Contributions to the General Geometry of Paths

Thesis by

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In Partial Fulfilment of the Requirements
for the Degree of Doctor of Philosophy

California Institute of Technology
Pasadena, California

1940

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Abstract

By general geometry of paths is here meant the study of the properties of a space of undefined elements having a Hausdorff topology and coordinates in a Banach space, and having as fundamental element of structure a class of distinguished subsets called paths. The paths are defined locally in terms of their mappings under certain coordinate systems (N.c.s.) analogous to normal coordinates. The topologic properties of the paths and their mappings in N.c.s. are studied in § 2.1, and it is shown that the map of the aggregate of paths through any point is itself an open set. In § 2.2 a class of allowable coordinate systems is introduced and it is shown that a linear connection (which was not postulated) can be defined and from it a curvature form. These are shown to have their usual properties. It is further shown that the space is locally flat. In § 2.3 a modification of Frechet differentiation is introduced and used to define another linear connection and curvature which do not necessarily imply a locally flat space. § 2.4 is devoted to examples illustrating the above in particular instances.

The second part (Chapter III) of the thesis consists of certain results extending the properties of the normal coordinate component of linear connection and the curvature form as studied by Michal and Hyers {Annali di Pisa 1938} and of a more extended study of the variety of uniform continuity of the Fréchet differential defined and introduced by Michal and the writer {Proceedings of N. A. S. 1939}.

Appreciation

It is difficult to express to those members of the Institute staff with whom I have worked my appreciation for the knowledge and inspiration I have gained from them, both in the classroom and out. Particularly do I thank Professor A. D. Michal for his help and encouragement during the progress of the work presented in this dissertation. From the time he first suggested the basic idea on which the major portion rests until the completion of the thesis his helpful suggestions and criticisms of my work have been invaluable. I hope that the results contained herein may be found worthy additions to the body of research in general differential geometry to which he and his students at the Institute have contributed so largely.

I

Introduction

The problem of the structure of a space in terms of straight lines or geodesics and the generalizations of these as paths has been the subject of extensive study by geometers. Modern work in the differential geometry of paths has been based on the definition of these curves in terms of one of their three fundamental properties. First, the path may be considered as a curve of minimum length in a metric space and studied by variational methods. Second, it may be defined as a self-parallel curve in a space in which parallel displacement has meaning. Third, as a solution of a certain second order differential equation. In each of these three methods there is used either directly or indirectly the concept of a linear connection defined in the space. If a Riemann metric is used the Christoffel symbols enter in, and in the parallelism of Levi-Civita again they play an important role. Even in abstract work, the linear connection is well nigh all important. These three approaches, however, have led to essentially equivalent geometries, and, among other things, to the notion of normal coordinates.

It is the writer's purpose to approach the study of the general geometry of paths (i.e. geometry of paths in a space with coordinates in an abstract (Banach) coordinate space) from a different viewpoint which will, in a sense, generalize the existing theory. Here we start with paths defined locally as curves with a representation suggested by the above mentioned normal coordinates. Then the properties of paths so defined are first studied from a topologic viewpoint, and later the differential properties are studied by the use of Fréchet differentiation, and the writer's modification of the Fréchet differential. In the latter aspect, it is shown possible to define a linear connection and a curvature form locally in a space of paths so defined, and that these geometric objects have their usual properties.

The second part (Ch. III) of this dissertation is devoted to the presentation of some additional results extending and generalizing certain aspects of a paper by Michal and Hyers {1938-1}* and another paper by Michal and the writer {1939-2, 1940-1} .

* The following numbering and reference conventions will be followed throughout this thesis:

i) The bibliography references are numbered according to year and are enclosed in braces.

ii) In all numbers the first digit represents the chapter and the second number, preceding the period, the section.

iii) Equations are numbered in sequence throughout each chapter, all other numbers are in sequence throughout each section.

iv) Internal references to hypotheses are given by number only, to definitions by number in parentheses, and to theorems and corollaries by numbers in square brackets.

II

GENERAL GEOMETRY OF PATHS

In the present chapter the geometric space will be taken to be a Hausdorff space H and the coordinate space will be taken to be a Banach space B. These two spaces will be assumed to be distinct, although this need not necessarily be the case. As mentioned before, the fundamental definition of the paths as distinguished subsets of neighborhoods in H, is suggested by analogy with the familiar normal coordinate representation of paths. No linear connection is assumed, but in §§ 2.2 and 2.3 this geometric object is defined and shown to exist under certain restrictions.

§ 2.1 Definitions and Topologic Properties.

By an open set in B we shall mean, as usual, a set U such that every $\alpha \in U$ may be enclosed in an open sphere S of radius $\rho > 0$, $(\|x - \alpha\| < \rho)$, lying entirely in U.

Definition 21.1 Coordinate system. A fixed homeomorphism (i.e. a fixed biunique bicontinuous functional relationship) between the elements of a Hausdorff neighborhood M and an open set X of B. This we shall denote by $x = x(P)$ where $P \in M$ and $x \in X$.

Definition 21.2 Curve. A subset C of the space H which has at least one map K under a particular coordinate system, such that K is the continuous transform in B of a single real variable (parameter) defined on a certain real interval (or set of intervals).

This yields as a necessary and sufficient condition for $C \subset H$ to be a curve that there exist a coordinate system $x = x(P)$ on a neighborhood $M \supset C$ such that $x(Q) = f(t)$ where $f(t)$ is a continuous function on $I \subset R$, (R is the real number system), to X and $Q \in C$. In the present discussion, the domain I will usually consist of a single open or closed interval in R.

Definition 21.3 Point of a curve. A point $P_1 \in H$ is on (or of) the curve C if and only if $P_1 \in C$.

Clearly, a necessary and sufficient condition that P_1 be on C is that $x(P_1) \in K$.

Definition 21.4 Path. A curve L in H for which there exists at least one coordinate system $y(P)$ whose geometric domain M contains L and such that the map λ of L in the coordinate domain Y is of the form

$$(2.1) \quad \lambda: \qquad y = s\xi^{\dagger} \qquad\qquad s \in J \subset R,$$

where $\xi^{\dagger} \in B$ is fixed and not the zero of B, and J is a single open interval having zero as an interior point.

It is well to point out in the above fundamental definition that in the representation λ the $\xi^{\dagger}$ is not uniquely specified even for a particular coordinate system $y(P)$, and that the interval J will depend on the $\xi^{\dagger}$ used.

Hypothesis 21.5 There exists a class Ω of coordinate systems such that every neighborhood $U \subset H$ is the geometric domain of at least one coordinate system of Ω.

