STRESS DISTRIBUTION DUE TO A TANGENTIAL CONCENTRATED LOAD IN A THIN CYLINDER

THESIS BY

DAVID HOLBROOK JARVIS

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MECHANICAL ENGINEER

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA

1946

STRESS DISTRIBUTION DUE TO TANGENTIAL CONCENTRATED LOAD IN A THIN CYLINDER

DAVID H. JARVIS

ABSTRACT:

A SOLUTION IS SHOWN FOR STRESS DISTRIBUTION IN AN INFINITE STRIP WITH WITH A TRANVERSE CONCENTRATED LOAD ACTING AT ITS CENTER. DEFORMATION RELATIONS ARE SATISFIED TO PERMIT THIS STRIP TO BE ROLLED TRANSVERSELY INTO AN INFINITE CYLINDER. EQUILIBRIUM CONDITIONS ARE THEN SATISFIED BY ADDING A PRESSURE DISTRIBUTION NORMAL TO THE SURFACE OF THIS CYLINDER. THE STRESS DUE TO THIS PRESSURE DISTRIBUTION IS SOLVED BY METHODS PREVIOUSLY DEVELOPED FOR THIN CYLINDERS AND SUPERPOSED ON THE STRIP STRESS. RESULTS SHOW THAT THE STRESSES ARE NEARLY THE SAME AS THOSE OBTAINED BY THE ELEMENTARY METHODS OF STRENGTH OF MATERIALS EXCEPT NEAR THE POINT OF APPLICATION OF THE LOAD, WHERE THERE ARE LOCALIZED BENDING STRESSES. THIS BENDING CAN BE EVALUATED BY THE THIN CYLINDER METHODS, BUT THE PROCESS IS VERY INVOLVED AND A SIMPLER APPROACH IS DESIRED. THIS IS FOUND BY CONSIDERING THE RESULTANT OF THE PRESSURE DISTRIBUTION TO BE A MOMENT APPLIED AT THE LOADING POINT. TWO METHODS OF ANALYZING STRESS IN A THIN CYLINDER LOADED WITH A CIRCUMFERENTIAL COUPLE ARE SHOWN AND DEFORMATION CURVES PLOTTED. THE STRESS CRITERION IS SHOWN TO BE THE NORMAL STRESS DEVELOPED NEAR THE LOAD IN THE INFINITE (UNROLLED) STRIP.

TABLE OF CONTENTS

PART	TITLE	PAGE
I	INTRODUCTION	1
II	STRESSES IN INFINITE STRIP	3
III	THIN CYLINDER SUBJECT TO PRESSURE AND BODY FORCES	9
IV	THIN CYLINDER SUBJECTED TO A CIRCUMFERENTIAL COUPLE	16
V	CONCLUSIONS	20

ACKNOWLEDGEMENTS

THE AUTHOR WISHES TO EXPRESS HIS GRATITUDE TO PROFESSOR R. R. MARTEL, DR. E. E. SECHLER, AND PROFESSOR R. L. DAUGHERTY WHO SUPERVISED THE WRITING OF THIS PAPER AT CALIFORNIA INSTITUTE OF TECHNOLOGY. HE IS ESPECIALLY GRATEFUL TO DR. G. W. HOUSNER OF THIS INSTITUTE FOR MANY VALUABLE SUGGESTIONS.

I. INTRODUCTION

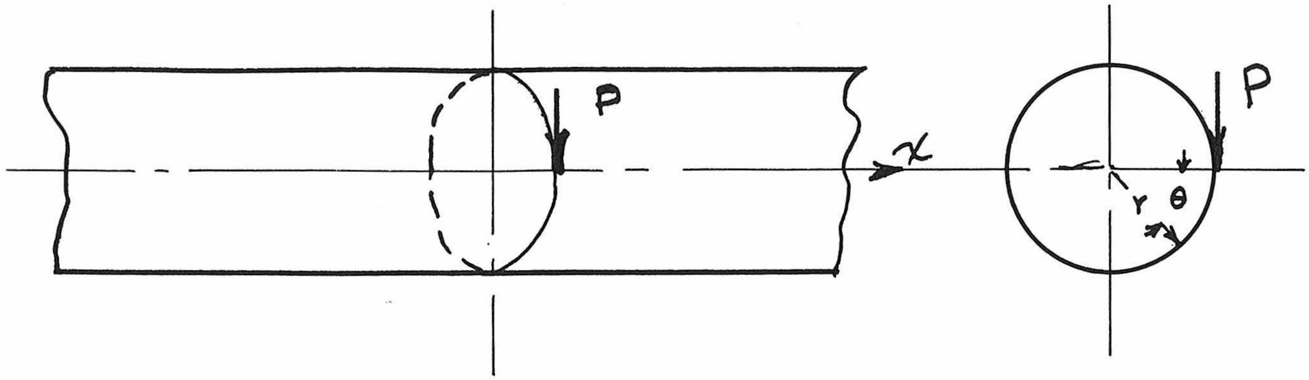
THE LOADING OF A PIPE OR PENSTOCK WITH A TANGENTIAL CONCENTRATED LOAD AS SHOWN IN FIGURE 1, HAS FREQUENT APPLICATION IN THE DESIGN, CONSTRUCTION AND INSTALLATION OF EQUIPMENT USED IN LARGE HYDRO-ELECTRIC POWER PLANTS.

AVAILABLE TECHNICAL LITERATURE DOES NOT GIVE AN EXACT SOLUTION OF THIS PROBLEM EXCEPT IN THE LIMITING CASE WHEN THE RADIUS OF THE CYLINDER APPROACHES INFINITY OR THICKNESS APPROACHES ZERO. THE SOLUTION IN THIS CASE IS THAT FOR A LOAD IN THE PLANE OF AN INFINITE PLATE OR STRIP, AND HAS BEEN DISCUSSED BY R. C. J. HOWLAND,[1] AND S. TIMOSHENKO.[2] IT IS REASONABLE TO ASSUME THAT THIS SOLUTION CAN BE APPLIED TO A CYLINDER OF FINITE RADIUS IF THE RESULTS OF DEFORMING THE STRIP CAN BE DETERMINED AND REMOVED BY A PROCESS WHICH GIVES STRESSES WITH THE SAME ORDER OF ACCURACY AS THOSE DETERMINED IN THE STRIP BEFORE DEFORMATION.

