ORTHOGONALITY IN NORMED LINEAR SPACES

Thesis by

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- SUMMARY -

Three possible definitions of orthogonality of elements of normed linear spaces are studied. For abstract Euclidean spaces, each is equivalent to the usual requirement that the inner product be zero. These definitions are never vacuous, since for each definition and any elements x and y of a normed linear space there is at least one number a for which $x \perp ax+y$.

It is shown that neither Pythagorean nor isosceles orthogonality can be either homogeneous or additive in a normed linear space unless that space is abstract Euclidean, although both of these types of orthogonality are symmetric. If $x \perp y$ in the isosceles sense, then $\|x+ky\| > \frac{1}{2}\|x\|$ for all k. This and other similar inequalities give a comparison of isosceles and spherical orthogonality. Spherical orthogonality is homogeneous in any normed linear space, but is neither additive nor symmetric in general. It can be additive and symmetric for a space of three or more dimensions only if the space is abstract Euclidean. For two-dimensional spaces, an inner product can be gotten from additivity and a strengthened form of symmetry, expressed in terms of the differential of the norm or of the limits, $f_{\pm}(x;y)$, of $\frac{\|x+hy\|-\|x\|}{h}$ as $h \rightarrow \pm 0$.

Because of the lack of symmetry of spherical orthogonality, the uniqueness of the number b for which $bx+y \perp x$ (left-uniqueness) does not imply that of the number a for which

$x \perp ax+y$ (right-uniqueness). This number b is unique if and only if the space is strictly normed, and can have any value for which $\|bx+y\|$ is minimum. The number a is unique if and only if spherical orthogonality is additive, or if and only if the norm is differentiable at all non-zero points, and can have any value satisfying $f_+(x;y) \leq -a\|x\| \leq f_-(x;y)$.

There are many connections between the theories of spherical orthogonality, of linear functionals, and of maximal linear subsets and supporting and tangent hyperplanes. For example, for a linear functional f, $|f(x)| = \|f\| \cdot \|x\|$ if and only if $x \perp h$ for all h satisfying $f(h) = 0$. For any element x there is a maximal linear subset H with $x \perp H$. Right- and left-uniqueness are expressed in terms of conditions on the elements at which a linear functional takes on its maximum in the unit sphere, and of like conditions on the points of contact of supporting hyperplanes of the unit sphere. If the norm of a Banach space is differentiable at non-zero points and some additional assumption such as weak compactness, regularity, or uniform convexity is made, then it is possible to give a general form for all linear functionals in terms of this differential or of spherical orthogonality. Then for a linear functional f there is an element x such that $f(y) = -a\|x\|^2$ for all y, where $x \perp ax+y$.

- NOTATION -

$\overline{U}$	Closure of the set U (see pg. 1).
$x \in U$	The element x is a member of the set U.
$U \subset V$	The set U is contained in the set V.
$U \cup V$	The set of all elements belonging to either U or V; the sum of U and V.
$U \cap V$	The set of all elements belonging to both U and V; the intersection of U and V.
$x+y$	Sum of the elements x and y (see Definitions 1.1-1.6).
$x+U$	The set of all elements $x+u$, where $u \in U$.
$U+V$	The set of all elements $u+v$, where $u \in U$ and $v \in V$.
ax	Product of the number a and element x (see Def. 1.4).
aU	The set of all elements of the form au, where $u \in U$.
$\|x\|$	Norm of the element x (see Def. 1.6).
$\|f\|$	Modulus of the linear functional f (see pg. 8).
T'	The conjugate space of T (see pg. 8).
(x,y)	Inner product of x and y (see Def. 1.9).
$x \perp y$	The element x is orthogonal to y. This may be in the sense of Definition 2.1, 2.2, or 2.3, or of Theorem 3.1, depending on the section in which it is used.
$x \perp U$	The element x is orthogonal to every element of U.
$f(x;y)$	The differential of the norm at x with increment y, $\lim_{h \to 0} \frac{\|x+hy\| - \|x\|}{h}$, unless specified otherwise (as in Def. 7.3 and 10.2). See Theorem 7.4 and Corollary 7.4.
$f_+(x;y)$	$\lim_{h \to +0} \frac{\|x+hy\| - \|x\|}{h}$ (see Corollary 6.6 and pg. 101).
$f_-(x;y)$	$\lim_{h \to -0} \frac{\|x+hy\| - \|x\|}{h}$.

- CONTENTS -

1. TOPOLOGICAL SPACES

The topological spaces used in this thesis will be normed linear spaces, Banach spaces, and abstract Euclidean spaces. However, these spaces are special cases of more general topological spaces which will be discussed first.

Definition 1.1. A Hausdorff topological space[1] is a set of elements T such that to each element x of T there are associated subsets of T, called neighborhoods of x, which satisfy the conditions:

(1). Each element belongs to all of its neighborhoods.

(2). For each of two distinct elements there exists a neighborhood which does not contain the other.

(3). If two neighborhoods U and V each contain an element x, then there exists a neighborhood of x contained in the intersection of U and V.

(4). If an element y belongs to a neighborhood U of x, then there exists a neighborhood of y contained in U.

A subset U of a Hausdorff topological space is said to be open if every element of U has a neighborhood contained in U. An element x is a limit point of a set U if every neighborhood of x contains an element of U (other than x). The closure, $\overline{U}$, of U is the sum of U and all of its limit points. A closed set is a set which is equal to its closure. Also:[2]

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1. Hausdorff (III), pp. 228-229; the space satisfies postulates (A), (B), (C), (5).
2. See Pontrjagin (VII), pp. 26-30.

(a). If U contains a finite number of elements, then $\overline{U} = U$.

(b). If U and V are any sets, then $\overline{U \cup V} \subset \overline{U} \cup \overline{V}$.

(c). For any set U, $\overline{\overline{U}} = \overline{U}$. Thus $\overline{U}$ is a closed set.

(d). The complement[1] of an open set is closed, and the complement of a closed set is open.

(e). The sum of a finite number of closed sets is closed, and the intersection of a finite number of open sets is open.

(f). The sum of any number of open sets is open, and the intersection of any number of closed sets is closed.

A Hausdorff space consists only of a set of elements and their neighborhoods, there being no operation between elements defined. Such an operation can be introduced by requiring that the set be an abstract group.

Definition 1.2. A set G of elements is called a group[2] if there is an operation in G which associates with each pair of elements x, y a third element (which will be called $x+y$) and this operation satisfies the conditions:

(1). $x+(y+z) = (x+y)+z$ for all x, y and z.

(2). There exists a zero element such that $x+0 = x$ for all x.

(3). For each x there is an inverse element $-x$ such that $x+(-x) = 0$.

Definition 1.3. A set G is called a topological group[3] if it is an abstract group and a Hausdorff topological space and the group operations are continuous. That is:

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1. The complement of a set consists of all elements not in the set.
2. Pontrjagin (VII), pg. 3.
3. Pontrjagin (VII), pp. 52-53.

(1). <u>If x and y are elements of G, then for any neighborhood W of $x+y$ there exist neighborhoods U and V of x and y such that $U+V\subset W$.</u>

(2). <u>If x is an element of G, then for any neighborhood V of $-x$ there exists a neighborhood U of x such that $-U\subset V$.</u>

A topological group is called a <u>topological Abelian group</u> if its abstract group is Abelian[1]. Some of the elementary properties of a topological group are:[2]

(a). If F is a closed set, U an open set, P an arbitrary set, and x any element of the space, then $x+F$, $F+x$, and $-F$ are closed sets, while $U+P$, $P+U$, and $-U$ are open sets.

(b). A topological group is homogeneous; that is, it is sufficient to verify its local properties for a single element only. For example, if the zero element has a neighborhood containing only zero, then every element x has a neighborhood containing the single element x.

(c). A topological group is <u>regular</u>; that is, for any neighborhood U of an element x there exists a neighborhood V of x such that $\overline{V}\subset U$.

In addition to the group operation between elements of a topological group, it will be useful to have multiplication by real numbers. Because of the group operation, multiplication of elements of a topological group by integers is defined. The multiplication by real numbers will be introduced by requiring that the space be a linear (vector) space.

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1. A group is Abelian if the group operation is commutative; i.e. if $x+y = y+x$ for all elements x and y.
2. Pontrjagin (VII), pp. 52-53.

Definition 1.4. A set L is called a linear (vector) over $\mathbb{R}$ space if it is an Abelian group and an operation (called multiplication) between real numbers and elements of L is defined and satisfies the conditions:

(1). $a(x+y)=ax+ay$ for all numbers a and elements x and y,

(2). $(a+b)x=ax+bx$,

(3). $a(bx)=(ab)x$,

(4). $1\cdot x=x$.

The identity (or zero) of the Abelian group of a linear space is $0\cdot x$, since $x+0\cdot x=1\cdot x+0\cdot x=x$ by (2) and (4) of Definition 1.4. From the theory of groups the element $0\cdot x$ must be the same for all x,[1] although this can be shown directly. For $1\cdot x+1\cdot y=1\cdot x+1\cdot y+0\cdot x=1\cdot x+1\cdot y+0\cdot y$ by (2) and the commutativity of addition. By subtracting the inverses of $1\cdot x$ and $1\cdot y$, it follows that $0\cdot x=0\cdot y$. Also, the inverse of x is $(-1)x$, since $x+(-1)x=1\cdot x+(-1)x=0\cdot x$ by (2) and (4). Furthermore:

(a). If $ax=0$, either $x=0$ or $a=0$. For if $ax=0$, and $a\neq 0$, then $\frac{1}{a}(ax)=0=x$ by (3) and (4).

(b). If $ax=ay$ and $a\neq 0$, then $x=y$. For if $ax=ay$, then $ax+a(-y)=0$, since $ay+a(-y)=0$ by (1). From (1), it also follows that $a(x-y)=0$ and hence from (a) that $x=y$.

(c). If $ax=bx$ and $x\neq 0$, then $a=b$. For if $ax=bx$, then $(a-b)x=0$ and $x=0$ by (a) if $a\neq b$.

1. Pontrjagin (VII), pg. 4.

Definition 1.5. A set L is called a linear topological space if it is a linear space and a Hausdorff topological space and the operations $a \cdot x$ and $x+y$ are continuous simultaneously in a and x, and in x and y, respectively.

Since a linear topological space is a topological Abelian group, it possesses all the properties discussed below Definitions 1.1 and 1.3. The following will also be useful:

(a). If U is an open set in a linear topological space, and $a \neq 0$, then aU is an open set. For suppose y is an element of aU. Then $y = ax$, for some x in U. Hence U contains $\frac{1}{a}(y) = x$, and from the continuity of the product at $(\frac{1}{a}, y)$ there exists a neighborhood V of y such that $\frac{1}{a}V \subset U$. Then $V \subset aU$, and since y was arbitrary it follows that aU is open.

(b). If U is any neighborhood of the zero of a linear topological space, T there is an open set V such that $aV \subset V \subset U$ for all $|a| \leq 1$. It follows from the continuity of the product ax at $(0, \vec{0})$ that for the nbd U of $\vec{0}$ there is a positive number δ and a neighborhood W of zero such that $aW \subset U$ if $|a| < \delta$. But by (a), aW is open if $a \neq 0$. Since the sum of open sets is open,[1] the sum V of all sets aW for which $|a| < \delta$ is open. Clearly $aV \subset V$ if $|a| \leq 1$, and $V \subset U$, because all sets $aW \subset U$.

By introducing the following generalization of absolute value in a linear (vector) space, one obtains a particular type of linear topological space--a normed linear space.

1. See (f) following Definition 1.1.

Definition 1.6. A set T is a normed linear space if it is a linear space and to each element x of T there corresponds a real number $\|x\|$, the norm of x, satisfying the conditions:

(1). $\|x\| > 0$ if $x \neq 0$,

(2). $\|x+y\| \leq \|x\| + \|y\|$,

(3). $\|ax\| = |a| \cdot \|x\|$ for all numbers a.

Theorem 1.1. A normed linear space is also a linear topological space.

Proof: If x is any element of the normed linear space, then for each number $\varepsilon > 0$ the set consisting of those elements y satisfying the inequality $\|x-y\| < \varepsilon$ will be said to be a neighborhood of x. Then: (1). An element belongs to each of its neighborhoods,[1] since $\|0\| = 0$ by (3) of Definition 1.6; (2) if $\|x-y\| = \varepsilon$, then the neighborhood of x defined by $\|x-y\| < \varepsilon$ does not contain y; (3) if x is in the neighborhoods of u and v defined by $\|u-y\| < \varepsilon_1$ and $\|v-y\| < \varepsilon_2$, then the neighborhood of x defined by $\|x-y\| < \varepsilon$, where ε is the smaller of $\varepsilon_1 - \|u-x\|$ and $\varepsilon_2 - \|v-x\|$, is contained in each of these neighborhoods and hence in their intersection. Thus a normed linear space is a Hausdorff topological space. The continuity of $x+y$ follows immediately from (2), and that of ax from (2) and (3), of Definition 1.6.

If the number $\|x-y\|$ is called the distance between the elements x and y, then it is easily seen that a normed linear space is also a metric space: i.e., the distance relation

1. See Definition 1.1.

$\|x-y\| = \rho(x,y)$ satisfies the conditions:

(1). $\rho(x,y) = 0$ if and only if $x = y$,

(2). $\rho(x,y) = \rho(y,x)$,

(3). $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

The neighborhoods of x defined by $\|x-y\| < \varepsilon$ are called <u>spheres</u> with x as center and of radius ε. If the distance from an element x to a set U is defined as the lower bound of $\|x-u\|$ for $u \in U$, then x is a limit point[1] of U if and only if this distance is zero. Likewise, a sequence of elements $x_1, x_2, \cdots$ is said to have the <u>limit x</u> if $\lim_{n\to\infty} \|x-x_n\| = 0$. A <u>Cauchy sequence</u> is a sequence $x_1, x_2, \cdots$ such that $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$. Such a sequence in a normed linear space does not necessarily have a limit. in the space.

<u>Definition 1.7.</u> <u>A normed linear space is said to be complete if all Cauchy sequences have a limit. A complete normed linear space is called a Banach space</u>[2].

A function is said to be a <u>functional</u> if its values are real numbers and its arguments are elements of a topological space. The norm of Definition 1.6 is an example of a functional in a normed linear space. Functionals which are linear in the argument will $f(x+y) = f(x) + f(y)$, $f(\alpha x) = \alpha f(x)$ be found useful in studying orthogonality in normed linear spaces.

<u>Definition 1.8.</u> <u>A functional $f(x)$ with argument in a topological Abelian group is linear if it is continuous</u>[3] <u>and</u>

1. See the discussion following Definition 1.1.
2. Banach (I), pg. 53.
3. That is, for any element x and number $\varepsilon > 0$ there exists a neighborhood U of x such that $|f(x)-f(u)| < \varepsilon$ if $u \in U$.

$f(x+y) = f(x)+f(y)$ for all elements x and y.

If a linear functional $f(x)$ has its argument in a normed linear space, it follows that there exists a number M such that $|f(x)| \leq M \cdot \|x\|$ for all elements x.[1] The lower bound of such numbers M is called the modulus of f and written $\|f\|$. It can be shown[2] that for any element x_0 of a normed linear space there is a linear functional $f(x)$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$. The set of all linear functionals defined on a normed linear space T is also a normed linear space (the conjugate space) if the norm of a linear functional is its modulus.

By assuming the existence of a bilinear and symmetric functional of two variables, and relating this to the norm, it is possible to get a generalization for normed linear spaces of the inner (scalar) product of vectors.

Definition 1.9. An abstract Euclidean space is a normed linear space such that to each ordered pair of elements x and y there can be associated a number (x,y) with the following properties:

(1). $(tx,y) = t(x,y)$ for all numbers t, (homogeneity)

(2). $(x,y) = (y,x)$, (symmetry)

(3). $(x,y+z) = (x,y)+(x,z)$,

(4). $\|x\|^2 = (x,x)$.

The number (x,y) is the (bilinear and symmetric) inner product of the elements x and y.[3] The assumption of such an inner product implies that any finite dimensional subset of the

1. Banach (I), pg. 54, Theorem 1.
2. Banach (I), pg. 55, Theorem 3.
3. See Fréchet (II), pg. 707.

space is equivalent[1] to a finite dimensional Euclidean space-- i.e. to a space with elements of the form $x = (x_1, x_2, \cdots)$ and $\|x\| = \sqrt{\Sigma(x_i^2)}$. For example, it is clear from (1) and (3) of Definition 1.~~8~~9 that for any abstract Euclidean space there exist elements x and y such that $\|x\| = \|y\| = 1$ and $(x,y) = 0$. See opposite

Then $\|ax+by\|^2 = (ax+by, ax+by) = a^2(x,x)+b^2(y,y) = a^2\|x\|^2+b^2\|y\|^2$. $= a^2 + b^2 = \|(a,b)\|^2$

Hence if the correspondence $ax+by \leftrightarrow (a,b)$ is set up, it follows that the operations of addition and multiplication by real numbers are preserved under the correspondence and that $\|ax+by\| = \|(a,b)\|$ for all a and b. Hence the abstract Euclidean space generated by the elements x and y is equivalent to the two dimensional Euclidean space. It is evident that the same argument can be used for any finite dimensional abstract Euclidean space.

ordered pair

→ ie it is an isomorphism of vector spaces

It will be found useful to express the inner product (x,y) directly in terms of the norm:

<u>Theorem 1.2.</u> <u>The inner product (x,y) of an abstract Euclidean space is equal to:</u>

(1). $\frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right]$, <u>or</u>

(2). $\pm\frac{1}{2}\left[\|x \pm y\|^2 - \|x\|^2 - \|y\|^2\right]$.

<u>Also</u>: (3). $\|x+ky\|^2 = \|x\|^2 + k^2\|y\|^2 + 2k(x,y)$.

<u>Proof</u>: From (4) of Definition 1.9, $\|x \pm y\|^2 = (x \pm y, x \pm y)$. The additivity and symmetry of (3) and (2) of Definition 1.8 make it possible to expand this as:

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1. In the sense that there exists a 1-to-1 bilinear correspondence which preserves the norm (see Banach (I), pg. 180).

$$\|x+y\|^2 = (x,x)+(y,y)+2(x,y),$$

and $$\|x-y\|^2 = (x,x)+(y,y)-2(x,y).$$

Using (4) of Definition 1.9, these become

$$(x,y) = \pm\tfrac{1}{2}\left[\|x\pm y\|^2-\|x\|^2-\|y\|^2\right].$$

Subtracting them gives $(x,y) = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right]$. Likewise,

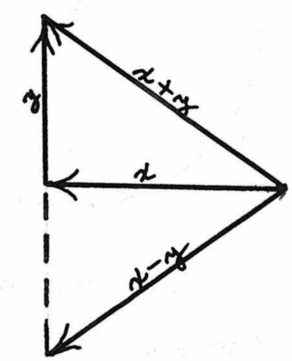
$$\|x+ky\|^2 = (x+ky,\ x+ky) = (x,x)+k^2(y,y)+2k(x,y)$$
$$= \|x\|^2+k^2\|y\|^2+2k(x,y).$$

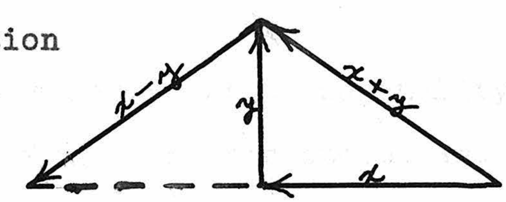
2. ORTHOGONALITY IN NORMED LINEAR SPACES

A definition of orthogonality of elements of a normed linear space has been suggested by B. D. Roberts[1]. It is that two elements x and y are orthogonal if and only if $\|x+ky\| = \|x-ky\|$ identically in k. Unfortunately, there exist normed linear spaces where two elements can be orthogonal by Roberts' definition only if one is zero.[2] In order to avoid this difficulty the following three definitions have been developed. They are equivalent for abstract Euclidean spaces,[3] but are not equivalent in a general normed linear space.

Definition 2.1. Isosceles orthogonality: An element x of a normed linear space is orthogonal to an element y $(x \perp y)$ if and only if $\|x+y\| = \|x-y\|$.

This is a generalization of the fact that two vectors x and y are perpendicular if and only if $x+y$ and $x-y$ are equal in length. The vectors $x+y$ and $x-y$ can then form the sides of an isosceles triangle with $2y$, or $2x$, as base. Because of the nature of geometrical addition of vectors, the latter could be taken as the condition for perpendicularity.





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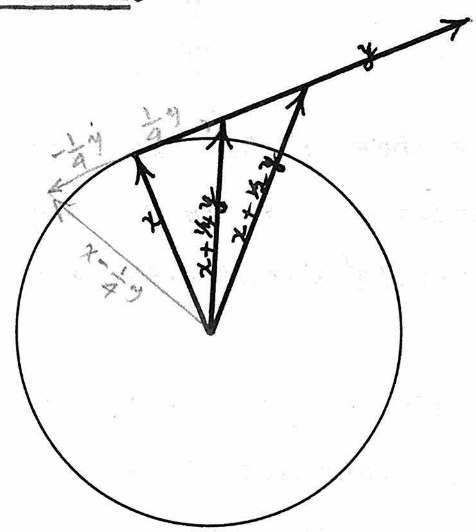
1. Roberts (VIII), pg. 56.
2. See Example 2.1.
3. See Theorem 3.1.

Definition 2.2. Pythagorean orthogonality: An element x of a normed linear space is orthogonal to an element y $(x \perp y)$ if and only if $\|x\|^2 + \|y\|^2 = \|x - y\|^2$.

If a right triangle in ordinary Euclidean space is defined, by virtue of the Pythagorean theorem, as one such that the square of the hypotenuse is the sum of the squares of the legs, then this is a generalization of the fact that perpendicular vectors can be placed so as to be the legs of a right triangle.

Definition 2.3. Spherical orthogonality: An element x of a normed linear space is orthogonal to an element y $(x \perp y)$ if and only if $\|x + ky\| \geq \|x\|$ for all numbers k.



This definition could have been stated: "$x \perp y$ if and only if every element $x + ky$ lies on or outside the spherical surface consisting of elements z satisfying $\|z\| = \|x\|$." It is a generalization of the fact that in ordinary Euclidean space two vectors are perpendicular if, when in the position illustrated, one is tangent to the circle with radius x--or that the vector $x + ky$ is longer than the radius of this circle for all k.

from the given defns

It is evident that for any of these concepts of orthogonality the zero element is orthogonal to every element of the space, and conversely. Also 0 is the only self orthogonal element. There are other elementary properties of perpendicular vectors whose generalizations will be interesting to investigate:

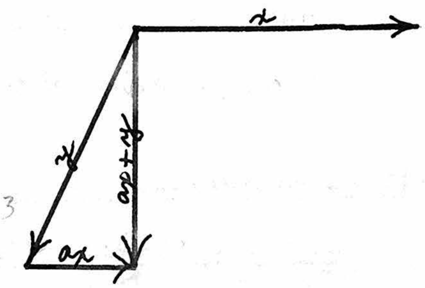
I. Symmetry: If $x \perp y$, then $y \perp x$.

II. Homogeneity: If $x \perp y$, then $ax \perp by$ for all numbers a and b.

III. Additivity: If $x \perp y$ and $x \perp z$, then $x \perp (y+z)$.

IV. If x and y are any two elements, then there exists a number a such that $x \perp (ax+y)$.

The last property is the most important, since it would keep the concept of orthogonality from being vacuous (in the it is an assurance of existence of non-zero orthogonal elements sense that conditions on the orthogonality would have no effect on the space). It is clearly satisfied for ordinary Euclidean space, and for a general normed linear space it will be shown to hold for all 3 types of ~~the~~ orthogonality of Definitions 2.1-2.3.



A slightly weaker statement of this property is: "If x and y are any two elements of a normed linear space, then x is orthogonal to some element in the plane of x and y"--where a plane is a two-dimensional linear subspace.[1]

Both isosceles and Pythagorean orthogonality are clearly symmetric, although it will be seen that spherical orthogonality is not. When the orthogonality is not symmetric, property IV could be stated either: "There exists an a such that $x \perp ax+y$", or "There exists an a such that $ax+y \perp x$". However, spherical orthogonality will be shown to satisfy either statement. Hence the one form will be used for simplicity.

All three types of orthogonality are equivalent if the space is abstract Euclidean, and have properties I-IV in this case.

1. That is, the plane of x and y consists of all elements $ax+by$.

Neither isosceles nor Pythagorean orthogonality is homogeneous or additive in a normed linear space, although they are both symmetric. The assumption of these properties will be shown to imply that the space is abstract Euclidean. Spherical orthogonality is clearly homogeneous, although it is not symmetric nor additive, and the assumptions of these properties will be shown to imply that the space is abstract Euclidean. The following example shows the independence of the above types of orthogonality in normed linear spaces, and that for Roberts' definition of orthogonality there exist normed linear spaces where at least one of two orthogonal elements must be zero.

Example 2.1. Let T be the normed linear space consisting of all continuous (polynomials) functions of the form $f = ax + bx^2$, where $\|ax+bx^2\|$ is the maximum of $|ax+bx^2|$ in the interval $(0,1)$ $[0,1]$. Then:

(1). Two elements of T are orthogonal by Roberts' definition if and only if one is zero; i.e. $\|f+kg\| = \|f-kg\|$ for all k only if $f = 0$ or $g = 0$. Consider the function $f = ax+bx^2$. If $f \perp g$, then $\|f+kg\| = \|f-kg\|$ for all k. If $|f|$ takes on its maximum only at $x = 1$, and $f \not\equiv 0$, then by taking k sufficiently small $|f+kg|$ and $|f-kg|$ can be made to take on their maxima as near $x = 1$ as desired, because of the continuity of f and g. Hence $g(1)$ must be zero if $\|f+kg\| = \|f-kg\|$ for k small, and g must be a multiple of $x - x^2$ if it is not zero. Thus if $g \not\equiv 0$, it (then $|g|$) takes on its maximum only at $x = \frac{1}{2}$, and by taking k large it follows similarly that $f(\frac{1}{2}) = 0$ and f is a multiple of $x - 2x^2$. But $\|f+4g\| \neq \|f-4g\|$, since $\|f+4g\| = \max_{(0,1)} |5x-6x^2| = \frac{25}{24}$ and $\|f-4g\| = \max_{(0,1)} |3x-2x^2| = \frac{9}{8}$.

Hence $f \perp g$ is impossible if either $|f|$ or $|g|$ takes on its maximum only at $x = 1$, unless f or g is zero. Now suppose $|f|$ takes on its maximum only at $x = \frac{-a}{2b}$, the only other possible point for the maximum, since $f' = a + 2bx$. Then reasoning as before, $f \perp g$ implies $g(\frac{-a}{2b}) = 0$ if $f \neq 0$, and that g is either zero or a multiple of $ax + 2bx^2$. Since $|f|$ takes on its maximum only for $x = \frac{-a}{2b}$, $|f+kg|$ must take on its maximum near $x = \frac{-a}{2b}$ if $|k|$ is sufficiently small. But $f \pm kg = a(1 \pm k)x + b(1 \pm 2k)x^2$, and for $|k|$ small $\|f+kg\| = \|f-kg\|$ therefore becomes

$$\left|\frac{a^2(1+k)^2}{4b(1+2k)}\right| = \left|\frac{a^2(1-k)^2}{4b(1-2k)}\right|.$$

But this implies $k = 0$ or $k^2 = 1/3$. Therefore $f \perp g$ is impossible if either of $|f|$ or $|g|$ takes on its maximum only in the interior of $(0,1)$, unless f or g is zero. The only remaining possibility is for $|f|$ and $|g|$ to take on their maxima at both an interior point of $(0,1)$ and at $x = 1$. But f and g would then both be multiples of $2(1-\sqrt{2})x + x^2$, which is clearly impossible.

(2). The elements $f_1 = x - x^2$ and $g_1 = x$ are orthogonal in the isosceles sense, since $\|x-(x-x^2)\| = \|x+(x-x^2)\| = 1$. However, they are not orthogonal in the Pythagorean sense, since $\|x-x^2\|^2 + \|x\|^2 = 17/16$, while $\|f_1 - g_1\|^2 = 1$. Also, $\|f_1 - \frac{1}{8}g_1\| = \|\frac{7}{8}x - x^2\| = (7/16)^2$, which is less than $\|f_1\| = \frac{1}{4}$. Thus f_1 and g_1 are not spherically orthogonal.

(3). The elements $f_2 = x$ and $g_2 = -2x + \frac{1}{4}\sqrt{65}\,x^2$ are orthogonal in the Pythagorean sense, since $\|f_2\|^2 = 1$, $\|g_2\|^2 = 16/65$, $\|f_2 - g_2\|^2 = 81/65$, and hence $\|f_2\|^2 + \|g_2\|^2 = \|f_2 - g_2\|^2$. They are not orthogonal in the isosceles sense, since

$\|f_2+g_2\|^2 = \frac{81}{16} - \frac{1}{2}\sqrt{65}$. Also, $\|f_2-\frac{1}{2}g_2\| = \max_{(0,1)} |2x - \frac{1}{8}\sqrt{65}\, x^2| = \frac{8}{\sqrt{65}}$, which is less than $\|f_2\| = 1$. Thus f_2 and g_2 are not spherically orthogonal.

(4). The elements $f_3 = x$ and $g_3 = 2(x-x^2)$ are spherically orthogonal, since $|f_3(1)+kg_3(1)| = 1$, and hence $\|f_3+kg_3\| \geq \|f_3\| = 1$, for all k. But $\|f_3+g_3\| = \|3x-2x^2\| = \frac{9}{8}$ and $\|f_3-g_3\| = \|x-2x^2\| = 1$. Therefore f_3 and g_3 are not orthogonal in the isosceles sense. They are not orthogonal in the Pythagorean sense since $\|f_3\|^2 + \|g_3\|^2 = \frac{5}{4}$, while $\|f_3-g_3\|^2 = 1$.

The properties I-IV above could have been stated differently. For example, homogeneity and additivity could have been combined, and one could have generalized the theorem: "Every plane <u>intersecting</u> (Containing)? a given line contains at least one line perpendicular to the given one". Some relations between this and properties I-IV for normed linear spaces are given by the following theorem.

<u>Theorem 2.1.</u> (1). <u>Homogeneity and additivity of orthogonality are equivalent to: "If $x \perp y$ and $x \perp z$, then $x \perp (ay+bz)$ for all numbers a and b.</u>

(2). <u>If orthogonality is of one of the types of Definitions 2.1-2.3, then homogeneity, additivity, and property IV</u>[1] <u>are equivalent to homogeneity, additivity, and: "If x, y and z are any three elements for which x is not orthogonal to y, there is a number a such that $x \perp ay+z$.</u>

1. See page 13.

Proof: (1). If $x \perp y$ and $x \perp z$ imply $x \perp (ay + bz)$ for all numbers a and b, then homogeneity can be gotten by taking $b = 0$ and additivity by taking $a = b = 1$. Conversely, if $x \perp y$ and $x \perp z$, homogeneity gives $x \perp ay$ and $x \perp bz$. Additivity then gives $x \perp (ay + bz)$. (2). If x, y and z are any three elements, property IV gives the existence of numbers b and c such that $x \perp bx + y$ and $x \perp cx + z$. If x is not orthogonal to y, then $b \neq 0$ and homogeneity gives $x \perp (-cx - \frac{c}{b} y)$. Additivity then gives $x \perp (-\frac{c}{b} y + z)$. The converse can be gotten by taking $y = x$, if x is not orthogonal to itself. If one of the types of orthogonality of Definitions 2.1-2.3 is used, x cannot be orthogonal to itself unless it is zero--in which case $x \perp (ax + y)$ for all a and y.

3. ORTHOGONALITY IN ABSTRACT EUCLIDEAN SPACES

The most obvious requirement for the orthogonality of two elements of an abstract Euclidean space[1] is that their inner product be zero. The relation between this and the above types of orthogonality follows immediately from Theorem 1.2.

Theorem 3.1. For abstract Euclidean spaces, isosceles, Pythagorean, and spherical orthogonality are equivalent, and two elements x and y are orthogonal if and only if their inner product $(x,y) = 0$.

Proof: It is evident from (1) and (2) of Theorem 1.2 that two elements x and y are orthogonal in the isosceles sense, or in the Pythagorean sense, if and only if $(x,y) = 0$.[2] If $(x,y) = 0$, it follows from (3) of Theorem 1.2 that $\|x+ky\| \geq \|x\|$ for all k, and hence that x and y are spherically orthogonal.[3] Conversely, it also follows from (3) that $\|x+ky\| \geq \|x\|$ implies $k^2\|y\|^2 + 2k(x,y) \geq 0$. Since this is zero for $k = 0$, the inequality can hold for all k only if the derivative with respect to k is zero at $k = 0$. That is, if $(x,y) = 0$.

ie passes thru origin. it will be ≥ 0 only if the tangent line at origin is the x-axis
ie only if slope at $k=0$ is zero.

