Appendix A

Governing equations for the generation of strain, vorticity, dissipation, and enstrophy

A.1 Generation of strain and dissipation

From the Navier-Stokes equations for incompressible flow ($\nabla \cdot \mathbf{u} = 0$):

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k},$$

(A.1)

decompose the velocity gradient tensor $\partial u_i/\partial x_j$ in its symmetric (strain-rate tensor, $S_{ij}$) and anti-symmetric (rotation-rate tensor, $\Omega_{ij}$) parts:

$$\frac{\partial u_i}{\partial x_j} = S_{ij} + \Omega_{ij}, \quad S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad \Omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right).$$

(A.2)

Take $[\partial(A.1)_i/\partial x_j + \partial(A.1)_j/\partial x_i]/2$ and use

$$\frac{1}{2} \left( \frac{\partial u_k}{\partial x_j} \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_j}{\partial x_k} \right) = S_{ik} S_{kj} + \Omega_{ik} \Omega_{kj}$$

(A.3)

to obtain:

$$\frac{DS_{ij}}{Dt} = -S_{ik} S_{kj} - \Omega_{ik} \Omega_{kj} - \rho^{-1} \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \frac{\partial^2 S_{ij}}{\partial x_k \partial x_k},$$

(A.4)
where $D/Dt$ denotes the substantial derivative operator ($D/Dt = ∂/∂t + u_k ∂/∂x_k$).

Multiply equation A.4 by $S_{ji}$ and use its symmetry to obtain:

$$\frac{D}{Dt} \left( \frac{1}{2} S_{ij} S_{ij} \right) = -S_{ik} S_{kj} S_{ji} - \Omega_{ik} \Omega_{kj} S_{ji} - \rho^{-1} S_{ji} \frac{∂^2 p}{∂x_i ∂x_j} + ν S_{ji} \frac{∂^2 S_{ij}}{∂x_k ∂x_k}. \quad (A.5)$$

To find the equivalent relations in terms of the vorticity, $ω ≡ \nabla × \mathbf{u}$, instead of the rotation-rate tensor, $Ω$, use $Ω_{ij} = -\epsilon_{ijk} ω_k / 2$, where $\epsilon_{ijk}$ is the Levi-Civita symbol and the contraction epsilon identity, $\epsilon_{lik} \epsilon_{jmk} = δ_{lj} δ_{im} - δ_{lm} δ_{ij}$, to express

$$Ω_{ik} Ω_{kj} = (ω_i ω_j - δ_{ij} ω_m ω_m) / 4. \quad (A.6)$$

Thus, equations A.4 and A.7 can be rewritten as:

$$\frac{DS_{ij}}{Dt} = -S_{ik} S_{kj} - \frac{1}{4} (ω_i ω_j - δ_{ij} ω_k ω_k) - ρ^{-1} \frac{∂^2 p}{∂x_i ∂x_j} + ν \frac{∂S_{ij}}{∂x_k ∂x_k}, \quad (A.7)$$

$$\frac{D}{Dt} \left( \frac{1}{2} S_{ij} S_{ij} \right) = -S_{ik} S_{kj} S_{ji} - \frac{1}{4} ω_i S_{ij} ω_j - ρ^{-1} S_{ji} \frac{∂^2 p}{∂x_i ∂x_j} + ν S_{ji} \frac{∂S_{ij}}{∂x_k ∂x_k}, \quad (A.8)$$

where the symmetry of $S_{ji}$ and the incompressibility condition ($\nabla · \mathbf{u} = \text{tr}(\nabla \mathbf{u}) = \text{tr}(\mathbf{S}) = δ_{ij} S_{ji} = 0$) have been considered.

