On multiple SLE systems and their deterministic limits

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ABSTRACT

In this thesis, we study multiple radial SLE(κ) systems —a family of random multi-curve systems in a simply-connected domain Ω, with marked boundary points $z_1, \ldots, z_n \in \partial\Omega$ and a marked interior point q, where parameter $\kappa > 0$ measures the randomness of the system. We also study the multiple radial SLE(0) systems as the deterministic limit of multiple radial SLE(κ) systems.

As a consequence of domain Markov property and conformal invariance, we derive that a multiple radial SLE(κ) system is characterized by a conformally covariant partition function satisfying the null vector equations–a second-order PDE system. On the other hand, using the Coulomb gas method inspired by conformal field theory, we construct four types of solutions to the null vector equations, which can be classified according to topological link patterns.

We construct the multiple radial SLE(0) systems from stationary relations by heuristically taking the classical limit of partition functions as $\kappa \to 0$. By constructing the field integrals of motion for the Loewner dynamics, we show that the traces of multiple radial SLE(0) systems are the horizontal trajectories of an equivalence class of quadratic differentials. These trajectories have limiting ends at the growth points and form a radial link pattern.

The stationary relations connect the classification of multiple radial SLE(0) systems to the enumeration of critical points of the master function of trigonometric Knizhnik-Zamolodchikov (KZ) equations.

For $\kappa > 0$, the partition functions of multiple radial SLE(κ) systems correspond to eigenstates of the quantum Calogero-Sutherland (CS) Hamiltonian beyond the fermionic states. In the deterministic case of $\kappa = 0$, we show that the Loewner dynamics with a common parametrization of capacity form a special class of classical CS systems, restricted to a submanifold of phase space defined by the Lax matrix.

PUBLISHED CONTENT AND CONTRIBUTIONS

Makarov, Nikolai and Jiaxin Zhang (2025a). *Multiple radial SLE(κ) and quantum Calogero-Sutherland system*. arXiv: 2505.14762 [math.PR]. URL: https://arxiv.org/abs/2505.14762.
J. Zhang formulated the main definitions, proved all the theorems, and wrote the entire manuscript.

– (2025b). *Multiple radial SLE(0) and classical Calogero-Sutherland System*. arXiv: 2410.21544 [math.PR]. URL: https://arxiv.org/abs/2410.21544.
J. Zhang formulated the main definitions, proved all the theorems, and wrote the entire manuscript.

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J. Zhang formulated the main definitions, proved all the theorems, and wrote the entire manuscript.

– (2025b). *Multiple chordal SLE(0) and classical Calogero-Moser system*. arXiv: 2505.17129 [math.PR]. URL: https://arxiv.org/abs/2505.17129.
J. Zhang formulated the main definitions, proved all the theorems, and wrote the entire manuscript.

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NOMENCLATURE

κ. SLE parameter measuring the roughness of the SLE curve.

$\mathbf{E}[\mathcal{X}\mathcal{O}_\beta]$. CFT correlation function involving product of fields $\mathcal{X}$.

$\mathcal{L}_j$. Null vector differential operator acting on θ_j.

$\mathcal{O}_\beta[\boldsymbol{\tau}]$. Chiral vertex operator with background charge $\boldsymbol{\beta}$ and charge $\boldsymbol{\tau}$ in CFT.

$\mathcal{Z}(\boldsymbol{\theta})$. Partition function of the multiple SLE(κ) system.

a. Charge at SLE growth point, defined by $a = \sqrt{\kappa/2}$.

b. Background charge in Coulomb gas formalism, $b = \sqrt{8/\kappa} - \sqrt{\kappa/2}$.

c. Central charge in CFT, defined by $c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$.

h. Conformal dimension of the field associated to the SLE growth point with charge a defined by $h = \frac{6-\kappa}{2\kappa}$.

$H_n(\beta)$. Quantum Calogero–Sutherland Hamiltonian with parameter $\beta = \frac{8}{\kappa}$.

Charge Relation. $2a(a+b) = 1$, relating vertex charge a and background charge b in Coulomb gas formalism.

Neutrality Condition (NC$_b$)**.** Total charge of a divisor is $2b$.

Chapter 1

INTRODUCTION

1.1 Background and main results

The Schramm-Loewner evolution SLE(κ) with $\kappa > 0$ is a one-parameter family of random conformally invariant curves in the plane describing interfaces within conformally invariant systems arising from statistical physics, as introduced in Schramm (2000), Lawler, Schramm, and Werner (2004), Smirnov (2006), Schramm (2006), and Schramm and Sheffield (2009). Conformal field theory (CFT), a quantum field theory invariant under conformal transformations, is also widely used to study critical phenomena, see J. L. Cardy (1996) and Friedrich and Kalkkinen (2004). SLE and the multiple SLE systems can be coupled to conformal field theories (CFT) through the SLE-CFT correspondence, which serves as a powerful tool for predicting phenomena and computing important quantities of SLE(κ) and multiple SLE(κ) systems from the CFT perspective, as demonstrated in references like Bauer and Bernard (2003), J. L. Cardy (2003), Friedrich and Werner (2003), Friedrich and Kalkkinen (2004), Dubédat (2015), and E. Peltola (2019). The parameter κ measures the roughness of these fractal curves and determines the central charge $c(\kappa) = (3\kappa - 8)(6 - \kappa)/2\kappa$ of the associated CFT.

In most recent years, there has been tremendous interest in multiple chordal SLE(κ) systems, as discussed in Dubédat (2006), Kozdron and Lawler (2007), Lawler (2009), S. Flores and Kleban (2015a), E. Peltola and Wu (2019), and Eveliina Peltola and H. Wang (2020). In contrast, multiple radial SLE(κ) systems have been less explored, with notable contributions including Healey and Lawler (2021), Y. Wang and Wu (2024), Nikolai Makarov and Zhang (2025b), and Nikolai Makarov and Zhang (2025a) and discussions in physics literature such as J. Cardy (2004), Doyon and J. Cardy (2007), Simmons et al. (2011), and S. M. Flores, Kleban, and Ziff (2012).

The core principle throughout our study of the multiple radial SLE(κ) system is the SLE-CFT correspondence. SLE and multiple SLE systems can be coupled to a conformal field in two key aspects:

- The level-two degeneracy equations for the conformal fields coincide with the null vector equations for the SLE partition functions.

- The correlation functions of the conformal fields serve as martingale observables for the SLE processes.

1.2 Multiple radial SLE(κ) systems with $\kappa > 0$

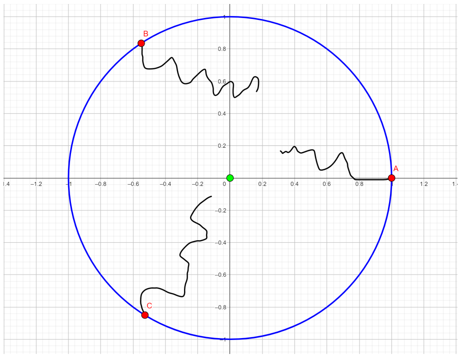

Figure 1.1: Multiple radial SLE(κ) systems in $\mathbb{D}$

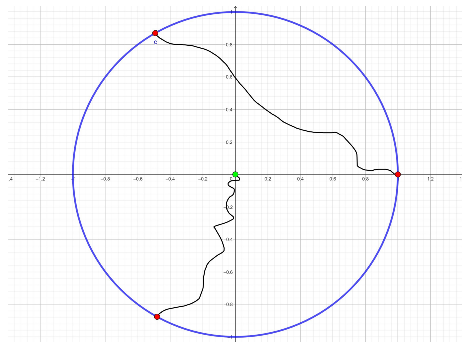

Figure 1.2: Multiple radial SLE(κ) in $\mathbb{D}$

In a simply connected domain Ω with boundary points $z_1, z_2, \ldots, z_n$ and a marked interior point q, we define a *local multiple radial SLE(κ) system* as a compatible family of probability measures

$$\mathbb{P}^{(U_1,U_2,\ldots,U_n)}_{(\Omega;z_1,z_2,\ldots,z_n,q)}$$

on n-tuples of continuous, non-self-crossing curves starting from z_i within a localization neighborhood U_i, none of which contains q. A more precise characterization of these measures is provided in Definition 1.2.1 and Definition 1.2.2.

Definition 1.2.1 (Localization of Measures)**.** *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain with an interior marked point $u \in \Omega$. Let $z_1, z_2, \ldots, z_n$ denote distinct prime ends of $\partial\Omega$, and let $U_1, U_2, \ldots, U_n$ be closed neighborhoods of $z_1, z_2, \ldots, z_n$ in Ω such that:*

- *$U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq n$,*
- *None of the U_j contain the interior point q.*

We consider the measures

$$\mathbb{P}^{(U_1,U_2,\ldots,U_n)}_{(\Omega;z_1,z_2,\ldots,z_n,q)}$$

defined on n-tuples of unparametrized continuous curves in Ω. Each curve $\eta^{(j)}$ begins at z_j and exits U_j almost surely.

A family of such measures indexed by different choices of $(U_1, U_2, \ldots, U_n)$ *is called* ***compatible*** *if for all* $U_j \subset U'_j$*, the measure*

$$\mathbb{P}^{(U_1,U_2,\ldots,U_n)}_{(\Omega;z_1,z_2,\ldots,z_n,q)}$$

is obtained by restricting the curves under

$$\mathbb{P}^{\left(U'_1,U'_2,\ldots,U'_n\right)}_{(\Omega;z_1,z_2,\ldots,z_n,q)}$$

to the portions of the curves that remain inside the subdomains U_j *before their first exit.*

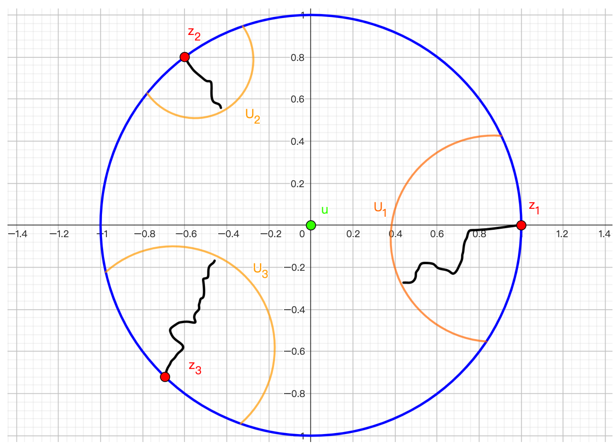

Figure 1.3: Localization of multiple radial SLE(κ)

Similar to the chordal case, multiple radial SLE(κ) systems are characterized by conformal invariance, the domain Markov property, and absolute continuity to independent standard SLE(κ) (see Y. Wang and Wu (2024)).

Definition 1.2.2 (Local multiple radial SLE(κ))**.** *The locally commuting n-radial* SLE(κ) *is a compatible family of measures*

$$\mathbb{P}^{(U_1,U_2,\ldots,U_n)}_{(\Omega;z_1,z_2,\ldots,z_n,q)}$$

on n-tuples of continuous, non-self-crossing curves $\left(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(n)}\right)$ *for all simply connected domains* Ω *with marked points* $(z_1, z_2, \ldots, z_n, q)$ *and target sets* $(U_1, U_2, \ldots, U_n)$*. These measures satisfy the following conditions:*

(i) ***Conformal invariance****: If* $\varphi : \Omega \to \Omega'$ *is a conformal map, then the pullback measure satisfies*

$$\varphi^*\mathbb{P}^{(\varphi(U_1),\varphi(U_2),\ldots,\varphi(U_n))}_{(\Omega';\varphi(z_1),\varphi(z_2),\ldots,\varphi(z_n),\varphi(q))} = \mathbb{P}^{(U_1,U_2,\ldots,U_n)}_{(\Omega;z_1,z_2,\ldots,z_n,u)}.$$

It suffices to describe the measure when $(\Omega; z_1, z_2, \ldots, z_n, q) = (\mathbb{D}; z_1, z_2, \ldots, z_n, 0)$. *The definition for arbitrary* Ω *with a marked interior point* q *can then be extended by pulling back via a conformal equivalence* $\varphi : \Omega \to \mathbb{D}$ *mapping* q *to* 0.

(ii) ***Domain Markov property**: Let* $\left(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(n)}\right) \sim \mathbb{P}^{(U_1, U_2, \ldots, U_n)}_{(\mathbb{D}; z_1, z_2, \ldots, z_n, q)}$, *and parametrize* $\gamma^{(j)}$ *by their own capacity in* $\mathbb{D}$. *For stopping times* $\mathbf{t} = (t_1, t_2, \ldots, t_n)$, *define*

$$\tilde{U}_j = U_j \setminus \gamma^{(j)}_{[0,t_j]}, \quad \tilde{\gamma}^{(j)} = \gamma^{(j)} \setminus \gamma^{(j)}_{[0,t_j]}, \quad \tilde{\Omega} = \mathbb{D} \setminus \bigcup_{j=1}^{n} \gamma^{(j)}_{[0,t_j]}.$$

Then, conditionally on the initial segments $\bigcup_{j=1}^{n} \gamma^{(j)}_{[0,t_j]}$, *we have*

$$\left(\tilde{\gamma}^{(1)}, \tilde{\gamma}^{(2)}, \ldots, \tilde{\gamma}^{(n)}\right) \sim \mathbb{P}^{(\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_n)}_{\left(\tilde{\Omega}; \gamma^{(1)}_{t_1}, \gamma^{(2)}_{t_2}, \ldots, \gamma^{(n)}_{t_n}, q\right)}.$$

(iii) ***Absolute continuity with respect to independent SLE(κ)**: Let* $\left(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(n)}\right) \sim \mathbb{P}^{(U_1, U_2, \ldots, U_n)}_{(\mathbb{D}; z_1, z_2, \ldots, z_n, 0)}$. *Let* $z_j(t) = e^{i\theta_j(t)}$, *the capacity-parametrized Loewner driving function* $t \mapsto \theta_j(t)$ *for* $\gamma^{(j)}$ *satisfies*

$$\begin{aligned} \mathrm{d}\theta_j(t) &= \sqrt{\kappa}\, \mathrm{d}B_j(t) + b_j\left(\boldsymbol{\theta}(t)\right) \mathrm{d}t, \\ \mathrm{d}\theta_k(t) &= \cot\left(\frac{\theta_k(t) - \theta_j(t)}{2}\right) \mathrm{d}t, \quad k \neq j, \end{aligned} \tag{1.2.1}$$

where $B_j(t)$ *are independent standard Brownian motions, and* $b_j(\boldsymbol{\theta})$ *are* C^2 *functions on the chamber*

$$\mathfrak{X}^n = \left\{(\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^n \mid \theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + 2\pi\right\}.$$

The domain Markov property implies that one can sequentially map out the curves $\gamma^{(i)}_{[0,t_i]}$ using $g^{(i)}_{t_i}$, or perform the mappings in reverse order. The resulting image has the same distribution regardless of the order. This property is known as the commutation relation or reparametrization symmetry (see Section 3.2).

In the following, we study how commutation relations and conformal invariance impose constraints on the drift terms $b_j(\boldsymbol{\theta})$.

We study the multiple radial SLE(κ) systems by exploring the following two aspects:

- Commutation relations and conformal invariance;

- Solution space of the null vector equations.

Extending the results in Dubédat (2007) on commutation relations (see also Y. Wang and Wu (2024) for the two radial case), we derive analogous commutation relations for multiple radial SLEs in the unit disk $\mathbb{D}$ with $z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2}, \ldots, z_n = e^{i\theta_n} \in \partial\mathbb{D}$ and one additional marked point $q = 0$, see section 3.2. The family of measure $\mathbb{P}_{(\theta_1,\ldots,\theta_n)}$ of a multiple radial SLE(κ) system is encoded by a partition function $\psi(\boldsymbol{\theta}) : \{(\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^n \mid \theta_1 < \theta_2 < \ldots < \theta_n < \theta_1 + 2\pi\} \to \mathbb{R}_{>0}$.

Theorem 1.2.3. *For a local multiple radial SLE(κ) system in the unit disk $\mathbb{D}$ with boundary points $z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2}, \ldots, z_n = e^{i\theta_n}$ and a marked point at $q = 0$, there exists a positive partition function $\psi(\boldsymbol{\theta})$ such that the drift term b_j in equation (1.2.1) satisfies*

$$b_j = \kappa\frac{\partial_j\psi}{\psi}, \quad j = 1, 2, \ldots, n. \tag{1.2.2}$$

Moreover, $\psi(\boldsymbol{\theta})$ satisfies the null vector equation

$$\frac{\kappa}{2}\partial_{ii}\psi + \sum_{j\neq i}\cot\left(\frac{\theta_j - \theta_i}{2}\right)\partial_i\psi + \left(1 - \frac{6}{\kappa}\right)\sum_{j\neq i}\frac{1}{4\sin^2\left(\frac{\theta_j-\theta_i}{2}\right)}\psi - h\psi = 0, \tag{1.2.3}$$

for some constant h.

Furthermore, there exists a real constant ω such that for all $\theta \in \mathbb{R}$,

$$\psi(\theta_1 + \theta, \ldots, \theta_n + \theta) = e^{-\omega\theta}\psi(\theta_1, \ldots, \theta_n). \tag{1.2.4}$$

Conversely, given a positive partition function $\psi(\boldsymbol{\theta})$ that satisfies both the null vector equation (1.2.3) and the rotation invariance condition (1.2.4), we define the multiple radial Loewner chain as a normalized conformal map $g_t = g_t(z)$. The evolution of g_t is governed by the following Loewner equation with the initial condition $g_0(z) = z$:

$$\partial_t g_t(z) = \sum_{j=1}^n \nu_j(t) g_t(z)\frac{z_j(t) + g_t(z)}{z_j(t) - g_t(z)}, \quad g_0(z) = z.$$

For the covering map $h_t(z)$, defined by $e^{ih_t(z)} = g_t(e^{iz})$, the evolution is given by

$$\partial_t h_t(z) = \sum_{j=1}^n \nu_j(t)\cot\left(\frac{h_t(z) - \theta_j(t)}{2}\right), \quad h_0(z) = z.$$

The driving functions $\theta_j(t)$, for $j = 1, \ldots, n$, evolve as

$$d\theta_j = \nu_j(t)\frac{\partial_j \log \psi(\boldsymbol{\theta})}{\partial \theta_j}dt + \sum_{k\neq j} \nu_k(t)\cot\left(\frac{\theta_j - \theta_k}{2}\right)dt + \sqrt{\kappa}dB_t,$$

where $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$ is a set of measurable functions, each $\nu_i : [0,\infty) \to [0,\infty)$. The process $h_t(z)$ thus defines a local multiple radial SLE(κ) system.

Proof. This is a summary of results derived from Theorem 3.2.1 and Theorem 3.2.2, which establish the commutation relations and conformal covariance properties of the partition function ψ. □

A significant difference between the multiple radial SLE(κ) systems and standard multiple chordal SLE(κ) systems arises when we study their conformal invariance properties. Although the multiple radial SLE(κ) systems are conformally invariant, the partition functions in its corresponding equivalence classes do not necessarily exhibit conformal covariance when we have an extra marked point.

We define two partition functions as *equivalent* if and only if they induce identical multiple chordal SLE(κ) systems. Equivalent partition functions differ by a multiplicative function $f(u)$.

$$\tilde{\psi} = f(u)\cdot\psi, \tag{1.2.5}$$

where $f(u)$ is an arbitrary positive real smooth function depending on the marked interior point u A simple example that violates conformal covariance is when $f(u)$ is not conformally covariant. However, within each equivalence class, it is still possible to find at least one conformally covariant partition function.

Theorem 1.2.4. *For a multiple radial SLE(κ) system with n SLEs starting at $(\theta_1, \theta_2, \dots, \theta_n) \in \mathfrak{X}^n(\boldsymbol{\theta})$ and a marked point $u \in \mathbb{D}$ not necessarily fixed at 0:*

(i) Two partition functions $\widetilde{\psi}$ and ψ are equivalent *if they differ by a multiplicative factor $f(u)$:*

$$\widetilde{\psi} = f(u)\cdot\psi,$$

where $f(u)$ is a smooth, positive function of u. Under this equivalence, $\widetilde{\psi}$ and ψ induce identical multiple radial SLE(κ) systems.

(ii) Within the equivalence class of partition functions, we can choose ψ to satisfy conformal covariance. Under $\tau \in \mathrm{Aut}(\mathbb{D})$, ψ transforms as:

$$\psi(\theta_1, \dots, \theta_n, u) = \left(\prod_{i=1}^{n} \tau'(\theta_i)^{\frac{6-\kappa}{2\kappa}}\right)\tau'(u)^{\lambda(u)}\overline{\tau'(u)}^{\overline{\lambda(u)}}\psi(\tau(\theta_1), \dots, \tau(\theta_n), \tau(u)).$$

(iii) The choice of a conformally covariant partition function is not unique. Let:

$$f(u) = (\mathrm{Rad}(u, \mathbb{H}))^{\alpha} = (i(\overline{u} - u))^{\alpha}, \quad \alpha \in \mathbb{R}.$$

Then for any conformally covariant ψ, $\widetilde{\psi} = f(u) \cdot \psi$ yields an equivalent solution with:

$$\widetilde{\lambda}(u) = \lambda(u) + \alpha.$$

Following S. Flores and Kleban (2015b) on solution space of the null vector equations for partition functions of multiple chordal SLE(κ), we construct four types of solutions to the null vector equations and Ward's identities for partition functions of multiple radial SLE(κ) via Coulomb gas integral method in conformal field theory.

Choosing charges σ_j for $j \in \{1, 2, \ldots, n\}$ and charges τ_k for $k \in \{1, 2, \ldots, m\}$, the following trigonometric Coulomb gas integral plays a crucial role in the theory of multiple radial SLE:

$$\oint \cdots \oint_{\Gamma} \prod_{1 \le i < j \le n} \left(\sin \frac{\theta_j - \theta_i}{2}\right)^{\sigma_i \sigma_j} \prod_{1 \le r < s \le m} \left(\sin \frac{\zeta_s - \zeta_r}{2}\right)^{\tau_r \tau_s} \prod_{\substack{1 \le i \le n \\ 1 \le r \le m}} \left(\sin \frac{\zeta_r - \theta_i}{2}\right)^{\tau_r \sigma_i} \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_m.$$

The integration variables $\zeta_1, \zeta_2, \ldots, \zeta_m$ are integrated along multiple contours Γ, corresponding to various topological link patterns. A detailed explanation can be found in Section 5.2.

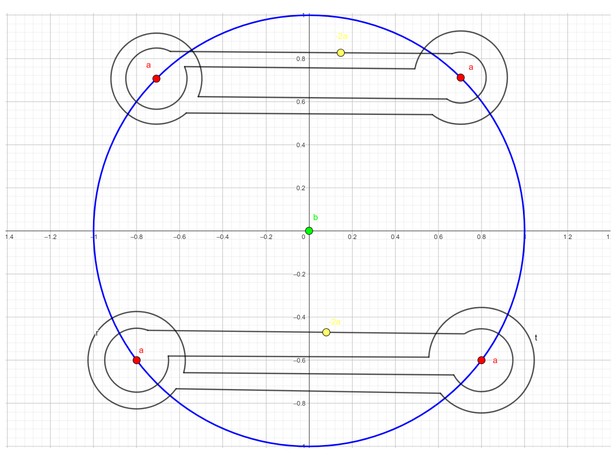

Figure 1.4: Integrate ζ_1, ζ_2 (yellow points) along two Pochhammer contour

Theorem 1.2.5. *The following four types of Coulomb gas integrals (see definitions in Section 5.2) solve the null vector equation (1.2.3) and the rotation equation (3.2.2):*

(1) *For any link pattern* $\alpha \in LP(n,m)$, *with* $m, n \in \mathbb{Z}$ *and* $1 \leq m \leq \frac{n}{2}$, *the Coulomb gas integral* $\mathcal{J}_\alpha^{n,m}(\boldsymbol{\theta})$ *defined in (5.2) solves the null vector equation* (1.2.3) *with*

$$h = \frac{1-(n-2m)^2}{2\kappa},$$

and the rotation equation (3.2.2) *with* $\omega = 0$.

(2) *For any link pattern* $\alpha \in LP(n,m)$, *with* $m, n \in \mathbb{Z}$, $1 \leq m \leq \frac{n}{2}$, *and n even, the corresponding Coulomb gas integrals* $\mathcal{K}_\alpha^{(m,n)}(\boldsymbol{\theta})$ *defined in (5.2.20) solve the null vector equation* (1.2.3) *with*

$$h = \frac{1-\left(n-2m+\frac{\kappa}{2}\right)^2}{2\kappa},$$

and the rotation equation (3.2.2) *with* $\omega = 0$.

(3) *For any link pattern* $\alpha \in LP(n,m)$, *with* $m, n \in \mathbb{Z}$ *and* $1 \leq m \leq \frac{n}{2}$, *the Coulomb gas integral* $\mathcal{J}_\alpha^{n,m}(\boldsymbol{\theta}, \eta)$ *solves the null vector equation* (1.2.3) *with*

$$h = -\frac{(n-2m)^2}{2\kappa} + \frac{1+\eta^2}{2\kappa},$$

and the rotation equation (3.2.2) *with*

$$\omega = \frac{\eta(n-2m)}{\kappa}.$$

(4) *For any link pattern* $\alpha \in LP(n, \frac{n}{2})$, *with n even, the Coulomb gas integral* $\mathcal{L}_\alpha^n(\boldsymbol{\theta})$ *solves the null vector equation* (1.2.3) *with*

$$h = \frac{(6-\kappa)(\kappa-2)}{8\kappa},$$

and the rotation equation (3.2.2) *with* $\omega = 0$.

Here, α denotes the integration contour, and $LP(n,m)$ represents the set of all possible multiple integration contours with n boundary points and m integration variables. The abbreviation LP stands for link pattern, which is defined in Section 5.2.

We will discuss the linear independence of these solutions in our forthcoming work. Understanding the complete classification of the solution space to the null vector equations and rotation equation remains an intriguing open question. The classification of the multiple radial SLE(κ) systems can be reduced to studying the positive solutions to the null vector equations and rotation equations.

Notably, in Y. Wang and Wu (2024), the authors introduce the following chordal SLE weighted by the conformal radius. This type of solutions can also be realized by screening. The corresponding partition function is given by

$$\mathcal{Z}_\alpha(\theta_1, \theta_2) = (\sin(\theta/2))^{\frac{\kappa-6}{\kappa}} \mathbb{E}_\theta \left[\mathrm{CR}(\mathbb{D}\backslash\gamma)^{-\alpha} \right].$$

In Remark 5.2.5, we explain how this type of solution—weighted by the conformal radius—corresponds to a sum of terms of the form $\mathcal{J}_\alpha^{2,1}(\boldsymbol{\theta}, \eta)$.

1.3 Multiple radial SLE(0) systems

We treat multiple radial SLE(0) curves as natural geometric objects without reference to multiple radial SLE(κ) systems.

The defining properties of this ensemble of curves are geometric commutation and conformal invariance.

Definition 1.3.1. *Let $\gamma_1, \ldots, \gamma_n$ be simple disjoint smooth curves starting from $\{z_1, z_2, \ldots, z_n\}$ which are n distinct points counterclockwise on the unit circle $\partial\mathbb{D}$.*

(i) Each curve can be individually generated by a Loewner chain. In angular coordinate, let $z_j = e^{i\theta_j}$, then the Loewner equation for the covering map $h_t(z)$ of $g_t(z)$ (i.e. $e^{ih_t(z)} = g_t(e^{iz})$) is given by

$$\partial_t h_t(z) = \cot\left(\frac{h_t(z) - \theta_j(t)}{2}\right), \quad h_0(z) = z, \tag{1.3.1}$$

and the driving function $\theta_j(t)$ evolve as

$$\begin{cases} \mathrm{d}\theta_j(t) = U_j\left(\theta_1(t), \theta_2(t), \ldots, \theta_j(t), \ldots, \theta_n(t)\right) \mathrm{d}t \\ \mathrm{d}\theta_k(t) = \cot\left(\left(\theta_k(t) - \theta_j(t)\right)/2\right) \mathrm{d}t, k \neq j \end{cases},$$

where $U_j(\boldsymbol{\theta}) : \mathfrak{X}^n \to \mathbb{R}$ is assumed to be smooth in the chamber

$$\mathfrak{X}^n = \{(\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^n \mid \theta_1 < \theta_2 < \ldots < \theta_n < \theta_1 + 2\pi\}.$$

(ii) The curves geometrically commute, meaning that the same collection of curves can be generated by applying the individual Loewner chains in any chosen order. For example, we can first map out $\gamma^{(i)}_{[0,t_i]}$ using $h^{(i)}_{t_i}$, then mapping out $h^{(i)}_{t_i}\left(\gamma^{(j)}_{[0,t_j]}\right)$, or vice versa. The images are the same regardless of the order in which we map out the curves.

(iii) Each curve γ_j is Möbius invariant in $\mathbb{D}$. This means that if γ_j is the curve generated by a Loewner flow and initial data $\boldsymbol{\theta}$, then its image $\phi\left(\gamma_j\right)$ under a conformal automorphism ϕ of $\mathbb{D}$ is, up to a time change, generated by the same flow with initial data $\phi(\boldsymbol{\theta}) = (\phi(\theta_1), \ldots, \phi(\theta_n))$. Our definition for multiple radial SLE(0) can be naturally extended to an arbitrary simply-connected domain Ω with a marked interior point u via a conformal uniformizing map $\phi : \Omega \to \mathbb{D}$, sending u to 0.

Under these dynamics, the driving function $\theta_j(t)$ evolves according to $U_j(\boldsymbol{\theta})$, while the points $\theta_k(t)$, for $k \neq j$, follow the Loewner chain generated by $\theta_j(t)$. We define a differential operator corresponding to the curve γ_j by

$$\mathcal{M}_j = U_j(\boldsymbol{\theta})\partial_j + \sum_{k\neq j} \cot\left(\frac{\theta_k - \theta_j}{2}\right)\partial_k, \quad j = 1, \ldots, n.$$

For $\kappa = 0$, we can also derive the commutation relations for the generators $\mathcal{M}_j$; see Section 3.3 for details.

Theorem 1.3.2. *Let $\gamma_1, \ldots, \gamma_n$ be simple curves that are generated by Loewner flows and $\mathcal{U}(\boldsymbol{\theta}) : \mathfrak{X}^n \to \mathbb{R}$ is C^2 smooth. We define the differential operator $\mathcal{M}_j = U_j\partial_j + \sum_{k\neq j}\cot(\frac{\theta_k-\theta_j}{2})\partial_k$. If the curves locally geometrically commute, then the vector fields $\mathcal{M}_j$ satisfy the commutation relations.*

$$[\mathcal{M}_i, \mathcal{M}_j] = \frac{1}{\sin^2(\frac{\theta_i-\theta_j}{2})}(\mathcal{M}_j - \mathcal{M}_i) \tag{1.3.2}$$

Moreover, under the additional assumption that $\partial_j U_k = \partial_k U_j$ for all j, k, then there exists a smooth function $\mathcal{U}(\boldsymbol{\theta})$ such that $U_j = \partial_j\mathcal{U}$. The commutation relations hold for $\mathcal{L}_j$ if and only if there exists a common constant h such that

$$\frac{1}{2}U_j^2 + \sum_{k\neq j}\cot(\frac{\theta_k - \theta_j}{2})U_k - \sum_{k\neq j}\frac{3}{2\sin^2(\frac{\theta_j-\theta_k}{2})} = h \tag{1.3.3}$$

However, it is important to emphasize a key distinction between the cases $\kappa > 0$ and $\kappa = 0$: in the case $\kappa = 0$, the conditions $\partial_j U_k = \partial_k U_j$ are not consequences of the commutation relations. These conditions are equivalent to the existence of a smooth potential function $\mathcal{U}(\boldsymbol{\theta}) : \mathfrak{X}^n \to \mathbb{R}$ such that

$$U_j = \partial_j\mathcal{U},$$

where the chamber $\mathfrak{X}^n$ is defined by

$$\mathfrak{X}^n = \{(\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^n \mid \theta_1 < \theta_2 < \ldots < \theta_n < \theta_1 + 2\pi\}.$$

If we view the multiple radial SLE(0) system as the classical limit of a random multiple radial SLE(κ) system, then for the latter, we have shown that the drift term $b_j(\boldsymbol{\theta})$ takes the form

$$b_j(\boldsymbol{\theta}) = \kappa \frac{\partial \log \mathcal{Z}(\boldsymbol{\theta})}{\partial \theta_j},$$

where $\mathcal{Z}(\boldsymbol{\theta})$ is a positive function satisfying the null vector equations. The idea is that, as $\kappa \to 0$, the limit

$$\lim_{\kappa \to 0} \mathcal{Z}(\boldsymbol{\theta})^{\kappa} = \mathcal{U}(\boldsymbol{\theta})$$

exists (at least for suitably chosen partition functions).

Therefore, we typically assume the existence of such a potential $\mathcal{U}(\boldsymbol{\theta})$ with $U_j(\boldsymbol{\theta}) = \partial_j \mathcal{U}(\boldsymbol{\theta})$ when defining a multiple radial SLE(0) system.

This observation also suggests that, in fact, not all multiple radial SLE(0) systems admit a quantization or arise as classical limits of random multiple SLE(κ) systems.

Extending the methods in Alberts et al. (2020), we establish the theory of the multiple radial SLE(0) systems. We investigate the structure of the multiple radial SLE(0) from four different perspectives:

- Stationary relations and critical points of master functions;
- Traces as horizontal trajectories of quadratic differentials forming link patterns;
- Enumeration and classification;
- Relations to classical Calogero-Sutherland system.

We construct multiple radial SLE(0) systems through stationary relations. We heuristically demonstrate how stationary relations naturally emerge when normalizing the partition function for the multiple radial SLE(κ) system as $\kappa \to 0$, as discussed in Section 6.1.

Definition 1.3.3 (Stationary relations)**.** *Let $z = \{z_1, z_2, \ldots, z_n\}$ be distinct points on the unit circle, and let $\boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_m\}$ be a set of involution-symmetric marked* *points. In the unit disk $\mathbb{D}$, the stationary relations are given by*

$$-\sum_{j=1}^{n}\frac{2}{\xi_k - z_j} + \sum_{l\neq k}\frac{4}{\xi_k-\xi_l} + \frac{n-2m+2}{\xi_k} = 0, \quad k = 1,2,\dots,m. \tag{1.3.4}$$

In angular coordinates, setting $z_i = e^{i\theta_i}$ for $i = 1,2,\dots,n$ and $\xi_k = e^{i\zeta_k}$ for $k = 1,2,\dots,m$, the stationary relations take the form

$$\sum_{j=1}^{n}\cot\left(\frac{\zeta_k-\theta_j}{2}\right) = \sum_{l\neq k} 2\cot\left(\frac{\zeta_k-\zeta_l}{2}\right), \quad k = 1,2,\dots,m. \tag{1.3.5}$$

Based on the stationary relations, we now define the multiple radial SLE(0) systems.

Definition 1.3.4 (Multiple radial SLE(0) Loewner chain)**.** *Given boundary points $\boldsymbol{z} = \{z_1, z_2, \dots, z_n\}$ on the unit circle, a marked interior point $u = 0$, and involution-symmetric screening charges $\{\xi_1, \xi_2, \dots, \xi_m\}$ that solve the **stationary relations**, we define the multiple radial SLE(0) Loewner chain as follows:*

Let $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$ be a set of parametrizations for the capacity, where each $\nu_i : [0,\infty) \to [0,\infty)$ is assumed to be measurable.

In the unit disk $\mathbb{D}$ with $u = 0$, we define the multiple radial SLE(0) Loewner chain as a normalized conformal map $g_t = g_t(z)$, with the initial condition $g_0(z) = z$ and the evolution given by the Loewner equation

$$\partial_t g_t(z) = \sum_{j=1}^{n}\nu_j(t) g_t(z)\frac{z_j(t)+g_t(z)}{z_j(t)-g_t(z)}, \quad g_0(z) = z. \tag{1.3.6}$$

The Loewner chain for the covering map $h_t(z) = -i\log(g_t(e^{iz}))$ is given by

$$\partial_t h_t(z) = \sum_{j=1}^{n}\nu_j(t)\cot\left(\frac{h_t(z)-\theta_j(t)}{2}\right), \quad h_0(z) = z. \tag{1.3.7}$$

The driving functions $\theta_j(t)$, for $j = 1,\dots,n$, evolve according to

$$\dot\theta_j = \nu_j(t)\frac{\partial \log \mathcal{Z}(\boldsymbol{\theta},\boldsymbol{\zeta})}{\partial\theta_j} + \sum_{k\neq j}\nu_k(t)\cot\left(\frac{\theta_j-\theta_k}{2}\right), \tag{1.3.8}$$

where the multiple radial SLE(0) master function is defined by

$$\mathcal{Z}(\boldsymbol{\theta},\boldsymbol{\zeta}) := \prod_{1\le j<k\le n}\sin^2\left(\frac{\theta_j-\theta_k}{2}\right)\prod_{1\le s<t\le m}\sin^8\left(\frac{\zeta_s-\zeta_t}{2}\right)\prod_{k=1}^{n}\prod_{l=1}^{m}\sin^{-4}\left(\frac{\theta_k-\zeta_l}{2}\right). \tag{1.3.9}$$

The logarithmic derivative of $\mathcal{Z}(\boldsymbol{\theta}, \boldsymbol{\zeta})$ with respect to θ_j (treating $\boldsymbol{\theta}$ and $\boldsymbol{\zeta}$ as independent variables) is given by

$$\frac{\partial \mathcal{Z}(\boldsymbol{\theta}, \boldsymbol{\zeta})}{\partial \theta_j} = \sum_{k \neq j} \cot\left(\frac{\theta_j - \theta_k}{2}\right) - 2\sum_{l} \cot\left(\frac{2}{\theta_j - \zeta_l}\right) \tag{1.3.10}$$

for $j = 1, \ldots, n$, The flow map g_t is well-defined up to the first time τ at which $z_j(t) = z_k(t)$ for some $1 \leq j < k \leq n$. For each $z \in \mathbb{C}$, the process $t \mapsto g_t(z)$ is well-defined up to the time $\tau_z \wedge \tau$, where τ_z is the first time at which $g_t(z) = z_j(t)$. The hull associated with this Loewner chain is denoted by

$$K_t = \left\{z \in \overline{\mathbb{D}} : \tau_z \leq t\right\}.$$

Remark 1.3.5. The above definition of the multiple radial SLE(0) system is dynamic and local. It allows us to define such a system for arbitrary initial configurations of involution-symmetric $\boldsymbol{\xi}$, without assuming the stationary relations. When $\boldsymbol{\xi}$ satisfies the stationary relations, we will show that, for any parametrization $\boldsymbol{\nu}(t)$, the traces are horizontal trajectories of a quadratic differential $Q(z), dz^2$, as stated in Theorem 1.3.7. This gives us an alternative perspective on the reparametrization symmetry (commutation relations) in the case $\kappa = 0$.

To characterize the traces of multiple radial SLE(0) systems, we introduce a class of quadratic differentials, denoted $\mathcal{QD}(\boldsymbol{z})$, with prescribed zeros at $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$. These quadratic differentials are defined on the Riemann sphere and exhibit involution symmetry.

Definition 1.3.6 (Quadratic differentials with prescribed zeros)**.** *Let $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$ be distinct points on the unit circle. The class of quadratic differentials, denoted by $\mathcal{QD}(\boldsymbol{z})$.*

1. *symmetric under the involution $z^* = \frac{1}{\bar{z}}$, meaning*

$$\overline{Q(z^*)}\overline{(dz^*)^2} = Q(z)dz^2.$$

2. *distinct zeros at $\{z_1, z_2, \ldots, z_n\}$, each of order 2.*

3. *distinct finite poles at $\{\xi_1, \ldots, \xi_m\}$, each of order 4, and the residues vanish (Residue-free condition):*

$$Res_{\xi_j}(\sqrt{Q(z)}dz) = 0, \quad \text{for } j = 1, \ldots, m.$$

4. *poles of order $n+2-2m$ at the marked points 0 and ∞. This ensures the total difference between the number of zeros and poles is -4.*

Here, the poles $\{\xi_1,\ldots,\xi_m\}$ are finite, meaning they do not coincide with 0 or ∞. The quadratic differential $Q(z) \in \mathcal{QD}(\boldsymbol{z})$ must take the following form:

$$Q(z) = \frac{\prod_{k=1}^m \xi_k^2}{\prod_{j=1}^n z_j} z^{2m-n-2} \frac{\prod_{j=1}^n (z-z_j)^2}{\prod_{k=1}^m (z-\xi_k)^4}.$$

By considering the primitive of $F(z) = \int \sqrt{Q(z)}dz$, we find that the residue-free quadratic differentials are natural generalization of rational functions, specifically designed to address the monodromy at 0; see section (6.3).

The geometry of the horizontal trajectories of a quadratic differential $Q(z) \in \mathcal{QD}(\boldsymbol{z})$ is described as follows.

In the main theorem (1.3.7), we show that the traces of the multiple radial SLE(0) systems correspond precisely to the horizontal trajectories of the class of residue-free quadratic differentials $Q(z) \in \mathcal{QD}(\boldsymbol{z})$ with limiting ends at $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$.

Theorem 1.3.7. *Let $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$ be distinct growth points on the unit circle and screening charges $\xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$ involution symmetric and solve the stationary relations.*

There exists an $Q(z) \in \mathcal{QD}(\boldsymbol{z})$ with $\boldsymbol{\xi}$ as poles and $\boldsymbol{z}$ as zeros, the hulls K_t generated by the Loewner flows with parametrization $\boldsymbol{\nu}(t)$ are subsets of the horizontal trajectories of $Q(z)dz^2$ with limiting ends at $\boldsymbol{z}$, up to any time t before the collisions of any poles or critical points. Up to any such time

$$Q(z) \circ g_t^{-1} \in \mathcal{QD}(\boldsymbol{z}(t)).$$

where $\boldsymbol{z}(t)$ is the location of the critical points at time t under the multiple radial Loewner flow with parametrization $\boldsymbol{\nu}(t)$.

The key ingredient in the proof of theorem (1.3.7) is the integral of motion for the Loewner flows. This integral of motion, denoted by $N_t(z)$, arises as the classical limit of a martingale observable inspired by conformal field theory.

Theorem 1.3.8. *In the unit disk $\mathbb{D}$, let $z_1, z_2, \ldots, z_n$ be distinct growth points on $\partial\mathbb{D}$. For each $z \in \overline{\mathbb{D}}$, define the following:*

$$
\begin{cases}
A(t) = \dfrac{\prod_{j=1}^{m} \xi_k^2(t)}{\prod_{k=1}^{n} z_k(t)}, \\
B_t(z) = e^{-(2m-n)\left(\int_0^t \sum_j \nu_j(s)\, ds\right)} g_t(z)^{2m-n-2} (g_t'(z))^2 \dfrac{\prod_{k=1}^{n} (g_t(z) - z_k(t))^2}{\prod_{j=1}^{m} (g_t(z) - \xi_j(t))^4}, \\
N_t(z) = A(t)B_t(z) = e^{-(2m-n)\left(\int_0^t \sum_j \nu_j(s)\, ds\right)} \dfrac{\prod_{j=1}^{m} \xi_k(t)^2}{\prod_{k=1}^{n} z_k(t)} g_t(z)^{2m-n-2} \\
\qquad (g_t'(z))^2 \dfrac{\prod_{k=1}^{n} (g_t(z) - z_k(t))^2}{\prod_{j=1}^{m} (g_t(z) - \xi_j(t))^4}.
\end{cases}
$$

Then, $A(t)$, $B_t(z)$, and $N_t(z)$ are field integrals of motion on the interval $[0, \tau_t \wedge \tau)$ for the multiple radial SLE(0) Loewner flows with parametrization $\nu_j(t)$, $j = 1, \ldots, n$.

Remark 1.3.9. $N_t(z)$ is a field integral of motion for arbitrary initial positions of screening charges $\boldsymbol{\xi}$ even without assuming stationary relations. The stationary relations imply the existence of a quadratic differential $Q(z)dz^2 \in \mathcal{QD}(z)$; see Theorem 1.3.7.

A detailed explanation of this construction can be found in Section 6.5. This approach can also be extended to various multiple SLE systems. For the systematic and rigorous study of such conformal field theories, please refer to Kang and N. Makarov (2013) and N-G. Kang and N. Makarov (2021).

The closure of horizontal trajectories of $Q(z) \in \mathcal{QD}(z)$ with limiting ends at zeros $\{z_1, z_2, \ldots, z_n\}$ form radial topological link patterns, see theorem (6.3.5) for detailed proof and section 6.7 and section 6.8 for figures illustrating the traces of the multiple radial SLE(0) systems.

The classification of multiple radial SLE(0) is linked to the enumeration of the master function for trigonometric KZ equations. This connection touches on a rich and intricate area of enumerative geometry.

As shown in Scherbak (2002a), Scherbak (2002b), and Scherbak and Varchenko (2003), these studies establish connections between the space of critical points and tensor products of Verma modules and other algebraic structures, providing deeper insights into the monodromy of the KZ equations. They also demonstrate that the critical points of the master function can be interpreted as solutions to the Bethe Ansatz equations, thereby linking the study of KZ equations with integrable systems.

Furthermore, when all $z_1, z_2, \ldots, z_n$ lie on the real line, it is shown in Mukhin, Tarasov, and Varchenko (2009) that the enumeration of these critical points is equivalent to the real Shapiro-Shapiro conjecture in real enumerative geometry.

This is part of our ongoing research, and based on Mukhin and Varchenko (2008), we propose several illuminating conjectures about the enumeration problem in section 6.6.

1.4 Relations to classical Calegero-Sutherland systems

From the Hamiltonian point of view, we show that the multiple radial SLE(0) Loewner growing with common parametrization of capacity (i.e. $\nu_j(t) = 1$) are a special type of classical Calogero-Sutherland system.

The **stationary relations** can be interpreted as initial conditions for the particles and n quadratic null vector equations as n null vector Hamiltonians, which are related to the classical Calegro-Sutherland Hamiltonian via the lax pair. Furthermore, these null vector Hamiltonians induce commuting Hamiltonian flows along the submanifolds defined as the intersection of their level sets.

Theorem 1.4.1. *From a dynamical system perspective, the driving functions of multiple radial SLE(0) systems are given by*

$$\dot{\theta}_j = U_j(\boldsymbol{\theta}) + \sum_{k\neq j} \cot\left(\frac{\theta_j - \theta_k}{2}\right),$$

where U_j satisfies the quadratic null vector equation for a constant h:

$$\frac{1}{2}U_j^2 + \sum_{k\neq j} \cot\left(\frac{\theta_k - \theta_j}{2}\right) U_k - \sum_{k\neq j} \frac{3}{2\sin^2\left(\frac{\theta_j - \theta_k}{2}\right)} = h. \tag{1.4.1}$$

(i) By introducing the momentum function p_j, defined as

$$p_j = U_j + \sum_{k\neq j} \cot\left(\frac{\theta_j - \theta_k}{2}\right), \tag{1.4.2}$$

we can reformulate the multiple radial SLE(0) system as a Calogero-Sutherland system. The momentum p_j satisfies the null vector Hamiltonian equation:

$$\mathcal{H}_j(\boldsymbol{\theta}, \boldsymbol{p}) = \frac{1}{2}p_j^2 - \sum_{k\neq j} (p_j + p_k) f_{jk} + \sum_k \sum_{l\neq k} f_{jk} f_{jl} - 2\sum_{k\neq j} f_{jk}^2 = h - \frac{3(n-1)}{2} - C_{n-1}^2. \tag{1.4.3}$$

The total null vector Hamiltonian $\mathcal{H} = \sum_j \mathcal{H}_j$ is equivalent to the classical Calogero-Sutherland Hamiltonian:

$$\mathcal{H} = \sum_j \frac{p_j^2}{2} - \sum_{1\le j<k\le n} \frac{4}{\sin^2\left(\frac{\theta_j-\theta_k}{2}\right)} = nh - \frac{n(n^2-1)}{6}. \tag{1.4.4}$$

(ii) The commutation relations between different growth pairs are expressed in terms of the Poisson bracket:

$$\left\{\mathcal{H}_j, \mathcal{H}_k\right\} = \frac{1}{f_{jk}^2}\left(\mathcal{H}_k - \mathcal{H}_j\right). \tag{1.4.5}$$

Consequently, the vector flows $X_{\mathcal{H}_j}$ induced by the Hamiltonians $\mathcal{H}_j$ commute along the submanifolds N_c:

$$N_c = \left\{(\boldsymbol{\theta}, \boldsymbol{p}) : \mathcal{H}_j(\boldsymbol{\theta}, \boldsymbol{p}) = c, \textit{for all } j\right\}. \tag{1.4.6}$$

This relationship is a classical analog of the relation between multiple radial $SLE(\kappa)$ and quantum Calogero-Sutherland system, first discovered in Doyon and J. Cardy (2007). Notably, in the $\kappa > 0$ case, the solutions to the null vector PDE system in section 5.2 yield eigenstates of the quantum Calogero-Sutherland system beyond the eigenstates built upon the fermionic ground states.

