Appendix C

Differential geometry background

Let M be a *regular surface*[1] parametrized by:

$$\boldsymbol{x}(\boldsymbol{u}) : \{x(u,v),\, y(u,v),\, z(u,v)\}. \tag{C.1}$$

Its *tangent plane* at any point P, $T_P M$, is defined by the tangent vectors $\{\boldsymbol{x}_u,\, \boldsymbol{x}_v\}$ contained on it, or, alternatively, by the normal unit vector $\mathbf{N}$ orthogonal to it:

$$\boldsymbol{x}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right), \qquad \boldsymbol{x}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right), \tag{C.2}$$

$$\boldsymbol{N} = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{||\mathbf{x}_u \wedge \mathbf{x}_v||}. \tag{C.3}$$

The *first fundamental form* of M at a point P is the inner product restricted to tangent vectors:

$$I(\boldsymbol{m}_P, \boldsymbol{n}_P) = \boldsymbol{m}_P \cdot \boldsymbol{n}_P, \tag{C.4}$$

where $\boldsymbol{m}_P$, $\boldsymbol{n}_P \in T_P M$ (tangent plane of M at P). The first fundamental form is independent of the surface representation, and therefore invariant under parameter transformations. It satisfies:

$$I(a\,\boldsymbol{x}_u + b\,\boldsymbol{x}_v, a\,\boldsymbol{x}_u + b\,\boldsymbol{x}_v) = E\,a^2 + 2\,F\,a\,b + G\,b^2, \tag{C.5}$$

[1] $M \subset \mathbb{R}^n$ is a *regular surface* if for each point $P \in M$ there exists a neighborhood of P, $V \in \mathbb{R}^n$, and a map $x : U \to M$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap M$ such that: (i) x is differentiable; (ii) $x : U \to V \cap M$ is a homeomorphism; (iii) each map $x : U \to M$ is a regular patch, that is, its Jacobian has rank 2 for all $(u,v) \in U$.

where E, F, G are the first fundamental coefficients:

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u = ||\boldsymbol{x}_u||^2, \tag{C.6}$$

$$F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v, \tag{C.7}$$

$$G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v = ||\boldsymbol{x}_v||^2. \tag{C.8}$$

These coefficients are not invariant under parameter transformations. Since $||\boldsymbol{x}_v \cdot \boldsymbol{x}_v|| < ||\boldsymbol{x}_u||||\boldsymbol{x}_v||$ (recall $\boldsymbol{x}_u \not\parallel \boldsymbol{x}_v$ in a regular surface for the tangent plane to be defined):

$$E\,G - F^2 = ||\boldsymbol{x}_u||^2||\boldsymbol{x}_v||^2 - (\boldsymbol{x}_v \cdot \boldsymbol{x}_v)^2 > 0, \tag{C.9}$$

and therefore, the first fundamental form is a positive definite quadratic form on the tangent plane of M at P ($T_P M$):

$$I(\boldsymbol{m},\boldsymbol{n}) = \begin{pmatrix} m_u & m_v \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} n_u \\ n_v \end{pmatrix}, \qquad \left\{ \begin{array}{c} m = m_u\,\boldsymbol{x}_u + m_v\,\boldsymbol{x}_v \\ n = n_u\,\boldsymbol{x}_u + n_v\,\boldsymbol{x}_v \end{array} \right\} \in T_P. \tag{C.10}$$

A property of the surface M that depends only on the first fundamental form of M is called an *intrinsic property* [2].

The *arc length*, $s(t)$, of a curve C on M (given by its parametrization $\vec{\alpha}(t) = (u(t), v(t)) = u(t)\,\boldsymbol{x}_u + v(t)\,\boldsymbol{x}_v$, $t \in [a,b]$), is:

$$s(t) = \int_a^t ||\vec{\alpha}'(r)||\,\mathrm{d}r, \qquad \vec{\alpha}'(r) \equiv \frac{\mathrm{d}\vec{\alpha}(r)}{\mathrm{d}r}. \tag{C.11}$$

[2] An *intrinsic property* of a surface is independent of the space in which the surface may be considered. Thus, a hypothetical "inhabitant" of the surface can measure it without knowing anything about the space in which the surface is embedded. On the other hand an *extrinsic property* of a surface depends on its embedding space, and therefore cannot be measured by "inhabitants" of the surface. In a more formal definition a property is called *intrinsic* if it is preserved by *local isometries*, and *extrinsic* otherwise. An *isometry* (or *congruence transformation*) is a bijective distance preserving map between two metric spaces.