### A.2 Generation of vorticity and enstrophy

Apply the curl operator to equation A.1, obtaining:

$$\frac{∂(\nabla × \mathbf{u})}{∂t} + \nabla × (\mathbf{u} · \nabla \mathbf{u}) = -\nabla × (\nabla p) + ν \nabla^2 (\nabla × \mathbf{u}). \quad (A.9)$$
From tensorial algebra, the following identities hold:

\[
\nabla \times (\nabla \phi) \equiv 0, \quad \nabla \cdot (\nabla \times \mathbf{u}) \equiv 0, \quad \nabla u^2/2 \equiv \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \times \nabla \times \mathbf{u}, \quad (A.10)
\]

\[
\nabla \times (\mathbf{A} \times \mathbf{B}) \equiv (\mathbf{B} \cdot \nabla + \mathbf{B} \times \nabla \times) \mathbf{A} + (\mathbf{A} \cdot \nabla + \mathbf{A} \times \nabla \times) \mathbf{B}, \quad (A.11)
\]

where \( \phi \) is a scalar field, \( \mathbf{u} \) is a vector field with modulus \( u \equiv \| \mathbf{u} \| \), and \( \mathbf{A} \) and \( \mathbf{B} \) are second-order tensor fields. Use these tensor identities and the incompressibility condition, \( \nabla \cdot \mathbf{u} = 0 \) to rewrite equation A.9 as:

\[
\frac{\partial \mathbf{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{\omega} = \mathbf{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{\omega}. \quad (A.12)
\]

The first term of the right-hand side of equation A.12 is responsible for the vortex-stretching.

Dot product equation A.12 with \( \mathbf{\omega} \), decompose the velocity gradient tensor as \( \nabla \mathbf{u} = \mathbf{S} + \mathbf{\Omega} \) in the first term of the right-hand side, and use the antisymmetry of \( \mathbf{\Omega} \) to obtain:

\[
\frac{D}{Dt} \left( \frac{1}{2} \omega_i \omega_i \right) = \omega_i S_{ik} \omega_k + \nu \omega_i \frac{\partial^2 \omega_i}{\partial x_k \partial x_k}. \quad (A.13)
\]

To express equation A.13 in terms of \( \mathbf{\Omega} \), particularize

\[
\Omega_{ij} \frac{\partial^2 \Omega_{kj}}{\partial x_p \partial x_p} = \frac{1}{4} \epsilon_{ijl} \omega_l \frac{\partial^2 \omega_m}{\partial x_p \partial x_p} = \frac{1}{4} \left( \omega_m \frac{\partial^2 \omega_m}{\partial x_p \partial x_p} \delta_{ik} - \omega_k \frac{\partial^2 \omega_i}{\partial x_p \partial x_p} \right) \quad (A.14)
\]

for \( k = i \), obtaining:

\[
\Omega_{ij} \frac{\partial^2 \Omega_{ij}}{\partial x_p \partial x_p} = \frac{1}{2} \omega_i \frac{\partial^2 \omega_i}{\partial x_p \partial x_p}. \quad (A.15)
\]

Then, from equation A.6 and the antisymmetry of \( \mathbf{\Omega} \) it results \( \omega_i \omega_i = 2 \Omega_{ij} \omega_j \), \( \omega_i \omega_j = 4 \Omega_{ik} \Omega_{kj} + 2 \delta_{ij} \Omega_{mn} \Omega_{mn} \), which can be substituted, along with the relation A.15, into equation A.13 to obtain:

\[
\frac{D}{Dt} \left( \frac{1}{2} \Omega_{ij} \Omega_{ij} \right) = 2 \Omega_{ik} \Omega_{kj} S_{ji} + \nu \Omega_{ij} \frac{\partial^2 \Omega_{ij}}{\partial x_k \partial x_k}. \quad (A.16)
\]

\(^1\)If \( \mathbf{A} \) is an antisymmetric second-order tensor, then \( \mathbf{a} \cdot \mathbf{A} \cdot \mathbf{a} = 0 \), for any vector field \( \mathbf{a} \).
where the symmetry of $S$ and the incompressibility relation expressed in terms of $S$, $\delta_{ij}S_{ij} = 0$, have been used to rewrite the first term of the right-hand side.