FEEDBACK SERVO-STABILIZATION OF A ROCKET DURING TAKE-OFF

Thesis by

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SUMMARY

When a rocket is launched, there is a short initial period of acceleration during which the rocket is unstable. As the flight velocity increases, the aerodynamic forces acting on the fins and stabilizers become large enough to give stability. Various methods have been employed to stabilize the rocket during this launching period. Guide rails, "zero length" launchers, booster rockets which produce high initial acceleration, and auto-pilot controlled nozzles are typical devices that have been used.

This is an investigation of the requirements of a nozzle control which would stabilize the rocket during the launching period. The configuration investigated is unique in that the nozzle of the rocket is mounted as a compound pendulum, and the movement of the pendulum is utilized to furnish the signal for the nozzle control servo-mechanism, thereby eliminating the need for gyroscopic elements in the control system. The pendulum motion of the nozzle caused by a change in flight attitude of the rocket is introduced into a computer which produces an output signal proportional to the attitude of the rocket. This attitude signal is fed back to the nozzle control, which positions the nozzle.

The results of the analysis showed that the rocket was unstable during the take-off period when the nozzle control acted on the rocket attitude signal alone. Stability over a narrow range of feedback gains was indicated for the system using a simple lead circuit as a nozzle control, or in other words, when the nozzle control acted on both the

attitude signal and the rate of change of attitude of the rocket. The damping characteristics of this system were poor. By changing the nozzle control function to include a response to the acceleration of the rocket attitude, the damping characteristics were improved and the range of feedback gains was widened.

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q_i generalized coordinate

$\mathcal{Q}_i$ generalized force

Q_i Laplace transformed coordinate

ℓ_i generalized length

T kinetic energy

M_1 mass of the rocket independent of the mass of the motor and nozzle

M_2 mass of the motor and nozzle

I_1 moment of inertia of the rocket about its center of gravity

I_2 moment of inertia of the motor about its center of gravity

F rocket thrust

D jet damping force

I_{sp} specific impulse of the rocket

k radius of gyration

g acceleration of gravity

I. INTRODUCTION

The rocket stability problem has existed for many centuries. Historians record that rockets were first used in battle by the Chinese in 1232 A.D. By 1400, rockets were in widespread use as major weapons by the warring nations of Europe. The poor stability of the rocket in flight was responsible for its gradual replacement by the cannon which gave more accurate trajectories. In the late 1700's, India employed rockets with improved stability characteristics against the British. When British soldiers attempted to defeat the forces of Hydar Ali, Prince of Mysore, a trained corps of rocket bombardiers launched rocket missiles at the charging British cavalry, inflicting many casualties. The Indian rockets were made of iron cylinders about eight inches long and two inches in diameter, with a bamboo shaft about ten feet long attached to give stability.

An English military officer, Colonel William Congreve, became interested in the Indian advances in rocket development, and worked to further improve the rocket for British use. Congreve devised a launching system consisting of a tube mounted at an angle on a tripod base, similar to the mortar. Long shafts were attached to the rockets to keep them flying in a straight line. The British employed rockets against Napoleon at Boulogne, and in other bombardments with great success.

During the War of 1812, Americans were first introduced to rocket warfare when the British bombarded Bladenburg, Maryland, and subsequently Ft. McHenry at Baltimore. An American by the name of

William Hale improved the stability of the Congreve rocket by replacing the wooden shaft with three curved fins on the rear part of the rocket body. The exhaust gases were deflected by the fins, causing the rocket to spin in flight. These rockets were used in the war with Mexico, and were found to have superior trajectories to the stick stabilized types. Again the rocket declined in popularity partly because of its inferior trajectory compared to that of the gun. No other outstanding developments in rocket stabilization were made following Hale's work until early in this century.

With the advent of the airplane, and an increase in knowledge in the field of aerodynamics, the stability problem of the rocket in flight became defined more clearly. The use of tail fins, or tail fins with a shroud ring as on the Bazooka, and later, the use of controlled airfoils solved the in-flight stability problem to a great extent.

In the first part of the rocket's flight path there is a destabilizing effect from the change in mass of the rocket as fuel is burned. The aerodynamic forces must counter this tendency in order to stabilize the rocket. During the launching period, the aerodynamic forces are too small to counteract the effect of the change in mass, and the rocket is unstable or at best neutrally stable. Consequently, stability during the critical take-off period must be attained by some means other than the use of aerodynamic surfaces.

Various means have been employed to stabilize the rocket during the launching period. Guide rails to hold the rocket to a straight flight path until it has gained sufficient speed to become aerodynamically stable have been used successfully. The length of the guide rail has been reduced to a minimum in some applications by firing the rocket with a very high initial acceleration. The German V-2 rocket was stabilized during the launch by an automatic pilot which controlled four carbon vanes in the jet stream behind the nozzle. The American "Viking" and the German "Enzian" employed a controllable nozzle positioned by an automatic pilot.

A unique means of launching a large rocket with a booster rocket mounted free to swing as a compound pendulum was investigated by J. C. Norris in 1951 following a proposal by H. S. Tsien of Cal Tech. (Ref. 1). For the system investigated, the booster rocket was mounted below the main rocket on gimbals off the center of gravity of the booster as shown in Fig. 1. A deviation of the main rocket from its course would cause motion of the booster relative to the main rocket causing a change in the direction of the thrust axis. It was hoped that a practical configuration could be devised to give launching stability. The results of the investigation were somewhat discouraging in that for a typical vehicle-booster combination for which stability was indicated, the required length of the tail boom was too great to be practical.

Since the basic idea of the pendulum mounted thrust unit appears sound, the possibility of utilizing the pendulum in conjunction with a servo-control is suggested.

In this thesis, the requirements of such a servo-control system which would stabilize the rocket during the launching period are investigated. The rocket has a controllable nozzle mounted as a compound pendulum; conceivably, the motor and nozzle of a liquid propellant

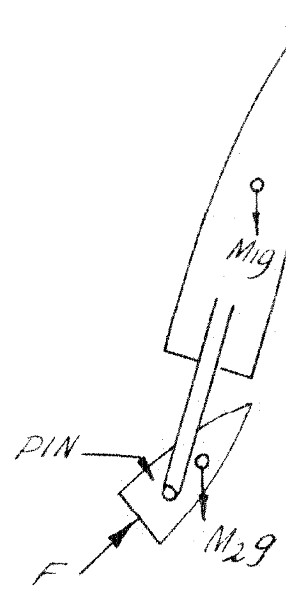

Fig. 1 Compound Pendulum Booster Rocket Suspension

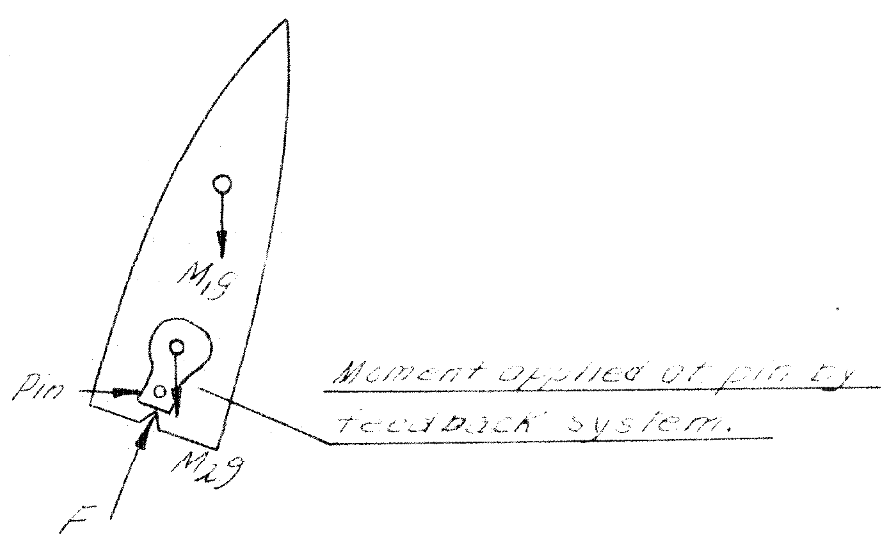

Fig. 2 Modified Pendulum Suspension with Feedback Control

rocket could be mounted as a unit as shown in Fig. 2. The motion of the pendulum with respect to the rocket is fed to a computer which calculates the attitude of the rocket. The attitude signal from the computer is fed to a nozzle control. The aim of this investigation is to determine the requirements of the nozzle control such that it will stabilize the rocket during the launching period.

The equations of motion of the configuration were first derived, assuming all aerodynamic forces to be negligibly small during the launching period. Motion in one plane only was considered. External forces considered were the rocket thrust, assumed to be constant; the jet damping force arising from rotation of the jet stream; and the gravity force. The effect of the change in mass of the rocket resulting from the burning of fuel was considered. The displacements of the rocket and nozzle about the initial condition were assumed to be small quantities for the period of the stability investigation.

Following the derivation, the equations involving the pitching motion of the rocket and the motion of the nozzle were transformed to simple algebraic equations by use of the Laplace Transform. From the transformed equations, it was possible to solve for the transfer functions of all of the components of the control system, as shown in Fig. 4, except that of the nozzle control. The form of the nozzle control transfer function was estimated, and the stability of the resulting system was investigated by use of the Nyquist Stability Criterion. The Evans Root Locus plot was used to find the damping coefficients of the modes of oscillation of the system. Successive estimates of the form of the nozzle control function were made in order to improve the stability of the system.

It was found that for a nozzle control function employing feedback of the rocket attitude signal only, the system was unstable. For a control function employing feedback of both the rocket attitude and rate of change of attitude, it was determined that a stable range existed between upper and lower limits of feedback gain; the limits depending on the constants of the control function. The damping of the system was poor.

The nozzle control function was revised by introducing additional lead circuits, or sensitivity to the accelerative change of attitude of the rocket. The damping was improved by this change, and the range of gains was widened.

II. ANALYSIS

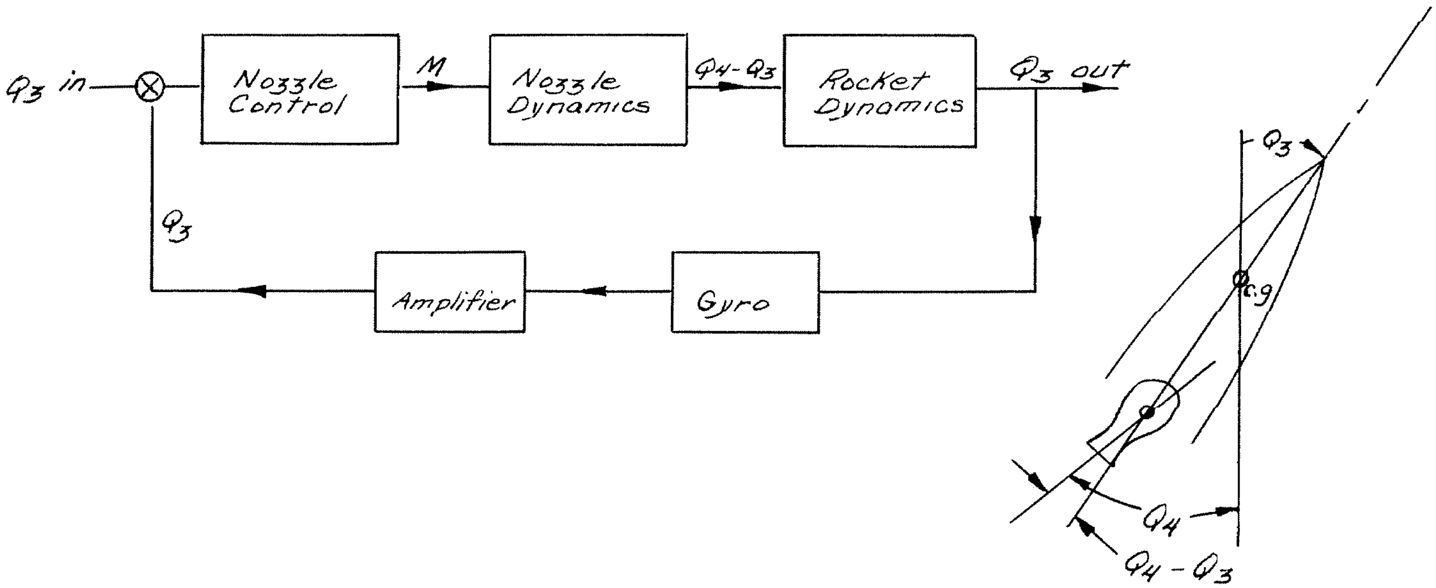

Fig. 3. Block Diagram of a Feedback Control System

A typical feedback control system which might be employed to guide a rocket of the controllable nozzle type is shown in Fig. 3 above. The attitude of the rocket, as indicated by the angle φ_3 which the rocket makes with the vertical, is measured by an attitude gyro. The gyro signal is matched with the desired rocket attitude signal and the error, or difference, is fed to a nozzle control. The nozzle control applies a moment "M" to the nozzle which results in a relative angular displacement "$\varphi_4 - \varphi_3$" of the nozzle with respect to the rocket. The thrust axis rotates with the nozzle, causing a change in the rocket attitude.

Now consider the case of a rocket with the motor and nozzle mounted as a compound pendulum as shown in Figs. 1 and 2. With any change in rocket attitude, the pendulum moves in a calculable manner. From a knowledge of the system dynamics and a time history of the pendulum motion, the attitude of the rocket may be computed. Suppose, then, that the angular displacement of the nozzle from the center line of the rocket, $q_4 - q_3$, is measured and fed as an electrical or mechanical signal into a computer with an output representing the attitude of the rocket q_3. The gyroscopic element is thereby eliminated from the circuit.

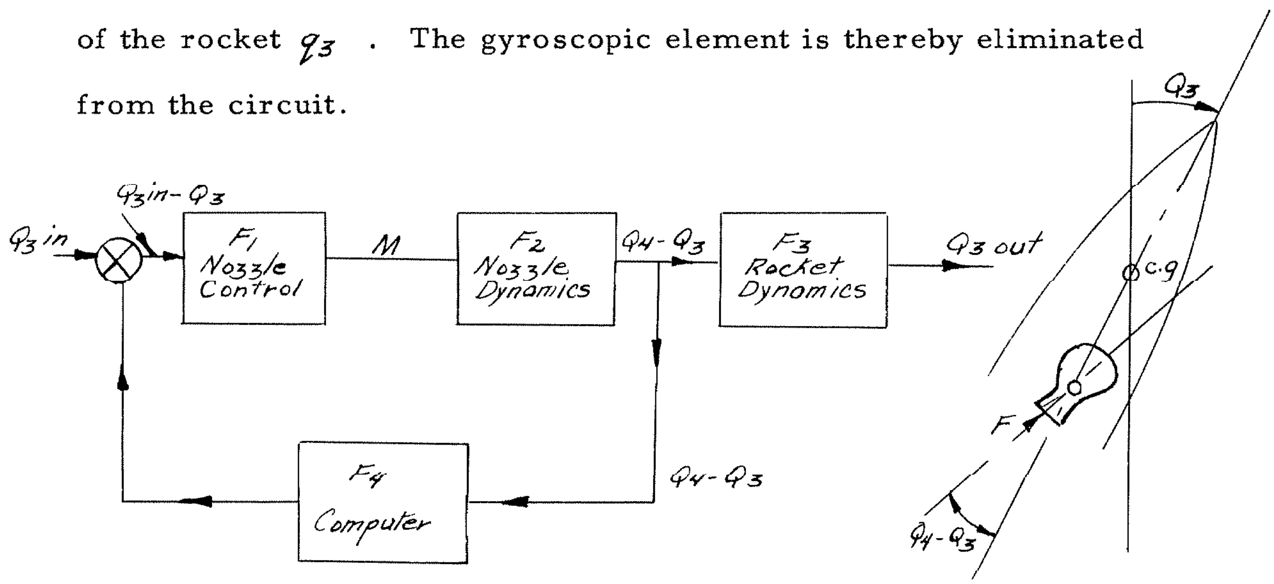

Fig. 4. Block Diagram of a Feedback Control System Employing a Compound Pendulum Nozzle

The block diagram of the proposed feedback control circuit as shown in Fig. 4 above may be interpreted as follows.

The desired input of the coordinate "q_3", or angle which the rocket makes with the vertical, is matched with a feedback signal representing the actual angle at any given time. The difference between the desired angle (in this case it is desired that $q_3 = 0$ for vertical takeoff) and the computed angle is fed into the nozzle control mechanism as an error signal, "$q_{3(o)} - q_3$". The error signal initiates the application of a moment "M" to the nozzle gimbals. The moment contributes to the nozzle dynamics to yield a certain position of the nozzle relative to the rocket, which is represented by $q_4'' - q_3''$. The thrust line is rotated off the center line of the rocket by movement of the nozzle; this contributes a moment and side force to the overall rocket dynamics, resulting in a final rocket position.

The functions F_2, F_3, and F_4 appearing in the block diagram of Fig. 4 are transfer functions derived from the equations of motion of the configuration which have been transformed by the use of the Laplace Transformation. The transfer functions are linear operators and can be defined by inspection of the block diagram of the system (i.e., $F_2 = \frac{Q_4 - Q_3}{M}$, $F_3 = F_4 = \frac{Q_3}{Q_4 - Q_3}$). The nozzle control transfer function F_1 is the unknown quantity which is the subject of this investigation.