Chapter 2

COULOMB GAS CORRELATION AND RATIONAL $SLE(\kappa)$

2.1 Schramm Loewner evolutions

In this section, we briefly recall the basic defintions and properties of the chordal and radial SLE. We will describe the radial Loewner chain in $\mathbb{D}$, where $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ and chordal Loewner chain in $\mathbb{H} = \{\mathrm{Im}(z) > 0\}$.

Definition 2.1.1 (Conformal radius)**.** *The conformal radius of a simply connected domain Ω with respect to a point $z \in \Omega$, defined as*

$$\mathrm{CR}(\Omega, z) := |f'(0)|,$$

where $f : \mathbb{D} \to \Omega$ is a conformal map from the open unit disk $\mathbb{D}$ onto Ω with $f(0) = z$.

Definition 2.1.2 (Capacity in $\mathbb{D}$)**.** *For any compact subset K of $\overline{\mathbb{D}}$ such that $\mathbb{D}\backslash K$ is simply connected and contains 0 , let g_K be the unique conformal map $\mathbb{D}\backslash K \to \mathbb{D}$ such that $g_K(0) = 0$ and $g'_K(0) > 0$. The conformal radius of $\mathbb{D}\backslash K$ is*

$$\mathrm{CR}(\mathbb{D}\backslash K) := \left(g'_K(0)\right)^{-1}.$$

The capacity of K is

$$\mathrm{cap}(K) = \log g'_K(0) = -\log \mathrm{CR}(\mathbb{D}\backslash K, 0).$$

Definition 2.1.3 (Capacity in $\mathbb{H}$)**.** *For any compact subset $K \subset \overline{\mathbb{H}}$ such that $\mathbb{H}\backslash K$ is a simply connected domain. The half-plane capacity of a hull K is the quantity*

$$\mathrm{hcap}(K) := \lim_{z\to\infty} z\,[g_K(z) - z],$$

where $g_K : \mathbb{H}\backslash K \to \mathbb{H}$ is the unique conformal map satisfying the hydrodynamic normalization $g(z) = z + O\left(\frac{1}{z}\right)$ as $z \to \infty$.

Definition 2.1.4 (Radial Loewner chain)**.** *Let g_t satisfies the radial Loewner equation*

$$\partial_t g_t(z) = g_t(z)\frac{e^{i\theta_t} + g_t(z)}{e^{i\theta_t} - g_t(z)}, \quad g_0(z) = z, \tag{2.1.1}$$

where $t \mapsto \theta_t$ is real continuous and called the driving function. Let K_t be the set of points z in $\mathbb{D}$ such that the solution $g_s(z)$ blows up before or at time t. K_t is called the radial SLE hull driven by θ_t.

Radial Loewner chain in arbitrary simply connected domain $\Omega \subsetneq \mathbb{C}$ with a marked interior point $u \in D$, is defined via a conformal map from $\mathbb{D}$ onto Ω sending 0 to u.

Definition 2.1.5 (Radial SLE(κ))**.** *For $\kappa \geq 0$, the radial SLE(κ) is the random Loewner chain in $\mathbb{D}$ from 1 to 0 driven by:*

$$\theta_t = \sqrt{\kappa} B_t, \tag{2.1.2}$$

where B_t is the standard Brownian motion.

Definition 2.1.6 (Characterization of radial SLE)**.** *The radial SLE is a family $\mathbb{P}(\mathbb{D}; \zeta, 0)$ of probability measures on curves $\eta : [0, \infty) \to \overline{\mathbb{D}}$ with $\eta(0) = \zeta$ and parametrized by capacity satisfies the following properties:*

- *(Conformal invariance) For all $a \in \mathbb{R}$, let $\rho_a(z) = \mathrm{e}^{\mathrm{i}a} z$ be the rotation map $\mathbb{D} \to \mathbb{D}$, the pullback measure $\rho_a^* \mathbb{P}(\mathbb{D}; \zeta, 0) = \mathbb{P}(\mathbb{D}; e^{-ia}\zeta, 0)$. From this, we may extend the definition to $\mathbb{P}(\Omega; a, b)$ in any simply connected domain Ω with an interior marked point u by pulling back using a uniformizing conformal map $\Omega \to \mathbb{D}$ sending u to 0.*

- *(Domain Markov property) given an initial segment $\gamma[0, \tau]$ of the radial SLE_κ curve $\gamma \sim \mathbb{P}(\Omega; x, y)$ up to a stopping time τ, the conditional law of $\gamma[\tau, \infty)$ is the law $\mathbb{P}\left(\Omega \backslash K_\tau; \gamma(\tau), 0\right)$ of the SLE_κ curve in the complement of the hull K_τ from the tip $\gamma(\tau)$ to 0.*

- *(Reflection symmetry) Let $\iota : z \mapsto \bar{z}$ be the complex conjugation, then $\mathbb{P}(\zeta, 0) \sim \iota^* \mathbb{P}(\overline{\zeta}, 0)$.*

Definition 2.1.7 (Chordal Loewner chain)**.** *Let g_t satisfies the chordal Loewner equation*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z, \tag{2.1.3}$$

where $t \mapsto \xi_t$ is continuous and called the driving function. Let K_t be the set of points z in $\mathbb{H}$ such that the solution $g_s(z)$ blows up before or at time t. K_t is called the chordal SLE hull driven by ξ_t

Chordal Loewner chain in arbitrary simply connected domain $\Omega \subsetneq \mathbb{C}$ from a to b, is defined via a uniformizing conformal map from Ω onto $\mathbb{H}$ sending a to 0 and b to ∞.

Definition 2.1.8 (Chordal SLE(κ))**.** *For $\kappa \geq 0$, the chordal SLE(κ) is the random Loewner chain in $\mathbb{H}$ from 0 to ∞ driven by*

$$\xi_t = \sqrt{\kappa} B_t, \tag{2.1.4}$$

where B_t is the standard Brownian motion.

Definition 2.1.9 (Characterization of Chordal SLE)**.** *Chordal SLE is a family of probability measures on curves $\mathbb{P}(\mathbb{H}; a, b)$ $\eta : [0, \infty] \to \overline{\mathbb{H}}$ with $\eta(0) = a, \eta(\infty) = b$ and parametrized by capacity satisfies the following properties:*

- *(Conformal invariance) $\rho(z) \in$ Aut$(\mathbb{H})$, the pullback measure $\rho^*\mathbb{P}(a, b) = \mathbb{P}(\mathbb{H}; \rho(a), \rho(b))$. From this, we may extend the definition of to $\mathbb{P}(\mathbb{H}; z_1, z_2)$ in any simply connected domain Ω with two boundary points z_1, z_2 by pulling back using a uniformizing conformal map $\Omega \to \mathbb{H}$ sending z_1 to a and z_2 to b.*
- *(Domain Markov property) given an initial segment $\gamma[0, \tau]$ of the SLE_κ curve $\gamma \sim \mathbb{P}(\Omega; x, y)$ up to a stopping time τ, the conditional law of $\gamma[\tau, \infty)$ is the law $\mathbb{P}(\Omega \backslash K_\tau; \gamma(\tau), y)$ of the SLE_κ curve in the complement of the hull K_τ from the tip $\gamma(\tau)$ to y.*

2.2 Coulomb gas correlation on Riemann sphere

To define more general SLE processes beyond the chordal and radial SLEs, we introduce the concept of Coulomb gas correlations. These correlations serve as partition functions for various SLE processes and play a central role in conformal field theory.

We define the Coulomb gas correlations as the (holomorphic) differentials with conformal dimensions $\lambda_j = \sigma_j^2/2 - \sigma_j b$ at z_j (including infinity) and with values

$$\prod_{\substack{j<k \\ z_j, z_k \neq \infty}} (z_j - z_k)^{\sigma_j \sigma_k}, \quad \left(z_j \in \widehat{\mathbb{C}}\right)$$

in the identity chart of $\mathbb{C}$ and the chart $z \mapsto -1/z$ at infinity. If $\sigma_j \sigma_k \notin 2\mathbb{Z}$, the Coulomb gas differential is multi-valued; in this case, we choose a single-valued branch. After explaining this definition, we prove that under the neutrality condition, $\sum \sigma_j = 2b$, the Coulomb gas correlation functions are conformally invariant with respect to the Möbius group $\mathrm{Aut}(\widehat{\mathbb{C}})$.

Definition 2.2.1 (Differential)**.** *A local coordinate chart on a Riemann surface M is a conformal map $\phi : U \to \phi(U) \subset \mathbb{C}$ on an open subset U of M. A differential f is an assignment of a smooth function $(f\|\phi) : \phi(U) \to \mathbb{C}$ to each local chart $\phi : U \to \phi(U)$. f is a differential of conformal dimensions $[\lambda, \lambda_*]$ if for any two overlapping charts ϕ and $\tilde{\phi}$, we have:*

$$(f\|\phi) = (h')^{\lambda} \left(\overline{h'}\right)^{\lambda_*} (\tilde{f} \circ h \|\tilde{\phi}), \tag{2.2.1}$$

where $h = \tilde{\phi} \circ \phi^{-1} : \phi(U \cap \tilde{U}) \to \tilde{\phi}(U \cap \tilde{U})$ is the transition map.

Definition 2.2.2 (Neutrality Condition)**.** *A divisor $\boldsymbol{\sigma} : \widehat{\mathbb{C}} \to \mathbb{R}$ is said to satisfy the* neutrality condition $(\mathrm{NC})_b$ *if*

$$\int \boldsymbol{\sigma} = 2b, \tag{2.2.2}$$

for some $b \in \mathbb{R}$. In the context of SLE_κ*, the parameter b is related to $\kappa > 0$ by*

$$b = \sqrt{\frac{8}{\kappa}} - \sqrt{\frac{\kappa}{2}}. \tag{2.2.3}$$

Definition 2.2.3 (Coulomb gas correlations for a divisor on the Riemann sphere)**.** *Let the divisor*

$$\boldsymbol{\sigma} = \sum \sigma_j \cdot z_j,$$

where $\{z_j\}_{j=1}^n$ is a finite set of distinct points on $\widehat{\mathbb{C}}$. The Coulomb gas correlation $C_{(b)}[\boldsymbol{\sigma}]$ is a differential of conformal dimension λ_j at z_j, given by

$$\lambda_j = \lambda_b\left(\sigma_j\right) \equiv \frac{\sigma_j^2}{2} - \sigma_j b, \tag{2.2.4}$$

where $\lambda_b(\sigma) = \frac{\sigma^2}{2} - \sigma b \quad (\sigma \in \mathbb{C})$ whose value is given by

$$C_{(b)}[\boldsymbol{\sigma}] = \prod_{j<k} \left(z_j - z_k\right)^{\sigma_j \sigma_k}, \tag{2.2.5}$$

where the product is taken over all finite z_j and z_k.

This defines a holomorphic function of $\boldsymbol{z}$ on the configuration space

$$\mathbb{C}^n_{\text{distinct}} = \left\{\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \,\middle|\, z_j \neq z_k \text{ for } j \neq k\right\}.$$

In general, the function is multivalued, and one must choose a single-valued branch for each factor $(z_j - z_k)^{\sigma_j \sigma_k}$*, except in special cases where all* σ_j *are integers. If all* σ_j *are even integers, the function becomes single-valued and independent of the ordering of the product. In the special case where* $\sigma_j = 1$ *for all* j*, the correlation function coincides with the Vandermonde determinant.*

Theorem 2.2.4 (see N-G. Kang and N. Makarov (2021) thm (2.2))**.** *Under the neutrality condition* (NC_b)*, the differentials* $C_{(b)}[\boldsymbol{\sigma}]$ *are Möbius invariant on* $\hat{\mathbb{C}}$*.*

2.3 Coulomb gas correlation in a simply connected domain

In this section, we define the Coulomb gas correlation differential in a simply connected domain.

Definition 2.3.1 (Symmetric Riemann surface)**.** *A symmetric Riemann surface is a pair* (S, j) *consisting of a Riemann surface* S *and an anticonformal involution* j *on* S*. The latter means that* $j : S \to S$ *is an anti-analytic map with* $j \cdot j = id$ *(the identity map).*

The principal example for us is the symmetric Riemann surface obtained by taking the Schottky double of a simply connected domain domain. The construction of this is briefly as follows. (See section 2.2, Schiffer and Spencer (1954), II.3E, Ahlfors and Sario (1960) for details.)

Definition 2.3.2 (Schottky double)**.** *Let* $\Omega \subsetneq \mathbb{C}$ *be a simply connected domain in* $\mathbb{C}$ *with* $\Gamma = \partial\Omega$ *consisting of prime ends. Take copy* $\tilde{\Omega}$ *of* Ω *and weld* Ω *and* $\tilde{\Omega}$ *together along* Γ *so that a compact topological surface* $\Omega^{Double} = \Omega \cup \Gamma \cup \tilde{\Omega}$ *is obtained. If* $z \in \Omega$ *let* $\tilde{z}$ *denote the corresponding point on* $\tilde{\Omega}$*. Then an involution* j *on* Ω^{Double} *is defined by*

$$\begin{aligned} j(z) &= \tilde{z} \quad \textit{and} \\ j(\tilde{z}) &= z \quad \textit{for } z \in \Omega, \\ j(z) &= z \quad \textit{for } z \in \Gamma. \end{aligned}$$

The conformal structure on $\tilde{\Omega}$ *will be the opposite to that on* Ω*, which means that the function* $\tilde{z} \mapsto \overline{z}$ *serves as a local variable on* $\tilde{\Omega}$*, and* j *becomes anti-analytic.*

For $p \in \partial\Omega$*, let* $\phi : U \subset \overline{\Omega} \to \phi(U)$ *be a local boundary chart at* p*, let* $\tilde{U}$ *be the corresponding subset in* $\tilde{\Omega}$*, then* $\tilde{\phi} : \tilde{U} \subset \tilde{\Omega} \to \overline{\phi}(\tilde{U})$ *is a local chart at* $\tilde{p}$*. Then we* *can define a local chart τ for Ω^{Double} at boundary point p by*

$$\tau(z) = \begin{cases} \phi(z), z \in U \\ \overline{\phi(z)}, z \in \tilde{U}. \end{cases}$$

Thus, the conformal structure on Ω^{Double}, inherited from $\mathbb{C}$, extends in a natural way across Γ to a conformal structure on all of Ω^{Double}. This makes Ω^{Double} into a symmetric Riemann sphere.

For example, we identify $\widehat{\mathbb{C}}$ with the Schottky double of $\mathbb{H}$ or that of $\mathbb{D}$. Then the corresponding involution j is $j_{\mathbb{H}} : z \mapsto z^* = \bar{z}$ for $\Omega = \mathbb{H}$ and $j_{\mathbb{D}} : z \mapsto z^* = 1/\bar{z}$ for $\Omega = \mathbb{D}$.

Definition 2.3.3 (Double divisor)**.** *Suppose Ω is a simply connected domain ($\Omega \subsetneq \mathbb{C}$).*

A double divisor $(\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$ is a pair of divisor in $\overline{\Omega}$

$$\boldsymbol{\sigma}^+ = \sum \sigma_j^+ \cdot z_j, \boldsymbol{\sigma}^- = \sum \sigma_j^- \cdot z_j. \tag{2.3.1}$$

We introduce an equivalence relation for double divisors:

$$(\boldsymbol{\sigma}_1^+, \boldsymbol{\sigma}_1^-) \sim (\boldsymbol{\sigma}_2^+, \boldsymbol{\sigma}_2^-) \tag{2.3.2}$$

if and only if

$$\boldsymbol{\sigma}_1^+ + \boldsymbol{\sigma}_1^- = \boldsymbol{\sigma}_2^+ + \boldsymbol{\sigma}_2^- \quad \text{on } \partial\Omega. \tag{2.3.3}$$

Thus, we may choose a representative $\boldsymbol{\sigma}^-$ from each equivalence class that is supported in Ω, i.e., $\sigma_j^- = 0$ if $z_j \in \partial\Omega$.

Definition 2.3.4. *Suppose Ω is a simply connected domain ($\Omega \subsetneq \mathbb{C}$), let $\partial\Omega$ be its Carathéodory boundary (prime ends) and consider the Schottky double $S = \Omega^{double}$, which equips with the canonical involution $\iota \equiv \iota_\Omega : S \to S, z \mapsto z^*$.*

Then, for a double divisor $(\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$, we define the associated divisor on the Schottky double S by

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^+ + \boldsymbol{\sigma}_*^-, \quad \text{where} \quad \boldsymbol{\sigma}_*^- := \sum \sigma_j^- \cdot z_j^*, \tag{2.3.4}$$

and each z_j^* denotes the image of z_j under the canonical involution ι of S. Accordingly, $\boldsymbol{\sigma}_*^-$ is the pushforward of $\boldsymbol{\sigma}^-$ under ι.

Definition 2.3.5 (Neutrality condition)**.** *A double divisor* $(\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$ *satisfies the neutrality condition* (NC_b) *if*

$$\int \boldsymbol{\sigma} = \int \boldsymbol{\sigma}^+ + \int \boldsymbol{\sigma}^- = 2b. \tag{2.3.5}$$

Definition 2.3.6 (Coulomb gas correlation for a double divisor in a simply connected domain)**.** *For a double divisor* $(\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$*, let* $\boldsymbol{\sigma} = \boldsymbol{\sigma}^+ + \boldsymbol{\sigma}^-_*$ *be its corresponding divisor in the Schottky double* S*, we define the Coulomb gas correlation of the double divisor* $(\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$ *by*

$$C_\Omega\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right](z) := C_S[\boldsymbol{\sigma}]. \tag{2.3.6}$$

We often omit the subscripts Ω, S *to simplify the notations.*

If the double divisor $(\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$ *satisfies the neutrality condition* (NC_b)*, then the Coulomb gas correlation function* $C_\Omega\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right]$ *is a well-defined differential on* Ω*, with conformal weights* $\left[\lambda_j^+, \lambda_j^-\right]$ *at each point* $z_j \in \Omega$*.*

If $z_j \in \partial\Omega$*, then the differential is with respect to a boundary chart: that is, a local conformal map from a neighborhood of* z_j *in* Ω *to the upper half-plane* $\mathbb{H}$*, sending* z_j *to a boundary point of* $\mathbb{H}$*. The derivative* ∂_{z_j} *is then defined as the holomorphic derivative in this local coordinate.*

$$\lambda_j^+ = \lambda_b\left(\sigma_j^+\right) \equiv \frac{(\sigma_j^+)^2}{2} - \sigma_j^+ b, \quad \lambda_j^- = \lambda_b\left(\sigma_j\right) \equiv \frac{(\sigma_j^-)^2}{2} - \sigma_j^- b. \tag{2.3.7}$$

By conformal invariance of the Coulomb gas correlation differential $C_S[\boldsymbol{\sigma}]$ *on the Riemann sphere under Möbius transformation, the Coulomb gas correlation differential* $C_\Omega\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right](z)$ *is invariant under* $Aut(\Omega)$*.*

Theorem 2.3.7 (see N-G. Kang and N. Makarov (2021) thm (2.4))**.** *Under the neutrality condition* (NC_b)*, the value of the differential* $C_\mathbb{H}\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right]$ *in the identity chart of* $\mathbb{H}$ *(and the chart* $z \mapsto -1/z$ *at infinity) is given by*

$$C_\mathbb{H}\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right] = \prod_{j<k} \left(z_j - z_k\right)^{\sigma_j^+\sigma_k^+} \left(\bar{z}_j - \bar{z}_k\right)^{\sigma_j^-\sigma_k^-} \prod_{j,k} \left(z_j - \bar{z}_k\right)^{\sigma_j^+\sigma_i^-}, \tag{2.3.8}$$

where the products are taken over finite z_j *and* z_k*.*

Example 2.3.8. *We have*

(i) if $\boldsymbol{\sigma}^- = \mathbf{0}$*, then (up to a phase)*

$$C_{\mathbb{H}}\left[\boldsymbol{\sigma}^+, \mathbf{0}\right] = \prod_{j<k} \left(z_j - z_k\right)^{\sigma_j^+ \sigma_k^+};$$

(ii) if $\boldsymbol{\sigma}^- = \overline{\boldsymbol{\sigma}^+}$*, then (up to a phase)*

$$C_{\mathbb{H}}\left[\boldsymbol{\sigma}^+, \overline{\boldsymbol{\sigma}^+}\right] = \prod_{j<k} \left|\left(z_j - z_k\right)^{\sigma_j^+ \sigma_k^+} \left(z_j - \bar{z}_k\right)^{\sigma_j^+ \overline{\sigma_k^+}}\right|^2 \prod_{\operatorname{Im} z_j > 0} \left(2 \operatorname{Im} z_j\right)^{\left|\sigma_j^+\right|^2};$$

(iii) if $\boldsymbol{\sigma}^- = -\overline{\boldsymbol{\sigma}^+}$*, then (up to a phase)*

$$C_{\mathbb{H}}\left[\boldsymbol{\sigma}^+, -\overline{\boldsymbol{\sigma}^+}\right] = \prod_{j<k} \left|\left(z_j - z_k\right)^{\sigma_j^+ \sigma_k^+} \left(z_j - \overline{z_k}\right)^{-\sigma_j^+ \overline{\sigma_k^+}}\right|^2 \prod_{\operatorname{Im} z_j > 0} \left(2 \operatorname{Im} z_j\right)^{-\left|\sigma_j^+\right|^2}.$$

where the products are taken over finite z_j *and* z_k*.*

Theorem 2.3.9 (see N-G. Kang and N. Makarov (2021) thm (2.5))**.** *Under the neutrality condition* (NC$_b$)*, the value of the differential* $C_{\mathbb{D}}\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right]$ *in the identity chart of* $\mathbb{D}$ *is given by*

$$C_{\mathbb{D}}\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right] = \prod_{j<k} \left(z_j - z_k\right)^{\sigma_j^+ \sigma_k^+} \left(\bar{z}_j - \bar{z}_k\right)^{\sigma_j^- \sigma_k^-} \prod_{j,k} \left(1 - z_j \bar{z}_k\right)^{\sigma_j^+ \sigma_k^-}, \tag{2.3.9}$$

where the product is taken over finite z_j *and* z_k*.*

2.4 Rational $SLE_\kappa[\boldsymbol{\sigma}]$

Definition 2.4.1 (Rational SLE)**.** *In the unit disk* $\mathbb{D}$*, let* $e^{i\theta} \in \partial\mathbb{D}$ *be the growth point, and let* $u_1 = e^{i\theta_1}, u_2 = e^{i\theta_2}, \ldots, u_k = e^{i\theta_n} \in \overline{\mathbb{D}}$ *be marked points. The symmetric double divisor* $(\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$ *assigns a charge distribution on* $e^{i\theta}$ *and* $\{u_1, u_2, \ldots, u_m\}$*, where* $\boldsymbol{\sigma}^+ = a \cdot e^{i\theta} + \sum \sigma_j \cdot u_j$ *and* $\boldsymbol{\sigma}^- = \overline{\boldsymbol{\sigma}^+}|_\Omega$*, satisfying the neutrality condition (*NC_b*).*

We define the rational $SLE_\kappa[\boldsymbol{\sigma}]$ *Loewner chain as a random normalized conformal map* $g_t = g_t(z)$*, with initial conditions* $g_0(z) = z$ *and* $g_t'(0) = e^{-t}$*. The evolution of* $g_t(z)$ *is governed by the Loewner differential equation:*

$$\partial_t g_t(z) = g_t(z) \frac{e^{i\theta(t)} + g_t(z)}{e^{i\theta(t)} - g_t(z)}, \quad g_0(z) = z.$$

Let $h_t(z)$ be the covering map of $g_t(z)$, i.e., $e^{ih_t(z)} = g_t(e^{iz})$. The evolution of $h_t(z)$ is described by:

$$\partial_t h_t(z) = \cot\left(\frac{h_t(z) - \theta(t)}{2}\right), \quad h_0(z) = z.$$

The driving function $\theta(t)$ evolves according to:

$$d\theta(t) = \frac{\partial \log \mathcal{Z}(\theta)}{\partial \theta}\, dt + \sqrt{\kappa}\, dB_t,$$

where the partition function

$$\mathcal{Z}(\boldsymbol{\theta}) = \prod_{j<k} \sin\left(\frac{\theta_j - \theta_k}{2}\right)^{\sigma_j\sigma_k} \prod_j e^{\frac{i}{2}\sigma_j(\sigma_0-\sigma_\infty)\theta_j}. \tag{2.4.1}$$

is defined in (2.4.1).

The flow map g_t is well-defined up to the first time τ at which $\zeta(t) = g_t(w)$ for some w in the support of $\boldsymbol{\sigma}$. For each $z \in \mathbb{C}$, the process $t \mapsto g_t(z)$ is well-defined up to $\tau_z \wedge \tau$, where τ_z is the first time at which $g_t(z) = e^{i\theta(t)}$. Denote $K_t = \left\{z \in \overline{\mathbb{H}} : \tau_z \leq t\right\}$ as the hull associated with this Loewner chain.

Furthermore, the law of the rational SLE(κ) Loewner chain is invariant under Möbius transformations Aut($\mathbb{D}$) (up to a time change), due to the conformal invariance of the Coulomb gas correlation. Consequently, we define rational $SLE_\kappa[\boldsymbol{\sigma}]$ in any simply connected domain Ω by pulling back via a conformal map $\phi : \Omega \to \mathbb{D}$.

In definition (2.4.1), we define the rational SLE from the perspective of the partition function. This approach helps us to understand the SLE within the framework of conformal field theory and can be naturally extended to various settings, including multiple SLE(κ) systems.

Example 2.4.2. *Double divisor for chordal and radial* SLE(κ, ρ)*, where ξ denotes the growth point and q is the marked boundary point (in the chordal case) or interior point (in the radial case).*

In addition to the aforementioned definition, another widely used equivalent is known as SLE(κ,ρ). We prove the equivalence between rational $SLE_\kappa[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-]$ and SLE(κ,ρ) in the following theorem.

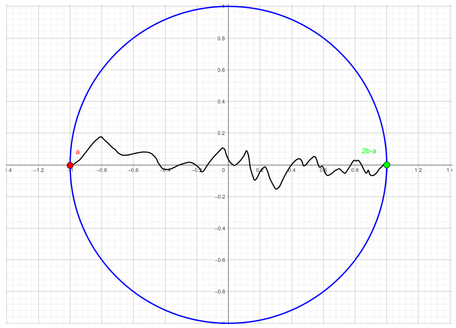

Figure 2.1: Chordal SLE(κ) $\boldsymbol{\sigma}^+ = a \cdot \xi + (2b-a)\cdot q$, $\boldsymbol{\sigma}^- = 0$

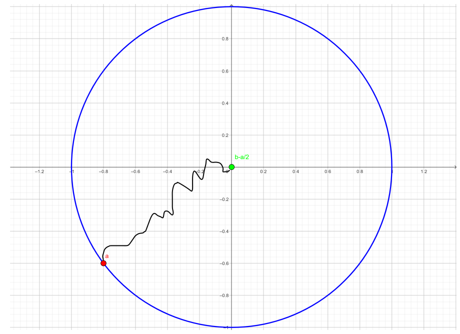

Figure 2.2: Radial SLE(κ) $\boldsymbol{\sigma}^+ = a \cdot \xi + (b-a)\cdot q$, $\boldsymbol{\sigma}^- = b \cdot q$

Definition 2.4.3 (Radial SLE(κ, ρ))**.** *Let ξ be the growth point on the unit circle, and let*

$$\boldsymbol{\rho} = \sum_{j=1}^{n} \rho_j \delta_{u_j} + \sigma_0 \cdot \delta_0 + \sigma_\infty \cdot \delta_\infty$$

be a divisor on $\widehat{\mathbb{C}}$, where $\rho_j \in \mathbb{C}$, and the divisor $\boldsymbol{\rho}$ is symmetric under inversion, i.e.,

$$\boldsymbol{\rho}(z) = \overline{\boldsymbol{\rho}\left(\frac{z}{|z|^2}\right)} \quad \textit{for all } z \in \widehat{\mathbb{C}}.$$

We say $\boldsymbol{\rho}$ satisfies the neutrality condition for SLE(κ, ρ) *if*

$$\int \boldsymbol{\rho} = \kappa - 6.$$

. Define the radial SLE$(\kappa, \xi, \boldsymbol{\rho})$ *Loewner chain by*

$$\partial_t g_t(z) = g_t(z)\frac{\xi(t) + g_t(z)}{\xi(t) - g_t(z)}, \quad g_0(z) = z. \tag{2.4.2}$$

Let $\xi(t) = e^{i\theta(t)}$, $u_j = e^{iq_j}$ and $h_t(z)$ be the covering map of $g_t(z)$ (i.e. $h_t(z) = g_t(e^{iz})$) , then the Loewner differential equation for $h_t(z)$ is given by

$$\partial_t h_t(z) = \cot(\frac{h_t(z) - \theta(t)}{2}), \quad h_0(z) = z, \tag{2.4.3}$$

the driving function $\theta(t)$ evolves as

$$d\theta(t) = \sqrt{\kappa} dB_t + \sum_j \rho_j \cot(\frac{\theta(t) - q_j(t)}{2}). \tag{2.4.4}$$

Note that although the lifts of $\theta(t)$ in universal cover are not unique, different lifts lead to the same differential equation for $h_t(z)$ by periodicity $\cot(z + k\pi) = \cot(z)$, $k \in \mathbb{Z}$.

Theorem 2.4.4. *For a symmetric double divisor* $\boldsymbol{\sigma}^+ = a \cdot \xi + \sum \sigma_j \cdot u_j$ *and* $\boldsymbol{\sigma}^- = \overline{\boldsymbol{\sigma}^+}|_\Omega$ *satisfying neutrality condition* (NC_b)*, let* $\boldsymbol{\rho} = \sum_{j=1}^m \rho_j \cdot u_j$ *where* $\rho_j = (\kappa a)\sigma_j$*. Then two definitions* $SLE_\kappa[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-]$ *and* $SLE(\kappa, \rho)$ *are equivalent.*

Proof. The equivalence in one chart can be verified by directly computing the drift term in the Loewner equation. The conformal invariance of SLE(κ, ρ) under the neutrality condition (NC_b), where the divisor $\boldsymbol{\rho}$ consists of real charges, is established in Schramm and D. Wilson (2005). Moreover, their argument extends naturally to the case where the charges $\boldsymbol{\rho}$ are complex. □

2.5 Classical limit of Coulomb gas correlation

Now, we extend our definition of Coulomb gas correlation to $\kappa = 0$ by normalizing the Coulomb gas correlation.

Definition 2.5.1 (Normalized Coulomb gas correlations for a divisor on the Riemann sphere)**.** *Let the divisor*

$$\boldsymbol{\sigma} = \sum \sigma_j \cdot z_j,$$

where $\{z_j\}_{j=1}^n$ *is a finite set of distinct points on* $\widehat{\mathbb{C}}$*. The normalized Coulomb gas correlation* $C[\boldsymbol{\sigma}]$ *is a differential of conformal dimension* λ_j *at* z_j *by*

Let $\lambda(\sigma) = \sigma^2 + 2\sigma \quad (\sigma \in \mathbb{R})$.

$$\lambda_j = \lambda_b(\sigma_j) \equiv \sigma_j^2 + 2\sigma_j, \tag{2.5.1}$$

whose value is given by

$$C[\boldsymbol{\sigma}] = \prod_{j<k} (z_j - z_k)^{2\sigma_j \sigma_k}, \tag{2.5.2}$$

where the product is taken over all finite z_j *and* z_k*.*

Remark 2.5.2. The normalized Coulomb gas correlation can be viewed as taking the $\kappa \to 0$ limit of the divisor $\sqrt{2\kappa}\boldsymbol{\sigma}$, the Coulomb gas correlation function $C_{(b)}[\boldsymbol{\sigma}]^\kappa$, and conformal dimension $\kappa\lambda_j$.

Definition 2.5.3 (Neutrality condition)**.** *A divisor* $\boldsymbol{\sigma} : \widehat{\mathbb{C}} \to \mathbb{R}$ *satisfies the neutrality condition if*

$$\int \boldsymbol{\sigma} = -2. \tag{2.5.3}$$

Theorem 2.5.4. *Under the neutrality condition $\int \boldsymbol{\sigma} = -2$, the normalized Coulomb gas correlation differentials $C[\boldsymbol{\sigma}]$ are Möbius invariant on $\hat{\mathbb{C}}$.*

Proof. By direct computation, similar to the $\kappa > 0$ case. □

Definition 2.5.5 (Coulomb gas correlation for a double divisor in a simply connected domain)**.** *For a double divisor $(\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$, let $\boldsymbol{\sigma} = \boldsymbol{\sigma}^+ + \boldsymbol{\sigma}^-_*$ be its corresponding divisor in the Schottky double S, we define the Coulomb gas correlation of the double divisor $(\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$ by:*

$$C_\Omega\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right](z) := C_S[\boldsymbol{\sigma}]. \tag{2.5.4}$$

We often omit the subscripts Ω, S to simplify the notations.

If the double divisor $(\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$ satisfies the neutrality condition, then $C\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right]$is a well-defined differential with conformal dimensions $\left[\lambda_j^+, \lambda_j^-\right]$at z_j.

$$\lambda_j^+ = \lambda\left(\sigma_j^+\right) \equiv \frac{(\sigma_j^+)^2}{2} + 2\sigma_j^+, \quad \lambda_j^- = \lambda\left(\sigma_j\right) \equiv \frac{(\sigma_j^-)^2}{2} + 2\sigma_j^-. \tag{2.5.5}$$

By conformal invariance of the Coulomb gas correlation differential $C_S[\boldsymbol{\sigma}]$ on the Riemann sphere under Möbius transformation, the Coulomb gas correlation differential $C_\Omega\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right](z)$ is invariant under $Aut(\Omega)$.

Definition 2.5.6 (Neutrality condition)**.** *A double divisor $(\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$ satisfies the neutrality condition if*

$$\int \boldsymbol{\sigma} = \int \boldsymbol{\sigma}^+ + \int \boldsymbol{\sigma}^- = -2. \tag{2.5.6}$$

Theorem 2.5.7. *Under the neutrality condition $\int \boldsymbol{\sigma}^+ + \int \boldsymbol{\sigma}^- = -2$, the value of the differential $C_{\mathbb{H}}\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right]$ in the identity chart of $\mathbb{H}$ (and the chart $z \mapsto -1/z$ at infinity) is given by*

$$C_{\mathbb{H}}\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right] = \prod_{j<k}\left(z_j - z_k\right)^{2\sigma_j^+\sigma_k^+}\left(\bar{z}_j - \bar{z}_k\right)^{2\sigma_j^-\sigma_k^-}\prod_{j,k}\left(z_j - \bar{z}_k\right)^{2\sigma_j^+\sigma_i^-}, \tag{2.5.7}$$

where the products are taken over finite z_j and z_k.

Theorem 2.5.8. *Under the neutrality condition $\int \boldsymbol{\sigma}^+ + \int \boldsymbol{\sigma}^- = -2$, the value of the differential $C_{\mathbb{D}}\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right]$ in the identity chart of $\mathbb{D}$ is given by*

$$C_{\mathbb{D}}\left[\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-\right] = \prod_{j<k} \left(z_j - z_k\right)^{2\sigma_j^+\sigma_k^+} \left(\bar{z}_j - \bar{z}_k\right)^{2\sigma_j^-\sigma_k^-} \prod_{j,k} \left(1 - z_j \bar{z}_k\right)^{2\sigma_j^+\sigma_k^-}, \quad (2.5.8)$$

where the product is taken over finite z_j and z_k.

2.6 Rational $SLE_0[\boldsymbol{\sigma}]$

Definition 2.6.1 (Rational SLE$_0$)**.** *In the unit disk $\mathbb{D}$, let $e^{i\theta} \in \partial\mathbb{D}$ be the growth point, and let $u_1, u_2, \ldots, u_m \in \overline{\mathbb{D}}$ be marked points. A symmetric double divisor $(\boldsymbol{\sigma}^+, \boldsymbol{\sigma}^-)$ assigns a charge distribution on $e^{i\theta}$ and $\{u_1, \ldots, u_k\}$, where*

$$\boldsymbol{\sigma}^+ = a \cdot e^{i\theta} + \sum_{j=1}^{k} \sigma_j \cdot u_j, \quad \textit{and} \quad \boldsymbol{\sigma}^- = \overline{\boldsymbol{\sigma}^+}|_{\mathbb{D}},$$

and the total charge satisfies the neutrality condition $\int \boldsymbol{\sigma} = -2$.

We define the rational SLE$_0[\boldsymbol{\sigma}]$ *Loewner chain as a normalized conformal map $g_t(z)$ with initial conditions $g_0(z) = z$ and $g_t'(0) = e^{-t}$. The evolution of g_t is governed by the Loewner differential equation:*

$$\partial_t g_t(z) = g_t(z) \frac{e^{i\theta(t)} + g_t(z)}{e^{i\theta(t)} - g_t(z)}, \quad g_0(z) = z.$$

In the angular coordinate, let $h_t(z)$ be the covering map of $g_t(z)$ defined by $e^{ih_t(z)} = g_t(e^{iz})$. Then $h_t(z)$ evolves according to

$$\partial_t h_t(z) = \cot\left(\frac{h_t(z) - \theta(t)}{2}\right), \quad h_0(z) = z.$$

The driving function $\theta(t)$ evolves according to

$$d\theta(t) = \frac{\partial \log \mathcal{Z}(\boldsymbol{\theta})}{\partial \theta}\, dt.$$

where the Coulomb gas partition function is

$$\mathcal{Z}(\boldsymbol{\theta}) = \prod_{j<k} \sin\left(\frac{\theta_j - \theta_k}{2}\right)^{\sigma_j\sigma_k} \cdot \prod_j e^{\frac{i}{2}\sigma_j(\sigma_0 - \sigma_\infty)\theta_j}. \quad (2.6.1)$$

The flow g_t is well-defined up to the first time τ at which $w(t) = g_t(w)$ for some w in the support of $\boldsymbol{\sigma}$. For each $z \in \overline{\mathbb{D}}$, the process $t \mapsto g_t(z)$ is well-defined up to $\tau_z \wedge \tau$, where τ_z is the first time such that $g_t(z) = e^{i\theta(t)}$. Denote

$$K_t = \left\{z \in \overline{\mathbb{D}} : \tau_z \leq t\right\}$$

as the hull associated with the Loewner chain.

Furthermore, the rational SLE_0 *Loewner chain is invariant under Möbius transformations in* $\mathrm{Aut}(\mathbb{D})$ *(up to a time reparameterization), due to the conformal invariance of the Coulomb gas correlation. Consequently, rational* $\mathrm{SLE}_0[\boldsymbol{\sigma}]$ *in any simply connected domain* Ω *is defined by pulling back via a conformal map* $\phi : \Omega \to \mathbb{D}$.

In definition (2.6.1), we introduce the definition of $SLE_0[\beta]$ as a natural extension of $SLE_\kappa[\beta]$ to $\kappa = 0$. The main ingredient in our definition is the normalized Coulomb gas as the partition function.

Now, we introduce another widely used definition $\mathrm{SLE}(0, \boldsymbol{\rho})$ which is a natural extension of $\mathrm{SLE}(\kappa, \boldsymbol{\rho})$ to $\kappa = 0$. We prove the equivalence between rational $SLE_0[\boldsymbol{\sigma}]$ and $\mathrm{SLE}(0, \boldsymbol{\rho})$ in the end.

Definition 2.6.2 ($\mathrm{SLE}(0, \boldsymbol{\rho})$)**.** *Let* w *be the growth point on* $\partial\mathbb{D}$ *and* $\boldsymbol{\rho} = \sum_{i=1}^n \rho_j \delta_{u_j} + \sigma_0 \cdot 0 + \sigma_\infty \cdot \infty$ *be a divisor on* $\widehat{\mathbb{C}}$ *that is symmetric under involution, i.e.* $\boldsymbol{\rho}(z) = \boldsymbol{\rho}(\frac{z}{|z|^2})$ *for all* z *and* $\int \boldsymbol{\rho} = -6$*. Define the radial* $\mathrm{SLE}(0, w, \boldsymbol{\rho})$ *Loewner chain by*

$$\partial_t g_t(z) = g_t(z)\frac{w(t) + g_t(z)}{w(t) - g_t(z)}, \quad g_0(z) = z, \tag{2.6.2}$$

where the driving function $w(t)$ *evolves as*

$$\dot{w}(t) = w(t)\sum_j \rho_j \frac{g_t(u_j) + w(t)}{g_t(u_j) - w(t)}, \quad z(0) = z_0. \tag{2.6.3}$$

In the angular coordinate, $w(t) = e^{i\theta(t)}$ *and* $u_j(t) = e^{iq_j(t)}$*, let* $h_t(z)$ *be the covering conformal map of* $g_t(z)$ *(i.e.* $e^{ih_t(z)} = g_t(e^{iz})$*).*

Then the Loewner differential equation for $h_t(z)$ *is*

$$\partial_t h_t(z) = \cot(\frac{h_t(z) - \theta_t}{2}), \quad h_0(z) = z, z \in \overline{\mathbb{H}}, \tag{2.6.4}$$

where the driving function θ_t *evolves as*

$$\dot{\theta}_t = \sum_j \rho_j \cot(\frac{\theta_t - q_j(t)}{2}), \quad x(0) = x_0. \tag{2.6.5}$$

Theorem 2.6.3. *For an involution symmetric divisor* $\boldsymbol{\sigma} = w + \sum_{j=1}^m \sigma_j \cdot z_j$ *satisfying neutrality condition* $\int \boldsymbol{\sigma} = -2$*, let* $\boldsymbol{\rho} = 2\sum_{j=1}^m \sigma_j \cdot z_j$*, then* $\int \boldsymbol{\rho} = -6$ *and two definitions* $SLE_0[\sigma]$ *and SLE(*$0, \rho$*) are equivalent.*

Proof. The equivalence in one chart can be verified by directly computing the drift term in the Loewner equation. The conformal invariance of $\mathrm{SLE}(\kappa, \rho)$ under the neutrality condition (NC_b), where the divisor $\boldsymbol{\rho}$ consists of real charges, is established in Schramm and D. Wilson (2005). Moreover, their argument extends naturally to the case where the charges $\boldsymbol{\rho}$ are complex. $\square$

Chapter 3

COMMUTATION RELATIONS AND CONFORMAL INVARIANCE FOR MULTIPLE SLE SYSTEMS

3.1 Transformation of Loewner flow under coordinate change

In this section we show that the Loewner chain of a curve, when viewed in a different coordinate chart, is a time reparametrization of the Loewner chain in the standard coordinate chart but with different initial conditions. This result serves as a preliminary step towards understanding the local commutation relations and the conformal invariance of multiple SLE(κ) systems.

Let us briefly review how Loewner chains transform under coordinate changes.

Theorem 3.1.1 (Deterministic Loewner chain under coordinate change)**.** *In angular (trigonometric) coordinates, suppose $\gamma(0) = \theta \in \mathbb{R}$ and let the marked points be $\theta_1, \theta_2, \ldots, \theta_n \in \mathbb{C}$. Let $\gamma(t)$ be the curve generated by the deterministic Loewner chain:*

$$\partial_t h_t(z) = \cot\left(\frac{h_t(z) - \theta_t}{2}\right), \qquad \dot{\theta}_t = b\left(\theta_t; h_t(\theta_1), \ldots, h_t(\theta_n)\right), \tag{3.1.1}$$

with initial condition $h_0(z) = z$ and $\theta_0 = \theta$, where $b : \mathbb{R} \times \mathbb{C}^n \to \mathbb{R}$ is a smooth vector field.

Let $\tau : \mathcal{N} \to \mathbb{H}$ be a conformal map defined on a neighborhood $\mathcal{N}$ of θ such that $\gamma[0,T] \subset \mathcal{N}$ and $\tau(\partial\mathcal{N} \cap \mathbb{R}) \subset \mathbb{R}$. Define the image curve $\widetilde{\gamma}(t) := \tau(\gamma(t))$ for $t \in [0,T]$, and let $\widetilde{h}_t$ denote the conformal map associated with $\widetilde{\gamma}[0,t]$.

Define the conformal coordinate change

$$\Psi_t := \widetilde{h}_t \circ \tau \circ h_t^{-1}.$$

Then the image conformal map $\widetilde{h}_t(z)$ satisfies the evolution equation:

$$\partial_t \widetilde{h}_t(z) = \cot\left(\frac{\widetilde{h}_t(z) - \widetilde{\theta}_t}{2}\right) \cdot \left[\Psi_t'(\theta_t)\right]^2, \qquad \widetilde{h}_0(z) = z, \tag{3.1.2}$$

where the new driving function is

$$\widetilde{\theta}_t := \widetilde{h}_t \circ \tau \circ h_t^{-1}(\theta_t) = \Psi_t(\theta_t), \quad \widetilde{\theta}_0 = \tau(\theta).$$

Moreover, the curve $\widetilde{\gamma}(t)$ is parameterized so that its unit disk capacity satisfies

$$\mathrm{hcap}(\widetilde{\gamma}[0,t]) = 2\sigma(t), \qquad \textit{where} \quad \sigma(t) := \int_0^t \left|\Psi_s'(\theta_s)\right|^2 ds. \tag{3.1.3}$$

Proof. See Section 4.6.2 in Lawler (2005). □

Theorem 3.1.2 (Stochastic Loewner chain under coordinate change)**.** *Suppose the driving function θ_t evolves according to the stochastic differential equation*

$$d\theta_t = \sqrt{\kappa}\, dB_t + b\left(\theta_t; \Psi_t(\theta_1), \ldots, \Psi_t(\theta_n)\right)\, dt, \tag{3.1.4}$$

where B_t is standard Brownian motion, and $\Psi_t := \widetilde{h}_t \circ \tau \circ h_t^{-1}$ is the conformal coordinate change defined as in Theorem 3.1.1.