Since two elements x and y of an abstract Euclidean space are orthogonal if and only if $(x,y) = 0$, it is clear that

1. See Definition 1.9.
2. See Definitions 2.1 and 2.2.
3. See Definition 2.3.

symmetry, homogeneity, and additivity of the orthogonality are equivalent to symmetry, homogeneity, and additivity of the inner product. Also, because of (1), (3) and (4) of Definition 1.9, $(x,ax+y) = a\|x\|^2 + (x,y)$. Hence if x and y are not orthogonal, then $x \perp ax+y$, where $a = -\frac{(x,y)}{\|x\|^2}$. Thus if $x \neq 0$, then the number a such that $x \perp ax+y$ is unique.

Theorem 3.2. The orthogonality of abstract Euclidean spaces is symmetric, homogeneous, and additive, and for any two elements $x\ (\neq 0)$ and y there exists a unique number a such that $x \perp ax+y$.

4. ISOSCELES ORTHOGONALITY

A number of relations can be obtained between orthogonal elements of general normed linear spaces. Isosceles orthogonality is obviously symmetric, and it can be shown that for any elements x and y there is a number a such that $x \perp ax+y$.[1] The assumption of homogeneity and additivity of the orthogonality[2] will be shown to imply that the space is abstract Euclidean.

If two vectors x and y of ordinary Euclidean space are perpendicular, then $\|x+ky\| > \|x\|$ for $k \neq 0$. While this is not true for isosceles orthogonality in normed linear spaces, it is possible to establish weaker inequalities. These give an idea of the degree of independence of isosceles and spherical orthogonality, as well as interesting inequalities resulting from the condition $\|x+y\| = \|x-y\|$.

Lemma 4.1. *If x and y are orthogonal elements of a normed linear space, then:*

(1). $\|x+ky\| \leq |k|\|x\pm y\|$ *and* $\|x\pm y\| \leq \|x+ky\|$, *if* $|k| \geq 1$,

(2). $\|x+ky\| \leq \|x\pm y\|$ *and* $|k|\|x\pm y\| \leq \|x+ky\|$, *if* $|k| \leq 1$.

Proof: The equation $x+ky = \frac{1}{2}(k+1)(x+y) - \frac{1}{2}(k-1)(x-y)$ is an identity in k. The triangular inequality of the norm[3] therefore gives $\|x+ky\| \leq \|\frac{1}{2}(k+1)(x+y)\| + \|\frac{1}{2}(k-1)(x-y)\|$. If $x \perp y$, then $\|x+y\| = \|x-y\|$. Hence if $|k| \geq 1$, then $\|x+ky\| \leq |k|\|x\pm y\|$.

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1. Theorem 4.5.
2. Whenever "orthogonality" is used in this section, it will mean the "isosceles orthogonality" of Definition 2.1.
3. Condition (2) of Definition 1.6.

If $|k| \leq 1$, then $\|x+ky\| \leq \|x \pm y\|$. Likewise, the relation $\|\frac{1}{2}(k+1)(x+y)\| \leq \|x+ky\| + \|\frac{1}{2}(k-1)(x-y)\|$ gives the other two inequalities of the Lemma. The inequalities (1) and (2) can also be obtained from each other by replacing k by $\frac{1}{k}$ and interchanging x and y, since isosceles orthogonality is symmetric.

<u>Theorem 4.1.</u> <u>If x and y are orthogonal elements of a normed linear space, then $\|x+ky\| \geq \|x\|$ for all k such that $|k| \geq 1$.</u>

<u>Proof</u>: From the identity $2x = (x+y)+(x-y)$, it follows that $2\|x\| \leq \|x+y\|+\|x-y\|$.[1] Since $\|x+y\| = \|x-y\|$, this gives $\|x\| \leq \|x \pm y\|$. But from (1) of Lemma 4.1, $\|x \pm y\| \leq \|x+ky\|$ if $|k| \geq 1$. Hence $\|x\| \leq \|x+ky\|$ if $|k| \geq 1$.

The result of Theorem 4.1 is not valid if $|k| < 1$. In fact, the following example shows that it is possible to have $x \perp y$ and $\|x+ky\|$ as near to $\frac{1}{2}\|x\|$ as desired--although it is not possible to have $\|x+ky\| = \frac{1}{2}\|x\|$.[2] These results, and those following, give a comparison between isosceles and spherical orthogonality.

<u>Example 4.1.</u> <u>Let T be the normed linear space consisting of all pairs (a,b) of real numbers, with $(a,b)+(c,d) = (a+c,b+d)$, $k(a,b) = (ka,kb)$, and $\|(a,b)\|$ as the larger of $|a|$ and $|b|$.</u> Let $x = (1,0)$ and $y = (n-1,n)$. Then $\|x\| = 1$, $(x-y) = (2-n,-n)$, and $(x+y) = (n,n)$. If $n \geq 1$, then

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1. Condition (2) of Definition 1.6.
2. Theorem 4.2.

$\|x - y\| = \|x+y\| = n$, and x and y are orthogonal. But

$$x - \frac{1}{2n}y = \left(\frac{n+1}{2n}, -\tfrac{1}{2}\right), \quad \text{and} \quad \left\|x - \frac{1}{2n}y\right\| = \frac{n+1}{2n},$$

which approaches $\frac{1}{2}$ as n becomes infinite. That is, given the element $x = (1,0)$ and $\varepsilon > 0$, there can be found an element y orthogonal to x and such that $\|x + ky\| < \frac{1}{2} + \varepsilon$ for some k. However, for no normed linear space can there exist orthogonal elements x ($\neq 0$) and y for which $\|x+ky\| \leq \frac{1}{2}\|x\|$ for some k. This is shown by the following theorem.

Theorem 4.2. *If x ($\neq 0$) and y are orthogonal elements of a normed linear space, then $\|x + ky\| > \frac{1}{2}\|x\|$ for all k.*

Proof: The equation $(k+1)(x-ky) = (1-k)(x+ky) + 2k(x-y)$ is an identity in k. Hence it follows from the triangular inequality of the norm[1] that $|k+1|\,\|x-ky\| \leq |1-k|\,\|x+ky\| + 2\,|k|\,\|x-y\|$. Suppose there exists a number k such that $\|x+ky\| = \frac{1}{2}\|x\|$. There is no loss of generality in taking $k \geq 0$, since otherwise y could be replaced by $-y$. It then follows from Theorem 4.1 that $0 \leq k < 1$. Also, $\|x - ky\| \geq \frac{3}{2}\|x\|$, since $2\|x\| \leq \|x - ky\| + \|x + ky\|$. Substituting in the above inequality, it follows that $\frac{3}{2}(k+1)\|x\| \leq \frac{1}{2}(1-k)\|x\| + 2k\|x-y\|$, or $(2k+1)\|x\| \leq 2k\|x-y\|$. But from (2) of Lemma 4.1, $k\|x-y\| \leq \|x+ky\|$, or $k\|x-y\| \leq \frac{1}{2}\|x\|$. Therefore $(2k+1)\|x\| \leq \|x\|$ and $k = 0$. But this is impossible if $\|x+ky\| = \frac{1}{2}\|x\|$. Hence there can not be a number k such that $\|x+ky\| = \frac{1}{2}\|x\|$. Since $\|x+ky\|$ is a continuous function of k, and therefore must take on all values between its maximum and minimum, it follows that $\|x+ky\| > \frac{1}{2}\|x\|$ for all k.

1. Condition (2) of Definition 1.6.

It has been shown by Example 4.1 that Theorem 4.2 is the best result of its type obtainable without further assumptions of some kind. However, in this example $\|y\|$ became infinite as $\|x+ky\|$ approached $\frac{1}{2}\|x\|$. By restricting the value of $\|y\|$ the inequality of Theorem 4.2 can be strengthened, the following theorem being an example of this.

<u>Theorem 4.3.</u> <u>If x and y are orthogonal elements of a normed linear space, and $\|y\| \leq \|x\|$, then $\|x+ky\| \geq (2\sqrt{2}-2)\|x\|$ for all k.</u>

<u>Proof</u>: Suppose $\|x+ky\| = r\|x\|$, where $r<1$. Take k to be positive, since otherwise y could be replaced by $-y$. Then $0<k<1$, since by Theorem 4.1, $\|x+ky\| \geq \|x\|$ if $|k| \geq 1$. Then, since $2\|x\| \leq \|x+ky\|+\|x-ky\|$,[1] we have $(2-r)\|x\| \leq \|x-ky\|$. But

$$(k+1)\|x-ky\| \leq (1-k)\|x+ky\|+2k\|x-y\|$$

follows from the identity $(k+1)(x-ky) = (1-k)(x+ky)+2k(x-y)$.[1] Hence $(k+1)(2-r)\|x\| \leq (1-k)r\|x\|+2k\|x-y\|$, or

$$(k+1)\|x\| \leq r\|x\|+k\|x-y\|.$$

Also, from $x+y = (x+ky)+(1-k)y$ and the orthogonality condition $\|x-y\| = \|x+y\|$, it follows that $\|x-y\| \leq \|x+ky\|+(1-k)\|y\|$. Using $\|x+ky\| = r\|x\|$ and $\|y\| \leq \|x\|$, this gives $\|x-y\| \leq (1-k+r)\|x\|$. Hence $(k+1)\|x\| \leq r\|x\|+k(1-k+r)\|x\|$, which gives $r \geq \frac{1+k^2}{1+k}$. But $\frac{1+k^2}{1+k}$ has a minimum (for $k \geq 0$) of $2\sqrt{2}-2$, when $k = \sqrt{2}-1$. Hence $r \geq 2\sqrt{2}-2$, and $\|x+ky\| \geq (2\sqrt{2}-2)\|x\|$ for all k.

As with Theorem 4.2, it is possible to give an example showing that the inequality of Theorem 4.3 can not be strengthened

1. Condition (2) of Definition 1.6.

without making further assumptions. That is, it is possible to have $x \perp y$, $\|y\| \leq \|x\|$, and $\|x+ky\| = (2\sqrt{2}-2)\|x\|$. As seen by the method of proof of Theorem 4.3,[1] it is then necessary that $|k| = \sqrt{2}-1$.

Example 4.2. Let T be the normed linear space consisting of all pairs (a,b) of real numbers, with $(a,b)+(c,d) = (a+c,b+d)$, $k(a,b) = (ka,kb)$, and $\|(a,b)\| = |a|+|b|$. Let $x = (1,s)$ and $y = (s,-1)$, where s will be chosen later. Then $\|x\| = \|y\| = |s|+1$. Also, $(x+y) = (1+s,s-1)$ and $(x-y) = (1-s,s+1)$. Hence $\|x+y\| = \|x-y\| = |1+s|+|1-s|$ and x and y are orthogonal. But $(x+ky) = (1+ks,s-k)$, and if k is taken equal to s, then $\|x+ky\| = 1+s^2 = \frac{1+s^2}{1+|s|}\|x\|$. If $s = \sqrt{2}-1$, this becomes $\|x+(\sqrt{2}-1)y\| = (2\sqrt{2}-2)\|x\|$. That is, if $x = (1,\sqrt{2}-1)$ and $y = (\sqrt{2}-1,-1)$, then $\|x+ky\| = (2\sqrt{2}-2)\|x\|$ if $k = \sqrt{2}-1$.

It is necessary to have $\|x\| = \|y\|$ in any example of the limiting case of Theorem 4.3. This is evident from the proof of that theorem, since for such an example we would have $r = \frac{1+k^2}{1+k}$, and hence every inequality used in the derivation of the inequality $r \geq \frac{1+k^2}{1+k}$ must be valid if made an equality. Among these was $\|y\| \leq \|x\|$.

Corollary 4.3. If x and y are orthogonal elements of a normed linear space, and $\|y\| \leq \|x\|$, then $\|x+ky\| > (2\sqrt{2}-2)\|x\|$ for all k.

There is an interesting (although very unusual) case where it can be shown that $\|x+ky\| \geq \|x\|$ for all k, and hence that

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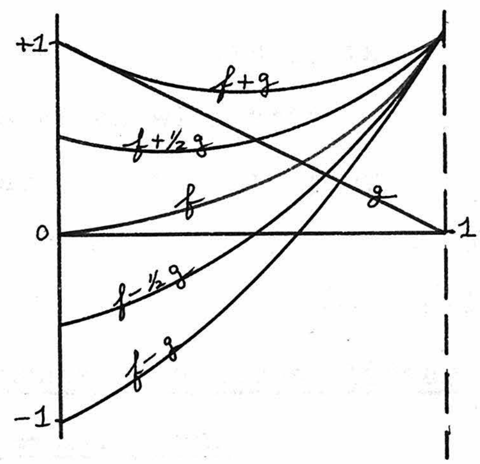
1. Because of the inequality $r \geq \frac{1+k^2}{1+k}$, and the following.

x and y are spherically orthogonal.[1] This is shown by the following theorem and examples.

Theorem 4.4. If x and y are two elements of a normed linear space, and $\|x-y\| = \|x+y\| = \|x\|$, then $\|x+ky\| = \|x\|$ if $|k| \leq 1$. Hence $\|x+ky\| \geq \|x\|$ for all k.

Proof: If $|k| \leq 1$ and $\|x-y\| = \|x+y\| = \|x\|$, it follows from (2) of Lemma 4.1 that $\|x \pm ky\| \leq \|x\|$. But from $2x = (x-ky) + (x+ky)$ and the triangular inequality of the norm,[2] it follows that $2\|x\| \leq \|x-ky\| + \|x+ky\|$. Hence $\|x+ky\| = \|x\|$ if $|k| \leq 1$. By Theorem 4.1, $\|x+ky\| \geq \|x\|$ if $|k| \geq 1$. Hence $\|x+ky\| \geq \|x\|$ for all k.

Example 4.3. Let T be the normed linear space consisting of all continuous functions in the interval $(0,1)$, with $\|f\| = \max. |f(x)|$. The elements $f = x^2$ and $g = 1-x$ satisfy the conditions of Theorem 4.4, since $\|f\| = \|g\| = \|f+g\| = \|f-g\| = 1$. Then (as must be true as a result of Theorem 4.4), $\|f+kg\| = 1$ if $|k| \leq 1$. That is, the maximum in $(0,1)$ of $|x^2 + k(1-x)|$ is 1 if $|k| \leq 1$, as is evident from the figure. Thus f is orthogonal to g by either Definition 2.1 or Definition 2.3.



Example 4.4. Consider the normed linear space of all pairs (a,b) of real numbers, with $\|(a,b)\|$ as the larger of $|a|$ and

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1. See Definition 2.3.
2. Condition (2) of Definition 1.6.

$|b|$.[1] Let $x = (1,0)$ and $y = (0,b)$, where $|b| \leq 1$. Then $\|x\| = \|x-y\| = \|x+y\| = 1$. Hence $\|x+ky\| = \|(1,kb)\| = \|x\| = 1$, if $|k| \leq 1$. Actually, $\|x+ky\| = \|x\|$ if $\|ky\| \leq \|x\|$, and $\|x+ky\| = \|ky\|$ if $\|ky\| \geq \|x\|$.

For orthogonality in normed linear spaces to have a useful meaning, it is necessary to know that there exist non-zero orthogonal elements. For in that case, assumptions such as homogeneity and additivity of the orthogonality are not vacuous. That is, such assumptions made on the orthogonality will have a restrictive effect on the space and will not necessarily be trivially satisfied, if it is known that there always exist non-zero orthogonal elements. For isosceles orthogonality, it can be shown that for any elements x and y of a normed linear space, there exists at least one number a such that $x \perp ax+y$. That is, for any elements x and y the element x is orthogonal to an element in the plane of x and y (and of the form $ax+y$). It is also possible to give limits to the value of a in terms of $\|x\|$ and $\|y\|$. Before establishing these results it is convenient to investigate certain limits which will arise numerous times in this thesis, particularly in the study of spherical orthogonality.

Lemma 4.5. If x and y are elements of a normed linear space, then $\lim_{n\to\infty} [\|(n+a)x+y\| - \|nx+y\|] = a\|x\|$ and $\lim_{n\to-\infty} [\|(n+a)x+y\| - \|nx+y\|] = -a\|x\|$.

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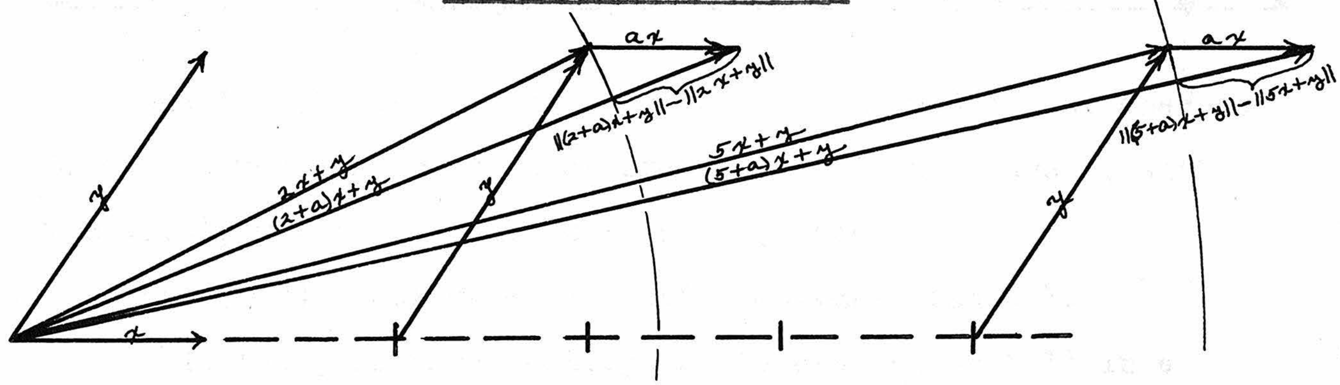
1. The same space as used for Example 4.1.

<u>Proof:</u> Since $\frac{n}{n+a} + \frac{a}{n+a} = 1$ identically, $\|(n+a)x+y\|$ can be written as $\|nx+\frac{ny}{n+a}\| + a\|x+\frac{y}{n+a}\|$, if n is positive and large enough that $n+a > 0$. Thus

$$[\|(n+a)x+y\| - \|nx+y\|] = [\|nx+\frac{ny}{n+a}\| - \|nx+y\|] + a\|x+\frac{y}{n+a}\|.$$

But $\left|\|nx+\frac{ny}{n+a}\| - \|nx+y\|\right| \leq \left|\frac{a}{n+a}\right| \|y\|$, which approaches zero as n becomes infinite. Also, $\lim_{n\to\infty} a\|x+\frac{y}{n+a}\| = a\|x\|$. Hence $\lim_{n\to\infty}[\|(n+a)x+y\| - \|nx+y\|] = a\|x\|$. Since the value of the limit is independent of y, it follows that $\lim_{n\to\infty}[\|(n-a)x-y\| - \|nx-y\|] = -a\|x\|$. But this is the same as $\lim_{n\to-\infty}[\|(n+a)x+y\| - \|nx+y\|]$, which is therefore equal to $-a\|x\|$.



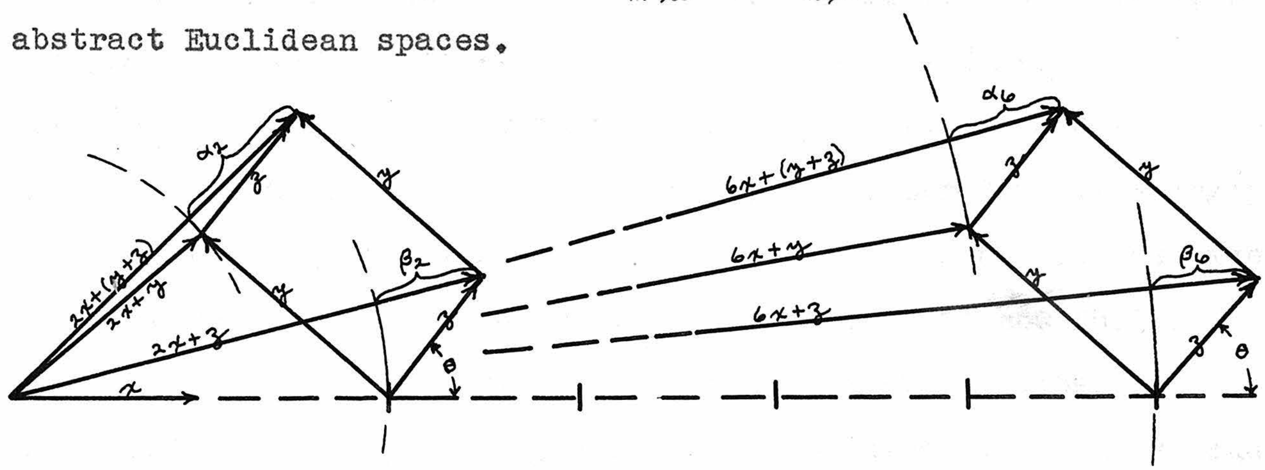
The above figure illustrates the validity of Lemma 4.5 for ordinary Euclidean space. However two similar results valid in ordinary Euclidean space can not be generalized to normed linear spaces. In an ordinary Euclidean space,

(1). $\lim_{n\to\infty}[\|nx+y\| + \|nx-y\| - 2\|nx\|] = 0.$

This is not true in general normed linear spaces. For example, consider the normed linear space of Example 4.2, consisting of ordered pairs of numbers (a,b) with $\|(a,b)\| = |a| + |b|$. Let $x = (1,0)$ and $y = (1,1)$. Then $\|nx+y\| = n+2$ if $n \geq 0$, and $\|nx-y\| = n$ if $n \geq 1$. Since $\|nx\| = n$, it follows that $\|nx+y\| + \|nx-y\| - 2\|nx\| = 2$ if $n \geq 1$. This is a special case of a more general relation which is valid in ordinary Euclidean space, but not in a general normed linear space. Namely:

(2). $\lim_{n\to\infty} [\|nx+(y+z)\| - \|nx+y\|] = \lim_{n\to\infty} [\|nx+z\| - \|nx\|]$.

Letting $\alpha_n = \|nx+(y+z)\| - \|nx+y\|$ and $\beta_n = \|nx+z\| - \|nx\|$, it is clear from the figure that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \|z\|\cos\theta$, for abstract Euclidean spaces.



The equality (2) is not valid for general normed linear spaces. If the normed linear space of Example 4.2 is used again, with $x = (1,0)$, $y = (1,1)$, and $z = (0,-1)$, then

$\|nx+(y+z)\| - \|nx+y\| = -1$ and $\|nx+z\| - \|nx\| = 1$.

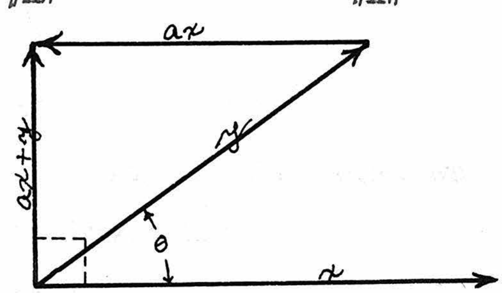
In fact, if the equality (2) is assumed to hold in a normed linear space, then a differential of the norm, $\lim_{h\to 0} \frac{\|x+hy\| - \|x\|}{h} = f(x;y)$, exists at each non-zero point of the space.[1] For $\lim_{n\to\infty} \|nx+y\| - \|nx\| = -\lim_{n\to\infty} \|nx-y\| - \|nx\|$ is equivalent to the existence of this differential, and results from (2) if z is replaced by $-y$. Conversely, the equation (2) is nothing more than an expression of the additivity of this differential[2]--which results from the assumption of the existence of $\lim_{h\to 0} \frac{\|x+hy\| - \|x\|}{h}$.[3] Thus the equality (2) holds (and the limits involved exist) if and only if this differential exists at each non-zero point of the space.

1. Such limits and differentials will be thoroughly studied in relation to spherical orthogonality.
2. That is, it is equivalent to $f(x;y)+f(x;z) = f(x;y+z)$.
3. This is shown by Mazur (VI), pg. 128.

<u>Theorem 4.5.</u> <u>If x and y are elements of a normed linear space, then there exists a number a such that $\|x+(ax+y)\| = \|x-(ax+y)\|$, ie, or $x \perp ax+y$.</u>

<u>Proof:</u> Define the real function of a real variable, $f(n)$, by $f(n) = \|x+(nx+y)\| - \|x-(nx+y)\| = \|(n+1)x+y\| - \|(n-1)x+y\|$. Then $\lim_{n\to\infty} f(n) = \lim_{n\to\infty}\left[\|(n+2)x+y\| - \|nx+y\|\right]$, which is equal to $2\|x\|$ by Lemma 4.5. Likewise, $\lim_{n\to-\infty} f(n) = \lim_{n\to-\infty}\left[\|(n+2)x+y\| - \|nx+y\|\right]$. Then Lemma 4.5 gives $\lim_{n\to-\infty} f(n) = -2\|x\|$. Since $f(n)$ is positive for some values of n and negative for others, it now follows from the continuity of $f(n)$ [1] that there exists a number a such that $f(a) = 0$, or $\|x+(ax+y)\| = \|x-(ax+y)\|$. Then x and $ax+y$ are orthogonal in the isosceles sense of Definition 2.1.

If two vectors x and ~~y~~ $ax+y$ of ordinary Euclidean space are orthogonal, then the inner product $(x,ax+y) = a\|x\|^2+(x,y)$ is zero. Also, $(x,y) = \|x\|\,\|y\|\cos\theta$, where θ is the angle between the vectors x and y. Hence $a = -\frac{\|y\|}{\|x\|}\cos\theta$, and $|a| \leq \frac{\|y\|}{\|x\|}$. It is also clear from the figure that $\|ax\| = \|y\|\,|\cos\theta|$, and hence that $|a| = \frac{\|y\|}{\|x\|}|\cos\theta|$. However, these relations do not carry over to normed linear spaces. While



$|a| \leq \frac{\|y\|}{\|x\|}$ is not valid in general, it is if $\|y\| \leq \|x\|$, and it can be shown that $|a| \leq \frac{2\|y\|-\|x\|}{\|x\|}$ if $\|y\| \geq \|x\|$.

<u>Theorem 4.6.</u> <u>If x $(\neq 0)$ and y are two elemeмts of a normed linear space and $\|y\| \leq \|x\|$, then $|a| \leq \frac{\|y\|}{\|x\|}$ if $x \perp ax+y$.</u>

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1. The continuity of the norm follows from (2) of Definition 1.6.

Proof: Define the real function of a real variable, $f(n)$, as $\|x+(nx+y)\| - \|x-(nx+y)\|$. Since $\|x+(nx+y)\| = \|(n+1)x+y\|$, it follows from the triangular inequality of the norm[1] that $\|x+(nx+y)\| \geq |n+1|\,\|x\| - \|y\|$. Likewise, $\|x-(nx+y)\| = \|(n-1)x+y\| \leq |n-1|\,\|x\| + \|y\|$. Hence $f(n) \geq \big[|n+1|\,\|x\| - \|y\|\big] - \big[|n-1|\,\|x\| + \|y\|\big]$, or $f(n) \geq \big[|n+1| - |n-1|\big]\|x\| - 2\|y\|$. If $0 \leq n \leq 1$, then $f(n) \geq 2(n\|x\| - \|y\|)$. Hence $f(n) > 0$ if $1 \geq n > \frac{\|y\|}{\|x\|}$. If $n > 1$, then $f(n) > 2(\|x\| - \|y\|)$. Since $\|y\| \leq \|x\|$, it follows that $f(n) > 0$ if $n > \frac{\|y\|}{\|x\|}$. Now consider $f(-n) = -\big[\|x+(nx-y)\| - \|x-(nx-y)\|\big]$. Since the above argument is valid if y is replaced by $-y$, the only assumption being $\|y\| \leq \|x\|$, it follows that $f(n) < 0$ if $n < -\frac{\|y\|}{\|x\|}$. Thus the only region for which $f(n)$ can be zero is for $|n| \leq \frac{\|y\|}{\|x\|}$. Hence if $f(a) = 0$, or $x \perp ax+y$, then $|a| \leq \frac{\|y\|}{\|x\|}$.

Theorem 4.7. If $x\ (\neq 0)$ and y are two elements of a normed linear space and $\|y\| \geq \|x\|$, then $|a| \leq \frac{2\|y\| - \|x\|}{\|x\|}$ if $x \perp ax+y$.

Proof: Take $f(n)$ as $\|x+(nx+y)\| - \|x-(nx+y)\|$. From the identity, $\frac{2}{n+1} + \frac{n-1}{n+1} = 1$, it follows that

$$f(n) = \left\|2x + \frac{2y}{n+1}\right\| + \left[\left\|(n-1)x + \frac{n-1}{n+1}y\right\| - \|(n-1)x+y\|\right]$$

if $n \geq 1$. But from the triangular inequality of the norm, we get $\left\|2x+\frac{2y}{n+1}\right\| \geq \|2x\| - \frac{2\|y\|}{n+1}$. It also follows that $\|(n-1)x+y\| \leq 2\frac{\|y\|}{n+1} + \left\|(n-1)x + \frac{n-1}{n+1}y\right\|$, and hence

$$\left\|(n-1)x + \frac{n-1}{n+1}y\right\| - \|(n-1)x+y\| \geq -2\frac{\|y\|}{n+1}.$$

Thus $f(n) \geq 2\|x\| - 4\frac{\|y\|}{n+1}$, and $f(n) > 0$ if $n > \frac{2\|y\| - \|x\|}{\|x\|}$. It was assumed that $n \geq 1$, but $\frac{2\|y\| - \|x\|}{\|x\|} \geq 1$ if $\|y\| \geq \|x\|$. Also, $f(-n) = -\big[\|x+(nx-y)\| - \|x-(nx-y)\|\big]$. This differs from $f(n)$ only in the sign of y. Since the above argument is valid if y is

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1. Condition (2) of Definition 1.6.

replaced by $-y$, the only assumption being $\|y\| \geq \|x\|$, it follows that $f(n) < 0$ if $n < -\frac{2\|y\| - \|x\|}{\|x\|}$. Since $f(n)$ can not be zero if $|n| > \frac{2\|y\| - \|x\|}{\|x\|}$, all values of a for which $f(a) = 0$, or $x \perp ax+y$, must be in the interval $|a| \leq \frac{2\|y\| - \|x\|}{\|x\|}$.

It was shown above Theorem 4.6 that for ordinary Euclidean space $|a| = \frac{\|y\|}{\|x\|}|\cos\theta|$ if $x \perp ax+y$, where θ is the angle between the vectors x and y. Thus it is possible to have $|a|$ as near to $\frac{\|y\|}{\|x\|}$ as desired, whether $\|y\| \leq \|x\|$ or not. It is $|a| \leq \|y\|/\|x\|$ therefore clear that the inequality of Theorem 4.6 cannot be strengthened without introducing some concept analogous to the angle between x and y. Moreover, the difficulty of doing this is illustrated by the following Example 4.5, which shows that for normed linear spaces it is possible to have $|a| = \frac{\|y\|}{\|x\|}$ without y being a multiple of x. Example 4.6 shows that Theorem 4.6 is not valid without the assumption $\|y\| \leq \|x\|$, and that the relation $|a| \leq \frac{2\|y\| - \|x\|}{\|x\|}$ of Theorem 4.7 can be an equality without having y a multiple of x.

Example 4.5. *Consider the normed linear space of all pairs (a,b) of real numbers, with $\|(a,b)\|$ as the larger of $|a|$ and $|b|$.*[1] Let $x = (1,0)$ and $y = (1,1)$. Then $x + (nx+y) = (2+n,1)$ and $x - (nx+y) = (-n,-1)$. Hence $n = -1$ is the only value of n for which $\|x + (nx+y)\| = \|x - (nx+y)\|$. But $\|x\| = \|y\| = 1$. Since in this case the number a of Theorem 4.5 is -1, it is possible to have $|a| = \frac{\|y\|}{\|x\|}$ without y being a multiple of x.

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1. The space of Example 4.1.

Example 4.6. Consider the normed linear space of all pairs (a,b) of real numbers, with $\|(a,b)\| = |a| + |b|$.[1] Let $x = (-1,1)$ and $y = (a+1,0)$. Then $\|x\| = 2$ and $\|y\| = |a+1|$. Also, $x + (ax+y) = (0,1+a)$ and $x - (ax+y) = (-2,1-a)$. If $a \geq 1$, it follows that $\|x + (ax+y)\| = \|x - (ax+y)\| = a+1$, and hence that $x \perp ax+y$. But $\frac{\|y\|}{\|x\|} = \frac{1}{2}(a+1)$, and hence $a > \frac{\|y\|}{\|x\|}$ if $a > 1$. This shows that Theorem 4.6 is not valid for normed linear spaces without the assumption $\|y\| \leq \|x\|$. Moreover, for this example it is seen that $a = \frac{2\|y\| - \|x\|}{\|x\|}$ if $a \geq 1$, which is the largest value allowed for $|a|$ by Theorem 4.7.

It has been shown (Th. 3.2) that orthogonality in abstract Euclidean spaces is symmetric, homogeneous, and additive, and that for any elements x and y there exists a number a such that $x \perp ax+y$.[2] For normed linear spaces, isosceles orthogonality is clearly symmetric, and the existence of such a number a is given by Theorem 4.5. The effect of assuming homogeneity and additivity will now be investigated, it being shown that isosceles orthogonality is homogeneous and additive only for abstract Euclidean spaces.

