I THEORETICAL INVESTIGATION OF THE REFLECTION OF IONIZING SHOCKS

II THEORETICAL STUDY OF SOUND AND SHOCK WAVES IN A TWO-PHASE FLOW

Thesis by

Antonio Crespo-Martínez

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ABSTRACT

PART I

The reflection of an ionizing shock from the end wall of a shock tube is studied theoretically following the experimental model of J. Smith. The observed perturbations in the wall pressure history are found to agree with this theory. To describe the first perturbation, a decrease in pressure due to the ionization part in the reflected shock structure, the flow equations are linearized but the rate equations are used in nonlinear form. The second perturbation, an increase in pressure due to the ionization part of the incident shock structure, is studied using Whitham's theory and assuming equilibrium behind the reflected shock.

PART II

The propagation of sound and shock waves in a two-phase medium is studied theoretically using the flow equations for each component. It is shown that the assumption of constant mass ratio during the sound propagation, used previously in the literature for the case of bubbles suspended in a liquid, is only valid for low frequencies. For high frequencies a larger sound speed is obtained. These two sound speeds give two different Mach numbers. It is found that when both Mach numbers are larger than one, the shock structure in a liquid containing bubbles is given by an initial increase of the pressure, followed by a region in which it oscillates around its final equilibrium value. When the low frequency Mach number is larger than one, and the high frequency Mach number smaller

than one, the oscillations disappear and the transition is uniform.

The speed of sound of the mixture is also calculated by evaluating the scattering by the suspended phase.

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I. THEORETICAL INVESTIGATION OF THE REFLECTION OF IONIZING SHOCKS

1.1 Introduction

In a series of shock tube experiments with xenon, Smith (1) found that the end wall pressure was not constant after the reflection of an ionizing shock. Similar results were found by Camac and Feinberg (2) for argon. A typical form of the pressure profiles measured by Smith is shown in fig. 1. In fig. 1 is also given a description of the model used to study this problem.

The incident shock wave propagating in the shock tube, leaves behind a gas at very high temperature. In the equilibrium state this gas must be ionized. However, as was shown by Petschek and Byron (3), this gas does not reach an equilibrium state immediately after the passage of the shock; there is a region (called 2 in fig. 1) in which the gas is almost in a frozen state with zero degree of ionization. After that, in region 6, the gas is supposed to have reached an equilibrium degree of ionization. Between 2 and 6 there is a thin ionization front (fig. 1) (refs. 1, 2, 3, 13).

Behind the reflected shock, a similar process takes place. First, in region 4 the gas is almost frozen and reaches equilibrium after a certain time in the region called 5 in fig. 1. Due to the higher temperatures behind the reflected shock, the equilibrium state is reached faster than behind the incident shock.

The first drop in the end wall pressure (fig. 1) observed by Smith (1), was associated with the transition from frozen to equilibrium state behind the reflected shock. This problem will be studied

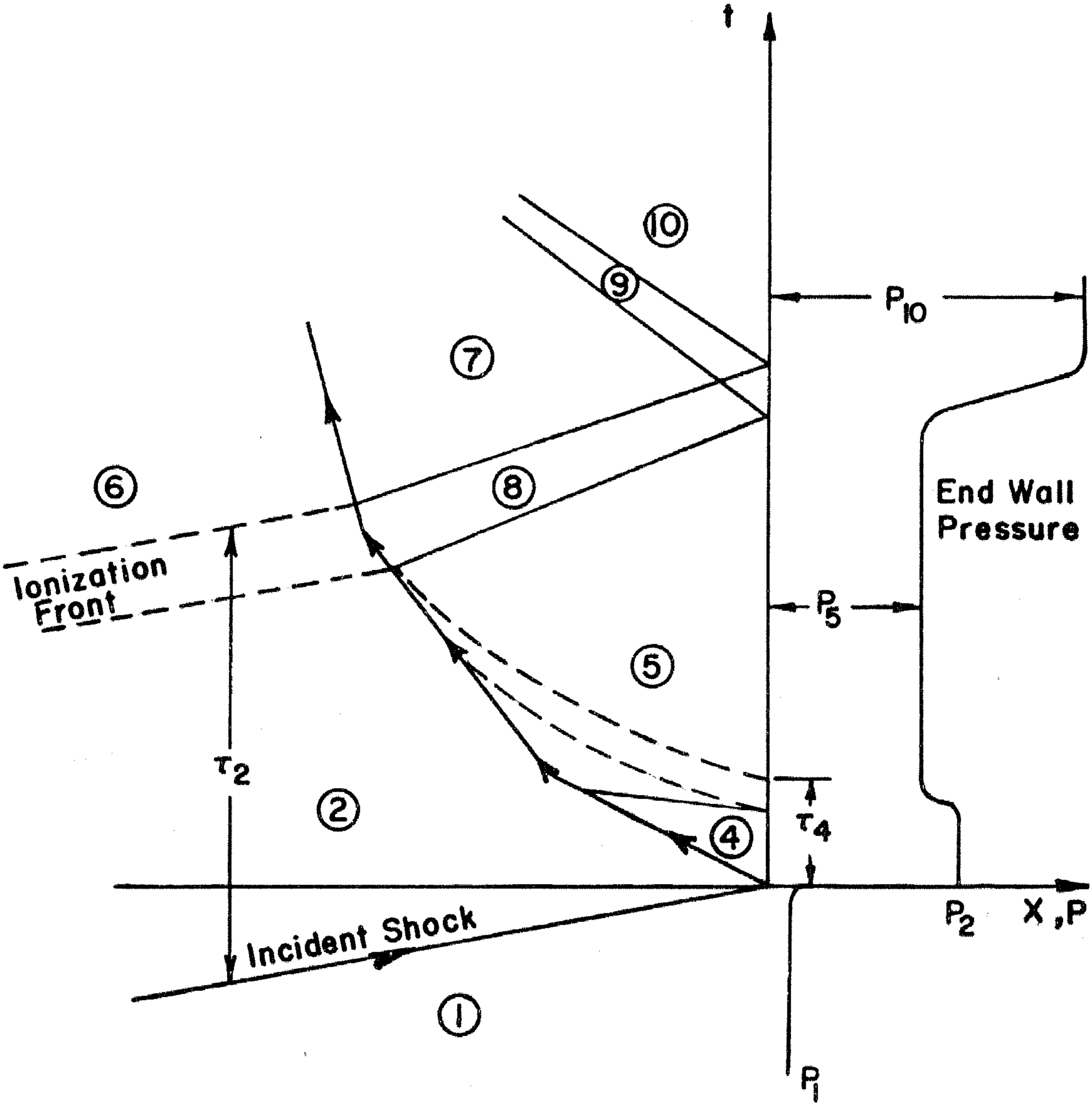


FIG. I (X - t) DIAGRAM SHOWING SHOCK TRAJECTORIES, IONIZATION FRONTS, AND THEIR INTERACTIONS. AT THE RIGHT IS SHOWN A TYPICAL FORM OF THE END WALL PRESSURE PROFILES MEASURED BY SMITH (I).

in 1.2. After the first drop in pressure, Smith (1) observed a region of almost constant pressure followed by an increase. He associated this increase in pressure with a shock wave produced by the interaction of the ionization front (behind the incident shock) with the reflected shock. It will be shown that sometimes, rather than a shock, this interaction produces a family of weak compression waves. In 1.3 we study this problem.

To study the transition from 4 to 5, it will be assumed that the equilibrium degree of ionization is so small that the equilibrium flow properties differ only in first order from the frozen flow properties. This will allow us to linearize the flow equations. However, it will not be possible to linearize the equations governing the production of ions and electrons. In these equations appear terms like "$\exp(-\theta_i/T)$" where θ_i is the ionization temperature and T the temperature of the gas. Terms like this can only be linearized for such small degrees of ionization that the properties of the flow will not experience any significant change. A study similar to this was made by Spence (4). He studied the problem of a piston moving in a gas with such a velocity that the vibrating states were excited (or dissociation produced); however, he linearized the rate equations.

The rate equations governing the production of electrons and ions will be the ones given by Morgan and Morrison (5). Their model can be described as follows: after the reflection of the shock there are no electrons; the first electrons are produced by atom-atom collision according to the model proposed by Harwell and Jahn (6); when enough electrons are produced, the dominating mechanism is electron atom collision (Petschek and Byron (3)). It will be shown that the perturbation in pressure at the wall decreases proportionally to the increase in degree of ionization according to the formula:

$$p_4 - p_{wall}(t) = \frac{2}{3} \frac{(\theta_i/T_4)p_4}{(1+\sqrt{3})} \alpha_{wall}(t)$$

where p_4 and T_4 are the frozen pressure and temperature about which we linearized, θ_i is the ionization temperature, and α is the degree of ionization at the wall. This behavior checks with Smith's results (1) that show a decrease in pressure in the region where α increases, and a constant pressure in the equilibrium region where α is constant.

Camac and Feinberg (10) say that this decrease in pressure is due to an expansion wave preceding the ionization front. If that were so the expansion wave would be in a frozen region and could be considered as an isentropic expansion. We do not think that this is a correct description of what really happens. According to our results the decrease in pressure is due to the fact that the process is not isentropic. The ionization acts like heat sinks distributed over the flow field. These heat sinks decrease the entropy of the atom gas. This decrease in entropy combined with the non-stationary character of the process are the factors that produce the decrease in pressure at the wall.

The compression wave between 5 and 7 is supposed to be at the equilibrium degree of ionization as given by the Saha equation. Our calculations are performed for degrees of ionization of the order of 30%. Under these conditions it is observed that temperature and degree of ionization do not experience any significant changes. This seems to be consistent with the fact that, at such degrees of ionization, γ, the ratio of the specific heats, is close to one. Whitham's rule (7) is used to calculate the second decrease of the reflected shock speed, during its interaction with the ionization front following the incident shock. An extension of Whitham's rule (7) was made by Lick (8); both Whitham's and Lick's rules are, in our problem, equivalent to assuming that the curvature of the shock does not produce a significant change in the entropy of the gas behind the shock. This approximation seems to be good. As the reflected shock curves, it sends a family of compression waves to the wall; if they intersect they produce a shock; it will be shown that sometimes this is not the case. The interaction of this family of compression waves with the wall is studied assuming that temperature and degree of ionization remain constant. It will be seen that this interaction produces an increase in pressure on the end wall. Our results are in good agreement with Smith's measurements regarding the total amount of the increase in pressure and the time at which it happens. But the characteristic time of the pressure rise is longer in Smith's measurements.

1.2 Relaxation Region Behind the Incident Shock

1.2.1 Introductory Remarks

In this section a study will be made of the transition region from 4 to 5 (fig. 1). Region 4 of frozen flow does not exist in reality. The transition to region 5 will start right after the reflection of the shock. However, in the initial stages the ionization is almost negligible, and we can talk about a certain frozen region behind the reflected shock. Similar arguments can be used for regions 2 and 6 behind the incident shock. This last transition is supposed to be much longer than the one behind the reflected shock. The frozen conditions 4 are calculated by the conventional methods of gas dynamics applied to a perfect monatomic gas ($\gamma = 5/3$). We know the conditions in 1 and the Mach number M_1 of the incident shock; conditions 2 will be given by the usual jump conditions across the shock:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)$$

$$u_2 = M_1 a_1 \left[1 - \frac{(\gamma-1)M_1^2+2}{(\gamma-1)M_1^2}\right]$$

p is the pressure, u the flow velocity (relative to the end wall of the shock tube), and a the speed of sound. Similar formulae hold for the other flow quantities in 2; see Liepmann and Roshko (9), Chapter 2. Once we know the flow quantities in 2, we apply the same jump conditions to calculate the flow properties in region 4 where the speed of the gas has to be zero. If c_4 is the speed of the reflected shock:

$$\frac{c_4 + u_2}{c_4} = \frac{1 + \frac{\gamma+1}{\gamma-1}\frac{p_4}{p_2}}{\frac{\gamma+1}{\gamma-1} + \frac{p_4}{p_2}}$$

$$c_4 + u_2 = a_2 \left(\frac{\gamma-1}{2\gamma} + \frac{\gamma+1}{2\gamma}\,\frac{p_4}{p_2}\right)^{\frac{1}{2}}$$

These two equations give us p_4 and c_4. The rest of the flow properties at 4 can be easily calculated from the other jump conditions. For large incident Mach numbers the limiting values of the conditions at 4 take a simpler form:

$$M_1 >> 1$$

$$\frac{p_4}{p_2} = \frac{3\gamma-1}{\gamma-1} = 6 \tag{1a}$$

$$\frac{T_4}{T_2} = \frac{3\gamma-1}{\gamma} = \frac{12}{5} \tag{1b}$$

$$\frac{a_4}{c_4} = \sqrt{\frac{3\gamma-1}{2(\gamma-1)}} = \sqrt{3} \tag{1c}$$

$$\frac{a_2}{c_4} = \sqrt{\frac{\gamma}{2(\gamma-1)}} = \sqrt{\frac{5}{4}} \tag{1d}$$

The movement of the gas in the transition from 4 to 5 will be considered one-dimensional and non-stationary. The x axis will be taken along the shock tube; the origin will be at the end wall. The origin of time will be the moment at which the incident shock hits the end wall (see fig. 1). The problem will be solved for such small degrees of ionization that the flow properties can be considered a small perturbation of the frozen flow properties.

$$\rho = \rho_4 + \rho'(x,t) \qquad \rho'/\rho_4 << 1 \tag{2a}$$

$$u = u'(x,t) \qquad u'/a_4 << 1 \tag{2b}$$

$$T = T_4 + T'(x,t) \qquad T'/T_4 << 1 \tag{2c}$$

$$p = p_4 + p'(x,t) \qquad p'/p_4 << 1 \tag{2d}$$

T is the temperature, and a_4 is the speed of sound of the frozen gas.

The effect of the ionization will be taken into account only in the energy equation. As the monatomic gas becomes ionized, the translational energy of its atoms is transformed into ionization energy. The ionization effect can be considered as a series of heat sinks distributed over the flow field.*

1.2.2 Flow Equations

They will be:

$$\frac{\partial \rho'}{\partial t} + \rho_4 \frac{\partial u'}{\partial t} = 0 \tag{3}$$

$$\rho_4 \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0 \tag{4}$$

$$T_4 \frac{\partial S'}{\partial t} = - \frac{E_{ion}}{m_A} \frac{\partial \alpha}{\partial t} \tag{5}$$

S' is the perturbed entropy of the gas:

$$S = S_4 + S'$$

α is the degree of ionization, E_{ion} is the energy required to ionize an atom, and m_A the mass of an atom. Equation (3) is the continuity equation, (4) the momentum equation, and (5) the energy equation.

The ionization temperature θ_i is defined by:

$$\theta_i = \frac{E_{ion}}{m_A R}$$

R is the perfect gas constant: $R = c_p - c_v$, $c_p/c_v = \gamma = 5/3$

*The electronic excitation of the atoms also acts like another heat sink. This effect could be important in the first stages of the ionization. It has been neglected in this analysis.

Using well-known thermodynamic relations, the energy equation can be written:

$$\frac{\partial \rho'}{\partial t} - \frac{1}{a_4^2}\frac{\partial p'}{\partial t} = \frac{\gamma-1}{\gamma}\,\rho_4\,\frac{\theta_i}{T_4}\,\frac{\partial \alpha}{\partial t} \tag{6}$$

This equation tells us how small α has to be in order for conditions (1) to be satisfied.

$$(\theta_i/T_4)\ \alpha \ll 1 \tag{7}$$

For argon θ_i is 183,000°K, and in our case T_4 is of the order of 10,000°K. Then (θ_i/T_4) is a large quantity, and α being small is not enough for conditions (1) to be satisfied.

If α were known, we would have three equations (3), (4) and (6) with three unknowns u', p', ρ'. We need to find more information about the degree of ionization α.

To obtain equation (6) we used the equation of state:

$$\frac{p}{p_4} = \frac{\rho}{\rho_4}\,\frac{T}{T_4}$$

where p, ρ, and T are the properties of the atom gas. It could be argued that this is not true because of the presence of the electron gas. However, in our case this will only introduce a lower order correction. We should remember that:

$$1 \gg \frac{\theta_i}{T_4}\,\alpha \gg \alpha$$

A similar argument can be used for the continuity equation. Since θ_i/T_4 is so large, the most important effect of the ionization is to take heat out of the atom gas, and any other effect is negligible compared to this.

1.2.3 Equations Determining Ionization Rates

It will be assumed that the mechanisms that determine ionization rates are three: atom-atom collision, electron-atom collision, and recombination. We will give a brief description of these three.

a) Atom-atom collisions. This process is the one that produces the first electrons. It was studied by Harwell and Jahn (6). Two atoms collide, and, if the collision energy is large enough, one of the atoms becomes excited and reaches a certain metastable energy level. An atom that has reached this energy level ionizes very easily. The rate of ionization is then determined by the rate at which the atoms reach this excited state. Morgan and Morrison (5) give an expression for the rate at which electrons are produced by this process:

$$\left(\frac{Dn_e}{Dt}\right)_{aa} = 4S_2\, n_a^2\, (\pi m_a)^{-\frac{1}{2}}\, (kT)^{3/2} \left(\frac{E_1}{2kT} + 1\right) \exp\left(-\frac{E_1}{kT}\right)$$

$$= n_a^2\, R_{aa} \tag{8}$$

$\frac{D}{Dt}$ is the derivative following the gas:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}$$

k is the Boltzmann constant, n_e is the number of electrons per unit volume, n_a the number of atoms per unit volume, m_a the atom mass, E_1 the excitation energy of the metastable state, and S_2 a constant determining the inelastic collision cross section. Harwell and Jahn found that the atom-atom cross section for excitation from the ground state is represented by

$$S_2(\varepsilon - E_1) \text{ for } \varepsilon > E_1; \qquad 0 \text{ for } \varepsilon < E_1 \tag{9}$$

where ε is the energy of the collision in electron volts in the center of mass system. For argon E_2 is 11.7 e.v. and S_2 is 7.1×10^{-19} cm^2/e.v. (according to Harwell and Jahn (6)). However, Morgan and Morrison (5) say that S_2 is perhaps 7.1×10^{-20} cm^2/e.v. Better and more recent experiments by Kelly (16) give an intermediate value of S_2 of 1.2×10^{-19} cm^2/e.v.

b) Electron-atom collisions. This mechanism is described in detail by Petschek and Byron (3). Once the first electrons are produced, this process, which is more effective than the atom-atom collision, takes place. An electron collides inelastically with an atom and the atom reaches a certain metastable state. Then the atom ionizes very easily as in the case of atom-atom collision. The electron loses energy in the inelastic collision and recovers it by elastic collision with atoms and ions. This requires the electron gas to have a smaller temperature than the atom gas. Both ion and atom gas are supposed to have the same temperature, and this temperature is the one that appears in the flow equations (1.2.2). Morgan and Morrison (5) give an expression for the rate of production of electrons by electron-atom collision:

$$\begin{aligned}\left(\frac{Dn_e}{Dt}\right) &= 8n_e n_a S_1 (2\pi m_e)^{-\frac{1}{2}} (kT_e)^{3/2} \left(\frac{E_1}{2kT_e} + 1\right)\exp\left(-\frac{E_1}{kT_e}\right) \\ &= n_e\, n_a\, Re_a \end{aligned} \tag{10}$$

T_e is the temperature of the electron gas. S_1 has a similar meaning to the one given by formula (9). m_e is the mass of the electron. Petschek and Byron (3) give for S_1 the value $7 \times 10^{-18} cm^2/e.v.$ (Argon).

c) Recombination. When the degree of ionization is close to equilibrium, the recombination makes the rate of production of electrons smaller and tend to zero as the degree of ionization tends to the equilibrium value. In this work we will use the three-body recombination model proposed by Camac and Feinberg (10). This process is the opposite of ionization by electron-atom collision:

$$e + e + A_r^+ \rightleftarrows e + A_r \tag{11}$$

Two electrons collide with an ion and produce an electron and an atom. We need the two electrons to satisfy the momentum conservation. For details see Camac and Feinberg (10). According to (11) the rate of production of electrons will be:

$$\frac{Dn_e}{Dt} = k_{ion}\, n_e n_a - k_{rec}\, n_e^2 n_i \tag{12}$$

where k_{rec} is the recombination rate constant and k_{ion} the ionization rate constant. The value of k_{ion} can be found from (10) and (8). In equilibrium

$$(k_{rec})_{eq} = (k_{ion})_{eq} \; \frac{(n_a)_{eq}}{(n_e)_{eq}(n_i)_{eq}}$$

Assuming that the electrical forces are large compared to the dynamic forces:

$$n_e \cong n_i$$

Also from condition (7)

$$n_e << n_a$$

$$\alpha = \frac{n_e}{n_e + n_a} \cong \frac{n_e}{n_a}$$

$$(k_{rec})_{eq} = (k_{ion})_{eq} \frac{1}{n_a} [\alpha_{eq}(p_{eq}, T_{eq})]^{-2} \tag{13}$$

α_{eq} can be determined from the Saha equation as a function of the flow properties at equilibrium, $\alpha_{eq}(p_{eq}, T_{eq})$. Camac and Feinberg (10) assume that relation (13) also holds in non-equilibrium situations, with α_{eq} given by the Saha equation, but with the pressure and the electron temperature that the gas really has in the non-equilibrium conditions:

$$k_{rec} = k_{ion} \frac{1}{n_a} [\alpha_{eq}(p, T_e)]^{-2} \tag{13a}$$

Because of conditions (2) n_a can be considered a constant. There is some arbitrariness in the fact that we have chosen the electron temperature instead of the atom temperature to determine this fictitious α_{eq}. Camac and Feinberg (10) say that the electrons are the ones that determine the ionization rate, and consequently the electron temperature is the significant one (also see Smith (1)).

Combining the three results obtained previously, (8), (10) and (13a), we get:

$$k_{ion} = \frac{n_a}{n_e} (R_{aa} + \alpha R_{ea})$$

$$\frac{Dn_e}{Dt} = n_a^2 (R_{aa} + \alpha R_{ea}) [1 - (\alpha/\alpha_{eq})^2] \tag{14}$$

We see that when α approaches its equilibrium value $\frac{Dn_e}{Dt}$ goes to zero as was expected. The Saha equation gives:

$$\frac{\alpha_{eq}^2}{1-\alpha_{eq}^2} = (3.35 \times 10^{-2} \frac{\text{Newton}}{m^2(^{o}K)^{5/2}}) \frac{2[C_1 + C_2 \exp(-\theta_a/T_e) + \ldots]}{[C_3 + C_4 \exp(-\theta_e/T_e) + \ldots]} \times$$

$$\times \frac{T_e^{5/2}}{p} \exp(-\theta_i/T_e) \tag{15}$$

where C_1 and C_2 are the degeneracies of ground state and first excited state of the ion, and C_3 and C_4 the ones of the atom. θ_a is the excitation temperature of the first excited state of the ion. The contributions of the other excited states are neglected. For the values of these constants see for example Witte (11).

We now face the problem of how to linearize equations (14) and (15). We will also assume

$$T_e = T_4 + T_e' \qquad\qquad T_e'/T_4 << 1 \tag{16}$$

This is to be expected because in equilibrium atoms and electrons have the same temperature, and because of conditions (2) the flow properties are a small perturbation of the equilibrium flow properties. This assumption will be checked later when an equation to determine T_e is found, and when the numerical results of the problem are obtained.

In expressions (8), (10), (14) and (15) appear terms like:

$$\exp(-\frac{\theta_i}{T_e}) = \exp(-\frac{\theta_i}{T_4 + T_e'}) \cong \exp(-\frac{\theta_i}{T_4}) \exp(\frac{\theta_i}{T_4} \frac{T_e'}{T_4})$$

and similar terms with T. The linearization of these terms would require:

$$\frac{\theta_i}{T_4}\,\frac{T'_e}{T_4} << 1 \qquad\qquad \frac{\theta_e}{T_4}\,\frac{T'}{T_4} << 1$$

But these are much stronger conditions than (2) and (7). Since the perturbations are expected to be of order $\alpha(\theta_i/T_4)$, the above conditions would require $(\theta_i/T_4)^2\alpha$ to be small compared to one. Under these circumstances the perturbed flow quantities will be so small that this analysis will become meaningless. We proceed supposing that:

$$\left(\frac{\theta_i}{T_4}\right)^2 \alpha \text{ is not necessarily small.} \tag{17}$$

We can linearize all the terms of equation (14) except the ones containing exponentials of the form indicated. We notice that:

$$\frac{E_1}{kT} = \frac{\theta_e}{T}$$

where θ_e is 135,000°K for argon, that is, of the same order of magnitude of θ_i.

The semi-linearized equation (14) becomes:

$$\frac{\partial\alpha}{\partial t} = A(e^{-\theta_e/T} + Be^{-\theta_e/T_e}\alpha)\,(1 - C\,\alpha^2 e^{\theta_i/T_e}) \tag{18}$$

where A, B, and C are constants

$$A = 4n_4S_2(\pi m_a)^{-\frac{1}{2}}(kT_4)^{3/2}\left[\frac{\theta_e}{2T_4} + 1\right]$$

$$B = 2(S_1/S_2)\,(m_a/2m_e)^{\frac{1}{2}}$$

$$C = \left[3.35 \times 10^{-2}\ \frac{\text{Newton}}{\text{meter}^2(^{\circ}\text{K})^{5/2}}\right]^{-1}\left[\frac{C_3}{2(C_1+C_2\exp(-\theta_a/T_4))}\right]\frac{p_4}{T_4^{5/2}}$$

θ_a is only $2,060°K$ (argon) so it is possible to linearize the exponential containing θ_a. The other terms of the partition functions are ignored because they are small.

We need now an equation to calculate T_e. It will be given by the conservation of energy of the electron-gas. The electrons lose their energy by inelastic collisions with the atoms and recover it by elastic collisions with the ions and atoms. This process was studied in detail by Petschek and Byron (3) and by Morgan and Morrison (5).

The rate of energy transfer by elastic collisions between electrons and ions was calculated by Landau and is given in Morgan and Morrison's paper (5):

$$Q_i = \frac{n_e^2 e^4}{m_i} \left(\frac{8\pi m_e}{kT_e}\right)^{\frac{1}{2}} \frac{(T_i - T_e)}{T_e} \ln\left[\frac{9(kT_e)^3}{8\pi n_e e^6}\right] \tag{18a}$$

where e is the electron charge in Gaussian units:

$$e = 4.8 \times 10^{-10} \text{ cm} \times \text{dyne}^{\frac{1}{2}}$$

T_i is the ion temperature that, in our case, is equal to T (the atom temperature).

The rate of energy transfer by elastic collisions between atoms and ions was calculated by Petschek and Byron (3). We use for it the expression given by Morgan and Morrison (5):

$$Q_a = 4.36 \times 10^{46} \frac{m_e^2}{m_a} \left(\frac{T}{T_e} - 1\right) n_a n_e 4\pi \left(\frac{m_e}{2\pi kT_e}\right)^{3/2} \times$$

$$\int_0^{\infty} S(\varepsilon)\, \varepsilon^{5/2} \exp\left(-\frac{11600\varepsilon}{T_e}\right) d\varepsilon^{\frac{1}{2}} \tag{18b}$$

where all quantities are in the c.g.s. system; except ε (translational energy of the electron) which is expressed in e.v.. $S(\varepsilon)$ is the electron-atom momentum transfer cross-section. For the numerical analysis the values of $S(\varepsilon)$ will be taken from Von Engel (12).

The energy balance for the electrons will be:

$$Q_i + Q_a = \left(\frac{Dn_e}{Dt}\right)_{ea} (E_{ion} + \frac{3}{2} kT_e) \tag{19}$$

(See Morgan and Morrison (5).) We now apply conditions (2) and (16) to this equation. We also assume that in our case the electron temperature is much smaller than the ionization temperature (18a, b, 19):

$$\left[D\alpha \, \ln \left(\frac{E}{\alpha}\right) + F\right] (T - T_e) = ABe^{-\theta_e/T_e}(1 - C\alpha^2 e^{\theta_i/T_e}) \tag{20}$$

where D, E, and F are new constants depending on the unperturbed conditions of region 4. A, B, and C are the constants appearing in equation (18).

We now have a system of four differential equations (3), (4), (6) and (18) with six unknowns p, ρ, u, T, T_e, and α. There are two additional algebraic equations, (20) and the equation of state:

$$\frac{p}{p_4} = \frac{\rho}{\rho_4} \frac{T}{T_4} \tag{21}$$

1.2.4 Boundary Conditions

At the wall the speed of the gas is zero.

$$x = 0, \quad u = 0 \tag{22}$$

Other boundary conditions are given by the Rankine-Hugoniot equations across the reflected shock. Upstream of the reflected shock the conditions are constant, and downstream a small perturbation of the constant frozen conditions 4. We expect that the speed of the reflected shock will be a small perturbation of the frozen speed of the shock c_4. Then, in first approximation, we can apply the boundary conditions on the frozen reflected shock $x = -c_4 t$.

On region 2 we have some fixed values of the flow quantities: p_2, ρ_2, T_2, u_2. The Rankine-Hugoniot equations give the conditions behind the reflected shock as a function of these fixed values and the speed of the reflected shock. Calling this speed c we have:

$$c = c_4 + c' \qquad c'/c_4 << 1$$

$$p_4 + p' = f_1(c_4 + c') \qquad p_4 = f_1(c_4)$$

$$\rho_4 + \rho' = f_2(c_4 + c') \qquad \rho_4 = f_2(c_4)$$

$$T_4 + T' = f_3(c_4 + c') \qquad T_4 = f_3(c_4)$$

$$u' = f_4(c_4 + c') \qquad 0 = f_4(c_4)$$

then:

$$p' = \left(\frac{df_1}{dc}\right)_{c=c_4} c' \tag{23a}$$

$$\rho' = \left(\frac{df_2}{dc}\right)_{c=c_4} c' \tag{23b}$$

$$T' = \left(\frac{df_3}{dc}\right)_{c=c_4} c' \tag{23c}$$

$$u' = \left(\frac{df_4}{dc}\right)_{c=c_4} c' \tag{23d}$$

Eliminating c':

$$p' = \frac{(df_1/dc)_{c=c_4}}{(df_4/dc)_{c=c_4}} u' \qquad \text{on } x = -c_4 t$$

$$T' = \frac{(df_3/dc)_{c=c_4}}{(df_1/dc)_{c=c_4}} p' \qquad \text{on } x = -c_4 t$$

We could obtain a further relation between p' and ρ', but that would be a combination between the last equation and the equation of state. After a lengthy but straightforward manipulation the last two equations reduce to:

$$p' = -k_1 a_4 \rho_4 u' \qquad \text{on } x = -c_4 t \tag{24}$$

$$T' = \frac{T_4}{p_4} k_2 p' \qquad \text{on } x = -c_4 t \tag{25}$$

where:

$$k_1 = \frac{[(\gamma+1)+(\gamma-1)(p_4/p_2)]^{\frac{1}{2}}}{[(p_4/p_2)(\gamma+1)+3\gamma-1]} \left[\frac{2}{\gamma(p_4/p_2)}\right]^{\frac{1}{2}} \left[(\gamma-1)+(\gamma+1)\frac{p_4}{p_2}\right] \tag{26}$$

$$k_2 = 1 + \frac{p_4/p_2}{[(\gamma+1)/(\gamma-1)]+(p_4/p_2)} - \frac{(p_4/p_2)(\frac{\gamma+1}{\gamma-1})}{1+(\frac{\gamma+1}{\gamma-1})(\frac{p_4}{p_2})} \tag{27}$$

For $\gamma = 5/3$

$$k_1 = \left[\frac{4+(p_4/p_2)}{5p_4/p_2}\right]^{\frac{1}{2}} \left[\frac{1+(4p_4/p_2)}{3+(2p_4/p_2)}\right] \tag{28}$$

$$k_2 = 1 + \frac{p_4/p_2}{4+(p_4/p_2)} - \frac{4p_4/p_2}{1+(4p_4/p_2)} \tag{29}$$

And for $M_1 >> 1$ we saw in (1) that p_4/p_2 tends to 6 and

$$k_1 = 0.96 \tag{28a}$$

$$k_2 = 0.64 \tag{29a}$$

The fact that k_1 is so close to one has a very important physical significance:

$$p' \cong -\rho_4 a_4 u' \quad \text{along the shock}$$

This means that the characteristics intersecting shock are absorbed by the shock, and they almost do not reflect. The shock almost behaves like a characteristic (cf. Whitham (7)). It should also be pointed out that k_1 remains close to unity for every value of p_4/p_2 between one and six (see (28)). In particular for $(p_4/p_2) = 1.0$, $k_1 = 1.0$.

Spence (4) used in his analysis the strong shock jump conditions, because he was working with an incident shock. His boundary condition cannot be applied in this work because the reflected shock can never be considered a strong shock ($p_4/p_2 = 6$).

Since the degree of ionization is zero in region 2 we have also:

$$\alpha = 0 \qquad \text{on } x = -c_4 t \tag{30}$$

A further boundary condition is that initially there is no ionization and all quantities are frozen.

$$\text{at } x = 0, \quad t = 0$$

$$\alpha = p' = \rho' = T' = u = 0 \tag{31}$$

Equations (22), (24), (25), (30) and (31) will be our boundary conditions.

If we were interested in knowing the perturbed speed of the shock, we could obtain it from (23a)

$$c' = [1/(df_1/dc)_{c=c_4}]\ p' = k_3 \frac{a_4}{p_4} p'$$

where:

$$k_3 = (2\gamma)^{\frac{1}{2}} \left(\frac{\gamma+1}{4\gamma}\right) \left(\frac{p_4}{p_2}\right)^{\frac{1}{2}} \left[(\gamma+1)+(\gamma-1)\frac{p_4}{p_2}\right]^{-\frac{1}{2}} \tag{32}$$

p' is the perturbed pressure behind the shock. For $\gamma = 5/3$

$$k_3 = \frac{2}{\sqrt{5}} \left(\frac{p_4}{p_2}\right)^{\frac{1}{2}} \left[4 + \frac{p_4}{p_2}\right]^{-\frac{1}{2}} \tag{32a}$$

And for $M_1 >> 1$, p_4/p_2 is 6;

$$k_3 = 0.692 \tag{32b}$$

1.2.5 Numerical Results

The system of equations (3), (4), (6), (18), (20) and (21) with boundary conditions (22), (24), (25), (30) and (31) was solved numerically. To do it the flow equations were put in characteristic form:

$$\left(\frac{\partial}{\partial t} + a_4 \frac{\partial}{\partial x}\right) (p' + a_4\rho_4 u') = -(\gamma-1) \frac{\theta_i}{T_4} p_4\, g(\alpha, T, T_e) \tag{33a}$$

$$\left(\frac{\partial}{\partial t} - a_4 \frac{\partial}{\partial x}\right) (p' - a_4\rho_4 u') = -(\gamma-1) \frac{\theta_i}{T_4} p_4\, g(\alpha, T, T_e) \tag{33b}$$

$$\frac{\partial}{\partial t} \left(\frac{\gamma}{\gamma-1} \frac{T'}{T_4} - \frac{p'}{p_4} + \frac{\theta_i}{T_4} \alpha\right) = 0 \tag{33d}$$

$$\frac{\partial \alpha}{\partial t} = g(\alpha, T, T_e) \tag{33e}$$

g is given in equation (18). The characteristics are the lines:

$x - a_4 t = \text{const.}$

$x + a_4 t = \text{const.}$

$t = \text{const.}$

In this problem the characteristics and the boundaries are fixed straight lines.

In figs. 2, 3, 4, 5 and 6 are represented some of the results obtained from the numerical analysis. As was pointed out before, the value of S_2 (formula (9)) is uncertain. So the calculations were performed for the two different values of S_2 given by Harwell and Jahn (6) and Morgan and Morrison (5).

None of Smith's (1) experiments was done for such low degrees of ionization that would allow linearization. Consequently we cannot compare them quantitatively with our calculations. However, we can see that there is a decrease in pressure that corresponds to an increase in the degree of ionization, and that after a certain time the degree of ionization reaches asymptotically its equilibrium value and the pressure also reaches a constant equilibrium value (figs. 2 and 3). When the equilibrium value is reached (fig. 4) electrons and atoms reach the same temperature; this was to be expected because of the energy balance in the electron gas, see equation (20). In fig. 5 is shown the shock trajectory; the ionization tends to decrease the shock speed; after a certain time the shock moves at a constant equilibrium speed.

A very interesting result is the one given in fig. 6, where the lines of constant degree of ionization are represented. These lines are very nearly parallel to the unperturbed shock trajectory.