Thus:

$$\left(\frac{\mathrm{d}s(t)}{\mathrm{d}t}\right)^2 = ||\vec{\alpha}'(t)||^2 = \vec{\alpha}'(t)\cdot\vec{\alpha}'(t)$$
$$= [u'(t)\,\boldsymbol{x}_u + v'(t)\,\boldsymbol{x}_v]\cdot[u'(t)\,\boldsymbol{x}_u + v'(t)\,\boldsymbol{x}_v]$$
$$= E\,u'(t)^2 + 2\,F\,u'(t)\,v'(t) + G\,v'(t)^2$$

and:

$$(\mathrm{d}s)^2 = E\,(\mathrm{d}u)^2 + 2\,F\,\mathrm{d}u\,\mathrm{d}v + G\,(\mathrm{d}v)^2 = I(\vec{\alpha}',\vec{\alpha}'). \tag{C.12}$$

Therefore, the arc length is an intrinsic property, since it depends only on the first fundamental form. The *area element*, $\mathrm{d}A$, of M at a point $P \in M$ is defined, in terms of its parametrization, as:

$$\mathrm{d}A = \sqrt{E\,G - F^2}\,\mathrm{d}u \wedge \mathrm{d}v, \tag{C.13}$$

where $\mathrm{d}u \wedge \mathrm{d}v$ is the wedge product.

The *second fundamental form* of a (three-dimensional regular) surface M at a point P is the symmetric bilinear[3] form on the tangent plane at P ($T_P M$) given by:

$$II(\boldsymbol{m}_P, \boldsymbol{n}_P) = S(\boldsymbol{m}_P)\cdot\boldsymbol{n}_P = \boldsymbol{m}_P\cdot S(\boldsymbol{n}_P), \tag{C.14}$$

where S is the *shape operator* (or *second fundamental tensor* or *Weingarten map*), which is defined, when operating on a vector $\boldsymbol{m}$, as the negative covariant derivative (along the direction of $\boldsymbol{m}$), D_m, of the unit normal vector field $\boldsymbol{N}$ of the surface M:

$$S(\boldsymbol{m}) = -D_{\boldsymbol{m}}\boldsymbol{N}. \tag{C.15}$$

[3]Equivalently quadratic in this context

The second fundamental form satisfies:

$$II(a\,\boldsymbol{x}_u + b\,\boldsymbol{x}_v, a\,\boldsymbol{x}_u + b\,\boldsymbol{x}_v) = e\,a^2 + 2\,f\,a\,b + g\,b^2, \tag{C.16}$$

where e, f, g are the second fundamental coefficients:

$$e = -\boldsymbol{N}_u \cdot \boldsymbol{x}_u = \boldsymbol{N} \cdot \boldsymbol{x}_{uu} = \frac{\det(\boldsymbol{x}_{uu}, \boldsymbol{x}_u, \boldsymbol{x}_v)}{\sqrt{E\,G - F^2}}, \tag{C.17}$$

$$f = -\boldsymbol{N}_v \cdot \boldsymbol{x}_u = \boldsymbol{N} \cdot \boldsymbol{x}_{uv} = \frac{\det(\boldsymbol{x}_{uv}, \boldsymbol{x}_u, \boldsymbol{x}_v)}{\sqrt{E\,G - F^2}} \tag{C.18}$$

$$= -\boldsymbol{N}_u \cdot \boldsymbol{x}_v = \boldsymbol{N} \cdot \boldsymbol{x}_{vu} = \frac{\det(\boldsymbol{x}_{vu}, \boldsymbol{x}_u, \boldsymbol{x}_v)}{\sqrt{E\,G - F^2}}, \tag{C.19}$$

$$g = -\boldsymbol{N}_v \cdot \boldsymbol{x}_v = \boldsymbol{N} \cdot \boldsymbol{x}_{vv} = \frac{\det(\boldsymbol{x}_{vv}, \boldsymbol{x}_u, \boldsymbol{x}_v)}{\sqrt{E\,G - F^2}}, \tag{C.20}$$

being:

$$\boldsymbol{x}_{\alpha\beta} \equiv \left(\frac{\partial^2 x}{\partial\alpha\partial\beta}, \frac{\partial^2 y}{\partial\alpha\partial\beta}, \frac{\partial^2 z}{\partial\alpha\partial\beta} \right). \tag{C.21}$$