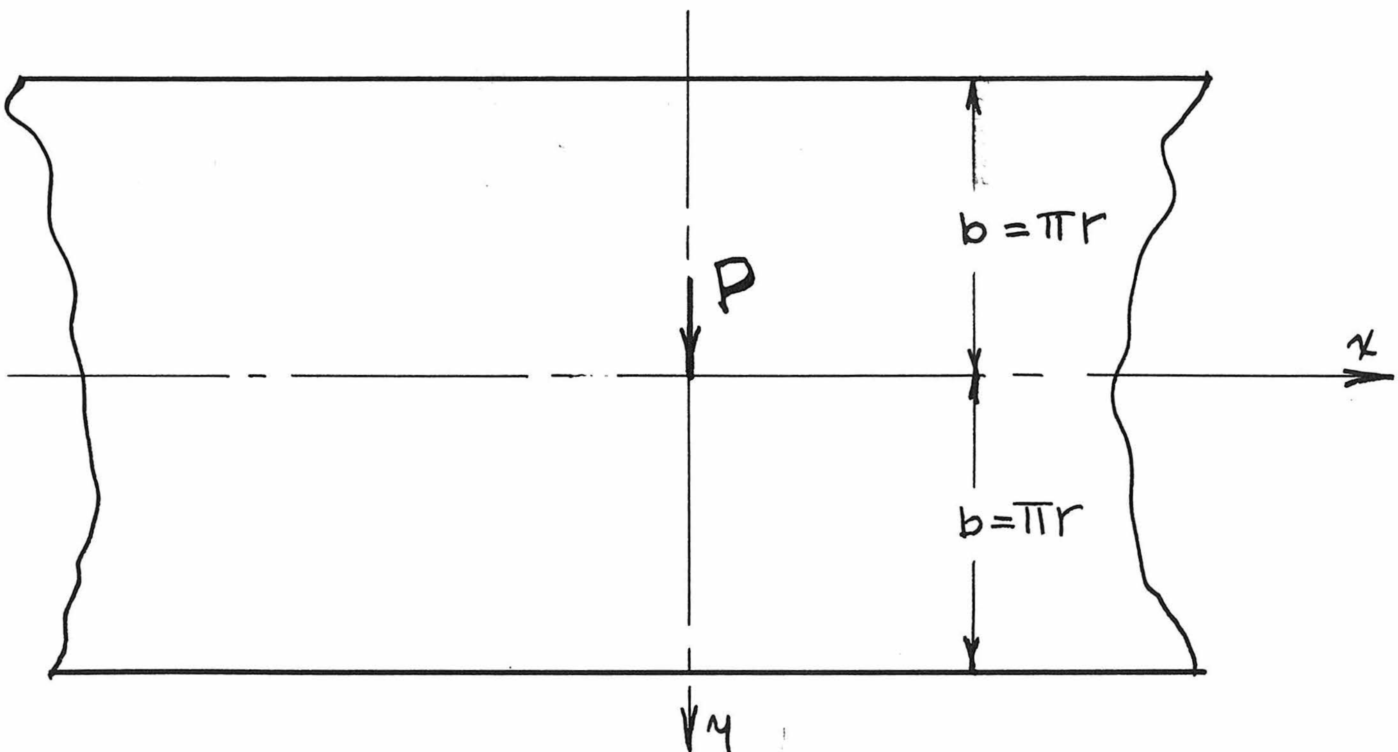
THE RATIO OF RADIUS TO THICKNESS IS ASSUMED TO BE OF THE ORDER OF 100 TO 1, SO NORMAL STRESSES ON A SECTION CAN BE CONSIDERED TO BE UNIFORMLY DISTRIBUTED ACROSS THE THICKNESS OF SECTION, AND BENDING STRESSES TO VARY LINEARLY ACROSS THE SECTION.

(1) HOWLAND, R. C. J., PROCEEDINGS OF THE ROYAL SOCIETY OF LONDON A VOL.124 P.89

(2) TIMOSHENKO, S., "THEORY OF ELASTICITY" MCGRAW-HILL BOOK CO., N.Y.,1934



a) TANGENTIAL LOAD ON INFINITE CYLINDER



b) TANGENTIAL LOAD ON INFINITE STRIP

FIGURE 1 LOADING

II. STRESSES IN INFINITE STRIP

THE FOLLOWING NOTATION IS USED IN THIS AND SUBSEQUENT CHAPTERS:

E - MODULUS OF ELASTICITY IN POUNDS PER SQUARE INCH

σ - INTENSITY OF STRESS IN POUNDS PER SQUARE INCH

μ - POISSONS RATIO

t - THICKNESS OF PLATE IN INCHES

b - HALF WIDTH OF STRIP $= \pi r$

r - RADIUS OF CYLINDER IN INCHES

θ - ANGULAR COORDINATE, A PURE NUMBER

x - AXIAL OR LONGITUDINAL COORDINATE IN INCHES

$y, r\theta$ - CIRCUMFERENTIAL OR TRANSVERSE COORDINATE IN INCHES

$X, Y, Z,$ - DISTRIBUTED LOADS PER UNIT AREA OF MIDDLE SURFACE IN POUNDS PER SQUARE INCH, POSITIVE AS SHOWN.

P - CONCENTRATED LOAD IN POUNDS $= 2bp$

p - CONCENTRATED LOAD PER UNIT WIDTH IN POUNDS PER INCH

ξ, η, ζ - DISPLACEMENTS HAVING SAME SENSES AS $X, Y, Z,$

S_x, S_y - NORMAL STRESS RESULTANTS PER UNIT LENGTH OF MIDDLE SURFACE, IN POUNDS PER INCH, POSITIVE AS SHOWN

S_{xy}, S_{yx} - SHEAR STRESS RESULTANTS PER UNIT LENGTH OF MIDDLE SURFACE IN POUNDS PER INCH, POSITIVE AS SHOWN

M_x, M_y - AXIAL AND TANGENTIAL BENDING MOMENTS OF CYLINDER PER UNIT LENGTH OF MIDDLE SURFACE IN POUNDS, POSITIVE WHEN TENDING TO CAUSE COMPRESSION IN OUTER FIBERS

M_{xy}, M_{yx} - AXIAL AND TANGENTIAL TWISTING MOMENTS OF PIPE OR CYLINDER PER UNIT OF MIDDLE SURFACE, POSITIVE AS SHOWN

V_x, V_y - RADIAL SHEAR STRESS RESULTANTS PER UNIT LENGTH OF MIDDLE SURFACE ON PLANES X= CONSTANT AND Y= CONSTANT IN POUNDS PER INCH, POSITIVE AS SHOWN

THE BOUNDARY CONDITIONS TO BE SATISFIED BY THE STRESS RELATIONS FOR AN INFINITE STRIP WITH TRANSVERSE CONCENTRATED LOAD AT THE CENTER ARE AS FOLLOWS:

1. NORMAL STRESS RESULTANTS MUST VANISH AT THE BOUNDARIES $y = b$ AND $y = -b$
2. SHEAR STRESS MUST VANISH AT $x = 0$
3. SHEAR STRESS RESULTANT AT ANY x = CONSTANT $\neq 0$ MUST, WHEN INTEGRATED BETWEEN $y = -b$ AND $y = b$ BE EQUAL TO $bp = P/2$
4. DISPLACEMENTS AT ANY POINT x = CONSTANT ON THE BOUNDARY $y = -b$ MUST EQUAL THE DISPLACEMENTS ON THE BOUNDARY $y = b$ FOR x = THE SAME CONSTANT.