Define the transformed driving function

$$\widetilde{\theta}_t := \Psi_t(\theta_t),$$

and introduce the reparameterized time

$$s(t) := \int_0^t |\Psi_u'(\theta_u)|^2\, du.$$

Then the process $\widetilde{\theta}_s := \Psi_{t(s)}(\theta_{t(s)})$ satisfies the following SDE:

$$d\widetilde{\theta}_s = \sqrt{\kappa}\, dB_s + \frac{b\left(\theta_s; \Psi_{t(s)}(\theta_1), \ldots, \Psi_{t(s)}(\theta_n)\right)}{\Psi_{t(s)}'(\theta_s)}\, ds + \frac{\kappa - 6}{2} \cdot \frac{\Psi_{t(s)}''(\theta_s)}{[\Psi_{t(s)}'(\theta_s)]^2}\, ds. \tag{3.1.5}$$

Proof. We apply Itô's formula to the composed process $\widetilde{\theta}_t = \Psi_t(\theta_t)$. Using the chain rule for semimartingales:

$$\begin{aligned}
d\widetilde{\theta}_t &= (\partial_t \Psi_t)(\theta_t)\, dt + \Psi_t'(\theta_t)\, d\theta_t + \frac{1}{2}\Psi_t''(\theta_t)\, d\langle\theta\rangle_t \\
&= (\partial_t \Psi_t)(\theta_t)\, dt + \Psi_t'(\theta_t)\left[\sqrt{\kappa}\, dB_t + b(\theta_t; \Psi_t(\theta_1), \ldots, \Psi_t(\theta_n))\, dt\right] + \frac{\kappa}{2}\Psi_t''(\theta_t)\, dt.
\end{aligned}$$

From Proposition 4.43 in Lawler (2005), we use the identity

$$(\partial_t \Psi_t)(\theta_t) = -3\Psi_t''(\theta_t).$$

Substituting into the equation:

$$d\widetilde{\theta}_t = \Psi_t'(\theta_t)\sqrt{\kappa}\, dB_t + \Psi_t'(\theta_t) b(\theta_t; \Psi_t(\theta_1), \ldots, \Psi_t(\theta_n))\, dt + \left(\frac{\kappa}{2} - 3\right) \Psi_t''(\theta_t)\, dt$$
$$= \Psi_t'(\theta_t)\sqrt{\kappa}\, dB_t + \Psi_t'(\theta_t) b(\theta_t; \Psi_t(\theta_1), \ldots, \Psi_t(\theta_n))\, dt + \frac{\kappa - 6}{2} \Psi_t''(\theta_t)\, dt.$$

Now, we reparameterize time via $s(t) = \int_0^t |\Psi_u'(\theta_u)|^2 du$. Under this change of time, we obtain the transformed SDE for $\widetilde{\theta}_s$ by dividing all drift and diffusion terms by $|\Psi_t'(\theta_t)|$:

$$d\widetilde{\theta}_s = \sqrt{\kappa}\, dB_s + \frac{b(\theta_s; \Psi_{t(s)}(\theta_1), \ldots, \Psi_{t(s)}(\theta_n))}{\Psi_{t(s)}'(\theta_s)}\, ds + \frac{\kappa - 6}{2} \cdot \frac{\Psi_{t(s)}''(\theta_s)}{[\Psi_{t(s)}'(\theta_s)]^2}\, ds.$$

□

Remark 3.1.3. By Theorem 3.1.2, under a conformal coordinate change τ, the drift term in the marginal law transforms as a *pre-Schwarzian form*. Specifically, if the driving function satisfies

$$d\theta_t = \sqrt{\kappa}\, dB_t + b(\theta_t)\, dt,$$

then the drift b transforms under τ as

$$b(\theta) = \tau'(\theta) \cdot \widetilde{b}(\tau(\theta)) + \frac{6 - \kappa}{2} \cdot (\log \tau'(\theta))'.$$

Here, $\widetilde{b}$ is the drift in the image coordinate $\widetilde{\theta} = \tau(\theta)$, and the second term is the pre-Schwarzian derivative of τ.

Corollary 3.1.4. *Let γ, $\tilde{\gamma}$ be two hulls starting at $e^{ix} \in \partial\mathbb{D}$ and $e^{iy} \in \partial\mathbb{D}$ with capacity ϵ and $c\epsilon$, let g_ϵ be the normalized map removing γ and $\tilde{\varepsilon} = \mathrm{hcap}(g_\epsilon \circ \gamma(t))$, then we have:*

$$\tilde{\varepsilon} = c\varepsilon \left(1 - \frac{\varepsilon}{\sin^2(\frac{x-y}{2})}\right) + o\left(\varepsilon^2\right). \tag{3.1.6}$$

Proof. Locally, we can define $h_t(z) = -i \log(g_t(e^{iz}))$. Then from the Loewner equation, $\partial_t h_t'(w) = -\frac{h_t'(w)}{2\sin^2\left(\frac{h_t(w) - x_t}{2}\right)}$, which implies $h_\varepsilon'(y) = 1 - \frac{\varepsilon}{2\sin^2(\frac{y-x}{2})} + o(\varepsilon)$. By applying conformal transformation $h_\epsilon(y)$, we get

$$\tilde{\varepsilon} = c\epsilon(h_\epsilon'(y)^2 + o(\epsilon)) = c\varepsilon \left(1 - \frac{\varepsilon}{\sin^2(\frac{x-y}{2})}\right) + o\left(\varepsilon^2\right).$$

□

3.2 Local commutation relation and null vector equations in $\kappa > 0$ case

In this section, we explore how the commutation relations (reparametrization symmetry) and conformal invariance impose constraints on the drift terms $b_j(\boldsymbol{z}, u)$ and equivalently impose constraints on the partition function $\psi(\boldsymbol{z}, u)$ derived from b_j.

The pioneering work on commutation relations was done in Dubédat (2007). The author studied the commutation relations for multiple SLEs in the upper half plane $\mathbb{H}$ with n growth points $z_1, z_2, \ldots, z_n \in \mathbb{R}$ and m additional marked points $u_1, u_2, \ldots, u_m \in \mathbb{R}$.

We extend this Dubedat's commutation argument to the case where there are n growth points $z_1, z_2, \ldots, z_n \in \partial\mathbb{D}$ and one interior marked point $u \in \mathbb{D}$; see theorem (3.2.1) and (3.2.5).

The commutation relations in the unit disk $\mathbb{D}$ with the marked point 0 are partially studied in Dubédat (2007) and Y. Wang and Wu (2024).

A significant difference between the multiple radial and standard chordal SLE(κ) systems (with no marked points) arises when we study their conformal invariance properties. Although the multiple radial SLE(κ) systems are conformally invariant, their corresponding partition functions form equivalence classes that do not necessarily exhibit conformal covariance. However, within each equivalence class, it is still possible to find at least one conformally covariant partition function.

Theorem 3.2.1 (Commutation Relations for $u = 0$)**.** *In the unit disk $\mathbb{D}$, consider n radial SLEs starting at $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n} \in \partial\mathbb{D}$, with a marked interior point $u = 0$.*

(i) Let the infinitesimal diffusion generators be

$$\mathcal{M}_i = \frac{\kappa}{2}\partial_{ii} + b_i(\theta_1, \theta_2, \ldots, \theta_n)\partial_i + \sum_{j\neq i} \cot\left(\frac{\theta_j - \theta_i}{2}\right)\partial_j,$$

where $\partial_i = \partial_{\theta_i}$. If the n SLEs locally commute, the associated infinitesimal generators satisfy

$$[\mathcal{M}_i, \mathcal{M}_j] = \frac{1}{\sin^2\left(\frac{\theta_j - \theta_i}{2}\right)}(\mathcal{M}_j - \mathcal{M}_i).$$

There exists a positive function $\psi(\boldsymbol{\theta})$, defined on $\mathfrak{X}^n(\boldsymbol{\theta})$, such that the drift term satisfies

$$b_i(\boldsymbol{\theta}) = \kappa\partial_i \log\psi,$$

and ψ satisfies the null vector equations:

$$\frac{\kappa}{2}\partial_{ii}\psi + \sum_{j\neq i}\cot\left(\frac{\theta_j-\theta_i}{2}\right)\partial_i\psi + \left(1-\frac{6}{\kappa}\right)\sum_{j\neq i}\frac{1}{4\sin^2\left(\frac{\theta_j-\theta_i}{2}\right)}\psi - h_j(\theta_j)\psi = 0, \tag{3.2.1}$$

for $i = 1, 2, \ldots, n$, with undetermined functions $h_j(\theta_j)$.

(ii) By analyzing the asymptotic behavior of two adjacent growth points θ_i and θ_{i+1} (with no marked points between them), we further deduce that $h_i(\theta) = h_{i+1}(\theta)$. Consequently, if all growth points are consecutive with no marked points between them, there exists a common function $h(\theta)$ such that

$$h(\theta) = h_1(\theta) = \cdots = h_n(\theta).$$

Theorem 3.2.2 (Conformal Invariance under $\mathrm{Aut}(\mathbb{D}, 0)$)**.** *For a rotation map ρ_θ, the drift term $b_i(\boldsymbol{\theta})$ is invariant under ρ_θ, i.e.,*

$$b_i = \widetilde{b}_i \circ \rho_\theta.$$

(i) The function $h(\theta)$ in the null vector equation (3.2.1) is rotation-invariant, and there exists a real constant h such that

$$h(\theta) = h. \tag{3.2.2}$$

(ii) There exists a real constant ω such that, for all $\theta \in \mathbb{R}$,

$$\psi(\theta_1+\theta, \ldots, \theta_n+\theta) = e^{-\omega\theta}\psi(\theta_1, \ldots, \theta_n). \tag{3.2.3}$$

Remark 3.2.3. Combining Theorem (3.2.1) with Theorem (3.2.2), we conclude that a multiple radial SLE(κ) system with fixed $u = 0$ is characterized by a partition function that satisfies the null vector equations (3.2.1) with a constant h and has a rotation constant ω.

Remark 3.2.4. The drift term $b_i(\theta_1, \ldots, \theta_n)$ is 2π-periodic and therefore well-defined on the unit circle S^1. However, for $\omega \neq 0$, the partition function ψ is not 2π-periodic, making it multivalued on S^1 and well-defined only on the real line $\mathbb{R}$, the universal cover of S^1.

Thus, for $\omega \neq 0$, the conformal invariance of partition functions requires the use of the group $\widetilde{\mathrm{Aut}}(\mathbb{D}, 0)$ with a group action on $\mathbb{R}$.

The discussion on commutation relations above extend to arbitrary $u \in \mathbb{D}$, as described in the following theorem.

Theorem 3.2.5 (Commutation Relations for $u \in \mathbb{D}$)**.** *In the unit disk $\mathbb{D}$, let n radial SLEs start at $e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n} \in \partial\mathbb{D}$, with a marked interior point $u \neq 0$.*

(i) For $u = e^{iv}$, let the infinitesimal diffusion generators be

$$\mathcal{M}_i = \frac{\kappa}{2}\partial_{ii} + b_i(\boldsymbol{\theta}, u)\partial_i + \sum_{j\neq i} \cot\left(\frac{\theta_j - \theta_i}{2}\right)\partial_j + \cot\left(\frac{v - \theta_i}{2}\right)\partial_v + \cot\left(\frac{\overline{v} - \theta_i}{2}\right)\overline{\partial}_v.$$

If the n SLEs locally commute, the generators satisfy

$$[\mathcal{M}_i, \mathcal{M}_j] = \frac{1}{\sin^2\left(\frac{\theta_j - \theta_i}{2}\right)}(\mathcal{M}_j - \mathcal{M}_i).$$

There exists a partition function $\psi(\boldsymbol{\theta}, u)$ such that the drift term $b_i(\boldsymbol{\theta}, u)$ is given by

$$b_i(\boldsymbol{\theta}, u) = \kappa\partial_i \log\psi,$$

and ψ satisfies the null vector equations:

$$\frac{\kappa}{2}\partial_{ii}\psi + \sum_{j\neq i}\frac{2}{\theta_j - \theta_i}\partial_i\psi + \frac{2}{v - \theta_i}\partial_v\psi + \frac{2}{\overline{v} - \theta_i}\overline{\partial}_v\psi + \left[\left(1 - \frac{6}{\kappa}\right)\sum_{j\neq i}\frac{1}{(\theta_j - \theta_i)^2} + h_i(\theta_i, u)\right]\psi = 0.$$

(ii) By analyzing the asymptotics of adjacent points θ_i and θ_{i+1}, we deduce that $h_i(\theta, u) = h_{i+1}(\theta, u)$. If all points are consecutive, there exists a common function $h(\theta, u)$ such that

$$h(\theta, u) = h_1(\theta, u) = \cdots = h_n(\theta, u).$$

Now, we discuss how $\mathrm{Aut}(\mathbb{D})$-invariance imposes constraints on the drift terms of a multiple radial SLE(κ) system and how to choose a conformally covariant partition function representative within its equivalence class.

Definition 3.2.6. *The conformal group $\mathrm{Aut}(\mathbb{D})$ satisfies the following properties (see Lang (1985)):*

- *$\mathrm{Aut}(\mathbb{D})$ is isomorphic to $PSL_2(\mathbb{R})$. Each element $\tau \in \mathrm{Aut}(\mathbb{D})$ can be written as*

$$\tau(z) = T_v \circ \rho_\theta(z),$$

where $\rho_\theta(z) = e^{i\theta}z$ *and* $T_v(z) = \frac{z-v}{1-\bar{v}z}$.

Geometrically, $\mathrm{Aut}(\mathbb{D})$ *is an* S^1*-bundle over* $\mathbb{H}$*, and it naturally acts on* $\partial\mathbb{D} \cong S^1$ *by extending the conformal maps to the boundary.*

- *The universal cover* $\widetilde{\mathrm{Aut}}(\mathbb{D})$ *is isomorphic to* $\widetilde{SL}_2(\mathbb{R})$*. Each element* $\tau \in \widetilde{\mathrm{Aut}}(\mathbb{D})$ *can be decomposed as*

$$\tau = T_v \circ A_\theta,$$

where $\theta \in \mathbb{R}$*, and* A_θ *represents addition by* θ *on* $\mathbb{R}$.

Geometrically, $\widetilde{\mathrm{Aut}}(\mathbb{D})$ *is an* $\mathbb{R}$*-bundle over* $\mathbb{H}$*, and it naturally acts on* $\mathbb{R}$*, the universal cover of* S^1.

  - *For* $x \in \mathbb{R}$*,* $A_\theta(x) = x + \theta$.
  - *For* $v \in \mathbb{D}$*, there exists a unique* $|y - x| < \pi$ *such that* $e^{iy} = T_v(e^{ix})$.

Theorem 3.2.7. *Let* $\tau \in \mathrm{Aut}(\mathbb{D})$*. The drift term* $b(\boldsymbol{\theta}, u)$ *is a pre-Schwarzian form, satisfying*

$$b_i = \tau' \widetilde{b}_i \circ \tau + \frac{6-\kappa}{2} (\log \tau')'.$$

(i) There exists a smooth function $F(\tau, u) : \widetilde{\mathrm{Aut}}(\mathbb{D}) \times \mathbb{D} \to \mathbb{R}$ *such that*

$$\log \psi - \log(\psi \circ \tau) + \frac{\kappa - 6}{2\kappa} \sum_i \log \tau'(\theta_i) = F(\tau, u),$$

where F *satisfies the functional equation*

$$F(\tau_1\tau_2, u) = F(\tau_1, \tau_2(u)) + F(\tau_2, u). \tag{3.2.4}$$

(ii) There exists a rotation constant ω *such that*

$$F(A_\theta, 0) = \omega\theta. \tag{3.2.5}$$

If $\omega = 0$*,* $\mathrm{Aut}(\mathbb{D})$ *suffices to describe conformal invariance, and* F *reduces to a map* $\mathrm{Aut}(\mathbb{D}) \times \mathbb{D} \to \mathbb{R}$.

(iii) Suppose $F_1(\tau, u)$ *and* $F_2(\tau, u)$ *correspond to partition functions* ψ_1 *and* ψ_2*. If their rotation constants* $\omega_1 = \omega_2$*, then there exists a function* $g(u)$ *such that*

$$\psi_2 = g(u) \cdot \psi_1.$$

(iv) For $\tau \in \widetilde{\mathrm{Aut}}(\mathbb{D})$*, let* $\tau(u) = v$*. Decompose* $\tau = T_v \circ A_\theta \circ T_u^{-1}$*, where* $u, v \in \mathbb{D}$ *and* $\theta \in \mathbb{R}$*. Using the relations* $T_u'(u) = \frac{1}{1-|u|^2}$ *and* $T_v'(0) = 1 - |v|^2$*, define*

$$\tau'(u)^{\lambda(u)}\overline{\tau'(u)}^{\overline{\lambda(u)}} := \left(\frac{1-|v|^2}{1-|u|^2}\right)^{2\,\mathrm{Re}(\lambda(u))} e^{-2\theta\,\mathrm{Im}(\lambda(u))}.$$

Then

$$F(\tau, u) = \log\left(\tau'(u)^{\lambda(u)}\overline{\tau'(u)}^{\overline{\lambda(u)}}\right),$$

satisfy the functional equation (3.2.4), with rotation constant $\omega = \mathrm{Im}(\lambda(u))$.

Remark 3.2.8. Combining Theorem (3.2.5) with Theorem (3.2.7), we show that a multiple radial SLE(κ) system with $u \in \mathbb{D}$ is described by a conformally covariant partition function that satisfies the null vector equations (3.2.1) with a constant h. The partition function has a rotation constant ω and a non-unique conformal dimension $\lambda(u)$, with $\omega = \Im(\lambda(u))$. Moreover, two distinct conformally covariant solutions differ by a multiplicative factor corresponding to a power of the conformal radius.

Proof of theorem (3.2.1) and theorem (3.2.5). The derivations of the commutation relations for $u = 0$ and arbitrary $u \neq 0$ are similar. We mainly discuss the case $u \neq 0$. The proof for $u = 0$ can be obtained by simply ignoring the u-dependence and related derivatives in the drift and diffusion generators since $u = 0$ is fixed by the Loewner flow.

(i) We first focus on the growth of two hulls from a specific pair of growth points. Consider the following scenario: we grow two hulls from e^{ix} and e^{iy} on the boundary $\partial\mathbb{D}$ and relabel the remaining growth points as e^{iz_j} the marked point $u = e^{iv}$.

Lemma 3.2.9. *In the angular coordinate, suppose two radial SLE hulls start from* $x, y \in \mathbb{R}$ *with marked points* $z_1, z_2, \ldots, z_n \in \mathbb{R}$ *and marked interior point* $u \in \mathbb{D}$*. If* $u = 0$ *is a marked point, we simply omit it since* $u = 0$ *is fixed by the Loewner flow. Let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$

$$\mathcal{M}_1 = \frac{\kappa}{2}\partial_{xx} + b(x, y \ldots)\partial_x + \cot(\frac{y-x}{2})\partial_y + \sum_{i=1}^{n} \cot(\frac{z_i - x}{2})\partial_i + \cot\left(\frac{v-x}{2}\right)\partial_v + \cot\left(\frac{\overline{v}-x}{2}\right)\overline{\partial}_v$$

$$\mathcal{M}_2 = \frac{\kappa}{2}\partial_{yy} + \tilde{b}(x, y, \ldots)\partial_y + \cot(\frac{x-y}{2})\partial_x + \sum_{i=1}^{n} \cot(\frac{z_i - y}{2})\partial_i + \cot\left(\frac{v-y}{2}\right)\partial_v + \cot\left(\frac{\overline{v}-y}{2}\right)\overline{\partial}_v$$

be the infinitesimal diffusion generators of two SLE hulls, where $\partial_i = \partial_{z_i}$.

If two SLEs commute, then the associated infinitesimal generators satisfy

$$[\mathcal{M}_1, \mathcal{M}_2] = \frac{1}{\sin^2(\frac{y-x}{2})}(\mathcal{M}_2 - \mathcal{M}_1). \tag{3.2.6}$$

Moreover, there exists a positive function $\psi(x, y, z, u)$ *such that*

$$b = \kappa \frac{\partial_x \psi}{\psi}, \tilde{b} = \kappa \frac{\partial_y \psi}{\psi}$$

and ψ *satisfies the null vector equations*

$$\begin{cases} \frac{\kappa}{2}\partial_{xx}\psi + \sum_i \cot(\frac{z_i - x}{2})\partial_i\psi + \cot(\frac{y-x}{2})\partial_y\psi + \cot\left(\frac{v-x}{2}\right)\partial_v\psi + \cot\left(\frac{\overline{v}-x}{2}\right)\overline{\partial}_v\psi \\ + \left(\left(1 - \frac{6}{\kappa}\right)\frac{1}{4\sin^2(\frac{y-x}{2})} + h_1(x, z)\right)\psi = 0 \\ \frac{\kappa}{2}\partial_{yy}\psi + \sum_i \cot(\frac{z_i - y}{2})\partial_i\psi + \cot(\frac{x-y}{2})\partial_x\psi + \cot\left(\frac{v-y}{2}\right)\partial_v\psi + \cot\left(\frac{\overline{v}-y}{2}\right)\overline{\partial}_v\psi \\ + \left(\left(1 - \frac{6}{\kappa}\right)\frac{1}{4\sin^2(\frac{x-y}{2})} + h_2(y, z)\right)\psi = 0 \end{cases}. \tag{3.2.7}$$

Proof. Consider a Loewner chain $(K_{s,t})_{(s,t)\in\mathcal{T}}$ with a double time index. The associated conformal equivalence are $g_{s,t}$. We also assume that $K_{s,t} = K_{s,0} \cup K_{0,t}$. If $s \leq s', t \leq t', (s', t') \in \mathcal{T}$, then $(s, t) \in \mathcal{T}$.

Let σ(resp. τ) be a stopping time in the filtration generated by $(K_{s,0})_{(s,0)\in\mathcal{T}}$ (resp. $\left((K_{0,t})_{(0,t)\in\mathcal{T}}\right)$. Let also $\mathcal{T}' = \{(s, t) : (s + \sigma, t + \tau) \in \mathcal{T}\}$ and $(K'_{s,t})_{(s,t)\in\mathcal{T}'} = \left(\overline{g_{\sigma,\tau}\left(K_{s+\sigma,t+\tau} \backslash K_{s,t}\right)}\right)$. Then $\left(K'_{s,0}\right)_{(s,0)\in\mathcal{T}'}$ is distributed as a stopped $\mathrm{SLE}_\kappa(b)$, i.e an SLE driven by

$$dx_s = \sqrt{\kappa}dB_s + b\left(x_s, h_s(y), \ldots, h_s(z_i), \ldots, e^{ih_s(v)}\right)dt,$$

where h_s is the covering map of Loewner map g_s, (i.e. $e^{ih_s(z)} = g_s(e^{iz})$).

Likewise $\left(K'_{0,t}\right)_{(0,t)\in\mathcal{T}'}$ is distributed as a stopped $\mathrm{SLE}_{\tilde{\kappa}}(\tilde{b})$, i.e an SLE driven by:

$$dy_t = \sqrt{\kappa}d\tilde{B}_t + \tilde{b}\left(\tilde{h}_t(x), y_t, \ldots, \tilde{h}_t(z_i), \ldots, e^{i\tilde{h}_t(v)}\right)dt,$$

where $\tilde{h}_t$ is the covering map of $\tilde{g}_t$ (i.e. $e^{i\tilde{h}_s(z)} = \tilde{h}_s(e^{iz})$).

Here $B, \tilde{B}$ are standard Brownian motions, $(g_s), (\tilde{g}_t)$ are the associated conformal equivalences, $b, \tilde{b}$ are some smooth, translation invariant functions.

Note that

$$(x_s, h_s(y), \ldots, h_s(z_i), \ldots, \operatorname{Re}(h_s(v)), \operatorname{Im}(h_s(v)))$$

is a Markov process with semigroup P and infinitesimal generator $\mathcal{M}_1$. Similarly,

$$\left(\tilde{h}_t(x), y_t, \ldots, \tilde{h}_t(z_i), \ldots, \operatorname{Re}\left(\tilde{h}_t(v)\right), \operatorname{Im}\left(\tilde{h}_t(v)\right)\right)$$

is a Markov process with semigroup Q and infinitesimal generator $\mathcal{M}_2$.

We denote A^x, A^y the unit disk capacity of hulls growing at e^{ix} and e^{iy}, and consider the stopping time $\sigma = \inf\left(s : A^x\left(K_{s,0}\right) \geq a^x\right), \tau = \inf\left(t : A^y\left(K_{0,t}\right) \geq a^y\right)$, where $a^x = \varepsilon, a^y = c\varepsilon$.

We are interested in the SLE hull $K_{\sigma,\tau}$. There are two natural ways to evolve from the initial configuration $K_{0,0}$ to $K_{\sigma,\tau}$:

- via $K_{0,0} \to K_{\sigma,0} \to K_{\sigma,\tau}$,
- or via $K_{0,0} \to K_{0,\tau} \to K_{\sigma,\tau}$.

We describe both paths using infinitesimal capacity expansions:

(i) Run the first SLE (i.e., $\mathrm{SLE}_\kappa(b)$) from the initial configuration $(x, y, \ldots, z_i, \ldots)$ until it accumulates capacity ε. Then, independently run the second SLE (i.e., $\mathrm{SLE}_\kappa(\tilde{b})$) in the transformed domain $g_\varepsilon^{-1}(\mathbb{D})$, stopping when it reaches capacity $c\varepsilon$, measured in the original unit disk. Let h_ε and $\tilde{h}_{\tilde{\varepsilon}}$ be the conformal maps removing the corresponding hulls, centered at x and y, respectively. Define $\phi = \tilde{h}_{\tilde{\varepsilon}} \circ h_\varepsilon$ as the normalized composition map. Then expand
$$\mathbb{E}\left(F\left(\tilde{h}_{\tilde{\varepsilon}}(X_\varepsilon), \tilde{Y}_{\tilde{\varepsilon}}\right)\right)$$
up to second order in ε. This expansion describes, in distribution, the evolution from $K_{0,0}$ to $K_{\sigma,0}$, and then to $K_{\sigma,\tau}$.

(ii) Begin by running the second SLE (i.e., $\mathrm{SLE}_\kappa(\tilde{b})$) until it reaches capacity $c\varepsilon$. Then, run the first SLE (i.e., $\mathrm{SLE}_\kappa(b)$) independently in the transformed domain $\tilde{g}_\varepsilon^{-1}(\mathbb{D})$, stopping when it accumulates capacity ε. Let $\tilde{h}_\varepsilon$ and $h_{\tilde{\varepsilon}}$ be the conformal maps removing the hulls at y and x, respectively, and define $\phi = h_{\tilde{\varepsilon}} \circ \tilde{h}_\varepsilon$. Then expand
$$\mathbb{E}\left(F\left(h_{\tilde{\varepsilon}}(\tilde{X}_\varepsilon), Y_{\tilde{\varepsilon}}\right)\right)$$

up to second order in ε. This describes the evolution from $K_{0,0}$ to $K_{0,\tau}$, and then to $K_{\sigma,\tau}$.

Note that under the conformal map h_ϵ, the capacity of $\tilde{\gamma}$ is not $c\epsilon$. According to lemma (3.1.4), the capacity is given by

$$\tilde{\varepsilon} = c\varepsilon\left(1 - \frac{\varepsilon}{\sin^2(\frac{x-y}{2})}\right) + o\left(\varepsilon^2\right). \tag{3.2.8}$$

Now, let F be a test function $\mathbb{R}^{n+2,2} \to \mathbb{R}$, and $c > 0$ be some constant and let

$$\begin{aligned}
w &= (x, y, \ldots, z_i, \ldots, v) \\
w' &= (X_\varepsilon, g_\varepsilon(y), \ldots g_\varepsilon(z_i), \ldots, g_\varepsilon(v)) \\
w'' &= \left(\tilde{g}_{\tilde{\varepsilon}}(X_\varepsilon), \tilde{Y}_{\tilde{\varepsilon}}, \ldots \tilde{g}_{\tilde{\varepsilon}} \circ g_\varepsilon(z_i), \ldots, \tilde{g}_{\tilde{\varepsilon}} \circ g_\varepsilon(v)\right)
\end{aligned} \quad . \tag{3.2.9}$$

We consider the conditional expectation of $F(w'')$ with respect to w.

$$\begin{aligned}
\mathbb{E}\left(F\left(w''\right) \mid w\right) &= \mathbb{E}\left(F\left(w''\right) |w'| \, w\right) = P_\varepsilon \mathbb{E}\left(Q_{\tilde{\varepsilon}} F \mid w'\right)(w) \\
&= P_\varepsilon \mathbb{E}\left(\left(1 + \varepsilon\mathcal{M}_1 + \frac{\varepsilon^2}{2}\mathcal{M}_1^2\right) F\left(w'\right)\right)(w) = P_\varepsilon Q_{c\varepsilon\left(1 - \frac{\varepsilon}{\sin^2(\frac{x-y}{2})}\right)} F(w) + o\left(\varepsilon^2\right) \\
&= \left(1 + \varepsilon\mathcal{M}_1 + \frac{\varepsilon^2}{2}\mathcal{M}_1^2\right)\left(1 + c\varepsilon\left(1 - \frac{\varepsilon}{\sin^2(\frac{x-y}{2})}\right)\mathcal{M}_2 + \frac{c^2\varepsilon^2}{2}\mathcal{M}_2^2\right) F(w) + o\left(\varepsilon^2\right) \\
&= \left(1 + \varepsilon(\mathcal{M}_1 + c\mathcal{M}_2) + \varepsilon^2\left(\frac{1}{2}\mathcal{M}_1^2 + \frac{c^2}{2}\mathcal{M}_2^2 + c\mathcal{M}_1\mathcal{M}_2 - \frac{c}{\sin^2(\frac{x-y}{2})}\mathcal{M}_2\right)\right) F(w) \\
&+ o\left(\varepsilon^2\right)
\end{aligned}$$

If we first grow a hull at y, then at x, one gets instead

$$\left(1 + \varepsilon(\mathcal{M}_1 + c\mathcal{M}_2) + \varepsilon^2\left(\frac{1}{2}\mathcal{M}_1^2 + \frac{c^2}{2}\mathcal{M}_2^2 + c\mathcal{M}_2\mathcal{M}_1 - \frac{c}{\sin^2(\frac{x-y}{2})}\mathcal{M}_1\right)\right) F(w) + o\left(\varepsilon^2\right).$$

Hence, the commutation condition reads

$$[\mathcal{M}_1, \mathcal{M}_2] = \frac{1}{\sin^2(\frac{y-x}{2})}(\mathcal{M}_2 - \mathcal{M}_1). \tag{3.2.10}$$

After simplifications, one gets

$$
\begin{aligned}
&[\mathcal{M}_1, \mathcal{M}_2] + \frac{1}{\sin^2(\frac{y-x}{2})}(\mathcal{M}_1 - \mathcal{M}_2) = \left(\kappa\partial_x\tilde{b} - \kappa\partial_y b\right)\partial_{xy} \\
&+ \left[\cot(\frac{y-x}{2})\partial_x b + \sum_i \cot(\frac{y-z_i}{2})\partial_i b + \cot\left(\frac{v-x}{2}\right)\partial_v b + \cot\left(\frac{\overline{v}-x}{2}\right)\overline{\partial}_v b\right. \\
&\left. -\tilde{b}\partial_y b + \frac{b}{2\sin^2(\frac{y-x}{2})} + \frac{\cos(\frac{x-y}{2})}{4\sin^3(\frac{x-y}{2})}(\kappa-6) - \frac{\kappa}{2}\partial_{yy}b\right]\partial_x \\
&- \left[\cot(\frac{x-y}{2})\partial_y\tilde{b} + \sum_i \cot(\frac{x-z_i}{2})\partial_i\tilde{b} - \cot\left(\frac{v-x}{2}\right)\partial_v\tilde{b} + \cot\left(\frac{\overline{v}-x}{2}\right)\overline{\partial}_v\tilde{b}\right. \\
&\left. -b\partial_x\tilde{b} + \frac{\tilde{b}}{2\sin^2(\frac{y-x}{2})} + \frac{\cos(\frac{y-x}{2})}{4\sin^3(\frac{y-x}{2})}(\kappa-6) - \frac{\kappa}{2}\partial_{xx}\tilde{b}\right]\partial_y
\end{aligned}
.
$$

So, the commutation condition reduces to three differential conditions involving b and $\tilde{b}$.

$$
\begin{cases}
\kappa\partial_x\tilde{b} - \kappa\partial_y b = 0 \\
\cot(\frac{y-x}{2})\partial_x b + \sum_i \cot(\frac{y-z_i}{2})\partial_i b + \cot\left(\frac{v-x}{2}\right)\partial_v b + \cot\left(\frac{\overline{v}-x}{2}\right)\overline{\partial}_v b \\
-\tilde{b}\partial_y b + \frac{b}{2\sin^2(\frac{y-x}{2})} + \frac{\cos(\frac{x-y}{2})}{4\sin^3(\frac{x-y}{2})}(\kappa-6) - \frac{\kappa}{2}\partial_{yy}b = 0 \\
\cot(\frac{x-y}{2})\partial_y\tilde{b} + \sum_i \cot(\frac{x-z_i}{2})\partial_i\tilde{b} \\
-b\partial_x\tilde{b} + \frac{\tilde{b}}{2\sin^2(\frac{y-x}{2})} + \frac{\cos(\frac{y-x}{2})}{4\sin^3(\frac{y-x}{2})}(\kappa-6) - \frac{\kappa}{2}\partial_{xx}\tilde{b} = 0
\end{cases}
. \tag{3.2.11}
$$

Now, from the first equation, one can write

$$
b = \kappa\frac{\partial_x\psi}{\psi}, \tilde{b} = \kappa\frac{\partial_y\psi}{\psi}
$$

for some non-vanishing function ψ (at least locally). The smoothness of b and $\tilde{b}$ implies the smoothness of ψ. It turns out that the second equation now writes

$$\kappa\partial_x\left(\frac{\frac{\kappa}{2}\partial_{yy}\psi+\sum_i\cot(\frac{z_i-y}{2})\partial_i\psi+\cot(\frac{x-y}{2})\partial_x\psi+}{\psi}\right.$$
$$\left.\frac{\cot\left(\frac{v-y}{2}\right)\partial_v\psi+\cot\left(\frac{\overline{v}-y}{2}\right)\overline{\partial}_v\psi+\left(1-\frac{6}{\kappa}\right)\frac{\psi}{4\sin^2(\frac{x-y}{2})}}{\psi}\right)=0.$$

Symmetrically, the last equation is

$$\kappa\partial_y\left(\frac{\frac{\kappa}{2}\partial_{xx}\psi+\sum_i\cot(\frac{z_i-x}{2})\partial_i\psi+\cot(\frac{y-x}{2})\partial_y\psi}{\psi}\right.$$
$$\left.\frac{+\cot\left(\frac{v-x}{2}\right)\partial_v\psi+\cot\left(\frac{\overline{v}-x}{2}\right)\overline{\partial}_v\psi+\left(1-\frac{6}{\kappa}\right)\frac{\psi}{4\sin^2(\frac{y-x}{2})}}{\psi}\right)=0$$

It turns out that two equations now write

$$\begin{cases}\frac{\kappa}{2}\partial_{xx}\psi+\sum_i\cot(\frac{z_i-x}{2})\partial_i\psi+\cot(\frac{y-x}{2})\partial_y\psi+\cot\left(\frac{v-x}{2}\right)\partial_v\psi+\cot\left(\frac{\overline{v}-x}{2}\right)\overline{\partial}_v\psi\\ +\left(\left(1-\frac{6}{\kappa}\right)\frac{1}{4\sin^2(\frac{y-x}{2})}+h_1(x,z,u)\right)\psi=0\\ \frac{\kappa}{2}\partial_{yy}\psi+\sum_i\cot(\frac{z_i-y}{2})\partial_i\psi+\cot(\frac{x-y}{2})\partial_x\psi+\cot\left(\frac{v-x}{2}\right)\partial_v\psi+\cot\left(\frac{\overline{v}-x}{2}\right)\overline{\partial}_v\psi\\ +\left(\left(1-\frac{6}{\kappa}\right)\frac{1}{4\sin^2(\frac{x-y}{2})}+h_2(y,z,u)\right)\psi=0.\end{cases}\tag{3.2.12}$$

□

Let us now begin our discussion on the multiple radial SLE(κ) systems with n distinct growth points $e^{i\theta_1},e^{i\theta_2},\ldots,e^{i\theta_n}$. We want to grow n infinitesimal hulls at $e^{i\theta_i}$, $i=1,2,\ldots,n$. We can either grow a hull K_{ε_i} at $e^{i\theta_i}$, and then another one at $e^{i\theta_j}$ in the perturbed domain $\mathbb{D}\backslash K_{\varepsilon_i}$, or proceed in any order. The coherence condition is that these procedures yield the same result.

We grow two SLE hulls from θ_i,θ_j, $i\neq j$ and treat the rest as marked points. By lemma (3.2.9), the commutation relation between two SLEs implies that the infinitesimal generator satisfies

$$[\mathcal{M}_i,\mathcal{M}_j]=\frac{1}{\sin^2(\frac{\theta_i-\theta_j}{2})}(\mathcal{M}_j-\mathcal{M}_i).\tag{3.2.13}$$

By expanding (3.2.6), we derive that

$$\kappa\partial_i b_j - \kappa\partial_j b_i = 0 \tag{3.2.14}$$

for all $1 \le i < j \le n$.

Since the chamber

$$\mathfrak{X}^n\times\mathbb{D} = \{(\theta_1, \theta_2, \dots, \theta_n, u) \in \mathbb{R}^n \times \mathbb{D} \mid \theta_1 < \theta_2 < \dots < \theta_n < \theta_1 + 2\pi, u \in \mathbb{D}\}$$

is simply connected (contractible). Equations (3.2.14) imply that we can integrate the differential form $\sum_j b_j(\boldsymbol{\theta}, u)d\theta_j$ with respect to $\theta_1, \theta_2, \dots, \theta_n$. Stoke's theorem implies that this integral is path-independent. Consequently, there exists a positive function $\psi(\boldsymbol{\theta}, u)$ such that

$$b_i(\boldsymbol{\theta}, u) = \kappa\frac{\partial_i\psi}{\psi} \tag{3.2.15}$$

and the null vector equations

$$\begin{cases} \frac{\kappa}{2}\partial_{ii}\psi + \sum_{k\neq i,j}\cot(\frac{\theta_k-\theta_i}{2})\partial_k\psi + \cot(\frac{\theta_j-\theta_i}{2})\partial_j\psi + \cot\left(\frac{v-\theta_i}{2}\right)\partial_v\psi + \cot\left(\frac{\overline{v}-\theta_i}{2}\right)\overline{\partial}_v\psi \\ + \left(\left(1-\frac{6}{\kappa}\right)\frac{1}{4\sin^2(\frac{\theta_j-\theta_i}{2})} + h_i(\boldsymbol{\theta}, u)\right)\psi = 0 \\ \frac{\kappa}{2}\partial_{jj}\psi + \sum_{k\neq i,j}\cot(\frac{\theta_k-\theta_j}{2})\partial_k\psi + \cot(\frac{\theta_i-\theta_j}{2})\partial_i\psi + \cot\left(\frac{v-\theta_i}{2}\right)\partial_v\psi + \cot\left(\frac{\overline{v}-x}{2}\right)\overline{\partial}_v\psi \\ + \left(\left(1-\frac{6}{\kappa}\right)\frac{1}{4\sin^2(\frac{\theta_i-\theta_j}{2})} + h_j(\boldsymbol{\theta}, u)\right)\psi = 0. \end{cases} \tag{3.2.16}$$

We may write the first equation in (3.2.16) as

$$\begin{aligned} &\frac{\kappa}{2}\partial_{ii}\psi + \sum_{k\neq i,j}\cot(\frac{\theta_k-\theta_i}{2})\partial_k\psi + \cot(\frac{\theta_j-\theta_i}{2})\partial_j\psi + \cot\left(\frac{v-\theta_i}{2}\right)\partial_v\psi + \cot\left(\frac{\overline{v}-\theta_i}{2}\right)\overline{\partial}_v\psi \\ &= -\left(\left(1-\frac{6}{\kappa}\right)\frac{1}{4\sin^2(\frac{\theta_j-\theta_i}{2})} + h_i(\boldsymbol{\theta}, u)\right)\psi, \end{aligned} \tag{3.2.17}$$

where h_i does not depend on θ_j. Since integrability conditions hold for all $j \neq i$, by subtracting all $\left(1-\frac{6}{\kappa}\right)\frac{1}{4\sin^2\left(\frac{\theta_j-\theta_i}{2}\right)}$ terms, we obtain

$$\frac{\kappa}{2}\partial_{ii}\psi + \sum_{k\neq i,j}\cot(\frac{\theta_k-\theta_i}{2})\partial_k\psi + \cot(\frac{\theta_j-\theta_i}{2})\partial_j\psi + \cot\left(\frac{v-\theta_i}{2}\right)\partial_v\psi + \cot\left(\frac{\overline{v}-\theta_i}{2}\right)\overline{\partial}_v\psi$$
$$= -\left(\left(1-\frac{6}{\kappa}\right)\frac{1}{4\sin^2(\frac{\theta_j-\theta_i}{2})} + h_i(\theta_i,u)\right)\psi \tag{3.2.18}$$

where $h_i = h_i(\theta_i, u)$ only depends on θ_i and u.

(ii)

Lemma 3.2.10. *For adjacent growth points $x, y \in \mathbb{R}$ (no marked points $\{z_1, z_2, \ldots, z_n\}$ are between x and y). If the system*

$$\begin{cases} \frac{\kappa}{2}\partial_{xx}\psi + \sum_i \cot(\frac{z_i-x}{2})\partial_i\psi + \cot(\frac{y-x}{2})\partial_y\psi + \left(\left(1-\frac{6}{\kappa}\right)\frac{1}{4\sin^2(\frac{y-x}{2})} + h_1(x,z)\right)\psi = 0 \\ \frac{\kappa}{2}\partial_{yy}\psi + \sum_i \cot(\frac{z_i-y}{2})\partial_i\psi + \cot(\frac{x-y}{2})\partial_x\psi + \left(\left(1-\frac{6}{\kappa}\right)\frac{1}{4\sin^2(\frac{x-y}{2})} + h_2(y,z)\right)\psi = 0 \end{cases} \tag{3.2.19}$$

admits a non-vanishing solution ψ, then: functions h_1, h_2 can be written as $h_1(x,z) = h(x,z), h_2(y,z) = h(y,z)$.

Proof. The problem is now to find functions h_1, h_2 such that the above system has solutions. So assume that we are given h_1, h_2, and a non-vanishing solution ψ of this system. Let:

$$\mathcal{L}_1 = \frac{\kappa}{2}\partial_{xx} + \sum_i \cot(\frac{z_i-x}{2})\partial_i + \cot(\frac{y-x}{2})\partial_y + \left(1-\frac{6}{\kappa}\right)\frac{1}{4\sin^2(\frac{y-x}{2})}$$
$$\mathcal{L}_2 = \frac{\kappa}{2}\partial_{yy} + \sum_i \cot(\frac{z_i-y}{2})\partial_i + \cot(\frac{x-y}{2})\partial_x + \left(1-\frac{6}{\kappa}\right)\frac{1}{4\sin^2(\frac{x-y}{2})}.$$

Then ψ is annihilated by all operators in the left ideal generated by $(\mathcal{L}_1 + h_1)$, $(\mathcal{L}_2 + h_2)$, including in particular their commutator:

$$
\begin{aligned}
\mathcal{L} &= [\mathcal{L}_1 + h_1, \mathcal{L}_2 + h_2] + \frac{1}{\sin^2(\frac{x-y}{2})}((\mathcal{L}_1 + h_1) - (\mathcal{L}_2 + h_2)) \\
&= [\mathcal{L}_1, \mathcal{L}_2] + \frac{1}{\sin^2(\frac{x-y}{2})}(\mathcal{L}_1 - \mathcal{L}_2) + ([\mathcal{L}_1, h_2] - [\mathcal{L}_2, h_1]) + \frac{(h_1 - h_2)}{\sin^2(\frac{x-y}{2})} \\
&= \left(\cot(\frac{y-x}{2})\partial_y + \sum_i \cot(\frac{z_i - x}{2})\partial_i + \cot\left(\frac{v-x}{2}\right)\partial_v\psi + \cot\left(\frac{\overline{v}-x}{2}\right)\overline{\partial}_v\psi\right) h_2 \\
&- \left(\cot(\frac{x-y}{2})\partial_x + \sum_i \cot(\frac{z_1 - y}{2})\partial_i + \cot\left(\frac{v-y}{2}\right)\partial_v\psi + \cot\left(\frac{\overline{v}-y}{2}\right)\overline{\partial}_v\psi\right) h_1 \\
&+ \frac{4\,(h_1 - h_2)}{\sin^2(\frac{x-y}{2})}.
\end{aligned}
$$

$\mathcal{L}$ is an operator of order 0, it is a function. Since $\mathcal{L}(\psi) = 0$ for a non-vanishing ψ, $\mathcal{L}$ must vanish identically.

Note that if the two growth points x and y are adjacent (no marked points $\{z_1, z_2, \ldots, z_n\}$ are between x and y), we consider the pole of $\mathcal{L}$ at $x = y$. The second-order pole must vanish, this implies $h_1(x, z) = h(x, z)$, $h_2(y, z) = h(y, z)$ for a common function h. □

By applying lemma (3.2.10) to adjacent θ_i and θ_{i+1}, we obtain that the function $h_i(\theta, u) = h_{i+1}(\theta, u)$ for each $1 \le i \le n-1$, which implies the existence of a common function $h(\theta, u)$.

□

We have already established the commutation relations and now we consider how conformal invariance imposes constraints on the drift term and partition functions.

The first case is the $\mathrm{Aut}(\mathbb{D}, 0)$ invariance of the multiple radial SLE(κ) with a marked point $u = 0$.

Proof of theorem (3.2.2). For a multiple radial SLE(κ) system with marked point $u = 0$. Note that by rotation invariance of the drift term b_i, under a rotation ρ_a, the functions $b_i(\theta_1, \theta_2, \ldots, \theta_n)$ satisfy

$$b_i(\theta_1, \theta_2, \ldots, \theta_n) = b_i(\theta_1 + a, \theta_2 + a, \ldots, \theta_n + a)$$

for $i = 1, 2, \ldots, n$.

(i) By equation (3.2.18), for $u = 0$, we simply omit the u-dependence and related derivatives we obtain that

$$\begin{aligned} h(\theta_i) &= -\frac{\kappa}{2}\frac{\partial_{ii}\psi}{\psi} - \sum_j \cot(\frac{\theta_j - \theta_i}{2})\frac{\partial_j \psi}{\psi} - \left(1 - \frac{6}{\kappa}\right)\sum_j \frac{1}{4\sin^2(\frac{\theta_j - \theta_i}{2})} \\ &= -\frac{\kappa}{2}(\partial_i b_i + b_i^2) - \sum_j \cot(\frac{\theta_j - \theta_i}{2}) b_j - \left(1 - \frac{6}{\kappa}\right)\sum_j \frac{1}{4\sin^2(\frac{\theta_j - \theta_i}{2})}. \end{aligned} \tag{3.2.20}$$

The rotation invariance of $b_i(\boldsymbol{\theta})$ implies the rotation invariance of $h(\theta_i)$. Thus, h must be a constant.

(ii) Since $b_i = \kappa \partial_i \log(\psi)$, by the rotation invariance of b_i, for rotation transformation ρ_a:

$$\partial_i \left(\log(\psi) - \log(\psi \circ \rho_a)\right) = 0.$$

for $i = 1, 2, \ldots, n$. Thus, independent of $\theta_1, \theta_2, \ldots, \theta_n$. We obtain that there exists a function $F(a) : \mathbb{R} \to \mathbb{R}$ such that

$$\log(\psi) - \log(\psi \circ \rho_a) = F(a).$$

Since for $a, b \in \mathbb{R}$, F satisfies the Cauchy functional equation

$$F(a) + F(b) = F(a + b).$$

The only solution for the Cauchy functional equation is linear. Thus, there exists $\omega \in \mathbb{R}$.

$$F(a) = \omega a.$$

By differentiating with respect to a,

$$\sum_i \partial_i \psi = \omega \psi.$$

□

Proof of theorem (3.2.7).