Hypothesis 21.6 There exists at least one coordinate system $y=y(P)$ of Ω for every $P_0 \in H$ such that every path L containing P_0 is mapped in the form (2.1) in this coordinate system.

The first of these two important hypotheses is introduced to make our theory non-vacuous and is automatically satisfied in the interesting particular cases of this theory. The second is of more far reaching import and deserves particular consideration.

Definition 21.7 N.c.s. associated with P_0. Any coordinate system $y=y(P)$ of Ω on the neighborhood M_0 of P_0 to the open set $Y_0 \subset B$ and satisfying the conditions of hypothesis 21.6 will be called an N-coordinate system or N.c.s. associated with P_0.

Definition 21.8 Equivalent paths. Any two paths L, $\bar{L}$ having a point P_0 in common are equivalent if they have a second point $P \neq P_0$ also in common.

Definition 21.9 Linearly dependent (independent) set in B. A set of elements $\xi_1, \xi_2, \cdots, \xi_n$ in B will be termed linearly dependent (independent) if there exists a set (exists no set) of n real numbers $a_1, a_2, \cdots, a_n$ not all zero such that $a_1\xi_1 + a_2\xi_2 + \cdots + a_n\xi_n = 0$.

Definition 21.10 n-dimensional space. The dimensionality n of a space will be taken to be the largest number of linearly independent elements in the space*.

From the above definitions and hypotheses follow a number of theorems which are given below.

Theorem 21.11 "Every path L containing a given point P_0 will be contained in the geometric domain M_0 of every N.c.s. associated with P_0".

Proof. An immediate consequence of hypothesis 21.6.

Theorem 21.12 "In the coordinate domain Y_0 of any N.c.s. $y=y(P)$ associated with P_0 there exists a sphere S of center zero and radius $\rho>0$ lying entirely in Y_0, and if $Y_0 \neq B$, then ρ has an upper bound ρ_0 which it attains."

Proof: 1) The existence of S follows from the definitions of open set and coordinate system (21.1)

2) If $B-Y_0$ is the non-empty complement of Y_0, then

$$\rho_0 = \text{g.l.b.}\ \|z\|, \quad z \in (B-Y_0),$$

is maximum radius of the sphere S.

(Q.E.D.)

Theorem 21.13 "Every $\xi^{+} \in S_0$, where S_0 is the sphere of maximum radius in Y_0, is the map of a point Q on a path L through P_0; and $y=s\xi^{+}$, $\xi^{+}=y(Q)$, $s \in J$ where J is $|s| < \rho_0/\|\xi^{+}\|$ is a mapping function of L."

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* For further definitions concerning n-dimensional subsets see §2.3.

Proof: $\mathcal{P}(y) = \mathcal{P}(s\xi^{+}) \qquad s \in J$

gives a set of points $L \subset M_0$ satisfying (21.4).

(Q.E.D.)

Theorem 21.14 "A necessary and sufficient condition that two paths $L, \bar{L}$ through P_0 be equivalent is that, in at least one N.c.s., in which their maps are respectively $y = s\xi^{+}$, $s \in J$ and $y = \bar{s}\bar{\xi}^{+}$, $\bar{s} \in \bar{J}$, the fixed elements ξ^{+} and $\bar{\xi}^{+}$ be linearly dependent."

Proof:

1) To prove sufficiency, assume $k\xi^{+} + \bar{k}\bar{\xi}^{+} = 0$ and $\bar{k} \neq 0$ and let J be $a < 0 < b$ and $\bar{J}$ be $\bar{a} < 0 < \bar{b}$

2) Since $\bar{\xi}^{+} \neq 0$ by (21.4), $k' = -\, k/\bar{k} \neq 0$

3) Let $s_1 \in \mathcal{R}$ satisfy $0 < |s_1| < b' = \min(-a, b, -|k'|\bar{a}, |k'|\bar{b})$.

4) Then $P_1 = \mathcal{P}(s_1\xi^{+}) = \mathcal{P}(\bar{s}_1\bar{\xi}^{+})$ is an element of L, for $s_1 \in J$ by step 3), and of $\bar{L}$, for $\bar{s} \in \bar{J}$ by step 3).

5) Now if $P_1 \neq P_0$, then $L \cong \bar{L}$ by (21.8). But if $P_1 = P_0$, then let $P_2 = \mathcal{P}(s_2\xi^{+})$ where $s_2 = \frac{1}{2}s_1$ and so $P_2 \neq P_1 = P_0$. Steps 3) and 4) apply for and (21.8) is again satisfied.

6) Conversely to prove necessity, assume $L \cong \bar{L}$ and let $Q_1 \neq P_0$ be a common point of L and $\bar{L}$.

7) Let $y(Q_1) = s_1\xi^{+}$ as a point of L, and $y(Q_1) = \bar{s}_1\bar{\xi}^{+}$ as a point of $\bar{L}$.

8) $\therefore$ $s_1\xi^{+} - \bar{s}_1\bar{\xi}^{+} = 0$ whence by (21.9) ξ^{+} and $\bar{\xi}^{+}$ are linearly dependent. (Q.E.D.)

<u>Corollary 21.15</u> "If $L \cong \bar{L}$, the fixed elements ξ^{+} and $\bar{\xi}^{+}$ of their maps in <u>any</u> N.c.s. are linearly dependent."

<u>Corollary 21.16</u> "Every pair of equivalent paths has infinitely many points in common and conversely."

<u>Corollary 21.17</u> "The equivalence of paths satisfies the four conditions of an equivalence relation."

<u>Theorem 21.18</u> <u>"Given an arbitrary $P_0 \in H$ and a fixed N.c.s. associated with P_0, then every $\xi \in B$ corresponds uniquely to a class of equivalent paths $\{L_1\}$ through P_0 in the sense that $\{L_1\}$ is specified by a representative L_1 whose map is $y = s\xi_1^{+}$, $s \in J_1$, where $\xi_1^{+} = \frac{\rho_0}{2\|\xi\|}\xi$, J_1: $-1 < s < 1$."</u>

<u>Proof:</u> 1) By theorems [21.12] and [21.13], $\rho_0 =$ radius of S_0 exists and $\xi_1^{+} \in S_0$, since $\|\xi_1^{+}\| = \left\|\frac{\rho_0}{2\|\xi\|}\xi\right\| = \frac{\rho_0}{2} < \rho_0$ so $y = s\xi_1^{+}$, $|s| < 1 < \frac{\rho_0}{\|\xi_1^{+}\|}$ is the map in the given N.c.s. of L_1.

2) To prove uniqueness, assume a second class $\{L_2\}$ of representative L_2, $\xi_2^{+} = \frac{\rho_0}{2\|\xi\|}\xi$ and $\{L_1\} \neq \{L_2\}$.