USING THE SOLUTION FOR A LOAD IN THE PLANE OF AN INFINITE PLATE, THE BOUNDARY CONDITIONS CAN BE SATISFIED BY THE FOLLOWING RELATIONS.

$$S_y = \frac{P}{4\pi} \sum_{n=0}^{\pm\infty} \frac{y+2nb}{x^2+(y+2nb)^2}\left[-(3+\mu)+2(1+\mu)\frac{x^2}{x^2+(y+2nb)^2}\right] \qquad (1)$$

$$S_x = \frac{P}{4\pi} \sum_{n=0}^{\pm\infty} \frac{y+2nb}{x^2+(y+2nb)^2}\left[1-\mu-2(1+\mu)\frac{x^2}{x^2+(y+2nb)^2}\right] \qquad (2)$$

$$S_{xy} = -\frac{P}{4\pi} \sum_{n=0}^{\pm\infty} \frac{x}{x^2+(y+2nb)^2}\left[1-\mu+2(1+\mu)\frac{(y+2nb)^2}{x^2+(y+2nb)^2}\right] \qquad (3)$$

THESE EXPRESSIONS MAY BE SIMPLIFIED BY SUBSTITUTING **2bp** FOR P AND USING A NEW VARIABLE λ SUCH THAT $\Delta\lambda = \frac{2b}{x}$ AND $\lambda = \frac{2nb}{x}$. FOR VERY LARGE VALUES OF x THEN, WE MAY WRITE.

$$S_y = \frac{p}{4\pi}\int \frac{\lambda}{1+\lambda^2}\left[-(3+\mu) + 2(1+\mu)\frac{1}{1+\lambda^2}\right]d\lambda$$

$$S_x = \frac{p}{4\pi}\int \frac{\lambda}{1+\lambda^2}\left[1-\mu - 2(1+\mu)\frac{1}{1+\lambda^2}\right]d\lambda$$

$$S_{xy} = -\frac{p}{4\pi}\int \frac{1}{1+\lambda^2}\left[1-\mu + 2(1+\mu)\frac{\lambda^2}{1+\lambda^2}\right]d\lambda$$

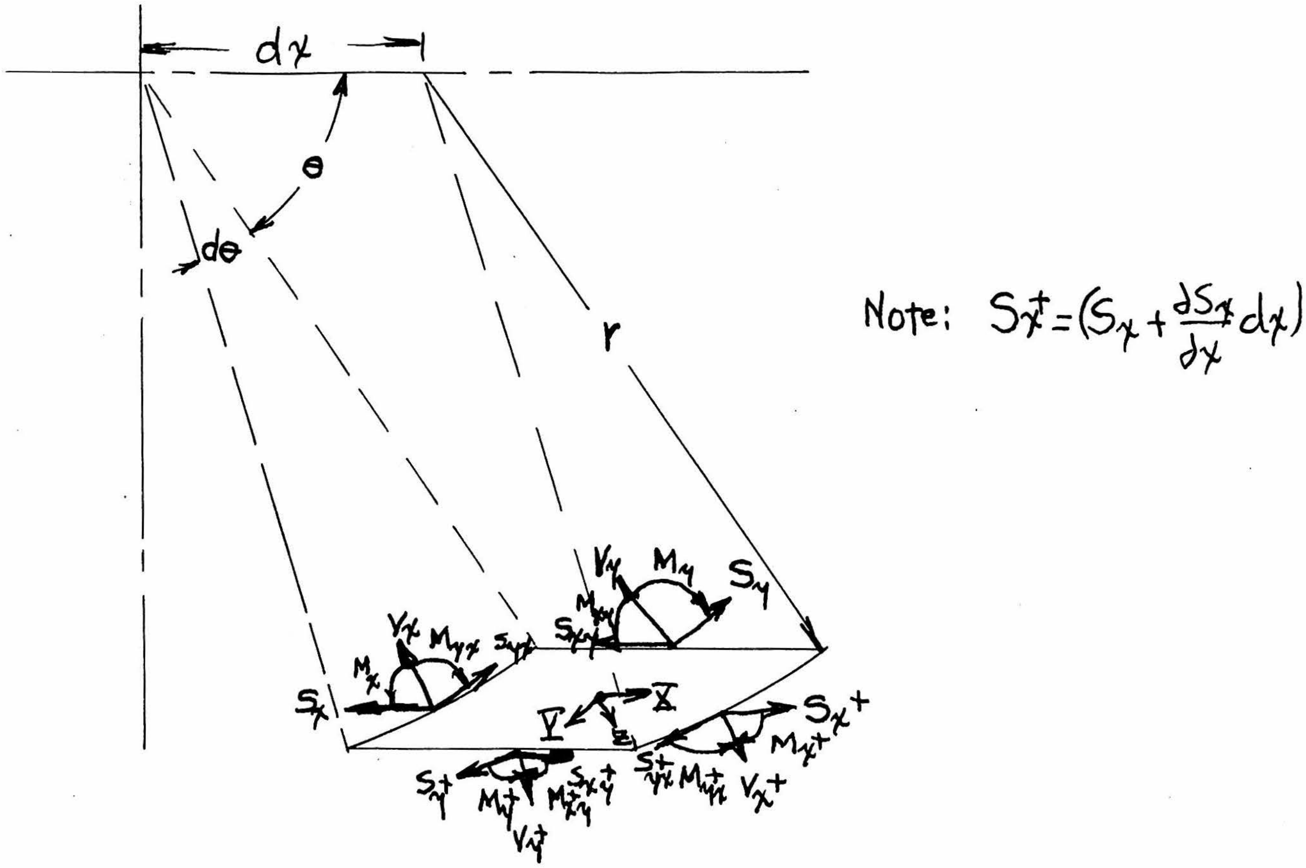
WHEN THE INTEGRALS ARE EVALUATED, AND CONSIDERED AT THE BOUNDARIES, THE FOLLOWING RESULTS ARE OBTAINED.