The equation relating the output of the system to the input may be derived as follows:

Following the circuit from $Q_{3\,IN}$ to $Q_{3\,OUT}$

$$Q_{3\,OUT} = \left[Q_{3\,IN} - Q_3\right] F_1 F_2 F_3$$

Around the loop

$$Q_3 = \left[Q_{3\,IN} - Q_3\right] F_1 F_2 F_4$$

$$Q_3 = \frac{F_1 F_2 F_4 Q_{3\,IN}}{1 + F_1 F_2 F_4}$$

Substituting for Q_3 in the first equation

$$\frac{Q_{3\,OUT}}{Q_{3\,IN}} = \frac{\text{Output}}{\text{Input}} = \left[1 - \frac{F_1 F_2 F_4}{1 + F_1 F_2 F_3}\right] F_1 F_2 F_3 = \frac{F_1 F_2 F_3}{1 + F_1 F_2 F_4}$$

The System Transfer Function therefore becomes:

$$K_s G_s = \frac{\text{OUTPUT}}{\text{INPUT}} = \frac{F_1 F_2 F_3}{1 + F_1 F_2 F_4}$$

The stability analysis thus involves 1) derivation of the equations of motion for the rocket and movable nozzle configuration, 2) conversion of these equations by the Laplace Transform to solve for the transfer functions F_2 , F_3 , and F_4 , and finally, 3) synthesis of a nozzle control function F_1 , and investigation of the complete system transfer function $K_s G_s$ for its stability characteristics.

The Nyquist Stability Criterion and the Evans Root Locus Method are applied in the stability investigation. Refs. (2) and (3) contain explanations of the use of these and other methods of analysis applicable in the field of stability and control.

III. EQUATIONS OF MOTION

The assumed rocket configuration with all external forces to be considered is shown in Fig. 5. A schematic diagram of the coordinate system and dimensions of the configuration are shown in Fig. 6.

Frictional and aerodynamic forces are neglected. The change of mass of the main rocket body with burning of fuel is considered. Motion in one plane only is considered.

It is convenient to employ coordinates q_1 through q_8 in the derivation, but of these, only four coordinates, q_1 through q_4 , are independent. The coordinates q_1 and q_2 measure the horizontal and vertical distances respectively to the center of gravity of the entire configuration, and have the dimensions of length. Coordinates q_3 and q_4 measure the angular displacement of the main rocket center line and the nozzle line respectively from the vertical, and have the dimensions of the radian.

The equations of motion in the four degrees of freedom, namely in the directions of coordinates q_1 , q_2 , q_3 , and q_4 are derived using the generalized Lagrangian momentum equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \qquad (1)$$

Elimination of Extraneous Coordinates

By taking moments about the center of gravity of the system as shown schematically in Fig. 7, the relation between the system center of gravity and the centers of gravity of the main rocket body and the nozzle may be found:

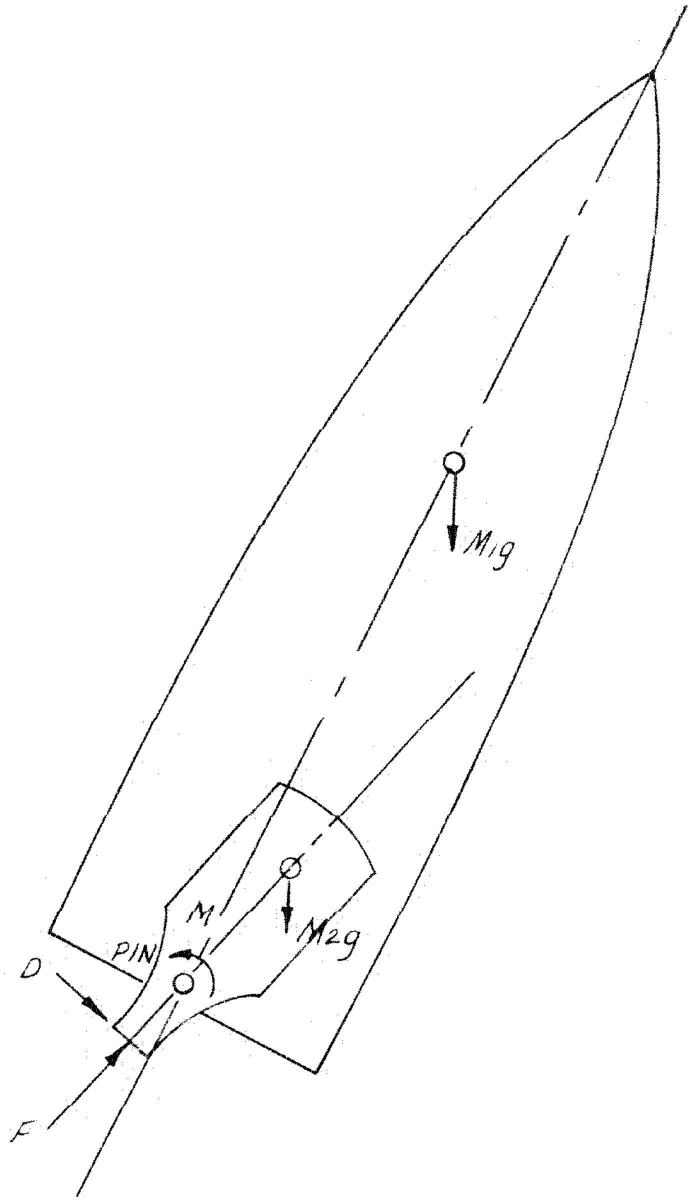

Fig. 5 Sketch Showing External Forces on Main Rocket Body and on Motor.

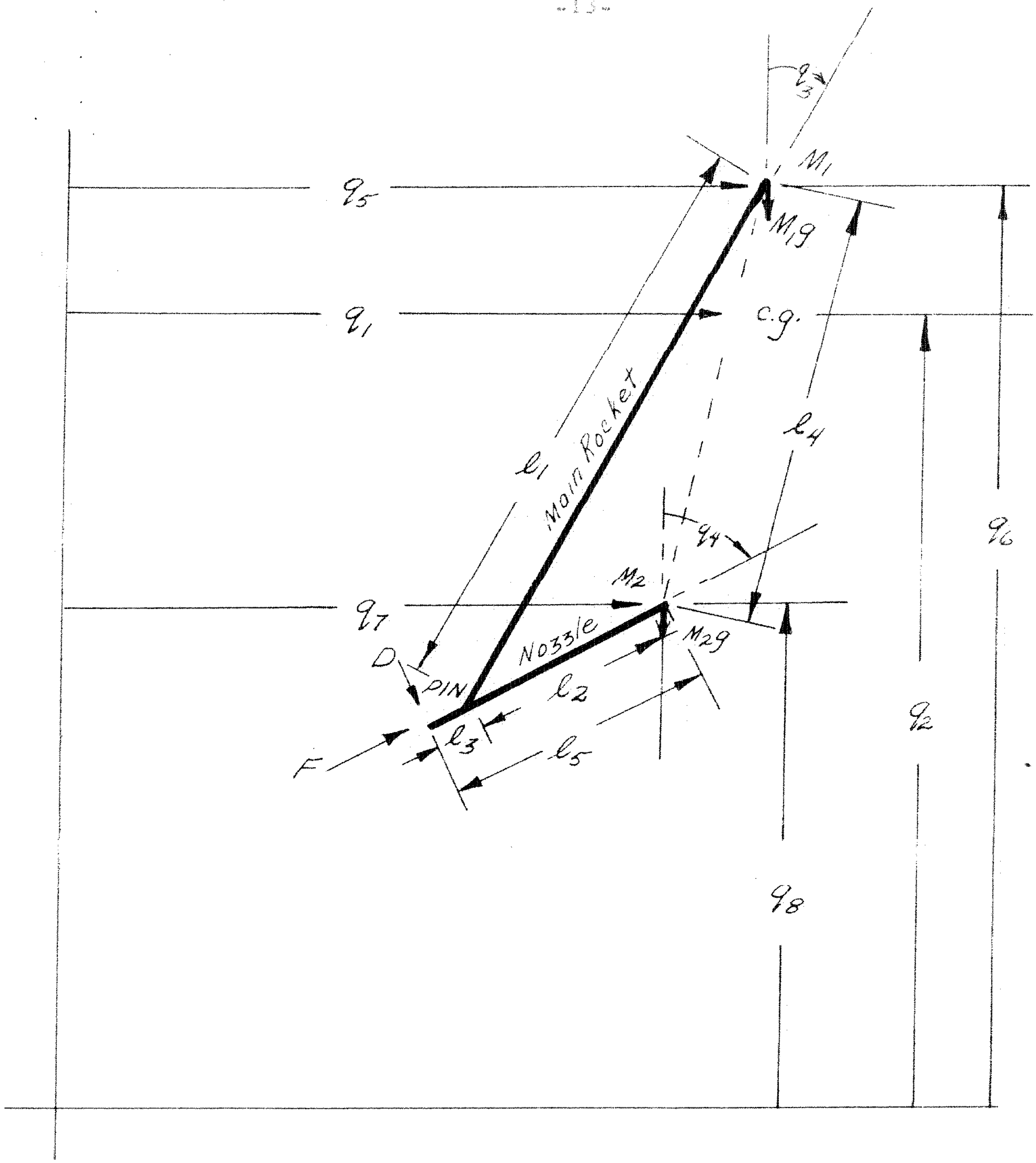

Fig. 6 Schematic Diagram of Coördinate System and Dimensions

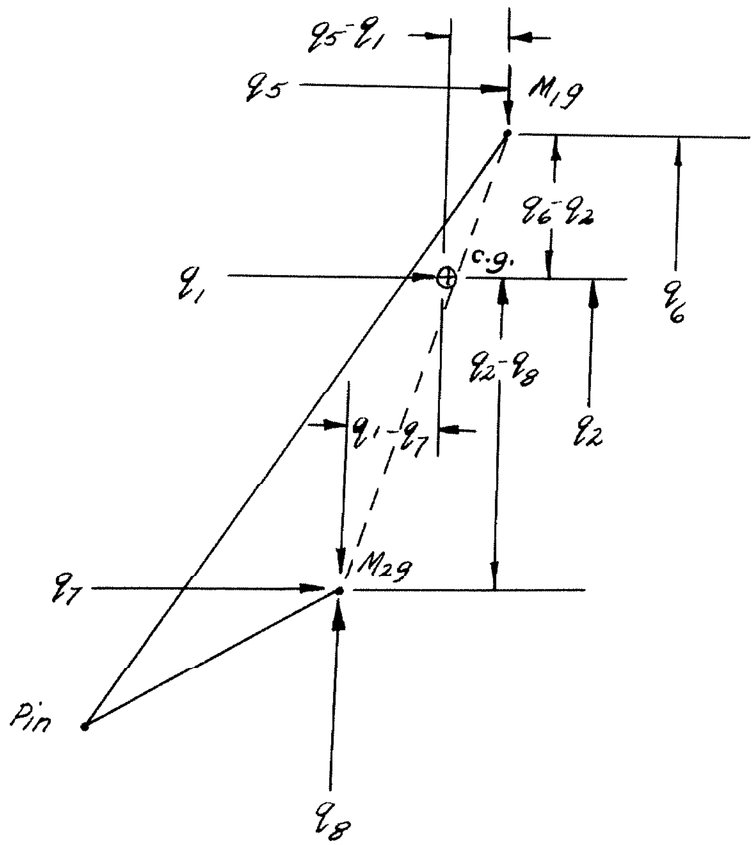

Fig. 7. Sketch Showing the Relation Between the Centers of Mass, M_1 and M_2, and the Center of Gravity of the System

$$\Sigma M_{c.g.} = M_1 g\,(q_5 - q_1) - M_2 g\,(q_1 - q_2) = 0$$

$$M_1\,(q_5 - q_1) = M_2\,(q_1 - q_7) \qquad (2)$$

From the geometry of the system as shown in Fig. 8 the following relations can be obtained:

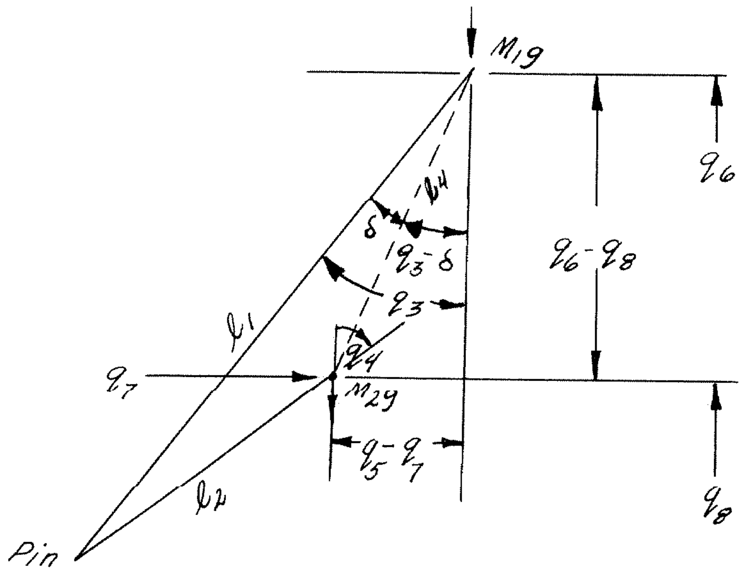

Fig. 8. Geometry of the Configuration

Equating the vertical distances between mass centers:

$$q_6 - q_8 = \ell_4 \cos(q_3 - \delta)$$

$$\ell_4 = \frac{q_6 - q_8}{\cos(q_3 - \delta)} \tag{3}$$

Equating the horizontal distances between mass centers:

$$q_5 - q_7 = \ell_4 \sin(q_3 - \delta)$$

$$\ell_4 = \frac{q_5 - q_7}{\sin(q_3 - \delta)} \tag{4}$$

By considering similar triangles in Fig. 8 it follows that:

$$\frac{q_6 - q_2}{q_5 - q_1} = \frac{q_2 - q_8}{q_1 - q_7}$$

$$\frac{M_1(q_6 - q_2)}{M_1(q_5 - q_1)} = \frac{M_2(q_2 - q_8)}{M_2(q_1 - q_7)} \tag{5}$$

Substituting Eq. (2) into (5), the result is

$$M_1(q_6 - q_2) = M_2(q_2 - q_8) \tag{6}$$

Eqs. (2), (3), (4), and (6) are four linear expressions containing the four extraneous coordinates q_5, q_6, q_7, q_8 and the principal coordinates q_1, q_2, q_3 and q_4. The former may be expressed in terms of the principal coordinates as follows:

From Eq. (3)

$$q_6 = l_4 \cos(q_3 - \delta) + q_8 \tag{7}$$

From Eq. (4)

$$q_7 = q_5 - l_4 \sin(q_3 - \delta) \tag{8}$$

Substituting (8) into (2)

$$q_5 = q_1 + \frac{M_2}{M_1 + M_2} l_4 \sin(q_3 - \delta) \tag{9}$$

Substituting (7) into (6)

$$q_8 = q_2 - \frac{M_1}{M_1 + M_2} l_4 \cos(q_3 - \delta) \tag{10}$$

Substituting (9) into (8)

$$q_7 = q_1 - \frac{M_1}{M_1 + M_2} l_4 \sin(q_3 - \delta) \tag{11}$$

Substituting (10) into (7)

$$q_6 = q_2 + \frac{M_2}{M_1 + M_2} l_4 \cos(q_3 - \delta) \tag{12}$$

It will be noted that the above equations contain the variables l_4, and δ from Fig. 7 which must be expressed in terms of the principal coordinates.

In Fig. 8 by the law of cosines

$$\ell_4 = \sqrt{\ell_1^2 + \ell_2^2 - 2\,\ell_1 \ell_2 \cos(q_4 - q_3)} \tag{13}$$

and by the law of sines

$$\frac{\ell_2}{\sin\delta} = \frac{\ell_4}{\sin(q_4 - q_3)}$$

$$\delta = \sin^{-1}\left[\frac{\ell_2}{\ell_4}\sin(q_4 - q_3)\right]$$

$$q_3 - \delta = q_3 - \sin^{-1}\left[\frac{\ell_2}{\ell_4}\sin(q_4 - q_3)\right] \tag{14}$$

To simplify Eqs. (13) and (14) consider the following approximations:

$$q_i \simeq \sin q_i \qquad\qquad \cos q_i \simeq 1$$

Then from Eq. (13)

$$\ell_4 \simeq \sqrt{\ell_1^2 + \ell_2^2 - 2\,\ell_1 \ell_2} \simeq \ell_1 - \ell_2$$

From Eq. (14)

$$q_3 - \delta \simeq q_3 - \frac{\ell_2}{\ell_4}(q_4 - q_3)$$

$$q_3 - \delta \simeq q_3 \frac{\ell_4 + \ell_2}{\ell_4} - \frac{\ell_2}{\ell_4} q_4$$

and since

$$\ell_4 \simeq \ell_1 - \ell_2$$

$$q_3 - \delta = \frac{\ell_1}{\ell_1 - \ell_2} q_3 - \frac{\ell_2}{\ell_1 - \ell_2} q_4 \simeq \sin(q_3 - \delta)$$

The approximations then are

$$\left.\begin{aligned} \sin(q_3-\delta) &\simeq \frac{\ell_1}{\ell_1-\ell_2}\,q_3 - \frac{\ell_2}{\ell_1-\ell_2}\,q_4 \\ \cos(q_3-\delta) &\simeq 1 \\ \ell_4 &\simeq \ell_1-\ell_2 \end{aligned}\right\} \tag{15}$$

Substituting (15) into (9), (12), (11), and (10) yields the following relations between the extraneous and the principal coordinates:

$$q_5 \simeq q_1 + \frac{M_2}{M_1+M_2}\left(\ell_1 q_3 - \ell_2 q_4\right) \tag{16}$$

$$q_6 \simeq q_2 + \frac{M_2}{M_1+M_2}\left(\ell_1 - \ell_2\right) \tag{17}$$

$$q_7 \simeq q_1 - \frac{M_1}{M_1+M_2}\left(\ell_1 q_3 - \ell_2 q_4\right) \tag{18}$$

$$q_8 \simeq q_2 - \frac{M_1}{M_1+M_2}\left(\ell_1 - \ell_2\right) \tag{19}$$

<u>The Kinetic Energy Term</u>

The kinetic energy T appearing in the Lagrangian equation of motion may be derived as the sum of kinetic energies of translation and rotation by referring to Fig. 6:

$$T = \tfrac{1}{2} I_1 \dot{q}_3^{\,2} + \tfrac{1}{2} I_2 \dot{q}_4^{\,2} + \tfrac{1}{2} M_1\left(\dot{q}_5^{\,2} + \dot{q}_6^{\,2}\right) + \tfrac{1}{2} M_2\left(\dot{q}_7^{\,2} + \dot{q}_8^{\,2}\right) \tag{20}$$

<u>Derivation of the Equation of Motion for the Coordinate "q_1"</u>

The first term of the Lagrangian equation may be obtained by first taking the derivative of Eq. (20) with respect to $\dot{q}_1$:

$$\frac{\partial T}{\partial \dot{q}_1} = M_1 \dot{q}_5 \frac{\partial \dot{q}_5}{\partial \dot{q}_1} + M_2 \dot{q}_7 \frac{\partial \dot{q}_7}{\partial \dot{q}_1}$$

The extraneous coordinates q_5 and q_7 appearing in the above expression may be eliminated by differentiating Eqs. (16) and (18) and substituting the results. Thus we obtain:

$$\frac{\partial T}{\partial \dot{q}_1} = (M_1 + M_2)\,\dot{q}_1 - \frac{M_2 \dot{M}_1}{M_1 + M_2}\left(l_1 q_3 - l_2 q_4\right)$$

The above expression may be differentiated with respect to time to yield:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = (M_1+M_2)\ddot{q}_1 + \dot{M}_1 \dot{q}_1 - \frac{M_2 \ddot{M}_1}{M_1+M_2}\left(l_1 \dot{q}_3 - l_2 \dot{q}_4\right) + \frac{M_2 \dot{M}_1^2}{(M_1+M_2)^2}\left(l_1 q_3 - l_2 q_4\right) \qquad (21)$$

The second term of the Lagrangian equation may be obtained by differentiating Eq. (20) with respect to   .

$$\frac{\partial T}{\partial q_1} = 0 \qquad (22)$$

The generalized force Q_i can be computed by considering the work done by the external forces as the coordinate q_i is varied, all other coordinates remaining fixed.

