Appendix D

Density functions on manifolds

Consider a function $\xi_M(P) : P \in M \mapsto I \subseteq \mathbb{R}$, that defines a local property ξ of an m-dimensional manifold M embedded in a n-dimensional space $\mathbb{R}^n$, $m < n$. At every point $P \in M$. Define $M_\xi^=$ as the set of points on M where $\xi_M(P)$ is equal to a particular value ξ, $M_\xi^= = \{P \in M \mid \xi_M(P) = \xi\}$, and $M_\xi^\leq$ as the set of points on M where $\xi_M(P)$ is less or equal to a particular value ξ, $M_\xi^\leq = \{P \in M \mid \xi_M(P) \leq \xi\}$. Consider the *measure spaces*[1] $(\mathbb{R}^n, \mathcal{F}(\mathbb{R}^n), \mu_1)$ and $(\mathbb{R}^n, \mathcal{F}(\mathbb{R}^n), \mu_2)$, where $\mathcal{F}(\mathbb{R}^n)$ is a σ-algebra of $\mathbb{R}^n$, and μ_1, μ_2 are two particular measures defined on $(\mathbb{R}^n, \mathcal{F}(\mathbb{R}^n))$.

Define the function $\Psi(\xi) : \xi \in I \mapsto \bar{\mathbb{R}}^+$ such that for every value $\xi \in I$ it returns the μ_1-measure of the set $M_\xi^= \subset M$:

$$\Psi(\xi) \equiv \mu_1\left(M_\xi^=\right) \equiv \int_{M_\xi^=} \mathrm{d}\mu_1 = \int_M \mathbf{1}_{[M_\xi^=]} \mathrm{d}\mu_1, \tag{D.1}$$

where the function $\mathbf{1}_{[M_\xi^=]}$ is the *characteristic function*[2] on $M_\xi^=$ and the integrals are defined in the generalized Lebesgue[3] sense.

---

[1] A *measure space* $(E, \mathcal{F}(E), \mu)$ is a measurable space, $(E, \mathcal{F}(E))$, with a non-negative measure, μ. A *measurable space*, $(E, \mathcal{F}(E))$, is a set E with a σ-algebra, $\mathcal{F}(E)$, on it. A σ-algebra $\mathcal{F}$ on a given set E is a nonempty collection of subsets of E such that: 1) $\emptyset \in \mathcal{F}(E)$; 2) if $A \in \mathcal{F}$ then $\bar{A} \in \mathcal{F}$, where $\bar{A}$ is the complement of A; 3) if A_n is a sequence of elements of $\mathcal{F}$, then $\bigcup A_n \in \mathcal{F}$. As a consequence: $E \in \mathcal{F}$. A *measure* μ, defined on a measurable space $(E, \mathcal{F}(E))$, is a function $\mu : \mathcal{F}(E) \mapsto \bar{\mathbb{R}}$ (where $\bar{\mathbb{R}}$ denotes the extended real numbers ($\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$)) such that: 1) $\mu(A) \geq 0$ for $A \in \mathcal{F}(E)$ (equality iff $A = \emptyset$), 2) $\mu(\cup_{n=0}^\infty A_n) = \sum_{n=0}^\infty \mu(A_n)$ for any sequence of disjoint sets $A_n \in \mathcal{F}(E)$ (countable additivity). If $\mu(E) = 1$ then μ is called a *probability measure* and the measure space $(E, \mathcal{F}(E), \mu)$ is called a *probability space*.

[2] The *characteristic function* or *indicator function*, of a subset $A \in E$ is a function $\mathbf{1}_A : E \mapsto \{0, 1\}$ defined as $\mathbf{1}_A = \{1, \text{if } P \in A; 0, \text{if } P \notin A\}$.

[3] The *Lebesgue integral* of a measurable function $f : E \mapsto \bar{\mathbb{R}}$ on a measure space $(E, \mathcal{F}(E), \mu)$, is defined through the following steps: 1) for the characteristic function, $\mathbf{1}_A$, $\int_E \mathbf{1}_A \,\mathrm{d}\mu \equiv \mu(A)$; 2) for a simple function (i.e., $s = \sum_{i=1}^n c_i \mathbf{1}_{A_i}$, $c_i \in \mathbb{R}$, for some finite collection $A_i \in \mathcal{F}(E)$), then: $\int_E f \,\mathrm{d}\mu \equiv \sum_{i=1}^n c_i \int_E \mathbf{1}_{A_i} \,\mathrm{d}\mu = \sum_{i=1}^n c_i \mu(A_i)$; 3) For a non-negative measurable function f (possibly attaining ∞ at some points), $\int_E f \,\mathrm{d}\mu \equiv \sup\left\{\int_E s \,\mathrm{d}\mu : s \leq f, s \text{ simple}\right\}$; 4) For any measurable function f (possibly attaining $\pm\infty$ at some points), $\int_E f \,\mathrm{d}\mu \equiv \int_E f^+ \,\mathrm{d}\mu - \int_E f^- \,\mathrm{d}\mu$, where $f^\pm \equiv \max(\pm f, 0)$, provided $\int_E |f| \,\mathrm{d}\mu = \int_E (f^+ + f^-) \,\mathrm{d}\mu < \infty$ (f is then said to be *Lebesgue integrable*). A function $f : E_x \mapsto E_y$ is *measurable* if $f^{-1}(\mathcal{F}(E_y)) \subseteq \mathcal{F}(E_x)$, where $(E_x, \mathcal{F}(E_x))$ and $(E_y, \mathcal{F}(E_y))$ are two measurable spaces. The generalized Lebesgue integral extends this concept of Lebesgue integral to measure spaces with generalized measures μ, not necessarily being Lebesgue measures (e.g., Hausdorff measures).