(i) Note that by corollary (3.1.3), under a conformal map $\tau \in \operatorname{Aut}(\mathbb{D})$, the drift term $b_i(\theta_1, \theta_2, \ldots, \theta_n, u)$ transforms as

$$b_i = \tau'(\theta_i)\,(b_i \circ \tau) + \frac{6-\kappa}{2}\,(\log \tau'(\theta_i))'\,.$$

Since $b_i = \kappa \partial_i \log(\psi)$

$$\kappa \partial_i \log(\psi) = \kappa \tau'(\theta_i) \partial_i \log(\psi \circ \tau) + \frac{6-\kappa}{2}\,(\log \tau'(\theta_i))'$$

which implies

$$\partial_i \left(\log(\psi) - \log(\psi \circ \tau) + \frac{\kappa - 6}{2\kappa} \sum_i \log(\tau'(\theta_i)) \right) = 0$$

for $i = 1, 2, \ldots, n$. Thus, independent of variables $\theta_1, \theta_2, \ldots, \theta_n$.

We obtain that there exists a function $F :\ \mathrm{Aut}(\mathbb{H}) \times \mathbb{H} \to \mathbb{C}$ such that

$$\log(\psi) - \log(\psi \circ \tau) + \frac{\kappa - 6}{2\kappa} \sum_i \log(\tau'(\theta_i)) = F(\tau, u).$$

By direct computation and the chain rule, we can show that

$$\begin{aligned} F(\tau_1\tau_2, u) &= \log(\psi) - \log(\psi \circ \tau_1\tau_2) + \frac{\kappa-6}{2\kappa} \sum_i \log((\tau_1\tau_2)'(\theta_i)) \\ &= \log(\psi) - \log(\psi \circ \tau_2) + \log(\psi \circ \tau_2) - \log(\psi \circ \tau_1\tau_2) \\ &+ \frac{\kappa-6}{2\kappa} \sum_i \log(\tau_2'(\theta_i)) + \frac{\kappa-6}{2\kappa} \sum_i \log(\tau_1'(\tau_2(\theta_i))) \\ &= F(\tau_1, \tau_2(u)) + F(\tau_2, u). \end{aligned}$$

(ii) By the functional equation (3.2.4) and $u = 0$ is the fixed point of the addition transformation $A_\theta(z)$, we obtain that

$$F(A_{\theta_1+\theta_2}, i) = F(A_{\theta_1}, A_{\theta_2}(i)) + F(A_{\theta_2}, i) = F(A_{\theta_1}, i) + F(A_{\theta_2}, i).$$

This is a Cauchy functional equation, the only solution is linear, thus there exists real constant β such that

$$F(A_\theta, i) = \beta\theta.$$

(iii) Let $v = \tau(u)$, let T_u be the conformal map:

$$T_u(z) = \frac{z - u}{1 - \bar{u}z}$$

then by the functional equation (3.2.4), we obtain that

$$F_i(\tau, u) = F_i(T_v \circ A_\theta \circ T^{-1}u, T_u(0)) = -F_i(T_u, 0) + F_i(T_v \circ A_\theta, 0) = F_i(T_v, 0) - F_i(T_u, 0) + \omega_i\theta$$

for $i = 1, 2.$ we define

$$f(u) = F_1(T_u, 0) - F_2(T_u, 0).$$

Now, suppose ψ_i are corresponding partition functions. By the definition of function $F(\tau, u)$, ψ_i satisfies the following functional equation

$$\log(\psi_i) - \log(\psi_i \circ \tau) + \frac{\kappa - 6}{2\kappa} \sum_j \log(\tau'(z_j)) = F_i(\tau, u). \tag{3.2.21}$$

Subtracting two equations, we obtain that

$$\log(\frac{\psi_1}{\psi_2}) - \log(\frac{\psi_1 \circ \tau}{\psi_2 \circ \tau}) = f(v) - f(u) + (\omega_1 - \omega_2)\theta.$$

Then if $\omega_1 = \omega_2$

$$\log(\frac{\psi_1}{\psi_2}) - \log(\frac{\psi_1 \circ \tau}{\psi_2 \circ \tau}) = f(v) - f(u).$$

which is equivalent to

$$\psi_2 = ce^{f(u)}\psi_1.$$

thus

$$g(u) = ce^{f(u)}.$$

where $c > 0$.

(iv) Now we verify that $F(\tau, u)$ defined in satisfy the functional equation (3.2.4).

Let $v = \tau_2(u)$, $w = \tau_1 \circ \tau_2(u)$,

$$\tau_2 = T_v \circ A_{\theta_2} \circ T_{-u}$$

$$\tau_1 = T_w \circ A_{\theta_1} \circ T_{-v}$$

$$\tau_1 \circ \tau_2 = T_w \circ A_{\theta_1 + \theta_2} \circ T_{-u}$$

then

$$F(\tau_1 \circ \tau_2, u) = 2\,\mathrm{Re}(\lambda(u)) \log\left(\frac{1 - |w|^2}{1 - |u|^2}\right) - 2(\theta_1 + \theta_2)\mathrm{Im}(\lambda(u))$$

$$F(\tau_1, \tau_2(u)) = F(\tau_1, v) = 2\,\mathrm{Re}(\lambda(u)) \log\left(\frac{1 - |w|^2}{1 - |v|^2}\right) - 2\theta_1\mathrm{Im}(\lambda(u))$$

$$F(\tau_2, u) = F(\tau_1, v) = 2\,\mathrm{Re}(\lambda(u)) \log\left(\frac{1 - |v|^2}{1 - |u|^2}\right) - 2\theta_2\mathrm{Im}(\lambda(u)).$$

Combining above three equations, we obtain that

$$F(\tau_1\tau_2, u) = F(\tau_1, \tau_2(u)) + F(\tau_2, u). \tag{3.2.22}$$

□

Theorem (Restatement of Theorem 1.2.4)**.** *For a multiple radial SLE(κ) system with n SLEs starting at $(\theta_1, \theta_2, \ldots, \theta_n) \in \mathfrak{X}^n(\boldsymbol{\theta})$ and a marked point $u \in \mathbb{D}$ not necessarily fixed at* 0*:*

(i) Two partition functions $\widetilde{\psi}$ and ψ are equivalent *if they differ by a multiplicative factor $f(u)$:*

$$\widetilde{\psi} = f(u) \cdot \psi,$$

where $f(u)$ is a smooth, positive function of u. Under this equivalence, $\widetilde{\psi}$ and ψ induce identical multiple radial SLE(κ) systems.

(ii) Within the equivalence class of partition functions, we can choose ψ to satisfy conformal covariance. Under $\tau \in \mathrm{Aut}(\mathbb{D})$, ψ transforms as:

$$\psi(\theta_1, \ldots, \theta_n, u) = \left(\prod_{i=1}^{n} \tau'(\theta_i)^{\frac{6-\kappa}{2\kappa}}\right) \tau'(u)^{\lambda(u)} \overline{\tau'(u)}^{\overline{\lambda(u)}} \psi(\tau(\theta_1), \ldots, \tau(\theta_n), \tau(u)).$$

(iii) The choice of a conformally covariant partition function is not unique. Let:

$$f(u) = (\mathrm{Rad}(u, \mathbb{H}))^{\alpha} = (i(\overline{u} - u))^{\alpha}, \quad \alpha \in \mathbb{R}.$$

Then for any conformally covariant ψ, $\widetilde{\psi} = f(u) \cdot \psi$ yields an equivalent solution with:

$$\widetilde{\lambda}(u) = \lambda(u) + \alpha.$$

Proof of Theorem (1.2.4). For a multiple radial SLE(κ) system with partition function $\psi(\boldsymbol{\theta}, u)$, we proceed as follows

(i) By equation (3.2.14), the drift term in the marginal law for multiple radial SLE(κ) systems is given by

$$b_i = \kappa \partial_j \log(\psi).$$

If two partition functions differ by a multiplicative function $f(u)$.

$$\tilde{\psi} = f(u) \cdot \psi,$$

where $f(u)$ is an arbitrary positive real smooth function depending on the marked interior point u. Note that

$$b_i = \kappa \partial_j \log(\psi) = \kappa \partial_j \log(\widetilde{\psi}) = \widetilde{b_i}.$$

Thus $\widetilde{\psi}$ and ψ induce identical multiple radial SLE(κ) system.

(ii) Let ω be the corresponding rotation constant. Define:

$$\psi(\theta_1, \theta_2, \ldots, \theta_n, 0) = \left(\prod_{i=1}^{n} T_u'(\theta_i)^{\frac{6-\kappa}{2\kappa}}\right) T_u'(u)^{\lambda(u)} T_u'(u^*)^{\lambda(u^*)} \tilde{\psi}\left(T_u(\theta_1), T_u(\theta_2), \ldots, T_u(\theta_n), u\right).$$

Here, $\tilde{\psi}$ and ψ share the same rotation constant ω. By (iii) of Theorem (3.2.7), there exists a function $f(u)$ such that

$$\tilde{\psi} = f(u) \cdot \psi.$$

(iii) Since $f(u)$ is given by:

$$f(u) = (1 - |u|^2)^{\alpha},$$

where α is the conformal dimension, we conclude that the partition function:

$$(1 - |u|^2)^{\alpha} \cdot \psi$$

has conformal dimension $\lambda(u) + \alpha$.

□

3.3 Local commutation relation and null vector equations in $\kappa = 0$ case

Proof of theorem 1.3.2. To study the commutation relations, we focus on growing two hulls from growth points x, y and relabeling other points as marked points z.

The definition of commutation implies that for sufficiently small $s, t > 0$ the normalizing map for the hull $\gamma_1[0, t] \cup \gamma_2[0, s]$ is the composition of the Loewner maps for each individual hull $\gamma_1[0, t]$ or $\gamma_2[0, s]$, when applied in either order. In removing $\gamma_1[0, t]$ we are considering the coordinate change $h_{1,t}$, which leads to

$$\sigma_{1,2}^{t,s} = \mathrm{hcap}\left(\tilde{\gamma}_2[0, s]\right) = \int_0^s \left(f_{1,2}^{t,u}\right)' (y(u))^2 \, du, \tag{3.3.1}$$

where $y(u)$ is the position of y at time u. In this case $f_{1,2}^{t,s} = \tilde{h}_{2,s} \circ h_{1,t} \circ h_{2,s}^{-1}$. With this notation in hand, commutation implies that

$$h_{2,\sigma_{1,2}^{t,s}} \circ h_{1,t} = h_{1,\sigma_{2,1}^{s,t}} \circ h_{2,s}.$$

On the left-hand side the driving function first evolves according to the dynamics of $\mathcal{L}_x$ for t units of time and then $\mathcal{L}_y$ for $\sigma_{1,2}^{t,s}$ units of time. The right-hand side is analogous. These Loewner maps can be the same only if the driving function move to the same position when the maps are applied in either order. In our setup, the motion of the driving functions is fully determined by the motion of the points in θ, so a necessary condition for these maps to be the same is that

$$e^{\sigma^{t,s}_{1,2}\mathcal{M}_y} e^{t\mathcal{M}_x} = e^{\sigma^{s,t}_{2,1}\mathcal{M}_x} e^{s\mathcal{M}_y}, \tag{3.3.2}$$

where $t \mapsto e^{t\mathcal{M}}$ denotes the flow map corresponding to the dynamics $\mathcal{M}$. The commutation relation (1.3.2), as we now explain, is a straightforward consequence of this identity. From (3.3.1) we obtain

$$\begin{aligned}\sigma^{t,s}_{1,2} = s\left(\left(f^{t,0}_{1,2}\right)'(y) + O(s)\right) = s\left(h'_{1,t}(x) + O(s)\right) &= s\left(1 - \frac{t}{\sin^2(\frac{x-y}{2})} + o(t) + O(s)\right) \\ &= s - \frac{st}{\sin^2(\frac{\theta_x-\theta_y}{2})} + o(st) + O\left(s^2\right),\end{aligned}$$

where the constants in the error terms may depend on x and y. Now use the above to expand (3.3.2) in powers of s and t, and compare coefficients of st, we obtain the desired commutation relations for generators

$$[\mathcal{M}_x, \mathcal{M}_y] = \frac{1}{\sin^2(\frac{x-y}{2})}(\mathcal{M}_y - \mathcal{M}_x). \tag{3.3.3}$$

Expanding the infinitesimal commutation relation:

$$\begin{aligned}&[\mathcal{M}_x, \mathcal{M}_y] + \frac{1}{\sin^2(\frac{y-x}{2})}(\mathcal{M}_x - \mathcal{M}_y) = \\ &\left[\cot(\frac{y-x}{2})\frac{\partial U_x}{\partial x} + \sum_i \cot(\frac{y-z_i}{2})\frac{\partial U_i}{\partial x} - \frac{1}{2}\frac{\partial (U_y)^2}{\partial x} + \frac{U_x}{2\sin^2(\frac{y-x}{2})} - \frac{3\cos(\frac{x-y}{2})}{2\sin^3(\frac{x-y}{2})}\right]\partial_x \\ &-\left[\cot(\frac{x-y}{2})\frac{\partial U_y}{\partial y} + \sum_i \cot(\frac{x-z_i}{2})\frac{\partial U_i}{\partial y} - \frac{1}{2}\frac{\partial (U_x)^2}{\partial y} + \frac{U_y}{2\sin^2(\frac{y-x}{2})} - \frac{3\cos(\frac{y-x}{2})}{2\sin^3(\frac{y-x}{2})}\right]\partial_y.\end{aligned} \tag{3.3.4}$$

The right hand side of (3.3.4) equal to 0 implies the null vector equations

$$\begin{cases} \frac{1}{2}U_x^2 + \sum_i \cot(\frac{z_i-x}{2})U_i + \cot(\frac{y-x}{2})U_y + \left(-\frac{3}{2\sin^2(\frac{y-x}{2})} + h_1(x,z)\right) = 0 \\ \frac{1}{2}U_y^2 + \sum_i \cot(\frac{z_i-y}{2})U_i + \cot(\frac{x-y}{2})U_x + \left(-\frac{3}{2\sin^2(\frac{y-x}{2})} + h_2(y,z)\right) = 0. \end{cases} \tag{3.3.5}$$

Note that $\partial_j U_k = \partial_k U_j$ do not naturally follow from the commutation relation. This condition is equivalent to the existence of a function $\mathcal{U}(\boldsymbol{\theta})$ such that

$$U_j = \partial_j \mathcal{U}(\boldsymbol{\theta}).$$

$\mathcal{U} : \mathfrak{X}^n \to \mathbb{R}$ is smooth in the chamber

$$\mathfrak{X}^n = \{(\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^n \mid \theta_1 < \theta_2 < \ldots < \theta_n < \theta_1 + 2\pi\}$$

Lemma 3.3.1. *Suppose there exists $\mathcal{U}$ such that $U_j = \partial_j \mathcal{U}$, then for adjacent x, y (no marked points are between x,y), if the system*

$$\begin{cases} \frac{1}{2}U_x^2 + \sum_i \cot(\frac{z_i - x}{2})U_i + \cot(\frac{y-x}{2})U_y + \left(-\frac{3}{2\sin^2(\frac{y-x}{2})} + h_1(x,z)\right) = 0 \\ \frac{1}{2}U_y^2 + \sum_i \cot(\frac{z_i - y}{2})U_i + \cot(\frac{x-y}{2})U_x + \left(-\frac{3}{2\sin^2(\frac{y-x}{2})} + h_2(y,z)\right) = 0. \end{cases} \tag{3.3.6}$$

admits a non-vanishing solution, then the functions h_1, h_2 can be written as $h_1(x,z) = h(x,z), h_2(y,z) = h(y,z)$

Define two operators $\mathcal{L}_1$,$\mathcal{L}_2$ by:

$$\mathcal{L}_1 = \frac{U_x}{2}\partial_x + \sum_i \cot(\frac{z_i - x}{2})\partial_i + \cot(\frac{y-x}{2})\partial_y - \frac{3}{2\sin^2(\frac{y-x}{2})}$$

$$\mathcal{L}_2 = \frac{U_y}{2}\partial_y + \sum_i \cot(\frac{z_i - y}{2})\partial_i + \cot(\frac{x-y}{2})\partial_x - \frac{3}{2\sin^2(\frac{x-y}{2})}.$$

$\mathcal{U}(\boldsymbol{\theta})$ is annihilated by all operators in the left ideal generated by $(\mathcal{L}_1 + h_1)$, $(\mathcal{L}_2 + h_2)$, including in particular their commutator:

$$\begin{aligned} \mathcal{L} &= [\mathcal{L}_1 + h_1, \mathcal{L}_2 + h_2] + \frac{1}{\sin^2(\frac{x-y}{2})}((\mathcal{L}_1 + h_1) - (\mathcal{L}_2 + h_2)) \\ &= \left(\left(\cot(\frac{y-x}{2})\partial_y + \sum_i \cot(\frac{z_i - x}{2})\partial_i\right) h_2(y,z) - \left(\cot(\frac{x-y}{2})\partial_x + \sum_i \cot(\frac{z_1 - y}{2})\partial_i\right) h_1(x,z)\right) \\ &\quad + \frac{4\,(h_1 - h_2)}{\sin^2(\frac{x-y}{2})}. \end{aligned}$$

The operator $\mathcal{L}$ is an operator of order 0, and a function that must vanish identically.

If x, y are adjacent (no marked points are between x,y), consider the pole of $\mathcal{L}$ at $x = y$. The second-order pole must vanish, and this implies $h_1(x,z) = h(x,z), h_2(y,z) = h(y,z)$ for some function h.

Let us return to the proof of the theorem (1.3.2) for multiple radial SLE(0) systems with n distinct growth points $z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2}, \ldots, z_n = e^{i\theta_n}$.

We grow two SLEs from θ_i, θ_j, $i \neq j$ and treat the rest as marked points denoted by z.

The commutation relation between two SLEs implies that the infinitesimal generator satisfies

$$[\mathcal{M}_i, \mathcal{M}_j] = \frac{1}{\sin^2(\frac{\theta_i - \theta_j}{2})}(\mathcal{M}_j - \mathcal{M}_i) \tag{3.3.7}$$

and the null vector equations

$$\begin{cases} \frac{1}{2}U_i^2 + \sum_k \cot(\frac{z_k-\theta_i}{2})U_k + \cot(\frac{\theta_j-\theta_i}{2})U_j + \left(\frac{3}{2\sin^2(\frac{\theta_j-\theta_i}{2})} + h_i(\theta_i, z)\right) = 0 \\ \frac{1}{2}U_j^2 + \sum_k \cot(\frac{z_k-\theta_j}{2})U_k + \cot(\frac{\theta_i-\theta_j}{2})U_i + \left(\frac{3}{2\sin^2(\frac{\theta_i-\theta_j}{2})} + h_j(\theta_j, z)\right) = 0. \end{cases} \tag{3.3.8}$$

We may write the first equation in (3.3.8) as

$$\frac{1}{2}U_i^2 + \sum_k \cot(\frac{z_k - \theta_i}{2})U_k + \cot(\frac{\theta_j - \theta_i}{2})U_j = -\frac{3}{2\sin^2(\frac{\theta_j-\theta_i}{2})} - h_i(\theta_i, z). \tag{3.3.9}$$

By the integrability condition theorem (3.3.1), $h_i(\theta_i, z)$ does not depend on θ_j,

Since integrability conditions hold for all $j \neq i$, by subtracting all $\frac{3}{2\sin^2(\frac{\theta_j-\theta_i}{2})}$ term, we obtain that

$$\frac{1}{2}U_i^2 + \sum_j \cot(\frac{\theta_j - \theta_i}{2})U_j = -\sum_j \frac{3}{2\sin^2(\frac{\theta_j-\theta_i}{2})} - h_i(\theta_i). \tag{3.3.10}$$

where $h_i = h_i(\theta_i)$ only depends on θ_i.

Moreover, by the integrability condition, $h_i = h_{i+1}$ for every pair of $1 \leq i \leq n-1$, this implies $h_1 = h_2 = \ldots = h_n = h$.

$$h(\theta_i) = -\frac{1}{2}U_i^2 - \sum_j \cot(\frac{\theta_j - \theta_i}{2})U_j - \sum_{j \neq i} \frac{3}{2\sin^2(\frac{\theta_j-\theta_i}{2})}. \tag{3.3.11}$$

Rotation invariance of U_i and U_j implies that $h(\theta_i)$ must also be rotation invariant and thus a constant. This completes the proof of the theorem (1.3.2).

$\square$

Chapter 4

CONFORMAL FIELD THEORY FOR MULTIPLE SLE(κ)

4.1 Vertex operators and level two degeneracy equations

From the perspective of conformal field theory (CFT), we derive the null vector equations, also known as the level-two degeneracy equations. This derivation relies on the SLE-CFT correspondence, which describes the coupling between multiple radial SLE(κ) and conformal field theory. For further reference on such conformal field theories, please refer to Kang and N. Makarov (2013) and N-G. Kang and N. Makarov (2021)

Definition 4.1.1 (Vertex Operator)**.** *Given a modified Gaussian free field $\Phi_{\boldsymbol\beta}$ with background charge $\boldsymbol\beta$ satisfying neutrality condition (NC_b) and a charge distribution $\boldsymbol\tau = \sum_j \tau_j \cdot z_j$ satisfying neutrality condition (NC_0), we define the* vertex operator *$\mathcal{O}_{\boldsymbol\beta}[\boldsymbol\tau]$ as the OPE-exponential of the chiral bosonic field $i\Phi^+_{\boldsymbol\beta}[\boldsymbol\tau]$:*

$$\mathcal{O}_{\boldsymbol\beta}[\boldsymbol\tau] := \frac{C_{(b)}[\boldsymbol\beta+\boldsymbol\tau]}{C_{(b)}[\boldsymbol\beta]}\, e^{\odot i\Phi^+_{\boldsymbol\beta}[\boldsymbol\tau]}, \tag{4.1.1}$$

where $\Phi^+_{\boldsymbol\beta}[\boldsymbol\tau] := \sum_j \tau_j \Phi^+_{\boldsymbol\beta}(z_j)$ is the chiral bosonic field and $\odot$ denotes Wick ordering.

Its expectation value yields

$$\mathbf{E}[\mathcal{O}_{\boldsymbol\beta}[\boldsymbol\tau]] = \frac{C_{(b)}[\boldsymbol\beta+\boldsymbol\tau]}{C_{(b)}[\boldsymbol\beta]}.$$

Definition 4.1.2 (Current Field)**.** *Given a modified Gaussian free field $\Phi_{\boldsymbol\beta}$ with background charge $\boldsymbol\beta$, the* current field *$J_{\boldsymbol\beta}(z)$ is defined as the holomorphic derivative*

$$J_{\boldsymbol\beta}(z) := i\,\partial\Phi_{\boldsymbol\beta}(z).$$

Definition 4.1.3 (Virasoro Field)**.** *The* Virasoro field *$T_{\boldsymbol\beta}(z)$, also called the stress-energy tensor, is defined in terms of the current field as*

$$T_{\boldsymbol\beta}(z) := -\frac{1}{2}J_{\boldsymbol\beta}(z)\odot J_{\boldsymbol\beta}(z) + ib\,\partial J_{\boldsymbol\beta}(z),$$

Let $\{J_n\}$ and $\{L_n\}$ denote the modes of the current field $J_{\boldsymbol\beta}$ and the Virasoro field $T_{\boldsymbol\beta}$ in $\mathcal{F}_{\boldsymbol\beta}$ theory, respectively:

$$J_n(z) := \frac{1}{2\pi i}\oint_{(z)} (\zeta - z)^n J_{\boldsymbol\beta}(\zeta)\mathrm{d}\zeta, \quad L_n(z) := \frac{1}{2\pi i}\oint_{(z)} (\zeta - z)^{n+1} T_{\boldsymbol\beta}(\zeta)\mathrm{d}\zeta \quad (4.1.2)$$

Recall that a field X is called Virasoro primary if X is a differential and if X is in the family $\mathcal{F}_{\boldsymbol\beta}$. It is well known that X in $\mathcal{F}_{\boldsymbol\beta}$ is Virasoro primary if and only if $L_n X = L_n\bar{X} = 0$ for all $n \geq 1$ and

$$\begin{aligned} L_{-1}X &= \partial X, \\ L_{-1}\overline{X} &= \partial\overline{X}, \\ L_0 X &= \lambda X, \\ L_0\overline{X} &= \lambda_*\overline{X}. \end{aligned} \tag{4.1.3}$$

for some numbers λ and λ_*. (These numbers are called conformal dimensions of X.) See Kang and N. Makarov (2013), Proposition 7.5 for this.

Definition 4.1.4. *A Virasoro primary field X is called current primary if $J_n X = J_n\bar{X} = 0$ for all $n \geq 1$ and*

$$J_0 X = -i\sigma X, \quad J_0\overline{X} = i\overline{\sigma}_*\overline{X}$$

for some numbers σ and σ_* (These numbers are charges of X.) It is well known that current primary fields with specific charges satisfy the level two degeneracy equations.

Theorem 4.1.5. *For a current primary field $\mathcal{O}$ in $\mathcal{F}_\beta$ with charges σ, σ_* at z, we have*

$$\left(L_{-2}(z) + \eta L_{-1}^2(z)\right)\mathcal{O} = 0$$

if $2\sigma(b+\sigma) = 1$ and $\eta = -\frac{1}{2\sigma^2}$.

Proof. See Kang and N. Makarov, 2013, Proposition 11.2. □

We now derive BPZ equations (Belavin-Polyakov-Zamolodchikov equations) on the Riemann sphere with background charges

$$\boldsymbol\beta = b \cdot u + b \cdot u^*$$

, where u^* is the conjugation of u.

Given $b = \sqrt{\frac{\kappa}{8}} - \sqrt{\frac{2}{\kappa}}$, let $a = \sqrt{\frac{2}{\kappa}}$ be one of the solutions to the quadratic equation $2x(x+b) = 1$ for x and let $\mathcal{O}_\beta(z) \equiv \mathcal{O}_\beta^{(a,\tau)}(z) := \mathcal{O}_{\boldsymbol\beta}[a \cdot z + \tau]$

$$\boldsymbol{\tau} = \sum \tau_j \cdot z_j$$

with the neutrality condition $(\mathrm{NC}_0) : a + \sum \tau_j = 0$.

Theorem 4.1.6. *If $z \notin \operatorname{supp} \boldsymbol\beta \cup \operatorname{supp} \tau$, then for any tensor product $\mathcal{X}_{\boldsymbol\beta}$ of fields X_j in $\mathcal{F}_{\boldsymbol\beta}$, we have in the $\widehat{\mathbb{C}}$-uniformization,*

$$\frac{1}{2a^2}\partial_z^2 \mathbf{E}\mathcal{O}_\beta(z)\mathcal{X}_\beta = \mathbf{E}T_\beta(z)\mathbf{E}\mathcal{O}_\beta(z)\mathcal{X}_\beta + \mathbf{E}\check{\mathcal{L}}^+_{k_z}\mathcal{O}_\beta(z)\mathcal{X}_\beta \tag{4.1.4}$$

where the vector field k_z is given by $k_z(\zeta) = 1/(z-\zeta)$ in the identity chart of $\mathbb{C}$ and the Lie derivative operator $\check{\mathcal{L}}^+_{k_x}$ does not apply to the z-variable (In general, $\check{\mathcal{L}}^+_v$ means that we differentiate with respect to $\mathcal{L}^+_v$ except for poles of v).

Proof. See Kang and N. Makarov, 2013, Theorem 10.9. □

Theorem 4.1.7 (Expanded Level-Two Degeneracy Equation for $\Phi(z)$)**.** *Let $a = \sqrt{2/\kappa}$ and assume the neutrality condition* (NC_0) *for $\boldsymbol\tau$. Let*

$$\Phi(z) := \mathbf{E}\left[\mathcal{O}_\beta(z)\right],$$

where $\mathcal{O}_\beta(z)$ is a current primary field of charge a at z in the Coulomb gas formalism.

Then $\Phi(z)$ satisfies the following second-order differential equation:

$$\begin{aligned}\frac{\kappa}{4}\partial_z^2\Phi(z) = &\sum_j \left(\frac{\lambda_j}{(z-z_j)^2} + \frac{\partial_{z_j}}{z-z_j}\right)\Phi(z) \\ &+ \left(\frac{\lambda_{(b)}(\sigma_u)}{(z-u)^2} + \frac{\lambda_{(b)}(\sigma_{u^*})}{(z-u^*)^2} + \frac{b^2}{(z-u)(z-u^*)}\right)\Phi(z) \\ &+ \left(\frac{\partial_u}{z-u} + \frac{\partial_{u^*}}{z-u^*}\right)\Phi(z),\end{aligned}$$

where $\lambda_j = \lambda_{(b)}(\sigma_j) = \frac{\sigma_j^2}{2} - \sigma_j b$, and similarly for $\lambda_{(b)}(\sigma_u), \lambda_{(b)}(\sigma_{u^*})$.

Proof. This follows from Theorem 4.1.6 with $\mathcal{X} \equiv 1$.

$$\frac{1}{2a^2}\partial_z^2\mathbf{E}\mathcal{O}_\beta(z) = \mathbf{E}T_\beta(z)\mathcal{O}_\beta(z) + \check{\mathcal{L}}^+_{k_z}\mathbf{E}\mathcal{O}_\beta(z),$$

we evaluate each term separately.

From the Lie derivative:

$$\check{\mathcal{L}}^{+}_{k_z}\mathbf{E}O_\beta(z) = \sum_j \left(\frac{\lambda_j}{(z-z_j)^2} + \frac{\partial_{z_j}}{z-z_j}\right)\mathbf{E}O_\beta(z) + \left(\frac{\lambda_{(b)}(\sigma_u) - \lambda_{(b)}(b)}{(z-u)^2} + \frac{\partial_u}{z-u}\right)\mathbf{E}O_\beta(z) + \left(\frac{\lambda_{(b)}(\sigma_{u^*}) - \lambda_{(b)}(b)}{(z-u^*)^2} + \frac{\partial_{u^*}}{z-u^*}\right)\mathbf{E}O_\beta(z).$$

From the expectation of the stress-energy tensor:

$$\mathbf{E}T_{\boldsymbol{\beta}}(z) = \frac{\lambda_{(b)}(b)}{(z-u)^2} + \frac{\lambda_{(b)}(b)}{(z-u^*)^2} + \frac{b^2}{(z-u)(z-u^*)}.$$

Adding both contributions and using $\Phi(z) = \mathbf{E}O_\beta(z)$ yields the claimed equation.

□

Remark 4.1.8. Let $\sigma_j = -2a$ or $\sigma_j = 2(a+b)$ such that $\lambda_j = 1$. Then

$$\left(\frac{\lambda_j}{\left(z-z_j\right)^2} + \frac{\partial_j}{z-z_j}\right)\Phi = \partial_j\left(\frac{\Phi}{z-z_j}\right) \tag{4.1.5}$$

which is in a closed form for z_j. By choosing appropriate closed contours Γ, such as the Pochhammer contour, to integrate Φ with respect to z_j, the right-hand side of (4.1.5) integrates to 0 along Γ. Consequently, the variable z_j is eliminated. This procedure to generate a new correlation function is referred to as screening.

Chapter 5

COULOMB GAS SOLUTIONS TO THE NULL VECTOR EQUATIONS

5.1 Coulomb gas correlation and Coulomb gas integral

Recall that the Coulomb gas correlation differential associated with a divisor $\boldsymbol{\sigma} = \sum_{j=1}^{n} \sigma_j \cdot z_j$ on the Riemann sphere $\hat{\mathbb{C}}$ is given by

$$C_{(b)}[\boldsymbol{\sigma}] = \prod_{j<k} (z_j - z_k)^{\sigma_j \sigma_k}, \tag{5.1.1}$$

where the product is taken over all finite z_j and z_k.

Definition 5.1.1 (Monodromy of Coulomb Gas Correlation Differential)**.** *Let $\boldsymbol{\sigma} = \sum_{j=1}^{n} \sigma_j \cdot z_j$ be a divisor on $\mathbb{C}$ with associated Coulomb gas correlation differential*

$$C_{(b)}[\boldsymbol{\sigma}] = \prod_{j<k} (z_j - z_k)^{\sigma_j \sigma_k}.$$

This function is multivalued due to the presence of non-integer exponents. Its multivaluedness is described by the monodromy representation *arising from analytic continuation around branch points.*

To illustrate the basic mechanism, consider the case $n = 2$. Then

$$C_{(b)}[\boldsymbol{\sigma}] = (z_1 - z_2)^{\sigma_1 \sigma_2},$$

which is analytic in z_1 on $\mathbb{C} \setminus \{z_2\}$. If we analytically continue this function as z_1 travels once counterclockwise around z_2, the function picks up a multiplicative factor of $e^{2\pi i \sigma_1 \sigma_2}$.

Thus, the monodromy representation

$$\rho : \pi_1(\mathbb{C} \setminus \{z_2\}, z_1) \longrightarrow \mathbb{C}^*, \quad \rho(C_2) = e^{2\pi i \sigma_1 \sigma_2}$$

captures how the function changes under analytic continuation around the singularity at z_2, where C_2 is the loop encircling z_2.

In general, for $\boldsymbol{\sigma} = \sum_{j=1}^{n} \sigma_j \cdot z_j$, the function $C_{(b)}[\boldsymbol{\sigma}]$ is analytic in z_1 on $\mathbb{C} \setminus \{z_2, \ldots, z_n\}$. The fundamental group $\pi_1(\mathbb{C} \setminus \{z_2, \ldots, z_n\}, z_1)$ is the free group *generated by loops C_j encircling each z_j ($j = 2, \dots, n$), and the monodromy representation*

$$\rho : \pi_1(\mathbb{C} \setminus \{z_2, \dots, z_n\}) \to \mathbb{C}^*, \quad \rho(C_j) = e^{2\pi i \sigma_1 \sigma_j}$$

describes the multiplicative factor acquired by $C_{(b)}[\boldsymbol{\sigma}]$ when z_1 loops around z_j once in the counterclockwise direction.

Definition 5.1.2 (Screening Charge)**.** *Let $\boldsymbol{\sigma}$ be a configuration of charges on the Riemann sphere, and let $C_{(b)}[\boldsymbol{\sigma}]$ denote the associated Coulomb gas correlation differential. The conformal dimension of a charge $\sigma \in \mathbb{C}$ inserted at a point z_j is defined by*

$$\lambda_b(\sigma) = \frac{\sigma^2}{2} - \sigma b. \tag{5.1.2}$$

The condition $\lambda_b(\sigma) = 1$ characterizes special charges whose insertions yield integrands of weight $(1, 0)$. Solving this quadratic equation yields two solutions:

$$\sigma = -2a, \quad \sigma = 2(a + b).$$

A charge $\tau \in \{-2a,\, 2(a + b)\}$ is called a screening charge. *Consider a divisor of the form*

$$\boldsymbol{\sigma} = \sum_i \sigma_i \cdot z_i + \sum_j \tau_j \cdot \xi_j, \tag{5.1.3}$$

where $\{\tau_j\}$ are screening charges inserted at positions $\{\xi_j\}$.

The resulting Coulomb gas differential on the Riemann sphere $\hat{\mathbb{C}}$ is given by

$$C_{(b)}[\boldsymbol{\sigma}] = \prod_{i<j} (z_i - z_j)^{\sigma_i \sigma_j} \prod_{i,k} (z_i - \xi_k)^{\sigma_i \tau_k} \prod_{j<k} (\xi_j - \xi_k)^{\tau_j \tau_k}, \tag{5.1.4}$$

where the products range over all distinct pairs of points.

Since each τ_j satisfies $\lambda_b(\tau_j) = 1$, the differential $C_{(b)}[\boldsymbol{\sigma}]\, d\xi_j$ transforms as a holomorphic 1-form in each variable ξ_j. This allows for the definition of contour integrals of the form

$$\int_\Gamma C_{(b)}[\boldsymbol{\sigma}]\, d\xi_1 \cdots d\xi_m,$$

where Γ is a suitable multidimensional integration cycle avoiding branch points.

This procedure is known as screening*, and it plays a fundamental role in the Coulomb gas formalism. By integrating out screening charges, one obtains new correlation functions that are conformally covariant and satisfy null vector differential equations, as required by conformal field theory.*

We now consider the simplest nontrivial case involving a single screening charge ξ. The corresponding Coulomb gas correlation differential takes the form

$$C_{(b)}[\boldsymbol{\sigma}] = \prod_{i<j}(z_i - z_j)^{\sigma_i\sigma_j}\prod_{j}(z_j - \xi)^{\sigma_j\tau}, \tag{5.1.5}$$

where $\{z_j\}$ are fixed insertion points with charges $\{\sigma_j\}$, and τ is the charge at the variable point ξ.

Let $\Gamma : [0,1] \to \mathbb{C} \setminus \{z_1, \ldots, z_n\}$ be a path with basepoint $p_0 = \Gamma(0)$. Due to the non-integer exponents, the integrand is multivalued in ξ, and its analytic continuation along Γ depends on the monodromy of the branches. Consequently, even if Γ is a closed loop, the contour integral

$$\int_{\Gamma} C_{(b)}[\boldsymbol{\sigma}]\, d\xi$$

is not necessarily single-valued and may depend on the homotopy class of Γ relative to the chosen branch at p_0.

This multivaluedness necessitates a more refined homological framework: the integration should be understood in the context of *twisted homology*, where chains are equipped with local system coefficients determined by the monodromy representation of the integrand. In this setting, valid integration cycles are twisted 1-cycles, which keep track of the phase accumulated during analytic continuation.

A canonical example of such an integration path is the *Pochhammer contour* $\mathscr{P}(z_i, z_j)$, which loops around two branch points z_i and z_j alternately. Though homologically trivial in ordinary homology, this contour generates a nontrivial class in twisted homology and yields a well-defined integral. These twisted cycles form the natural domain of integration for Coulomb gas differentials with screening charges.

Definition 5.1.3 (Pochhammer Contour)**.** *Let $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{C}$ be distinct points. The punctured plane $\mathbb{C}\setminus\{z_1, \ldots, z_n\}$ is homotopy equivalent to a bouquet of n circles, $\bigvee_{i=1}^{n} S^1$, and its fundamental group is the free group:*

$$\pi_1\left(\mathbb{C} \setminus \{z_1, \ldots, z_n\}\right) \cong *_{i=1}^{n}\mathbb{Z},$$

generated by simple loops C_i encircling each puncture z_i in the positive (counter-clockwise) direction.

The Pochhammer contour *associated with a pair of points* (z_i, z_j) *is defined as the commutator of the generators* C_i *and* C_j*:*

$$\mathscr{P}(z_i, z_j) := C_i C_j C_i^{-1} C_j^{-1}. \tag{5.1.6}$$

Geometrically, this contour first winds around z_i*, then around* z_j*, and then retraces both loops in reverse order.*

Although $\mathscr{P}(z_i, z_j)$ *is null-homologous in ordinary homology, it typically represents a nontrivial class in* twisted homology*, where chains are valued in a local system determined by the monodromy of a multivalued function. Such contours are essential for defining well-posed integrals of Coulomb gas correlation differentials, which exhibit nontrivial monodromy around insertion points.*

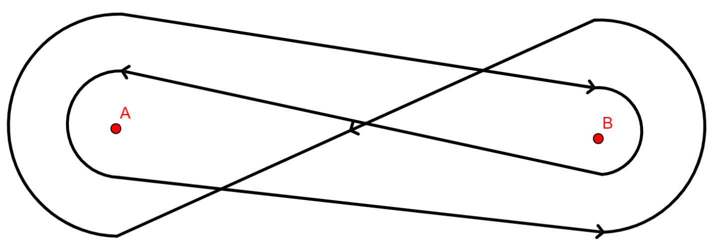

Figure 5.1: The Pochhammer contour $\mathscr{P}(z_i, z_j)$: a commutator loop around z_i and z_j.

We now analyze the role of the Pochhammer contour in defining single-valued integrals of multivalued Coulomb gas differentials.

By definition, the Pochhammer contour $\mathscr{P}(z_i, z_j) := C_i C_j C_i^{-1} C_j^{-1}$ is a commutator of simple loops C_i, C_j around z_i and z_j, respectively. Since the winding numbers of a loop and its inverse cancel, the total winding number of $\mathscr{P}(z_i, z_j)$ around any puncture vanishes:

$$\operatorname{wind}(\mathscr{P}(z_i, z_j), z_k) = 0, \qquad \text{for all } k = 1, \dots, n. \tag{5.1.7}$$

In particular, $\mathscr{P}(z_i, z_j)$ encircles neither z_i nor z_j in total:

$$\operatorname{wind}(\mathscr{P}(z_i, z_j), z_i) = \operatorname{wind}(\mathscr{P}(z_i, z_j), z_j) = 0. \tag{5.1.8}$$

As a consequence, when the integrand is of the form

$$C_{(b)}[\boldsymbol{\sigma}] = \prod_{k=1}^{n} (z_k - \xi)^{\sigma_k \tau},$$

the analytic continuation of $C_{(b)}[\boldsymbol{\sigma}]$ along $\mathscr{P}(z_i, z_j)$ returns to the original branch, and the monodromy along this loop is trivial:

$$\rho(\mathscr{P}(z_i, z_j)) = 1. \tag{5.1.9}$$

Theorem 5.1.4 (Base Point Independence)**.** *Let $\Gamma = \mathscr{P}(z_i, z_j)$ be a Pochhammer contour, and let $p_0 = \Gamma(0)$ denote its base point. Then the integral*

$$\int_\Gamma C_{(b)}[\boldsymbol{\sigma}]\, d\xi \tag{5.1.10}$$

is independent of the choice of base point p_0.

Proof. Let p_0' be another base point, and let γ be a path from p_0' to p_0. Define the conjugated loop $\Gamma' = \gamma \cdot \Gamma \cdot \gamma^{-1}$. Since the integrand is single-valued along Γ, and $\rho(\Gamma) = 1$, we have

$$\int_{\Gamma'} C_{(b)}[\boldsymbol{\sigma}]\, d\xi = \int_\Gamma C_{(b)}[\boldsymbol{\sigma}]\, d\xi.$$

Hence the integral is independent of the base point. □

Remark 5.1.5. The Pochhammer contour is a canonical example of a nontrivial twisted cycle, but the base point independence property extends to any closed contour Γ satisfying:

(i) $\mathrm{wind}(\Gamma, z_k) = 0$ for all $k = 1, \ldots, n$;

(ii) Γ represents a nontrivial class in the twisted homology group.

Under these conditions, Γ lies in the twisted homology group $H_1(\mathbb{C}\backslash\{z_1, \ldots, z_n\}; \mathbb{C}_\rho)$, where $\mathbb{C}_\rho$ is the rank-one local system determined by the monodromy representation ρ of the integrand.

This framework generalizes naturally to the case of m screening charges $\xi_1, \ldots, \xi_m$. In that setting, the integration domain is the product of m twisted cycles $\Gamma_1 \times \cdots \times \Gamma_m$, with each Γ_j lying in $H_1(\mathbb{C} \setminus \{z_1, \ldots, z_n\}; \mathbb{C}_\rho)$ and chosen, for instance, as pairwise non-intersecting Pochhammer contours. The resulting integral

$$\int_{\Gamma_1} \cdots \int_{\Gamma_m} C_{(b)}[\boldsymbol{\sigma}]\, d\xi_1 \cdots d\xi_m \tag{5.1.11}$$

defines a well-posed conformally covariant correlation function.

Theorem 5.1.6 (Conformal Invariance of the Coulomb Gas Integral)**.** *Let $\boldsymbol{\sigma} = \sum_{i=1}^{n} \sigma_i \cdot z_i + \sum_{j=1}^{m} \tau_j \cdot \xi_j$ be a divisor on the Riemann sphere $\hat{\mathbb{C}}$, where each screening charge $\tau_j \in \{-2a, 2(a+b)\}$ is chosen so that its conformal dimension satisfies $\lambda_b(\tau_j) = 1$. Let $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation, and define*

$$\zeta_j := h(\xi_j), \quad h(\boldsymbol{\sigma}) := \sum_{i=1}^{n} \sigma_i \cdot h(z_i) + \sum_{j=1}^{m} \tau_j \cdot \zeta_j.$$

Then the Coulomb gas integral transforms covariantly under h as

$$\left(\prod_{i=1}^{n} h'(z_i)^{\lambda_b(\sigma_i)}\right) \oint_{h(\Gamma)} C_{(b)}[h(\boldsymbol{\sigma})]\, d\zeta_1 \cdots d\zeta_m = \oint_{\Gamma} C_{(b)}[\boldsymbol{\sigma}]\, d\xi_1 \cdots d\xi_m, \tag{5.1.12}$$

where $h(\Gamma)$ denotes the image of the integration contour Γ under h.

Each differential $d\zeta_j$ transforms as $d\zeta_j = h'(\xi_j)\, d\xi_j$, and since $\lambda_b(\tau_j) = 1$, the integrand $C_{(b)}[\boldsymbol{\sigma}]\, d\xi_1 \cdots d\xi_m$ is invariant under pullback by h, up to the multiplicative factor $\prod_i h'(z_i)^{\lambda_b(\sigma_i)}$ determined by the insertion points.

Proof. The conformal invariance of the Coulomb gas integral naturally comes from the conformal invariance of the Coulomb gas correlation differential.

$$C_{(b)}[\boldsymbol{\sigma}] = \left(\prod_i h'(z_i)^{\lambda_i}\right)\left(\prod_j h'(\xi_j)\right) C_{(b)}\, [h(\boldsymbol{\sigma})] \tag{5.1.13}$$

Since $\zeta_j = h(\xi_j)$, then $d\xi_j = \frac{d\zeta_j}{h'(\xi_j)}$, we have

$$\begin{aligned} &\left(\prod_i h'(z_i)^{\lambda_j}\right) \oint_{h(\Gamma)} C_{(b)}\, [h(\boldsymbol{\sigma})]\, d\zeta_1 d\zeta_2 \dots d\zeta_m = \\ &\oint_{\Gamma} C_{(b)}\, [\boldsymbol{\sigma}]\, \frac{d\zeta}{\left(\prod_j h'(\xi_j)\right)} = \oint_{\Gamma} C_{(b)}\, [\boldsymbol{\sigma}]\, d\xi_1 d\xi_2 \dots d\xi_m \end{aligned} \tag{5.1.14}$$

□

Corollary 5.1.7. *The Coulomb gas integral $\mathcal{J}(z) = \oint_{C_1} \dots \oint_{C_m} \Phi_\kappa(\boldsymbol{z}, \boldsymbol{\xi}) d\xi_m \dots d\xi_1$*

$\Phi_\kappa(\boldsymbol{z}, \boldsymbol{\xi})$ is a Coulomb gas correlation function of conformal dimension $\lambda_i = \lambda_i(\sigma_i)$ at z_i, and screening charges ξ_j of conformal dimension 1.

satisfy the following conformal Ward's indentities:

$$\left[\sum_{i=1}^{n} \partial_{z_i}\right] \mathcal{J}(z) = 0,$$
$$\left[\sum_{i=1}^{n} \left(z_i \partial_{z_i} + \lambda_i(\sigma_i)\right)\right] \mathcal{J}(z) = 0, \tag{5.1.15}$$
$$\left[\sum_{i=1}^{n} \left(z_i^2 \partial_{z_i} + 2\lambda_i(\sigma_i) z_i\right)\right] \mathcal{J}(z) = 0.$$

Proof. The Ward identities follow from the invariance of the Coulomb gas integral $\mathcal{J}(z)$ under Möbius transformations. Consider the following three one-parameter families of conformal maps:

$$h_\epsilon^{(1)}(z) = z + \epsilon, \quad h_\epsilon^{(2)}(z) = (1+\epsilon)z, \quad h_\epsilon^{(3)}(z) = \frac{z}{1+\epsilon z},$$

which correspond to translations, dilations, and special conformal transformations, respectively.