In general,

$$\text{Work}_i = Q_i \,\Delta q_i \quad ; \quad \text{or} \quad Q_i = \frac{\text{Work}_i}{\Delta q_i}$$

In particular, for coordinate q_1, the force Q_1 becomes, referring to Fig. 6:

$$Q_1 = \frac{\text{Work}_1}{\Delta q_1} = \frac{(F \sin q_4 + D \cos q_4)\,\Delta q_1}{\Delta q_1}$$

With the approximations of Eq. (15), the above expression becomes

$$Q_1 \simeq F q_4 + D \qquad (23)$$

Substituting Eqs. (21), (22), and (23) into Eq. (1), the equation of motion in the q_1 direction becomes, finally:

$$(M_1+M_2)\ddot{q}_1 + \dot{M}_1 \dot{q}_1 - \frac{M_2 \dot{M}_1}{M_1+M_2}(l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_2 \dot{M}_1^2}{(M_1+M_2)^2}(l_1 q_3 - l_2 q_4)$$

$$= F q_4 + D \tag{24}$$

<u>Derivation of the Equation of Motion for the Coordinate "q_2"</u>

In a similar manner, differentiating Eq. (20):

$$\frac{\partial T}{\partial \dot{q}_2} = M_1 \dot{q}_6 \frac{\partial \dot{q}_6}{\partial \dot{q}_2} + M_2 \dot{q}_8 \frac{\partial \dot{q}_8}{\partial \dot{q}_2}$$

and, differentiating Eq. (17):

$$\dot{q}_6 = \dot{q}_2 - \frac{M_2 \dot{M}_1}{(M_1+M_2)^2}(l_1 - l_2)$$

$$\frac{\partial \dot{q}_6}{\partial \dot{q}_2} = 1$$

Similarly, differentiating Eq. (19)

$$\dot{q}_8 = \dot{q}_2 - \frac{M_2 \dot{M}_1}{(M_1+M_2)^2}(l_1 - l_2)$$

$$\frac{\partial \dot{q}_8}{\partial \dot{q}_2} = 1$$

Combining the above results:

$$\frac{\partial T}{\partial \dot{q}_2} = (M_1+M_2)\dot{q}_2 - \frac{M_2 \dot{M}_1}{M_1+M_2}(l_1 - l_2)$$

By differentiating this expression with respect to time the first term of the Lagrangian equation becomes:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = (M_1+M_2)\ddot{q}_2 + \dot{M}_1\dot{q}_2 + \frac{M_2\dot{M}_1^2}{(M_1+M_2)^2}(\ell_1-\ell_2) \tag{25}$$

The second term of the Lagrangian equation is:

$$\frac{\partial T}{\partial q_2} = 0 \tag{26}$$

The generalized force $\mathcal{Q}_2$, referring to Fig. 5, may be expressed as:

$$\mathcal{Q}_2 = \frac{\text{Work}_2}{\Delta q_2} = \frac{[-(M_1+M_2)g + F\cos q_4 - D\sin q_4]\,\Delta q_2}{\Delta q_2}$$

With the approximations of Eq. (15)

$$\mathcal{Q}_2 \simeq -(M_1+M_2)g + F - Dq_4 \tag{27}$$

Substituting Eqs. (25), (26), and (27) into (1), the equation of motion in the q_2 direction becomes:

$$(M_1+M_2)\ddot{q}_2 + \dot{M}_1\dot{q}_1 + \frac{M_2\dot{M}_1^2}{(M_1+M_2)^2}(\ell_1-\ell_2) = F - g(M_1+M_2) - Dq_4 \tag{28}$$

<u>Derivation of the Equation of Motion for the Coordinate "q_3"</u>

The first term of the Lagrangian equation is obtained in an identical manner to that employed for coordinates q_1 and q_2 :
Differentiating Eq. (20):

$$\frac{\partial T}{\partial \dot{q}_3} = I_1\dot{q}_3 + M_1\dot{q}_5\frac{\partial \dot{q}_5}{\partial \dot{q}_3} + M_2\dot{q}_7\frac{\partial \dot{q}_7}{\partial \dot{q}_3} \tag{29}$$

and, differentiating Eq. (16):

$$\dot{q}_5 = \dot{q}_1 + \frac{M_2}{M_1+M_2}(\ell_1\dot{q}_3 - \ell_2\dot{q}_4) - \frac{M_2\dot{M}_1}{(M_1+M_2)^2}(\ell_1 q_3 - \ell_2 q_4)$$

$$\frac{\partial \dot{q}_5}{\partial \dot{q}_3} = \frac{M_2\ell_1}{M_1+M_2}$$

also, by differentiating Eq. (18):

$$\dot{q}_7 = \dot{q}_1 - \frac{M_1}{M_1+M_2}\left(\ell_1\dot{q}_3 - \ell_2\dot{q}_4\right) - \frac{M_2\dot{M}_1}{(M_1+M_2)^2}\left(\ell_1 q_3 - \ell_2 q_4\right)$$

$$\frac{\partial\dot{q}_7}{\partial\dot{q}_3} = -\frac{M_1\ell_1}{M_1+M_2}$$

Substituting into (29)

$$\frac{\partial T}{\partial\dot{q}_3} = \left[I_1 + \frac{M_1M_2\ell_1^2}{M_1+M_2}\right]\dot{q}_3 - \frac{M_1M_2\ell_1\ell_2}{M_1+M_2}\dot{q}_4$$

Differentiating the above expression with respect to time, the first term of the Lagrangian equation becomes:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_3}\right) = I_1\ddot{q}_3 + \dot{I}_1\dot{q}_3 + \frac{M_1M_2\ell_1}{M_1+M_2}\left(\ell_1\ddot{q}_3 - \ell_2\ddot{q}_4\right) + \frac{M_2^2\dot{M}_1}{(M_1+M_2)^2}\left(\ell_1\dot{q}_3 - \ell_2\dot{q}_4\right) \qquad (30)^*$$

From Eq. (20)

$$\frac{\partial T}{\partial q_3} = M_1\left(\dot{q}_5\frac{\partial\dot{q}_5}{\partial q_3} + \dot{q}_6\frac{\partial\dot{q}_6}{\partial q_3}\right) + M_2\left(\dot{q}_7\frac{\partial\dot{q}_7}{\partial q_3} + \dot{q}_8\frac{\partial\dot{q}_8}{\partial q_3}\right) \qquad (31)$$

By differentiating Eq. (16):

$$\dot{q}_5 = \dot{q}_1 + \frac{M_2}{M_1+M_2}\left(\ell_1\dot{q}_3 - \ell_2\dot{q}_4\right) - \frac{M_2\dot{M}_1}{(M_1+M_2)^2}\left(\ell_1 q_3 - \ell_2 q_4\right)$$

$$\frac{\partial\dot{q}_5}{\partial q_3} = -\frac{M_2\dot{M}_1\ell_1}{(M_1+M_2)^2} \qquad (32)$$

By differentiating Eq. (18):

$$\dot{q}_7 = \dot{q}_1 - \frac{M_1}{M_1+M_2}\left(\ell_1\dot{q}_3 - \ell_2\dot{q}_4\right) - \frac{M_2\dot{M}_1}{(M_1+M_2)^2}\left(\ell_1 q_3 - \ell_2 q_4\right)$$

$$\frac{\partial\dot{q}_7}{\partial q_3} = -\frac{M_2\dot{M}_1\ell_1}{(M_1+M_2)^2} \qquad (33)$$

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*The term "$\dot{I}\dot{q}_3$" which accounts for the propellent mass transfer out of the rocket is not correct, inasmuch as this moment should be calculated from the velocity with which the propellent leaves the transfer tubes and enters the motor. Corresponding errors appear in subsequent equations. The numerical effect of the resulting terms is negligible.

In order to obtain $\frac{\partial \dot{q}_6}{\partial q_3}$ and $\frac{\partial \dot{q}_8}{\partial q_3}$, it becomes necessary to differentiate the exact expressions for q_6 and q_8, since the q_3 term has vanished in the linearization of the exact expressions. Eqs. (10) and (12) will be differentiated and the results then linearized.

From Eq. (12)

$$q_6 = q_2 + \frac{M_2}{M_1+M_2} \ell_4 \cos\theta \quad ; \qquad (\theta = q_3 - \delta) \tag{12}$$

$$\dot{q}_6 = \dot{q}_2 - \frac{M_2}{M_1+M_2} \ell_4 \sin\theta\, \dot{\theta} - \frac{M_2 \dot{M}_1}{(M_1+M_2)^2} \ell_4 \cos\theta + \frac{M_2}{M_1+M_2} \dot{\ell}_4 \cos\theta$$

$$\frac{\partial \dot{q}_6}{\partial q_3} = -\frac{M_2}{M_1+M_2} \frac{\partial \ell_4}{\partial q_3} \sin\theta\, \dot{\theta} - \frac{M_2}{M_1+M_2} \ell_4 \frac{\partial \sin\theta}{\partial q_3} \dot{\theta} - \frac{M_2}{M_1+M_2} \ell_4 \sin\theta \frac{\partial \dot{\theta}}{\partial q_3}$$
$$- \frac{M_2 \dot{M}_1}{(M_1+M_2)^2} \frac{\partial \ell_4}{\partial q_3} \cos\theta - \frac{M_2 \dot{M}_1}{(M_1+M_2)^2} \ell_4 \frac{\partial \cos\theta}{\partial q_3} + \frac{M_2}{M_1+M_2} \frac{\partial \dot{\ell}_4}{\partial q_3} \cos\theta$$
$$+ \frac{M_2}{M_1+M_2} \dot{\ell}_4 \frac{\partial \cos\theta}{\partial q_3} \tag{34}$$

From Eq. (13):

$$\ell_4 = \sqrt{\ell_1^2 + \ell_2^2 - 2\ell_1 \ell_2 \cos(q_4 - q_3)}$$

By differentiation:

$$\frac{\partial \ell_4}{\partial q_3} = - \frac{\ell_1 \ell_2 \sin(q_4 - q_3)}{\ell_4} \tag{35}$$

$$\frac{\partial \ell_4}{\partial t} = \dot{\ell}_4 = \frac{\ell_1 \ell_2 [\sin(q_4 - q_3)][\dot{q}_4 - \dot{q}_3]}{\ell_4} \tag{36}$$

$$\frac{\partial \dot{\ell}_4}{\partial t} = \frac{-\ell_1 \ell_2 (\dot{q}_4 - \dot{q}_3) \cos(q_4 - q_3)}{\ell_4} - \text{(2nd order term)}$$

$$\frac{\partial \dot{\ell}_4}{\partial q_3} \simeq - \frac{\ell_1 \ell_2}{\ell_4} (\dot{q}_4 - \dot{q}_3) \tag{37}$$

From Eq. (14):

$$\theta = q_3 - \sin^{-1}\left[\frac{l_2}{l_4}\ \sin(q_4 - q_3)\right]$$

By differentiation:

$$\dot{\theta} = \dot{q}_3 - \frac{\frac{l_2}{l_4}\cos(q_4-q_3)(\dot{q}_4-\dot{q}_3) - (\text{2nd order term})}{\sqrt{1-(\text{2nd order term})}}$$

$$\dot{\theta} \simeq \dot{q}_3 - \frac{l_2}{l_4}(\dot{q}_4 - \dot{q}_3) \tag{38}$$

$$\frac{\partial\dot{\theta}}{\partial q_3} = -\frac{l_1 l_2^2}{l_4^3}(q_4 - q_3)(\dot{q}_4 - \dot{q}_3) \tag{39}$$

$$\frac{\partial\theta}{\partial q_3} = 1 - \frac{l_2}{\sqrt{1-\left[\frac{l_2\sin(q_4-q_3)}{l_4}\right]^2}} \cdot \frac{-l_4\cos(q_4-q_3) - \left[\sin(q_4-q_3)\right]\left[-\frac{l_1 l_2\sin(q_4-q_3)}{l_4}\right]}{l_4^2}$$

$$\frac{\partial\theta}{\partial q_3} \simeq 1 + \frac{l_2}{l_4} \simeq \frac{l_1}{l_4} \tag{40}$$

Also:

$$\frac{\partial\sin\theta}{\partial q_3} = \cos\theta\frac{\partial\theta}{\partial q_3} \simeq \frac{l_1}{l_4} \tag{41}$$

$$\frac{\partial\cos\theta}{\partial q_3} = -\sin\theta\frac{\partial\theta}{\partial q_3} \simeq \frac{l_1(l_1 q_3 - l_2 q_4)}{l_4^2} \tag{42}$$

By substituting Eqs. (35) through (42) into (34) and ignoring terms of second order or higher in the small quantities:

$$\frac{\partial\dot{q}_6}{\partial q_3} \simeq -\frac{M_2}{M_1+M_2}\, l_1\dot{q}_3 + \frac{M_2\dot{M}_1}{(M_1+M_2)^2}\, l_1 q_3 \tag{43}$$

In a similar manner, from Eq. (10):

$$\dot{q}_8 = \dot{q}_2 + \frac{M_1}{M_1+M_2} l_4\sin\theta\,\dot{\theta} - \frac{M_1}{M_1+M_2}\cos\theta\frac{\partial l_4}{\partial t} - \frac{\dot{M}_1}{M_1+M_2} l_4\cos\theta + \frac{M_1\dot{M}_1}{(M_1+M_2)^2} l_4\cos\theta$$

By differentiating this expression:

$$\frac{\partial \dot{q}_8}{\partial q_3} = \frac{M_1}{M_1+M_2}\left[\ell_4 \sin\theta \frac{\partial \dot{\theta}}{\partial q_3} + \ell_4 \dot{\theta} \frac{\partial \sin\theta}{\partial q_3} + \sin\theta\, \dot{\theta} \frac{\partial \ell_4}{\partial q_3} \right]$$

$$- \frac{M_1}{M_1+M_2}\left[\cos\theta \frac{\partial \dot{\ell}_4}{\partial q_3} - \dot{\ell}_4 \sin\theta \frac{\partial \theta}{\partial q_3} \right] - \frac{\dot{M}_1}{M_1+M_2}\left[\frac{\partial \ell_4}{\partial q_3} \cos\theta - \sin\theta\, \ell_4 \frac{\partial \theta}{\partial q_3} \right]$$

$$+ \frac{M_1 \dot{M}_1}{(M_1+M_2)^2}\left[\frac{\partial \ell_4}{\partial q_3} \cos\theta - \sin\theta\, \ell_4 \frac{\partial \theta}{\partial q_3} \right] \tag{44}$$

Again substituting Eqs. (35) through (42):

$$\frac{\partial \dot{q}_8}{\partial q_3} = \frac{M_1}{M_1+M_2} \ell_1 \dot{q}_3 + \frac{M_2 \dot{M}_1}{(M_1+M_2)^2} \ell_1 q_3 \tag{45}$$

Having evaluated $\frac{\partial \dot{q}_6}{\partial q_3}$ and $\frac{\partial \dot{q}_8}{\partial q_3}$ from the exact expressions, $\frac{\partial T}{\partial q_3}$ can now be obtained by substituting these values in Eq. (31).

$$\frac{\partial T}{\partial q_3} \simeq - \frac{M_2 \dot{M}_1}{M_1+M_2} \ell_1 \dot{q}_1 + \frac{M_2^2 \dot{M}_1^2}{(M_1+M_2)^3} (\ell_1 q_3 - \ell_2 q_4)$$

$$- \frac{M_2^2 \dot{M}_1^2}{(M_1+M_2)^3} \ell_1 (\ell_1 - \ell_2) q_3 + \frac{M_2 \dot{M}_1 \ell_1}{M_1+M_2} q_3 \dot{q}_2 \tag{46}$$

The generalized force

$$\mathcal{Q}_3 = \frac{\text{Work } 3}{\Delta q_3}$$

Then, referring to Fig. 6

$$\mathcal{Q}_3 = - M_1 g \frac{\partial q_6}{\partial q_3} - M_2 g \frac{\partial q_8}{\partial q_3} + F \sin q_4 \frac{\partial}{\partial q_3}\left[q_7 - (\ell_2 + \ell_3) \sin q_4 \right]$$

$$+ F \cos q_4 \frac{\partial}{\partial q_3}\left[q_8 - (\ell_2 + \ell_3) \cos q_4 \right] + D \cos q_4 \frac{\partial}{\partial q_3}\left[q_7 - (\ell_2 + \ell_3) \sin q_4 \right]$$

$$- D \sin q_4 \frac{\partial}{\partial q_3}\left[q_8 - (\ell_2 + \ell_3) \cos q_4 \right]$$