Define also the function $\eta(\xi) : \xi \in I \mapsto \bar{\mathbb{R}}^+$ such that for every value $\xi \in I$ it returns the μ_2-measure of the set $M_\xi^\leq \subset M$:

$$\eta(\xi) \equiv \mu_2\left(M_\xi^\leq\right) \equiv \int_{M_\xi^\leq} \mathrm{d}\mu_2 = \int_M \mathbf{1}_{\left[M_\xi^\leq\right]} \mathrm{d}\mu_2, \tag{D.2}$$

where the function $\mathbf{1}_{\left[M_\xi^\leq\right]}$ is the characteristic function on $M_\xi^\leq$. Let $\delta\eta(\xi, \mathrm{d}\xi)$ be the difference between the values of η at $\xi + \mathrm{d}\xi$ and ξ:

$$\begin{aligned}
\delta\eta(\xi, \mathrm{d}\xi) &\equiv \eta(\xi + \mathrm{d}\xi) - \eta(\xi) \equiv \mu_2\left(M_{\xi+\mathrm{d}\xi}^\leq\right) - \mu_2\left(M_\xi^\leq\right) \equiv && \text{(D.3)}\\
&\equiv \left(\int_{M_{\xi+\mathrm{d}\xi}^\leq} - \int_{M_\xi^\leq}\right)\mathrm{d}\mu_2 = \int_M \left(\mathbf{1}_{\left[M_{\xi+\mathrm{d}\xi}^\leq\right]} - \mathbf{1}_{\left[M_\xi^\leq\right]}\right)\mathrm{d}\mu_2 && \text{(D.4)}
\end{aligned}$$

and define formally $\mathrm{d}\eta(\xi)/\mathrm{d}\xi$ as the limit of $\delta\eta(\xi, \mathrm{d}\xi)/\mathrm{d}\xi$ when $\mathrm{d}\xi \to 0$:

$$\begin{aligned}
\frac{\mathrm{d}\eta(\xi)}{\mathrm{d}\xi} &\equiv \lim_{\mathrm{d}\xi\to 0} \frac{\delta\eta(\xi, \mathrm{d}\xi)}{\mathrm{d}\xi} = && \text{(D.5)}\\
&= \lim_{\mathrm{d}\xi\to 0} \frac{\left(\int_{M_{\xi+\mathrm{d}\xi}^\leq} - \int_{M_\xi^\leq}\right)\mathrm{d}\mu_2}{\mathrm{d}\xi} = && \text{(D.6)}\\
&= \lim_{\mathrm{d}\xi\to 0} \frac{\int_M \left(\mathbf{1}_{\left[M_{\xi+\mathrm{d}\xi}^\leq\right]} - \mathbf{1}_{\left[M_\xi^\leq\right]}\right)\mathrm{d}\mu_2}{\mathrm{d}\xi} = && \text{(D.7)}\\
&= \lim_{\mathrm{d}\xi\to 0} \int_M \frac{\mathbf{1}_{\left[M_{\xi+\mathrm{d}\xi}^\leq\right]} - \mathbf{1}_{\left[M_\xi^\leq\right]}}{\mathrm{d}\xi}\mathrm{d}\mu_2 = && \text{(D.8)}\\
&= \int_M \lim_{\mathrm{d}\xi\to 0} \left(\frac{\mathbf{1}_{\left[M_{\xi+\mathrm{d}\xi}^\leq\right]} - \mathbf{1}_{\left[M_\xi^\leq\right]}}{\mathrm{d}\xi}\right)\mathrm{d}\mu_2. && \text{(D.9)}
\end{aligned}$$

Define the generalized function:

$$\delta_{\left[M_\xi^=\right]} \equiv \lim_{\mathrm{d}\xi\to 0}\left(\frac{\mathbf{1}_{\left[M_{\xi+\mathrm{d}\xi}^\leq\right]} - \mathbf{1}_{\left[M_\xi^\leq\right]}}{\mathrm{d}\xi}\right) \equiv \frac{\mathrm{d}\mathbf{1}_{\left[M_\xi^=\right]}}{\mathrm{d}\xi}. \tag{D.10}$$

It can be considered as an operator such that, when applied to a function $f(P, \xi)$ defined on M, it returns the variation of $f(P, \xi)$ in the direction normal to the tangent space of $M_\xi^=$ on M at each point $P \in M_\xi^=$. Then, equation D.9 results:

$$\frac{\mathrm{d}\eta(\xi)}{\mathrm{d}\xi} = \int_M \delta_{[M_\xi^=]} \mathrm{d}\mu_2. \tag{D.11}$$

Consider, in particular, μ_i, $i = 1, 2$, to be the α_i-dimensional *Hausdorff measure*[4], $\mathcal{H}^{\alpha_i}$ on $\mathbb{R}^n$, such that $\alpha_2 > \alpha_1$, and $\mathrm{d}\mu_2 = \mathrm{d}\mu_1 \mathrm{d}(\mu_2/\mu_1)$, where μ_2/μ_1 is the quotient of μ_2 by μ_1. Then, for a regular and smooth[5] manifold M:

$$\begin{aligned} \frac{d\eta(\xi)}{\mathrm{d}\xi} &= \int_M \delta_{[M_\xi^=]} \mathrm{d}\mu_2 = \int_M \left[\delta_{[M_\xi^=]} d(\mu_2/\mu_1)\right] \mathrm{d}\mu_1 \equiv f(\xi) \int_{M_\xi^=} \mathrm{d}\mu_1 && \text{(D.12)} \\ &= f(\xi)\mu_1(M_\xi^=) \equiv f(\xi)\Psi(\xi), && \text{(D.13)} \end{aligned}$$

where the function $f(\xi)$ is defined according the Mean-Value Theorem (applicable since the manifold is regular and smooth):

$$f(\xi) \equiv \frac{\displaystyle\int_M \left[\delta_{[M_\xi^=]} d(\mu_2/\mu_1)\right] \mathrm{d}\mu_1}{\displaystyle\int_{M_\xi^=} \mathrm{d}\mu_1} \equiv \frac{\overline{\mathrm{d}[\mu_2/\mu_1|_{M_\xi^=}}}{\mathrm{d}\xi}. \tag{D.14}$$

