Appendix E

Gauss-Bonnet theorem in the shape index, curvedness space

For any compact two-dimensional Riemann manifold without boundaries, M, the Gauss–Bonnet theorem states that the integral of the Gaussian curvature, K, over the manifold with respect to area, A, equals 2π times its *Euler characteristic*, χ:

$$\int_M K \mathrm{d}A = 2\pi\chi(M). \tag{E.1}$$

This formula relates the geometry of the surface (given by the integration of the Gaussian curvature, a differential-geometry property) to its topology (given by the Euler characteristic). The Euler characteristic of a surface is related to its *genus* [1] by $\chi = 2 - 2g$. From the relation among shape index, curvedness, and mean and Gaussian curvatures stated in Appendix C (see equations C.38 and C.39), the following relation can be obtained:

$$K = -\Lambda^2 \cos(\pi\Upsilon). \tag{E.2}$$

Then, the Gauss-Bonnet theorem can be restated in terms of the shape index and curvedness as

$$\int_M \Lambda^2 \cos(\pi\Upsilon)\, \mathrm{d}A = 4\pi[g(M) - 1]. \tag{E.3}$$

[1] The genus of an orientable surface is a topological invariant (as is the Euler characteristic) defined as the largest number of non-intersecting simple closed curves that can be drawn on the surface without disconnecting it.

Furthermore, considering the non-dimensionalization of the curvedness introduced in §2.2, $C = \mu\Lambda$ ($\mu \equiv 3V/A$, where V is the volume and A the area of the surface) and taking into account that cosine is a symmetric function and thus $\cos(\pi\Upsilon) = \cos(\pi|\Upsilon|) \equiv \cos(\pi S)$, then equation E.1 can be rewritten as

$$\int_M C^2 \cos(\pi S)\, \mathrm{d}A = 4\pi\mu^2[g(M) - 1]. \tag{E.4}$$

The left-hand side can be expressed in terms of the $\{S, C\}$ area-based joint probability density function of the surface, $\mathcal{P}(S, C)$:

$$\int_M C^2 \cos(\pi S)\, \mathrm{d}A = A \cdot \int\int C^2 \cos(\pi S)\, \mathcal{P}(S, C)\, \mathrm{d}S\, \mathrm{d}C. \tag{E.5}$$

Considering the stretching parameter, $\lambda \equiv \sqrt[3]{36\pi}(V^{2/3}/A)$, also introduced in §2.2, the Gauss-Bonnet theorem finally results in an integral relation between the $\{S, C\}$ area-based joint probability density function, $\mathcal{P}$, the stretching parameter, λ, and the genus of the surface, g:

$$\int\int C^2 \cos(\pi S)\, \mathcal{P}(S, C)\, \mathrm{d}S\, \mathrm{d}C = \lambda^3[g(M) - 1]. \tag{E.6}$$