Substituting in values of q_7 and q_8 from Eqs. (11) and (10)

$$\mathcal{Q}_3 = \left[F \sin q_4 + D\cos q_4\right] \frac{\partial}{\partial q_3}\left[q_1 - \frac{M_1}{M_1+M_2} \ell_4 \sin\theta - (\ell_2+\ell_3)\sin q_4\right]$$

$$+ \left[F\cos q_4 - D\sin q_4\right] \frac{\partial}{\partial q_3}\left[q_2 - \frac{M_1}{M_1+M_2} \ell_4 \cos\theta - (\ell_2+\ell_3)\cos q_4\right]$$

Performing the differentiation indicated:

$$\mathcal{Q}_3 = \left[F\sin q_4 + D\cos q_4\right]\left[\frac{-M_1}{M_1+M_2}\ell_4 \frac{\partial \sin\theta}{\partial q_3} + \sin\theta \frac{\partial \ell_4}{\partial q_3}\right]$$

$$+\left[F\cos q_4 - D\sin q_4\right]\left[-\frac{M_1}{M_1+M_2}\ell_4 \frac{\partial \cos\theta}{\partial q_3} + \cos\theta \frac{\partial \ell_4}{\partial q_3}\right]$$

Now, substituting Eqs. (35), (41), and (42):

$$\mathcal{Q}_3 \simeq \left(Fq_4 + D\right)\left\{-\frac{M_1}{M_1+M_2}\left[\ell_1 + \left(\frac{\ell_1 q_3 - \ell_2 q_4}{\ell_4}\right)\left(-\frac{\ell_1\ell_2(q_4-q_3)}{\ell_4}\right)\right]\right\}$$

$$+ \left(F - Dq_4\right)\left\{-\frac{M_1}{M_1+M_2}\left[-\frac{\ell_1(\ell_1 q_3 - \ell_2 q_4)}{-\ell_4} - \frac{\ell_1\ell_2(q_4-q_3)}{\ell_4}\right]\right\}$$

Finally, retaining only first order terms, the expression reduces to:

$$\mathcal{Q}_3 \simeq -\frac{M_1\ell_1}{M_1+M_2}\left[F(q_4-q_3)+D\right] \tag{47}$$

The equation of motion for the coordinate q_3 can then be obtained by substituting Eqs. (30), (46), and (47) into Eq. (1).

$$I_1\ddot{q}_3 + \frac{M_1M_2\ell_1}{M_1+M_2}\left(\ell_1\ddot{q}_3 - \ell_2\ddot{q}_4\right) + \frac{M_2^2\dot{M}_1\ell_1}{(M_1+M_2)^2}\left(\ell_1\dot{q}_3 - \ell_2\dot{q}_4\right)$$

$$+\dot{I}_1\dot{q}_3 + \frac{M_2\dot{M}_1\ell_1}{M_1+M_2}\dot{q}_1 - \frac{M_2\dot{M}_1\ell_1}{M_1+M_2}q_3\dot{q}_2 - \frac{M_2^2\dot{M}_1^2\ell_1}{(M_1+M_2)^3}\left(\ell_1 q_3 - \ell_2 q_4\right)$$

$$+\frac{M_2^2\dot{M}_1^2\ell_1(\ell_1-\ell_2)}{(M_1+M_2)^3}q_3 = -\frac{M_1\ell_1}{M_1+M_2}\left[F(q_4-q_3)+D\right] \tag{48}$$

Derivation of the Equation of Motion for the Coordinate "q_4"

By differentiating Eq. (20) with respect to: "$\dot{q}_4$"

$$\frac{\partial T}{\partial \dot{q}_4} = I_2 \dot{q}_4 + M_1 \dot{q}_5 \frac{\partial \dot{q}_5}{\partial \dot{q}_4} + M_2 \dot{q}_7 \frac{\partial \dot{q}_7}{\partial \dot{q}_4} \tag{49}$$

From Eq. (16):

$$\dot{q}_5 = \dot{q}_1 + \frac{M_2}{M_1+M_2}(\ell_1 \dot{q}_3 - \ell_2 \dot{q}_4) - \frac{M_2 \dot{M}_1}{(M_1+M_2)^2}(\ell_1 q_3 - \ell_2 q_4)$$

Differentiating:

$$\frac{\partial \dot{q}_5}{\partial \dot{q}_4} = -\frac{M_2 \ell_2}{M_1+M_2}$$

From Eq. (18):

$$\dot{q}_7 = \dot{q}_1 - \frac{M_1}{M_1+M_2}(\ell_1 \dot{q}_3 - \ell_2 \dot{q}_4) - \frac{M_2 \dot{M}_1}{(M_1+M_2)^2}(\ell_1 q_3 - \ell_2 q_4)$$

Differentiating:

$$\frac{\partial \dot{q}_7}{\partial \dot{q}_4} = \frac{M_1 \ell_2}{M_1+M_2}$$

Substituting these quantities into Eq. (49)

$$\frac{\partial T}{\partial \dot{q}_4} = I_2 \dot{q}_4 - \frac{M_1 M_2 \ell_2}{M_1+M_2}(\ell_1 \dot{q}_3 - \ell_2 \dot{q}_4)$$

The first term of the Lagrange equation is obtained by differentiation of the above expression:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_4}\right) = I_2 \ddot{q}_4 - \frac{M_1 M_2 \ell_2}{M_1+M_2}(\ell_1 \ddot{q}_3 - \ell_2 \ddot{q}_4) - \frac{M_2^2 \dot{M}_1 \ell_2}{(M_1+M_2)^2}(\ell_1 \dot{q}_3 - \ell_2 \dot{q}_4) \tag{50}$$

From Eq. (20)

$$\frac{\partial T}{\partial q_4} = M_1\left(\dot{q}_5 \frac{\partial \dot{q}_5}{\partial q_4} + \dot{q}_6 \frac{\partial \dot{q}_6}{\partial q_4}\right) + M_2\left(\dot{q}_7 \frac{\partial \dot{q}_7}{\partial q_4} + \dot{q}_8 \frac{\partial \dot{q}_8}{\partial q_4}\right) \tag{51}$$

By differentiating Eq. (16):

$$\dot{q}_5 = \dot{q}_1 + \frac{M_2}{M_1+M_2}(\ell_1 \dot{q}_3 - \ell_2 \dot{q}_4) - \frac{M_2 \dot{M}_1}{(M_1+M_2)^2}(\ell_1 q_3 - \ell_2 q_4)$$

$$\frac{\partial \dot{q}_5}{\partial q_4} = \frac{M_2 \dot{M}_1 \ell_2}{(M_1+M_2)^2}$$

And by differentiating Eq. (18):

$$\dot{q}_7 = \dot{q}_1 + \frac{M_2}{M_1+M_2}\left(\ell_1 \dot{q}_3 - \ell_2 \dot{q}_4\right) - \frac{M_2 \dot{M}_1}{(M_1+M_2)^2}\left(\ell_1 q_3 - \ell_2 q_4\right)$$

$$\frac{\partial \dot{q}_7}{\partial q_4} = \frac{M_2 \dot{M}_1 \ell_2}{(M_1+M_2)^2}$$

In order to evaluate $\frac{\partial \dot{q}_6}{\partial q_4}$ and $\frac{\partial \dot{q}_8}{\partial q_4}$ it is necessary, as in the case of the q_3 coordinate, to differentiate the exact expressions, and then make simplifying approximations:

From Eq. (12):

$$\dot{q}_6 = \dot{q}_2 - \frac{M_2}{M_1+M_2}\ell_4 \sin\theta\, \dot{\theta} - \frac{M_2 \dot{M}_1}{(M_1+M_2)^2}\ell_4 \cos\theta + \frac{M_2}{M_1+M_2}\dot{\ell}_4 \cos\theta$$

Differentiating:

$$\frac{\partial \dot{q}_6}{\partial q_4} = -\frac{M_2}{M_1+M_2}\frac{\partial \ell_4}{\partial q_4}\sin\theta\,\dot{\theta} - \frac{M_2}{M_1+M_2}\ell_4 \frac{\partial \sin\theta}{\partial q_4}\dot{\theta} - \frac{M_2}{M_1+M_2}\ell_4 \sin\theta \frac{\partial \dot{\theta}}{\partial q_4}$$
$$- \frac{M_2 \dot{M}_1}{(M_1+M_2)^2}\frac{\partial \ell_4}{\partial q_4}\cos\theta - \frac{M_2 \dot{M}_1}{(M_1+M_2)^2}\ell_4 \frac{\partial \cos\theta}{\partial q_4} + \frac{M_2}{M_1+M_2}\frac{\partial \dot{\ell}_4}{\partial q_4}\cos\theta$$
$$+ \frac{M_2}{M_1+M_2}\dot{\ell}_4 \frac{\partial \cos\theta}{\partial q_4} \tag{52}$$

By differentiating Eq. (13):

$$\frac{\partial \ell_4}{\partial q_4} \simeq \frac{\ell_1 \ell_2 (q_4 - q_3)}{\ell_4}$$

Also by differentiating Eq. (36):

$$\frac{\partial \dot{\ell}_4}{\partial q_4} \simeq \frac{\ell_1 \ell_2 (\dot{q}_4 - \dot{q}_3)}{\ell_4}$$

By differentiating Eq. (14):

$$\frac{\partial \theta}{\partial q_4} \simeq - \frac{l_2}{l_4}$$

And by differentiating Eq. (38):

$$\frac{\partial \dot{\theta}}{\partial q_4} = \frac{l_1 l_2^2 (q_4 - q_3)(\dot{q}_4 - \dot{q}_3)}{l_4^3}$$

Then $\frac{\partial \dot{q}_6}{\partial q_4}$ may be evaluated by the substitution of the above expressions and Eqs. (35) through (42) into Eq. (52):

$$\frac{\partial \dot{q}_6}{\partial q_4} = \frac{M_2}{M_1 + M_2} l_2 \dot{q}_4 - \frac{M_2 \dot{M}_1 l_2}{(M_1 + M_2)^2} q_4 \tag{53}$$

From Eq. (10):

$$\dot{q}_8 = \dot{q}_2 + \frac{M_1}{M_1 + M_2} l_4 \sin\theta\, \dot{\theta} - \frac{M_1}{M_1 + M_2} \cos\theta \frac{\partial l_4}{\partial t} - \frac{M_2 \dot{M}_1}{(M_1 + M_2)^2} l_4 \cos\theta$$

Differentiating:

$$\frac{\partial \dot{q}_8}{\partial q_4} = \frac{M_1}{M_1 + M_2} \left\{ l_4 \sin\theta \frac{\partial \dot{\theta}}{\partial q_4} + l_4 \dot{\theta} \frac{\partial \sin\theta}{\partial q_4} + \sin\theta\, \dot{\theta} \frac{\partial l_4}{\partial q_4} - \frac{\partial l_4}{\partial t} \frac{\partial \cos\theta}{\partial q_4} \right.$$
$$\left. - \cos\theta \frac{\partial^2 l_4}{\partial t \partial q_4} \right\} - \frac{M_2 \dot{M}_1}{(M_1 + M_2)^2} \left\{ l_4 \frac{\partial \cos\theta}{\partial q_4} + \cos\theta \frac{\partial l_4}{\partial q_4} \right\} \tag{54}$$

Then $\frac{\partial \dot{q}_8}{\partial q_4}$ may be evaluated in a similar manner to $\frac{\partial \dot{q}_6}{\partial q_4}$ by substituting for the extraneous terms in Eq. (54):

$$\frac{\partial \dot{q}_8}{\partial q_4} = - \frac{M_1 l_2}{M_1 + M_2} \dot{q}_4 - \frac{M_2 \dot{M}_1 l_2}{(M_1 + M_2)^2} q_4 \tag{55}$$

The second term of the Lagrangian equation may be evaluated by substituting the derivatives obtained in the foregoing steps into Eq. (51):

$$\frac{\partial T}{\partial q_4} = \frac{M_2 \dot{M}_1 l_2}{M_1 + M_2} \dot{q}_1 - \frac{M_2 \dot{M}_1 l_2}{M_1 + M_2} q_4 \dot{q}_2$$

$$- \frac{M_2^2 \dot{M}_1^2 l_2}{(M_1 + M_2)^3} (l_1 q_3 - l_2 q_4) + \frac{M_2^2 \dot{M}_1^2 l_2}{(M_1 + M_2)^3} (l_1 - l_2) q_4 \tag{56}$$

The generalized force

$$\mathcal{Q}_4 = \frac{\text{Work}_4}{\Delta q_4}$$

By reference to Fig. 6, it is evident that:

$$\mathcal{Q}_4 = (F \sin q_4 + D \cos q_4) \frac{\partial}{\partial q_4} \left[q_7 - (l_2 + l_3) \sin q_4 \right]$$

$$+ (F \cos q_4 - D \sin q_4) \frac{\partial}{\partial q_4} \left[q_8 - (l_2 + l_3) \cos q_4 \right]$$

$$- M_2 g \frac{\partial q_8}{\partial q_4} - M_1 g \frac{\partial q_6}{\partial q_4}$$

Substituting for the extraneous coordinates and simplifying, as in the derivation of Eq. (47):

$$\mathcal{Q}_4 \simeq -D \left(\frac{M_2 l_2}{M_1 + M_2} + l_3 \right) \tag{57}$$

Introducing Eqs. (50), (51), and (57) into Eq. (1), the equation of motion for coordinate q_4 becomes, finally:

$$I_2 \ddot{q}_4 - \frac{M_1 M_2 l_2}{M_1 + M_2} (l_1 \ddot{q}_3 - l_2 \ddot{q}_4) - \frac{M_2^2 \dot{M}_1 l_2}{(M_1 + M_2)^2} (l_1 \dot{q}_3 - l_2 \dot{q}_4)$$

$$- \frac{M_2 \dot{M}_1 l_2}{M_1 + M_2} \dot{q}_1 + \frac{M_2 \dot{M}_1 l_2}{M_1 + M_2} q_4 \dot{q}_2 + \frac{M_2^2 \dot{M}_1^2 l_2}{(M_1 + M_2)^3} (l_1 q_3 - l_2 q_4)$$

$$- \frac{M_2^2 \dot{M}_1^2 l_2}{(M_1 + M_2)^3} (l_1 - l_2) q_4 = -D \left(\frac{M_2 l_2}{M_1 + M_2} + l_3 \right) \tag{58}$$

IV. COMPUTATION OF SYSTEM COMPONENT TRANSFER FUNCTIONS

Only the two equations involving the rotation of the rocket (coordinate q_3) and the motion of the nozzle (coordinate q_4) are considered since the stability investigation is made for rotation only. First, the thrust, jet damping, and moment of inertia terms are substituted in terms of the equation variables and constants.

Rocket thrust:

$$F = -\dot{M}_1 g I_{sp}$$

Jet damping force:

$$D = -\dot{M}_1 \ell_5 \dot{q}_4$$

(The negative sign is necessary in order to make the term positive, since $\dot{M}_1$ is a negative quantity.)

Moment of Inertia derivative:

$$I_1 = M_1 k_1^2 \qquad (k = \text{radius of gyration})$$

$$\dot{I}_1 = \dot{M}_1 k_1^2 + 2 k_1 M_1 \dot{k}_1$$

$$\dot{I}_1 \simeq \dot{M}_1 k_1^2$$

(since $\dot{k}_1$ will be small for most configurations)

Substituting the above terms in Eq. (48):

$$I_1 \ddot{q}_3 + \dot{M}_1 k_1^2 \dot{q}_3 + \frac{M_1 M_2 \ell_1}{M_1 + M_2}\left(\ell_1 \ddot{q}_3 - \ell_2 \ddot{q}_4\right) + \frac{M_2^2 \dot{M}_1 \ell_1}{(M_1+M_2)^2}\left(\ell_1 \dot{q}_3 - \ell_2 \dot{q}_4\right)$$
$$- \frac{M_2^2 \dot{M}_1^2 \ell_1}{(M_1+M_2)^3}\left(\ell_1 q_3 - \ell_2 q_4\right) + \frac{M_2^2 \dot{M}_1^2 \ell_1}{(M_1+M_2)^3}\left(\ell_1 - \ell_2\right) q_3 + \frac{M_2 \dot{M}_1 \ell_1}{M_1 + M_2}\dot{q}_1$$
$$- \frac{M_2 \dot{M}_1 \ell_1}{M_1 + M_2} q_3 \dot{q}_2 = - \frac{M_1 \ell_1}{M_1 + M_2}\left[- \dot{M}_1 g I_{sp} \left(q_4 - q_3\right) - \dot{M}_1 \ell_5 \dot{q}_4 \right] \tag{59}$$

Again, substituting for F , D , and I_1 , and introducing the applied moment* $\mathfrak{M}$ into Eq. (58):

$$I_2 \ddot{q}_4 + \frac{M_2^2 \dot{M}_1^2 l_2}{(M_1+M_2)^3}(l_1 q_3 - l_2 q_4) - \frac{M_2^2 \dot{M}_1 l_2}{(M_1+M_2)^2}(l_1 \dot{q}_3 - l_2 \dot{q}_4)$$

$$- \frac{M_2 M_1 l_2}{M_1+M_2}(l_1 \ddot{q}_3 - l_2 \ddot{q}_4) - \frac{M_2^2 \dot{M}_1^2 l_2}{(M_1+M_2)^3}(l_1 - l_2) q_4 - \frac{M_2 \dot{M}_1}{M_1+M_2} l_2 \dot{q}_1$$

$$+ \frac{M_2 \dot{M}_1}{M_1+M_2} l_2 q_4 \dot{q}_2 = \dot{M}_1 l_5 \dot{q}_4 \left(l_3 + \frac{M_2}{M_1+M_2} l_2\right) - \mathfrak{M} \tag{60}$$

The Laplace Transformation is applied to the equations prior to the stability analysis. Ref. (2) explains in detail the use of this transformation in such analyses. In essence, the transformation converts the differential equations to simple algebraic equations.