Lemma 4.8. If isosceles orthogonality is additive in a normed linear space T, then for any two elements $x\ (\neq 0)$ and y of T there is a unique number a such that $x \perp ax+y$.

Proof: By Theorem 4.5 there is at least one number a such that $x \perp ax+y$. Suppose it is also true that $x \perp bx+y$. Then since $\|x + (bx+y)\| = \|x - (bx+y)\|$, it is clear that $x \perp -(bx+y)$. Additivity

1. The space of Example 4.2.
2. See page 13.

then gives $x \perp (a-b)x$. That is, $\|x+(a-b)x\| = \|x-(a-b)x\|$, or $|1+(a-b)| = |1-(a-b)|$. But this can only be true if $a-b = 0$, or $a = b$.

<u>Lemma 4.8'.</u> <u>If isosceles orthogonality is homogeneous and additive in a normed linear space T and $x \perp ax+y$ and $y \perp by+x$, where x and y are any elements of T, then $b\|y\|^2 = a\|x\|^2$.</u>

<u>Proof:</u> Clearly a non-zero element can not be orthogonal to a non-zero multiple of itself. Hence if $y = 0$, then $a = 0$ unless $x = 0$; and if $x = 0$, then $b = 0$ unless $y = 0$. If neither x nor y is zero and either a or b is zero, then $x \perp y$ and the other is zero because of the uniqueness given by Lemma 4.8. Hence the Lemma is true if one of a, b, x, or y is zero, and it will be supposed hereafter that none are zero. It will be shown first that $a = b$ if $\|x\| = \|y\|$. If $x \perp ax+y$, it follows from homogeneity that $x \perp \frac{ax+y}{a}$, or $\|x+\frac{1}{a}(ax+y)\| = \|x-\frac{1}{a}(ax+y)\|$. Thus $\|y\| = \|2ax+y\|$. Similarly, $y \perp \frac{by+x}{b}$, and $\|x\| = \|x+2by\|$. Thus if $\|x\| = \|y\|$, we get

$$\|2ax+y\| = \|x+2by\|.$$

But also, $\|(x+y)+(x-y)\| = 2\|x\|$ and $\|(x+y)-(x-y)\| = 2\|y\|$. Hence if $\|x\| = \|y\|$, it follows that $(x+y) \perp (x-y)$. From homogeneity, $(2a+1)(x+y) \perp (2a-1)(x-y)$. That is, $\|(2a+1)(x+y)+(2a-1)(x-y)\| = \|(2a+1)(x+y)-(2a-1)(x-y)\|$, or $\|2ax+y\| = \|x+2ay\|$. Since it was shown that $\|2ax+y\| = \|x+2by\|$, it now follows that

$$\|x+2ay\| = \|x+2by\|.$$

But $x+2ay = [x+(a+b)y]+(a-b)y$ and $x+2by = [x+(a+b)y]-(a-b)y$, and hence $(a-b)y \perp [x+(a+b)y]$. If $a-b \neq 0$, homogeneity gives $y \perp [x+(a+b)y]$. But $y \perp by+x$, and the uniqueness given by

Lemma 4.8 implies $a = 0$, contrary to assumption. Thus $a = b$ if $\|y\| = \|x\|$. But if $x \perp ax+y$ and $y \perp by+x$, then because of homogeneity, $rx \perp \frac{a}{r}(rx)+y$ and $y \perp (br)y+rx$. Hence if r is chosen so that $\|rx\| = \|y\|$, then $\frac{a}{r} = br$, or $a\|x\|^2 = b\|y\|^2$.

Theorem 4.8. If isosceles orthogonality is homogeneous and additive in a normed linear space T, then T is an abstract Euclidean space.

Proof: Define the inner product (x,y) as $-a\|x\|^2$, where x and $ax+y$ are orthogonal. It is only necessary to show that this inner product satisfies the conditions of Definition 1.9:

(1). $(tx,y) = t(x,y)$. If $x \perp ax+y$, and $t \neq 0$, then $tx \perp \frac{a}{t}(tx)+y$ if the orthogonality is homogeneous. Thus $(tx,y) = -\frac{a}{t}\|tx\|^2 = -at\|x\|^2$. Hence $(tx,y) = t(x,y)$. If $t = 0$, the proof is trivial.

(2). $(x,y)=(y,x)$. If $x \perp ax+y$, and $y \perp by+x$, then $(x,y) = -a\|x\|^2$ and $(y,x) = -b\|y\|^2$. These are equal by Lemma 4.8'.

(3). $(x,y)+(x,z) = (x, y+z)$. Suppose $x \perp ax+y$ and $x \perp bx+z$. Then $x \perp [(a+b)x+(y+z)]$ if the orthogonality is additive. Hence $(x,y) = -a\|x\|^2$, $(x,z) = -b\|x\|^2$, and $(x,y+z) = -(a+b)\|x\|^2$.

(4). $(x,x) = \|x\|^2$. Since $\|x+(-x+x)\| = \|x-(-x+x)\|$, $x \perp (-1)x+x$, and $(x,x) = \|x\|^2$.

It can be shown that isosceles orthogonality is homogeneous if it is additive. This makes it possible to simplify the assumptions used in Theorem 4.8, assuming only additivity.

Homogeneity alone can also be shown to be sufficient by using Ficken's condition for the existence of an inner product.

<u>Theorem 4.9.</u> <u>If isosceles orthogonality is homogeneous or additive in a normed linear space T, then T is an abstract Euclidean space.</u>

<u>Proof</u>: It has been proved by Ficken[1] that a normed linear space is an abstract Euclidean space if and only if $\|ax+y\| = \|x+ay\|$ for all numbers a and elements x and y for which $\|x\| = \|y\|$. If for elements x and y of T we have $\|x\| = \|y\|$, then $\|(x+y)+(x-y)\| = \|(x+y)-(x-y)\|$ and $(x+y) \perp (x-y)$.[2] If isosceles orthogonality is homogeneous in T, then

$$\|(a+1)(x+y)+(a-1)(x-y)\| = \|(a+1)(x+y)-(a-1)(x-y)\|,$$

or $\|ax+y\| = \|x+ay\|$. Thus a normed linear space for which isosceles orthogonality is homogeneous is an abstract Euclidean space. If $x \perp y$, then $x \perp -y$ and $y \perp x$ because of the nature of the condition for isosceles orthogonality.[2] Hence if the orthogonality is additive, then $nx \perp my$ for all integers m and n. Thus $\|nx+my\| = \|nx-my\|$ and $\|x+\frac{m}{n}(y)\| = \|x-\frac{m}{n}(y)\|$. Since the norm is continuous,[3] it follows that $\|x+ty\| = \|x-ty\|$ for all numbers t. That is, $x \perp ty$ for all t. Thus isosceles orthogonality is homogeneous if it is additive, and a normed linear space for which isosceles orthogonality is additive is an abstract Euclidean space.

<u>Corollary 4.9.</u> <u>Isosceles orthogonality is additive in a normed linear space T if and only if it is homogeneous in T.</u>

1. Ficken (XIV).
2. Since $x \perp y$ if and only if $\|x+y\| = \|x-y\|$.
3. As follows from condition (2) of Definition 1.6.

5. PYTHAGOREAN ORTHOGONALITY

While Pythagorean orthogonality is perhaps the most obvious means of introducing orthogonality into normed linear spaces, it is more difficult to use than the other types[1] because of the squaring of the norms. There is also a lack of symmetry resulting from the possibility of defining x and y to be orthogonal if $\|x\|^2+\|y\|^2 = \|x-y\|^2$, or if $\|x\|^2+\|y\|^2 = \|x+y\|^2$. One sign is as good as the other, although the two statements are not equivalent. More simply, $x \perp y$ does not imply $x \perp -y$, as is the case with isosceles and spherical orthogonality.

Pythagorean orthogonality is clearly symmetric, and it will be shown that the assumption that such orthogonality is homogeneous in a normed linear space T implies T is an abstract Euclidean space. Thus homogeneity of the orthogonality implies its additivity. It can also be shown that there exist non-zero orthogonal[2] elements in any normed linear space.

Theorem 5.1. If x and y are elements of a normed linear space, then there exists a number a such that $\|x\|^2+\|ax+y\|^2 = \|x-(ax+y)\|^2$, or $x \perp (ax+y)$.

Proof: Define the real valued function of a real variable, $f(n)$, as $\|x\|^2+\|nx+y\|^2 - \|x-(nx+y)\|^2$ or $\|x\|^2+\|nx+y\|^2 - \|(n-1)x+y\|^2$. Using the identity $\left(\frac{n-1}{n}\right)^2 + \frac{2n-1}{n^2} = 1$, we get:

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1. See Definitions 2.1-2.3.
2. Whenever "orthogonality" is used in this section, it will mean the "Pythagorean Orthogonality" of Definition 2.2.

$$f(n) = \|x\|^2 + \frac{2n-1}{n^2}\|nx+y\|^2 + \left[\|(n-1)x+\left(\tfrac{n-1}{n}\right)y\|^2 - \|(n-1)x+y\|^2\right],$$

$$= \|x\|^2 + (2n-1)\|x+\tfrac{1}{n}y\|^2 +$$

$$\left[\|(n-1)x+\left(\tfrac{n-1}{n}\right)y\| - \|(n-1)x+y\|\right]\left[\|(n-1)x+\left(\tfrac{n-1}{n}\right)y\| + \|(n-1)x+y\|\right].$$

The triangular inequality of the norm[1] gives

$$\left|\|(n-1)x+\left(\tfrac{n-1}{n}\right)y\| - \|(n-1)x+y\|\right| \leq \|\tfrac{1}{n}y\|, \text{ and}$$

$$f(n) \geq \|x\|^2 + (2n-1)\|x+\tfrac{1}{n}y\|^2 - \left|\tfrac{1}{n}\right|\|y\|\left[\|(n-1)x+\left(\tfrac{n-1}{n}\right)y\| + \|(n-1)x+y\|\right],$$

$$= \|x\|^2 + (2n-1)\|x+\tfrac{1}{n}y\|^2 - \|y\|\left[\|\left(\tfrac{n-1}{n}\right)x+\left(\tfrac{n-1}{n^2}\right)y\| + \|\left(\tfrac{n-1}{n}\right)x+\tfrac{1}{n}y\|\right].$$

If $n \geq 0$, it follows by using the triangular inequality of the norm that $\|x\| \leq \frac{1}{n}\|y\| + \|x+\frac{1}{n}y\|$, and $\|x+\frac{1}{n}y\|^2 \geq \left[\|x\| - \frac{1}{n}\|y\|\right]^2$. Likewise if $n \geq 1$, $\|\left(\frac{n-1}{n}\right)x+\left(\frac{n-1}{n^2}\right)y\| + \|\left(\frac{n-1}{n}\right)x+\frac{1}{n}y\| \leq 2\left(\frac{n-1}{n}\right)\|x\| + \frac{2n-1}{n^2}\|y\|$.

Hence

$$f(n) \geq 2n\|x\|^2 + \frac{2n-1}{n^2}\|y\|^2 - 2\left(\frac{2n-1}{n}\right)\|x\|\,\|y\| - \|y\|\left[2\left(\frac{n-1}{n}\right)\|x\| + \frac{2n-1}{n^2}\|y\|\right],$$

$$= 2n\|x\|^2 - 2\left(\frac{3n-2}{n}\right)\|x\|\,\|y\|,$$

$$= 2\|x\|\left[n\|x\| - \left(\frac{3n-2}{n}\right)\|y\|\right],$$ which can be made greater than zero by taking n sufficiently large. Now consider

$$f(-n) = \left[\|x\|^2 + \|nx-y\|^2 - \|(n+1)x-y\|^2\right].$$

Using the identity $\left(\frac{n+1}{n}\right)^2 - \frac{2n+1}{n^2} = 1$, we get:

$$f(-n) = \|x\|^2 - \frac{2n+1}{n^2}\|nx-y\|^2 + \left[\|(n+1)x-\left(\tfrac{n+1}{n}\right)y\|^2 - \|(n+1)x-y\|^2\right],$$

$$= \|x\|^2 - (2n+1)\|x-\tfrac{1}{n}y\|^2 + \left[\|(n+1)x-\left(\tfrac{n+1}{n}\right)y\| - \|(n+1)x-y\|\right] \times$$

$$\left[\|(n+1)x-\left(\tfrac{n+1}{n}\right)y\| + \|(n+1)x-y\|\right],$$

$$\leq \|x\|^2 - (2n+1)\|x-\tfrac{1}{n}y\|^2 + \left|\tfrac{1}{n}\right|\|y\|\left[\|(n+1)x-\left(\tfrac{n+1}{n}\right)y\| + \|(n+1)x-y\|\right],$$

$$= \|x\|^2 - (2n+1)\|x-\tfrac{1}{n}y\|^2 + \|y\|\left[\|\left(\tfrac{n+1}{n}\right)x-\left(\tfrac{n+1}{n^2}\right)y\| + \|\left(\tfrac{n+1}{n}\right)x-\tfrac{1}{n}y\|\right].$$

If $n > 0$, then $\|x-\frac{1}{n}y\|^2 \geq \|x\|^2 - \frac{2}{n}\|x\|\,\|y\| + \frac{1}{n^2}\|y\|^2$, and

$$f(-n) \leq -2n\|x\|^2 + 2\left(\frac{2n+1}{n}\right)\|x\|\,\|y\| - \frac{2n+1}{n^2}\|y\|^2 + \|y\|\left[\frac{2(n+1)}{n}\|x\| + \frac{2n+1}{n^2}\|y\|\right],$$

$$= -2\|x\|\left[n\|x\| - \frac{3n+2}{n}\|y\|\right].$$

Thus $f(-n)$ is negative if n is sufficiently large. Since

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1. Condition (2) of Definition 1.6.

$f(n)$ is continuous and has positive and negative values, it follows that $f(n) = 0$ for some value a of n. That is, $\|x\|^2 + \|ax+y\|^2 = \|x-(ax+y)\|^2$ for some number a.

While restrictions on the value of a are not as simple as for isosceles orthogonality, it is possible to obtain them from inequalities derived in the proof of Theorem 5.1. Thus

$$f(n) \geq 2\|x\|\left[n\|x\| - \frac{3n-2}{n}\|y\|\right] \quad \text{and} \quad f(-n) \leq -2\|x\|\left[n\|x\| - \frac{3n+2}{n}\|y\|\right].$$

When $\|x\| = \|y\|$, these become:

$$f(n) \geq \frac{2}{n}\|x\|^2\left[n^2-3n+2\right] \quad \text{and} \quad f(-n) \leq -\frac{2}{n}\|x\|^2\left[n^2-3n-2\right].$$

For this case it then follows that a must be between $-\frac{1}{2}(3+\sqrt{17})$ and 2, and that there is a possible value of a between $-\frac{1}{2}(3+\sqrt{17})$ and 1. The lack of symmetry of these limits is due to the condition for Pythagorean orthogonality not being equivalent to $\|x\|^2 + \|y\|^2 = \|x+y\|^2$. That is, $x \perp y$ does not imply $x \perp -y$. However, Theorem 5.1 is valid if this change is made.

<u>Corollary 5.1</u>. <u>If x and y are elements of a normed linear space, then there exists a number a such that $\|x\|^2 + \|ax+y\|^2 = \|x+(ax+y)\|^2$.</u>

<u>Proof</u>: By Theorem 5.1, there exists a number b for the elements x and $-y$ such that $\|x\|^2 + \|bx-y\|^2 = \|x-(bx-y)\|^2$. Let $a = -b$. Then $\|x\|^2 + \|ax+y\|^2 = \|x+(ax+y)\|^2$.

However, it is not always possible to find a number a for arbitrary x and y such that

$$\|x\|^2 + \|ax+y\|^2 = \|x-(ax+y)\|^2 = \|x+(ax+y)\|^2.$$

This is shown by the following example:

Example 5.1. Consider the space of all number pairs (a,b), where $\|(a,b)\|$ is the larger of $|a|$ and $|b|$.[1] Let $x = (1,0)$ and $y = (0,1)$. Then $ax+y = (a,1)$ and $x \pm (ax+y) = (1 \pm a, \pm 1)$. Consider the equation $\|x\|^2 + \|ax+y\|^2 = \|x-(ax+y)\|^2$, or $1 + \|(a,1)\|^2 = \|(1-a,-1)\|^2$. This equation must have at least one solution because of Theorem 5.1.

If $0 \leq a \leq 1$, it becomes $2 = 1$, which has no solution.

If $1 \leq a \leq 2$, it becomes $1+a^2 = 1$, which has no solution in the interval $(1,2)$.

If $a \geq 2$, it becomes $1+a^2 = 1-2a+a^2$, which has no solution for $a \geq 2$.

If $-1 \leq a \leq 0$, it becomes $2 = 1-2a+a^2$, or $a = 1-\sqrt{2}$.

If $a \leq -1$, it becomes $1+a^2 = 1-2a+a^2$, which has no solution for $a \leq -1$.

Therefore $a = 1-\sqrt{2}$ is the only value of a such that $\|x\|^2 + \|ax+y\|^2 = \|x-(ax+y)\|^2$. Similarly, $a = \sqrt{2}-1$ is the only value of a such that $\|x\|^2 + \|ax+y\|^2 = \|x+(ax+y)\|^2$. Hence there is no value of a such that $\|x\|^2 + \|ax+y\|^2 = \|x-(ax+y)\|^2 = \|x+(ax+y)\|^2$.

Pythagorean orthogonality is clearly symmetric, and for any x and y there exists a number a such that $x \perp ax+y$. However, such orthogonality is neither homogeneous nor additive, and can be homogeneous only if the normed linear space is abstract Euclidean. Thus Pythagorean orthogonality is additive if it is homogeneous. The converse of this is also true, and it therefore follows that such orthogonality can be additive in a normed linear space only if the space is abstract Euclidean.

1. The space used for Example 4.1.

Theorem 5.2. If Pythagorean orthogonality is homogeneous in a normed linear space T, then T is an abstract Euclidean space.

Proof: It will be shown that the condition given by Jordan and Neumann[1] for the existence of an inner product in a normed linear space is satisfied, namely:

$$\|x+y\|^2+\|x-y\|^2 = 2\left[\|x\|^2+\|y\|^2\right]$$

for all elements x and y. Because of Theorem 5.1, it is known that for any elements x and y of T a number a exists such that $\|x\|^2+\|ax+y\|^2 = \|x-(ax+y)\|^2$. Assuming homogeneity, it follows that $k^2\|x\|^2+\|ax+y\|^2 = \|kx-(ax+y)\|^2$ for all numbers k. Set k equal to $a\pm 1$. This gives $\|x\mp y\|^2 = (a\pm 1)^2\|x\|^2+\|ax+y\|^2$, and:

$$\|x+y\|^2+\|x-y\|^2 = 2\|ax+y\|^2+2(a^2+1)\|x\|^2.$$

But using homogeneity again, it follows that $ax\perp ax+y$, and hence $\|ax+y\|^2+a^2\|x\|^2 = \|y\|^2$. Therefore $\|x+y\|^2+\|x-y\|^2 = 2\left[\|x\|^2+\|y\|^2\right]$. It has been shown that the condition for the existence of an inner product is satisfied for all elements x and y. Thus the conditions of Definition 1.9 are satisfied and T is an abstract Euclidean space.

The inner product (x,y) known to exist by Theorem 5.2 is equal to $\frac{1}{4}\left[\|x+y\|^2-\|x-y\|^2\right]$.[2] It is interesting to note that, by using $\|x\mp y\|^2 = (a\pm 1)^2\|x\|^2+\|ax+y\|^2$, as shown in the proof of Theorem 5.2, this reduces to $(x,y) = \frac{1}{4}\left[(a-1)^2\|x\|^2-(a+1)^2\|x\|^2\right]$, or

$$(x,y) = -a\|x\|^2.$$

This is the definition of the inner product used in establishing

1. See Jordan and Neumann (IV).
2. See Theorem 1.2.

Theorem 4.8, which is the analogy of Theorem 5.2 for isosceles orthogonality.

It was shown that isosceles orthogonality is homogeneous in any normed linear space in which it is additive.[1] This can be strengthened for Pythagorean orthogonality:

<u>Theorem 5.3.</u> <u>The properties of homogeneity and additivity of Pythagorean orthogonality are equivalent for normed linear spaces.</u>

<u>Proof</u>: If Pythagorean orthogonality is homogeneous in a normed linear space T, then T is an abstract Euclidean space because of Theorem 5.2. The orthogonality is then additive because of Theorem 3.2. Conversely, suppose $x \perp y$, where x and y are arbitrary elements of a normed linear space. Then Theorem 5.1 gives the existence of a number a such that $x \perp ax - y$. If the orthogonality is additive, then $x \perp ax$, and hence $a = 0$ if $x \neq 0$. Thus $x \perp -y$. Also, $y \perp x$, because of the nature of the condition for Pythagorean orthogonality.[2] Using additivity, it now follows that $nx \perp my$ for all integers m and n. Thus $\|nx\|^2 + \|my\|^2 = \|nx - my\|^2$, or

$$\|x\|^2 + \|\tfrac{m}{n} y\|^2 = \|x - \tfrac{m}{n} y\|^2.$$

Since the norm is continuous,[3] it follows that $\|x\|^2 + \|ty\|^2 = \|x - ty\|^2$ for all numbers t, or $x \perp ty$ for all t. Thus Pythagorean orthogonality is homogeneous if it is additive.

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1. Lemma 4.9.
2. That $\|x\|^2 + \|y\|^2 = \|x - y\|^2$.
3. As follows from Condition (2) of Definition 1.6.

Clearly Theorem 5.2 is still valid if y is replaced by $-y$ throughout. This and the combination of Theorems 5.2 and 5.3 give the following Corollaries:

Corollary 5.2. If two elements x and y of a normed linear space are said to be orthogonal if and only if

$$\|x\|^2+\|y\|^2=\|x+y\|^2,$$

and if such orthogonality is homogeneous or additive in a normed linear space T, then T is an abstract Euclidean space.

Corollary 5.3. If Pythagorean orthogonality is additive in a normed linear space T, then T is an abstract Euclidean space.

6. SPHERICAL ORTHOGONALITY IN GENERAL NORMED LINEAR SPACES.

It is clear from the forms of their definitions that both isosceles and Pythagorean orthogonality are symmetric, and that spherical orthogonality is homogeneous.[1] It has been shown that a normed linear space in which either isosceles or Pythagorean orthogonality is additive is an abstract Euclidean space,[2] while it will be shown that a normed linear space in which spherical orthogonality is symmetric and additive is an abstract Euclidean space.[3]

As with isosceles and Pythagorean orthogonality, it follows that for any elements x and y of a normed linear space there exists a number a such that x is orthogonal to $ax+y$.[4] Actually, a stronger theorem than this can be established by relating spherical orthogonality and the theory of linear functionals.[5] Thus for any element x of a normed linear space there is a maximal linear subset H such that $x \perp h$ for all h in H.[6] This "hyperplane" is not necessarily unique. In fact, it will be shown in the next section that its uniqueness is equivalent to the existence of a differential of the norm, as well as to additivity of the orthogonality.[7] In Section 10 the relations between spherical

1. See Definitions 2.1-2.3. Homogeneity, symmetry, and additivity of orthogonality are discussed on page 13.
2. Theorem 4.9 and Corollary 5.3.
3. See Section 8.
4. Whenever "orthogonality" is used in this section, it will mean the "spherical orthogonality" of Definition 2.3.
5. See Definition 1.8 and the following discussion.
6. Theorem 6.2.
7. See Corollaries 7.3 and 7.4.

orthogonality and linear functionals will be used in finding an evaluation of linear functionals for certain types of normed linear spaces.

The present section will deal primarily with showing the existence of this hyperplane and a discussion of the resulting number a for which $x \perp ax+y$, and with showing the existence, for general normed linear spaces, of limits whose equality would give the existence of a type of differential of the norm.[1]

Definition 6.1. An element x of a normed linear space T is orthogonal to a set U $(x \perp U)$ if and only if $x \perp u$ for all $u \varepsilon U$.

Definition 6.2. A maximal linear subset of a normed linear space is a proper linear subset U which is not properly contained in any other linear subset. Such a subset U, or any translation $x+U$, is called a hyperplane.[2]

words, if V is a subspace and $V \supsetneq U$ then $V = T$.

The properties of linear functionals which will be needed here are discussed on page 8 of this thesis. In particular, if the modulus $\|f\|$ of the linear functional f is 1, then $|f(x)| \leq \|x\|$ for all x, and elements x^* can be found for which $|f(x^*)|$ is as near to $\|x^*\|$ as desired. The equality may not actually hold for any element, the following theorem showing that it will hold if and only if there is an element x orthogonal to the hyperplane consisting of elements $h \in T$ for which $f = 0$.[3] $f(h) = 0$ This theorem also

1. See Corollary 6.6 and Theorem 6.7.
2. Hyperplanes are discussed by Mazur (IX), pg. 71.
3. That the elements form a hyperplane is shown implicitly in the proof of Theorem 6.1.

enables one to show the existence, for any element x, of a hyperplane H for which $x \perp H$.

Theorem 6.1. If f, with $\|f\| = 1$, is a linear functional on a normed linear space T, then $|f(x)| = \|x\|$ if and only if $x \perp h$ for all elements h of T such that $f(h) = 0$.

Proof: Let H be the linear subset consisting of all elements h such that $f(h) = 0$.[1] This linear subset is maximal, since, for any two elements x and y not in H, $f(x+ky) = 0$ if $k = -\frac{f(x)}{f(y)}$. Thus $x+ky \in H$, and hence any linear set containing x and H contains y and therefore all of T. Suppose $|f(x)| = \|x\|$. Since $\|f\| = 1$, it follows that $f(h) = 0$ implies $|f(x+h)| = |f(x)| \leq \|x+h\|$. Hence $\|x\| \leq \|x+h\|$ for all $h \in H$. Since H is closed under multiplication, this gives $x \perp h$ if $h \in H$.

Conversely, suppose $x \perp H$. Take $|f(x)| = p\|x\|$, where $p \leq 1$ since $\|f\| = 1$. Then $\|x\| \leq \|x+h\|$ if $h \in H$, and

$$|f(x+h)| = |f(x)| = p\|x\| \leq p\|x+h\|.$$

Since H is maximal, every element for which f is not zero is of the form $k(x+h)$, where $h \in H$. Thus $|f(y)| \leq p\|y\|$ for all $y \in T$. Then $\|f\| \leq p$ and $p = 1$, or $|f(x)| = \|x\|$.

Theorem 6.2. If x is any element of a normed linear space, then there is a maximal linear subset H such that $x \perp H$.

Proof: Let x be an arbitrary element of a normed linear space T, and f be a linear functional such that $f(x) = \|x\|$

1. H is closed under addition because of the linearity of f (see Definition 1.8), and is closed under multiplication by real numbers because of the homogeneity of f which results from its additivity and continuity.

and $\|f\| = 1.$[1] Take H as the maximal linear subset consisting of all elements h such that $f(h) = 0$. Then by Theorem 6.1, $x \perp H$.

The converse of Theorem 6.2 is not true. The most obvious failure is the case when H is dense in its space.[2] This is the case for Hilbert space[3] when the maximal linear subset H contains the set S of elements with only a finite number of non-zero components, since any element $x = (x_1, x_2, x_3, x_4, \cdots)$ is the limit of the elements $(x_1, 0, 0, \cdots)$, $(x_1, x_2, 0, 0, \cdots)$ $(x_1, x_2, x_3, 0, 0, \cdots)$, $\cdot\ \cdot\ \cdot$. The converse of Theorem 6.2 will be further investigated in Section 10. It is closely related to the problem of finding an evaluation of linear functionals for normed linear spaces.

Theorem 6.2 is stronger than the analogous theorems proven for isosceles and Pythagorean orthogonality.[4] However, while for any elements x and y of a normed linear space there is a number a such that $x \perp ax+y$,[5] or $\|x + k(ax+y)\| \geq \|x\|$ for all k, it is not always possible to choose a such that

$$\|x + k(ax+y)\| > \|x\|$$

for all $k \neq 0$ —as can be done for abstract Euclidean spaces. This can be shown by using the normed linear space consisting of all pairs of real numbers (a,b), with $\|(a,b)\|$ as the larger of

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1. The existence of such a functional f is given by Banach (I), pg. 55, Theorem 3.
2. H is dense in T if $\bar{H} = T$.
3. The space of all sequences $x = (x_1, x_2, x_3, \cdots)$ for which Σx_i^2 is convergent, with $\|x\| = \sqrt{\Sigma (x_i)^2}$.
4. Theorems 4.5 and 5.1.
5. Theorem 6.3, below.

$|a|$ and $|b|$.[1] For let $x = (1,0)$ and $y = (0,1)$. Then $\|x + k(ax+y)\| = \|(1+ka,k)\|$, and is less than $\|x\| = 1$ if $|k| < 1$ and $-2 < ka < 0$. Thus $\|x + k(ax+y)\| \geq \|x\|$ for all k only if $a = 0$. Thus $x \perp ax+y$ only if $a = 0$. But $\|x+ky\| = \|x\|$ if $|k| \leq 1$, and hence there is no number a such that $\|x + k(ax+y)\| > \|x\|$ for all $k \neq 0$.

Theorem 6.3. *If x and y are any two elements of a normed linear space, then there exists a number a such that $\|x + k(ax+y)\| \geq \|x\|$ for all k. That is, such that $x \perp ax+y$.*[2]

Proof: If $x = 0$, then clearly $x \perp ax+y$ for all values of a. If $x \neq 0$, then Theorem 6.2 gives the existence of a maximal linear subset H such that $x \perp H$. From here on, see opposite Since x is not in H, $ax+y$ must be in H for some number a. For if not, the linear subset gotten by adding y to H would not contain x, since $x = by+h$ implies $-\frac{1}{b}x+y = -\frac{1}{b}h \in H$. Thus H would be properly contained in another linear subset $\neq T$ (because it does not contain x) and hence would not be maximal. If a is taken so that $ax+y \in H$, then $x \perp ax+y$.

For ordinary Euclidean space, it has been shown that $|a| \leq \frac{\|y\|}{\|x\|}$ if $x \perp ax+y$.[3] It was shown that this inequality was valid for isosceles orthogonality in normed linear spaces if $\|y\| \leq \|x\|$, but had to be weakened for $\|y\| > \|x\|$.[4] The following Corollary shows its validity for spherical orthogonality, without any assumption about the relative sizes of $\|y\|$ and $\|x\|$.

1. The space of Example 4.1, pg. 21.
2. In the sense of Definition 2.3.
3. See the discussion and figure on page 29.
4. Theorems 4.6 and 4.7 and Example 4.6.

Corollary 6.1. If $x\ (\neq 0)$ and y are elements of a normed linear space, and $x \perp ax+y$, then $|a| \leq \frac{\|y\|}{\|x\|}$.

Proof: By definition,[1] $x \perp ax+y$ if and only if

$$\|x + k(ax+y)\| \geq \|x\|$$

for all k. If $k = -\frac{1}{a}$, this gives $\|\frac{1}{a}\, y\| \geq \|x\|$, or $|a| \leq \frac{\|y\|}{\|x\|}$.

Unlike isosceles and Pythagorean orthogonality, spherical orthogonality is not symmetric in general normed linear spaces. That is, $x \perp y$ does not imply $y \perp x$. This lack of symmetry is shown by the normed linear space consisting of number pairs (a,b), with $\|(a,b)\| = |a| + |b|$.[2] For let $x = (1,0)$ and $y = (1,1)$. Then $\|x+ky\| = \|(1+k,k)\| = |1+k| + |k| \geq 1 = \|x\|$. Thus $\|x+ky\| \geq \|x\|$ for all k, and $x \perp y$. But $\|y+kx\| = \|(1+k,1)\| = |1+k| + 1$. Since $\|y\| = 2$, $\|y+kx\| < \|y\|$ if $-2 < k < 0$. Thus y is not orthogonal to x.

Because of this lack of symmetry, Theorem 6.3 does not give the existence of a number a such that $ax+y \perp x$. This is shown by Theorem 6.4. Likewise, the uniqueness of the number a such that $x \perp ax+y$ is independent of that of the number b such that $bx+y \perp x$. These types of uniqueness will be studied in Section 7, and will be shown to be equivalent, respectively, to Gateaux differentiability of the norm and to the condition for a normed linear space to be strictly normed.[3]

1. Definition 2.3.
2. The space of Example 4.2, page 24.
3. See Definitions 7.3 and 7.4, and Theorems 7.3 and 7.8.

<u>Theorem 6.4.</u> <u>If x and y are any two elements of a normed linear space, then there exists a number a such that $ax+y \perp x$.</u> <u>This number a is a value of k' for which $\|k'x+y\|$ takes on its absolute minimum.</u>

<u>Proof</u>: By definition,[1] $ax+y \perp x$ if and only if

$$\|(ax+y)+kx\| \geq \|ax+y\|$$

for all k. Thus $ax+y \perp x$ if and only if $\|ax+y\|$ is the smallest value of $\|k'x+y\|$. Since $\|k'x+y\|$ is a continuous function of variable k' with its value in [0,+∞ and becomes positive infinite ~~at~~ as k→ $+\infty$ and $-\infty$, it must take on its minimum. The number a can then be any value of k' for which $\|k'x+y\|$ takes on its absolute minimum.