The last equality in equations C.17–C.20 for the coefficients comes from rewriting the normal vector as $\boldsymbol{N} = (\boldsymbol{x}_u \wedge \boldsymbol{x}_v)/||\boldsymbol{x}_u \wedge \boldsymbol{x}_v|| = (\boldsymbol{x}_u \wedge \boldsymbol{x}_v)/\sqrt{E\,G - F^2}$ where the result $||\boldsymbol{x}_u \wedge \boldsymbol{x}_v|| = \sqrt{E\,G - F^2}$ is a consequence of the Lagrange identity $((\boldsymbol{a} \cdot \boldsymbol{b})^2 + ||\boldsymbol{a} \wedge \boldsymbol{b}||^2 = ||\boldsymbol{a}||^2||\boldsymbol{b}||^2)$ and recalling that $(\boldsymbol{a} \wedge \boldsymbol{b}) \cdot \boldsymbol{c} = \det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$. Unlike the first fundamental form, the second fundamental form is not necessarily positive or definite.

The *normal curvature* of a regular surface M in the direction of a unit tangent vector $\boldsymbol{t}_P$ at a point $P \in M$ is formally defined as:

$$\kappa_N(\boldsymbol{t}_P) = S(\boldsymbol{t}_P) \cdot \boldsymbol{t}_P, \tag{C.22}$$

S being the shape operator. From the previous definition of the second fundamental form, we can express the normal curvature κ_N as:

$$\kappa_N(\boldsymbol{t}_P) = II(\boldsymbol{t}_P, \boldsymbol{t}_P), \tag{C.23}$$

and, for a generic non-unitary tangent vector, $\boldsymbol{t}_P^{'}$:

$$\kappa_N(\boldsymbol{t}_P^{'}) = \frac{S(\boldsymbol{t}_P^{'}) \cdot \boldsymbol{t}_P^{'}}{\boldsymbol{t}_P^{'} \cdot \boldsymbol{t}_P^{'}} = \frac{II(\boldsymbol{t}_P^{'}, \boldsymbol{t}_P^{'})}{I(\boldsymbol{t}_P^{'}, \boldsymbol{t}_P^{'})}. \tag{C.24}$$

The maximum (κ_1) and minimum (κ_2) values of the normal curvature at a point $P \in M$ are called *principal curvatures*. The directions defined by the tangent vectors associated with those principal curvatures are called *principal directions* and are orthogonal. Formally, the principal curvatures at a point P are defined as the eigenvalues (κ_1, κ_2) of the shape operator $S(P)$, and the principal directions correspond to the associated (orthogonal) eigenvectors ($\boldsymbol{e}_1$,$\boldsymbol{e}_2$). The normal curvature at P along $\boldsymbol{t}_P$ is then given by *Euler's formula*:

$$\kappa_N = \kappa_1 \cos^2(\theta) + \kappa_2 \sin^2(\theta), \tag{C.25}$$

where θ is the angle between $\boldsymbol{e}_1$ and $\boldsymbol{t}_P$. The *Gaussian curvature* of a regular surface M at a point $P \in M$ is formally defined as the determinant of the shape operator S at that point:

$$K(P) = \det(S(P)). \tag{C.26}$$

Gauss' *Theorema Egregium* proves that the Gaussian curvature of a regular surface M is invariant under local isometry. In other words, it is an intrinsic property of the surface, and therefore it only depends on its first fundamental form (at every point $P \in M$). This is a remarkable result since the formal definition of the Gaussian curvature involves the second fundamental form directly (and, therefore, the embedding of the surface M). In terms of the first fundamental form only, the Gaussian curvature is written as:

$$K = \frac{1}{\sqrt{E\,G - F^2}} \left[\frac{\partial}{\partial v} \left(\frac{\sqrt{E\,G - F^2}}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial u} \left(\frac{\sqrt{E\,G - F^2}}{E} \Gamma_{12}^2 \right) \right], \tag{C.27}$$

where Γ_{ij}^k are the Christoffel symbols of the second kind, which can be expressed in terms of the first fundamental coefficients as:

$$\Gamma_{11}^2 = \frac{2\,E\,F_u - E\,E_v - F\,E_u}{2\,(E\,G - F^2)}, \qquad \Gamma_{12}^2 = \frac{E\,G_u - F\,E_v}{2\,(E\,G - F^2)}. \tag{C.28}$$

