.7759	0.018087	4-
0.6	455.25	62.965	2.7408	0.017945	4-
0.8	4353.3	611.99	26.923	0.17747	5-
1.0	4108.9	589.96	26.312	0.17497	5-
1.2		564.06	25.584	0.17195	5-
1.4		534.83	24.748	0.16845	5-
1.6		502.88	23.816	0.16450	5-
1.8		468.86	22.800	0.16013	5-
2.0		4334.2	217.15	1.5538	6-
2.2			205.73	1.5029	6-
2.4			193.89	1.4490	6-
2.6			181.76	1.3925	6-
2.8			169.48	1.3340	6-
3.0			157.18	1.2738	6-
3.2			144.99	1.2124	6-
3.4			1330.1	11.502	7-
3.6			1213.4	10.876	7-
3.8			1100.8	10.251	7-
4.0			993.00	9.6294	7-
4.2				9.0160	7-
4.4				8.4136	7-
4.6				78.252	8-
4.8				72.536	8-
5.0				67.010	8-
5.2				61.694	8-
5.4				56.606	8-
5.6				51.758	8-
5.8				47.162	8-
6.0				428.23	9-
6.2				387.47	9-
6.4				349.34	9-
6.6				313.83	9-
6.8				280.91	9-
7.0				250.53	9-