Because of the difference in the methods of evaluation and interpretation of the numbers for which $x \perp ax+y$, and for which $ax+y \perp x$, it is interesting to consider the effect of assuming symmetry of spherical orthogonality. The following corollaries follow immediately from Theorem 6.4, using Theorems 6.1 and 6.2. A further result of this type is given in the next section by Corollary 7.5.

<u>Corollary 6.2.</u> <u>If a normed linear space is such that spherical orthogonality is symmetric, and f is a linear functional with $\|f\| = 1$ and $|f(x)| = \|x\|$, then $\|kx+y\|$ is minimum if $f(kx+y) = 0$, or</u> $k = -\frac{f(y)}{f(x)}$.

<u>Corollary 6.3.</u> <u>If a normed linear space is such that spherical orthogonality is symmetric, then for any element x there exists a maximal linear subset H such that $x \perp H$ and $H \perp x$.</u>

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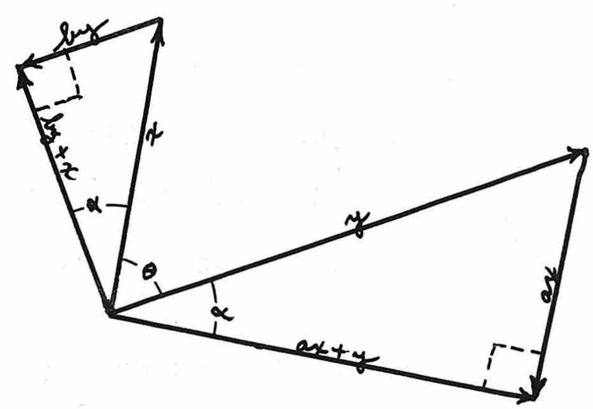
1. Definition 2.3.

Symmetry also aids in a further investigation of the relation between the numbers a and b, where $x \perp ax+y$ and $y \perp by+x$. Such a relation was very important for isosceles orthogonality, it having been shown that $b\|y\|^2 = a\|x\|^2$ if isosceles orthogonality is homogeneous and additive.[1] This was the key to the proof of Theorem 4.8, which showed that isosceles orthogonality can be homogeneous and additive in a normed linear space only if the space is abstract Euclidean.

Theorem 6.5. If x and y are any two nonzero elements of a normed linear space, and $x \perp ax+y$ and $y \perp by+x$, then $|ab| \leq 1$. If the space is such that spherical orthogonality is symmetric, then $0 \leq ab \leq 1$.

Proof: By Corollary 6.1, $|a| \leq \frac{\|y\|}{\|x\|}$ and $|b| \leq \frac{\|x\|}{\|y\|}$. Hence $|ab| \leq 1$. If $x \perp ax+y$, then $ax+y \perp x$ if the orthogonality is symmetric. Hence $\|(ax+y)+k_1 x\| \geq \|ax+y\|$ for all k_1. If $y \perp by+x$, then $\|y+k_2(by+x)\| \geq \|y\|$ for all k_2. If $k_1 = -a$, we get $\|y\| \geq \|ax+y\|$. If $k_2 = \frac{a}{1-ab}$, we get $\|ax+y\| \geq |1-ab|\ \|y\|$. Hence $\|y\| \geq |1-ab|\ \|y\|$, and $1 \geq |1-ab|$. Thus $ab \geq 0$, if spherical orthogonality is symmetric.

The validity of Theorem 6.5 for ordinary Euclidean space can be easily shown. From the figure, it is evident that a and b are both negative or both positive



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1. Lemma 4.8', page 33.

according as θ is acute or obtuse. Also, $\sin\alpha = \frac{\|by\|}{\|x\|} = \frac{\|ax\|}{\|y\|}$. Then $\sin^2\alpha = |ab|$, and $|ab| \leq 1$. Since a and b are of the same sign, $0 \leq ab \leq 1$.

Suppose $x \perp ax+y$ and $y \perp by+x$. By assigning particular values to k_1 and k_2 in $\|x + k_1(ax+y)\| \geq \|x\|$ and $\|y + k_2(by+x)\| \geq \|y\|$, a number of interesting inequalities can be derived:[1]

$$\begin{array}{lll}
 & \text{If } k_1 = \pm\frac{1}{b}: \|(a\pm b)x+y\| \geq |b|\,\|x\|, & \text{if } k_2 = \pm\frac{1}{a}: \|x+(b\pm a)y\| \geq |a|\,\|y\|, \\
 & k_1 = \frac{1}{a}: \|2ax+y\| \geq |a|\,\|x\|, & k_2 = \frac{1}{b}: \|x+2by\| \geq |b|\,\|y\|, \\
\underline{(1).} & k_1 = \frac{\pm 1}{1\mp a}: \|x\pm y\| \geq |1\mp a|\,\|x\|, & k_2 = \frac{\pm 1}{1\mp b}: \|x\pm y\| \geq |1\mp b|\,\|y\|, \\
 & k_1 = \frac{b}{1-ab}: \|by+x\| \geq |1-ab|\,\|x\|, & k_2 = \frac{a}{1-ab}: \|ax+y\| \geq |1-ab|\,\|y\|, \\
 & k_1 = \frac{1}{b-a}: \|bx+y\| \geq |a-b|\,\|x\|, & k_2 = \frac{1}{a-b}: \|ay+x\| \geq |a-b|\,\|y\|.
\end{array}$$

If spherical orthogonality is symmetric, it also follows that $ax+y \perp x$ and $by+x \perp y$. Thus $\|x + k_1(ax+y)\| \geq |k_1|\,\|ax+y\|$ and $\|y + k_2(by+x)\| \geq |k_2|\,\|by+x\|$, and other inequalities can be gotten by giving values to k_1 and k_2. But because of Theorem 6.4, these are nothing more than consequences of $\|ax+y\|$ and $\|by+x\|$ being minimum. Thus by making the same substitutions as above, one gets the following inequalities--all of which are obvious results of this minimum condition, but are interesting in that they are the variations in equations (1) above made possible by the assumption of symmetry.

$$\begin{array}{lll}
 & \|(a\pm b)x+y\| \geq \|ax+y\|, & \|x+(b\pm a)y\| \geq \|by+x\|, \\
\underline{(2).} & \|2ax+y\| \geq \|ax+y\|, & \|x+2by\| \geq \|by+x\|, \\
 & \|x\pm y\| \geq \|ax+y\|, & \|x\pm y\| \geq \|by+x\|,
\end{array}$$

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1. $k_1 = -\frac{1}{a}$ (or analogously, $k_2 = -\frac{1}{b}$) was used in proving Corollary 6.1.

(2) (Cont.) $\|by+x\| \geq |b|\,\|ax+y\|$, $\|ax+y\| \geq |a|\,\|by+x\|$,

$\|bx+y\| \geq \|ax+y\|$, $\|ay+x\| \geq \|by+x\|$.

If x and y are elements of a normed linear space, and $x \perp y$, then $\|x+ky\| \geq \|x\|$ for all k, or See opposite $\|nx+y\| \geq \|nx\|$ for all n. It thus seems interesting to investigate the limit of $\|nx+y\| - \|nx\|$ as n becomes infinite, both when x and y are orthogonal and when they are not. This limit will be shown to exist, and to give a condition for orthogonality, as well as a determination of all numbers a such that $x \perp ax+y$.[4] It can then be related to the limits of $\frac{\|x+hy\| - \|x\|}{h}$ as h approaches zero from the right and left, and to the Gateaux differential of the norm[1] if these limits are equal.

Theorem 6.6. If x and y are any two elements of a normed linear space, then $\lim_{n\to\infty} \|nx+y\| - \|nx\|$ exists.

Proof: Because of the triangular inequality of the norm,[2] $\big|\,\|nx+y\| - \|nx\|\,\big| \leq \|y\|$. Thus $\|nx+y\| - \|nx\|$ is bounded as n becomes infinite. Assuming that $\lim_{n\to\infty} \|nx+y\| - \|nx\|$ does not exist, let r and s be two limit points of $\|nx+y\| - \|nx\|$ as n becomes infinite. Since $\|nx+y\| - \|nx\|$ is a continuous function of n,[3] there must be arbitrarily large values of n such that $\|nx+y\| - \|nx\| = \frac{r+s}{2}$. If such a value of n is greater than $\frac{|r+s|}{2}$, then, for that n, $\|nx+y\| = \left\|\left(n+\frac{r+s}{2\|x\|}\right)x\right\|$, or

$$\left\|\left[\left(n+\frac{r+s}{4\|x\|}\right)x+\tfrac{1}{2}y\right]+\left[-\frac{r+s}{4\|x\|}x+\tfrac{1}{2}y\right]\right\| = \left\|\left[\left(n+\frac{r+s}{4\|x\|}\right)x+\tfrac{1}{2}y\right]-\left[-\frac{r+s}{4\|x\|}x+\tfrac{1}{2}y\right]\right\|.$$

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1. See Definition 7.3.
2. Condition (2) of Definition 1.6.
3. The continuity of the norm follows from Condition (2) of Definition 1.6.
4. The existence of this limit also follows from results of Ascoli (XV), pp. 53-55.

Theorem 4.1 then gives

$$\left\|\left[\left(n+\frac{r+s}{4\|x\|}\right)x+\tfrac{1}{2}y\right]+k\left[-\frac{r+s}{4\|x\|}x+\tfrac{1}{2}y\right]\right\| \geq \left\|\left(n+\frac{r+s}{4\|x\|}\right)x+\tfrac{1}{2}y\right\|$$

for $|k| \geq 1$. If k is replaced by $2m$ and this inequality divided by $|m|$,[1] it gives

$$\left\|\left(\frac{n}{m}+\frac{r+s}{4m\|x\|}\right)x+\frac{1}{2m}y-\frac{r+s}{2\|x\|}x+y\right\| \geq \left\|\left(\frac{n}{m}+\frac{r+s}{4m\|x\|}\right)x+\frac{1}{2m}y\right\|.$$

Now take any number p, and let n become infinite, keeping $m=\frac{n}{p}$. This gives

$$\lim_{n\to\infty}\left\|\left(p+\frac{p(r+s)}{4n\|x\|}\right)x+\frac{p}{2n}y-\frac{r+s}{2\|x\|}x+y\right\| \geq \lim_{n\to\infty}\left\|\left(p+\frac{p(r+s)}{4n\|x\|}\right)x+\frac{p}{2n}y\right\|,$$

or

$$\left\|\left(p-\frac{r+s}{2\|x\|}\right)x+y\right\| \geq \|px\|,$$

for all p. From Lemma 4.5, it follows that all limit points of $\|px+y\|-\|px\|$ as $p\to\infty$ are greater than or equal to $\frac{r+s}{2\|x\|}$. But this is impossible if r and s are both limit points. Hence the assumption that $\lim\limits_{n\to\infty}\|nx+y\|-\|nx\|$ does not exist is false.

<u>Corollary 6.6.</u> <u>If x and y are any two elements of a normed linear space, then $\lim\limits_{h\to+0}\frac{\|x+hy\|-\|x\|}{h}$ and $\lim\limits_{h\to-0}\frac{\|x+hy\|-\|x\|}{h}$ exist.</u>

<u>Proof:</u> Theorem 6.6 gives the existence of $\lim\limits_{n\to\infty}\|nx+y\|-\|nx\|$. Setting $n=\frac{1}{h}$, this becomes $\lim\limits_{h\to+0}\frac{\|x+hy\|-\|x\|}{h}$. The existence of $\lim\limits_{n\to\infty}\|nx-y\|-\|nx\|$ is also given by Theorem 6.6, since the element y was arbitrary and can therefore be replaced by $-y$. This is equal to $\lim\limits_{h\to+0}\frac{\|x-hy\|-\|x\|}{h}$, or $-\lim\limits_{h\to-0}\frac{\|x+hy\|-\|x\|}{h}$.

Theorem 6.6 and its Corollary are essentially the same, but the proof is easier to follow if the limits are kept as in the theorem. The two forms will therefore be carried together.

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1. Using Condition (3) of Definition 1.6.

Conditions under which the limits of Corollary 6.6 are equal will be studied in Section 7.[1]

The following theorem gives an evaluation of the limits known to exist because of Theorem 6.6. The proof is very much like that of Theorem 6.6, and it could have been revised so as not to make use of that theorem and to give it as a corollary. However, the proof is much more complicated in that case and it therefore seems advisable to include both.

Theorem 6.7. If x and y are any two elements of a normed linear space, with $x \neq 0$, then $\lim_{n\to\infty} \|nx+y\| - \|nx\| = -A\|x\|$, where A is the algebraically smallest number such that $x \perp Ax+y$. Also, $\lim_{n\to\infty} \|nx-y\| - \|nx\| = B\|x\|$, where B is the largest number such that $x \perp Bx+y$.

Proof: The existence of $\lim_{n\to\infty} \|nx+y\| - \|nx\|$ is given by Theorem 6.6. But, for any number a, this limit is equal to $\lim_{n\to\infty} \|(n-a)x+ax+y\| - \|nx\|$, or because of Lemma 4.5, to

$$\lim_{n\to\infty} \|nx + (ax+y)\| - \|nx\| - a\|x\|.$$

If $x \perp ax+y$, then $\|nx+(ax+y)\| \geq \|nx\|$, and hence $\lim_{n\to\infty} \|nx+y\| - \|nx\| \geq -a\|x\|$. Thus if A is the greatest lower bound of all numbers a such that $x \perp ax+y$, then $\lim_{n\to\infty} \|nx+y\| - \|nx\| \geq -A\|x\|$. Suppose $\lim_{n\to\infty} \|nx+y\| - \|nx\| = -r\|x\|$. Let ε be any positive number. Then there must exist arbitrarily large values of n such that $\big|\|nx+y\| - \|nx\| + r\|x\|\big| < \varepsilon\|x\|$. If $n > |r|$, this can be written $\big|\|nx+y\| - \|(n-r)x\|\big| < \varepsilon\|x\|$. There is then a number e such that $|e| < \varepsilon$ and $\|nx+y\| - \|(n-r)x\| - e\|x\| = 0$,

1. See Theorems 7.2, 7.3, and Corollary 7.4.

or $\|nx+y\| = \|(n-r+e)x\|$. But this can be written:

$$\left\|\left[(n-\tfrac{1}{2}r+\tfrac{1}{2}e)x+\tfrac{1}{2}y\right] + \left[(\tfrac{1}{2}r-\tfrac{1}{2}e)x+\tfrac{1}{2}y\right]\right\| = \left\|\left[(n-\tfrac{1}{2}r+\tfrac{1}{2}e)x+\tfrac{1}{2}y\right] - \left[(\tfrac{1}{2}r-\tfrac{1}{2}e)x+\tfrac{1}{2}y\right]\right\|.$$

Theorem 4.1 then gives:

$$\left\|\left[(n-\tfrac{1}{2}r+\tfrac{1}{2}e)x+\tfrac{1}{2}y\right] + k\left[(\tfrac{1}{2}r-\tfrac{1}{2}e)x+\tfrac{1}{2}y\right]\right\| \geq \left\|(n-\tfrac{1}{2}r+\tfrac{1}{2}e)x+\tfrac{1}{2}y\right\|$$

for $|k| \geq 1$. If k is replaced by $2m$ and the inequality divided by $|m|$,[1] this gives

$$\left\|\left[(\frac{n}{m}-\frac{r}{2m}+\frac{e}{2m})x+\frac{1}{2m}y\right]+(r-e)x+y\right\| \geq \left\|(\frac{n}{m}-\frac{r}{2m}+\frac{e}{2m})x+\frac{1}{2m}y\right\|.$$

Now take any number p, and let n become infinite, keeping $m = \frac{n}{p}$. This gives

$$\lim_{n\to\infty}\left\|\left[(p-\frac{pr}{2n}+\frac{ep}{2n})x+\frac{p}{2n}y\right]+(r-e)x+y\right\| \geq \lim_{n\to\infty}\left\|(p-\frac{rp}{2n}+\frac{ep}{2n})x+\frac{p}{2n}y\right\|,$$

or $\|(p+r-e)x+y\| \geq \|px\|$ for all p. Since $|e|<\varepsilon$ and ε was arbitrary, this gives $\|px+(rx+y)\| \geq \|px\|$ for all p. It has thus been shown that if $\lim_{n\to\infty} \|nx+y\|-\|nx\| = -r\|x\|$, then $x \perp rx+y$. But it was shown that $\lim_{n\to\infty} \|nx+y\|-\|nx\| \geq -A\|x\|$, where A is the greatest lower bound of all numbers a such that $x \perp ax+y$. It now follows that $\lim_{n\to\infty} \|nx+y\|-\|nx\| = -A\|x\|$, and $x \perp Ax+y$. If B is the largest number such that $x \perp Bx+y$, then $-B$ is the smallest number such that $x \perp -Bx-y$.[2] Hence $\lim_{n\to\infty} \|nx-y\|-\|nx\| = B\|x\|$, where B is the largest number such that $x \perp Bx+y$.

<u>Corollary 6.7.</u> <u>If x and y are any two elements of a normed linear space, and $x \neq 0$, then $\lim_{h\to+0} \frac{\|x+hy\|-\|x\|}{h} = -A\|x\|$, and $\lim_{h\to-0} \frac{\|x+hy\|-\|x\|}{h} = -B\|x\|$, where A and B are the algebraically smallest and largest of the numbers a for which $x \perp ax+y$.</u>

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1. Using Condition (3) of Definition 1.6.
2. Since spherical orthogonality is homogeneous. See page 13 and Definition 2.3.

<u>Proof</u>: This follows immediately by replacing n by $\frac{1}{h}$ in $\lim_{n\to\infty} \|nx+y\| - \|nx\| = -A\|x\|$, and n by $-\frac{1}{h}$ in $\lim_{n\to\infty} \|nx-y\| - \|nx\| = B\|x\|$, both of which are given by Theorem 6.7.

For elements x and y of a normed linear space, Theorem 6.7 gives the upper and lower bounds of all numbers a for which $x \perp ax+y$. The following results show that $x \perp ax+y$ for all numbers a between these bounds, and hence give an evaluation of all such numbers. This gives a necessary and sufficient condition for the spherical orthogonality of elements of a normed linear space.

<u>Lemma 6.8. If x and y are any two elements of a normed linear space, and $x \perp Ax+y$ and $x \perp Bx+y$, then $x \perp ax+y$ if a is a number between A and B.</u>

<u>Proof</u>: For definiteness, assume ~~$A<a<B$.~~ $A<B$. If $x \perp Ax+y$, then $\|x+k(Ax+y)\| \geq \|x\|$ for all k. Likewise, if $x \perp Bx+y$, then $\|x+k(Bx+y)\| \geq \|x\|$ for all k. Let a be any number such that $A \leq a \leq B$. Consider $\|x+k(ax+y)\|$. If $k \geq 0$, then

$$\|x+k(ax+y)\| = \left\|[1+k(a-A)]\,x+k(Ax+y)\right\| \geq \left\|[1+k(a-A)]\,x\right\| \geq \|x\|.$$

If $k \leq 0$, then

$$\|x+k(ax+y)\| = \left\|[1+k(a-B)]\,x+k(Bx+y)\right\| \geq \left\|[1+k(a-B)]\,x\right\| \geq \|x\|.$$

Hence $\|x+k(ax+y)\| \geq \|x\|$ for all k, and therefore $x \perp ax+y$.

Theorem 6.8. If x and y are any two elements of a normed linear space, then $x \perp ax+y$ if and only if

$$-\underline{\lim}_{h\to+0} \frac{\|x+hy\| - \|x\|}{h} \leq a\|x\| \leq -\overline{\lim}_{h\to-0} \frac{\|x+hy\| - \|x\|}{h}$$

Proof: It follows from Corollary 6.7 that the inequality of Theorem 6.8 is satisfied if a is such that $x \perp ax+y$. Also $x \perp Ax+y$ and $x \perp Bx+y$, where $A = -\underline{\lim}_{h\to+0} \frac{\|x+hy\| - \|x\|}{h\|x\|}$ and $B = -\underline{\lim}_{h\to-0} \frac{\|x+hy\| - \|x\|}{h\|x\|}$. Thus by Lemma 6.8, $x \perp ax+y$ if $A \leq a \leq B$.

Corollary 6.8. If x and y are any two elements of a normed linear space, then $x \perp y$ if and only if $\underline{\lim}_{h\to+0} \frac{\|x+hy\| - \|x\|}{h} \geq 0$ and $\overline{\lim}_{h\to-0} \frac{\|x+hy\| - \|x\|}{h} \leq 0$.[1]

It follows from Theorem 6.7 that $\lim_{n\to\infty} \left[\|nx+y\| - \|nx\|\right] \geq \lim_{n\to\infty} \left[\|nx\| - \|nx-y\|\right]$, or $\lim_{n\to\infty} \left[\|nx+y\| + \|nx-y\| - 2\|nx\|\right] \geq 0$. This is also evident from the triangular inequality of the norm,[2] which gives $2\|nx\| \leq \|nx+y\| + \|nx-y\|$. This limit was investigated before.[3] It is zero in ordinary Euclidean space, and because of Theorem 6.7 (or 6.8) it being zero is necessary and sufficient for the uniqueness of the number a for which $x \perp ax+y$ and for the existence of the Gateaux differential of the norm.[4]

For any elements x and y of a normed linear space, Theorem 6.8 enables one to determine all numbers a such that $x \perp ax+y$. How this can be done is illustrated by the following example:

1. Also see Corollary 7.2.
2. Condition (2) of Definition 1.6.
3. Page 27 of this thesis.
4. See Definition 7.3.

Example 6.1. Let T be the normed linear space consisting of all pairs (a,b) of real numbers, with $\|(a,b)\|$ as the larger of $|a|$ and $|b|$.[1] Let $x = (1,1)$ and $y = (0,1)$. Then $\|x+hy\| = \|(1,1+h)\|$ and $\frac{\|x+hy\| - \|x\|}{h} = 1$ if $h > 0$ and zero if $h < 0$ and small. Thus $\lim_{h\to+0} \frac{\|x+hy\| - \|x\|}{h} = 1$ and $\lim_{h\to-0} \frac{\|x+hy\| - \|x\|}{h} = 0$. Then by Theorem 6.8, $x \perp ax+y$ if and only if $-1 \leq a \leq 0$. That is, $(1,1) \perp (a,a+1)$ if and only if $-1 \leq a \leq 0$. This conclusion can also be verified directly. By definition,[2] $x \perp ax+y$ if and only if $\|x + k(ax+y)\| \geq \|x\|$, or

$$\left\| [1+ka, 1+k(a+1)] \right\| \geq \|(1,1)\| = 1$$

for all k. Clearly this is true for $|k|$ small if and only if a and $a+1$ are not of the same sign; that is, if and only if $-1 \leq a \leq 0$.

The two limits of Theorem 6.7, and of Corollary 6.7, can be added. This has the advantage of making it possible to let $h \to 0$, since the right and left limits are then equal. It will also be valuable in interpreting the following results.

Theorem 6.9. If x and y are any two elements of a normed linear space, then $\lim_{n\to\infty} \|nx+y\| - \|nx-y\| = -(A+B)\|x\|$, or $\lim_{h\to 0} \frac{\|x+hy\| - \|x-hy\|}{h} = -(A+B)\|x\|$, where A and B are the smallest and largest of the numbers a for which $x \perp ax+y$.

Proof: Theorem 6.7 gave $\lim_{n\to\infty} \|nx+y\| - \|nx\| = -A\|x\|$, and $\lim_{n\to\infty} \|nx-y\| - \|nx\| = B\|x\|$. Subtracting these gives $\lim_{n\to\infty} \|nx+y\| - \|nx-y\| = -(A+B)\|x\|$. If n is replaced by $\frac{1}{h}$, this

1. The space of Example 4.1.
2. Definition 2.3.

becomes $\lim_{h\to+0} \frac{\|x+hy\| - \|x-hy\|}{h}$, which is obviously equal to $\lim_{h\to-0} \frac{\|x+hy\| - \|x-hy\|}{h}$.

The limits of Theorem 6.9 are twice the differential of the norm[1] if it exists. For ordinary Euclidean space, these limits being zero is clearly necessary and sufficient for the orthogonality of x and y. The following theorem shows that even for normed linear spaces there is always a unique number a for which $x \perp ax+y$ and $\lim_{n\to\infty} \|nx + (ax+y)\| - \|nx - (ax+y)\| = 0$.

Theorem 6.10. If x and y are any two elements of a normed linear space, and A and B are the smallest and largest of the numbers a for which $x \perp ax+y$, then $x \perp \frac{1}{2}(A+B)x+y$ and $\lim_{n\to\infty} \|nx + (ax+y)\| - \|nx - (ax+y)\| = 0$ if and only if $a = \frac{1}{2}(A+B)$.

Proof: It follows immediately from Lemma 6.8 that $x \perp \frac{1}{2}(A+B)x+y$. But since A and B are the smallest and largest of the numbers a for which $x \perp ax+y$, it follows that $\frac{1}{2}(A-B)$ and $\frac{1}{2}(B-A)$ are the smallest and largest of the numbers a' for which $x \perp a'x + \frac{1}{2}(A+B)x + y$. Hence it follows from Theorem 6.9 that $\lim_{n\to\infty} \|nx + [\frac{1}{2}(A+B)x+y]\| - \|nx - [\frac{1}{2}(A+B)x+y]\| = -[\frac{1}{2}(A-B) + \frac{1}{2}(B-A)] = 0$. From this and Lemma 4.5, it is clear that $\lim_{n\to\infty} \|nx + (ax+y)\| - \|nx - (ax+y)\| = 0$ if and only if $a = \frac{1}{2}(A+B)$.

It is now interesting to look at Example 6.1 again. For the x and y of this example, the number A of Theorem 6.10 is -1, and $B = 0$. Thus $\frac{1}{2}(A+B) = -\frac{1}{2}$. Hence Theorem 6.10 shows

1. See Definition 7.3.

that we must have $\lim_{n\to\infty} \|nx+(-\frac{1}{2}x+y)\|-\|nx-(-\frac{1}{2}x+y)\| = 0$, or $\lim_{n\to\infty} \|(n-\frac{1}{2},n+\frac{1}{2})\| - \|(n+\frac{1}{2},n-\frac{1}{2})\| = \lim_{n\to\infty} 0 = 0$.

While isosceles and spherical orthogonality[1] are not equivalent in general normed linear spaces, it is possible to establish a relation between them. Thus the unique number $\frac{1}{2}(A+B)$ of Theorem 6.10 can be shown to be $\lim_{n\to\infty} a_n$, where $nx \perp a_n x+y$ in the isosceles sense. The following Lemma used in establishing this is a strengthened form of Lemma 4.5.

Lemma 6.11. For any normed linear space T and any positive numbers E and ε there exists a number N such that for all elements x and y of T $\left|\|(n+a)x+y\|-\|nx+y\|-a\|x\|\right|<\varepsilon$ if $|a|<E$ and $n>N[\|y\|+1]$.

Proof: Since $\frac{n}{n+a}+\frac{a}{n+a}=1$ identically, $\|(n+a)x+y\|$ can be written as $\|nx+(\frac{n}{n+a})y\|+a\|x+(\frac{1}{n+a})y\|$, if n is positive and large enough that $n+a>0$. Thus

$$\|(n+a)x+y\|-\|nx+y\| = \left[\|nx+(\tfrac{n}{n+a})y\|-\|nx+y\|\right]+a\|x+(\tfrac{1}{n+a})y\|.$$

But from the triangular inequality of the norm,

$$\left|\|nx+(\tfrac{n}{n+a})y\|-\|nx+y\|\right|\le\left|\tfrac{a}{n+a}\right|\|y\|,$$

and

$$\left|a\|x+(\tfrac{1}{n+a})y\|-a\|x\|\right|\le\left|\tfrac{a}{n+a}\right|\|y\|.$$

If N is large enough that $N+a>0$, then for $n>N[\|y\|+1]$, $\left|\frac{a}{n+a}\right|\|y\|<\left|\frac{a\|y\|}{N\|y\|+N+a}\right|<\left|\frac{a}{N}\right|$ for all y. For positive numbers E and ε, choose N large enough that $\left|\frac{a}{N}\right|<\frac{\varepsilon}{2}$ and $N+a>0$ if $|a|<E$. Then $\left|\frac{a}{n+a}\right|\|y\|<\varepsilon/2$ if $n>N[\|y\|+1]$ and $|a|<E$. Thus $\left|\|(n+a)x+y\|-\|nx+y\|-a\|x\|\right|<\varepsilon$ if $|a|<E$ and $n>N[\|y\|+1]$.

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1. See Definitions 2.1 and 2.3.

Theorem 6.11. If x and y are any two elements of a normed linear space, and the numbers a_n are such that $\|nx+(a_nx+y)\| = \|nx-(a_nx+y)\|$, then $\lim_{n\to\infty} a_n = a$ exists and is such that $x \perp ax+y$. Also, $\lim_{n\to\infty} \|nx+y\|-\|nx-y\| = -2a\|x\|$.[1]

Proof: The existence of $\lim_{n\to\infty} \|nx+y\|-\|nx-y\|$ is known.[2] But

$$\|nx+y\|-\|nx-y\| = \big[\|nx+y\|-\|nx+(a_nx+y)\|\big]+\big[\|nx-(a_nx+y)\|-\|nx-y\|\big].$$

If n is large enough that $\|y\| \le \|nx\|$, then Theorem 4.6 gives $\left|\frac{a_n}{n}\right| \le \frac{\|y\|}{\|nx\|}$, or $|a_n| \le \frac{\|y\|}{\|x\|}$. This, with a double application of Lemma 6.11, gives the existence, for any positive number ε, of a number N such that $n > N$ implies

$$\big|\|(n+a_n)x+y\|-\|nx+y\|-a_n\|x\|\big| < \tfrac{1}{2}\varepsilon$$

and

$$\big|\|(n-a_n)x-y\|-\|nx-y\|+a_n\|x\|\big| < \tfrac{1}{2}\varepsilon.$$

Hence $\big|\|nx+y\|-\|nx-y\|+2a_n\|x\|\big| < \varepsilon$ if $n > N$. Since $\lim_{n\to\infty} \|nx+y\|-\|nx-y\|$ exists, it how follows that $\lim_{n\to\infty} a_n = a$ exists, and that $\lim_{n\to\infty} \|nx+y\|-\|nx-y\| = -2a\|x\|$. From Theorem 6.9, it follows that $a = \frac{1}{2}(A+B)$, where A and B are the smallest and largest of the numbers b such that $x \perp bx+y$. The orthogonality of x and $ax+y$ now follows from Lemma 6.8.

Corollary 6.11. If x and y are any two elements of a normed linear space, then $\lim_{h\to 0} \frac{\|x+hy\|-\|x-hy\|}{h}$ exists and is equal to $\lim_{n\to\infty} -2a_n\|x\|$, where $\|nx+(a_nx+y)\| = \|nx-(a_nx+y)\|$.

It is interesting to note that while the number a such that $x \perp ax+y$ is not unique in general, the particular a

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1. For any value of n, the existence of such a number a_n is given by Theorem 4.5.
2. See Theorem 6.9.

of Theorem 6.11 is unique. In any normed linear space, $\lim_{h\to 0} \frac{\|x+hy\|-\|x-hy\|}{h} = -2a\|x\|$, where a is the unique number of Theorem 6.11, and $x \perp ax+y$. If the number a for which $x \perp ax+y$ is assumed to be unique, then $\lim_{h\to 0} \frac{\|x+hy\|-\|x\|}{h}$ exists and is equal to $-a\|x\|$.[1]

Furthermore, Theorem 6.8, or Theorem 6.7 and Lemma 6.8, gives a simple evaluation of all numbers a for which $x \perp ax+y$. The largest such number is $\lim_{n\to\infty} \frac{\|nx-y\|-\|nx\|}{\|x\|}$, or $-\lim_{h\to -0} \frac{\|x+hy\|-\|x\|}{h\|x\|}$; the smallest is $-\lim_{n\to\infty} \frac{\|nx+y\|-\|nx\|}{\|x\|}$, or $-\lim_{h\to +0} \frac{\|x+hy\|-\|x\|}{h\|x\|}$. The numbers between these two bounds are the totality of other numbers a for which $x \perp ax+y$. The mean of these limits is the <u>unique</u> number a for which $\lim_{n\to\infty} \|nx+y\|-\|nx-y\| = 0$, or $\lim_{h\to 0} \frac{\|x+hy\|-\|x-hy\|}{h} = 0$, which are obvious conditions for orthogonality if x and y are elements of ordinary Euclidean space.

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1. See Theorems 7.2 and 7.3.

7. SPHERICAL ORTHOGONALITY, DIFFERENTIABILITY OF THE NORM, AND STRICTLY NORMED SPACES.

In the previous section, spherical orthogonality was investigated for general normed linear spaces. This section will consider the effect on normed linear spaces of assuming certain types of uniqueness of orthogonal elements. For elements x and y, these types of uniqueness will be defined in terms of the number a for which $x \perp ax+y$.[1] Since spherical orthogonality is not symmetric,[2] $x \perp ax+y$ does not imply $ax+y \perp x$, and this uniqueness can take the following forms:

Definition 7.1. Orthogonality is right-unique if for any elements x $(\neq 0)$ and y there exists at most one number a such that $x \perp ax+y$.