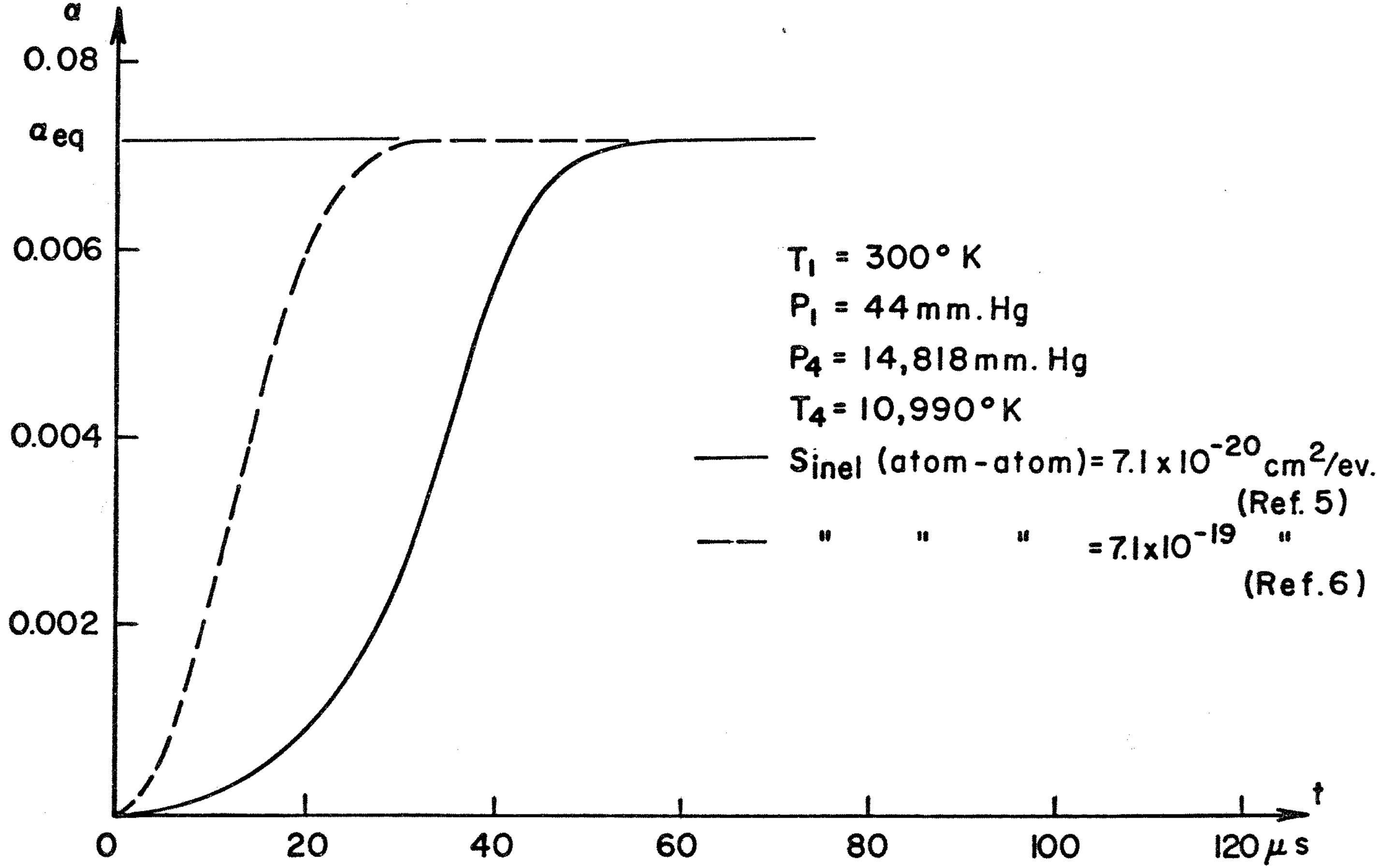


FIG. 2 DEGREE OF IONIZATION AT THE WALL

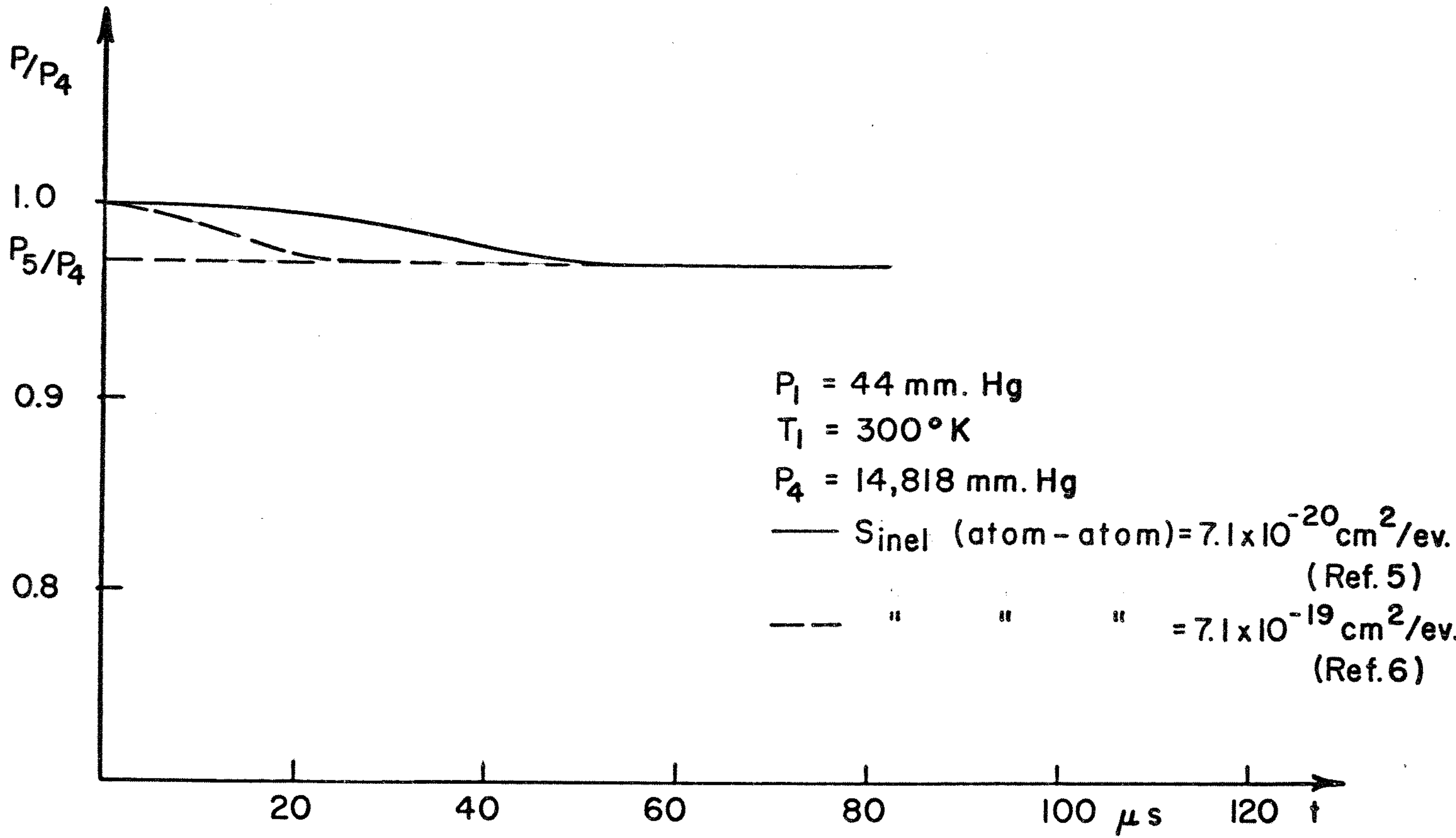


FIG. 3 END WALL PRESSURE DISTRIBUTION

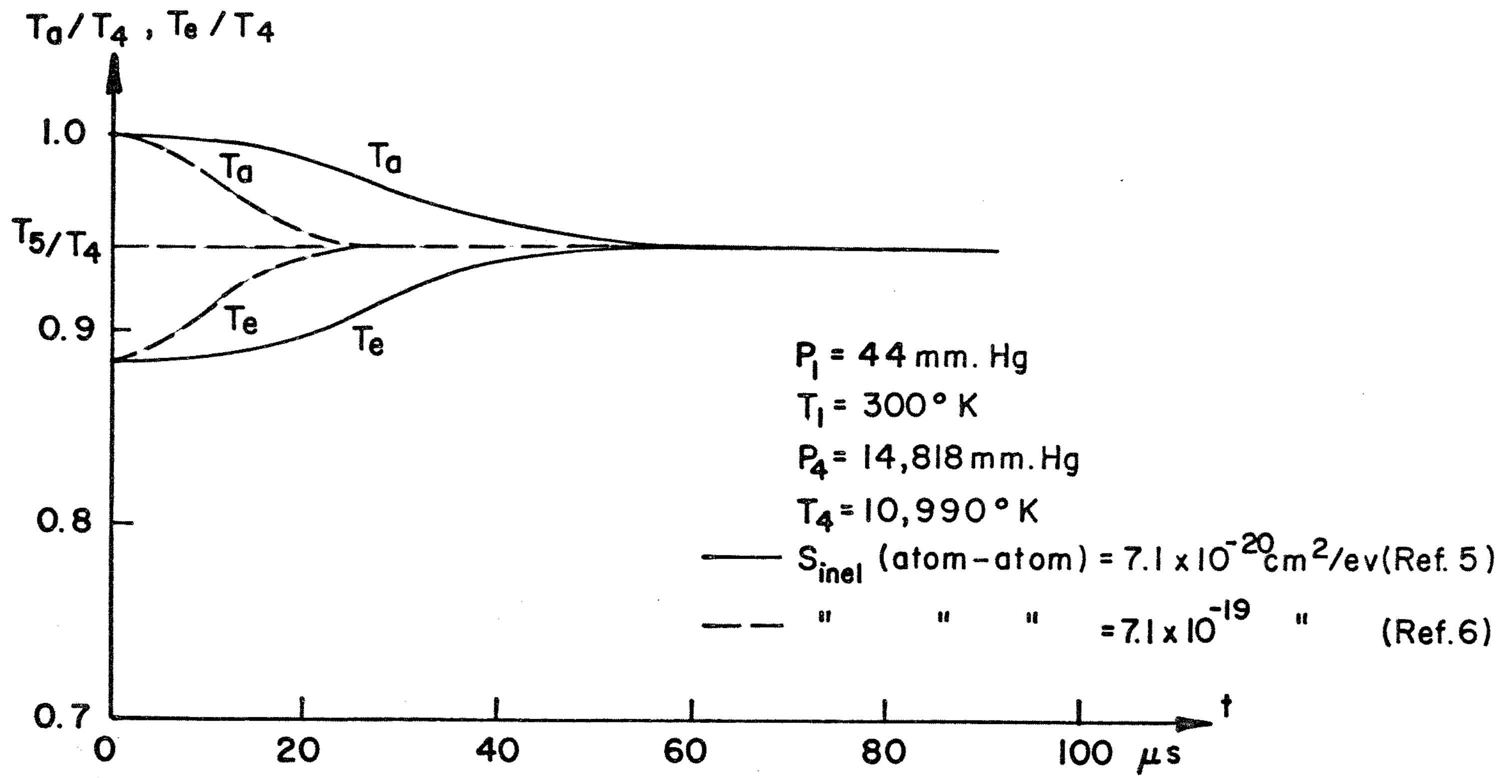


FIG. 4 ELECTRON AND ATOM TEMPERATURES AT THE END WALL

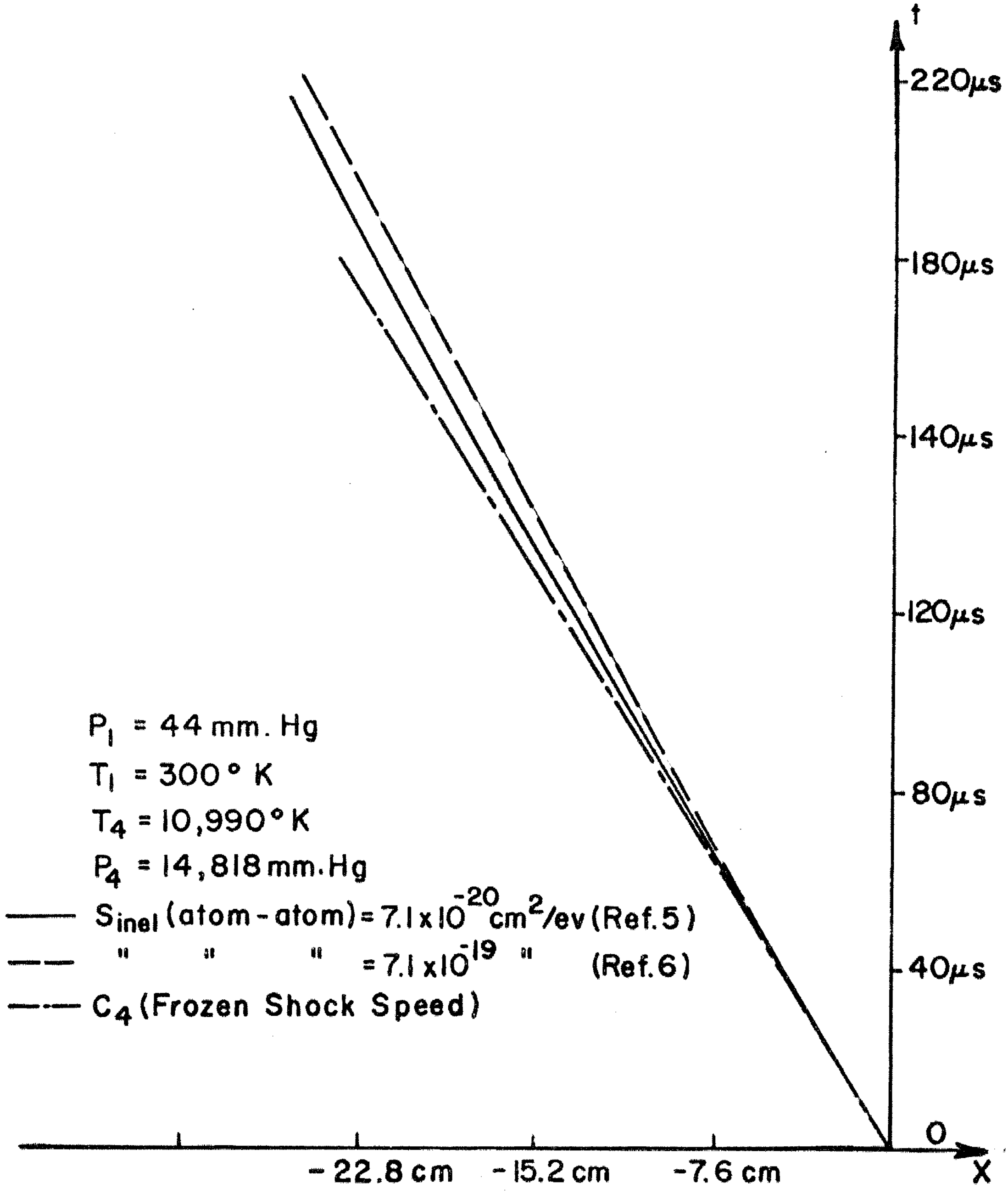


FIG. 5 REFLECTED SHOCK TRAJECTORY

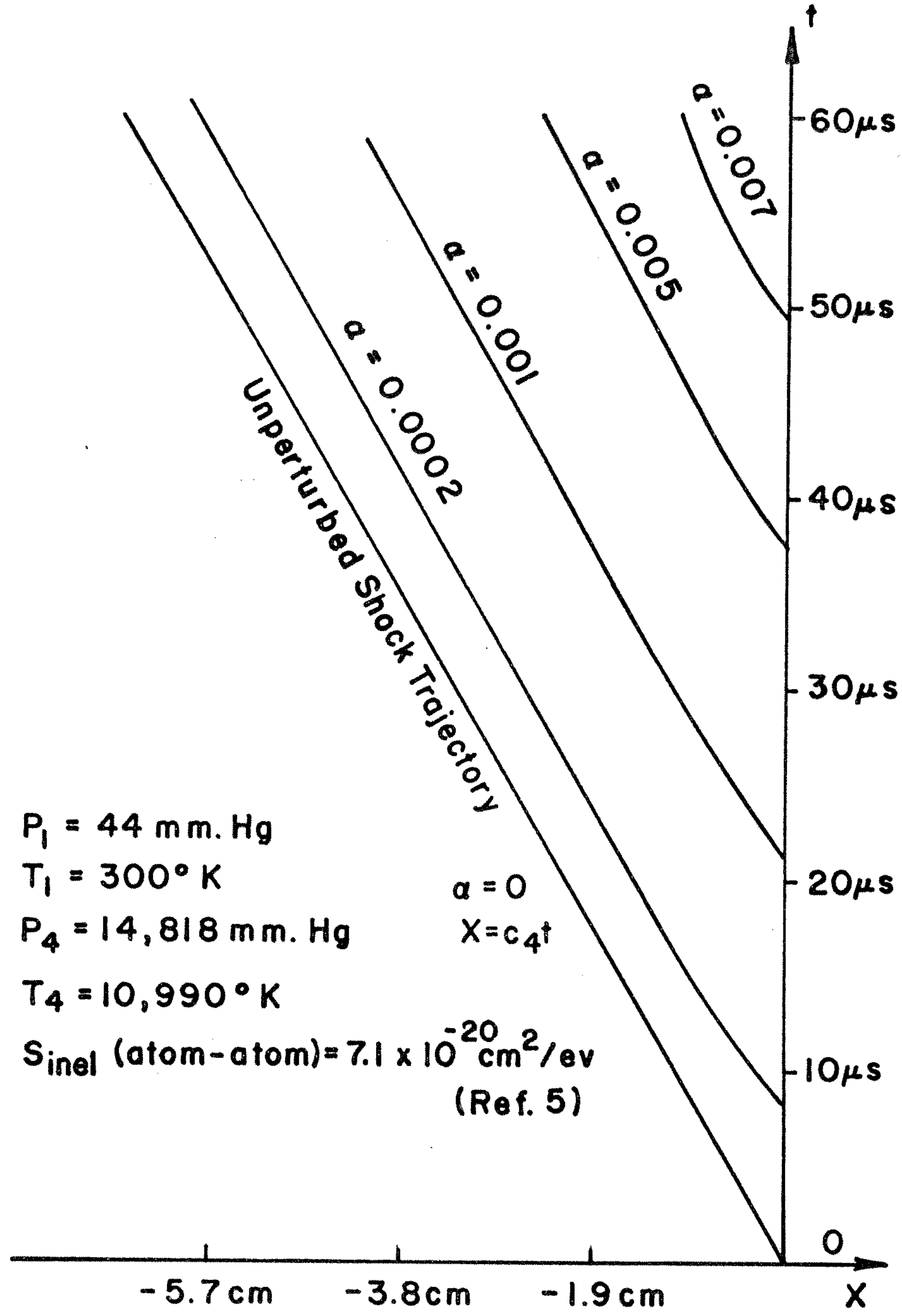


FIG. 6 DISTRIBUTION OF DEGREE OF IONIZATION IN THE X, t PLANE

The reason for this is not very clear. It could be argued that the exponentials of equation (18) can be linearized, then we have:

$$\frac{\partial \alpha}{\partial t} = \text{function } (\alpha)$$

This equation can be integrated independently of the others, and by using the boundary condition (30) we obtain:

$$\alpha = \text{function } (x + c_4 t)$$

This result is similar to the one of fig. 6. However, it can be shown that by doing this, we obtain values of α that are two or three times larger than the ones obtained without linearizing the exponentials. Linearization of the exponentials is incorrect for our problem.

Camac and Feinberg (10) extrapolated the experimental data of Petschek and Byron (3) in order to calculate the relaxation time behind the reflected shock; according to them:

$$\tau_4 = \frac{0.156}{p_1} \left(\frac{\rho_2}{\rho_1}\right) \left(\frac{\rho_2}{\rho_4}\right) \exp \left(\frac{87,000^\circ K}{T_4}\right) \ \mu\text{sec.}$$

p_1 is in mm. Hg. For the conditions of fig. 2 this formula gives a value of 17 μsec. for τ_4 (the relaxation time behind the reflected shock). This value of τ_4 seems to be in good agreement with the profile of fig. 2 that corresponds to

$$S_2 = 7.1 \times 10^{-19} \ \text{cm}^2/\text{e.v.} \qquad \text{(Harwell and Jahn)}$$

In this case our value of τ_4 is about 20 μs. However, in Petschek and Byron's experiments impurities probably played an important role, and they have been ignored in the present analysis.

1.2.6 Approximate Analysis of the Problem

Although the linearization of the exponentials of equation (18) is incorrect in our problem, we can still make an analysis similar to Spence's (4) by using for α the form:

$$\alpha = \alpha\left(t + \frac{x}{c_4}\right) \tag{34}$$

By eliminating ρ' from (6) and using (3) we obtain:

$$\frac{1}{a_4^2}\frac{\partial p'}{\partial t} + \rho_4 \frac{\partial u'}{\partial x} = -\frac{\gamma-1}{\gamma}\frac{\theta_i}{T_4}\rho_4\frac{\partial \alpha}{\partial t} \tag{35}$$

By using (35) and (4) we obtain two separate equations for p' and u'

$$\frac{1}{a_4^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = -\frac{\gamma-1}{\gamma}\frac{\theta_i}{T_4}\rho_4\frac{\partial^2 \alpha}{\partial t^2} \tag{36a}$$

$$\frac{1}{a_4^2}\frac{\partial^2 u'}{\partial t^2} - \frac{\partial^2 u'}{\partial x^2} = \frac{\gamma-1}{\gamma}\frac{\theta_i}{T_4}\frac{\partial^2 \alpha}{\partial t \partial x} \tag{36b}$$

If α were known, the solution of these equations would be:

$$p' = (\gamma-1)\frac{\theta_i}{T_4}\rho_4\frac{c_4^2}{a_4^2-c_4^2}\,\alpha\left(t+\frac{x}{c_4}\right) + \rho_4 a_4 f_5\left(t-\frac{x}{a_4}\right) - \rho_4 a_4 f_6\left(t+\frac{x}{a_4}\right) \tag{37}$$

$$u' = -\frac{(\gamma-1)}{\gamma}\frac{\theta_i}{T_4}\frac{a_4^2 c_4}{(a_4-c_4)^2}\,\alpha\left(t+\frac{x}{c_4}\right) + f_5\left(t-\frac{x}{a_4}\right) + f_6\left(t+\frac{x}{a_4}\right) \tag{38}$$

where we used also the lower order equations (35) and (4) and the functional form (34) for α. f_5 and f_6 are two arbitrary functions to be determined by the boundary conditions. Using (22), (24) and (30), we obtain:

$$f_5(t) + f_6(t) = \frac{\gamma-1}{\gamma} \frac{\theta_i}{T_4} \frac{a_4^2 c_4}{(a_4^2-c_4^2)} \alpha(t)$$

$$\rho_4 a_4 f_5[(1 + \frac{c_4}{a_4})t] - \rho_4 a_4 f_6 [(1 - \frac{c_4}{a_4})t] =$$

$$= -k_1 \rho_4 a_4 f_5 [(1 - \frac{c_4}{a_4})t] - k_1 \rho_4 a_4 f_6 [(1 - \frac{c_4}{a_4})t]$$

These two equations can be put in the form:

$$f_5(z) = \frac{1-k_1}{1+k_1} f_6 [(1 - \frac{c_4^2}{a_4^2}) z]$$

$$\frac{(1-k_1)}{(1+k_1)} f_6 [(1 - \frac{c_4^2}{a_4^2})z] + f_6(z) = \frac{\gamma-1}{\gamma} \frac{\theta_i}{T_4} \frac{a_4^2 c_4}{(a_4^2-c_4^2)} \alpha(z)$$

The solution of these equations is:

$$f_6(z) = \frac{\gamma-1}{\gamma} \frac{\theta_i}{T_4} \frac{a_4^2 c_4}{(a_4^2-c_4^2)} \sum_0^\infty (-1)^n \frac{(1-k_1)^n}{(1+k_1)^n} \alpha[(1 - \frac{c_4^2}{a_4^2})^n z] \tag{39a}$$

$$f_5(z) = \frac{\gamma-1}{\gamma} \frac{\theta_i}{T_4} \frac{a_4^2 c_4}{(a_4^2-c_4^2)} \sum_1^\infty (-1)^{n-1} \frac{(1-k_1)^n}{(1+k_1)^n} \alpha[(1 - \frac{c_4^2}{a_4^2})^n z] \tag{39b}$$

These series can be shown to be uniformly convergent for any finite interval of z. The values of p' and u' are given as functions of x and t by (37), (38) and (39) if α is known.

Our case is different from Spence's (4), because we are considering monatomic gases ($\gamma = 5/3$), and because the reflected shock can never be strong. This makes k_1 very close to one (28a) and allows us to make some simplifications. We can consider f_5 (40) zero and take only the first term in the series expansion for f_6 (39). Then:

$$p' = (\gamma-1)\frac{\theta_i}{T_4}\frac{c_4^2\, p_4}{(a_4^2-c_4^2)}\,\alpha(t+\frac{x}{c_4}) - (\gamma-1)\frac{\theta_i}{T_4}\,p_4\,\frac{a_4c_4}{(a_4^2-c_4^2)}\,\alpha(t+\frac{x}{a_4}) \tag{40}$$

$$u' = -\frac{(\gamma-1)}{\gamma}\frac{\theta_i}{T_4}\frac{a_4^2c_4}{(a_4^2-c_4^2)}\,\alpha\,(t+\frac{x}{c_4}) + \frac{\gamma-1}{\gamma}\frac{\theta_i}{T_4}\frac{a_4^2c_4}{(a_4^2-c_4^2)}\,\alpha(t+\frac{x}{a_4}) \tag{41}$$

and on the wall where x is zero:

$$p'_{wall}(t) = -(\gamma-1)\frac{(\theta_i/T_4)p_4}{(1+a_4/c_4)}\,\alpha(t) \tag{42}$$

$$u' = 0$$

and on the shock:

$$x = -c_4 t$$

$$p'_{shock}(t) = -(\gamma-1)\frac{\theta_i}{T_4}\frac{p_4a_4c_4}{(a_4^2-c_4^2)}\,\alpha[(1-\frac{c_4}{a_4})t]$$

where p'_{shock} is the pressure behind the shock. Using (32) we have for the perturbed speed of the shock:

$$c'(t) = -k_3(\gamma-1)\frac{\theta_i}{T_4}\frac{c_4}{(1-c_4^2/a_4^2)}\,\alpha[(1-\frac{c_4}{a_4})t] \tag{43}$$

Then the speed of the shock is:

$$c = c' + c_4$$

These results will be interpreted in the next section.

1.2.7 Interpretation of the Results

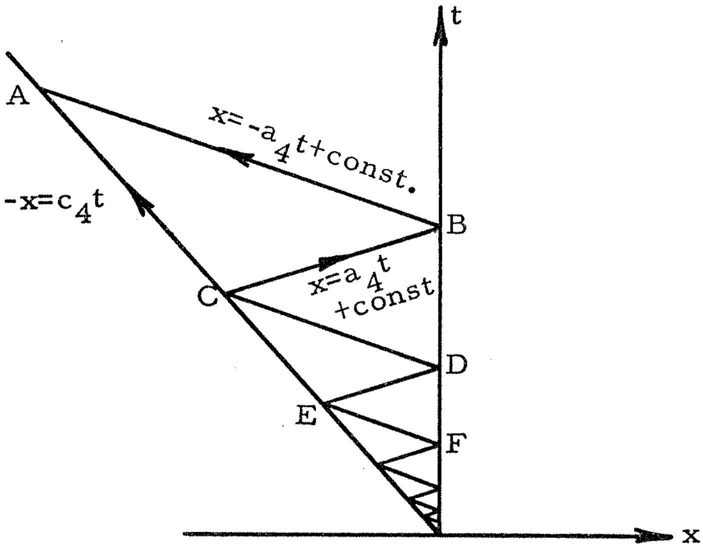
Using for α the functional form (34), equations (33a) and (33b) become:

$$\frac{d}{dt}\left[p' + a_4\rho_4 u' + (\gamma-1)\frac{\theta_i}{T_4}\frac{p_4}{\left(1+\frac{a_4}{c_4}\right)}\alpha\right] = 0 \quad \text{on } x = a_4 t + \text{const.} \tag{44}$$

$$\frac{d}{dt}\left[p' - a_4\rho_4 u' - (\gamma-1)\frac{\theta_i}{T_4}\frac{p_4}{\left(\frac{a_4}{c_4}-1\right)}\alpha\right] = 0 \quad \text{on } x = -a_4 t + \text{const.} \tag{45}$$

Suppose we want to calculate the conditions at point B of the diagram below. B is at the wall and u_B is zero. Using (44) along CB we get:

$$p'_B = (p'_C + \rho_4 a_4 u'_C) - (\gamma-1)\frac{\theta_i}{T_4}\frac{p_4}{\left(1+\frac{a_4}{c_4}\right)}\alpha_B \tag{45a}$$



since α_C is zero. At C (in the shock) we know a relation between p' and u' (24). Using (45) along CD and (24) at C we get:

$$p'_C = \frac{k_1}{1+k_1}\left[p'_D - (\gamma-1)\frac{\theta_i}{T_4}\;\frac{p_4}{\frac{a_4}{c_4}-1}\;\alpha_D\right]$$

We can continue along the characteristics DE, EF, etc. until we get arbitrarily close to the origin where α is zero. It is easy to check that the values of p_B or p_C obtained in this way correspond to the series expansions obtained previously ((37), (39) and (40)). However, the boundary condition (24) tells us that:

$$p'_C + \rho_4 a_4 u'_C \simeq 0$$

We don't need to calculate the values of the flow quantities at C, D, E, F, etc. All the information we need is contained in the characteristic CB; the characteristic CD is almost totally absorbed by the shock and does not affect what happens along CB. This simplification corresponds to taking only the first term in the series expansion (39). Formula (45a) gives the same result obtained in the previous section (42). This tells us that the perturbation in pressure at the wall is proportional to the degree of ionization that there is at that time on the wall, and that the pressure decreases as the ionization increases. These results are in agreement with Smith's (1) experiments.

A similar argument can be used to calculate the conditions at A behind the shock. In that case the only relevant characteristics will be AB and BC, because of the above arguments. So the conditions at A can only be affected by the degree of ionization at B;

this checks with formula (43) which can be written:

$$c'_A = -k_3\ (\gamma-1)\,\frac{\theta_i}{T_4}\,\frac{c_4}{\left(1-\frac{c_4^2}{a_4^2}\right)}\,\alpha_B$$

We can say that the perturbation in the speed of the shock is proportional to the degree of ionization at the point of the wall that is on the same C^- characteristic as the point in the shock.

In all these formulae we could use approximations (1). For example, c_4/a_4 is $1/\sqrt{3}$.

It is also interesting to note that all perturbations of the flow quantities are proportional to the factor $(\theta_i/T_4)\alpha$. This checks the validity of the perturbation method and assumption (7).

We must finally note that the analysis of sections 1.2.6 and 1.2.7 is subject to the condition that $\alpha(x+c_4t)$ is known. To do this we think the only possible method is the numerical solution indicated in 1.2.5.

1.3 Interaction of the Reflected Shock with the Ionization Front Following the Incident Shock

1.3.1 Introductory Remarks

We saw in part 1.2 that the reflected shock, after a certain time, reaches an equilibrium speed and leaves behind an ionized gas in equilibrium. There is experimental evidence that the ionization time behind the reflected shock is shorter than behind the incident shock. The reason is that behind the reflected shock the temperature is higher. (Smith (1), Camac and Feinberg (10)). To study this problem we will assume that the gas behind the incident shock is relaxing and behind the reflected shock is in equilibrium. In the transition from 2 to 6 (fig. 1) the gas is relaxing, and in 5, 7, 8, 9 and 10 the gas is in equilibrium. This approximation will be exact for τ_1/τ_4 going to infinity. The experiments of Smith (1) and Camac and Feinberg (10) seem to indicate that this ratio is only of order five. However, as the ionization degree increases behind the incident shock new electrons are produced, and as the reflected shock advances it meets regions where there are more electrons. We saw in part 1.2 that the initial and much longer stage of the ionization process is by atom-atom collision, and that this process ceases to be important as soon as there are enough electrons for the electron-atom collision process to take over. Consequently it is to be expected that when the reflected shock meets regions where there are already electrons, the ionization time will be much smaller than τ_4. Then the assumption of equilibrium conditions behind the reflected shock is greatly improved.

We now want to calculate the structure of the ionization front behind the incident shock. These fronts have been studied theoretically and experimentally by many people (Morgan and Morrison (5), Smith (1), Petschek and Byron (3) and Wong and Bershader (13)). This problem is similar to the one worked out in section 1.2, but simpler, because by taking coordinates fixed to the shock the flow becomes stationary. Wong and Bershader (13) measured the ionization profile behind the incident shock. They give $\alpha(\tau)$; τ is the distance in laboratory time behind the incident shock

$$\tau = t - x/c_1$$

c_1 is the speed of the incident shock (fig. 1). In the numerical analysis we will use the experimental values of α obtained by Wong and Bershader (13) for Argon and Smith's (1) theoretical results for Xenon.*

Since we know the values of α behind the incident shock, we can calculate the rest of the flow properties there. Petschek and Byron (3) give these expressions:

$$p_a = p_2 \tag{46}$$

$$T_a = T_2 - \frac{2}{5}\alpha_a \theta_i \tag{47}$$

$$\rho_a = \rho_2 \left(1 - \frac{2}{5}\alpha_a \frac{\theta_i}{T_2}\right)^{-1} \tag{48}$$

$$u_a = M_1 a_1 \left(\frac{3}{4} + \frac{1}{10}\alpha_a \frac{\theta_i}{T_2}\right) \tag{49}$$

These equations represent conservation of momentum (46), energy (47), and mass (49). By using the equation of state we obtain (48). The degree of ionization is supposed to be small, but not the products $\frac{\theta_i}{T_2}\alpha_a$ (compare with formula 2). The subscript "a" represents any point behind the incident shock, and subindex 2 represents frozen conditions right after the passage of the shock. Conditions 2 are calculated by the usual Rankine-Hugoniot relations (see section 1.2.1).

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*Some of these results were given to us by Smith in a private communication.

Region 2 does not exist in reality. The ionization degree is never zero except immediately after the passage of the shock; however, it is close to zero for some time. This allows us to define region 2.

We know all the information about the ionization front following the incident shock and can study its interaction with the reflected shock.

1.3.2 Trajectory of the Reflected Shock

We have the problem of a shock propagating in a region of known but not constant properties, and we want to calculate the trajectory of the shock and the properties of the gas behind it.

Across the shock we have conservation of mass, momentum and energy.

$$\rho_a(u_a+c) = \rho(u+c) \tag{50}$$

$$p_a+\rho_a(u_a+c)^2 = p + \rho(u+c)^2 \tag{51}$$

$$h_a + \tfrac{1}{2}(u_a+c)^2 = h + \tfrac{1}{2}(u+c)^2 \tag{52}$$

p_a, ρ_a, h_a and u_a are known as functions of the position of the shock. In particular (see (46), (47), (48) and (49)):

$$h_a = c_pT_a + R\theta_i\alpha_a = h_2 = \text{const.} \tag{53}$$

ρ, u, p and h are the density, velocity, pressure and enthalpy of the gas behind the reflected shock. c is the speed of the reflected shock. These five quantities are unknown.

The gas behind the reflected shock is in equilibrium, consequently its properties are related by:

$$h = (1+\alpha)\, c_p T + \theta_i\, \alpha R \tag{54}$$

$$p = (1+\alpha)\, R\rho T \tag{55}$$

$$\alpha = \alpha(p, T) \tag{56}$$

The last equation is the Saha equation, similar to equation (15). (In our case the gas is in equilibrium and electrons and atoms have the same temperature.) Behind the reflected shock the degree of ionization is higher than behind the incident shock; then α is not necessarily small behind the reflected shock.

For argon the coefficients of equations (54) and (55) are

$$R = \frac{2}{5} c_p = 2.08299 \times 10^2 \; \frac{m^2}{sec^2\; {}^oK}$$

To find a further condition along the shock we are going to use Whitham's theory (7). Whitham says that behind a shock that is propagating in a non-uniform region, the same relation that holds along the characteristics intersecting the shock also holds along the shock. In our case (see, for example, Courant and Friedrichs (14), Chapter III):

$$dp - \rho a\, du = 0 \tag{57}$$

where "a" is the speed of sound of the ionized gas in equilibrium. This relation is not exactly true. Let us consider fig. 7. During the time that the shock meets the frozen gas of region 2 which has constant properties, there is a uniform region 5 behind the shock.

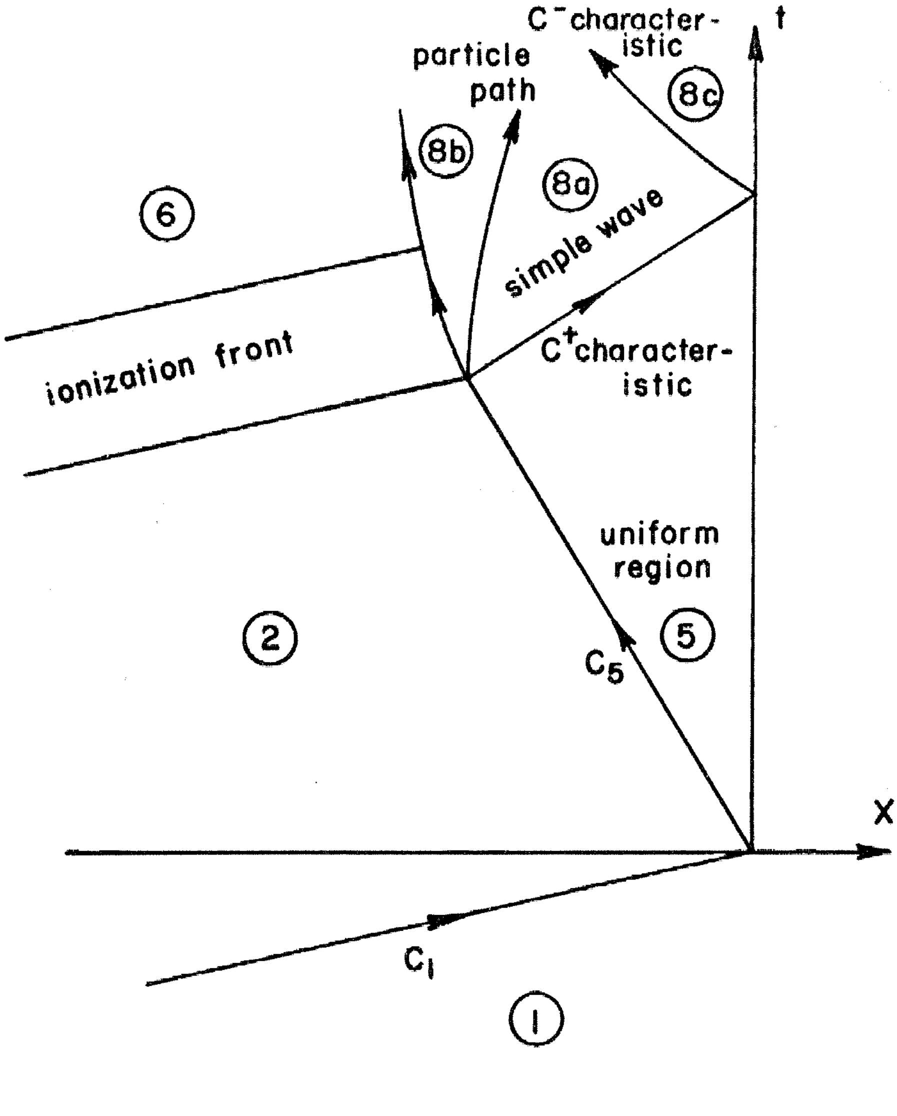


FIG. 7 X - t DIAGRAM THAT DESCRIBES THE INTERACTION OF THE IONIZATION FRONT WITH THE REFLECTED SHOCK

As the shock meets the transition region behind the incident shock it starts to bend. We then have, as indicated in fig. 7, region 8a which is a simple-wave, region 8b which is non-isentropic because of the curvature of the shock, and region 8c which is isentropic but not a simple wave. Along the C^- characteristics relation (57) holds, and since region 8a is isentropic we can integrate equation (57):

$$u - \int_{p_5}^{p} \frac{dp}{\rho a} = 0 \tag{58}$$

This is satisfied over the whole region 8a. The integral should be calculated at constant entropy; we will explain later how this can be done in the case of ionized gases in equilibrium; at the moment we just notice that for constant entropy ρ and "a" can be expressed as functions only of p. Since equation (58) holds over the whole region 8a, differentiating it we find that (57) also holds over the whole region 8a, because p_5 is constant. However, (57) does not hold on the whole region 8b, but holds only along the C^- characteristics of that region, because 8b is not isentropic and the integral relation (58) does not hold there. Relation (57) proposed by Whitham (7) is then only an approximation. Lick (8) extended Whitham's theory and said that relation (57) holds over the whole non-isentropic region behind the shock. Lick's theory is equivalent in our case to assuming that the region adjoining the shock is also a simple wave. The justification of these assumptions is not very clear. For further information the reader is referred to the works of Whitham (7) and Lick (8). Our numerical results will also confirm these assumptions.