Considering the explanation of the character of $\delta_{[M_\xi^=]}$, the function $f(\xi)$ can be interpreted as the average value of the variation with ξ of μ_2/μ_1 on the set $M_\xi^=$ (expressed as $\overline{\mathrm{d}[\mu_2/\mu_1|_{M_\xi^=}/\mathrm{d}\xi}$). Therefore, in order to measure sets $M_\xi^= \subset M \subset \mathbb{R}^n$ in the μ_1 Hausdorff measure, $\mu_1 \equiv \mathcal{H}^{\alpha_1}$, it is possible to use alternatively the μ_2 Hausdorff measure, $\mu_2 \equiv \mathcal{H}^{\alpha_2}$, on the set $M_{\xi,\mathrm{d}\xi} = M_{\xi+\mathrm{d}\xi}^{\leq} \cap M_\xi^{\leq} =$

---

[4] Let (E, d) be a metric space (with a distance d defined on the set E). The α-dimensional *Hausdorff measure* of the set $A \subset E$, $\mathcal{H}^\alpha(A) \in [0, +\infty]$, is defined as $\mathcal{H}^\alpha(A) \equiv \lim_{\delta \to 0^+} \mathcal{H}^\alpha_\delta(A)$, being $\mathcal{H}^\alpha_\delta(A) \equiv \inf\{\sum_{j=0}^\infty \omega_\alpha \left(\mathrm{diam}(B_j^\delta)/2\right)^\alpha : B_j^\delta \subset E,\ \bigcup_{j=0}^\infty B_j^\delta \supset A,\ \mathrm{diam}(B_j^\delta) \leq \delta, \forall j = 0, 1, \ldots\}$, where $\mathrm{diam}(B_j^\delta) \equiv \sup_{x,y \in B_j^\delta} d(x, y)$, $\omega_\alpha = \pi^{\alpha/2}/\Gamma(\alpha/2+1)$, ($\Gamma(x)$ is the Gamma Function), $\delta > 0$, $\alpha \geq 0$ and the infimum is taken over all possible enumerable families of sets $\{B_0^\delta, B_1^\delta, \ldots, B_j^\delta, \ldots\}$ which are sufficiently small ($\mathrm{diam}(B_j) \leq \delta$) and which cover A. The limit exists since the function $\mathcal{H}^\alpha_\delta(E)$ is decreasing in δ: $\delta' < \delta \Rightarrow \bigcup_{i=0}^\infty B_i^{\delta'} \subset \bigcup_{j=0}^\infty B_j^\delta \Rightarrow \mathcal{H}^\alpha_{\delta'}(E) < \mathcal{H}^\alpha_\delta(E)$. The Hausdorff measure is a Borel external measure on $\mathbb{R}^n$ that generalizes the concept of length, area, and volume of sets in $\mathbb{R}^n$. For the particular case of a m-dimensional regular manifold $M \subset \mathbb{R}^n$, $\mathcal{H}^m(M)$ is the m-dimensional area of M. For $m = n$, $\mathcal{H}^n$ is the Lebesgue measure on $\mathbb{R}^n$. But as an external measure, $\mathcal{H}^\alpha$ is defined on every subset of $\mathbb{R}^n$, not only on regular manifolds.

[5] A smooth manifold is infinitely differentiable. In particular, a two-dimensional surface parametrized by variables (u, v) is smooth if the tangent vectors in the u and v directions satisfy: $\boldsymbol{t}_u \wedge \boldsymbol{t}_v \neq 0$.

$\{P \in M \mid \xi \leq \xi_M(P) < \xi + \mathrm{d}\xi\}$ divided by $f(\xi)\,\mathrm{d}\xi$ and then take the limit $\mathrm{d}\xi \to 0$:

$$\mu_1(M_\xi^=) = \lim_{\mathrm{d}\xi \to 0} \frac{\mu_2(M_{\xi,\mathrm{d}\xi})}{f(\xi)\mathrm{d}\xi}, \tag{D.15}$$

derived from equations D.6, D.13, and the relation $\mu_2(M_{\xi+\mathrm{d}\xi}^{\leq} \cap M_\xi^{\leq}) = \mu_2(M_{\xi+\mathrm{d}\xi}^{\leq}) - \mu_2(M_\xi^{\leq})$.

The *Hausdorff dimension*[6], α, of the sets $M_\xi^=, M_{\xi,\mathrm{d}\xi} \subset M$, satisfies:

$$\Delta\alpha \equiv \alpha(M_{\xi,\mathrm{d}\xi}) - \alpha(M_\xi^=) \geq 0. \tag{D.16}$$

Thus, equation D.15 implicitly indicates a reduction in the Hausdorff dimension of $\mu_2(M_{\xi,\mathrm{d}\xi})$ by taking the limit of it after dividing by $f(\xi)\,\mathrm{d}\xi$, obtaining $\mu_1(M_\xi^=)$ as a result.

