SIMPLE GROUPS OF ORDER $2^a q^b r^2$

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ABSTRACT

In this thesis we study simple groups of order $p^aq^br^2$ (p, q, and r prime numbers). John Thompson has shown that, in any simple $\{p,q,r\}$-group, the primes dividing the group order are 2, 3, and an element of $\{5, 7, 13, 17\}$. Richard Brauer and David Wales have classified simple groups of order p^aq^br. In the case of interest here, known results permit us to write the group order as $2^aq^br^2$ unless the group is isomorphic to A_5. We shall deal primarily with the case $r = 3$.

Let G be a simple group with $|G| = 2^aq^b3^2$. Recent work of W. J. Wong on the relation between blocks and exceptional characters provides the key to obtaining information about the principal 3-block of G. Using the block-section orthogonality relations, we obtain diophantine equations for the degrees of the irreducible characters in this block. Methods are developed for solving the type of equation which arises. Finally, we perform a detailed analysis of the solutions; the most important technique is restriction of characters to 3-local subgroups.

Let $k_3(G)$ denote the number of conjugacy classes of 3-elements in G. It is proven that, if $k_3(G) \geq 3$, then G must be isomorphic to one of the linear groups $PSL_2(8)$ or $PSL_2(17)$. If $k_3(G) = 2$, either G is isomorphic to the alternating group A_6 or else the principal 3-block of G has one of three explicit forms. Results for the case $k_3(G) = 1$ are weaker; however, they permit us to show that $|G| \neq 2^aq^23^2$ and that, with known exceptions, $|G| > 10^6$. The latter result serves to prove

the nonexistence of a simple group for a number of previously unresolved orders.

In deriving the above theorems, we establish a number of preliminary results valid in any simple group of order $2^a q^b r^2$. We also prove a number of theorems on 3-blocks of defect 2 and groups of order $3^2 t$, $(3,t) = 1$.

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CHAPTER I. INTRODUCTION

An elementary theorem of finite group theory states that a group of prime power order is solvable and hence not simple. In 1904, W. Burnside [10] used character theory to derive the same conclusion for a group of order $p^a q^b$, p and q distinct primes. A group of order $p^a q^b r^c$ may be simple; eight examples are known, namely:

Group	Order	Group	Order
A_5	$2^2 \cdot 3 \cdot 5$	$PSL_2(8)$	$2^3 \cdot 3^2 \cdot 7$
A_6	$2^3 \cdot 3^2 \cdot 5$	$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	$PSL_3(3)$	$2^4 \cdot 3^3 \cdot 13$
$PSL_2(7)$	$2^3 \cdot 3 \cdot 7$	$PSL_2(17)$	$2^4 \cdot 3^2 \cdot 17$

The determination of all simple groups whose order is divisible by exactly three distinct prime factors is a longstanding, unsolved problem.

Recently John Thompson [28] classified minimal simple groups -- that is, simple groups all of whose proper subgroups are solvable. It follows easily that, if there exists a simple group whose order is divisible only by the primes p, q, and r, then there exists a minimal simple group with this property. By Thompson's result, the minimal simple group must be A_5, $PSL_2(7)$, $PSL_2(8)$, $PSL_3(3)$, or $PSL_2(17)$. Thus a simple group whose order is divisible by only three distinct primes must have order $2^a 3^b p^c$, where $p \in \{5, 7, 13, 17\}$.

Simple groups of this type in which one of the primes appears to the first power only have been classified completely. The difficult case, that in which $c = 1$, is due to R. Brauer for $p = 5$ [6] and

D. Wales for $p = 7$, 13, and 17 [29, 30, 31]. Exactly the eight groups listed on the preceding page arise.

We shall investigate simple groups of order $p^a q^b r^2$. If $r = 2$, the Sylow 2-subgroups would have order 4; all simple groups with this property are known [15-Chapter 16], and the only one with exactly three distinct prime factors in the order is A_5. Hence we shall write the group order as $2^a q^b r^2$, with $r \in \{3, 5, 7, 13, 17\}$. We shall deal primarily with the case $r = 3$. $PSL_2(8)$ and $PSL_2(17)$ will be characterized as the only simple groups of order $2^a q^b 3^2$ having more than two conjugacy classes of 3-elements. We shall come close to characterizing A_6 as the unique simple group of order $2^a q^b 3^2$ with two classes of 3-elements. Weaker results will be proven for the case of a single class of 3-elements.

Chapter II will set forth some notation and conventions.

Chapter III will present results on the r-local structure of a simple group of order $2^a q^b r^2$. We shall show how to apply exceptional character theory to obtain from these results information about the principal r-block of the full group.

Chapter IV will deal with fusion and 3-block structure in an arbitrary simple group of order $3^2 t$, $(3,t) = 1$.

In Chapter V, we shall combine the results of III and IV to produce the fragment of the character table of a simple group of order $2^a q^b 3^2$ corresponding to the principal 3-block and the 3-singular elements. The block-section orthogonality relations then yield diophantine equations for the degrees of the characters in the principal 3-block.

In Chapter VI, methods will be developed for solving the diophantine equations.

In Chapter VII, some results about the structure of the 3-local subgroups will be proven. We shall then consider the restriction of characters of the full group to these subgroups in an attempt to determine whether the solutions to the equations actually occur as the principal 3-block degrees of a simple group.

Chapter VIII will give applications to the problem of determining all simple groups of order less than one million. It will be shown that no unknown simple group of this type can have order $2^a q^b 3^2$; this result eliminates six cases from the list of previously unresolved orders.

Chapter IX contains a summary of the major results on simple groups of order $2^a q^b 3^2$.

CHAPTER II. NOTATION AND CONVENTIONS

The notation employed here will be primarily the standard notation of finite group theory, such as appears in Gorenstein's *Finite Groups* [15]. A list of symbols appears in Appendix I; in this chapter, we shall introduce only a few of the less standard ones.

Let G be a group, H a subgroup, and p a prime number. $B_0(p,H)$ will denote the principal p-block of H. Frequently we shall write $B_0(p)$ in place of $B_0(p,G)$. If S is any set containing an element which is in some sense trivial, S* will denote the set of nontrivial elements of S. Thus H* is the set of nonidentity elements of H and $B_0(p,H)^*$ is the set of nonidentity characters in $B_0(p,H)$. If g and u are group elements, g^u will mean $u^{-1}gu$. If K and L are groups, K·L will denote any semidirect product of K by L, that is, any split extension of K by L; in general, K·L is not uniquely defined.

To shorten the statement of theorems, we shall adopt several conventions. A "group" will always mean a *finite* group. Unless stated otherwise, a "simple group" will mean a *noncyclic* simple group. A "p-element" will be a *nonidentity* element of order a power of p. In a phrase such as "a group of order $p^a q^b r^2$," it will always be understood that p, q, and r are distinct prime numbers and that a and b are arbitrary nonnegative integers.

Groups will always be denoted by capital letters. Lower case Latin letters will be used for group elements. Characters of groups will be denoted by Greek letters. The symbol ϵ, often subscripted, will stand for a sign, that is, a variable which takes on values ± 1.

CHAPTER III. THE r-LOCAL SUBGROUPS OF A SIMPLE GROUP OF ORDER $2^a q^b r^2$

Let G be a simple group of order $2^a q^b r^2$. In this chapter, we shall derive results on the r-subgroups of G and their centralizers and normalizers. In Section 4, we shall show how these results may, in principle, be applied to obtain a fragment of the character table of G corresponding to the block $B_0(r)$ and an r-section.

1. The Sylow r-subgroups

The Sylow r-subgroups of G are either cyclic or elementary abelian. In the former case, E. C. Dade's theory of blocks with cyclic defect groups yields enough information to obtain, at least for certain values of r, a complete characterization of G. In fact, the general classification theorems do not require that the exponent of r be 2. As a corollary, we obtain:

Theorem 3.1 Let G be a simple group of order $2^a q^b r^2$ (r = 3, 5, 7, 13, or 17) in which the Sylow r-subgroups are cyclic. Then

1) r = 3 only if G is $PSL_2(8)$ or $PSL_2(17)$
2) $r \neq 5$.

Proof (1) is a corollary of a theorem of Herzog [19-Theorem 1]. (2) follows from a theorem of the author [22-Theorem 1].

Since our interest is primarily in the case r = 3, we shall assume, when necessary, that the Sylow r-subgroups are elementary abelian.

2. Sylow intersections

A subgroup K of a group H is called strongly self-centralizing if $C_H(k) = K$ whenever $k \in K^*$.

Theorem 3.2 Let G be a simple group of order $2^a q^b r^2$. Either

(1) G contains two Sylow r-subgroups with nonidentity intersection

or (2) The Sylow r-subgroups of G are strongly self-centralizing.

If r = 3, alternative (2) occurs only if G is A_6, $PSL_2(8)$, or $PSL_2(17)$.

Proof Suppose alternative (2) does not occur. Let g be a nonidentity r-element and u be a nonidentity r'-element such that gu = ug. Without loss of generality, u has prime order. Assume u has order q; if u has order 2, the same proof holds with 2 and q interchanged.

$\sum_{B_0(q)} \chi(1)\,\chi(gu) = 0$ since gu is q-singular [3, II-7C]. We may write this as $1 + \sum_{B_0(q)^*} \chi(1)\,\chi(gu) = 0$. It follows that $B_0(q)$ contains a nonidentity character η with $\eta(1) \not\equiv 0 \pmod 2$ and $\eta(gu) \neq 0$. The first condition implies that $\eta(1) = q^m r^n$, where n is 0, 1, or 2. If n = 2, η would lie in an r-block of defect 0 and hence vanish on the r-singular element gu [3, I], contrary to assumption. If n = 0, the degree of η would be a power of q, contradicting $\eta \in B_0(q)$ by a theorem of Brauer and Tuan [9-Lemma 2]. Hence n = 1 and η lies in an r-block of defect 1 [1-Theorem 3]; the defect group has order r. But the defect group is the intersection of two Sylow subgroups [16-Theorem 2]. Thus alternative (1) holds.

Suppose alternative (2) holds and r = 3. If the Sylow 3-subgroups of G are cyclic, then by Theorem 3.1 the only possibilities for G are $PSL_2(8)$ and $PSL_2(17)$. Both of these groups do in fact have strongly self-centralizing Sylow 3-subgroups. If the Sylow 3-subgroups of G are elementary abelian, G must be A_6 by the complete classification of all simple groups with a strongly self-centralizing, elementary abelian Sylow subgroup of order 9 [19-Theorems 13.3, 13.5].

Clearly alternatives (1) and (2) of Theorem 3.2 are mutually exclusive.

If (2) holds, theorems of Brauer and Leonard [8] yield detailed information concerning the r-block structure of G. Also, the number n_r of Sylow r-subgroups satisfies $n_r \equiv 1 \pmod{r^2}$; this follows from the proof of the Third Sylow Theorem [17-Theorem 4.2.3].

If (1) holds, combinatorial methods give some information. Let $K_1, K_2, \ldots, K_m$ be representatives of the conjugacy classes of subgroups of order r, all chosen in a fixed Sylow r-subgroup R. Let:

$n_r = 1 + rs_r$ = number of Sylow r-subgroups of G

k_i = number of conjugates of K_i in R

$H_i = N_G(K_i)$

$1 + rb_i$ = number of Sylow r-subgroups of H_i
= number of Sylow r-subgroups of G containing K_i.

We may order the K_i so that $b_1, b_2, \ldots, b_{m'}$ are positive and $b_{m'+1}, \ldots, b_m$ are zero. The numbers $1 + rb_i$ must be of the form $2^u q^v$. Alternative (1) holding is equivalent to $m' > 0$; if (1) fails to hold, the formulas which follow are trivially valid but yield no information.

The equations below are attributed to Richard Brauer. Proofs appear in [18-Section 3, part 15].

<u>Lemma 3.3</u> (a) $n_r k_i = |G:H_i|(1 + rb_i)$ for $i = 1, 2, \ldots, m$.

(b) $\sum_{i=1}^{m'} k_i b_i \equiv s_r \pmod{r}$.

Using Lemma 3.3 and other theorems on the permutation representation of G on the cosets of H_i, the author has investigated simple groups of order $2^a 3^b 5^2$ in which the number of Sylow 5-subgroups does not exceed 485. The method seems to break down as the number of Sylow 5-groups becomes larger. Lemma 3.3 will be used in Chapter VIII, in which we treat simple groups of order less than one million.

3. <u>Centralizers and normalizers of subgroups of order r</u>

Let G be a group of order $2^a q^b r^2$. The theorems of this section will not require that G be simple. Let D be a subgroup of order r. We shall characterize $C(D)/O_{r'}(C(D))$ and $N(D)/O_{r'}(N(D))$.

Simple groups of order $2^2 q^b r$ have been classified completely. For r = 5, the classification is due to Richard Brauer [6]; for r = 7, 13, and 17, to David Wales [29, 30, and 31]. If r = 3, it follows from [6-Proposition 1] that the Sylow 3-group is self-centralizing, and Feit and Thompson have classified all simple groups with a self-centralizing subgroup of order 3 [14-page 186]. Alternatively, the classification for r = 3 may be obtained easily by direct application of Brauer's theory of blocks in groups whose order contains a prime to the first power [2].

The automorphism groups of the simple groups of order $2^a q^b r$ are also known [27]. We shall be interested in groups H such that $S \subseteq H \subseteq \mathrm{Aut}(S)$ for some simple group S of order $2^a q^b r$, that is, in extensions of simple groups of order $2^a q^b r$ by subgroups of their outer automorphism groups. Table 1 lists the simple groups of order $2^a q^b r$ and the groups H of the type just mentioned. This type of group will arise frequently, so we make the following definition.

Definition 3.4 $\mathscr{N}(r) = \{ H \mid H$ is a group and $S \subseteq H \subseteq \mathrm{Aut}(S)$ for some simple group S of order $2^a q^b r \}$.

$\mathscr{N}(r)$ may be obtained immediately from Table 1. For example, $\mathscr{N}(3) = \{ A_5, S_5, PSL_2(7), PGL_2(7) \}$. Of course, $\mathscr{N}(r)$ is empty if $r \notin \{3, 5, 7, 13, 17\}$.

Lemma 3.5 If H is any group of order $2^a q^b r$, then either

(1) $H/O_{r'}(H)$ is a Frobenius group of order rd, $d \mid (r-1)$, with Frobenius kernel of order r

or (2) $H/O_{r'}(H) \in \mathscr{N}(r)$.

Proof Denote $H/O_{r'}(H)$ by $\overline{H}$. Let K be a minimal normal subgroup of $\overline{H}$. K is characteristically simple and hence is a direct product of (possibly cyclic) isomorphic simple groups [15-Theorem 2.1.4]. The order of $\overline{H}$ and the condition $O_{r'}(\overline{H}) = 1$ imply that either $|K| = r$ or else K is a simple group of order $2^m q^n r$. Since K is normal in $\overline{H}$, $\overline{H}/C_{\overline{H}}(K)$ is isomorphic to a group of automorphisms of K.

Suppose $|K| = r$. $C_{\overline{H}}(K) = K \times V$ for some r'-group V [15-Theorem 7.6.5]. V is characteristic in $C_{\overline{H}}(K)$ and hence normal in $\overline{H}$. Since

TABLE 1

Simple groups S of order $2^a q^b r$
and groups H such that $S \subseteq H \subseteq Aut(S)$

r	S	|S|	|Aut(S):S|	H
3,5	A_5	60	2	i) A_5 ii) S_5
3,7	$PSL_2(7)$	168	2	ii) $PSL_2(7)$ ii) $PGL_2(7)$
5	A_6	360	4	i) A_6 ii) S_6 iii) $PGL_2(9)$ iv) An extension of $PSL_2(9)$ by a field automorphism of order 2 v) An extension of $PGL_2(9)$ by a field automorphism of order 2, written $P\Gamma L_2(9)$
5	$U_4(2)$	25920	2	i) $U_4(2)$ ii) An extension of $U_4(2)$ by a field automorphism of order 2
7	$PSL_2(8)$	504	3	i) $PSL_2(8)$ ii) An extension of $PSL_2(8)$ by a field automorphism of order 3, written $P\Gamma L_2(8)$
7	$U_3(3)$	6048	2	i) $U_3(3)$ ii) An extension of $U_3(3)$ by a field automorphism of order 2, isomorphic to $G_2(2)$
13	$PSL_3(3)$	5616	2	i) $PSL_3(3)$ ii) An extension of $PSL_3(3)$ by a graph automorphism of order 2
17	$PSL_2(17)$	2448	2	i) $PSL_2(17)$ ii) $PGL_2(17)$

$O_{r'}(\overline{H}) = 1$, $V = 1$. Thus $\overline{H}/K$ is isomorphic to a group of automorphisms of K and hence has order dividing r-1. It follows easily that $\overline{H}$ is a Frobenius group with kernel K. Hence (1) holds.

Suppose K is a simple group of order $2^m q^n r$. $C_{\overline{H}}(K)$ is an r'-group, for suppose an element g of order r centralized K. $g \notin K$, so $\langle g \rangle \times K$ would be a subgroup of $\overline{H}$, contrary to r^2 dividing $|\langle g \rangle \times K|$. Hence $C_{\overline{H}}(K) = 1$. Thus $\overline{H}$ is isomorphic to a group of automorphisms of K. It follows that $\overline{H} \in \mathcal{X}(r)$ and (2) holds.

Lemma 3.6 If H is any group and D is a p-subgroup, then $O_{p'}(C(D)) = O_{p'}(N(D))$.

Proof $O_{p'}(C(D))$ is characteristic in C(D) and hence normal in N(D). It follows that $O_{p'}(C(D)) \subseteq O_{p'}(N(D))$.

D is normal in N(D) and $O_{p'}(N(D))$ is normal in N(D). Hence $[D, O_{p'}(N(D))] \subseteq D \cap O_{p'}(N(D)) = \{1\}$. Thus $O_{p'}(N(D))$ centralizes D. It follows that $O_{p'}(N(D)) \subseteq O_{p'}(C(D))$.

Denoting $C(D)/O_{r'}(N(D))$ and $N(D)/O_{r'}(N(D))$ by $\overline{C(D)}$ and $\overline{N(D)}$ respectively, we obtain:

Theorem 3.7 Let G be a group of order $2^a q^b r^2$ with elementary abelian Sylow r-subgroups. If D is a subgroup of order r, then

$$\overline{C(D)} \simeq D \times U$$
$$\text{and } \overline{N(D)} \simeq D \cdot T,$$

where either $U \in \mathcal{X}(r)$ or U is a Frobenius group of order rd, d dividing r-1, and T is an extension of U by a cyclic group of order dividing r-1.

Proof C(D) has elementary abelian Sylow r-subgroups of order r^2. It follows from a theorem of Gaschütz [24-Theorem IV.8.b] that C(D) is a split extension of D. Thus there exists a group V of order $2^m q^n r$ such that $C(D) = D \times V$. Since D is an r-group, $O_{r'}(C(D)) = O_{r'}(V)$. Thus $\overline{C(D)} \simeq D \times V/O_{r'}(V)$. Applying Lemma 3.5 to the group V gives the required form for $\overline{C(D)}$, with $U = V/O_{r'}(V)$.

N(D) has elementary abelian Sylow r-subgroups, so by the theorem of Gaschütz, N(D) is a split extension of D, that is, there exists a subgroup W such that $N(D) = D \cdot W$. V is the subgroup of W centralizing D. $O_{r'}(N(D)) \subseteq W$, since $O_{r'}(N(D)) = O_{r'}(C(D)) = O_{r'}(V)$. It follows that $\overline{N(D)} \simeq D \cdot (W/O_{r'}(N(D)))$. Set $T = W/O_{r'}(N(D))$. Then $T/U = (W/O_{r'}(N(D))) \;/\; (V/O_{r'}(N(D))) \simeq W/V \simeq D \cdot W/D \cdot V = N(D)/C(D)$. $N(D)/C(D)$ is isomorphic to a group of automorphisms of D and hence is a cyclic group of order dividing r-1. Thus the same holds for T/U.

4. The use of exceptional character theory

Let G be a group of order $2^a q^b r^2$ with elementary abelian Sylow r-subgroups. Let D be a subgroup of order r. From Theorem 3.7, we can obtain all possibilities for $\overline{C(D)}$ and $\overline{N(D)}$. For $\overline{C(D)}$, there are 6, 12, 10, 8, or 7 alternatives according as to whether r is 3, 5, 7, 13, or 17; for $\overline{N(D)}$ the number of alternatives is somewhat larger. In principle, at least, we can write down the character table for each possibility for $\overline{N(D)}$.

From the character table of $\overline{N(D)}$, we can obtain a fragment of the character table of G whose rows correspond to the block $B_0(r)$. The following two principles are used:

1) $N(D)$ and $\overline{N(D)}$ have the same principal r-block.

2) Induction of exceptional characters from $N(D)$ to G "respects blocks" and, in particular, "preserves" the principal r-block.

We shall now make these notions precise. To each ordinary irreducible character η of $\overline{N(D)}$, there corresponds an ordinary irreducible character η' of $N(D)$ defined by $\eta'(g) = \eta(O_{r'}(N(D))g)$. The mapping $\eta \to \eta'$ is a one-to-one mapping of the irreducible characters of $\overline{N(D)}$ onto the set of irreducible characters of $N(D)$ having $O_{r'}(N(D))$ in the kernel. Brauer [4, I-pp. 155-156] shows that the principal r-block of $\overline{N(D)}$ is mapped onto the principal r-block of $N(D)$.

The relation between blocks and exceptional characters is presented by W. J. Wong in [32]. We shall be interested only in what Wong terms the "ordinary" case -- that is, the case in which Π (a set of primes) contains all the primes dividing the group order. This case is discussed in [32-Section 4]; a brief summary follows.

Let p be a fixed prime. Let G be a finite group, H a subgroup of G, and S an invariant subset of H. $M_H(S)$ denotes the module of generalized characters of H vanishing on H-S. $M_H(S,b)$ denotes the submodule of $M_H(S)$ consisting of linear combinations of irreducible characters in the p-block b of H.

For the ordinary case, S is <u>special</u> in G if

i) Two elements of S conjugate in G are conjugate in H

and ii) $C_G(s) \subseteq H$ for all $s \in S$.

S is <u>complete</u> if it is the union of p-sections of H. S is <u>closed</u> if

$s^i \in S$ whenever $s \in S$ and i is prime to the order of s.

Assume that S is a closed complete special invariant subset of the subgroup H of G. Let B be a block of G with irreducible characters $\chi_1, \chi_2, \ldots, \chi_m$. Let $\eta_1, \eta_2, \ldots, \eta_n$ be the irreducible characters of H contained in blocks b of H such that $b^G = B$. Let $\Phi_1, \Phi_2, \ldots, \Phi_t$ be a basis of $\sum_{b^G=B} M_H(S,b)$. Define the $t \times n$ matrix $A = (a_{ij})$ by $\Phi_i = \sum_j a_{ij}\eta_j$. By [32-Theorem 7], the induced characters Φ_i^G are linear combinations of the irreducible characters in B; thus there exists a $t \times m$ matrix $C = (c_{ij})$ of integers such that $\Phi_i^G = \sum_j c_{ij}\chi_j$. By [32-Theorem 10], there exists an $n \times m$ matrix $X = (x_{ij})$ of integers such that that $AX = C$, and for <u>any</u> such matrix X,

$$\chi_i(s) = \sum_j x_{ji}\eta_j(s) \qquad \text{for all } s \in S.$$

Also, $AA^T = CC^T$ since the mapping $\Phi_i \to \Phi_i^G$ is an isometry, that is, $\langle \Phi_i^G, \Phi_j^G \rangle = \langle \Phi_i, \Phi_j \rangle$.

Now suppose that B is the principal block $B_0(p)$ of G and that H is the normalizer of a p-group D. If b is a block of H with defect group D', then $D \subseteq D'$ [3, I-9F]. Hence $C_G(D') \subseteq C_G(D) \subseteq H$, and it follows from [4-Theorem 3] that $b^G = B$ if and only if $b = B_0(p,H)$. Thus the characters $\eta_1, \eta_2, \ldots, \eta_n$ are precisely the characters in the principal p-block of H.

Let G be a group and g an element of prime order p. Let $H = N(\langle g \rangle)$, $S = \{h \in H \mid h^i = g \text{ for some integer } i\}$, and $B = B_0(p)$. Then S is a closed complete special invariant subset of H, and all the results of the preceding two paragraphs can be applied. In particular, the results may be applied in the case in which G is a group of order $2^a q^b r^2$ with elementary abelian Sylow r-subgroups, D is a subgroup of order r, and H = N(D). In this case, Theorem 3.7 enables us to obtain the characters $\eta_1, \eta_2, \ldots, \eta_h$ and the basis $\Phi_1, \Phi_2, \ldots, \Phi_t$ for $M_H(S, B_o(r,H))$. In Chapter V, the process will be carried out for the case r = 3.

CHAPTER IV. SIMPLE GROUPS OF ORDER 3^2t

Thus far, we have investigated simple groups of order $2^a q^b r^2$, where $r \in \{3, 5, 7, 13, 17\}$. Henceforth we shall restrict ourselves to the case $r = 3$. A number of results on fusion and block structure depend only on the group order being of the form 3^2t, $(3,t) = 1$. In most cases, the group need not be simple; it suffices for there to be no normal subgroup of index 3, that is, for $|G:G'|$ to be prime to 3.

1. Fusion of 3-elements

Theorem 4.1 Let G be a group of order 3^2t, $(3,t) = 1$, with elementary abelian Sylow 3-subgroups. Assume G does not contain a normal subgroup of index 3. Let P be a Sylow 3-subgroup. Then, for appropriately chosen generators a and b of P, fusion of 3-elements of P occurs in one of the following ways:

a) $a \sim a^2 \sim b \sim b^2 \sim ab \sim a^2b^2 \sim a^2b \sim ab^2$ (T, Q_8, Z_8)

b) $a \sim a^2 \sim b \sim b^2$ $ab \sim a^2b^2 \sim a^2b \sim ab^2$ (D_8, Z_4)

c) $a \sim a^2$ $b \sim b^2$ $ab \sim a^2b^2 \sim a^2b \sim ab^2$ (D_4)

d) $a \sim a^2$ $b \sim b^2$ $ab \sim a^2b^2$ $a^2b \sim ab^2$ (Z_2).