The speed of sound of an ionized gas in equilibrium is (Witte (11)):

$$a^2 = \left(\frac{dp}{d\rho}\right)_{S=\text{const}} = \frac{c_p}{\left[c_p\left(\frac{\partial\rho}{\partial p}\right)_T - \frac{T}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_p\right]}$$

$$= \frac{5}{3}\,\frac{p}{\rho}\,\frac{1+\frac{5}{4}\left(1+\frac{2}{5}\frac{\theta_i}{T}\right)^2\alpha(1-\alpha) + \frac{1}{2}\left(1+\frac{2}{5}\frac{\theta_i}{T}\right)\alpha(1-\alpha)\psi(T)}{1+\left[\frac{5}{4}+\frac{\theta_i}{T}+\frac{1}{3}\left(\frac{\theta_i}{T}\right)^2\right]\alpha(1-\alpha)+\frac{1}{2}\left(1+\frac{2}{3}\frac{\theta_i}{T}\right)\alpha(1-\alpha)\psi(T)} \tag{59}$$

where

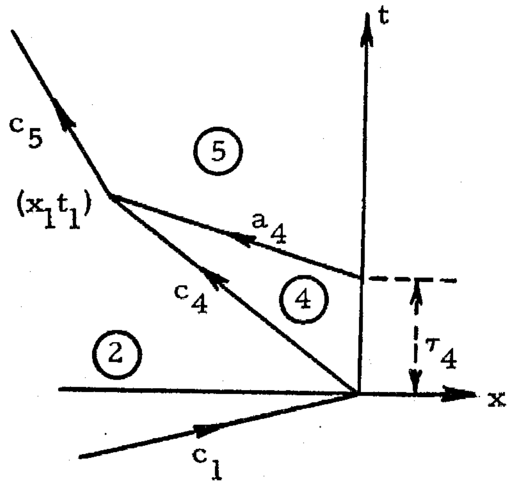
$$\psi(T) = \frac{C_2\theta_a}{T}\,\frac{\exp(-\theta_a/T)}{C_1+C_2\exp(-\theta_a/T)} - \frac{C_4\theta_e}{T}\,\frac{\exp(-\theta_e/T)}{C_3+C_4\exp(-\theta_e/T)}$$

The term in $\psi(T)$ does not appear in Witte's paper; it is given by the contributions of the first excited states of the atom and of the ion atom. Formula (59) is obtained by using (55) and (56).

The position of the shock is given by

$$dx/dt = -c \tag{60}$$

In fig. 7 it is assumed that the reflected shock propagates from the beginning with a speed c_5 that leaves behind a gas in equilibrium and at rest. We know that this is not true. There is a certain time, τ_4, during which the ionization degree at the wall is zero. The first particles that are ionized are the ones close to the wall, because they are the ones that have been for a longer time in region 4. The shock will propagate with the speed c_4 that leaves behind



a gas frozen and at rest until it feels the effect of the ionization. This will happen when the frozen characteristic leaving the wall at the time τ_4 meets the shock. Instead of supposing that the reflected shock starts propagating with speed c_5 from the origin, we will suppose that it propagates with that velocity from a point of coordinates $(x_1\ t_1)$:

$$t_1 = \frac{\tau_4}{\left(1 - \frac{1}{\sqrt{3}}\right)} \tag{61a}$$

$$x_1 = \frac{c_4\,\tau_4}{\left(1 - \frac{1}{\sqrt{3}}\right)} \tag{61b}$$

$$a_4/c_4 = \sqrt{3} \qquad \text{(see 1c)}$$

The values of τ_4 were taken from Smith (1) for xenon and Camac and Feinberg (2) for argon. This approximation cannot be thought to be a rigorous estimation of the effect of ionization behind the reflected shock. It should be regarded rather as an improvement over considering the shock propagating with the equilibrium speed from the origin. The way to calculate c_4 and a_4 was indicated in 1.2.1.

To calculate the initial speed of the shock c_5 and the initial flow properties ρ_5, p_5, h_5, we just solve the algebraic equations (50), (51), (52), (54), (55) and (56), taking the initial velocity u_5 equal to zero and the properties ahead of the shock p_a, ρ_a, h_a, u_a equal to the frozen values p_2, ρ_2, h_2, u_2. The initial conditions will be:

$$t = t_1,\ x = x_1,\ c = c_5,\ \rho = \rho_5,\ h = h_5,\ p = p_5,\ \alpha = \alpha_5,$$
$$T = T_5,\ a = a_5,\ u = 0$$

We can now solve the system of equations (50), (51), (52), (54), (55), (56), (57), (59) and (60) for x, c, ρ, h, p, α, T, a and u as functions of time with the initial conditions given above. This was done numerically. The initial conditions were also calculated numerically.

In fig. 8 are represented some of the calculated shock trajectories. In Table I are given the calculated flow properties behind the shock.

An interesting result is that behind the reflected shock the values of the degree of ionization, temperature, enthalpy, and speed of sound remain nearly constant, and consequently the ratio p/ρ also remains constant. This result seems to hold for all the numerical cases solved. The values of the degree of ionization varied between 20% and 40% for the different examples worked out.

Along the shock the pressure behind it increased with time. In the different cases considered, the total increase in pressure

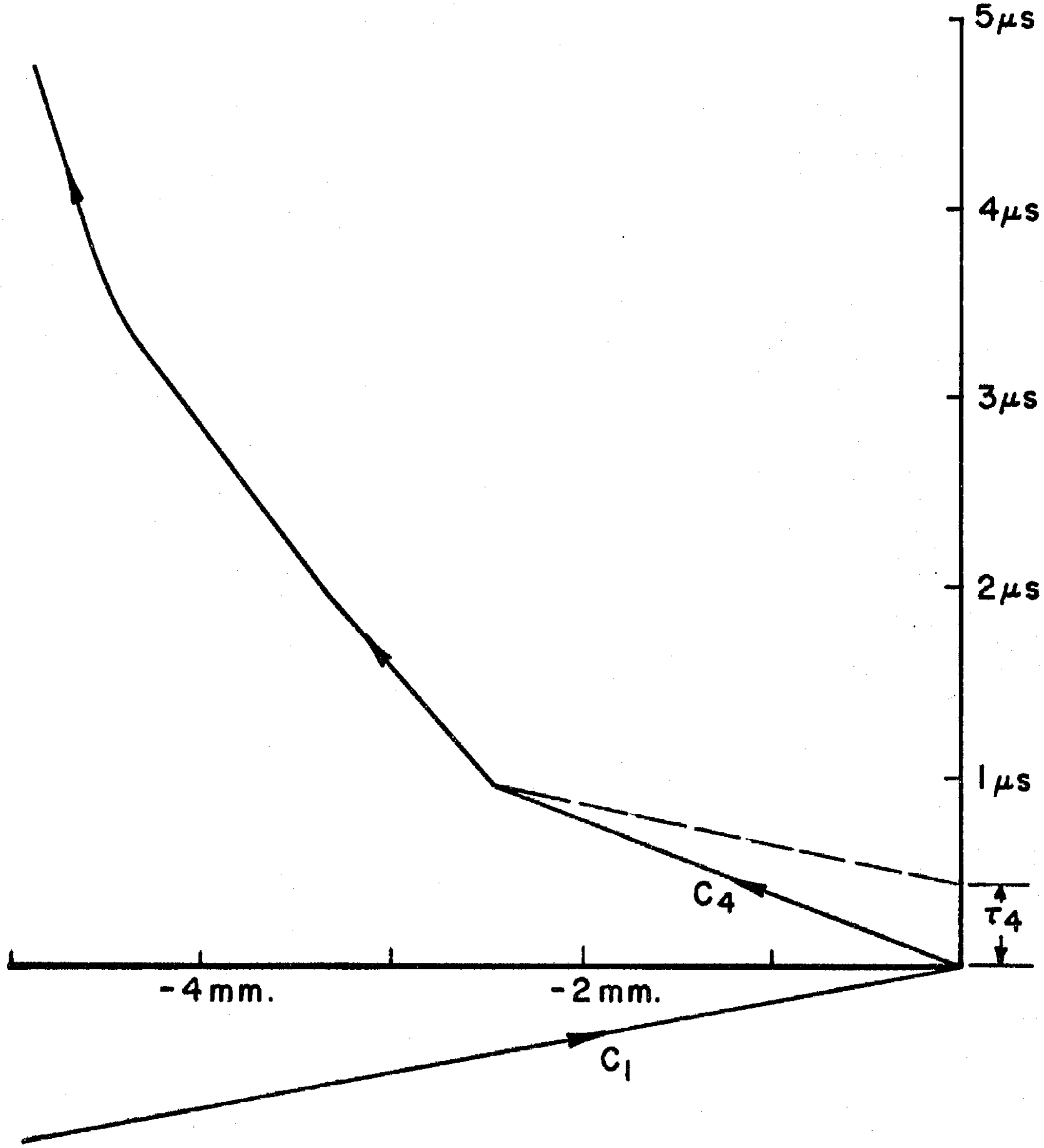


FIG. 8a REFLECTED SHOCK TRAJECTORY, a TAKEN FROM REF. 13, $M_1 = 16.3$, $P_1 = 3$mm. Hg, $T_1 = 300$°K, ARGON

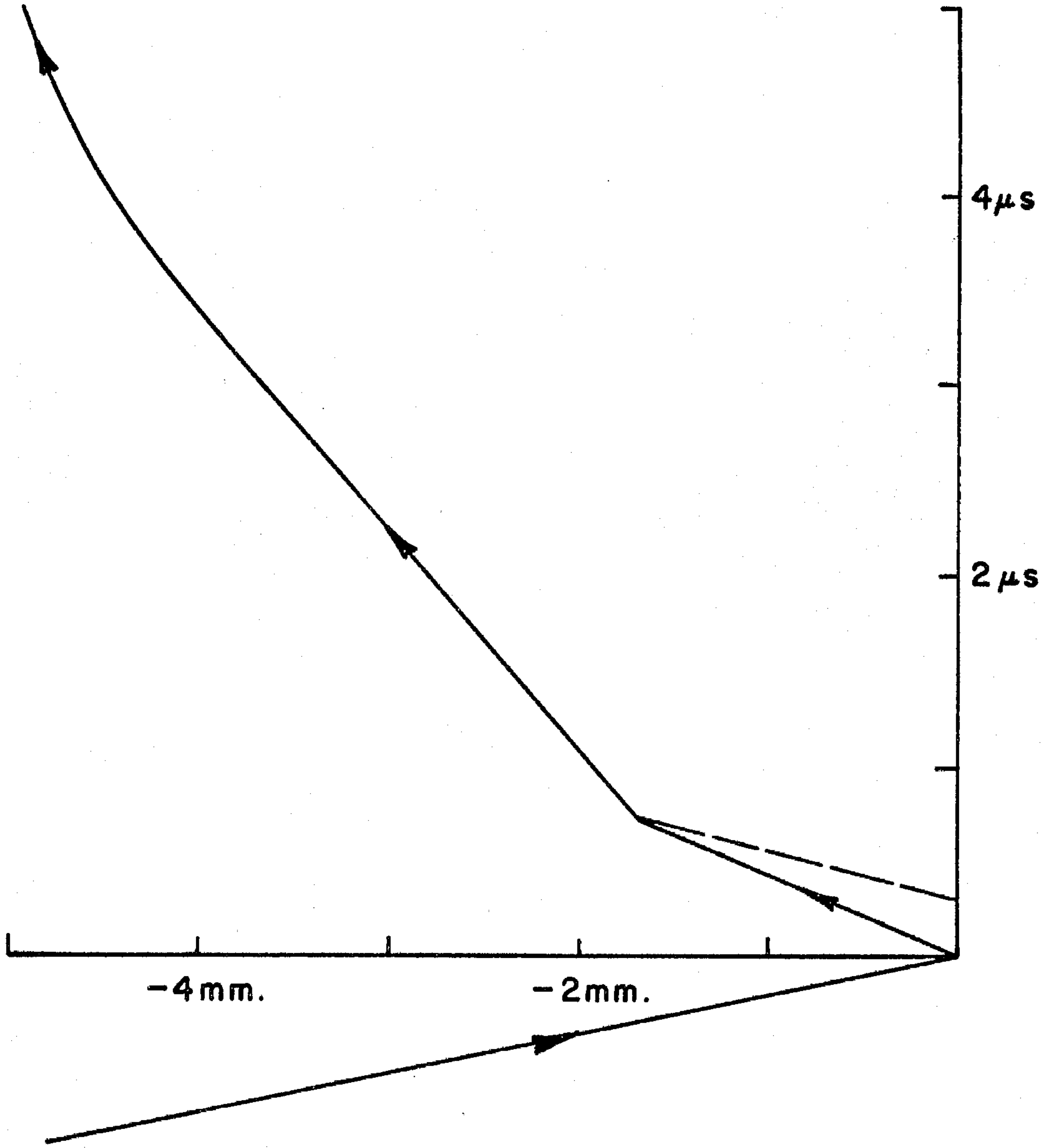


FIG. 8b REFLECTED SHOCK TRAJECTORY, a TAKEN FROM REF. 13. M_1 = 15.0, P_1 = 5mm. Hg, ARGON

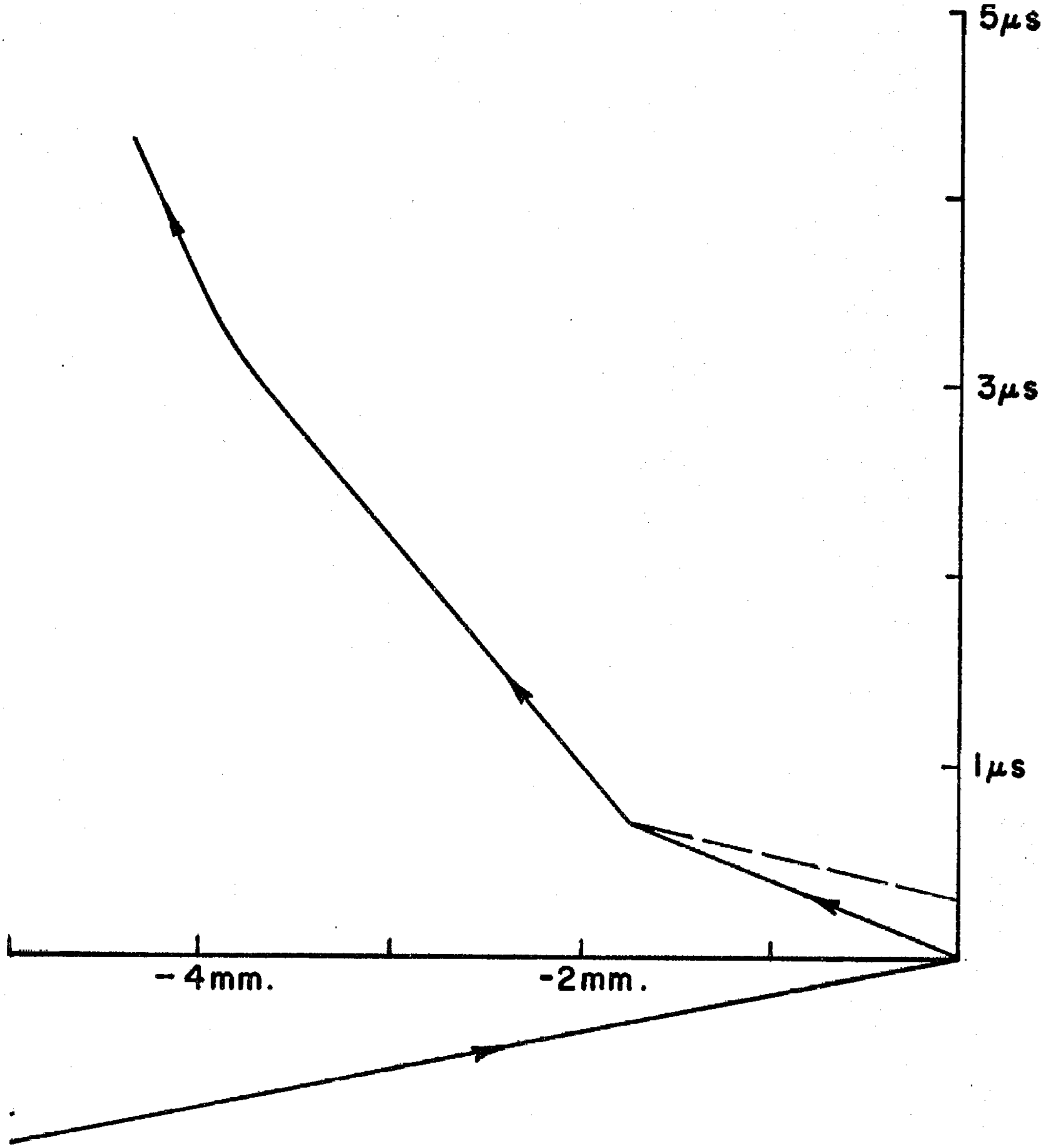


FIG. 8c REFLECTED SHOCK TRAJECTORY, a TAKEN FROM REF. 13. $M_1 = 15.4$, $P_1 = 5$mm. Hg, ARGON

Table I. Flow properties behind the reflected shock as functions of the shock position

$M_1 = 16.3$, $p_1 = 3$ mm. Hg., Argon

x mm.	t μs	u mm/μs	p mm.Hg.	α	a mm/μs	T °K
2.75	1.27	0	4644	0.348	2.25	15,509
3.53	2.20	0.06	4782	0.348	2.25	15,539
4.01	2.79	0.10	4895	0.348	2.25	15,562
4.30	3.18	0.20	5152	0.347	2.26	15,610
4.41	3.35	0.40	5704	0.343	2.26	15,696
4.55	3.66	0.67	6538	0.336	2.26	15,795
4.58	3.74	0.79	6942	0.332	2.26	15,834

$M_1 = 15.4$, $p_1 = 5$ mm. Hg., Argon

2.27	1.30	0	6991	0.297	2.21	15,567
2.97	2.11	0.05	7186	0.297	2.21	15,594
3.46	2.69	0.08	7299	0.296	2.22	15,609
3.69	2.98	0.15	7554	0.296	2.22	15,641
3.86	3.22	0.35	8387	0.292	2.22	15,729
3.92	3.32	0.42	8727	0.291	2.22	15,760
3.97	3.41	0.58	9464	0.287	2.22	15,816
4.00	3.49	0.65	9802	0.285	2.23	15,840

Table I (Cont'd)

$M_1 = 15.1$, $p_1 = 0.5$ mm. Hg., Xenon

x mm.	t μs	u mm/μs	p mm.Hg.	α	a mm./μs	T °K
7.96	6.13	0	655	0.410	1.08	11,400
8.23	6.85	0.03	679	0.410	1.08	11,428
8.39	7.30	0.10	730	0.409	1.09	11,478
8.46	7.56	0.27	869	0.402	1.09	11,582
8.47	7.64	0.37	967	0.396	1.09	11,636
8.48	7.70	0.41	1004	0.393	1.09	11,654
8.49	8.11	0.46	1060	0.390	1.09	11,679

varied between 20% and 80% of the initial pressure.

The speed of the shock decreases with time. The interaction of the ionization front with the shock bends the shock towards the wall (because the density is increasing ahead of the shock).

The speed of the gas behind the shock, which initially is zero, is towards the wall and increases in magnitude as the shock bends. This means that there is a mechanism to stop the gas, because the velocity at the end wall must be zero. To study this phenomena we have to consider regions 8a, 8b and 8c (fig. 7); this will be done in the following sections.

1.3.3 Flow Field Behind the Reflected Shock

It is well known that an ionized gas in equilibrium tends to behave as if γ, the ratio of the specific heats, were close to one. This means that, if the gas behaves isentropically, the classical relation:

$$p/\rho^{\gamma} = \text{const.}$$

will tend to (p/ρ) equal to constant. If the degree of ionization and the temperature of the gas remain constant, it is clear (from (55)) that this relation holds. The entropy of equilibrium ionized argon is: (Witte (11))

$$S/R = (1+\alpha)\,\ell n\,\frac{T^{5/2}}{p} + \ell n\left[\frac{(1+\alpha)^{1+\alpha}}{(1-\alpha)^{1-\alpha}}\ \alpha^{-2\alpha}\right] - 10.354\alpha \quad (62)$$

We claim that for moderate changes in pressure (80%) and high degrees of ionization, constant entropy means constant temperature and constant degree of ionization. From the expression above we see that the term that gives larger changes in entropy is 10.354α. So if the entropy is going to be constant, α should also remain almost constant. On the other hand, if we differentiate the Saha equation (15) we obtain:

$$d\alpha = \left(\frac{1-\alpha^2}{2}\right)\left[\frac{\theta_i}{T} + \frac{5}{2} + \psi(T)\right]\frac{dT}{T} - \left(\frac{1-\alpha^2}{2}\right)\alpha\,\frac{dp}{p}$$

$\psi(T)$ is given in (59). By far the biggest changes in this expression will come from the term containing (θ_i/T). This term corresponds to the exponential of the Saha equation. If α is going to be constant then T also has to be almost constant. Even more, if we examine equation (59) of the speed of sound and take limits for (θ_i/T) large, and assume that α is not small, we get:

$$a^2 \rightarrow p/\rho$$

But if we assume that at constant entropy α and T are constants we get also:

$$\frac{p}{\rho} = R\ (1+\alpha)\ T = \text{const.}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_S = \frac{p}{\rho} = a^2$$

This can also be checked by looking at fig. 5 of Witte (11).

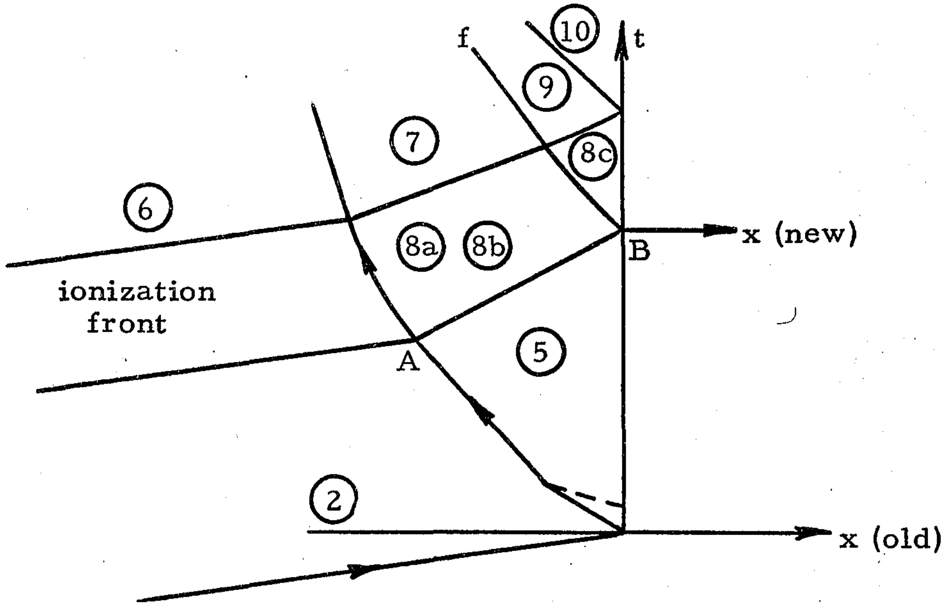
Now we can see that Whitham's (7) and Lick's theories give us a very good approximation. In part 2.2 we saw that behind the reflected shock the relation:

$$\frac{p}{\rho} = \text{const.}$$

held, and that α, T and "a" remained almost constant. This means that the curvature of the shock does not change very much the entropy behind the shock. Then regions 8a and 8b can be considered as a single simple-wave region where the C^+ characteristics are straight lines along which the flow properties remain constant. Since the conditions behind the shock are known, we also know all the flow properties in regions 8a and 8b.

The C^+ characteristics are convergent lines because their slope is given by (a+u), and u increases along the shock. However for the moment we will assume that the characteristics do not intersect in 8a or 8b. We will study this case later. This family of C^+ characteristics forms a compression wave going to the wall.

1.3.4 Interaction of the Compression Wave with the Wall



Let us suppose that at point A of the diagram the shock starts to bend by a significant amount. The characteristic leaving A hits the wall at a certain point B and then reflects. Let us call f the reflected characteristic. The trajectory of f and the flow properties along it are known by the results of the previous section. Between f and the wall is the region 8c that we want to study.

Let us choose B as our new origin of coordinates. Region 8c is isentropic because all the particle paths crossing this region come from 5 where the entropy is constant. On this region then:

$$u + \int \frac{dp}{\rho a} = \text{const. along } \frac{dx}{dt} = u + a$$

$$u - \int \frac{dp}{\rho a} = \text{const. along } \frac{dx}{dt} = u - a$$

The integral has to be evaluated at constant entropy (see for example Courant and Friedrichs (14), Chapter III). The evaluation of this integral, although straightforward, is extremely messy, and it does not look as though we would be able to obtain it in closed form. From (55), (56), (59) and (62) eliminating α and T (constant entropy) we can obtain ρ and a in terms of p. Instead of doing this, we are going to use the approximations pointed out before in section 1.3.3. We saw that in the case of an ionized gas in equilibrium with high degrees of ionization and with moderate changes in pressure involved in the process, the constant entropy condition is equivalent to taking:

$$\alpha = \text{const.}, \quad T = \text{const.}, \quad \frac{p}{\rho} = a^2 = \text{const.}$$

Then:

$$\int \frac{dp}{\rho a} \simeq a \,\ln\left(\frac{p}{p_5}\right) + \text{const.}$$

The constant pressure p_5 was added for convenience. Then in region 8c:

$$u + a \,\ln \frac{p}{p_5} = 2r = \text{const. along } \frac{dx}{dt} = u + a \tag{63}$$

$$u - a \,\ln \frac{p}{p_5} = -2s = \text{const. along } \frac{dx}{dt} = u - a \tag{64}$$

r and s are the Riemann invariants. We now change the independent variables to r and s. Equations (63) and (64) become:

$$u = r - s \tag{65}$$

$$a \,\ln \frac{p}{p_5} = r + s \tag{66}$$

$$x_s = (r-s+a)t_s \tag{67}$$

$$x_r = (r-s-a)t_r \tag{68}$$

The subindices mean derivatives. Eliminating x between (67) and (68) we obtain:

$$2a\, t_{sr} + t_s + t_r = 0 \tag{69}$$

This equation combined with (67) and (68) (and the boundary conditions) will give us $x(r,s)$, $t(r,s)$. Inverting these functions we get $s(x,t)$ and $r(x,t)$ and by using (65) and (66) $u(x,t)$ and $p(x,t)$. The inversion of $x(r,s)$ and $t(r,s)$ is possible if the Wronskian of the transformation is different from zero. In simple-wave regions this Wronskian is zero; we will discuss later under what other circumstances this could also happen; for the moment we assume that this is not the case in region 8c.

We need some boundary conditions to solve (69). At the wall from (65) we have:

$$u = 0 \qquad x = 0$$

$$r = s \qquad x = 0$$

On the line $r = s$, x has to be constant (equal to zero):

$$x_r dr + x_s ds = (x_r + x_s)\, dr = 0$$

$$x_r = -x_s$$

Using (67) and (68) we get the boundary condition:

$$t_r = t_s \text{ on } r = s \tag{70}$$

Another boundary condition is given by the fact that we know the flow properties along the C^- characteristic f (see diagram) and its trajectory. Along f:

$$s = \text{const.} = 0$$

because on B u is zero and p is p_5. We then have:

$$t = T(r) \text{ on } s = 0 \tag{71}$$

where $T(r)$ is known by the calculations of section 1.3.3. A typical form of $T(r)$ is given in fig. 9. When the reflected shock has crossed the ionization front following the incident shock, the conditions behind

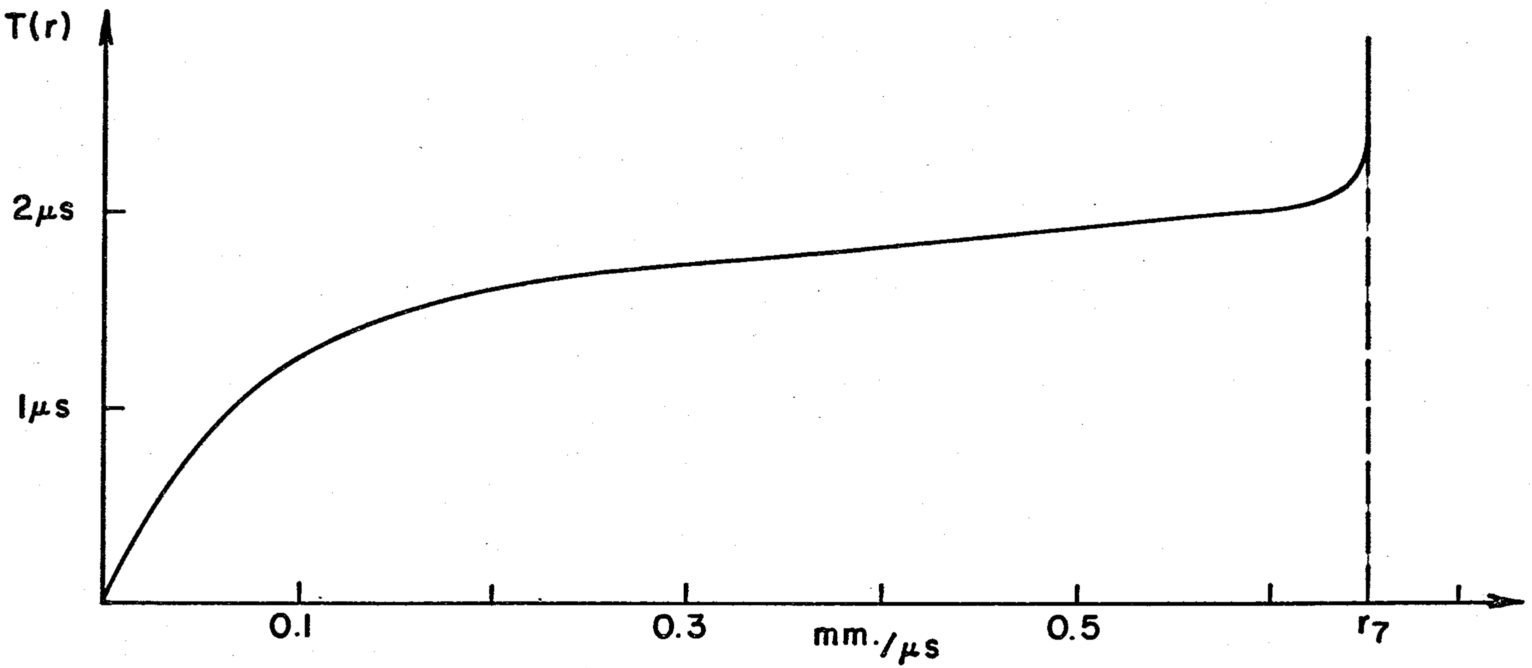


FIG. 9 VALUES OF THE RIEMANN INVARIANT ALONG THE LEADING C^- CHARACTERISTIC $M_1 = 15.0$, $P_1 = 5$ mm.Hg, ARGON

the reflected shock (region 7) are constant. Region 7 is uniform, and region 9 is a simple wave; there our theory is no longer correct. The value of r_7 is:

$$r_7 = \tfrac{1}{2}(u_7 + a \ln \frac{p_7}{p_5}) \tag{72}$$

Condition (71) should rather be:

$$t = T(r) \text{ on } s = 0 \text{ for } 0 \leqslant r \leqslant r_7 \tag{73}$$

We now make the transformation:

$$t = e^{-\frac{1}{2a}(s+r)} w \tag{74}$$

Equation (69) becomes:

$$4a^2 w_{rs} - w = 0 \tag{75}$$

and the boundary conditions (70) and (73):

$$w_r = w_s \quad \text{on} \quad r = s \tag{76}$$

$$w = T(r)e^{r/2a} \quad \text{on} \quad s = 0 \text{ and } 0 \leqslant r \leqslant r_7 \tag{77}$$

Since the problem is completely symmetrical we can substitute boundary condition (76) by:

$$w = T(s)\, e^{s/2a} \text{ on } r = 0 \text{ and } 0 \leqslant s \leqslant r_7 \tag{78}$$

It is evident that equation (75) with boundary conditions (77) and (78) will give us a solution that also satisfies (76).

The Riemann function of equation (75) is (see Courant and Hilbert (15), Chapter V §5):

$$R(x,y,\xi,\eta) = J_o\left[\sqrt{-\frac{1}{a^2}(x-\xi)(y-\eta)}\right] = I_o\left[\frac{1}{a}\sqrt{(\xi-x)(\eta-y)}\right]$$

where I_o is the modified Bessel function of order zero. The solution of equation (75) with boundary conditions (77) and (78) is then:

$$w(r,s) = \int_0^s I_o\left(\frac{1}{a}\sqrt{r(s-z)}\right)\beta'(z)\,dz + \int_0^r I_o\left(\frac{1}{a}\sqrt{s(r-z)}\right)\beta'(z)dz$$

$$r \geqslant s \quad \text{and} \quad 0 \leqslant r \leqslant r_7 \tag{79}$$

Condition $r \geqslant s$ means that we are at the left of or at the wall. The value of β is:

$$\beta(z) = T(z)\,e^{z/2a}$$

It can be checked directly by substitution that (79) is a solution of (75), (76) and (77).

We are interested in calculating the pressure distribution at the wall where u is zero. From (65) and (66)

$$\frac{p_{wall}}{p_5} = e^{2r/a} \tag{80}$$

and from (79) and (74) we get:

$$t_{wall}(r) = 2e^{-r/a}\int_0^r I_o\left(\frac{\sqrt{r(r-z)}}{a}\right)\beta'(z)\,dz \tag{81}$$

Formulae (80) and (81) give p(t) at the wall in parametric form with r as a parameter. We will give later a simpler interpretation of this result.

In the examples that we worked out the ratio r/a was never larger than 0.4 and the modified Bessel function oscillated between 1 and 1.04. A very good approximation to (81) could be obtained by taking I_o equal to one:

$$t_{wall} = 2\, e^{-r/2a}\, T(r) \tag{82}$$

The reason for this is that the expansion of I_o is

$$I_o = 1 + O(r^2/4a^2) \tag{82a}$$

r/a can be regarded as the Mach number of the flow. In fig. 10 we have p(t) at the wall for the different examples calculated by this theory. We did not find any experimental data to compare the results of fig. 10. Later we will try to check Smith's (1) experimental results; we cannot do it with this theory because, as we will see, there may be shock formation in these cases. However we can see that the interaction of the ionization front with the reflected shock produces an increase in pressure at the end wall of the shock tube as the experiments seem to indicate.