Consider the function $\Psi(\xi)$, introduced above, with the particular choice of the measure μ_1 as being a α-dimensional Hausdorff measure, $\mathcal{H}^\alpha$. Also, consider the density function $\tilde{\Psi}(\xi) \equiv f(\xi)\,\Psi(\xi) : \xi \in I \mapsto \mathbb{R}^+$ with the choice of μ_1 and μ_2 as Hausdorff measures of dimension α and $\alpha + \Delta\alpha$, respectively, $\mu_1 = \mathcal{H}^\alpha$ and $\mu_2 = \mathcal{H}^{\alpha+\Delta\alpha}$. From the definition of $\tilde{\Psi}(\xi)$ and $\eta(\xi)$, (equations D.1, D.2), and the relation between them given by equation D.13 it results:

$$\begin{aligned} \int_{\xi_{\min}}^{\xi_{\max}} \tilde{\Psi}(\xi)\,\mathrm{d}\xi &= \int_{\xi_{\min}}^{\xi_{\max}} f(\xi)\,\Psi(\xi)\,\mathrm{d}\xi = \int_{\xi_{\min}}^{\xi_{\max}} \frac{\mathrm{d}\eta(\xi)}{\mathrm{d}\xi}\,\mathrm{d}\xi = \int_{\xi_{\min}}^{\xi_{\max}} \mathrm{d}\eta(\xi) = & \text{(D.17)} \\ &= \eta(\xi_{\max}) - \eta(\xi_{\min}) = \mu_2(M) = \mathcal{H}^{\alpha+\Delta\alpha}(M). & \text{(D.18)} \end{aligned}$$

Therefore, the integral of the density function $\tilde{\Psi}(\xi)$ of M over the range I of ξ is the $(\alpha + \Delta\alpha)$-dimensional Hausdorff measure of M. It can be normalized to obtain the corresponding probability

---

[6] The *Hausdorff dimension*, $\alpha(A) \geq 0$, of a subset A of a metric space (E, d), is defined as $\alpha(A) = \inf\{D \mid \lim_{r\to 0}[H_r^D(A)]\}$ being $H_r^D(A) = \inf\sum_{i\in I}\left(\mathrm{diam}(B_i^r)\right)^D$ where $\{B_i^r, i \in I, I \text{ countable set}\}$ is a countable r-cover of A and the infimum in H_r^D is over all countable r-covers of A. If A is a subset of $\mathbb{R}^n$ with any restricted norm-induced metric, this definition is equivalent to $\alpha(A) = -\lim_{r\to 0}[\log N_A(r)/\log r]$, where $N_A(r)$ is the minimum number of balls of radius r required to cover A. For a fixed set $A \subset E$ there exists at most one value α such that the α-dimensional Hausdorff measure of A, $\mathcal{H}^\alpha(A)$ is finite and positive. For $\alpha' > \alpha$, $\mathcal{H}^{\alpha'}(A) = 0$, whereas for $\alpha' < \alpha$, $\mathcal{H}^{\alpha'}(A) \to +\infty$. This result can be used equivalently to define the dimension of a set A, $\alpha(A)$ as the value for which its associated α-dimensional Hausdorff measure, $\mathcal{H}^\alpha(A)$, is finite and positive. For example, the Hausdorff dimension of a regular two-dimensional surface $M \subset \mathbb{R}^n$ is two, and $\mathcal{H}^2(M)$ (which coincides with the area of the surface) will be finite and positive, while $\mathcal{H}^1(M)$ (length of M) will be infinite, and $\mathcal{H}^3(M)$ (volume of M) will be zero. The Hausdorff dimensions of a set need not be integer (e.g., most fractals have a non-integer Hausdorff dimension).

density function:

$$\mathcal{P}(\xi) \equiv \frac{\tilde{\Psi}(\xi)}{\mathcal{H}^{\alpha+\Delta\alpha}(M)}, \qquad \text{with} \qquad \int_{\xi_{\min}}^{\xi_{\max}} \mathcal{P}(\xi)\,\mathrm{d}\xi = 1. \tag{D.19}$$

Depending on the distribution of the local property ξ throughout the manifold M, it will be appropriate to choose particular values of α and $\Delta\alpha$ for measuring the sets $M_\xi^=$ and $M_{\xi,\mathrm{d}\xi}$ in order to obtain a relevant $\tilde{\Psi}(\xi)$. For example, for a surface M in a three-dimensional euclidean space:

- If ξ is distributed mainly in patches of constant ξ, then a dimension $\alpha = 2$ with $\Delta\alpha = 0$ ($\Rightarrow \mu_1 = \mu_2$) would be appropriate: $\tilde{\Psi}(\xi)$ would then give the area of those patches for the particular values of ξ at which they appear (see Figure D.1). The sum of all those values would be the area of M ($\sum_i \tilde{\Psi}(\xi_i) = \mathcal{H}^2(M)$). By using this measure, subsets of Hausdorff dimension less than two (curves of constant ξ or isolated points of constant ξ) would not be reflected in $\tilde{\Psi}(\xi)$, since their associated $\mathcal{H}^2$ measure is null.

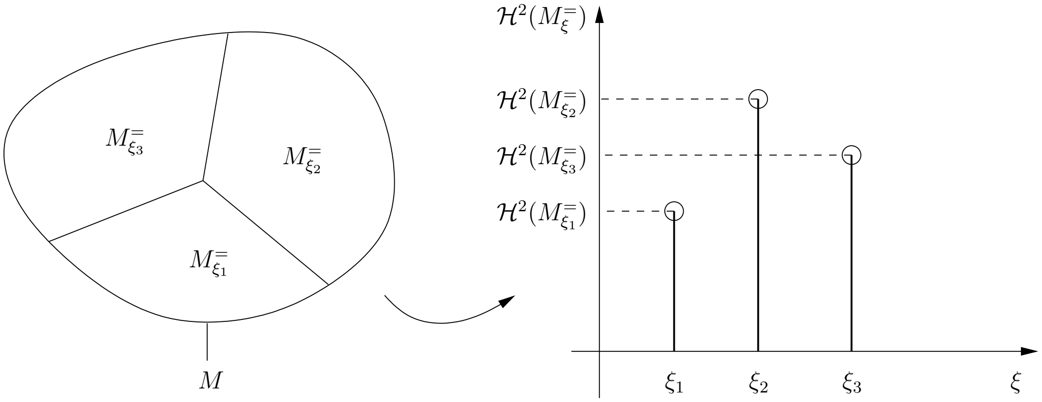

Figure D.1: $\Psi(\xi) = \tilde{\Psi}(\xi)$ (right) with $\alpha = 2$ for a surface M (left) with the local property ξ distributed in patches of constant ξ. Each point of that function (right) represents the area (two-dimensional Hausdorff measure) of the associated patch. Their discrete sum equals the total area of the surface M