For each case, the possible structures of $N(P)/C(P)$ are given in parentheses (T denotes the Sylow 2-subgroup of $GL_2(3)$).

Corollary 4.2 Let G be a group of order 3^2t, $(3,t) = 1$, which does not contain a normal subgroup of index 3. Then every 3-element of G is conjugate to its inverse.

<u>Corollary 4.3</u> Let G be a group of order 3^2t, $(3,t) = 1$, which does not contain a normal subgroup of index 3. Then every character of G is integer-valued on the elements of order 3.

<u>Corollary 4.4</u> Let G be a group of order 3^2t, $(3,t) = 1$, which does not contain a normal subgroup of index 3. Then $\chi(g) \equiv \chi(1) \pmod 3$ for all characters χ and all elements g of order 3.

<u>Proof of Theorem 4.1</u> P is abelian; hence two elements of P are conjugate in G if and only if they are conjugate in N(P) [17-Lemma 14.3.1]. N(P)/C(P) is isomorphic to a group of automorphisms of P and hence to a group of permutations of P*. Denote this permutation group by A. Two elements of P* are conjugate if and only if they lie in the same orbit of A.

Suppose A has an orbit of length 1, say $\{a\}$. Computing the transfer $V_{G\to P}$ exactly as in the proof of Burnside's Theorem on normal p-complements [17-Theorem 14.3.1], we obtain $V_{G\to P}(a) = a$. Thus $V_{G\to P}(G) = \langle a \rangle$ or P. In the first case, $\mathrm{Ker}(V_{G\to P})$ is a normal subgroup of G of index 3; in the second, the inverse image of $\langle a \rangle$ is a normal subgroup of index 3. This contradicts the hypothesis of the theorem. Hence no orbit of A has length 1.

We may consider P as a 2 dimensional vector space over GF(3). Aut(P) is isomorphic to $GL_2(3)$. $|GL_2(3)| = 48$. A is isomorphic to a subgroup of $GL_2(3)$ of order prime to 3, hence to a subgroup of a Sylow 2-group of $GL_2(3)$. The matrices $u = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

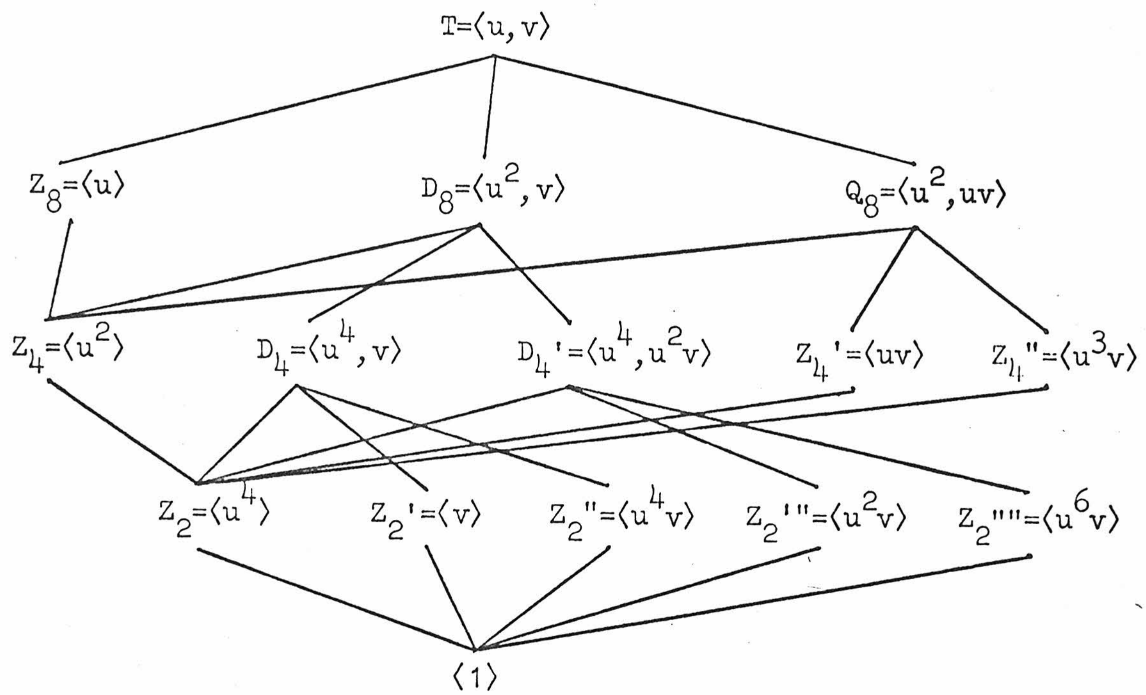
with elements in GF(3) satisfy $u^8 = 1$, $v^2 = 1$, $u^v = u^3$. Therefore, $T = \langle u,v \rangle$ is a Sylow 2-subgroup of $GL_2(3)$. T may be considered as a permutation group on P*; then A is a subgroup of some conjugate of T. The lattice of subgroups of T is shown in Figure 1 on the next page. T, Q_8, and Z_8 act transitively on P*. D_8, Z_4, Z_4', and Z_4'' have orbits $\{a,b,a^2,b^2\}$ and $\{ab,a^2b,a^2b^2,ab^2\}$ on P*, where a is (1,0) and b is (0,1), (0,1), (1,1), and (1,2) respectively for the four groups. D_4 and D_4' have orbits $\{a,a^2\}$, $\{b,b^2\}$ and $\{ab,a^2b,a^2b^2,ab^2\}$ on P*, with a = (1,0) and b = (0,1) for D_4 and a = (1,1) and b = (1,2) for D_4'. Z_2 has orbits $\{a,a^2\}$, $\{b,b^2\}$, $\{ab,a^2b^2\}$, and $\{a^2b,ab^2\}$ on P*, with a = (1,0) and b = (0,1). The remaining subgroups of T fix an element of P*. A does not fix an element of P* by the last paragraph; hence, if A is a subgroup of T, one of the alternatives of the theorem holds for an appropriate choice of the generators a and b of P. Clearly the same is true if A is a subgroup of a conjugate of T.

Proof of Corollary 4.2 If the Sylow 3-subgroups of G are elementary abelian, Corollary 4.2 follows immediately from Theorem 4.1. If they are cyclic, Aut(P) is a cyclic group of order 6. N(P)/C(P) is a 3'-group of automorphisms of P and hence has order 1 or 2. In the first case, G would have a normal 3-complement [17-Theorem 14.3.1]; hence G would have a normal subgroup of index 3, contrary to assumption. Therefore $|N(P)/C(P)| = 2$, and N(P)/C(P) induces the automorphism of P of order 2, which maps every element to its inverse.

Proof of Corollary 4.3 Let g be an element of order 3. It follows from Corollary 4.2 that $g \sim g^m$ whenever $(m,3) = 1$. Hence every

FIGURE 1

Lattice of subgroups of a Sylow 2-group of $GL_2(3)$



$$u = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \qquad v = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Z_n denotes a cyclic group of order n

Q_n denotes a quaternion group of order n

D_n denotes a dihedral group of order n character of G is rational on g.

Proof of Corollary 4.4 Let ξ be a primitive cube root of unity. Let $D = \langle g\rangle$ and let η be the character of D defined by $\eta(g^i) = \xi^i$. Since $g \sim g^2$, $\langle \chi|_D, \eta\rangle = \frac{1}{3}(\chi(1) + \xi\chi(g) + \xi^2\chi(g^2)) = \frac{1}{3}(\chi(1) - \chi(g))$. $\langle \chi|_D, \eta\rangle$ is an integer; hence the corollary holds.

2. Congruences for the degrees of the characters

Corollary 4.4 gives a congruence between the degrees of the characters and their values on elements of order 3. In this section, we shall obtain stronger congruences.

Theorem 4.5 Let P be an elementary abelian p-group of order p^2. Let χ be a rational (possibly reducible) character of P. If $g_1, g_2, \ldots, g_{p+1}$ are generators of the p+1 subgroups of order p, then:

$$(p + 1)\chi(1) \equiv \sum_{i=1}^{p+1} \chi(g_i) \pmod{p^2}.$$

Proof Since χ is rational, $\chi(g) = \chi(g^i)$ whenever $g \in P$ and $(i,p) = 1$.

Let η_i be a linear character of P having kernel $\langle g_i\rangle$; η_i can be obtained from a nonidentity character of $P/\langle g_i\rangle$. If $j \neq i$, $\{\eta_i(g_j{}^k) \mid k = 1,2,\ldots,p-1\} = \{$primitive p'th roots of unity$\}$. The sum of the primitive p'th roots of unity is -1; hence we obtain $\sum_{k=1}^{p-1} \eta_i(g_j{}^k) = \begin{cases} p-1 & \text{if } i = j \\ -1 & \text{if } i \neq j .\end{cases}$ Computing the inner product of χ and η_i, we obtain:

$$p^2 \langle \chi, \eta_i\rangle = \chi(1) + \sum_{j=1}^{p+1} \chi(g_j) \sum_{k=1}^{p-1} \eta_i(g_j{}^k)$$

$$= \chi(1) + (p - 1)\chi(g_i) - \sum_{j\neq i} \chi(g_j).$$

Adding the equations for i = 1, 2, ..., p+1 gives:

$$p^2 \sum_{i=1}^{p+1} \langle \chi, \eta_i \rangle = (p + 1)\,\chi(1) - \sum_{i=1}^{p+1} \chi(g_i).$$

As the left side of the above equation is divisible by p^2, so is the right side; this proves the theorem.

Corollary 4.6 Let G be a group of order 3^2t satisfying the hypotheses of Theorem 4.1. Let χ be any character of G. Corresponding to the four alternatives of Theorem 4.1, we have the following congruences:

a) $\chi(1) \equiv \chi(g_1) \pmod 9$

b) $2\,\chi(1) \equiv \chi(g_1) + \chi(g_2) \pmod 9$

c) $4\,\chi(1) \equiv 2\,\chi(g_1) + \chi(g_2) + \chi(g_3) \pmod 9$

d) $4\,\chi(1) \equiv \chi(g_1) + \chi(g_2) + \chi(g_3) + \chi(g_4) \pmod 9$.

Here the g_i are representatives of the conjugacy classes of 3-elements, and in (c), g_1 is in the class intersecting a Sylow 3-group in 4 elements.

Proof By Corollary 4.3, χ is rational on the elements of a Sylow 3-subgroup. Corollary 4.6 then follows immediately from Theorems 4.1 and 4.5.

3. Results on 3-blocks

This section will present some results on blocks, and in particular 3-blocks, which will be applied in Chapter 5.

The following notation will be used. B will be a p-block of defect d containing m ordinary irreducible characters $\chi_1, \chi_2, \ldots, \chi_m$ and n modular irreducible characters $\varphi_1, \varphi_2, \ldots, \varphi_n$. The number of ordinary and modular characters in B will also be denoted by k(B) and $\ell(B)$ respectively. x_i will denote $\chi_i(1)$. D will be the decomposition matrix of B and C the Cartan matrix. Define the $m \times m$ matrix $A = (a_{ij})$ by $a_{ij} = (p^d/|G|)\left(\sum_{g \text{ p-reg}} \chi_i(g)\,\overline{\chi_j(g)}\right)$.

Brauer and Feit [7] show that the matrix A has the following properties:

> A is a symmetric $m \times m$ matrix of integers.
>
> $A = p^d DC^{-1}D^T$.
>
> $A^2 = p^d A$.
>
> $a_{kk} \not\equiv 0 \pmod p$ for some k, and $a_{ij} \equiv \dfrac{x_i x_j}{x_k^2}\, a_{kk} \pmod p$
>
> for all i,j, congruences being in the ring of local integers for p.

If $E = (e_{ij})$ is a matrix, then $E_{i_1, i_2, \ldots, i_w}$ will denote the matrix formed from row i_1, row i_2, ..., row i_w of E in that order, and $E^{i_1, i_2, \ldots, i_w}$ will denote the matrix formed from column i_1, column i_2, ..., column i_w of E. $E_{\hat{i}}$ and $E^{\hat{i}}$ will denote the matrices formed from E by deleting row i and column i respectively. Thus, if E is a 3×3 matrix, $E^{\hat{2}}_{3,2} = \begin{bmatrix} e_{31} & e_{33} \\ e_{21} & e_{23} \end{bmatrix}$.

A well-known theorem states that, if F is $k \times w$ and E is $w \times k$, then $\det(EF) = \sum_{1 \le i_1 < i_2 < \ldots < i_w \le k} \det(E^{i_1, \ldots, i_w})\, \det(F_{i_1, \ldots, i_w})$.

Lemma 4.7 Let $S_{ij} = \{ (k_1,k_2,\ldots,k_{n-1}) \mid 1 \leq k_1 < k_2 < \ldots < k_{n-1} \leq m,$ $k_t \neq i,j$ for $t = 1,2,\ldots,n-1 \}$. Then

$$a_{ij} = \frac{p^d}{\det(C)} \Big(\sum_{S_{ij}} \det(D_{i,k_1,\ldots,k_{n-1}}) \det(D_{j,k_1,\ldots,k_{n-1}} \Big) .$$

Proof Let $(u_{st}) = C^{-1}$. $A = p^d DC^{-1}D^T$; hence $a_{ij} = p^d \sum_{s,t} d_{is}u_{st}d_{jt}$.

Now $u_{st} = \frac{1}{\det(C)} (-1)^{s+t} \det(C^{\hat{s}}_{\hat{t}})$. $C^{\hat{s}}_{\hat{t}} = (D^{\hat{t}})^T D^{\hat{s}}$. Hence

$\det(C^{\hat{s}}_{\hat{t}}) = \sum_S \det(D^{\hat{t}}_{k_1,\ldots,k_{n-1}}) \det(D^{\hat{s}}_{k_1,\ldots,k_{n-1}})$, where

$S = \{ (k_1,k_2,\ldots,k_{n-1}) \mid 1 \leq k_1 < k_2 < \ldots < k_{n-1} \leq m \}$. Combining these formulas yields:

$$a_{ij} = \frac{p^d}{\det(C)} \sum_{s,t} d_{is} \Big((-1)^{s+t} \sum_S \det(D^{\hat{t}}_{k_1,\ldots,k_{n-1}}) \det(D^{\hat{s}}_{k_1,\ldots,k_{n-1}}) \Big) d_{jt}$$

$$= \frac{p^d}{\det(C)} \sum_S \sum_s (-1)^s d_{is} \det(D^{\hat{s}}_{k_1,\ldots,k_{n-1}}) \sum_t (-1)^t d_{jt} \det(D^{\hat{t}}_{k_1,\ldots,k_{n-1}})$$

$$= \frac{p^d}{\det(C)} \sum_S \det(D_{i,k_1,\ldots,k_{n-1}}) \det(D_{j,k_1,\ldots,k_{n-1}}).$$

In the above formula, the sum need run only over the terms of S_{ij}, as all other terms are zero.

If G is a group and B is a p-block of defect d, Brauer [3,I] has shown that $\ell(b) \leq k(B) - 1$ and that one elementary divisor of C is p^d and the others are lower powers of p. As an application of Lemma 4.7, we shall derive a stronger result on the elementary divisors for the case $\ell(B) = k(B) - 1$.

<u>Theorem 4.8</u> Let G be a group and let B be a p-block of defect d such that $\ell(B) = k(B) - 1$. Let C denote the Cartan matrix of B. Then one elementary divisor of C is p^d and the others all are 1. $\mathrm{Det}(C) = p^d$.

<u>Proof</u> Let D be the decomposition matrix of B. Let $w_i = \det(D_{\hat{i}})$. By Lemma 4.7, $a_{ij} = (p^d/\det(C))v_{ij}$, where $v_{ii} = \sum_{k \neq i} w_k^2$ and $v_{ij} = -(-1)^{i+j} w_i w_j$ for $i \neq j$.

Suppose p^{d+1} divides $\det(C)$. Since A is a matrix of integers, p must divide v_{ij} for all i and j. Suppose p does not divide w_{i_0}. $p \mid w_{i_0} w_j$ for all $j \neq i_0$; hence $p \mid w_j$ for all $j \neq i_0$. Let $j_0 \neq i_0$. $p \mid \sum_{k \neq j_0} w_k^2$, a contradiction since p fails to divide exactly one term in the sum. Thus p must divide w_i for all i. Let $n = \ell(B)$. We have shown that p divides the determinant of every $n \times n$ submatrix of D; hence p divides the n'th determinantal divisor of D, contrary to a theorem of Brauer [11-Corollary 84.18] stating that the determinantal divisors of D are 1.

Thus p^{d+1} does not divide $\det(C)$. $\mathrm{Det}(C)$ is the product of the elementary divisors of C, all of which are powers of p [3, I-5B] and one of which is p^d [3, I-6C]. It follows that the remaining elementary divisors all must be 1 and that $\det(C) = p^d$.

Henceforth we shall investigate 3-blocks.

<u>Theorem 4.9</u> In any 3-block of positive defect, the number of irreducible characters of height 0 is a multiple of 3.

Proof Denote the block by B and the height of a character χ in B by $\lambda(\chi)$. Choose r such that $a_{rr} \not\equiv 0 \pmod 3$. $a_{rj} \equiv (x_j/x_r)a_{rr} \pmod 3$. It follows that $a_{rj} \equiv 0 \pmod 3$ if and only if $\lambda(\chi_j) > 0$. Since $A^2 = 9A$, $\sum_j a_{rj}^2 = 9a_{rr} \equiv 0 \pmod 3$. Now a_{rj}^2 is congruent to 0 modulo 3 if $\lambda(\chi_j) > 0$ and a_{rj}^2 is congruent to 1 modulo 3 if $\lambda(\chi_j) = 0$. Thus $\sum_{\lambda(\chi_j)=0} 1 = 0$. It follows that the number of characters of height zero is a multiple of 3.

Theorem 4.10 A 3-block of defect 2 contains 3, 6, or 9 irreducible characters. If the block is principal, the number of characters is 6 or 9.

Proof The first statement follows from Theorem 4.9, the fact that a block of defect 2 contains no characters of positive height [7-Theorem 2], and the fact that a p-block of defect 2 contains at most p^2 irreducible characters [7-Theorem 1*].

Let $B_0(3)$ be the principal 3-block of G, where G is a group of order 3^2t, $(3,t) = 1$. We must prove that $k(B_0(3)) \neq 3$. Assume the contrary. If $B_0(3)$ contains only one modular irreducible character, namely the identity character, then a theorem of Brauer [4, I-Cor. 3] shows that G has a normal 3-complement H. $H = O_{3'}(G)$; hence $G/O_{3'}(G)$ is a group of order 9. $G/O_{3'}(G)$ has 9 characters in its principal 3-block. Since the principal 3-blocks of G and $G/O_{3'}(G)$ are identical [4, I-p. 156], $B_0(3)$ must contain 9 characters, contrary to assumption. We conclude that $B_0(3)$ contains 2 modular characters.

By [3, II-7D] and [4, I- Th. 3], $k(B_0(3)) = \sum_g \ell(B_0(3, C(g)))$, where g ranges over the identity and a set of representatives of the conjugacy classes of 3-elements. The term on the left is 3, and the term on the right corresponding to $g = 1$ is 2. We conclude that there is only one conjugacy class of 3-elements and that, if g is in this class, $\ell(B_o(3,C(g)))=1$. Hence $C(g)$ has a normal 3-complement [4, I-Cor. 3]. If H is any group, define $\lambda(H,x) = \sum_{B_o(p,H)} |\chi(x)|^2$. By [4, II-Page 310], $\lambda(G,g) = \lambda(C(g),1) = \lambda(C(g)/O_{3'}(C(g)),1) = 9$; the last equality follows because $C(g)/O_{3'}(C(g))$ is a group of order 9. All the 3-elements of G are conjugate; hence every character of G is rational on g. It follows that the fragment of the character table of G corresponding to $B_o(3)$ and the 3-singular elements has the form:

	1	g
χ_1	1	1
χ_2	f_2	± 2
χ_3	f_3	± 2

The block-section orthogonality relations [3, II-7C] imply $1 \pm 2f_2 \pm 2f_3 = 0$. This contradiction completes the proof.

<u>Theorem 4.11</u> Let G be a group of order 3^2t, $(3,t) = 1$, not containing a normal subgroup of index 3. If the principal 3-block of G contains exactly 2 modular irreducible characters, then it contains exactly 6 ordinary irreducible characters.

<u>Proof</u> In view of Theorem 4.10, it suffices to show that $B_o(3)$ does not contain 9 ordinary irreducible characters. Assume the contrary. Let D be the 9×2 decomposition matrix of $B_o(3)$. The rows and columns may be

arranged so that the first row corresponds to the identity character and the first column to the identity modular character. We shall use the results of Brauer and Feit [7] on the matrix A, defined earlier. If $j \geq 2$, then $d_{j2} > 0$, for suppose d_{j2} were 0; then $\chi_j(x) = d_{j1}$ for all 3-regular elements x; the kernel K of χ_j would contain all 3-regular elements of G; hence G/K would have order 3 or 9, and in either case G would contain a normal subgroup of index 3, contrary to assumption.

By Lemma 4.7, $a_{11} = \frac{9}{\det(C)} \sum_{j=2}^{9} d_{j2}^2$. Det(C) is either 9 or 27 according as to whether the smaller elementary divisor of C is 1 or 3 [3, I-6C]. $0 < a_{ii} \leq 9$ by [7-page 363]. [7-Equation 7] yields, for the case $p = 3$, $a_{ii} \equiv a_{jj} \not\equiv 0 \pmod 3$ for all i and j. It follows that $a_{11} \in \{1, 2, 4, 5, 7, 8\}$ and $\sum_{j=2}^{9} d_{j2}^2 \in \{1,2,4,5,7,8,3,6,12,15,21,24\}$. It is straightforward to check that, among the numbers in the last set, only 8 and 24 are sums of exactly 8 nonzero squares. Hence $\sum_{j=2}^{9} d_{j2}^2 = 8$ or 24; in either case $a_{11} = 8$. Since $B_0(3)$ has 2 modular irreducibles, $tr(A) = 9 \cdot 2 = 18$ (This follows from [5, II-5C] upon noting that A is p^d times the matrix $M^{(1)}$ of [5]). From the condition $a_{ii} \equiv a_{11} \equiv 2 \pmod 3$, we obtain $a_{ii} \geq 2$ for all i. It follows that $tr(A) = \Sigma\, a_{ii} \geq 8 + 8 \cdot 2 = 24$, contrary to $tr(A) = 18$. Hence the theorem holds.

As an application of Theorem 4.11, we shall obtain the principal 3-block of a group of order 3^2t in which the centralizer of a Sylow 3-group has index 2 in the normalizer.

Theorem 4.12 Let G be a group of order 3^2t, $(3,t) = 1$, not containing a normal subgroup of index 3. Let P denote a Sylow 3-subgroup. Assume that P is elementary abelian and $|N(P):C(P)| = 2$. Then there exists a sign ϵ ($\epsilon = \pm 1$) and an integer f such that $f \equiv 7\epsilon \pmod 9$ and the principal 3-block of G has the following form:

	1	$S(g_1)$	$S(g_2)$	$S(g_3)$	$S(g_4)$
χ_1	1	1	1	1	1
χ_2	f	ϵ	ϵ	ϵ	-2ϵ
χ_3	f	ϵ	ϵ	-2ϵ	ϵ
χ_4	f	ϵ	-2ϵ	ϵ	ϵ
χ_5	f	-2ϵ	ϵ	ϵ	ϵ
χ_6	$f+\epsilon$	$-\epsilon$	$-\epsilon$	$-\epsilon$	$-\epsilon$

Here g_1, g_2, g_3 and g_4 denote representatives of the four classes of 3-elements, and $S(g_i)$ denotes the 3-section of g_i. The characters χ_2, χ_3, χ_4, and χ_5 agree on 3-regular elements.

Proof By Theorem 4.10, $B_0(3)$ contains 6 or 9 ordinary irreducible characters. Assume the latter alternative occurs. By [3, II-7D] and [4, I- Th. 3], $k(B_0(3)) = \ell(B_0(3)) + \sum_{i=1}^{4} \ell(B_0(3, C(g_i)))$.

$\ell(B_0(3)) \neq 1$ since the hypotheses imply that G does not have a normal 3-complement [4, I-Corollary 3]. $\ell(B_0(3)) \neq 2$ by Theorem 4.11. It follows that $\ell(B_0(3)) \geq 3$ and $\sum_{i=1}^{4} \ell(B_0(3, C(g_i))) \leq 6$. Hence there are at least two distinct classes of 3-elements for which the centralizer has a single modular character in its principal 3-block and hence has a normal 3-complement; let g and g' be representatives of two such classes. $C(g)/O_{3'}(C(g))$ and $C(g')/O_{3'}(C(g'))$ are groups of order 9. Defining the function λ as in [4, II-Section 3], we obtain

$\lambda(G,g) = \lambda(C(g)) = \lambda(C(g)/O_{3'}(C(g))) = 9.$

Thus $\sum_{\chi \in B_0(3)} |\chi(g)|^2 = 9$. By Corollary 4.3, $\chi(g)$ is rational for all characters χ; hence $\chi(g)$ is integral. Hence for each character χ in $B_0(3)$, there is a sign ϵ_χ such that $\chi(g) = \epsilon_\chi$. Similarly, there is a sign ϵ_χ' such that $\chi(g') = \epsilon_\chi'$. By Corollary 4.4, $\chi(g) \equiv \chi(1) \equiv \chi(g') \pmod 3$; it follows that $\epsilon_\chi = \epsilon_\chi'$. Block-section orthogonality [3, II-7C] gives: $\sum_{\chi \in B_0(3)} \epsilon_\chi \epsilon_\chi' = 0$. Since each term in the sum is 1, this is a contradiction.