As we said before, once the shock has crossed the ionization front it leaves behind a uniform region 7; adjoining this region is the simple-wave 9, and between this and the wall the uniform region 10. After the compression wave has hit the wall, the pressure at the wall remains constant (fig. 10). This pressure can be easily determined. Along a C^+ characteristic going through 7, 9 and 10 we have (see diagram and fig. 1)

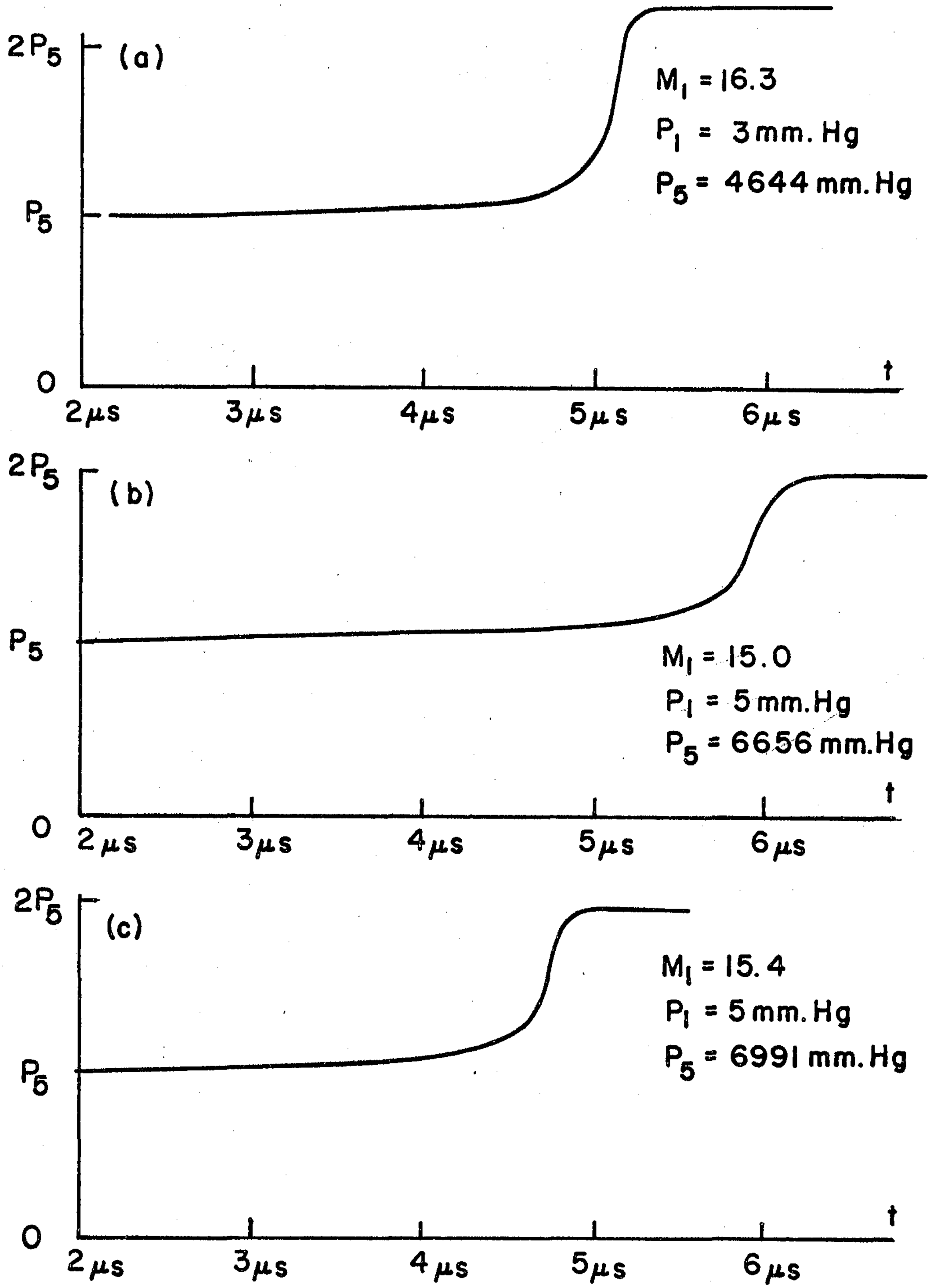


FIG. 10 END WALL PRESSURE INCREASE AS A RESULT OF THE INTERACTION OF THE IONIZATION FRONT WITH THE REFLECTED SHOCK (ARGON)

$$a \, \ln \frac{p_{10}}{p_5} = 2r_7 = u_7 + a \, \ln \frac{p_7}{p_5}$$

and along a C^- characteristic going from 5 to 7:

$$0 = u_7 - a \, \ln \frac{p_7}{p_5}$$

These equations give us:

$$p_{10} = p_5 \left(\frac{p_7}{p_5}\right)^2 \tag{83}$$

A result similar to this can be obtained, instead of formula (80), for the region of changing pressure. Using the relations (63) and (64) along a C^+ characteristic going from the shock to the wall and along the shock (that behaves like a C^- characteristic):

$$u_{sh} - a \, \ln \frac{p_{sh}}{p_5} = 0$$

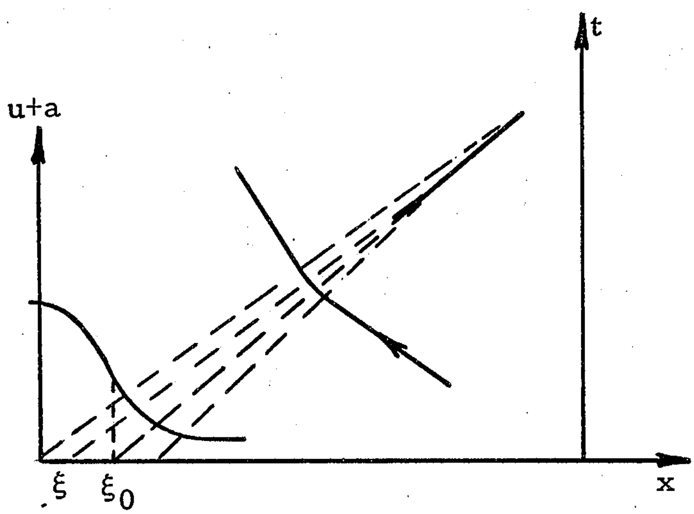
$$u_{sh} + a \, \ln \frac{p_{sh}}{p_5} = a \, \ln \frac{p_{wall}}{p_5}$$

$$p_{wall} = p_5 \left(\frac{p_{sh}}{p_5}\right)^2 \tag{84}$$

p_{sh} is the pressure behind the shock at a point that is on the same C^+ characteristic as the point at the wall. The fact that (p_{sh}/p_5) is elevated to the power two means that the pressure at the wall is greater than behind the shock, because behind the shock the gas has a speed towards the wall. Although this is a very simple interpretation, we must not forget that we still need to know the trajectory of the C^+ characteristic and in particular the point at which it hits the wall. This is given by formula (81).

1.3.5 Cases for Which a Shock Wave is Formed in the Simple-Wave Region

If the ionization front behind the incident shock is very thin the reflected shock will bend very sharply. It is then possible that the characteristics leaving the reflected shock will intersect. We are going to assume that this intersection occurs somewhere in the simple-wave regions 8a or 8b (fig. 7).



Characteristic equation:

$$x = \xi + c(\xi)t$$

envelope of characteristics:

$$\left.\begin{aligned} \frac{\partial x}{\partial \xi} &= 1 + c'(\xi)t = 0 \\ \xi + c(\xi)t &= x \end{aligned}\right\}$$

cusp of the envelope:

$$\frac{dt}{d\xi} = \frac{c''(\xi)}{c'(\xi)^2} = 0 \quad c''(\xi) = 0$$

To obtain the point on which the shock starts to form, we prolong the characteristics to the left and calculate the coordinates ξ where they intersect the x axis (as indicated in the diagram). Then we represent the speeds of the characteristics, $c = u+a$, as a function of ξ. The characteristic where $c''(\xi_o)$ is zero is the one on which the cusp of the envelope of the characteristics lies. The coordinates of the cusp are:

$$t_c = -\frac{1}{c'(\xi_o)}$$

$$x_c = \xi_o - \frac{c(\xi_o)}{c'(\xi_o)}$$

Now we have to continue constructing the shock. To do this we examine first the conditions that the gas satisfies on both sides of the shock. From energy conservation we get:

$$c_pT_a(1+\alpha_a)+R\theta_i\alpha_a+\tfrac{1}{2}(U-u_a)^2 = c_pT_b(1+\alpha_b)+R\theta_i\alpha_b + \tfrac{1}{2}(U-u_b)^2$$

U is the shock speed. Subindices a and b mean conditions on both sides of the shock. The velocities are of the same order as c_pT. Since θ_i is much larger than T, the dominant term in this expression is $\theta_i\alpha$; then across the shock α is approximately constant. Using the same arguments of section 1.3.3 we can conclude that α, T, a, and p/ρ are constants over the whole field, independently of the presence of this shock. The entropy will also be a constant* (section 1.3.3). The mechanical conditions across the shock are:

$$\rho_a\,(U-u_a) = \rho_b(U-u_b) = m \tag{85a}$$

$$p_a+m(U-u_a) = p_b+m(U-u_b) \tag{85b}$$

and from the above arguments:

$$\frac{p_a}{\rho_a} = \frac{p_b}{\rho_b} = a^2 = \text{const.} \tag{85c}$$

From these three equations we get:

$$U = \frac{u_a+u_b+\sqrt{(u_a-u_b)^2+4a^2}}{2} = \frac{(u_a+a)+(u_b+a)}{2} + \frac{1}{8}\,\frac{(u_a-u_b)^2}{a} + \ldots \tag{86}$$

*There should be an increase in entropy across the shock, but it is small and does not affect the flow field.

where the expansion is made for small values of the ratio $(u_a-u_b)^2/4a^2$ (see formula (82a). The first term of the expansion is the classical result that the speed of the shock is the average of the speeds of the characteristics on both sides of the shock. The first two terms of the expansion (85) are the same ones obtained by Courant and Friedrichs ((14), paragraph 72) for a perfect gas with constant specific heats. However, their method cannot be applied for γ equal to one. Using equations (85) and (86) it is easy to show that the s Riemann invariant also remains constant across the shock (in first approximation):

$$p_b-p_a = m(u_b-u_a)$$

$$\frac{p_b}{p_a} = 1 + \frac{m}{p_a}(u_b-u_a) = 1 + \frac{1}{a}(u_b-u_a) + O((\frac{u_a}{a})^2)$$

$$\frac{p_b}{p_a} = \exp\left(\frac{u_b-u_a}{a}\right) + O\left[(\frac{u_a}{a})^2\right]$$

$$\ell n \frac{p_b}{p_a} = \frac{u_b-u_a}{a} + O\left[(\frac{u_a}{a})^2\right]$$

$$u_b - a\,\ell n \frac{p_b}{p_5} = u_a - \ell n \frac{p_a}{p_5} + O\left[(\frac{u_a}{a})^2\right] \tag{87}$$

This means that the region behind the shock is also a simple-wave with straight C^+ characteristics along which the flow properties are constants. With this latter result and formula (86), we can continue constructing the shock and calculate all the flow field (see fig. 11). However, the method fails as soon as the effect of the wall is felt. This happens when the non-simple wave region 8c is reached.

We could study region 8c by using an analysis similar to the one of section 1.3.4 and applying on the shock the conditions (86) and (87), but in the examples that we are going to deal with we can apply a further simplification. We assume that by the time the shock arrives at region 8c the conditions behind and after the shock are almost uniform; they are a small perturbation of uniform conditions 5 upstream of the shock and of conditions 7 downstream. All the characteristics, across which the velocity and pressure suffer large changes, are absorbed by the shock before it arrives at the non-simple wave region adjoining the wall. We can apply the linearized theory of sound in this region. The characteristics upstream of the shock propagate with speeds $\pm a$ (u_5 is zero) and downstream with $u_7 \pm a$. The shock moves with velocity:

$$U = a + \frac{u_7}{2}$$

This shock reflects at the wall leaving behind a region 10 (fig. 11) that is at rest in first approximation; the characteristics move with velocities $\pm a$ in that region. Across this reflected shock the Riemann invariant r of the C^+ characteristics remains constant. The speed of this shock is given by the average of the speeds of the C^- characteristics on 7 and 10.

$$|U| = a - \frac{u_7}{2}$$

Since we know the location of all the characteristics we can apply equation (84): (see section 1.3.4)

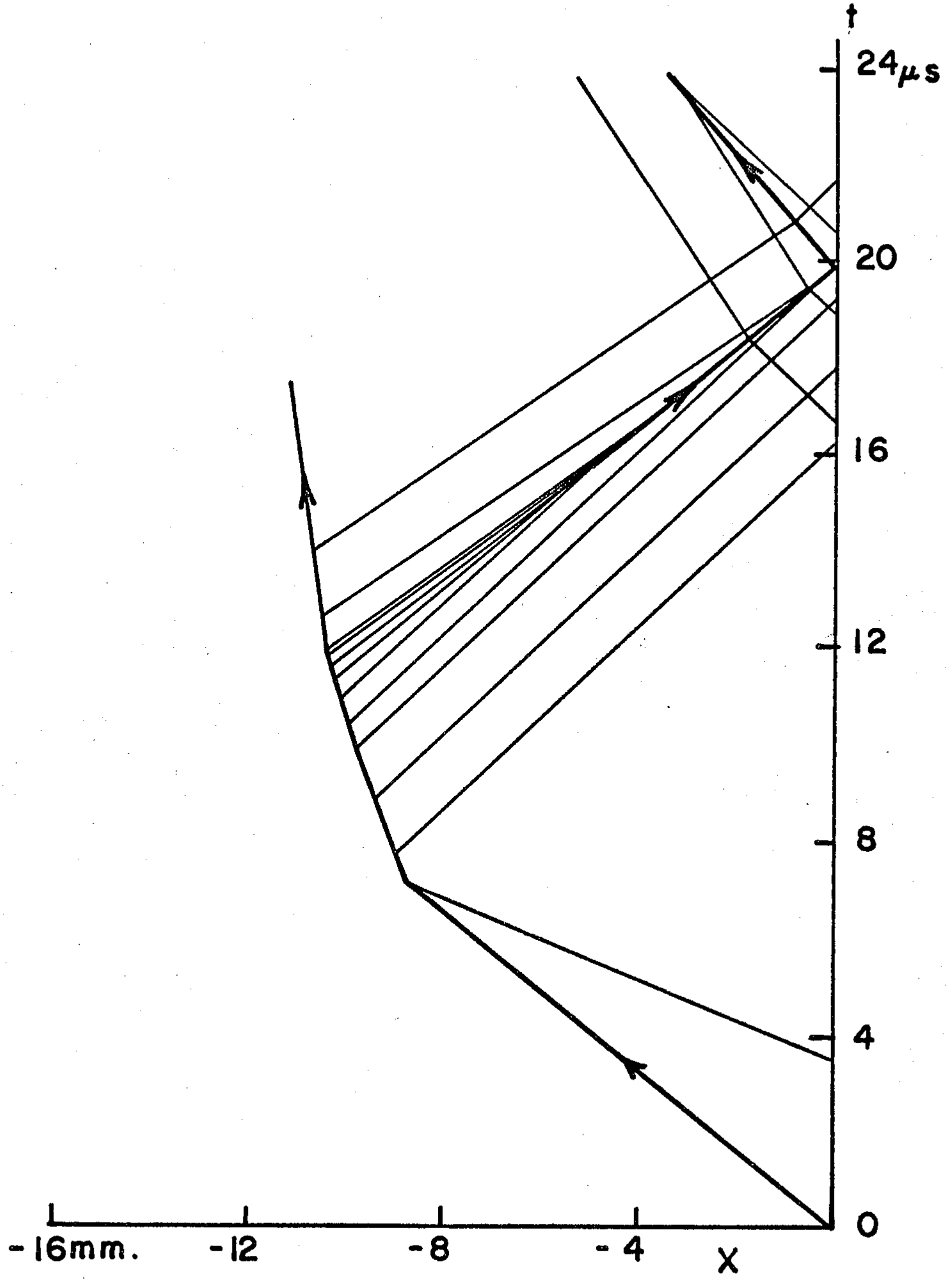


FIG. 11 (X - t) DIAGRAM SHOWING THE FORMATION OF THE SHOCK AND THE REFLECTED SHOCK TRAJECTORY. $M_1 = 14.0$, $P_1 = 0.5$ mm. Hg (XENON)

$$p_{wall} = \left(\frac{p_{sh}}{p_5}\right)^2 p_5$$

p_{sh} is the pressure behind the reflected shock (that interacts with the ionization front) at a point that is on the same C^+ characteristic as the point at the wall. Since we also know the location of the secondary shocks, we can calculate the pressure distribution at the wall.

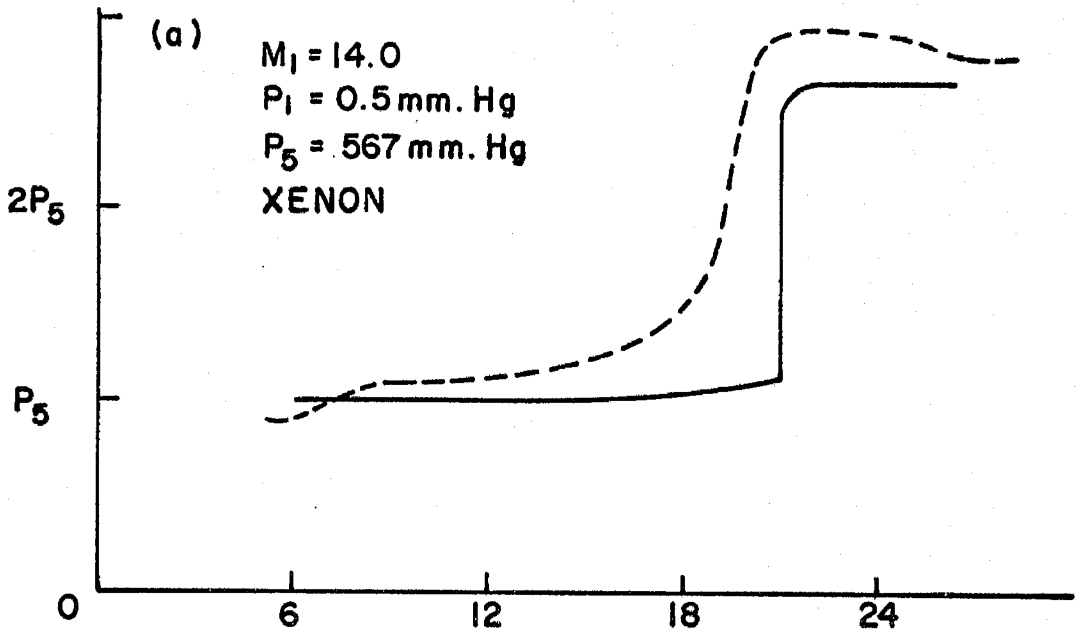
In figs. 12 are represented the calculated pressure distributions at the wall. In those figures are also shown Smith's (1) measurements. These seem to be much more spread out than our theoretical results, and their form seems to be more like the examples presented on fig. 10 on which there was no shock formation. Perhaps impurities played an important role in the process of ionization behind the incident shock, and the actual ionization front was thicker than the one resulting from the theoretical model proposed by Smith (1). It could also be that our assumption of equilibrium degree of ionization is not a good one, and there are relaxation and end wall boundary layer effects that produce this spreading.

The agreement with Smith's (1) experiments regarding the location and the amount of the jump in pressure seems to be good.

1.3.6 Formation of the Shock Inside the Region Adjoining the Wall

We now want to discuss the possibility of shock formation in region 8c. In 1.3.4 we saw that the coordinates x and t could be expressed in the forms $x(r, s)$ and $t(r, s)$, where r and s are the Riemann invariants. r can be expressed as $r(t, s)$, and the family of C^- characteristics can be put in the parametric form:

$$x = x(r(t, s), s) = x(t, s)$$



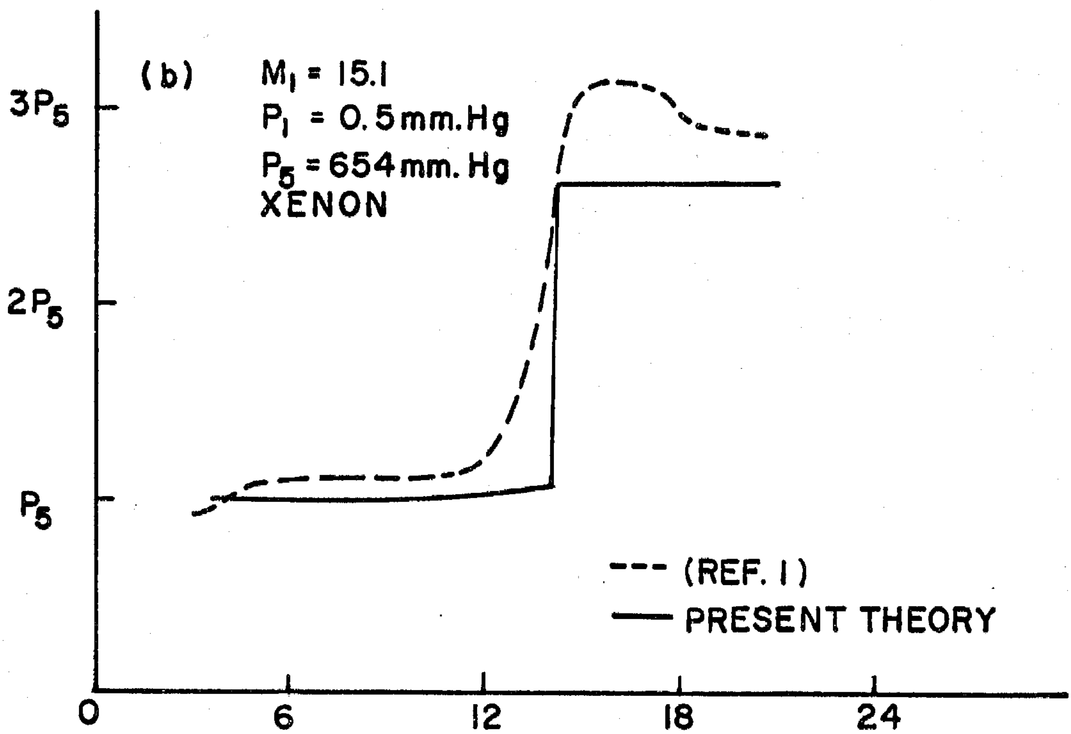


FIG. 12 END WALL PRESSURE INCREASE AS A RESULT OF THE IONIZATION FRONT WITH THE REFLECTED SHOCK

If these C^- characteristics intersect they will form an envelope of equations:

$$x = x(t, s)$$

$$\left(\frac{\partial x}{\partial s}\right)_t = 0$$

By a simple manipulation we get:

$$\left(\frac{\partial x}{\partial r}\right)_s \left(\frac{\partial r}{\partial s}\right)_t + \left(\frac{\partial x}{\partial s}\right)_r = 0$$

$$\left(\frac{\partial r}{\partial s}\right)_t = - \frac{(\partial t/\partial s)_r}{(\partial t/\partial r)_s}$$

$$\left(\frac{\partial x}{\partial r}\right)_s \frac{(\partial t/\partial s)_r}{(\partial t/\partial r)_s} - \left(\frac{\partial x}{\partial s}\right)_r = 0$$

and by using (67) and (68) this last equation reduces to:

$$(r-s-a)t_s - (r-s+a)t_s = 0$$

$$(\partial t/\partial s)_r = 0 \tag{88}$$

In the same way the envelope of the C^+ characteristics is given by

$$(\partial t/\partial r)_s = 0 \tag{89}$$

The procedure used in 1.3.4 was correct as long as the Wronskian was different from zero:

$$\left(\frac{\partial x}{\partial r}\right)_s \left(\frac{\partial t}{\partial s}\right)_r - \left(\frac{\partial x}{\partial s}\right)_r \left(\frac{\partial t}{\partial r}\right)_s \neq 0$$

If we use equations (67) and (68) we see that for the Wronskian to be zero we need:

$$x_r t_s - x_s t_r = 0$$

$$(r-s+a)t_r t_s - (r-s+a)t_s t_r = -2a t_r t_s = 0$$

For this to be true either (88) or (89) has to be satisfied. If the Wronskian vanishes in the non-simple wave region 8c there is shock formation in that region.

Using (79), (74) and the approximation (82a):

$$t = e^{-r/2a}\, T(s) + e^{-s/2a}\, T(r) \qquad r \geqslant s$$

Taking this expression into (88) and (89) we get:

$$T(r) = 2a\, e^{\frac{s-r}{2a}}\, T'(s) \qquad r \geqslant s \tag{90}$$

$$T(s) = 2a\, e^{\frac{r-s}{2a}}\, T'(r) \qquad r \geqslant s \tag{91}$$

Equations (90) and (91) together with the functions x(r, s) and t(r, s) give us the envelopes of the C^- and C^+ characteristics respectively. T is a known function.

The values of r and s that satisfy one of the equations (90) or (91) and give a minimum value of t correspond to the cusp of the envelope. If we consider that (90) gives us r(s), then the cusp will satisfy:

$$t_r \frac{dr}{ds} + t_s = 0$$

and since t_s is zero on the envelope of the C^- characteristics:

$$\frac{dr}{ds} = 0$$

$$T'(s) + 2aT''(s) = 0 \tag{92}$$

The values of r and s that correspond to the cusp of the envelope of the C^- characteristics will be given by equations (90) and (92) and the ones corresponding to the cusp of the envelope of the C^+ characteristics by (91) and:

$$T'(r) + 2a\,T''(r) = 0 \tag{93}$$

We should first check if the C^+ characteristics have an envelope. If that is so, on the cusp of the envelope a shock wave will start to form. This shock will propagate towards the wall and will reflect from it. If the C^+ characteristics do not collapse, we look for the envelope of the C^- characteristics.

In none of the cases studied in 1.3.4 were (90) or (91) satisfied at any point of 8c.

It must be remembered that since the C^- characteristics are convergent lines, they must collapse somewhere in the simple-wave region 9 (fig. 1) if they did not do it before. However, in our examples this shock will be of moderate strength and will not affect the previous results.

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II. THEORETICAL STUDY OF SOUND AND SHOCK WAVES IN A TWO-PHASE FLOW

2.1 Introduction

When the speed of sound is calculated in a liquid containing bubbles, it is generally assumed that the mass ratio of the gas and the liquid remains constant during the sound propagation (Wood (1), Campbell and Pitcher (2), Eddington (3), Murray (4), Plesset (5)). We will show that this is not true, unless during the sound propagation liquid and bubbles move together at the same velocity. When the dynamic forces acting over the bubble are not negligible compared to the viscous forces, liquid and bubble move at different velocities; then the formulae for the speed of sound given in the previous references are incorrect.

In the case of heavy particles in suspension in a gas a similar phenomenon takes place. However, in this case the consequences of this behavior are better known. For example, in the papers by Rudinger (6) and Marble (7) appear two different sound speeds. Rudinger (6) calls them the equilibrium and frozen sound speeds. The equilibrium sound speed is obtained when particles and fluid move together at the same velocity, and it can be obtained by using Wood's (1) formula (if heat transfer effects are taken into account). The frozen sound speed is obtained when the particles do not move, and this sound speed is just the speed of sound of the gas phase.

In this work we will find an expression for the speed of sound valid for any relative velocity between the phases and any type of suspension (drops, bubbles, or particles) in a fluid, and we will show that the assumption of constant mass ratio can lead to significant error. Heat transfer and surface tension effects will also be included.

To account for the relative velocity between the phases we will define K as the ratio of the speed of the suspension (drop, particle, or bubble) to the speed of the fluid at a certain point, and to account for heat transfer effects we will define K' as the ratio of the perturbations in temperatures. In order for these definitions to be meaningful, we have to assume that the general properties of the flow do not change significantly over distances comparable to the dimensions of the particle. The expression for the speed of sound will appear as a function of K and K'. In certain cases we are able to calculate K and K' explicitly and sometimes we obtain values of the speed of sound that are quite different from the ones obtained in references (1) to (5). These results will be shown to be valid only for low frequency sound.

Dobbins and Temkin (8) studied the case of sound propagation in a fluid with solid particles in suspension, but they considered only the low frequency limit in which the drag between the phases is due to viscous forces; their result checks with ours (in that limit). Soo (9) (10) also considered the case of sound propagating in a fluid with solid particles in suspension; however, we think that his method is incorrect because he assumed that the volume fraction occupied by the particles was constant during the sound propagation. He did not obtain the heat transfer effects in closed form and his results do not check with our analysis.

For finite values of viscosity and heat conduction (or finite frequencies), we obtain complex values of the speed of sound; the imaginary part gives the damping of the sound wave. The damping rate calculated in this way coincides with the one obtained by Carhart (11) calculated by considering the dissipation terms in the energy equation.

In part 2.2.6 we consider a general fluid in which there is a thin slab with spheres of a different material (particles, drops or bubbles). We suppose there is a plane sound wave whose front is parallel to the slab. We then calculate at a certain point behind the slab the sum of all the intensities of the scattered waves produced by the spheres. Comparing this sum with the difference in phase produced by the fact that the sound wave travels at a different speed in the slab, we obtain the speed of sound in the slab. The speed of sound calculated in this way is the same as the one calculated in sections 2.2.1 and 2.2.2. This demonstrates that the physical mechanism of sound speed change is scattering by the suspended phase.

In section 2.3 the structure of a shock wave in a liquid containing bubbles will be studied. We will consider the two-phase flow to be a compressible fluid in which the compressibility is given by the bubbles (the liquid is incompressible). On the other hand the inertia of the mixture is provided by the liquid phase (the gas density is much smaller than the liquid density). We will neglect interactions between the bubbles.

The Rankine-Hugoniot relations for this type of shock are given by Campbell and Pitcher (2); they also solved them and checked their results experimentally. From these results they concluded that the condition for shock existence is that the Mach number (ratio of the speed of the mixture to the speed of sound of the mixture) has to be larger than one upstream of the shock. However, as we said before, their speed of sound is valid only in the low frequency limit; from the calculations of section 2.2.1 we obtain a larger sound speed (and a lower Mach number) in the high frequency limit. When both Mach numbers are larger than one, we will show that the shock wave structure is given by an exponential rise in pressure followed by a relaxation region in which the pressure oscillates around the final equilibrium value. The oscillations are damped by viscous and thermal dissipation. These oscillations are produced by the fact that the bubble behaves like a harmonic oscillator or a spring on which the mass is provided by the liquid outside the bubble and the recovery force by the bubble pressure. The frequency of these oscillations is similar to the classical resonant frequency (Silberman (19)) corrected in our case by the effect of the virtual mass of the bubble.

In the case that the Mach number referred to the low frequency speed of sound is larger than one and the high frequency Mach number smaller than one, the shock structure changes. The oscillations of the previous case disappear, and the shock takes a more uniform and conventional shape. In this case all the regions of the shock have the same character and the entire shock structure is determined by viscous and heat transfer effects, whereas in the previous case the exponential rise in pressure was not influenced by these dissipative mechanisms. The shock thickness is also larger in this case. Under the latter conditions no numerical examples were calculated.

The pressure gradient across the shock is maintained by the drag between the phases. This drag is created by viscous and inertial forces (virtual mass of the bubble).

We will take into account the heat transfer between the phases. The temperature of the liquid is assumed to be constant but not the temperature of the bubbles*.

For large relative velocity between the phases the bubbles are supposed to take the umbrella-like shape described by Davies and Taylor (12).

Eddington (3) measured the structure of this type of shock. The thicknesses of the shocks that he measured are about two times larger than the ones calculated by us.

The shock thicknesses that we obtained are large compared to the size of bubble.

*This assumption should be considered carefully because it could lead to the result of constant entropy across the shock (Campbell and Pitcher (2)).

2.2 Sound Propagation in Two-Phase Flow

2.2.1 Calculation of a General Expression of the Speed of Sound

Let us suppose that we have a suspension (drops, bubbles or particles) in a fluid. The suspension is characterized by the subscript 1, and the fluid by the subscript 0. The volume fraction occupied by the suspension is X, and the one occupied by the fluid 1-X. Now we assume there is a sound wave propagating in this mixture. We denote the perturbations of the flow properties by a tilde. These perturbations are supposed to be small and the equations will be linearized.

The continuity equations of each component are:

$$\frac{\partial(\rho_0+\tilde{\rho}_0)(1-X-\tilde{X})}{\partial t} + \frac{\partial(\rho_0+\tilde{\rho}_0)(1-X-\tilde{X})\tilde{u}_0}{\partial x} = 0$$

$$\frac{\partial(X+\tilde{X})(\rho_1+\tilde{\rho}_0)}{\partial t} + \frac{\partial(\rho_1+\tilde{\rho}_1)(X+\tilde{X})\tilde{u}_1}{\partial x} = 0$$

$\tilde{u}_0$ and $\tilde{u}_1$ are the velocities of both components (that are perturbed quantities) and ρ_1 and ρ_0 are the densities. The linearized continuity equations are:

$$(1-X)\frac{\partial\tilde{\rho}_0}{\partial t} - \rho_0\frac{\partial\tilde{X}}{\partial t} + (1-X)\rho_0\frac{\partial\tilde{u}_0}{\partial x} = 0 \tag{1a}$$

$$X\frac{\partial\tilde{\rho}_1}{\partial t} + \rho_1\frac{\partial\tilde{X}}{\partial t} + X\rho_1\frac{\partial\tilde{u}_1}{\partial x} = 0 \tag{1b}$$

We define the average density, ρ, and velocity, u, as:

$$\rho + \tilde{\rho} = (\rho_0+\tilde{\rho}_0)(1-X-\tilde{X}) + (\rho_1+\tilde{\rho}_1)(X+\tilde{X}) \tag{2a}$$

$$\rho = \rho_0(1-X) + \rho_1 X \tag{2b}$$

$$\tilde{\rho} = \tilde{\rho}_0(1-X) - \tilde{X}\rho_0 + \tilde{X}\rho_1 + \tilde{\rho}_1 X \tag{2c}$$

$$\tilde{u} = \frac{\rho_0(1-X)\tilde{u}_0 + \rho_1 X\tilde{u}_1}{\rho_0(1-X)+\rho_1 X} \tag{3}$$

Adding equations (1), and using (2) and (3) we get:

$$\frac{\partial\tilde{\rho}}{\partial t} + \rho\,\frac{\partial\tilde{u}}{\partial x} = 0 \tag{4}$$

The linearized momentum equation of the mixture is:

$$\rho_1 X\,\frac{\partial\tilde{u}_1}{\partial t} + (1-X)\,\rho_0\,\frac{\partial\tilde{u}_0}{\partial t} + \frac{\partial\tilde{p}}{\partial x} = 0\,^*$$

where p is the pressure of phase 0. The above equation becomes:

$$\rho\,\frac{\partial\tilde{u}}{\partial t} + \frac{\partial\tilde{p}}{\partial x} = 0 \tag{5}$$

on using (2) and (3). We take for p the external pressure p_0. We can justify this by taking a control surface that does not go through any particle and applying to it the momentum equation.

Equations (4) and (5) are the usual sound equations. The speed of sound will be $(dp/d\rho)^{\frac{1}{2}}$ if p can be defined as a function of ρ only.

Let us define the mass fraction μ and velocity ratio K:

$$(\mu+\tilde{\mu}) = \frac{(1-X-\tilde{X})(\rho_0+\tilde{\rho}_0)}{(X+\tilde{X})(\rho_1+\tilde{\rho}_1)} \tag{6a}$$

---

*The viscosity of the mixture (as a single fluid) is neglected in comparison to the drag between the phases. This drag cancels when we add the momentum equations for each component.

$$\mu = \frac{(1-X)\rho_0}{X\rho_1} \tag{6b}$$

$$\tilde{\mu} = \frac{(1-X)\tilde{\rho}_0 - \rho_0\tilde{X}}{X\rho_1} - \frac{(1-X)\rho_0}{X\rho_1}\left(\frac{\tilde{X}}{X} + \frac{\tilde{\rho}_1}{\rho_1}\right) \tag{6c}$$

$$K = \tilde{u}_1/\tilde{u}_0 \tag{7}$$

We assume that K is independent of x and t. Then from equations (1) (eliminating u_1 and u_0) we get:

$$\frac{K}{\rho_0}\frac{\partial\tilde{\rho}_0}{\partial t} - \frac{K}{(1-X)}\frac{\partial\tilde{X}}{\partial t} - \frac{1}{\rho_1}\frac{\partial\tilde{\rho}_1}{\partial t} - \frac{1}{X}\frac{\partial\tilde{X}}{\partial t} = 0$$

Using equations (2) and (6) to eliminate $(X\tilde{\rho}_1 + \rho_1\tilde{X})$ and $[(1-X)\tilde{\rho}_0 - \rho_0\tilde{X}]$ in terms of $\tilde{\mu}$ and $\tilde{\rho}$ we obtain:

$$\frac{\partial\tilde{\mu}}{\partial t} = \frac{(1-K)\mu(1+\mu)}{(K+\mu)}\frac{1}{\rho}\frac{\partial\tilde{\rho}}{\partial t} \tag{8}$$

$\tilde{\rho}$ will be a solution of the system (4) (5) and will have a form (if $(dp/d\rho)$ can be defined):

$$\tilde{\rho} = \text{const. x} \exp\left[i\omega(t - \frac{x}{c})\right] \text{ where } c = (dp/d\rho)^{\frac{1}{2}} \tag{8a}$$

$\tilde{\mu}$ will have the same form and we can write (8):

$$\tilde{\mu} = \frac{(1-K)\mu(1+\mu)}{(K+\mu)}\frac{1}{\rho}\tilde{\rho} \tag{9}$$

$\tilde{\mu}$ is zero if K is unity (both phases move together) and not otherwise. We will see that in many cases this condition is not satisfied, and the assumption of considering the mass ratio constant (which is usual in the literature) is erroneous.

From (2a) and (6a) we get (eliminating $(X+\tilde{X})$):

$$\frac{1+(\mu+\tilde{\mu})}{(\rho+\tilde{\rho})} = \frac{1}{(\rho_1+\tilde{\rho}_1)} + \frac{(\mu+\tilde{\mu})}{\rho_0+\tilde{\rho}_0} \tag{10a}$$

$$\frac{1+\mu}{\rho} = \frac{1}{\rho_1} + \frac{\mu}{\rho_0} \tag{10b}$$

$$\frac{\tilde{\mu}}{\rho} - \frac{(1+\mu)}{\rho^2}\tilde{\rho} = -\frac{\tilde{\rho}_1}{\rho_1^2} + \frac{\tilde{\mu}}{\rho_0} - \frac{\mu}{\rho_0^2}\tilde{\rho}_0 \tag{10c}$$

If p_1 is the pressure of the suspended phase and we assume local mechanical equilibrium:

$$p_1 = p_0 + \frac{2\tau}{a} \tag{11}$$

$$p_1 + \tilde{p}_1 = p_0 + \tilde{p}_0 + \frac{2\tau}{a+\tilde{a}} \tag{11a}$$

$$\tilde{p}_1 = \tilde{p}_0 - \frac{2\tau}{a^2}\tilde{a} \tag{12}$$

τ is the surface tension that is supposed to be constant. We also assume that the mass of the bubble (or drop) remains constant:

$$(a+\tilde{a})^3 (\rho_1+\tilde{\rho}_1) = a^3 \rho_1$$

$$3\tilde{a}\rho_1 + a\tilde{\rho}_1 = 0 \tag{13}$$

From (12) and (13) (eliminating $\tilde{a}$) we get

$$\frac{dp_1}{dp_0} = \frac{\tilde{p}_1}{\tilde{p}_0} = \left[1 - \frac{2\tau}{3a}\,\frac{1}{\rho_1(dp_1/d\rho_1)}\right]^{-1} \tag{14}$$

where:

$$\frac{dp_1}{d\rho_1} = \frac{\tilde{p}_1}{\tilde{\rho}_1}$$

In the same way we can write:

$$\frac{dp_0}{d\rho_0} = \frac{\tilde{p}}{\tilde{\rho}_0} \qquad \frac{dp}{d\rho} = \frac{\tilde{p}}{\tilde{\rho}}$$

(as we said before we take for $\tilde{p}_0$ the pressure of the mixture $\tilde{p}$).