$$S_y = S_x = 0 \qquad \text{at} \quad y = \pm b$$

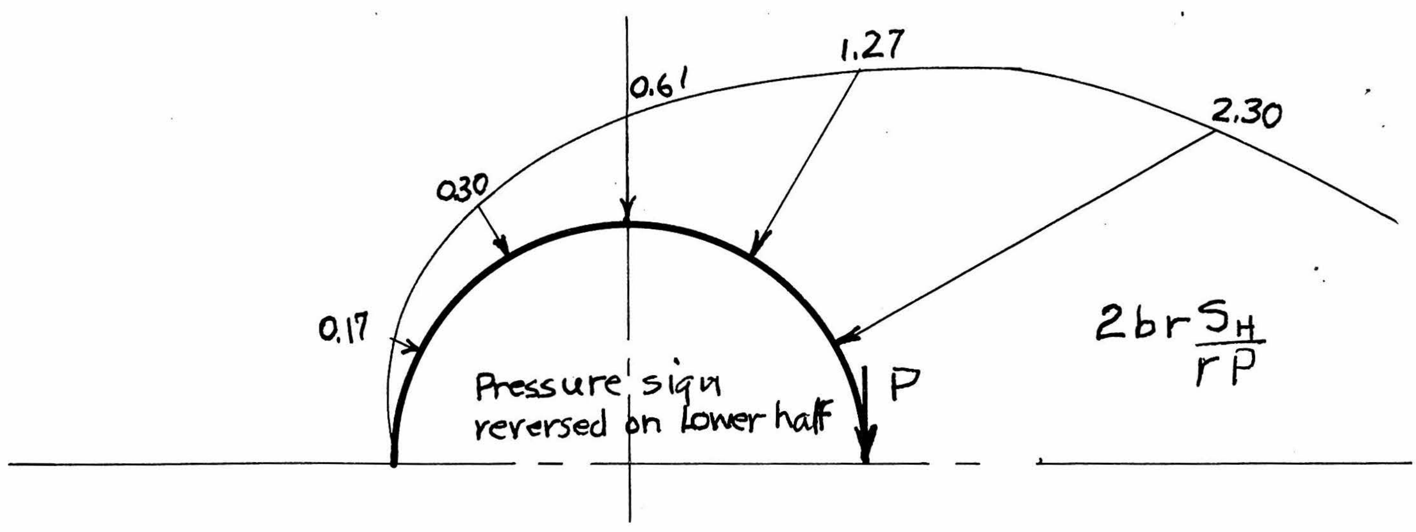
$$S_{xy} = -\frac{p}{2}, \quad \text{then} \int_{-b}^{b} S_{xy}\,dy = -pb = -P/2$$

THE FIRST THREE BOUNDARY CONDITIONS ARE THUS SEEN TO BE SATISFIED. THE FOURTH CONDITION IS OBVIOUSLY SATISFIED IF THE METHOD OF DERIVING THE SOLUTION IS CONSIDERED. IT IS APPARENT THAT EXPRESSIONS 1, 2, AND 3 REPRESENT THE SUMMATION OF AN INFINITELY LARGE NUMBER OF FORCES ACTING ALONG A STRAIGHT LINE WHICH REPRESENTS THE Y AXIS AT INTERVALS OF $2b$ IN AN INFINITE PLANE. THUS, THE RESULTS GIVE AN INFINITE NUMBER OF IDENTICAL STRIPS OF THE TYPE WE ARE CONSIDERING. THERE CAN BE NO DIS-CONTINUITIES IN THIS PLANE, SO DEFORMATIONS AT OPPOSITE BOUNDARIES MUST BE IDENTICAL.

CONSIDERATION OF EQUILIBRIUM OF THE STRIP AFTER IT HAS BEEN ROLLED INTO A CYLINDER DISCLOSES THAT A PRESSURE MUST BE EFFECTIVELY DISTRIBUTED OVER THE SURFACE OF THE CYLINDER. THE MAGNITUDE OF THIS PRESSURE AT ANY POINT IS S_y/r . SINCE THE PRESSURE IS ESSENTIAL FOR SATISFACTION OF EQUILIBRIUM RELATIONS, AND CAN ONLY BE RESISTED BY STRESS AND DEFORMATION COMBINATIONS IN THE CYLINDER, IT IS NECESSARY TO DETERMINE THE STRESS AND DEFORMATION IN ANY CYLINDER FOR ANY PRESSURE LOADING.



FORCES AND MOMENTS ON ELEMENT OF CYLINDER



EQUILIBRATING PRESSURE AT $x = 0$

FIGURE - 2

THIN CYLINDERS SUBJECT TO PRESSURE AND BODY FORCES

FIGURE 2 SHOWS A DIFFERENTIAL ELEMENT OF SHELL WITH THE FORCE AND COUPLES ACTING ON IT. USING THE VALUES SHOWN, SIX INDEPENDENT EQUATIONS OF EQUILIBRIUM CAN BE OBTAINED. THEY ARE,

$$\frac{\partial S_x}{\partial x} + \frac{\partial S_{xy}}{\partial y} + X = 0$$

$$\frac{\partial S_y}{\partial y} + \frac{\partial S_{yx}}{\partial x} + \frac{V_y}{r} + Y = 0$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} - \frac{S_y}{r} + Z = 0$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + V_x = 0$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{yx}}{\partial x} + V_y = 0$$

$$S_{xy} - S_{yx} - \frac{M_{xy}}{r} = 0$$

THE NEXT STEP IN THE ANALYSIS IS TO OBTAIN SIX STRESS STRAIN RELATIONS FOR THE SHELL. THE USUAL THIN CYLINDER ASSUMPTIONS AND SIMPLIFICATIONS ARE MADE AND THE RELATIONS OBTAINED FOR STRAIN SOLVED IN TERMS OF NORMAL STRESS RESULTANTS AND BENDING MOMENTS. THEY ARE,

$$M_x = \frac{Et^3}{12(1-\mu^2)}\left[\frac{\partial^2 \zeta}{\partial x^2} + \mu\left(\frac{\partial^2 \zeta}{\partial y^2} + \frac{\zeta}{r^2}\right)\right]$$

$$M_y = \frac{Et^3}{12(1-\mu^2)}\left[\mu\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\zeta}{r^2}\right]$$

$$S_x = \frac{Et}{1-\mu^2}\left[\frac{\partial \xi}{\partial x} + \mu\left(\frac{\partial \eta}{\partial y} + \frac{\zeta}{r}\right)\right]$$

$$S_y = \frac{Et}{1-\mu^2}\left[\frac{\partial \eta}{\partial y} + \frac{\zeta}{r} + \mu\frac{\partial \xi}{\partial x}\right]$$

$$M_{yx} = M_{xy} = \frac{Et^3}{12(1-\mu^2)}\frac{\partial}{\partial x}\left(\frac{\partial \zeta}{\partial y} - \frac{\eta}{r}\right)$$

$$S_{yx} + S_{xy} = \frac{Et}{1+\mu}\left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right)$$

SIMULTANEOUS SOLUTION OF THESE TWELVE PARTIAL DIFFERENTIAL EQUATIONS IS THEORETICALLY POSSIBLE, BUT PRACTICAL SOLUTIONS ARE OBTAINED ONLY BY MAKING ESSENTIAL SIMPLIFICATIONS.