Definition 7.2. Orthogonality is left-unique if for any elements x $(\neq 0)$ and y there exists at most one number a such that $ax+y \perp x$.

For a type of orthogonality which is homogeneous,[3] these two concepts of uniqueness are equivalent, respectively, to:

1. The existence of this number a is given by Theorem 6.3.
2. This follows from the independence of the types of uniqueness given by Definitions 7.1 and 7.2, as shown by Theorems 7.3 and 7.8, taken with Examples 7.2 and 7.3. A simple example could also be given to show this.
3. See page 13 of this thesis. Spherical orthogonality is homogeneous by the nature of its definition ($x \perp y$ if and only if $\|x+ky\| \geq \|x\|$ for all k) and property (3) of Definition 1.6.

"If $x \neq 0$, then any plane[1] containing an element x contains at most one element y such that $x \perp y$", and: "If $x \neq 0$, then any plane[1] containing an element x contains at most one element y such that $y \perp x$".

If the orthogonality is symmetric, then right and left-uniqueness are equivalent. This was the case in normed linear spaces for the other two types of orthogonality which have been investigated, as is clear from their definitions.[2] Right and left-uniqueness of spherical orthogonality are not equivalent, and will be related, respectively, to the concepts of Gateaux differentiability of the norm and strictly normed spaces as given in the following definitions. For isosceles and Pythagorean orthogonality, additivity implies uniqueness (left and right).[3] For normed linear spaces, it will be shown that spherical orthogonality is right-unique if and only if it is additive, and additive if and only if the Gateaux differential of the norm exists at all non-zero points.[4] Also, spherical orthogonality is left-unique if and only if the normed linear space is strictly normed.[5] This gives a relation between these and other well known concepts applicable to normed linear spaces, and a means of investigating them by use of spherical orthogonality.

1. "Plane" here means a two-dimensional linear subset.
2. Definitions 2.1 and 2.2.
3. Lemma 4.8 and Corollary 5.3. The uniqueness in abstract Euclidean spaces follows from Theorem 3.2.
4. Theorems 7.3 and 7.4.
5. Theorem 7.8.

Definition 7.3. A functional[1] f defined on a normed linear space T is Gateaux differentiable[2] at a point x if $\lim_{h\to 0} \frac{f(x+hy)-f(x)}{h}$ exists for all elements y of T. If this limit exists it is the Gateaux differential at x with increment y, and is written $f(x;y)$.

Definition 7.4. A normed linear space is strictly normed[3] if from the equality $\|x\| + \|y\| = \|x+y\|$ $(y \neq 0)$ it follows that there is a number t for which $x = ty$.

An abstract Euclidean space is strictly normed. For if $\|x\| + \|y\| = \|x+y\|$, then $\|x+y\|^2 - \|x\|^2 - \|y\|^2 = 2\|x\| \cdot \|y\|$. If $y \neq 0$, then a positive number t can be chosen such that $\|x\| = \|ty\|$. Then (2) of Theorem 1.2 gives $\|x\| \cdot \|y\| = (x,y)$, and $\|x\| \cdot \|ty\| = (x,ty)$.[4] Applying (2) of Theorem 1.2 now gives $\|x-ty\|^2 = \|x\|^2 + \|ty\|^2 - 2\|x\| \cdot \|ty\|$. Thus $\|x-ty\| = 0$ and $x = ty$. Therefore any abstract Euclidean space is strictly normed. That the norm of such a space is Gateaux differentiable will be shown by Corollary 7.3'.

Right-uniqueness of spherical orthogonality can be readily applied to the results of the previous section. The following theorem is not as important as some later ones, but is interesting in that it involves limits which exist and have been evaluated without assuming right-uniqueness of spherical orthogonality.

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1. See page 7.
2. Sometimes called weakly differentiable, as by Mazur (VI).
3. Called strictly normalized by Smulian (XI), and strictly convex by Clarkson (XII).
4. See Condition (1) of Definition 1.9.

Theorem 7.1. If a normed linear space is such that spherical orthogonality is right-unique, then, for any elements x and y, $\lim_{n\to\infty} \|nx+y\| - \|nx-y\| = -2a\|x\|$, and $\lim_{h\to 0} \frac{\|x+hy\| - \|x-hy\|}{h} = -2a\|x\|$, where a is the number such that $x \perp ax+y$.

Proof: This is a restatement of Theorem 6.9, making use of the assumption of right-uniqueness of the orthogonality.

Corollary 7.1. If a normed linear space is such that spherical orthogonality is right-unique, and $nx \perp a_n x + y$ in the isosceles sense, then $x \perp ax+y$ in the spherical sense if and only if $\lim_{n\to\infty} a_n = a$.[1]

Proof: This follows easily from Theorem 6.11.

Corollary 7.1 gives an interesting relation between spherical and isosceles orthogonality.[1] The number a of this Corollary is the mean of the largest and smallest of the numbers b for which $x \perp bx+y$ in the spherical sense when such orthogonality is not right-unique,[2] but is the only such number if the orthogonality is right-unique. In any case, it is the unique limit of the numbers a_n for which $nx \perp a_n x+y$ in the isosceles sense.[3]

Theorem 7.2. If a normed linear space is such that spherical orthogonality is right-unique, then, for any elements

1. See Definitions 2.1 and 2.3.
2. This follows from Theorems 6.9 and 6.11.
3. That is, $\|nx + (a_n x+y)\| = \|nx - (a_n x+y)\|$ for all n. The existence of such a number a_n for each n is given by Theorem 4.5

<u>x $(\neq 0)$ and y,</u> $\lim_{n\to\infty} \|nx+y\| - \|nx\| = -a\|x\|$ <u>and</u> $\lim_{h\to 0} \frac{\|x+hy\| - \|x\|}{h} = -a\|x\|$, <u>where a is the number such that $x \perp ax+y$.</u>

<u>Proof</u>: This is merely a restatement of Theorem 6.7 and Corollary 6.7, making use of right-uniqueness. That is, using $A = B = a$.

<u>Corollary 7.2.</u> <u>If a normed linear space is such that spherical orthogonality is right-unique, then necessary and sufficient conditions that two elements x $(\neq 0)$ and y be orthogonal are:</u>

$$(\underline{a}).\quad \lim_{n\to\infty} \|nx+y\| - \|nx\| = 0,$$

<u>and</u>

$$(\underline{b}).\quad \lim_{h\to 0} \frac{\|x+hy\| - \|x\|}{h} = 0.$$

If spherical orthogonality is right-unique, it is then possible to define orthogonality in terms of the conditions of Corollary 7.2. Spherical orthogonality is homogeneous,[1] and symmetry and additivity can then be restated as follows:

<u>Symmetry</u> ($x \perp y$ implies $y \perp x$): "If $\lim_{n\to\infty} \|nx+y\| - \|nx\| = 0$, or $\lim_{h\to 0} \frac{\|x+hy\| - \|x\|}{h} = 0$, then $\lim_{n\to\infty} \|ny+x\| - \|ny\| = 0$ and $\lim_{h\to 0} \frac{\|y+hx\| - \|y\|}{h} = 0$.

<u>Additivity</u> ($x \perp y$ and $x \perp z$ imply $x \perp y+z$): "If $\lim_{h\to 0} \frac{\|x+hy\| - \|x\|}{h} = 0$ and $\lim_{h\to 0} \frac{\|x+hz\| - \|x\|}{h} = 0$, then $\lim_{h\to 0} \frac{\|x+h(y+z)\| - \|x\|}{h} = 0$.

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1. By virtue of its definition (Definition 2.3) and Condition (3) of Definition 1.6.

This statement of additivity is equivalent to the additivity of the Gateaux differential[1] of the norm, which follows from the existence of this differential.[2] This and the preceding theorems can be neatly restated by use of this concept.

Theorem 7.3. A normed linear space is such that spherical orthogonality is right-unique[3] if and only if the norm is Gateaux differentiable[1] at each non-zero point. If the norm is Gateaux differentiable, then the differential, $f(x;y)$, is equal to $-a\|x\|$, where a is the number such that $x \perp ax+y$.

Proof: If spherical orthogonality is right-unique, then Theorem 7.2 gives the existence and evaluation of the Gateaux differential of the norm. Conversely, if the Gateaux differential of the norm exists at each non-zero point, then the numbers A and B of Corollary 6.7 must be equal. For any elements x $(\neq 0)$ and y there can then be only one number a such that $x \perp ax+y$, and hence spherical orthogonality is right-unique.

Corollary 7.3. The norm of a normed linear space is Gateaux differentiable at each non-zero point if and only if for each non-zero element x there is a unique maximal linear subset H such that $x \perp H$.

Proof: For any non-zero element x, Theorem 6.2 gives the existence of a maximal linear subset H such that $x \perp H$. But if y is any element not a multiple of x, then any maximal

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1. See Definition 7.3.
2. This is evident from Theorems 7.3 and 7.4, or see Mazur (VI), pages 129-130.
3. See Definition 7.1.

linear subset not containing x must contain $ax+y$ for some number a.[1] Thus if spherical orthogonality is right-unique,[2] then the hyperplane H is unique. The Corollary now follows from Theorem 7.3.

<u>Corollary 7.3'.</u> <u>The norm of an abstract Euclidean space[3] is Gateaux differentiable at each non-zero point, and the Gateaux differential $f(x;y)$ of the norm is equal to</u> $\frac{(x,y)}{\|x\|}$. <u>An element x is orthogonal to $ax+y$ if and only if</u> $a = \frac{-(x,y)}{\|x\|^2}$.

<u>Proof</u>: The ratio $\frac{\|x+hy\| - \|x\|}{h}$ can be written as $\frac{\|x+hy\|^2 - \|x\|^2}{h(\|x+hy\| + \|x\|)}$. Using Definition 1.9, this becomes

$$\frac{(x+hy,x+hy)-(x,x)}{h[\|x+hy\| + \|x\|]} = \frac{2(x,y)+h(y,y)}{\|x+hy\| + \|x\|}$$

Letting h approach zero, it is seen that the Gateaux differential of the norm at x with increment y $\left(\lim_{h \to 0} \frac{\|x+hy\| - \|x\|}{h}\right)$ is equal to $\frac{(x,y)}{\|x\|}$. The rest of the Corollary follows from Theorem 7.3.

<u>Corollary 7.3".</u> <u>If the Gateaux differential of the norm exists at each non-zero point of a normed linear space T, then an element x of T is orthogonal to an element y if and only if the Gateaux differential $f(x;y)$ is zero.</u>

<u>Proof</u>: This is a restatement of Corollary 7.2, and is also immediate from Theorem 7.3.

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1. See Definition 6.2.
2. See Definition 7.1.
3. See Definition 1.9.

<u>Theorem 7.4. In a normed linear space, spherical orthogonality is additive if and only if it is right-unique.</u>

Let $x \neq 0$

<u>Proof</u>: Suppose spherical orthogonality is additive and that there exist numbers a and b such that $x \perp ax+y$ and $x \perp bx+y$. Then from the homogeneity of spherical orthogonality, $x \perp -bx-y$. Hence additivity gives $x \perp (a-b)x$, or $\|x+k(a-b)x\| \geq \|x\|$ for all k. This is clearly not true if $k = -\frac{1}{a-b}$. Thus $a = b$ and spherical orthogonality is right-unique if it is additive.

Conversely, if spherical orthogonality is right-unique, Theorem 7.3 gives the existence of the Gateaux differential of the norm. If $x \perp y$ and $x \perp z$ and the Gateaux differential of the norm at x is denoted by $f(x;y)$, then $f(x;y)+f(x;z) = 0$.[1] But Mazur[2] has shown that the Gateaux differential of the norm is linear if it exists. Hence $f(x;y+z) = 0$, and by Corollary 7.3" $x \perp y+z$. Thus right-uniqueness implies additivity.

That right-uniqueness of spherical orthogonality implies its additivity can also be proved nicely directly, and the proof seems interesting enough to be included. Let x be any non-zero element. Then there exists a linear functional f such that $f(x) = \|x\|$ and $\|f\| = 1$.[3] Let y be any element such that x is orthogonal to y. Since $\|f\| = 1$, $|f(x+ky)| \leq \|x+ky\|$. If k is taken so that $kf(y)$ is positive, then $|f(ky)| \leq \|x+ky\| - \|x\|$. Dividing by $|k|$, $|f(y)| \leq \left\|\frac{1}{k}x+y\right\| - \left\|\frac{1}{k}x\right\|$. Letting k spproach zero and assuming uniqueness, it follows from Corollary 7.2 and the assumption $x \perp y$ that $f(y) = 0$. Thus $x \perp y$ implies $f(y) = 0$, and likewise $x \perp z$ implies $f(z) = 0$. Since f is

1. See Corollary 7.3".
2. Mazur (VI), pages 129-130.
3. Banach (I), pg. 55, Theorem 3. Also see page 8 of this thesis.

linear, it follows that $f(y+z) = 0$. If $x \perp ax+(y+z)$, then $f[ax+(y+z)] = 0$. But $f(y+z) = 0$. Hence $a = 0$ and $x \perp (y+z)$. Thus additivity has been shown to follow from right-uniqueness.

Corollary 7.4. Spherical orthogonality is additive in a normed linear space if and only if the Gateaux differential of the norm exists at each non-zero point.

Proof: This follows immediately by using Theorems 7.3 and 7.4 together.

By making use of the Gateaux differential $f(x;y)$, the statements of symmetry and additivity of spherical orthogonality when right-unique, as given prior to Theorem 7.3, can now be stated simply:

Symmetry: "If $f(x;y) = 0$, then $f(y;x) = 0$.[1]

Additivity: "If $f(x;y) = 0$ and $f(x;z) = 0$, then $f(x;y+z) = 0$.

It has been shown[2] that additivity of spherical orthogonality follows from the assumption of right-uniqueness or of Gateaux differentiability of the norm, but symmetry does not. This added assumption of symmetry will be studied in Section 8. Any normed linear space of more than two dimensions will be shown to be an abstract Euclidean space if spherical orthogonality is symmetric and unique.

It is interesting to verify the evaluation of the Gateaux differential for a two-dimensional Euclidean space. Suppose x

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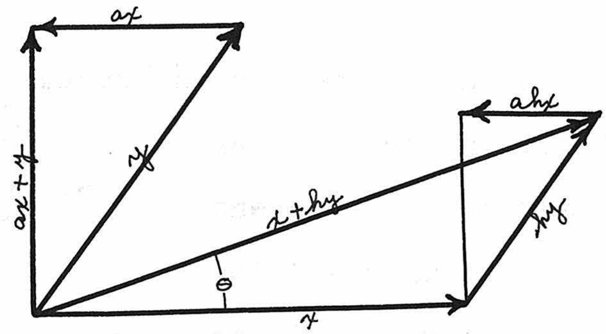
1. See Corollary 7.5'. Also, Theorem 8.5 shows that a strengthening of this condition can imply that the space is abstract Euclidean.
2. Theorems 7.3 and 7.4.



and y are situated as in the figure, with $x \perp ax+y$. In this position, a is negative, and for simplicity h will be taken as positive. Then
$\|x+hy\| = \|x\| \sec\theta + \|ahx\| \sec\theta = \|x\| \sec\theta - ah\|x\| \sec\theta$. Hence

$$\frac{\|x+hy\| - \|x\|}{h} = \frac{\|x\|(\sec\theta - 1)}{h} - a\|x\| \sec\theta .$$

But h and θ are clearly of the same order of magnitude. That is, $\lim_{h\to 0} \frac{\sec\theta - 1}{h} = 0$. Therefore $\lim_{h\to 0} \frac{\|x+hy\| - \|x\|}{h} = -a\|x\|$.

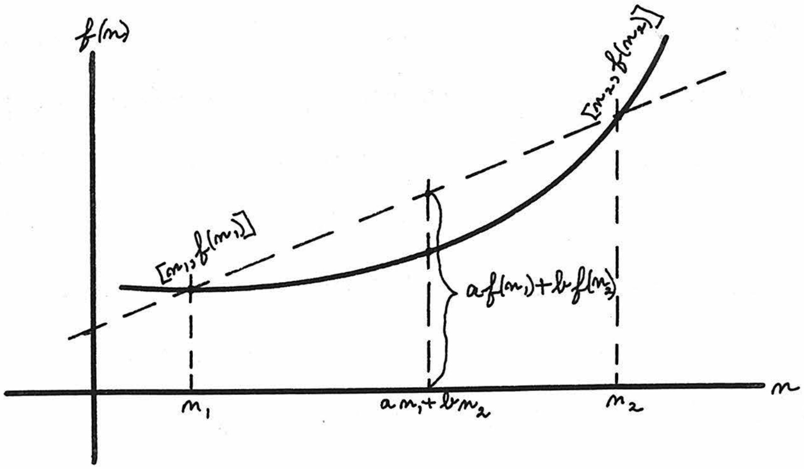
It has been shown that right-uniqueness of spherical orthogonality is equivalent to Gateaux differentiability of the norm.[1] It also follows that such orthogonality is left-unique[2] if and only if the normed linear space is strictly normed.[3] It will first be shown that the concepts of left-uniqueness and strictly normed spaces can be interpreted in terms of functions of the form $f(n) = \|x+ny\|$. These concepts will then be investigated for spaces of continuous functions, and their independence shown by means of such examples.

<u>Theorem 7.5.</u> <u>If x and y are any two elements of a normed linear space, then the curve $f(n) = \|x+ny\|$ is concave up except for possible straight line sections.</u>

<u>Proof</u>: To prove the theorem, it is sufficient to show that for any straight line intersecting the curve in two points, the segment of the curve cut off by these two points lies below or

1. Theorem 7.3.
2. See Definition 7.2.
3. See Definition 7.4 and Theorem 7.8.

on the line. If the points of intersection are for values n_1 and n_2 of n, then the ordinate of the point having abscissa $an_1 + bn_2$ and lying on the straight line joining these points is $af(n_1) + bf(n_2)$.



To establish the theorem, it is then sufficient to show that for all numbers a and b such that $a+b = 1$, it follows that

$$f(an_1 + bn_2) \leq af(n_1) + bf(n_2),$$

or
$$\|x + (an_1 + bn_2)y\| \leq a\|x + n_1 y\| + b\|x + n_2 y\|.$$

Since $\|x + (an_1 + bn_2)y\| = \|a(x+n_1 y) + b(x+n_2 y)\|$, this follows from the triangular inequality of the norm.[1]

<u>Theorem 7.6.</u> <u>If a normed linear space is such that the Gateaux differential of the norm exists at all non-zero points, then $\frac{d\|x+ny\|}{dn}$ is a continuous monotonic increasing function of n equal to $-a_n\|y\|$, where $x+ny \perp a_n(x+ny) + y$.</u>[2]

<u>Proof</u>: It follows immediately from Theorem 7.5 that $\frac{d\|x+ny\|}{dn}$ is a monotonic increasing function of n if it exists. To show its continuity, let $f(n) = \|x+ny\|$. Then the Gateaux differential $f(x+ny;y)$ at $x+ny$ with increment y is $\lim_{h \to 0} \frac{\|(x+ny) + hy\| - \|x+ny\|}{h}$, or $f'(n)$. Thus by Theorem 7.3, $f'(n) = -a_n\|y\|$, where $(x+ny) \perp a_n(x+ny) + y$. Hence $f'(n)$ is

1. Condition (2) of Definition 1.6.
2. Also see Corollary 7.5". The existence of such a number a_n for each value of n is given by Theorem 6.3.

continuous if and only if a_n is a continuous function of n. Suppose a_n is not continuous for $n = n_1$. By Corollary 6.1, $|a_n| \leq \frac{\|y\|}{\|x+ny\|}$, and hence a_n is bounded and must have at least one limit point $a \neq a_{n_1}$ as n approaches n_1. Since the Gateaux differential of the norm exists, spherical orthogonality is right-unique[1] and it is impossible for $x+n_1y$ to be orthogonal to $a(x+n_1y)+y$.[2] Therefore there exists a number k such that

$$\|(x+n_1y)+k[a(x+n_1y)+y]\| < \|x+n_1y\|.$$

But since a is a limit point of a_n as n approaches n_1, it would then be possible to select a number n_2 such that $|n_2-n_1|$ and $|a_{n_2}-a|$ are small enough that[3]

$$\|(x+n_2y)+k[a_{n_2}(x+n_2y)+y]\| < \|x+n_2y\|,$$

which is impossible if $(x+n_2y) \perp a_{n_2}(x+n_2y)+y$. Therefore $f'(n)$ is a continuous function of n.

Theorems 7.5 and 7.6 raise the question of whether it is possible for a curve $f(n) = \|x+ny\|$ to have a straight line section.[4] The following example shows that this is possible even when the norm is Gateaux differentiable. In other words, $\frac{d\|x+ny\|}{dn}$ can be constant over an interval.

Example 7.1. Let T be the space consisting of all number pairs (a,b), with $(a,b)+(c,d) = (a+c,b+d)$, $k(a,b) = (ka,kb)$, and

$$\|(a,b)\| = \begin{cases} |b| & \text{if } |a| \leq |b|, \\ \frac{1}{2}\left|a+\frac{b^2}{a}\right| & \text{if } |a| \geq |b|. \end{cases}$$

Clearly Conditions (1) and (3) of Definition 1.6 are satisfied.

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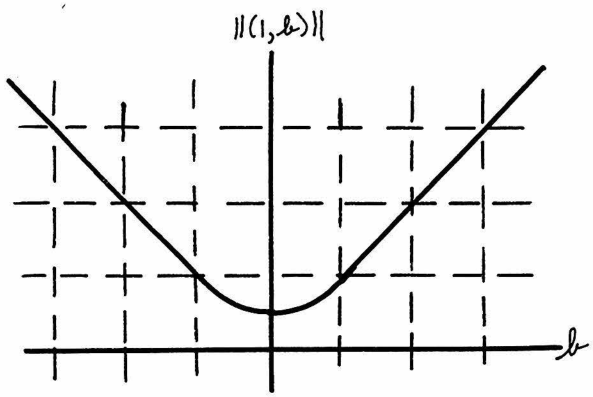
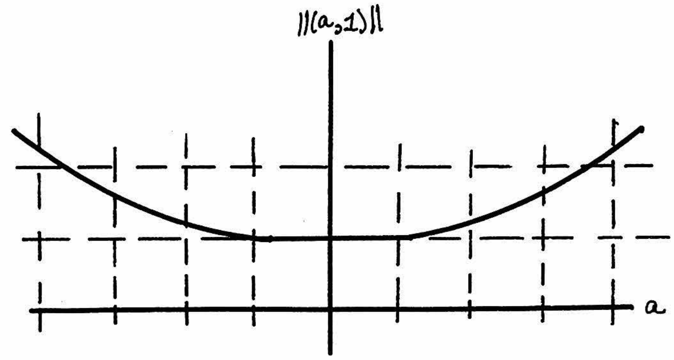
1. Theorem 7.3.
2. Since $x+n_1y \perp a_{n_1}(x+n_1y)+y$.
3. It is possible to do this beaause of the continuity of the norm, which follows from Condition (2) of Definition 1.6.
4. That this is possible was implied by Theorem 4.4, and shown by Example 4.3. However, the norm of Ex. 4.3 is not Gateaux differentiable.

Thus T is a normed linear space if Condition (2) is satisfied. That this "triangular inequality" is satisfied can be shown by considering the following curves:

$$\|(1,b)\| = \begin{cases} |b| & \text{if } |b| \geq 1, \\ \frac{1}{2}(1+b^2) & \text{if } |b| \leq 1, \end{cases} \quad \text{or} \quad \|(a,1)\| = \begin{cases} 1 & \text{if } |a| \leq 1, \\ \frac{1}{2}\left|a+\frac{1}{a}\right| & \text{if } |a| \geq 1. \end{cases}$$





Clearly the curve $\|(1,b)\|$ is never concave downward, and hence

$$\frac{r\,\|(1,b)\| + s\,\|(1,c)\|}{r+s} \geq \left\|\left(1, \frac{rb+sc}{r+s}\right)\right\|$$

for all r and s. The equality holds if b and c are both greater than 1 or less than -1; i.e. if only values of $\|(1,b)\|$ are considered which lie on a straight line. If rb is replaced by r' and sc by s', it follows from the above inequality that

$$\|(r+s, r'+s')\| \leq \|(r,r')\| + \|(s,s')\|.$$

Thus $\|x+y\| \leq \|x\|+\|y\|$ for all x and y of T, and T is a normed linear space. The Gateaux differential of the norm of T exists at each non-zero point. This follows from the following relations, which are derived from the definition of $\|(a,b)\|$:

$$\frac{d\,\|(a,b)\|}{db} = \begin{cases} 1 & \text{if } |a| \leq |b| \text{ and } b \geq 0, \\ -1 & \text{if } |a| \leq |b| \text{ and } b \leq 0, \\ \frac{b}{|a|} & \text{if } |a| \geq |b|. \end{cases}$$

$$\frac{d\,\|(a,b)\|}{da} = \begin{cases} 0 & \text{if } |a| \leq |b|, \\ \frac{1}{2}\left(1 - \frac{b^2}{a^2}\right) & \text{if } |a| \geq |b| \text{ and } a \geq 0, \\ -\frac{1}{2}\left(1 - \frac{b^2}{a^2}\right) & \text{if } |a| \geq |b| \text{ and } a \leq 0. \end{cases}$$

By using these relations (or more simply by direct calculation):

$$\lim_{h\to 0} \frac{\|(a,b)+h(A,B)\| - \|(a,b)\|}{h} = A\frac{d\|(a,b)\|}{da} + B\frac{d\|(a,b)\|}{db},$$

$$= \frac{bB}{|b|} \quad \text{if} \quad |a| \leq |b|,$$

$$= \frac{bB}{|a|} - \tfrac{1}{2}\frac{b^2 A}{a|a|} + \tfrac{1}{2}\frac{aA}{|a|} \quad \text{if} \quad |a| \geq |b|.$$

This is the Gateaux differential of the norm at (a,b) with increment (A,B).

Since $\|(a,1)\| = 1$ if $|a| \leq 1$, it follows that $ax+y \perp x$ for $|a| \leq 1$ if $x = (1,0)$ and $y = (0,1)$. Thus spherical orthogonality is not left-unique for the space of Example 7.1, even though it is right-unique by virtue of the existence of the Gateaux differential of the norm.[1] Another illustration of this will be given later,[2] as well as an example[3] to show the converse. Before doing this it is to advantage to further investigate left-uniqueness of spherical orthogonality. The following theorem is different from Lemma 6.8 because of the lack of symmetry of spherical orthogonality. The most interesting difference is that $\|x+ky\|$ has the same value for all k for which $x+ky \perp y$. That this is not true for all k for which $x \perp kx+y$ can be seen from Example 6.1.

<u>Theorem 7.7.</u> <u>If x and y are elements of a normed linear space T, then $\|x+ky\|$ has the same value for all numbers k between or equal to the smallest and largest numbers a for which $x+ay \perp y$. Also, $x+ky \perp y$ for all numbers k in this interval.</u>

<u>Proof</u>: If $x+ay \perp y$, then $\|(x+ay)+ky\| \geq \|y\|$ for all k. Hence $\|x+ay\|$ is the minimum value of $\|x+ky\|$ if and only if

1. See Theorem 7.3.
2. Example 7.2.
3. Example 7.3.

$x+ay \perp y$. Therefore if $x+a_1y \perp y$ and $x+a_2y \perp y$, then $\|x+a_1y\| = \|x+a_2y\|$. Obviously the set of numbers a for which $\|x+ay\|$ is minimum is bounded.[1] From the continuity[1] of $\|\cdot\|$, the it follows that there exist smallest and largest numbers A and B of this set. Then $\|x+Ay\| = \|x+By\|$, and these are both minimum values of $\|x+ky\|$. It now follows from Theorem 7.5 that $\|x+ky\|$ is constant for $A \leq k \leq B$, and that $x+ky \perp y$ for all k in this interval.

Theorem 7.7 essentially states that spherical orthogonality is left-unique if and only if for all x and $y \in$ space no curve of the form $f(n) = \|x+ny\|$ has a "flat bottom" like that of $\|(a,1)\|$ of Example 7.1. But if any such curve has a straight line segment, it can be shown that there is another with a "flat bottom". This follows from Theorem 7.8, and is also the basic idea of the proof of that theorem. Thus it is shown that[2] $\|x\|+\|y\| = \|x+y\|$ implies the curve $\|(1-k)x+ky\|$ has a straight line segment (as illustrated below). It then follows that elements x' and y' can be found such that the curve $\|x'+ky'\|$ has a "flat bottom" similar to that of $\|(a,1)\|$ of Example 7.1, and thus that spherical orthogonality is not left-unique.

Theorem 7.8. A necessary and sufficient condition that a normed linear space be strictly normed[3] is that spherical orthogonality be left-unique.

Proof: Suppose spherical orthogonality is not left-unique. There then exist elements x and y ($y \neq 0$) and unequal

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1. This follows from Condition (2) of Definition 1.6.
2. The following assumes the space is not strictly normed (see Def. 7.4).
3. See Definitions 7.2 and 7.4.

numbers A and B such that $x+Ay \perp y$ and $x+By \perp y$. It then follows from Theorem 7.7 that $x+\frac{1}{2}(A+B)y \perp y$ and

$$\|x+Ay\| = \|x+By\| = \|x+\tfrac{1}{2}(A+B)y\|.$$

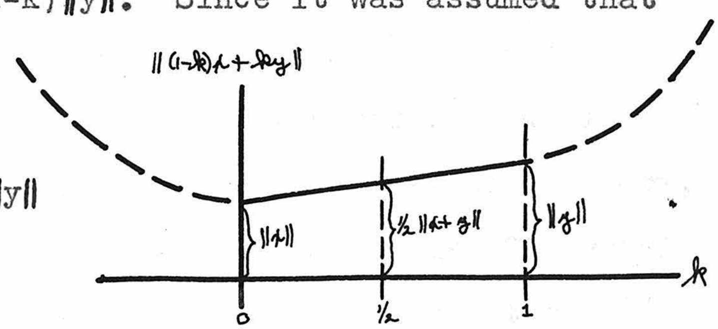
Thus $$\|x+Ay\| + \|x+By\| = \|(x+Ay)+(x+By)\|.$$

If the space is strictly normed, there then exists a number t such that $x+Ay = t(x+By)$.[1] If $t = 1$, then either $y = 0$ or $A = B$, contrary to assumption. If $t \neq 1$, then x is either zero or a multiple of y, which is impossible since it would then be necessary that $x+Ay = x+By = 0$,[2] and hence that $A = B$. Hence a normed linear space in which spherical orthogonality is not left-unique cannot be strictly normed, and spherical orthogonality is left-unique in any normed linear space which is strictly normed.

Conversely, suppose the normed linear space T is not strictly normed. There then exist elements x and y $(y \neq 0)$ such that $x \neq ty$ for any t and $\|x\| + \|y\| = \|x+y\|$. Take any number k such that $0 \leq k \leq 1$. Then

$$\|(1-k)x+ky\| \leq (1-k)\|x\| + k\|y\|$$

because of the triangular inequality of the norm[3]. But from $x+y = [(1-k)x+ky] + kx + (1-k)y$, it also follows that $\|x+y\| \leq \|(1-k)x+ky\| + k\|x\| + (1-k)\|y\|$. Since it was assumed that $\|x\| + \|y\| = \|x+y\|$, this gives $\|(1-k)x+ky\| \geq (1-k)\|x\| + k\|y\|$. Thus $\|(1-k)x+ky\| = (1-k)\|x\| + k\|y\|$ for $0 \leq k \leq 1$. Therefore if



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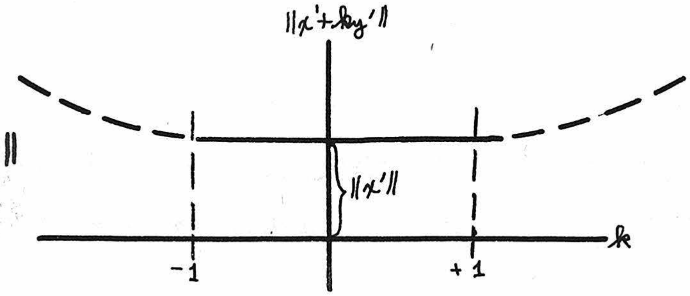
1. See Definition 7.4.
2. Since $ky \perp y$ if and only if $k = 0$ (see Def. 2.3), and $x+Ay \perp y$ and $x+By \perp y$.
3. Condition (2) of Definition 1.6.

$k = \frac{\|x\|}{\|x\|+\|y\|}$ and $x' = (1-k)x+ky$ and $y' = (1-k)x - ky$, then $\|x'\| = \frac{2\|x\|\cdot\|y\|}{\|x+y\|}$, and $\|x'+y'\| = \|x'-y'\| = \|x'\|$. Hence from Theorem 7.5,[1] $\|x'+ky'\| = \|x'\|$ for $|k| \leq 1$. This with Theorem 4.1 gives $\|x'+ky'\| \geq \|x'\|$ for all k. Thus $\|x'\|$ must be the minimum of $\|x'+ky'\|$,



and $\|(x'+ay')+ky'\| \geq \|x'+ay'\| = \|x'\|$ for all k, if $|a| \leq 1$. Thus $x'+ay' \perp y'$ if $|a| \leq 1$. Since $y' = \frac{\|y\|x+\|x\|y}{\|x\|+\|y\|}$, y' can not be zero because $x \neq ty$ for any t. Therefore spherical orthogonality is not left-unique, and a normed linear space for which spherical orthogonality is left-unique is strictly normed.