Since the vertical and horizontal velocities $\dot{q}_2$ and $\dot{q}_1$ are quite small during the launch, the terms involving $\dot{q}_2$ and $q_3 \dot{q}_1$ are neglected in the equations.

Application of the Laplace Transformation to Eq. (59) yields:

$$Q_3 \left\{ \left[I_1 + \frac{M_1 M_2 l_1^2}{M_1+M_2} \right] S^2 + \left[\dot{M}_1 k_1^2 + \frac{M_2^2 \dot{M}_1 l_1^2}{(M_1+M_2)^2} \right] S + \left[\frac{M_1 \dot{M}_1 g I_{sp} l_1}{M_1+M_2} - \frac{M_2^2 \dot{M}_1^2 l_1 l_2}{(M_1+M_2)^3} \right] \right\}$$

$$= Q_4 \left\{ \left[\frac{M_1 M_2 l_1 l_2}{M_1+M_2} \right] S^2 + \left[\frac{M_2^2 \dot{M}_1 l_1 l_2}{(M_1+M_2)^2} + \frac{M_1 \dot{M}_1 l_1 l_5}{M_1+M_2} \right] S \right.$$

$$\left. + \left[\frac{M_1 \dot{M}_1 l_1 g I_{sp}}{M_1+M_2} - \frac{M_2^2 \dot{M}_1^2 l_1 l_2}{(M_1+M_2)^3} \right] \right\} \tag{61}$$

where Q_i represents the transformed coordinate q_i , and S is the complex variable. Similarly, transforming Eq. (60):

---

*The effect of the control moment $\mathfrak{M}$ on the main rocket is neglected in Eq. (59). For a system in which the nozzle is of comparable size and weight to the main rocket, the term should be included.

$$Q_3\left\{\left[\frac{M_1M_2\ell_1\ell_2}{M_1+M_2}\right]s^2+\left[\frac{M_2^2\dot{M}_1\ell_1\ell_2}{(M_1+M_2)^2}\right]s-\left[\frac{M_2^2\dot{M}_1^2\ell_1\ell_2}{(M_1+M_2)^3}\right]\right\}=M_{(s)}+$$

$$Q_4\left\{\left[I_2+\frac{M_1M_2\ell_2^2}{M_1+M_2}\right]s^2+\left[\frac{M_2^2\dot{M}_1\ell_2^2}{(M_1+M_2)^2}-\dot{M}_1\ell_5\left(\ell_3+\frac{M_2\ell_2}{M_1+M_2}\right)\right]s-\left[\frac{M_2^2\dot{M}_1^2\ell_1\ell_2}{(M_1+M_2)^3}\right]\right\} \quad (62)$$

where M represents the transformed moment $\mathcal{M}$.

In order to solve for the transfer functions F_2 and F_3 it is necessary to rearrange the equations slightly; Eq. (61) becomes

$$Q_3\left\{\left[I_1+\frac{M_1M_2\ell_1(\ell_1-\ell_2)}{M_1+M_2}\right]s^2+\left[\dot{M}_1k_1^2+\frac{M_2^2\dot{M}_1\ell_1(\ell_1-\ell_2)}{(M_1+M_2)^2}-\frac{M_1\dot{M}_1\ell_1\ell_5}{M_1+M_2}\right]s\right\}$$

$$=\left[Q_4-Q_3\right]\left\{\left[\frac{M_1M_2\ell_1\ell_2}{M_1+M_2}\right]s^2+\left[\frac{M_2^2\dot{M}_1\ell_1\ell_2}{(M_1+M_2)^2}+\frac{M_1\dot{M}_1\ell_1\ell_5}{M_1+M_2}\right]s+\left[\frac{M_1\dot{M}_1\ell_1 gI_{sp}}{M_1+M_2}-\frac{M_2^2\dot{M}_1\ell_1\ell_2}{(M_1+M_2)^3}\right]\right\} \quad (63)$$

Similarly, Eq. (62) becomes:

$$Q_3\left\{\left[\frac{M_1M_2\ell_2(\ell_1-\ell_2)}{M_1+M_2}-I_2\right]s^2+\left[\frac{M_2^2\dot{M}_1\ell_2(\ell_1-\ell_2)}{(M_1+M_2)^2}+\dot{M}_1\ell_5\left(\ell_3+\frac{M_2\ell_2}{M_1+M_2}\right)\right]s\right\}$$

$$=\left[Q_4-Q_3\right]\left\{\left[I_2+\frac{M_2M_1\ell_2^2}{M_1+M_2}\right]s^2+\left[\frac{M_2^2\dot{M}_1\ell_2^2}{(M_1+M_2)^2}-\dot{M}_1\ell_5\left(\ell_3+\frac{M_2\ell_2}{M_1+M_2}\right)\right]s\right.$$

$$\left.-\left[\frac{M_2^2\dot{M}_1^2\ell_1\ell_2}{(M_1+M_2)^2}\right]\right\}+M_{(s)} \quad (64)$$

Then the transfer function F_3 from Eq. (63) becomes

$$F_3=\frac{Q_3}{Q_4-Q_3}=\frac{\frac{M_1M_2\ell_1\ell_2}{M_1+M_2}s^2+\left[\frac{M_2^2\dot{M}_1\ell_1\ell_2}{(M_1+M_2)^2}+\frac{M_1\dot{M}_1\ell_1\ell_5}{M_1+M_2}\right]s+\left[\frac{M_1\dot{M}_1 g\ell_1 I_{sp}}{M_1+M_2}-\frac{M_2^2\dot{M}_1^2\ell_1\ell_2}{(M_1+M_2)^3}\right]}{\left[I_1+\frac{M_1M_2\ell_1(\ell_1-\ell_2)}{M_1+M_2}\right]s^2+\left[\dot{M}_1k_1^2+\frac{M_2^2\dot{M}_1\ell_1(\ell_1-\ell_2)}{(M_1+M_2)^2}-\frac{M_1\dot{M}_1\ell_1\ell_5}{M_1+M_2}\right]s} \quad (65)$$

By substituting Eq. (63) into (64), the function F_2 may be written as

$$F_2 = \frac{Q_4 - Q_3}{M} = \frac{1}{F_3 \cdot \left\{\left[\frac{M_1 M_2 \ell_2(\ell_1-\ell_2)}{M_1+M_2} - I_2\right] S^2 + \left[\frac{M_2^2 \dot{M}_1 \ell_2(\ell_1-\ell_2)}{(M_1+M_2)^2} + \dot{M}_1 \ell_5\left(\ell_3 + \frac{M_2 \ell_2}{M_1+M_2}\right)\right] S\right\} - \left\{\left[I_2 + \frac{M_1 M_2 \ell_2^2}{M_1+M_2}\right] S^2 + \left[\frac{M_2^2 \dot{M}_1 \ell_2^2}{(M_1+M_2)^2} - \dot{M}_1 \ell_5\left(\ell_3 + \frac{M_2 \ell_2}{M_1+M_2}\right)\right] S - \left[\frac{M_2^2 \dot{M}_1^2 \ell_1 \ell_2}{(M_1+M_2)^3}\right]\right\}} \qquad (66)$$

As an aid in the numerical computations for the stability analysis, the transfer functions are symbolized as follows:

$$F_3 = \frac{Q_3}{Q_4 - Q_3} = \frac{X}{Y} \qquad (67)$$

$$F_2 = \frac{Q_4 - Q_3}{M} = \frac{1}{\left(\frac{X}{Y}\right) Z - W} \qquad (68)$$

where:

$$X = \left\{\left[\frac{M_1 M_2 \ell_1 \ell_2}{M_1+M_2}\right] S^2 + \left[\frac{M_2^2 \dot{M}_1 \ell_1 \ell_2}{(M_1+M_2)^2} + \frac{M_1 \dot{M}_1 \ell_1 \ell_5}{M_1+M_2}\right] S + \left[\frac{M_1 \dot{M}_1 \ell_1 g J_{sp}}{M_1+M_2} - \frac{M_2^2 \dot{M}_1^2 \ell_1 \ell_2}{(M_1+M_2)^3}\right]\right\}$$

$$Y = \left\{\left[I_1 + \frac{M_1 M_2 \ell_1(\ell_1-\ell_2)}{M_1+M_2}\right] S^2 + \left[\dot{M}_1 k_1^2 + \frac{M_2^2 \dot{M}_1 \ell_1(\ell_1-\ell_2)}{(M_1+M_2)^2} - \frac{M_1 \dot{M}_1 \ell_1 \ell_5}{M_1+M_2}\right] S\right\}$$

$$Z = \left\{\left[\frac{M_1 M_2 \ell_2(\ell_1-\ell_2)}{M_1+M_2} - I_2\right] S^2 + \left[\frac{M_2^2 \dot{M}_1 \ell_2(\ell_1-\ell_2)}{(M_1+M_2)^2} + \dot{M}_1 \ell_5\left(\ell_3 + \frac{M_2 \ell_2}{M_1+M_2}\right)\right] S\right\}$$

$$W = \left\{\left[I_2 + \frac{M_1 M_2 \ell_2^2}{M_1+M_2}\right] S^2 + \left[\frac{M_2^2 \dot{M}_1 \ell_2^2}{(M_1+M_2)^2} - \dot{M}_1 \ell_5\left(\ell_3 + \frac{M_2 \ell_2}{M_1+M_2}\right)\right] S - \left[\frac{M_2^2 \dot{M}_1^2 \ell_1 \ell_2}{(M_1+M_2)^3}\right]\right\}$$

V. SYNTHESIS OF THE NOZZLE CONTROL TRANSFER FUNCTION "F_1", AND STABILITY INVESTIGATION

To illustrate the determination of the nozzle control transfer function, a specific numerical example will be worked out using the German V-2 as a typical vehicle.

As indicated in the Analysis, the system transfer function may be written as

$$K_S G_S = \frac{F_1 F_2 F_3}{1 + F_1 F_2 F_4} \tag{69}$$

Also, the computer synthesizes the dynamics of the rocket to give

$$F_3 = F_4 \tag{70}$$

Substituting Eq. (70) into (69)

$$K_S G_S = \frac{F_1 F_2 F_3}{1 + F_1 F_2 F_3} \tag{71}$$

Let

$$KG = F_1 F_2 F_3 \tag{72}$$

Substituting for F_2 and F_3 from Eqs. (67) and (68):

$$KG = F_1 \left(\frac{X}{XZ - YW} \right) \tag{73}$$

As a numerical example, consider the V-2 rocket with pivoted rocket motor and nozzle. The parameters X , Y , Z , and W have the following values as calculated in the Appendix:

$$\left.\begin{aligned} X &= 2675\,s^2 - 161\,s - 1{,}044{,}000 \\ Y &= 92{,}380\,s^2 - 110\,s \\ Z &= 2030\,s^2 - 11.65\,s \\ W &= 646\,s^2 + 10.85\,s \end{aligned}\right\} \tag{74}$$

The problem now is to determine a function F_1 which will give the system suitable stability characteristics during the launch. The procedure will be to:

1. Assume a function F_1 .
2. Check the system for stability by the Nyquist criterion.
3. Having established a stable system, determine the roots of the system transfer function by means of the Evans Root Locus Method.
4. Knowing the roots, at a typical operating gain of the feedback system, investigate the damping of the system.
5. If the resulting motion is unsatisfactory, improve the choice of F_1 and repeat the process.

In explanation of the methods to be used to investigate the stability of the feedback control system, it will be noted that the equations of motion are linear differential equations with constant coefficients. The solution for the particular coordinate in question can then be represented in the general form

$$q_i = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \cdots$$

where "s" is a simple root of the equation. In order to have a damped motion which will subside with time, it is obvious that the complex roots "s_i" must all have negative real parts. Another way of stating this requirement of a stable system is that all the roots "s_i" of the system must appear in the left or negative half of the complex plane.

<u>The Nyquist Criterion</u>

The Nyquist Criterion is an application to the simple feedback network of the Cauchy Theorem in the theory of complex variables. This theorem may be employed to show that if "$G_{(s)}$" is the vector representing the function of a complex variable "s", the function having "m" poles and "n" zeros within a closed contour "C" in the "s" plane, that as the point "s" moves around the contour once in a clockwise direction, the vector "$G_{(s)}$" carries out "$n-m$" clockwise revolutions about the origin. If the contour "C" is chosen to include the entire right half plane, and the function in question is plotted as the complex variable encircles the right half plane, the difference between the number of zeros and poles of the function that exist in the right half plane will be indicated by the number of encirclements of the origin. Either the number of zeros, or poles existing within the contour must be known previously.

In applying the theorem to the feedback network, the function in question is usually expressed as

$$K_S G_S = \frac{KG(s)}{1 + KG_{(s)}}$$

where the function "$KG_{(s)}$" can be factored; thereby determining the zeros of the function "$K_S G_S$". The denominator "$1 + KG$" in general cannot be factored so readily, but application of the Cauchy Theorem will show the number of roots of the denominator existing in the contour chosen, or the right half plane.

In order to obtain a plot independent of the gain "K", the inverse of the function may be considered.

$$\frac{1}{K_S G_S} = \frac{1}{KG} + 1$$

Then if the vector $\frac{1}{K_S G_S}$ encircles the origin, the vector $\frac{1}{KG}$ will encircle the point "-1", as the variable "s" moves around the contour of the right half plane. Or similarly the vector $\frac{1}{G}$ will encircle the point "$-K$". The function "$\frac{1}{G}$" is then plotted as "s" encircles the right half plane. By application of this Nyquist Criterion, the stability or instability of the system can readily be determined.

If the nozzle control transfer function "F_1" is first assumed to be equal to a constant $(F_1 = K_1)$ the system transfer function takes the form

$$K_S G_S = \frac{KG}{1+KG} = \frac{K_1\left[\frac{X}{XZ-YW}\right]}{1+K_1\left[\frac{X}{XZ-YW}\right]} \tag{75}$$

For the Nyquist diagram, the inverse of "G" is plotted as the vector "S" encircles the right half of the "S" plane. The function to be plotted then is

$$\frac{1}{G_{(s)}} = \frac{XZ-YW}{X} \tag{76}$$

In order to use the Nyquist stability criterion it is necessary to know whether the function "KG" has any zeros in the right half plane. By factoring Eq. (73) the following expression is obtained:

$$KG_{(s)} = K_1[-49.2\times10^{-6}]\left\{\frac{[S+19.79][S-19.61]}{S[S-.006][S-(-.0204+i6.25][S-(-.0204-i6.25)]}\right. \quad (77)$$

Since the function "KG" has one zero in the positive real half of the plane, the Nyquist Criterion requires that the plot of $\frac{1}{G_{(s)}}$ must encircle the point $-K$ once in a counter-clockwise direction as the S vector circles the right half of the S plane once in a clockwise direction.

Fig. 9 is the Nyquist diagram of Eq. (76). As the frequency increases from zero to positive infinity, thence to negative infinity and back to zero, there is one clockwise encirclement of all gains from 0 to ∞. The net encirclement may be considered to be two clockwise rotations, indicating two roots of the system transfer function existing in the right half plane. The system is unstable for all gains.

Next, assume the nozzle control transfer function to be of the form $F_1 = K_1(1+TS)$; then

$$K_S G_S = \frac{K_1(1+T_S)\left(\frac{X}{XZ-YW}\right)}{1+K_1(1+T_S)\left(\frac{X}{XZ-YW}\right)} \quad (78)$$

The Nyquist diagram for the above equation is shown in Figs. 10 and 11. Again the number of zeros of "KG" existing in the right half plane determines the Nyquist criterion. The factored equation now becomes

$$KG_{(s)} = TK_1[-49.2\times10^{-6}]\left\{\frac{[S-(-\frac{1}{T})][S+19.79][S-19.61]}{S[S-.006][S-(-.0204+i6.25)][S-(-.0204-i6.25)]}\right\} \quad (79)$$

Since there is one zero in the right half plane, again it is desired to have one counter-clockwise encirclement of the gain "K" as an indication of stability.

It was found necessary to plot the Nyquist diagram to several scales to indicate clearly the crossing points of the negative real axis. The enlarged portions were plotted for the case when $T=1$, and appear as Figs. 12 and 13. From these plots it is clear that there is one counter-clockwise encirclement of all gains from $K=12$ to $K=75$. The configuration is stable for gains in this range.

The Evans Root Locus Method

This method is a graphical solution for the variation of the roots of the feedback system transfer function "$K_S G_S$" as the gain "K" is varied from zero to infinity, where

$$K_S G_S = \frac{KG_{(s)}}{1+KG_{(s)}} = \frac{N_{(s)}}{D_{(s)}}$$

The plot is constructed by observing the following rules (Ref. 2):

1. For $K=0$, the poles of "$K_S G_S$" are the poles of $G_{(s)}$.
2. For $K \rightarrow \infty$, the poles of "$K_S G_S$" are the zeros of $G_{(s)}$, with any missing poles taken at "$s=\infty$".
3. The product of the distances from a root to the poles of "KG" divided by the product of the distances from the root to the zeros of "KG" is equal to the gain "K".
4. The sum of the angles from the zeros of "KG" to a root, less the sum of the angles from the poles of "KG" to the root must equal "π", or "π" plus an integer multiplied by 2π.

The locus of the roots of "$K_S G_S$" can be determined graphically by trial and error, observing the above rules. The gain scale can be plotted along the path of the roots, and for a chosen operating gain, the roots can be determined by inspection.