- If ξ is smoothly distributed throughout M, the appropriate dimension to use is $\alpha = 1$ ($\mu_1 \equiv \mathcal{H}^1$), with $\Delta\alpha = 1$ since the sets $M_\xi^=$ will be curves of constant ξ (unitary Hausdorff dimension) or isolated points (null Hausdorff dimension). $\tilde{\Psi}(\xi)$ will be continuous and its integral with respect to ξ will be $\mathcal{H}^2(M)$ (according to equation D.18), that is, the area of M. If ξ is piecewise smoothly distributed throughout M, that is, smooth except in the boundaries of patches of M with constant ξ (see Figure D.2), these patches will be reflected in $\Psi(\xi)$ as delta functions at the corresponding value of ξ associated with each patch, such that the integral with respect to ξ equals (in the limit $\mathrm{d}\xi \to 0$) the area of the patch. The shape of $\tilde{\Psi}(\xi)$ (see Figure D.3)

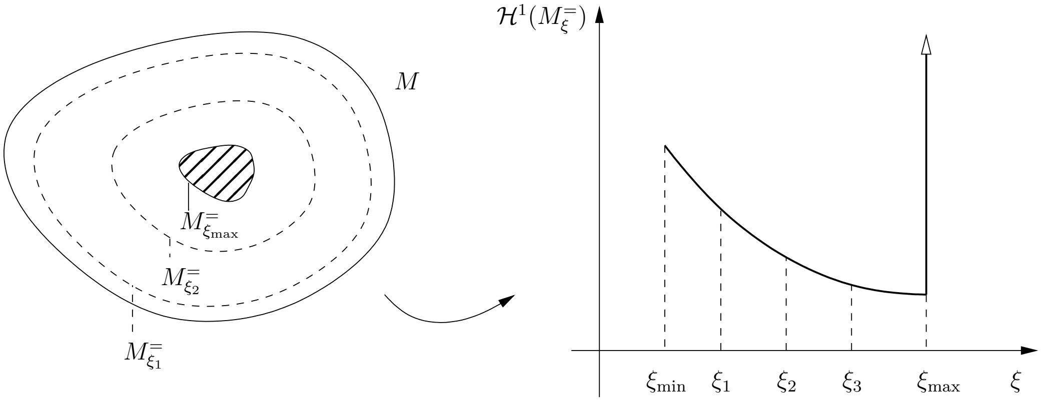

Figure D.2: $\Psi(\xi)$ (right) with $\alpha = 1$ for a surface M with ξ smoothly distributed throughout M (except one patch of constant $\xi = \xi_{\max}$). In the left diagram, dashed lines represent line contours of constant $\xi = \xi_1, \xi_2$, which have an associated finite value of $\Psi(\xi)$ (since their Hausdorff dimension equals the dimension of the measure used to obtain $\Psi(\xi)$, $\alpha(M^{=}_{\xi_{1,2}}) = 1$), whereas the central patch (filled with oblique lines pattern) of constant $\xi = \xi_{\max}$ has an associated value $\mathcal{H}^1(M^{=}_{\xi_{\max}}) \to \infty$, since its Hausdorff dimension is $\alpha(M^{=}_{\xi_{\max}}) = 2$

will depend on the function $f(\xi)$, that represents how 'distant' two different sets (curves, in general), $M^{=}_{\xi}$ and $M^{=}_{\xi+\mathrm{d}\xi}$, are. That distance, for each point $P \in M^{=}_{\xi}$, is measured along the coordinate n of the tangent plane at P normal to the arc length s of $M^{=}_{\xi}$, and then averaged over the whole set, thus resulting in a function of ξ only. Large values of $f(\xi)$ indicate that the property ξ varies slowly along n in average, whereas small values of $f(\xi)$ correspond to a rapid averaged variation of ξ with n.

The resulting $\tilde{\Psi}(\xi)$ (and, alternatively, $\mathcal{P}(\xi)$) of M can be regarded as a non-local characterization of the distribution of ξ throughout M.