<u>Corollary 7.5</u>. <u>If a normed linear space is such that spherical orthogonality is symmetric, then the space is strictly normed if and only if the norm is Gateaux differentiable at each non-zero point.</u>

If spherical orthogonality is symmetric, then such orthogonality is clearly left-unique if and only if it is right-unique.[2] The above corollary then follows from the equivalence of left-uniqueness to the space being strictly normed and of right-uniqueness to Gateaux differentiability of the norm.[3] But it has also been shown that if the Gateaux differential of the norm, $f(x;y)$, at x with increment y exists, then x is orthogonal to y if and only if $f(x;y) = 0$.[4] Thus in this case the

1. Or from Theorem 4.4, which is now a corollary of Theorem 7.5.
2. See Definitions 7.1 and 7.2.
3. Theorems 7.3 and 7.8.
4. Corollary 7.3".

orthogonality is symmetric if and only if $f(x;y) = 0$ implies $f(y;x) = 0$. This gives the following corollary.

Corollary 7.5'. If the norm of a normed linear space T is Gateaux differentiable, then T is strictly normed if $f(x;y) = 0$ implies $f(y;x) = 0$ for all non-zero elements x and y of T.[1]

Corollary 7.5". If the norm of a normed linear space T is Gateaux differentiable, then T is strictly normed if and only if the Gateaux differential of the norm, $f(x+ny;y)$, at $x+ny$ with increment y, is an increasing function of n.[2]

Proof: As in the proof of Theorem 7.6, it is clear that $f(x+ny;y) = \frac{d\|x+ny\|}{dn}$. It also follows from Theorems 7.7 and 7.8, or directly from the condition for a normed linear space to be strictly normed,[3] that such a space is strictly normed if and only if no curve $f(n) = \|x+ny\|$ $(y \neq 0)$, for which $x \neq ty$ for any t, contains a straight line segment. The conclusion of the corollary then follows from Theorem 7.6.

It is interesting to investigate the effect on spaces of continuous functions of the assumption of right- or left-uniqueness of spherical orthogonality. This will be seen to give an easy means of establishing the independence of these types of uniqueness of orthogonality, and hence also the independence of Gateaux differentiability and strict normedness.

1. The stronger relation $f(x;\frac{y}{\|y\|}) = f(y;\frac{x}{\|x\|})$ holds in abstract Euclidean spaces, but by Theorem 8.5 only in such spaces.
2. As follows from Theorem 7.6, this difference is continuous in n if it exists.
3. As in the proof of Theorem 7.8.

Definition 7.5. The space (C) of continuous functions consists of all functions which are continuous in the closed interval $[0,1]$, with $\|f\|$ defined as $\max_{0 \leq x \leq 1} |f(x)|$.[1]

Theorem 7.9. A subspace[2] T of the space of continuous functions is strictly normed if and only if all non-zero functions of T whose absolute values take on their maximum at a common point are multiples of each other.

Proof: Suppose there exist non-zero functions f and g such that $|f|$ and $|g|$ are both maximum at some number a. That is, $\|f\| = |f(a)|$ and $\|g\| = |g(a)|$. Then either $|f(a)+g(a)|$ is the maximum of $|f+g|$, or $|f(a)-g(a)|$ is the maximum of $|f-g|$. Hence either $\|f\|+\|g\| = \|f+g\|$ or $\|f\|+\|g\| = \|f-g\|$. If T is strictly normed, there then exists a non-zero[3] number t such that $f = tg$.

Conversely, if T is not strictly normed, there exist non-zero functions f and g such that $\|f\|+\|g\| = \|f+g\|$ and $f \neq tg$ for any number t. Suppose $\|f\| = |f(a)|$, $\|g\| = |g(b)|$, and $\|f+g\| = |f(c)+g(c)|$. But

$$|f(c)+g(c)| \leq |f(c)| + |g(c)|.$$

Also, $|f(c)| \leq |f(a)|$ and $|g(c)| \leq |g(b)|$. By assumption,

$$|f(c)+g(c)| = |f(a)| + |g(b)|.$$

It follows from this that $|f(c)| = |f(a)|$ and $|g(c)| = |g(b)|$. That is, both $|f|$ and $|g|$ take on their absolute maximum at c.

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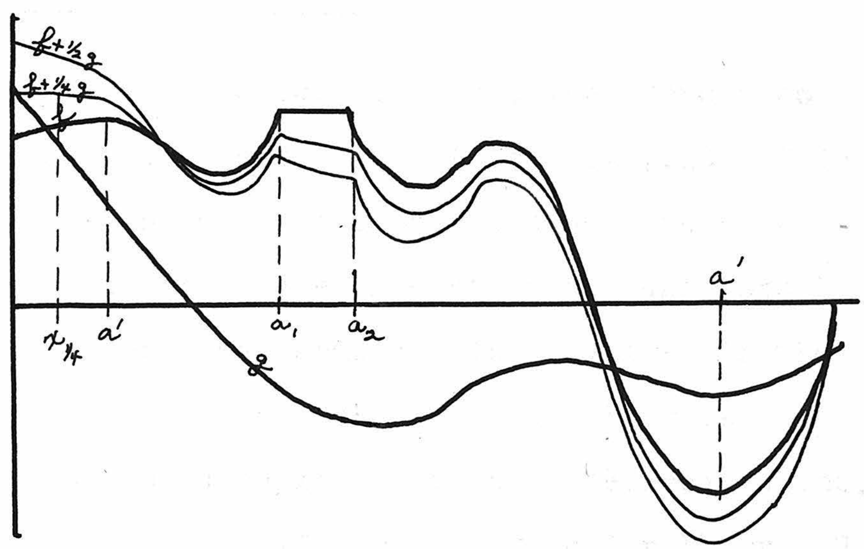
1. See Banach (I), page 11.
2. A subspace here means a subset which is also a normed linear space.
3. Non-zero since $f \neq 0$.

The above theorem enables one to tell readily whether a space of continuous functions is strictly normed. In order to find a simple condition that the norm of such a space be Gateaux differentiable, the limits of Corollary 6.7 will first be evaluated. Conditions for differentiability of the norm can then be deduced from these limits.

Theorem 7.10. If T is a subspace of the space of continuous functions, and $f(\neq 0)$ and g are elements of T, then $\lim_{k\to+0} \frac{\|f+kg\|-\|f\|}{k} = \max_U \left(\frac{g\cdot f}{|f|}\right)$ and $\lim_{k\to-0} \frac{\|f+kg\|-\|f\|}{k} = \min_U \left(\frac{g\cdot f}{|f|}\right)$, where U is the set of all numbers for which $|f|$ is maximum.[1]

Proof: Because of the continuity of f, the set of numbers for which $|f|$ is maximum is closed, and hence there exists a number a' of this set for which $\frac{g\cdot f}{|f|}$ takes on its maximum.[2] Because of the way a' was chosen, it is a member of the set U of numbers for which $|f|$ is maximum. Since $\frac{g(x)f(x)}{|f(x)|}$ is maximum for



$x = a'$, it follows that $|f(a')+kg(a')|$ is the maximum value of $|f(a)+kg(a)|$ for $a \varepsilon U$, provided k is positive and small

1. The only function of the factor $\frac{f}{|f|}$ is in determining the sign of $\left(\frac{g\cdot f}{|f|}\right)$.
2. In the figure, a' can be either of the indicated points; U consists of the points marked a' and the points between a_1 and a_2; and $x_{\frac{1}{4}}$ is the point at which $\|f+\frac{1}{4}g\|$ takes on its maximum.

enough that $|kg(a)| \leq |f(a)|$ for all $a \varepsilon U$. If x_k is a number for which $|f+kg|$ is maximum, then since $g(x)$ is bounded and the norm of T is continuous,[1] all limit points of x_k as k approach zero belong to U. Then for k small and positive:

$$\|f+kg\| = |f(x_k)+kg(x_k)| = |f(x_k)| + \frac{kg(x_k)f(x_k)}{|f(x_k)|} \leq |f(a')| + \frac{kg(x_k)f(x_k)}{|f(x_k)|},$$

and $$\lim_{k\to+0} \frac{\|f+kg\|-\|f\|}{k} \leq \lim_{k\to+0} \frac{kg(x_k)f(x_k)}{k|f(x_k)|} = \frac{g(a)f(a)}{|f(a)|},$$

for the number a of U which is a limit point of x_k as k approaches zero. But $\|f+kg\| \geq |f(a')+kg(a')|$,[2] and hence

$$\lim_{k\to+0} \frac{\|f+kg\|-\|f\|}{k} \geq \lim_{k\to+0} \frac{|f(a')+kg(a')|-|f(a')|}{k} = \frac{g(a')f(a')}{|f(a')|} \geq \frac{g(a)f(a)}{|f(a)|}$$

for all $a \varepsilon U$, because of the way a' was chosen. Hence

$$\lim_{k\to+0} \frac{\|f+kg\|-\|f\|}{k} = \frac{g(a')f(a')}{|f(a')|} = \max_U . \frac{g(a)f(a)}{|f(a)|}.$$

From this,

$$\lim_{k\to-0} \frac{\|f+kg\|-\|f\|}{k} = \lim_{k\to+0} \frac{\|f-kg\|-\|f\|}{-k} = -\max_U . \frac{-g(a)f(a)}{|f(a)|} = \min_U . \frac{g(a)f(a)}{|f(a)|}.$$

It is interesting that Corollary 6.6 (and therefore also Theorem 6.6) could be given as a corollary of Theorem 7.10. This follows from the fact that any separable Banach space can be represented as a subspace of the space of continuous functions,[3] in effect only only two-dimensional spaces being involved in the proof of this type of theorem. The following corollary is Theorem 7.10 restated with norms replaced by their defined values.[4] Corollary 7.7 is gotten by applying Theorem 7.10 to Corollary 6.7.

---

1. $g(x)$ is bounded since it is continuous, and the norm of T is continuous because of Condition (2) of Definition 1.6.
2. See Definition 7.5.
3. Banach (I), page 185, Theorem 9. A separable space is one which contains a countable subset whose closure is the whole space. Clearly any finite dimensional normed linear space is separable.
4. See Definition 7.5.

Corollary 7.6. If f and g are continuous functions defined over a closed interval, then

$$\lim_{k\to+0} \frac{\max.|f+kg|-\max.|f|}{k} = \max_{U}.\ \frac{g\cdot f}{|f|}, \quad \text{and}$$

$$\lim_{k\to-0} \frac{\max.|f+kg|-\max.|f|}{k} = \min_{U}.\ \frac{g\cdot f}{|f|},$$

where U is the set of numbers for which $|f|$ is maximum.

Corollary 7.7. If T is a subspace of the space of continuous functions, and f and g are elements of T, then the smallest and largest numbers A and B such that $f\perp Af+g$ and $f\perp Bf+g$ are $A = -\max_{U}.\ \frac{g}{f}$ and $B = -\min_{U}.\ \frac{g}{f}$, where U is the set of numbers for which $|f|$ is maximum.

Corollary 7.7 gives a simple means of finding all numbers a for which $f\perp af+g$, if f and g are elements of a space of continuous functions. If the orthogonality is right-unique, then the norm is Gateaux differentiable at all non-zero points.[1] It is possible to use Theorem 7.10 to find a condition for the existence of the Gateaux differential of the norm, and a means of evaluating this differential if it exists. This leads to simple conditions for orthogonality of elements of a space of continuous functions.

Theorem 7.11. If T is a subspace of the space of continuous functions, and $f\ (\neq 0)$ and g are elements of T, then the Gateaux differential, $\lim_{k\to 0} \frac{\|f+kg\|-\|f\|}{k}$, exists if and only if $|g(a)|$ has the same value for all numbers a for which $|f|$

1. Theorem 7.3.

is maximum, and $g(a)$ and $f(a)$ are either of the same sign for all such a, or of opposite sign for all such a. If it exists, this limit equals $\frac{g(a)f(a)}{|f(a)|}$, where a is any number for which $|f|$ is maximum.

Proof: From Theorem 7.10, this Gateaux differential exists if and only if $\max_U.\left(\frac{g\cdot f}{|f|}\right) = \min_U.\left(\frac{g\cdot f}{|f|}\right)$, where U is the set of numbers for which $|f|$ is maximum. But $|f(a)|$ has the same value for all $a \varepsilon U$. Hence for this equality to hold, $g(a)\cdot f(a)$ must have the same sign for all $a \varepsilon U$, and $|g(a)|$ must have the same value for all $a \varepsilon U$.

Corollary 7.8. If T is a subspace of the space of continuous functions, and the norm of T is Gateaux differentiable at all non-zero points, then for elements f and g of T, $f \perp -\frac{g(a)}{f(a)} f + g$, where a is any number for which $|f|$ is maximum.

Proof: This follows immediately from Corollary 7.7, since spherical orthogonality is right-unique if the norm is Gateaux differentiable[1]. It also follows from Theorems 7.3 and 7.11.

Theorem 7.12. If T is a subspace of the space of continuous functions and the norm of T is Gateaux differentiable at all non-zero points, then an element f is spherically orthogonal to an element g if and only if $g(a) = 0$ for numbers a for which $|f|$ is maximum.

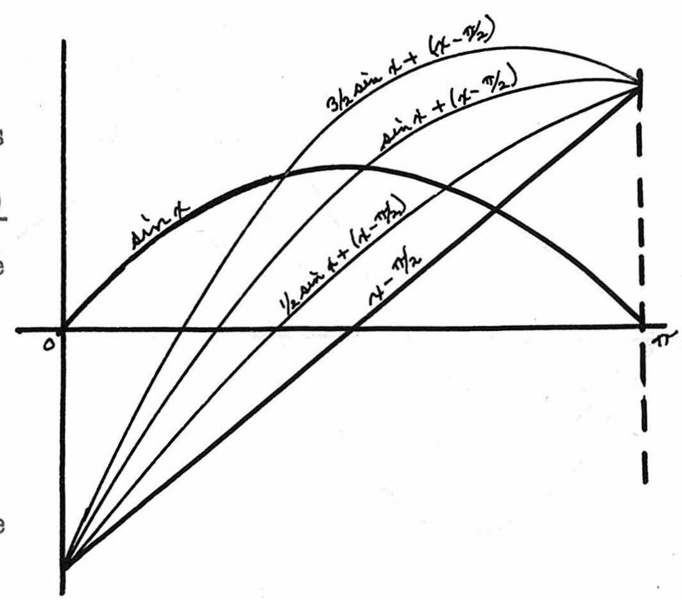
This theorem is really a weak form of Corollary 7.8, but provides a very simple means of testing the orthogonality of

1. See Theorem 7.3.

elements of a space of continuous functions in which the norm is Gateaux differentiable. If the orthogonality is also symmetric, then two elements are orthogonal if and only if each is zero at every point where the absolute value of the other is maximum. This will be used extensively in the next section. Its application is made possible by the fact that any separable Banach space[1] is equivalent[2] to a subspace of the space of continuous functions.[3]

The above results enable one to construct simple examples to show the independence of Gateaux differentiability of the norm and the condition for strictly normed spaces--and hence of right- and left-uniqueness of spherical orthogonality. Example 7.1 showed that Gateaux differentiability of the norm does not imply the space is strictly normed. This is shown more simply by Example 7.2, while the converse is given by Example 7.3.

<u>Example 7.2.</u> <u>A normed linear space whose norm is Gateaux differentiable, but which is not strictly normed.</u> Let T be the space of all functions of the form $f = a\sin x + b(x-\frac{\pi}{2})$ in the interval $(0,\pi)$. Since $f' = a\cos x + b$, the slope of the curve f is of the same sign throughout $(0,\pi)$ if $|a| \leq |b|$. Hence in this case

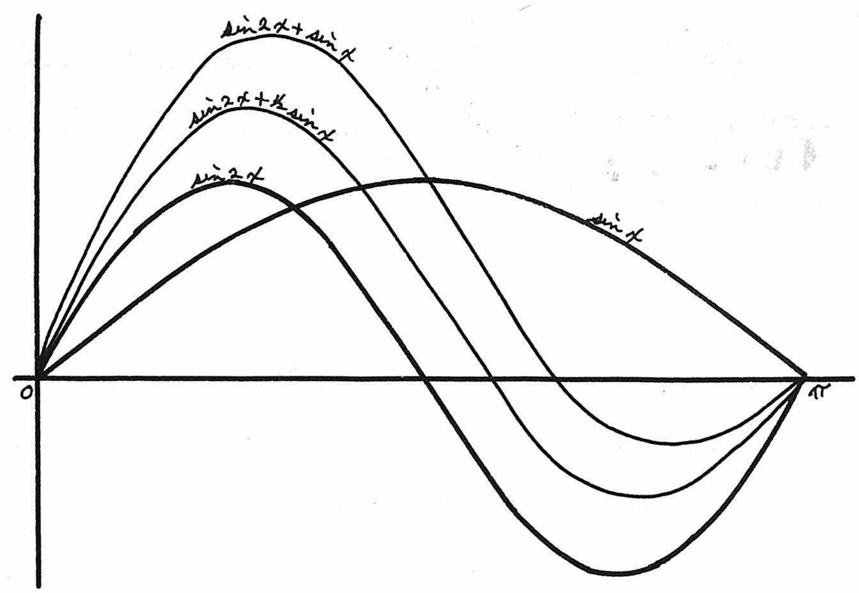


1. See Definition 1.7. A space T is separable if it contains a countable subset U such that $\overline{U} = T$.
2. Two normed linear spaces are <u>equivalent</u> if there is a 1-1 correspondence between them which preserves the norm and the operations of addition and multiplication by real numbers.
3. Banach (I), page 185, Theorem 9.

the maximum of $|f|$ is attained at 0 and π. Thus by Theorem 7.9, T is not strictly normed.

If $|a| > |b|$, then f' is of opposite signs at the ends of the interval and $|f|$ must take on a larger value than that of $\frac{\pi}{2}|b|$ which it has at 0 and π. Thus if $|a| > |b|$, $|f|$ takes on its maximum at the unique point for which $\cos x = -\frac{b}{a}$. Hence if $|a| > |b|$, then $|f|$ takes on its maximum at only one point. If $|a| \leq |b|$, $|f|$ takes on its maximum at both 0 and π. Since $\sin 0 = \sin \pi = 0$, the values at 0 and π of any other function $A \sin x + B(x - \frac{\pi}{2})$ are $\mp\frac{\pi}{2}B$. Since the corresponding values of f are $\mp\frac{\pi}{2}b$, it now follows from Theorem 7.11 that the norm of T is Gateaux differentiable.

<u>Example 7.3.</u> <u>A normed linear space which is strictly normed, but whose norm is not Gateaux differentiable.</u> Let T be the space of all functions of the form $f = a \sin x + b \sin 2x$ in the interval $(0, \pi)$. Then $\sin 2x$ has a maximum of 1 at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, being of opposite signs at these points. But $\sin x$ has the same sign at both points. Hence by Theorem 7.11,



$$\lim_{k \to 0} \frac{\|\sin 2x + k \sin x\| - \|\sin 2x\|}{k}$$

does not exist, and the norm of T is not Gateaux differentiable at the "point" $\sin 2x$.

The function $f = a \sin x + b \sin 2x$ is zero at both ends of the interval $(0, \pi)$, and hence the maximum of $|a \sin x + b \sin 2x|$ must be taken on for a number x_0 such that

$$a \cos x_0 + 2b \cos 2x_0 = 0.$$

If the absolute value of another function $F = A \sin x + B \sin 2x$ takes on its maximum at the same point x_0, then

$$A \cos x_0 + 2B \cos 2x_0 = 0.$$

Since $\cos x_0$ and $\cos 2x_0$ cannot be zero simultaneously, it follows that $Ab = Ba$ and that any functions f and F are multiples of each other if they take on their absolute maxima at the same point. It then follows from Theorem 7.9 that T is strictly normed.

Theorem 7.13. A normed linear space can be strictly normed without its norm being Gateaux differentiable, and conversely.

Theorem 7.14. There are normed linear spaces for which spherical orthogonality is right-unique and not left-unique, and others for which such orthogonality is left-unique and not right-unique.

These theorems are immediate consequences of the above Examples 7.2 and 7.3, the second making use of the equivalence of Gateaux differentiability of the norm and right-uniqueness of spherical orthogonality, and of the equivalence of strict normedness and left-uniqueness of spherical orthogonality.[1]

1. Theorems 7.3 and 7.8.

8. SPHERICAL ORTHOGONALITY AND THE EXISTENCE OF INNER PRODUCTS

It has been shown that if either isosceles or Pythagorean orthogonality is either additive or homogeneous in a normed linear space T, then T is an abstract Euclidean space.[1] These types of orthogonality are symmetric in any normed linear space, while spherical orthogonality[2] is homogeneous, but neither symmetric nor additive in general normed linear spaces. Two conditions for the existence of an inner product[3] in normed linear spaces have been used in this thesis as an aid in proving the existence of inner products from assumptions on the orthogonality. In connection with Pythagorean orthogonality, it was convenient to use the condition given by Jordan and Neumann[4] that a normed linear space is abstract Euclidean if and only if

$$\|x+y\|^2 + \|x-y\|^2 = 2[\|x\|^2 + \|y\|^2]$$

for all elements x and y. With isosceles orthogonality, a condition given by Ficken was found useful:[5] That a normed linear space is abstract Euclidean if and only if

$$\|ax+y\| = \|x+ay\|$$

for all numbers a and elements x and y for which $\|x\| = \|y\|$. It will be seen that it is sufficient to assume only that

$$\lim_{n\to\infty}[\|nx+y\| - \|x+ny\|] = 0 \quad \text{if } \|x\| = \|y\|.^{6}$$

1. Theorems 4.9 and 5.2 and Corollary 5.3. Also see page 13 of this thesis.
2. See Definitions 2.1-2.3.
3. The existence of an inner product is equivalent to the space being abstract Euclidean. See Definition 1.9.
4. See page 40 of this thesis, and Jordan and Neumann (IV).
5. See page 35 of this thesis, and Ficken (XIV).
6. Theorem 8.4.

Garrett Birkhoff has shown that a normed linear space of three or more dimensions is abstract Euclidean if and only if spherical orthogonality is symmetric and unique, and that this is not true for spaces of two dimensions.[1] It has been shown that right-uniqueness of spherical orthogonality is equivalent to additivity of the orthogonality and to Gateaux differentiability of the norm of the space, and that left-uniqueness is equivalent to the space being strictly normed.[2] Thus Birkhoff's theorem can be stated in any of the following forms:

Theorem 8.1. A normed linear space of three or more dimensions is an abstract Euclidean space if and only if it is such that spherical orthogonality is symmetric and additive.

Theorem 8.2. A normed linear space of three or more dimensions is an abstract Euclidean space if and only if it is strictly normed and spherical orthogonality is symmetric.

Theorem 8.3. A normed linear space of three or more dimensions is an abstract Euclidean space if and only if its norm is Gateaux differentiable at each non-zero point and spherical orthogonality is symmetric.

Examples have been given[3] of normed linear spaces which show the independence of the condition for strict normedness and the

1. Birkhoff (XIII). Conditions which are sufficient for the existence of an inner product for two-dimensional spaces are given by Theorems 8.4 and 8.5. The uniqueness assumed by Birkhoff can be either left or right, because of the assumption of symmetry (See Definitions 7.1 and 7.2.).
2. Theorems 7.3, 7.4, and 7.8.
3. Examples 7.1, 7.2, and 7.3.

existence of the Gateaux differential of the norm at non-zero points, making use of the equivalence of these to left- and right-uniqueness of spherical orthogonality. Since symmetry of spherical orthogonality clearly implies the equivalence of these properties,[1] it is interesting to consider whether the assumption of both left- and right-uniqueness implies symmetry of spherical orthogonality. That is, whether spherical orthogonality is symmetric in a strictly normed space whose norm is Gateaux differentiable at all non-zero points. That this is not true is shown by Example 8.1, and also by the following analysis: Suppose a particular linear representation of a normed linear space T is taken and the unit pseudo-sphere S constructed as representing the points whose norms are unity[2]. Then it can be seen that T is strictly normed, or spherical orthogonality is left-unique in T, if and only if S does not contain a straight line segment. Also, the norm of T is Gateaux differentiable at non-zero points, or spherical orthogonality is right-unique in T, if and only if S has a tangent hyper-plane at each point.[3] Clearly these two conditions can be easily met without satisfying the condition for symmetry: namely, that a diameter drawn through the origin parallel to a tangential hyperplane at any point p of S cut S in points at which the tangential hyperplane is parallel to the diameter through p.

Example 8.1. A normed linear space which is strictly normed and whose norm is Gateaux differentiable at all non-zero points,

1. See Corollary 7.6.
2. See Birkhoff (XIII), page 169. The pseudo-sphere S can be in an Euclidean space of any number of dimensions, while Birkhoff uses a circle.
3. See Definition 9.2. This will be discussed in more detail in the next section.

<u>but for which spherical orthogonality is not symmetric</u>. Let T be the normed linear space of all functions of the form $F = af+bg$ in the interval $(0,2)$, where $\|F\| = \max_{(0,2)} |F|$ and

$$f = x(2-x) \quad \text{and} \quad g = \begin{cases} 1-x^2 & \text{for } 0 \leq x \leq 1, \\ (x-2)^2-1 & \text{for } 1 \leq x \leq 2. \end{cases}$$

The function f is positive throughout $(0,2)$ and symmetric about the line $x = 1$, while g is positive in $(0,1)$ and negative in $(1,2)$ and symmetric about the point $(1,0)$. Thus if a and b are of the same sign, then $|F|$ has its maximum in $(0,1)$--at the point $x = \frac{a}{a+b}$. Likewise, if a and b are of opposite sign, then the maximum is in $(1,2)$--at the point $x = \frac{2b-a}{b-a}$. Clearly then the absolute values of two functions $F = af+bg$ and $G = Af+Bg$ can take on their maxima at the same point only if they are proportional, and the absolute value of a function can take on its maximum at only one point. Thus T is strictly normed and the norm of T is Gateaux differentiable at every non-zero point.[1] Spherical orthogonality is therefore symmetric in T if and only if the absolute value of the function G which is zero where $|F|$ is maximum, is maximum where F is zero.[2] If a and b are of the same sign, then $|F|$ has its maximum at $x = \frac{a}{a+b}$ and $F \perp G$ if and only if $G(\frac{a}{a+b}) = A\left[\frac{a}{a+b}(2-\frac{a}{a+b})\right]+B\left[1-\frac{a^2}{(a+b)^2}\right] = 0$, or

$$(1). \quad Aa(a+2b)+Bb(2a+b) = 0.$$

But then A and B are of opposite sign and $|G|$ has its maximum at $x = \frac{2B-A}{B-A}$. If $G \perp F$, then $F(\frac{2B-A}{B-A}) = 0$, or

$$(2). \quad Aa(2B-A)+Bb(B-2A) = 0.$$

But (1) and (2) are both satisfied if and only if $Ab = -aB$, and hence (1) and (2) are not equivalent in general. Thus spherical orthogonality is not symmetric in T.

1. See Theorems 7.9 and 7.11.
2. See Theorem 7.12.

It has been shown that for the three types of orthogonality of Definitions 2.1-2.3, and any elements x and y of a normed linear space, there exists a number a for which $x \perp ax+y$.[1] For isosceles and Pythagoren orthogonality it has been shown that the assumption of either homogeneity or additivity of the orthogonality implies the normed linear space is an abstract Euclidean space.[2] The assumption of both homogeneity and additivity of spherical orthogonality in normed linear spaces of three or more dimensions implies the space is abstract Euclidean, but this is not true for two dimensional spaces.[3] However, several conditions can be given which are valid for two-dimensional spaces. The first of these is similar to the condition of Ficken's,[4] and makes use of orthogonality concepts only in the method of proof.

<u>Lemma 8.4.</u> <u>If for a normed linear space T $\|x\| = \|y\|$ implies $\lim_{n\to\infty} \big|\|nx+y\| - \|x+ny\|\big| = 0$, then spherical orthogonality is symmetric, additive, and unique in T, and $x \perp ax+y$ and $y \perp by+x$ imply $a\|x\|^2 = b\|y\|^2$.</u>

<u>Proof</u>: Let x and y be elements of T, A and B be the algebraically smallest and largest of the numbers a for which $x \perp ax+y$, and A' and B' be the algebraically smallest and largest of the numbers b for which $y \perp by+x$. If $\|x\| = \|y\|$, then

$$\lim_{n\to\infty} \big[\|nx+y\| - \|x+ny\|\big] = \lim_{n\to\infty} \big[\|nx+y\| - \|nx\|\big] - \lim_{n\to\infty} \big[\|x+ny\| - \|ny\|\big] = 0,$$

since these limits are known to exist. Their existence follows from Theorem 6.7, which also gives

1. That is, there is a number a for each type of orthogonality, but not necessarily the same number.
2. Theorems 4.9, 5.2, and Cor. 5.3. Also see page 13 of this thesis.
3. See Theorem 8.1 and the preceding discussion.
4. That $\|ax+y\| = \|x+ay\|$ if $\|x\| = \|y\|$. Ficken (XIV).

$$\lim_{n\to\infty}\left[\|nx+y\|-\|x+ny\|\right] = -(A-A')\|x\| = 0,$$

and hence that $A = A'$. Likewise, $\|x\| = \|y\|$ implies

$$\lim_{n\to\infty}\left[\|nx-y\|-\|x-ny\|\right] = \lim_{n\to\infty}\left[\|nx-y\|-\|nx\|\right] - \lim_{n\to\infty}\left[\|x-ny\|-\|ny\|\right] = 0.$$

Another application of Theorem 6.7 then gives

$$\lim_{n\to\infty}\left[\|nx-y\|-\|x-ny\|\right] = (B-B')\|x\| = 0,$$

and $B = B'$. Now suppose $x \perp y$ and $\|x\| = \|y\|$. Then clearly $A \leq 0 \leq B$, and from the above $A' \leq 0 \leq B'$. That $y \perp x$ now follows from Lemma 6.8 and the definitions of A' and B'. Since spherical orthogonality is homogeneous, there was no loss of generality in assuming $\|x\| = \|y\|$. It has thus been shown that spherical orthogonality is symmetric in a normed linear space for which $\|x\| = \|y\|$ implies $\lim_{n\to\infty} \|nx+y\| - \|x+ny\| = 0$. Now suppose spherical orthogonality is not unique in such a normed linear space. There are then elements x and y and a positive number ε for which $\|x\| = \|y\|$ and $x \perp ax+y$ for $|a| < \varepsilon$.[1] From the symmetry that has been proved, it now follows that $ax+y \perp x$ for $|a| < \varepsilon$, and hence from Theorem 7.7 that $\|ax+y\| = \|y\| = \|x\|$ if $|a| < \varepsilon$. Then it follows from the original assumption that

$$\lim_{n\to\infty}\left[\|nx+(ax+y)\| - \|x+n(ax+y)\|\right] = 0 \quad \text{if} \quad |a| < \varepsilon.$$

But $\|x+n(ax+y)\| = \|(1+na)x+ny\| = \|ny\|$ if n is large enough that $|a+1/n| < \varepsilon$. Also, by Lemma 4.5, $\lim_{n\to\infty}\left[\|nx+(ax+y)\|-\|nx+y\|\right] = a\|x\|$. Thus $\lim_{n\to\infty}\left[\|nx+(ax+y)\| - \|x+n(ax+y)\|\right] = \lim_{n\to\infty}\left[\|nx+y\|-\|ny\|\right] + a\|x\|$, and therefore can clearly not be zero for all values of a with $|a| < \varepsilon$. The assumption that spherical orthogonality was not right-unique in T is therefore false. Such orthogonality is then right-unique in T, and it follows from Theorem 7.4 that it

1. That this is true if spherical orthogonality is not right-unique follows from Lemma 6.8 and the homogeneity of spherical orthogonality. The symmetry makes left- and right-uniqueness equivalent (see Definitions 7.1 and 7.2).

is also additive. It now follows that if $\|x\| = \|y\|$, then not only does $A = A'$ and $B = B'$, but because of the uniqueness just proved, $A = B = A' = B'$. Thus $x \perp ax+y$ implies $y \perp ay+x$ if $\|x\| = \|y\|$. If $\|x\| \neq \|y\|$, take r such that $\|rx\| = \|y\|$. If $x \perp ax+y$ and $y \perp by+x$, it follows from the homogeneity of spherical orthogonality that $rx \perp \frac{a}{r}(rx)+y$ and $y \perp (br)y+rx$. Then $\frac{a}{r} = br$, or $a\|x\|^2 = b\|y\|^2$.