IT CAN BE SHOWN THAT EXCEPT NEAR A RESTRAINT, STRAINS DUE TO M_x AND M_y ARE OF THE ORDER OF MAGNITUDE OF t/r COMPARED WITH STRAINS DUE TO S_x AND S_y. THEREFORE, A SOLUTION OF THESE EQUATIONS MAY BE SIMPLIFIED BY TAKING AN EXTREME CASE WHERE FLEXURAL STRAINS ARE NEGLIGIBLE COMPARED TO EXTENSIONAL STRAINS. THIS SOLUTION, DESIGNATED THE MEMBRANE SOLUTION CONTAINS THE BODY FORCES. THE RELATIONS OBTAINED ARE,

$$
\begin{aligned}
S_y &= rZ \\
S_{xy} &= -\int_0^x \left(\frac{\partial S_y}{\partial y} + Y \right) dx + F_1 \\
S_x &= -\int_0^x \left(\frac{\partial S_{xy}}{\partial y} + X \right) dx + F_2 \\
\xi &= \int_0^x \frac{S_x - \mu S_y}{Et}\, dx + F_3 \\
\eta &= \int_0^x \frac{2(1+\mu)}{Et} S_{xy}\, dx - \int_0^x \frac{\partial \xi}{\partial y}\, dx + F_4 \\
\zeta &= \frac{r}{Et}\left(S_y - \mu S_x \right) - r \frac{\partial \eta}{\partial y}
\end{aligned}
\tag{4}
$$

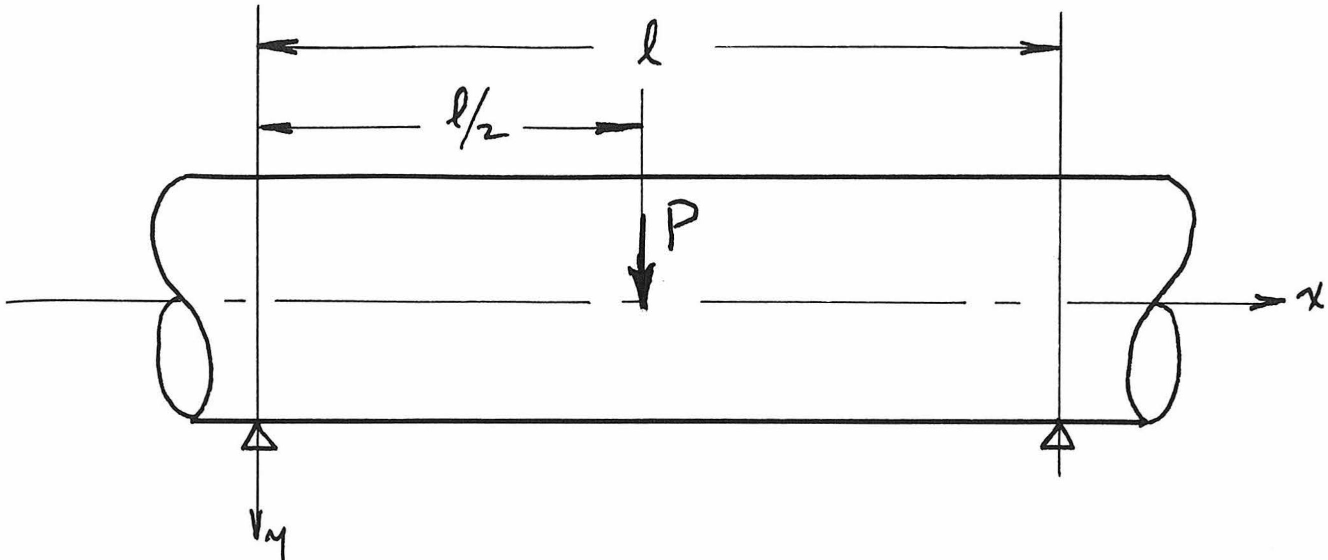
THE VALUES OF F ARE DETERMINED FROM THE BOUNDARY AND LOADING CONDITIONS. WHEN THE STRESSES OBTAINED BY THIS METHOD ARE SUPERPOSED ON THE STRIP STRESSES, THE RESULTS OBTAINED APPROACH THE STRESS DISTRIBUTION SHOWN IN FIGURE 3. THIS IS THE DISTRIBUTION OBTAINED BY USING ELEMENTARY BENDING THEORY AND THE STATICALLY EQUIVALENT LOADING SHOWN.

THIS CASE HAS NO VALUE IN THE DETERMINATION OF LOCAL STRESSES CAUSED BY THE CONCENTRATED LOAD, BUT GIVES A SOLUTION WHICH IS USEFUL AT A CONSIDERABLE DISTANCE FROM THE RESTRAINTS. IT ALSO DEMONSTRATES THAT THE METHOD SATISFIES EQUILIBRIUM CONDITIONS.