Gaussian curvature can also be expressed in terms of the first and second fundamental coefficients in a more compact way:

$$K = \frac{e\,g - f^2}{E\,G - F^2}. \tag{C.29}$$

In terms of the principal curvatures, the Gaussian curvature is expressed as:

$$K = \kappa_1 \kappa_2. \tag{C.30}$$

Points with positive/negative Gaussian curvature are called *elliptic/hyperbolic*. A point is *parabolic* if the Gaussian curvature is zero but not the shape operator. At *planar* points both the Gaussian curvature and the shape operator are zero. In a *synclastic/anticlastic* surface, all its points are elliptic/hyperbolic.

The *mean curvature* of a regular surface M at a point $P \in M$ is formally defined as the trace of the shape operator S at that point:

$$H(P) = \mathrm{tr}(S(P)). \tag{C.31}$$

In terms of the first and second fundamental coefficients, the mean curvature is:

$$H = \frac{e\,G - 2\,f\,F + g\,E}{2\,(E\,G - F^2)}. \tag{C.32}$$

Unlike Gaussian curvature, which is intrinsic, the mean curvature is an extrinsic property of the surface, that is, it depends on the embedding [4]. The mean curvature coincides with the mean of the principal curvatures:

$$H = \frac{\kappa_1 + \kappa_2}{2}. \tag{C.33}$$

[4]For instance, a cylinder and a plane are locally isometric but the mean curvature of a plane is zero while that of a cylinder is nonzero. Their Gaussian curvature is zero.

Combining equations C.30 and C.33, the principal curvatures can be obtained from the Gaussian and mean curvatures:

$$\begin{cases} \kappa_1 = H + \sqrt{H^2 - K} \\ \kappa_2 = H - \sqrt{H^2 - K} \end{cases} . \quad \text{(C.34)}$$

Gaussian and mean curvature satisfy:

$$H^2 - K = \left(\frac{\kappa_1 - \kappa_2}{2} \right)^2 \geq 0. \quad \text{(C.35)}$$

Points where $H^2 = K$ (that is, $\kappa_1 = \kappa_2$ and, therefore, the normal curvature is the same in any direction) are called *umbilical points*. A surface is defined as (locally) *minimal* if its mean curvature (locally) vanishes ($H = 0$, that is, $\kappa_1 = -\kappa_2$).

The *shape index*, Υ, and *curvedness*, Λ, of a regular surface M at a point P are defined (Koenderink & van Doorn, 1992) by:

$$\Upsilon = -\frac{2}{\pi} \arctan\left(\frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} \right), \quad \text{(C.36)}$$

$$\Lambda = \sqrt{\frac{\kappa_1^2 + \kappa_2^2}{2}}. \quad \text{(C.37)}$$

In terms of the Gaussian and mean curvatures, the shape index and curvedness can be expressed as:

$$\Upsilon = -\frac{2}{\pi} \arctan(\frac{H}{\sqrt{H^2 - K}}), \quad \text{(C.38)}$$

$$\Lambda = \sqrt{2H^2 - K}. \quad \text{(C.39)}$$

Shape index is dimensionless, while curvedness has the dimensions of a reciprocal length. The planar patch, for which $\kappa_1 = \kappa_2 = 0$, has null curvedness and an indeterminate shape index. All other regular patches of a regular surface M map on the domain $(\Upsilon, \Lambda) \in [-1, +1] \times \mathbb{R}^+$. $\{\rho, \phi\} \equiv \{\sqrt{2}\Lambda, -\pi\Upsilon/2\}$ are polar coordinates in the semi-plane[5] of axes $\{\kappa_1 - \kappa_2\}^+$ and $\{\kappa_1 + \kappa_2\}$ (see Figure C.1). Some other properties of the shape index are summarized below (extracted

[5]Note that, by definition, $\kappa_1 - \kappa_2 > 0$.