We have shown that $B_0(3)$ must contain 6 ordinary irreducible characters. As before, $k(B_0(3)) = \ell(B_0(3)) + \sum_{i=1}^{4} \ell(B_0(3, C(g_i)))$. It follows that $\ell(B_0(3)) = 2$ and $\ell(B_0(3, C(g_i))) = 1$ for i = 1, 2, 3, 4. $C(g_i)$ has a normal 3-complement and, as above, $\sum_{\chi \in B_0(3)} |\chi(g_i)|^2 = 9$.

For each i, there must be one character χ in $B_0(3)$ for which $|\chi(g_i)| = 2$; for the remaining characters, $|\chi(g_i)| = 1$. The columns of the character table corresponding to g_1, g_2, g_3, and g_4 must fit together in such a way as to satisfy the block-section orthogonality relations [3, II-7C] and the condition $\chi(g_i) \equiv \chi(g_j)$ $(i, j = 1,2,3,4)$; apart from a permutation of the rows, this can occur only in the way given in the statement of the theorem. The column corresponding to 1 must have the form given in order to be orthogonal to the columns of g_1, g_2, g_3, and g_4. The condition $f \equiv 7\epsilon \pmod 9$ follows immediately from Corollary 4.6. Since $C(g_i)$ has a normal 3-complement, [4, I-Corollary 5] implies that each character in $B_o(3)$ is constant on the 3-section of g_i. Finally, if x is any 3-regular element, orthogonality between the column of x and those of the g_i shows that $\chi_2(x) = \chi_3(x) = \chi_4(x) = \chi_5(x)$.

CHAPTER V. SIMPLE GROUPS OF ORDER $2^a q^b 3^2$: OBTAINING THE DEGREE EQUATION

Let G denote a simple group of order $2^a q^b 3^2$. In this chapter, we shall determine all possibilities for the fragment of the character table of G corresponding to the principal 3-block and the 3-singular elements. A number of alternatives will arise, corresponding to different fusion patterns (Theorem 4.1) and different structures for $\overline{N(D)}$ (Theorem 3.7). For each alternative, the block-section orthogonality relations will yield diophantine equations for the degrees of the principal 3-block characters; equations obtained in this manner will be referred to as degree equations.

1. Preliminary results

Lemma 5.1 If G is a simple group of order $2^a q^b 3^2$ not isomorphic to $PSL_2(17)$, then q is 5 or 7.

Proof We remarked earlier that Thompson's classification of minimal simple groups [28] implies that $q \in \{5, 7, 13, 17\}$. If q were 13, G would involve the minimal simple group $PSL_3(3)$, a contradiction since 3^3 divides $|PSL_3(3)|$. If q is 17, G involves the minimal simple group $PSL_2(17)$. $PSL_2(17)$ has cyclic Sylow 3-groups of order 9 [21-8.10]; hence G has cyclic Sylow 3-groups, and Theorem 3.1 implies that $G \simeq PSL_2(17)$.

Theorem 3.1 also tells us that, if G is not isomorphic to $PSL_2(8)$ or $PSL_2(17)$, the Sylow 3-subgroups are elementary abelian.

The final step in many subsequent arguments will be application of the following lemma.

Lemma 5.2 Let G be a simple group of order $2^a q^b 3^2$. Suppose the principal 3-block contains two irreducible characters whose degrees are consecutive integers. Then G is isomorphic to $PSL_2(8)$ or $PSL_2(17)$.

Proof If q is 17, then Lemma 5.1 tells us that G is isomorphic to $PSL_2(17)$. The principal 3-block of $PSL_2(17)$ does in fact contain two consecutive degrees, namely, 16 and 17.

Assume q is 5 or 7. Let the consecutive degrees be 2^x and q^y. Then $2^x - q^y = \pm 1$. For low values of q, the solutions to this equation are well-known (The methods of Chapter VI, developed to solve a much more general type of equation, are applicable in this case). For $q = 5$, $(2^x, q^y)$ is (2,1) or (4,5); for $q = 7$, $(2^x, q^y)$ is (2,1) or (8,7). Thus G must have an irreducible representation of degree 2, 4, or 7. All such simple groups are known [13-Section 8.5]; the only one whose order has the form $2^a q^b 3^2$ is $PSL_2(8)$, which has an irreducible representation of degree 7. In fact, this representation is in the principal 3-block, and the principal 3-block also contains a representation of degree 8.

By Theorem 4.1, a simple group of order $2^a q^b 3^2$ has 1, 2, 3, or 4 conjugacy classes of 3-elements. The difficulty of the classification problem will vary inversely with the number of classes; if the number is 4, the following theorem provides a complete classification.

Theorem 5.3 A simple group of order $2^a q^b 3^2$ having 4 conjugacy classes of 3-elements is isomorphic to $PSL_2(8)$ or $PSL_2(17)$.

Proof By Theorem 4.12, the principal 3-block of the group must contain two characters of consecutive degrees. Hence Lemma 5.2 gives the desired conclusion.

2. Applying the exceptional character theory

In this section, we shall apply the exceptional character theory outlined in Chapter III, Section 4 to a group of order $2^a q^b 3^2$. As in Chapter III, $\overline{N(D)}$ and $\overline{C(D)}$ will denote respectively $N(D)/O_{3'}(N(D))$ and $C(D)/O_{3'}(N(D))$. More generally, if S is any subset of N(D), $\overline{S}$ will denote the image of S under the natural mapping from N(D) to $\overline{N(D)}$. $\overline{D}$ is isomorphic to D; hence the two groups will be identified.

Theorem 5.4 Let G be a group of order $2^a q^b 3^2$, q equal 5 or 7, with elementary abelian Sylow 3-subgroups. Assume G does not contain a normal subgroup of index 3. If $D = \langle g \rangle$ is a subgroup of order 3, then one of the following alternatives occurs for $\overline{C(D)}$ and $\overline{N(D)}$:

	$\overline{C(D)}$	$\overline{N(D)}$
i.	$D \times A_3$	$S_3 \times A_3$ or $D \cdot S_3$
ii.	$D \times S_3$	$S_3 \times S_3$
iii.	$D \times A_5$	$S_3 \times A_5$ or $D \cdot S_5$
iv.	$D \times S_5$	$S_3 \times S_5$
v.	$D \times PSL_2(7)$	$S_3 \times PSL_2(7)$ or $D \cdot PGL_2(7)$
vi.	$D \times PGL_2(7)$	$S_3 \times PGL_2(7)$

Here D·H denotes a semidirect product of D by H which is not direct; for the cases arising here, it is uniquely determined.

Proof The alternatives for $\overline{C(D)}$ follow immediately from Theorem 3.7. The alternatives listed for $\overline{N(D)}$ are those obtained from Theorem 3.7 for which $\overline{N(D)}$ does not centralize D. Let D = $\langle g \rangle$. By Corollary 4.2, there exists $x \in N(D)$ such that $g^x = g^{-1}$. $\overline{g}^{\overline{x}} = \overline{g}^{-1}$. $\overline{g}$ has order 3; hence $\overline{N(D)}$ does not centralize D.

Appendix II contains fragments of the character tables of the various alternatives for $\overline{N(D)}$, except $S_3 \times A_3$ and $D \cdot S_3$ (Tables for these two groups will not be needed). The fragments are those corresponding to the principal 3-blocks. Each fragment may also be considered as a fragment of the character table of N(D) corresponding to the principal 3-block, provided each column represents a set of conjugacy classes of N(D) mapped to a single class of $\overline{N(D)}$. We shall denote such a set by brackets; specifically, [x] will denote $\{y \in N(D) \mid \overline{y}$ is conjugate to $\overline{x}$ in $N(D)\}$. Using the results of Wong [32], summarized in Chapter III, Section 4, we obtain:

Theorem 5.5 Let G be a group of order 3^2t, $(3,t) = 1$, for which the conclusion of Theorem 5.4 holds. Then there are seven possible cases for the fragment of the character table of G corresponding to the principal 3-block and the 3-section of a 3-element g; the cases are given on the next two pages. Cases i and i' correspond to alternative i of Theorem 5.4; cases ii, iii, iv, v, and vi correspond to alternatives ii, iii, iv, v, and vi of Theorem 5.4 respectively. ϵ_i denotes a sign.

Case i

	1	[g]
X_1	1	1
X_2	f_2	ϵ_2
X_3	f_3	ϵ_3
X_4	f_4	ϵ_4
X_5	f_5	ϵ_5
X_6	f_6	ϵ_6
X_7	f_7	ϵ_7
X_8	f_8	ϵ_8
X_9	f_9	ϵ_9

Case i'

	1	[g]
X_1	1	1
X_2	f_2	ϵ_2
X_3	f_3	ϵ_3
X_4	f_4	ϵ_4
X_5	f_5	ϵ_5
X_6	f_6	$2\epsilon_6$

Case ii

	1	[g]	[gv]
X_1	1	1	1
X_2	f_2	ϵ_2	ϵ_2
X_3	f_3	ϵ_3	ϵ_3
X_4	f_4	$-\epsilon_4$	ϵ_4
X_5	f_5	$-\epsilon_5$	ϵ_5
X_6	f_6	$-\epsilon_6$	ϵ_6
X_7	f_7	$2\epsilon_7$	0
X_8	f_8	$2\epsilon_8$	0
X_9	f_9	$2\epsilon_9$	0

$\overline{v}$ is an element of order 2 in S_3. $\overline{v}$ is in $\overline{C(D)}$.

Case iii

	1	[g]	[gv]	[gv^2]	[gw]
X_1	1	1	1	1	1
X_2	f_2	ϵ_2	ϵ_2	ϵ_2	ϵ_2
X_3	f_3	ϵ_3	ϵ_3	ϵ_3	ϵ_3
X_4	f_4	$-4\epsilon_4$	ϵ_4	ϵ_4	0
X_5	f_5	$-4\epsilon_5$	ϵ_5	ϵ_5	0
X_6	f_6	$-4\epsilon_6$	ϵ_6	ϵ_6	0
X_7	f_7	$5\epsilon_7$	0	0	ϵ_7
X_8	f_8	$5\epsilon_8$	0	0	ϵ_8
X_9	f_9	$5\epsilon_9$	0	0	ϵ_9

$\overline{v}$ and $\overline{w}$ are elements of A_5

$\overline{v} = (12345)$

$\overline{w} = (12)(34)$

Case iv

	1	[g]	[gv]	[gw]	[gx]	[gy]
x_1	1	1	1	1	1	1
x_2	f_2	ϵ_2	ϵ_2	ϵ_2	ϵ_2	ϵ_2
x_3	f_3	ϵ_3	ϵ_3	ϵ_3	ϵ_3	ϵ_3
x_4	f_4	$-4\epsilon_4$	ϵ_4	0	0	$2\epsilon_4$
x_5	f_5	$-4\epsilon_5$	ϵ_5	0	0	$2\epsilon_5$
x_6	f_6	$-4\epsilon_6$	ϵ_6	0	0	$2\epsilon_6$
x_7	f_7	$5\epsilon_7$	0	ϵ_7	ϵ_7	$-\epsilon_7$
x_8	f_8	$5\epsilon_8$	0	ϵ_8	ϵ_8	$-\epsilon_8$
x_9	f_9	$5\epsilon_9$	0	ϵ_9	ϵ_9	$-\epsilon_9$

$\bar{v}$, $\bar{w}$, $\bar{x}$, and $\bar{y}$ are elements of S_5

$\bar{v} = (12345)$

$\bar{w} = (12)(34)$

$\bar{x} = (1234)$

$\bar{y} = (12)$

Case v

	1	[g]	[gv]	$[gv^2]$	[gw]	$[gw^6]$
x_1	1	1	1	1	1	1
x_2	f_2	ϵ_2	ϵ_2	ϵ_2	ϵ_2	ϵ_2
x_3	f_3	ϵ_3	ϵ_3	ϵ_3	ϵ_3	ϵ_3
x_4	f_4	$-7\epsilon_4$	ϵ_4	ϵ_4	0	0
x_5	f_5	$-7\epsilon_5$	ϵ_5	ϵ_5	0	0
x_6	f_6	$-7\epsilon_6$	ϵ_6	ϵ_6	0	0
x_7	f_7	$8\epsilon_7$	0	0	ϵ_7	ϵ_7
x_8	f_8	$8\epsilon_8$	0	0	ϵ_8	ϵ_8
x_9	f_9	$8\epsilon_9$	0	0	ϵ_9	ϵ_9

$\bar{v}$ and $\bar{w}$ are elements of $PSL_2(7)$

$\bar{v}$ has order 4

$\bar{w}$ has order 7

Case vi

	1	[g]	[gv]	[gx]	[gy]	[gu]	[gw]	[gz]
x	1	1	1	1	1	1	1	1
x_2	f_2	ϵ_2	ϵ_2	ϵ_2	ϵ_2	ϵ_2	ϵ_2	ϵ_2
x_3	f_3	ϵ_3	ϵ_3	ϵ_3	ϵ_3	ϵ_3	ϵ_3	ϵ_3
x_4	f_4	$-7\epsilon_4$	ϵ_4	ϵ_4	ϵ_4	ϵ_4	0	$-\epsilon_4$
x_5	f_5	$-7\epsilon_5$	ϵ_5	ϵ_5	ϵ_5	ϵ_5	0	$-\epsilon_5$
x_6	f_6	$-7\epsilon_6$	ϵ_6	ϵ_6	ϵ_6	ϵ_6	0	$-\epsilon_6$
x_7	f_7	$8\epsilon_7$	0	0	0	0	ϵ_7	$2\epsilon_7$
x_8	f_8	$8\epsilon_8$	0	0	0	0	ϵ_8	$2\epsilon_8$
x_9	f_9	$8\epsilon_9$	0	0	0	0	ϵ_9	$2\epsilon_9$

$\bar{v}$, $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{w}$, and $\bar{u}$ are elements of $PGL_2(7)$

$\bar{w}^7 = 1$

$\bar{v}^8 = \bar{z}^2 = 1$

$\bar{v}^{\bar{z}} = \bar{v}^3$

$\bar{x} = \bar{v}^7$

$\bar{y} = \bar{v}^2$

$\bar{u} = \bar{v}^4$

Proof Assume first that alternative ii, iii, iv, v, or vi of Theorem 5.4 holds. We shall use the notation introduced in Chapter III, Section 4. The set S of special classes will be $\{x \in N(D) \mid x^i = g$ for some $i\}$. In each case, S has a basis $\{\Phi_1, \Phi_2\}$, where $\Phi_1 = \eta_1 + \eta_2 - \eta_3 - \eta_4 - \eta_5 + \eta_6$ and $\Phi_2 = \eta_1 + \eta_2 - \eta_3 + \eta_7 + \eta_8 - \eta_9$. The matrix A is as follows:

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

The matrix C is an integral matrix satisfying the equation $AA^T = CC^T$. Each entry in the first column of C is 1, for χ_1 and η_1 are the identity characters of G and N(D) respectively; hence $c_{i1} = \langle \Phi_i^G, \chi_1 \rangle = \langle \Phi_i, \eta_1 \rangle = 1$. By Theorem 4.10, $B_0(3)$ contains 6 or 9 irreducible characters; hence C has 6 or 9 columns. Suppose C has a column consisting entirely of zeroes. Let X be an integral matrix such that AX = C. If column j of C contains only zeros, we may replace column j of X by a column of zeroes and still maintain the equality. Then $\chi_j(g) = \sum_i x_{ij}\eta_i(g) = 0$. By Corollary 4.3, $\chi_j(1) \equiv 0 \pmod 3$, contrary to χ_j being in a principal 3-block of defect 2. We conclude that no column of C consists entirely of zeroes. Apart from a permutation of the rows and columns, the only possibility for the matrix C is the following:

$$C = \begin{bmatrix} 1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 & 0 & 0 & 0 \\ 1 & \epsilon_2 & \epsilon_3 & 0 & 0 & 0 & \epsilon_7 & \epsilon_8 & \epsilon_9 \end{bmatrix}.$$

The ϵ_i are signs (i.e. $\epsilon_i = \pm 1$). A matrix X satisfying the equation AX = C is the following:

$$X = \begin{bmatrix} 1 & \epsilon_2 & \epsilon_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_4 & -\epsilon_5 & -\epsilon_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon_7 & \epsilon_8 & \epsilon_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If s is in a special class, $\chi_i(s)$ can be computed from the formula: $\chi_i(s) = \sum_j x_{ji}\eta_j(s)$. We obtain the fragments of the character table of G given in the statement of the theorem.

Now assume that alternative i of Theorem 5.4 holds. In this case, the desired conclusion can be obtained without using exceptional characters. As in [4, II-Section III], let the function λ be defined on G by $\lambda(G,x) = \sum_{\chi \in B_0(3)} |\chi(x)|^2$. $\overline{C(D)}$ is a group of order 9; hence, using the results of [4, II], we obtain

$$\lambda(G,g) = \lambda(C(g),1) = \lambda(\overline{C(g)},1) = 9 .$$

By Theorem 4.10, $B_0(3)$ contains 6 or 9 characters. Corollary 4.3 implies that $\chi(g)$ is rational. It follows easily that one of the two alternatives in the statement of the theorem holds.

<u>Definition 5.6</u> The alternatives of Theorem 5.5 will be referred to as the "cases for g" or "cases for the class of g." For example, we will say that case i' occurs for g if $C(g)/O_{3'}(C(g)) \simeq \langle g \rangle \times A_3$ and if the fragment of the character table of G corresponding to $B_0(3)$ and the 3-section of g is that of case i' of Theorem 5.5.

3. A single conjugacy class of 3-elements

For a simple group of order $2^a q^b 3^2$ with a single conjugacy class of 3-elements, Theorem 5.5 gives the fragment of the character table corresponding to the principal 3-block and the 3-singular elements. The block-section orthogonality relations and the congruences developed in Chapter IV, Section 2 yield the following equations for the degrees in $B_0(3)$.

Theorem 5.7 Let G be a simple group of order $2^a q^b 3^2$ containing a single conjugacy class of 3-elements. Depending on which case occurs in Theorem 5.5, the degrees f_i of the irreducible characters in the principal 3-block of G satisfy one of the following sets of equations:

Case i: $1 + \sum_{i=2}^{9} \epsilon_i f_i = 0$ $\quad f_i \equiv \epsilon_i \pmod 9$ for $i = 2,3,\ldots,9$

Case i': $1 + \sum_{i=2}^{5} \epsilon_i f_i + 2\epsilon_6 f_6 = 0$ $\quad f_i \equiv \epsilon_i \pmod 9$ for $i = 2,3,4,5$; $f_6 \equiv 2\epsilon_6 \pmod 9$

Case ii: $1 + \sum_{i=2}^{6} \epsilon_i f_i = 0$

$1 + \sum_{i=2}^{3} \epsilon_i f_i + \sum_{i=7}^{9} \epsilon_i f_i = 0$

$f_i \equiv \epsilon_i \pmod 9$ for $i = 2,3$; $f_i \equiv 8\epsilon_i \pmod 9$ for $i = 4,5,6$; $f_i \equiv 2\epsilon_i \pmod 9$ for $i = 7,8,9$

Case iii and Case iv: $1 + \sum_{i=2}^{6} \epsilon_i f_i = 0$

$1 + \sum_{i=2}^{3} \epsilon_i f_i + \sum_{i=7}^{9} \epsilon_i f_i = 0$

$f_i \equiv \epsilon_i \pmod 9$ for $i = 2,3$; $f_i \equiv 5\epsilon_i \pmod 9$ for $i = 4,5,6$; $f_i \equiv 5\epsilon_i \pmod 9$ for $i = 7,8,9$

Case v and Case vi:

$$1 + \sum_{i=2}^{6} \epsilon_i f_i = 0$$

$$1 + \sum_{i=2}^{3} \epsilon_i f_i + \sum_{i=7}^{7} \epsilon_i f_i = 0$$

$f_i \equiv \epsilon_i \pmod 9$ for $i=2,3$
$f_i \equiv 2\epsilon_i \pmod 9$ for $i=4,5,6$
$f_i \equiv 8\epsilon_i \pmod 9$ for $i = 7,8,9$.

In each case, $\epsilon_i = \pm 1$ and $f_i = 2^{x_i} q^{y_i}$ for some integers x_i and y_i.

Proof $PSL_2(8)$ and $PSL_2(17)$ have more than one conjugacy class of 3-elements. Hence Theorem 3.1 and Lemma 5.1 imply that q is 5 or 7 and that G has elementary abelian Sylow 3-subgroups. Theorem 5.5 gives fragments of the character table of G. $\sum_{\chi \in B_0(3)} \chi(1)\,\chi(h) = 0$ for all 3-singular elements h [3, II-7C]. Setting h = g, we obtain the equations for cases i and i'. With h = gv, we obtain the first equation for cases ii, iii, iv, v, and vi; with h = gw, we get the second equation for cases iii, iv, v, and vi. To derive the second equation for case ii, we add the equations obtained with h = g and h = gv and divide the sum by 2.

The congruences follow immediately from Corollary 4.6.

4. More than one conjugacy class of 3-elements

Let G be a simple group of order $2^a q^b 3^2$. Assume that G is not isomorphic to $PSL_2(8)$ or $PSL_2(17)$; then Theorem 5.5 is applicable. If G has more than one class of 3-elements, we must consider how the fragments of the character table corresponding to the different classes fit together. First we make the following definition.

Definition 5.8 Let $\chi_1, \chi_2, \ldots, \chi_n$ be the irreducible characters in the principal 3-block of G. For $h \in G$, $Y(h)$ will denote the n-dimensional column vector with entries $\chi_1(h), \chi_2(h), \ldots, \chi_n(h)$.

Lemma 5.9 Let G be a simple group of order $2^a q^b 3^2$ not isomorphic to $PSL_2(8)$ or $PSL_2(17)$. Assume G has more than one conjugacy class of 3-elements; let g_1 and g_2 be representatives of two such classes. Then:

1) If case i' occurs for g_1, case i' must also occur for g_2.
2) If case iii or case iv occurs for g_1, case v or case vi cannot occur of g_2.
3) $Y(h_1)\cdot Y(h_2) = 0$ (i.e. $Y(h_1)$ and $Y(h_2)$ are orthogonal) whenever h_1 is in the 3-section of g_1 and h_2 is in the 3-section of g_2.
4) $\chi(g_1) \equiv \chi(g_2) \pmod 3$. Hence, if $\chi(g_1) = m_1\epsilon_1$ and $\chi(g_2) = m_2\epsilon_2$, then $\epsilon_2 = \epsilon_1$ provided $m_2 \equiv m_1 \pmod 3$ and $\epsilon_2 = -\epsilon_1$ provided $m_2 \equiv 2m_1 \pmod 3$.

Proof If case i' occurs for g_1, then $B_0(3)$ contains 6 irreducible characters; it follows from Theorem 5.5 that case i' must also occur for g_2.

If case iii or case iv occurs for g_1, then $|G| = 2^a 5^b 3^2$ since A_5 is involved in G. Case v or case vi cannot occur for g_2, since in these cases, $|G| = 2^a 7^b 3^2$.

(3) follows immediately from the block-section orthogonality equation $\sum_{\chi \in B_0(3)} \chi(h_1)\,\chi(h_2) = 0$ [3, II-7C].

By Corollary 4.4, $\chi(g_i) \equiv \chi(1) \pmod 3$ for $i = 1, 2$. Hence (4) holds.

Lemma 5.10 Let G be a simple group of order $2^a q^b 3^2$. Then case i occurs for at most one class of 3-elements.

Proof Suppose case i occurs for g_1 and g_2, where g_1 and g_2 are non-conjugate 3-elements. Theorem 5.5 gives fragments of the character table. For i = 2, 3, ..., 9, there exist signs ϵ_i and ϵ_i' such that $\chi_i(g_1) = \epsilon_i$ and $\chi_i(g_2) = \epsilon_i'$. It follows from Lemma 5.9, Part 4 that $\epsilon_i = \epsilon_i'$ for all i. $Y(g_1)\cdot Y(g_2) = 1 + \sum_{2}^{9} \epsilon_i\epsilon_i' = 9$, contrary to Lemma 5.9, Part 3.

If fragments corresponding to different classes of 3-elements do fit together in such a way that the conditions of Lemma 5.9 are satisfied, we may obtain diophantine equations for the degrees f_i in $B_0(3)$ by setting $Y(1)\cdot[\sum_i a_i Y(h_i)] = 0$ for any real numbers a_i and 3-singular elements h_i (not necessarily in different 3-sections). Also, congruences may be obtained from Corollary 4.6.

In each of cases iii, iv, v, and vi, the character table has a fragment of the form

	g	gv	gw
χ_1	1	1	1
χ_2	ϵ_2	ϵ_2	ϵ_2
χ_3	ϵ_3	ϵ_3	ϵ_3
χ_4	$-\beta\epsilon_4$	ϵ_4	0
χ_5	$-\beta\epsilon_5$	ϵ_5	0
χ_6	$-\beta\epsilon_6$	ϵ_6	0
χ_7	$(\beta+1)\epsilon_7$	0	ϵ_7
χ_8	$(\beta+1)\epsilon_8$	0	ϵ_8
χ_9	$(\beta+1)\epsilon_9$	0	ϵ_9

with $\beta = 4$ in cases iii and iv and $\beta = 7$ in cases v and vi. In case ii, the character table almost has a fragment of this form with $\beta = 1$; only the column of gw is missing. We may obtain a column vector identical to the column of gw by taking one half the sum of the columns of g and of gv; this column is orthogonal to any column of another 3-section. Hence we shall assume that, in case ii, there is a fragment of the form just given with $\beta = 1$; the "column of gw" will be used only in orthogonality relations.

In considering how fragments corresponding to different classes of 3-elements can fit together, we need to consider the various fusion patterns (Theorem 4.1) and, for a given fusion pattern, the various cases (Theorem 5.5) which occur for each class of 3-elements. The following notation will be useful.