3) This is obviously a contradiction, as any $L_1' \in \{L_1\}$ is equivalent to any $L_2' \in \{L_2\}$ by [21.17].

(Q.E.D.)

<u>Corollary 21.19</u> "If $L \cong \bar{L}$, then any pair $\xi, \bar{\xi} \in B$ to which they correspond in the same N.c.s. are linearly dependent."

<u>Theorem 21.20</u> <u>"The dimensionalities of B and the coordinate domain Y_0 of any N.c.s. are the same."</u>

Proof: 1) Since $Y_0 \subset B$ the dimensionality n of B is not less than the dimensionality m of Y_0. (21.10)

2) For n to be greater than m, we should have to have at least one $\xi \in B$ not linearly dependent on any element or set of elements in Y_0, contrary to [21.18].

(Q.E.D.)

Theorem 21.21 "If B is one-dimensional, then every point $P_0 \in B$ has only one equivalence class of paths containing it. If B has dimensionality $n \geqslant 2$ then every $P_0 \in B$ has ∞^{n-1} equivalence classes of paths through it, where n is finite or denumerably infinite".

Proof: 1) The first part follows at once from [21.18], [21.19], and [21.20].

2) If $n \geqslant 2$ is finite or denumerably infinite, then the number q of non-equivalent paths cannot exceed the number of straight lines through the origin in n-dimensional unitary space or classical Hilbert space respectively, and is in fact equal to this number.

(Q.E.D.)

Theorem 21.22 "If $n \geqslant 2$, the only point common to all non-equivalent paths through P_0 is P_0 and the map of P_0 in all N.c.s. associated with P_0 is zero. Hence zero is an element of the intersection of the coordinate domains of all N.c.s. associated with any $P_0 \in H$."

Proof: 1) By definition (21.8) non-equivalent paths cannot have two distinct points in common.

2) Since $n \geqslant 2$ there exists at least two non-equivalent paths L, $\bar{L}$ thru any $P_0 \in H$. (21.10), [21.14].

3) The maps λ and $\bar{\lambda}$ in any N.c.s. $y=y(P)$ associated with P_0 have the zero element in common. (21.14)

4) Hence by (21.1) $y(P_0)=0$.

(Q.E.D.)

Nothing so far has been assumed which would make the N-coordinate systems discussed above unique. In fact it is quite obvious that if $y=y(P)$ is an N.c.s. associated with P_0, then $\bar{y}=\bar{y}(P)=ky(P)$, $k \in R$ $k \neq 0$, is also an N.c.s. associated with P_0. In the following discussion of coordinate systems it will be convenient to have a fixed and unambiguous notation to be used whenever applicable. Hence, unless otherwise specified, the following symbols will henceforth be used only with the meanings stated.

Consider the schematic representation of the situations in the geometric space H and the coordinate space B under two N.c.s. associated with a single point $P_0 \in B$, figure 1.

Definitions 21.23 (a) $M_0, \bar{M}_0$ are neighborhoods of P_0 and geometric domains of the N.c.s. $y(P)$ and $\bar{y}(P)$ respectively. $M_0' = M_0 \cdot \bar{M}_0$ which, in general, will not be a neighborhood and hence not a geometric domain for any coordinate system.

(b) $Y_0, \bar{Y}_0$ are open sets containing zero $\in B$ and are the coordinate domains of $y(P)$ and $\bar{y}(P)$ respectively. $Y_0', \bar{Y}_0'$ are

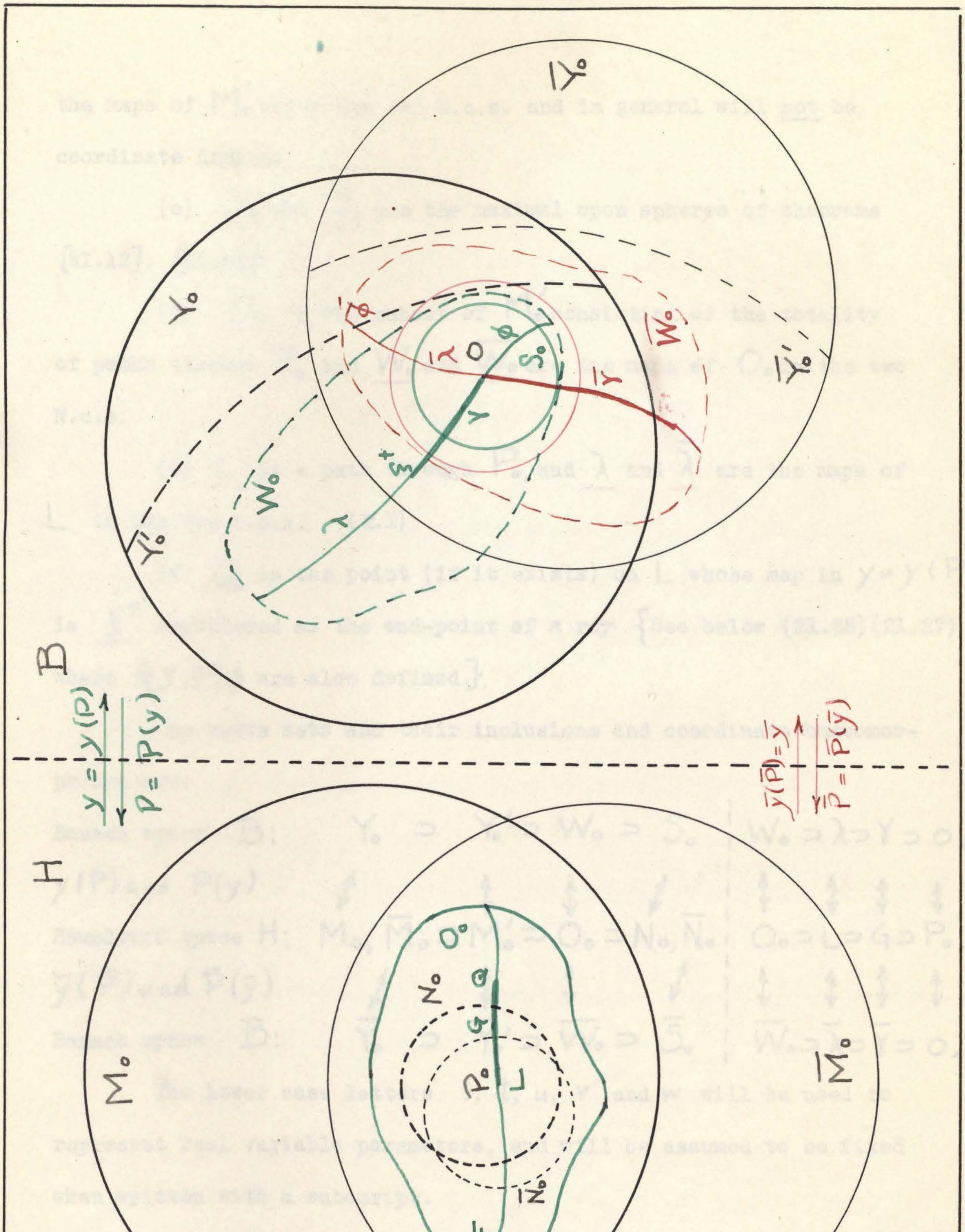
B
H
$y = y(P)$
$P = P(y)$
$\bar{y}(\bar{P}) = \bar{y}$
$\bar{P} = \bar{P}(\bar{y})$
Y_o
Y_o'
W_o
S_o
M_o
M_o'
N_o
P_o
F
L

Figure 1.

the maps of M_0' under the two N.c.s. and in general will <u>not</u> be coordinate domains.