Then (10c) becomes:

$$\left[\frac{\mu}{\rho_0^2}\frac{1}{(dp_0/d\rho_0)} - \frac{1+\mu}{\rho^2}\frac{1}{(dp/d\rho)} + \frac{1}{\rho_1^2}\frac{1}{(dp_1/d\rho_1)}\left(\frac{dp_1}{dp_0}\right)\right]\tilde{p} =$$

$$= \left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)\tilde{\mu} = \left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)\frac{(1-K)\mu(1+\mu)}{(K+\mu)}\frac{1}{\rho}\tilde{\rho}$$

on using (9). This can be put in the form:

$$\left[\frac{\mu}{\rho_0^2}\frac{1}{(dp_0/d\rho_0)} + \frac{1}{\rho_1^2}\frac{1}{(dp_1/d\rho_1)}\left(\frac{dp_1}{dp_0}\right)\right]\left(\frac{dp}{d\rho}\right) = \left[1 + \left(\frac{\rho-\rho_0}{\rho_0}\right)\frac{(1-K)\mu}{K+\mu}\right]\frac{(1+\mu)}{\rho^2}$$

Using (6b) and (2b) to eliminate μ and ρ, and taking for (dp_1/dp_0) the value given in (14) we get:

$$c^2 = \frac{dp}{d\rho} = \frac{\alpha\, \rho_0\rho_1(dp_1/d\rho_1)(dp_0/d\rho_0)}{[X\beta(dp_0/d\rho_0)\rho_0+(1-X)(dp_1/d\rho_1)\rho_1]\cdot[(1-X)\rho_0+X\rho_1]} \tag{15}$$

where

$$\alpha = 1 + \frac{X(\rho_1-\rho_0)(1-K)(1-X)}{K\rho_1 X + \rho_0(1-X)} \tag{16a}$$

$$\beta = \left[1 - \frac{2\tau}{3a} \frac{1}{\rho_1 (dp_1/d\rho_1)}\right]^{-1} = \frac{p_1/p_0}{[1 - \frac{2\tau}{3a} \frac{1}{\rho_1 (dp_1/d\rho_1)}](1 + \frac{2\tau}{ap_0})} \tag{16b}$$

The values of $(dp_0/d\rho_0)$ and $(dp_1/d\rho_1)$ have not yet been calculated. If we neglect surface tension β becomes unity; the second expression of β is obtained by using (11). If (τ/ap_0) is small and the fluid forming the suspension behaves isothermally and is calorifically perfect, the denominator of (16b) becomes $(1 + 4\tau/3ap_0)$; this is the usual surface tension correction found in the literature for the case of bubbles (Eddington (3)). If the speeds of both components are the same α becomes unity. α is then the correction factor to the incorrect expressions of the speed of sound found in references (1) to (5).

We are now going to calculate the quantities $(dp_1/d\rho_0)$ of formula (15). If the substances forming the phases are such that their state law is of the form $p(\rho)$, the values of those quantities are evident. However, this is not so if the state law is of the form $p(\rho, T)$ (T is the temperature); in this case we have to know whether the phases behave isentropically, isothermally, or in any other way.

The following thermodynamic relations are supposed to hold for each component:

$$c_{pi} = T_i(\partial s_i/\partial T_i)_{p_i} \tag{17a}$$

$$c_{vi} = T_i(\partial s_i/\partial T_i)_{\rho_i} \tag{17b}$$

$$c_i^2 = (\partial p_i/\partial \rho_i)_s = \gamma_i c_i^{*2} = \gamma_i(\partial f_i/\partial \rho_i) \tag{17c}$$

$$p_i = f_i(\rho_i, T_i) \tag{17d}$$

$$c_{pi}/c_{vi} = \gamma_i \tag{17e}$$

$$\alpha_{vi} = -(1/\rho_i)(\partial\rho_i/\partial T_i)_{p_i} \tag{17f}$$

$$T_i c_i^2 \alpha_{vi}^2 = (\gamma_i - 1)c_{pi} \tag{17g}$$

$$ds_i = -[\alpha_{vi}/\rho_i(\gamma_i - 1)](dp_i - c_i^2 d\rho_i) \tag{17h}$$

$i = 0, 1$

These relations are supposed to hold for very general types of components (even a solid or a liquid). c_p and c_v are the specific heats, s is the specific entropy, c the adiabatic speed of sound, c* the isothermal speed of sound, and α_v the compressibility. For further justification of relations (17) see Appendix A.

We now express the conservation of energy of the system by the linearized equation:

$$T_0(1-X)\rho_0 \frac{\partial \tilde{s}_0}{\partial t} + T_0 X \rho_1 \frac{\partial \tilde{s}_1}{\partial t} = 0 \tag{18}$$

where we assumed that in the unperturbed state T_0 is equal to T_1. The heat conduction of the mixture (considered as if it were a single fluid) is neglected in comparison to the heat exchange between the phases. This heat exchange does not appear in (18) because it cancels when we add the energy equations for each component. To higher order we cannot consider that $\tilde{T}_0$ is equal to $\tilde{T}_1$, then in equation (18) the heat transfer would appear like a dissipative term. The dissipation due to the drag between the phases will also appear as a higher order effect. These dissipative mechanisms would produce an increase in the entropy of the system (to a higher order); although they are negligible in our linearized analysis, they have a definite physical significance; by evaluating them Carhart (11) obtained the imaginary part of the speed of sound (damping), and his results will be shown to coincide with ours.

From (17) and (18) (and using an argument similar to the one in (8a)) we get:

$$(1-X)\frac{\alpha_{v0}}{(\gamma_0-1)}(\tilde{p}_0 - c_0^2\,\tilde{\rho}_0) + X\frac{\alpha_{v1}}{(\gamma_1-1)}(\tilde{p}_1 - c_1^2\,\tilde{\rho}_1) = 0 \tag{19}$$

We now assume that the perturbations in the temperatures are such that:

$$\tilde{T}_1/\tilde{T}_0 = K' \tag{20}$$

where K' is supposed to be a function of the unperturbed quantities and of the frequency of sound.

From the state equation (17d) we get:

$$\tilde{p}_1 = \left(\frac{\partial f_1}{\partial \rho_1}\right)\tilde{\rho}_1 + \left(\frac{\partial f_1}{\partial T_1}\right)\tilde{T}_1 \tag{21a}$$

$$\tilde{p}_0 = \left(\frac{\partial f_1}{\partial \rho_0}\right)\tilde{\rho}_0 + \left(\frac{\partial f_0}{\partial T_0}\right)\tilde{T}_0 \tag{21b}$$

We now have a system of six homogeneous equations (12), (13), (19), (20), (21a) and (21b) for the seven perturbed quantities $\tilde{\rho}_0$, $\tilde{p}_0$, $\tilde{\rho}_1$, $\tilde{p}_1$, $\tilde{T}_0$, $\tilde{T}_1$ and $\tilde{a}$. We can then obtain after some algebraic manipulation (using also relations (17)):

$$\frac{dp_0}{d\rho_0} = \frac{\tilde{p}_0}{\tilde{\rho}_0} = c_0^2 \left\{ \frac{1+X[\frac{\alpha_{v1}}{\alpha_{v0}} \frac{1}{(1-\gamma_1 A)} - 1] + \frac{X}{(1-\gamma_1 A)}[K' \frac{\rho_1}{\rho_0} \frac{c_{p1}}{c_{p0}} (1-A) - \frac{\alpha_{v1}}{\alpha_{v0}}]}{1+X[\frac{\alpha_{v1}}{\alpha_{v0}} \frac{1}{(1-\gamma_1 A)} - 1] + \gamma_0 \frac{X}{(1-\gamma_1 A)}[K' \frac{\rho_1}{\rho_0} \frac{c_{p1}}{c_{p0}} (1-A) - \frac{\alpha_{v1}}{\alpha_{v0}}]} \right\} \tag{22a}$$

$$\frac{dp_1}{d\rho_1} = \frac{\tilde{p}_1}{\tilde{\rho}_1} = c_1^2 \left\{ \frac{1+(1-X)[\frac{\alpha_{v0}}{\alpha_{v1}} (1-\gamma_1 A) - 1] + (1-X)[\frac{1}{K'} \frac{\rho_0}{\rho_1} \frac{c_{p0}}{c_{p1}} - \frac{\alpha_{v0}}{\alpha_{v1}} (1-A)]}{1+(1-X)[\frac{\alpha_{v0}}{\alpha_{v1}} (1-\gamma_1 A) - 1] + \gamma_1 (1-X)[\frac{1}{K'} \frac{\rho_0}{\rho_1} \frac{c_{p0}}{c_{p1}} - \frac{\alpha_{v0}}{\alpha_{v1}} (1-A)]} \right\} \tag{22b}$$

where $A = 2\tau/3a\rho_1 c_1^2$.

For details of the calculation see Appendix A.

If:

$$K' = \frac{\alpha_{v1}}{\alpha_{v0}} \frac{\rho_0}{\rho_1} \frac{c_{p0}}{c_{p1}} \frac{1}{(1-A)}$$

both species behave isentropically and:

$$dp_1/d\rho_1 = c_1^2 \qquad dp_0/d\rho_0 = c_0^2$$

On the other hand, the isothermal behavior is given by:

$$K' \frac{\rho_1}{\rho_0} \frac{c_{p1}}{c_{p0}} (1-A) - \frac{\alpha_{v1}}{\alpha_{v0}} >> 1 \qquad \frac{dp_0}{d\rho_0} = \frac{c_0^2}{\gamma_0} = c_0^{*2}$$

$$\frac{1}{K'} \frac{\rho_0}{\rho_1} \frac{c_{p0}}{c_{p1}} - \frac{\alpha_{v0}}{\alpha_{v1}} (1-A) >> 1 \qquad \frac{dp_1}{d\rho_1} = \frac{c_1^2}{\gamma_1} = c_1^{*2}$$

We see that if one of the species has a much higher density and specific heat and much lower compressibility than the other, then the latter species would have a tendency to behave isothermally.

In the case of a liquid or a solid both specific heats are the same, γ is close to one, and from (22) we see that: (the numerator and denominator are equal)

$$dp/d\rho = c^2 \text{ (independent of } K'\text{)}$$

Equations (15), (16) and (22) give us the speed of sound of the mixture as a function of K and K'.

We expect that for low sound frequency the species will be at the same temperature (K' = 1) and that for high frequency they will behave isentropically. At high frequencies the process is so fast that there is no time for the heat to be transferred from one phase to the other. At low frequencies the process is so slow that there is enough time for the temperatures to become equal. These assumptions will be justified in section 2.2.3 for the case in which X is so small that the speed of sound of the mixture is a perturbation of the speed of sound of the external flow; however, we are going to assume that they hold for any X much smaller than one. The results of section 2.2.3 are:

$$K' = 1 \tag{23}$$

$$\left(\frac{\omega a^2 \rho_1 c_{p1}}{\sigma_1}\right)^{\frac{1}{2}} << 1 \quad \text{and} \quad \left(\frac{\omega a^2 \rho_1 c_{p1}}{\sigma_1}\right) << \frac{\sigma_0}{\sigma_1}$$

σ is the heat conductivity and ω the sound frequency. In the case of bubbles the first condition is more restrictive, and in the case of drops or particles the second condition should be used. For high frequencies:

$$K' = \frac{\alpha_{v1}}{\alpha_{v0}} \frac{\rho_0}{\rho_1} \frac{c_{p0}}{c_{p1}} \tag{24}$$

if $\left(\frac{\omega a^2 \rho_1 c_{p1}}{\sigma_1}\right)^{\frac{1}{2}} >> 1$

Equation (24) gives the isentropic law (see (22)) (surface tension is neglected).

At low frequencies both components are expected to move together (K = 1). Lamb (13, section 363) gives for the case of particles in suspension:

$$K = \frac{\tilde{u}_1}{\tilde{u}_0} = 1 \quad \text{if} \quad \frac{\omega a^2}{\nu_0} << \frac{\rho_0}{\rho_1} \tag{25a}$$

However, it is not difficult to redo his analysis for the case of bubbles (see section 2.2.5):

$$K = \frac{\tilde{u}_1}{\tilde{u}_0} = 1 \quad \text{if} \quad \frac{\omega a^2}{\nu_0} << 1 \quad \text{(bubbles)} \tag{25b}$$

These results hold for X << 1. For high frequencies (see Lamb (13, section 298) and Landau and Lifshitz (14, p. 36)):

$$K = \frac{\tilde{u}_1}{\tilde{u}_0} = \frac{3\rho_0}{2\rho_1 + \rho_0} \tag{26}$$

if $\frac{\omega a^2}{\nu_0} >> 1$ and $X << 1$

These results will also be obtained in 2.2.3.* We see that only for low frequencies can it be assumed that both species move at the same velocity. In this case the viscous forces are more important

*Under more restrictive conditions for X than the inertia forces, and the external fluid is able to drive along the suspension. The results obtained in references (1) to (5) are correct only for low frequencies.

Let us examine the case of air bubbles in water. For low frequencies we saw that K' is one. Then (22) gives:

$$(dp_1/d\rho_1) = c_1^2 \frac{1+(1-X)(\rho_0 c_{p0})/(\rho_1 c_{p1})}{1+\gamma_1(1-X)(\rho_0 c_{p0})/(\rho_1 c_{p1})}$$

(Surface tension has been neglected.) Since the density and specific heat of the water are much larger than the density and specific heat of the air ($\rho_0 c_{p0} >> \rho_1 c_{p1}$) the above formula reduces to:

$$(dp_1/d\rho_1) = \frac{c_1^2}{\gamma_1} = c_1^{*2} = \frac{p_1}{\rho_1} \tag{27}$$

$$\text{for } \frac{\omega a^2 \rho_1 c_{p1}}{\sigma_1} << 1$$

The last equality holds if the air can be considered a perfect gas. At low frequencies the bubbles behave isothermally. For high frequencies we saw (formula (24)) that:

$$\frac{dp_1}{d\rho_1} = c_1^2 = \frac{\gamma_1 p_1}{\rho_1} \tag{28}$$

$$\text{for } \frac{\omega a^2 \rho_1 c_{p1}}{\sigma_1} >> 1$$

The bubbles behave isentropically at high frequencies. For air and water we have:

$$\left(\frac{\sigma_1}{\rho_1 c_{p1}}\right)_{\text{air}} = 0.19 \text{ cm}^2/\text{sec.}$$

$$(\nu_0)_{\text{water}} = 0.01 \text{ cm}^2/\text{sec.}$$

then

$$\frac{\sigma_1}{\rho_1 c_{p1}} >> \nu_0$$

We can distinguish the following cases:

(i) $\omega a^2 << \nu_0 << \sigma_1/(\rho_1 c_{p1})$

In this case Wood's (1) formula is correct and the bubbles behave isothermally. (See fig. 1).

$$\alpha = 1, \quad dp_1/d\rho_1 = c_1^{*2} = \frac{p_1}{\rho_1} \tag{29a}$$

(ii) $\nu_0 << \omega a^2 << \sigma_1/(\rho_1 c_{p1})$

The bubble still behaves isothermally, but Wood's formula is incorrect. The correction factor α (using (16a) and (26) with $\rho_0 >> \rho_1$) is: (See fig. 1)

$$\alpha = 1 + 2X, \quad dp_1/d\rho_1 = c_1^{*2} = \frac{p_1}{\rho_1} \tag{29b}$$

For values of X of 10% we obtain a 20% correction.

(iii) $\nu_0 << (\sigma_1/\rho_1 c_{p1}) << \omega a^2$

The bubbles behave isentropically: (See fig. 1)

$$\alpha = 1+2X, \quad dp_1/d\rho_1 = c_1^2 = \gamma p_1/\rho_1 \tag{29c}$$

The value of $(dp_0/d\rho_0)^{\frac{1}{2}}$ (speed of sound in water) is a constant. Taking formulae (29) in (15) we obtain the sound speed of the mixture.

Let us examine the case of drops or particles in suspension in a gas. In this case we will discuss only high and low frequency limits without considering intermediate situations. In this case:

$$\rho_1 >> \rho_0, \quad dp_1/d\rho_1 >> dp_0/d\rho_0, \quad X << 1 \tag{30a}$$

For high frequencies (26) holds, and (16a) gives:

$$\alpha = 1 + X\frac{\rho_1}{\rho_0} - \frac{5}{2}X + O(X^2\rho_1/\rho_0) \tag{30b}$$

For high frequencies the gas phase behaves isentropically (see formula (24)). Taking (30b) into (15) and using the above inequalities we get:

$$c^2 = c_0^2\,[1-(X/2) + O(X^2\rho_1/\rho_0)] = \frac{\gamma p_0}{\rho_0}[1-(X/2)+O(X^2\rho_1/\rho_0)]$$

$$\text{if} \qquad (\omega a^2/\nu_0)^{\frac{1}{2}} >> 1, \; (\omega a^2\rho_1 c_{p1}/\sigma_1)^{\frac{1}{2}} >> 1 \tag{30c}$$

This means that the speed of sound of the mixture is only in first approximation affected by the presence of drops or particles. Wood's formula will give us a quite different result if the factor $(X\rho_1/\rho_0)$ turns out to be of order unity or larger (see formula (30b)). In the low frequency limit particles and gas move together at the same velocity ($K = 1$) (formula (25a)); the temperatures are also the same ($K' = 1$) (formula (23)). Then (22a) gives

$$\frac{dp_0}{d\rho_0} = c_0^2\;\frac{1+X\dfrac{\rho_1 c_{p1}}{\rho_0 c_{p0}}}{1+\gamma_0 X\dfrac{\rho_1 c_{p1}}{\rho_0 c_{p0}}}$$

where we neglected surface tension ($\tau = 0$) and assumed that the compressibility of the gas, α_{v0}, is much higher than that of the solid, α_{v1}. Since K is one, our correction factor α will be one, and formula (15) gives (using inequalities (30a)):

$$c^2 = \frac{dp_0}{d\rho_0}\,\frac{1}{[1+X\frac{\rho_1}{\rho_0}]} = \frac{\gamma_0 p_0}{\rho_0}\,\frac{1+X\frac{\rho_1 c_{p0}}{\rho_0 c_{p0}}}{1+\gamma_0 X\frac{\rho_1 c_{p1}}{\rho_1 c_{p0}}}\,\frac{1}{1+X\frac{\rho_1}{\rho_0}}$$

$$= \left(\frac{c_{p0}+X\frac{\rho_1}{\rho_0}c_{p1}}{c_{v0}+X\frac{\rho_1}{\rho_0}c_{p1}}\right)\left(\frac{p_0}{\rho_0+X\rho_1}\right) \qquad (31)$$

if $(\omega a^2/\nu_0)^{\frac{1}{2}} \ll \rho_0/\rho_1$ and $(\omega a^2\rho_1 c_{p1}/\sigma_1)^{\frac{1}{2}} \ll \frac{\sigma_0}{\sigma_1}$

In formula (31) we used the relation $\gamma_0 = c_{p0}/c_{v0}$. The first factor of (31) can be interpreted as the ratio of the average specific heats of the mixture, and the second as the ratio of the pressure over the effective density of the mixture.

These two limits of high and low frequency give us the frozen and equilibrium sound speeds obtained by Rudinger (6) (see also Marble (7)).

2.2.2 Dispersion Relation

In 2.2.1 we obtained an expression for the speed of sound that is supposed to hold under completely general conditions. However, in that expression there are two constants, K and K', whose values are not always known. We estimated them only for some limiting values of the sound frequency. We now want to obtain K and K' for any sound frequency and any type of suspension. However, to solve this problem we will have to put even more restrictive conditions on the volume fraction X. We will assume that X is so small that the speed of sound of the mixture is only in first approximation affected by the presence of the suspended phase.

Suppose that we have a spherical particle (drop or bubble) suspended in a fluid in which there is a plane sound wave propagating. We want to calculate the perturbed velocity and temperature of such suspension and compare them with the perturbed velocity and temperature of the fluid; this will give us K and K'. Carhart (11) calculated the flow fields inside and outside a particle under these conditions. First we are going to give a summary of Carhart's (11) results:

Suppose $\varphi_i e^{-i\omega t}$ is the incident wave, where:

$$\varphi_i = e^{ik_0 x} = e^{ik_0 r\cos\theta} = \sum_{n=0}^{\infty} i^n(2n+1)P_n(\cos\theta)j_n(k_0 r) \tag{32}$$

$j_n(z)$ are the spherical Bessel functions:

$$j_0(z) = \frac{\sin z}{z}\ , \quad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}\ ,$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z \ldots\ldots \tag{32a}$$

The values of P_n are given in formula (37d). k_0 is the wave number:

$$k_0 = \omega/c_0 \tag{33}$$

r is the radial distance from the center of the suspended sphere that we take as the origin of coordinates. The two other spherical coordinates are θ and ψ (azimuthal coordinate). Carhart (11) found that the velocity, pressure and temperature fields outside the particle are:

$$\underline{v}_0^* = [-\nabla(\varphi_i + \varphi_0 + \varphi_0') + \nabla \times \underline{A}_0]\, e^{-i\omega t} \tag{34a}$$

$$p_0^* = -i\omega \rho_0 [\varphi_i + \varphi_0 + \gamma_0' \varphi_0']\, e^{-i\omega t} \tag{34b}$$

$$T_0^* = [\alpha_0(\varphi_i + \varphi_0) + \alpha_0' \varphi_0']\, e^{-i\omega t} \tag{34c}$$

The scattered waves φ_0, φ_0', and $\underline{A}_0$ are:

$$\varphi_0 = \sum_{n=0}^{\infty} i^n (2n+1) P_n(\cos\theta)\, B_{n0}\, h_n(k_0 r) \tag{35a}$$

$$\varphi_0' = \sum_{n=0}^{\infty} i^n (2n+1) P_n(\cos\theta)\, C_{n0}\, h_n(k_0' r) \tag{35b}$$

$$A_{0\psi} = \sum_{n=1}^{\infty} i^n (2n+1) P_n^1(\cos\theta)\, D_{n0}\, h_n(k_0'' r) \tag{35c}$$

$$A_{0\theta} = A_{0r} = 0 \tag{35d}$$

The coefficients determining p_0^* and T_0^* are:

$$\gamma_0' = 1 - 4\left(1 + \frac{\eta_0}{2\mu_0}\right) \nu_0 \rho_0\, c_{p0}/3\sigma_0 \tag{36a}$$

$$\alpha_0 = -i\omega(\gamma_0 - 1)/(\alpha_{v0} c_0^2) \tag{36b}$$

$$\alpha_0' = -(\rho_0 c_{p0})/(\alpha_{v0} \sigma_0) \tag{36c}$$

η and μ are the normal and shear viscosities respectively. The wave numbers corresponding to the scattered waves are:

$$k_0' = (1+i) \left(\frac{\omega \rho_0 c_{p0}}{2\sigma_0}\right)^{\frac{1}{2}} \tag{37a}$$

$$k''_0 = (1+i)\left(\frac{\omega}{2\nu_0}\right)^{\frac{1}{2}} \tag{37b}$$

$h_n(z)$ are the spherical Hankel functions:

$$h_0(z) = -i\,\frac{e^{iz}}{z}\ , \quad h_1(z) = -e^{iz}\left(\frac{i}{z^2} + \frac{1}{z}\right)$$

$$h_2(z) = -e^{iz}\left(\frac{3i}{z^3} + \frac{3}{z^2} - \frac{i}{z}\right), \ \ldots\ldots. \tag{37c}$$

and $P_n^m(z)$ are the Legendre functions:

$$P_0(z) = 1\ , \quad P_1(z) = z,\ \ldots\ldots.$$

$$P_n^1(z) = -(1-z^2)(dP_n/dz) \tag{37d}$$

The expansions (35) are supposed to be made for small values of a_0:

$$a_0 = k_0 a << 1 \tag{38}$$

The wavelength is much longer than the radius of the particle. The coefficients of the expansions (35) decay with n like increasing powers of a_0; consequently, only the first coefficients will be needed:

$$B_{00} = \frac{ia_0^3}{3}\left[\left(\frac{\rho_0}{\rho_1}\right)\left(\frac{a_1}{a_0}\right)^2 - 1\right] + \left(\frac{\sigma_0}{\sigma_1}\,\frac{\alpha'_0}{\alpha'_1} - 1\right) i a_0 h_1(a'_0) a'_0 C_{00} =$$

$$= O(a_0^3) \tag{39a}$$

$$C_{00} = \frac{\alpha_0}{\alpha'_0}\ \frac{\left(\frac{\rho_0}{\rho_1}\,\frac{\alpha_1}{\alpha_0} - 1\right)}{h_0(a'_0)\left[1 - \frac{\sigma_0}{\sigma_1}\,\frac{j_0(a'_1)}{a'_1 j_1(a'_1)}\quad \frac{a'_0 h_1(a'_0)}{h_0(a'_0)}\right]} = O(a_0^2) \tag{39b}$$

$$B_{10} = \frac{ia_0^3}{3}\left(1 - \frac{\rho_0}{\rho_1}\right)\left[\frac{b_0 h_2(b_0)\,\Theta + \Gamma}{\left[-\left(2+\frac{\rho_0}{\rho_1}\right) + 9\,\frac{\rho_0}{\rho_1}\,h_1(b_0)\right]\Theta + \left(2 + \frac{\rho_0}{\rho_1}\right)\Gamma}\right] = O(a_0^3)^{*}$$

(39c)

where $\Theta = \left(1 - \frac{\mu_0}{\mu_1}\right) b_1\, j_2(b_1) - \tfrac{1}{2} b_1^2\, j_1(b_1)$

$$\Gamma = \left(\tfrac{1}{2}\,\frac{\rho_0}{\rho_1}\, b_1^2\right) b_1\, j_2\,(b_1)\, h_1\,(b_0)$$

$$C_{10} = \frac{\sigma_0}{\sigma_1}\;\frac{\alpha_0'}{\alpha_1'}\;\frac{a_0' h_1(a_0')}{a_1' j_1(a_1')}\; C_{00} = O(a_0^3) \tag{39d}$$

$$D_{10} = O(a_0)$$

where:

$$b_0 = ak_0'' = (1+i)\,(\omega a^2/(2\nu_0))^{\frac{1}{2}} \tag{40a}$$

$$a_0' = ak_0' = (1+i)\,(\omega a^2 \rho_0 c_{p0}/(2\sigma_0))^{\frac{1}{2}} \tag{40b}$$

The coefficients γ_1', α_1, α_1', k_1, k_1'', a_1 and a_1' have the same form as the corresponding terms with subindex zero [(36), (37), (38) and (40)], but instead of being defined by the external fluid properties, they are defined by the properties of the substance forming the suspension (subindex one).

---

* We think that there is a mistake in Carhart's result. Before the second term of the numerator Carhart has a minus sign that we think should be a plus.

The velocity, pressure, and temperature fields inside the suspended sphere are given also by Carhart (11):

$$\underline{v}_1^* = [-\nabla(\varphi_1 + \varphi_1) + \nabla x\, \underline{A}_1] e^{-i\omega t} \tag{41a}$$

$$p_1^* = -i\omega\rho_1 (\varphi_1 + \gamma_1' \varphi_1') e^{-i\omega t} \tag{41b}$$

$$T_1^* = (\alpha_1 \varphi_1 + \alpha_1' \varphi_1') e^{-i\omega t} \tag{41c}$$

where

$$\varphi_1 = \sum_{n=0}^{\infty} i^n (2n+1) P_n(\cos\theta) B_{n1} j_n(k, r) \tag{42a}$$

$$\varphi_1' = \sum_{n=0}^{\infty} i^n (2n+1) P_n(\cos\theta) C_{n1}\, j_n(k_1' r) \tag{42b}$$

$$A_1 = \sum_{n=1}^{\infty} i^n (2n+1) P_n^1(\cos\theta) D_{n1}\, j_n(k_1'' r) \tag{42c}$$

$$A_{1\theta} = A_{1r} = 0 \tag{42d}$$

These expansions are supposed to be for small values of a_1:

$$a_1 = \omega a / c_1 << 1$$

The coefficients decay with n like increasing powers of a_1. The values of the coefficients are:

$$B_{01} = \rho_0/\rho_1 = O(1) \tag{43a}$$

$$C_{01} = \frac{\sigma_0}{\sigma_1} \frac{\alpha_0'}{\alpha_1'} \frac{a_0' h_1(a_0')}{a_1' h_1(a_1')} C_{00} = O(a_1^2) \tag{43b}$$

$$B_{11} = - \frac{9(\rho_0/\rho_1)i}{[1-(\rho_0/\rho_1)]} \frac{B_{10}}{a_1 a_0^2} = O(1) \tag{43c}$$

$$C_{11} = O(a_1^3) \tag{43d}$$

$$D_{11} = O(a_1) \tag{43e}$$

At this point Carhart (11) proceeds to calculate the dissipation terms of the energy equation, and from this he obtains the damping. Instead of doing that, we are going to calculate K and K', take these values into (15), (16) and (22), and obtain the speed of sound of the mixture. The imaginary part of our sound speed will give the same damping calculated by Carhart (11). Since the radius of the sphere is small compared to the wavelength, we can say that the changes in the flow field created by the presence of the suspended sphere will die away over distances that are small compared to the wavelength. We then define the velocity and temperature of the external flow as the velocity and temperature at the origin given by the incident wave only (the velocity and temperature of the external fluid at the origin when the suspended sphere is absent). The velocity will have only the x component. From (34):

$$\tilde{u}_0 = - \left[\frac{\partial}{\partial x} (\varphi_i e^{-i\omega t})\right]_{x=0} = - \frac{\partial}{\partial x} \left[e^{i(k_0 x - \omega t)}\right]_{x=0}$$

$$= -ik_0 e^{-i\omega t} \tag{44a}$$

$$\tilde{T}_0 = \alpha_0 \left[e^{i(k_0 x - \omega t)}\right]_{x=0} = \alpha_0 e^{-i\omega t} \tag{44b}$$

For the velocity and temperature of the suspended particle we take the average of these values inside the particle (drop or bubble):

$$\underset{\sim}{\tilde{u}}_1 = \frac{1}{(4\pi a^3/3)} \int_{\substack{\text{volume}\\ \text{particle}}} \underline{v}_1^* \, dV \tag{45a}$$

$$\tilde{T}_1 = \frac{1}{(4\pi a^3/3)} \int_{\substack{\text{volume}\\ \text{particle}}} T_1^* \, dV \tag{45b}$$

Looking at (42) and (43) we see that the B_{01} and C_{01} terms give a potential that depends only on the radial distance; consequently, they only produce symmetrical radial velocities that do not give any net velocity of the particle when the integration (45a) is carried out. The lowest order term that gives a contribution to the velocity of the sphere that does not cancel when we perform the integration (45a) is B_{11} (of expression (42a)). Using for j_1 the value given in (32a) we get:

$$\varphi_1 + \varphi_1' = 3i \cos\theta \, B_{11} \left(\frac{\sin k_1 r}{(k, r)^2} - \frac{\cos k_1 r}{k_1 r} \right) + \text{symmetrical terms} + O(a_1)$$

The vector potential $\underline{A}_1$ gives smaller order contributions. Since $k_1 r$ is at most $k_1 a$, we can expand the above expression for small values of $k_1 a$:

$$\varphi_1 + \varphi_1' \simeq 3i \cos\theta \, B_{11} \left[\frac{1}{k_1 r} - \frac{k_1 r}{6} \, \ldots\ldots - \frac{1}{k_1 r} + \frac{k_1 r}{2} + \ldots. \right]$$

$$\simeq i B_{11} \, k_1 r \cos\theta = i B_{11} \, k_1 x$$

This potential produces a uniform velocity in the x direction (see (41a)):

$$V_{1x}^{*} = -i\, B_{11}\, k_1\, e^{-i\omega t}, \quad v_{1y}^{*} = v_{1z}^{*} = 0$$

Since this quantity is constant, the integration (45a) gives the same value for the velocity of the particle, and using (44a) we get

$$K = \frac{\tilde{u}_1}{\tilde{u}_0} = \frac{-iB_{11}k_1 e^{-i\omega t}}{-ik_0\, e^{-i\omega t}} = B_{11}\,\frac{k_1}{k_0} \tag{46}$$

The temperature T_1^* is given in (41c). Using (45b) we get:

$$T_1^{*} = \frac{e^{-i\omega t}}{(4\pi a^3/3)} \int_{\substack{\text{volume}\\ \text{particle}}} (\alpha_1\varphi_1 + \alpha_1'\varphi_1')\, dV =$$

$$= \frac{e^{-i\omega t}}{(4\pi a^3/3)} \int_{\substack{\text{volume}\\ \text{particle}}} [\alpha_1 B_{01} j_0(k_1 r) + \alpha_1' C_{01} j_0(k_1' r)]\, dV + O(a_1^3)$$

The term B_{11} although of the same order as B_{01} will not give any net contribution (because of the $\cos\theta$ term) when integrated over the volume of the sphere. Since $k_1 r$ is small we have: (see (32a))

$$j_0(k_1 r) \simeq 1$$

However, $k_1' r$ is not small:

$$\int_{\substack{\text{volume}\\ \text{particle}}} j_0(k_1' r)\, dV = \int_0^a \frac{\sin(k_1' r)}{k_1' r}\, 4\pi r^2 dr =$$

$$= \frac{4\pi \sin(k_1' a)}{{k_1'}^3} - \frac{4\pi a \cos k_1' a}{{k_1'}^2}$$

Then (since $k_1' a$ is equal to a_1' (40b))

$$\tilde{T}_1 = [\alpha_1 B_{01} + \frac{3\alpha_1' C_{01}}{a_1'^3} (\sin a_1' - a_1' \cos a_1')] e^{-i\omega t}$$

From this result and (44b) we get:

$$K' = \frac{\tilde{T}_1}{\tilde{T}_0} = \frac{\alpha_1}{\alpha_0} B_{01} + \frac{3\alpha_1' C_{01}}{a_1'^3 \alpha_0} (\sin a_1' - a_1' \cos a_1') \qquad (47)$$

(46) and (47) give the values of K and K' as functions of the unperturbed properties of the mixture and the frequency of sound. Putting these values in (15), (16) and (22) we get the dispersion relation that we are looking for.

However, when we calculated K and K', we assumed that at infinity there was a sound wave propagating with the speed of sound of the external flow; because of that assumption this method is expected to be valid only if the speed of sound of the mixture is a perturbation of the speed of sound of the external flow. Expanding (15), (16) and (22) for small X, and then putting into this expansion the values of K and K' previously calculated (46) (47) (using also relations (39) and (43)) we get, after some algebraic manipulation:

$$c = c_0 [1 + \frac{2\pi n i}{(\omega/c_0)^3} (B_{00} + 3B_{10}) + O(X^2)] \qquad (48)$$

$$X = \frac{4}{3} \pi a^3 n \qquad (49)$$

n is the number of suspended particles (or bubbles) per unit volume of mixture. For details of the calculation see Appendix B. Formula (48) gives the speed of sound of the mixture as a function of the coefficients of the scattered waves in the external flow field*. In section 2.2.6 we will give an interpretation of this result.

It is also easy to check that to a first approximation the assumption of mechanical equilibrium ((12) with surface tension neglected) holds. From (41b) and (43) we get:

$$\tilde{p}_1 = \langle p_1^* \rangle = -i\omega\rho_1 \, B_{01} \, e^{-i\omega t} + O(a_1) \simeq - i\omega\rho_0 e^{-i\omega t} = \tilde{p}_0$$

$\langle \, \rangle$ means the average value over the particle (see formulae (45)). The term $[B_{11} \, P_1(\cos\theta)]$ of expansion (42) gives zero average. The term $(-i\omega\rho_0)e^{-i\omega t}$ is the pressure at the origin when there is no drop or bubble there (see (34b) and (32)).

The damping will be given by the imaginary part of the speed of sound (48)

$$c = c_{real} + i \, c_{imag.} = [c_0 + O(X)] + i \, c_{imag.}$$

$$c_{imag.} = O(X), \quad c_{imag.} << c_0$$

The properties of the mixture will have the functional form:

$$\exp[i\omega(\frac{x}{c} - t)] = \exp\left[i\omega\left(\frac{x}{c_0 + ic_{imag}} - t\right)\right] \simeq$$

$$\simeq \exp\left[i\omega\left(\frac{x}{c_0} - t\right)\right] \exp\left[\omega \frac{x \, c_{imag}}{c_0^2}\right]$$

The damping factor is then:

---

*K and K' were obtained as functions of the coefficients of the scattered waves inside the particle; after the algebraic manipulation we obtained c (48) as a function of the coefficients of the scattered waves outside the particle.

$$\alpha_d = \frac{-\omega\, c_{imag}}{c_0^2} \text{ per unit length} \tag{50}$$

Using (48), (17g) and (39a,b) we get (see also Appendix B):

$$\alpha_d = \frac{2\pi n a}{c_0}\,\frac{\sigma_0}{\rho_0 c_{p0}}\,(\gamma_0 - 1)\left(1 - \frac{\rho_0}{\rho_1}\,\frac{\alpha_1}{\alpha_0}\right)^2 \text{Real}\left\{\frac{a_0' h_1(a_0')/h_0(a_0')}{1 - \frac{\sigma_0}{\sigma_1}\,\frac{j_0(a_1')}{a_1' j_1(a_1')}\,\frac{a_0' h_1(a_0')}{h_0(a_0')}}\right\}$$

$$- \frac{2\pi n}{(\omega/c_0)^2}\,\text{Real}\left\{3B_{10}\right\} \tag{51}$$

which coincides with Carhart's (11) result. The energy dissipation is obtained by multiplying the above result by two:

$$\text{Energy} \sim \left\{\exp\left[i\omega\left(\frac{x}{c} - t\right)\right]\right\}^2 = \exp\left[2i\omega\left(\frac{x}{c} - t\right)\right]$$

The first term of (51) gives the thermal damping, and the second the viscous damping.