<u>Definition 5.11</u> Let $z \in \{a,b,c,d\}$ and let $x_j \in \{i,i',ii,iii,iv,v,vi\}$ for $j = 1, 2, \ldots, n$. A simple group of order $2^a q^b 3^2$, not isomorphic to $PSL_2(8)$ or $PSL_2(17)$, will be said to be of <u>type z-$x_1, x_2, \ldots, x_n$</u> provided that fusion pattern z (Theorem 4.1) occurs and that cases $x_1, x_2, \ldots,$ and x_n respectively occur for the n conjugacy classes of 3-elements (n must be 1, 2, 3, or 4 according as to whether z is a, b c, or d).

For example, G is of type b-i, iv if alternative b of Theorem 4.1 occurs and if case i occurs for one class of 3-elements and case iv for the other.

5. Two conjugacy classes of 3-elements

Throughout this section, G will denote a simple group of order $2^a q^b 3^2$ with exactly two conjugacy classes of 3-elements. Representatives of the two classes will be denoted by g and g'. A "prime" following a symbol will indicate that the object is associated with the second class; for example, if case iii occurs for the second class, the elements v and w of Theorem 5.5 will be written v' and w'.

We shall treat separately the following three alternatives:

1) Case i' occurs for both classes, that is, G is of type b-i',i'.

2) Case i occurs for one class and case ii, iii, iv, v, or vi occurs for the other.

3) A case numbered ii or higher occurs for each class.

Theorem 5.12 Let G be a simple group of order $2^a q^b 3^2$ with two classes of 3-elements. If case i' occurs for both classes, the character table has the following fragment:

	1	g	g'
χ_1	1	1	1
χ_2	f_2	ϵ_2	ϵ_2
χ_3	f_3	ϵ_3	ϵ_3
χ_4	f_4	ϵ_4	ϵ_4
χ_5	f_5	$-\epsilon_5$	$2\epsilon_5$
χ_6	f_5	$2\epsilon_5$	$-\epsilon_5$

The degrees in $B_o(3)$ satisfy the conditions:

$$1 + \epsilon_2 f_2 + \epsilon_3 f_3 + \epsilon_4 f_4 + \epsilon_5 f_5 = 0$$

$$f_i \equiv \epsilon_i \pmod 9 \text{ for } i = 2,3,4, \quad f_5 \equiv 5\epsilon_5 \pmod 9).$$

Proof Using Lemma 5.9, Parts 3 and 4, we obtain easily that there is a fragment of the type:

	1	g_1	g_2
χ_1	1	1	1
χ_2	f_2	ϵ_2	ϵ_2
χ_3	f_3	ϵ_3	ϵ_3
χ_4	f_4	ϵ_4	ϵ_4
χ_5	f_5	$-\epsilon_5$	$2\epsilon_5$
χ_6	f_6	$2\epsilon_6$	$-\epsilon_6$

$Y(1)\cdot[Y(g_1)-Y(g_2)] = 0$; hence $\epsilon_5 f_5 - \epsilon_6 f_6 = 0$. Since f_5 and f_6 are positive, ϵ_5 and ϵ_6 are both positive or both negative; hence $\epsilon_5 = \epsilon_6$. It follows that $f_5 = f_6$, and the character table has a fragment of the type in the statement of the theorem.

$Y(1)\cdot[Y(g_1)+Y(g_2)] = 0$; hence we obtain the equation in the theorem. The congruences follow immediately from Corollary 4.6.

Theorem 5.13 There are no simple groups of order $2^a q^b 3^2$ with two classes of 3-elements, in which case i occurs for one class and case ii, iii, iv, v, or vi occurs for the other class.

Proof Assume that G is a group of this type. We may assume that case i occurs for the second class. It follows from Lemma 5.9, Part 4 that the character table has a fragment:

	1	g	gv	gw	g'
χ_1	1	1	1	1	1
χ_2	f_2	ϵ_2	ϵ_2	ϵ_2	ϵ_2
χ_3	f_3	ϵ_3	ϵ_3	ϵ_3	ϵ_3
χ_4	f_4	$-\beta\epsilon_4$	ϵ_4	0	$-\epsilon_4$
χ_5	f_5	$-\beta\epsilon_5$	ϵ_5	0	$-\epsilon_5$
χ_6	f_6	$-\beta\epsilon_6$	ϵ_6	0	$-\epsilon_6$
χ_7	f_7	$(\beta+1)\epsilon_7$	0	ϵ_7	$-\epsilon_7$
χ_8	f_8	$(\beta+1)\epsilon_8$	0	ϵ_8	$-\epsilon_8$
χ_9	f_9	$(\beta+1)\epsilon_9$	0	ϵ_9	$-\epsilon_9$

Here β is 1 if case ii occurs for g, 4 if case iii or case iv occurs for g, and 7 if case v or case vi occurs for g. $1 + \epsilon_2 f_2 + \epsilon_3 f_3 = 0$ since $Y(1)\cdot[Y(gv)+Y(gw)+Y(g')] = 0$. On the other hand, Corollary 4.6 implies $f_i \equiv \epsilon_i \pmod 9$ for i = 2, 3; hence $1 + \epsilon_2 f_2 + \epsilon_3 f_3 \equiv 3 \pmod 9$.

<u>Theorem 5.14</u> Let G be a simple group of order $2^a q^b 3^2$ with two conjugacy classes of 3-elements. Suppose that a case numbered ii or higher occurs for each class. Then the character table of G has a fragment:

	1	g	gv	gw	g'	g'v'	g'w'
χ_1	1	1	1	1	1	1	1
χ_2	f_2	ϵ_2	ϵ_2	ϵ_2	$\beta'\epsilon_2$	$-\epsilon_2$	0
χ_3	f_3	ϵ_3	ϵ_3	ϵ_3	$-(\beta'+1)\epsilon_3$	0	[illegible]
χ_4	f_4	$-\beta\epsilon_4$	ϵ_4	0	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	ϵ_5	0
χ_6	[illegible]	[illegible]	ϵ_6	0	$(\beta'+1)\epsilon_6$	0	ϵ_6
χ_7	f_7	$(\beta+1)\epsilon_7$	0	ϵ_7	$-\epsilon_7$	$-\epsilon_7$	$-\epsilon_7$
χ_8	f_8	$(\beta+1)\epsilon_8$	0	ϵ_8	$-\beta'\epsilon_8$	ϵ_8	0
χ_9	f_9	$(\beta+1)\epsilon_9$	0	ϵ_9	$(\beta'+1)\epsilon_9$	0	ϵ_9

Here β is 1 if case ii occurs for the first class, 4 if case iii or case iv occurs, and 7 if case v or case vi occurs; β' depends in the same way on which case occurs for the second class. The following equations and congruences hold:

$$1 + \epsilon_2 f_2 + \epsilon_6 f_6 + \epsilon_9 f_9 = 0$$
$$1 - \epsilon_4 f_4 + \epsilon_8 f_8 + \epsilon_9 f_9 = 0$$
$$1 + \epsilon_5 f_5 + \epsilon_6 f_6 - \epsilon_7 f_7 = 0$$
$$1 + \epsilon_3 f_3 + \epsilon_5 f_5 + \epsilon_8 f_8 = 0$$

$$\begin{aligned} f_2 &\equiv 5(1+\beta')\epsilon_2 \\ f_3 &\equiv 4\beta'\epsilon_3 \\ f_4 &\equiv 4(1+\beta)\epsilon_4 \\ f_5 &\equiv 4(\beta+\beta')\epsilon_5 \\ f_6 &\equiv 5(1-\beta+\beta')\epsilon_6 \\ f_7 &\equiv 5\beta\epsilon_7 \\ f_8 &\equiv 5(1+\beta-\beta') \\ f_9 &\equiv 5(2+\beta+\beta'). \end{aligned}$$

Proof $\chi_1, \chi_2, \ldots, \chi_9$ are the characters in $B_0(3)$. By convention, χ_1 is always the identity character. $\{1,2,3,4,5,6,7,8,9\}$ may be partitioned into three sets X_1, X_2, and X_3 as follows:

$$X_1 = \{\ i: |\chi_i(g)| = 1,\ \chi \text{ is constant on the 3-section of } g\ \}$$
$$X_2 = \{\ i: |\chi_i(g)| = \beta,\ \chi \text{ is not constant on the 3-section of } g\ \}$$
$$X_3 = \{\ i: |\chi_i(g)| = \beta + 1\ \}.$$

$\{1,2,3,4,5,6,7,8,9\}$ may also be partitioned into sets X_1', X_2', and X_3', defined as above except with β' and g' replacing β and g.

$|X_k| = |X_k'| = 3$ for $k = 1, 2, 3$. Let $u_{jk} = |X_j \cap X_k'|$ for $j,k = 1,2,3$. The matrix $U = (u_{jk})$ is a 3×3 matrix of nonnegative integers in which each row sum and each column sum is 3; also $u_{11} > 0$. The key observation is the following: The matrix U completely determines how the fragments corresponding to the sections of g and g' fit together, that is, the matrix U determines, apart from the order of the rows and from signs $\epsilon_2, \ldots, \epsilon_9$, the fragment of the character table corresponding to $B_o(3)$ and the 3-singular elements. In particular, given U, we can compute $Y(h) \cdot Y(h')$ whenever h is in the 3-section of g and h' is in the 3-section of g'. The only matrix U for which all these inner products vanish is the matrix in which each entry is 1. In this case, we may assume the sets are:

$$X_1 = \{1,2,3\} \qquad X_2 = \{4,5,6\} \qquad X_3 = \{7,8,9\}$$
$$X_2' = \{1,4,7\} \qquad X_2' = \{2,5,8\} \qquad X_3' = \{3,6,9\}.$$

We obtain the fragment given in the statement of the theorem. The four equations follow respectively from:

$$Y(1) \cdot [Y(gv)+Y(gw)-Y(g'v')+2Y(g'w')] = 0$$
$$Y(1) \cdot [-Y(gv)+2Y(gw)+Y(g'v')+Y(g'w')] = 0$$
$$Y(1) \cdot [2Y(gv)-Y(gw)+Y(g'v')+Y(g'w')] = 0$$
$$Y(1) \cdot [Y(gv)+Y(gw)+2Y(g'v')-Y(g'w')] = 0.$$

Finally, the congruences are a direct consequence of Corollary 4.6.

The four equations of Theorem 5.14 contain all the information obtainable from the orthogonality relations; any degrees satisfying the four equations will satisfy all the orthogonality relations.

6. Three conjugacy classes of 3-elements

Theorem 5.15 There are no simple groups of order $2^a q^b 3^2$ containing exactly 3 conjugacy classes of 3-elements

Proof In view of Lemmas 5.9 and 5.10, it suffices to consider the following cases:

1) Case i' occurs for each class.
2) Case i occurs for one class and cases ii, iii, iv, v or vi occur for the other classes.
3) Cases ii, iii, iv, v, or vi occur for each class.

Assume case i' occurs for each class. By Lemma 5.9, parts 3 and 4, the character table has a fragment:

	1	g	g'	g"
χ_1	1	1	1	1
χ_2	f_2	ϵ_2	ϵ_2	ϵ_2
χ_3	f_3	ϵ_3	ϵ_3	ϵ_3
χ_4	f_4	ϵ_4	ϵ_4	$-2\epsilon_4$
χ_5	f_5	ϵ_5	$-2\epsilon_5$	ϵ_5
χ_6	f_6	$-2\epsilon_6$	ϵ_6	ϵ_6

$Y(1)\cdot[Y(g)+Y(g')+Y(g")] = 0$; hence $1 + \epsilon_2 f_2 + \epsilon_3 f_3 = 0$. On the other hand, Corollary 4.6 implies $f_i \equiv \epsilon_i \pmod 9$ for $i = 2, 3$; hence $1 + \epsilon_2 f_2 + \epsilon_3 f_3 \equiv 3 \pmod 9$.

Assume that a case numbered ii or higher occurs for the first and second classes and that case i occurs for the third class. The fragments for the second and third classes must fit together as in Theorem 5.13 (The assumption that G had only two conjugacy classes of

3-elements was not used in deriving the fragment given in Theorem 5.13, although it was needed to obtain the congruences). As in the proof of Theorem 5.13, block-section orthogonality yields the equation: $1 + \epsilon_2 f_2 + \epsilon_3 f_3 = 0$. This contradicts Lemma 5.2.

Finally, assume that a case numbered ii or higher occurs for each of the 3 classes. The fragments corresponding to any two of the classes must fit together as in Theorem 5.14. As in the proof of Theorem 5.14, we define the sets X_1, X_2, X_3, the sets X_1', X_2', X_3', and the sets X_1'', X_2'', X_3'' (primed objects are associated with the second class and double-primed objects with the third class). As in Theorem 5.14, no generality is lost in assuming:

$$X_1 = \{1,2,3\} \quad X_2 = \{4,5,6\} \quad X_3 = \{7,8,9\}$$
$$X_1' = \{1,4,7\} \quad X_2' = \{2,5,8\} \quad X_3' = \{3,6,9\}.$$

For i = 1, 2, 3, the set X_i'' must be such that $|X_i'' \cap X_j| = 1$ and $|X_i'' \cap X_j'| = 1$ (j = 1, 2, 3). Also, $1 \in X_1''$. There are four alternatives for the sets X_1'', X_2'', X_3'', namely:

Alternative	X_1''	X_2''	X_3''
1	{1,5,9}	{2,6,7}	{3,4,8}
2	{1,5,9}	{3,4,8}	{2,6,7}
3	{1,6,8}	{2,4,9}	{3,5,7}
4	{1,6,8}	{3,5,7}	{2,4,9}

For each alternative, we obtain a fragment of the character table of the group corresponding to $B_0(3)$ and the 3-singular elements. For alternative 1, the fragment is:

	1	g	gv	gw	g'	g'v'	g'w'	g"	g"v"	g"w"
x_1	1	1	1	1	1	1	1	1	1	1
x_2	f_2	ϵ_2	ϵ_2	ϵ_2	$\beta'\epsilon_2$	$-\epsilon_2$	0	$\beta''\epsilon_2$	$-\epsilon_2$	0
x_3	f_3	ϵ_3	ϵ_3	ϵ_3	$-(\beta'+1)\epsilon_3$	0	$-\epsilon_3$	$-(\beta''+1)\epsilon_3$	0	$-\epsilon_3$
x_4	f_4	$-\beta\epsilon_4$	ϵ_4	0	$-\epsilon_4$	$-\epsilon_4$	$-\epsilon_4$	$(\beta''+1)\epsilon_4$	0	ϵ_4
x_5	f_5	$-\beta\epsilon_5$	ϵ_5	0	$-\beta'\epsilon_5$	ϵ_5	0	$-\epsilon_5$	$-\epsilon_5$	$-\epsilon_5$
x_6	f_6	$-\beta\epsilon_6$	ϵ_6	0	$(\beta'+1)\epsilon_6$	0	ϵ_6	$-\beta''\epsilon_6$	ϵ_6	0
x_7	f_7	$(\beta+1)\epsilon_7$	0	ϵ_7	$-\epsilon_7$	$-\epsilon_7$	$-\epsilon_7$	$-\beta''\epsilon_7$	ϵ_7	0
x_8	f_8	$(\beta+1)\epsilon_8$	0	ϵ_8	$-\beta'\epsilon_8$	ϵ_8	0	$(\beta''+1)\epsilon_8$	0	ϵ_8
x_9	f_9	$(\beta+1)\epsilon_9$	0	ϵ_9	$(\beta'+1)\epsilon_9$	0	ϵ_9	$-\epsilon_9$	$-\epsilon_9$	$-\epsilon_9$

Here β is 1 if case ii occurs for the first class, 4 if case iii or case iv occurs, and 7 if case v or case vi occurs. β' and β'' depend similarly on which case occurs for the second and third classes. The orthogonality relation $Y(1)\cdot[Y(gw)+Y(g'v')+Y(g''w'')] = 0$ yields $1 + \epsilon_8 f_8 = 0$; hence $f_8 = 1$, contrary to the assumption that the group is simple.

The other alternatives are handled similarly. For alternative 2, the orthogonality relation $Y(1)\cdot[Y(gv)+Y(g'w')+Y(g''w'')] = 0$ yields the equation $1 + \epsilon_6 f_6 = 0$. For alternative 3, the relation $Y(1)\cdot[Y(gv)+Y(g'v')+Y(g''w'')] = 0$ implies that $1 + \epsilon_5 f_5 = 0$. For alternative 4, $Y(1)\cdot[Y(gw)+Y(g'w')+Y(g''w'')] = 0$ implies $1 + \epsilon_9 f_9 = 0$.

7. Summary of the degree equations

Thus far, we have proven that a simple group of order $2^a q^b 3^2$, not isomorphic to $PSL_2(8)$ or $PSL_2(17)$, has one or two conjugacy classes of 3-elements. Theorems 5.7, 5.12, and 5.14 give the degree equations; a summary of the equations follows.

Type a-i: $1 + \sum_{i=2}^{9} \epsilon_i f_i = 0 \qquad f_i \equiv \epsilon_i \pmod 9$

Type a-i': $1 + \epsilon_2 f_2 + \epsilon_3 f_3 + \epsilon_4 f_4 + \epsilon_5 f_5 + 2\epsilon_6 f_6 = 0$

$f_i \equiv \epsilon_i \pmod 9$ for $i = 2, 3, 4, 5$
$f_6 \equiv 2\epsilon_6 \pmod 9$

Types a-ii, a-iii, a-iv, a-v, and a-vi:

$$1 + \epsilon_2 f_2 + \epsilon_3 f_3 + \epsilon_4 f_4 + \epsilon_5 f_5 + \epsilon_6 f_6 = 0$$

$$1 + \epsilon_2 f_2 + \epsilon_3 f_3 + \epsilon_7 f_7 + \epsilon_8 f_8 + \epsilon_9 f_9 = 0$$

$f_i \equiv \epsilon_i \pmod 9$ for $i = 2, 3$
$f_i \equiv n\epsilon_i \pmod 9$ for $i = 4, 5, 6$
$f_i \equiv n'\epsilon_i \pmod 9$ for $i = 7, 8, 9$

n is 8 for a-ii, 5 for a-iii and a-iv, and 2 for a-v and a-vi

n' is 2 for a-ii, 5 for a-iii and a-iv, and 8 for a-v and a-vi

Type b-i',i': $1 + \epsilon_2 f_2 + \epsilon_3 f_3 + \epsilon_4 f_4 + \epsilon_5 f_5 = 0$

$f_i \equiv \epsilon_i \pmod 9$ for $i = 2, 3, 4, 5$
$f_6 \equiv 5\epsilon_6 \pmod 9$

Types b-ii,ii; b-ii,iii; b-ii,iv; b-ii,v; b-ii,vi; b-iii,iii; b-iii,iv; b-iv,iv; b-v,v; b-v,vi; and b-vi,vi:

$$1 + \epsilon_2 f_2 + \epsilon_6 f_6 + \epsilon_9 f_9 = 0$$

$$1 + (-\epsilon_4) f_4 + \epsilon_8 f_8 + \epsilon_9 f_9 = 0$$

$$1 + \epsilon_5 f_5 + \epsilon_6 f_6 + (-\epsilon_7) f_7 = 0$$

$$1 + \epsilon_3 f_3 + \epsilon_5 f_5 + \epsilon_8 f_8 = 0$$

$f_i \equiv n_i \epsilon_i \pmod 9$ for $i = 2, 3, 5, 6, 8, 9$
$f_i \equiv n_i(-\epsilon_i) \pmod 9$ for $i = 4, 7$

n_i as given in the table below

	b-ii,ii	b-ii,iii b-ii,iv	b-ii,v b-ii,vi	b-iii,iii b-iii,iv b-iv,iv	b-v,v b-v,vi b-vi,vi
n_2	1	7	4	7	4
n_3	4	7	1	7	1
n_4	1	1	1	7	4
n_5	8	2	5	5	2
n_6	5	2	8	5	5
n_7	4	4	4	7	1
n_8	5	8	2	5	5
n_9	2	8	5	5	8

CHAPTER VI. SIMPLE GROUPS OF ORDER $2^a q^b 3^2$: SOLVING THE DEGREE EQUATIONS

In this chapter, we shall develop methods for solving the degree equations obtained in Chapter V. It will turn out that many, if not all, of the equations have an "essentially" finite number of solutions.

1. The equations

Let Π be a fixed, finite set of primes. A Π-number will be a nonzero integer each of whose prime factors lies in Π. Let $m_2, m_3, \ldots, m_k$ be fixed nonzero integers. We shall consider the equation:

$$1) \qquad 1 + m_2 f_2 + m_3 f_3 + \ldots + m_k f_k = 0$$

$$f_2, f_3, \ldots, f_k \text{ unknown } \Pi\text{-numbers.}$$

In addition to equation (1), we may have congruences for the unknowns:

$$2) \qquad f_i \equiv c_i \pmod{z} \quad \text{for } i = 2, 3, \ldots, k.$$

Here $z, c_2, \ldots, c_k$ are fixed integers. In fact, we may assume that we always have the congruences (2), for they hold trivially with $z = 1$, $c_2 = \ldots = c_k = 0$.

The methods to be developed are applicable for any finite set Π of primes; however, for simplicity, we shall assume $|\Pi| = 2$ and denote the primes in Π by p and q. The f_i of equation (1) can be written as $p^{x_i} q^{y_i}$; the unknowns become $x_2, \ldots, x_k, y_2, \ldots, y_k$.

Each of the degree equations in Chapter V, Section 7 breaks up into a finite number of equations of the form (1); one equation of this type is obtained for each fixed choice of the signs ϵ_i. We also have congruences of the form (2) with $z = 9$.

Equation (1) and the congruences (2) may have an infinite number of solutions. Suppose that $(f_2, f_3, \ldots, f_k)$ is any solution for which there exists a proper subset S of $\{2, 3, \ldots, k\}$ such that $1 + \sum_{i \in S} m_i f_i = 0$. Let S^c denote the complement of S in $\{2, 3, \ldots, k\}$. Then $1 + \sum_{i \in S} m_i f_i + \sum_{i \in S^c} m_i(\gamma f_i) = 0$ for any integer γ. Thus we obtain a solution to equation (1) for each integer γ of the form $p^x q^y$; if $\gamma \equiv 1 \pmod{z}$, the congruences (2) also hold.

The author conjectures that this is the only way in which the number of solutions may be infinite. To make this statement more precise, we define a solution $(f_2, f_3, \ldots, f_k)$ to equation (1) to be <u>basic</u> if, for each proper subset S of $\{2, 3, \ldots, k\}$, $1 + \sum_{i \in S} m_i f_i \neq 0$. The conjecture then becomes: Equation (1) has only a finite number of basic solutions.

We shall show that the conjecture is valid for a number of specific choices of the constants p, q, k, $m_2, \ldots, m_k$. Also, in Section 3, it will be proven that equation (1) has only a finite number of basic solutions for which $x_i \leq M$ for all i, M being any fixed integer.

In many cases, the congruences (2) imply that every solution of (1) is basic. For example, in the equation $1 + \sum_2^9 \epsilon_i f_i = 0$ with the congruences $f_i \equiv \epsilon_i \pmod{9}$, every solution is basic; for if S is a proper subset of $\{2, 3, \ldots, 9\}$, $1 + \sum_S \epsilon_i f_i \equiv |S| + 1 \pmod{9}$.

2. The method of congruences

In this section, we shall present a method of determining all solutions of equation (1) and the congruences (2).

The quantity $1 + m_2 p^{x_2} q^{y_2} + m_3 p^{x_3} q^{y_3} + \ldots + m_k p^{x_k} q^{y_k}$ will be denoted by (*). The key to applying congruences is the following elementary observation: If $p^t \equiv q^{t'} \equiv 1 \pmod r$, then (*) can be computed modulo r from the values of $x_2, x_3, \ldots, x_k$ modulo t and of $y_2, y_3, \ldots, y_k$ modulo t'.

There are two stages to the procedure. In the first stage, we determine all possibilities for the exponents $x_2, x_3, \ldots, x_k$ modulo N and the exponents $y_2, y_3, \ldots, y_k$ modulo N', where N and N' are appropriately chosen integers; in the second, we determine the exponents absolutely.

For the first stage, we need three sequences as follows:

i) A sequence $\{u_i\}_{i=0}^{n}$ of positive integers such that $u_0 = 1$, $u_n = N$, and $u_i | u_{i+1}$ for all i.

ii) A sequence $\{v_i\}_{i=0}^{n}$ of positive integers such that $v_0 = 1$, $v_n = N'$, and $v_i | v_{i+1}$ for all i.

iii) A sequence $\{\mathcal{P}_i\}_{i=1}^{n}$, where $\mathcal{P}_i$ is a set of primes with the property that $p^{u_i} \equiv q^{v_i} \equiv 1 \pmod r$ for all $r \in \mathcal{P}_i$.

The process is inductive. Let $1 \leq J \leq n$. Assume that, after step J-1, we know all possibilities for the x_i modulo u_{J-1} and the y_i modulo v_{J-1} -- that is, assume that we have determined an integer L_{J-1} and a set $\{(x_{2s},x_{3s},\ldots,x_{ks},y_{2s},y_{3s},\ldots,y_{ks}) \mid s = 1,2,\ldots,L_{J-1}\}$ such that, if $(x_2,\ldots,x_k,y_2,\ldots,y_k)$ is any solution of (1), there exists an integer s such that $x_i \equiv x_{is} \pmod{u_{J-1}}$ and $y_i \equiv y_{is} \pmod{v_{J-1}}$. We shall say that we have L_{J-1} cases for the exponents, each case corresponding to a value of s. In step J, each case is divided into subcases according to the values of the x_i modulo u_J and the y_i modulo v_J. Each case divides into $(u_J/u_{J-1})^{k-1}(v_J/v_{J-1})^{k-1}$ subcases as x_i ranges over the values x_{is}, $x_{is} + u_{J-1}$, ..., $x_{is} + (u_J/u_{J-1} - 1)u_{J-1}$ and y_i over the values y_{is}, $y_{is} + v_{J-1}$, ..., $y_{is} + (v_J/v_{J-1} - 1)v_{J-1}$. For each subcase, (*) is computed modulo each prime in $\mathcal{P}_J$; if a nonzero quantity is obtained for any one of the primes, the subcase is discarded. This completes step J; the subcases which remain form the set of cases with which we begin step J+1 unless J = n, in which case the remaining subcases give all possibilities for the exponents x_i modulo N and the exponents y_i modulo N'.