Theorem 8.4. A normed linear space T is an abstract Euclidean space if and only if $\lim_{n\to\infty} \|nx+y\| - \|x+ny\| = 0$ whenever $\|x\|=\|y\|$.[1]

Proof: Define the inner product (x,y) as $-a\|x\|^2$, where $x \perp ax+y$. This value of the inner product is unique because of Lemma 8.4, and it is only necessary to show that it satisfies the conditions of Definition 1.9:

(1). $(tx,y) = t(x,y)$. If $x \perp ax+y$, and $t \neq 0$, then $tx \perp \frac{a}{t}(tx)+y$ because of the homogeneity of spherical orthogonality. Thus $(tx,y) = -\frac{a}{t}\|tx\|^2 = -at\|x\|^2$. Hence $(tx,y) = t(x,y)$. If $t = 0$, the proof is trivial.

(2). $(x,y) = (y,x)$. If $x \perp ax+y$, and $y \perp by+x$, then $(x,y) = -a\|x\|^2$ and $(y,x) = -b\|y\|^2$. These are equal by Lemma 8.4.

(3). $(x,y)+(x,z) = (x,y+z)$. Suppose $x \perp ax+y$ and $x \perp bx+z$. Then $x \perp (a+b)x+(y+z)$ because of the additivity given by Lemma 8.4. Hence $(x,y) = -a\|x\|^2$, $(x,z) = -b\|y\|^2$, and $(x,y+z) = -(a+b)\|x\|^2$.

(4). $(x,x) = \|x\|^2$. This is immediate from $x \perp (-1)x+x$.

Conversely, it follows from the definition of an inner product[2] that for abstract Euclidean spaces $\|nx+y\| - \|x+ny\| = 0$ for all n and elements x and y for which $\|x\| = \|y\|$ or $(x,x) = (y,y)$.

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1. See Theorem 9.3 for an equivalent statement.
2. Definition 1.9.

It has been pointed out that if the norm of a normed linear space is Gateaux differentiable at each non-zero point, then this differential gives a convenient statement of properties of spherical orthogonality.[1] It was seen that the existence of this differential of the norm is a necessary and sufficient condition for spherical orthogonality to be additive. If the differential of the norm at x with increment y is denoted by $f(x;y)$, this means that $f(x;y) = 0$ and $f(x;z) = 0$ imply $f(x;y+z) = 0$.[2] Thus spherical orthogonality is additive and symmetric and the normed linear space is strictly normed if and only if the norm is Gateaux differentiable at all non-zero points and $f(x;y) = 0$ implies $f(y;x) = 0$. This is a necessary and sufficient condition for the existence of an inner product in a normed linear space of three or more dimensions.[3] The following theorem shows how this can be strengthened to give a like condition for two-dimensional spaces.

Theorem 8.5. A necessary and sufficient condition for a normed linear space to be an abstract Euclidean space is that the Gateaux differential of the norm exist at each non-zero point, and $f(x;\frac{y}{\|y\|}) = f(y;\frac{x}{\|x\|})$ for all elements x and y, where $f(x;y)$ is the Gateaux differential of the norm at x with increment y.[4]

Proof: To prove the sufficiency, define the inner product (x,y) of Definition 1.9 as $-a\|x\|^2$, where $x \perp ax+y$. If the norm is Gateaux differentiable, then the uniqueness of (x,y) is given by Theorem 7.3. It will be shown that it satisfies the

1. See page 71 and Corollary 7.5'.
2. Corollary 7.4. That linearity of the differential of the norm follows from its existence was shown by Mazur (VI), pp. 129-130.
3. Theorem 8.3.
4. Also see Theorem 9.3.

conditions of Definition 1.9.

(1). $\underline{(tx,y) = t(x,y)}$. If $x \perp ax+y$, and $t \neq 0$, then $tx \perp \frac{a}{t}(tx)+y$ because of the homogeneity of spherical orthogonality. Thus $(tx,y) = -\frac{a}{t}\|tx\|^2 = -at\|x\|^2 = t(x,y)$. If $t = 0$, the proof is trivial.

(2). $\underline{(x,y) = (y,x)}$. Theorem 7.3 gave $f(x;y) = -a\|x\|$, where $x \perp ax+y$. Likewise, $f(y;x) = -b\|y\|$, where $y \perp by+x$. Then $x \perp \frac{1}{\|y\|}(ax+y)$ and $y \perp \frac{1}{\|x\|}(by+x)$, because of the homogeneity of spherical orthogonality. Therefore $f(x;\frac{y}{\|y\|}) = f(y;\frac{x}{\|x\|})$ becomes $-\frac{a}{\|y\|}\|x\| = -\frac{b}{\|x\|}\|y\|$, or $-a\|x\|^2 = -b\|y\|^2$. This is the same as $(x,y) = (y,x)$.

(3). $\underline{(x,y)+(x,z) = (x,y+z)}$. Suppose $x \perp ax+y$ and $x \perp bx+z$. It follows from the existence of the Gateaux differential of the norm that spherical orthogonality is additive,[1] and that $x \perp (a+b)x+(y+z)$. That is, $(x,y) = -a\|x\|^2$; $(x,z) = -b\|x\|^2$; and $(x,y+z) = -(a+b)\|x\|^2$.

(4). $\underline{\|x\|^2 = (x,x)}$. This is immediate from $x \perp (-1)x+x$.

Conversely, if a normed linear space is an abstract Euclidean space, its norm can be defined by a (bilinear and symmetric) inner product (x,y).[2] Then:

$$\frac{\|x+hy\|-\|x\|}{h} = \frac{\|x+hy\|^2-\|x\|^2}{h\left[\|x+hy\|+\|x\|\right]} = \frac{(x+hy,x+hy)-(x,x)}{h\left[\|x+hy\|+\|x\|\right]} = \frac{2(x,y)+h(y,y)}{\|x+hy\|+\|x\|}.$$

Hence the Gateaux differential $f(x;y)$ of the norm exists and is equal to $\frac{(x,y)}{\|x\|}$. Then $f(x;\frac{y}{\|y\|}) = \frac{(x,\frac{y}{\|y\|})}{\|x\|} = \frac{(x,y)}{\|x\|\cdot\|y\|}$. But also $f(y;\frac{x}{\|x\|}) = \frac{(y,\frac{x}{\|x\|})}{\|y\|} = \frac{(y,x)}{\|x\|\cdot\|y\|}$. Therefore $f(x;\frac{y}{\|y\|}) = f(y;\frac{x}{\|x\|})$.

1. Corollary 7.4.
2. See Definition 1.9.

9. SPHERICAL ORTHOGONALITY, HYPERPLANES, AND FUNCTIONALS

It has been shown that for any elements x and y of a normed linear space there can be found at least one number a for which $x \perp ax+y$, and a number b for which $bx+y \perp x$.[1] An evaluation of all such numbers a and b has also been given, and their uniqueness has been shown to be equivalent to Gateaux differentiability of the norm and to strict normedness, respectively.[2] Spherical orthogonality is homogeneous, and the properties of additivity and symmetry were also studied and related to strict normedness and to Gateaux differentiability of the norm and used to get conditions for the existence of an inner product.[3] In this section, some of these results will be interpreted and extended by use of functionals and the concepts of conjugate spaces and supporting and tangential hyperplanes.

If a linear functional f takes on its maximum in the unit sphere at a point x, then x is orthogonal to the maximal linear subset consisting of all elements for which f is zero.[4] Thus while spherical orthogonality is not additive in general normed linear spaces, there is (for each element x) always at least one maximal linear subset H with $x \perp H$.[5] The problem of finding an x orthogonal to a given linear set H will be studied in the next section, and will be related to the problem of finding elements on the unit sphere for which linear functionals take on their

1. Theorems 6.3 and 6.4. In this section, "orthogonality" is the spherical orthogonality of Definition 2.3.
2. Theorems 6.8 and 7.7; 7.3 and 7.8.
3. Corollaries 7.4 and 7.5 and Theorems 8.1-8.3.
4. Theorem 6.1.
5. Theorem 6.2.

maximum values. If the Gateaux differential of the norm exists, then it is linear. If for an element x there is a unique linear functional f for which $f(x) = \|x\|$ and $\|f\| = 1$, then f can be evaluated in terms of the differential of the norm and hence in terms of spherical orthogonality.

Spherical orthogonality and linear functionals have been related by showing that $x \perp H$ if a linear functional f, with $\|f\| = 1$ and $f(x) = \|x\|$, is zero for all elements of H.[1] The following theorem establishes a reciprocal relationship.

__Theorem 9.1.__ __If an element x_0 of a normed linear space T is orthogonal to a linear subset H of T, then there exists a linear functional f for which $f(x_0) = \|f\|\,\|x_0\|$ and $f(h) = 0$ if $h \varepsilon H$. There is also a maximal linear subset M of T such that $x_0 \perp M$ and $H \subset M$.__

__Proof__: Let G be the linear subset of T generated by x_0 and H. Define the functional F by $F(ax_0 + h) = a\|x_0\|$, where $h \varepsilon H$. Then F is clearly linear over G. But also, $|F(ax_0+h)| = \|ax_0\|$ and $\|ax_0\| \leq \|ax_0+h\|$, since $x_0 \perp h$ if $h \varepsilon H$. Thus $|F(ax_0+h)| \leq \|ax_0+h\|$. Since $F(x_0) = \|x_0\|$, it follows that $\|F\| = 1$. But there then exists a linear functional f defined over all of T and such that $\|f\| = 1$ and $F(x) = f(x)$ if $x \varepsilon G$.[2] Then $f(x_0) = \|x_0\|$ and $f(h) = 0$ if $h \varepsilon H$. The maximal linear subset M of all elements m for which $f = 0$ contains H and is such that $x \perp M$.

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1. That is, if at least one such functional is zero on H. See Theorem 6.1.
2. This is shown by Banach (I), pg. 55, Theorem 2.

The following concepts of tangential and supporting hyperplanes of the unit sphere of a normed linear space will be found to be closely related to spherical orthogonality.[1]

Definition 9.1. A hyperplane[2] H of a normed linear space T is a supporting hyperplane of the unit sphere S if the distance between H and S is zero and H contains no interior points of S.

Definition 9.2. A hyperplane[2] H of a normed linear space T is the tangential hyperplane of the unit sphere S at the boundary point x_0 of S if H is the only supporting hyperplane containing the point x_0.

Theorem 9.1 and the preceding remark state that any element x of a normed linear space is orthogonal to some hyperplane through the origin, and that any hyperplane H through the origin for which there exists an element $x_0 \perp H$ is the set of elements for which $f = 0$, for some linear functional f with $\|f\| = 1$ and $f(x_0) = \|x_0\|$. But then $f(x) = 1$ defines a hyperplane through x_0 which contains no interior points of the unit sphere. This hyperplane is thus a supporting hyperplane of the unit sphere. Hence there is a supporting hyperplane through every point of the unit sphere.[3]

1. See Mazur (IX), pp. 71-73 and 77, for a more general discussion of supporting hyperplanes (Stützhyperebene) and tangential hyperplanes.
2. See Definition 6.2.
3. This also follows from Mazur (IX), page 73, Theorem 1.

<u>Theorem 9.2.</u> <u>If</u> x_0 <u>is a boundary point of the unit sphere</u> <u>S of a normed linear space and y is any element, then</u> <u>$x_0 \perp x_0 - y$ if and only if S has a supporting hyperplane containing</u> <u>x_0 and y.</u>

<u>Proof</u>: Suppose the supporting hyperplane H contains x_0 and y. Then the set $L = H - x_0$ is linear, where $H - x_0$ consists of all elements of the form $h - x_0$ with $h \varepsilon H$.[1] Therefore $k(x_0 - y) \varepsilon L = H - x_0$ for all k, and $x_0 + k(x_0 - y) \varepsilon H$. Since H contains no interior points of S, $\|x_0 + k(x_0 - y)\| \geq \|x_0\| = 1$ and $x_0 \perp x_0 - y$. Conversely, if $x_0 \perp x_0 - y$, then by Theorem 9.1 there exists a linear functional f for which $f(x_0) = \|f\| \cdot \|x_0\|$ and $f(x_0 - y) = 0$. If x_0 is a boundary point of S, then $\|x_0\| = 1$ and $f(y) = \|f\|$. Thus y is in the supporting hyperplane defined by $f(x) = \|f\|$.

It is known that if spherical orthogonality is right-unique or additive, then the Gateaux differential of the norm exists at all non-zero points.[2] However, $\lim_{h \to +0} \frac{\|x+hy\| - \|x\|}{h}$ exists for all elements x and y of a general normed linear space and can be evaluated in terms of spherical orthogonality. Thus if $x \neq 0$ and this limit is denoted by $f_+(x;y)$, then:

$$(1). \qquad A = -\frac{f_+(x;-y)}{\|x\|} \quad \text{and} \quad B = \frac{f_+(x;y)}{\|x\|},$$

where A and B are the smallest and largest of the numbers a for which $x \perp ax - y$.[3] If spherical orthogonality is additive, then this function f_+ is the Gateaux differential of the norm

1. See Def. 6.2. H is a coset of the subgroup L of the abstract Abelian group of the normed linear space.
2. Theorems 7.3 and 7.4.
3. Corollary 6.7.

and is hence linear.[1] It is not linear in general, but does satisfy the following weakened linearity conditions:[2]

$$(2). \quad f_+(x;y+z) \leq f_+(x;y) + f_+(x;z),$$

$$(3). \quad f_+(x;ty) = t \cdot f_+(x;y) \quad \text{for } t \geq 0,$$

$$(4). \quad f_+(x;y) \leq \|y\|.$$

If B is the largest of the numbers a for which $x \perp ax-sy$, then $B+r$ is the largest of the numbers b for which $x \perp bx-(rx+sy)$. Using this, and $f_+(x;rx) = r\|x\|$, it follows from (1) that:

$$(5). \quad f_+(x;rx+sy) = f_+(x;rx) + f_+(x;sy),$$
$$= r\|x\| + s \cdot f_+(x;y), \text{ for } s \geq 0 \text{ and all } r.$$

It is known[3] that, for a linear functional F with $F(x) = \|x\|$ and $\|F\| = 1$, it follows that $-f_+(x;-y) \leq F(y) \leq f_+(x;y)$ for all $y \neq 0$. Also, if $-f_+(x;-y) \leq a \leq f_+(x;y)$, then there is such a linear functional F for which $F(y) = a$. The following is a statement of this in terms of spherical orthogonality,[4] but follows more easily directly from the evaluation of $f_+(x;-y)$ and $f_+(x;y)$ given by (1) above:

$$(6). \quad x \perp ax-y \text{ if and only if } -f_+(x;-y) \leq a\|x\| \leq f_+(x;y).$$

In the previous section, it was possible to establish the existence of an inner product by assuming Gateaux differentiability

1. That $\lim_{h \to 0} (\|x+hy\| - \|x\|)/h$ is linear if it exists has been noted by Mazur (VI), pp. 129-130, but also follows from the additivity of spherical orthogonality resulting from its right-uniqueness or from Gateaux differentiability of the norm (see Theorems 7.3 and 7.4).
2. These three relations were given by Ascoli (XV), pp. 53-55. The first follows trivially from the triangular inequality of the norm, while the others can be proved easily from the theory of spherical orthogonality. Thus (3) follows immediately from $f_+(x;ty)/\|x\|$ and $f_+(x;y)/\|x\|$ being the largest of the numbers a and b, respectively, for which $x \perp ax-ty$ and $x \perp bx-y$. The inequality (4) follows from $x \perp [f_+(x;y)/\|x\|]\, x - y$ and Corollary 6.1.
3. Mazur (IX), pg. 75, statement 7.
4. The equivalence follows from Theorems 6.1 and 9.1.

of the norm and a type of symmetry of this differential.[1] This is also possible without explicitly assuming Gateaux differentiability of the norm.

Theorem 9.3. A normed linear space T is an abstract Euclidean space if and only if $f_+(x;\frac{y}{\|y\|}) = f_+(y;\frac{x}{\|x\|})$ for all non-zero elements x and y.[2]

Proof: Using the definition of $f_+(x;y)$,[2] it follows that:

$$\begin{aligned} f_+(x;\tfrac{y}{\|y\|}) - f_+(y;\tfrac{x}{\|x\|}) &= \lim_{h\to+0} \frac{\|x+h\frac{y}{\|y\|}\|-\|x\|}{h} - \lim_{h\to+0}\frac{\|y+h\frac{x}{\|x\|}\|-\|y\|}{h}, \\ &= \lim_{h\to+0}\frac{\|x+\frac{h\|x\|}{\|y\|}y\|-\|x\|}{h\|x\|} - \lim_{h\to+0}\frac{\|y+\frac{h\|y\|}{\|x\|}x\|-\|y\|}{h\|y\|}, \\ &= \lim_{h\to+0}\frac{\|\frac{x}{\|x\|}+h\frac{y}{\|y\|}\|-\|\frac{y}{\|y\|}+h\frac{x}{\|x\|}\|}{h}, \\ &= \lim_{n\to\infty}\|n\frac{x}{\|x\|}+\frac{y}{\|y\|}\| - \|\frac{x}{\|x\|}+n\frac{y}{\|y\|}\|. \end{aligned}$$

But by Theorem 8.4 this is zero if and only if the space is abstract Euclidean.

The above theorem does not explicitly assume Gateaux differentiability of the norm. However, it must follow from the equality of $f_+(x;\frac{y}{\|y\|})$ and $f_+(y;\frac{x}{\|x\|})$ for all non-zero elements x and y, since the norm of an abstract Euclidean space is Gateaux differentiable.[3] This gives the following corollary.[4]

Corollary 9.1. If $f_+(x;\frac{y}{\|y\|}) = f_+(y;\frac{x}{\|x\|})$ for all non-zero elements x and y of a normed linear space T,[2] then the norm of T is Gateaux differentiable at all non-zero points and $f(x;\frac{y}{\|y\|}) = f(y;\frac{x}{\|x\|})$, where $f(x;y)$ is the differential at x with increment y.

1. Theorem 8.5.
2. Where $f_+(x;y) = \lim_{h\to+0}\left[\|x+hy\|-\|x\|\right]/h$.
3. Corollary 7.3'.
4. Compare with Theorem 8.5.

If spherical orthogonality is symmetric in a normed linear space T, then the orthogonality is right-unique if and only if it is left-unique.[1] The space T is then strictly normed if and only if the norm is Gateaux differentiable at all non-zero points.[2] It was also seen that spherical orthogonality is symmetric in a normed linear space whose norm is Gateaux differentiable if and only if $f(x;y) = 0$ implies $f(y;x) = 0$, where $f(x;y)$ is the differential at x with increment y. If $f(x;y)$ is replaced by $f_+(x;y)$, this condition implies symmetry and the existence of $f(x;y)$.

<u>Theorem 9.4.</u> <u>Spherical orthogonality is symmetric and additive in a normed linear space T if $f_+(y;x) = 0$ whenever $f_+(x;y) = 0$,[3] where x and y are any non-zero elements.</u>

<u>Proof</u>: If $x \neq 0$ and $x \perp y$, then all the numbers a for which $x \perp ax-y$ must be in the interval (A,B), where $A \leq 0 \leq B$ and $A = -\frac{f_+(x;-y)}{\|x\|}$ and $B = \frac{f_+(x;y)}{\|x\|}$.[4] Then the largest number a for which $x \perp (A-a)x-y$, or $x \perp ax-(Ax-y)$, must be zero. Hence $f_+(x;Ax-y) = 0$.[4] Then $f_+(Ax-y;x) = 0$ and $Ax-y \perp x$. Likewise, the smallest number a for which $x \perp (B-a)x-y$, or $x \perp ax-(Bx-y)$, must be zero. Therefore $f_+(x;-(Bx-y)) = 0$.[4] Then $f_+(-(Bx-y);x) = 0$ and $Bx-y \perp x$. Since $Ax-y \perp x$ and $Bx-y \perp x$, it follows from Theorem 7.7 and $A \leq 0 \leq B$ that $y \perp x$. Thus spherical orthogonality is symmetric if $f_+(x;y) = 0$ implies $f_+(y;x) = 0$ for all non-zero elements x and y. For elements

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1. See Definitions 7.1 and 7.2.
2. Corollary 7.5.
3. Where $f_+(x;y) = \lim_{h \to 0^+} [\|x+hy\| - \|x\|]/h$.
4. See (1), page 101 of this thesis.

$x\ (\neq 0)$ and y, suppose that $x \perp ax-y$ if $A' \leq a \leq B'$. Then because of symmetry, $ax-y \perp x$ and (from Theorem 7.7) $\|ax-y\|$ is minimum and constant for $A' \leq a \leq B'$. If $A' < a' < B'$, then $\|(a'x-y)+hx\|-\|a'x-y\| = 0$ if $|h| < a'-A'$ and $|h| < B'-a'$. Thus $f_+(a'x-y;x) = f_+(-(a'x-y);x) = 0$, and hence $f_+(x;a'x-y) = f_+(x;-(a'x-y)) = 0$. But with (6) of page 102 this implies that $A' = B'$, and that there is a unique number a for which $x \perp ax-y$. The additivity of spherical orthogonality follows from Theorem 7.4.

Corollary 9.2. *A normed linear space of three or more dimensions is an abstract Euclidean space if and only if $f_+(x;y) = 0$ implies $f_+(y;x) = 0$ for all non-zero elements x and y.*[1]

This corollary follows easily from Theorem 8.1 and Theorem 9.4.

It was shown that spherical orthogonality is additive, or right-unique, in a normed linear space T if and only if the norm of T is Gateaux differentiable.[2] This differential is a linear functional if it exists, and hence furnishes a link between spherical orthogonality and the theory of linear functionals.[3] Thus the fact that for every element x_0 of a normed linear space there is a linear functional f with $f(x_0) = \|f\|\cdot\|x_0\|$ was used to prove the existence of a hyperplane H such that $x_0 \perp H$, H consisting of all elements for which $f = 0$.[4]

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1. Where $f_+(x;y) = \lim_{h\to +0} [\|x+hy\| - \|x\|]/h$.
2. See Theorems 7.3 and 7.4.
3. The linearity of this differential has been noted by Mazur (VI), pp. 129-130. It also follows from Theorem 7.3 and the additivity of spherical orthogonality given by Corollary 7.4.
4. Theorem 6.2. That f exists is shown by Banach (I), pg. 55, Theorem 3.

If $\|x_o\| = 1$, then the hyperplane defined by $f(x) = \|f\|$ can not contain interior points of the unit sphere $\|x\| \leqslant 1$ and is therefore a supporting hyperplane at x_o.[1] This hyperplane is unique, or also a tangential hyperplane, if and only if there is a unique linear functional f with $\|f\| = 1$ and $f(x_o) = \|x_o\| = 1$. The tangential hyperplane then consists of all elements y satisfying $f(y) = 1$.[2] But f is then unique if and only if there is a unique hyperplane H for which $x_o \perp H$,[3] and hence if and only if spherical orthogonality is right-unique or the norm is Gateaux differentiable at x_o. This tangent hyperplane is then defined by $f(x_o;y) = \|x_o\|$, where $f(x_o;y) = \lim\limits_{h\to 0} \frac{\|x_o+hy\| - \|x_o\|}{h}$.[4] These results can be stated in terms of spherical orthogonality to give the following:

<u>Theorem 9.5.</u> <u>Spherical orthogonality is right-unique in a normed linear space T if and only if for any element x_o of T there is a unique linear functional f with $\|f\| = 1$ and $f(x_o) = \|x_o\|$. This functional f is then defined by $f(y) = -a\|x_o\|$, where $x_o \perp ax_o + y$.</u>

<u>Proof:</u> If the number a for which $x_o \perp ax_o+y$ is not unique for each element y, then Theorem 9.1 shows that f is not unique. If f is not unique, then Theorem 6.1 shows that a is not unique for all y. That $f(y) = -a\|x_o\|$ is a linear functional of y follows from its being equal to the Gateaux

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1. Since $f(x_o) = \|f\|$, the elements x satisfying $f(x) = \|f\|$ are of the form x_o+h, where $f(h) = 0$ and the elements h form a maximal linear subset. See Def. 9.1 and the discussion on page 100 of this thesis.
2. This has been shown by Mazur (VI), page 130.
3. See Theorem 6.1.
4. Also shown by Mazur (VI), page 130.

differential of the norm at x_0 with increment y, the linearity of this differential of the norm following from its existence.[1] Since $x_0 \perp (-1)x_0 + x_0$, it follows that $f(x_0) = \|x_0\|$ if $f(y) = -a\|x_0\|$ for all y, where $x_0 \perp ax_0 + y$.

Corollary 9.3. Spherical orthogonality is right-unique in a normed linear space T if and only if there is a tangent hyperplane at each boundary point of the unit sphere of T.

Theorem 9.6. There is a hyperplane tangent to the unit sphere of a normed linear space T at the boundary point x_0 if and only if spherical orthogonality is additive at x_0.[2] Such a tangential hyperplane consists of all elements y for which $x_0 \perp x_0 - y$.

Proof: Since Theorem 9.2 gives $x_0 \perp x_0 - y$ if and only if y is in a supporting hyperplane at x_0, it follows that if the tangent hyperplane exists it must consist of all y for which $x_0 \perp x_0 - y$. But by definition, there is a tangent hyperplane if and only if there is a unique supporting hyperplane, or by Theorem 9.2 if and only if the totality of elements y satisfying $x_0 \perp x_0 - y$ is a hyperplane. That is, if and only if the set of elements $x_0 - y$ forms a linear set.[3]

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1. See Theorem 7.3. The linearity of this differential has been noted by Mazur (VI), pp. 129-130. It also follows from Theorem 7.3 and the additivity of spherical orthogonality given by Corollary 7.4.
2. Orthogonality is additive at x_0 if $x_0 \perp y$ and $x_0 \perp z$ imply $x_0 \perp y+z$ for all elements y and z. See III, page 13 of of this thesis.
3. Such a set is necessarily maximal. See Th. 6.3 and Def. 6.2.

In the above, the equivalence of Gateaux differentiability of the norm to right-uniqueness of spherical orthogonality in normed linear spaces was used. Analogous results can be gotten by using the equivalence of left-uniqueness and strict normedness. Thus the norm of a normed linear space is Gateaux differentiable if and only if for each element x there is a unique linear functional f with $\|f\| = 1$ and $f(x) = \|x\|$, while it can be shown that a normed linear space is strictly normed if and only if for no linear functional f with $\|f\| = 1$ is there more than one point x (with $\|x\| = 1$) for which $f(x) = \|x\|$ —that is, no linear functional takes on its maximum on the unit sphere $\|x\| \leq 1$ at more than one point. This has been shown by Smulian,[1] but also follows easily from the theory of spherical orthogonality:

For suppose f $(\|f\| = 1)$ takes on its maximum in the unit sphere at the points x_1 and x_2. Then $f(x_1) = \|x_1\|$ and $f(x_2) = \|x_2\|$, where $\|x_1\| = \|x_2\| = 1$. Thus $f(x_1 - x_2) = 0$ and $f(x_1 + k(x_2 - x_1)) = 1$. If $\|f\| = 1$, then $1 \leq \|x_1 + k(x_2 - x_1)\|$ for all k. But it also follows from the triangular inequality of the norm[2] that $\|x_1 + k(x_2 - x_1)\| \leq 1$ if $0 \leq k \leq 1$. Thus $\|x_1 + k(x_2 - x_1)\| = 1$ if $0 \leq k \leq 1$, and it follows from Theorem 6.1 that $x_1 + k(x_2 - x_1) \perp x_2 - x_1$ if $0 \leq k \leq 1$. Thus spherical orthogonality is not left-unique, and hence the space is not strictly normed.

Conversely, if a normed linear space is not strictly normed, then spherical orthogonality is not left-unique and for some

1. Smulian (XI), Theorem 6.
2. Condition (2) of Definition 1.6.

elements x and y and number a, $y \perp x$ and $ax+y \perp x$.[1] Theorem 9.1 gives the existence of a linear functional f with $f(y) = \|y\|$, $f(x) = 0$, and $\|f\| = 1$. But it then follows from $\|ax+y\| = \|y\|$ that $f(ax+y) = \|ax+y\|$.[2] Thus f takes on its maximum in the unit sphere at the two points $\frac{y}{\|y\|}$ and $\frac{ax+y}{\|ax+y\|}$.

For every hyperplane H there is a linear functional f and a number c such that H consists of all x satisfying $f(x) = c$, and conversely all x satisfying an equation of the form $f(x) = c$ form a hyperplane.[3] If $\|f\| = 1$ and $f(x_0) = \|x_0\| = 1$, then the hyperplane $f(x) = 1$ contains no interior points of the unit sphere $\|x\| \leq 1$ and is therefore a supporting hyperplane. Thus the above can be restated: "A normed linear space is strictly normed if no supporting hyperplane of the unit sphere contains more than one boundary point of the unit sphere." Using the equivalence of strict normedness and left-uniqueness of spherical orthogonality gives the following theorem.

<u>Theorem 9.7.</u> <u>Spherical orthogonality is left-unique in a normed linear space if and only if no supporting hyperplane of the unit sphere S contains more than one boundary point of S.</u>

While right- and left-uniqueness of spherical orthogonality have been related to Gateaux differentiability of the norm and to strict normedness, no direct relationship has been developed between these concepts except the analogy between Corollary 9.3 and Theorem 9.7, and the equivalence which results

1. See Theorem 7.8 and Definition 7.2.
2. $\|ax+y\| = \|y\|$ because of Theorem 7.7. Actually, $f(Ax+y) = \|Ax+y\|$ if $0 \leq A \leq a$.
3. Mazur (IX), page 71.

from the assumption of symmetry of the orthogonality.[1]

However, it has been shown by Smulian that if every linear functional defined in a normed linear space T attains its maximum in the unit sphere $\|x\| \leq 1$, then the conjugate space T' is strictly normed if and only if the norm of T is Gateaux differentiable.[2] With this condition, it can also be shown that T is strictly normed if the norm of T' is Gateaux differentiable. These statements are equivalent to the following theorem,[1] which will be proved by using the theory of spherical orthogonality.

Theorem 9.8. *If every linear functional defined on a normed linear space T attains its maximum in the unit sphere $\|x\| \leq 1$, then spherical orthogonality is left-unique in T' if and only if it is right-unique in T, and it is left-unique in T if it is right-unique in T'.*

Proof: Suppose spherical orthogonality is not left-unique in T'. Then using Theorem 7.7 it follows that there are non-zero elements f and g of T' and a positive number e for which $af+g \perp f$ if $|a| < e$, and that $\|af+g\| = \|g\|$ for $|a| < e$. Take $\|g\| = 1$. Since g attains its maximum in the unit sphere, there is an element x_0 for which $\|x_0\| = 1$ and $g(x_0) = 1$. But there are both positive and negative values of a for which $\|af+g\| = 1$. Therefore $f(x_0) = 0$, and $af+g$ attains its maximum in the unit sphere at the point x_0 if $|a| < e$. Because of Theorem 6.1, x_0 is orthogonal to all elements for which $g = 0$ and also (if $|a| < e$) to all elements for which $af+g = 0$. Since

1. As a result of Theorems 7.3 and 7.8.
2. Smulian (XI), Theorem 8.

the maximal linear subsets of T consisting of elements for which f and g are zero, respectively, do not coincide, it follows that spherical orthogonality is not right-unique in T. Thus spherical orthogonality is left-unique in T' if it is right-unique in T.

Suppose spherical orthogonality is not right-unique in T. Then it follows from Theorem 9.5 that there is an element x_o of T for which there are two linear functionals f and g with $\|f\| = \|g\| = 1$ and $f(x_o) = g(x_o) = \|x_o\|$. But then $f(x_o)+k[f(x_o)-g(x_o)] = \|x_o\|$, and therefore $\|f+k(f-g)\| \geq 1 = \|f\|$ for all k. Thus $f \perp f-g$. Likewise, $\|g+k(f-g)\| \geq \|g\|$ and $g \perp f-g$. Hence $f+k(f-g) \perp f-g$ if $k = 0$ or $k = -1$, and spherical orthogonality is not left-unique in T'. Thus spherical orthogonality is right-unique in T if it is left-unique in T'.

Suppose spherical orthogonality is not left-unique in T. Then it follows from Theorem 7.7 that there are non-zero elements x and y and a positive number e for which $ax+y \perp x$ if $|a| < e$, and that $\|ax+y\| = \|y\|$ for $|a| < e$. From Theorem 9.1, there is then a linear functional f with $\|f\| = 1$, $f(y) = \|y\|$, and $f(x) = 0$. Let g be a linear functional with $g(y) = 0$ and $g(x) \neq 0$.[1] Then

$$f(ax+y)+k[bf(ax+y)+g(ax+y)] = f(y)+k[bf(y)+ag(x)].$$

Thus if $|b| \leq \left|\frac{eg(x)}{f(y)}\right|$, then a can be chosen so that

$$\left|f(ax+y)+k[bf(ax+y)+g(ax+y)]\right| \geq |f(ax+y)| = \|ax+y\|.$$

Therefore $\|f+k(bf+g)\| \geq \|f\| = 1$ if $|b| \leq \left|\frac{e[g(x)]}{f(y)}\right| \neq 0$, and

1. The existence of g follows from Banach (I), pg. 57, or from Theorem 9.1 and the existence of a non-zero number a for which $ay+x \perp y$, as given by Theorem 6.3.

spherical orthogonality is not right-unique in T'. Hence spherical orthogonality is left-unique in T if it is right-unique in T'.