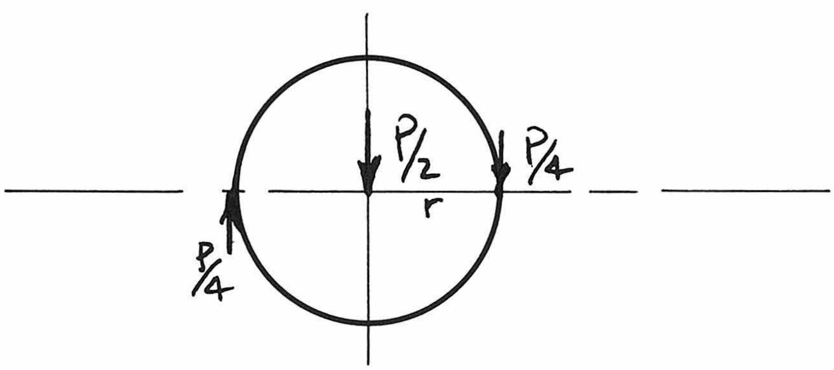
IT IS APPARENT THAT BENDING MOMENTS ARE IMPORTANT NEAR THE LOAD, AND SHOULD NOT BE NEGLECTED. A SOLUTION WHICH INCLUDES BENDING MOMENTS IS OBTAINABLE IF THE BODY FORCES ARE NEGLECTED. THIS SOLUTION IS REACHED BY APPROXIMATING THE LOADING CONDITION AND DEFLECTION BY A SUITABLE FOURIER SERIES. THE RESULT OF THIS SOLUTION SHOWS THAT DISPLACEMENTS NEAR A RESTRAINT TAKE THE FORM OF A RAPIDLY DAMPED OSCILLATORY WAVE OF WHICH THE THE WAVE LENGTH IS NEARLY

$$L = \frac{2\pi}{q}$$

where $$q = \sqrt[4]{\frac{3(1-\mu^2)}{r^2 t^2}}$$



MEMBRANE ANALYSIS FOR THESE END CONDITIONS APPROACH THE LOADING BELOW



STATICALLY EQUIVALENT LOAD AT $x = \frac{\ell}{2}$

$S_y = 0$, $S_x = \frac{Px}{2\pi r^2} \sin\theta$, $S_{xy} = -\frac{P}{4b} - \frac{P}{2b} \cos\theta$

THESE RESULTS DERIVE FOR SIMPLE BENDING

FIGURE - 3

THIS EFFECT IS ALMOST COMPLETELY DAMPED OUT IN ONE WAVE LENGTH. RATHER THAN ATTEMPT TO SOLVE THE HIGHLY COMPLICATED AND LABORIOUS EXPRESSIONS RESULTING FROM THIS ANALYSIS, ANOTHER SIMPLIFICATION IS ATTEMPTED IN WHICH THE STRESS DUE TO RESTRAINT ALONE IS CONSIDERED AND THE RESTRAINT AT THE LOAD IS ESTIMATED. THE RESULTS OF THIS APPROACH ARE,

$$
\left.
\begin{aligned}
S_y &= 2q^2 r e^{-qx}\left[\left(M-\frac{V}{q}\right)\cos qx - M\sin qx\right]\\
Et\xi &= -\mu q r e^{-qx}\left[\frac{V}{q}\cos qx + \left(2M-\frac{V}{q}\right)\sin qx\right]\\
Et\zeta &= 2q^2 r^2 e^{-qx}\left[\left(M-\frac{V}{q}\right)\cos qx - M\sin qx\right]\\
Et\frac{\partial\zeta}{\partial x} &= -2q^3 r^2 e^{-qx}\left[\left(2M-\frac{V}{q}\right)\cos qx - \frac{V}{q}\sin qx\right]\\
M_x &= e^{-qx}\left[M\cos qx + \left(M-\frac{V}{q}\right)\sin qx\right]\\
V_x &= q e^{-qx}\left[\frac{V}{q}\cos qx + \left(2M-\frac{V}{q}\right)\sin qx\right]
\end{aligned}
\right\} \quad (5)
$$

WHERE M AND V REPRESENT THE ESTIMATED MOMENTS AND SHEARS.

A METHOD OF CHECKING ANY ESTIMATED RESTRAINING MOMENT AND SHEAR IS REQUIRED. CONSIDERING THE CHARACTER OF THE EQUILIBRATING PRESSURE DISTRIBUTION IT IS OBSERVED THAT ITS RESULTANT IS A MOMENT ABOUT $y = 0$.

$$M_{y=0} = \int_{-\infty}^{\infty} dx \int_{-b}^{b} \frac{s_y}{r}\, y\, dy = Pb = Pr\pi$$

ON THE FLAT PLATE. THIS WILL REDUCE TO

$$M_0 = Pr$$

WHEN THE STRIP IS ROLLED INTO A CYLINDER.

THIS MOMENT WITH THE RESULTANT IN THE DIRECTION OF LOAD, ALSO SATISFIES THE EQUILIBRIUM CONDITIONS FOR THE CYLINDER. THERE ARE TWO METHODS FOR DETERMINING THE DEFORMATIONS DUE TO A CIRCUMFERENTIAL COUPLE ON A THIN CYLINDER. THE COUPLE IS CONSIDERED TO BE APPLIED AT THE POINT OF APPLICATION OF LOAD. THIS MAKES A PHYSICAL PICTURE THAT SEEMS MORE NEARLY REASONABLE THAN THAT OF THE DISTRIBUTED PRESSURE.

IV. THIN CYLINDER SUBJECTED TO A CIRCUMFERENTIAL COUPLE

THE DEFORMATION DUE TO A CIRCUMFERENTIAL COUPLE ACTING ON A THIN CYLINDER MAY BE DERIVED FROM RAYLEIGHS THEORY OF INEXTENSIONAL DEFORMATION[(1)] OR FROM AN EXACT SOLUTION FOR NORMAL CONCENTRATED LOADS ON A THIN CYLINDER DEVELOPED BY SHAO WEN YUAN.[(2)] IN EITHER CASE, THE RADIAL DEFORMATION IS DETERMINED FOR TWO EQUAL AND OPPOSITE FORCES ACTING ALONG A DIAMETER. USING THIS EXPRESSION FOR ζ IT IS READILY SEEN THAT TWO EQUAL AND OPPOSITE FORCES OF OPPOSITE SIGN FROM THE FIRST PAIR CAN BE MADE TO OPERATE ALONG A DIAMETER DISPLACED A DISTANCE $r\Delta\theta$ FROM THE FIRST DIAMETER (TANGENTIALLY). THE DISPLACEMENT OR DEFORMATION OF THE CYLINDER DUE TO THE COUPLE $Pr\Delta\theta$ IS THEN GIVEN BY THE SUM OF THE DISPLACEMENTS AT A GIVEN POINT.

$$\zeta_T = \zeta(x, y) - \zeta(x, y - r\Delta\theta)$$

WHEN $\Delta\theta$ IS VERY SMALL, THIS APPROACHES THE VALUE

$$\zeta_T = \frac{d\,\zeta(x, y)}{dy}\,\Delta y \qquad (6)$$

AND THE DEFORMATION DUE TO ANY COUPLE CAN BE CALCULATED FROM THE FIRST DEFORMATION CURVE.