Of course, some of the cases may not correspond to actual solutions. To obtain meaningful results, it is necessary to produce enough primes in the sets $\mathcal{P}_j$ so that all or nearly all of the extraneous cases are eliminated.

In the second stage, congruences modulo powers of p and q are used to determine some or all of the exponents absolutely. We shall describe a check modulo q^β. Assume that stage 1 has been performed for integers N and N' such that $p^N \equiv 1 \pmod{q^\beta}$ and $N' \geq \beta$. We have a set $\{(x_{2s},\ldots,x_{ks},y_{2s},\ldots,y_{ks}) \mid s = 1,2,\ldots,L_n\}$ such that, if $(x_2,\ldots,x_k,y_2,\ldots,y_k)$ is any solution of (1) and (2), there exists s such that $x_i \equiv x_{is} \pmod{N}$ and $y_i \equiv y_{is} \pmod{N'}$ for i = 2, 3, ..., k.

Suppose that $(x_2,\ldots,x_k,y_2,\ldots,y_k)$ is any solution. For some s, we may write $y_i = y_{is} + t_iN'$ for $i = 2, 3, \ldots, n$. p^{x_i} may be computed modulo q^β; in fact, $p^{x_i} \equiv p^{x_{is}} \pmod{q^\beta}$. $q^{y_i} \equiv q^{y_{is}} \pmod{q^\beta}$ if $t_i = 0$ and $q^{y_i} \equiv 0 \pmod{q^\beta}$ if $t_i > 0$. Thus, for a given case (i.e. for a given value of s), there are at most 2^{k-1} possibilities for (*) modulo q^β, depending on which of the integers $t_2, t_3, \ldots, t_k$ are zero. If none of the alternatives for (*) is zero, the case is discarded. If a zero value for (*) is obtained only for alternatives in which $t_i = 0$, we have shown that $y_i = y_{is}$, that is, we have determined the exponent x_i absolutely (for the case numbered s). Frequently, a check modulo q^β for sufficiently high β suffices to determine $y_2, y_3, \ldots, y_k$ absolutely. To determine the y_i absolutely in case s, it is necessary that β be greater than the maximum of the integers
$y_{2s}, y_{3s}, \ldots, y_{ks}$; further discussion of this point appears is [22-Section 3].

Similarly, a check modulo p^γ may be used to determine some or all of the exponents $x_2, x_3, \ldots, x_k$ absolutely. If $y_2, y_3, \ldots, y_k$ are already known absolutely, q^{y_i} may be computed absolutely for $i = 2, 3, \ldots, k$. Hence we may use any γ for which $N \geq \gamma$.

Having outlined the method of congruences, we now present some modifications. The congruences (2) may be used to eliminate cases in the first stage of the procedure. Let e_p and e_q be the exponents of p and q modulo z respectively. Let J be the smallest subscript such that $e_p | u_J$ and $e_q | v_J$. Then, in step J, we may check the conditions $p^{x_i}q^{y_i} \equiv c_i \pmod{z}$ for $i = 2, 3, \ldots, k$ and discard any subcase for which one or more of the congruences fails.

In equation (1), the first term is 1; it follows that there must be another term relatively prime to p and another term relatively prime to q. Thus there exist j and j' such that $x_j = y_{j'} = 0$; this reduces the number of subcases into which each case is divided in step J to $(u_J/u_{J-1})^{k-2}(v_J/v_{J-1})^{k-2}$. Symmetry assumptions may be used to reduce the number of cases further.

The method of congruences is easily programmed on a computer. The steps are represented by nested "do" loops. The number of terms in the equation must be reasonably small -- no greater than six or seven -- in order for the program to run in a matter of minutes. Also, in order to determine all solutions absolutely, the number of solutions must in fact be finite; in particular, there must be no non-basic solutions.

2. A second method for solving the degree equations

In this section, we shall present a method for determining all basic solutions of equation (1) and the congruences (2) for which $x_i \leq M$, M being some fixed integer, and y_i is arbitrary $(i = 2,3,\ldots,k)$. This method will be used in cases in which the method of congruences breaks down.

Lemma 6.1 Let M be a nonnegative integer. The equation

$$1 + m_2p^{x_2}q^{y_2} + m_3p^{x_3}q^{y_3} + \ldots + m_kp^{x_k}q^{y_k} = 0$$

has only a finite number of basic solutions for which $x_i \leq M$ for all i.

Proof If j is a nonzero integer, let $\rho(j)$ denote the power to which p occurs in the prime factorization of j and let $\sigma(j)$ denote the power to which q occurs. For $i = 2, 3, \ldots, k$, let $S_i = \{m_i p^x q^y \mid x \leq M,$ x and y are nonnegative integers$\}$. Let S be the union of $S_2, S_3, \ldots,$ and S_k. Rearranging the terms, we may write the equation as:

i) $1 + w_2 + w_3 + \ldots + w_k = 0, \quad 1 + \sum_{i \in T} w_i \neq 0$ if $T \not\subset \{2,..,k\}$

ii) $w_i \in S$ for all i

iii) $\sigma(w_2) \leq \sigma(w_3) \leq \ldots \leq \sigma(w_k)$

iv) There is a permutation β of $\{2, 3, \ldots, k\}$ such that $w_i \in S_{\beta(i)}$ for each i.

We shall show that the number of solutions is finite even without the last condition.

The elements of S may be arranged in a sequence $\{s_i\}_{i=1}^{\infty}$ such that $\sigma(s_1) \leq \sigma(s_2) \leq \sigma(s_3) \leq \ldots,$ for there are only a finite number of elements of S for which σ takes on values less than any fixed integer. Define a function h on the nonnegative integers by setting $h(j)$ equal to the largest integer i for which $\sigma(s_i) \leq j$.

Let $\Sigma_i = \{(1,w_2,\ldots,w_i) \mid$ there exist $w_{i+1},\ldots,w_k$ such that $w_2,w_3,\ldots,w_k$ satisfies i, ii, and iii above$\}$; Σ_i is defined for $i = 1, 2, \ldots, k$. Clearly Σ_1 is finite; our objective is to show that Σ_k is finite. Assume by induction that Σ_i is finite. Suppose $(1,w_2,\ldots,w_i,w_{i+1}) \in \Sigma_{i+1}$. Then $(1,w_2,\ldots,w_i) \in \Sigma_i$. Let $v = 1 + w_2 + \ldots + w_i$. $v \neq 0$ by condition (i) above. If $(w_2,\ldots,w_k)$ satisfies i, ii, and iii above, then $\sigma(w_{i+1} + \ldots + w_k) = \sigma(-v) = \sigma(v)$; by condition (iii), $\sigma(w_{i+1}) \leq \sigma(v)$; hence $w_{i+1} = s_j$ for some j such that $j \leq h(\sigma(v))$. Thus there are only a finite number of possibilities for an element $(1,w_2,\ldots,w_{i+1})$ in Σ_{i+1} for which $(1,w_2,\ldots,w_i)$ is a given element of Σ_i. Since Σ_i is finite, so is Σ_{i+1}. By induction

Σ_k is finite.

The proof of Lemma 6.1 gives a constructive method for obtaining the solutions. In place of the infinite sequence $\{s_i\}_{i=1}^{\infty}$, we use a sufficiently long finite sequence. The sets Σ_i are constructed by setting $\Sigma_{j+1} = \{(1,w_2,\ldots,w_j,w_{j+1}) \mid (1,w_2,\ldots,w_j) \in \Sigma_j$ and $w_{j+1} = s_n$ for some n with $r \leq n \leq h(\sigma(1+w_2+\ldots+w_j))$, where r is the integer such that $w_j = s_r\}$. Finally, we obtain the solutions by selecting those elements of Σ_k for which $1 + w_2 + \ldots + w_k = 0$.

The method outlined above is easily programmed for a computer and turns out to be very efficient. Even if a bound for the exponents y_i is given, considerable computer time can be saved by ordering the possible values for the unknowns as in Lemma 6.1.

4. The solutions

In this section, solutions are given for degree equations obtained in Chapter V for simple groups of order $2^a q^b 3^2$ with two classes of 3-elements. When the method of congruences is used, the sequences $\{u_i\}$, $\{v_i\}$, $\{\theta_i\}$ are given, as are the powers of 2 and q used at the final stage to determine the exponents absolutely. For the case of a single class of 3-elements, solutions will be given in Chapter VII. For future reference, the equations which follow will be labeled A, B, C, D, E, F, and G.

(A) $1 + \epsilon_2 f_2 + \epsilon_3 f_3 + \epsilon_4 f_4 + \epsilon_5 f_5 = 0$, $f_i = 2^{x_i} q^{y_i}$ with $q = 5$ or 7

$f_i \equiv \epsilon_i \pmod 9$ for $i = 2,3,4$ $\quad f_5 \equiv 5\epsilon_5 \pmod 9$

i	u_i	v_i	θ_i
0	1	1	
1	6	6	7 (if q=5)
2	12	12	13, 5 (if q=7)
3	36	36	19, 37
4	72	144	17, 73
5	144	288	97
6	288	576	577, 193
7	288	1152	1153
8	288	2304	257

(Check congruences between f_i and ϵ_i)

Check sum modulo 2^{10} and then modulo q^5

Solutions to (A) for q = 5

1) $1 + 1 + 1 + 1 - 4 = 0$
2) $1 + 1 + 1 - 8 + 5 = 0$
3) $1 + 1 - 8 + 10 - 4 = 0$
4) $1 + 1 + 10 - 512 + 500 = 0$
5) $1 + 1 - 800 - 1250 + 2048 = 0$
6) $1 + 64 + 64 - 125 - 4 = 0$
7) $1 - 125 - 512 + 640 - 4 = 0$
8) $1 - 8 - 8 + 10 + 5 = 0$
9) $1 + 10 + 64 - 80 + 5 = 0$
10) $1 - 8 + 100 - 125 + 32 = 0$
11) $1 + 10 + 64 - 125 + 50 = 0$
12) $1 - 8 - 8 + 640 - 625 = 0$
13) $1 + 64 + 100 - 125 - 40 = 0$
14) $1 + 64 - 80 + 640 - 625 = 0$

Solutions to (A) for q = 7

1) $1 + 1 + 1 + 1 - 4 = 0$
2) $1 + 1 - 8 - 8 + 14 = 0$
3) $1 - 8 + 28 + 28 - 49 = 0$
4) $1 - 8 - 8 + 64 - 49 = 0$
5) $1 - 224 - 512 + 784 - 49 = 0$

Note For solution (5) to equation (A) for $q = 5$, checking the sum modulo 2^{10} and 5^5 is not sufficient to determine all the exponents absolutely; however, it does show that $x_2 = y_2 = 0$ and consequently $f_2 = 1$; hence the solution cannot occur for a simple group. Further steps are required to determine all the exponents absolutely.

(B) $1 + \epsilon_a f_a + \epsilon_b f_b + \epsilon_c f_c = 0, \quad f_i = 2^{x_i} q^{y_i}$ for $i = a,b,c$, $q = 5$ or 7

$$f_a \equiv \epsilon_a \pmod 9 \qquad f_b \equiv 2\epsilon_b \pmod 9 \qquad f_c \equiv 5\epsilon_c \pmod 9$$

i	u_i	v_i	θ_i
0	1	1	
1	6	6	
2	12	12	13, 5 (if q=7), 7 (if q = 5)
3	36	36	19, 37
4	72	144	17, 73
5	144	288	97
6	288	576	193, 577

(Check congruences between f_i and ϵ_i)

Check sum modulo 2^8 and then modulo q^4

Solutions to (B) for q = 5

1) $1 + 1 + 2 - 4 = 0$
2) $1 - 8 + 2 + 5 = 0$
3) $1 - 125 + 128 - 4 = 0$
4) $1 + 10 - 16 + 5 = 0$
5) $1 - 8 - 25 + 32 = 0$
6) $1 + 640 - 16 - 625 = 0$
7) $1 + 64 - 25 - 40 = 0$

Solutions to (B) for q = 7

1) $1 + 1 + 2 - 4 = 0$
2) $1 + 1 - 16 + 14 = 0$
3) $1 - 8 - 7 + 14 = 0$
4) $1 - 8 + 56 - 49 = 0$
5) $1 + 64 - 16 - 49 = 0$

$\underline{(C)}$ $\quad 1 + \epsilon_a f_a + \epsilon_b f_b + \epsilon_c f_c = 0, \quad f_i = 2^{x_i} 5^{y_i}$ for $i = a,b,c$

$$f_a \equiv 2\epsilon_a \pmod 9 \qquad f_b \equiv 2\epsilon_b \pmod 9 \qquad f_c \equiv 4\epsilon_c \pmod 9$$

The method of solution is exactly the same as that for (B)

Solutions to (C)

1) $1 + 2 + 2 - 5 = 0$

2) $1 + 20 - 25 + 4 = 0$

3) $1 - 16 + 20 - 5 = 0$

4) $1 - 16 - 25 + 40 = 0$

$\underline{(D)}$ $\quad 1 + \epsilon_a f_a + \epsilon_b f_b + \epsilon_c f_c = 0, \quad f_i = 2^{x_i} 5^{y_i}$ for $i = a,b,c$

$$f_a \equiv 5\epsilon_a \pmod 9 \qquad f_b \equiv 5\epsilon_b \pmod 9 \qquad f_c \equiv 7\epsilon_c \pmod 9$$

i	u_i	v_i	θ_i
0	1	1	
1	6	6	
2	12	12	7, 13
3	36	36	19, 37
4	72	144	17, 73
5	144	288	97
6	288	576	193, 577
7	32·9·7	64·9·7	29, 43, 113, 127
8	32·9·7	256·9·7	257, 1153
9	128·9·7	256·9·7	769
10	257·9·7	512·9·7	10753

(Check congruences between f_i and ϵ_i)

Check sum modulo 2^{11} and then modulo 5^5

Solutions to (D)

1) $1 - 4 + 5 - 2 = 0$

2) $1 + 5 - 256 + 250 = 0$

3) $1 - 400 - 625 + 1024 = 0$

(E) $1 + \epsilon_a f_a + \epsilon_b f_b + \epsilon_c f_c = 0, \quad f_i = 2^{x_i} 5^{y_i}$ for $i = a,b,c$

$$f_a \equiv 8\epsilon_8 \pmod 9 \quad f_b \equiv 5\epsilon_b \pmod 9 \quad f_c \equiv 4\epsilon_c \pmod 9$$

The method of Section 3 was used to find solutions with $y_i \leq 7$ for all i

Solutions to (E) with $y_i \leq 7$ for all i

1) $1 - 1 + t - t = 0$ (any appropriate value of t)

2) $1 + 8 - 4 - 5 = 0$

3) $1 - 10 + 5 + 4 = 0$

(F) $1 + \epsilon_a f_a + \epsilon_b f_b + \epsilon_c f_c = 0, \quad f_i = 2^{x_i} 5^{y_i}$ for $i = a,b,c$

$$f_a \equiv 8\epsilon_a \pmod 9 \quad f_b \equiv \epsilon_b \pmod 9 \quad f_c \equiv 8\epsilon_c \pmod 9$$

The method of Section 3 was used to find solutions with $y_i \leq 4$ for all i

Solutions to (F) with $y_i \leq 4$ for all i

1) $1 - 1 + t - t = 0$ (any appropriate value of t)

2) $1 + 8 + 1 - 10 = 0$

(G) $1 + \epsilon_a f_a + \epsilon_b f_b + \epsilon_c f_c = 0, \quad f_i = 2^{x_i} 7^{y_i}$ for $i = a,b,c$

$$f_a \equiv 8\epsilon_a \pmod 9 \qquad f_b \equiv 5\epsilon_b \pmod 9 \qquad f_c \equiv 4\epsilon_c \pmod 9$$

The method of Section 3 was used to find solutions with $y_i \leq 2$ for all i

Solutions to (G) with $y_i \leq 2$ for all i

1) $1 - 1 + t - t = 0$ (any appropriate value of t)
2) $1 - 64 + 14 + 49 = 0$
3) $1 - 64 - 49 + 112 = 0$

CHAPTER VII. SIMPLE GROUPS OF ORDER $2^a q^b 3^2$: ANALYSING THE SOLUTIONS

In this chapter, we shall attempt to determine whether the solutions obtained in Chapter VI actually occur as the degrees of the principal 3-block characters of a simple group. The most important technique will be restriction of the characters to 3-local subgroups. In the first few sections, we shall derive some properties of these subgroups and some congruences for the characters.

1. Fixed-point-free automorphisms of order 3

Lemma 7.1 Let P be a p-group. If σ is a fixed-point-free automorphism of P of order 3, then:

1) If $p \equiv 2 \pmod 3$, σ leaves invariant an elementary abelian subgroup of order p^2.
2) If $p \equiv 1 \pmod 3$, σ leaves invariant an elementary abelian subgroup of order p or p^3.

Proof Let g be an element of order p. Let $K = \langle g, g\sigma, g\sigma^2 \rangle$. K is invariant under σ since σ permutes the elements of $\{g, g\sigma, g\sigma^2\}$. By [15-Theorem 10.1.5], the elements g, $g\sigma$, and $g\sigma^2$ commute; it follows that K is an elementary abelian p-group of order at most p^3.

Suppose $p \equiv 2 \pmod 3$. σ permutes the nonidentity elements of K in orbits of length 3; hence $|K| \equiv 1 \pmod 3$. It follows that $|K| = p^2$.

Suppose $p \equiv 1 \pmod 3$ and $|K| = p^2$. K contains p+1 subgroups of order p; σ permutes these in orbits of length 3 or 1. Since $p+1 \not\equiv 0 \pmod 3$, σ leaves invariant a subgroup of order p.

2. Congruences for the characters

Lemma 7.2 Let P be an elementary abelian p-group of order p^2. Let χ be a rational (possibly reducible) character of P. If $g_1, \ldots, g_{p+1}$ are generators of the p+1 subgroups of order p, then:

$$(p + 1)\, \chi(1) \equiv \sum_{i=1}^{p+1} \chi(g_i) \pmod{p^2}.$$

Proof Restatement of Theorem 4.5.

Lemma 7.3 Let P be a cyclic group of order p^n. Let χ be a rational (possibly reducible) character of P. If $g \in P$, $g^p = 1$, then:

$$\chi(1) \equiv \chi(g) \pmod{p^n}.$$

Proof Let h be a generator of P such that $h^{p^{n-1}} = g$. Let ξ be a primitive p^n'th root of unity. Let η be the character of P defined by $\eta(h^i) = \xi^i$. Π_k will denote $\{i \mid \xi^i$ is a primitive p^k'th root of unity, $0 \leq i < p^n\}$. If $i, j \in \Pi_k$, then $\chi(h^i) = \chi(h^j)$ since χ is rational.

$$\begin{aligned} p^n \langle \chi, \eta \rangle &= \sum_{u \in P} \chi(u)\, \eta(u) \\ &= \sum_{i=0}^{n} \sum_{j \in \Pi_i} \chi(h^j)\, \eta(h^j) \\ &= \sum_{i=0}^{n} \chi(h^{p^{n-i}}) \sum_{j \in \Pi_i} \xi^j \\ &= \chi(1) - \chi(g). \end{aligned}$$

The last equality follows from the fact that the sum of the p^i'th roots of unity is -1 if i = 1 and 0 if i > 1. Since $\langle \chi, \eta \rangle$ is an integer, the assertion of the lemma follows.

If $|P| = 2$, the congruence of Lemma 7.3 can be strengthened. The following lemma was suggested to the author by Prof. Marshall Hall, Jr.

Lemma 7.4 If χ is a character of a simple group G and if t is an involution of G, then $\chi(1) \equiv \chi(t) \pmod 4$.

Proof Let χ be the character of the representation ρ. Let $\rho(t)$ have a eigenvalues +1 and b eigenvalues -1. Then $\chi(1) = a + b$ and $\chi(t) = a - b$, so $\chi(1) - \chi(t) = 2b$. It suffices to prove that b is even.

Suppose b is odd. Then $\det(\rho(t)) = -1$. Let $N = \{g \in G \mid \det(\rho(g)) = 1\}$. N is a proper, normal subgroup of G. G/N is isomorphic to a multiplicative group of roots of unity and hence is abelian. This contradicts the simplicity of G; for if $N = \langle 1 \rangle$, G is abelian, and if $N \neq \langle 1 \rangle$, G contains a nontrivial, proper, normal subgroup.

The next sequence of lemmas will deal with characters of a group of order 3^2t which are constant on an appropriate set of 3-singular elements.

Lemma 7.5 Suppose G is a group of order 3^2t, $(3,t) = 1$, with a Sylow 3-subgroup P. Suppose u is an element of prime order r ($r \neq 3$) in C(P). If χ is a (possibly reducible) character of G which is rational on u and is constant on the 3-singular elements of $P \times \langle u \rangle$, then $\chi(1) \equiv \chi(u) \pmod{9r}$.

If, in addition, G is simple and $r = 2$, $\chi(1) \equiv \chi(u) \pmod{36}$.

Proof Let ξ be a primitive r'th root of unity, and let η be the character of $P \times \langle u \rangle$ defined by $\eta(gu^i) = \xi^i$ for $g \in P$. By hypothesis, $\chi(gu^i) = \chi(g)$ for all i and all $g \in P^*$. Also, since χ is rational, $\chi(u^i) = \chi(u)$ for $1 \leq i \leq r-1$. Hence:

$$9r \langle \chi, \eta \rangle = \sum_{g \in P} \sum_{i=0}^{r-1} \chi(gu^i)\, \eta(gu^i)$$

$$= \chi(1) + \sum_{i=1}^{r-1} \chi(u^i)\, \eta(u^i) + \sum_{g \in P^*} \sum_{i=0}^{r-1} \chi(gu^i)\, \eta(gu^i)$$

$$= \chi(1) + \chi(u) \sum_{i=1}^{r-1} \xi^i + \sum_{g \in P^*} \chi(g) \sum_{i=0}^{r-1} \xi^i$$

$$= \chi(1) - \chi(u).$$

Since $\langle \chi, \eta \rangle$ is an integer, the assertion $\chi(1) \equiv \chi(u) \pmod{9r}$ holds. If, in addition, G is simple and $r = 2$, the second assertion follows immediately from Lemma 7.4 and the first part of Lemma 7.5.

Lemma 7.6 Suppose G is a group of order 3^2t. Suppose g is an element of order 3 and u is an element of prime order r such that $u \in C(g)$. If χ is a (possibly reducible) character of G which is rational on u and is constant on the 3-singular elements of $\langle g \rangle \times \langle u \rangle$, then $\chi(1) \equiv \chi(u) \pmod{3r}$.

If, in addition, G is simple and $r = 2$, $\chi(1) \equiv \chi(u) \pmod{12}$.

Proof The proof is analogous to that of Lemma 7.5; the group P is replaced by the group $\langle g \rangle$.

Lemma 7.7 Let G be a group of order 3^2t, $(3,t) = 1$, and let g be an element of order 3 contained in a Sylow 3-subgroup P. Suppose that 5 divides $|O_{3'}(C(g))|$ and that 5 does not divide $|C(P)|$. Then C(g) contains a subgroup $\langle g \rangle \times V$, where V is an elementary abelian group of order 25 contained in $O_{3'}(C(g))$ and normalized by P. V contains elements u and v of order 5 such that, if χ is any character of G rational on V and constant of the 3-singular elements of $\langle g \rangle \times V$, then

$$2\,\chi(1) \equiv \chi(u) + \chi(v) \pmod{75}.$$

χ takes on the value $\chi(u)$ on 12 elements of V and the value $\chi(w)$ on 12 elements of V.

Proof Choose $h \in P$, $h \notin \langle g \rangle$. Let σ be the inner automorphism of C(g) defined by $x\sigma = x^h$. σ leaves invariant the characteristic subgroup $O_{3'}(C(g))$. Hence σ permutes the Sylow 5-subgroups of $O_{3'}(C(g))$ in orbits of length 3 or 1. The number of subgroups divides $|O_{3'}(C(g))|$ and hence is prime to 3; it follows that σ leaves invariant a Sylow 5-subgroup of $O_{3'}(C(g))$. σ acts fixed-point-free on this subgroup since the hypotheses imply that no element of the subgroup centralizes P. By Lemma 7.1, σ leaves invariant an elementary abelian group V of order 25 contained in the Sylow 5-subgroup. Clearly V is normalized by P.

V contains 6 subgroups of order 5. σ permutes these groups in two orbits of length 3. Choose u and v such that $\langle u \rangle$ and $\langle v \rangle$ are in different orbits. Since χ is rational, it is constant on the 12 nonidentity elements of the 3-subgroups in the orbit of $\langle u \rangle$; a similar statement holds for $\langle v \rangle$. Let η_1 be the character of $\langle g \rangle \times V$ defined by $\eta_1(g^i u^j v^k) = \xi^j$, where ξ is a primitive 5'th root of unity. Define η_2 by

$\eta_2(g^i u^j v^k) = \xi^k$. Computing $\langle \chi, \eta_1 \rangle$ and $\langle \chi, \eta_2 \rangle$ as in the proof of Lemma 7.5, we obtain:

$$75(\langle \chi, \eta_1 \rangle + \langle \chi, \eta_2 \rangle) = 2\,\chi(1) - \chi(u) - \chi(v).$$

Thus all the assertions of the lemma hold.

Lemma 7.8 Let G be a group of order $3^2 t$, $(3,t) = 1$, and let g be an element of order 3 contained in a Sylow 3-subgroup P. Suppose that 2 divides $|O_{3'}(C(g))|$ and that 2 does not divide $|C(P)|$. Then C(g) contains a subgroup $\langle g \rangle \times V$, where V is a four-group in $O_{3'}(C(g))$ normalized by P. If χ is any character of G constant on the 3-singular elements of $\langle g \rangle \times V$, then $\chi(1) \equiv \chi(u) \pmod{12}$ for each nonidentity element u of V.

