Appendix E

Gauss-Bonnet theorem in the shape index, curvedness space

For any compact two-dimensional Riemann manifold without boundaries, $M$, the Gauss–Bonnet theorem states that the integral of the Gaussian curvature, $K$, over the manifold with respect to area, $A$, equals $2\pi$ times its Euler characteristic, $\chi$:

$$\int_M K\,dA = 2\pi\chi(M). \quad (E.1)$$

This formula relates the geometry of the surface (given by the integration of the Gaussian curvature, a differential-geometry property) to its topology (given by the Euler characteristic). The Euler characteristic of a surface is related to its genus $^1$ by $\chi = 2 - 2g$. From the relation among shape index, curvedness, and mean and Gaussian curvatures stated in Appendix C (see equations C.38 and C.39), the following relation can be obtained:

$$K = -\Lambda^2 \cos(\pi \Upsilon). \quad (E.2)$$

Then, the Gauss-Bonnet theorem can be restated in terms of the shape index and curvedness as

$$\int_M \Lambda^2 \cos(\pi \Upsilon) \,dA = 4\pi[g(M) - 1]. \quad (E.3)$$

$^1$The genus of an orientable surface is a topological invariant (as is the Euler characteristic) defined as the largest number of non-intersecting simple closed curves that can be drawn on the surface without disconnecting it.
Furthermore, considering the non-dimensionalization of the curvedness introduced in §2.2, \(C = \mu \Lambda\) (\(\mu \equiv 3V/A\), where \(V\) is the volume and \(A\) the area of the surface) and taking into account that cosine is a symmetric function and thus \(\cos(\pi \Upsilon) = \cos(|\Upsilon|) \equiv \cos(\pi S)\), then equation E.1 can be rewritten as

\[
\int_M C^2 \cos(\pi S) \, dA = 4\pi \mu^2 [g(M) - 1].
\]  
(E.4)

The left-hand side can be expressed in terms of the \(\{S, C\}\) area-based joint probability density function of the surface, \(\mathcal{P}(S, C)\):

\[
\int_M C^2 \cos(\pi S) \, dA = A \cdot \int \int C^2 \cos(\pi S) \, \mathcal{P}(S, C) \, dS \, dC.
\]  
(E.5)

Considering the stretching parameter, \(\lambda \equiv \sqrt[3]{\frac{3\pi V^2}{A}}\), also introduced in §2.2, the Gauss-Bonnet theorem finally results in an integral relation between the \(\{S, C\}\) area-based joint probability density function, \(\mathcal{P}\), the stretching parameter, \(\lambda\), and the genus of the surface, \(g\):

\[
\int \int C^2 \cos(\pi S) \, \mathcal{P}(S, C) \, dS \, dC = \lambda^3 [g(M) - 1].
\]  
(E.6)