(c) $\underline{S_0}$ and $\underline{\bar{S}_0}$ are the maximal open spheres of theorems [21.12], [21.13].

(d) $\underline{O_0}$ is the subset of M_0' consisting of the totality of paths through P_0, and $\underline{W_0}$ and $\underline{\bar{W}_0}$ are the maps of O_0 in the two N.c.s.

(e) $\underline{L}$ is a path through P_0, and $\underline{\lambda}$ and $\underline{\bar{\lambda}}$ are the maps of L in the two N.c.s. (2.1)

(f) $\underline{Q}$ is the point (if it exists) on L whose map in $y = y(P)$ is $\underline{\xi^+}$ considered as the end-point of a ray $\{$See below (21.26)(21.27) where G, Y, F, ϕ are also defined.$\}$.

The above sets and their inclusions and coordinate homeomorphisms are:

$$
\begin{array}{l|l}
\text{Banach space } B: \quad Y_0 \supset Y_0' \supset W_0 \supset S_0 & W_0 \supset \lambda \supset Y \supset 0, \ \xi^+ \\
y(P) \text{ and } P(y) \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow & \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \\
\text{Hausdorff space } H: \quad M_0, \bar{M}_0 \supset M_0' \supset O_0 \supset N_0, \bar{N}_0 & O_0 \supset L \supset G \supset P_0, \ Q \\
\bar{y}(\bar{P}) \text{ and } \bar{P}(\bar{y}) \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow & \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \\
\text{Banach space } B: \quad \bar{Y}_0 \supset \bar{Y}_0' \supset \bar{W}_0 \supset \bar{S}_0 & \bar{W}_0 \supset \bar{\lambda} \supset \bar{Y} \supset 0.
\end{array}
$$

The lower case letters s, t, u, v and w will be used to represent real variable parameters, and will be assumed to be fixed when written with a subscript.

<u>Theorem 21.24</u> <u>"Under (21.23) there exists a homeomorphism $y(\bar{y}), \bar{y}(y)$ between Y_0' and $\bar{Y}_0'$, and for all $y \in W_0$ $(y \neq 0)$, $\frac{d\bar{y}(y)}{d\xi} = \bar{\xi}^+ \neq 0$,</u>

and similarly for $y(\bar{y})$.

Proof: 1) By the transitivity of homeomorphisms - {Sierpinski 1934-1, pp. 21-22; Michal 1937-1} the existence of $y(\bar{y})$ and $\bar{y}(y)$ follows from (21.1).

2) All $y \in W_0, y \neq 0$ can be written $y = s\xi^+, \; s \neq 0,$
and $\bar{y} \in \overline{W}_0, \bar{y} \neq 0$ " " " $\bar{y} = \bar{s}\bar{\xi}^+, \; \bar{s} \neq 0$
so $\frac{d}{d\bar{s}}\bar{y}(y) = \frac{d}{d\bar{s}}\bar{s}\bar{\xi}^+ = \bar{\xi}^+$ and $\frac{d}{ds}y(\bar{y}) = \xi^+$.

(Q.E.D.)

Theorem 21.25 "The set Y_0' is an open set."

Proof: 1) Assume it is not an open set, then there exists at least one $y_1 \in Y_0'$ such that every sphere about y_1 contains at least one point $y' \bar{\in} Y_0'$.

2) Since Y_0 is open we may assume $\rho_1 < \rho_0$ so $y' \in Y_0$.

3) The coordinate mapping $y(P)$ is continuous, hence for every neighborhood U of $P_1 = P(y_1)$ there exists an open set (and hence an open sphere S of radius $\rho > 0$) about y_1 such that $y_2 = y(P_2) \in S$ implies $P_2 \in U$.

4) But, by steps, 1), 2), every such S has a y' such that $P' = P(y') \bar{\in} M_0'$.

5) P' is an element of every neighborhood U of P_1 by step 3), contradicting the second Hausdorff postulate in H.

(Q.E.D.)

Likewise $\overline{Y}_0'$ is open, and the homeomorphism of theorem [21.24] carries an open set into an open set. Further, this property is true in general for all coordinate systems and not only for N.c.s.

Definition 21.26 Ray - A closed subset (segment) G of a path L containing P_0 such that its map γ in the N.c.s. $y(P)$ is of the form

$$(2.2) \quad \gamma: \quad y = s\xi^{+} \qquad s \in I$$

where I is the closed interval $0 \leq s \leq 1$.

It is clear that this definition of a ray depends on the particular mapping λ in $y(P)$ that is chosen. To avoid ambiguity we shall avoid the equivalence of rays under definition (21.8) and shall speak only of their equality in the sense of logical identity.

Definition 21.27 End points of a ray - The points P_0 and Q of a ray G whose maps are 0 and ξ^{+} respectively.

In view of the preceding remark, two rays are equal if and only if their end points coincide.

Definition 21.28 Half path - A subset (segment) F of a path L containing P_0 such that its map ϕ is of the form

$$(2.3) \quad \phi: \quad y = s\xi^{+} \qquad s \in K$$

where K is either $0 \leq s < b$ or $a < s \leq 0$

and J of (2.1) is $a < 0 < b$.

Definition 21.29 Metric mapping of a path, ray and half-path - Mappings λ, γ, ϕ of definitions (21.4), (21.26) and (21.28) respectively which satisfy the additional condition that $\|\xi^{+}\| = 1$.

Theorem 21.30 "In any given N.c.s. there exists a metric mapping of every path and half-path through P_0 which is unique to within a factor of ±1, while at most two rays on any path (one on any half-path) possess a metric mapping."