Proof Analogous to the proof of Lemma 7.7.

3. Miscellaneous lemmas on characters

Lemma 7.9 If χ is a character of a group G which is rational on an element x of prime order r, then $\chi(x) \geq \frac{-1}{r-1}\chi(1)$.

Proof Let ξ be a primitive r'th root of unity. Let ξ have multiplicity b as an eigenvalue of $\rho(x)$, where ρ is a representation having χ as its character. Since $\chi(x)$ is rational, ξ^i also has multiplicity b whenever $(i,r) = 1$. Let 1 have multiplicity c as an eigenvalue of $\rho(x)$. Then $\chi(1) = c + (r-1)b$ and $\chi(x) = c - b$. Hence $\chi(1) \geq (r-1)b \geq -(r-1)\chi(x)$.

Lemma 7.10 If the character χ of the group G is rational on g, then $\chi(1)$ divides $|G:C(g)|\chi(g)$.

Proof $|G:C(g)|\chi(g)/\chi(1)$ is an algebraic integer [17-Theorem 16.8.3] and a rational number; hence it is a rational integer.

The next lemma gives a variation of a well-known formula, proven in [20-Section 2].

Lemma 7.11 Let G be a simple group of order $2^a q^b 3^2$. Assume that, for some 3-element g, C(g) has a normal 3-complement. Let u and v be arbitrary elements of G. Then

$$0 \leq \sum_{\chi \in B_0(3)} \frac{\chi(u)\chi(v)\chi(g)}{\chi(1)} \leq 9 .$$

Proof The proof which follows is essentially identical to the derivation of [20-Equation 2.1], except that we shall use only the characters of $B_0(3)$ rather than all irreducible characters.

Let U and V be the class sums (in the complex group algebra) of the classes of u and v respectively. Let $C_1, C_2, \ldots, C_s, \ldots, C_r$ be the complete set of class sums, $C_1, C_2, \ldots, C_s$ being the sums corresponding to classes in the 3-section of g. We may write $UV = \sum_1^r k_i C_i$, where k_i is the number of ways that an element in the class corresponding to C_i can be written as the product of a conjugate of u and a conjugate of v. Let c_i denote an element in the class corresponding to C_i. If χ is an irreducible character of G, then $C_i \to |C_i|\chi(c_i)/\chi(1)$ is a homomorphism from the center of the complex group algebra to the complex numbers.

Hence $|U||V|\ \chi(u)\ \chi(v) = \chi(1) \sum_1^r |C_i|\ \chi(c_i)\ k_i$. Multiplying by $\overline{\chi(g)}/\chi(1)$ and summing over all characters in $B_0(3)$, we obtain:

$$\sum_{B_0(3)} |U||V|\chi(u)\chi(v)\overline{\chi(g)}/\chi(1) = \sum_{i=1}^{r} |C_i|k_i \left[\sum_{B_0(3)} \chi(c_i)\overline{\chi(g)} \right].$$

The block-section orthogonality relations [3, II-7C] imply that the quantity in brackets is zero unless c_i is in the 3-section of g, that is, unless $i \leq s$. Since C(g) has a normal 3-complement, case i or case i' (of Theorem 5.5) occurs for g, and it follows from Theorem 5.5 that each principal 3-block character is constant on the section of g and that $\sum_{B_0(3)} |\chi(g)|^2 = 9$. It follows that:

$$\sum_{B_0(3)} \chi(u)\chi(v)\chi(g)/\chi(1) = \sum_{i=1}^{s} 9|C_i|k_i \ /\ |U||V| .$$

Since $|U||V| = \sum_{i=1}^{r} |C_i|k_i$, $0 \leq \sum_{i=1}^{s} |C_i|k_i \ /\ |U||V| \leq 1$. Hence the assertion of the theorem holds.

<u>Lemma 7.12</u> (<u>Block separation</u>) Let G be a simple group of order $2^a q^b 3^2$ not containing elements of order 3q. Let $1 + \sum_{i \in S} m_i \epsilon_i f_i = 0$ be a degree equation for G, obtained in Chapter V (All the degree equations are of this form; in fact, except in the equation for type a-i', each m_i is 1). Write $f_i = 2^{x_i} q^{y_i}$. Let $T = \{i \in S \mid x_i > 0,\ y_i < b-1\}$. Let 2^T denote the power set of T. Then there exists $U \in 2^T$ such that $1 + \sum_{i \in U} m_i \epsilon_i f_i \equiv 0 \pmod{q^b}$.

<u>Proof</u> By [9-Lemma 3], $\sum_{B_0(3) \cap B_0(q)} \chi(1)\ \chi(h) \equiv 0 \pmod{q^b}$ whenever h is a 3-singular element. In Definition 5.8, we defined the column

vector $Y(h)$ to be the vector whose components are the values of the principal 3-block characters on h. Let $Z(h)$ be the column vector obtained from $Y(h)$ by deleting entries corresponding to characters not in $B_0(q)$. The degree equations were obtained from othogonality relations of the form $Y(1)\cdot[a_1Y(h_1)+\ldots+a_kY(h_k)] = 0$, where each h_i is 3-singular. These relations remain valid if Y is replaced by Z, provided that the equality is replaced by a congruence modulo q^b. It follows that $1 + \sum_{i \in U} m_i\epsilon_i f_i \equiv 0 \pmod{q^b}$, where U denotes $\{i \mid \chi_i \in B_0(3) \cap B_0(q)\}$. (To obtain the congruence from the orthogonality relation, it is necessary to divide by some integer n; however, in all cases, n is relatively prime to q). Now $U \subseteq T$ since $B_0(q)$ contains no characters of degree a power of q [9-Lemma 2] and no characters of degree divisible by q^{b-1} [1-Theorem 3].

4. Two classes of 3-elements

This section will deal with simple groups of order $2^a q^b 3^2$ containing exactly two conjugacy classes of 3-elements. The possibilities for the principal 3-block will be determined explicitly. One such group is known, namely A_6.

Theorems 5.12 and 5.14 give fragments for the character table. The equations and congruences for the principal 3-block characters are summarized in Chapter V, Section 7. Solutions are given in Chapter VI, Section 4.

An algebraic conjugate of a character in $B_0(3)$ is in $B_0(3)$ [4, I-Lemma 2]. Hence, if χ is the only character of a given degree in

$B_o(3)$, then χ is rational. More generally, if $\chi \in B_o(3)$ and if, for any other character χ' in $B_o(3)$, there is an element g (depending on χ') such that $\chi(g)$ and $\chi'(g)$ are distinct rational numbers, then χ is a rational character. Using the fragment given in Theorem 5.14, we obtain that, if G is of type b-x,y with x,y $\geq$ ii, all characters in $B_o(3)$ are rational. G will denote a simple group of order $2^a q^b 3^2$.

Lemma 7.13 G is not of type b-ii,ii.

Proof If G were of type b-ii,ii, equation (B) of Chapter VI, Section 4 would hold for f_2, f_9, and f_6 (i.e. $1 + \epsilon_2 f_2 + \epsilon_9 f_9 + \epsilon_6 f_6 = 0$, $f_2 \equiv \epsilon_2$ (mod 9), $f_9 \equiv 2\epsilon_9$ (mod 9), $f_6 \equiv 5\epsilon_6$ (mod 9)). If q = 5, $B_o(3)$ contains a character of degree at most 25; if q = 7, it contains a character of degree at most 16. By the remark above, these characters are rational. A theorem of Schur [25] shows, in the former case, that $b \leq [25/4] + [25/20] = 7$, and, in the latter case, that $b \leq [16/6] = 2$.

If q = 5, then f_5, f_6, and f_7 satisfy equation (E) and $y_i \leq 7$ for all i. Hence $B_o(3)$ contains a nonidentity character of degree at most 5, contrary to known results [13-Section 8.5].

If q = 7, then f_5, f_6, and f_7 satisfy equation (G) and $y_i \leq 2$ for all i. Only solutions (2) and (3) are possible. In either case, $B_o(3)$ contains an irreducible character χ of degree 49 and $|\chi(g)| = 1$ or 2 if g is a 3-element. By Lemma 7.10, 49 divides $|G:C(g)|$ for all 3-elements g; hence G contains no element of order 21. Both solutions fail to satisfy the 7-block separation criteria (Lemma 7.12).

Lemma 7.14 G is not of type b-ii,iii or type b-ii,iv.

Proof Assume the contrary. $|G| = 2^a5^b3^2$. Equation (C) holds for f_5, f_6, and f_7. Hence G has a rational character of degree at most 16, and the theorem of Schur shows that $b \leq [16/4] = 4$. Then f_8, f_4, and f_9 satisfy equation (F) and $y_i \leq 4$ for all i. It follows that G has a nonidentity character of degree 1, contrary to the simplicity of G.

Lemma 7.15 G is not of type b-ii,v or type b-ii,vi.

Proof Assume the contrary. $|G| = 2^a7^b3^2$. Equation (B) holds for f_4, f_8, and f_9. G has a rational character of degree at most 16 and consequently $b \leq [16/6] = 2$. Since G has no character of degree less than 7, solution (4) or (5) of equation (B) must occur. In either case, $f_9 = 49$. Let g be an element of order 3 such $C(g)/O_{3'}(C(g))$ is $\langle g \rangle \times PSL_2(7)$ or $\langle g \rangle \times PGL_2(7)$. By Theorem 5.14, $\chi_9(g) = \pm 8$. Lemma 7.10 implies that 49 divides $|G:C(g)|$. Since 7 divides $|C(g)|$, this contradicts $b \leq 2$.

Lemma 7.16 G is not of type b-v,v, type b-v,vi, or type b-vi,vi.

Proof Assume the contrary. $|G| = 2^a7^b3^2$. f_3, f_5, and f_8 satisfy equation (B). As in the proof of Lemma 7.15, $b \leq 2$ and $f_8 = 49$. By Theorem 5.14, $\chi_8(g) = \pm 8$ for the elements g in one of the classes of elements of order 3. Then 49 divides $|G:C(g)|$ and 7 divides $|C(g)|$, contrary to $b \leq 2$.

Lemma 7.17 Suppose G is of type b-iii,iii, type b-iii,iv, or type b-iv,iv. Then the degrees f_2, f_3, ..., f_9 are either 1024, 1024, 1024, 400, 625, 1024, 625, 400 respectively or else the same except with 400 and 625 interchanged.

Proof Equation (D) holds for f_6, f_9, and f_2; for f_8, f_9, and f_4; for f_5, f_6, and f_7; and for f_5, f_8, and f_3. Since G does not have a representation of degree 5 or less, only solution (3) of equation (D) can occur. We obtain the two alternatives given in the lemma.

The two alternatives of Lemma 7.17 are not symmetric, since χ_9 is the only character in $B_0(3)$ which takes on the value ± 5 on each element of order 3 (Theorem 5.14).

Lemma 7.18 If G is of type b-i',i', then one of the following holds:

1) $G \simeq A_6$.

2) $q = 5$, and the degree equation is $1 + 64 - 80 + 640 - 625 = 0$.

3) $q = 7$, and the degree equation is $1 - 224 - 512 + 784 - 49 = 0$.

Proof Assume first that $q = 5$. The degree equation is equation (A) of Chapter VI, Section 4. Solution (8) occurs if $G \simeq A_6$; otherwise solutions (1) through (9) cannot occur since G has no representation of degree 5 or less. If solution (10) or (11) occured, G would have a rational character of degree at most 10, implying that $b \leq [10/4] = 2$; this contradicts the fact that, in either case, G has an irreducible character of degree 5^3.

Suppose that solution (12) occurs. $B_0(3)$ contains two characters of degree 8, namely χ_2 and χ_3. Let $\chi = \chi_2 + \chi_3$; χ is a rational character of degree 16. It follows that $b \leq [16/4] = 4$. χ_5 has degree 5^4 and $\chi_5(g) = 1$ or -2 if g is a 3-element. By Lemma 7.10, 5^4 divides

$|G:C(g)|$ if g is a 3-element. Hence G contains no elements of order 15. Let P denote a Sylow 3-group containing the element g of order 3. Suppose t is an element of order 2 in $C(P)$. By Lemma 7.5, $\chi_2(t) \equiv 8 \pmod{36}$; hence $\chi_2(t) \leq -28$, a contradiction. Thus $C(P) = P$. Suppose 2 divides $|O_{3'}(C(g))|$. By Lemma 7.8, $O_{3'}(C(g))$ contains a four-group V, and if v is a nonidentity element of V, $\chi_2(v) \equiv 8 \pmod{12}$; hence $\chi_2(v) = -4$. Let η be the identity character of V. Then $\langle \chi_2|_V, \eta\rangle = \frac{1}{4}(8-4-4-4) = -1$, a contradiction. We conclude that $O_{3'}(C(g)) = 1$. Since case i' occurs for g, $C(g) = P$. Thus P is strongly self-centralizing, contrary to Theorem 3.2.

Suppose that solution (13) occurs. Let g and g' be representatives of the two classes of elements of order 3. Let $t \in G$. By Lemma 7.11, $0 \leq \sum_{B_0(3)} \frac{\chi(t)^2\,\chi(g)}{\chi(1)} \leq 9$ and $0 \leq \sum_{B_0(3)} \frac{\chi(t)^2\,\chi(g')}{\chi(1)} \leq 9$. Taking one-half the sum of these inequalities and substituting the values of $\chi(g)$, $\chi(g')$, and $\chi(1)$ given in Theorem 5.12, we obtain:

$$0 \leq 1 + \sum_{i=2}^{5} \frac{\epsilon_i\chi_i(t)^2}{f_i} \leq 9.$$

Let g and g' be contained in a Sylow 3-subgroup P. Suppose that t is an element of order 5 in $C(P)$. From Lemmas 7.5 and 7.9, $\chi_i(t) \equiv f_i \pmod{45}$ and $-f_i/4 \leq \chi_i(t) < f_i$ for $i = 2, 3, 4, 5$. The block-section orthogonality relations imply that $1 + \epsilon_2\chi_2(t) + \epsilon_3\chi_3(t) + \epsilon_4\chi_4(t) + \epsilon_5\chi_5(t) = 0$. There are only two vectors $(\chi_2(t),\chi_3(t),\chi_4(t),\chi_5(t))$ which satisfy the criteria given in the last two sentences, namely $(19,55,80,-5)$ and $(19,10,35,-5)$. For these two alternatives, $1 + \sum_{B_0(3)} \frac{\epsilon_i\chi_i(t)^2}{f_i}$ is respectively -14.93 and

-2.78. In either case, the inequality obtained from Lemma 7.11 is violated. We conclude that 5 does not divide $|C(P)|$. Suppose 5 divides $|O_{3'}(C(g))|$. Lemma 7.7 is applicable. Let V, u, and v be as in Lemma 7.7. We assert that the integers $\chi_i(u)$ and $\chi_i(v)$ $(i = 2,3,4,5)$ satisfy the following conditions:

a) $-f_i/4 \leq \chi_i(u) < f_i, \quad -f_i/4 \leq \chi_i(v) < f_i$

b) $\chi_i(u) \equiv \chi_i(v) \equiv f_i \pmod{15}$

c) $\chi_i(u) + \chi_i(v) \equiv 2f_i \pmod{75}$

d) $1 + \sum_{i=2}^{5} \epsilon_i\chi_i(u) = 0, \quad 1 + \sum_{i=2}^{5} \epsilon_i\chi_i(v) = 0$

e) $f_i + 12\chi_i(u) + 12\chi_i(v) \geq 0$

f) $f_i + 2\chi_i(u) - 3\chi_i(v) \geq 0$

g) $f_i - 3\chi_i(u) + 2\chi_i(v) \geq 0$

h) $0 \leq 1 + \sum_{B_0(3)} \frac{\epsilon_i\chi_i(u)^2}{f_i} \leq 9, \quad 0 \leq \sum_{B_0(3)} \frac{\epsilon_i\chi_i(v)^2}{f_i} \leq 9.$

(a), (b), (c), and (d) follow respectively from Lemma 7.9, Lemma 7.6, Lemma 7.7, and the block-section orthogonality relations. (h) was proven on the last page. Let η_1 be the identity character of V, η_2 the character defined by $\eta_2(u^k v^j) = \xi^j$, where ξ is a primitive 5'th root of unity, and η_3 the character defined by $\eta_3(u^k v^j) = \xi^k$. Then (e), (f), and (g) follow from the fact that $\langle \chi_i|_V, \eta_1\rangle$, $\langle \chi_i|_V, \eta_2\rangle$, and $\langle \chi_i|_V, \eta_3\rangle$ are nonnegative.

Now a compter was used to determine all integers $\chi_i(u)$ and $\chi_i(v)$ ($i = 2,3,4,5$) which satisfy (a), (b), (c), and (d). For each alternative, conditions (e), (f), (g), and (h) were checked; in every case, at least one of these conditions failed. Hence 5 does not divide $|O_{3'}(C(g))|$. Similarly, 5 does not divide $|O_{3'}(C(g'))|$. Since $C(g)$ and $C(g')$ have normal 3-complements, this implies that G contains no elements of order 15. Then 5-block separation (Lemma 7.12) provides a contradiction.

Thus far we have proven that, if $q = 5$, alternative (1) or (2) of Lemma 7.18 holds.

Assume now that $q = 7$. Solutions (1) and (2) of equation (A) cannot occur since G does not have a nonidentity linear character. If solution (3) occured, G would have a rational character of degree 8, implying that $b \leq [8/6] = 1$, contrary to known results [29]. Suppose that solution (4) occurs. The argument is similar to that for solution (12) with $q = 5$. G has a rational character of degree 16. Hence $b \leq [16/6] = 2$. The existence of an irreducible character of degree 49 implies, using Lemma 7.10, that G has no elements of order 21. If G contained elements of order 6, we would obtain a contradiction exactly as in the case of solution (12) for $q = 5$. Hence the Sylow 3-groups of G are strongly self-centralizing, contrary to Theorem 3.2. We conclude that only solution (5) can occur, that is, alternative (3) of Lemma 7.18 must hold.

The results derived in this section are summarized in the following two theorems. As in Chapter V, $[x]$ denotes $\{y \in N(\langle g\rangle) \mid \bar{y}$ is conjugate to $\bar{x}$ in $\overline{N(\langle g\rangle)}$ }. Here a bar over a symbol denotes the image modulo $O_{3'}(N(\langle g\rangle))$.

Theorem 7.19 Let G be a simple group such that:

1) $|G| = 2^a q^b 3^2$, q prime.

2) G has two conjugacy classes of 3-elements.

3) All 3-local subgroups of G are solvable.

Let g and g' be nonconjugate elements of order 3.

Then C(g) and C(g') have normal 3-complements, and one of the following holds:

a) G is isomorphic to A_6.

b) q = 5, and the principal 3-block of G has the following form:

	1	[g]	[g']
χ_1	1	1	1
χ_2	64	1	1
χ_3	80	-1	-1
χ_4	640	1	1
χ_5	625	1	-2
χ_6	625	-2	1

c) q = 7, and the principal 3-block of G has the following form:

	1	[g]	[g']
χ_1	1	1	1
χ_2	224	-1	-1
χ_3	512	-1	-1
χ_4	784	1	1
χ_5	49	1	-2
χ_6	49	-2	1

Theorem 7.20 Let G be a simple group such that:

1) $|G| = 2^a q^b 3^2$, q prime.

2) G has two conjugacy classes of 3-elements.

3) Some 3-local subgroup of G is nonsolvable.

Let g and g' be nonconjugate elements of order 3.

Then $C(g)/O_{3'}(C(g)) \simeq \langle g \rangle \times A_5$ or $\langle g \rangle \times S_5$, and $C(g')/O_{3'}(C(g')) \simeq \langle g' \rangle \times A_5$ or $\langle g' \rangle \times S_5$. In particular, q = 5. The principal 3-block of G either has the form

	1	[g]	[gv]	[gw]	[gx]	[gy]	[g']	[g'v']	[g'w']	[g'x']	[g'y']
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1024	1	1	1	1	1	4	-1	0	0	-2
χ_3	1024	1	1	1	1	1	-5	0	-1	-1	1
χ_4	1024	4	-1	0	0	-2	1	1	1	1	1
χ_5	400	4	-1	0	0	-2	4	-1	0	0	-2
χ_6	625	4	-1	0	0	-2	-5	0	-1	-1	1
χ_7	1024	-5	0	-1	-1	1	1	1	1	1	1
χ_8	625	-5	0	-1	-1	1	4	-1	0	0	-2
χ_9	400	-5	0	-1	-1	1	-5	0	-1	-1	1

or else has the same form with 400 and 625 interchanged. v, w, x, and y are elements of S_5, defined in Theorem 5.5, case iv. If $C(g)/O_{3'}(C(g)) \simeq A_5$, the columns of [gx] and [gy] are absent, and the column of [gv] splits into two identical columns; a similar statement holds with g replaced by g'.

The methods developed in this chapter may be used to investigate whether the fragments given by Theorems 7.19 and 7.20 actually occur in the character table of a simple group of the appropriate type. A considerable amount of information can be obtained; however, the author has not been able to reach a contradiction in any of the cases.

5. A character of degree 8

Theorem 7.21 If G is a simple group of order $2^a q^b 3^2$ with an irreducible character χ of degree 8 in its principal 3-block, then G is isomorphic to A_6 or $PSL_2(8)$.

Proof Assume that G is not isomorphic to A_6 or $PSL_2(8)$. In view of Theorems 5.3, 5.15, 7.19, and 7.20, we may assume that G has a single class of elements of order 3. Let g have order 3. Corollary 4.6 shows that $\chi(g) = -1$. From the fragments given in Theorem 5.5, we see that either

1) χ is constant on 3-singular elements

or 2) Case ii occurs for g, and χ is χ_4, χ_5, or χ_6.

Suppose that alternative (2) occurs. Let P be a Sylow 3-group containing g. If t is a 3'-element of C(P), then $\overline{t}$ is a 3'-element of $C(\overline{P})$. $\overline{P}$ is the Sylow 3-group of $\langle g \rangle \times S_3$; hence $\overline{t} = 1$ and $t \in O_{3'}(C(g))$. Since $\chi \in B_0(3)$, Theorem 5.5 shows that χ is constant on the 3-singular elements of $P \times \langle t \rangle$.

Assume first that q = 5. Let ξ be a primitive 5'th root of unity, and let σ be an automorphism of the field of $|G|$'th roots of unity such that $\sigma(\xi) = \xi^2$. Let $\chi' = \chi + \chi^{\sigma} + \chi^{\sigma^2} + \chi^{\sigma^3}$. χ' is rational on elements of order 5.

Suppose that t and u are elements of C(P) of orders 2 and 5 respectively. By Lemma 7.5, $\chi(t) \equiv 8 \pmod{36}$ and $\chi'(u) \equiv 32 \pmod{45}$.

Hence $\chi(t) \leq -28$ and $\chi'(u) \leq -13$, both contrary to Lemma 7.9. It follows that $C(P) = 1$.

Suppose 2 divides $|O_{3'}(C(g))|$. Lemma 7.8 is applicable. $O_{3'}(C(g))$ contains a four-group V. If u is a nonidentity element of V, $\chi(u) \equiv 8 \pmod{12}$; hence $\chi(u) = -4$. Let η be the identity character of V. $\langle \chi|_V, \eta\rangle = \frac{1}{4}(8-4-4-4) = -1$. This contradiction shows that $O_{3'}(C(g))$ has odd order.

Suppose 5 divides $|O_{3'}(C(g))|$. Lemmas 7.6 and 7.7 are applicable. Let V, u, and v be as in Lemma 7.7. $\chi'(u) + \chi'(v) = 64 \pmod{75}$. We may assume $\chi'(u) \leq \chi'(v)$. $\chi'(u) + \chi'(v) \leq -11$; hence $\chi'(u) \leq -6$. By Lemma 7.6, $\chi'(u) \equiv 32 \pmod{15}$; hence $\chi'(u) \leq -13$. This contradicts Lemma 7.9. We conclude that $O_{3'}(C(g)) = 1$.

If case iii or case iv occurs for g, then $C(g) \simeq \langle g\rangle \times A_5$ or $\langle g\rangle \times S_5$. In either case, $C(g)$ contains a four-group V. Lemma 7.6 shows that $\chi(u) = -4$ if u is a nonidentity element of V. As above, $\langle \chi|_V, \eta\rangle = -1$. If case i or case i' occurs for g, then $C(g) = P$ and hence P is strongly self-centralizing, contrary to Theorem 3.2. If case ii occurs for g, then $C(g) \simeq \langle g\rangle \times S_3$. $C(g)$ contains a single Sylow 3-group; hence any two Sylow 3-groups of G intersect in the identity only, contrary to Theorem 3.2. This completes the proof of the lemma if $q = 5$.

Now assume that $|G| = 2^a 7^b 3^2$. Some unpublished work of J. H. Lindsay on representations of degree 8 shows that $b \leq 1$; applying the results of [29] completes the proof of the lemma. Alternatively, a direct proof can be given, similar to that for the case $q = 5$. However, it is somewhat more difficult to show that 7 does not divide

$|O_{3'}(C(g))|$; an analogue of Lemma 7.1 for elementary abelian groups of order p^3 is needed.

6. Simple groups of order $2^a q^2 3^2$

In this section, we shall show that there are no simple groups of order $2^a q^2 3^2$. Apparently this result was proven previously by Richard Brauer but was not published. It serves to show the nonexistence of a simple group for a number of orders less than one million which were previously unresolved.

Theorem 7.22 There are no simple groups of order $2^a q^2 3^2$, q prime.

Proof Assume that G is a counterexample. Suppose that the principal 3-block of G has the form given in alternative (c) of Theorem 7.19. Application of Lemma 7.10 to χ_5 shows that 49 divides $|G:C(g)|$ and 49 divides $|G:C(g')|$; hence G contains no elements of order 21. 7-block separation (Lemma 7.12) yields a contradiction. Hence, in view of Theorems 5.3, 5.15, 7.19, and 7.20, we may assume that G has a single class of 3-elements.