(1) RAYLEIGH 'PROC. LONDON MATH. SOC.', VOL. 13, 1881

(2) YUAN, S. W. ' THIN CYLINDRICAL SHELLS SUBJECTED TO CONCENTRATED LOADS' QUARTERLY OF APPLIED MATHEMATICS VOL 4, APRIL 1946 P.13

Rayleigh's results for the deformation due to simple concentrated loading are obtained by neglecting strain in the middle surface. The solution is simple and is obtained by application of energy methods and virtual displacements. The result is,

$$\varphi = \frac{P r^3}{\pi l E t^3} 12(1-\mu^2) \sum_{n=2,4,6\cdots} \left\{ \frac{1}{(n^2-1)^2} + \frac{n^2 c x}{(n^2-1)^2 \left[\frac{1}{3} n^2 l^2 + 2(1-\mu) r^2\right]} \right\} \cos n\theta$$

c = eccentricity , l = length

Yuan arrives at his exact result by developing a unique differential equation for the cylinder in terms of the radial displacement and solving it by the use of Fourier series and integrals for the loading, evaluated by Cauchy's theorem of residues. His expression for the deformation due to circumferential couples is

$$\frac{\delta_T / t}{T_c / E t^2} = \frac{6(1-\mu^2)}{\pi} \left(\frac{r}{t}\right)^2 \sum_{n=3,4,6\cdots}^{\infty} \frac{\cos n\, y/r}{n^4 \sqrt{1+(1/4n^2)^2}} \Big\{ e^{-Bx/r} \cos\left(\frac{Ax}{r}\right) \big[A(\phi G - \eta C)$$

$$- B(\phi C + \eta G)\big] - e^{-Bx/r} \sin\left(\frac{Ax}{r}\right) \big[(\phi C + \eta G) A + B(\phi G - \eta C)\big] + e^{-Gx/r} \cos\left(\frac{Cx}{r}\right)$$

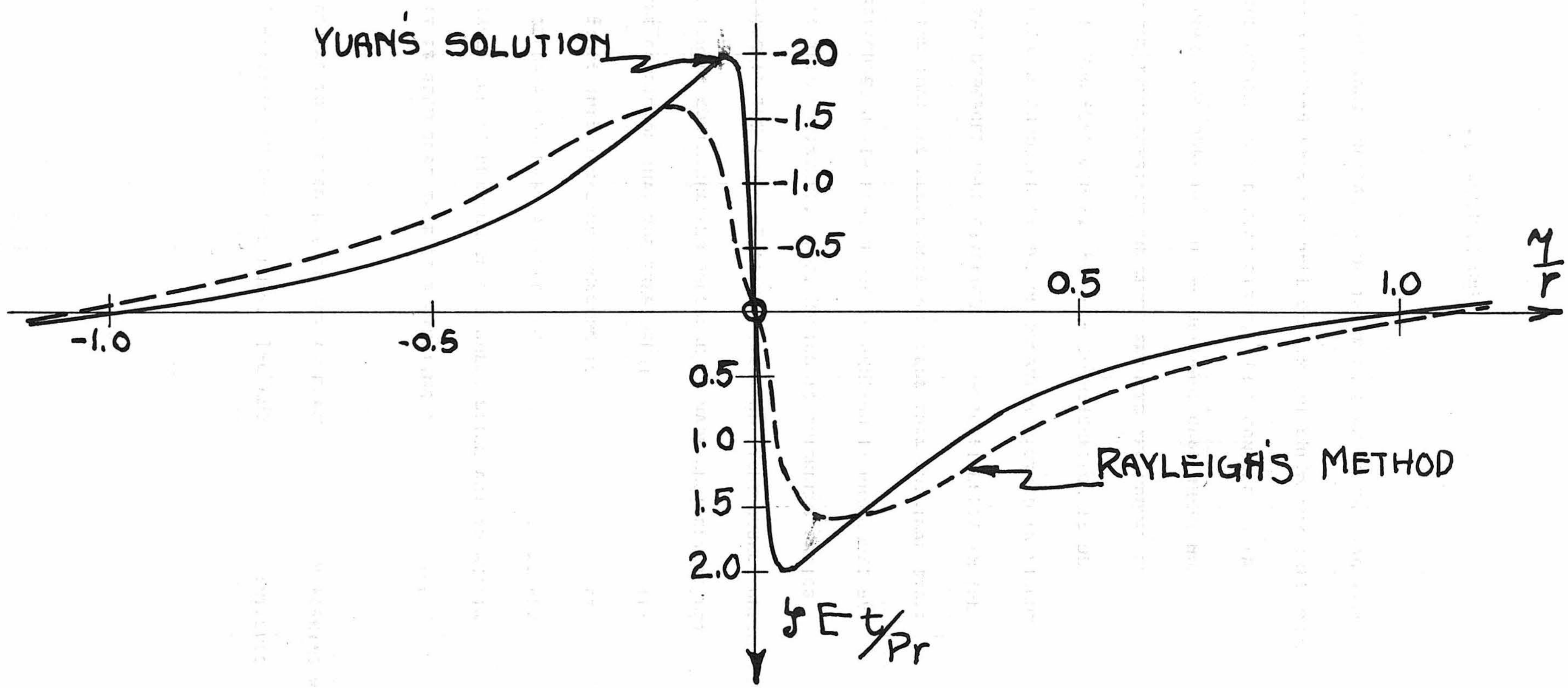
$$\big[C(\eta A + \phi B) - G(\phi A - \eta B)\big] - e^{-Gx/r} \sin\left(\frac{Cx}{r}\right) \big[C(\phi A - \eta B) + G(\eta A + \phi B)\big] \Big\} \qquad (7)$$

WHERE $\pm A \pm iB, \pm C \mp iG$ ARE THE COMPLEX ROOTS OF $\lambda^4 + J^2 = 0$

$$J^2 = 12(1-\mu^2)\left(\frac{r}{t}\right)^2 \quad , \quad \phi = \sqrt{\frac{1}{2}\left(R^2 + \frac{1}{4}J^2\right)}$$

$$\eta = \sqrt{\frac{1}{2}\left(R_2 - \frac{1}{4}J^2\right)} \quad , \quad R_2 = n^2 J\sqrt{1 + \left(J/4n^2\right)^2}$$

THE RESULTS OF BOTH METHODS ARE PLOTTED ON FIGURE 4, WHERE IT IS SEEN THAT RAYLEIGHS METHOD IS ADEQUATE EXCEPT VERY CLOSE TO THE LOAD.