The following theorem serves to complete Theorem 9.8. It makes use of the concepts of weak convergence and weak compactness, both of which will be used extensively in the next section.

Definition 9.3. A sequence $\{x_n\}$ of elements of a normed linear space T is weakly convergent if $\lim_{n\to\infty} f(x_n)$ exists for all linear functionals f defined in T. Such a sequence converges weakly to an element x if $\lim_{n\to\infty} f(x_n) = f(x)$ for all linear functionals f defined in T.

Definition 9.4. A subset S of a normed linear space T is weakly compact if every sequence of elements of S contains a sequence which converges weakly to an element of T.

Theorem 9.9. If the unit sphere of a normed linear space T is weakly compact, then spherical orthogonality is right-unique in T' if it is left-unique in T.

Proof: Suppose spherical orthogonality is not right-unique in T'. Then spherical orthogonality is not additive in T',[1] and there therefore exist linear functionals f, g, and h, defined in T, for which $f \perp g$ and $f \perp h$, but with f not orthogonal to g+h. Thus $\|f+kg\| \geq \|f\|$ and $\|f+kh\| \geq \|f\|$

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1. See Theorem 7.4.

for all k, but there is a value k_1 of k for which

$$\|f+k_1(g+h)\| < \|f\|.$$

Without loss of generality, $\|f\|$ can be taken as unity and k_1 to be positive. Then $\|f+k(g+h)\| < \|f\|$ for $0<k\leq k_1$.[1] Now let $\{x_n\}$ $(n = 1,2,3,\cdots)$ be a sequence of elements of T for which $\|x_n\| = 1$, $f(x_n)\geq 0$, and

$$\left|f(x_n)+\tfrac{1}{n}g(x_n)\right| \geq \|f\| = 1$$

for each n, and let $\{y_n\}$ be such that $\|y_n\| = 1$, $f(y_n)\geq 0$, and

$$\left|f(y_n)+\tfrac{1}{n}h(y_n)\right| \geq \|f\| = 1$$

for each n. Since $|g(x_n)|\leq \|g\|$ and $|h(y_n)|\leq \|h\|$, it follows from $0\leq f(x_n)\leq 1$ that $g(x_n)\geq 0$ and $h(y_n)\geq 0$ if $n>\|g\|$ and $n>\|h\|$. If n is this large and $\frac{1}{n}\leq k_1$, then $g(x_n)$ and $h(x_n)$ must be of opposite sign, since otherwise

$$\left|f(x_n)+\tfrac{1}{n}\left[g(x_n)+h(x_n)\right]\right|$$

would not be less than $\|f\| = 1$. Likewise, $g(y_n)$ and $h(y_n)$ would also be of opposite sign. Since T is weakly compact, there are subsequences of $\{x_n\}$ and of $\{y_n\}$ which converge weakly to elements x and y of T, respectively. Clearly $f(x_n)\to 1$ and $f(y_n)\to 1$. Thus $f(x) = f(y) = 1$. Since $\|f\| = 1$, it follows that $1\leq\|x\|$ and $1\leq \|y\|$. If F is a linear functional for which $\|F\| = 1$ and $F(x) = \|x\|$, then $|F(x_n)|\leq \|x_n\| = 1$. But since $F(x_n)\to F(x)$, it follows that $F(x)\leq 1$ and thus that $\|x\|\leq 1$. Likewise $\|y\|\leq 1$. Hence $\|x\| = \|y\| = 1$, and f takes on its maximum in the unit sphere at both x and y. But $g(x)\geq 0$ and $h(x)\leq 0$, while $g(y)\leq 0$ and $h(y)\geq 0$. Thus $x=y$ is possible only if

1. This follows from the triangular inequality of the norm, or from Theorem 7.5.

$g(x) = g(y) = h(x) = h(y) = 0$, and hence only if $\left|f(x) + k_1\left[g(x)+h(x)\right]\right| = \|f\|$. But this would contradict the assumption that $\|f + k_1(g+h)\| < \|f\|$. Since $x \neq y$, f takes on its maximum in the unit sphere at two distinct points and spherical orthogonality is not left-unique in T.[1] Therefore spherical orthogonality is right-unique in T' if it is left-unique in T.

If the unit sphere of a normed linear space is weakly compact, then each linear functional attains its maximum in the unit sphere.[2] It is then possible to combine Theorems 9.8 and 9.9. However, the concepts of uniform convexity and regularity[3] are also related to weak compactness, and hence to the question of whether a linear functional attains its maximum. It is known that a uniformly convex Banach space is regular,[4] and that a regular Banach space is weakly compact.[5] Thus any of these three concepts can be used for the following theorem.

<u>Theorem 9.10.</u> <u>If a Banach space T is uniformly convex, regular, or its unit sphere is weakly compact, then spherical orthogonality is left-unique in T' if and only if it is right-unique in T, and it is right-unique in T' if and only if it is left-unique in T.</u>

1. See the discussion on page 108 of this thesis.
2. This is implicitly shown in the proof of Theorem 9.9.
3. These concepts are defined and used in the next section. See Definitions 10.1 and 10.3.
4. Milman (XVI), Theorem 2. Also see Pettis (XIX).
5. Gantmakher and Smulian (XVIII), Theorem 1. Also, see Milman (XVI), page 244.

10. EVALUATIONS OF LINEAR FUNCTIONALS

An essential step in finding an evaluation of a linear functional F defined in a normed linear space T is finding an element x for which $F(x) = \|F\|\cdot\|x\|$. This is equivalent to the problem of finding an element x orthogonal to a given maximal linear subset H of T with $\overline{H} \neq T$:[1]

<u>Theorem 10.1.</u> <u>A necessary and sufficient condition that there exist an element orthogonal to each closed linear subset H of a normed linear space T is that for each linear functional F defined in T there is an element x with $F(x) = \|F\|\cdot\|x\|$.</u>

<u>Proof</u>: If $\|F\| \neq 0$, then the set H of elements for which $F = 0$ is a linear subset with $\overline{H} \neq T$. It follows from Theorem 6.1 that any element x orthogonal to this set is such that $F(x) = \|F\|\cdot\|x\|$. Conversely, suppose H is any closed linear subset of T. Define the functional f as being zero for elements of H and unity for some element x_0 not in H. This functional is clearly additive over the space gotten by adjoining x_0 to H, and its continuity follows from H being closed. It is then possible to extend f to all of T,[2] Theorem 6.1 then showing that an element x is orthogonal to H if $f(x) = \|f\|\cdot\|x\|$.

A normed linear space has the properties of Theorem 10.1 if and only if each supporting hyperplane of the unit sphere S

1. See Def. 1.8 and the following discussion, and Def. 6.2. In this section, "orthogonality" is "spherical orthogonality" of Definition 2.3.
2. Banach (I), page 55, Theorem 2.

contains a boundary point of S. It is known that there is a plane of support at each boundary point of S, and that the equation $F(x) = \|F\|$ defines a supporting hyperplane.[1] This hyperplane contains the boundary point x_0 ($\|x_0\| = 1$) if and only if $F(x_0) = \|F\|$.

It is known that for any element x there is a maximal linear subset H with $x \perp H$, a linear functional F with $F(x) = \|F\| \cdot \|x\|$,[2] and a supporting hyperplane to the unit sphere at the point $\frac{x}{\|x\|}$. The problem of this section is an investigation of the conditions under which the converse of this is true and of the resulting evaluations of linear functionals.

It should be recalled that a normed linear space is strictly normed if and only if no supporting hyperplane of the unit sphere S contains more than one boundary point of S, or if and only if for no linear functional F is there more than one element x with $F(x) = \|F\| \cdot \|x\|$.[3] Likewise, the norm is Gateaux differentiable if and only if there is a tangent hyperplane at each boundary point of S, or if and only if for every element x there is a unique linear functional F with $F(x) = \|F\| \cdot \|x\|$.[4] The latter will make it possible to evaluate linear functionals in terms of Gateaux differentials and spherical orthogonality for normed linear spaces in which the norm is Gateaux differentiable and linear functionals take on their maximum in the unit sphere.

1. See Definition 9.1, and the discussions following Definition 9.2 and on page 109.
2. Theorem 6.2 and Banach (I), page 55, Theorem 3.
3. See Theorem 9.7 and the discussion of page 108, remembering that spherical orthogonality is left-unique if and only if the space is strictly normed (Theorem 7.8).
4. Corollary 9.3 and Theorem 9.5, using the equivalence of Gateaux differentiability of the norm and right-uniqueness of spherical orthogonality (Theorem 7.3). Also, Mazur (VI), page 130.

The evaluation of linear functionals has been done by Löwig for complete abstract Euclidean spaces. He showed that if F is a linear functional defined in a complete abstract Euclidean space, then there is a unique element x_0 such that $F(y) = (x_0, y)$, where (x_0, y) is the inner product of x_0 and y.[1] Then $F(x_0) = (x_0, x_0) = \|x_0\|^2 = \|F\| \cdot \|x_0\|$.[2] The functional F thus takes on its maximum in the unit sphere only at the point $\frac{x_0}{\|x_0\|}$, and is the only functional taking on its maximum at that point. As follows from the above discussion,[3] x_0 is orthogonal to the set H of all h satisfying $(x_0, h) = 0$. It follows that for every maximal linear subset H (with $\overline{\overline{H}} \neq T$) in a complete abstract Euclidean space T there is a unique element x_0 for which $x_0 \perp H$.[4] It is also worthwhile to recall that

$$f(x_0;y) = \frac{(x_0, y)}{\|x_0\|} = -a\|x_0\|,$$

where $f(x_0;y)$ is the Gateaux differential of the norm at x_0 and $x_0 \perp ax_0 + y$.[5] Thus for any linear functional $F(y)$ (with $\|F\| = 1$) in a complete abstract Euclidean space, there is a unique element x_0 such that $F(y)$ is equal to $f(x_0;y)$, $\frac{(x_0, y)}{\|x_0\|}$, or $-a\|x_0\|$. All but the form using the inner product can be extended to certain normed linear spaces.

There are a number of restrictions which can be put on normed linear spaces and which are sufficient to assure linear functionals taking on their maximum in the unit sphere, or to give an element orthogonal to any given closed linear subset. Mazur has shown that for any linear functional F in a normed linear space

1. Lowig (V), page 11, Theorem 11; (x,y) is defined by Def. 1.9.
2. See (4) of Def. 1.9. That $\|f\| = \|x_0\|$, or $|(x_0,y)| \leq \|x_0\| \cdot \|y\|$, follows from (1) of Theorem 1.2 and (2) of Def. 1.6.
3. Also see Theorem 3.1.
4. This is given by Löwig (V), page 15, Theorem 16.
5. Theorem 7.3 and Corollary 7.3'.

with a weakly compact[1] unit sphere there is an element x for which $F(x) = \|F\| \cdot \|x\|$, and equivalently that every supporting hyperplane of the unit sphere S in such a space contains a boundary point of S. This is equivalent to the following theorem.[2]

Theorem 10.2. If the unit sphere of a normed linear space T is weakly compact, then for each closed linear subset H of T there is at least one element $x \perp H$.

The following conditions are more restrictive than that used above, but are worthwhile in that they are of quite different form and still give the desired conclusion of Theorem 10.2.

Definition 10.1. A normed linear space T is regular if and only if for every linear functional F defined over the conjugate space T' there is an element x_0 of T such that $F(f) = f(x_0)$ for all elements f of T'.[3]

Definition 10.2. A functional f defined on a normed linear space T possesses the Fréchet differential $f(x;y)$ at x if for any $\varepsilon > 0$ there is a $\delta > 0$ for which

$$\left| \frac{f(x+hy)-f(x)}{h} - f(x;y) \right| < \varepsilon \|y\|$$

if $|h| < \delta$; $f(x;y)$ is a "uniform Fréchet differential" if the number δ can be chosen independently of x for $\|x\| = 1$.[4]

1. Mazur (VI), pp. 129-130. Also see Definition 9.4.
2. See Theorem 10.1.
3. Some authors use "reflexive" in place of "regular". Conjugate spaces are defined on page 8 of this thesis.
4. These are Gateaux differentials (Def. 7.3) for which the convergence to $f(x;y)$ is uniform in y when $\|y\| \leq 1$, and uniform in x,y when $\|x\| = 1$ and $\|y\| \leq 1$.

It is known that the unit sphere of a regular normed linear space is weakly compact,[1] and that a Banach space is regular if its norm is uniformly Fréchet differentiable at each non-zero point.[2] Thus either of these conditions can be used in place of the weak compactness of Theorem 10.2, if completeness is assumed with the differentiability.[3] Uniform Fréchet differentiability of the norm has the added advantage of implying Gateaux differentiability, which will be either directly or implicitly needed in the evaluations of linear functionals.

Theorem 10.3. If the normed linear space T is regular, or if T is complete and the norm of T is uniformly Fréchet differentiable at non-zero points, then for each closed linear subset H of T there is at least one element $x \perp H$. If F is a linear functional defined in T, then there is at least one element x for which $F(x) = \|F\| \cdot \|x\|$.

Definition 10.3. A normed linear space is uniformly convex if for every $\varepsilon > 0$ there is a number $\delta > 0$ for which $\|x\| = \|y\| = 1$ and $\|x-y\| > \varepsilon$ imply $\|x+y\| < 2-\delta$.[4]

It has been shown by Milman that a uniformly convex Banach space is regular.[5] It therefore follows that the regularity of

1. Gantmakher and Smulian (XVIII), Theorem 1. Also see Milman (XVI), page 244. Using these and knowing that the conjugate space of a normed linear space T is the same as the conjugate space of the Banach space gotten by completing T, it follows that a regular normed linear space is a Banach space and its unit sphere is weakly compact.
2. Smulian (XX), page 648.
3. See Theorem 10.2 and the preceding discussion.
4. Originally given by Clarkson (XII), Definition 1.
5. Milman (XVI), Theorem 2. Also see Pettis (XIX).

Theorem 10.3 can be replaced by uniform convexity and completeness. Furthermore, if F is a linear functional defined in a uniformly convex Banach space T, then there is a unique element x for which $F(x) \stackrel{\wedge}{=} \|F\| \cdot \|x\|$ and there is one and only one element orthogonal to a given maximal linear subset of T.[1]

<u>Theorem 10.4</u>. <u>If a Banach space T is uniformly convex, then for each maximal linear subset H with $\bar{H} \neq T$ there is a unique element $x \perp H$. If F is a linear functional defined in T, then there is a unique element x for which $F(x) = \|F\| \cdot \|x\|$.</u>

If the Gateaux differential of the norm at a point x_0,

$$f(x_0;y) = \lim_{h \to 0} \frac{\|x_0+hy\| - \|x_0\|}{h},$$

exists in a normed linear space T, then it is a linear functional of y.[2] For this linear functional $F(y) = f(x_0;y)$, it is seen that $F(x_0) \stackrel{\wedge}{=} \|x_0\|$ and that $\|F\| = 1$.[3] But the norm of a normed linear space is Gateaux differentiable at x_0 if and only if there is a unique linear functional F with $\|F\| = 1$ and $F(x_0) = \|x_0\|$.[4] Hence if the Gateaux differential of the norm exists at each non-zero point, then any linear functional F, with $\|F\| = 1$ and for which there is an element x such that $F(x) = \|x\|$, is equal to the Gateaux differential $f(x;y)$.

1. The first statement is given by Pettis (XIX), Lemma 1. The two statements can be shown to be equivalent by the same reasoning as used to prove Theorem 10.1. They then follows from the equivalence of left-uniqueness of spherical orthogonality and strict normedness, a uniformly convex space being strictly normed.
2. That $f(x_0;y)$ is linear if it exists has been noted by Mazur (VI), pp. 129-130, but also follows from the additivity of spherical orthogonality resulting from its right-uniqueness or from the existence of $f(x_0;y)$. See Theorems 7.3 and 7.4.
3. Since $\big|\|x_0+hy\| - \|x_0\|\big| \leq \|hy\|$, because of (2) of Definition 1.6.
4. Mazur (VI), page 130. This also follows from Theorem 6.1 and the equivalence of Gateaux differentiability of the norm and right-uniqueness of spherical orthogonality (Theorem 7.3).

Using Theorem 7.3, it is seen that such a functional is also equal to $-a\|x\|$, where $x \perp ax+y$. The following theorem now follows easily from Theorems 10.2 and 10.3.

Theorem 10.5. *If the norm of a normed linear space T is Gateaux differentiable at each non-zero point and:*

(1) *the unit sphere of T is weakly compact,*

or (2) *T is regular,*

then for a linear functional F with $\|F\| = 1$ there is at least one element x for which

$$F(y) = f(x;y) = -a\|x\|$$

for all elements y, where $f(x;y)$ is the differential of the norm at x and $x \perp ax+y$.

However, it was shown that a normed linear space is Gateaux differentiable at each non-zero point if and only if spherical orthogonality is additive.[1] This gives the following corollary.

Corollary 10.1. *If spherical orthogonality is additive in a normed linear space T and:*

(1) *the unit sphere of T is weakly compact,*

or (2) *T is regular,*

then for a linear functional F with $\|F\| = 1$ there is at least one element x such that it follows from $x \perp ax+y$ that, for all y,

$$F(y) = -a\|x\|.$$

If the norm of a normed linear space is uniformly Fréchet differentiable, then it is clearly Gateaux differentiable. It

1. Corollary 7.4.

therefore follows from Theorem 10.3 that:[1]

<u>Theorem 10.6</u>. <u>If the norm of a Banach space is uniformly Fréchet differentiable, then for a linear functional F with $\|F\| = 1$ there is at least one element x for which</u>

$$F(y) = f(x;y) = -a\|x\|$$

<u>for all elements y, where $f(x;y)$ is the differential of the norm at x and $x \perp ax+y$</u>.

In Theorem 10.5, 10.6, and Corollary 10.1 it was not possible to say that there is a <u>unique</u> element x for which $F(y) = f(x;y) = -a\|x\|$. However, it is known that there can not be more than one such x for any linear functional F if, and only if, the space is strictly normed.[2] Therefore if this condition were added to either of these theorems, it would be possible to conclude the existence of one and <u>only one</u> element x for which $F(y) = f(x;y) = -a\|x\|$ for all y. Rather than assuming the space to be strictly normed, this can be included in uniform convexity and the following gotten from Theorem 10.4.[3]

<u>Theorem 10.7</u>. <u>If F is a uniformly convex Banach space whose norm is Gateaux differentiable at all non-zero points, then for a linear functional F with $\|F\| = 1$ there is a unique element x such that for all elements y</u>

$$F(y) = f(x;y) = -a\|x\|,$$

<u>where $f(x;y)$ is the differential of the norm at x and $x \perp ax+y$</u>.

1. By the same reasoning as preceded Theorem 10.5.
2. Smulian XI, Theorem 6. Also see page 108 of this thesis.
3. See Definitions 7.4 and 10.3. The reasoning is the same as that preceding Theorem 10.5.

<u>Corollary 10.2.</u> <u>If spherical orthogonality is additive in a uniformly convex Banach space, then for a linear functional F with $\|F\| = 1$ there is a unique element x for which</u>

$$F(y) = -a\|x\|$$

<u>for all elements y, where $x \perp ax+y$.</u>

This theorem would also give the result of Löwig[1] as a corollary, since an abstract Euclidean space is uniformly convex and its norm is Gateaux differentiable--the inner product (x,y) being equal to $\|x\| \cdot f(x;y)$.[2]

However, the conditions of Theorem 10.7 are rather severe. Gateaux differentiability of the norm implies that spherical orthogonality is additive,[3] while uniform convexity is a stronger condition than strict normedness.[4] It is therefore seen from Theorems 8.1-8.3 that requiring spherical orthogonality to be symmetric in a space of three or more dimensions satisfying the conditions of Theorem 10.7 makes the space abstract Euclidean.

Because of this, it seems worthwhile to consider what kind of a generalized inner product can be defined in normed linear spaces. It was seen that for abstract Euclidean spaces the inner product can be evaluated as $(x,y) = -a\|x\|^2$, where $x \perp ax+y$.[5] This can also be stated, equivalently, that if H is the maximal linear subset for which $x \perp H$ and a is the number for which $y = ax-h$ with $h \in H$, then $(x,y) = -a\|x\|^2$.[6]

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1. Lowig (V), Theorem 11. See page 117 of this thesis.
2. See Corollary 7.3'.
3. Corollary 7.4.
4. See Definitions 7.4 and 10.3.
5. See page 19 of this thesis.
6. The existence of H is given by Theorem 6.2; it must be unique since the number a is unique.

<u>Theorem 10.8. It is possible to assign to each pair of elements x,y of a normed linear space T a real number $[x,y]$ satisfying</u>:

(1). $[tx,y] = t[x,y]$ and $[x,ty] = t[x,y]$,

(2). $[x,y+z] = [x,y] + [x,z]$,

(3). $[x,y]$ is continuous in y,

(4). $[x,x] = \|x\|^2$ and $|[x,y]| \leq \|x\|\cdot\|y\|$.

<u>There is a unique possible definition of $[x,y]$ satisfying (1)-(4) if and only if orthogonality is additive.</u>[1] <u>In that case</u>:

(5). $[x,y]$ is continuous in x,

(6). $x \perp y$ if and only if $[x,y] = 0$.

<u>Proof</u>: For each element x of T for which $\|x\| = 1$ choose a linear functional F_x with $\|F_x\| = 1$ and $F_x(x) = 1$, and let H_x be the corresponding maximal linear subset with $x \perp H_x$.[2] Define $[kx,y]$ as $k\|x\|\cdot F_x(y)$ or as $-ak\|x\|^2$, where $ax+y \in H_x$. Since $F_x(ax+y) = 0$,[3] these definitions are the same and clearly satisfy (1)-(4), the second part of (4) following either from $\|F_x\| = 1$ or from $|a| \leq \frac{\|y\|}{\|x\|}$.[4]

Spherical orthogonality is additive in T if and only if for any elements x ($\neq 0$) and y there is a unique number a for which $x \perp ax+y$.[5] From the way $[x,y]$ was defined, it is clear that

1. Other conditions equivalent to additivity of spherical orthogonality are: (1) Right-uniqueness of spherical orthogonality; (2) Gateaux differentiability of the norm at non-zero points; (3) Uniqueness for each element x of the linear functional f with $\|f\| = 1$ and $f(x) = \|x\|$; (4) The existence of a tangent hyperplane at each point of the unit sphere. Any of these could replace spherical orthogonality in Theorem 10.8. (See Theorems 7.4, 7.3, 9.5, and Corollary 9.3).
2. See Theorem 6.2 and its proof.
3. Theorem 6.1.
4. The latter is given by Corollary 6.1.
5. Theorem 7.4.

the uniqueness of such definitions of $[x,y]$ implies the uniqueness of the number a for which $x \perp ax+y$, and therefore also implies additivity of spherical orthogonality. Also, if $[x,y] = 0$, then $[x,x+ky] = [x,x]$ and (4) gives $\|x\|^2 \le \|x\| \cdot \|x+ky\|$ or $\|x+ky\| \ge \|x\|$. Thus $[x,y] = 0$ implies $x \perp y$, and $[x,y] = -a\|x\|^2$ implies $x \perp ax+y$. Hence if spherical orthogonality is additive, then the only value $[x,y]$ can have is $-a\|x\|^2$, and $x \perp y$ if and only if $[x,y] = 0$. It now only remains to establish (6): Suppose $[x,y]$ is not continuous in x, and that $[x,y] = -a\|x\|^2$. Then for some sequence $\{x_i\}$ with $\|x_i\| = 1$ and $x_i \to x$, it isn't true that $[x_i,y] \to -a\|x\|^2$. Since (4) shows that $[x_i,y]$ is bounded, there is a subsequence $\{x_j\}$ of $\{x_i\}$ and a number A such that $[x_j,y] = -A_j\|x\|^2$ and $A_j \to A$. Then, as above, $x_j \perp A x_j + y$, or $\|x_j + k(A_j x_j + y)\| \ge \|x_j\| = 1$ for all k. Since the norm is continuous, it follows that $\|x + k(Ax+y)\| \ge \|x\| = 1$ for all k and that $x \perp Ax+y$. But the number a for which $x \perp ax+y$ is unique if spherical orthogonality is additive. Hence $a = A$ and $[x,y]$ is continuous in x if spherical orthogonality is additive.

This result gives another form in which to express Theorems 10.5-10.7, by replacing $f(x;y) = -a\|x\|$ by $\frac{[x,y]}{\|x\|}$.

If in addition to additivity of spherical orthogonality its symmetry is also assumed, then $[x,y] = 0$ implies $[y,x] = 0$. All normed linear spaces of three or more dimensions satisfying these conditions are abstract Euclidean.[1] For a two dimensional space it is necessary to make the assumption $[x,y] = [y,x]$, and

1. Theorem 8.1.

hence have all the conditions of the definition of an inner product.[1]

In the following examples, the generalized inner product $[x,y]$ is expressed in terms of the general form of the linear functionals defined on these spaces (for $p>1$), and interpreted in terms of spherical orthogonality (for $p\geq 1$).

Example 10.1. Consider the Banach space l_p $(p\geq 1)$ of all sequences $x=(x_1,x_2,\cdots)$ for which $\sum_{i=1}^{\infty}|x_i|^p$ is convergent, where $\|x\|=\left[\sum_{i=1}^{\infty}|x_i|^p\right]^{1/p}$.[2] By definition, $x\perp y$ if and only if $\|x+ky\|-\|x\|\geq 0$ for all k. If $y=(y_1,y_2,\cdots)$, this becomes $\left[\sum_{i=1}^{\infty}|x_i+ky_i|^p\right]^{1/p}-\left[\sum_{i=1}^{\infty}|x_i|^p\right]^{1/p}\geq 0$, or $\sum_{i=1}^{\infty}\left[|x_i+ky_i|^p-|x_i|^p\right]\geq 0$. Thus if $x\perp y$, then

$$\lim_{k\to+0}\frac{\sum_{i=1}^{\infty}\left[|x_i+ky_i|^p-|x_i|^p\right]}{k}\geq 0.$$

But this clearly implies that $\lim_{k\to+0}\frac{\|x+ky\|-\|x\|}{k}\geq 0$ and that $\|x+ky\|\geq\|x\|$ for $k\geq 0$.[3] Using the convergence of the series $\sum_{i=1}^{\infty}|x_i|^p$ and $\sum_{i=1}^{\infty}|y_i|^p$, it is not difficult to show that for any $\varepsilon>0$ there is an integer n for which $\left|\frac{\sum_{i=n}^{\infty}|x_i+ky_i|^p-\sum_{i=n}^{\infty}|x_i|^p}{k}\right|<\varepsilon$ for all k with $|k|<1$. Thus

$$\lim_{k\to+0}\frac{\sum_{i=1}^{\infty}|x_i+ky_i|^p-\sum_{i=1}^{\infty}|x_i|^p}{k}=\sum_{i=1}^{\infty}\lim_{k\to+0}\frac{|x_i+ky_i|^p-|x_i|^p}{k},$$

and $\|x+ky\|\geq\|x\|$ for $k\geq 0$ if and only if neither of these is less than zero. As $k\to+0$, $\frac{|x_i+ky_i|^p-|x_i|^p}{k}$ approaches $p|x_i|^{p-2}(x_iy_i)$, if $p>1$ or $p=1$ and $x_i\neq 0$, and it approaches $|y_i|$ if $p=1$ and $x_i=0$. Hence $\|x+ky\|\geq\|x\|$ for $k\geq 0$ if and only if

1. Definition 1.9.
2. This space is defined by Banach (I), page 12.
3. See Theorem 7.5.

$$(1). \quad \sum_{i=1}^{\infty} |x_i|^{p-2}(x_i y_i) \geq 0, \text{ if } p>1;$$

or
$$(2). \quad \sum_{i}^{(x_i \neq 0)} \frac{x_i y_i}{|x_i|} + \sum_{i}^{(x_i=0)} |y_i| \geq 0, \text{ if } p = 1.$$

Putting $ax+y$ and $-(ax+y)$ in (1) in place of y shows that $x \perp ax+y$ if and only if $\left[a\|x\|^p + \sum_{i=1}^{\infty} |x_i|^{p-2}(x_i y_i)\right] = 0$, and that $x \perp -\left[\frac{\sum_{i=1}^{\infty} |x_i|^{p-2}(x_i y_i)}{\|x\|^p}\right]x + y$.[1] From Theorem 7.3, it follows that

$$(3). \quad f(x;y) = \frac{\sum_{i=1}^{\infty} |x_i|^{p-2}(x_i y_i)}{\|x\|^{p-1}},$$

where $f(x;y)$ is the Gateaux differential of the norm at x.

Thus the norm of a space l_p with $p>1$ is Gateaux differentiable and spherical orthogonality is right-unique and additive in l_p.[2] A space l_p $(p>1)$ is uniformly convex, and hence if $[x,y]$ is defined by

$$(4). \quad [x,y] = \frac{\sum_{i=1}^{\infty} |x_i|^{p-2}(x_i y_i)}{\|x\|^{p-2}},$$

then for any linear functional F defined in l_p there is an unique element x_0 for which $F(y) = [x_0,y]$ for all y.[3] Also, this "generalized inner product" satisfies conditions (1)-(6) of Theorem 10.8.

For $p = 1$, putting $ax-y$ and $-(ax-y)$ in place of y in (2) shows that $x \perp ax-y$ if and only if $\pm\left[a\|x\| - \sum_{i}^{(x_i \neq 0)} \frac{x_i y_i}{|x_i|}\right] + \sum_{i}^{(x_i=0)} |y_i| \geq 0$,

or
$$(5). \quad \sum_{i}^{(x_i \neq 0)} \frac{x_i y_i}{|x_i|} - \sum_{i}^{(x_i=0)} |y_i| \leq a\|x\| \leq \sum_{i}^{(x_i \neq 0)} \frac{x_i y_i}{|x_i|} + \sum_{i}^{(x_i=0)} |y_i|.$$

Thus spherical orthogonality is right-unique, or additive, at an element $x = (x_1, x_2, \cdots)$ of the space l_1 if and only if $x_i \neq 0$ for any i. This condition is also necessary and

1. There must be at least one number a for which $x \perp ax+y$. See Theorem 6.3.
2. These three concepts are equivalent in any normed linear space. See Theorems 7.3 and 7.4.
3. The uniform convexity of l_p $(p>1)$ was shown by Clarkson (XII). The rest follows from Theorem 10.7, and has been known before, e.g. as a direct consequence of Pettis (XIX), Lemma 1, and the satisfying of (1)-(4) of Theorem 10.8 by $[x,y]$.

sufficient for the existence of the Gateaux differential at x and for the unique evaluation of the linear functional F with $\|F\| = 1$ and $F(x) = \|x\|$. Thus if $x_i \neq 0$ for any i, then the differential of the norm is given by (3) and $F(y) = [x,y]$ for all y, where $[x,y]$ is given by (4). Comparing (5) above with (6) of page 102 gives:

$$f_+(x;y) = \sum_i^{(x_i\neq 0)} \frac{x_i y_i}{|x_i|} + \sum_i^{(x_i=0)} |y_i|, \text{ and } f_-(x;y) = \sum_i^{(x_i\neq 0)} \frac{x_i y_i}{|x_i|} - \sum_i^{(x_i=0)} |y_i|.$$ [1]

<u>Example 10.2.</u> <u>Consider the Banach space</u> L_p $(p \geq 1)$ <u>of all functions</u> $x(t)$ <u>in</u> $(0,1)$ <u>for which</u> $\int_0^1 |x(t)|^p dt$ <u>exists, where</u> $\|x\| = \left[\int_0^1 |x(t)|^p dt\right]^{1/p}$. An L_p space is uniformly convex if $p>1$,[2] and for a linear functional F defined on L_p $(p>1)$ there is a unique element x for which $F(y) = [x,y]$ for all y, where

$$[x,y] = \int_0^1 |x(t)|^{p-2} x(t)\cdot y(t)\, dt.$$ [3]

This "generalized inner product" satisfies (1)-(6) of Theorem 10.8.

For the space L_1 the following is analogous to the corresponding equations for the space l_1, U being the set of all numbers t in $(0,1)$ for which $x(t) \neq 0$ and $1-U$ the complement of U.

$$f_+(x;y) = \int_U \frac{x(t)y(t)}{|x(t)|}dt + \int_{1-U} |y(t)|\, dt;\ f_-(x;y) = \int_U \frac{x(t)y(t)}{|x(t)|}dt - \int_{1-U} |y(t)|\, dt.$$

Since $x \perp ax+y$ if and only if $-f_+(x;y) \leq a\|x\| \leq -f_-(x;y)$, spherical orthogonality is right-unique, or additive, at x and there is a Gateaux differential of the norm at x, if and only if the set of numbers t for which $x(t) = 0$ is of measure zero.

1. Where $f_\pm(x;y) = \lim_{h\to\pm 0} \left[\|x+hy\| - \|x\|\right]/h$.
2. Clarkson (XII).
3. This follows from Theorem 10.7 and the evaluation of the differential of the norm given by Mazur (VI), pg. 132. It is also immediate from Pettis (XIX), Lemma 1, and the satisfying of (1)-(4) of Theorem 10.8 by $[x,y]$.

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