The root locus plot of Eq. (18) for the case when $T=1$ is shown in Fig. 14. The roots emerge from the poles of KG at zero gain and proceed along the loci indicated to the zeros of KG at a gain of infinity. Gains at intermediate points can be approximated by dividing the product of the distances to the poles of KG by the product of the distances to the zeros, and by the constant gain factor in the equation (49.2×10^{-6}). It is apparent from the root locus plot that the range for a stable configuration is from the gain where the roots "$s_{1,3}$" cross into the stable half plane to the gain where the roots "$s_{2,4}$" cross into the unstable half plane, referring to Fig. 14. The loci in this range are plotted to an enlarged scale and appear in Figs. 15 and 16. As an example of the computation, approximate expressions for the gain along the loci in this range are:

For root ____

$$K_{(s_1)} = \frac{(6.25-\omega)(6.25+\omega)(\omega^2)}{(49.2\times 10^{-6})(19.79)(19.61)\sqrt{1+\omega^2}} \simeq 2050\,\omega^2 \quad \text{(for "}\omega\text{" small)}$$

For root ____

$$K_{(s_2)} \simeq \frac{(\lambda + .0204)(6.25)(12.5)(6.25)}{(\sqrt{39.1+1})(\sqrt{39.1+385})(\sqrt{39.1+391})\,49.2\times 10^{-6}} \quad \text{(for "}\lambda\text{" small)}$$

$$K_{(s_2)} \simeq 3670(\lambda + .0204)$$

It will be noted that the gains at the limits of the stable range check within limits of accuracy of the scale used on the graph as shown in the table below for the Nyquist and the Evans Root Locus plots.

Method	Upper Limit of K	Lower Limit of K
Nyquist plot	75.0	12.0
Evans Root Locus	74.80	11.85

Having established the gain scale along the root loci in the range of positive stability, referring to Figs. 15 and 16, it is possible to choose an operating gain and determine the values of the roots "$S_{1,3}$" and "$S_{2,4}$" at that gain. Some possible operating points have been tabulated below

Roots	K = 50	K = 60	K = 70
S_1	-.0031 + i .156	-.00325 + i .171	-.0033 + i .185
S_3	-.0031 - i .156	-.00325 - i .171	-.0033 - i .185
S_2	-.0068 + i 6.25	-.004 + i 6.25	-.0003 + i 6.25
S_4	-.0068 - i 6.25	-.004 - i 6.25	-.0003 - i 6.25

The low frequency roots "$S_{1,3}$" are associated with the motion of the main rocket, whereas the high frequency roots "$S_{2,4}$" are associated with the motion of the nozzle. The time to damp to half amplitude and the period may be computed for the modes quite simply; the results of such a computation for a gain of $K=60$ are tabulated below.

K = 60	Rocket Oscillation	Nozzle Oscillation
Time to damp to half amplitude	210 seconds	171 seconds
Period of one cycle	36.8 seconds	1 second
Number of cycles to half amplitude	5.7 cycles	171 cycles

It is apparent that the damping is quite poor, and an improvement is desirable. The acceleration of the V-2 at take-off is roughly equal to 30 ft./sec.^{2}. If it is assumed that at a velocity of 200 ft./sec. the aerodynamic forces stabilize the rocket, the period of take-off instability is approximately 7 seconds long. For the foregoing example, the rocket conceivably could complete only one-fifth of a cycle of slightly damped motion in the take-off period as a result of a unit disturbance at the instant of firing. The nozzle oscillations are so rapid as to conceivably cause overloading of the control device.

In order to improve the damping characteristics of the system, referring to the root locus plot, Fig. 14, the addition of one or more zeros along the negative real axis would serve to pull the roots S_1 and S_3 into the negative half plane more quickly, since zeros attract the roots much as lines of force are drawn to magnetic poles. This should increase the magnitude of the negative real part of the roots at a given gain. Also, the angle of departure of the root S_2 would be increased counter-clockwise as a result of the angle condition of the root locus plot. The root would then be forced to travel a longer path before passing into the unstable half plane.

The addition of zeros in the left half plane involves adding additional lead networks, which in real components have transfer functions of the type $\frac{(1+TS)}{(1+\epsilon TS)}$ where $\epsilon \simeq \frac{1}{10}$ to $\frac{1}{20}$.

Assuming a new nozzle control transfer function to be of the form

$$F_1 = \frac{K(1+TS)(1+TS)(1+TS)}{(1+\frac{T}{20}S)(1+\frac{T}{20}S)}$$

where $T=1$, the factored expression for "KG" is now

$$KG = -.0968\left\{\frac{(S+1)(S+1)(S+1)(S-19.79)(S+19.61)}{S(S-.006)(S+20)(S+20)[S-(-.0204+i\,6.25)][S-(-.0204-i\,6.25)]}\right\} \quad (83)$$

The root locus plot of Eq. (83) is shown in Fig. 17. The range of gain for stability is now from K = 4 to K = 918.

By approximating the loci of the roots in the stable range by straight lines, the roots associated with various gains can be estimated conservatively. Some representative values are tabulated below.

Root	K = 600
S_1	-.0446 + i .541
S_3	-.0446 - i .541
S_2	-.02 + i 15.25
S_4	-.02 - i 15.25

If the damping time and period are computed for the latter system, the following results are obtained.

K = 600	Rocket Oscillation	Nozzle Oscillation
Time to damp to half amplitude	15.3 seconds	34.2 seconds
Period of one cycle	11.6 seconds	.412 seconds
Number of cycles to half amplitude	1.3 cycles	83 cycles

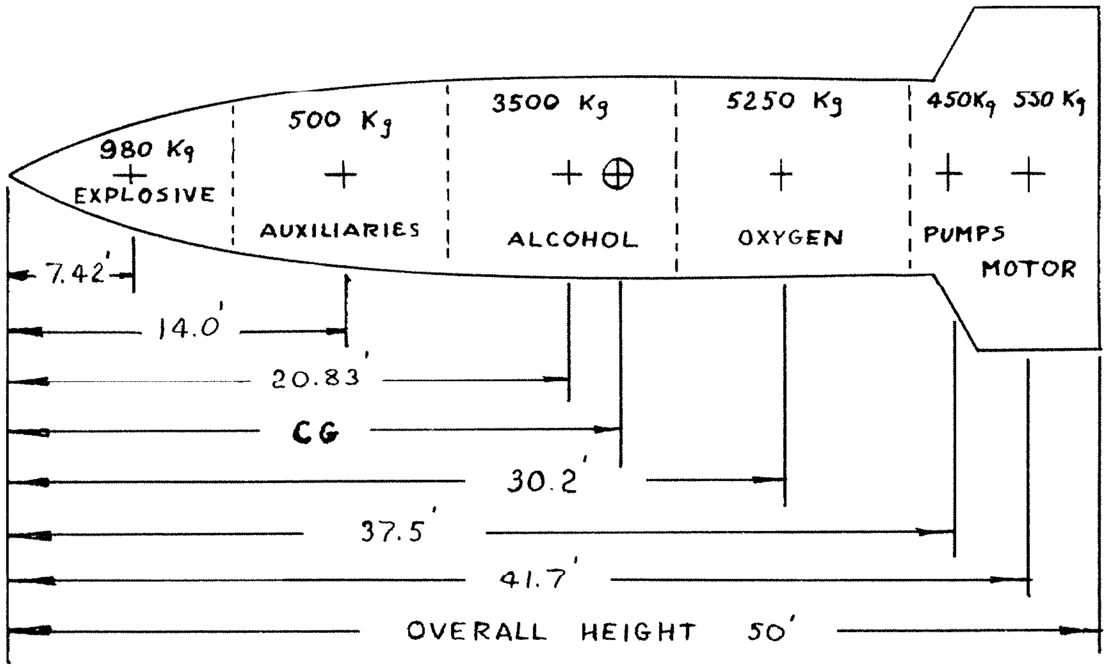
Interpreting the results as before, the rocket now completes approximately two-thirds of a cycle of oscillation during the launching period, but the effect of a unit disturbance is damped to a much greater extent. The nozzle oscillations occur at a higher frequency but the damping has been improved considerably.

VI. CONCLUDING REMARKS

The analysis shows the feasibility of stabilizing a rocket during the launching period by utilizing the motion of a compound pendulum suspended nozzle as the input to a feedback servo-control system. It was determined that the nozzle control for such a system must include response to both the attitude and the rate of change of attitude of the rocket, as computed from the pendulum motion, in order to obtain stability. It was further determined that changing the nozzle control to include response to accelerative motions of the rocket improved the damping characteristics of the system.

APPENDIX

The V-2 Rocket



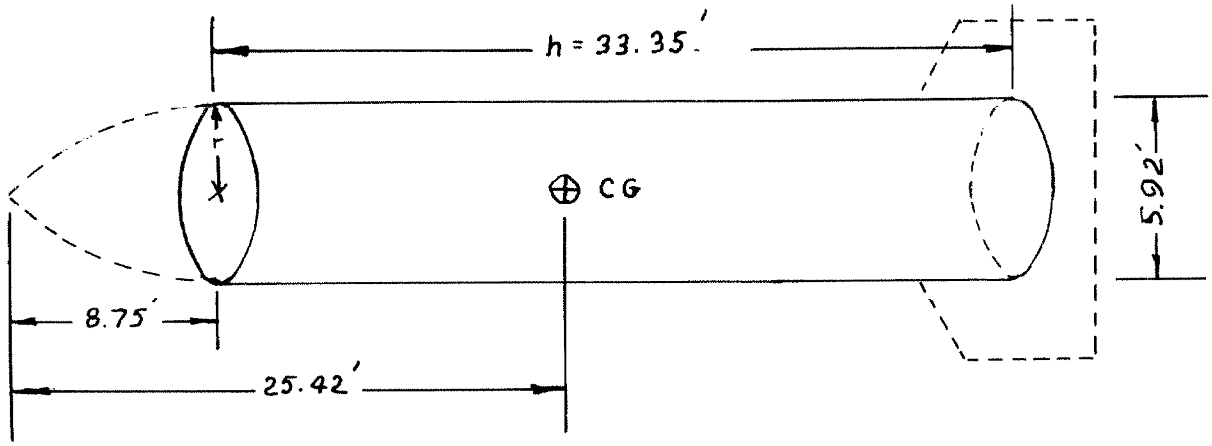
Weight and Balance Table

	Wt. in Kg.	Wt. in Lbs.	Moment Arm, ft.	Moment lb. ft.
1. Explosive	980	2160	7.42	16,000
2. Fuselage	1750	3860	26.65	103,000
3. Pump	450	993	37.5	37,200
4. Motor	550	1214	41.7	50,600
5. Auxiliaries	300	662	14.0	9,300
6. Alcohol	3500	7720	20.83	161,000
7. Oxygen	5250	11600	30.2	250,000
Totals	12,980	28,620		727,100

$$\text{Moment arm of C.G.} = \frac{727,100}{28,620} = 25.4 \text{ ft.}$$

Note: Data was obtained from Ref. (4) and is only approximately correct.

Calculation of the Moment of Inertia



Assume the rocket is a solid cylinder as shown in the above sketch.

Radius of gyration of a solid cylinder about its C.G. is

$$k_g = \sqrt{\frac{3r^2 + h^2}{12}}$$

$$k_1 = \sqrt{\frac{(3)(2.96)^2 + (33.35)^2}{12}} = \sqrt{94.7\ \text{ft}^2} = 9.72 \text{ ft.}$$

Additional Data from Ref. (4)

F = Total Thrust = 27.2 tons = 54, 400 lbs.

W = Total Weight = 12, 980 Kg. = 28, 620 lbs.

$\dot{M}_1$ = Mass Flow $= -\dfrac{19,320 \text{ lbs.}}{100 \text{ sec.}} = -$ 193.2 lb./sec. = − 6 lb.sec./ft.

I_{sp} = Specific Impulse $= \dfrac{F}{\dot{M}_1\, g} = \dfrac{54,400}{193.2}$ = 282 sec.

Calculations for the Constants of Section III

$M_1 = 27{,}406 \text{ lbs} = 851 \dfrac{\text{lb sec}^2}{\text{ft}}$

$M_2 = 1{,}214 \text{ lbs} = 37.7 \dfrac{\text{lb sec}^2}{\text{ft}}$

$\dot{M}_1 = -193.2 \dfrac{\text{lb}}{\text{sec}} = -6 \dfrac{\text{lb sec}}{\text{ft}}$

$k_1 = 9.7 \text{ ft}$

$k_2 \simeq 2 \text{ ft}$ (radius of gyration of motor)

$\ell_1 = 20 \text{ ft}$

$\ell_2 = 3.7 \text{ ft}$

$\ell_3 = 0.3 \text{ ft}$

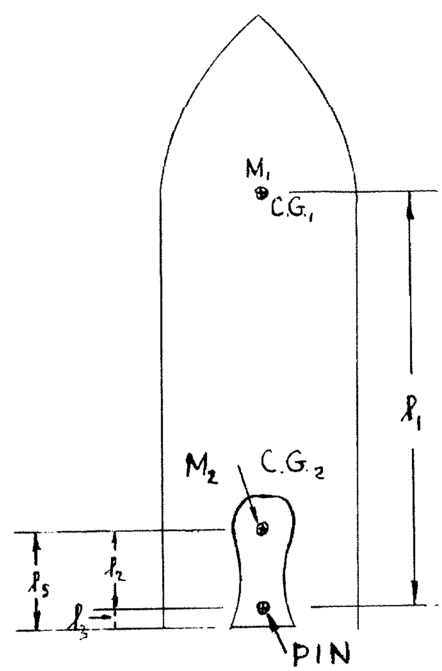
$\ell_5 = 4.0 \text{ ft}$

$I_1 = M_1 k_1^2 = 80{,}600 \text{ lb ft sec}^2$

$I_2 = M_2 k_2^2 = 151 \text{ lb ft sec}^2$

$I_{SP} = 282 \text{ secs}$

$M_1 + M_2 = 888.7 \dfrac{\text{lb sec}^2}{\text{ft}}$



$$X = \frac{M_1 M_2 \ell_1 \ell_2}{M_1 + M_2} s^2 + \left(\frac{M_2^2 \dot{M}_1 \ell_1 \ell_2}{(M_1 + M_2)^2} + \frac{M_1 \dot{M}_1 \ell_1 \ell_2}{M_1 + M_2} \right) s + \left(\frac{M_1 \dot{M}_1 \ell_1 g I_{SP}}{M_1 + M_2} - \frac{M_2^2 \dot{M}_1^2 \ell_1 \ell_2}{(M_1 + M_2)^3} \right)$$

$$X = \left[\frac{(27{,}406\ \text{lbs})(37.7\ \frac{\text{lb sec}}{\text{ft}})^2(20\ \text{ft})(3.7\ \text{ft})}{28{,}620\ \text{lbs}}\right] s^2 \; +$$

$$\left[\frac{(37.7\ \frac{\text{lb sec}^2}{\text{ft}})^2(-6.0\ \frac{\text{lb sec}}{\text{ft}})(20\ \text{ft})(3.7\ \text{ft})}{(888.7\ \frac{\text{lb sec}^2}{\text{ft}})^2} + \frac{(851\ \frac{\text{lb sec}^2}{\text{ft}})(-6.0\ \frac{\text{lb sec}}{\text{ft}})(20\ \text{ft})(4.0\ \text{ft})}{888.7\ \frac{\text{lb sec}^2}{\text{ft}}}\right] s$$

$$+ \left[\frac{(851\ \frac{\text{lb sec}^2}{\text{ft}})(-6.0\ \frac{\text{lb sec}}{\text{ft}})(20\ \text{ft})(32.2\ \frac{\text{ft}}{\text{sec}^2})(282\ \text{sec})}{888.7\ \frac{\text{lb sec}^2}{\text{ft}}} - \frac{(37.7\ \frac{\text{lb sec}^2}{\text{ft}})^2(-6.0\ \frac{\text{lb sec}}{\text{ft}})^2(20\ \text{ft})(3.7\ \text{ft})}{(888.7\ \frac{\text{lb sec}^2}{\text{ft}})^3}\right]$$

$$\underline{X = \left[2675\ s^2 - 461\ s - 1.044 \times 10^6\right] \quad (\text{lb ft})}$$

$$Y = \left[\left(I_1 + \frac{M_1 M_2 \ell_1(\ell_1 - \ell_2)}{M_1 + M_2}\right) s^2 + \left(\dot{M}_1 K_1^2 + \frac{M_2^2 \dot{M}_1 \ell_1(\ell_1 - \ell_2)}{(M_1 + M_2)^2} - \frac{M_1 \dot{M}_1 \ell_1 \ell_5}{M_1 + M_2}\right) s\right]$$

$$Y = \left[\left(80{,}600\ \text{lb ft sec}^2 + \frac{(851\ \frac{\text{lb sec}^2}{\text{ft}})(37.7\ \frac{\text{lb sec}^2}{\text{ft}})(20\ \text{ft})(20 - 3.7)\ \text{ft}}{(888.7\ \frac{\text{lb sec}^2}{\text{ft}})^2}\right) s^2 \; + \right.$$

$$\left(\left(-6.0\ \frac{\text{lb sec}}{\text{ft}}\right)(9.7\ \text{ft})^2 + \frac{(37.7\ \frac{\text{lb sec}^2}{\text{ft}})^2(-6.0\ \frac{\text{lb sec}}{\text{ft}})(20\ \text{ft})(20 - 3.7)\ \text{ft}}{(888.7\ \frac{\text{lb sec}^2}{\text{ft}})^2} - \right.$$

$$\left.\left.\frac{(851\ \frac{\text{lb sec}^2}{\text{ft}})(-6.0\ \frac{\text{lb sec}}{\text{ft}})(20\ \text{ft})(4.0\ \text{ft})}{888.7\ \frac{\text{lb sec}^2}{\text{ft}}}\right) s\right]$$

$$\underline{Y = \left[92{,}380\ s^2 - 110\ s\right]}$$

$$Z = \left[\left(\frac{M_1 M_2 \ell_2(\ell_1 - \ell_2)}{M_1 + M_2} - I_2\right) s^2 + \left(\frac{M_2^2 \dot{M}_1 \ell_2(\ell_1 - \ell_2)}{(M_1 + M_2)^2} + \dot{M}_1 \ell_5\left(\ell_3 + \frac{M_2}{M_1 + M_2}\ell_2\right)\right) s\right]$$