DEFORMATION AT $x = 0$ UNDER COUPLE $T_c = Pr$

FIGURE - 4

V. CONCLUSIONS

THE MOST IMPORTANT INDICATION OF THE WORK DONE SO FAR HAS BEEN TO SHOW THAT THE LIMITING CASE AND SAFE DESIGN CRITERION FOR THIS TYPE OF LOADING IS THE STRESS IN THE FLAT STRIP. THIS CONCLUSION IS DERIVED FROM THE CONDITION OBSERVED IN THE MEMBRANE CASE WHERE NORMAL STRESSES DUE TO THE EQUILIBRATING PRESSURE REDUCED THE NORMAL STRESS DUE TO THE LOAD IN THE FLAT STRIP. FURTHER CORROBORATION IS OBTAINED BY CALCULATING STRAINS PRODUCED BY THE DEFORMATION PLOTTED IN FIGURE 4, SUPERPOSING THE STRESSES THUS OBTAINED ON NORMAL STRESSES IN THE STRIP AND OBSERVING THAT THE RESULTANTS ARE LESS THAN ORIGINAL STRIP STRESSES. OVERSTRESSED OUTER FIBERS DUE TO BENDING IS PRECLUDED BY THE LIMITATION TO THIN CYLINDERS. I.E. RADIUS TO THICKNESS RATIOS IN THE NEIGHBORHOOD OF 100 TO 1. THAT THE CYLINDER MIGHT BUCKLE UNDER THE EFFECTIVE EXTERNAL EQUILIBRATING PRESSURE IS MADE UNLIKELY IF NOT IMPOSSIBLE BY THE CONDITION THAT THE PRESSURE IS AN INVERSE FUNCTION OF THE RADIUS. SINCE THIS BUCKLING PRESSURE IS ENTIRELY REMOVED AT THE PLACE WHERE DEFORMATION HAS CANCELED THE CYLINDER' CURVATURE, AND THE RESTORING FORCES DUE TO BENDING OF THE SHELL STILL ACT TO RETAIN THE CURVATURE, IT IS SEEN THAT THIS IS A LIMITING CONDITION ON THE DEFORMATION.

IT WOULD APPEAR TO BE POSSIBLE TO GET AN EXACT SOLUTION BY WORKING WITH THE PRESSURE DISTRIBUTION ON THE CYLINDER. THE AMOUNT OF LABOR INVOLVED

IS OUTSIDE THE SCOPE OF THE AVERAGE DESIGN ENGINEER, AND THE RESULTS ARE NO MORE USEFUL TO HIM THAN RESULTS WHICH CAN BE READILY OBTAINED. FOR THE CASE WHERE NO HOLES OR DISCONTINUITIES ARE CONTEMPLATED IN THE CYLINDER NEAR THE LOAD, IT IS RECOMMENDED THAT STRESS IN THE MAJOR PART OF THE CYLINDER NOT NEAR THE LOAD BE CALCULATED ON THE BASIS OF SIMPLE BENDING AND TORSION, AND THAT THE POINT OF LOAD APPLICATION BE REINFORCED AS IT WOULD BE TO STAND THE STRESSES DEVELOPED BY THE LOAD IN THE FLAT STRIP.

WHEN IT IS NECESSARY TO DETERMINE STRESS CONCENTRATIONS AT HOLES OR DISCONTINUITIES NEAR THE LOAD, RAYLEIGHS METHOD CAN BE APPLIED SAFELY. YUANS SOLUTION WILL GIVE BETTER RESULTS VERY CLOSE TO THE LOAD, BUT THE NUMERICAL WORK IS MULTIPLIED.

RAYLEIGHS RESULTS FOR DEFORMATION IN ALL THREE DIRECTIONS ARE

$$\xi = \frac{Pr^3}{\pi l E t^3} 12(1-\mu^2) \sum_{n=2,4,6\cdots} \frac{rc\cos n\theta}{(n^2-1)^2\left[\frac{1}{3}n^2l^2+2(1-\mu)r^2\right]}$$

$$\eta = \frac{Pr^3}{\pi l E t^3} 12(1-\mu^2) \sum_{n=2,4,6\cdots} \left\{\frac{1}{n(n^2-1)^2} + \frac{ncx}{(n^2-1)^2\left[\frac{1}{3}n^2l^2+2(1-\mu)r^2\right]}\right\}\sin n\theta \qquad (8)$$

$$\zeta = \frac{Pr^3}{\pi l E t^3} 12(1-\mu^2) \sum_{n=2,4,6\cdots} \left\{\frac{1}{(n^2-1)^2} + \frac{n^2cx}{(n^2-1)^2\left[\frac{1}{3}n^2l^2+2(1-\mu)r^2\right]}\right\}\cos n\theta$$

AND THE EXPRESSIONS FOR STRAINS AND CHANGES IN CURVATURE FROM WHICH THE STRESSES CAN BE CALCULATED ARE

$$\epsilon_x = \frac{\partial \xi}{\partial x} \quad , \quad \epsilon_\theta = \frac{1}{r}\frac{\partial \eta}{\partial \theta} - \frac{\zeta}{r}$$

$$\gamma_{x\theta} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{r\partial \theta} \quad , \quad \chi_x = \frac{\partial^2 \zeta}{\partial x^2}$$

$$\chi_\theta = \frac{1}{r^2}\frac{\partial^2 \zeta}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial \eta}{\partial \theta} \quad , \quad \chi_{x\theta} = \frac{1}{r}\frac{\partial^2 \zeta}{\partial x \partial \theta} + \frac{1}{r}\frac{\partial \eta}{\partial x}$$

BIBLIOGRAPHY

1. S. TIMOSHENKO, 'THEORY OF ELASTICITY ' MCGRAW-HILL BOOK CO. NEW YORK 1934
2. S. TIMOSHENKO, 'THEORY OF PLATES AND SHELLS' MCGRAW-HILL BOOK CO. N.Y. 1940
3. T. VON KARMAN AND M. A. BIOT, 'MATHEMATICAL METHODS IN ENGINEERING' MCGRAW-HILL BOOK CO., NEW YORK, 1940
4. 'PENSTOCK ANALYSIS AND STIFFENER DESIGN' BOULDER CANYON PROJECT FINAL REPORT PART V, BULLETIN 5, U. S. DEPARTMENT OF INTERIOR, BUREAU OF RECLAMATION
5. S. W. YUAN, 'THIN CYLINDRICAL SHELLS SUBJECTED TO CONCENTRATED LOADS', QUARTERLY OF APPLIED MATHEMATICS, VOLUME IV. APRIL 1946, PP.13