

Appendix A

Governing equations for the generation of strain, vorticity, dissipation, and enstrophy

A.1 Generation of strain and dissipation

From the Navier-Stokes equations for incompressible flow ($\nabla \cdot \mathbf{u} = 0$):

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k}, \quad (\text{A.1})$$

decompose the velocity gradient tensor $\partial u_i / \partial x_j$ in its symmetric (strain-rate tensor, S_{ij}) and anti-symmetric (rotation-rate tensor, Ω_{ij}) parts:

$$\frac{\partial u_i}{\partial x_j} = S_{ij} + \Omega_{ij}, \quad S_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad \Omega_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \quad (\text{A.2})$$

Take $[\partial(A.1)_i / \partial x_j + \partial(A.1)_j / \partial x_i] / 2$ and use

$$\frac{1}{2} \left(\frac{\partial u_k}{\partial x_j} \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_j}{\partial x_k} \right) = S_{ik} S_{kj} + \Omega_{ik} \Omega_{kj} \quad (\text{A.3})$$

to obtain:

$$\frac{DS_{ij}}{Dt} = -S_{ik} S_{kj} - \Omega_{ik} \Omega_{kj} - \rho^{-1} \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \frac{\partial^2 S_{ij}}{\partial x_k \partial x_k}, \quad (\text{A.4})$$

where D/Dt denotes the substantial derivative operator ($D/Dt = \partial/\partial t + u_k \partial/\partial x_k$).

Multiply equation A.4 by S_{ji} and use its symmetry to obtain:

$$\frac{D}{Dt} \left(\frac{1}{2} S_{ij} S_{ij} \right) = -S_{ik} S_{kj} S_{ji} - \Omega_{ik} \Omega_{kj} S_{ji} - \rho^{-1} S_{ji} \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu S_{ji} \frac{\partial^2 S_{ij}}{\partial x_k \partial x_k}. \quad (\text{A.5})$$

To find the equivalent relations in terms of the vorticity, $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}$, instead of the rotation-rate tensor, $\boldsymbol{\Omega}$, use $\Omega_{ij} = -\epsilon_{ijk} \omega_k / 2$, where ϵ_{ijk} is the Levi-Civita symbol and the contraction epsilon identity, $\epsilon_{lik} \epsilon_{jmk} = \delta_{lj} \delta_{im} - \delta_{lm} \delta_{ij}$, to express

$$\Omega_{ik} \Omega_{kj} = (\omega_i \omega_j - \delta_{ij} \omega_m \omega_m) / 4. \quad (\text{A.6})$$

Thus, equations A.4 and A.7 can be rewritten as:

$$\frac{DS_{ij}}{Dt} = -S_{ik} S_{kj} - \frac{1}{4} (\omega_i \omega_j - \delta_{ij} \omega_k \omega_k) - \rho^{-1} \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \frac{\partial S_{ij}}{\partial x_k \partial x_k}, \quad (\text{A.7})$$

$$\frac{D}{Dt} \left(\frac{1}{2} S_{ij} S_{ij} \right) = -S_{ik} S_{kj} S_{ji} - \frac{1}{4} \omega_i S_{ij} \omega_j - \rho^{-1} S_{ji} \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu S_{ji} \frac{\partial S_{ij}}{\partial x_k \partial x_k}, \quad (\text{A.8})$$

where the symmetry of S_{ji} and the incompressibility condition ($\nabla \cdot \mathbf{u} = \text{tr}(\nabla \mathbf{u}) = \text{tr}(\mathbf{S}) = \delta_{ij} S_{ji} = 0$) have been considered.