2.2.3 Limiting Values of K and K'

As indicated in section 2.2.1, it is expected that (if the volume fraction occupied by the particles is small) for low frequencies both components will move at the same velocity ($K = 1$), and they will have the same perturbed temperature ($K' = 1$). For high frequencies it is expected that the species will move according to formula (26) and that they will behave isentropically (no heat transfer between the phases). We are going to see that the results of the previous section follow these rules.

The value of K given in (46) can be put in the form (after using

(43c), (39c) and the identity $\frac{\mu_0}{\mu_1} b_0^2 = \frac{\rho_0}{\rho_1} b_1^2$ (40a))

$$K = \frac{3\rho_0}{2\rho_1+\rho_0}\left[\frac{b_0 h_2(b_0)+b_0^2 h_1(b_0)G}{\left(\frac{9\rho_0}{2\rho_1+\rho_0}\right)h_1(b_0)-b_0 h_0(b_0)+b_0^2 h_1(b_0)G}\right] \tag{52}$$

where

$$G=\tfrac{1}{2}\left(\frac{\mu_0}{\mu_1}\right) b_1 j_2(b_1)/[(1-\frac{\mu_0}{\mu_1})b_1 j_2(b_1)-\tfrac{1}{2} b_1^2 j_1(b_1)] \tag{53}$$

It is easy to check that G can never be large. For such small frequencies that:

$$\left(\frac{\omega a^2}{2\nu_0}\right)^{\frac{1}{2}} << 1 \qquad \text{or} \qquad |b_0| << 1 \tag{54a}$$

we have (see (37c)):

$$b_0\, h_2\,(b_0) \simeq -3/b_0^3 \simeq 3\, h_1(b_0) \tag{54b}$$

$$h_n(b_0) = O(1/b_0^{n+1}) \tag{54c}$$

In order to neglect the term $(b_0 h_0(b_0))$ of the denominator of (52), we have to assume also:

$$|b_0^2| << \rho_0/(2\rho_1 + \rho_0) \tag{54d}$$

Using relations (54) we obtain (G can never be large):

$$K = 1 \tag{55}$$

if

$$\left(\frac{\omega a^2}{\nu_0}\right)^{\frac{1}{2}} << 1\ , \qquad \text{and } \frac{\omega a^2}{\nu_0} << \frac{\rho_0}{2\rho_1+\rho_0}$$

In the case of bubbles both conditions are equivalent ($\rho_0 >> \rho_1$)

$$\left(\frac{\omega a^2}{\nu_0}\right)^{\frac{1}{2}} << 1 \qquad \text{(bubbles)}$$

In the case of drops or solid particles the second condition is more restrictive:

$$\left(\frac{\omega a^2}{\nu_0}\right) << \frac{\rho_0}{\rho_1} << 1 \quad \text{(drops or solid particles)}$$

These two results mean that it is easier for a heavy liquid to drive along a bubble than for a light gas to move a heavy particle.

In the high frequency limit we assume that:

$$\left(\frac{\omega a^2}{\nu_0}\right)^{\frac{1}{2}} >> 1 \quad \text{or} \quad |b_0| >> 1$$

Then (see (37c)):

$$h_n(b_0) = O(1/b_0)$$

$$b_0 h_0(b_0) \simeq -i\, e^{ib_0} \simeq -b_0 h_2(b_0)$$

and since $(9\rho_0/(2\rho_1+\rho_0))$ can never be large, the numerator and denominator of the bracket of (50) become the same, and:

$$K = \frac{3\rho_0}{2\rho_1+\rho_0} \tag{56}$$

if

$$\left(\frac{\omega a^2}{2\nu_0}\right)^{\frac{1}{2}} >> 1$$

This result checks with Lamb's (13, sect. 298).

Now let us examine K'. By using (36b) and (17f) we get:

$$\frac{\alpha_1}{\alpha_0}\frac{\rho_0}{\rho_1} = \frac{\alpha_{v1}}{\alpha_{v0}}\frac{c_{p0}}{c_{p1}}\frac{\rho_0}{\rho_1}$$

taking this into (47) and using (43a, b) and (39b) we get:

$$K' = \frac{\alpha_{v1}c_{p0}\rho_0}{\alpha_{v0}c_{p1}\rho_1} + H\left(\frac{\alpha_{v1}}{\alpha_{v0}}\frac{c_{p0}}{c_{p1}}\frac{\rho_0}{\rho_1} - 1\right) \tag{57a}$$

where

$$H = \frac{3\sigma_0}{\sigma_1}\frac{a'_0 h_1(a'_0)}{a_1'^2 h_0(a'_0)}\frac{1}{\left[1 - \frac{\sigma_0}{\sigma_1}\frac{j_0(a'_1)}{a'_1 j_1(a'_1)}\frac{a'_0 h_1(a'_0)}{h_0(a'_0)}\right]} \tag{57b}$$

Let us examine the low frequency limit. We assume:

$$|a'_1| << 1 \quad \text{or} \quad \left(\frac{\omega a^2 \rho_1 c_{p1}}{2\sigma_1}\right)^{\frac{1}{2}} << 1$$

Then from (32a):

$$j_1(a'_1) = \frac{\sin a'_1}{a_1'^2} - \frac{\cos a'_1}{a'_1} \simeq \frac{a'_1}{3} + O(a_1'^2)$$

$$j_0(a'_1) = 1 + O(a'_1)$$

$$\frac{j_0(a'_1)}{a'_1\, j_1(a'_1)} \simeq \frac{3}{a_1'^2} + O(1/a'_1) \tag{58}$$

We assume also that:

$$\frac{\sigma_0}{\sigma_1}\frac{1}{|a'_1|^2} >> 1$$

Then the second term in the bracket of the denominator of (57b) is much larger than one (it is easy to show that $[a'_0 h_1(a'_0)/h_0(a'_0)]$ can never be small). Using this and (58) we get:

$$H = -1$$

and (57a) gives:

$$K' = 1 \tag{59}$$

if $(\omega a^2 \rho_1 c_{p1}/\sigma_1)^{\frac{1}{2}} << 1$

and $(\omega a^2 \rho_1 c_{p1}/\sigma_1) << \sigma_0/\sigma_1$

In the case of bubbles the first condition is more restrictive. The heat conductivity of a gas is much lower than that of a liquid or a solid. In the case of solid particles in suspension, $\sigma_0 << \sigma_1$, the second condition should be used. This means that it is easier for a liquid to communicate its temperature to a bubble than for a gas to heat or cool a heavy particle. At sufficiently low frequencies both species have the same temperature.

In the high frequency limit we assume:

$$|a_1'| >> 1 \quad \text{or} \quad (\omega a^2 \rho_1 c_p/\sigma_1)^{\frac{1}{2}} >> 1$$

Then from (32a) we get:

$$\frac{j_0(a_1')}{a_1' j_1(a_1')} \simeq \frac{\tan a_1'}{a_1'} \qquad \text{for} \quad |a_1'| >> 1$$

and the value of H (57b) becomes:

$$H \simeq \frac{1}{a_1'} \; \frac{3Z}{a_1' - Z\tan a_1'}$$

where $Z = \dfrac{\sigma_0}{\sigma_1} \dfrac{a'_0 h_1(a'_0)}{h_0(a'_0)}$

Clearly, H goes always to zero for large $|a'_1|$, because if Z becomes large then it would cancel with the term in the denominator

$$H = 0$$

and from (57a):

$$K' = \frac{\alpha_{v1} c_{p0} \rho_0}{\alpha_{v0} c_{p1} \rho_1} \tag{60}$$

$$\text{if } (\omega a^2 \rho_1 c_{p1}/\sigma_1)^{\frac{1}{2}} >> 1$$

This value of K' gives us the isentropic behavior (see (22) with surface tension neglected). At high frequencies there is no heat transfer between the phases, and both species behave adiabatically.

2.2.4 Case of Drops or Solid Particles in Suspension

In this case we assume:

$$\rho_0 << \rho_1, \quad \mu_0 << \mu_1, \quad \sigma_0 << \sigma_1, \quad \alpha_{v0} >> \alpha_{v1} \tag{61}$$

We want to calculate the value of the speed of sound of the mixture (48) under these conditions. Formula (48) is only valid for X so small that:

$$c \simeq c_0 + O(X) \tag{62}$$

Under assumptions (61) formula (48) gives:

$$c = c_0 \left\{ 1 + \frac{2}{3}\pi na^3 - \pi na^3(\gamma_0 - 1)\left[\frac{4z_0^3 + 6\frac{\rho_0}{\rho_1}\frac{c_{p0}}{c_{p1}}(1+z_0+z_0^2) + 4(1+z_0)z_0^2 i}{4z_0^4 + 12\frac{\rho_0}{\rho_1}\frac{c_{p0}}{c_{p1}}z_0^3 + 9\left(\frac{\rho_0}{\rho_1}\frac{c_{p0}}{c_{p1}}\right)^2}\right]\right.$$

$$\left. -2\pi na^3\left[\frac{8y^4 + 12y^3 + 27\left(\frac{\rho_0}{\rho_1}\right)(2y^2+2y+1) + 12y^2(y+1)i}{16y^4 + 72\left(\frac{\rho_0}{\rho_1}\right)y^3 + 81\left(\frac{\rho_0}{\rho_1}\right)^2(1+2y+2y^2)}\right]\right\} \quad (63)$$

where

$$y = \left(\frac{\omega a^2}{2\nu_0}\right)^{\frac{1}{2}} \quad z_0 = \left(\frac{\omega a^2 \rho_0 c_{p0}}{2\sigma_0}\right)^{\frac{1}{2}} \quad X = \frac{4}{3}\pi a^3 n$$

The details of the calculation are shown in Appendix C. n is the number of drops per unit volume.

In the low frequency limit we have (see 2.2.3):

$$y << \rho_0/\rho_1$$

$$z_1^2 << \sigma_0/\sigma_1$$

but since:

$$z_1^2 = \frac{\rho_1 c_{p1}\sigma_0}{\rho_0 c_{p0}\sigma_1}\, z_0^2 \quad (64)$$

we have:

$$z_0^2 << \frac{\rho_0 c_{p0}}{\rho_1 c_{p1}} << 1$$

and (63) reduces to:

$$c = c_0[1 + \frac{X}{2} - (\gamma_0 - 1)\,\frac{3X}{4}\,(\frac{2}{3}\,\frac{\rho_1 c_{p1}}{\rho_0 c_{p0}}) - \frac{X}{2}\,\frac{\rho_1}{\rho_0}]$$

and since $\rho_1 >> \rho_0$, we get:

$$c = c_0 [1 - \frac{X}{2} (\gamma_0 - 1) \frac{\rho_1 c_{p1}}{\rho_0 c_{p0}} - \frac{X}{2} \frac{\rho_1}{\rho_0}]$$

which coincides with formula (31) and Marble's (7) result (if X is so small that expansion (62) can be made).

In the high frequency limit (see 2.2.3):

$$y >> 1$$

$$z_1 >> 1$$

Using (64) we have:

$$z_0^2 >> \frac{\rho_0 \, c_{p0} \sigma_1}{\rho_1 \, c_{p1} \sigma_0} >> \frac{\rho_0 c_{p0}}{\rho_1 c_{p1}}$$

If we assume that $(\rho_0 c_{p0})/(\rho_1 c_{p1})$ is of the order (σ_1/σ_0) we have also:

$$z_0 >> 1$$

Then (63) becomes:

$$c = c_0 \, (1 + \frac{X}{2} - \frac{X}{4}) = c_o (1 - \frac{X}{4}) \tag{66}$$

which coincides with formula (30c). At high frequencies the speed of sound of the mixture (for drops or solid particles in suspension) is only affected (in first approximation) by the volume occupied by the suspension. This means that, in this limit, expansion (62) is valid for any X much smaller than one.

Under assumptions (61) the values of K (52) and K' (56) are (see Appendix C):

$$K = \frac{3\rho_0}{\rho_1}\left[\frac{8y^4+12y^3+27\delta(2y^2+y+1)+12y^2(y+1)i}{16y^4+72\delta y^3+81\delta^2(1+2y+2y^2)}\right] \tag{67}$$

$$K' = \frac{\alpha_{v1}c_{p0}\rho_0}{\alpha_{v0}c_{p1}\rho_1} + H\left(\frac{\alpha_{v1}}{\alpha_{v0}}\;\frac{c_{p0}}{c_{p1}}\;\frac{\rho_0}{\rho_1} - 1\right) \tag{68a}$$

where

$$H = -\frac{3}{2}\frac{\rho_0}{\rho_1}\frac{c_{p0}}{c_{p1}}\left[\frac{4z_0^3+6\,\frac{\rho_0 c_{p0}}{\rho_1 c_{p1}}\,(1+z_0+z_0^2)+4(1+z_0)z_0^2\, i}{4z_0^4+12\,\frac{\rho_0 c_{p0}}{\rho_1 c_{p1}}\,z_0^3+9(\frac{\rho_0 c_{p0}}{\rho_1 c_{p1}})^2}\right] \tag{68b}$$

This value of K can be obtained from Lamb (13, sect. 363, formulae (36) (39)) after a straightforward manipulation. His result applies under only the condition:

$$X \ll 1 \tag{68c}$$

If we could calculate K' using only assumption (68c), then by putting K and K' in (15), (16) and (22) we would be able to obtain speeds of sound of the mixture that would be significantly different from the speed of sound of the external flow, c_0. However, to calculate K' in general, we have to use the more restrictive assumption (62).

The damping is: (formula 50))

$$\alpha_d = -\frac{\omega c_{imag}}{c_0^2} = \frac{2\pi ma}{c_0}(\gamma_0-1)\left(\frac{\sigma_0}{\rho_0 c_{p0}}\right)(1+z_0)$$

$$\left[\frac{4z_0^4}{4z_0^4+12\,\frac{\rho_0 c_{p0}}{\rho_1 c_{p1}}\,z_0^3+9(\frac{\rho_1 c_{p0}}{\rho_1 c_{p1}})^2}\right] + \left(\frac{2\pi ma}{c_0}\right)\nu_0\left(\frac{3}{2}\right)(1+y)\;\;\Lambda \tag{69}$$

where $\Lambda = \left[\dfrac{16y^4}{16y^4+72\delta y^3+81\delta^2(1+2y+2y^2)}\right]$

which coincides with Carhart's (11) result.

2.2.5 Case of Bubbles in Suspension

In this case:

$$\rho_0 >> \rho_1, \quad \mu_0 >> \mu_1, \quad \sigma_0 >> \sigma_1, \quad \alpha_{v0} << \alpha_{v1}, \quad c_0 >> c_1 \qquad (70)$$

We want to see the form that the expression of the speed of sound (48) takes under these conditions. Expression (48) is only valid if the volume concentration of bubbles, X, is so small that the speed of sound of the mixture is a small perturbation of the speed of sound of the liquid. From (48) and (70) we get:

$$c = c_0 \Bigg\{ 1 - \frac{2\pi n a^3}{3}\left(\frac{\rho_0 c_0^2}{\rho_1 c_1^2}\right) - \pi n a^3 (\gamma_1 - 1)\left(\frac{\rho_0 c_0^2}{\rho_1 c_1^2}\right)$$

$$\left[\frac{1}{z_1}\,\frac{\sinh 2z_1 - \sin 2z_1}{\cosh 2z_1 - \cos 2z_1} + \frac{i}{z_1^2}\left(z_1\,\frac{\sinh 2z_1 + \sin 2z_1}{\cosh 2z_1 - \cos 2z_1} - 1\right)\right] +$$

$$+ 2\pi n a^3 \left[\frac{(2y^6+6y^5+9y^4+24y^3+54y^2+54y+27)-(12y^4+24y^3+18y^2)i}{2y^6+6y^5+9y^4+36y^3+162y^2+162y+81}\right]\Bigg\} \qquad (71)$$

where:

$$y = (\omega a^2/2\nu_0)^{\frac{1}{2}}, \quad z_1 = (\omega a^2 \rho_1 c_{p1}/2\sigma_1)^{\frac{1}{2}},$$

$$X = \frac{4}{3}\pi a^3 n$$

For details of the calculation see Appendix D. For low frequencies (see 2.2.3):

$$z_1 << 1$$
$$y << 1$$

Expanding sinh, cosh, sin, and cos to third order in z, we get:

$$c = c_0 \left(1 - \frac{X}{2} \frac{\rho_0 c_0^2}{\rho_1 c_1^{*2}} + \frac{X}{2}\right) \tag{72}$$

where $c_1^{*2} = c_1^2/\gamma_1$ (c_1^* isothermal sound speed)

For high frequencies ($z_1 >> 1$, $y_1 >> 1$) we get from (71):

$$c = c_0 \left(1 - \frac{X}{2} \frac{\rho_0 c_0^2}{\rho_1 c_1^2} + \frac{3X}{2}\right) \tag{73}$$

Formula (72) can be obtained also from (15), (16) and (22) by taking for the speed of sound of the bubbles the isothermal limit, c_1^*, and assuming that bubbles and liquid move at the same velocity. Formula (73) can be obtained by using the adiabatic speed of sound and taking for K the value three. Formulae (15), (16) and (22) have to be expanded in powers of X in order to obtain (72) and (73). This requires such small values of X that in (72) and (73) we can neglect the third term of the bracket compared to the second.

The values of K and K' given in (52) and (56) become (on using (70)):

$$K = 3 \left[\frac{(2y^6+6y^5+9y^4+24y^3+54y^2+54y+27) - i(12y^4+24y^3+18y^2)}{2y^6+6y^5+9y^4+36y^3+162y+81} \right] \tag{74}$$

$$K' = \frac{\alpha_{v1} c_{p0} \rho_0}{\alpha_{v0} c_{p1} \rho_1} + H \left(\frac{\alpha_{v1} c_{p0} \rho_0}{\alpha_{v0} c_{p1} \rho_1} - 1\right) \tag{75a}$$

$$\text{where} \quad H = \frac{3}{2}\left\{- \frac{1}{z_1}\left[\frac{\sinh 2z_1 - \sin 2z_1}{\cosh 2z_1 - \cos 2z_1}\right] + i\left[\frac{1}{z_1^2} - \frac{1}{z_1}\,\frac{\sinh 2z_1 + \sin 2z_1}{\cosh 2z_1 - \cos 2z_1}\right]\right\} \tag{75b}$$

The value of K given in (74) can be deduced by doing an analysis similar to the one made by Lamb (13, section 361). Lamb considers a sphere moving like a rigid body under the beating of a sound wave propagating through a fluid. He assumes that at the surface of the sphere the tangential fluid velocity is the same as the one of the sphere. If we do the same analysis, but assuming instead that at the surface of the bubble the tangential stress is zero, we get the value of K given in (74). In the analysis made by Lamb (13) it is not necessary to assume that the sound wave is propagating at infinity with the speed of sound of the fluid; consequently his analysis is valid for any X much smaller than one (and not so small that condition (62) is satisfied). If we could calculate K' assuming only that:

$$X \ll 1 \tag{75c}$$

then by putting K and K' in (15), (16) and (22), we would be able to get values of the speed of sound of the mixture quite different from the speed of sound of the liquid, c_0. However, the value of K' given in (75a) can in general be used only in cases for which:

$$c = c_0 + O(X)$$

This condition is more restrictive than (75c).

The damping is (formulae (50), (71)):

$$\alpha_d = -\frac{\omega c_{imag}}{c_0^2} = \frac{2\pi na}{c_0}\left(\frac{\rho_1 c_{p1}}{\sigma_1}\right)(\gamma_1 - 1)\left(\frac{\rho_0 c_0^2}{\rho_1 c_1^2}\right)\left[z_1 \frac{\sinh 2z_1 + \sin 2z_1}{\cosh 2z_1 - \cos 2z_1} - 1\right]$$
$$+ \frac{4\pi na}{c_0}\,\nu_0\left(\frac{12y^6+24y^5+18y^4}{2y^6+6y^5+9y^4+36y^3+162y^2+162y+81}\right) \qquad (76)$$

which coincides with Carhart's (11) result except for the numerator of the second term (viscous dissipation). We think that the reason for this is a numerical mistake made by Carhart (11) in the evaluation of the coefficient B_{10} (see formula (39c) and footnote). This viscous damping is determined by K, and as mentioned before we calculated K also by using Lamb's (13) analysis.

2.2.6 Alternative Derivation of the Speed of Sound

The results of section 2.2 and references (1) to (9) show that the sound speed in the mixture may be appreciably less than that of each component separately. In order to understand this paradoxical result, we consider an alternative approach valid for very small X (so that the speed of sound of the mixture can be expanded in powers of X) and based on a derivation of refractive index in optics (see Feynman (17, Chap. 31)).

Let us suppose that we have a plane sound wave travelling in a medium with speed c_0:

$$\varphi = \exp\left[ik(x - c_0 t)\right] , \qquad (77)$$

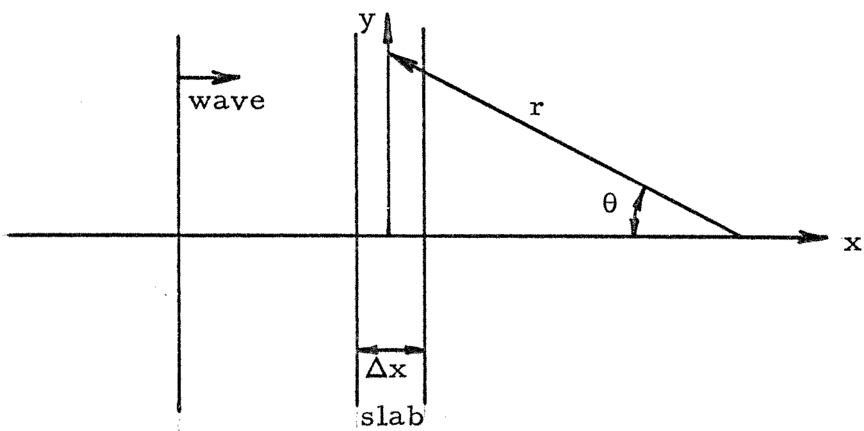
and there is a portion of spherical material (drop or bubble) in its way. The radius of the sphere is small compared to the wavelength. Then the intensity of the scattered wave in a direction θ and at a distance r (from the center of the sphere) large compared to the wavelength will be (formulae (34) and (35)):

$$\varphi'' = \varphi_0 = [B_{00}h_0(kr) + 3i\, B_{10}\cos\theta h_1(kr)]\, e^{-i\omega t} + o(k^3a^3) \qquad (78)$$

where we have neglected the viscous (35c) and thermal (35b) scattered waves because they decay exponentially (the term included in (78) decays algebraically as r increases).* Since (kr) is supposed to be large (see (37c)):

$$\varphi'' = \frac{-i\, e^{ikr}}{kr}\,(B_{00} + 3B_{10}\cos\theta)\, e^{-i\omega t} \qquad (79)$$

Let us suppose now that we have a slab with a density n of these spheres per unit volume. The volume fraction occupied by them is so small that we can in first approximation suppose that the sound wave is not affected by their presence. The thickness of the



*We should also notice that all the flow properties are defined in terms of φ_0 in exactly the same way as for the incident wave. This is not true for φ_0' and $A_{0\psi}$ (see formulae (34)), but fortunately we could neglect them.

slab is Δx and is much smaller than the wavelength. We can suppose that the slab occupies the plane $x = 0$. y is the radial cylindrical coordinate. We want to know what will be the additional flow field φ' produced by the scattering of these spheres at a point x of the x axis. Using (79):

$$\varphi' = \int_0^\infty \varphi'' \, n \, \Delta x \, 2\pi y \, dy$$

$$r^2 = x^2 + y^2$$

$$ydy = rdr$$

$$\cos\theta = \frac{x}{r}$$

$$\varphi' = - \left[n \, \Delta x \, 2r \int_x^\infty (B_{00} + 3B_{10} \frac{x}{r}) \frac{i\, e^{ikr}}{k} \, dr \right] e^{-i\omega t}$$

This integral is non-convergent. To avoid this we use the standard method of multiplying the integrand by a factor $[\exp[-\varepsilon(r-x)]]$ and let ε go to zero.

$$\lim_{\varepsilon \to 0} \int_x^\infty e^{ikr - \varepsilon(r-x)} \, dr = - \frac{e^{ikx}}{ik}$$

$$\lim_{\varepsilon \to 0} \int_x^\infty e^{ikr - \varepsilon(r-x)} \, \frac{x}{r} \, dr = - \frac{e^{ikx}}{ik} + O\left[(kx)^{-1}\right]$$

We assume that the distance x is large compared to the wavelength ($kx >> 1$). Then

$$\varphi' = \frac{2\pi n \Delta x}{k^2} (B_{00} + 3B_{10}) \, e^{i(kx - \omega t)} \tag{80}$$

The total field at the point considered is:

$$\varphi_T = \varphi' + \varphi \tag{81}$$

We are going to assume now that the slab has a different sound speed than the medium. Let us call it c. Then the field observed at the point x of the x axis (see diagram) is:

$$\varphi_T = e^{ikx} \, e^{i(\frac{\omega}{c} - k)\Delta x} \, e^{-i\omega t}, \qquad k = \omega/c_0$$

where the term $\Delta x(\frac{\omega}{c} - k)$ takes care of the difference in phase produced by the fact that the wave travels at a different speed in the slab. Since $\Delta x(\frac{\omega}{c} - k)$ is a small quantity:

$$\varphi_T \simeq e^{i(kx-\omega t)}[1 + i\Delta x\,(\frac{\omega}{c} - \frac{\omega}{c_0})]$$

Comparing this result with (77), (80) and (81):

$$[\frac{2\pi n\Delta x}{k^2}(B_{00} + 3B_{10}) + 1]\,e^{i(kx-\omega t)} = e^{i(kx-\omega t)}[1+i\Delta x(\frac{\omega}{c} - \frac{\omega}{c_0})]$$

this gives:

$$c = c_0\,[1 + \frac{2\pi n i}{(\omega/c_0)^3}(B_{00} + 3\,B_{10})] \tag{82}$$

where we used the fact that:

$$|(c-c_0)/c_0| << 1$$

The sound speed given in (82) is the same one obtained in section 2.2.2 (48).

2.2.7 Conclusions and Comparison with Experiments

The main result of this work is contained in formulae (15), (16) and (22) that give the correct expression of the speed of sound in a mixture. The results of sections 2.2.2 to 2.2.6 seem to check the correctness of our theory from a theoretical point of view.

Zink and Delsasso (18) measured the damping of sound waves propagating in a gas with particles in suspension; their measurements agree with Carhart's (11) results (and consequently with ours too). They also elaborated a theory to calculate the dispersion and checked their results experimentally. Their theory coincides with ours for low sound frequencies. In this limit our calculations also agree with the more recent theoretical and experimental results of Dobbins and Temkin (8,21). (Their results are supposed to hold only for low frequencies). Dobbins and Temkin's (8,21) experiments cover a wider range than Zink and Delsasso's (18).

In a paper by Silberman (19) are reproduced some measurements of sound speeds for water with volume concentrations of air bubbles of around 10%; these measurements were made by Campbell and Pitcher (we could not find the original reference). In fig. 1 are reproduced these measurements and the results of our theoretical calculations for low and high frequencies (formulae (15), (16) and (29a,b,c)). Formulae (15), (16) with (29a) give Wood's (1) results; with (29b,c) we obtain the results of the present theory for high frequencies. The experimental results seem to be between all these curves. Campbell and Pitcher used small amplitude pulses that will contain many different frequencies, and according to our theory

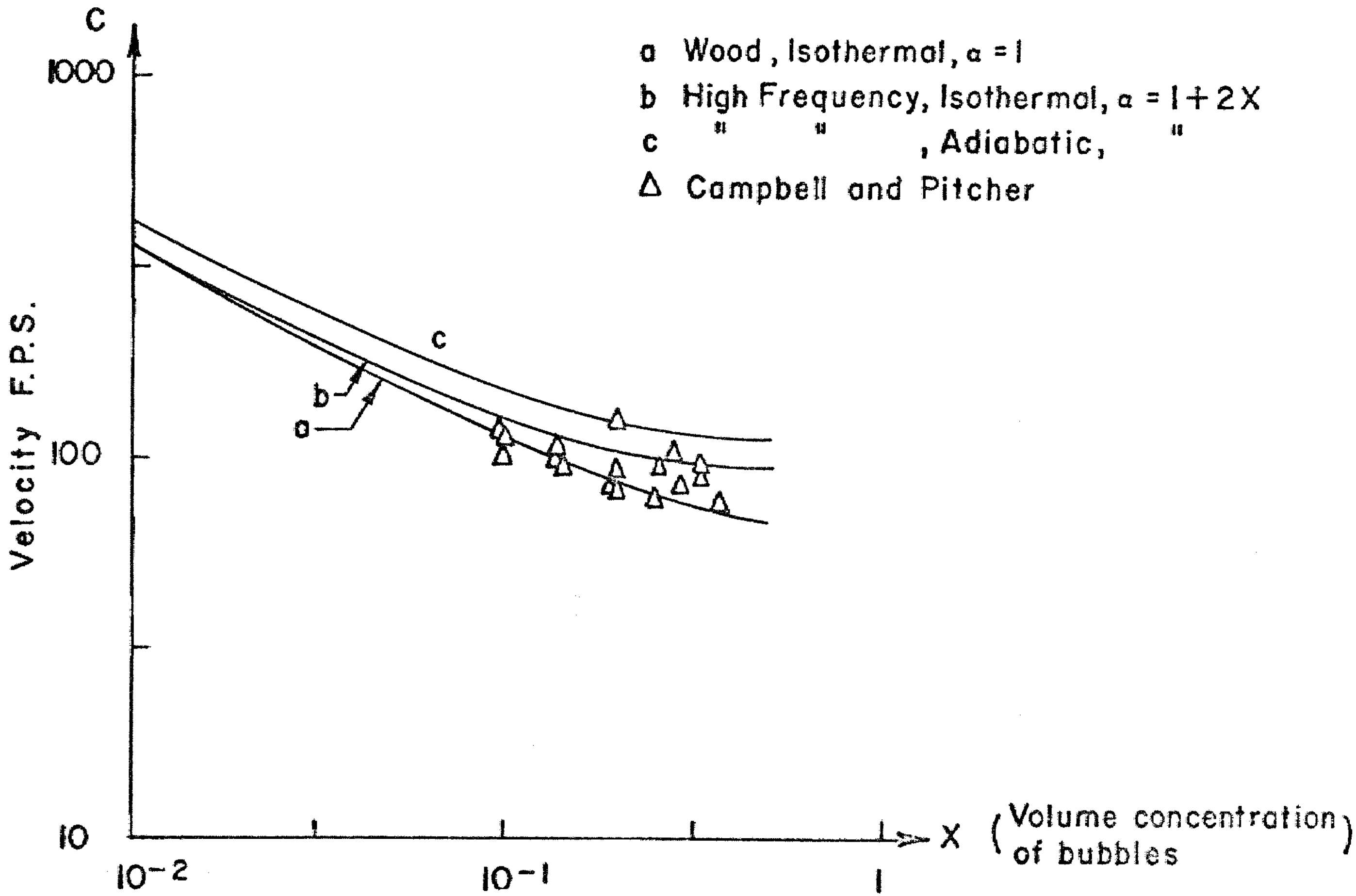


FIG. 1 SPEED OF SOUND IN WATER WITH BUBBLES

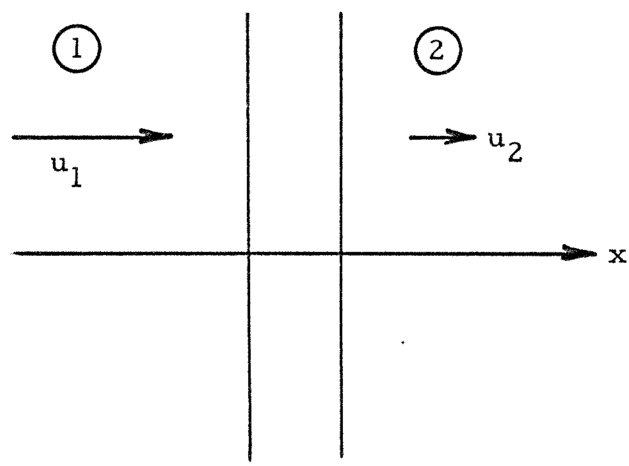
they will be dispersed. Since we do not know the conditions of their experiment, it is difficult to interpret their measurements.

Formulae (15), (16) and (22) are completely general, although we do not know K and K' for every X, but they could possibly be determined empirically or by another method.

2.3 Shock Wave Structure in a Liquid Containing Bubbles

2.3.1 One-dimensional Stationary Equations of Motion

Let us suppose that we have some transition region separating two uniform states. In the uniform regions we assume that liquid and



gas have the same velocity, temperature, and pressure. We denote the uniform region upstream of the shock by subindex 1 and the one downstream by subindex 2. We now assume that in the transition region the flow is one-dimensional and stationary; all quantities depend only on the x coordinate.

The conservation of mass of the gas phase is given by:

$$\rho_{g1}\, u_1 X_1 = \rho_g\, u_g X = \rho_{g2}\, u_2 X_2 \tag{83}$$

ρ_g is the gas density, X is the volume fraction occupied by the gas, and u_g is the velocity of a bubble. The conservation of mass of the liquid phase is (assuming that the liquid is incompressible):

$$(1-X_1)u_1 = (1-X)u_\ell = (1-X_2)u_2 \tag{84}$$

u_ℓ is the velocity of the liquid. As we said before it is assumed:

$$u_{\ell 1} = u_{g1} = u_1, \quad u_{\ell 2} = u_{g2} = u_2 \tag{84a}$$

Let us suppose that the liquid produces over a bubble a certain force F (in addition to the hydrostatic force dp_ℓ/dx). Then the force per unit volume of gas is F/V; V is the volume of a bubble. The equation of conservation of momentum of the gas phase is then:

$$\rho_g u_g \frac{du_g}{dx} + \frac{dp_\ell}{dx} = \frac{F}{V} \tag{85}$$

p_ℓ is the pressure of the liquid phase. The bubble will produce a force -F on the liquid. The number of bubbles per unit volume of mixture is X/V, and the number of bubbles per unit volume of liquid is X/[(1-X)V]. The conservation of momentum of the liquid phase is then:

$$\rho_\ell u_\ell \frac{du_\ell}{dx} + \frac{dp_\ell}{dx} = -\frac{XF}{V(1-X)} \tag{86}$$

ρ_ℓ is the density of the liquid. By multiplying (85) by X and (86) by (1-X), adding them, and then integrating (using (83) and (84)), we obtain the conservation of momentum of the mixture:

$$\rho_g u_g^2 X + \rho_\ell u_\ell^2 (1-X) + p_\ell = \text{const.}$$

Assuming that u_g is of the same order as u_ℓ (this assumption will be confirmed later by the numerical results) and that:

$$\rho_\ell >> \rho_g ,$$

the previous result reduces to:

$$\rho_\ell u_\ell^2(1-X)+p_\ell = \rho_\ell u_1^2(1-X)+p_1 = \rho_\ell u_2^2(1-X_2)+p_2 \tag{87}$$

This assumption also means that in equation (85) we can neglect the convective term so that

$$\frac{dp}{dx} = \frac{F}{V} \tag{88}$$

The energy conservation of the mixture is given by:

$$u_\ell \rho_\ell (1-X)\left[c_\ell T_\ell + \tfrac{1}{2}u_\ell^2\right] + u_g \rho_g X\left[c_v T_g + \tfrac{1}{2}u_g^2\right]$$

$$+ p_\ell u_\ell (1-X) + p_\ell u_g X = \text{const.}$$

T_ℓ is the liquid temperature, T_g the gas temperature, c_ℓ the specific heat of the liquid, and c_v the specific heat at constant volume of the gas phase (that is supposed to be calorifically perfect). In our calculations we will be involved with velocities of the order of 100 m./sec. and temperatures of about 300°K. The specific heat of water is:

$$c_\ell = 1\ \text{cal/gm}^\circ\text{K} = 4.186\ \frac{\text{joules}}{\text{gm}^\circ\text{K}} = 4.186 \times 10^3\ \frac{\text{m}^2}{\text{sec}^2}\ \frac{1}{^\circ\text{K}}$$

then:

$$c_\ell T \sim 10^6\ \text{m}^2/\text{sec}^2 >> u^2 \sim 10^4\ \text{m}^2/\text{sec}^2$$

Using this latter result, equation (87), and the fact that:

$$\rho_\ell >> \rho_g ,$$

the energy equation reduces to:

$$T_\ell = \text{const.} = T_1 = T_2 = T_0 \qquad (89a)$$

The temperature of the liquid remains constant in the transition region. The result that both temperatures (T_1 and T_2) are almost the same was also found by Campbell and Pitcher (2).* Since the temperature of the gas is the same as the temperature of the liquid in the uniform regions, we have:

$$T_{g1} = T_{g2} = T_0 \qquad (89b)$$

$$\frac{p_1}{\rho_{g1}} = \frac{p_2}{\rho_{g2}} \qquad (90)$$

However, inside the shock we cannot make this assumption; we should use there the full equation of state:

$$T_g = T_g(x)$$

$$p_g = R\rho_g T_g \qquad (91)$$

R is the perfect gas constant. p_g is the pressure inside the bubble.