The degree equations are given in Chapter 5, Section 7. In each equation, the first term is 1 and the sum of the terms on the left is 0; it follows that at least one additional term is odd. Hence, in each equation, there is an i such that $f_i = q^{y_i}$; moreover, in the equation for case i', we may assume that $i \neq 6$. $y_i > 1$ since the simple groups with a character of degree q (q = 5 or 7) are known [13-Section 8.5], and none has order $2^a q^2 3^2$. It follows that $y_i = 2$.

Chapter 5, Section 7 also gives congruences modulo 9 relating the integers f_i and ϵ_i. For every possible value of i and ϵ_i in each degree equation (except for f_6 in the equation for case i'), the congruence fails to hold when $f_i = q^2$ for the appropriate value(s) of q. Hence there are no simple groups of order $2^a q^2 3^2$.

7. Simple groups of order $2^a q^3 3^2$

In this section, we shall determine explicitly the solutions to the degree equations which could occur in a simple group of order $2^a q^3 3^2$. No such simple group is known.

Lemma 7.23 Suppose that G is a simple group of order $2^a q^3 3^2$ and that alternative (c) of Theorem 7.19 does not hold for G. Then G has one class of 3-elements, G contains no elements of order 3q, and case i, case i', or case ii occurs for the 3-elements.

Proof If alternative (b) of Theorem 7.19 held for G, or if the conclusion of Theorem 7.20 held for G, G would have a character of degree 5^4, a contradiction. Hence Theorems 5.3, 5.15, 7.19, and 7.20 show that G contains a single class of 3-elements.

The degree equations for G are given in Chapter V, Section 7. In proving Lemma 7.22, we remarked that each degree equation must have a term f_i which is a power of q, say q^{y_i}, and that $i \neq 6$ if case i' occurs for the class of 3-elements. As in Lemma 7.22, we obtain a contradiction if $y_i \leq 2$. Hence $y_i = 3$.

Let g be an element of order 3. Let χ_i be a principal 3-block character such that $f_i = \chi_i(1) = q^3$. If case iii or case iv occurs for g, choose χ_i such that f_i appears in the first degree equation; if case v or case vi occurs, choose χ_i such that f_i appears in the second degree equation. From the fragments given in Theorem 5.5, we obtain that $(\chi_i(g),q) = 1$. Lemma 7.10 shows that q^3 divides $|G:C(g)|$. Hence G contains no elements of order 3q.

Cases iii, iv, v, and vi are excluded immediately, since in each of these cases, $|C(g)|$ must be divisible by q.

The method described in Chapter VI, Section 3 was used to solve each of the degree equations for a simple group G of order $2^a q^3 3^2$ with a single class of 3-elements. In view of Theorem 7.21, the computer was programmed to find only those solutions in which the degree of each nonidentity character was greater than 8. For each solution, the computer checked the q-block separation criteria given in Lemma 7.12; if a contradiction was obtained, the solution was discarded. A relatively small number of solutions remained; these were examined by hand.

Using the remark in the third paragraph of Section 4 of this Chapter, we see that, if any degree equation contains a single character of some given degree, then that character is rational. In particular, any solution to a degree equation in which a single term is 10 or 14 can be excluded, since a theorem of Schur [25] implies that the exponent b of q in $|G|$ is at most 2.

In cases ii, iii, iv, v, and vi, there are two degree equations. The solutions to the two equations must fit together properly -- that is, if $(f_2', f_3', f_4', f_5', f_6')$ is a solution to the first equation and

$(f_2'', f_3'', f_7'', f_8'', f_9')$ is a solution to the second equation, then $\{f_2', f_3'\} = \{f_2'', f_3''\}$.

The solutions not eliminated by any of these means are presented in the theorem which follows.

Theorem 7.24 Let G be a simple group of order $2^a q^3 3^2$, q prime. Then G has no elements of order 3q (unless 7.19(c) holds), every 3-local subgroup of G is solvable, and the alternatives for the type of G (Definition 5.11) and for the degree equation(s) (Chapter V, Section 7) are:

q	type	Degree equation(s)
5	a-i	1 - 125 - 125 - 125 - 125 - 125 - 80 + 64 + 640 = 0
"	"	1 - 125 + 10 + 10 + 10 + 10 + 10 + 10 + 64 = 0
"	"	1 - 125 + 10 + 10 + 1000 + 64 + 64 - 512 - 512 = 0
"	"	1 - 125 + 10 + 10 + 1000 + 64 + 64 - 5120 + 4096 = 0
"	"	1 - 125 + 10 + 10 + 1000 + 640 - 512 - 512 - 512 = 0
"	"	1 - 125 + 10 + 10 + 1000 + 640 - 512 - 5120 + 4096 = 0
"	"	1 - 125 + 10 + 10 + 1000 + 640 + 64000 - 32768 - 32768 = 0
"	"	1 - 125 + 10 + 10 + 1000 + 640 + 64000 - 327680 + 262144 = 0
"	"	1 - 125 + 100 + 100 + 100 - 80 - 800 + 64 + 640 = 0
"	"	1 - 125 + 100 + 1000 - 80 - 800 - 800 + 64 + 640 = 0
"	"	1 - 125 + 100 + 1000 - 80 + 64 - 8000 + 640 + 6400 = 0
5	a-i'	1 - 125 + 10 + 10 + 64 + 2·20 = 0
"	"	1 - 125 - 80 + 64 + 640 - 2·250 = 0
5	a-ii	1 - 125 + 64 - 10 - 10 + 80 = 0, 1 - 125 + 64 + 20 + 20 + 20 = 0
7	a-i	1 + 343 + 28 + 28 + 28 + 28 + 28 + 28 - 512 = 0
"	"	1 + 343 - 2744 - 224 - 224 - 224 + 1792 + 1792 - 512 = 0
"	"	1 + 343 - 2744 - 224 - 224 - 224 - 512 - 512 + 4096 = 0
"	"	1 + 343 - 2744 - 224 + 64 + 1792 + 1792 - 512 - 512 = 0
"	"	1 + 343 - 2744 - 224 + 64 - 512 - 512 - 512 + 4096 = 0

7 a-i' $1 + 343 + 28 + 28 - 512 - 2\cdot 250 = 0$

7 a-ii $1 + 343 - 512 - 28 - 28 + 224 = 0$, $1 + 343 - 512 + 56 + 56 + 56 = 0$

7 b-i',i' $1 - 224 - 512 + 784 - 49 = 0$.

Corollary 7.25 A simple group of order $2^a 5^3 3^2$ has a rational character of degree 20 or 64.

CHAPTER VIII. SIMPLE GROUPS OF ORDER LESS THAN ONE MILLION

In this chapter, we shall prove that there are no unknown simple groups of order less than one million in which the group order has the form $2^a q^b 3^2$.

Several years ago, Marshall Hall, Jr. initiated a project to determine all simple groups with order less than one million. The techniques are described in [18]; when that article was written, approximately 100 orders remained for which the existence of a simple group was unresolved. Subsequent work reduced the number of unresolved orders to 28. Of these orders, six have the form $2^a q^b 3^2$, namely:

$$2^6 5^3 3^2 = 72000$$
$$2^7 5^3 3^2 = 144000$$
$$2^8 5^3 3^2 = 288000$$
$$2^9 5^3 3^2 = 576000$$
$$2^6 5^4 3^2 = 360000$$
$$2^7 5^4 3^2 = 720000.$$

A seventh order, $2^6 3^2 5^2 7^2 = 705600$, can be handled by techniques similar to those used in this thesis.

Let G be a simple group of one of the six orders listed above. $|G| \geq \sum_{B_0(3)} \chi(1)^2$. It follows that alternative (b) of Theorem 7.19 can not hold for G and that the conclusion of Theorem 7.20 can not hold for G. Theorems 5.3. 5.15, 7.19, and 7.20 show that G must have a single class of 3-elements.

Lemma 8.1 Let G be a simple group of order $2^a q^b 3^2$ with a single class of 3-elements. If g is an element of order 3, then $|G:C(g)| \equiv 8 \pmod 9$.

Proof We shall apply Lemma 3.3. Adopting the notation used in Lemma 3.3, we have $r = 3$, $m' = 1$, $k_1 = 4$ since an elementary abelian group of order 9 has four subgroups of order 3, and $|G:C(g)| = 2|G:H_1|$ since g is conjugate to its inverse.

Part (b) of Lemma 3.3 shows that $4b_1 \equiv s_3 \pmod 3$; it follows that $n_3 = 1 + 3s_3 \equiv 1 + 3(4b_1) \equiv 1 + 3b_1 \pmod 9$. Applying part (a), we obtain $|G:C(g)| = 2|G:H_1| = 2 \cdot 4 \cdot n_3/(1+3b_1) \equiv 2 \cdot 4 \cdot 1 \equiv 8 \pmod 9$.

Theorem 8.2 There are no simple groups of order 72000, 144000, 288000, 576000, 360000, or 720000.

Proof Assume that G is a simple group of one of the orders listed above. Let g denote an element of order 3. Case i, case i', case ii, case iii, or case iv must occur for g since $|G| = 2^a 5^b 3^2$. By Theorem 7.24, cases iii and iv can occur only if $b \geq 4$, that is, if G has order 360000 or 720000.

The degree equations are given in Chapter V, Section 7. The proof of Theorem 7.22 shows that each degree equation contains a term $f_i = 5^{y_i}$ with $y_i \geq 3$; if case i' occurs, we may assume that $i \neq 6$. f_i must satisfy a congruence of the form $f_i \equiv \pm n_i \pmod 9$; this determines the exponent y_i modulo 3. Similarly, each degree equation contains a term $f_j = 2^{x_j}$, and the congruence $f_j \equiv \pm n_j \pmod 9$ determines x_j modulo 3. In view of Theorem 7.21, $x_j \geq 4$. Using these facts, we obtain the following:

1) If case i occurs for g, there are characters χ_i and and χ_j such that $f_i = 125$, $f_j = 64$ or 512, $\chi_i(g) = -1$, and $\chi_j(g) = \pm 1$.

2) If case i' occurs for g, there are characters χ_i and χ_j such that $f_i = 125$, $f_j = 16$, 64, 128, or 512, $\chi_i(g) = -1$, and $\chi_j(g) = \pm 1$ or ± 2; the values ± 2 occur only if f_j is 16 or 128.

3) If case ii occurs for g, the first degree equation contains terms f_i and f_j such that $f_i = 125$ and $f_j = 64$ or 512. $\chi_i(g) = \pm 1$ and $\chi_j(g) = \pm 1$.

4) If case iii or case iv occurs for g, the first degree equation contains a term $f_i = 125$ or 625 such that $(\chi_i(g), 5) = 1$. The second degree equation contains a term $f_j = 32$ or 64 such that $(\chi_j(g), 2) = 1$.

Suppose case i' occurs for g. Theorem 7.24 shows that the alternative $f_j = 16$ does not occur if $|G| = 2^a 5^3 3^2$. For a group of order $2^a 5^4 3^2$, $a \leq 7$, the degree equation of case i' was solved completely; no solution has a term equal to 16 and all terms greater than 8.

Lemma 7.10 shows that $2^5 5^3$ divides $|G:C(g)|$. $|G:C(g)|$ must divide $2^9 5^3$ or $2^7 5^4$. Also, Lemma 8.1 states that $|G:C(g)| \equiv 8 \pmod 9$. The only possibilities for $|G:C(g)|$ are $2^6 5^3$ and $2^7 5^4$. If $|G:C(g)| = 2^7 5^4$, $|G| = 2^7 5^4 3^2$ and $|G:C(g)| = 9$; hence a Sylow 3-subgroup of G is strongly self-centralizing, contrary to Theorem 3.2. We conclude that $|G:C(g)| = 2^6 5^3$.

Suppose $|G|$ is $2^6 5^3 3^2$, $2^7 5^3 3^2$, $2^6 5^4 3^2$, or $2^7 5^4 3^2$. Then $|C(g)|$ is is 9, 18, 45, or 90, and C(g) has a single Sylow 3-subgroup [17-Theorem 9.3.1, Part 4]. Hence any two Sylow 3-subgroups of G intersect in the identity only, contrary to Theorem 3.2. We conclude that $|G| = 2^8 5^3 3^2$ or $2^9 5^3 3^2$.

Theorem 7.24 implies that case i, case i', or case ii occurs for g. If case i or case i' occurs, Lemma 7.10 shows that no character in the principal 3-block has degree divisible by 2^7. It follows from Theorem 7.24 that the degree equation is one of the following:

<u>Case for g</u>

i	$1 - 125 + 64 + 10 + 10 + 10 + 10 + 10 + 10 = 0$
i'	$1 - 125 + 64 + 10 + 10 + 2\cdot 20 = 0$
ii	$1 - 125 + 64 - 10 - 10 + 80 = 0, \quad 1 - 125 + 64 + 20 + 20 + 20 = 0.$

G has a character χ of degree 10. Let P be a Sylow 3-group containing g. Suppose t is an involution in C(P). It follows easily that $t \in O_{3'}(C(g))$; hence χ is constant on the 3-singular elements of $P \times \langle t \rangle$. Then Lemma 7.5 yields a contradiction. We conclude that $C(P) = P$.

Suppose that $|G| = 2^8 5^3 3^2$. Then $|C(g)| = 36$. Lemma 7.8 shows that $O_{3'}(C(g))$ contains an elementary abelian subgroup V of order 4. In fact, $O_{3'}(C(g)) = V$, $|C(g)/O_{3'}(C(g))| = 9$, and case i or case i' occurs for g. It is straightforward to show that C(g) is isomorphic to $\langle g \rangle \times A_4$ and that $N(\langle g \rangle)$ is isomorphic to $S_3 \times A_4$ or $\langle g \rangle \cdot S_4$ ($\langle g \rangle \cdot S_4$ denotes the semi-direct product of $\langle g \rangle$ by S_4 which is not direct). $|N(P)/C(P)|$ has order 8 or 16 (Theorem 4.1). In the latter case, $|G:N(P)| = 2^4 5^3$, contrary to the Sylow Theorems. Hence $|N(P)/C(P)| = 8$ and $N(P)/C(P)$ is isomorphic to Q_8 or Z_8; in particular, $N(P)/C(P)$ acts as a fixed-point-free group of automorphisms of P. This implies that $N(\langle g \rangle)$ is not isomorphic to

$S_3 \times A_4$, for if it were, an involution w in S_3 would induce a nontrivial automorphism of P fixing the elements of a subgroup of order 3. We conclude that $N(\langle g\rangle) \simeq \langle g\rangle \cdot S_4$.

Next suppose that $|G| = 2^9 5^3 3^2$. $|C(g)| = 72$. $C(P) = P$; it follows that, if $h \in P - \langle g\rangle$, the mapping $x \to x^h$ is a fixed-point-free automorphism of $O_{3'}(C(g))$. Since a group of order 8 cannot have a fixed-point-free automorphism of order 3, $O_{3'}(C(g))$ is elementary abelian of order 4, and $C(g)/O_{3'}(C(g))$ has order 18; consequently case ii occurs for G. It is straightforward to show that $C(g) \simeq \langle g\rangle \times S_4$ and $N(\langle g\rangle) \simeq S_3 \times S_4$.

We shall denote the groups $\langle g\rangle \cdot S_4$ and $S_3 \times S_4$ by N_1 and N_2 respectively. The character tables appear on the following page. N_1 has two classes of 3-singular elements; g and gt are class representatives. N_2 has three classes of 3-singular elements; g, gt, and gv are representatives. $\overline{gt}$ is conjugate to $\overline{g}$ in $\overline{N(\langle g\rangle)}$; neither element is conjugate to $\overline{gv}$. We note that t is one of the three conjugate involutions of the four-group $O_{3'}(N(\langle g\rangle))$.

As in [20-Section 2], $\#(a^{\cdot}b^{\cdot}=c)$ will denote the number of ways that c can be written as the product of a conjugate of a and a conjugate of b. If a, b, and c are contained in a subgroup H, $\#_H(a^{\cdot}b^{\cdot}=c)$ will denote the number of ways that c can be written as the product of a conjugate under H of a and a conjugate under H of b. We shall be interested in $\#(t^{\cdot}t^{\cdot}=g)$ and $\#(t^{\cdot}t^{\cdot}=gt)$.

If $t^x t^y = g$, then $\langle g, t^x, t^y\rangle$ is a dihedral group of order 6 [20-Theorem 2]; it follows that t^x and t^y are involutions of $N(\langle g\rangle) - C(g)$ and are conjugate in $N(\langle g\rangle)$. It $t^x t^y = gt$, then $\langle gt, t^x, t^y\rangle$ is a dihedral

TABLE 2

Character tables of N_1 and N_2

N_1

h(x)	1	2	8	8	8	3	6	18	18
c(x)	72	36	9	9	9	24	12	4	4
x	1	g	h	gh	g^2h	t	gt	u	v
η_1	1	1	1	1	1	1	1	1	1
η_2	1	1	1	1	1	1	1	-1	-1
η_3	2	2	-1	-1	-1	2	2	0	0
η_4	2	-1	-1	-1	2	2	-1	0	0
η_5	2	-1	-1	2	-1	2	-1	0	0
η_6	2	-1	2	-1	-1	2	-1	0	0
η_7	3	3	0	0	0	-1	-1	1	-1
η_8	3	3	0	0	0	-1	-1	-1	1
η_9	6	-3	0	0	0	-2	1	0	0

$N_1 = \langle g \rangle \cdot S_4$

$N_2 = S_3 \times S_4$

S_3: $g^3 = w^2 = 1$

$g^w = g^{-1}$

S_4: $h = (123)$

$t = (12)(34)$

$v = (12)$

$u = (1234)$

N_2

h(x)	1	2	3	3	6	9	8	16	24	6	12	18	6	12	18
c(x)	144	72	48	48	24	16	18	9	6	24	12	8	24	12	8
x	1	g	w	t	gt	wt	h	gh	wh	u	gu	wu	v	gv	wv
η_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
η_2	1	1	-1	1	1	-1	1	1	-1	1	1	-1	1	1	-1
η_3	2	-1	0	2	-1	0	2	-1	0	2	-1	0	2	-1	0
η_4	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
η_5	1	1	-1	1	1	-1	1	1	-1	-1	-1	1	-1	-1	1
η_6	2	-1	0	2	-1	0	2	-1	0	-2	1	0	-2	1	0
η_7	2	2	2	2	2	2	-1	-1	-1	0	0	0	0	0	0
η_8	2	2	-2	2	2	-2	-1	-1	1	0	0	0	0	0	0
η_9	4	-2	0	4	-2	0	-2	1	0	0	0	0	0	0	0
η_{10}	3	3	3	-1	-1	-1	0	0	0	1	1	1	-1	-1	-1
η_{11}	3	3	-3	-1	-1	1	0	0	0	1	1	-1	-1	-1	1
η_{12}	6	-3	0	-2	1	0	0	0	0	2	-1	0	-2	1	0
η_{13}	3	3	3	-1	-1	-1	0	0	0	-1	-1	-1	1	1	1
η_{14}	3	3	-3	-1	-1	1	0	0	0	-1	-1	1	1	1	-1
η_{15}	6	-3	0	-2	1	0	0	0	0	-2	1	0	2	-1	0

group of order 12, and t^x and t^y are involutions of $N(\langle g\rangle) - C(g)$. Hence $\#(t^{\cdot}t^{\cdot}=g)$ and $\#(t^{\cdot}t^{\cdot}=gt)$ are determined completely by the structure of $N(\langle g\rangle)$ and the manner in which involutions of $N(\langle g\rangle) - C(g)$ fuse with t in the group G; specifically, $\#(t^{\cdot}t^{\cdot}=g) = \sum_z \#_{N(\langle g\rangle)}(z^{\cdot}z^{\cdot}=g)$ and $\#(t^{\cdot}t^{\cdot}=gt) = \sum_{z,s} \#_{N(\langle g\rangle)}(z^{\cdot}s^{\cdot}=gt)$, where z and s range over representatives of the classes of involutions of $N(\langle g\rangle)$ not contained in $C(g)$ and conjugate under G to t.

N_1 contains a single class of involutions not centralizing g; a class representative is v. N_2 contains three classes of involutions not centralizing g; representatives are w, wt and wv. We assert that the elements t, w, and wt are not all conjugate in G. Assume the contrary. $B_0(3)$ contains an irreducible character χ of degree 10. By Lemma 7.6, $\chi(t) = -2$. Let $U = O_{3'}(N(\langle g\rangle)) \times \langle w\rangle$. U is an elementary abelian group of order 8; if z is a nonidentity element of U, z is conjugate to t in G and hence $\chi(z) = -2$. It follows that $\langle \chi|_U, \eta\rangle = -\frac{1}{2}$, where η is the identity character of U. This contradiction establishes the assertion.

We have the following alternatives for fusion and for $\#(t^{\cdot}t^{\cdot}=g)$ and $\#(t^{\cdot}t^{\cdot}=gt)$:

$N(\langle g\rangle)$	Elts. conj. to t	$\#(t^{\cdot}t^{\cdot}=g)$	$\#(t^{\cdot}t^{\cdot}=gt)$
N_1	-	0	0
"	v	18	6
N_2	-	0	0
"	w	3	0
"	wt	9	6
"	wv	18	6
"	w,wv	21	6
"	wt,wv	27	12

Assume that $|G| = 2^8 5^3 3^2$; then $N(\langle g \rangle) = N_1$. Let S denote $\sum_{\chi \in B_0(3)} \frac{\chi(t)^2 \chi(g)}{\chi(1)}$. The integers $\chi(g)$ and $\chi(1)$ depend only on which of the two possible solutions to the degree equation occurs. We assert that the integers $\chi(t)$ satisfy:

1) $-\frac{1}{3}\chi(1) \leq \chi(t) < \chi(1)$

2) $\chi(t) \equiv \chi(1) \pmod{12}$

3) $\sum_{B_0(3)} \chi(t)\,\chi(g) = 0$

4) $0 \leq S < 9$

5) $\sum_{B_0(3)} \chi(t)^2 \leq 2^{2+\nu_2(\chi_3(t))} \cdot 3 \cdot 5^{\nu_5(\chi_2(t))}$.

Here $\nu_2(n)$ denotes the largest integer k such that 2^k divides n; ν_5 is defined analogously. χ is constant on the three conjugate nonidentity elements of the four-group $O_{3'}(C(g))$; the first inequality in (1) follows from the fact that the inner product of $\chi|_{O_{3'}(C(g))}$ and the identity character of $O_{3'}(C(g))$ is nonnegative. (2) follows from Lemma 7.6 and (3) from the block-section orthogonality relations. Lemma 7.11 shows that $0 \leq S \leq 9$. Let T be the class sum (in the complex group algebra) of the class of t, and let r, s, C_i, and k_i be as in the proof of Lemma 7.11. $T = C_m$ for some $m > s$. Letting $U = V = T$ in the proof of Lemma 7.11, we obtain:

$$|T|^2 = \sum_{i=1}^{r} |C_i| k_i \geq \sum_{i=1}^{s} |C_i| k_i + |C_m| k_m > \sum_{i=1}^{s} |C_i| k_i$$

The last inequality is strict because $k_m > 0$ since $t = t^x t^y$, where t, t^x, and t^y are the distinct involutions of the four-group $O_{3'}(C(g))$. The proof of Lemma 7.11 then shows that $S < 9$.

P is not centralized by t; hence 3 divides $|G:C(t)|$. In either solution to the degree equation, $\chi_3(1) = 2^6$; Lemma 7.10 shows that $2^{6-\nu_2(\chi_3(t))}$ divides $|G:C(t)|$. Similarly, $5^{3-\nu_5(\chi_2(t))}$ divides $|G:C(t)|$. Since $|G| = 2^8 5^3 3^2$, $|C(t)|$ must divide the right side of (5). This establishes (5). Hence all the assertions hold.

All integers $\chi(t)$ ($\chi \in B_0(3)$) satisfying (1), (2), and (3) were found; solutions for which (4) or (5) failed were then discarded. Two solutions remained, corresponding to the following fragments:

	1	g	gt	t
χ_1	1	1	1	1
χ_2	125	-1	-1	5
χ_3	64	1	1	16
χ_4	10	1	1	-2
χ_5	10	1	1	-2
χ_6	10	1	1	-2
χ_7	10	1	1	-2
χ_8	10	1	1	-2
χ_9	10	1	1	-2

	1	g	gt	t
χ_1	1	1	1	1
χ_2	125	-1	-1	5
χ_3	64	1	1	16
χ_4	10	1	1	-2
χ_5	10	1	1	-2
χ_6	20	2	2	-4

For each of the above fragments, we may compute $S = \sum_{B_0(3)} \frac{\chi(t)^2\chi(g)}{\chi(1)}$. In each case, $S = 36/5$. Setting $U = V = T$ in the proof of Lemma 7.11, we obtain: $S = 9 \sum_{i=1}^{s} |C_i| k_i \,/\, |T|^2$. In the present case, $s = 2$; g and gt are representatives of the two classes of 3-singular elements. Hence:

$$\frac{36}{5} = 9\,\frac{\#(t^{\bullet}t^{\bullet}=g)\ |G:C(g)| + \#(t^{\bullet}t^{\bullet}=gt)\ |G:C(gt)|}{|G:C(t)|^2}.$$

$|G:C(g)| = 2^6 5^3$ and $|G:C(gt)| = 2^6 5^3 3$. The two possibilities for $\#(t^{\bullet}t^{\bullet}=g)$ and $\#(t^{\bullet}t^{\bullet}=gt)$ were given earlier. If t is not conjugate to v, the right hand side of the equation is 0, a contradiction. If t is conjugate to v, we compute that $|G:C(t)| = 600$ and $|C(t)| = 480$.

Let n_3 denote the number of Sylow 3-subgroups of $C(t)$. Any involution centralizing a group of order 3 is conjugate to t. We count incidences of an involution in the centralizer of a group of order 3. t has 600 conjugates, each of which centralizes n_3 groups of order 3. Every group of order 3 is conjugate to $\langle g\rangle$; hence there are $2^8 5^3 3^2/72 = 4000$ subgroups of order 3, each of which is centralized by three involutions. Hence $600n_3 = 4000\cdot 3$, and $n_3 = 20$. This contradicts the Third Sylow Theorem. It follows that there is no simple group of order $2^8 5^3 3^2$.