Proof: 1) Let the path L (half-path F) be arbitrary and let

$$\lambda\ (\phi):\ y = s\xi^+,\quad s \in J\ (s \in K),\quad a < s < b \quad \begin{pmatrix} 0 \leq s < b \\ a < s \leq 0 \end{pmatrix},$$

be a particular mapping in the N.c.s. $y = y(P)$.

2) Then $\underline{\lambda}\ (\underline{\phi}):\ y = \underline{s}\,\underline{\xi}^+,\ \underline{s} \in \underline{J}\ \ (\underline{s} \in \underline{K})$

will also be mappings of $L\,(F)$ in the same N.c.s. where

$$\underline{\xi}^+ = \frac{1}{\|\xi^+\|}\xi^+,\qquad \underline{J}:\ a\|\xi^+\| < s < b\|\xi^+\| \quad \left(\underline{K}:\ \begin{matrix} 0 < s \leq b\|\xi^+\| \\ a\|\xi^+\| < s \leq 0 \end{matrix}\right).$$

3) Hence $\underline{\lambda}\ (\underline{\phi})$ will exist and satisfy (21.29) whether or not $\xi^+ \in W_0$

4) $\underline{\lambda}\ (\underline{\phi})$ will be unique to within sign, for suppose $\underline{\lambda}_1\ (\underline{\phi}_1)$ is a distinct metric mapping of $L\ (F)$, then there exist $k,\ k_1 \neq 0$ such that $k\underline{\xi}^+ + k_1\underline{\xi}_1^+ = 0$ [21.14]. But

$$1 = \|\underline{\xi}^+\| = \left\|-\frac{k_1}{k}\underline{\xi}_1^+\right\| = \left|\frac{k_1}{k}\right|\|\underline{\xi}_1^+\| = \left|\frac{k_1}{k}\right| \quad \text{or} \quad k_1 = \pm k,$$

hence $\underline{\xi}_1^+ = \pm\underline{\xi}^+$.

5) From the above, only two points $Q_1, Q_2 \in L$ (one point $Q \in F$) is the geometric map of $\pm\underline{\xi}^+$. Hence only two points (one) which can map metrically end-points of a ray. These points are quite independent of the original mapping $\lambda\ (\phi)$.

(Q.E.D.)

In the following theorems no special notation will be used to indicate metric mappings.

Theorem 21.31 "In every equivalence class $\{L\}$ of paths through P_o there exists one 'maximal path' L_o which contains all distinct paths of $\{L\}$ as proper subsets."

Proof: 1) Let y_1 be a fixed point of an arbitrary path $L \in \{L\}$ and

$$\lambda: \quad y = s\xi^+, \quad s \in J \qquad \xi^+ = \frac{1}{\|y_1\|} y_1, \quad J: \; a < s < b$$

be a particular (metric) mapping of

2) Let $\phi(s \in K: 0 \leq s < b)$ and $\phi'(s \in K': a < s \leq 0)$ be (2.3) be (metric) mappings of the two half-paths into which it can be separated.

3) Every other path of $\{L\}$ will be mapped metrically with the same ξ^+, and is separable into two half-paths equivalent to ϕ and ϕ' respectively.

4) Let $\{F\}$ and $\{F'\}$ be the two equivalence classes of half-paths established in 3). Now in $\{F\}$, there exists a l.u.b. b_o of the corresponding values of b, and in $\{F'\}$ a g.l.b. a_o of the set of values a. (These may be finite or infinite).

5) Clearly F_o, whose map is $\phi_o(s \in K_o: 0 \leq s < b_o)$, is in $\{F\}$, and F_o', whose map is $\phi_o'(s \in K_o': a_o < s \leq 0)$ is in $\{F'\}$.

6) Then $\lambda_o: \quad y = s\xi^+, \quad s \in J_o \qquad J_o: \; a_o < s < b_o$

is the map of a path L_0 which satisfies the theorem by (21.4) and (21.29).

(Q.E.D.)

Henceforth, whenever "the path determined by ξ" or "the path corresponding to ξ" is spoken of without further restriction, the "maximal path" of the above theorem is to be understood. The same condition will also apply to half-paths.

Theorem 21.32 "The set W_0 is open"

Proof: 1) Assume W_0 is not open. That is, assume that there exist one or more elements y_1 of W_0 with the property that in every sphere S_1 with center y_1 and radius $\delta > 0$ there exists at least one point y_3 not an element of W_0.

2) Given any $y_1 \in W_0$, there is determined a half-path F_1 through P_0 and P_1 with metric map

$$\phi_1: \quad y = s\xi_1^+, \quad s \in K_1 \quad \xi_1^+ = \frac{1}{\|y_1\|} y_1 \quad \|\xi_1^+\| = 1$$

$$K_1: \quad 0 < s < a,$$

and $y_1 = s_1 \xi_1^+, \quad s_1 = \|y_1\| > 0.$

3) Three possibilities now exist on F_1:

(A) There exist a finite number of elements of type y_1 on F_1.

(B) There exists an infinite number of elements of type y_1 on F_1 of which one (call it y_2) has a norm $\|y_2\| < \|y_1\|$ for all other elements of type y_1 on F_1.

(C) There exist an infinite number of elements of type y_1 on F_1, but none of least norm.

4) Consider cases (A) and (B) where y_2 is the element of type y_1 of least norm on F_1. Then since $y_2 = s_2 \xi_1^+ \in Y_0$, and Y_0 is open there exists a $\delta_2 > 0$ $(\delta_2 < s_2)$ such that

$$\| y_2 - y \| < \delta_2 \quad :\supset: \quad y \in Y_0 .$$

5) Let $y_0 = s_0 \xi_1^+$ where $s_0 = s_2 - \frac{1}{8}\delta_2$

6) Now, there exists a δ_0 $(0 < \delta_0 \leqslant \frac{1}{8}\delta_2)$ such that

$$\| y_0 - y \| < \delta_0 \leqslant \tfrac{1}{8}\delta_2 \quad :\supset: \quad y \in W_0$$

(for if such δ_0 did not exist, then y_0 would be of type y_1 and $\|y_0\| < \|y_2\|$ contrary to 4).

7) Let $\bar{\xi}^+$ be any element of B such that

i) $\|\xi_1^+ - \bar{\xi}^+\| \leqslant \dfrac{\delta_0}{4 s_0}$

ii) $\|\bar{\xi}^+\| \leqslant 1$.

(These conditions are compatible since i) $:\supset:$ $1 - \frac{\delta_0}{4s_0} \leqslant \|\bar{\xi}^+\| \leqslant 1 + \frac{\delta_0}{4s_0}$.)