By Theorem 5.1.6, the Coulomb gas integral transforms covariantly under Möbius maps:

$$\left(\prod_{i=1}^{n} h'(z_i)^{\lambda_i}\right) \mathcal{J}(h_\epsilon(z)) = \mathcal{J}(z).$$

Taking the derivative with respect to ϵ at $\epsilon = 0$, we obtain infinitesimal constraints corresponding to conformal Ward identities.

- Translation: For $h_\epsilon(z) = z + \epsilon$, we have $h'(z) = 1$, so:

  $$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{J}(z_1 + \epsilon, \ldots, z_n + \epsilon) = 0.$$

  By the chain rule, this gives:

  $$\sum_{i=1}^{n} \partial_{z_i} \mathcal{J}(z) = 0.$$

- Dilation: For $h_\epsilon(z) = (1+\epsilon)z$, we have $h'(z) = 1 + \epsilon$, so:

  $$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left(\prod_{i=1}^{n} (1+\epsilon)^{\lambda_i} \cdot \mathcal{J}((1+\epsilon)z)\right) = 0.$$

  Differentiating yields:

  $$\sum_{i=1}^{n} \left(z_i \partial_{z_i} + \lambda_i\right) \mathcal{J}(z) = 0.$$

- Shearing: For $h_\epsilon(z) = \frac{z}{1+\epsilon z}$, we compute

$$h'(z) = \frac{1}{(1+\epsilon z)^2} \approx 1 - 2\epsilon z + o(\epsilon), \quad h_\epsilon(z) \approx z - \epsilon z^2 + o(\epsilon).$$

  Plugging into the covariance relation and differentiating gives:

$$\sum_{i=1}^{n} \left(z_i^2 \partial_{z_i} + 2\lambda_i z_i \right) \mathcal{J}(\boldsymbol{z}) = 0.$$

This establishes the three global conformal Ward identities in (5.1.15). □

5.2 Classification and link pattern

Throughout this section, we modify the notation by setting $z_{n+1} = u$ and $z_{n+2} = u^*$. As usual, let $z_1 < z_2 < \ldots < z_{n-1} < z_n$.

We begin by considering the charge $\boldsymbol{\sigma} = \sum_{j=1}^{n+m+2} \sigma_j \cdot z_j$ and the Coulomb gas correlation:

$$C_{(b)}[\boldsymbol{\sigma}] = \Phi\left(z_1, \ldots, z_{n+2+m}\right) = \prod_{i<j}^{n+2+m} \left(z_j - z_i\right)^{\sigma_i \sigma_j}.$$

Our strategy is to choose the σ_i (i.e., the charges associated with the divisor in the Coulomb gas correlation) such that for $1 \le i \le n$, and $\lambda_j = \frac{\sigma_j^2}{2} - \sigma_j b$:

$$\begin{aligned}\Bigg[\frac{\kappa}{4}\partial_i^2 + \sum_{j\neq i}^{n} \left(\frac{\partial_j}{z_j - z_i} - \frac{\lambda_j}{(z_j - z_i)^2} \right) + \frac{\partial_{n+1}}{z_{n+1} - z_i} + \frac{\partial_{n+2}}{z_{n+2} - z_i} \\ - \frac{\lambda_{n+1}}{(z_{n+1} - z_i)^2} - \frac{\lambda_{n+2}}{(z_{n+2} - z_i)^2}\Bigg]\Phi \\ = \sum_{k=n+3}^{n+2+m} \partial_k(\ldots),\end{aligned} \tag{5.2.1}$$

Theorem 5.2.1. *(i) If we choose $\sigma_j = a = \sqrt{\frac{2}{\kappa}}$, and $\lambda_j = \frac{a^2}{2} - ab = \frac{6-\kappa}{2\kappa}$ for $1 \le j \le n$, and $\lambda_j = \frac{\sigma_j^2}{2} - \sigma_j b$ for $n+1 \le j \le n+2$, then we obtain the following null vector equation:*

$$\left[\frac{\kappa}{4}\partial_j^2+\sum_{k\neq j}^{n}\left(\frac{\partial_k}{z_k-z_j}-\frac{(6-\kappa)/2\kappa}{(z_k-z_j)^2}\right)+\frac{\partial_{n+1}}{z_{n+1}-z_j}\right.$$
$$\left.+\frac{\partial_{n+2}}{z_{n+2}-z_j}-\frac{\lambda_{n+1}}{(z_{n+1}-z_j)^2}-\frac{\lambda_{n+2}}{(z_{n+2}-z_j)^2}\right]\Phi \tag{5.2.2}$$
$$=\sum_{k=n+3}^{n+2+m}\partial_k\left(-\frac{\Phi(z_1,\ldots,z_{n+m+2})}{z_k-z_j}\right)$$

for all $j\in\{1,2,\ldots,n\}$*. Thus, we attain the desired form (5.2.1) for all* $j\in\{1,2,\ldots,n\}$*.*

Currently, the number of screening charges m *and the values of* $\sigma_k=2a$ *or* $\sigma_k=2(a+b)$ *for* $k\in\{n+3,n+4,\ldots,n+m+2\}$ *remain unspecified. The charges* $\sigma_{n+1}=\overline{\sigma_{n+2}}$ *are chosen such that* $\boldsymbol{\sigma}=\sum_j\sigma_j\cdot z_j$ *satisfies the neutrality condition (*NC_b*).*

(ii) If $n=2k$*,* $m=k-1$*, and we choose* $\sigma_j=a$ *for all* $j\in\{1,2,\ldots,n-1\}$*,* $\sigma_n=2b-a$*, and the sign of* $\sigma_k=-2a$ *for all* $k\in\{n+1,n+2,\ldots,n+m\}$*, then we have the following null vector equation for* $j\in\{1,2,\ldots,n-1\}$*:*

$$\left[\frac{\kappa}{4}\partial_n^2+\sum_{k=1}^{n-1}\left(\frac{\partial_k}{z_k-z_n}-\frac{(6-\kappa)/2\kappa}{(z_k-z_n)^2}\right)\right]\Phi$$
$$=\sum_{k=n+1}^{n+m-1}\partial_k\left(-\frac{\Phi(z_1,\ldots,z_{n+m})}{z_k-z_n}\right) \tag{5.2.3}$$
$$+\frac{1}{2}\sum_{k=n+1}^{n+m-1}\partial_k\left[\frac{8-\kappa}{z_k-z_n}\left(\prod_{s=1}^{n-1}\frac{z_k-z_s}{z_n-z_s}\prod_{\substack{t=n+1\\ t\neq k}}^{n+m}\left(\frac{z_n-z_t}{z_k-z_t}\right)^2\right)\Phi\right].$$

Since the right-hand side of (5.2.3) consists of derivatives with respect to z_k *for* $k\in\{n+1,n+2,\ldots,n+m\}$*, we obtain the desired form (5.2.1) for* $j=n$ *as well. Therefore, the null vector equations are satisfied for all* $1\le j\le n$*.*

Then we will integrate $z_{n+3},\ldots,z_{n+2+m}$ on both sides of (5.2.1) around nonintersecting closed contours $\Gamma_1,\ldots,\Gamma_m$. On the left side, the integrand is a smooth function of $z_1,\ldots,z_{n+m+2}$ because the contours do not intersect.

Integration on the right side is expected to give zero. To attain this, we carefully choose the integration contour for $z_{n+3},\ldots,z_{n+2+m}$. A commonly used integration

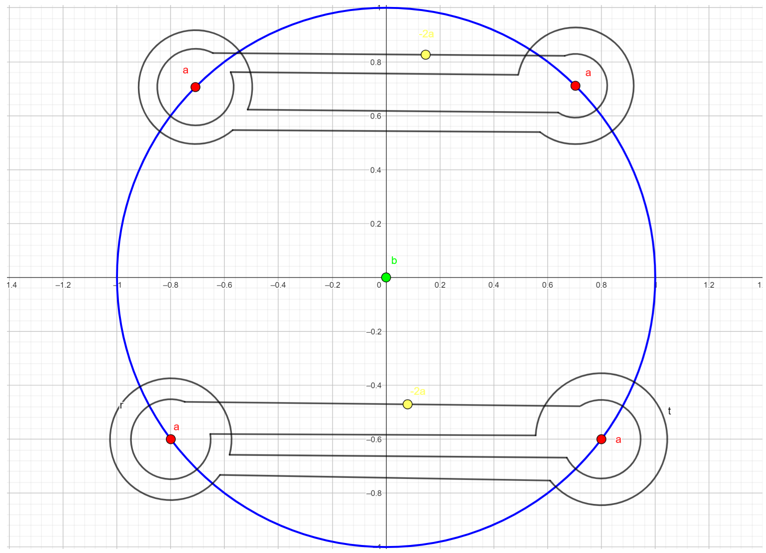

Figure 5.1: Example: z_1, z_2, z_3, z_4 with 2 screening charges ξ_1, ξ_2

contour is the Pochhammer contour encircling two points z_i and z_j, denoted by $\mathscr{P}\left(z_i, z_j\right)$.

Because either side of (5.2.1) is absolutely integrable on each path, we may perform these integrations in any order according to Fubini's theorem. Integrating the right side of (5.2.1) therefore gives zero. Finally, because the contours do not intersect, we have sufficient continuity to use the Leibniz rule of integration to exchange the order of differentiation and integration on the left side of (5.2.1). (If Γ_p intersects Γ_q but $\sigma_p \sigma_q > 0$, then the contour integral $\oint \Phi$ is not improper. In this event, we may still use the Leibniz rule to perform this last step as long as we may continuously deform these contours so they do not intersect.) We, therefore, find that the Coulomb gas integral $\mathcal{J} := \oint \Phi$ satisfies the null vector equations (5.2.4).

The Coulomb gas integral $\mathcal{J}(z, u)$ satisfies the following system of differential equations. For each $j = 1, \ldots, n$, the function $\mathcal{J}(z, u)$ satisfies the null vector equation:

$$\left[\frac{\kappa}{4}\partial_j^2 + \sum_{k \neq j}^{n} \left(\frac{\partial_k}{z_k - z_j} - \frac{(6-\kappa)/2\kappa}{(z_k - z_j)^2}\right) + \frac{\partial_u}{u - z_j} + \frac{\partial_{u^*}}{u^* - z_j} - \frac{\lambda_{(b)}(u)}{(u - z_j)^2} - \frac{\lambda_{(b)}(u^*)}{(u^* - z_j)^2}\right] \mathcal{J}(z, u) = 0. \tag{5.2.4}$$

In addition, $\mathcal{J}$ satisfies the following global conformal Ward identities, as given in

Corollary (5.1.7):

$$
\begin{aligned}
&\left[\sum_{i=1}^{n} \partial_{z_i} + \partial_u + \partial_{u^*}\right] \mathcal{J}(z,u) = 0,\\
&\left[\sum_{i=1}^{n} \left(z_i\partial_{z_i} + \frac{6-\kappa}{2\kappa}\right) + u\partial_u + \lambda_{(b)}(u)u + u^*\partial_{u^*} + \lambda_{(b)}(u^*)u^*\right] \mathcal{J}(z,u) = 0,\\
&\left[\sum_{i=1}^{n} \left(z_i^2\partial_{z_i} + \frac{6-\kappa}{\kappa}z_i\right) + u^2\partial_u + 2\lambda_{(b)}(u)u + (u^*)^2\partial_{u^*} + 2\lambda_{(b)}(u^*)u^*\right] \mathcal{J}(z,u) = 0.
\end{aligned}
\tag{5.2.5}
$$

Here, $\lambda_{(b)}(u)$ and $\lambda_{(b)}(u^*)$ denote the conformal weights of the screening charges located at u and u^*, respectively. We also use the standard notation, for $u = v + iw$, where $v, w \in \mathbb{R}$, the complex derivatives are given by

$$
\partial_u = \frac{1}{2}(\partial_v - i\partial_w), \qquad \partial_{u^*} = \frac{1}{2}(\partial_v + i\partial_w).
$$

Next, we explain how to construct $\mathcal{J}(z,u)$ by choosing the appropriate sets of integration contours. In what follows, we describe the choice of screening charges and contours that give rise to four distinct types of screening solutions; see Theorem 1.2.5. We conjecture that these screening solutions span the full solution space to the null vector equations (5.2.4) and the Ward identities (5.2.5).

To proceed, we begin by introducing the notion of link patterns, which encode the topological types of the integration contours.

To do this, let's begin by defining the link patterns that characterize the topology of integration contours.

Definition 5.2.2 (Radial Link Pattern)**.** *Given* $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$ *on the unit circle, a* radial link pattern *is a homotopy class (up to non-crossing deformation) of non-intersecting curves in the unit disk, consisting of:*

- m links *(or arcs), each connecting a distinct pair of boundary points, and*
- $n - 2m$ rays*, each connecting a boundary point to the origin.*

Such a configuration is called a radial (n,m)-link*, and the set of all such patterns is denoted by* $\mathrm{LP}(n,m)$.

The number of such patterns is given by

$$
|\mathrm{LP}(n,m)| = \binom{n}{m},
$$

Proof. Each radial (n, m)-link pattern corresponds to a configuration of m non-crossing arcs and $n - 2m$ rays connecting marked points $z_1, \ldots, z_n \in \partial\mathbb{D}$ to the origin. We describe such configurations by a class of discrete functions encoding their nesting structure.

Define a function

$$f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}_{\geq 0}$$

satisfying:

- $|f(x+1) - f(x)| = 1$ for all $x \in \mathbb{Z}/n\mathbb{Z}$,
- $\min f = 0$.

Such functions are called periodic Dyck walks, and they encode a height profile along the circle, rising and falling in steps of ± 1, returning to the starting height, and remaining non-negative throughout. Each local minimum at height 0 corresponds to a ray (i.e., a line from a boundary point to the origin), while each matching of an up-step followed by a down-step corresponds to an arc.

For a radial (n, m)-link, we require: - Exactly $2m$ of the n positions to participate in arcs (encoded by up/down steps), - The remaining $n - 2m$ steps form rays (flat local minima).

To count such walks: - Choose the m positions (out of n) at which the rays will attach to the origin — each such position corresponds to a local minimum (a peak that immediately rises or falls). - This uniquely determines the link structure (since the rest must form a fixed non-crossing matching of $2m$ points). - Thus, the number of such patterns is given by:

$$|\mathrm{LP}(n, m)| = \binom{n}{m}.$$

□

Definition 5.2.3 (Chordal Link Pattern)**.** *Given* $\mathbf{z} = \{z_1, z_2, \ldots, z_n\}$ *on the real line, a* chordal link pattern *is a homotopy class (up to non-crossing deformation) of non-intersecting curves in the upper half-plane, consisting of:*

- m links*, each connecting a distinct pair of boundary points, and*
- $n - 2m$ rays*, each connecting a boundary point to infinity.*

Such a configuration is called a chordal (n, m)-link*, and the set of all such patterns is denoted by* $\mathrm{LP}(n, m)$.

The number of such patterns is given by

$$|\mathrm{LP}(n, m)| = \binom{n}{m+1} - \binom{n}{m},$$

for all integers $n \geq 2m$.

Proof. Each (n, m)-link corresponds to an increasing path on $\mathbb{Z}^2$ from $(0, 0)$ to $(n - m, m)$, using only steps to the right $(1, 0)$ and upward $(0, 1)$, such that the path never crosses the diagonal $x = y$.

To reach $(n - m, m)$, the path must take its final step from either:

- $(n - m - 1, m)$, via a horizontal step, or
- $(n - m, m - 1)$, via a vertical step.

Since valid paths must stay strictly below the diagonal $x = y$ (except possibly at the start), any valid path to $(n - m, m)$ must be built by extending a valid path to one of these two predecessor points.

Therefore, the number of such paths satisfies the recursion

$$d_{n,m} = d_{n-1,m} + d_{n-1,m-1}.$$

This completes the proof. □

By part (ii) of Theorem 5.2.1, when all screening charges are taken to be $\sigma_i = a$ for $1 \leq i \leq n$, the null vector equations are satisfied provided the screening charges are chosen from the set $\{-2a, 2(a+b)\}$, and the total configuration satisfies the neutrality condition (NC_b). That is, each screening charge σ_k for $k \in \{n + 3, \ldots, n + m + 2\}$ may be assigned either $-2a$ or $2(a + b)$, independently, so long as the total sum of charges is $2b$.

- (Radial ground solutions) In the upper half plane $\mathbb{H}$, we assign charge a to $z_1, z_2, \ldots, z_n$, charge $-2a$ to $\xi_1, \ldots, \xi_m$ and charge $\sigma_u = \sigma_{u^*} = b - \frac{(n-2m)a}{2}$ to marked points u and u^* to maintain neutrality condition (NC_b).

$$\begin{aligned}\Phi_\kappa\left(z_1,\ldots,z_n,\xi_1,\xi_2,\ldots,\xi_m,u\right)=&\prod_{i<j}\left(z_i-z_j\right)^{a^2}\prod_{j<k}\left(z_j-\xi_k\right)^{-2a^2}\prod_{j<k}\left(\xi_j-\xi_k\right)^{4a^2}\\&\prod_j(z_i-u)^{a(b-\frac{(n-2m)a}{2})}\prod_j(z_i-u^*)^{a(b-\frac{(n-2m)a}{2})}\\&\prod_j(\xi_j-u)^{-2a(b-\frac{(n-2m)a}{2})}\prod_j(\xi_j-u^*)^{-2a(b-\frac{(n-2m)a}{2})}.\end{aligned}\tag{5.2.6}$$

In the unit disk $\mathbb{D}$, if we set $u=0$, then we have

$$\begin{aligned}\Phi_\kappa\left(z_1,\ldots,z_n,\xi_1,\xi_2,\ldots,\xi_m\right)=&\prod_{i<j}\left(z_i-z_j\right)^{a^2}\prod_{j<k}\left(z_j-\xi_k\right)^{-2a^2}\prod_{j<k}\left(\xi_j-\xi_k\right)^{4a^2}\\&\prod_j z_i^{a(b-\frac{(n-2m)a}{2})}\prod_j\xi_j^{-2a(b-\frac{(n-2m)a}{2})}\end{aligned}\tag{5.2.7}$$

(1) $(-2a)\cdot a=-\frac{4}{\kappa}$. $\xi_i=z_j$ is a singular point of the type $\left(\xi_i-z_j\right)^{-4/\kappa}$;

(2) $(-2a)\cdot(-2a)=\frac{8}{\kappa}$. $\xi_i=\xi_j$ is a singular point of of the type $(\xi_i-\xi_j)^{\frac{8}{\kappa}}$;

(3) $(-2a)\cdot(b-\frac{(n-2m)a}{2})=\frac{2(n-2m+2)}{\kappa}$. $\xi=u$ and $\xi=u^*$ are singular points of the type $(\xi_i-u)^{\frac{2(n-2m+2)}{\kappa}}$ and $(\xi_i-u^*)^{\frac{2(n-2m+2)}{\kappa}}$.

In this case, for $m\leq\frac{n+2}{2}$ and a (n,m) radial link pattern α, we can choose p non-intersecting Pochhammer contours $C_1,C_2,\ldots,C_m$ surrounding pairs of points (which correspond to links in a radial link pattern); see (5.2.2) to integrate Φ_κ. We obtain

$$\mathcal{J}_\alpha^{(m,n)}(z):=\oint_{C_1}\ldots\oint_{C_m}\Phi_\kappa(z,\xi)d\xi_m\ldots d\xi_1.\tag{5.2.8}$$

In particular, if $m=0$, we call Φ_κ the fermionic ground solution.

Note that the charges at u and u^* are given by $\sigma_u=\sigma_{u^*}=b-\frac{(n-2m)a}{2}$, and thus

$$\lambda_{(b)}(u)=\lambda_{(b)}(u^*)=\frac{(n-2m)^2a^2}{8}-\frac{b^2}{2}=\frac{(n-2m)^2}{4\kappa}-\frac{(\kappa-4)^2}{16\kappa}.$$

The radial ground solution $\mathcal{J}_\alpha^{(m,n)}$ satisfies the null vector equations (5.2.4) and Ward's identities (5.2.5) with above $\lambda_{(b)}(u)$ and $\lambda_{(b)}(u^*)$.

- (Radial excited solutions) In the upper half plane $\mathbb{H}$, we assign charge a to $z_1, z_2, \ldots, z_n$, charge $-2a$ to $\xi_1, \ldots, \xi_m$ and charge $2(a+b)$ to $\zeta_1, \ldots, \zeta_q$. Then, we assign charge $\sigma_u = \sigma_{u^*} = b - \frac{(n-2m)a+2q(a+b)}{2}$ to marked points u and u^* to maintain neutrality condition (NC$_b$).

$$\begin{aligned}
&\Phi_\kappa\left(z_1,\ldots,z_n,\xi_1,\xi_2,\ldots,\xi_m,\zeta_1,\zeta_2,\ldots,\zeta_q,u\right) = \\
&\prod_{i<j}\left(z_i-z_j\right)^{a^2}\prod_{j<k}\left(z_j-\xi_k\right)^{-2a^2}\prod_{j<k}\left(\xi_j-\xi_k\right)^{4a^2} \\
&\prod_{j<k}\left(z_j-\zeta_k\right)^{2a(a+b)}\prod_{j<k}\left(\zeta_j-\zeta_k\right)^{4(a+b)^2} \\
&\prod_{j}(z_i-u)^{a\sigma_u}\prod_{j}(z_i-u^*)^{a\sigma_{u^*}} \\
&\prod_{j}(\xi_j-u)^{-2a\sigma_u}\prod_{j}(\xi_j-u^*)^{-2a\sigma_{u^*}} \\
&\prod_{j}(\zeta_j-u)^{2(a+b)\sigma_u}\prod_{j}(\zeta_j-u^*)^{2(a+b)\sigma_{u^*}}.
\end{aligned} \tag{5.2.9}$$

In the unit disk $\mathbb{D}$, if we set $u=0$, then we have

$$\begin{aligned}
\Phi_\kappa\left(z_1,\ldots,z_n,\xi_1,\xi_2,\ldots,\xi_m\right) = &\prod_{i<j}\left(z_i-z_j\right)^{a^2}\prod_{j<k}\left(z_j-\xi_k\right)^{-2a^2}\prod_{j<k}\left(\xi_j-\xi_k\right)^{4a^2} \\
&\prod_{j} z_i^{a(b-\frac{(n-2m)a}{2})}\prod_{j}\xi_j^{-2a(b-\frac{(n-2m)a}{2})}
\end{aligned} \tag{5.2.10}$$

(1) $(-2a)\cdot a = -\frac{4}{\kappa}$. $\xi_i = z_j$ is a singular point of the type $\left(\xi_i - z_j\right)^{-4/\kappa}$;

(2) $(-2a)\cdot(-2a) = \frac{8}{\kappa}$. $\xi_i = \xi_j$ is a singular point of of the type $(\xi_i-\xi_j)^{\frac{8}{\kappa}}$;

(3) $(-2a)\cdot(b-\frac{(n-2m)a}{2}-q(a+b)) = \frac{2(n-2m+2)}{\kappa}+q$. $\xi=u$ and $\xi=u^*$ are singular points of the type $(\xi_i-u)^{\frac{2(n-2m+2)}{\kappa}+q}$ and $(\xi_i-u^*)^{\frac{2(n-2m+2)}{\kappa}+q}$;

(4) $2(a+b)\cdot(b-\frac{(n-2m)a}{2}-q(a+b)) = \frac{(1-q)\kappa}{4}+\frac{-n+2m-2}{2}$. $\xi=u$ and $\xi=u^*$ are singular points of the type $(\xi_i-u)^{\frac{(1-q)\kappa}{4}+\frac{-n+2m-2}{2}}$ and $(\xi_i-u^*)^{\frac{(1-q)\kappa}{4}+\frac{-n+2m-2}{2}}$.

For $q=1$, $\zeta_1=u$ and $\zeta_1=u^*$ are two singular points of degree $\frac{-n+2m-2}{2}$. We have two choices for screening contours to integrate ζ_1

- n odd, Pochhammer contour $\mathscr{P}(u,u^*)$ surrounding u and u^*, however,

$$\int_{\mathscr{P}(u,u^*)}\Phi_\kappa d\zeta = 0.$$

- n even, the circle $C(0,\epsilon)$ around 0 with radius ϵ, this gives the excited solution

  In this case, for $m \leq \frac{n+2}{2}$ and a (n,m) radial link pattern α, we can choose p non-intersecting Pochhammer contours $C_1, C_2, \ldots, C_m$ surrounding pairs of points (which correspond to links in a radial link pattern) to integrate Φ_κ, we obtain

$$\mathcal{K}_\alpha^{(m,n)}(z) := \oint_{C_1} \cdots \oint_{C_m} \oint_{C(0,\epsilon)} \Phi_\kappa(z, \boldsymbol{\xi}) d\xi_m \ldots d\xi_1 d\zeta_1. \tag{5.2.11}$$

  In particular, if $p = 0$, we call Φ_κ the fermionic excited solution.

Note that the charges at u and u^* are given by $\sigma_u = \sigma_{u^*} = \frac{(2m-n-2)a}{2}$.

$$\lambda_{(b)}(u) = \lambda_{(b)}(u^*) = \frac{(n-2m+\frac{\kappa}{2})^2}{4\kappa} - \frac{(\kappa-4)^2}{16\kappa}$$

The radial excited solution $\mathcal{K}_\alpha^{(m,n)}$ satisfies the null vector equations (5.2.4) and Ward's identities (5.2.5) with above $\lambda_{(b)}(u)$ and $\lambda_{(b)}(u^*)$.

For $q \geq 2$, since u and u^* are the only singular points for screening charges, it is impossible to choose two non-intersecting contours for $\{\zeta_1, \zeta_2, \ldots, \zeta_q\}$.

We refer to it as a *radial excited state* because introducing a screening charge of $-(a+b)$ leaves the conformal weight at z_i unchanged, while producing a Virasoro descendant rather than a new primary field.

- (Radial ground solutions with spin η) In the upper half plane $\mathbb{H}$, we assign charge a to $z_1, z_2, \ldots, z_n$, charge $-2a$ to $\xi_1, \ldots, \xi_m$. Then, we assign charge $\sigma_u = b - \frac{(n-2m)a}{2} - \frac{i\eta a}{2}$, $\sigma_{u^*} = b - \frac{(n-2m)a}{2} + \frac{i\eta a}{2}$ to marked points u and u^* to maintain neutrality condition (NC_b).

$$\begin{aligned}
&\Phi_\kappa\left(z_1, \ldots, z_n, \xi_1, \xi_2, \ldots, \xi_m, u\right) = \\
&\prod_{i<j} \left(z_i - z_j\right)^{a^2} \prod_{j<k} \left(z_j - \xi_k\right)^{-2a^2} \prod_{j<k} \left(\xi_j - \xi_k\right)^{4a^2} \\
&\prod_{j} (z_i - u)^{a(b - \frac{(n-2m)a}{2} - \frac{i\eta a}{2})} \prod_{j} (z_i - u^*)^{a(b - \frac{(n-2m)a}{2} + \frac{i\eta a}{2})} \\
&\prod_{j} (\xi_j - u)^{-2a(b - \frac{(n-2m)a}{2} - \frac{i\eta a}{2})} \prod_{j} (\xi_j - u^*)^{-2a(b - \frac{(n-2m)a}{2} + \frac{i\eta a}{2})}
\end{aligned} \tag{5.2.12}$$

In the unit disk $\mathbb{D}$, if we set $u = 0$, then we have

$$\Phi_\kappa(z_1,\dots,z_n,\xi_1,\xi_2,\dots,\xi_m) = \prod_{i<j}(z_i-z_j)^{a^2}\prod_{j<k}(z_j-\xi_k)^{-2a^2}\prod_{j<k}(\xi_j-\xi_k)^{4a^2} \prod_j z_i^{a(b-\frac{(n-2m)a}{2}-\frac{i\eta a}{2})}\prod_j \xi_j^{-2a(b-\frac{(n-2m)a}{2}+\frac{i\eta a}{2})} \tag{5.2.13}$$

(1) $(-2a)\cdot a = -\frac{4}{\kappa}$. $\xi_i = z_j$ is a singular point of the type $(\xi_i - z_j)^{-4/\kappa}$.

(2) $(-2a)\cdot(-2a) = \frac{8}{\kappa}$. $\xi_i = \xi_j$ is a singular point of of the type $(\xi_i-\xi_j)^{\frac{8}{\kappa}}$

(3) $(-2a)\cdot(b-\frac{(n-2m)a}{2}) = \frac{2(n-2m+2)}{\kappa}$. $\xi = u$ and $\xi = u^*$ are singular points of the type $(\xi_i - u)^{\frac{2(n-2m+2)}{\kappa}}$ and $(\xi_i-u^*)^{\frac{2(n-2m+2)}{\kappa}}$

In this case, for $p \leq \frac{n+2}{2}$ and a (n,p) radial link pattern α, we can choose p non-intersecting Pochhammer contours $C_1, C_2, \dots, C_p$ surrounding pairs of points (which correspond to links in a radial link pattern), see (5.2.2) to integrate Φ_κ, we obtain

$$\mathcal{J}_\alpha^{(m,n,\eta)}(z) := \oint_{C_1}\dots\oint_{C_m}\Phi_\kappa(z,\xi)d\xi_m\dots d\xi_1. \tag{5.2.14}$$

Note that the charges at u and u^* are given by $\sigma_u = b - \frac{(n-2m)a}{2} - \frac{i\eta a}{2}$, $\sigma_{u^*} = b - \frac{(n-2m)a}{2} + \frac{i\eta a}{2}$.

$$\lambda_{(b)}(u) = \frac{(n-2m+i\eta)^2a^2}{8} - \frac{b^2}{2} = \frac{(n-2m+i\eta)^2}{4\kappa} - \frac{(\kappa-4)^2}{16\kappa}$$

$$\lambda_{(b)}(u^*) = \frac{(n-2m-i\eta)^2a^2}{8} - \frac{b^2}{2} = \frac{(n-2m-i\eta)^2}{4\kappa} - \frac{(\kappa-4)^2}{16\kappa}$$

The radial ground solution with spin η, $\mathcal{J}_\alpha^{(m,n,\eta)}$ satisfies the null vector equations (5.2.4) and Ward's identities (5.2.5) with above $\lambda_{(b)}(u)$ and $\lambda_{(b)}(u^*)$

As shown in theorem (5.2.1), if we attach charge a for $z_1,\dots,z_{n-1}$ and $2b-a$ for z_c, where $n = 2k$. This corresponds to the charge distribution for multiple chordal SLE(κ) as discussed in S. Flores and Kleban (2015b). In this case, we can only assign charge $-2a$ to the $k-1$ screening charges and assign no spin at u, u^*; otherwise, the null vector equation at z_c will generally not be satisfied.

- (Chordal solutions) In the upper half plane $\mathbb{H}$, we assign charge a to $z_1, z_2, \ldots, z_{n-1}$ and charge $2b - a$ to z_c, charge $-2a$ to $\xi_1, \ldots, \xi_m$, where $n = 2k$, $m = k - 1$, then assign the charge $\sigma_u = \sigma_{u^*} = 0$.

$$\Phi_\kappa(z_1, \ldots, z_{n-1}, z_c, \xi_1, \ldots, \xi_m, u) = \prod_{i<j} \left(z_i - z_j\right)^{a^2} \prod_{j<k} \left(z_j - \xi_k\right)^{-2a^2} \prod_{j<k} \left(\xi_j - \xi_k\right)^{4a^2} \prod_i (z_i - z_c)^{a(2b-a)} \prod_j (\xi_j - z_c)^{-2a(2b-a)} \tag{5.2.15}$$

  - $(-2a) \cdot a = -\frac{4}{\kappa}$. $\xi_i = z_j$ is a singular point of the type $\left(\xi_i - z_j\right)^{-4/\kappa}$;
  - $(-2a) \cdot (2b - a) = \frac{12}{\kappa} - 2$. $\xi_i = z_c$ is a singular point of the type $(\xi_i - z_c)^{\frac{12}{\kappa}-2}$;
  - $(-2a) \cdot (-2a) = \frac{8}{\kappa}$. $\xi_i = \xi_j$ is a singular point of the type $(\xi_i - \xi_j)^{\frac{8}{\kappa}}$.

  In this case, for a $(2k, k)$ chordal link pattern, we choose $m = k - 1$ non-intersecting Pochhammer contours $C_1, C_2, \ldots, C_{k-1}$ surrounding pairs of points except z_c (which correspond to links in a chordal link pattern not connected to z_c) see S. Flores and Kleban (2015b) for detailed explanation. We obtain:

$$\mathcal{L}_\alpha^n(z) := \oint_{C_1} \ldots \oint_{C_{k-1}} \Phi_\kappa(z, \xi) d\xi_{k-1} \ldots d\xi_1. \tag{5.2.16}$$

Note that the charges at u and u^* are given by $\sigma_u = \sigma_{u^*} = 0$

$$\lambda_{(b)}(u) = \lambda_{(b)}(u^*) = 0.$$

The chordal solution $\mathcal{J}_\alpha^{(m,n)}$ satisfies the null vector equations (5.2.4) and Ward's identities (5.2.5) with above $\lambda_{(b)}(u)$ and $\lambda_{(b)}(u^*)$.

We can also construct the Coulomb gas integral solutions in angular coordinates. Consider the following Coulomb gas correlation in the angular coordinate

$$\Phi(z_1, z_2, \ldots, z_{n+m}) = \prod_{1 \le j < k \le n+m} \left(\sin \frac{z_j - z_k}{2}\right)^{\sigma_j \sigma_k}.$$

Then, similar computations show the following.

Theorem 5.2.4. *If we choose* $\sigma_j = a = \sqrt{\frac{2}{\kappa}}$, $\lambda_j = \frac{a^2}{2} - ab = \frac{6-\kappa}{2\kappa}$, $1 \le j \le n$ *then we have*

$$\begin{aligned}
&\left[\frac{\kappa}{2}\partial_j^2 + \sum_{k\neq j}\left(\cot\left(\frac{z_k - z_j}{2}\right)\partial_k - \frac{(6-\kappa)/2\kappa}{2\sin^2\left(\frac{z_k - z_j}{2}\right)}\right)\right]\Phi\left(z_1, z_2, \ldots, z_{n+m+2}\right)\\
&= \sum_{k=n+1}^{n+m}\partial_k\left(\cot\left(\frac{z_k - z_j}{2}\right)\Phi\left(z_1, z_2, \ldots, z_{n+m+2}\right)\right)\\
&-\left[\frac{1}{2\kappa}\left(n - 2p + \frac{\kappa}{2}q\right)^2 - \frac{1}{2\kappa}\right]\Phi\left(z_1, z_2, \ldots, z_{n+m+2}\right)
\end{aligned} \tag{5.2.17}$$

for all $j \in \{1, 2, \ldots, n\}$. *The number of screening charges* $\sigma_k = 2a$ *is given by* p, *and the number of screening charges* $\sigma_k = 2(a + b)$ *is given by* q, *with* $m = p + q$.

Now, we Coulomb gas integral solutions based on the theorem (5.2.4).

- Radial ground solutions:

$$\Phi_\kappa(\boldsymbol{\theta}, \boldsymbol{\zeta}) = \prod_{1\le i<j\le n}\left(\sin\frac{\theta_i - \theta_j}{2}\right)^{a^2}\prod_{1\le i<j\le m}\left(\sin\frac{\zeta_i - \zeta_j}{2}\right)^{4a^2}\prod_{i=1}^{n}\prod_{j=1}^{m}\left(\sin\frac{\theta_i - \zeta_j}{2}\right)^{-2a^2}. \tag{5.2.18}$$

  In this case, for $m \le \frac{n+2}{2}$ and a (n, m) radial link pattern α, we can choose p non-intersecting Pochhammer contours $C_1, C_2, \ldots, C_m$ surrounding pairs of points (which correspond to links in a radial link pattern); see (5.2.2) to integrate Φ_κ. We obtain

$$\mathcal{J}_\alpha^{(m,n)}(\boldsymbol{\theta}) := \oint_{C_1}\ldots\oint_{C_m}\Phi_\kappa(\boldsymbol{\theta}, \boldsymbol{\zeta})d\zeta_m \ldots d\zeta_1. \tag{5.2.19}$$

  By integration formula (5.2.4), $\mathcal{J}_\alpha^{(m,n)}(\boldsymbol{\theta})$ satisfies the null vector equations (1.2.3) with constant

$$h = \frac{(6-\kappa)(\kappa-2)}{8\kappa} - \lambda_b(0) - \overline{\lambda_b(0)} = \frac{1-(n-2m)^2}{2\kappa}$$

  and the conformal dimension at 0 is given by

$$\lambda_b(0) = \frac{(n-2m)^2a^2}{8} - \frac{b^2}{2} = \frac{(n-2m)^2}{4\kappa} - \frac{(\kappa-4)^2}{16\kappa}.$$

  The rotation constant $\omega = 0$

$$\sum_{j=1}^{n}\partial_j\mathcal{J}_\alpha^{(m,n)}(\boldsymbol{\theta}) = 0.$$

- Radial excited solutions:

$$\Phi_\kappa(\boldsymbol{\theta},\boldsymbol{\zeta}) = \prod_{1\le i<j\le n}\left(\sin\frac{\theta_i-\theta_j}{2}\right)^{a^2}\prod_{1\le i<j\le m}\left(\sin\frac{\zeta_i-\zeta_j}{2}\right)^{4a^2}\prod_{i=1}^{n}\prod_{j=1}^{m}\left(\sin\frac{\theta_i-\zeta_j}{2}\right)^{-2a^2} \cdot \prod_{i=1}^{n}\left(\sin\frac{\theta_i-\omega}{2}\right)^{\frac{(2m-n-2)}{2}} \tag{5.2.20}$$

In this case, for $m \le \frac{n+2}{2}$ and a (n,m) radial link pattern α, we can choose p non-intersecting Pochhammer contours $C_1, C_2, \ldots, C_m$ surrounding pairs of points (which correspond to links in a radial link pattern) to integrate $\zeta_1, \zeta_2, \ldots, \zeta_m$ and a vertical line from A to $A+2\pi i$ to integrate ω (which corresponds to a circle surrounds the origin), we obtain

$$\mathcal{K}_\alpha^{(m,n)}(\boldsymbol{\theta}) := \oint_{C_1}\ldots\oint_{C_m}\int_A^{A+2\pi i}\Phi_\kappa(\boldsymbol{\theta},\boldsymbol{\zeta})d\zeta_m\ldots d\zeta_1 d\omega. \tag{5.2.21}$$

By integration formula (5.2.4), $\mathcal{J}_\alpha^{(m,n)}(\boldsymbol{\theta})$ satisfies the null vector equations (1.2.3) with constant

$$h = \frac{(6-\kappa)(\kappa-2)}{8\kappa} - \lambda_b(0) - \overline{\lambda_b(0)} = \frac{1-(n-2m+\frac{\kappa}{2})^2}{2\kappa}$$

and conformal dimension at 0 is given by

$$\lambda_{(b)}(0) = \frac{(n-2m)^2}{4\kappa} - \frac{(\kappa-4)^2}{16\kappa}.$$

The rotation constant $\omega = 0$,

$$\sum_{j=1}^{n}\partial_j\mathcal{K}_\alpha^{(m,n)}(\boldsymbol{\theta}) = 0.$$

- Radial ground solutions with spin η:

$$\Phi_\kappa(\boldsymbol{\theta},\boldsymbol{\zeta}) = \prod_{1\le i<j\le n}(\sin\frac{\theta_i-\theta_j}{2})^{a^2}\prod_{1\le i<j\le m}(\sin\frac{\zeta_i-\zeta_j}{2})^{4a^2}\prod_{i=1}^{n}\prod_{j=1}^{m}\left(\sin\frac{\theta_i-\zeta_j}{2}\right)^{-2a^2} \prod_{i=1}^{n}e^{\frac{\eta a^2}{2}\theta_i}\prod_{j=1}^{m}e^{-\eta a^2\zeta_j}. \tag{5.2.22}$$

In this case, for $p \le \frac{n+2}{2}$ and a given (n,p) radial link pattern α, we can choose p non-intersecting Pochhammer contours $C_1, C_2, \ldots, C_p$, each surrounding a pair of points corresponding to a link in the radial link pattern (see (5.2.2)). Integrating Φ_κ along these contours, we obtain

$$\mathcal{J}_\alpha^{(m,n,\eta)}(\boldsymbol{\theta}) := \oint_{C_1} \dots \oint_{C_m} \Phi_\kappa(\boldsymbol{\theta}, \boldsymbol{\zeta})\, d\zeta_m \dots d\zeta_1. \tag{5.2.23}$$

By integration formula (5.2.4), $\mathcal{J}_\alpha^{(m,n,\eta)}(\boldsymbol{\theta})$ satisfies the null vector equations (1.2.3) with constant

$$h = \frac{(6-\kappa)(\kappa-2)}{8\kappa} - \lambda_b(0) - \overline{\lambda_b(0)} = -\frac{(n-2m)^2}{2\kappa} + \frac{1+\eta^2}{2\kappa}$$

and conformal dimension at 0 is given by

$$\lambda_{(b)}(0) = \frac{(n-2m+i\eta)^2}{4\kappa} - \frac{(\kappa-4)^2}{16\kappa}.$$

The rotation constant $\omega = \frac{\eta(n-2m)}{\kappa}$,

$$\sum_{j=1}^{n} \partial_j \mathcal{J}_\alpha^{(m,n,\eta)}(\boldsymbol{\theta}) = \frac{\eta(n-2m)}{\kappa} \mathcal{J}_\alpha^{(m,n,\eta)}(\boldsymbol{\theta}).$$

- Chordal solutions, for $n = 2k$ and $m = k-1$:

$$\begin{aligned}\Phi_\kappa(\theta_1, \dots, \theta_{n-1}, \theta_c, \zeta_1, \dots, \zeta_m) = &\prod_{1\le i<j\le n} \left(\sin\frac{\theta_i-\theta_j}{2}\right)^{a^2} \prod_{1\le i<j\le m} \left(\sin\frac{\zeta_i-\zeta_j}{2}\right)^{4a^2} \\ &\prod_{i=1}^{n}\prod_{j=1}^{m} \left(\sin\frac{\theta_i-\zeta_j}{2}\right)^{-2a^2} \prod_{i=1}^{n-1} \left(\sin\frac{\theta_i-\omega}{2}\right)^{a(2b-a)} \\ &\prod_{j=1}^{m} \left(\sin\frac{\theta_i-\omega}{2}\right)^{-2a(2b-a)}.\end{aligned} \tag{5.2.24}$$

In this case, for a $(2k,k)$ chordal link pattern, we choose $m = k-1$ non-intersecting Pochhammer contours $C_1, C_2, \dots, C_{k-1}$ surrounding pairs of points except z_c (which correspond to links in a chordal link pattern not connected to z_c); see S. Flores and Kleban (2015b) for detailed explanation. We obtain

$$\mathcal{L}_\alpha^n(\boldsymbol{\theta}) := \oint_{C_1} \dots \oint_{C_{k-1}} \Phi_\kappa(\boldsymbol{\theta}, \boldsymbol{\zeta}) d\zeta_{k-1} \dots d\zeta_1. \tag{5.2.25}$$

By rewriting the chordal null vector equations in angular coordinate, $\mathcal{J}_\alpha(\boldsymbol{\theta})$ satisfies the null vector equations (1.2.3) with constant

$$h = \frac{(6-\kappa)(\kappa-2)}{8\kappa} - \lambda_b(0) - \overline{\lambda_b(0)} = \frac{(6-\kappa)(\kappa-2)}{8\kappa}$$

and conformal dimension at 0 is given by

$$\lambda_b(0) = 0.$$

The rotation constant $\omega = 0$,

$$\sum_{j=1}^{n} \partial_j \mathcal{L}_\alpha^n(\boldsymbol{\theta}) = 0.$$

Remark 5.2.5. In Y. Wang and Wu (2024), the authors define the following Chordal SLE weighted by the conformal radius. The corresponding partition function is given by

$$\mathcal{Z}_\alpha(\theta_1, \theta_2) = (\sin(\theta/2))^{\frac{\kappa-6}{\kappa}} \mathbb{E}_\theta \left[\mathrm{CR}(\mathbb{D}\backslash\gamma)^{-\alpha}\right],$$

where $\theta = \theta_1 - \theta_2$, $\mathrm{CR}(\mathbb{D} \setminus \gamma)$ denotes the conformal radius of the domain $\mathbb{D} \setminus \gamma$ as seen from the origin, and $\mathbb{E}_\theta$ denotes the expectation with respect to the law of γ.

It is shown in Schramm, Sheffield, and D. B. Wilson (2009) and Y. Wang and Wu (2024) that this partition function satisfies the null vector equation (3.2.1) with constant

$$h = \frac{(6-\kappa)(\kappa-2)}{8\kappa} - \alpha$$

and is rotation invariant.

For radial ground solution with spin η, if $n = 2m$ (n is the number of growth points and m the number of screening charges) then in fact $\mathcal{J}^{(m,n,\eta)}$ is rotation invariant and independent of the value of η :

$$\sum_{j=1}^{n} \partial_j \mathcal{J}^{(m,n,\eta)}(\boldsymbol{\theta}) = \frac{\eta(n-2m)}{\kappa} \mathcal{J}^{(m,n,\eta)}(\boldsymbol{\theta}) = 0.$$

This function also satisfies the null vector equation (3.2.1) with conformal dimension:

$$\lambda_b(0) = \frac{(n-2m+i\eta)^2}{4\kappa} - \frac{(\kappa-4)^2}{16\kappa},$$

$$h = \frac{(6-\kappa)(\kappa-2)}{8\kappa} - \lambda_b(0) - \overline{\lambda_b(0)} = -\frac{(n-2m)^2}{2\kappa} + \frac{1+\eta^2}{2\kappa} = \frac{1+\eta^2}{2\kappa}.$$

In the special case where $n = 2$ and $m = 1$, there are two distinct topological link patterns that correspond to two screening solutions denoted by $\mathcal{Z}_1^\eta(\theta)$ and $\mathcal{Z}_2^\eta(\theta) = \mathcal{Z}_1^\eta(2\pi - \theta)$. Then $\mathcal{Z}_1^\eta + \mathcal{Z}_2^\eta$ is rotation invariant and interchangeable.

We match the constant by setting

$$\frac{(6-\kappa)(\kappa-2)}{8\kappa} - \alpha = \frac{1}{2\kappa} + \frac{\eta^2}{2\kappa}.$$

The uniqueness lemma (lemma A1 in Y. Wang and Wu (2024)) implies that there exists a constant c:

$$\mathcal{Z}_1^\eta + \mathcal{Z}_2^\eta = c\mathcal{Z}_\alpha.$$

Chapter 6

MULTIPLE RADIAL SLE(0) SYSTEM

6.1 Classical Limit of the Multiple Radial SLE(κ) System

In this section, we construct multiple radial SLE(0) systems as classical limits of the multiple radial SLE(κ) systems in the regime $\kappa \to 0$. Our construction is self-contained and does not rely on the rigorous resolution of the limiting procedure; rather, it is motivated by variational principles arising from the Coulomb gas formalism and the method of steepest descent.

A key object in this setting is the *stationary relation*, which emerges naturally from the asymptotic normalization of partition functions. For a multiple radial SLE(κ) system with n marked boundary points $z_j = e^{i\theta_j}$ on $\partial\mathbb{D}$, the drift term $b_j(\boldsymbol{\theta})$ of the driving function satisfies

$$b_j(\boldsymbol{\theta}) = \kappa \frac{\partial}{\partial \theta_j} \log \mathcal{Z}(\boldsymbol{\theta}),$$

where $\mathcal{Z}(\boldsymbol{\theta})$ is a positive solution to the system of null vector equations (see equation (1.2.3)).

To obtain a meaningful limit as $\kappa \to 0$, one must suitably renormalize the partition function. For a well-chosen $\mathcal{Z}(\boldsymbol{\theta})$, we expect the limit $\mathcal{Z}(\boldsymbol{\theta})^\kappa$ to exist and be finite as $\kappa \to 0$.