Next assume $|G| = 2^9 5^3 3^2$; then $N(\langle g\rangle) = N_2$. As before, let $S = \sum_{B_o(3)} \frac{\chi(t)^2\,\chi(g)}{\chi(1)}$. Let $S' = \sum_{B_o(3)} \frac{\chi(t)^2\,\chi(gv)}{\chi(1)}$. Conditions (1) and (2) remain valid; we assert that (3), (4) and (5) may be replaced by:

3') $\sum_{B_o(3)} \chi(g)\chi(t) = 0, \quad \sum_{B_o(3)} \chi(gv)\chi(t) = 0$

4') $0 \le S \le 18, \quad 0 \le S' \le 6$

5') $\sum_{B_o(3)} \chi(t)^2 \le 2^{3+\nu_2(\chi_3(t))}\cdot 3\cdot 5^{\nu_5(\chi_2(t))}$.

(3') follows from the block-section orthogonality relations. The proof of (4') is identical to that of Lemma 7.11 except for the evaluation of $\sum_{B_o(3)} \chi(c_i)\overline{\chi(g)}$; from the fragment in case ii of Theorem 5.5, we obtain that $\sum_{B_o(3)} \chi(c_i)\overline{\chi(g)}$ is 18 if c_i is conjugate to g or gt and 0 if c_i is conjugate to gv; of course, the sum is 0 if c_i is 3-regular. It follows that $0 \le S \le 18$. If replace g by gv in the proof of Lemma 7.11 and use the fact that $\sum_{B_o(3)} \chi(c_i)\overline{\chi(gv)}$ is 6 if c_i is conjugate to gv and 0 otherwise, we obtain $0 \leq S' \leq 6$. Hence (4') holds. The proof of (5') is completely analogous to that of (5).

A computer was used to determine all integers $\chi(t)$ satisfying (1), (2) and (3'); for each solution, conditions (4') and (5') were checked. Two alternatives for the $\chi(t)$ remained:

	1	g	gt	gv	t (alt. i)	t (alt. ii)
χ_1	1	1	1	1	1	1
χ_2	125	-1	-1	-1	5	5
χ_3	64	1	1	1	4	-8
χ_4	10	1	1	-1	-2	-2
χ_5	10	1	1	-1	-2	-2
χ_6	80	-1	-1	1	-4	8
χ_7	20	2	2	0	8	8
χ_8	20	2	2	0	-4	8
χ_9	20	2	2	0	-4	-4

For the two alternatives above, S is respectively 45/4 and 81/5.

Proceeding as in the proof of Lemma 7.11, we obtain:

$$S = 18\,\frac{\#(t^{\cdot}t^{\cdot}=g)\ |G:C(g)| + \#(t^{\cdot}t^{\cdot}=gt)\ |G:C(gt)|}{|G:C(t)|^2} .$$

$|G:C(g)| = 2^6 5^3$ and $|G:C(gt)| = 2^6 5^2 3$. There are two alternatives for S; the six alternatives for $\#(t^{\cdot}t^{\cdot}=g)$ and $\#(t^{\cdot}t^{\cdot}=gt)$ were listed earlier. Using the above formula, we obtain 12 alternatives for $|G:C(t)|^2$; however, only one of the twelve is a perfect square; specifically, if S = 45/4 and t is conjugate to wv but not to w or wt, then $|G:C(t)|^2 = 57600$ and $|G:C(t)| = 240$. Lemma 7.10 applied to χ_2 shows that 5^2 divides $|G:C(t)|$. This contradiction shows that there is no simple group of order $2^9 5^3 3^2$.

CHAPTER IX. SUMMARY OF RESULTS

This chapter contains a summary of the principal results on simple groups of order $2^a q^b 3^2$. These results appear in Theorems 5.3, 5.15, 7.19, 7.20, 7.21, 7.22, 7.24, and 8.2.

Main Theorem, Part 1 Let G be a simple group such that:

1) $|G| = 9\cdot 2^a \cdot q^b$, q prime.

2) G has at least three conjugacy classes of 3-elements.

Then G is isomorphic to $PSL_2(8)$ or $PSL_2(17)$.

Main Theorem, Part 2 Let G be a simple group such that:

1) $|G| = 9\cdot 2^a \cdot q^b$, q prime.

2) G has two conjugacy classes of 3-elements.

3) All 3-local subgroups of G are solvable.

Let g and g' be nonconjugate elements of order 3.

Then C(g) and C(g') have normal 3-complements, and one of the following holds:

a) G is isomorphic to A_6.

b) q = 5, and the principal 3-block of G has the following form:

	1	[g]	[g']
χ_1	1	1	1
χ_2	64	1	1
χ_3	80	-1	-1
χ_4	640	1	1
χ_5	625	1	-2
χ_6	625	-2	1

c) $q = 7$, and the principal 3-block of G has the following form:

	1	[g]	[g']
χ_1	1	1	1
χ_2	224	-1	-1
χ_3	512	-1	-1
χ_4	784	1	1
χ_5	49	1	-2
χ_6	49	-2	1

Main Theorem, Part 3 Let G be a simple group such that:

1) $|G| = 9 \cdot 2^a \cdot q^b$, q prime.

2) G has two conjugacy classes of 3-elements.

3) Some 3-local subgroup of G is nonsolvable.

Let g and g' be nonconjugate 3-elements.

Then $q = 5$, $C(g)/O_{3'}(C(g)) \simeq \langle g \rangle \times A_5$ or $\langle g \rangle \times S_5$, and $C(g')/O_{3'}(C(g')) \simeq \langle g' \rangle \times A_5$ or $\langle g' \rangle \times S_5$. The principal 3-block of G has either the form

	1	[g]	[gv]	[gw]	[gx]	[gy]	[g']	[g'v']	[g'w']	[g'x']	[g'y']
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1024	1	1	1	1	1	4	-1	0	0	-2
χ_3	1024	1	1	1	1	1	-5	0	-1	-1	1
χ_4	1024	4	-1	0	0	-2	1	1	1	1	1
χ_5	400	4	-1	0	0	-2	4	-1	0	0	-2
χ_6	625	4	-1	0	0	-2	-5	0	-1	-1	1
χ_7	1024	-5	0	-1	-1	1	1	1	1	1	1
χ_8	625	-5	0	-1	-1	1	4	-1	0	0	-2
χ_9	400	-5	0	-1	-1	1	-5	0	-1	-1	1

or else the same form with 400 and 625 interchanged. $\bar{v}$, $\bar{w}$, $\bar{x}$, $\bar{y}$, $\overline{v'}$, $\overline{w'}$,

$\overline{x'}$, and $\overline{y'}$ denote elements of S_5: $\overline{v} = (12345)$, $\overline{w} = (12)(34)$, $\overline{x} = (1234)$, $\overline{y} = (12)$; $\overline{v'}$, $\overline{w'}$, $\overline{x'}$, and $\overline{y'}$ are defined analogously. If $C(g)/O_{3'}(C(g)) \simeq \langle g \rangle \times A_5$, the columns of [gx] and [gy] are not present, and the column of [gv] splits into two identical columns; an analogous statement holds with g replaced by g'.

<u>Main Theorem, Part 4</u> Let G be a simple group such that:

1) $|G| = 9 \cdot 2^a \cdot q^b$, q prime.

2) G has one conjugacy class of 3-elements.

Then $q = 5$ or $q = 7$. The principal 3-block of G does not contain a character of degree 8. $b \geq 3$; if $b = 3$, G has no elements of order $3q$, every 3-local subgroup of G is solvable, and there are 21 possibilities for the degrees of the principal 3-block characters (listed in Theorem 7.24). Finally, $|G| > 10^6$.

APPENDIX I. NOTATION

Groups, subgroups, elements: G and H denote groups, K and L denote subgroups of G, and g, g_1, g_2, g_3, ..., g_k denote elements of G

1	Identity element of G, or subgroup of G containing the identity only
$\langle g_1, g_2, \ldots, g_k \rangle$	Subgroup of G generated by g_1, g_2, ..., g_k
$\lvert G \rvert$	Order of G
$\lvert G:H \rvert$	Index in G of H
$g_1^{\,g}$	$g^{-1}g_1g$
$g_1 \sim g_2$	g_1 conjugate to g_2
$G \simeq H$	G isomorphic to H
$C_L(K)$	Centralizer in L of K
$N_L(K)$	Normalizer in L of K
$C_L(g)$	$C_L(\langle g \rangle)$
$C(K)$, $N(K)$, $C(g)$	$C_G(K)$, $N_G(K)$, $C_G(g)$
$Z(G)$	Center of G
G'	Commutator subgroup of G
$O_{r'}(G)$	Largest normal subgroup of G of order relatively prime to r
Aut(G)	Automorphism group of G
$G \times H$	Direct product of G and H
$G \cdot H$	A semidirect product of G by H (G is normal in $G \cdot H$, and H is homomorphic to a subgroup of Aut(G))
G^*	$G - \{1\}$
$V_{G \to K}$	Transfer of G into K

$\overline{g_1}$ — The image of g_1 under the natural homomorphism of $N(\langle g\rangle)$ onto $N(\langle g\rangle)/O_{r'}(N(\langle g\rangle))$. Definition requires that r be a prime, g be an r-element, and g_1 be an element of $N(\langle g\rangle)$.

$\overline{L}$ — $\{\overline{x} \mid x \in L\}$

$[g_1]$ — $\{x \in N(\langle g\rangle) \mid \overline{x} \sim \overline{g_1} \text{ in } \overline{N(\langle g\rangle)}\}$

Special groups

Z_i — Cyclic group of order i

Q_i — Quaternion group of order i

D_i — Dihedral group of order i

S_n — Symmetric group on n letters

A_n — Alternating group on n letters

$GL_n(q)$ — General linear group of degree n over GF(q)

$SL_n(q)$ — Special linear group of degree n over GF(q)

$PGL_n(q)$ — Projective general linear group of degree n over GF(q)

$PSL_n(q)$ — Projective special linear group of degree n over GF(q)

$U_n(q)$ — Projective special unitary group of degree n over $GF(q^2)$

Characters and blocks: χ and χ' denote characters of a group G, and η denotes a character of a subgroup H

χ^σ — Character of G defined by $\chi^\sigma(g) = \sigma(\chi(g))$ (σ a field automorphism)

$\chi|_H$ — Restriction of χ to H

η^G — Induced character

$\langle \chi, \chi' \rangle$	Inner product of χ and χ' $= \frac{1}{\lvert G \rvert} \sum_{g \in G} \chi(g)\overline{\chi'(g)}$
$\langle \chi, \eta \rangle$	$\langle \chi\rvert_H, \eta \rangle$
$B_0(p, H)$	Principal p-block of H
$B_0(p)$	Principal p-block of G
$B_0(p)^*$	Set of nonidentity characters of $B_0(p)$
$M_H(S)$	Module of generalized characters of H vanishing off the set S
$M_H(S, b)$	Submodule of $M_H(S)$ consisting of linear combinations of characters in the block b of H
$k(B)$	Number of ordinary irreducible characters in the block B
$\ell(B)$	Number of modular irreducible characters in the block B
$\lambda(\chi)$	Height of the character χ (defined in [7])
$\lambda(G, g)$	$\sum_{\chi \in B_0(p)} \lvert \chi(g) \rvert^2$ (p a given prime)
$\lambda(G)$	$\lambda(G, 1)$

<u>Number theory</u>

(m, n)	Greatest common divisor of m and n
$[s]$	Largest integer less than or equal to s
p, q, r	Prime numbers
$m \mid n$	m divides n
$\equiv$	Congruence
$\nu_p(n)$	Greatest integer k such that p^k divides n
ϵ	± 1
$\lvert s \rvert$	Absolute value of s

APPENDIX II. CHARACTER TABLES

This section contains partial character tables of groups arising as alternatives for $\overline{N(D)}$ in Theorem 5.4. Only the principal 3-block characters are given. Also, the tables for the groups in alternative i of the theorem are omitted.

The number of conjugates of an element t is denoted by h(t). In each case, g is a generator of the group D. The special classes are underlined; here a class is special if, for each element x in the class, some power of x equals g.

$S_3 \times S_3$

S_3: $g^3 = s^2 = 1$, $g^s = g^{-1}$

S_3: $h^3 = v^2 = 1$, $h^v = h^{-1}$

h(t)	1	2	3	2	4	6	3	6	9
t	1	<u>g</u>	s	h	gh	sh	v	<u>gv</u>	sv
η_1	1	1	1	1	1	1	1	1	1
η_2	1	1	-1	1	1	-1	1	1	-1
η_3	2	-1	0	2	-1	0	2	-1	0
η_4	1	1	1	1	1	1	-1	-1	-1
η_5	1	1	-1	1	1	-1	-1	-1	1
η_6	2	-1	0	2	-1	0	-2	1	0
η_7	2	2	2	-1	-1	-1	0	0	0
η_8	2	2	-2	-1	-1	1	0	0	0
η_9	4	-2	0	-2	1	0	0	0	0

$\underline{S_3 \times A_5}$ (principal 3-block) $\quad S_3$: $g^3 = s^2 = 1$, $g^s = g^{-1}$

A_5: $h = (123)$, $v = (12345)$, $w = (12)(34)$

h(t)	1	2	3	20	40	60	12	24	36	12	24	36	15	30	45
t	1	$\underline{g}$	s	h	gh	sh	v	$\underline{gv}$	sv	v^2	$\underline{gv^2}$	sv^2	w	$\underline{gw}$	sw
η_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
η_2	1	1	-1	1	1	-1	1	1	-1	1	1	-1	1	1	-1
η_3	2	-1	0	2	-1	0	2	-1	0	2	-1	0	2	-1	0
η_4	4	4	4	1	1	1	-1	-1	-1	-1	-1	-1	0	0	0
η_5	4	4	-4	1	1	-1	-1	-1	1	-1	-1	1	0	0	0
η_6	8	-4	0	2	-1	0	-2	1	0	-2	1	0	0	0	0
η_7	5	5	5	-1	-1	-1	0	0	0	0	0	0	1	1	1
η_8	5	5	-5	-1	-1	1	0	0	0	0	0	0	1	1	-1
η_9	10	-5	0	-2	1	0	0	0	0	0	0	0	2	-1	0

$\underline{D \cdot S_5}$ (principal 3-block) $\quad D$: $g^3 = 1$

S_5: $h = (123)$, $v = (12345)$, $w = (12)(34)$,

$x = (45)$, $y = (1234)$, $k = hx$

$g^t = g^{-1}$ if $t \in S_5$, $t \notin A_5$

h(t)	1	2	20	40	24	24	24	15	30	30	90	60
t	1	$\underline{g}$	h	gh	v	$\underline{gv}$	$\underline{gv^2}$	w	$\underline{gw}$	x	y	k
η_1	1	1	1	1	1	1	1	1	1	1	1	1
η_2	1	1	1	1	1	1	1	1	1	-1	-1	-1
η_3	2	-1	2	-1	2	-1	-1	2	-1	0	0	0
η_4	4	4	1	1	-1	-1	-1	0	0	-2	0	1
η_5	4	4	1	1	-1	-1	-1	0	0	2	0	-1
η_6	8	-4	2	-1	-2	1	1	0	0	0	0	0
η_7	5	5	-1	-1	0	0	0	1	1	1	-1	1
η_8	5	5	-1	-1	0	0	0	1	1	-1	1	-1
η_9	10	-5	-2	1	0	0	0	2	-1	0	0	0

$S_3 \times S_5$ (principal 3-block) S_3: $g^3 = s^2 = 1$, $g^s = g^{-1}$

S_5: $h = (345)$, $v = (12345)$, $w = (12)(34)$, $x = (1234)$, $y = (12)$, $k = hy$

h(t)	1	2	3	20	40	60	24	48	72	15	30	45	30	60	90	10	20	30	20	40	60
t	1	$\underline{g}$	s	h	gh	sh	v	$\underline{gv}$	sv	w	$\underline{gw}$	sw	x	$\underline{gx}$	sx	y	$\underline{gy}$	sy	k	gk	sk
η_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
η_2	1	1	-1	1	1	-1	1	1	-1	1	1	-1	1	1	-1	1	1	-1	1	1	-1
η_3	2	-1	0	2	-1	0	2	-1	0	2	-1	0	2	-1	0	2	-1	0	2	-1	0
η_4	4	4	4	1	1	1	-1	-1	-1	0	0	0	0	0	0	-2	-2	-2	1	1	1
η_5	4	4	-4	1	1	-1	-1	-1	1	0	0	0	0	0	0	-2	-2	2	1	1	-1
η_6	8	-4	0	2	-1	0	-2	1	0	0	0	0	0	0	0	-4	2	0	2	-1	0
η_7	5	5	5	-1	-1	-1	0	0	0	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
η_8	5	5	-5	-1	-1	1	0	0	0	1	1	-1	1	1	-1	-1	-1	1	-1	-1	1
η_9	10	-5	0	-2	1	0	0	0	0	2	-1	0	2	-1	0	-2	1	0	-2	1	0

$S_3 \times PSL_2(7)$ (principal 3-block) S_3: $g^3 = s^2 = 1$, $g^s = g^{-1}$

$PSL_2(7)$: h, v, and w are elements order 3, 4, and 7 respectively

h(t)	1	2	3	56	112	168	42	84	126	21	42	63	24	48	72	24	48	72
t	1	$\underline{g}$	s	h	gh	sh	v	$\underline{gv}$	sv	v^2	$\underline{gv}^2$	sv^2	w	$\underline{gw}$	sw	w^6	$\underline{gw}^6$	sw^6
η_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
η_2	1	1	-1	1	1	-1	1	1	-1	1	1	-1	1	1	-1	1	1	-1
η_3	2	-1	0	2	-1	0	2	-1	0	2	-1	0	2	-1	0	2	-1	0
η_4	7	7	7	1	1	1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0
η_5	7	7	-7	1	1	-1	-1	-1	1	-1	-1	1	0	0	0	0	0	0
η_6	14	-7	0	2	-1	0	-2	1	0	-2	1	0	0	0	0	0	0	0
η_7	8	8	8	-1	-1	-1	0	0	0	0	0	0	1	1	1	1	1	1
η_8	8	8	-8	-1	-1	1	0	0	0	0	0	0	1	1	-1	1	1	-1
η_9	16	-8	0	-2	1	0	0	0	0	0	0	0	2	-1	0	2	-1	0

$D \cdot PGL_2(7)$ (principal 3-block) D: $g^3 = 1$

$PGL_2(7)$: h, w, r, and z are elements of order 3, 7, 8, and 2 respectively, with $r^z = r^3$. Let $v = r^2$.

h(t)	1	2	56	112	42	84	21	42	48	48	48	168	84	126	126
t	1	$\underline{g}$	h	gh	v	$\underline{gv}$	v^2	$\underline{gv^2}$	w	$\underline{gw}$	$\underline{gw^6}$	hz	z	r	r^7
η_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
η_2	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
η_3	2	-1	2	-1	2	-1	2	-1	2	-1	-1	0	0	0	0
η_4	7	7	1	1	-1	-1	-1	-1	0	0	0	1	1	-1	-1
η_5	7	7	1	1	-1	-1	-1	-1	0	0	0	-1	-1	1	1
η_6	14	-7	2	-1	-2	1	-2	1	0	0	0	0	0	0	0
η_7	8	8	-1	-1	0	0	0	0	1	1	1	-1	2	0	0
η_8	8	8	-1	-1	0	0	0	0	1	1	1	1	-2	0	0
η_9	16	-8	-2	1	0	0	0	0	2	-1	-1	0	0	0	0

$S_3 \times PGL_2(7)$ (prin. 3-block) S_3: $g^3 = s^2 = 1$, $g^s = g^{-1}$

$PGL_2(7)$: $h^3 = 1$, $w^7 = 1$, $v^8 = z^2 = 1$, $v^z = v^3$

h(t)	1	2	3	56	112	168	42	84	126	42	84	126
t	1	$\underline{g}$	s	h	gh	sh	v	$\underline{gv}$	sv	v^7	$\underline{gv^7}$	sv^7
η_1	1	1	1	1	1	1	1	1	1	1	1	1
η_2	1	1	-1	1	1	-1	1	1	-1	1	1	-1
η_3	2	-1	0	2	-1	0	2	-1	0	2	-1	0
η_4	7	7	7	1	1	1	-1	-1	-1	-1	-1	-1
η_5	7	7	-7	1	1	-1	-1	-1	1	-1	-1	1
η_6	14	-7	0	2	-1	0	-2	1	0	-2	1	0
η_7	8	8	8	-1	-1	-1	0	0	0	0	0	0
η_8	8	8	-8	-1	-1	1	0	0	0	0	0	0
η_9	16	-8	0	-2	1	0	0	0	0	0	0	0

continued on next page

$S_3 \times PGL_2(7)$ (continued)

42	84	126	28	56	84	48	96	144	21	42	63	56	112	168
v^2	$\underline{gv}^2$	sv^2	z	$\underline{gz}$	sz	w	$\underline{gw}$	sw	v^4	$\underline{gv}^4$	sv^4	hz	ghz	shz
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	-1	1	1	-1	1	1	-1	1	1	-1	1	1	-1
2	-1	0	2	-1	0	2	-1	0	2	-1	0	2	-1	0
-1	-1	-1	1	1	1	0	0	0	-1	-1	-1	1	1	1
-1	-1	1	1	1	-1	0	0	0	-1	-1	1	1	1	-1
-2	1	0	2	-1	0	0	0	0	-2	1	0	2	-1	0
0	0	0	2	2	2	1	1	1	0	0	0	-1	-1	-1
0	0	0	2	2	-2	1	1	-1	0	0	0	-1	-1	1
0	0	0	4	-2	0	2	-1	0	0	0	0	-2	1	0

REFERENCES

1. R. Brauer, Investigations on group characters, Ann. of Math. 42 (1941), 936-958

2. R. Brauer, On groups whose order contains a prime number to the first power, part I, Amer. J. Math. 64 (1942), 401-420.

3. R. Brauer, Zur Darstellungstheorie der Gruppen endlicher Ordnung, part I, Math. Zeit. 63 (1956), 406-444; part II, Math. Zeit. 72 (1959), 25-46.

4. R. Brauer, Some applications of the theory of blocks of characters of finite groups, part I, J. Algebra 1 (1964), 152-167; part II, J. Algebra 1 (1964), 307-334.

5. R. Brauer, On blocks and sections in finite groups, part I, Amer. J. Math. 89 (1967), 1115-1136; part II, Amer. J. Math. 90 (1968), 895-925.

6. R. Brauer, On simple groups of order $5 \cdot 3^a \cdot 2^b$, Bull. Amer. Math. Soc. 74 (1968), 900-903.

7. R. Brauer and W. Feit, On the number of irreducible characters of finite groups in a given block, Proc. Nat. Acad. Sci. 45 (1959), 361-365.

8. R. Brauer and H. S. Leonard, Jr., On finite groups with an abelian Sylow group, Canad. J. Math. 14 (1962), 436-450.

9. R. Brauer and H. F. Tuan, On simple groups of finite order, Bull. Amer. Math. Soc. 51 (1945), 756-766.

10. W. Burnside, "The Theory of Groups," 2nd ed., Cambridge University Press, Cambridge, 1911.

11. C. W. Curtis and I. Reiner, "Representation Theory of Finite Groups and Associative Algebras," Interscience, New York, 1962.

12. E. C. Dade, Blocks with cyclic defect groups, Ann. of Math. 84 (1966), 20-48.

13. W. Feit, The current situation in the theory of finite simple groups, to appear.

14. W. Feit and J. G. Thompson, Finite groups which contain a self-centralizing subgroup of order 3, Nagoya J. Math. 21 (1962), 185-197.

15. D. Gorenstein, "Finite Groups," Harper and Row, New York, 1968.

16. J. A. Green, Blocks of modular representations, Math. Zeit. 79 (1962), 100-115.

17. M. Hall, Jr., "The Theory of Groups," Macmillan, New York, 1959.

18. M. Hall, Jr., A search for simple groups of order less than one million, in "Computational Problems in Abstract Algebra," ed. by John Leech, Pergamon Press, Oxford, 1969.

19. M. Herzog, On groups of order $2^{\alpha}3^{\beta}p^{\gamma}$ with a cyclic Sylow 3-subgroup, Proc. Amer. Math. Soc. 24 (1970).

20. G. Higman, Odd characterizations of finite simple groups, Michigan Notes, Summer, 1968.

21. B. Huppert, "Endliche Gruppen I," Springer-Verlag, Berlin, 1967.

22. J. Leon, On simple groups of order $2^a3^b5^c$ containing a cyclic Sylow subgroup, to appear in J. Algebra.

23. W. F. Reynolds, Isometries and principal blocks of group characters, Math. Zeit. 107 (1968), 264-270.

24. E. Schenkman, "Group Theory," Van Nostrand, Princeton, 1965.

25. I. Schur, Uber eine Klasse von endlichen Gruppen linearer Substitutionen, Sitz. der Preussischen Akad. Berlin (1905), 77-91.

26. R. G. Stanton, The Mathieu groups, Canad. J. Math. 3 (1951), 164-174.

27. R. Steinberg, Automorphisms of finite linear groups, Canad. J. Math. 12 (1960), 606-615.

28. J. G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable, Bull. Amer. Math. Soc. 74 (1968), 383-437.

29. D. Wales, Simple groups of order $7\cdot3^a\cdot2^b$, J. Algebra 16 (1970), 575-596.

30. D. Wales, Simple groups of order $17\cdot3^a\cdot2^b$, to appear in J. Algebra.

31. D. Wales, Simple groups of order $13\cdot3^a\cdot2^b$, to appear in J. Algebra.

32. W. J. Wong, Exceptional character theory and the theory of blocks, Math. Zeit. 91 (1966), 363-379.