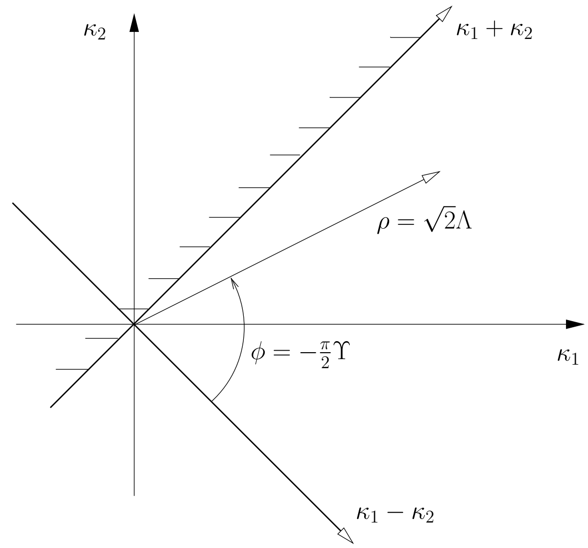

Figure C.1: Transformation from (κ_1, κ_2) to (Υ, Λ)

from Koenderink & van Doorn (1992)) (see Figure C.2 for a graphical explanation):

- Points where $|\Upsilon| = 1$ are umbilical points and represent locally spherical shapes ("cup" ($\Upsilon = -1$) or "cap" ($\Upsilon = +1$))[6]. Points where $0.5 < |\Upsilon| < 1$ are elliptic points and represent locally ellipsoidal shapes, tending toward the spherical shape when $|\Upsilon| \to 1$ and towards the cylindrical shape when $|\Upsilon| \to 0.5$. Points where $|\Upsilon| = 0.5$ are parabolic points and represent cylindrical shapes ("rut" ($\Upsilon = -0.5$), and "ridge" ($\Upsilon = +0.5$)). Points where $0 < |\Upsilon| < 0.5$ are hyperbolic points and represent locally hyperbolic shapes, tending toward the cylindrical shapes when $|\Upsilon| \to 0.5$ and towards the symmetrical saddle when $|\Upsilon| \to 0$.

- The range $\Upsilon \in (-1, -0.5)$ represents the concavities (concave "ruts" or "trough" shapes). The range $\Upsilon \in (-0.5, +0.5)$ represents the saddle-like shapes ("saddle-ruts" ($\Upsilon \in (-0.5, 0)$) and "saddle-ridges" ($\Upsilon \in (0, +0.5)$)), with the symmetrical saddle at $\Upsilon = 0$. The range $\Upsilon \in (+0.5, +1)$ represents the convexities (convex "ridges", or "dome-shapes").

- Generically, umbilicals ($|\Upsilon| = 1$) occur only at isolated points on the surface. Parabolic points ($|\Upsilon| = 0.5$) occur on curves of two distinct types ($\Upsilon = \pm 0.5$), which are smooth, closed

[6] The following convention has been chosen: a regular surface M is locally concave/convex at a given point $P \in M$ if the point P is a local minimum/maximum in the reference system with vertical axis pointing towards the outward normal at P.

loops on closed regular surfaces, and such that never intersect (although they can be nested or juxtaposed). Symmetrical saddles ($\Upsilon = 0$) also occur on curves. Ellipsoid patches with different sign(Υ) ("domes" and "troughs") are never adjacent, being necessarily separated by saddle-like patches.

- Two shapes with opposite shape indices represent complementary pairs (matching each other as "mold" and "stamp", when appropriately scaled).

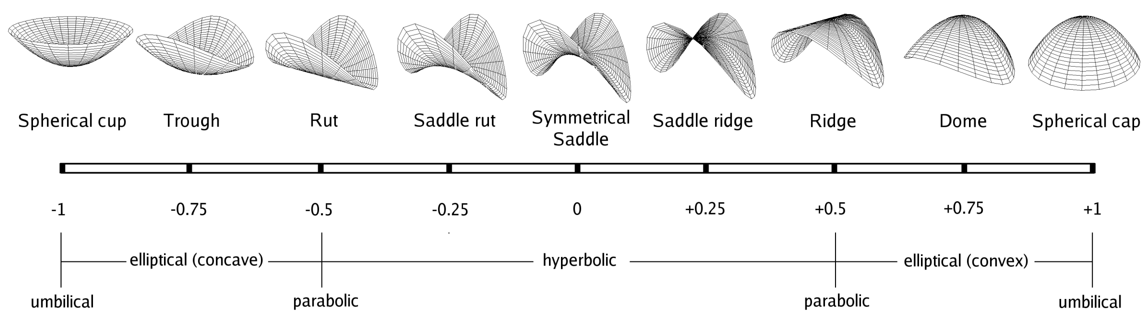

Figure C.2: Range of shape index (Υ), with its most representative associated local shapes (figure based on Koenderink & van Doorn (1992))