A.2 Generation of vorticity and enstrophy

Apply the curl operator to equation A.1, obtaining:

$$\frac{\partial(\nabla \times \mathbf{u})}{\partial t} + \nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla \times (\nabla p) + \nu \nabla^2 (\nabla \times \mathbf{u}). \quad (\text{A.9})$$

From tensorial algebra, the following identities hold:

$$\nabla \times (\nabla \phi) \equiv 0, \quad \nabla \cdot (\nabla \times \mathbf{u}) \equiv 0, \quad \nabla u^2/2 \equiv \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \times \nabla \times \mathbf{u}, \quad (\text{A.10})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) \equiv (\mathbf{B} \cdot \nabla + \mathbf{B} \times \nabla \times) \mathbf{A} + (\mathbf{A} \cdot \nabla + \mathbf{A} \times \nabla \times) \mathbf{B}, \quad (\text{A.11})$$

where ϕ is a scalar field, \mathbf{u} is a vector field with modulus $u \equiv \|\mathbf{u}\|$, and \mathbf{A} and \mathbf{B} are second-order tensor fields. Use these tensor identities and the incompressibility condition, $\nabla \cdot \mathbf{u} = 0$ to rewrite equation A.9 as:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}. \quad (\text{A.12})$$

The first term of the right-hand side of equation A.12 is responsible for the vortex-stretching.

Dot product equation A.12 with $\boldsymbol{\omega}$, decompose the velocity gradient tensor as $\nabla \mathbf{u} = \mathbf{S} + \boldsymbol{\Omega}$ in the first term of the right-hand side, and use the antisymmetry of $\boldsymbol{\Omega}$ ¹ to obtain:

$$\frac{\text{D}}{\text{D}t} \left(\frac{1}{2} \boldsymbol{\omega}_i \boldsymbol{\omega}_i \right) = \boldsymbol{\omega}_i S_{ik} \boldsymbol{\omega}_k + \nu \boldsymbol{\omega}_i \frac{\partial^2 \boldsymbol{\omega}_i}{\partial x_k \partial x_k}. \quad (\text{A.13})$$

To express equation A.13 in terms of $\boldsymbol{\Omega}$, particularize

$$\Omega_{ij} \frac{\partial^2 \Omega_{kj}}{\partial x_p \partial x_p} = \frac{1}{4} \epsilon_{ijl} \omega_l \epsilon_{kjm} \frac{\partial^2 \omega_m}{\partial x_p \partial x_p} = \frac{1}{4} \left(\omega_m \frac{\partial^2 \omega_m}{\partial x_p \partial x_p} \delta_{ik} - \omega_k \frac{\partial^2 \omega_i}{\partial x_p \partial x_p} \right) \quad (\text{A.14})$$

for $k = i$, obtaining:

$$\Omega_{ij} \frac{\partial^2 \Omega_{ij}}{\partial x_p \partial x_p} = \frac{1}{2} \omega_i \frac{\partial^2 \omega_i}{\partial x_p \partial x_p}. \quad (\text{A.15})$$

Then, from equation A.6 and the antisymmetry of $\boldsymbol{\Omega}$ it results $\omega_i \omega_i = 2 \Omega_{ij} \Omega_{ij}$, $\omega_i \omega_j = 4 \Omega_{ik} \Omega_{kj} + 2 \delta_{ij} \Omega_{mn} \Omega_{mn}$, which can be substituted, along with the relation A.15, into equation A.13 to obtain:

$$\frac{\text{D}}{\text{D}t} \left(\frac{1}{2} \Omega_{ij} \Omega_{ij} \right) = 2 \Omega_{ik} \Omega_{kj} S_{ji} + \nu \Omega_{ij} \frac{\partial^2 \Omega_{ij}}{\partial x_k \partial x_k}, \quad (\text{A.16})$$

¹If \mathbf{A} is an antisymmetric second-order tensor, then $\mathbf{a} \cdot \mathbf{A} \cdot \mathbf{a} = 0$, for any vector field \mathbf{a} .

where the symmetry of \mathbf{S} and the incompressibility relation expressed in terms of \mathbf{S} , $\delta_{ij}S_{ij} = 0$, have been used to rewrite the first term of the right-hand side.