Let us suppose that the liquid communicates to the bubble an amount of heat, Q, per unit time. Then the energy conservation of the bubble is given by:

---

*However, as they point out, this increase in temperature should not be neglected if we are interested in calculating entropy changes.

$$Q = mT_g\left(\frac{ds_g}{dt}\right)_{\substack{\text{following}\\ \text{the bubble}}} =$$

$$= mT_g u_g \frac{ds_g}{dx} = \frac{mu_g}{\rho_g}\frac{1}{(\gamma-1)}\left(\frac{dp_g}{dx} - \frac{\gamma p_g}{\rho_g}\frac{d\rho_g}{dx}\right) \quad (92)$$

m is the mass of a bubble that is assumed to be constant.

$$m = V\rho_g = V_1\ \rho_{g1} = V_2\ \rho_{g2} \quad (93)$$

We still need an equation to relate the bubble pressure to the pressure of the liquid and two expressions for the quantities F and Q. This will be done in the following sections.

2.3.2 Flow Properties Downstream of the Shock

From the conservation laws across the shock we can get enough information to calculate the flow properties downstream of the shock as a function of the properties upstream. The conservations of mass (83), (84), momentum (87) and energy (89), (90) give:

$$\rho_{g1}X_1u_1 = \rho_{g2}X_2u_2 \quad (94)$$

$$(1-X_1)u_1 = (1-X_2)u_2 \quad (95)$$

$$\rho_\ell u_1^2(1-X)+p_1 = \rho_\ell u_2^2(1-X_2)+p_2 \quad (96)$$

$$p_1/\rho_{g1} = p_2/\rho_{g2} \quad (97)$$

This is a system of four equations for the four unknowns ρ_{g2}, X_2, u_2, and p_2. Eliminating ρ_{g2}, X_2, and u_2 we get an equation for p_2 only:

$$\rho_\ell\, u_1^2\, X_1(1-X_1)\ \frac{(p_2-p_1)}{p_2} = p_2-p_1$$

One of the solutions is:

$$p_1 = p_2$$

as is obvious by looking at the original system. The non-trivial solution is:

$$p_2 = \rho_\ell u_1^2(1-X_1)X_1 \tag{98}$$

This solution was also found by Campbell and Pitcher (2) and Eddington (3). The other flow properties are given by:

$$u_2 = (1-X_1)\, u_1 + \frac{p_1}{\rho_\ell u_1(1-X_1)} \tag{99}$$

$$X_2 = \frac{p_1}{\rho_\ell u_1^2(1-X_1)^2+p_1} \tag{100}$$

$$\rho_2 = \rho_1\ \frac{\rho_\ell (u_1^2 X_1(1-X_1)}{p_1} \tag{101}$$

Campbell and Pitcher (2) have shown that the condition for shock existence (increase of entropy across the shock) is:

$$p_2 > p_1$$

using (98) this condition becomes:

$$\frac{p_2}{p_1} = \frac{u_1^2}{p_1/[\rho_\ell(1-X_1)X_1]} > 1$$

The quantity in the denominator is the square of the speed of sound for low frequencies; this can be seen by substituting (29a) into (15) and (16)* and assuming that:

$$(dp_\ell/d\rho_\ell) >> (dp_g/d\rho_g)$$

$$\rho_\ell >> \rho_g$$

$$\tau = 0$$

then we get:

$$c_{L1}^2 = \frac{p_1}{X_1(1-X_1)\rho_\ell} \qquad (102)$$

This is also the speed of sound found by Campbell and Pitcher (2). The subindex L means low frequency. The condition found by Campbell and Pitcher (2) can be expressed as:

$$\frac{p_2}{p_1} = M_{L1}^2 = \frac{u_1^2}{c_1^2} > 1 \qquad (103)$$

M_{L1} is the low frequency Mach number. In section 2.2.1 we also found that for high frequencies the sound speed is different (29c) and so is the corresponding Mach number. In section 2.3.5 we will see the meaning of this other sound speed.

2.3.3 Force on a Bubble

In this section we are going to calculate the force F (88).

Let us define the Reynolds number of the bubble as:

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*The subscripts 1, 0 in 2.2 correspond to g and ℓ in this section.

$$\mathrm{Re} = \frac{|u_\ell - u_g| 2a\rho_\ell}{\mu_\ell} \tag{104}$$

where

$$a = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} = \left(\frac{3m}{4\pi\rho_g}\right)^{\frac{1}{3}} \tag{105}$$

a is the equivalent radius of the bubble, μ_ℓ is the liquid viscosity, and m is the mass of a bubble that is supposed to be constant (93).

Levich (15) says that for moderately high Reynolds numbers (air bubbles in water):

$$1 \leq \mathrm{Re} \leq 700 \text{ or } 800$$

and stationary flow, the bubble remains spherical, and the viscous effects are confined to a thin boundary layer that almost does not separate. Outside the boundary layer there is ideal potential flow. This is not a boundary layer in the classical sense. At the boundary between liquid and gas the tangential stress must vanish (viscosity of the gas is negligible); the boundary layer required, for this condition to be satisfied, produces only a small perturbation of the potential flow (of the order $(\mathrm{Re})^{-\frac{1}{2}}$) and does not change the pressure field. Levich also shows that the drag coefficient due to viscous forces is:

$$c_D = 48/\mathrm{Re}, \qquad 1 \ll \mathrm{Re} < 700 \text{ or } 800 \tag{106}$$

In our case the relative velocity between bubble and liquid is changing at the rate:

$$\frac{d(u_\ell - u_g)}{dt} = u_g \frac{d(u_\ell - u_g)}{dx}$$

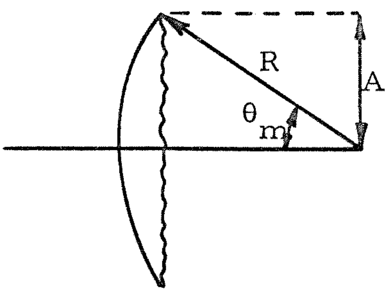
This does not happen in Levich's (15) problem. However, the non-stationary effects do not change the ideal potential velocity field around a body; they only change the pressure field when Bernoulli's equation is applied. Consequently the structure of the boundary layer found by Levich (15) is not expected to change because of the relative acceleration, and the viscous drag given in (106) applies also in our case. Since the boundary layer does not change the pressure field, we also have acting over the bubble a force produced by the ideal potential pressure distribution. This force is (see for example Landau and Lifshitz (14), section 11):

$$F = \tfrac{1}{2} \rho_\ell V u_g \frac{d(u_\ell - u_g)}{dx} \tag{107}$$

$\frac{1}{2} \rho_\ell V$ is the virtual mass of the bubble. The total force will then be:

$$F = \tfrac{1}{2} \rho_\ell V u_g \frac{d(u_\ell - u_g)}{dx} + \tfrac{1}{2} \rho_\ell (u_\ell - u_g) \left| u_\ell - u_g \right| \pi a^2 c_D \tag{108}$$

Davies and Taylor (12) studied the problem of a large bubble rising at constant large velocity in a fluid under gravity force. They



found experimentally that the bubble deforms and takes the shape of a spherical cap. In the front of the bubble the flow is laminar and on the back turbulent. The radius of the cap is such that the dynamic pressure produced by the stationary ideal flow at the front of the cap balances the hydrostatic pressure (created by gravity), and consequently there is a constant pressure at the front of the bubble. They found that the distance A was (see diagram):

$$A \simeq V^{\frac{1}{3}} \tag{109}$$

θ_m did not change very much for different cases and has an average value:

$$\theta_m \simeq 55^{o} \tag{110a}$$

The value of the coefficient of resistance referred to the distance A was around one; if we refer it to the equivalent radius "a" (105):

$$c_D = A^2/a^2 = (4\pi/3)^{\frac{2}{3}} = 2.6 \tag{110b}$$

This latter value is in excellent agreement with the experimental results of Haberman and Morton (these results are reproduced in a paper by Moore (16)) for Reynolds numbers higher than 5,300 (air bubbles in water). We want to apply Davies' and Taylor's (12) results to our problem. The gravity force of their problem could be interpreted in our case as the pressure gradient of the liquid (see formula (88)). However, the main inconvenience resides in the fact that Davies and Taylor (12) considered the case of a bubble moving at constant velocity. We will assume that the non-stationary effects produce the same force as the one acting over the spherical bubble of the same volume (see (107)). The total force acting over the bubble will be given by (108) with the value of c_D appearing in (110b).

In the transition region between Reynolds number equal to 700 and 5300 the experimental results of Haberman and Morton (see Moore (16)) seem to indicate that we can interpolate between the values found by Levich (15) and Davies and Taylor (12). We then have:

$$c_D = 48/Re^*, \qquad Re < 700 \tag{111a}$$

$$c_D = 2.6\,, \qquad Re > 5300 \tag{111b}$$

$$c_D = \frac{48}{700} + \frac{(2.6 - 48/700)}{4600}(Re-700)\,, \qquad 700 < Re < 5300 \tag{111c}$$

The value of the force on the bubble will be given by (108) with the values of c_D appearing in (111).

2.3.4 <u>Heat Transfer to a Bubble</u>

In this section we will try to calculate Q (92).

Let us examine first the case on which the bubble is spherical. We will assume that there is a uniform constant velocity of the liquid, $u_\ell - u_g$, around the bubble. This velocity is supposed to be large and the thermal effects will be shown to be confined to a thin thermal layer of thickness:

$$\delta \sim a\, Pr^{-\frac{1}{2}}\, Re^{-\frac{1}{2}}$$

---

*For Reynolds numbers much smaller than one the value of c_D is given by 16/Re (see Landau and Liftshitz (14, p. 70) and the value of the accelerating force is slightly changed (Basset effect, see Landau and Lifshitz, p. 97). However, as we will see, in this problem Re will be almost always larger than one.

where Pr (the Prandtl number of the liquid):

$$Pr = c_\ell \mu_\ell / \sigma_\ell = 6.75 \text{ (water)}$$

The heat equation for an incompressible fluid is:

$$\frac{\partial T}{\partial t} + \underline{q} \cdot \text{grad } T = \chi \nabla^2 T$$

where

$$\chi = \sigma_\ell / (\rho_\ell c_\ell) \tag{112}$$

and $\underline{q}$ is the liquid velocity. Let us choose spherical coordinates with origin at the center of the bubble. Using the thermal layer approximations:

$$\frac{1}{a}\frac{\partial T}{\partial \theta} \ll \frac{\partial T}{\partial r}, \qquad r = a + r', \quad r' \ll a$$

and neglecting the non-stationary term we get:

$$q_r \frac{\partial T}{\partial r} + q_\theta \frac{1}{a}\frac{\partial T}{\partial \theta} = \chi \frac{\partial^2 T}{\partial r^2} \tag{113}$$

As indicated in section 2.2.3 the boundary layer only produces a small perturbation of the ideal potential velocity field. This potential flow field is given by:

$$q_r = -\frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} \tag{114a}$$

$$q_\theta = \frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r} \tag{114b}$$

where ψ is the stream function:

$$\psi = \tfrac{1}{2} U\, r^2 \sin^2\theta \left(1 - \frac{a^3}{r^3}\right) \tag{114c}$$

$$U = u_\ell - u_g$$

Changing coordinates: (von Mises's transformation)

$$(\theta, r) \rightarrow (\theta, \psi)$$

equation (113) reduces to (using the boundary layer approximation):

$$\frac{2}{3Ua^3 \sin^3\theta} \frac{\partial T}{\partial \theta} = \chi \frac{\partial^2 T}{\partial \psi^2} \tag{115}$$

The boundary conditions are:

$$T = T_g, \quad r = a \tag{116a}$$

$$T = T_0, \quad r = \infty \tag{116b}$$

for all $0 \leqslant \theta < \pi$. See formula (89). The solution of (115), (116) is:

$$T = (T_0 - T_g)\, \mathrm{erf}\left[\frac{3U \sin^2\theta\, r'}{4\sqrt{\frac{3Ua}{2}\chi f(\theta)}}\right] + T_g \tag{117a}$$

where:

$$f(\theta) = \cos\theta \left(\tfrac{1}{3}\cos^2\theta - 1\right) + \tfrac{2}{3} \tag{117b}$$

The thickness of the thermal layer will be given by the value of r' that makes the bracket (of (117)) of order unity:

$$\delta \sim a\, Pr^{-\frac{1}{2}}\, Re^{-\frac{1}{2}} \frac{\sqrt{f(\theta)}}{\sin^2\theta}$$

It is easy to show that the last factor is of order unity for:

$0 \leqslant \theta < \pi$ *

The heat transfer will be:

$$Q = \int_0^{\pi} \sigma_\ell \, 2\pi a^2 \sin\theta \, \left(\frac{\partial T}{\partial r}\right)_{r=a} d\theta \; =$$

$$= 4\sqrt{2\pi} \;\; (T_0 - T_g) \, \sigma_\ell \, a \sqrt{\frac{Ua}{\chi}} \tag{118}$$

The corresponding Nusselt number will be:

$$Nu = \frac{2aQ}{(T_0 - T_g)\sigma_\ell 4\pi a^2} = 1.13 \; Pr^{\frac{1}{2}} \, Re^{\frac{1}{2}} \tag{119}$$

In the case of a solid body the dependence is:

$$Nu \sim Pr^{\frac{1}{3}} \, Re^{\frac{1}{2}}$$

(See for example Landau and Lifshitz (14) p. 208). In the case of bubbles we get a larger Nusselt number because there is no (to first approximation) boundary layer; consequently, the velocities around the bubble are larger and the convective heating is more effective.

For Reynolds numbers larger than 5,300 (air bubbles in water) we saw in the previous section that the bubble takes the shape of a spherical cap. The average angle of the cap was shown to be around 55°, and the flow on the front of the cap was shown to be laminar (Davies and Taylor (12)). If we assume that this laminar flow is

---

*For $\theta = \pi$, δ becomes infinity; this means that there is a wake and that our solution is not valid there. However, Leyich (15) also shows that the boundary layer separates for $\theta - \pi = O(Re^{-\frac{1}{2}})$, so this solution is not valid there anyway.

the same as the flow around the complete sphere, we can use the previous theory to calculate the heat transfer to the front of the cap. Using (118):

$$Q_{lam} = \int_0^{55^o} 2\pi R^2 \sigma \sin\theta \left(\frac{\partial T}{\partial r}\right)_{r=R} d\theta$$

$$R = \frac{(4/3\pi)^{\frac{1}{3}}}{\sin 55^o} a$$

for T we take the value given in (117a) (with R instead of a). We then get

$$Nu_{lam} = \frac{Q \times 2a}{\sigma_\ell (T_0 - T_g) 4\pi a^2} = 1.56 \, Pr^{\frac{1}{2}} \, Re^{\frac{1}{2}} \tag{120}$$

Let us calculate now the turbulent heat transfer to the back of the spherical cap. Richardson (20) gives for the local heat transfer to the back of a bluff body:

$$\overset{*}{Nu} = 0.19 \, \overset{*}{Re}^{\frac{2}{3}} \, Pr^{\frac{1}{3}}$$

It seems that this result is in good agreement with previous experiments. These Reynolds and Nusselt numbers are referred to the transversal dimension of the body, in our case:

$$2A = (4\pi/3)^{\frac{1}{3}} \, 2a$$

Then Richardson's (20) result referred to the surface and radius of the equivalent sphere is:

$$Nu_{turb} = \frac{0.19}{4} \left(\frac{4\pi}{3}\right)^{5/9} Re^{\frac{2}{3}} \, Pr^{\frac{1}{3}} = 0.1 \, Re^{\frac{2}{3}} \, Pr^{\frac{1}{3}} \tag{121}$$

Then the total heat transfer is given by:

$$Nu = Nu_{lam} + Nu_{turb} = 1.56\ Pr^{\frac{1}{2}}\ Re^{\frac{1}{2}} + 0.1\ Re^{\frac{2}{3}}\ Pr^{\frac{1}{3}}$$

$$\text{if } Re > 5{,}300 \qquad (122a)$$

For intermediate Reynolds numbers we interpolate between the previous result and (119):

$$Nu = \left[1.13 + \frac{1.56-1.13}{4{,}600}\ (Re-700)\right] Pr^{\frac{1}{2}} Re^{\frac{1}{2}} + $$

$$+ \frac{0.1}{4{,}600}\ (Re-700)\ Re^{\frac{2}{3}}\ Pr^{\frac{1}{2}}$$

$$\text{if } 700 < Re < 5{,}300 \qquad (122b)$$

When the bubble remains spherical (119):

$$Nu = 1.13\ Pr^{\frac{1}{2}}\ Re^{\frac{1}{2}} + 2.0 \qquad (122c)$$

$$\text{if } Re < 700$$

The term 2.0 gives the pure heat conduction to a sphere at rest. (See for example Rudinger (6)), for large Re this term is negligible.

The heat transfer has always been non-dimensionalized by the same parameters (119).

2.3.5 Bubble Dynamics

In 2.3.1 we said that the pressure inside the bubble could be different from the pressure of the liquid. Let us see how that is possible. As the bubble changes its volume it communicates a certain speed to the liquid outside; since the bubble could change its volume in an arbitrary manner, the liquid outside the bubble could be accelerating. A difference in pressure between the liquid (at ∞) and the bubble is necessary to provide this acceleration.

Let us suppose first that the bubble remains spherical. Lamb (13) calculated the flow field created by a spherical bubble immersed in liquid whose radius is increasing at the rate:

$$da/dt = \mathring{a} \ ,$$

and whose pressure is p_g. The liquid far from the bubble is at rest and has a pressure p_ℓ. The potential function satisfying these conditions is:

$$\varphi = -a^2\mathring{a}/r \qquad \underline{v} = \nabla\varphi \tag{125a}$$

and at the boundary of the bubble:

$$\frac{p_g - p_\ell}{\rho_\ell} = 4\nu_\ell \frac{\mathring{a}}{a} + \frac{3}{2}\mathring{a}^2 + \ddot{a}a \tag{125b}$$

The viscous term does not appear in Lamb's (13) formula; it just represents the viscous force at the interface* (viscosity of the gas neglected). The addition of this term in the boundary condition (125b) does not affect the validity of Lamb's (13) solution. As is well known, any solution of the potential equation is also a solution of Navier Stokes equations (with viscosity included). In our case the

---

*The normal viscous stress is:

$$\tau_{rr} = 2\mu_\ell \, \partial v_r/\partial r$$

in our problem $v_r r^2$ is constant. $\left(\frac{\partial v_r}{\partial r}\right)_{r=a} = -\left(\frac{2v_r}{r}\right)_{r=a} = -\frac{2\mathring{a}}{a}$

liquid is moving at a uniform velocity $(u_\ell - u_g)$ relative to the bubble. However, as we said before, this relative velocity produces a potential flow field (neglecting the boundary layer) around the sphere. Since Lamb's (13) solution also satisfies Laplace's equation (which is linear), we can superimpose both solutions.

For high Reynolds numbers the sphere deforms, and the flow becomes turbulent at its back. In this case the dynamics of the bubble are not known, and we will assume that we can still apply (125b) taking for "a" the equivalent radius of the bubble (105). We base this assumption on the fact that possibly the liquid at a large distance from the bubble is insensible to its shape and behaves as if the bubble were spherical.

In (125b) the derivatives with respect to time should be interpreted like derivatives following the bubble:

$$\frac{d}{dt} = u_g \frac{d}{dx} \tag{126}$$

Equation (125b) is an equation for the volume of the bubble or for its density (93), (105):

$$a^3 \rho_g = \text{const.} = a_1^3 \rho_{g1} \tag{127}$$

$$\frac{da}{dx} = - \frac{a}{3\rho_g} \frac{d\rho_g}{dx} \tag{128}$$

2.3.6 Preliminary Study of the Equations

By using the momentum (87) and continuity (84) equations, we can express the liquid velocity, u_ℓ, and volume fraction, X, as functions of the liquid pressure, p_ℓ:

$$u_\ell = u_1 \left(\frac{p_1 X_1 - p_\ell X_1 + p_2}{p_2}\right) \tag{129}$$

$$X = X_1 \left(\frac{p_1 + p_2 - p_\ell}{p_1 X_1 + p_2 - p_\ell X_1}\right) \tag{130}$$

By using the equation of conservation of mass of the gas (83) and the previous result (130) we get:

$$u_g = u_1 \left(\frac{\rho_{g1}}{\rho_g}\right) \left(\frac{p_1 X_1 + p_2 - p_\ell X_1}{p_1 + p_2 - p_\ell}\right) \tag{131}$$

The temperature of the gas is given by the equation of state (91) as a function of the pressure and density of the gas. The volume and equivalent radius of a bubble are (93):

$$V = \frac{4}{3}\pi a^3 = \frac{m}{\rho} \tag{132}$$

We can express all our variables as functions of the liquid pressure, p_ℓ, gas density, ρ_g, and gas pressure, p_g, and for these three quantities we have three differential equations: the momentum equation of the gas (88), the energy equation of a bubble (92), and the equation describing the dynamics of the bubble (125b). By putting in the momentum equation (88) the value of F given in (108), and then calculating the derivatives of u_ℓ and u_g by using (129) and (131), we get:

$$A \frac{dp_\ell}{dx} - B \frac{d\rho_g}{dx} = D \tag{133a}$$

where:

$$A = 1 + \tfrac{1}{2}\ \rho_\ell u_g u_1 \left[\frac{X_1}{p_2} + \frac{u_g(1-X_1)}{u_\ell(p_1+p_2-p)} \right] \tag{133b}$$

$$B = \tfrac{1}{2}\ \rho_\ell \frac{u_g^2}{\rho_g} \tag{133c}$$

$$D = \frac{3}{8}\ \rho_\ell \frac{(u_\ell - u_g)\,|u_\ell - u_g|\,c_D}{a} \tag{133d}$$

These three quantities can be expressed as functions of p_g, ρ_g, p_ℓ by using (129) to (132) and (111).

The energy conservation of a bubble (92) is:

$$\frac{dp_g}{dx} - \frac{\gamma p_g}{\rho_g}\ \frac{d\rho_g}{dx} = \frac{(\gamma-1)\rho_g Q}{m\,u_g} = N \tag{134}$$

N can be expressed as a function of p_g, ρ_g, p_ℓ by using (122) and the previous relations ((129) to (132)).

The equation describing the bubble dynamics (125b), (126) can be expressed as:

$$p_\ell - p_g = O(\rho_\ell u_g^2 \frac{a^2}{x^2})$$

Since we expect the radius of the bubble to be small compared to the thickness of the shock, it is possible that this equation would just mean that both pressures are the same. However, we are going to show that this pressure difference cannot always be neglected.

If p_ℓ and p_g are equal, equations (133), (134) can be written:

$$\frac{dp_\ell}{dx} = \frac{\frac{\gamma p}{\rho_g} D - BN}{\Delta} \tag{135a}$$

$$\frac{d\rho_g}{dx} = \frac{D - AN}{\Delta} \tag{135b}$$

where

$$\Delta = \frac{A\gamma p}{\rho_g} - B \tag{135c}$$

It is easy to show that Δ is positive when we are in region 2. From (133b, c):

$$\frac{\gamma p_2}{\rho_2} A_2 > \gamma B_2 \left(X_1 + \frac{(1-X_1)p_2}{p_1} \right) > B_2 \tag{135d}$$

On the other hand, Δ has in region 1 the value (using (133b, c)):

$$\Delta_1 = \frac{\gamma p_1}{\rho_1} \left(1 + \frac{1}{2X_1(1-X_1)} \right) (1-M_{H1}^2) \tag{136}$$

where:

$$M_{H1}^2 = \frac{u_1^2}{c_{H1}^2} = \frac{p_2}{\gamma[1+2X_1(1-X_1)]p_1} \tag{137a}$$

$$c_{H1}^2 = \frac{[1+2X_1(1-X_1)]\,\gamma p_1}{\rho_\ell X_1(1-X_1)} \tag{137b}$$

This value of c_H corresponds to the high frequency speed of sound in a liquid with bubbles that we calculated in 3.2.1 (see formulae (15), (16) and (29c)), except for the factor $(1-X_1)$ multiplying $2X_1$. The reason for this difference is probably that formula (26) (from which (29c) is deduced) is only valid for small values of X_1*. M_H is then the high frequency Mach number. Comparing c_H with c_L (102) we have:

*This analysis itself is only valid for small values of X; however, we expect that as the gas is compressed, the value of X will become very small even if X_1 is not small. This is the reason why we kept the $(1-X_1)$ term.

$$c_H > c_L \,, \qquad M_{H1} < M_{L1}$$

If:

$$M_{L1} > M_{H1} > 1 \tag{138}$$

Δ_1 (136) is negative; and since Δ_2 is always positive, Δ should become zero somewhere in the transition region. At the point where that happens, the function $p_\ell(x)$ (see (135)) will have an infinite slope that will change sign. This means that close to that point the function $p_\ell(x)$ will be double valued, and this is physically unacceptable. We can give a physical interpretation to this; if in equation (133a) we neglect the dissipative terms we get:

$$\left(\frac{dp_\ell}{d\rho_g}\right)_1 = \frac{B_1}{A_1} > \frac{\gamma p_1}{\rho_{g1}} \quad \text{if} \quad M_H > 1$$

If p_ℓ and p_g are the same, the bubble is subject to changes in pressure larger than the ones allowed by the isentropic law. Because of the form of our heat equation (Q proportional to $(T_0 - T_g)$), this is not possible. This means that the bubble cannot follow the pressure changes of the liquid imposed by the momentum equation. We should let the pressure of bubble and liquid be different and consider the full equation describing the bubble dynamics.

2.3.7 Case for Which Both Mach Numbers are Larger than One

We are going to study the case for which:

$$M_{H1} > 1 \;, \qquad A_1 \frac{\gamma p_1}{\rho_1} < B_1$$

In this case we have the equations of momentum (133), energy of the bubble (134), and dynamics of the bubble (125b) for the three unknowns p_ℓ, p_g and ρ_g. Let us examine first the behavior of these equations when we are close to the uniform region 1. We represent p_g, ρ_g and p_ℓ by:

$$p_g = p_1 + \tilde{p}_g \, e^{\lambda x} \tag{139a}$$

$$p_\ell = p_1 + \tilde{p}_\ell \, e^{\lambda x} \tag{139b}$$

$$\rho_g = \rho_{g1} + \tilde{\rho}_g \, e^{\lambda x} \tag{139c}$$

$\tilde{p}_g$, $\tilde{p}_\ell$ and $\tilde{\rho}_g$ are supposed to be small constants (so that the equations can be linearized). By using (129), (131) and (91) we get:

$$u_\ell - u_g = u_1 \left(\frac{\tilde{\rho}_\ell}{\rho_1} - \frac{\tilde{p}_\ell}{p_2} \right) e^{\lambda x} \tag{139d}$$

$$T_0 - T_g = T_0 \left(\frac{\tilde{\rho}_g}{\rho_1} - \frac{\tilde{p}_g}{p_1} \right) e^{\lambda x} \tag{139e}$$

Equation (133a) becomes (using (111a))

$$\lambda(A_1 + \varepsilon_1) \, \tilde{p}_\ell - (\lambda B_1 + \varepsilon_2) \, \tilde{\rho}_g = 0 \tag{140a}$$

where ε_1 and ε_2 are parameters that correspond to the viscous drag:

$$\varepsilon_1 = \frac{\rho_1}{p_2} \quad \varepsilon_2 = \frac{9 \, \rho_\ell u_1^2}{a_1 \, Re_1} \, \frac{1}{p_2} \tag{140b}$$

$$Re_1 = \frac{2 a_1 u_1}{\nu_\ell} \tag{140c}$$

For the cases that we are going to solve Re_1 is of the order of 10^5; we can then assume:

$$\varepsilon_1 \ll A_1 \lambda, \qquad \varepsilon_2 \ll \lambda B_1 \tag{140d}$$

because as will be shown:

$$\lambda \sim 1/a_1$$

The energy equation of the bubble becomes (using (134), (122c))

$$(\lambda+\varepsilon_3)\tilde{p}_g - (\lambda \frac{\gamma p_1}{\rho_{g1}} + \varepsilon_4)\,\tilde{\rho}_g = 0 \qquad (141a)$$

where ε_3 and ε_4 are coefficients that correspond to the heat transfer:

$$\varepsilon_3 = \frac{\rho_1}{p_1}\,\varepsilon_4 = \frac{6\gamma}{a_1}\,(\frac{c_\ell}{c_p})\,(\frac{\rho_\ell}{\rho_{g1}})\,\frac{1}{PrRe_1} \qquad (141b)$$

In this case the product $(\frac{c_\ell}{c_p})\,(\frac{\rho_\ell}{\rho_{g1}})$ is large (of the order of 10^4); however Pr Re is of the order of 10^6; we can then assume:

$$\varepsilon_3 << \lambda \qquad (141c)$$

$$\varepsilon_4 << \lambda\, p_1/\rho_1 \qquad (141d)$$

The equation of the dynamics of the bubble (125b) becomes:

$$\lambda^2 \frac{a_1^2 u_1^2}{3\rho_1}\,\tilde{\rho}_g + \varepsilon_5 \lambda \tilde{\rho}_g = \frac{\tilde{p}_\ell - \tilde{p}_g}{\rho_\ell} \qquad (142a)$$

on using (126), (127), (128). ε_5 is small and represents the normal viscous stress acting at the interface of gas and liquid:

$$\varepsilon_5 = \frac{8}{3}\,\frac{u_1^2 a_1}{\rho_1 Re_1} << \lambda\,\frac{a_1 u_1^2}{3\rho_1} \qquad (142b)$$

The homogeneous system (140), (141), (142) has a non-trivial solution if λ satisfies:

$$\begin{vmatrix} \lambda A_1+\varepsilon_1 & 0 & -(B_1\lambda+\varepsilon_2) \\ 0 & \lambda+\varepsilon_3 & -(\lambda \frac{\gamma p_1}{\rho_1} + \varepsilon_4) \\ -\frac{1}{\rho_\ell} & \frac{1}{\rho_\ell} & \lambda^2 \frac{a_1^2 u_1^2}{3\rho_{g1}} + \lambda\varepsilon_5 \end{vmatrix} = 0 \qquad (143)$$

Let us assume that we can neglect the dissipative terms $(\varepsilon_1 \ldots, \varepsilon_5)$. Then (143) becomes:

$$\lambda^4 A_1 \frac{a_1^2 u_1^2}{3\rho_{g1}} - \lambda^2 \frac{B_1}{\rho_\ell} + \lambda^2 \frac{A_1}{\rho_\ell} \frac{\gamma p_1}{\rho_1} = 0$$

This equation has the solutions:

$$\lambda = 0 \qquad (144a)$$

$$\lambda = \pm \frac{1}{a_1} \sqrt{\frac{3X_1(1-X_1)}{1+2X_1(1-X_1)}} \left(1 - \frac{1}{M_{H1}^2}\right)^{\frac{1}{2}} \qquad (144b)$$

where we used (133b,c) and (137). The solution λ equal to zero is not correct, because we cannot then make the assumptions (140d), (141c,d), (142b), and consequently the ε terms cannot be neglected in (143). This solution will really be (see next section) two real roots of the same sign, or possibly two imaginary roots; however, these roots have small modulus, and when taken into (142) they give a negligible difference between p_g and p_ℓ, and we find the same type of difficulty as when we tried to solve the problem taking the same value for both pressures (system (135)). As a matter of fact, these roots will be shown (in the next section) to correspond to the solution of (135). We then have to take one of the solutions (144b). Looking at (139) we see that we have to pick the positive value of λ. This will give us an exponential initial increase of all the quantities. This increase is very fast, $O(a_1)^*$, and does not depend on viscosity or thermal conductivity (see (144b)).

Let us examine now how the equations behave when we are close to the uniform region 2. In this case we have:

$$p_g = p_2 + \tilde{p}_g \, e^{\lambda x} \tag{145a}$$

$$p_\ell = p_2 + \tilde{p}_\ell \, e^{\lambda x} \tag{145b}$$

$$\rho_g = \rho_{g2} + \tilde{\rho}_g \, e^{\lambda x} \tag{145c}$$

and from (130), (131) and (91) we get:

$$u_\ell - u_g = u_2 \left(\frac{\tilde{\rho}_g}{\rho_{g2}} - \frac{\tilde{p}_\ell}{p_2} \right) \tag{145d}$$

$$T_0 - T_g = T_0 \left(\frac{\tilde{\rho}_g}{\rho_2} - \frac{\tilde{p}_g}{p_2} \right) \tag{145e}$$

By doing the same analysis that we did before we get:

$$\begin{vmatrix} \lambda A_2 + \varepsilon'_1 & 0 & -(B_2\lambda + \varepsilon'_2) \\ 0 & \lambda + \varepsilon'_3 & -(\lambda \dfrac{\gamma p_2}{\rho_2} + \varepsilon'_4) \\ -\dfrac{1}{\rho_\ell} & \dfrac{1}{\rho_\ell} & \lambda^2 \dfrac{a_2^2 u_2^2}{3\rho_2} + \lambda \varepsilon'_5 \end{vmatrix} = 0 \tag{146}$$

---

*This is not exactly true. The factor multiplying $(1/a_1)$ in (144b) is small and $1/\lambda$ is of the order of several bubble lengths. This is another indication of the fact that this analysis is strictly speaking only valid for small values of X_1. A similar argument can be applied to formula (148b) appearing in the next pages.

where:

$$\varepsilon_1' = \frac{\rho_2}{p_1} \quad \varepsilon_2' = \frac{9\rho_\ell u_2^2}{a_2 Re_2} \frac{1}{p_1} \tag{147a}$$

$$Re_2 = \frac{2a_2 u_2}{\nu_\ell} \simeq 10^5 \tag{147b}$$

$$\varepsilon_3' = \frac{\rho_2}{p_2} \quad \varepsilon_4' = \frac{6}{a_2} \left(\frac{c_\ell}{c_p}\right) \left(\frac{\rho_\ell}{\rho_{g2}}\right) \frac{p_1}{Pr\, Re_2} \frac{1}{\rho_2} \tag{147c}$$

ε_1', ..., ε_5' are small. In this case we are going to keep to first order the dissipative terms. The solution of (146) is:

$$\lambda = \lambda_{real} + i\, \lambda_{imag} \tag{148a}$$

where:

$$\lambda_{imag} = \pm \frac{1}{u_2 a_2} \left(\frac{3\gamma p_2}{\rho_\ell} - \frac{3B_2\rho_2}{A_2\rho_\ell} \right)^{\frac{1}{2}} \tag{148b}$$

$$\lambda_{real} = - \frac{3\rho_1}{\lambda_{imag}^2 a_2^2 u_2^2 A_2} \left[\delta\varepsilon_2 + (\gamma-1) \frac{A_2}{\rho_\ell} \varepsilon_4 \right] \tag{148c}$$

$$\delta = \frac{1}{\rho_\ell} \left(\frac{1}{A_2} + \frac{p_2}{\rho_\ell u_2^2} + \frac{1}{2} \frac{(p_2 - p_1)}{A_2 p_2} \left\{ \frac{(1-X_1)\rho_\ell u_2^2}{2p_1} \left[1 + \frac{1-X_1}{2} \left(\frac{p_2}{p_1} - 1\right) \right] + \frac{u_2}{(1-X_1)u_1} + \right.\right.$$

$$\left.\left. + X_1 \right\} \right) + \frac{1}{\rho_\ell} \left[\frac{1}{18} + \left(\frac{8\gamma}{9} - 1\right) \frac{A_2 p_2}{\rho_\ell u_2^2} \right] \tag{148d}$$

The first term in the bracket of (148c) represents the viscous damping (drag and normal stress) and the second the thermal damping. Condition (135d) insures that λ_{imag} is always real. λ_{real} is zero if we neglect the dissipative terms. In general λ_{real} is negative.

When we are close to region 2 all the variables have the functional dependence

$$e^{\lambda_{real} x} \sin(\lambda_{imag} x) , \qquad |\lambda_{imag}| >> |\lambda_{real}|$$

The flow properties oscillate around the values that they are supposed to take in region 2. Since λ_{real} is small and negative, these oscillations are slowly damped, and the flow properties tend to reach their final values (see figs. 3). Since both components move approximately at the same velocity, u_2, the frequency of these oscillations can be defined as:

$$f = \frac{u_2 \lambda_{imag}}{2\pi} = \frac{1}{2\pi a_2} \left(\frac{3\gamma p_2}{\rho_\ell} - \frac{3B_2 \rho_{g2}}{A_2 \rho_\ell} \right)^{\frac{1}{2}} \tag{149}$$

This is the classical resonant frequency found by Spitzer* corrected in our case by a term due to the virtual mass of the bubble. For very large values of the pressure ratio across the shock, p_2/p_1, the last term disappears and we recover the resonant frequency usually found in the literature.

The system of equations (133), (134), and (125) was solved numerically. The last equation (dynamics of the bubble) was put for convenience in the integral form:

$$\frac{a_1}{\rho_1}\left[\frac{d\rho_g}{dx} - \left(\frac{d\rho_g}{dx}\right)_{x_o}\right] = \frac{3(\rho_g/\rho_1)^{11/6}}{(u_g/u_1)} X_1(1-X_1)\int_{x_o}^{x} \frac{(p_\ell - p_g)u_1}{p_0(\rho_g/\rho_1)^{1/6} u_g} \frac{dx}{a_1}$$
$$- \frac{48(\rho_g/\rho_1)^{4/3}(\rho_g/\rho_1)^{1/6}}{Re} \left(\frac{u_\ell - u_g}{u_g}\right)\Bigg|_{x_o}^{x} \tag{150}$$

*We could not find this reference. See Silberman (19) instead.

Initially we give a certain arbitrary and very small value to $\rho_{g0}-\rho_{g1}$. Since we know λ close to region 1 (144), we also know:

$$(d\rho_g/dx)_{x_o} = \lambda(\rho_{g0}-\rho_{\rho 1})$$

and by using (140) and (141) we get $(dp_g/dx)_0$, $(dp_\ell/dx)_0$, p_{g0}, and $p_{\ell 0}$. Once we know all the flow properties and their derivatives at the initial point, we give a small increment to x and calculate the flow properties at a second point; at this second point we calculate the values of the derivatives by using (150), (133) and (134) and so on.