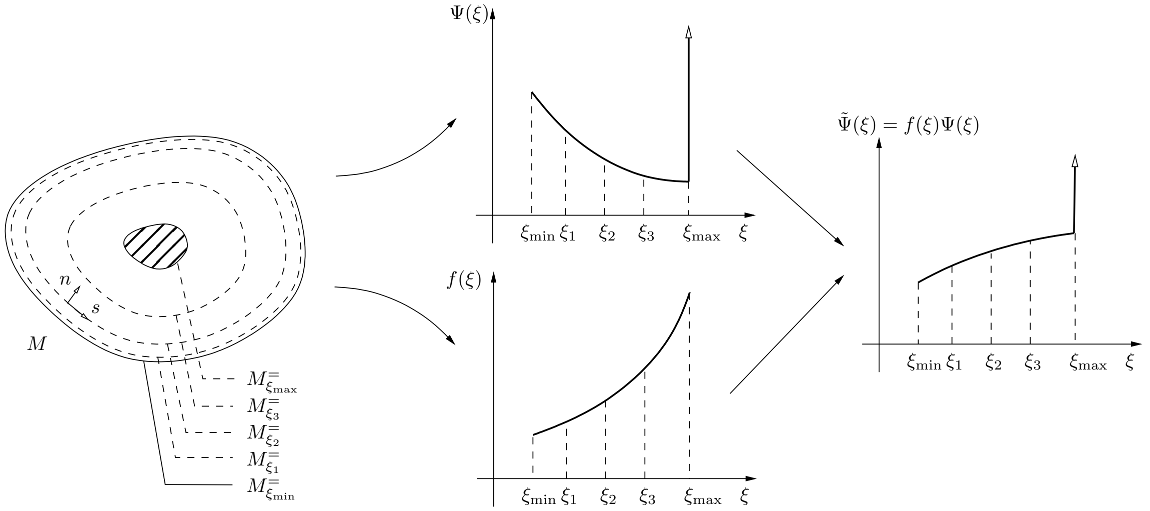

Figure D.3: $\tilde{\Psi}(\xi)$ (right) for a surface M with ξ smoothly distributed throughout M (except one patch of constant $\xi = \xi_{\max}$). In the left diagram, dashed lines represent line contours of constant $\xi = \xi_{\min}, \xi_1, \xi_2, \xi_3, \xi_{\max}$, such that they differ in a constant $\Delta\xi$. The central part of the figure represents $\Psi(\xi)$ (top) and $f(\xi)$ (bottom). $f(\xi)$ increases with ξ since the averaged distance between contours (normal to them along n) increases with ξ. On the right, the multiplication of both gives $\tilde{\Psi}(\xi)$

D.1 Conditions for existence of an explicit analytical solution

Consider an explicit parametrization of the surface M in terms of two parameters (u, v), and also an explicit parametrization of the local property ξ on M in terms of the same two parameters:

$$M : (u, v) \in (I_u, I_v) \mapsto \mathbb{R}^n, \tag{D.20}$$

$$\xi : (u, v) \in (I_u, I_v) \mapsto \mathbb{R}. \tag{D.21}$$

By choosing μ_2 to be the two-dimensional Hausdorff measure, the integrals in the function $\delta\eta$ defined above can be expressed in terms of the parametrization as:

$$\delta\eta(\xi, \mathrm{d}\xi) = \int_M \left(\mathbf{1}_{\left[M^{\leq}_{\xi+\mathrm{d}\xi}\right]} - \mathbf{1}_{\left[M^{\leq}_{\xi}\right]}\right)\mathrm{d}\mu_2 = \int_{(I_u, I_v)} \left(\mathbf{1}_{\left[M^{\leq}_{\xi+\mathrm{d}\xi}\right]} - \mathbf{1}_{\left[M^{\leq}_{\xi}\right]}\right)\theta(u, v)\,\mathrm{d}u\,\mathrm{d}v, \tag{D.22}$$

where $\mathrm{d}\mu_2 = \theta(u, v)\,\mathrm{d}u\,\mathrm{d}v$, and $\theta(u, v)$ depends on the parametrization of the surface.

Under the following constraints imposed on the parametrization, the function $\tilde{\Psi}(\xi) = \lim_{\mathrm{d}\xi \to 0} (\delta\eta(\xi, \mathrm{d}\xi)/\mathrm{d}\xi)$ can be obtained explicitly in terms of the parameters (u, v), providing interesting analytical solutions of $\tilde{\Psi}(\xi)$ for certain surfaces:

1. If the functions $\xi(u,v)$ and $\theta(u,v)$ are both independent of one (the same one) of the two parameters (u,v) (the parameter v has been chosen for that purpose in this development without loss of generality), $\xi(u,v) \equiv \xi(u)$, $\theta(u,v) \equiv \theta(u)$, then $\tilde{\Psi}(\xi)$ can be written as:

$$\begin{aligned} \tilde{\Psi}(\xi(u)) &= \lim_{\mathrm{d}\xi \to 0} \left(\int_{I_v} \mathrm{d}v \right) \left(\int_{I_u} \left(\mathbf{1}_{[M^{\leq}_{\xi+\mathrm{d}\xi}]} - \mathbf{1}_{[M^{\leq}_{\xi}]} \right) \theta(u)\,\mathrm{d}u \right) && \text{(D.23)} \\ &= \Delta v\, \frac{\theta(u)\,\mathrm{d}u}{\mathrm{d}\xi} = \Delta v\, \frac{\theta(u)\,\mathrm{d}u}{\dfrac{\mathrm{d}\xi}{\mathrm{d}u}\,\mathrm{d}u} = \Delta v \frac{\theta(u)}{\dfrac{\mathrm{d}\xi}{\mathrm{d}u}(u)}, && \text{(D.24)} \end{aligned}$$

where $\Delta v = \int_{I_v} \mathrm{d}v$ is a constant.

2. If the map $\xi(u) : u \in I_u \subset \mathbb{R} \mapsto \xi \in I_\xi \subset \mathbb{R}$ is invertible (i.e., bijective[7]), there exists the inverse map $u = u(\xi) : \xi \in I_\xi \subset \mathbb{R} \mapsto u \in I_u \subset \mathbb{R}$ and equation D.24 can be finally written as an explicit analytical result:

$$\tilde{\Psi}(\xi) = \Delta v \frac{\theta(u(\xi))}{\dfrac{\mathrm{d}\xi}{\mathrm{d}u}(u(\xi))}. \tag{D.25}$$

Note that the invertibility condition on the map $\xi(u)$ implies[8] that the first derivative $\mathrm{d}\xi/\mathrm{d}u$ exists and is non-zero $\forall u \in I_u$. Therefore, the function $\tilde{\Psi}(\xi)$ (that has $\mathrm{d}\xi/\mathrm{d}u$ in the denominator) is defined $\forall \xi \in I_\xi$.