$$Z = \left[\left(\frac{(851)(37.7)(3.7)(20-3.7)}{888.7} - 151\right) s^2 + \left(\frac{(37.7)^2(-6)(3.7)(20-3.7)}{(888.7)^2} + (-6)(4)\left(.3 + \frac{(37.7)(3.7)}{888.7}\right)\right) s\right]$$

$$Z = \left[(2181 - 151) s^2 + (-.652 - 11) s \right] \quad 10 \text{ ft}$$

$$\underline{Z = \left[2030 s^2 - 11.65 s \right]}$$

$$W = \left[\left(I_2 + \frac{M_1 M_2 \ell_2^2}{M_1 + M_2}\right) s^2 + \left(\frac{M_2^2 \dot{M}_1 \ell_2^2}{(M_1+M_2)^2} - \dot{M}_1 \ell_5\left(\ell_3 + \frac{M_2}{M_1+M_2} \ell_2\right)\right) s - \frac{M_2^2 \dot{M}_1^2 \ell_1 \ell_2}{(M_1+M_2)^3}\right]$$

$$W = \left[\left(151 + \frac{(851)(37.7)(3.7)^2}{888.7}\right) s^2 + \left(\frac{(37.7)^2(-6)(3.7)^2}{(888.7)^2} - (-6)(4)\left(.3 + \frac{(37.7)(3.7)}{888.7}\right)\right) s - \frac{(37.7)^2(-6)^2(20)(3.7)}{(888.7)^3}\right]$$

$$W = (151 + 495) s^2 + (-.15 + 11) s - .0054$$

$$\underline{W \approx \left[646 s^2 + 10.85 s \right]}$$

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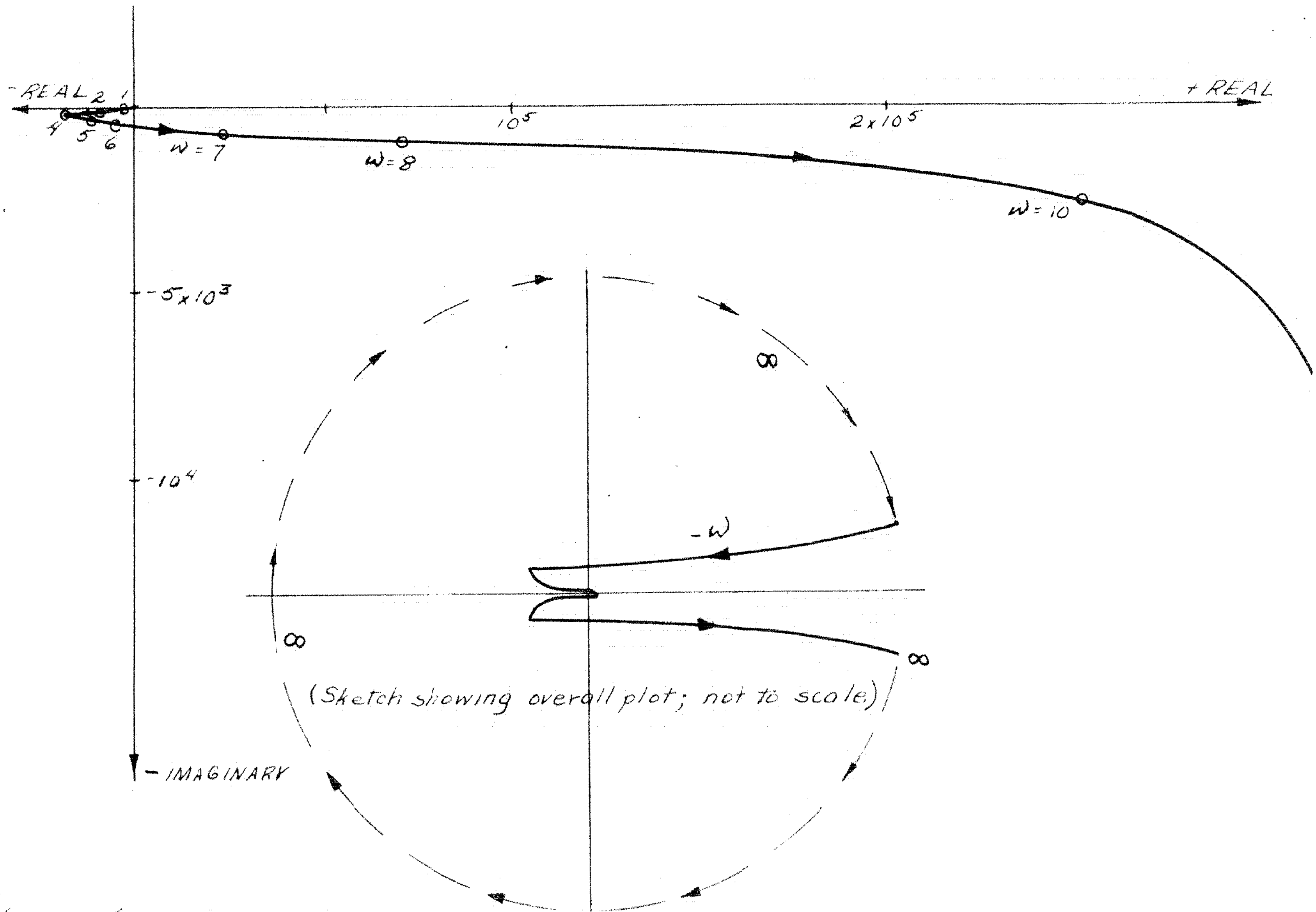
3. Brown, G. S. and Campbell, D. P.: Principles of Servo-Mechanisms. John Wiley and Sons, New York, 1948.

4. Kooy, J. M. J. and Uytenbogaart, J. W. H.: Ballistics of the Future. McGraw Hill Book Co., New York, 1946.



$$\frac{1}{KG} = \frac{S(S-.006)\left[S-(-.0204+i6.25)\right]\left[S-(-.0204-i6.25)\right]}{(-49.2 \times 10^{-6})(S+19.61)(S-19.79)}$$

Fig. 9 Nyquist Plot of Unstable Configuration

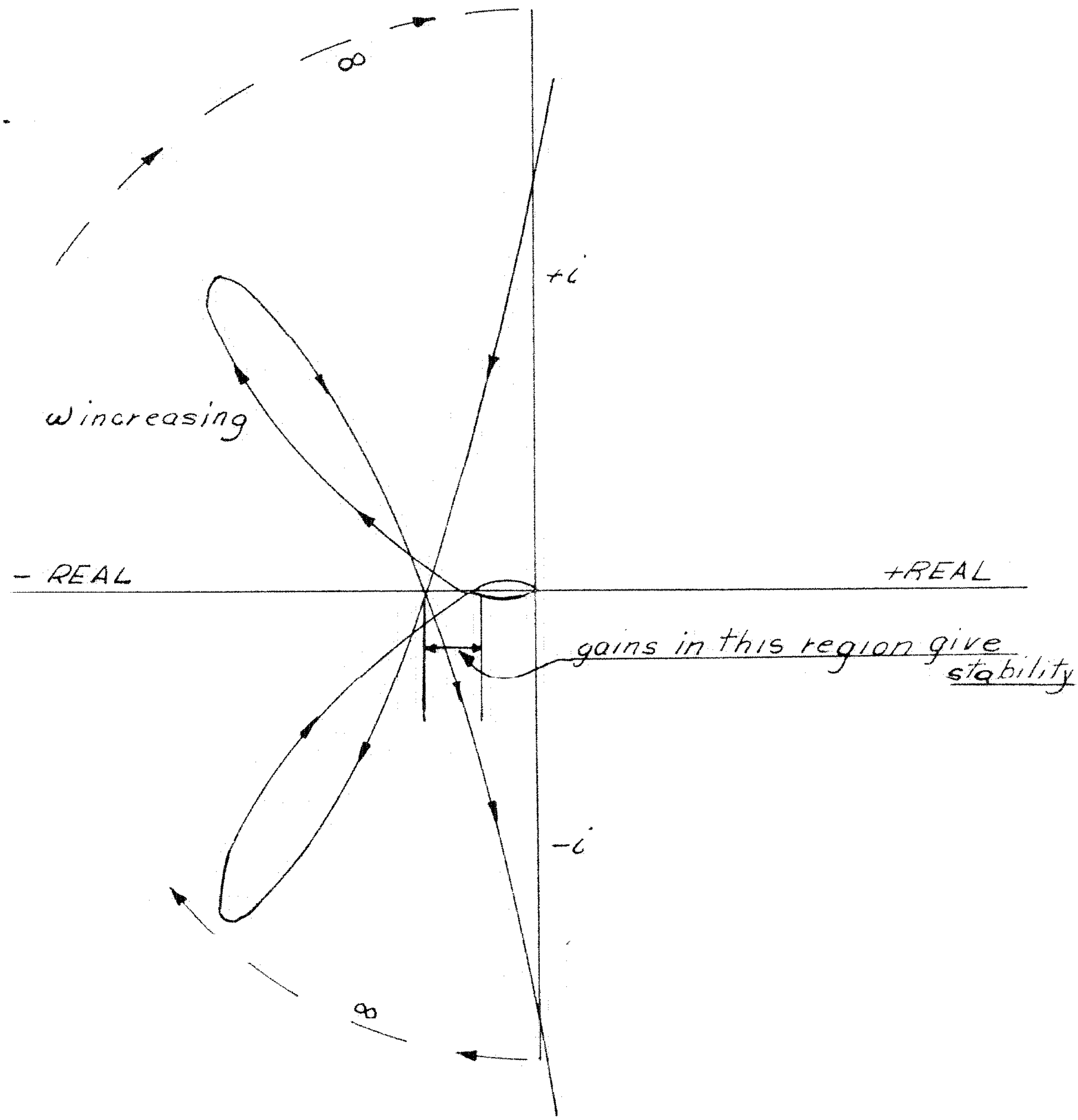
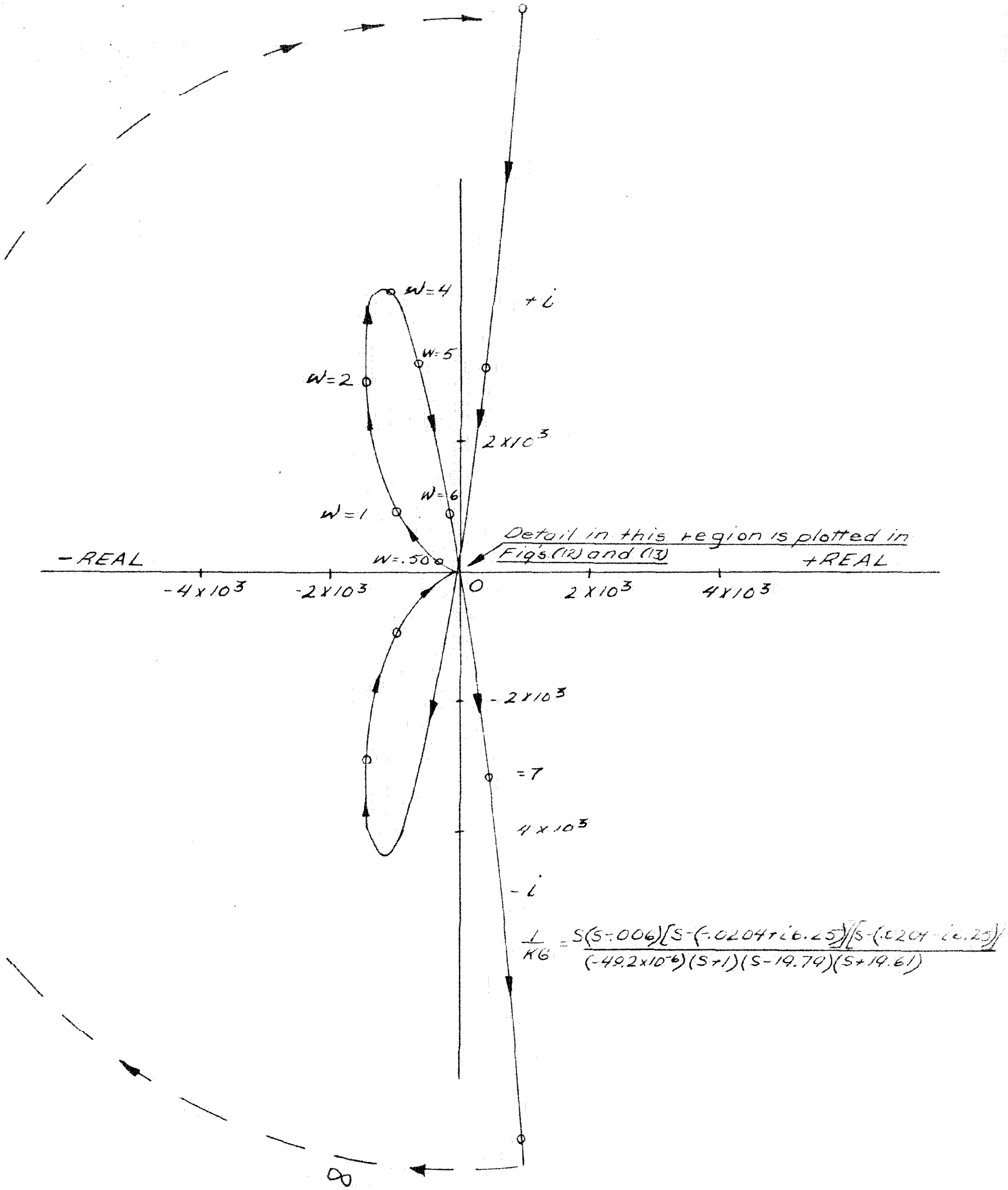


Fig.(10) Sketch of Nyquist diagram of stable configuration; not to scale.



$$\frac{1}{KG} = \frac{S(S-.006)[S-(-.0204+i6.25)][S-(-.0204-i6.25)]}{(-49.2\times10^{-6})(S+1)(S-19.79)(S+19.61)}$$