Recall that the Coulomb gas integral solutions associated with a link pattern α are given by

$$\mathcal{J}_\alpha(\boldsymbol{\theta}) = \oint_{C_1} \cdots \oint_{C_m} \Phi_\kappa(\boldsymbol{\theta}, \boldsymbol{\zeta})\, d\zeta_m \cdots d\zeta_1, \tag{6.1.1}$$

where Φ_κ is the SLE(κ) master function (see definition (5.2.18)), and the contours $C_1, \ldots, C_m$ are non-intersecting Pochhammer contours. The partition function $\mathcal{Z}(\boldsymbol{\theta})$ is a linear combination of such integrals.

Applying the method of steepest descent heuristically, we consider the asymptotic behavior:

$$\lim_{\kappa \to 0} \mathcal{Z}(\boldsymbol{\theta})^\kappa = \lim_{\kappa \to 0} \left(\oint_{C_1} \cdots \oint_{C_m} \Phi(\boldsymbol{\theta}, \boldsymbol{\zeta})^{\frac{1}{\kappa}}\, d\boldsymbol{\zeta} \right)^\kappa, \tag{6.1.2}$$

where $\Phi(\boldsymbol{\theta}, \boldsymbol{\zeta})$ is the SLE(0) master function (see definition (1.3.9)). In this limit, the integral is asymptotically dominated by the contribution from critical points of $\Phi(\boldsymbol{\theta}, \boldsymbol{\zeta})$, i.e., the points where the gradient with respect to $\boldsymbol{\zeta}$ vanishes.

This leads to the following conjecture, which provides the foundation for the deterministic structure of the SLE(0) system:

Conjecture 6.1.1. Let α be a link pattern and $\mathcal{Z}_\alpha(\boldsymbol{\theta})$ the corresponding pure partition function. Then, in the classical limit $\kappa \to 0$, the quantity $\mathcal{Z}_\alpha(\boldsymbol{\theta})^\kappa$ converges to the evaluation of the master function at a critical point:

$$\lim_{\kappa \to 0} \mathcal{Z}_\alpha(\boldsymbol{\theta})^\kappa = \Phi(\boldsymbol{\theta}, \boldsymbol{\zeta}), \tag{6.1.3}$$

where $\boldsymbol{\zeta}$ is a critical point of the SLE(0) master function $\Phi(\boldsymbol{\theta}, \boldsymbol{\zeta})$.

6.2 Stationary Relations Imply Commutation Relations in the $\kappa = 0$ Case

Our construction of the multiple radial SLE(0) system treats the positions of the screening charges $\boldsymbol{\xi}$ and the marked boundary points $\boldsymbol{z}$ as part of a coupled dynamical system. In this formulation, the stationary relations are imposed as constraints on the initial configuration of the screening charges.

In this section, we show that the stationary relations determine a partition function $\mathcal{Z}(\boldsymbol{\theta})$, depending only on the boundary data $\boldsymbol{\theta}$, and that the associated drift vector field $U_j = \partial_j \log \mathcal{Z}$ satisfies the $\kappa = 0$ null vector equations as well as the conformal Ward identities. Consequently, the induced evolution of $\boldsymbol{\theta}$ defines a dynamical system compatible with the structure of multiple SLE(0) as specified in Definition (1.3.1).

Theorem 6.2.1. *Let $\boldsymbol{z} = \{z_1, \ldots, z_n\} \subset \partial\mathbb{D}$ be distinct boundary points, and let $\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_m\} \subset \mathbb{D}$ denote the positions of screening charges. In angular coordinates, write $z_j = e^{i\theta_j}$ and $\xi_k = e^{i\zeta_k}$ for $j = 1, \ldots, n$, $k = 1, \ldots, m$. Assume that $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\theta})$ is a smooth solution of the stationary relations.*

Define the partition function

$$\mathcal{Z}(\boldsymbol{\theta}) := \prod_{1 \le j < k \le n} \sin^2\left(\frac{\theta_j - \theta_k}{2}\right) \prod_{1 \le s < t \le m} \sin^8\left(\frac{\zeta_s(\boldsymbol{\theta}) - \zeta_t(\boldsymbol{\theta})}{2}\right) \prod_{j=1}^{n} \prod_{l=1}^{m} \sin^{-4}\left(\frac{\theta_j - \zeta_l(\boldsymbol{\theta})}{2}\right).$$

Then $\mathcal{Z}(\boldsymbol{\theta})$ is smooth, strictly positive, and invariant under global rotation.

Define the drift vector field $\mathcal{U} = \log \mathcal{Z}(\boldsymbol{\theta})$, and let

$$U_j := \frac{\partial \mathcal{U}}{\partial \theta_j} = \sum_{k \neq j} \cot\left(\frac{\theta_j - \theta_k}{2}\right) - 2 \sum_{l=1}^{m} \cot\left(\frac{\theta_j - \zeta_l(\boldsymbol{\theta})}{2}\right).$$

Then each U_j is real-valued and satisfies the following second-order differential identity:

$$\frac{1}{2}U_j^2 + \sum_{k\neq j} \cot\left(\frac{\theta_k - \theta_j}{2}\right) U_k - \sum_{k\neq j} \frac{3}{2\sin^2\left(\frac{\theta_j-\theta_k}{2}\right)} = -\frac{(2m-n)^2}{2} + \frac{1}{2}. \tag{6.2.1}$$

Theorem 6.2.2 (Ward Identity)**.** *The drift components satisfy the constraint*

$$\sum_{j=1}^{n} U_j = 0.$$

Proof of Theorem 6.2.1. The function $\mathcal{Z}(\boldsymbol{\theta})$ is manifestly positive for distinct real θ_j, and the $\zeta_k(\boldsymbol{\theta})$ occurring in complex conjugate pairs preserve real-valuedness of the logarithmic derivative.

Smoothness follows from the fact that $\boldsymbol{\zeta}(\boldsymbol{\theta})$ solves the stationary relations, and hence depends smoothly on $\boldsymbol{\theta}$ by the implicit function theorem. Direct computation of $\partial_j \log \mathcal{Z}$ yields:

$$\begin{aligned}\partial_j \log \mathcal{Z} &= \sum_{k\neq j} \cot\left(\frac{\theta_j - \theta_k}{2}\right) + 2\sum_{i=1}^{m} \cot\left(\frac{\zeta_i - \theta_j}{2}\right) \\ &\quad + 2\sum_{k=1}^{n}\sum_{l=1}^{m} \cot\left(\frac{\theta_k - \zeta_l}{2}\right)\frac{\partial \zeta_l}{\partial \theta_j} + 4\sum_{1\leq l<s\leq m} \cot\left(\frac{\zeta_l - \zeta_s}{2}\right)\left(\frac{\partial \zeta_l}{\partial \theta_j} - \frac{\partial \zeta_s}{\partial \theta_j}\right).\end{aligned}$$

Applying the stationary relation

$$\sum_{k=1}^{n} \cot\left(\frac{\theta_k - \zeta_l}{2}\right) = 2\sum_{s\neq l} \cot\left(\frac{\zeta_l - \zeta_s}{2}\right),$$

one sees that the last two terms cancel, yielding $\partial_j \log \mathcal{Z} = U_j$ as claimed.

To derive the null vector identity (6.2.1), define

$$u_j := U_j - \sum_{k\neq j} \cot\left(\frac{\theta_j - \theta_k}{2}\right) = -2\sum_{l=1}^{m} \cot\left(\frac{\theta_j - \zeta_l}{2}\right).$$

A lengthy but straightforward computation (using trigonometric identities and the stationary relation again) shows that:

$$\frac{1}{2}U_j^2 + \sum_{k\neq j} \cot\left(\frac{\theta_k - \theta_j}{2}\right) U_k - \sum_{k\neq j} \frac{3}{2\sin^2\left(\frac{\theta_j-\theta_k}{2}\right)} = -\frac{(2m-n)^2}{2} + \frac{1}{2}.$$

□

Proof of Theorem 6.2.2. Summing over j yields:

$$\sum_{j=1}^{n} U_j = \sum_{j\neq k} \cot\left(\frac{\theta_j - \theta_k}{2}\right) - 2\sum_{j=1}^{n}\sum_{l=1}^{m} \cot\left(\frac{\theta_j - \zeta_l}{2}\right).$$

The first term vanishes by antisymmetry. For the second term, switching the order of summation and using the stationary relation again, we find:

$$\sum_{j=1}^{n} U_j = -4 \sum_{1\leq k<l\leq m} \cot\left(\frac{\zeta_k - \zeta_l}{2}\right) = 0,$$

since each term appears with opposite sign in the pair (k, l) and (l, k). □

6.3 Residue-free quadratic differentials with prescribed zeros

The locus of real rational functions characterizes the traces of multiple chordal SLE(0) systems. However, in the radial case, rational functions alone are insufficient to fully describe these traces.

To address this limitation, we introduce an equivalence class of residue-free quadratic differentials (Definition 1.3.6), denoted by $\mathcal{QD}(\boldsymbol{z})$. This extended class is designed to capture the behavior near the origin.

Definition 1 (Restatement of definition (1.3.6))**.** *Let $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$ be distinct points on the unit circle. The class of quadratic differentials, denoted by $\mathcal{QD}(\boldsymbol{z})$.*

1. *symmetric under the involution $z^* = \frac{1}{\bar{z}}$, meaning*

$$\overline{Q(z^*)}\overline{(dz^*)^2} = Q(z)dz^2.$$

2. *distinct zeros at $\{z_1, z_2, \ldots, z_n\}$, each of order 2.*

3. *distinct finite poles at $\{\xi_1, \ldots, \xi_m\}$, each of order 4, and the residues vanish (Residue-free condition):*

$$Res_{\xi_j}(\sqrt{Q(z)}dz) = 0, \quad for\ j = 1, \ldots, m.$$

4. *poles of order $n + 2 - 2m$ at the marked points 0 and ∞. This ensures the total difference between the number of zeros and poles is -4.*

Here, the poles $\{\xi_1, \ldots, \xi_m\}$ are finite, meaning they do not coincide with 0 or ∞.

A key analytical condition is that the associated differential $\sqrt{Q(z)}\,dz$ is residue-free at each pole, which turns out to be equivalent to the stationary relations introduced in Section 6.2.

Theorem 6.3.1 (Stationary Relations and Residue-Free Condition)**.** *The following statements are equivalent:*

1. *The points $\boldsymbol{\xi}$ are symmetric under the involution $z^* = \frac{1}{\bar{z}}$, and the zeros $\boldsymbol{z}$ on the unit circle satisfy the stationary relations.*

2. *There exists a quadratic differential $Q(z)dz^2 \in \mathcal{QD}(\boldsymbol{z})$ with zeros at $\boldsymbol{z}$ and poles at $\boldsymbol{\xi}$.*

Proof of Theorem 6.3.1. We prove the equivalence by analyzing the structure of the associated quadratic differential. The result follows from the following structural lemma.

Lemma 6.3.2. *Let $n \geq 1$ and $\boldsymbol{z} = \{z_1, \dots, z_n\} \subset \partial\mathbb{D}$ be distinct points on the unit circle. Then the following statements hold:*

(i) Up to a real constant multiple, any $Q(z)\,dz^2 \in \mathcal{QD}(\boldsymbol{z})$ admits the factorized form

$$Q(z) = \frac{\prod_{k=1}^m \xi_k^2}{\prod_{j=1}^n z_j} \cdot z^{2m-n-2} \cdot \frac{\prod_{j=1}^n (z - z_j)^2}{\prod_{k=1}^m (z - \xi_k)^4},$$

where $\boldsymbol{\xi} = \{\xi_1, \dots, \xi_m\}$ consists of distinct points in $\mathbb{C} \setminus \{0, \infty\}$ symmetric under inversion $z \mapsto 1/\bar{z}$.

(ii) The square root $\sqrt{Q(z)}$ takes the form

$$\sqrt{Q(z)} = \frac{\prod_{k=1}^m \xi_k}{\sqrt{\prod_{j=1}^n z_j}} \cdot z^{m-\frac{n}{2}-1} \cdot \frac{\prod_{j=1}^n (z - z_j)}{\prod_{k=1}^m (z - \xi_k)^2}.$$

If the poles ξ_k are pairwise distinct, then $\sqrt{Q(z)}$ has a Laurent expansion near each ξ_k:

$$\sqrt{Q(z)} = \frac{A_k}{(z - \xi_k)^2} + \frac{B_k}{z - \xi_k} + \textit{(holomorphic terms)},$$

where

$$A_k = \xi_k^{m-\frac{n}{2}-1} \cdot \frac{\prod_{j=1}^n (\xi_k - z_j)}{\prod_{l \neq k} (\xi_k - \xi_l)^2}, \tag{6.3.1}$$

$$B_k = A_k \cdot \left(\sum_{j=1}^n \frac{1}{\xi_k - z_j} - 2 \sum_{l \neq k} \frac{1}{\xi_k - \xi_l} - \frac{\frac{n}{2} - m + 1}{\xi_k} \right). \tag{6.3.2}$$

(iii) *The condition that $\sqrt{Q(z)}\, dz$ has vanishing residue at each pole ξ_k is equivalent to $B_k = 0$ for all $k = 1, \ldots, m$, which in turn is equivalent to the stationary relations.*

Proof of Lemma 6.3.2. (i) The global structure of $Q(z)\, dz^2$ is determined by its zeros and poles: - double zeros at $\boldsymbol{z}$, - poles of order four at $\boldsymbol{\xi}$, - and behavior at $z = 0$ and $z = \infty$ ensuring that the total degree of the meromorphic differential on $\widehat{\mathbb{C}}$ is -4.

Thus, we may write:

$$Q(z) = \lambda z^b \cdot \frac{\prod_{j=1}^n (z - z_j)^2}{\prod_{k=1}^m (z - \xi_k)^4}.$$

The involution symmetry of Q under $z \mapsto 1/\bar{z}$ implies

$$Q(z) = \overline{Q\left(\frac{1}{\bar{z}}\right)} \cdot z^{-4}.$$

Computing both sides, we find

$$Q(z) = \overline{\lambda} z^{4m-2n-b-4} \cdot \frac{\prod_{j=1}^n (1 - \overline{z_j} z)^2}{\prod_{k=1}^m (1 - \overline{\xi_k} z)^4}.$$

Matching powers of z gives $b = 2m - n - 2$. Comparing constants yields

$$\lambda = (\text{real constant}) \cdot (-1)^{2m-n-1} \cdot \frac{\prod_{k=1}^m \xi_k^2}{\prod_{j=1}^n z_j}.$$

(ii) From the explicit form of $Q(z)$, the square root is:

$$\sqrt{Q(z)} = C \cdot z^{m-\frac{n}{2}-1} \cdot \frac{\prod_{j=1}^n (z - z_j)}{\prod_{k=1}^m (z - \xi_k)^2},$$

with constant $C = \prod \xi_k / \sqrt{\prod z_j}$.

Near each pole ξ_k, the function $\sqrt{Q(z)}$ has a second-order pole and therefore expands as:

$$\sqrt{Q(z)} = \frac{A_k}{(z-\xi_k)^2} + \frac{B_k}{z-\xi_k} + g(z),$$

where g is holomorphic near ξ_k. This expansion implies:

$$\sqrt{Q(z)}(z-\xi_k)^2 = A_k + B_k(z-\xi_k) + g(z)(z-\xi_k)^2.$$

Differentiating both sides and evaluating at $z = \xi_k$ gives:

$$B_k = \frac{d}{dz}\left[\sqrt{Q(z)}(z-\xi_k)^2\right]\bigg|_{z=\xi_k}.$$

Substituting the explicit form of $\sqrt{Q(z)}$ yields the claimed expressions for A_k and B_k.

(iii) By definition, $\operatorname{Res}_{\xi_k}\sqrt{Q(z)}\,dz = B_k$. Hence, the condition that $Q(z)\,dz^2$ is residue-free is equivalent to $B_k = 0$ for all k. Equation (6.3.1) shows that these conditions coincide with the stationary relations.

□

Combining items (ii) and (iii), we conclude that the residue-free condition is equivalent to the stationary relations, and hence Theorem 6.3.1 follows. □

To further understand the structure of residue-free quadratic differentials in the class $\mathcal{QD}(\boldsymbol{z})$, we associate to each such differential a multivalued analytic function $F(z)$ with nontrivial monodromy at the origin.

For any $Q(z)\,dz^2 \in \mathcal{QD}(\boldsymbol{z})$, there exists a (locally defined) primitive $F(z)$ of $\sqrt{Q(z)}$, unique up to a real additive constant, which is involution symmetric and meromorphic away from $z = 0$. The nontriviality of the monodromy around $z = 0$ reflects the multivalued nature of $F(z)$.

Theorem 6.3.3. *Let $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\} \subset \partial\mathbb{D}$ be distinct boundary points, and let $Q(z)\,dz^2 \in \mathcal{QD}(\boldsymbol{z})$ be a residue-free quadratic differential. Then:*

- *If n is even, there exists a unique (up to real additive constant)*

$$F(z) = R(z) + ic\log z,$$

where $R(z)$ is a rational function and $c \in \mathbb{R}$, such that:

1. *$F(z)$ is involution symmetric, i.e., $F(z^*) = \overline{F(z)}$ where $z^* = 1/\bar{z}$,*
2. *$Q(z)\,dz^2 = (F'(z))^2\,dz^2$,*
3. *the finite critical points of F are exactly $\mathbf{z}$.*

- *If n is odd, there exists a unique*

$$F(z) = \sqrt{z}R(z),$$

where $R(z)$ is a rational function, such that:

1. *$F(z)$ is involution symmetric,*
2. *$Q(z)\,dz^2 = (F'(z))^2\,dz^2$,*
3. *the finite critical points of F are exactly $\mathbf{z}$.*

Although $F(z)$ is multivalued, the critical points where $F'(z) = 0$ are well defined, since: - for $F(z) = R(z) + ic\log z$, all branches differ by multiples of $2\pi ic$, so $F'(z)$ is single-valued, - for $F(z) = \sqrt{z}R(z)$, branches differ by sign, so $F'(z)$ differs by sign as well.

The odd-n case can be reduced to the even case by passing to the double cover via the change of variable $z = u^2$.

Proof. Let $Q(z)\,dz^2 \in \mathcal{QD}(\mathbf{z})$ with poles $\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_m\}$. Then by Lemma 6.3.2, we have:

$$\sqrt{Q(z)} = C \cdot z^{m-\frac{n}{2}-1} \cdot \frac{\prod_{j=1}^{n}(z - z_j)}{\prod_{k=1}^{m}(z - \xi_k)^2},$$

for some constant $C \in \mathbb{C}$.

Case 1: n even, $2m \le n$. In this case, $\sqrt{Q(z)}$ has a pole at $z = 0$, and its primitive $F(z)$ must contain a logarithmic singularity:

$$F(z) = R(z) + ic\log z,$$

with $R(z)$ rational. Since all residues vanish by the stationary relations, the only possible monodromy arises from the logarithmic term, and involution symmetry of F implies that $c \in \mathbb{R}$.

Case 2: n even, $2m > n$. Here, $z = 0$ is a removable singularity or zero of $\sqrt{Q(z)}$, so its primitive $F(z)$ is a single-valued rational function:

$$F(z) = R(z),$$

again involution symmetric and with critical points at $\boldsymbol{z}$.

Case 3: n **odd.** Let $z = u^2$ and define

$$\sqrt{Q(z)}\,dz = 2u \cdot \underbrace{u^{2m-n-1} \cdot \frac{\prod_{j=1}^{n}(u^2 - z_j)}{\prod_{k=1}^{m}(u^2 - \xi_k)^2}}_{:=S(u^2)}\,du.$$

This defines a rational 1-form $S(u^2)\,du$ on the double cover. Since the residues at all $\pm\sqrt{\xi_k}$ vanish, and the form is even in u, we also have zero residue at $u = 0$. Therefore, the primitive of $S(u^2)$ can be written as

$$\int S(u^2)\,du = u \cdot R(u^2),$$

with R rational. Returning to $z = u^2$, we obtain

$$F(z) = \sqrt{z}R(z),$$

as desired. $\square$

Lemma 6.3.4. *Let $F(z)$ be the multivalued analytic function associated to a residue-free quadratic differential $Q(z)\,dz^2 \in \mathcal{QD}(\boldsymbol{z})$. Then its real locus*

$$\Gamma(F) := \{z \in \mathbb{C} \mid F(z) \in \widehat{\mathbb{R}}\}$$

is well defined as a subset of $\mathbb{C} \setminus \{0\}$, despite the multivaluedness of $F(z)$.

Proof. We consider the two cases according to the parity of n.

Case 1: n **even.** In this case, $F(z) = R(z) + ic\log z$, where $R(z)$ is rational and $c \in \mathbb{R}$. Since $\log z$ is multivalued, $F(z)$ is naturally defined on the universal cover of $\mathbb{C} \setminus \{0\}$. Let $\rho : \theta \mapsto e^{i\theta}$ be the covering map, and define the lifted function

$$\widetilde{F}(\theta) := F(e^{i\theta}) = R(e^{i\theta}) - c\theta.$$

Then

$$\widetilde{F}(\theta + 2\pi) = \widetilde{F}(\theta) - 2\pi c.$$

Since the shift is real, the condition $\widetilde{F}(\theta) \in \widehat{\mathbb{R}}$ is preserved under translation by 2π. Therefore, the real locus

$$\Gamma(\widetilde{F}) := \{\theta \in \mathbb{R} \mid \widetilde{F}(\theta) \in \widehat{\mathbb{R}}\}$$

is invariant under $\theta \mapsto \theta + 2\pi$. As the projection $\rho(\theta) = e^{i\theta}$ is also 2π-periodic, the image $\Gamma(F) := \rho(\Gamma(\widetilde{F})) \subset \mathbb{C} \setminus \{0\}$ is well defined.

Case 2: n **odd.** In this case, $F(z) = \sqrt{z}R(z)$, where $R(z)$ is rational. The square root introduces a two-sheeted branch structure. As before, we lift F to the universal cover using $\rho(\theta) = e^{i\theta}$, and define

$$\widetilde{F}(\theta) = \sqrt{e^{i\theta}}\, R(e^{i\theta}) = e^{i\theta/2} R(e^{i\theta}).$$

This satisfies

$$\widetilde{F}(\theta + \pi) = -\widetilde{F}(\theta), \quad \widetilde{F}(\theta + 2\pi) = \widetilde{F}(\theta).$$

Hence the real locus $\Gamma(\widetilde{F}) := \{\theta \in \mathbb{R} \mid \widetilde{F}(\theta) \in \widehat{\mathbb{R}}\}$ is 2π-periodic, and its image under ρ defines a well-defined subset $\Gamma(F) \subset \mathbb{C} \setminus \{0\}$. □

Next, we characterize the geometry of the horizontal trajectories of $Q(z)dz^2 \in \mathcal{QD}(\boldsymbol{z})$.

Theorem 6.3.5. *Let $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$ be distinct points on the unit circle. Consider a quadratic differential $Q(z) \in \mathcal{QD}(\boldsymbol{z})$ defined by*

$$Q(z) = \frac{\prod_{k=1}^{m} \xi_k^2}{\prod_{j=1}^{n} z_j} z^{2m-n-2} \frac{\prod_{j=1}^{n}(z - z_j)^2}{\prod_{k=1}^{m}(z - \xi_k)^4},$$

where $\{\xi_1, \ldots, \xi_m\}$ are involution-symmetric finite poles of $Q(z)$, and $\xi_k \neq 0, \infty$ for $k = 1, 2, \ldots, m$.

The horizontal trajectories of $Q(z)$, denoted as $\Gamma(Q)$, are the trajectories of $Q(z)dz^2 \in \mathcal{QD}(\boldsymbol{z})$ that with limiting ends at the zeros $\{z_1, z_2, \ldots, z_n\}$. These trajectories satisfy the following properties:

(1) If $2m \leq n$, the trajectories $\Gamma(Q)$ form a topological radial link pattern in $LP(n, m)$.

(2) If $2m > n$, the trajectories $\Gamma(Q)$ form a topological radial link pattern in $LP(n, n - m)$.

To prepare for this, we first introduce some fundamental concepts from the theory of quadratic differentials.

Definition 6.3.6. *For a quadratic differential $Q(z)dz^2$ on a Riemann surface S, we denote the zeros and simple poles of $Q(z)$ by set C and poles of order at least 2 by set H.*

Definition 6.3.7 (F-set)**.** *A set K on a Riemann surface S is called an F-set (with respect to $Q(z)dz^2$) if any trajectory of $Q(z)dz^2$ which meets K lies entirely in K.*

Definition 6.3.8 (Inner closure)**.** *By the inner closure of a set on $\mathfrak{R}$ we mean the interior of the closure of the set. The inner closure of a set K will be denoted by $\hat{K}$.*

In the following four definitions, we understand in each case S to be a finite oriented Riemann surface, $Q(z)dz^2$ to be a quadratic differential on S.

Definition 6.3.9 (End domain)**.** *An end domain U (relative to $Q(z)dz^2$)) is a maximal connected open F-set on S with the properties:*

(i) U contains no critical point of $Q(z)dz^2$.

(ii) U is swept out by trajectories of $Q(z)dz^2$ each of which has a limiting end point in each of its possible senses at a given point A in H.

(iii) U is mapped by $F(z) = \int (Q(z))^{1/2}dz$ conformally onto an upper or lower half-plane.

Definition 6.3.10 (Strip domain)**.** *A strip domain U (relative to $Q(z)dz^2$) is a maximal connected open F-set on S with the properties:*

(i) U contains no critical point of $Q(z)dz^2$.

(ii) U is swept out by trajectories of $Q(z)dz^2$ each of which has at one point A in H in the one sense a limiting end point and at another (possibly coincident) point B in H in the other sense a limiting end point.

(iii) U is mapped by $F(z) = \int (Q(z))^{1/2}dz$ conformally onto a strip $a < \mathrm{Im}F < b$, a, b are finite real numbers, $a < b$.

Definition 6.3.11 (Circle domain)**.** *A circle domain U (relative to $Q(z)dz^2$) is a maximal connected open F-set on S with the properties:*

(i) U contains a single double pole A of $Q(z)dz^2$,

(ii) $U - A$ is swept out by trajectories of $Q(z)dz^2$ each of which is a Jordan curve separating A from the boundary of $\mathcal{S}$,

(iii) For a suitably chosen purely imaginary constant c the function

$$w = \exp\left\{c \int (Q(z))^{1/2} dz\right\}$$

extended to have the value zero at A maps U conformally onto a circle $|w| < R$, A going into the point $w = 0$.

Definition 6.3.12 (Ring domain)**.** *A ring domain U (relative to $Q(z)dz^2$) is a maximal connected open F-set on S with the properties:*

(i) U contains no critical point of $Q(z)dz^2$.

(ii) U is swept out by trajectories of $Q(z)dz^2$ each of which is a Jordan curve.

(iii) for a suitably chosen purely imaginary constant c the function

$$w = \exp\left\{c \int (Q(z))^{1/2} dz\right\}$$

maps U conformally onto a circular ring

$$r_1 < |w| < r_2 \, (0 < r_1 < r_2) \, .$$

In Jenkins (2012) thm 3.5, the author proves a general result for positive quadratic differentials on finite Riemann surface S. In our setting, we only need to consider a special case $S = \widehat{\mathbb{C}}$ where all quadratic differentials are positive.

Theorem 6.3.13 (Basic Structure Theorem, thm 3.5 in Jenkins (2012))**.** *Let S be a Riemann sphere and $Q(z)dz^2$ a quadratic differential on S where we exclude the following possibilities and all configurations obtained from them by conformal equivalence:*

(i) S the z-sphere, $Q(z)dz^2 = dz^2$.

(ii) S the z-sphere, $Q(z)dz^2 = Ke^{ix}dz^2/z^2$, α real, K positive.

Let $\Gamma(Q)$ denote the union of all trajectories which have a limiting end point at a point of C (see definition (6.3.6). Then *(i)* *$S - \Gamma(Q)$ consists of a finite number of end, strip, circle and ring domains,*

(ii) *Each such domain is bounded by a finite number of trajectories together with the points at which the latter meet; every boundary component of such a domain contains a point of C; for a strip domain the two boundary elements arising from points of H (see definition (6.3.6)) divide the boundary into two parts on each of which is a point of C,*

(iii) *Every pole of $Q(z)dz^2$ of order m greater than two has a neighborhood covered by the inner closure of $m-2$ end domains and a finite number (possibly zero) of strip domains,*

(iv) *Every pole of $Q(z)dz^2$ of order two has a neighborhood covered by the inner closure of a finite number of strip domains or has a neighborhood contained in a circle domain,*

(v) *The inner closure $\widehat{\Gamma(Q)}$ of $\Gamma(Q)$ is an F-set consisting of a finite number of domains on S each with a finite number (possibly zero) of boundary components,*

(vi) *Each boundary component of such a domain is a piecewise analytic curve composed of trajectories and their limiting end points in C.*

Proof of Theorem 6.3.5. We characterize the geometry of $\Gamma(Q)$ by considering the following cases:

(i) **Case n even,** $2m < n$: The poles of $Q(z)$ at 0 and ∞ are of order $n+2-2m \geq 4$. By the basic structure theorem (Theorem 6.3.13), the complement $\widehat{\mathbb{C}} \setminus \Gamma(Q)$ consists of a finite collection of end, strip, circle, and ring domains.

We first show that there can be no strip or ring domains. For $Q(z)\,dz^2 \in \mathcal{QD}(z)$, by Lemma 6.3.3 and 6.3.4, the function $F(z) = \int \sqrt{Q(z)}\,dz$ takes real values on $\Gamma(Q)$. Hence, $F(z)$ cannot map a domain U to a strip or a ring domain, because in such cases the imaginary part $\operatorname{Im} F$ takes two different values on ∂U, contradicting the level line structure.

Moreover, since the pole order at $z = 0$ is at least 4, there are no circle domains either. Thus, all domains are end domains. We denote the finite ones by $\{U_1, U_2, \ldots, U_s\}$, bounded away from 0 and ∞, and the infinite ones by $\{V_1, V_2, \ldots, V_t\}$, whose closures contain 0 or ∞.

For each finite domain U_i, the map $F(z)$ sends it to the upper or lower half-plane and extends continuously to the boundary. Therefore, there must be a pole ξ_l on ∂U_i, and each pole ξ_l lies on the boundary of exactly two adjacent finite domains. Since there are exactly m poles $\{\xi_1, \xi_2, \ldots, \xi_m\}$, we have $s = 2m$ finite domains.

Each U_i is bounded by disjoint arcs connecting pairs of zeros $\{z_1, \ldots, z_n\}$. By involution symmetry, there are m such domains in $\mathbb{D}$, giving exactly m arcs in $\Gamma(Q)$ connecting m pairs of zeros.

Since the pole at $z = 0$ is of order $n + 2 - 2m$, by Theorem 6.3.13 there are $n - 2m$ infinite domains with closure containing 0, and by symmetry, $n - 2m$ domains containing ∞, so $t = 2(n - 2m)$.

Hence, $\Gamma(Q)$ consists of m arcs connecting pairs of zeros and $n - 2m$ trajectories ending at 0, whose tangents are equally spaced at angles $\frac{2\pi}{n-2m}$. This defines a radial (n, m) link pattern.

(ii) **Case n even,** $n = 2m$: In this case, the poles of $Q(z)$ at 0 and ∞ are both of order 2. Again, by Theorem 6.3.13, $\widehat{\mathbb{C}} \setminus \Gamma(Q)$ consists of end, strip, circle, and ring domains.

As before, we exclude the possibility of strip or ring domains using Lemmas 6.3.3 and 6.3.4, since $F(z)$ takes real values on $\Gamma(Q)$ and cannot map U to such domains.

With pole order exactly 2, there is precisely one circle domain centered at 0, and by symmetry, one at ∞. The remaining domains are end domains.

Let the end domains be $\{U_1, \ldots, U_s\}$, each bounded away from 0 and ∞. By the same argument as before, each domain boundary contains a pole ξ_l, and we again find $s = 2m$ such domains.

By involution symmetry, half of these lie in $\mathbb{D}$, giving m arcs in $\Gamma(Q)$ connecting m pairs of zeros. Thus, $\Gamma(Q)$ forms a radial (n, m) link pattern.

(iii) **Case n even,** $2m > n$: In this case, by Theorem 6.3.3, the primitive $F(z) = \int \sqrt{Q(z)}\, dz$ is a rational function.

The degree of $F(z)$ at 0 is $m - \frac{n}{2} > 0$, so the real locus $\Gamma(F)$ is regular near 0. There are $2m - n$ trajectories ending at 0, with limiting tangents forming equal angles of $\frac{2\pi}{2m-n}$.

Hence, $\Gamma(F)$ forms a radial $(n, n - m)$ link pattern.

(iv) **Case n odd**: In this case, the primitive of $\sqrt{Q(z)}$ takes the form $F(z) = \int \sqrt{Q(z)}\, dz = \sqrt{z}R(z)$, where $R(z)$ is rational.

Passing to the double cover, $F(z^2) = zR(z^2)$ is rational. The degree of $F(z^2)$ at 0 is $|2m - n|$, so there are $|2n - 4m|$ trajectories ending at 0.

Projecting back, $\Gamma(F(z))$ has $|n - 2m|$ trajectories ending at 0, with tangents forming equal angles $\frac{2\pi}{|n-2m|}$.

Therefore, if $n > 2m$, $\Gamma(F)$ forms a radial (n, m) link pattern; if $n < 2m$, it forms a radial $(n, n - m)$ link pattern.

□

6.4 Field integral of motion and horizontal trajectories as flow lines

In this section, we show that the traces of a multiple radial SLE(0) system coincide with the horizontal trajectories of $Q(z)dz^2 \in \mathcal{QD}(z)$ with ends at $\{z_1, z_2, \ldots, z_n\}$ (which are double zeros of $Q(z)$).

From the dynamical point of view, the key ingredient in our proof of the main theorem (1.3.7) is the field of integral of motions for the multiple radial Loewner flow. This field integral of motion can be heuristically derived as the classical limit of a martingale observable constructed via conformal field theory, see section 6.5.

Lemma 6.4.1. *Let $z_1, z_2, \ldots, z_n$ be distinct growth points in the unit circle $\partial\mathbb{D}$, and let $\xi_1, \xi_2, \ldots, \xi_m$ be marked points. Let $g_t(z)$ be the solution to the multiple radial Loewner equation with a driving measure supported on $\{z_j(t)\}$, and assume that only the j-th curve is growing, that is, $\nu_j(t) = 1$ and $\nu_k(t) = 0$ for $k \neq j$.*

Define the following quantities for $z \in \overline{\mathbb{D}}$:

$$\begin{cases} A(t) = \dfrac{\prod_{j=1}^{m} \xi_j(t)^2}{\prod_{k=1}^{n} z_k(t)}, \\ B_t(z) = e^{-(2m-n)t}\, g_t(z)^{2m-n-2}\, (g_t'(z))^2\, \dfrac{\prod_{k=1}^{n}(g_t(z) - z_k(t))^2}{\prod_{j=1}^{m}(g_t(z) - \xi_j(t))^4}, \\ N_t(z) = A(t) \cdot B_t(z). \end{cases} \tag{6.4.1}$$

Then, for each $z \in \mathbb{D}$, the quantity $N_t(z)$ is constant on the time interval $[0, \tau_z \wedge \tau)$, where τ is the first time t at which $g_t(w) = z_j(t)$ for some $w \in \{\xi_1, \ldots, \xi_m\}$, and τ_z is the first time such that $g_t(z) = z_j(t)$.

Proof. Proof of lemma (6.4.1)

By the Loewner equation, the following identities hold:

$$\begin{cases} \dfrac{dz_j(t)}{dt} = \sum_{k\neq j} z_j(t)\dfrac{z_j(t)+z_k(t)}{z_k(t)-z_j(t)} - 2\sum_{l} z_j(t)\dfrac{z_j(t)+\xi_l(t)}{\xi_l(t)-z_j(t)} \\ \dfrac{dz_k(t)}{dt} = z_k(t)\dfrac{z_j(t)+z_k(t)}{z_j(t)-z_k(t)}, k\neq j \\ \dfrac{d\xi_l(t)}{dt} = \xi_l(t)\dfrac{z_j(t)+\xi_l(t)}{z_j(t)-\xi_l(t)} \end{cases} \tag{6.4.2}$$

By substituting above equations into $\frac{d\log A(t)}{dt}$, we obtain that

$$\log A(t) = 2\sum_{j=1}^{m}\log\xi_k(t) - \sum_{k=1}^{n}\log z_k(t) \tag{6.4.3}$$

$$\begin{aligned} \frac{d\log A(t)}{dt} &= 2\sum_{l=1}^{m}\frac{d\log\xi_l(t)}{dt} - \sum_{j=1}^{n}\frac{d\log z_j(t)}{dt} \\ &= \underbrace{2\sum_{l=1}^{m}\frac{z_j(t)+\xi_l(t)}{z_j(t)-\xi_l(t)} + \sum_{k\neq j}\frac{z_j(t)+z_k(t)}{z_j(t)-z_k(t)} + \sum_{k\neq j}\frac{z_j(t)+z_k(t)}{z_k(t)-z_j(t)} - 2\sum_{l}\frac{z_j(t)+\xi_l(t)}{\xi_l(t)-z_j(t)}}_{A^j(t)} \\ &= 0. \end{aligned} \tag{6.4.4}$$

We denote the sum in the second line of the equation (6.4.4) by $A^j(t)$. After simplifying, it is clear that the sum $A^j(t) = 0$.

Again, by Loewner equation, the following identities hold

$$\begin{cases} \dfrac{dg_t(z)}{dt} = g_t(z)\dfrac{z_j(t)+g_t(z)}{z_j(t)-g_t(z)} \\ \dfrac{d\log g_t'(z)}{dt} = \dfrac{z_j(t)+g_t}{z_j(t)-g_t} + \dfrac{2z_j(t)g_t}{(z_j(t)-g_t)^2} \\ \dfrac{d\log g_t(z)}{dt} = \dfrac{z_j(t)+g_t(z)}{z_j(t)-g_t(z)} \\ \dfrac{d\log(g_t'(z)/g_t)}{dt} = \dfrac{2g_t(z)z_j(t)}{(z_j(t)-g_t)^2} = \dfrac{g_t(z)(g_t(z)+z_j(t))}{(z_j(t)-g_t(z))^2} + \dfrac{g_t(z)}{z_j(t)-g_t(z)}. \end{cases} \tag{6.4.5}$$

$$
\begin{cases}
\dfrac{d\log(z_k(t)-g_t(z))}{dt} = \dfrac{1}{z_k-g_t}(z_k(t)\dfrac{z_j(t)+z_k(t)}{z_j(t)-z_k(t)} - g_t(z)\dfrac{z_j(t)+g_t(z)}{z_j(t)-g_t(z)}) \\
= -\dfrac{z_j(t)(z_j(t)+z_k(t))}{(z_j(t)-g_t(z))(z_k(t)-z_j(t))} + \dfrac{g_t}{z_j(t)-g_t(z)}, k \neq j \\
\dfrac{d\log(\xi_l(t)-g_t(z))}{dt} = \dfrac{1}{\xi_l-g_t}(\xi_l(t)\dfrac{z_j(t)+\xi_l(t)}{z_j(t)-\xi_l(t)} - g_t(z)\dfrac{z_j(t)+g_t(z)}{z_j(t)-g_t(z)}) \\
= -\dfrac{z_j(t)(z_j(t)+\xi_l(t))}{(z_j(t)-g_t(z))(\xi_l(t)-z_j(t))} + \dfrac{g_t}{z_j(t)-g_t(z)}, \\
\dfrac{d\log(z_j(t)-g_t(z))}{dt} = \dfrac{1}{z_j(t)-g_t}\left(z_j(t)\sum_{k\neq j}\dfrac{z_j(t)+z_k(t)}{z_k(t)-z_j(t)} - 2\sum_l z_j(t)\dfrac{z_j(t)+\xi_l(t)}{\xi_l(t)-z_j(t)} - g_t(z)\dfrac{z_j(t)+g_t(z)}{z_j(t)-g_t}\right).
\end{cases}
\tag{6.4.6}
$$

$$
\log B_t(z) = -(2m-n)t+(2m-n-2)\log g_t(z)+2\log(g_t'(z))+2\sum_{k=1}^{n}\log(g_t(z)-z_k(t))-4\sum_{j=1}^{m}\log(g_t(z)-\xi_j(t))
\tag{6.4.7}
$$

By substituting equations (6.4.6) into equation (6.4.7),

$$
\frac{d\log B_t(z)}{dt} = \underbrace{\begin{aligned}
&-(2m-n)+(2m-n-2)\frac{z_j(t)+g_t(z)}{z_j(t)-g_t(z)}+2\left(\frac{z_j(t)+g_t}{z_j(t)-g_t}+\frac{2z_j(t)g_t}{(z_j(t)-g_t)^2}\right)+\\
&+2\sum_{k=1}^{n}\left(-\frac{z_j(t)(z_j(t)+z_k(t))}{(z_j(t)-g_t(z))(z_k(t)-z_j(t))}+\frac{g_t}{z_j(t)-g_t(z)}\right)\\
&-4\sum_{j=1}^{m}\frac{1}{z_j(t)-g_t}\left(z_j(t)\sum_{k\neq j}\frac{z_j(t)+z_k(t)}{z_k(t)-z_j(t)}-2\sum_l z_j(t)\frac{z_j(t)+\xi_l(t)}{\xi_l(t)-z_j(t)}-g_t(z)\frac{z_j(t)+g_t(z)}{z_j(t)-g_t}\right)
\end{aligned}}_{B_t^j(z)}
\tag{6.4.8}
$$

We denote the sum on the right hand side of (6.4.8) by $B_t^j(z)$. By direct computations we obtain that all terms canceled out,

$$
\frac{d\log B_t(z)}{dt} = B_t^j(z) = 0
$$

which implies

$$
\frac{d\log N_t(z)}{dt} = \frac{d\log A(t)}{dt} + \frac{d\log B_t(z)}{dt} = 0.
$$

$\square$

Theorem (Restatement of theorem 1.3.8)**.** *In the unit disk $\mathbb{D}$, let $z_1, z_2, \ldots, z_n$ be distinct growth points on $\partial\mathbb{D}$. For each $z \in \overline{\mathbb{D}}$, define the following:*

$$\begin{cases} A(t) = \dfrac{\prod_{j=1}^m \xi_k^2(t)}{\prod_{k=1}^n z_k(t)}, \\ B_t(z) = e^{-(2m-n)\left(\int_0^t \sum_j \nu_j(s)\,ds\right)} g_t(z)^{2m-n-2}(g_t'(z))^2 \dfrac{\prod_{k=1}^n (g_t(z)-z_k(t))^2}{\prod_{j=1}^m (g_t(z)-\xi_j(t))^4}, \\ N_t(z) = A(t)B_t(z) = e^{-(2m-n)\left(\int_0^t \sum_j \nu_j(s)\,ds\right)} \dfrac{\prod_{j=1}^m \xi_k(t)^2}{\prod_{k=1}^n z_k(t)} g_t(z)^{2m-n-2} \\ \qquad (g_t'(z))^2 \dfrac{\prod_{k=1}^n (g_t(z)-z_k(t))^2}{\prod_{j=1}^m (g_t(z)-\xi_j(t))^4}. \end{cases}$$

Then, $A(t)$, $B_t(z)$, and $N_t(z)$ are field integrals of motion on the interval $[0, \tau_z \wedge \tau)$ for the multiple radial SLE(0) Loewner flows with parametrization $\nu_j(t)$, $j = 1, \ldots, n$.

Proof of theorem (1.3.8). Note that for $\nu(t)$ parametrization

$$\partial_t g_t(z) = \sum_{j=1}^n \nu_j(t) g_t(z) \frac{z_j(t)+g_t(z)}{z_j(t)-g_t(z)}, \quad g_0(z) = z,$$

$$\begin{cases} \dfrac{dz_j(t)}{dt} = \nu_j(t)\left(\sum_{k\neq j} z_j(t)\dfrac{z_j(t)+z_k(t)}{z_k(t)-z_j(t)} + 2\sum_l z_j(t)\dfrac{z_j(t)+\xi_l(t)}{\xi_l(t)-z_j(t)}\right) + \sum_{k\neq j}\nu_k(t)\left(z_j(t)\dfrac{z_j(t)+z_k(t)}{z_k(t)-z_j(t)}\right) \\ \dfrac{d\xi_l(t)}{dt} = \sum_j \nu_j(t)\left(\xi_l(t)\dfrac{z_j(t)+\xi_l(t)}{z_j(t)-\xi_l(t)}\right) \end{cases}.$$

$$\begin{cases} \dfrac{dg_t}{dt} = \sum_j \nu_j(t)\left(g_t(z)\dfrac{z_j(t)+g_t(z)}{z_j(t)-g_t(z)}\right) \\ \dfrac{d\log g_t'(z)}{dt} = \sum_j \nu_j(t)\left(\dfrac{z_j(t)+g_t}{z_j(t)-g_t} + \dfrac{2z_j(t)g_t}{(z_j(t)-g_t)^2}\right) \\ \dfrac{d\log g_t(z)}{dt} = \sum_j \nu_j(t)\left(\dfrac{z_j(t)+g_t(z)}{z_j(t)-g_t(z)}\right) \\ \dfrac{d\log(g_t'(z)/g_t)}{dt} = \sum_j \nu_j(t)\dfrac{2g_t(z)z_j(t)}{(z_j(t)-g_t)^2} = \sum_{j=1} \nu_j(t)\left(\dfrac{g_t(z)(g_t(z)+z_j(t))}{(z_j(t)-g_t(z))^2} + \dfrac{g_t(z)}{z_j(t)-g_t(z)}\right). \end{cases}$$

$$\begin{cases} \dfrac{d\log(\xi_l(t)-g_t(z))}{dt} = \dfrac{1}{\xi_l - g_t}\displaystyle\sum_j \nu_j(t)\left(\xi_l(t)\frac{z_j(t)+\xi_l(t)}{z_j(t)-\xi_l(t)} - g_t(z)\frac{z_j(t)+g_t(z)}{z_j(t)-g_t(z)}\right) \\ = \displaystyle\sum_j \nu_j(t)\left(\frac{z_j^2+g_tz_j+\xi_lz_j-\xi_lg_t}{(z_j(t)-\xi_l(t))(z_j(t)-g_t)}\right) = \sum_j \nu_j(t)\left(-\frac{z_j(t)(z_j(t)+\xi_l(t))}{(z_j(t)-g_t(z))(\xi_l(t)-z_j(t))} + \frac{g_t}{z_j(t)-g_t(z)}\right) \\ \dfrac{d\log(z_j(t)-g_t(z))}{dt} = \dfrac{1}{z_j(t)-g_t}\nu_j(t)\left(z_j(t)\displaystyle\sum_{k\neq j}\frac{z_j(t)+z_k(t)}{z_k(t)-z_j(t)} - 2\sum_l z_j(t)\frac{z_j(t)+\xi_l(t)}{\xi_l(t)-z_j(t)}\right. \\ \left. -g_t\dfrac{z_j(t)+g_t(z)}{z_j(t)-g_t}\right) + \dfrac{1}{z_j(t)-g_t}\displaystyle\sum_{k\neq j}\nu_k(t)\left(z_j(t)\frac{z_j(t)+z_k(t)}{z_k(t)-z_j(t)} - g_t\frac{z_j(t)+g_t(z)}{z_j(t)-g_t}\right). \end{cases}$$

By plugging in these identities, we obtain that

$$\frac{d\log A(t)}{dt} = \sum_{j=1}^n \nu_j(t)A^j(t) = 0$$

$$\frac{d\log B_t(z)}{dt} = \sum_{j=1}^n \nu_j(t)B_t^j(z) = 0$$

$$\frac{d\log N_t(z)}{dt} = \sum_{j=1}^n \nu_j(t)\frac{d\log N_t^j(z)}{dt} = 0.$$

□

$N_t(z)$ is a field integral of motion for arbitrary initial positions of screening charges $\boldsymbol{\xi}$ even without assuming stationary relations. The stationary relations imply the existence of a quadratic differential $Q(z)dz^2 \in \mathcal{QD}(\boldsymbol{z})$, see Theorem 1.3.7.