This invertibility condition can be relaxed still obtaining explicit analytical solution in those cases (see Figure D.4) in which there exists a countable number of **l**ocal **e**xtrema, $S_{le} = \{u^*_{le,p} \in I_u; p = 1, \ldots, N_{le}\}$, and a countable number of (surjective) subintervals, $S^{\mathrm{surj}}_I = \{I^{\mathrm{surj}}_{u,q} = [u^*_{\min,q}, u^*_{\max,q}] \subset I_u; q = 1, \ldots, N^{\mathrm{surj}}_I\}$ (the associated set of **e**xtreme **p**oints of those subintervals is called $S_{ep} = \{(u^*_{\min,q}, u^*_{\max,q}); q = 1, \ldots, N_{le}\}$), where the first derivative $\mathrm{d}\xi/\mathrm{d}u$ is null (i.e., $\mathrm{d}\xi/\mathrm{d}u\ |_{u^*} = 0,\ u^* \in S_{le} \cup S^{\mathrm{surj}}_I$). Define the set of points $S_P = \{u^*_j \in \{S_{le} \cup S_{ep} \cup \{u_{\min}, u_{\max}\}\}; j = 1, \ldots, (N_{le} + 2N^{\mathrm{surj}}_I)\}$ ordered such that $u^*_j < u^*_{j+1}$. Define also

---

[7] A map $f : a \in A \leftrightarrow b \in B$ is *bijective* $(\forall a \in A\ \exists!\ b \in B \mid b = f(a))$ if it is *injective* $(\forall a \in A\ \exists\ b \in B \mid b = f(a))$ and *surjective* $(\forall b \in B\ \exists\ a \in A \mid b = f(a))$.

[8] The *inverse function theorem* states that a continuous function $f : x \in I_x \subset \mathbb{R} \mapsto y \in I_y\mathbb{R}$ is (locally) invertible (at $x' \in I_x$) if its first derivative is non-null, $df/dx \neq 0$ (at x'), that is, if f is strictly monotonic (at x'). f is invertible in I_x if it is locally invertible $\forall x \in I_x$

the set of (bijective) subintervals $S_I^{\text{bij}} = \{I_{u,r}^{\text{bij}} =]u_j^*, u_k^*[, r = 1, \ldots, N_I^{\text{bij}}; u_j^*, u_k^* \in S_P\}$ where there exists an invertible map $u_{I_{u,r}^{\text{bij}}}(\xi)$ (inverse function of $\xi(u)$ in the interval I_u^j). Note that $S_I^{\text{surj}} \cap S_I^{\text{bij}} = \emptyset$, and $I_u = S_I^{\text{bij}} \cup S_I^{\text{surj}} \cup S_{le}$. In that case, and assuming that there exists an explicit analytical expression for $u_j^* \in S_P$ in terms of ξ (which depends on the solvability of the equation $(\mathrm{d}\xi/\mathrm{d}u)(u) = 0$), then the function $\tilde{\Psi}(\xi)$ can still be explicitly obtained by the following analytical expression:

$$\tilde{\Psi}(\xi) = \Delta v \left[\sum_{r=1}^{N_I^{\text{bij}}} \frac{\theta(u_{I_{u,r}^{\text{bij}}}(\xi))}{\frac{\mathrm{d}\xi}{\mathrm{d}u}(u_{I_{u,r}^{\text{bij}}}(\xi))} + \sum_{p=1}^{N_{le}} \theta(u_{le,p}^*)\, \delta_0(\xi(u_{le,p}^*)) + \sum_{q=1}^{N_I^{\text{surj}}} \left(\int_{I_{u,q}^{\text{surj}}} \theta(u) \mathrm{d}u \right) \delta_1(\xi(I_{u,q}^{\text{surj}})) \right], \tag{D.26}$$

where the generalized functions $\delta_0(\xi)$ and $\delta_1(\xi)$ are zero everywhere except at ξ, where their value is an infinite with null and unitary total integral, respectively. The subintervals $I_{u,q}^* \in S_I^{\text{surj}}$ correspond to patches of the surface with constant ξ, which have a Hausdorff dimension of two, and therefore their one-dimensional measure is an integrable infinite such that, when integrated, it results the area of the patch (i.e., $\left(\int_{I_{u,q}^*} \theta(u)\, du\right) \Delta v = \int_{I_v} \int_{I_{u,q}^*} \theta(u)\, \mathrm{d}u\, \mathrm{d}v$).