Fig. 11 Scale Plot of Nyquist Diagram

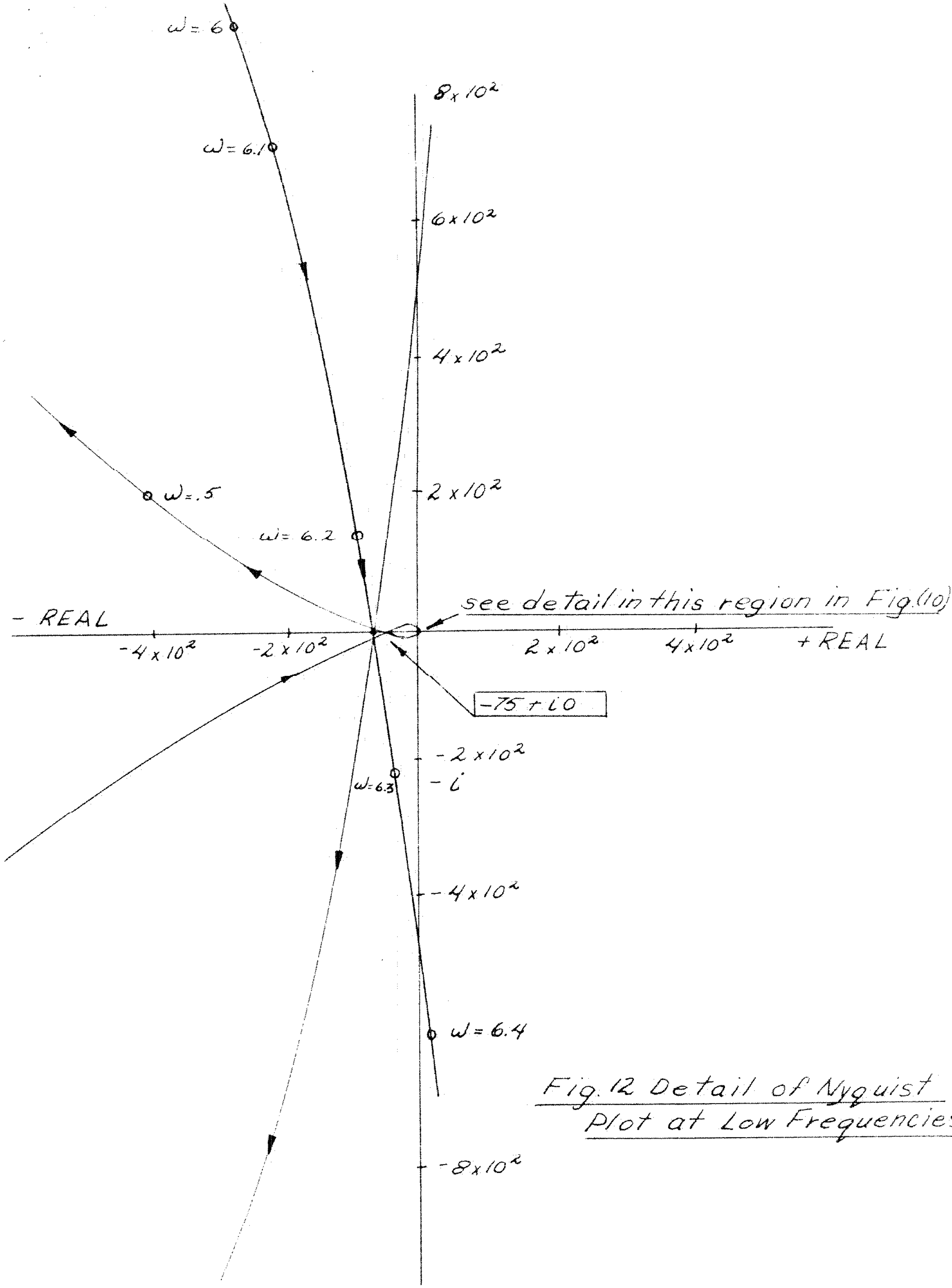


Fig. 12 Detail of Nyquist Plot at Low Frequencies

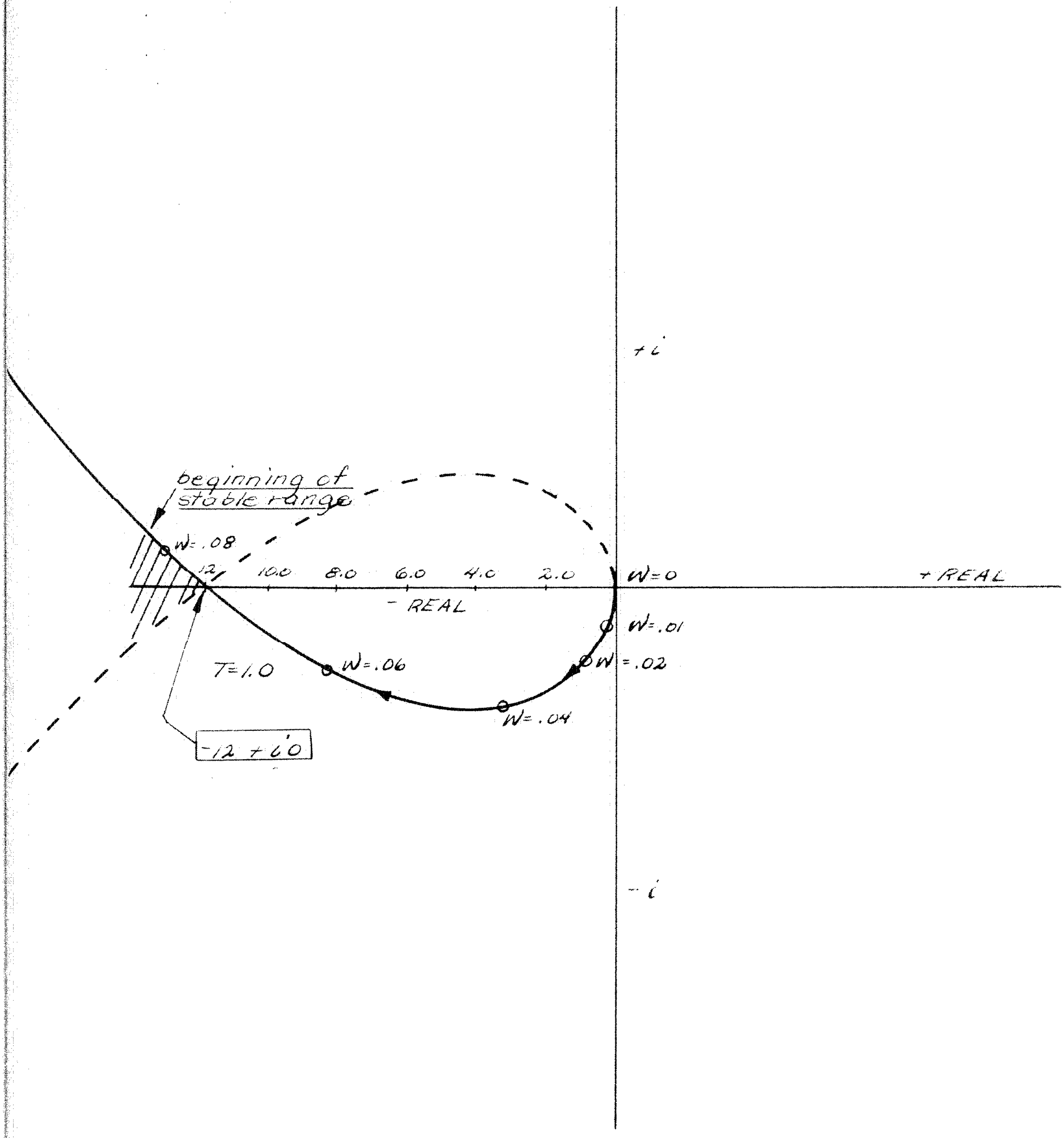
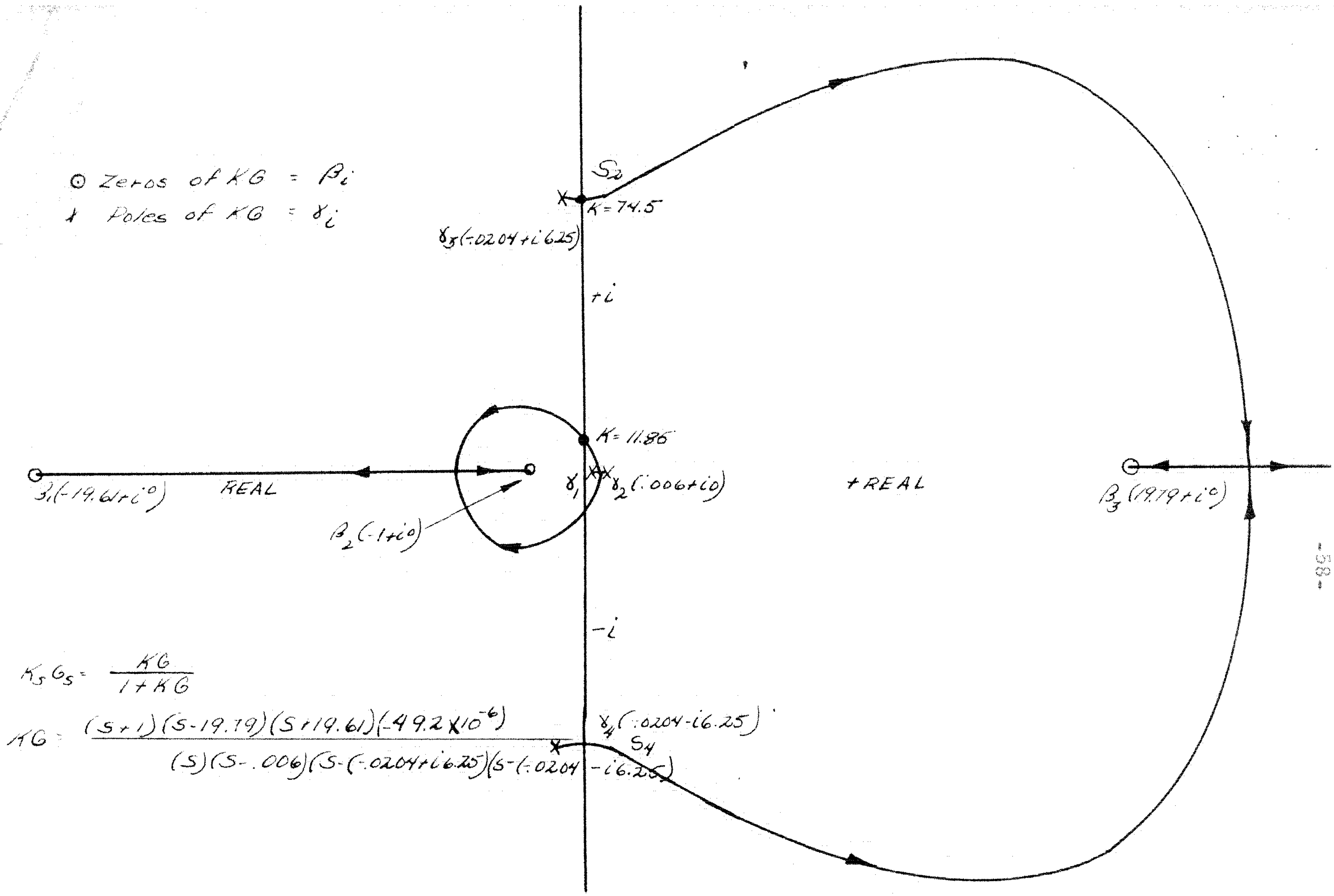


Fig. 13 Detail of Nyquist Diagram for Very Low Frequencies



$$K_5 G_5 = \frac{KG}{1+KG}$$

$$KG = \frac{(S+1)(S-19.79)(S+19.61)(-49.2\times10^{-6})}{(S)(S-.006)(S-(-.0204+i6.25))(S-(-.0204-i6.25))}$$

Fig. 14 Root Locus Plot of Function (not to scale)

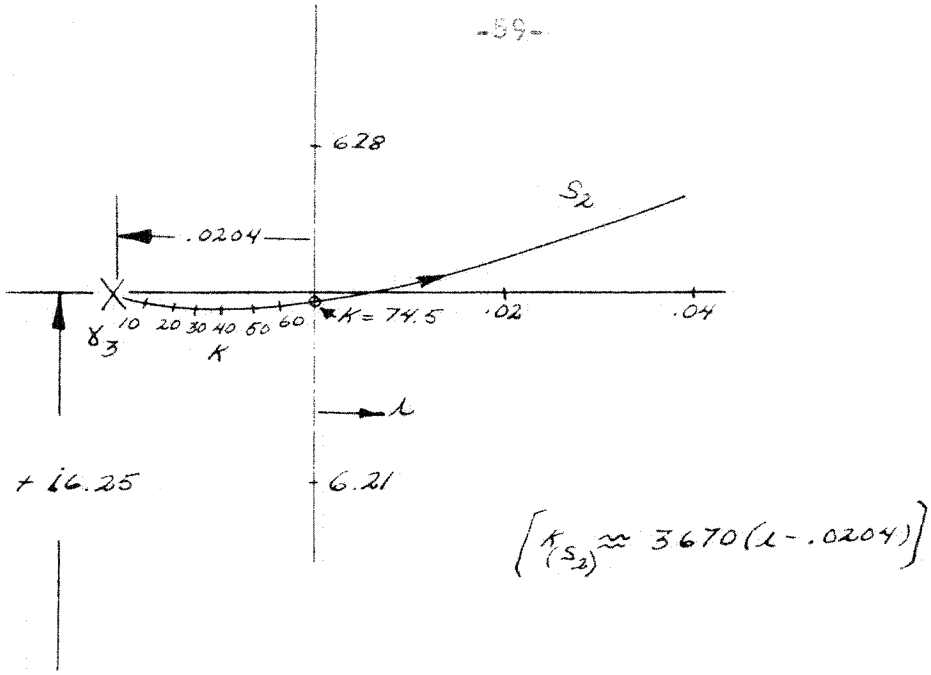


Fig. 15 Detail of Root (S_2) at Low Gains

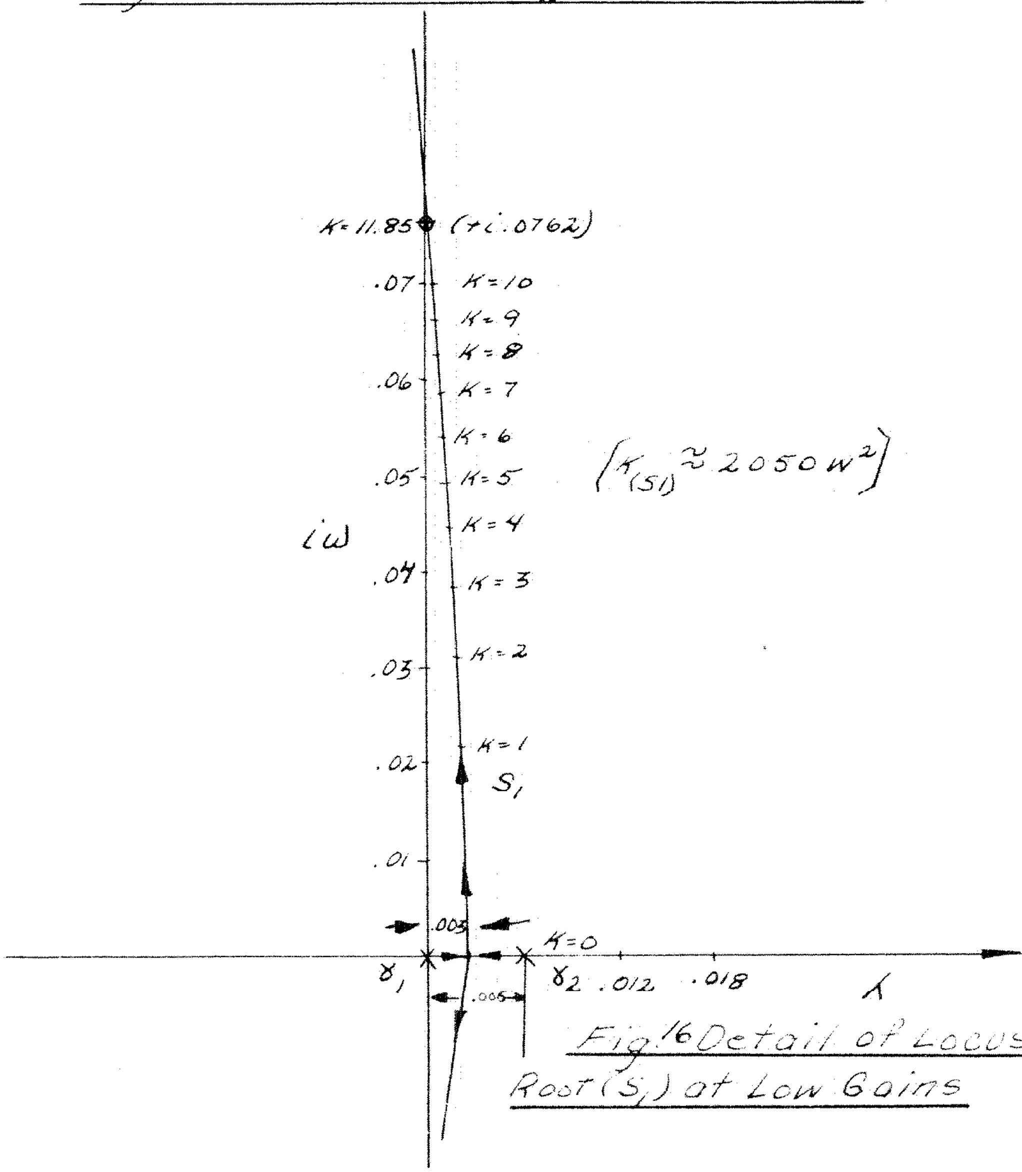


Fig. 16 Detail of Locus of Root (S_1) at Low Gains

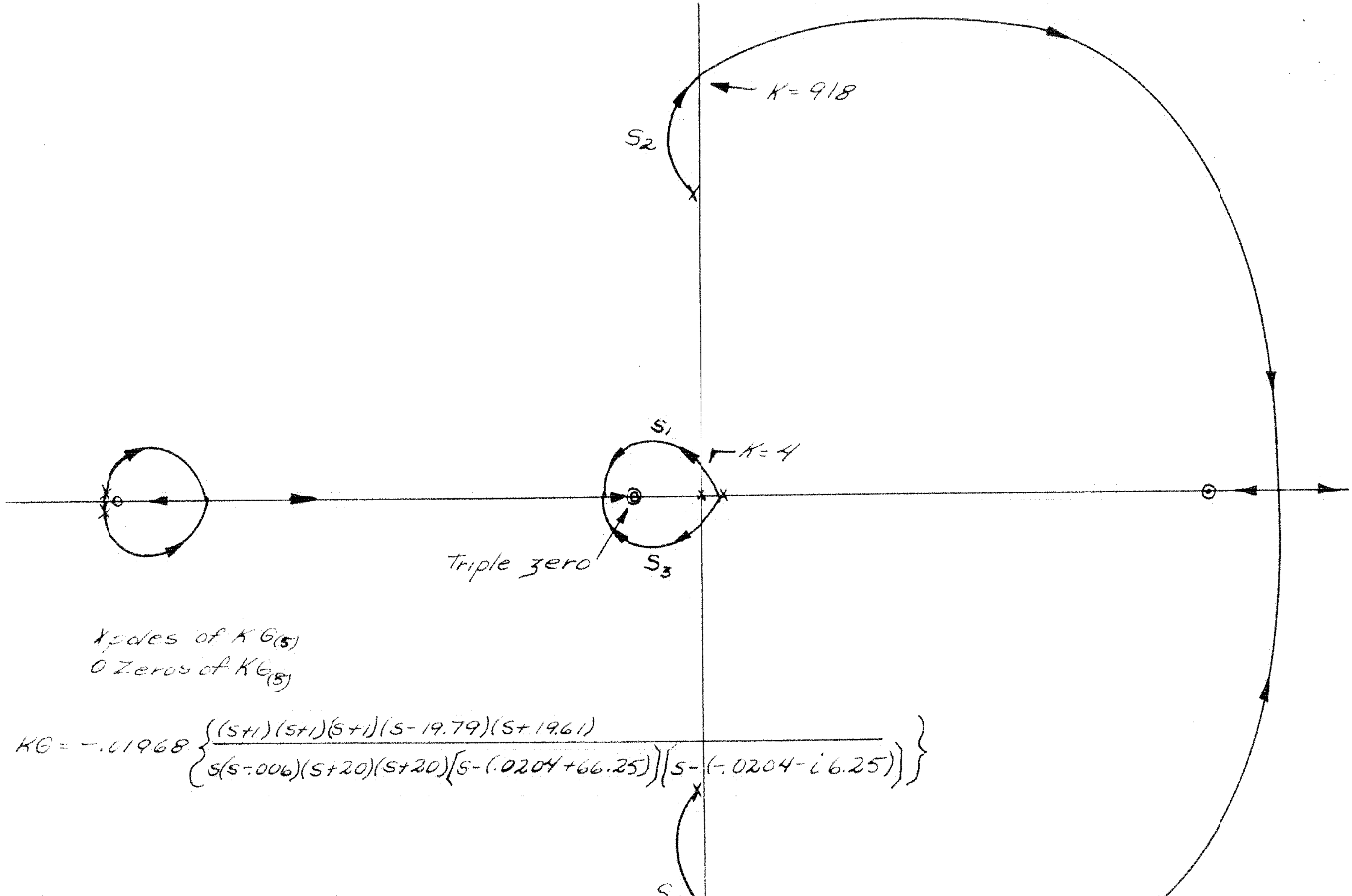


Fig. 17 Root Locus Plot of Function

(not to scale)