The integral of motion is motivated by a martingale observable in conformal field theory. For a field $\mathcal{X}$ in the OPE family F_β,

$$\hat{\mathbf{E}}[\mathcal{X}] := \frac{\mathbf{E}[\mathcal{X}O_\beta]}{\mathbf{E}[O_\beta]}$$

is a martingale observable where O_β is a vertex field. In our situation, we choose $\mathcal{X}$ to be the chiral vertex field and take the classical limit as $\kappa \to 0$. The martingale observable degenerates to the integral of motion. We will discuss the construction of the field $\mathcal{X}$ in Section 6.5.

In the proof of Theorem 1.3.7, we also need to consider $\sqrt{N_t(z)}$ as a field integral of motion. However, an obstacle arises in the expression

$$\sqrt{N_t(z)} = e^{-(m-\frac{n}{2})\left(\int_0^t \sum_j \nu_j(s)\,ds\right)} g_t(z)^{m-\frac{n}{2}-1} g_t'(z) \frac{\prod_{k=1}^n (g_t(z)-z_k(t))}{\prod_{j=1}^m (g_t(z)-\xi_j(t))^2},$$

where the term $g_t(z)^{m-\frac{n}{2}-1}$ becomes multivalued when n is an odd integer, thus $\sqrt{N_t(z)}$ is in fact not well defined. To resolve this technical problem, we introduce the angular coordinate.

Corollary 6.4.2. *In the angular coordinate, by changing variables, let $\xi_k = e^{i\zeta_k}$, $z_k = e^{i\theta_k}$, and $h_t(z)$ be the covering map of $g_t(z)$ (i.e., $e^{ih_t(z)} = g_t(e^{iz})$). For each $z \in \overline{\mathbb{H}}$, we define:*

$$\begin{cases} A^{ang}(t) = \dfrac{\prod_{j=1}^m e^{i\zeta_k(t)}}{\prod_{k=1}^n e^{i\frac{\theta_k(t)}{2}}}, \\ B_t^{ang}(z) = e^{-(m-\frac{n}{2})\left(\int_0^t \sum_j \nu_j(s)\,ds\right)} g_t(z)^{m-\frac{n}{2}-1} e^{i(m-\frac{n}{2}-1)h_t(z)} h_t'(z) e^{ih_t(z)} \dfrac{\prod_{k=1}^n \left(e^{ih_t(z)} - e^{i\theta_k(t)}\right)}{\prod_{j=1}^m \left(e^{ih_t(z)} - e^{i\zeta_j(t)}\right)^2}, \\ N_t^{ang}(z) = A^{ang}(t) B_t^{ang}(z) \\ \qquad = e^{-(m-\frac{n}{2})\left(\int_0^t \sum_j \nu_j(s)\,ds\right)} \dfrac{\prod_{j=1}^m e^{i\zeta_k(t)}}{\prod_{k=1}^n e^{i\frac{\theta_k(t)}{2}}} e^{i(m-\frac{n}{2}-1)h_t(z)} h_t'(z) e^{ih_t(z)} \dfrac{\prod_{k=1}^n \left(e^{ih_t(z)} - e^{i\theta_k(t)}\right)}{\prod_{j=1}^m \left(e^{ih_t(z)} - e^{i\zeta_j(t)}\right)^2}. \end{cases} \tag{6.4.9}$$

Then, $A^{ang}(t)$, $B_t^{ang}(z)$, and $N_t^{ang}(z)$ are field integrals of motion on the interval $[0, \tau_z \wedge \tau)$ for the multiple radial SLE(0) Loewner flows with parametrization $\nu_j(t)$, $j = 1, \ldots, n$.

Theorem (Restatement of theorem (1.3.7))**.** *Let $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$ be distinct growth points on the unit circle and screening charges $\xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$ involution symmetric and solve the stationary relations.*

There exists an $Q(z) \in \mathcal{QD}(\boldsymbol{z})$ with $\boldsymbol{\xi}$ as poles and $\boldsymbol{z}$ as zeros, the hulls K_t generated by the Loewner flows with parametrization $\boldsymbol{\nu}(t)$ are subsets of the horizontal trajectories of $Q(z)dz^2$ with limiting ends at $\boldsymbol{z}$, up to any time t before the collisions of any poles or critical points. Up to any such time

$$Q(z) \circ g_t^{-1} \in \mathcal{QD}(\boldsymbol{z}(t)).$$

where $\boldsymbol{z}(t)$ is the location of the critical points at time t under the multiple radial Loewner flow with parametrization $\boldsymbol{\nu}(t)$.

Proof of Theorem (1.3.7). We first prove that $Q(z) \circ g_t^{-1}$ is in $\mathcal{QD}(\boldsymbol{z}(t))$.

Since at $t = 0$, the screening charges $\boldsymbol{\xi}$ are assumed to be involution symmetric and solve the stationary relations, stationary relations- residue free theorem guarantees the existence of an $Q_0(z)dz^2 \in \mathcal{QD}(\boldsymbol{z}(0))$ with $\boldsymbol{\xi}(0)$ as poles and $\boldsymbol{z}(0)$ as zeros.

Moreover, $Q_0(z)$ factors as

$$Q_0(z) = \frac{\prod_{k=1}^m \xi_k^2(0)}{\prod_{j=1}^n z_j(0)} z^{2m-n-2} \frac{\prod_{j=1}^n \left(z - z_j(0)\right)^2}{\prod_{k=1}^m \left(z - \xi_k(0)\right)^4}. \tag{6.4.10}$$

Using the integral of motion $N_t(z)$, we have

$$\begin{aligned} N_t(z) =& \frac{\prod_{j=1}^m \xi_k^2(t)}{\prod_{k=1}^n z_k(t)} e^{-(2m-n-2)(\int_0^t \sum_j \nu_j(s)ds)} g_t^{2m-n-2}(z)(g_t'(z))^2 \frac{\prod_{j=1}^n \left(g_t(z) - z_j(t)\right)^2}{\prod_{k=1}^m \left(g_t(z) - g_t(\xi_k)\right)^4} \\ =& \frac{\prod_{k=1}^m \xi_k^2(0)}{\prod_{j=1}^n z_j(0)} z^{2m-n-2} \frac{\prod_{j=1}^n \left(z - y_j\right)^2}{\prod_{k=1}^m \left(z - \xi_k\right)^4} = N_0(z). \end{aligned} \tag{6.4.11}$$

Denote the constant $\mu(t) = e^{-(m-\frac{n}{2}-1)(\int_0^t \sum_j \nu_j(s)ds)}$.

Let $f_t = g_t^{-1}$, and since the above holds everywhere evaluate it at $f_t(z)$ to obtain

$$\begin{aligned} &\mu^2(t) \frac{\prod_{j=1}^m \xi_k^2(t)}{\prod_{k=1}^n z_k(t)} z^{2m-n-2} \frac{\prod_{j=1}^n \left(z - z_j(t)\right)^2}{\prod_{k=1}^m \left(z - g_t(\xi_k)\right)^4} \\ =& \frac{\prod_{k=1}^m \xi_k^2(0)}{\prod_{j=1}^n z_j(0)} f_t'(z)^2 f_t^{2m-n-2}(z) \frac{\prod_{j=1}^n \left(f_t(z) - z_j\right)^2}{\prod_{k=1}^m \left(f_t(z) - \zeta_t\right)^4} = f_t'(z)^2 Q_0\left(f_t(z)\right). \end{aligned} \tag{6.4.12}$$

The left-hand side is exactly

$$\mu^2(t) z^{m-\frac{n}{2}-1} \frac{\prod_{j=1}^n \left(z - z_j(t)\right)}{\prod_{k=1}^m \left(z - g_t(\xi_k)\right)^2} = \mu(t) Q_t(z). \tag{6.4.13}$$

we obtain that

$$\pm\mu(t) Res_{\xi(t)}(\sqrt{Q_t(z)}) = Res_{\xi(0)}(\sqrt{Q_0(f_t(z))} = 0. \tag{6.4.14}$$

The stationary relations at $t = 0$ give the last equality. Thus, the residue-free condition holds, and clearly, the involution symmetry of $\boldsymbol{\xi}$ is preserved, which implies $Q(z) \circ g_t^{-1} \in \mathcal{QD}(\boldsymbol{z}(t))$.

Finally, we prove that the hull K_t is a subset of the horizontal trajectories of $Q(z)dz^2$ with limiting ends at $\{z_1, z_2, \ldots, z_n\}$. By theorem (6.3.4), equivalently, we can show that K_t is a subset of the real locus of the $F(z) = \int \sqrt{Q(z)}dz$ (which is well defined as shown in lemma (6.3.4)).

Note that $F(z)$ is a multivalued function. To deal with the multi-valuedness, let $\rho(z) = e^{iz}$ be the exponential covering map, and we consider $h_t(z)$ the lifting map of $g_t(z)$ (i.e. $e^{ih_t(z)} = g_t(e^{iz})$). We denote the lifting of $F(z)$ by $\tilde{F}(z)$, which is now a single-valued function, and the lifting of $Q_t(z)$ by $\tilde{Q}_t(z) = -Q_t(e^{iz})e^{2iz}dz$, the lifting of the hull K_t by $\tilde{K}_t$.

By the integral of motion in angular coordinate (6.4.2)

$$\begin{aligned} N_t^{ang}(z) =& \mu(t)\frac{\prod_{j=1}^m e^{i\zeta_k(t)}}{\prod_{k=1}^n e^{i\frac{\theta_k(t)}{2}}} e^{i(m-\frac{n}{2}-1)h_t(z)} h_t'(z) e^{ih_t(z)} \frac{\prod_{k=1}^n (e^{ih_t(z)} - e^{i\theta_k(t)})}{\prod_{j=1}^m (e^{ih_t(z)} - e^{i\zeta_j(t)})^2} \\ =& \frac{\prod_{j=1}^m e^{i\zeta_k(0)}}{\prod_{k=1}^n e^{i\frac{\theta_k(0)}{2}}} e^{i(m-\frac{n}{2}-1)z} e^{iz} \frac{\prod_{k=1}^n (e^{iz} - e^{i\theta_k(0)})}{\prod_{j=1}^m (e^{iz} - e^{i\zeta_j(0)})^2} = N_0^{ang}(z). \end{aligned} \tag{6.4.15}$$

Let $s_t = h_t^{-1}$, and since the above holds everywhere evaluate it at $s_t(z)$ to obtain

$$\begin{aligned} \mu(t)\sqrt{\tilde{Q}_t(z)} =& \mu(t)\frac{\prod_{j=1}^m e^{i\zeta_k(t)}}{\prod_{k=1}^n e^{i\frac{\theta_k(t)}{2}}} e^{i(m-\frac{n}{2}-1)z} e^{iz} \frac{\prod_{k=1}^n (e^{iz} - e^{i\theta_k(t)})}{\prod_{j=1}^m (e^{iz} - e^{i\zeta_j(t)})^2} \\ =& \frac{\prod_{j=1}^m e^{i\zeta_k(0)}}{\prod_{k=1}^n e^{i\frac{\theta_k(0)}{2}}} e^{i(m-\frac{n}{2}-1)s_t(z)} e^{is_t(z)} \frac{\prod_{k=1}^n (e^{is_t(z)} - e^{i\theta_k(0)})}{\prod_{j=1}^m (e^{is_t(z)} - e^{i\zeta_j(0)})^2} (h_t^{-1}(z))' \\ =& \sqrt{Q_0(s_t(z))}(s_t(z))' \\ =& (\tilde{F}(s_t(z)))'. \end{aligned} \tag{6.4.16}$$

Since $\mu(t)$ is a real constant, $\tilde{F}(s_t(z))$ is the primitive of $\sqrt{\tilde{Q}_t(z)}$ (up to a real multiplicative constant). It suffices to show that $\tilde{K}_t$ is the real locus of $\tilde{F}(z)$

Recall that g_t is the unique conformal map from $\mathbb{D}\backslash K_t$ onto $\mathbb{D}$ with the hydrodynamic normalization. Therefore, $g_t(z)$ maps the subset K_t to $\partial\mathbb{D}$ and $h_t(z)$ maps the subset $\tilde{K}_t$ to the real line.

Since $\tilde{F}(s_t(z))$ is the primitive of $\sqrt{\tilde{Q}_t(z)}$ (up to a real multiplicative constant), the real line is part of the real locus $\tilde{F}(s_t(z))$. Since $h_t(z)$ maps the subset $\tilde{K}_t$ to the real line, it follows that $\tilde{K}_t$ is a subset of the real locus of $\tilde{F}(z)$. □

Remark 6.4.3. The underlying principle is that for $Q(z) \in \mathcal{QD}(z)$, the function $\sqrt{Q(z)}$ admits a local meromorphic primitive in $\mathbb{D} \setminus \{0\}$. As a result, the residues of $\sqrt{Q(z)}$ at all nonzero poles must vanish.

When the poles are distinct from the critical points, this principle manifests algebraically as the *stationary relation*. However, when a pole coincides with a critical point (in which case the pole must be of order two), the partial fraction expansion of $\sqrt{Q(z)}$ becomes more intricate, and so does the associated algebra. Nevertheless, the same fundamental principle continues to govern the structure.

Example 6.4.4. *The traces of n-braids multiple SLE(0) in $\mathbb{D}$.*

Proof. For n-braids multiple radial SLE(0), $m = 0$, and there are no poles, thus $Q(z)dz^2$ is given by

$$Q(z) = cz^{-n-2} \prod_{j=1}^{n} \left(z - z_j\right)^2 \tag{6.4.17}$$

The n-braids multiple SLE(0) has n trajectories with limiting ends at $z = 0$, and with limiting tangential directions that form an equal $\frac{2\pi}{n}$ with each other.

□

6.5 Classical limit of martingale observables*

In this section, we discuss how the field integral of motion is heuristically derived as the classical limit of martingale observables constructed via conformal field theory.

Based on the SLE-CFT correspondence, the multiple radial SLE(κ) system can be coupled to a conformal field theory. We will construct this conformal field theory using vertex operators, following the approach in Kang and N. Makarov (2013) and N-G. Kang and N. Makarov (2021).

Definition 6.5.1 (Vertex operator)**.** *For a background charge $\boldsymbol{\beta} = \sum_k \beta_k \cdot q_k$ with the neutrality condition* (NC$_b$) *and divisor $\boldsymbol{\tau} = \sum_j \tau_j \cdot z_j$ with the neutrality condition* (NC$_0$)*. We define the vertex operator $\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ as*

$$\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}] := \frac{C_{(b)}[\boldsymbol{\tau} + \boldsymbol{\beta}]}{C_{(b)}[\boldsymbol{\beta}]} e^{\odot i\Phi^+[\boldsymbol{\tau}]}. \tag{6.5.1}$$

where $\Phi^+[\boldsymbol{\tau}] := \sum \tau_j \Phi^+\left(z_j\right)$ is the chirdal bosonic field and $\odot$ is the wick product.

Definition 6.5.2 (n-leg operator with screening charges)**.** *Consider the following charge distribution on the Riemann sphere.*

$$\boldsymbol{\beta} = b\delta_0 + b\delta_\infty$$

$$\boldsymbol{\tau_1} = \sum_{j=1}^{n} a\delta_{z_j} - \sum_{k=1}^{m} 2a\delta_{\xi_k} - (\frac{n-2m}{2})a\delta_0 - (\frac{n-2m}{2})a\;\delta_\infty$$

The n-leg operator with screening charges $\boldsymbol{\xi}$ and background charge $\boldsymbol{\beta}$ is given by the OPE exponential:

$$O_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}] = \frac{C_{(b)}[\boldsymbol{\tau_1}+\boldsymbol{\beta}]}{C_{(b)}[\boldsymbol{\beta}]}\mathrm{e}^{\odot i\Phi[\boldsymbol{\tau_1}]}. \tag{6.5.2}$$

For each link pattern α, we can choose closed contours $C_1,\ldots,C_n$ along which we may integrate the $\boldsymbol{\xi}$ variables to screen the vertex fields. Let $\mathcal{S}$ be the screening operator. We define the screening operation as

$$\mathcal{S}_\alpha O_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}] = \oint_{C_1}\ldots\oint_{C_n} O_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]. \tag{6.5.3}$$

We integrate the correlation function $\mathbf{E}O_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}] = \Phi_\kappa(z,\boldsymbol{\xi})$, the conformal dimension is 1 at the $\boldsymbol{\xi}$ points, i.e. since $\lambda_b(-2a) = 1$. This leads to the partition function for the corresponding multiple radial SLE(κ) system:

$$\mathcal{J}_\alpha(z) := \mathbf{E}\mathcal{S}_\alpha O_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}] = \oint_{C_1}\ldots\oint_{C_n}\Phi_\kappa(z,\boldsymbol{\xi})d\xi_n\ldots d\xi_1. \tag{6.5.4}$$

Theorem 6.5.3 (Martingale observable)**.** *For any tensor product X of fields in the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$ of $\Phi_{\boldsymbol{\beta}}$,*

$$M_t(X) = \frac{\mathbf{E}(\mathcal{S}_\alpha O_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]X)}{\mathbf{E}\mathcal{S}_\alpha O_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]}\|g_t^{-1} \tag{6.5.5}$$

is a local martingale, where $g_t(z)$ is the Loewner map for multiple radial SLE(κ) system associated to $\mathcal{J}_\alpha(z) = \mathbf{E}\mathcal{S}O_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]$

Remark 6.5.4. The structure of multiple radial SLE(κ) systems is not yet fully understood. We do not provide a rigorous justification of this theorem in this paper, nor is the validity of our results dependent on this. In particular, the integral of motion used in our arguments can be directly verified independently.

Moreover, the Martingale Observable Theorem can be extended to linear combinations of screening fields of the form

$$\mathcal{S}O_{\boldsymbol{\beta}} := \sum_\alpha \sigma_\alpha\,\mathcal{S}_\alpha O_{\boldsymbol{\beta}}[\boldsymbol{\tau}_1],$$

where each $\mathcal{S}_\alpha$ corresponds to a distinct choice of integration contours associated with a link pattern α, and $\sigma_\alpha \in \mathbb{R}$ are real coefficients.

Corollary 6.5.5. *Let the divisor* $\boldsymbol{\tau_2} = -\frac{\sigma}{2}\delta_0 - \frac{\sigma}{2}\,\delta_\infty + \sigma\delta_z$ *where the parameter* $\sigma = \frac{1}{a}$, *and insert* $X = \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_2}]$

$$M_{t,\kappa}(z) = \frac{\mathbf{E}\mathcal{S}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_2}]}{\mathbf{E}\mathcal{S}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]}\|g_t^{-1} \tag{6.5.6}$$

is local martingale where $g_t(z)$ *is the Loewner map for multiple radial SLE(*κ*) system associated to* $\mathcal{Z}_\kappa(z) = \mathbf{E}\mathcal{S}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]$.

Explicit computation shows that

$$\begin{aligned}
&\mathbf{E}\mathcal{S}_\alpha\mathcal{O}_\beta[\boldsymbol{\tau_1}] = \mathbf{E}\oint_{C_1}\dots\oint_{C_n}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_2}]\\
&= \oint_{C_1}\dots\oint_{C_n}\prod_{1\le i<j\le n}(z_i-z_j)^{a^2}\prod_{1\le i<j\le m}(\xi_i-\xi_j)^{4a^2}\prod_{i=1}^{n}\prod_{j=1}^{m}\left(z_i-\xi_j\right)^{-2a^2}\\
&\prod_j z_j^{a(b-\frac{n-2m}{2}a-\frac{\sigma}{2})}\prod_k \xi_k^{-2a(b-\frac{n-2m}{2}a-\frac{\sigma}{2})}z^{\sigma(b-\frac{n-2m}{2}a)}g'(z_j)^{\lambda_b(a)}g'(z)^{\lambda_b(\sigma)}\\
&(z-z_j)^{\sigma a}(z-\xi_k)^{-2\sigma a}|g'(0)|^{2\lambda_b(b+\frac{2m-n}{2}a-\frac{\sigma}{2})}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{E}\oint_{C_1}\dots\oint_{C_n}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]\\
&= \oint_{C_1}\dots\oint_{C_n}\prod_{1\le i<j\le n}(z_i-z_j)^{a^2}\prod_{1\le i<j\le m}(\xi_i-\xi_j)^{4a^2}\prod_{i=1}^{n}\prod_{j=1}^{m}\left(z_i-\xi_j\right)^{-2a^2}\\
&\prod_j z_j^{a(b-\frac{n-2m}{2})}\prod_k \xi_k^{-2a(b-\frac{n-2m}{2}a)}z^{\sigma(b-\frac{n-2m}{2}a)}g'(z_j)^{\lambda_b(a)}|g'(0)|^{2\lambda_b(b-\frac{2m-n}{2}a)}.
\end{aligned}$$

Conjecture 6.5.6. As $\kappa \to 0$, the contour integral concentrate on the critical points of the master function.

$$\begin{aligned}
N_t(z) = &M_{t,0}(z) = \lim_{\kappa\to 0}M_{t,\kappa}(z) = \lim_{\kappa\to 0}\frac{\mathbf{E}\oint_{C_1}\dots\oint_{C_n}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_2}]}{\mathbf{E}\oint_{C_1}\dots\oint_{C_n}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]}\\
&= |g'(0)|^{-(m-\frac{n}{2})}\frac{\prod_{j=1}^m\xi_k}{\sqrt{\prod_{k=1}^n z_k}}z^{m-\frac{n}{2}-1}g'(z)\frac{\prod_{k=1}^n(z-z_k)}{\prod_{j=1}^m(z-\xi_j)^2},
\end{aligned} \tag{6.5.7}$$

where $\boldsymbol{\xi}$ solve the **stationary relations**. This is exactly the integral of motion $N_t(z)$ in the proof of the theorem.

$M_{t,\kappa}(z)$ is a $(\lambda_b(\sigma), 0)$ differential with respect to z, where $\lambda_b(\sigma) = \frac{1}{2a^2} - \frac{b}{a}$. By taking the limit $\kappa \to 0$, $\lim_{\kappa\to 0} \lambda_b(\sigma) = 1$, and thus $M_{t,0}(z)$ is a (1,0) differential.

Remark 6.5.7. The integral of motion $N_t(z)$ can be verified through direct computation. This heuristic argument provides valuable insight and motivation for constructing the integral of motion. It is worth noting that our paper is self-consistent and does not depend on the detailed clarification of the classical limits.

6.6 Enumerative algebraic geometry and link pattern*

In this section, we propose several illuminating conjectures for the classification of the quadratic differential $Q(z)dz^2 \in \mathcal{QD}(\boldsymbol{z})$ and equivalently the critical points of the trigonometric KZ equations:

In the chordal setting, multiple chordal SLE(0) systems have been constructed and analyzed in detail by Eveliina Peltola and H. Wang (2020), and the corresponding stationary (commutation) relations have been completely solved and classified in Scherbak and Varchenko (2003), Scherbak (2002a), and Scherbak (2002b). Motivated by these results, we propose the following conjectures concerning the structure of multiple radial SLE(0) systems.

Conjecture 6.6.1 (n even). Let $Q(z)\,dz^2 \in \mathcal{QD}(\boldsymbol{z})$ be an involution symmetric meromorphic quadratic differential with n simple zeros located on the unit circle (with even n) and m poles. Then, up to multiplication by a nonzero real constant, the horizontal trajectory $\Gamma(Q)$ with limiting ends at $\boldsymbol{z}$ satisfies:

- (Underscreening) If $m \le \frac{n}{2}$, then $\Gamma(Q)$ consists of m disjoint arcs connecting distinct pairs of zeros, forming a radial (n, m)-link. For each such link pattern, there exists a unique differential $Q \in \mathcal{QD}(\boldsymbol{z})$ (up to scaling) whose horizontal trajectories form this pattern.
- (Overscreening) If $\frac{n+1}{2} \le m \le n$, then $\Gamma(Q)$ consists of $n - m$ disjoint arcs connecting pairs of zeros, forming a radial $(n, n-m)$-link. For each such link pattern, there exists a continuous family of differential $Q \in \mathcal{QD}(\boldsymbol{z})$ (up to scaling) whose horizontal trajectories form this pattern.
- (Upper bound) If $m > n$, there exists no such quadratic differential $Q \in \mathcal{QD}(\boldsymbol{z})$.

Conjecture 6.6.2 (n odd). Let $Q(z)\,dz^2 \in \mathcal{QD}(\boldsymbol{z})$ be an involution symmetric meromorphic quadratic differential with n simple zeros located on the unit circle (with odd n) and m poles. Then, up to multiplication by a nonzero real constant, the horizontal trajectory $\Gamma(Q)$ with limiting ends at $\boldsymbol{z}$ satisfies:

- (Underscreening) If $m \leq \frac{n}{2}$, then $\Gamma(Q)$ consists of m disjoint arcs connecting distinct pairs of zeros, forming a radial (n, m)-link. For each such link pattern, there exists a unique differential $Q \in \mathcal{QD}(\boldsymbol{z})$ (up to scaling) whose horizontal trajectories form this pattern.

- (Overscreening) If $\frac{n+1}{2} \leq m \leq n$, then $\Gamma(Q)$ consists of $n - m$ disjoint arcs connecting pairs of zeros, forming a radial $(n, n-m)$-link. For each such link pattern, there exists a unique differential $Q \in \mathcal{QD}(\boldsymbol{z})$ (up to scaling) whose horizontal trajectories form this pattern.

- (Upper bound) If $m > n$, there exists no such quadratic differential $Q \in \mathcal{QD}(\boldsymbol{z})$.

Remark 6.6.3. When n is an even integer, in the overscreening case, $Q(z)dz^2 = R'(z)^2 dz^2$, where $R(z)$ is an involution symmetric rational function with $\boldsymbol{z}$ as critical points. In this case, the continuous family of solutions can be obtained by post-composition with Möbius transformations.

We can equivalently reformulate our conjectures concerning the critical points of the master functions.

Conjecture 6.6.4. For generic $\boldsymbol{z}$ on the unit circle, critical points $\boldsymbol{\xi}$ of the master function $\Phi_{m,n}(\boldsymbol{z}, \boldsymbol{\xi})$ are involution symmetric.

Conjecture 6.6.5 (n even). For generic $\boldsymbol{z}$ on the unit circle:

- (Underscreening) If $m \leq \frac{n}{2}$, $\Phi_{m,n}(\boldsymbol{z}, \boldsymbol{\xi})$ has exactly $|LP(n, m)|$ isolated critical points.

- (Overscreening) If $\frac{n+1}{2} \leq m \leq n$, $\Phi_{m,n}(\boldsymbol{z}, \boldsymbol{\xi})$ has non-isolated critical points.

  Let $\lambda_1 = \sum \xi_i, \lambda_2 = \sum \xi_i \xi_j, \ldots, \lambda_m = \xi_1 \cdots \xi_m$ be the standard symmetric functions of $\xi_1, \ldots, \xi_m$. Denote $\mathbb{C}^m_\lambda$ the space with coordinates $\lambda_1, \ldots, \lambda_m$. Then written in symmetric coordinates $\lambda_1, \ldots, \lambda_m$, the critical points consist of $|LP(n, n-m)|$ straight lines in the space $\mathbb{C}^m_\lambda$.

- (Upperbound) If $m > n$, $\Phi_{m,n}(\boldsymbol{z}, \boldsymbol{\xi})$ has no critical points.

Conjecture 6.6.6 (n odd). For $\boldsymbol{\xi}$ and generic $\boldsymbol{z}$ on the unit circle:

- (Underscreening) If $m \le \frac{n}{2}$, $\Phi_{m,n}(\boldsymbol{z}, \boldsymbol{\xi})$ has $|LP(n,m)|$ isolated critical points.
- (Overscreening) If $\frac{n+1}{2} \le m \le n$, $\Phi_{m,n}(\boldsymbol{z}, \boldsymbol{\xi})$ has $|LP(n, n-m)|$ isolated critical points.
- (Upperbound) If $m > n$, $\Phi_{m,n}(\boldsymbol{z}, \boldsymbol{\xi})$ has no critical points.

6.7 Examples: underscreening

In this section, we provide a series of figures to illustrate the trace configurations arising from various multiple radial SLE(0) systems.

For multiple radial SLE(0) system with growth points $\boldsymbol{z}$ and screening charges $\boldsymbol{\xi}$, the corresponding quadratic differential is given by

$$Q(z)dz^2 = \frac{\prod_{j=1}^m \xi_k^2}{\prod_{k=1}^n z_k} z^{2m-n-2} \frac{\prod_{k=1}^n (z-z_k)^2}{\prod_{j=1}^m (z-\xi_j)^4} dz^2.$$

Lemma 6.7.1. *Given $Q(z) \in \mathcal{QD}(\boldsymbol{z})$ associate to it a vector field v_Q on $\widehat{\mathbb{C}}$ defined by*

$$v_Q(z) = \frac{1}{\sqrt{Q(z)}} \tag{6.7.1}$$

where

$$\sqrt{Q(z)} = \frac{\prod_{j=1}^m \xi_k}{\sqrt{\prod_{k=1}^n z_k}} z^{m-\frac{n}{2}-1} \frac{\prod_{k=1}^n (z-z_k)}{\prod_{j=1}^m (z-\xi_j)^2}$$

The flow lines of $\dot{z} = v_Q(z)$ are the horizontal trajectories of $Q(z)dz^2$.

Remark 6.7.2. This lemma provides an elementary way to plot the horizontal trajectories of $Q(z)dz^2$.

In the following figures, the zeros are marked in red, the poles in yellow, and the marked point $u = 0$ in green.

Figure 6.1: $n = 2, m = 1, z_1 = i, z_2 = -i$. The SLE(0) curves connect z_1 and z_2 to the origin.

$$\sqrt{Q(z)} = iz^{-2}(z-i)(z+i)$$

Figure 6.2: $n = 2, m = 1, z_1 = i, z_2 = -i, \xi_1 = 1$. The SLE(0) curve connects z_1 and z_2, and does not surround the origin.

$$\sqrt{Q(z)} = iz^{-1}\frac{(z-i)(z+i)}{(z-1)^2}$$

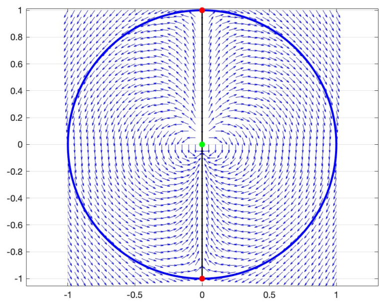

Figure 6.1: $z_1 = i$, $z_2 = -i$

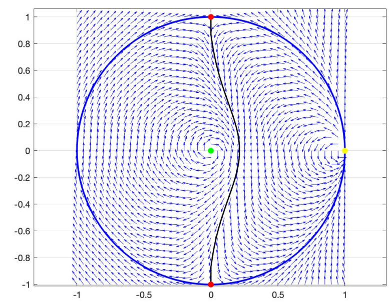

Figure 6.2: $z_1 = i$, $z_2 = -i$, $\xi_1 = -1$

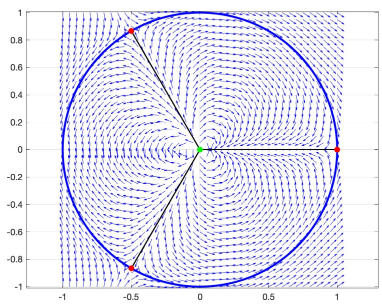

Figure 6.3: $z_k = e^{2\pi i k/3}$, $k = 0, 1, 2$

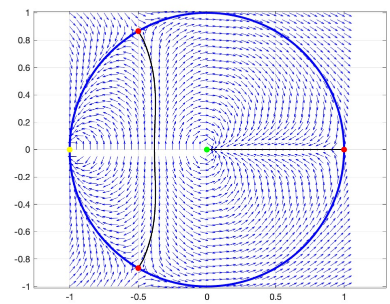

Figure 6.4: $z_k = e^{2\pi i k/3}$, $k = 0, 1, 2$, $\xi_1 = -1$

Figure 6.3: $n = 3, m = 0$. The SLE(0) curves connect all three z_k to the origin.

$$\sqrt{Q(z)} = iz^{-5/2}(z-1)(z-e^{2\pi i/3})(z-e^{4\pi i/3})$$

Figure 6.4: $n = 3, m = 1$, with pole $\xi = -1$. The SLE(0) curves connect z_2 and z_3, and connect z_1 to 0.

$$\sqrt{Q(z)} = iz^{-3/2}\frac{(z-1)(z-e^{2\pi i/3})(z-e^{4\pi i/3})}{(z-1)^2}$$

Figure 6.5: $n = 4, m = 0$. All z_k are connected to the origin.

$$\sqrt{Q(z)} = iz^{-3}\prod_{k=0}^{3}(z-e^{(2k+1)\pi i/4})$$

Figure 6.6: $n = 4, m = 1$, with pole $\xi = 1$. The SLE(0) curves connect z_3 and z_4 to 0, and connect z_1 to z_2.

$$\sqrt{Q(z)} = -iz^{-2}\frac{\prod_{k=0}^{3}(z-e^{(2k+1)\pi i/4})}{(z-1)^2}$$

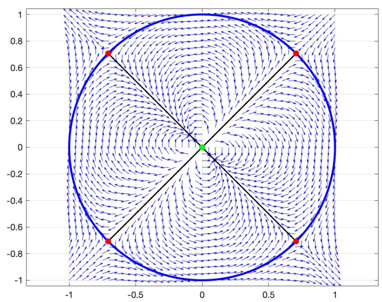

Figure 6.5: $z_k = e^{(2k+1)\pi i/4}$, $k = 0, 1, 2, 3$

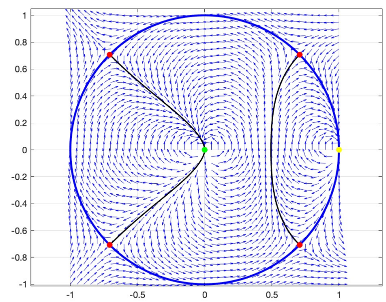

Figure 6.6: $z_k = e^{(2k+1)\pi i/4}$, $k = 0, 1, 2, 3$, $\xi_1 = 1$

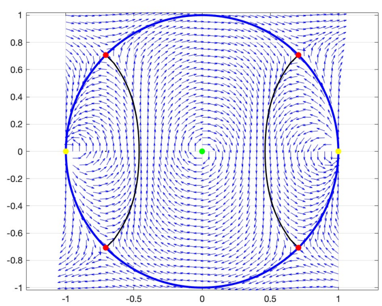

Figure 6.7: $z_k = e^{k\pi i/4}$, $k = 1, 2, 3, 4$, $\xi_1 = 1$, $\xi_2 = -1$

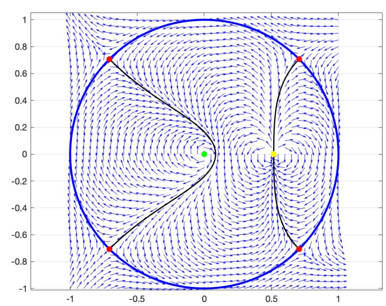

Figure 6.8: $z_k = e^{(2k+1)\pi i/4}$, $k = 0, 1, 2, 3$, $\xi_1 = -1$

Figure 6.7: $n = 4, m = 2$, $z_k = e^{(2k+1)\pi i/4}$, $\xi_1 = -1, \xi_2 = 1$. The SLE(0) curves connect z_1 and z_4, and z_2 and z_3.

$$\sqrt{Q(z)} = iz^{-1}\frac{\prod_{k=0}^{3}(z - e^{(2k+1)\pi i/4})}{(z-1)^2(z+1)^2}$$

Figure 6.8: $n = 4, m = 2$, $z_k = e^{(2k+1)\pi i/4}$, $\xi_1 = \sqrt{2-\sqrt{3}}, \xi_2 = \sqrt{2+\sqrt{3}}$. The SLE(0) curves connect z_3 and z_4 to 0, and connect z_1 to z_2.

$$\sqrt{Q(z)} = iz^{-1}\frac{\prod_{k=0}^{3}(z - e^{(2k+1)\pi i/4})}{(z-\sqrt{2-\sqrt{3}})^2(z-\sqrt{2+\sqrt{3}})^2}$$

6.8 Examples: overscreening

Let us recall the definition of the trace quadratic differential

Definition 2. *Restatement of definition (1.3.6)*

Let $z = \{z_1, z_2, \ldots, z_n\}$ be distinct points on the unit circle, a class of quadratic differentials with prescribed zeros denoted by $\mathcal{QD}(z)$:

(1) involution symmetric: $\overline{Q(z^*)(dz^*)^2} = Q(z)dz^2$, *where* $z^* = \frac{1}{\bar{z}}$;

(2) zeros of order 2 at $\{z_1, z_2, \ldots, z_n\}$;

(3) $\{\xi_1, \ldots, \xi_m\}$ *are poles of order 4 and* $Res_{\xi_j}(\sqrt{Q}dz) = 0$, $j = 1, \ldots, m$ *(Residue-free)*;

(4) poles of order $n + 2 - 2m$ *at marked points* 0 *and* ∞.

Note that when $m > \frac{n}{2} + 1$, the poles at 0 and ∞ are in fact zeros. $m = \frac{n}{2} + 1$ is a threshold for screening.

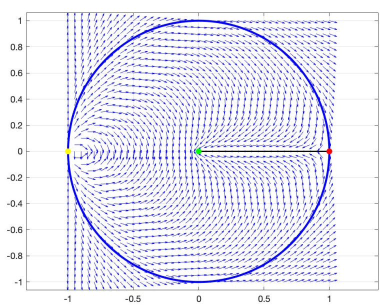

Figure 6.1: $z_1 = 1, \xi_1 = -1$

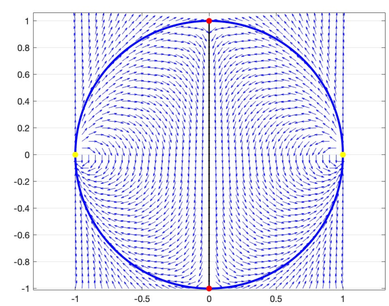

Figure 6.2: $z_1 = i$, $z_2 = -i$, $\xi_1 = 1$, $\xi_2 = -1$

Figure 4.9: $n = 1, m = 1$, $z_1 = 1$, $\xi_1 = -1$. The SLE(0) curve connects z_1 to 0.

$$\sqrt{Q(z)} = z^{-1/2}\frac{z-1}{(z-i)^2}$$

Figure 4.10: $n = 2, m = 2$, $z_1 = -i$, $z_2 = i$, $\xi_1 = -1$, $\xi_2 = 1$. The SLE(0) curve connects z_1 and z_2.

$$\sqrt{Q(z)} = i\frac{(z-i)(z+i)}{(z-1)^2(z+1)^2}$$

Figure 4.11: $n = 3, m = 2$, $z_k = e^{2k\pi i/3}$, $\xi_1 = -\frac{3}{2} + \frac{\sqrt{5}}{2}, \xi_2 = -\frac{3}{2} - \frac{\sqrt{5}}{2}$. SLE(0) curves connect two of the z_k's to 0, and the third pair together.

$$\sqrt{Q(z)} = z^{-1/2}\frac{(z-1)(z-e^{2\pi i/3})(z-e^{4\pi i/3})}{(z+\frac{3}{2}-\frac{\sqrt{5}}{2})^2(z+\frac{3}{2}+\frac{\sqrt{5}}{2})^2}$$

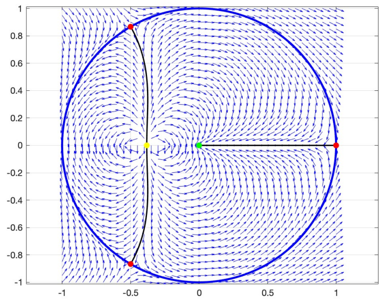

Figure 6.3: $z_k = e^{2k\pi i/3}$, $\xi_1 = -\frac{3}{2} + \frac{\sqrt{5}}{2}$, $\xi_2 = -\frac{3}{2} - \frac{\sqrt{5}}{2}$

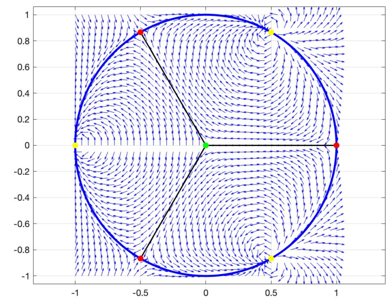

Figure 6.4: $z_k = e^{2k\pi i/3}$, $\xi_k = e^{(2k-1)\pi i/3}$

Figure 4.12: $n = 3, m = 3$, $z_k = e^{2k\pi i/3}$, $\xi_k = e^{(2k-1)\pi i/3}$. The SLE(0) curve connects two of the z_k's to 0, and the remaining two together.

$$\sqrt{Q(z)} = z^{1/2} \frac{(z-1)(z-e^{2\pi i/3})(z-e^{4\pi i/3})}{\prod_{k=0}^{2}(z-e^{(2k+1)\pi i/3})^2}$$

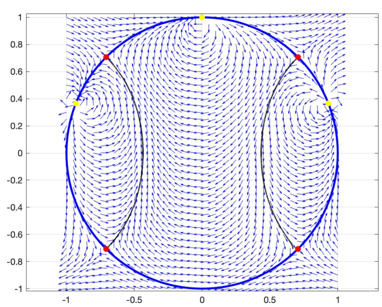

Figure 6.5: $z_k = e^{(2k+1)\pi i/4}$, $\xi_1 = i$, $\xi_{2,3}$ complex conjugates

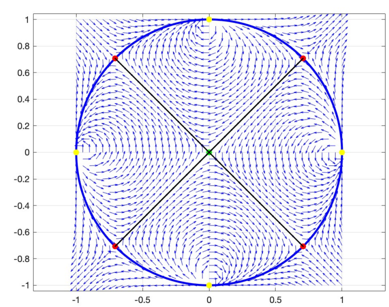

Figure 6.6: $z_k = e^{(2k+1)\pi i/4}$, $\xi_k = e^{k\pi i/2}$

Figure 4.13: $n = 4, m = 3$, $z_k = e^{(2k+1)\pi i/4}$, $\xi_1 = i$, $\xi_2 = -\frac{\sqrt[4]{3}}{\sqrt{2}} + \frac{-1+\sqrt{3}}{2}i$, $\xi_3 = \frac{\sqrt[4]{3}}{\sqrt{2}} + \frac{-1+\sqrt{3}}{2}i$. SLE(0) curves connect z_1 and z_4, and z_2 and z_3.

$$\sqrt{Q(z)} = \frac{z^4+1}{(z-i)^2(z-\xi_2)^2(z-\xi_3)^2}$$

Figure 4.14: $n = 4, m = 3$, $z_k = e^{(2k+1)\pi i/4}$, $\xi_k = e^{k\pi i/2}, k = 0, 1, 2, 3$. SLE(0) curves connect each z_k to the origin.

$$\sqrt{Q(z)} = z \cdot \frac{z^4+1}{z^4-1}$$

Chapter 7

MULTIPLE RADIAL SLE(0) SYSTEM WITH SPIN

7.1 Residue-free quadratic differentials with prescribed zeros

Definition 7.1.1 (Quadratic differentials with prescribed zeros and spin)**.** *Let $\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be distinct points on the unit circle $\partial\mathbb{D}$. We define $\mathcal{QD}(\boldsymbol{\theta})$ to be the class of meromorphic quadratic differentials on $\mathbb{C}$ of the form*

$$Q(\theta)\, d\theta^2 = \frac{\prod_{k=1}^m e^{2i\zeta_k}}{\prod_{j=1}^n e^{i\theta_j}}\, e^{i(2m-n)\theta}\, \frac{\prod_{j=1}^n \left(e^{i\theta} - e^{i\theta_j}\right)^2}{\prod_{k=1}^m \left(e^{i\theta} - e^{i\zeta_k}\right)^4}\, d\theta^2,$$

satisfying the following conditions:

1. *symmetric under the involution $\theta^* = \bar{\theta}$, meaning*
$$\overline{Q(\theta^*)}\overline{(d\theta^*)^2} = Q(\theta) d\theta^2.$$

2. *distinct zeros at $\{\theta_1, \theta_2, \ldots, \theta_n\}$, each of order 2.*

3. *distinct finite poles at $\{\zeta_1, \ldots, \zeta_m\}$, each of order 4, and the residues vanish (Residue-free condition):*
$$Res_{\zeta_j}(\sqrt{Q(\theta)}d\theta) = 0, \quad \textit{for } j = 1, \ldots, m.$$

Here, the poles $\{\zeta_1, \ldots, \zeta_m\}$ are finite, meaning they do not coincide with ∞.

Theorem 7.1.2 (Traces as horizontal trajectories in angular coordinates)**.** *Let $\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be distinct angular coordinates on the unit circle, i.e., $z_j = e^{i\theta_j} \in \partial\mathbb{D}$, and let $\boldsymbol{\zeta} = \{\zeta_1, \zeta_2, \ldots, \zeta_m\}$ be positions of poles satisfying the conjudgation symmetry $\zeta_k^* = -\zeta_k$ and the stationary relations.*

Then there exists a quadratic differential $Q(\theta)\, d\theta^2 \in \mathcal{QD}(\boldsymbol{\theta})$, with double zeros at $\theta_1, \ldots, \theta_n$ and poles of order 4 at $\zeta_1, \ldots, \zeta_m$, such that the hulls K_t generated by the multiple radial Loewner flow with driving functions $\boldsymbol{\theta}(t)$ and screening charges $\boldsymbol{\zeta}(t)$ are a subset of the horizontal trajectories of $Q(\theta)\, d\theta^2$ whose limiting ends are at $\boldsymbol{\theta}$, up to any time t prior to a collision among poles and zeros.

Moreover, for such times t,

$$Q(\boldsymbol{\theta}) \circ h_t^{-1} \in \mathcal{QD}(\boldsymbol{\theta}(t)),$$

where h_t denotes the covering map associated with the Loewner evolution, and $\boldsymbol{\theta}(t)$ are the angles of the time-evolved growth points.

Proof. The proof proceeds by adapting the argument used in Theorem 1.3.7, now expressed entirely in angular coordinates. Specifically, we apply the angular version of the integral of motion (see Corollary 6.4.2) to show that the time-evolved hulls K_t remain embedded in the horizontal trajectories of a quadratic differential $Q(\theta)d\theta^2 \in \mathcal{QD}(\boldsymbol{\theta})$, as claimed. $\square$

7.2 Field integral of motion and horizontal trajectories as flow lines

In this section, we generalize the integral of motion for multiple radial SLE(0) systems to the case where the spin η is non-zero.

We begin by considering the following integral of motion $N_t(z)$: let $z_1, z_2, \ldots, z_n$ be distinct points on the unit circle, and $z \in \overline{\mathbb{D}}$. Let

$$N_t(z) = e^{-(m-\frac{n}{2})(\int_0^t \sum_j \nu_j(s)ds)} g_t(z)^{m-\frac{n}{2}-1-\frac{\eta i}{2}} g_t'(z) \frac{\prod_{k=1}^n (g_t(z) - z_k(t))}{\prod_{j=1}^m (g_t(z) - \xi_j(t))^2}.$$

However, the term $g_t(z)^{m-\frac{n}{2}-1-\frac{\eta i}{2}}$ is multivalued and $N_t(z)$ is in fact not well-defined. To resolve this technical issue, we will write this expression in angular coordinates.