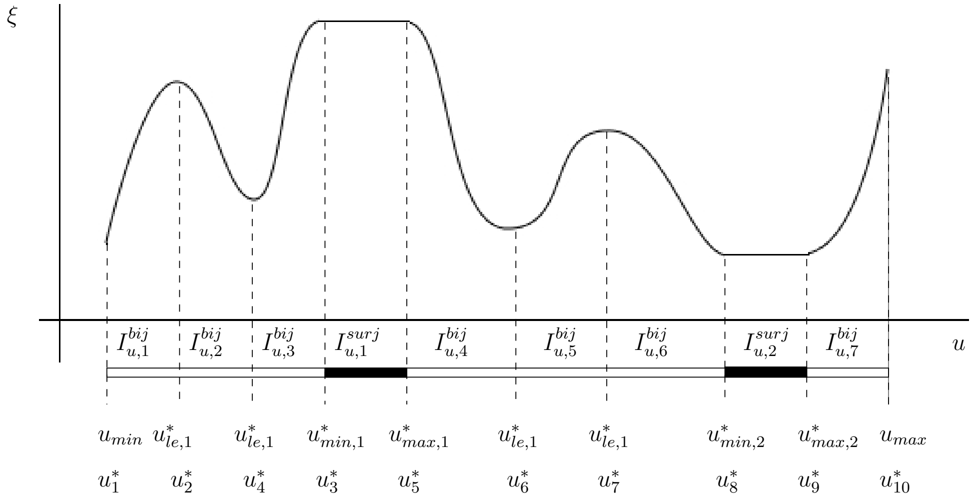

Figure D.4: Example of non-invertible $\xi(u)$ map

Common cases of existence of explicit analytical solution (complying with these two sufficient conditions) arise for cylindrical surfaces and surfaces of revolution, such that the property ξ preserves the cylindrical nature (being independent of the variable along the cylindrical axis) or the axisymmetric character of the surface (being independent of the azimuthal coordinate), and the invertibility of the function relating ξ and the other variable of the parametrization is guaranteed, either globally or along subintervals.

D.2 Extension to multiple dimensions

A parallel development can be followed to define multi-variable density functions on a manifold M. For two local properties, ξ and ζ, we define $M^{=}_{\xi,\zeta}$ as the set of points P of M where $(\xi_M,\zeta_M)(P) = (\xi,\zeta)$, and $M^{=}_{\xi+\mathrm{d}\xi,\zeta+\mathrm{d}\zeta}$ as:

$$\begin{aligned} M^{=}_{\xi+\mathrm{d}\xi,\zeta+\mathrm{d}\zeta} &= \left(M^{\leq}_{\xi+\mathrm{d}\xi} \cap M^{\leq}_{\xi}\right) \cap \left(M^{\leq}_{\zeta+\mathrm{d}\zeta} \cap M^{\leq}_{\zeta}\right) = && \text{(D.27)} \\ &= \{P \in M \mid \xi \leq \xi_M(P) < \xi + \mathrm{d}\xi, \zeta \leq \zeta_M(P) < \zeta + \mathrm{d}\zeta\}\,. && \text{(D.28)} \end{aligned}$$

$\Psi(\xi,\zeta)$ is now defined as $\Psi(\xi,\zeta) \equiv \mu_1\left(M^{=}_{\xi,\zeta}\right)$. Instead of $\mathrm{d}\eta(\xi)/\mathrm{d}\xi$, we have Jacobian determinant

$$J(\xi,\zeta) \equiv \left|\frac{\partial(\eta_\xi,\eta_\zeta)}{\partial(\xi,\zeta)}\right| = f(\xi,\zeta)\,\Psi(\xi,\zeta), \tag{D.29}$$

with $f(\xi,\zeta) \equiv \int_M \left[\delta_{\left[M^{=}_{\xi,\zeta}\right]} d(\mu_2/\mu_1)\right] \mathrm{d}\mu_1 \bigg/ \int_{M^{=}_{\xi,\zeta}} \mathrm{d}\mu_1$. Then

$$\mu_1(M^{=}_{\xi,\zeta}) = \lim_{\mathrm{d}\xi,\mathrm{d}\zeta\to 0} \frac{\mu_2(M_{\xi,\zeta;\mathrm{d}\xi,\mathrm{d}\zeta})}{f(\xi,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta}. \tag{D.30}$$

We also define $\Delta\alpha \equiv \alpha(M_{\xi,\mathrm{d}\xi;\zeta,\mathrm{d}\zeta}) - \alpha(M_{\xi,\zeta})$ and $\tilde{\Psi}(\xi,\zeta) \equiv f(\xi,\zeta)\,\Psi(\xi,\zeta)$. Therefore:

$$\begin{aligned} \int_{\xi_{\min}}^{\xi_{\max}} \int_{\zeta_{\min}}^{\zeta_{\max}} \tilde{\Psi}(\xi,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta &= \int_{\xi_{\min}}^{\xi_{\max}} \int_{\zeta_{\min}}^{\zeta_{\max}} f(\xi,\zeta)\,\Psi(\xi,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta = && \text{(D.31)} \\ &= \int_{\xi_{\min}}^{\xi_{\max}} \int_{\zeta_{\min}}^{\zeta_{\max}} J(\xi,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta = \mu_2(M) = \mathcal{H}^{\alpha+\Delta\alpha}(M), && \text{(D.32)} \end{aligned}$$

and the corresponding joint probability density function can be obtained by normalization as:

$$\mathcal{P}(\xi,\zeta) \equiv \frac{\tilde{\Psi}(\xi,\zeta)}{\mathcal{H}^{\alpha+\Delta\alpha}(M)}, \qquad \text{with} \qquad \int_{\xi_{\min}}^{\xi_{\max}} \int_{\zeta_{\min}}^{\zeta_{\max}} \mathcal{P}(\xi,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta = 1. \tag{D.33}$$

The one-dimensional (probability) density functions in terms of each variable can be directly obtained from the multi-dimensional one by integration with respect to the rest of variables. They are named *marginal* (probability) density functions. For the two-dimensional case, the corresponding marginal probability density functions are $\mathcal{P}_{\xi}(\xi) = \displaystyle\int_{\zeta_{\min}}^{\zeta_{\max}} \mathcal{P}(\xi,\zeta)\,\mathrm{d}\zeta$ and $\mathcal{P}_{\zeta}(\zeta) = \displaystyle\int_{\xi_{\min}}^{\xi_{\max}} \mathcal{P}(\xi,\zeta)\,\mathrm{d}\xi$.

An example of application is the use of the joint and/or marginal probability density functions of two differential-geometry properties of a surface M, such as the principal curvatures (κ_1, κ_2) or the shape index and curvedness (Υ, Λ), in terms of area-coverage on M, to provide a non-local geometrical characterization of such surface M.