In figs. 2 and 3 are shown some of the results of the numerical calculations. In these figures we see that the numerical analysis gives the oscillatory character (when we are close to region 2) predicted by our linearized analysis. In fig. 2 are also given the results of experiments made by Eddington (3). These experiments seem to give shock thicknesses about two times larger than the ones calculated here. He did not measure enough points inside the shock to determine whether our oscillations really exist or not. On the other hand, he did not use a uniform bubble size, and, as indicated (149), the frequency of the oscillations depends on the radius of the bubble.

We can interpret the results of this section as follows. When we are close to region 1, an increase in the density of the bubble produces an increase in its pressure, but it also produces a larger increase in the pressure of the liquid (because of the virtual mass of the bubble acting through the momentum equation); as a result, the bubble is compressed more and we get an exponential increase of the bubble density. When we are close to region 2 we

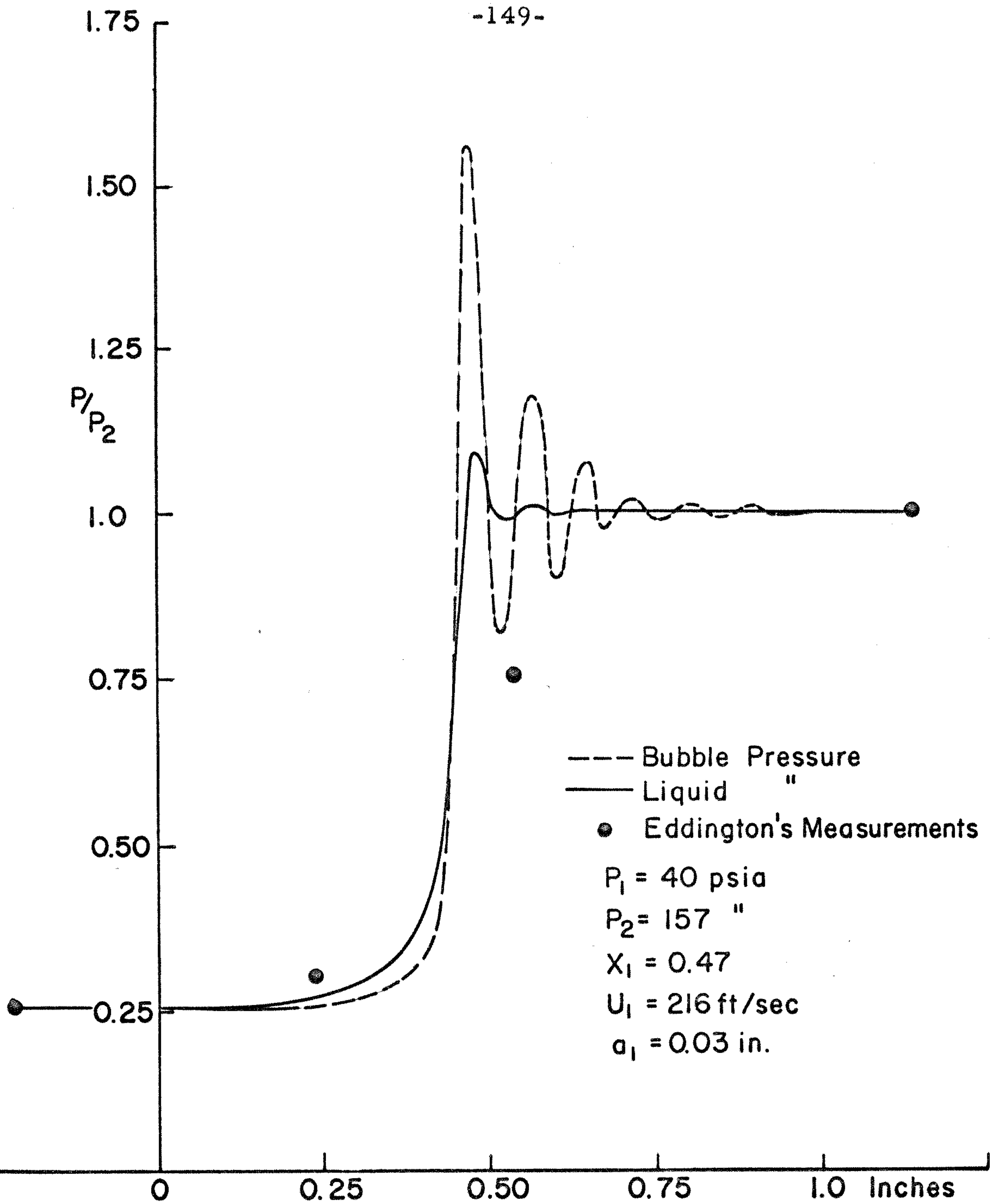


FIG. 2a PRESSURE DISTRIBUTION ACROSS THE SHOCK

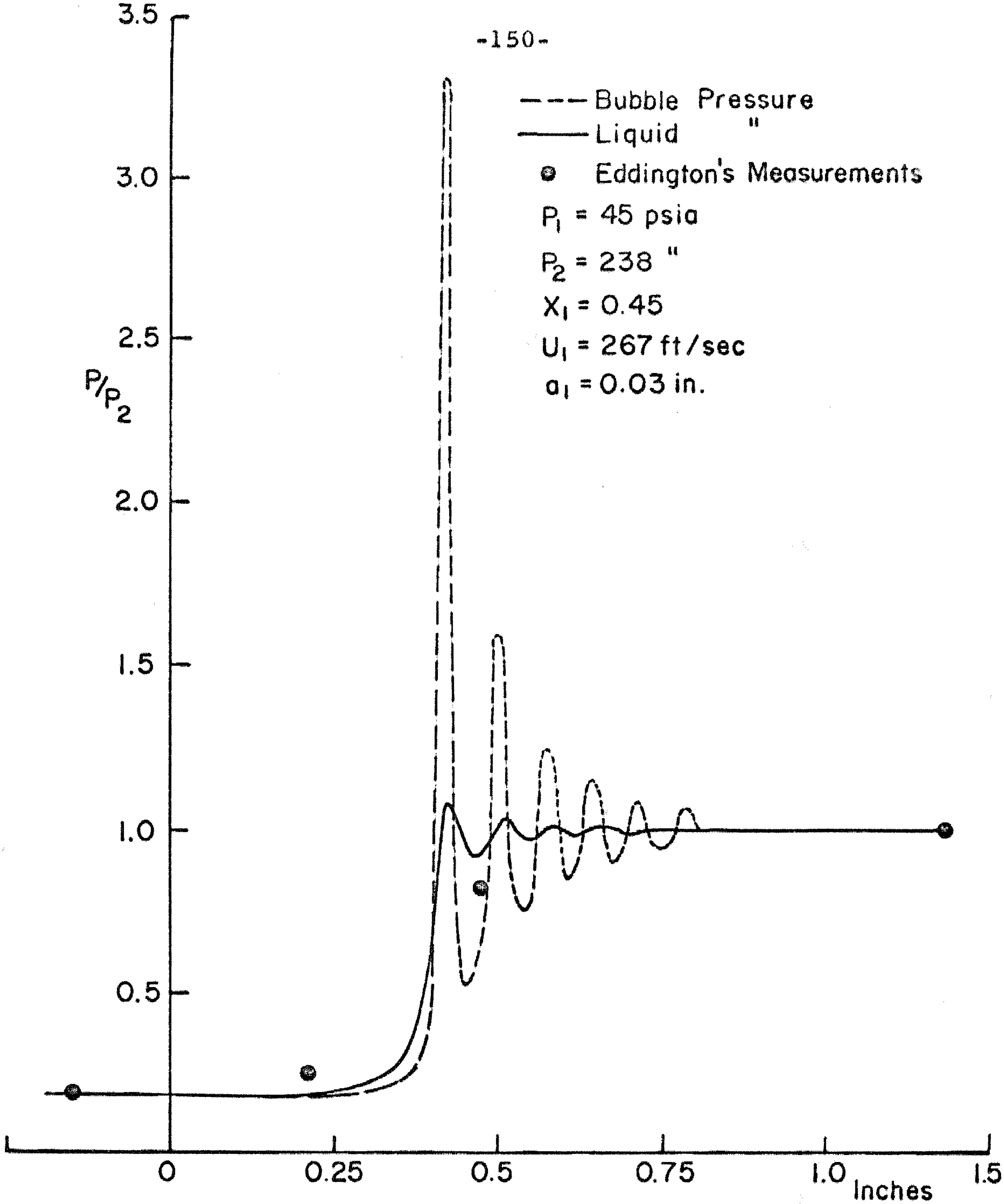


FIG. 2b PRESSURE DISTRIBUTION ACROSS THE SHOCK

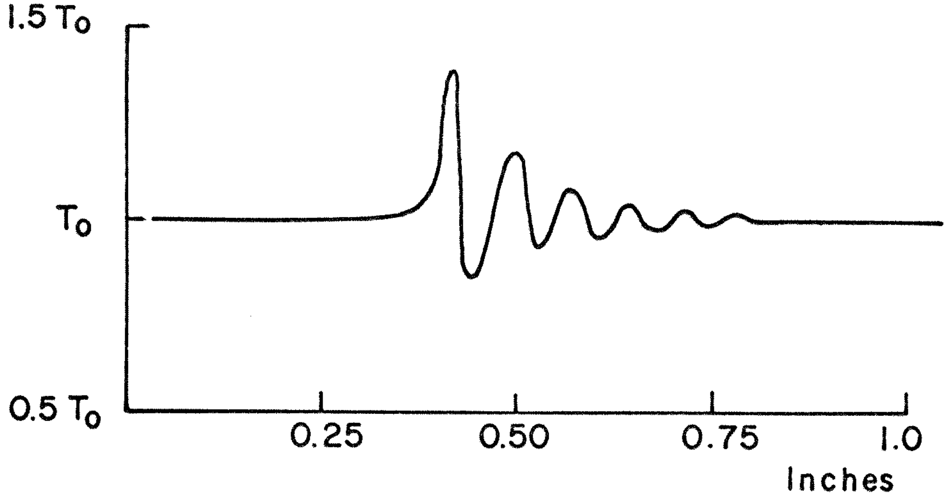


FIG. 3a GAS TEMPERATURE ACROSS THE SHOCK
P_1 = 45.0 psia, P_2 = 238.27 psia, a_1 = 0.03 in.

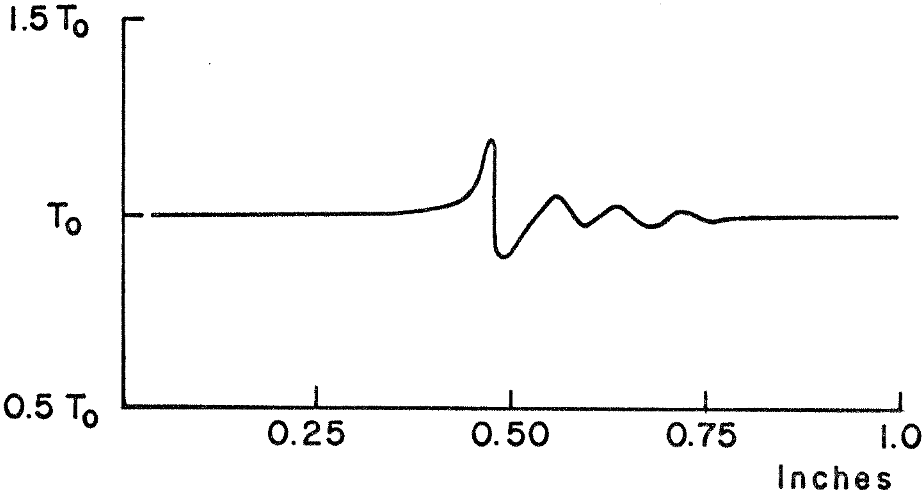


FIG. 3b GAS TEMPERATURE ACROSS THE SHOCK
P_1 = 40.0 psia, P_2 = 156.7 psia, a_1 = 0.03 in.

have the opposite behavior; a compression of the bubble produces an increase in its pressure larger than the increase in pressure of the liquid; consequently the bubble tends to behave like a harmonic oscillator.

2.3.8 Case for which the High Frequency Mach Number is Smaller than One

In this case we suppose:

$$M_{L1}^2 = \frac{p_2}{p_1} > 1 \tag{151a}$$

$$M_{H1}^2 = \frac{p_2}{\gamma\left[1+2X_1(1-X_1)\right]p_1} < 1 \tag{151b}$$

In this case there are no difficulties if we try to solve the problem considering that the pressure of liquid and gas are the same (see section 2.3.6 (135), (136)).

Let us examine the behavior of the flow properties when we are close to region 1. By doing the same analysis of the previous section we get the determinant (143) that gives us four values of λ. The values of λ given in (144b) (which were the only ones we could use in the last section) become imaginary in this case, and by keeping to first order the dissipative terms, it is easy to show that they become complex with negative real part. These values of λ give damped oscillations; that means that we will never be able to leave region 1. We have to go back to (143) and try to calculate the two other roots. In the previous section we obtained for these roots the value zero, but as we said there, this value is erroneous because we cannot neglect the dissipation terms when calculating it. However, the values of λ are expected to be small, and as we will show:

$$a\lambda \sim (\varepsilon_2 \varepsilon_3 \frac{\rho_1}{p_1})^{\frac{1}{2}} \sim \frac{1}{Re\sqrt{Pr}} \left(\frac{\rho_\ell}{\rho_{g1}}\right)^{\frac{1}{2}} \left(\frac{c_\ell}{c_p}\right)^{\frac{1}{2}} << 1 \tag{152}$$

If that is so, in (143) the third term of the last row can be neglected, but that is exactly the same as neglecting the left hand side of the equation of bubble dynamics (142a) and assuming that the pressure of liquid and gas are equal! Equation (143) with that term neglected gives:

$$\lambda^2 \left(\frac{\gamma p_1 A_1}{\rho_1} - B_1\right) + \lambda \left(-B_1 \varepsilon_3 - \varepsilon_2 + A_1 \varepsilon_4 + \frac{\gamma p_1}{\rho_1} \varepsilon_1\right) - - \varepsilon_2 \varepsilon_3 \left(\frac{p_2}{p_1} - 1\right) \doteq 0 \tag{153}$$

where we also used (140b) and (141b). It is easy to check that the solutions of this equation satisfy the condition (152). The coefficient of λ^2 is positive (135c, 136) and the independent term is negative. This equation then has two real roots of opposite sign*.

In the same way when we are close to region 2 we get (using (146) with the third term of the third row neglected):

$$\lambda^2 \left(\frac{\gamma p_2}{\rho_2} A_2 - B_2\right) + \lambda \left(-B_2 \varepsilon'_3 - \varepsilon'_2 + A_2 \varepsilon'_4 + \frac{\gamma p_2}{\rho_2} \varepsilon'_1\right) + + \varepsilon'_2 \varepsilon'_3 \left(1 - \frac{p_1}{p_2}\right) = 0 \tag{154}$$

*In the case of the previous section the coefficient of λ^2 would be negative and the roots would either be real (of the same sign) or imaginary.

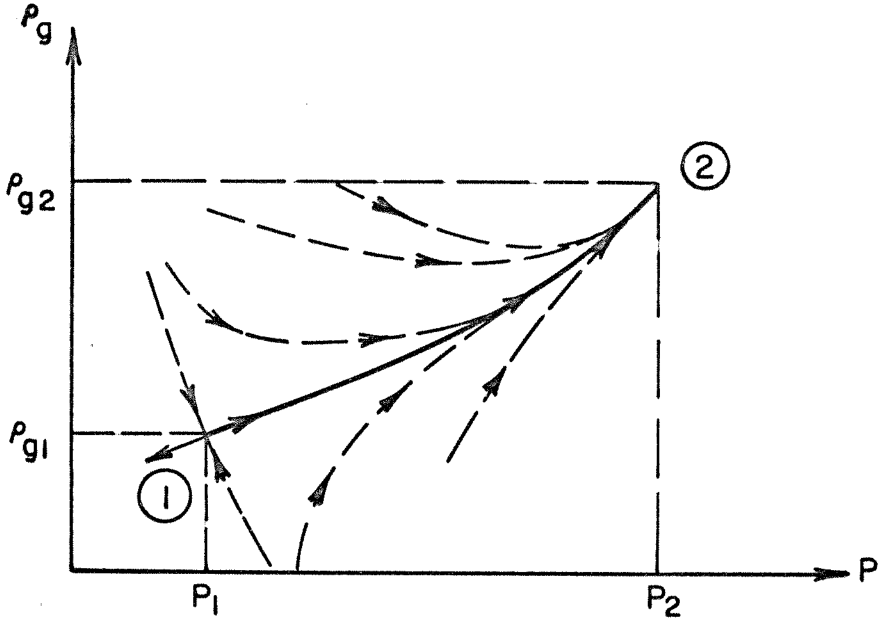


FIG. 4 PHASE PLANE REPRESENTATION OF THE GAS DENSITY AND THE PRESSURE FOR THE CASE $M_{H1} < 1$

where we also used (147a, c). The two coefficients and the independent term of this equation are positive (135d), (147a, c), (151a, b). It is also easy to check that the roots of this equation are real. This equation has then two negative real roots.

The solution of this problem can be calculated considering only the momentum (133) and energy equations (134) and assuming that the pressures of liquid and gas are equal. This is the same as using the system of equations (135). Dividing (135a) by (135b) we get:

$$\frac{dp}{d\rho_g} = \frac{(\gamma p/\rho_g)D - BN}{D - AN} \tag{155}$$

This equation does not involve x. We can then draw a phase plane representation for p and ρ_g. The singular points of (155) are obviously $(p_1 \rho_1)$ and $(p_2 \rho_2)$. The behavior of (155) at these points is given by (153), (154). Equation (153) has two real roots of opposite sign; this means that $(p_1 \rho_1)$ is a saddle point. Equation (154) has two negative real roots; this means that $(p_2 \rho_2)$ is a stable node. In fig. 4 a phase plane representation of equation (155) is given.

In this case the transition is uniform, and the oscillations of the previous case do not appear. In the previous case the dissipative terms did not have influence in the first stages of the shock structure; in this case the whole shock structure is determined by viscous and heat transfer effects. In this case when we are close to region 1 we have (152) $\lambda \gg 1/a$, whereas in the previous case $\lambda \sim 1/a$. This means that the shock is thicker now.

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APPENDIX A

Thermodynamic Properties of the Phases

Let us suppose that each phase has a state law:

$$p = f(\rho, T) \tag{A-1}$$

then:

$$c^{*2} = \left(\frac{\partial p}{\partial \rho}\right)_T = \frac{\partial f}{\partial \rho} \tag{A-2}$$

$$dQ = Tds = T\left(\frac{\partial s}{\partial T}\right)_\rho dT + \left(\frac{\partial s}{\partial \rho}\right)_T d\rho = T\left(\frac{\partial s}{\partial T}\right)_p + \left(\frac{\partial s}{\partial p}\right)_T dp \quad \text{A-3)}$$

$$c_p = T\left(\frac{\partial s}{\partial T}\right)_p , \qquad c_v = T\left(\frac{\partial s}{\partial T}\right)_\rho \tag{A-4}$$

from (A-1) and (A-3) we get:

$$Tds = \frac{c_v}{f_T} dp - \left[c_v \frac{f_\rho}{f_T} - T\left(\frac{\partial s}{\partial \rho}\right)_T\right] d\rho \tag{A-5}$$

$$= \left[\frac{c_p}{f_T} + T\left(\frac{\partial s}{\partial p}\right)_T\right] dp - c_p \frac{f_\rho}{f_T} d\rho$$

$$\left(\frac{\partial s}{\partial \rho}\right)_T = - \frac{(c_p - c_v)}{T} \; \frac{f_\rho}{f_T} \tag{A-6}$$

and from (A-5) and (A-6)

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right) = \frac{(c_v f_\rho / f_T) - T(\partial s/\partial \rho)_T}{(c_v/f_T)} = \frac{c_p}{c_v} f_\rho = \gamma c^{*2} \tag{A-7}$$

where $\gamma = c_p/c_v$

then:

$$Tds = \frac{c_v}{f_T}\ (dp - c^2 d\rho) \qquad \text{(A-8)}$$

The compressibility is defined as:

$$\alpha_v = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p = \frac{-1}{\rho}\ \frac{f_T}{f_\rho} = \frac{-1}{\rho}\ \frac{\gamma}{c^2} f_T \qquad \text{(A-9)}$$

One of the Maxwell reciprocity relations is:

$$\left(\frac{\partial s}{\partial \rho}\right)_T = -\frac{1}{\rho^2}\left(\frac{\partial p}{\partial T}\right)_\rho = -\frac{1}{\rho^2}\ f_T$$

Using (A-6) and (A-9):

$$Tc^2 \alpha_v^2 = c_p(\gamma - 1) \qquad \text{(A-10)}$$

Then (A-8) becomes:

$$ds = -\frac{\alpha_v}{(\gamma-1)\rho}\ (dp - c^2\ d\rho) \qquad \text{(A-11)}$$

We have justified relations (17) of section 2.2.1.

Relations (19), (20), (12) and (13) of section 2.2.1 give:

$$(1-X)\ \frac{\alpha_{v0}}{(\gamma_0 - 1)}\ (dp_0 - c_0^2\ d\rho_0) + X\ \frac{\alpha_{v1}}{(\gamma-1)}\ (dp_1 - c_1^2 d\rho_1) = 0 \qquad \text{(A-12)}$$

$$dT_1 = K' dT_0 \qquad \text{(A-13)}$$

$$dp_1 = -\frac{2\tau}{a^2}\ da + dp_0 \qquad \text{(A-14)}$$

$$d\rho_1 = -\frac{3\rho_1}{a}\ da \qquad \text{(A-15)}$$

From (A-1) we have:

$$dp_1 = \left(\frac{\partial f_1}{\partial \rho_1}\right) d\rho_1 + \left(\frac{\partial f_1}{\partial T_1}\right) dT_1 \qquad \text{(A-16)}$$

$$dp_0 = \left(\frac{\partial f_0}{\partial \rho_0}\right) d\rho_0 + \left(\frac{\partial f}{\partial T_0}\right) dT_0 \qquad \text{(A-17)}$$

Using (A-14) and (A-15), (A-12) becomes:

$$-(dp_0 - c_0^2 d\rho_0) = \frac{X}{1-X} \frac{(\gamma_0 - 1)}{(\gamma_1 - 1)} \frac{\alpha_{v1}}{\alpha_{v0}} \left[dp_0 - (c_1^2 - \frac{2\tau}{3a\rho_1}) d\rho_1\right] \qquad \text{(A-18)}$$

and (A-16):

$$dp_0 = \left[(\frac{\partial f_1}{\partial \rho_1}) - \frac{2\tau}{3a\rho_1}\right] d\rho_1 + (\frac{\partial f_1}{\partial T_1})\, dT_1 \qquad \text{(A-19)}$$

Eliminating the temperature dependence by using (A-13), (A-17) and (A-19) we get:

$$(K' f_{1T_1} - f_{0T_0})\, dp_0 = K' f_{0\rho_0} f_{1T_1} d\rho_0 - (f_{1\rho_1} - \frac{2\tau}{3a\rho_1}) f_{0T_0} d\rho_1 \qquad \text{(A-20)}$$

by eliminating $d\rho_1$ between (A-18) and (A-20) and using (A-10) we can easily get (22a) of section 2.2.1. By eliminating $d\rho_0$ we get (22b).

APPENDIX B

Calculation of the Dispersion and Attenuation of Sound

Expanding (15), (16) and (22) for small X and neglecting surface tension:

$$c^2 = \frac{dp_0}{d\rho_0}\,\alpha\left[1 + X\left(1 - \frac{\rho_0(dp_0/d\rho_0)}{\rho_1(dp_1/d\rho_1)}\right) + X\left(1 - \frac{\rho_1}{\rho_0}\right)\right]$$

$$\alpha = 1 + X\,\frac{(\rho_1-\rho_0)(1-K)}{\rho_0}$$

$$\frac{dp_0}{d\rho_0} = c_0^2\left[1 - X(\gamma_0-1)\left(K'\,\frac{\rho_1}{\rho_0}\,\frac{c_{p1}}{c_{p0}} - \frac{\alpha_{v1}}{\alpha_{v0}}\right)\right]$$

$$\frac{dp_1}{d\rho_1} = c_1^2\;\frac{\rho_0 c_{p0}/(K'\rho_1 c_{p1})}{\dfrac{\alpha_{v0}}{\alpha_{v1}} + \gamma_1\left(\dfrac{1}{K'}\,\dfrac{\rho_0}{\rho_1}\,\dfrac{c_{p0}}{c_{p1}} - \dfrac{\alpha_{v0}}{\alpha_{v1}}\right)} + O(X)$$

From these three formulae we get:

$$c^2 = c_0^2\left[1 - KX\,\frac{(\rho_1-\rho_0)K}{\rho_0} - X\left(K'\,\frac{\rho_1}{\rho_0}\,\frac{c_{p1}}{c_{p0}}\,\frac{\alpha_{v0}}{\alpha_{v1}} - 1\right)\right.$$
$$\left.\left((\gamma_0-1)\,\frac{\alpha_{v1}}{\alpha_{v0}} - (\gamma_1-1)\,\frac{c_0^2\rho_0}{c_1^2\rho_1}\right) + X\left(1 - \frac{c_0^2\rho_0}{c_1^2\rho_1}\right)\right] \qquad \text{(B-1)}$$

Using (39c) and (43c) (and the relations for the wave numbers (33) and (38)) we get:

$$-X\,\frac{(\rho_1-\rho_0)K}{\rho_0} = \frac{12\pi n\, B_{10}}{(\omega/c_0)^3}\; i \qquad \text{(B-2)}$$

where K is given in (46) and:

$$X = \frac{4}{3}\,\pi n^3 \qquad \text{(B-3)}$$

In the same way: (using (33) and (38)):

$$X\left(1-\frac{c_0^2\rho_0}{c_1^2\rho_1}\right)=\frac{4\pi n}{(\omega/c_0)^3}\; i\left[\frac{ia_0^3}{3}\left(\frac{\rho_0}{\rho_1}\frac{a_0^2}{a_1^2}-1\right)\right] \tag{B-4}$$

Using (57a):

$$K'\frac{\alpha_{v0}c_{p1}\rho_1}{\alpha_{v1}c_{p0}\rho_0}-1=H\left(1-\frac{\alpha_{v0}c_{p1}\rho_1}{\alpha_{v1}c_{p0}\rho_0}\right) \tag{B-5}$$

From relations (36) and the identity (17g) we get:

$$\frac{\alpha_{v0}c_{p1}\rho_1}{\alpha_{v1}c_{p0}\rho_0}=\frac{\alpha_0\rho_1}{\alpha_1\rho_0}=\frac{\alpha_1'\sigma_1}{\alpha_0'\sigma_0} \tag{B-6}$$

$$(\gamma_0-1)\frac{\alpha_{v1}}{\alpha_{v0}}-(\gamma_1-1)\frac{c_0^2\rho_0}{c_1^2\rho_1}=\frac{(\gamma_0-1)\alpha_{v1}}{\alpha_{v0}}\left(1-\frac{\alpha_1\rho_0}{\alpha_0\rho_1}\right) \tag{B-7}$$

Using (B-5), (B-6) and (B-7) we get:

$$\left(K'\frac{\rho_1}{\rho_0}\frac{c_{p1}}{c_{p0}}\frac{\alpha_{v0}}{\alpha_{v1}}-1\right)\left[(\gamma_0-1)\frac{\alpha_{v1}}{\alpha_{v0}}-(\gamma_1-1)\frac{c_0^2\rho_0}{c_1^2\rho_1}\right]=$$

$$=H\left(1-\frac{\alpha_1'\sigma_1}{\alpha_0'\sigma_0}\right)\frac{(\gamma_0-1)\alpha_{v1}}{\alpha_{v0}}\left(1-\frac{\alpha_1\rho_0}{\alpha_0\rho_1}\right)$$

Using (17g), (B-6) and the value of H (57b) we get for the last expression:

$$=-\frac{3ic_0^3}{\omega^3a^3}\left[B_{00}-\frac{ia_0^3}{3}\left(\frac{\rho_0a_1^2}{\rho_1a_0^2}-1\right)\right] \tag{B-8}$$

where B_{00} is given in (39a).

Putting (B-8), (B-4) and (B-2) in (B-1) we get:

$$c^2 = c_0^2 \left[1 + \frac{4n\pi i}{(\omega/c_0)^3} \left(B_{00} + 3B_{10} \right) \right] \tag{B-9}$$

This is the result given in (48).

From (B-9) we get:

$$c_{imag} = \frac{2n\pi}{(\omega/c_0)^3} c_0 \text{ Real } [B_{00} + 3B_{10}] \tag{B-10}$$

From (39a) and (B-6)

$$\text{Real } (B_{00}) = -\text{Real} \left[\left(1 - \frac{\alpha_1 \rho_0}{\alpha_0 \rho_1} \right) i \; a_0 \, h_1(a_0') \, C_{00} \right] \tag{B-11}$$

Using (36), (38) and (33):

$$i \, a_0 \frac{\alpha_0}{\alpha_0'} = - \frac{\omega^2 (\gamma_0 - 1)}{c_0^3 \rho_0 c_{p0}} a \tag{B-12}$$

Then (B-11) becomes (using (39a, b))

$$\text{Real } (B_{00}) = \left(1 - \frac{\alpha_1 \rho_0}{\alpha_0 \rho_1} \right)^2 \frac{\omega^2 (\gamma_0 - 1) a \sigma_0}{c_0^3 \rho_0 c_{p0}} \text{ Real} \left\{ \frac{a_0' h_1(a_0')/h_0(a_0')}{1 - \frac{\sigma_0}{\sigma_1} \frac{j_0(a_1')}{a_1' j_1(a_1')} \frac{a_0' h_1(a_0')}{h_0(a_0')}} \right\}$$

Putting this result in (B-10) and then using (50) we get (51).

APPENDIX C

Case of Drops or Solid Particles

In this appendix we want to calculate the limiting value of the expression of the speed of sound given in (48) when:

$$c_1 \gg c_0,\ \rho_1 \gg \rho_0,\ \mu_1 \gg \mu_0,\ \sigma_1 \gg \sigma_0,\ \alpha_{v1} \ll \alpha_{v0} \qquad \text{(C-1)}$$

This means that we have to obtain the limiting values of the coefficients B_{00} and B_{10} (39a,c). Carhart (11) calculated the limiting value of the damping factor (51). His results can be easily applied to our problem.

Carhart says that the bracket of the denominator of (39b):

$$1 - \frac{\sigma_0}{\sigma_1}\ \frac{j_0(a_1')}{a_1' j_1(a_1')}\ \frac{a_0' h_1(a_0')}{h_0(a_0')}$$

should be expanded for small values of $|a_1'|$. The reason for this is that σ_0/σ_1 is small (C-1), and consequently the second term is important only for large values of $j_0(a_1')/a_1' j_1(a_1')$.

$$\frac{j_0(a_1')}{a_1' j_1(a_1')} = \frac{1}{1 - a_1' \operatorname{cotg} a_1'} \simeq \frac{45}{a_1'^2\ (15 + a_1'^2)} \qquad \text{(C-2)}$$

From (40b) we get:

$$a_0' = (1+i)\left(\frac{\omega a^2 \rho_0 c_{p0}}{2\sigma_0}\right)^{\frac{1}{2}} = (1+i) z_0 \qquad \text{(C-3)}$$

$$a_1' = (1+i) z_1 \qquad \text{(C-4)}$$

This means that:

$$z_1^2 = \frac{\sigma_0}{\sigma_1} \frac{\rho_1}{\rho_0} \frac{c_{p1}}{c_{p0}} z_0^2 \tag{C-5}$$

$$\frac{\sigma_0}{\sigma_1} \frac{z_0^2}{z_1^2} = \frac{\rho_0}{\rho_1} \frac{c_{p0}}{c_{p1}} << 1 \tag{C-6}$$

Using (C-2) to (C-6) and the values of spherical Hankel functions (37c) we get:

$$\frac{a_0' h_1(a_0')}{h_0(a_0')} \frac{1}{1 - \frac{\sigma_0}{\sigma_1} \frac{j_0(a_1')}{a_1' j_1(a_1')} \frac{a_0' h_1(a_0')}{h_0(a_0')}} =$$

$$= \frac{4(1+z_0)z_0^4 - iz_0^2\left[4z_0^3 + 6\left(\frac{\rho_0}{\rho_1} \frac{c_{p0}}{c_{p1}}\right)(1+z_0+z_0^2)\right]}{4z_0^2 + 12\, z_0^3\left(\frac{\rho_0}{\rho_1} \frac{c_{p0}}{c_{p1}}\right) + 9\left(\frac{\rho_0}{\rho_1} \frac{c_{p0}}{c_{p1}}\right)^2} \tag{C-7}$$

In Appendix B we saw that:

$$\frac{\alpha_1 \rho_0}{\alpha_0 \rho_1} = \frac{\alpha_0' \sigma_0}{\alpha_1' \sigma_1} = \frac{\alpha_{v1} c_{p0} \rho_0}{\alpha_{v0} c_{p1} \rho_1} << 1 \tag{C-8}$$

The inequality is obtained using (C-1). Also from (38):

$$\frac{\rho_0}{\rho_1} \left(\frac{a_1}{a_0}\right)^2 = \frac{\rho_0 c_0^2}{\rho_1 c_1^2} < < 1 \tag{C-9}$$

Taking (C-7), (C-8) and (C-9) in (39a,b) we obtain a value of B_{00}; substituting this value of B_{00} in (48) we obtain the first two terms of the bracket of (63). Similarly taking (C-7) in (57) we obtain the value of K' given in (68).

Using (C-1) the value of B_{10} (39c) becomes:

$$B_{10} = i\,\frac{a_0^3}{3}\left[\frac{b_0 h_2(b_0)}{9\,\frac{\rho_0}{\rho_1}\,h_1(b_0) - 2b_0 h_0(b_0)}\right] \tag{C-10}$$

where:

$$b_0 = (1+i)\left(\frac{\omega a_0^2}{2\nu}\right)^{\frac{1}{2}} = (1+i)y \tag{C-11}$$

(C-10) becomes:

$$B_{10} = \frac{ia_0^3}{3}\left[\frac{8y^4+12y^3+27\delta(2y^2+2y+1)+12y^2(y+1)i}{16y^4+72\delta y^3+81\delta^2(1+2y+2y^2)}\right] \tag{C-12}$$

on using (37c). Taking this value of B_{10} in (48) we get the third term of the bracket of (63). Substituting (C-12) in (43c) and using (46) we obtain the value of K (67).

APPENDIX D

Case of Bubbles

We want to calculate the limiting value of the expression of the speed of sound (48) when:

$$c_0 \gg c_1, \quad \rho_0 \gg \rho_1, \; \mu_0 \gg \mu_1, \quad \sigma_0 \gg \sigma_1, \quad \alpha_{v0} \ll \alpha_{v1} \qquad \text{(D-1)}$$

Using relations (B-6) and (D-1) we get:

$$\frac{\alpha_1 \rho_0}{\alpha_0 \rho_1} = \frac{\alpha'_0 \sigma_0}{\alpha'_1 \sigma_1} = \frac{\alpha_{v1} c_{p0} \rho_0}{\alpha_{v0} c_{p1} \rho_1} \gg 1 \qquad \text{(D-2)}$$

Also from (38):

$$\frac{\rho_0}{\rho_1} \left(\frac{a_1}{a_0}\right)^2 = \frac{\rho_0 c_0^2}{\rho_1 c_1^2} > > 1 \qquad \text{(D-3)}$$

In this case (using D-1):

$$\frac{a'_0 h_1(a'_0)}{h_0(a'_0)} \quad \frac{1}{\left[1 - \dfrac{\sigma_0}{\sigma_1} \dfrac{j_0(a'_1)}{a'_1 j_1(a'_1)} \dfrac{a'_0 h_1(a'_0)}{h_0(a'_0)}\right]} \simeq$$

$$\simeq \frac{\sigma_1}{\sigma_0} \frac{a'_1 j_1(a'_1)}{j_0(a'_1)} =$$

and using the values of the spherical Bessel functions (32a)

$$= \frac{\sigma_1}{\sigma_0} (1 - a'_1 \operatorname{cotg} a'_1) \qquad \text{(D-4)}$$

Using the value of a'_1 given in (C-4):

$$1 - a'_1 \operatorname{cotg} a'_1 = 1 - z_1 \frac{\sinh 2z_1 + \sin 2z_1}{\cosh 2z_1 - \cos 2z_1} + i z_1 \frac{\sinh 2z_1 - \sin 2z_1}{\cosh 2z_1 - \cos 2z_1}$$

Taking (D-4) in (39a, b) and using the inequalities (D-2) and (D-3) we obtain B_{00}. This value of B_{00} (using (48)) gives the first two terms of the bracket of (71). Similarly, taking (D-4) in (57) and using relations (36) we get the value of K' given in (75).

From (40a) we get:

$$b_0^2 \frac{\mu_0}{\rho_0} = (1+i)^2 \frac{\omega a^2}{2} = b_1^2 \frac{\mu_1}{\rho_1} \tag{D-5}$$

Formula (39c) becomes:

$$B_{10} = -\frac{ia_0^3}{3} \left[\frac{-b_0 h_2(b_0) + \frac{1}{2} b_0^2 h_1(b_0)}{b_0 h_0(b_0) + \frac{1}{2} b_0^2 h_1(b_0)} \right] =$$

$$= \frac{-ia_0^3}{3} \left[\frac{2y^6+6y^5+9y^4+24y^3+54y^2+54y+27-i(12y^4+24y^3+18y^2)}{2y^6+6y^5+9y^4+36y^3+162y^2+162y+81} \right]$$

on using (D-1), (D-5), (37c) and (C-11). This value of B_{10} (using (48)) gives the last term of the bracket of (71). By using (43c) and (46) we get the value of K given in (74).