8) Then $\|s_0 \xi_1^+ - s_0 \bar{\xi}^+\| = \|s_0 \xi_1^+ - (s_0 + u\delta_0)\bar{\xi}^+ + u\delta_0 \bar{\xi}^+\| \leqslant \dfrac{\delta_0}{4}$

Let $\bar{s} = s_0 + u\,\delta_0$ and $|u| < \frac{3}{4}$.

9) $$\|s_0 \xi_1^+ - \bar{s}\,\bar{\xi}^+\| - \|u\,\delta_0 \bar{\xi}^+\| \leqslant \frac{\delta_0}{4}$$

or $\|s_0 \xi_1^+ - \bar{s}\,\bar{\xi}^+\| \leqslant (|u| + \frac{1}{4})\,\delta_0 < \delta_0$

and hence by step 6), $\bar{y} = \bar{s}\,\bar{\xi}^{+} \in W_0$, $s_0 - \frac{3}{4}\delta_0 < \bar{s} < s_0 + \frac{3}{4}\delta_0$.

10) $y \in Y_0 \; :\supset . \; \|y_2 - y\| < \delta_2$ step 4)

$.\supset . \; \|y_2 - y_0 + y_0 - \bar{y} + \bar{y} - y\| < \delta_2$

$.\supset . \; \|y_2 - y_0\| + \|y_0 - \bar{y}\| + \|\bar{y} - y\| < \delta_2$

$.\supset . \; \|\bar{y} - y\| < \delta_2 - \{\|y_2 - y_0\| + \|y_0 - \bar{y}\|\}$

$.\supset . \; \|\bar{y} - y\| < \delta_2 - \{\frac{1}{8}\delta_2 + (|u| + \frac{1}{4})\frac{1}{8}\delta_2\}$

$.\supset : \; \|\bar{y} - y\| < \frac{3}{4}\delta_2$

11) Now by steps 7) and 9) we can define the half-path $\bar{F}$ whose map is

$\bar{\phi}: \; y = v\,\bar{\xi}^{+}$ where $v \in \bar{K}$ and $\bar{K}: \; 0 \leq v < \bar{s}$.

12) But by step 10) we know that the real interval $\bar{K}$ can be increased to at least $0 \leq v < \bar{s} + \frac{3}{4}\delta_2$ which can be written as

$$\bar{K}: \qquad 0 \leq v < s_0 + \frac{3}{4}\delta_2 - \frac{3}{4}\delta_0 = s_0 + \frac{21}{32}\delta_2$$

by the definitions of $\bar{s}$ and δ_0.

13) By our assumption of step 1) we can always choose a $y_3 \,\bar{\in}\, W_0$ satisfying the conditions:

i) $y_3 = s_3\,\xi_3^{+}$, $s_3 > 0$ $\|\xi_3^{+}\| = 1$

ii) $\|y_2 - y_3\| = \|s_2\,\xi_1^{+} - s_3\,\xi_3^{+}\| < \frac{1}{16}\delta_0$

14) Hence

i) $|s_2| - |s_3| = s_2 - s_3 < \frac{1}{16}\delta_0$

and $|s_3| - |s_2| = s_3 - s_2 < \frac{1}{16}\delta_0$

ii) and either $0 \leqslant s_2 - s_3$ i.e. $s_3 \leqslant s_2$

or $0 > s_2 - s_3$ i.e. $s_3 < s_2 + \frac{1}{16}\delta_0$

or $s_3 < s_0 + \frac{1}{8}\delta_2 + \frac{1}{16}\frac{1}{8}\delta_2$

by steps 5) and 6)

iii) and $\therefore$ $s_3 \in \overline{K}$ by step 12).

15) i) Suppose $s_3 = s_2 + \frac{1}{16}\delta_0 - w$, $0 < w < \frac{1}{8}\delta_0$,

ii) $\| s_2 \xi_1^+ - s_3 \xi_3^+ \| = \| s_2 (\xi_1^+ - \xi_3^+) - (\frac{1}{16}\delta_0 - w)\xi_3^+ \| < \frac{1}{16}\delta_0$

iii) $|s_2| \cdot \| \xi_1^+ - \xi_3^+ \| < \frac{1}{16}\delta_0 + |\frac{1}{16}\delta_0 - w| < \frac{1}{8}\delta_0$.

iv) Since $|s_2| = s_2 = s_0 + \frac{1}{8}\delta_2$ by step 5),

$\| \xi_1^+ - \xi_3^+ \| < \dfrac{\delta_0}{8 s_2} = \dfrac{1}{8}\left(\dfrac{1}{s_0 + 1/8\,\delta_2}\right) < \dfrac{\delta_0}{8 s_0}$

v) and $\therefore$ ξ_3^+ satisfies step 7) and is a $\overline{\xi}^+$.

16) By step 11) y_3 is on a half-path $\overline{F}$ and hence is an element of W_0 contrary to step 1).

17) Consider now case C of step 3).

i) No y_3 can lie in S_0 of theorems [21.12], and [21.13] hence $\| y_3 \| \geqslant \rho_0 > 0$.

ii) Let F_1 be as in step 2) and $y_0 = \rho_0' \xi_1^+$ where $\rho_0' = \rho_0 + \text{g.l.b.}\, \| y_1 \|$ where y_1 ranges over all elements y_1 of type y_1 on F_1.

iii) Hence, y_0 is an accumulation point of elements of type y_1 on F_1, and therefore is itself of type y_1.

iv) This denies the assumption of step 3 C) since we have shown an element of type y_1 of least norm on L_1, and case C cannot arise.

(Q.E.D.)

An analogous method of proof is used in the following

Theorem 21.33 "The set $Z_0 = Y_0 - W_0$ is open."

Proof: 1) Assume Z_0 is not open. That is assume there exists at least one element z_1 of Z_0 such that in every sphere with z_1 as center there is a point z_3 not an element of Z_0.

2) Every such z_1 determines a subset ψ_1 of Z_0 of the form

$$\psi_1 : \quad z = t\,\zeta_1^+, \quad t \in T_1 \qquad \zeta_1^+ = \frac{1}{\|z_1\|} z_1$$

and $\bar{T}_1 > T_1 :\supset: \bar{z} = \bar{t}\,\zeta_1^+ \bar{\in} Z_0$ for some $\bar{t} \in \bar{T}$

where T_1 is a real interval of one of the forms

i) $c_1 < t < d_1$

ii) $c_1 \leq t < d_1$

iii) $c_1 < t \leq d_1$

iv) $c_1 \leq t \leq d_1$

3) Also every such z_1 determines a path whose metric map in W_0 is of the form 21.18

$$\lambda : \quad y = s\,\zeta_1^+, \quad s \in J_1 \qquad J_1 : a_1 < s < b_1 .$$

4) Suppose $c_1 \leq b_1$ and T_1 is of form ii) or iv), then

$$\bar{\lambda}: \quad y = s\,\rho_1^+, \quad s \in \bar{J}_1 \qquad \bar{J}_1: \; a_1 < s < d_1$$

satisfies definition (21.4) and is the metric map of a path $\bar{L}$ of O_0 and hence

$z_1 = \|z_1\|\,\rho_1^+$ is in W_0 contrary to step 1).