Theorem 7.2.1. *In angular coordinates, let $\xi_k = e^{i\zeta_k}$, $z_k = e^{i\theta_k}$, and let $h_t(z)$ be the covering map of the radial Loewner flow $g_t(z)$, i.e.,($e^{ih_t(z)} = g_t(e^{iz})$.) Then for each $z \in \overline{\mathbb{H}}$, define the observable*

$$N_t^{\mathrm{ang}}(z) = e^{-(m-\frac{n}{2})t} \cdot \frac{\prod_{j=1}^m e^{i\zeta_j(t)}}{\prod_{k=1}^n e^{i\frac{\theta_k(t)}{2}}} \cdot e^{i(m-\frac{n}{2}-1)h_t(z)} \cdot e^{\frac{\eta}{2}h_t(z)} \cdot h_t'(z) \cdot \frac{\prod_{k=1}^n (e^{ih_t(z)} - e^{i\theta_k(t)})}{\prod_{j=1}^m (e^{ih_t(z)} - e^{i\zeta_j(t)})^2} \cdot e^{-iz}. \tag{7.2.1}$$

Then $N_t^{\mathrm{ang}}(z)$ is an integral of motion on the time interval $[0, \tau_z \wedge \tau)$, where τ is the first collision time of any poles or critical points, and τ_z is the swallowing time of the point z under the multiple radial Loewner flow with parametrization $\nu_j(t) = 1$, $\nu_k(t) = 0$ for $k \neq j$.

Proof. The expression $N_t^{\mathrm{ang}}(z)$ can be factorized as the product of a part depending only on time,

$$A^{\mathrm{ang}}(t) = \frac{\prod_{j=1}^m e^{i\zeta_j(t)}}{\prod_{k=1}^n e^{i\frac{\theta_k(t)}{2}}} \cdot e^{-(m-\frac{n}{2})t},$$

and a part depending on z,

$$B_t^{\text{ang}}(z) = e^{i(m-\frac{n}{2}-1)h_t(z)} \cdot e^{\frac{\eta}{2}h_t(z)} \cdot h_t'(z) \cdot \frac{\prod_{k=1}^{n}(e^{ih_t(z)} - e^{i\theta_k(t)})}{\prod_{j=1}^{m}(e^{ih_t(z)} - e^{i\zeta_j(t)})^2} \cdot e^{-iz}.$$

By direct computation,

$$\frac{d}{dt}\log A^{\text{ang}}(t) = -\frac{i\eta}{2}, \quad \frac{d}{dt}\log B_t^{\text{ang}}(z) = \frac{i\eta}{2}.$$

These terms cancel, and hence

$$\frac{d}{dt}\log N_t^{\text{ang}}(z) = \frac{d}{dt}\log A^{\text{ang}}(t) + \frac{d}{dt}\log B_t^{\text{ang}}(z) = 0.$$

Therefore, $N_t^{\text{ang}}(z)$ is conserved under the flow. □

Theorem 7.2.2. *In angular coordinates, define $\xi_k = e^{i\zeta_k}$, $z_k = e^{i\theta_k}$, and let $h_t(z)$ be the covering map of the Loewner flow $g_t(z)$, i.e., $e^{ih_t(z)} = g_t(e^{iz})$. For any $z \in \overline{\mathbb{H}}$, define:*

$$A^{\text{ang}}(t) = \frac{\prod_{j=1}^{m} e^{i\zeta_j(t)}}{\prod_{k=1}^{n} e^{i\frac{\theta_k(t)}{2}}}, \tag{7.2.2}$$

$$B_t^{\text{ang}}(z) = e^{-(2m-n)\int_0^t \sum_j \nu_j(s)ds} \cdot g_t(z)^{2m-n-2} \cdot e^{i(m-\frac{n}{2}-1+\frac{\eta}{2})h_t(z)} \cdot h_t'(z) \cdot e^{ih_t(z)} \cdot \frac{\prod_{k=1}^{n}\left(e^{ih_t(z)} - e^{i\theta_k(t)}\right)}{\prod_{j=1}^{m}\left(e^{ih_t(z)} - e^{i\zeta_j(t)}\right)^2}, \tag{7.2.3}$$

$$N_t^{\text{ang}}(z) = A^{\text{ang}}(t) \cdot B_t^{\text{ang}}(z). \tag{7.2.4}$$

Then $N_t^{\text{ang}}(z)$ defines a field integral of motion for the multiple radial SLE(0) Loewner flows with driving weights $\nu_j(t)$, on the interval $[0, \tau_t \wedge \tau)$, where τ is the first collision time among the poles or driving points.

Proof. The computation is a deformation of the zero-spin case ($\eta = 0$), with the additional spin term contributing to the angular prefactor. By direct differentiation:

$$\frac{d}{dt}\log A^{\text{ang}}(t) = -\frac{i\eta}{2}\sum_{j=1}^{n}\nu_j(t), \tag{7.2.5}$$

$$\frac{d}{dt}\log B_t^{\text{ang}}(z) = \frac{i\eta}{2}\sum_{j=1}^{n}\nu_j(t), \tag{7.2.6}$$

$$\frac{d}{dt}\log N_t^{\text{ang}}(z) = \frac{d}{dt}\log A^{\text{ang}}(t) + \frac{d}{dt}\log B_t^{\text{ang}}(z) = 0. \tag{7.2.7}$$

Hence, $N_t^{\text{ang}}(z)$ is preserved under the flow and is therefore a field integral of motion. □

7.3 Classical limit of martingale observables*

In this section, we discuss how the field integral of motion is heuristically derived as the classical limit of martingale observables constructed as the correlation functions of conformal fields.

Based on the SLE-CFT correspondence, we can couple the multiple radial SLE(κ) system to a conformal field theory constructed via vertex operators, following the approach outlined in Kang and N. Makarov (2013) and N-G. Kang and N. Makarov (2021)

Definition 7.3.1 (n-leg operator with screening charges)**.** *Consider the following charge distribution on the Riemann sphere*

$$\boldsymbol{\beta} = b\delta_0 + b\delta_\infty$$

$$\boldsymbol{\tau_1} = \sum_{j=1}^{n} a\delta_{z_j} - \sum_{k=1}^{m} 2a\delta_{\xi_k} + (b+(m-\frac{n}{2})a - \frac{i\eta a}{2})\delta_0 + (b+(m-\frac{n}{2})a + \frac{i\eta a}{2})\delta_\infty \quad (7.3.1)$$

$$\boldsymbol{\tau_2} = -\frac{\sigma}{2}\delta_0 - \frac{\sigma}{2}\,\delta_\infty + \sigma\delta_z,$$

where the parameter $\sigma = \frac{1}{a}$.

The n-leg operator with screening charges $\boldsymbol{\xi}$ *and background charge* $\boldsymbol{\beta}$ *is given by the OPE exponential:*

$$\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}] = \frac{C_{(b)}[\boldsymbol{\tau_1} + \boldsymbol{\beta}]}{C_{(b)}[\boldsymbol{\beta}]} \mathrm{e}^{\odot i\Phi[\boldsymbol{\tau_1}]}. \quad (7.3.2)$$

Definition 7.3.2 (Screening fields)**.** *For each link pattern* α*, we can choose closed contours* $C_1, \ldots, C_n$ *along which we may integrate the* $\boldsymbol{\xi}$ *variables to screen the vertex fields. Let* $\mathcal{S}$ *be the screening operator, we define the screening operation as*

$$\mathcal{S}_\alpha \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}] = \oint_{C_1} \ldots \oint_{C_n} \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}].$$

Meanwhile, we integrate the correlation function $\mathbf{E}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}] = \Phi_\kappa(z, \boldsymbol{\xi})$ *, the conformal dimension is 1 at the* $\boldsymbol{\xi}$ *points, i.e. since* $\lambda_b(-2a) = 1$*. This leads to the partition function for the corresponding multiple radial SLE(*κ*) system:*

$$\mathcal{J}_\alpha^\eta(z) := \mathbf{E}\mathcal{S}_\alpha \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}] = \oint_{C_1} \ldots \oint_{C_n} \Phi_\kappa(z, \boldsymbol{\xi}) d\xi_n \ldots d\xi_1.$$

Theorem 7.3.3 (Martingale observable)**.** *For any tensor product X of fields in the OPE family $\mathcal{F}_\beta$ of Φ_β,*

$$M_{t,\kappa}(X) = \frac{\mathbf{E}\mathcal{S}_\alpha O_\beta[\boldsymbol{\tau_1}]X}{\mathbf{E}\mathcal{S}_\alpha O_\beta[\boldsymbol{\tau_1}]} \| g_t^{-1} \tag{7.3.3}$$

is a local martingale, where $g_t(z)$ is the Loewner map for multiple radial SLE(κ) system associated to $\mathcal{J}_\alpha^\eta(z) = \mathbf{E}\mathcal{S}_\alpha O_\beta[\boldsymbol{\tau_1}]$.

Corollary 7.3.4. *Let the divisor $\boldsymbol{\tau_2} = -\frac{\sigma}{2}\delta_0 - \frac{\sigma}{2}\ \delta_\infty + \sigma\delta_z$ where the parameter $\sigma = \frac{1}{a}$, and insert $X = O_\beta[\boldsymbol{\tau_2}]$.*

$$M_{t,\kappa}(z) = \frac{\mathbf{E}\mathcal{S}O_\beta[\boldsymbol{\tau_1}]O_\beta[\boldsymbol{\tau_2}]}{\mathbf{E}\mathcal{S}O_\beta[\boldsymbol{\tau_1}]} \| g_t^{-1} \tag{7.3.4}$$

is local martingale where $g_t(z)$ is the Loewner map for multiple radial SLE(κ) system associated to $\mathcal{Z}_\kappa(z) = \mathbf{E}\mathcal{S}O_\beta[\boldsymbol{\tau_1}]$.

Explicit computation shows that

$$\begin{aligned}
&\mathbf{E}\oint_{C_1}\cdots\oint_{C_n} O_\beta[\boldsymbol{\tau_1}]O_\beta[\boldsymbol{\tau_2}] \\
&= \oint_{C_1}\cdots\oint_{C_n} \prod_{1\le i<j\le n}(z_i-z_j)^{a^2}\prod_{1\le i<j\le m}(\xi_i-\xi_j)^{4a^2}\prod_{i=1}^{n}\prod_{j=1}^{m}\left(z_i-\xi_j\right)^{-2a^2} \\
&\prod_j z_j^{a(b-\frac{n-2m}{2}a-\frac{i\eta a}{2}-\frac{\sigma}{2})}\prod_k \xi_k^{-2a(b-\frac{n-2m}{2}a-\frac{i\eta a}{2}-\frac{\sigma}{2})} z^{\sigma(b-\frac{n-2m}{2}a-\frac{i\eta a}{2}-\frac{\sigma}{2})} g'(z_j)^{\lambda_b(a)} g'(z)^{\lambda_b(\sigma)} \\
&(z-z_j)^{\sigma a}(z-\xi_k)^{-2\sigma a}|g'(0)|^{\lambda_b(b+\frac{2m-n}{2}a+\frac{i\eta a}{2}-\frac{\sigma}{2})+\lambda_b(b+\frac{2m-n}{2}a-\frac{i\eta a}{2}-\frac{\sigma}{2})}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{E}\oint_{C_1}\cdots\oint_{C_n} O_\beta[\boldsymbol{\tau_1}] \\
&= \oint_{C_1}\cdots\oint_{C_n} \prod_{1\le i<j\le n}(z_i-z_j)^{a^2}\prod_{1\le i<j\le m}(\xi_i-\xi_j)^{4a^2}\prod_{i=1}^{n}\prod_{j=1}^{m}\left(z_i-\xi_j\right)^{-2a^2}. \\
&\prod_j z_j^{a(b-\frac{n-2m}{2}-\frac{i\eta a}{2})}\prod_k \xi_k^{-2a(b-\frac{n-2m}{2}a-\frac{i\eta a}{2})} g'(z_j)^{\lambda_b(a)} \\
&|g'(0)|^{\lambda_b(b-\frac{2m-n}{2}a+\frac{i\eta a}{2})+\lambda_b(b-\frac{2m-n}{2}a-\frac{i\eta a}{2})}
\end{aligned}$$

Conjecture 7.3.5. As $\kappa \to 0$, the Coulomb gas contour integrals concentrate on the critical points of the master function.

$$
\begin{aligned}
N_t(z) = & M_{t,0}(z) = \lim_{\kappa\to 0} M_{t,\kappa}(z) = \lim_{\kappa\to 0} \frac{\mathbf{E} \oint_{C_1} \cdots \oint_{C_n} O_\beta[\boldsymbol{\tau_1}] O_\beta[\boldsymbol{\tau_2}]}{\mathbf{E} \oint_{C_1} \cdots \oint_{C_n} O_\beta[\boldsymbol{\tau_1}]} \\
= & |g'(0)|^{-(m-\frac{n}{2})} \frac{\prod_{j=1}^m \xi_k}{\sqrt{\prod_{k=1}^n z_k}} z^{m-\frac{n}{2}-1-\frac{\eta i}{2}} g'(z) \frac{\prod_{k=1}^n (z-z_k)}{\prod_{j=1}^m (z-\xi_j)^2}
\end{aligned}
\tag{7.3.5}
$$

which is exactly the integral of motion we use.

$M_{t,\kappa}(z)$ is a $(\lambda_b(\sigma), 0)$ differential with respect to z, where $\lambda_b(\sigma) = \frac{1}{2a^2} - \frac{b}{a}$. By taking the limit $\kappa \to 0$, $\lim_{\kappa\to 0} \lambda_b(\sigma) = 1$, thus $M_{t,0}(z)$ is a (1,0) differential.

Remark 7.3.6. The integral of motion $N_t(z)$ can be verified through direct computation.

7.4 Examples: spin

In this section, we provide a series of figures to illustrate the trace configurations arising from various multiple radial SLE(0) systems with spin.

Remark 7.4.1. In the case of multiple radial SLE(0) with spin η, for $\boldsymbol{z}$ and $\boldsymbol{\xi}$, the quadratic differential $Q(z)dz^2$ can be written as

$$
Q(z)dz^2 = \frac{\prod_{j=1}^m \xi_k^2}{\prod_{k=1}^n z_k} z^{2m-n-2-\eta i} \frac{\prod_{k=1}^n (z-z_k)^2}{\prod_{j=1}^m (z-\xi_j)^4} dz^2.
$$

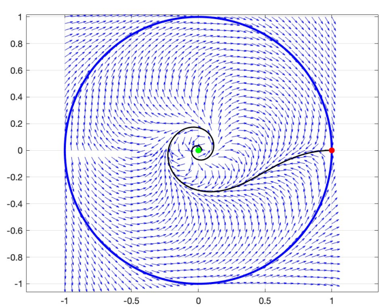

Figure 7.1: $n = 1, \eta = -4$

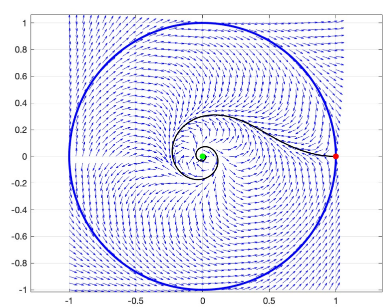

Figure 7.2: $n = 1, \eta = 4$

Figure 7.1: $n = 1$, $z_1 = 1$, $\eta = -4$. A clockwise spiral connects z_1 to 0.

$$
\sqrt{Q(z)} = z^{-3/2+2i}(z-1)
$$

Figure 7.2: $\eta = 4$. A counterclockwise spiral connects z_1 to 0.

$$
\sqrt{Q(z)} = z^{-3/2-2i}(z-1)
$$

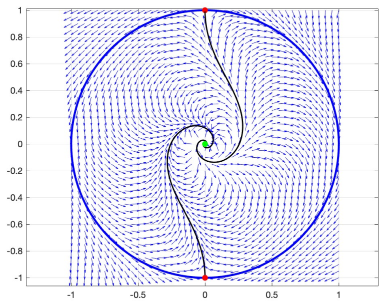

Figure 7.3: $n = 2$, $m = 0$, $\eta = -4$

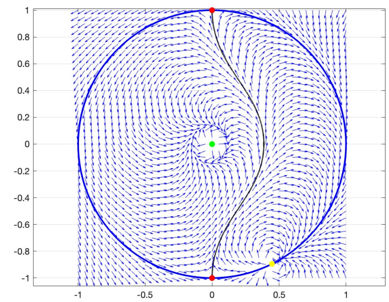

Figure 7.4: $n = 2$, $m = 1$, $\eta = -4$

—

Figure 7.3: $z_1 = i, z_2 = -i$, two spirals connect z_1, z_2 to 0.

$$\sqrt{Q(z)} = z^{-2+2i}(z-i)(z+i)$$

Figure 7.4: Pole $\xi = \frac{\sqrt{-4-2i}}{\sqrt{4-2i}}$. The link pattern remains stable under spin perturbation; the pole moves clockwise as $\eta < 0$. A closed orbit is observed.

$$\sqrt{Q(z)} = iz^{-1+2i}\frac{(z-i)(z+i)}{\left(z - \frac{\sqrt{-4-2i}}{\sqrt{4-2i}}\right)^2}$$

—

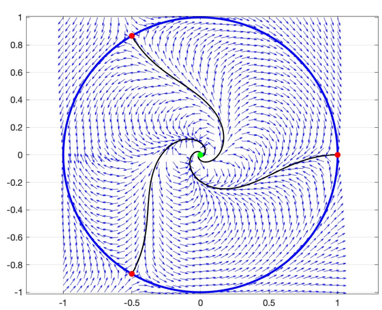

Figure 7.5: $n = 3$, $m = 0$, $\eta = -4$

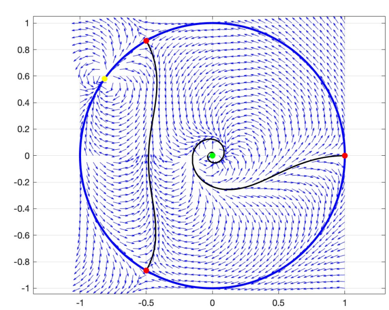

Figure 7.6: $n = 3$, $m = 1$, $\eta = -4$

Figure 7.5: $z_k = e^{2k\pi i/3}$. Three spirals connect each z_k to 0.

$$\sqrt{Q(z)} = z^{-5/2+2i}(z-1)(z-e^{2\pi i/3})(z-e^{4\pi i/3})$$

Figure 7.6: Pole $\xi = \frac{(4+3i)^{1/3}}{(4-3i)^{1/3}}$.

$$\sqrt{Q(z)} = z^{-3/2+2i}\frac{(z-1)(z-e^{2\pi i/3})(z-e^{4\pi i/3})}{(z-\xi)^2}$$

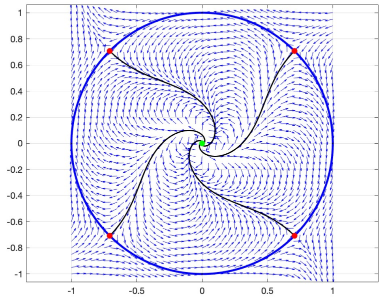

Figure 7.7: $n = 4$, $m = 0$, $\eta = -4$

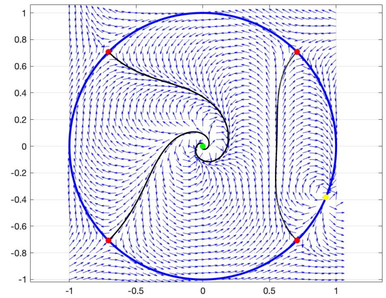

Figure 7.8: $n = 4$, $m = 1$, $\eta = -4$

—

Figure 7.7: $z_k = e^{(2k+1)\pi i/4}$, $\frac{\eta}{2} = 2$.

$$\sqrt{Q(z)} = z^{-3+2i} \prod_{k=0}^{3} \left(z - e^{(2k+1)\pi i/4}\right)$$

Figure 7.8: Pole $\xi = \frac{(-4-4i)^{1/4}}{(4-4i)^{1/4}}$.

$$\sqrt{Q(z)} = -i\frac{(-4-4i)^{1/4}}{(4-4i)^{1/4}} z^{-2+2i} \frac{\prod_{k=0}^{3}(z - e^{(2k+1)\pi i/4})}{(z-\xi)^2}$$

—

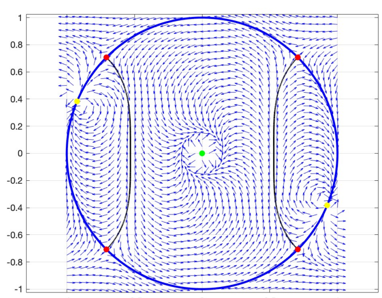

Figure 7.9: $n = 4$, $m = 2$, $\eta = -4$

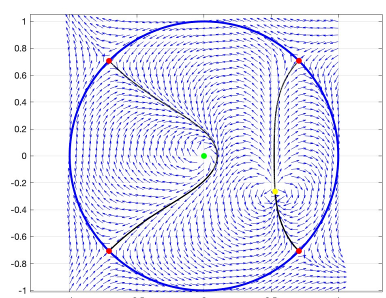

Figure 7.10: $n = 4$, $m = 2$, $\eta = -1$

Figure 7.9: Poles at

$$\xi_1 = \frac{(-4-4i)^{1/4}}{(4-4i)^{1/4}}, \quad \xi_2 = -\frac{(-4-4i)^{1/4}}{(4-4i)^{1/4}}$$

The link pattern is stable under perturbation; a closed orbit appears.

$$\sqrt{Q(z)} = -i\frac{(-4-4i)^{1/2}}{(4-4i)^{1/2}} z^{-2+i} \frac{\prod_{k=0}^{3}(z - e^{(2k+1)\pi i/4})}{(z-\xi_1)^2(z-\xi_2)^2}$$

Figure 7.10: $\eta = -1$. Let $\xi_1 = 0.5299 - 0.2650i$, $\xi_2 = 1.5097 - 0.7549i$ be the roots of

$$\sum_{k=0}^{3} \frac{\xi - e^{(2k+1)\pi i/4}}{\xi + e^{(2k+1)\pi i/4}} + i = 2\frac{\xi + 1/\xi^*}{\xi - 1/\xi^*}$$

Then

$$\sqrt{Q(z)} = (0.8 + 0.6i)z^{-1+i/2} \frac{\prod_{k=0}^{3}(z - e^{(2k+1)\pi i/4})}{(z - \xi_1)^2(z - \xi_2)^2}$$

Chapter 8

RELATIONS TO CALOGERO-SUTHERLAND SYSTEM

8.1 Multiple radial SLE(0) and classical Calogero-Sutherland system

In this section, we study the relations between the multiple radial SLE(0) and classical Calogero-Sutherland system.

Theorem 8.1.1. *Let $\boldsymbol{\theta} = \{\theta_1, \dots, \theta_n\}$ be distinct real points and $\zeta = \{\zeta_1, \dots, \zeta_m\}$ closed under conjugation and solve the stationary relation. Let $\boldsymbol{\theta}(t)$ and $\boldsymbol{\zeta}(t)$ evolve according to multiple radial SLE(0) system with a common parametrization of capacity ($i.e. v_j(t) = 1$).*

(i) The pair $(\boldsymbol{\theta}(t), \boldsymbol{\zeta}(t))$ forms the closed dynamical system satisfying

$$\dot{\theta}_j = 2\left(\sum_{k\neq j} \cot(\frac{\theta_j - \theta_k}{2}) - \sum_{k=1}^{m} \cot(\frac{\theta_j - \zeta_k}{2})\right), \tag{8.1.1}$$

and

$$\dot{\zeta}_k = 2\left(-\sum_{l\neq k} \cot(\frac{\zeta_k - \zeta_l}{2}) + \sum_{j=1}^{n} \cot(\frac{\zeta_k - \theta_j}{2})\right). \tag{8.1.2}$$

(ii) $\boldsymbol{\theta}(t)$ evolve according to the classical Calegero-Sutherland Hamiltonian, in other words:

$$\ddot{\theta}_j = -\sum_{k\neq j} \frac{\cos(\frac{\theta_j-\theta_k}{2})}{\sin^3(\frac{\theta_j-\theta_k}{2})}.$$

(iii) ζ_k follows the second-order dynamics.

$$\ddot{\zeta}_k = -\sum_{l\neq k} \frac{\cos(\frac{\zeta_k-\zeta_l}{2})}{\sin^3(\frac{\zeta_k-\zeta_l}{2})}. \tag{8.1.3}$$

(iv) The energy of the system is given by

$$\mathcal{H}(\boldsymbol{\theta}, \boldsymbol{p}) = -\frac{n(2m-n)^2}{2} + \frac{n}{2} - \frac{n(n^2-1)}{6}.$$

Proof of theorem (8.1.1).

(i) The evolution of $\theta_j(t)$ is

$$\dot{\theta}_j = \left(\sum_{k\neq j} \cot(\frac{\theta_j-\theta_k}{2}) - 2\sum_{k=1}^{m} \cot(\frac{\theta_j-\zeta_k}{2}) + \sum_{k\neq j} \cot(\frac{\theta_j-\theta_k}{2}) \right)$$
$$= 2\sum_{k\neq j} \cot(\frac{\theta_j-\theta_k}{2}) - 2\sum_{k=1}^{m} \cot(\frac{\theta_j-\zeta_k}{2}).$$

On the other hand, since the poles follow the Loewner flow we have $\zeta_k(t) := g_t(\zeta_k(0))$, and therefore

$$\dot{\zeta}_k = \dot{g}_t(\zeta_k(0)) = \sum_{j=1}^{n} \cot(\frac{g_t(\zeta_k(0))-\theta_j}{2}) = \sum_{j=1}^{n} \cot(\frac{\zeta_k-\theta_j}{2}).$$

The stationary relation implies that

$$\dot{\zeta}_k = 2\sum_{l\neq k} \cot(\frac{\zeta_k-\zeta_l}{2}) = -2\sum_{l\neq k} \cot(\frac{\zeta_k-\zeta_l}{2}) + 2\sum_{j=1}^{n} \cot(\frac{\zeta_k-\theta_j}{2}).$$

(ii) By differentiating, we have

$$\ddot{\theta}_j = -\sum_{k\neq j} \frac{\dot{\theta}_j-\dot{\theta}_k}{\sin^2(\frac{\theta_j-\theta_k}{2})} + \sum_{l} \frac{\dot{\theta}_j-\dot{\zeta}_l}{\sin^2(\frac{\theta_j-\zeta_l}{2})}.$$

Using the formula (8.1.1) for $\dot{\theta}_j, \dot{\theta}_k$ and the equality (8.1.2) for $\dot{\zeta}_l$ we obtain

$$\ddot{\theta}_j = -\frac{1}{2}\sum_{k\neq j} \frac{1}{\sin^2(\frac{\theta_j-\theta_k}{2})} \left(2\cot(\frac{\theta_j-\theta_k}{2})\right) +$$
$$-\frac{1}{2}\sum_{k\neq j} \frac{1}{\sin^2(\frac{\theta_j-\theta_k}{2})} \left(\sum_{l\neq j,k} \left(\cot(\frac{\theta_j-\theta_l}{2}) - \cot(\frac{\theta_k-\theta_l}{2})\right) + \sum_{l} \left(\cot(\frac{\zeta_l-\theta_j}{2}) - \cot(\frac{\zeta_l-\theta_k}{2})\right)\right)$$
$$+\frac{1}{2}\sum_{l} \frac{1}{\sin^2(\frac{\theta_j-\zeta_l}{2})} \left(\sum_{k\neq j} \cot(\frac{\theta_j-\theta_k}{2}) + \sum_{s=1}^{m} \cot(\frac{\zeta_s-\theta_j}{2}) + \sum_{s\neq l} \cot(\frac{\zeta_l-\zeta_s}{2}) - \sum_{k=1}^{n} \cot(\frac{\zeta_l-\theta_k}{2}) \right).$$

Rearranging terms gives

$$\ddot{\theta}_j + \sum_{k\neq j} \frac{\cos(\frac{\theta_j-\theta_k}{2})}{\sin^3(\frac{\theta_j-\theta_k}{2})} - \frac{1}{2}\sum_{k\neq j}\sum_{l\neq j,k} \frac{1}{\sin(\frac{\theta_j-\theta_k}{2})\sin(\frac{\theta_j-\theta_l}{2})\sin(\frac{\theta_k-\theta_l}{2})}$$
$$= -\frac{1}{2}\sum_{k\neq j}\sum_{l} \frac{1}{\sin\left(\frac{\theta_j-\theta_k}{2}\right)\sin\left(\frac{\theta_j-\zeta_l}{2}\right)\left(\frac{\theta_k-\zeta_l}{2}\right)} +$$
$$+\frac{1}{2}\sum_{l} \frac{1}{\sin\left(\frac{\theta_j-\zeta_l}{2}\right)^2} \left(\sum_{k\neq j} \cot(\frac{\theta_j-\theta_k}{2}) + \sum_{m} \cot(\frac{\zeta_m-\theta_j}{2}) - \sum_{m\neq l} \cot(\frac{\zeta_l-\zeta_m}{2}) \right).$$

The last term on the right hand side used the stationary relation and then use the stationary relation again to obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{l}\frac{1}{\sin\left(\frac{\theta_j-\zeta_l}{2}\right)^2}\left(\sum_{k\neq j}\cot(\frac{\theta_j-\theta_k}{2})+\sum_{m}\cot(\frac{\zeta_m-\theta_j}{2})-\sum_{m\neq l}\cot(\frac{\zeta_l-\zeta_m}{2})\right)\\
&=\frac{1}{2}\sum_{l}\frac{1}{\sin\left(\frac{\theta_j-\zeta_l}{2}\right)^2}\sum_{m\neq l}\left(\cot(\frac{\zeta_l-\zeta_m}{2})+\cot(\frac{\zeta_m-\theta_j}{2})\right)\\
&=\frac{1}{2}\sum_{l}\sum_{m\neq l}\frac{1}{\sin\left(\frac{\theta_j-\zeta_l}{2}\right)\sin\left(\frac{\theta_j-\zeta_m}{2}\right)\sin\left(\frac{\zeta_l-\zeta_m}{2}\right)}.
\end{aligned}$$

Combining all of the above, we obtain

$$\begin{aligned}
\ddot{\theta}_j+\sum_{k\neq j}\frac{\cos(\frac{\theta_j-\theta_k}{2})}{\sin^3(\frac{\theta_j-\theta_k}{2})}=&\frac{1}{2}\sum_{k\neq j}\sum_{l\neq j,k}\frac{1}{\sin\left(\frac{\theta_j-\theta_k}{2}\right)\sin\left(\frac{\theta_j-\theta_l}{2}\right)\sin\left(\frac{\theta_k-\theta_l}{2}\right)}\\
&+\frac{1}{2}\sum_{l}\sum_{m\neq l}\frac{1}{\sin\left(\frac{\theta_j-\zeta_l}{2}\right)\sin\left(\frac{\theta_j-\zeta_m}{2}\right)\sin\left(\frac{\zeta_l-\zeta_m}{2}\right)}.
\end{aligned}$$

The right-hand side is canceled by symmetry.

(iii) Differentiating the equality (8.1.2), we have

$$\ddot{\zeta}_k=-\sum_{l\neq k}\frac{\dot{\zeta}_k-\dot{\zeta}_l}{\sin^2(\frac{\zeta_k-\zeta_l}{2})}.$$

Now by using the first equality of (6.4) again for $\dot{\zeta}_k,\dot{\zeta}_l$ we obtain

$$\ddot{\zeta}_k=-\frac{1}{2}\sum_{l\neq k}\frac{1}{\sin^2(\frac{\zeta_k-\zeta_l}{2})}\left(\cot(\frac{\zeta_k-\zeta_l}{2})+\sum_{m\neq k,l}\cot(\frac{\zeta_k-\zeta_m}{2})-\cot(\frac{\zeta_l-\zeta_k}{2})-\sum_{m\neq k,l}\cot(\frac{\zeta_l-\zeta_m}{2})\right)$$

Rearranging terms gives

$$\ddot{\zeta}_k=-\sum_{l\neq k}\frac{\cos(\frac{\zeta_k-\zeta_l}{2})}{\sin^3(\frac{\zeta_k-\zeta_l}{2})}+\frac{1}{2}\sum_{l\neq k}\sum_{m\neq k,l}\frac{1}{\sin(\frac{\zeta_k-\zeta_l}{2})\sin(\frac{\zeta_k-\zeta_m}{2})\sin(\frac{\zeta_l-\zeta_m}{2})}.$$

The last term is canceled by symmetry.

(iv) For a multiple radial SLE(0) system with n growth points and m screening charges that solve the stationary relations, by equation (6.2.1) in the proof of the theorem (6.2.1),

$$U_j=\sum_{k\neq j}\cot(\frac{\theta_j-\theta_k}{2})-2\sum_{k=1}^{m}\cot(\frac{\theta_j-\zeta_k}{2})$$

satisfies the null vector equation (1.3.3) with constant

$$h_{m,n} = -\frac{(2m-n)^2}{2} + \frac{1}{2}.$$

Plugging into equation (8.1.4) and equation (8.1.6), we obtain the desired result.

□

Proof of theorem (1.4.1). For multiple radial SLE(0) system with common parametrization of capacity (i.e. $\nu_j(t) = 1$ for $j = 1, 2, \ldots, n$), let $\left\{\left(\theta_j, U_j\right), j = 1, \ldots, n\right\}$ are related to $\left\{\left(\theta_j, p_j\right), j = 1, \ldots, n\right\}$ via

$$p_j = \left(U_j + \sum_{k \neq j} \cot(\frac{\theta_j - \theta_k}{2})\right),$$

where U_j solves the null vector equations (1.3.3).

(i) Solving for U_j and inserting the result into the left-hand side of the null vector equation leads to the identity.

$$\begin{aligned} h =& \frac{1}{2}U_j^2 + \sum_k f_{kj}U_k - \sum_k \frac{3}{2}(1 + f_{jk}^2) \\ =& \frac{1}{2}p_j^2 - \sum_k \left(p_j + p_k\right) f_{jk} + \sum_k \sum_{l \neq k} f_{jk} f_{jl} - 2\sum_k f_{jk}^2 + C_{n-1}^2 + \frac{3}{2}(n-1) \\ =& \mathcal{H}_j(\boldsymbol{\theta}, \boldsymbol{p}) + C_{n-1}^2 + \frac{3}{2}(n-1), \end{aligned} \tag{8.1.4}$$

where

$$f_{jk} = f_{jk}(\boldsymbol{\theta}) = \begin{cases} 0, & j = k \\ \cot(\frac{\theta_j - \theta_k}{2}), & j \neq k \end{cases}.$$

Therefore, $\mathcal{H}_j$ is preserved under the Loewner flow.

Futheremore, for each $c \in \mathbb{R}$, the submanifolds defined by the null vector Hamiltonian

$$N_c = \left\{(\boldsymbol{\theta}, \boldsymbol{p}) : \mathcal{H}_j(\boldsymbol{\theta}, \boldsymbol{p}) = c \text{ for all } j\right\} \tag{8.1.5}$$

are invariant under the Loewner flow

By direct computation, $\mathcal{H}_j$ is related to the Calogero-Sutherland Hamiltonian $\mathcal{H}$ by:

$$\sum_j \mathcal{H}_j = \mathcal{H}. \tag{8.1.6}$$

Our next result shows that null vector Hamiltonian $\mathcal{H}_j$ has a nice interpretation in terms of the Lax pair for the Calogero-Sutherland system.

Theorem 8.1.2. *The Lax pair is two square matrices $L = L(\boldsymbol{\theta}, \boldsymbol{p})$ and $M = M(\boldsymbol{\theta}, \boldsymbol{p})$ each of size $n \times n$, and by Moser (1975) the entries are given by*

$$L_{jk} = \begin{cases} p_j, & j = k, \\ 2f_{jk}, & j \neq k, \end{cases} \quad \text{and} \quad M_{jk} = \begin{cases} -\sum_l f_{jl}^2, & j = k \\ f_{jk}^2, & j \neq k \end{cases}.$$

This leads to the following representation of $\mathcal{H}_j$ in terms of L^2.

$$\mathcal{H}_j = \frac{1}{2}\mathrm{e}_j' L^2 \mathbf{1}, \tag{8.1.7}$$

where e_j' is the transpose of the j th standard basis vector and 1 is the vector of all ones.

Consequently, the U_j, $j = 1, \ldots, n$, defined by solving the null vector equations for a given $\boldsymbol{\theta}$ iff the $\boldsymbol{p}$ variables satisfy $L^2(\boldsymbol{\theta}, \boldsymbol{p})\mathbf{1} = \mathbf{0}$.

Proof. Write $L = P - X_1$, where $P = P(\boldsymbol{p}) = \mathrm{diag}(\boldsymbol{p})$ is the square matrix with entries of $\boldsymbol{p}$ along its diagonal, and $X_1 = X_1(\boldsymbol{\theta})$ is the square matrix with entries $(X_1)_{jk} = f_{jk}$. Note that P is symmetric and X_1 is anti-symmetric. Then

$$L^2 = P^2 - PX_1 - X_1P + X_1^2.$$

It is straightforward to compute the entries of $P^2 - PX_1 - X_1P$ and see that they give the first two terms on the right-hand side of the Hamiltonian. For

X_1^2 we have

$$
\begin{aligned}
\mathrm{e}_j' X_1^2 1 = \sum_k \left(X_1^2\right)_{jk} &= -4(\sum_k \sum_i f_{jl} f_{kl}) \\
&= -4\left(\sum_l f_{jl}^2 + \sum_{k\neq j} \sum_{l\neq j} f_{jl} f_{kl}\right) \\
&= -4\left(\sum_l f_{j1}^2 + \frac{1}{2} \sum_{k\neq j} \sum_{l\neq k} \left(f_{jl} f_{kl} + f_{jk} f_{lk}\right)\right) \\
&= -4\left(\sum_l f_{j1}^2 - \frac{1}{2} \sum_{k\neq j} \sum_{l\neq k} f_{jk} f_{jl} + \frac{1}{2} C_{n-1}^2\right).
\end{aligned}
$$

□

(ii)

Definition 8.1.3 (Poisson Bracket)**.** *For any smooth function $F = F(\boldsymbol{x}, \boldsymbol{p})$ defined on phase space, the associated vector field is given by*

$$X_F = \sum_{j=1}^{n} \frac{\partial F}{\partial p_j} \partial_{x_j} - \sum_{j=1}^{n} \frac{\partial F}{\partial x_j} \partial_{p_j}.$$

Given two smooth functions $F = F(\boldsymbol{x}, \boldsymbol{p})$ and $G = G(\boldsymbol{x}, \boldsymbol{p})$, the commutator of their associated vector fields satisfies

$$[X_F, X_G] = X_{\{F,G\}},$$

where $\{F, G\}$, the Poisson bracket of F and G, is defined by

$$\{F, G\} = \sum_{j=1}^{n} \left(\frac{\partial F}{\partial p_j} \frac{\partial G}{\partial x_j} - \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial p_j}\right).$$

By direct computation, for all j, k, the null vector Hamiltonians $\mathcal{H}_j$ and $\mathcal{H}_k$ satisfy the Poisson bracket identity

$$\{\mathcal{H}_j, \mathcal{H}_k\} = \frac{1}{f_{jk}^2} \left(\mathcal{H}_k - \mathcal{H}_j\right).$$

By the definition of N_c, we have $\{\mathcal{H}_j, \mathcal{H}_k\} = 0$ along N_c.

Thus, the vector fields $X_{\mathcal{H}_j}$ induced by the Hamiltonians $\mathcal{H}_j$ commute along the submanifolds N_c.

□

8.2 Null vector equations and quantum Calogero-Sutherland system

In this section, we obtain parallel relations between multiple radial SLE(κ) systems and the quantum Calogero-Sutherland system. We show that a partition function satisfying the null vector equations corresponds to an eigenfunction of the quantum Calogero-Sutherland Hamiltonian, as first discoverd in J. Cardy (2004).

Theorem 8.2.1. *The multiple radial SLE(κ) is described by the partition function $\mathcal{Z}(\boldsymbol{\theta})$, which satisfies the following relation:*

$$\mathcal{L}_j \mathcal{Z}(\boldsymbol{\theta}) = h\mathcal{Z}(\boldsymbol{\theta}), \tag{8.2.1}$$

where $\mathcal{L}_j$ is the null vector differential operator given by:

$$\mathcal{L}_j = \frac{\kappa}{2}\left(\frac{\partial}{\partial\theta_j}\right)^2 + \sum_{k\neq j}\left(\cot\left(\frac{\theta_k-\theta_j}{2}\right)\frac{\partial}{\partial\theta_k} - \frac{6-\kappa}{2\kappa}\frac{1}{2\sin^2\left(\frac{\theta_k-\theta_j}{2}\right)}\right). \tag{8.2.2}$$

(i) *By transforming the partition function $\mathcal{Z}(\boldsymbol{\theta})$ using the Coulomb gas correlation factor $\Phi^{-1}_{\frac{1}{\kappa}}(\boldsymbol{\theta})$, we obtain*

$$\tilde{\mathcal{Z}}(\boldsymbol{\theta}) = \Phi^{-1}_{\frac{1}{\kappa}}(\boldsymbol{\theta})\mathcal{Z}(\boldsymbol{\theta}), \tag{8.2.3}$$

where

$$\Phi_r(\boldsymbol{\theta}) = \prod_{1\leq j<k\leq n}\left(\sin\frac{\theta_j-\theta_k}{2}\right)^{-2r}.$$

The transformed partition function $\tilde{\mathcal{Z}}(\boldsymbol{\theta})$ satisfies

$$\left(\Phi^{-1}_{\frac{1}{\kappa}}\cdot\mathcal{L}_j\cdot\Phi_{\frac{1}{\kappa}}\right)\tilde{\mathcal{Z}}(\boldsymbol{\theta}) = h\tilde{\mathcal{Z}}(\boldsymbol{\theta}),$$

where the differential operator $\Phi^{-1}_{\frac{1}{\kappa}}\cdot\mathcal{L}_j\cdot\Phi_{\frac{1}{\kappa}}$ is given by

$$\begin{aligned}\Phi^{-1}_{\frac{1}{\kappa}}\cdot\mathcal{L}_j\cdot\Phi_{\frac{1}{\kappa}} =& \frac{\kappa}{2}\partial_j^2 - F_j\partial_j + \frac{1}{2\kappa}F_j^2 - \frac{1}{2}F_j' \\ &- \sum_{k\neq j}\left(f_{jk}\left(\partial_k - \frac{1}{\kappa}F_k\right) - \frac{6-\kappa}{2\kappa}f_{jk}'\right).\end{aligned} \tag{8.2.4}$$

The sum of the null vector differential operators is

$$\Phi_{-\frac{1}{\kappa}}\cdot\mathcal{L}\cdot\Phi_{\frac{1}{\kappa}} = \kappa H_n\left(\frac{8}{\kappa}\right) - \frac{n(n^2-1)}{6\kappa}, \tag{8.2.5}$$

where $H_n(\beta)$, with $\beta = \frac{8}{\kappa}$, is the quantum Calogero-Sutherland Hamiltonian:

$$H_n(\beta) = \sum_{j=1}^{n}\frac{1}{2}\frac{\partial^2}{\partial\theta_j^2} - \frac{\beta(\beta-2)}{16}\sum_{1\leq j<k\leq n}\frac{1}{\sin^2\left(\frac{\theta_j-\theta_k}{2}\right)}.$$

(ii) The commutation relation between the null vector operators $\mathcal{L}_j$ and $\mathcal{L}_k$ is

$$[\mathcal{L}_j, \mathcal{L}_k] = \frac{1}{\sin^2\left(\frac{\theta_j-\theta_k}{2}\right)}(\mathcal{L}_k - \mathcal{L}_j).$$

As a result:

$$[\mathcal{L}_j, \mathcal{L}_k]\mathcal{Z}(\boldsymbol{\theta}) = \frac{1}{\sin^2\left(\frac{\theta_j-\theta_k}{2}\right)}(\mathcal{L}_k - \mathcal{L}_j)\mathcal{Z}(\boldsymbol{\theta}) = 0.$$

Proof of theorem (8.2.1). Recall that the null vector differential operator $\mathcal{L}_j$ is given by

$$\mathcal{L}_j = \frac{\kappa}{2}\left(\frac{\partial}{\partial\theta_j}\right)^2 + \sum_{k\neq j}\left(\cot\left(\frac{\theta_k-\theta_j}{2}\right)\frac{\partial}{\partial\theta_k} + \left(1-\frac{6}{\kappa}\right)\frac{1}{4\sin^2\left(\frac{\theta_k-\theta_j}{2}\right)}\right). \tag{8.2.6}$$

Then, the null vector equations for $\psi(\boldsymbol{\theta})$ can be written as

$$\mathcal{L}_j\psi(\boldsymbol{\theta}) = h\psi(\boldsymbol{\theta}) \tag{8.2.7}$$

for $j = 1, 2, \ldots, n$.

(i) To simplify the formula, we introduce the notation

$$f(x) = \cot\left(\frac{x}{2}\right), \quad f_{jk} = f\left(\theta_j - \theta_k\right), \quad F_j = \sum_{k\neq j} f_{jk}.$$

$$f'(x) = -\frac{1}{2}\frac{1}{\sin^2(\frac{x}{2})}, \quad f'_{jk} = f'\left(\theta_j - \theta_k\right), \quad F'_j = \sum_{k\neq j} f'_{jk}.$$

Using this notation, we have

$$\mathcal{L}_j = \frac{\kappa}{2}\partial_j^2 + \sum_{k\neq j} f_{kj}\partial_k + \sum_{k\neq j}(1-\frac{6}{\kappa})f'_{jk}$$

with $\partial_j = \frac{\partial}{\partial\theta_j}$ and the Calogero-Sutherland hamiltonian can be written as

$$H_n(\beta) = -\sum_j\left(\frac{1}{2}\partial_j^2 + \frac{\beta(\beta-2)}{16}F'_j\right). \tag{8.2.8}$$

where $\beta = \frac{8}{\kappa}$,

To relate the null-vector equations to the Calogero-Sutherland system, we sum up the null-vector operators. Let

$$\mathcal{L} = \sum_j \mathcal{L}_j = \frac{\kappa}{2} \sum_j \partial_j^2 + \sum_j \left(F_j \partial_j + h F_j' \right). \tag{8.2.9}$$

Then the partition functions $\psi(\boldsymbol{\theta})$ are eigenfunctions of $\mathcal{L}$ with eigenvalue nh.

$$\mathcal{L}\psi(\boldsymbol{\theta}) = nh\psi(\boldsymbol{\theta}) \tag{8.2.10}$$

Recall that

$$\Phi_r(\boldsymbol{\theta}) = \prod_{1 \le j < k \le n} \left(\sin \frac{\theta_j - \theta_k}{2} \right)^{-2r}.$$

From the properties $\partial_j \Phi_r = -r \Phi_r F_j$ and $\sum_j F_j^2 = -2 \sum_j F_j' - \frac{n\left(n^2-1\right)}{3}$, we can check that

$$\Phi_{-\frac{1}{\kappa}} \cdot \mathcal{L} \cdot \Phi_{\frac{1}{\kappa}} = \kappa H_n \left(\frac{8}{\kappa} \right) + \frac{n\left(n^2 - 1\right)}{6\kappa}$$

which implies

$$\tilde{\psi}(\boldsymbol{\theta}) = \Phi_{\frac{1}{\kappa}}^{-1}(\boldsymbol{\theta}) \psi(\boldsymbol{\theta})$$

is an eigenfunction of the Calogero-Sutherland hamiltonian $H_n\left(\frac{8}{\kappa}\right)$, with eigenvalue

$$E = \frac{n}{\kappa} \left(-h + \frac{(n^2 - 1)}{6\kappa} \right). \tag{8.2.11}$$

(ii) This is exactly the commutation relations of generators proved in theorem (3.2.1).

□

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