Similarly if $c_1 < b_1$ and T_1 is of form i) or iii)

5) This shows that $b_1\rho_1^+$ is not in W_0 or Z_0 and hence is in $B - Y_0$.

6) Therefore, as in the previous proof, there exists an element z_2 of type z_1 on ψ_1 and with least norm, $\|z_2\| > b_1 > 0$

7) If T_1 is of form ii) or iv), $\|z_2\| > c_1$; and T_1 cannot be of form iii) or iv), for otherwise every sphere about $d_1\rho_1^+$ would contain points of either $B-Y_0$ (which is impossible since Y_0 is open) or points of W_0 of form $k\,\rho_1^+$ which would imply $\psi \subset \lambda \subset W_0$ contrary to hypothesis.

8) Hence there exists $\epsilon > 0$ such that if we take T_2 as

$$0 < \|z_2\| - \epsilon < t < \|z_2\| + \epsilon$$

$$t \in T_2 \quad :\supset: \quad z = t\,\rho_1^+ \in Z_0 ;$$

9) Let $\rho_2 > 0$ satisfy i) $\|z_2\|\rho_2 < \epsilon$, ii) $\rho_2 < 1$ iii) $\|z_2 - \rho\| < \rho_2 :\supset: y \in Y_0$ (Y_0 is open).

10) Let $z_4 = t_4\,\rho_1^+$ where $t_4 = \|z_2\|\{1 - \frac{1}{4}\rho_2\}$

whence $\|z_4\| < \|z_2\|$ and z_4 is not of type z_1

and we can take $\rho_4 > 0$ such that

$$\|z_4 - y\| < \rho_4 \leqslant \frac{\rho_2}{4} \quad :\supset: \quad y \in Z_0 \qquad (\text{or } y \bar{\epsilon} W_0)$$

11) Now, by step 1), there exists some z_3 such that

$$\|z_2 - z_3\| < \frac{\rho_4}{4} \quad \text{and} \quad z_3 \bar{\epsilon} Z_0 \qquad (\text{or } z_3 \in W_0)$$

12) Let λ_3 be the metric map of the path $L_3 \subset O_0$ defined by P_3 and P_0

$$z_3 = \|z_3\| \, \zeta_3^+, \qquad \zeta_3^+ = \frac{1}{\|z_3\|} z_3.$$

13) From step 11), $\|z_3\| < \|z_2\| + \frac{\rho_4}{4}$, $\|z_3\| = \|z_2\| + \theta \frac{\rho_4}{4}$, $(|\theta| < 1)$

whence

$$\|z_2\| \cdot \|\zeta_1^+ - \zeta_3^+\| - |\theta| \frac{\rho_4}{4} \|\zeta_3^+\| \leqslant \left\| \|z_2\| \zeta_1^+ - \|z_2\| \zeta_3^+ - \theta \frac{\rho_4}{4} \zeta_3^+ \right\|$$

$$\text{or} \quad \|z_2\| \cdot \|\zeta_1^+ - \zeta_3^+\| - |\theta| \frac{\rho_4}{4} \|\zeta_3^+\| \leqslant \left\| \|z_2\| \zeta_1^+ - \|z_3\| \zeta_3^+ \right\| < \frac{\rho_4}{4}$$

giving $\|z_2\| \cdot \|\zeta_1^+ - \zeta_3^+\| < \rho_4/4$.

14) Let $\bar{y} = \left\{ \|z_3\| - \frac{\rho_2}{4} \|z_2\| \right\} \zeta_3^+$, then

$$\|z_4 - \bar{y}\| = \left\| \left(\|z_2\| - \frac{\rho_2}{4} \|z_2\| \right) \zeta_1^+ - \left(\|z_3\| - \frac{\rho_2}{4} \|z_2\| \right) \zeta_3^+ \right\|$$

$$\leqslant \|z_2 - z_3\| + \frac{\rho_2}{4} \|z_2\| \cdot \|\zeta_1^+ - \zeta_3^+\|$$

$$< \frac{\rho_4}{4} + \frac{\rho_2}{4} \frac{\rho_4}{2} < \rho_4$$

15) From its definition, $\bar{y} \in \lambda_3$ and hence $\bar{y} \in W_0$ contradicting step 10).

(Q.E.D.)

Theorem 21.34 " $B - W_0$ has no isolated points."

Proof: 1) Assume $z \in (B - W_0)$ is isolated. That is assume there exists $\rho > 0$ such that

$$\| z - y \| < \rho \quad :\supset: \quad y \in W_0 \qquad\qquad y \neq z$$

2) Then $y = (1 + \frac{\rho}{2}) z$ is in W_0, and hence is the end-point of a ray thru O containing z.

3) This implies $z \in W_0$ contradicting step 1).

(Q.E.D.)

In the study of the properties of the set O_0 we make some rather obvious generalizations of familiar concepts in finite dimensional geometry. These will be useful in further study of the above sets, and the geometry of the local space of paths O_0 about P_0.

Definition 21.35 Pencil of path - The totality of paths in O_0 corresponding in the N.c.s. $y = y(P)$ to elements of B of the form $\xi^+ = r_1 \xi_1^+ + r_2 \xi_2^+$, $(r_1^2 + r_2^2 > 0)$ where ξ_1^+ and ξ_2^+ are fixed linearly independent non-zero elements of B and r_1 and r_2 are real parameters, will be called the pencil of paths determined by the paths L_1 and L_2 through P_0 corresponding to ξ_1^+ and ξ_2^+.

Clearly all pairs (r_1, r_2) having the same ratio $r_0 = \frac{r_2}{r_1}$

or $r_0' = \frac{r_1}{r_2}$ determine the same path L_0 in the pencil, and L_0 will have the mappings

$$(2.4)\begin{cases} \lambda_0: & y = r(\rho_1^+ + r_0 \rho_2^+) \quad r \in J_0 \quad J_0: a < r < b \quad r_0' \neq 0 \\ \lambda_0': & y = r(r_0' \rho_1^+ + \rho_2^+) \quad r \in J_0' \quad J_0': a' < r < b' \quad r_0 \neq 0 \end{cases}$$

$$(2.5)\begin{cases} \bar{\lambda}_0: & y = r \dfrac{\rho_1^+ + r_0 \rho_2^+}{\|\rho_1^+ + r_0 \rho_2^+\|} \quad r \in \bar{J}_0 \\ \bar{\lambda}_0': & y = r \dfrac{r_0' \rho_1^+ + \rho_2^+}{\|r_0' \rho_1^+ + \rho_2^+\|} \quad r \in \bar{J}_0' \end{cases}$$

We shall speak of L_0 as being "between" L_1 and L_2 if $r_0, r_0' > 0$. The maps of such paths are more conveniently written in the form:

$$(2.6)\begin{cases} \tilde{\lambda}_0: & y = s(u_0 \rho_1^+ + (1-u_0)\rho_2^+) \quad s \in \tilde{J}_0 \\ \tilde{\lambda}_0': & y = s((1-u_0')\rho_1^+ + u_0' \rho_2^+) \quad s \in \tilde{J}_0' \end{cases} \qquad \text{for fixed } u_0, u_0' \quad 0 < u_0, u_0' < 1$$

Definition 21.36 Path surface - The two dimensional subset Σ of elements of O_0 mapping in the form $y(r, r_0)$ of (2.4) {metric path surface if $y(r, r_0)$ is of form (2.5)} for $r \in J(r_0), J'(r_0')$ $\{r \in \bar{J}(r_0), \bar{J}'(r_0')\}$ and $-\infty < r_0, r_0' < \infty$.

Definition 21.37 Included path surface - A portion of path surface Σ mapping in the form $y(s, u_0)$ of (2.6) where $s \in \tilde{J}(u_0), \tilde{J}'(u_0')$ where $0 \leq u_0, u_0' \leq 1$.

Theorem 21.38 "A path surface (and an included path surface) is defined uniquely through any given P_0 by:

i) Two non-zero elements $\mathfrak{f}_1, \mathfrak{f}_2 \in B$ not linearly dependent

ii) Two non-equivalent paths, of given mapping, in O_0.

iii) Any path L_1, of given mapping, in O_0 and a point $P_2 \in O_0$ not on L_1.

iv) Any two points $P_1, P_2 \in O_0$ not lying on the same path through P_0.

Proof follows at once from our definitions.

Theorem 21.39 "Given an N.c.s. associated with P_0 and any element y_1 of W_0 there exists a $\delta > 0$ such that for any y_2, $\|y_1 - y_2\| < \delta$, there exists a path L lying in the path surface Σ and containing the points $P_1 = P(y_1)$ and $P_2 = P(y_2)$.

Proof: 1) Since W_0 is open [21.32] there exists $\delta_1 > 0$ such that $\|y_1 - y\| < \delta_1 \ :\supset:\ y \in W_0$. Let $\delta_1 = \delta$, then $y_2 \in W_0$.

2) If y_1 and y_2 are linearly dependent, then there exists a path through P_0 containing P_1, P_2 and the theorem is proved, so let y_1, y_2 be linearly independent, and hence P_1, P_2 not on the same path through P_0.

3) Let the path surface Σ determined [21.38] map in the form S: $y(s,u) = s((1-u)y_1 + uy_2)$

$$s \in J(u) \qquad -\infty < u < \infty.$$

4) Denote by χ' the subset of S of form

$$\lambda' : \quad y(u) = (1-u)y_1 + u\, y_2 , \qquad u \in K$$

$$K: \quad -\epsilon < u < 1+\epsilon$$

where $\epsilon = \theta\left(\frac{\delta}{\|y_1 - y_2\|} - 1\right)$ $\quad 0<\theta<1$ $\quad$ (θ fixed)

5) $\|y_1 - y(u)\| = |1-u| \cdot \|y_1 - y_2\| < \delta$ for all $u \in K$.

6) Define $\bar{y} = \bar{y}(P) = y(P) - y_1$. Now, $\bar{y}(P)$ is defined for all P in M_0 and its set of values $\bar{Y}_0$ is an open set in B. Hence $\bar{y}(P)$ is a coordinate system (21.1).

7) In $\bar{y}(P)$, $L' = \bar{P}(\bar{\lambda}')$ has the map

$$\bar{\lambda}' : \quad \bar{y} = u(y_1 - y_2) = u\bar{\xi}^+ \qquad u \in K$$

where K is an open interval of R with zero as an inner point - hence L' is the path required (21.4).

(Q.E.D.)

Corollary 21.40 "The map in O_0 of the sphere of radius $\delta_1 > 0$ about y_1 as center in N.c.s. $y = y(P)$ lies in the domain of paths O_1 associated with the point $P_1 = P(y_1)$ for any y_1 of W_0."

Theorem 21.41 "If P_1 and P_2 are any two points of a path surface Σ in O_0 mapping in the form

$$S: \quad y(s,u) = s((1-u)y_1 + u y_2)$$

such that $y(u) = (1-u)y_1 + u\, y_2$ is in S for all u of the included path surface, then there exists a path L containing P_1 and P_2."

Proof: 1) Since W_0 is open, there exists

$\delta_1 > 0$ such that $\|y_1 - y\| < \delta_1 \;:\supset:\; y \in W_0$; and

$\delta_2 > 0$ such that $\|y_2 - y\| < \delta_2 \;:\supset:\; y \in W_0$.

Let $\delta = \min(\delta_1, \delta_2)$.

2) The remainder of the proof is the same as for [21.39]

(Q.E.D.)

Theorem 21.42 "If P_1 is in O_0, then P_0 is in O_1."

Proof: Steps 6) and 7) of the proof of [21.39] establish this theorem.

§2.2 Allowable Coordinate Systems.

Up to this point, only a very restricted subset of the general class Ω of coordinate systems has been studied. We now introduce a wider class of coordinate and make use of these to study the differential properties of the local geometry of paths. In the present section the space arrived at will necessarily be "locally flat".

Again a diagramatic representation is introduced to present more vividly the possible relations of the sets and domains discussed. (Fig. 2). It should be born in mind, however, that these two-dimensional schemata may be misleading if taken too literally, though they serve well enough for the limited number of sets considered here. We call attention to the fact that when dealing with infinite dimensional or dimensionless spaces, the boundaries and intersections of boundaries of sets (assuming sets which have boundaries!) will in general themselves be infinite dimensional or dimensionless sets.

All notation used in figure 1 will be used with the same meaning here except S_0 and $\bar{S}_0$. All other notation not made clear in the following definition and theorem will be explained later.

Definition 22.1 L-coordinate system (L.c.s.) Any coordinate system of Ω of the form $x(P) = x(y(P))$ on a neighborhood S_0 of P_0, where $S_0 \subset M_0$, and whose values form the open set X_0 of B. Furthermore, the function $x = x(y)$ by means of which $x(P)$ is defined

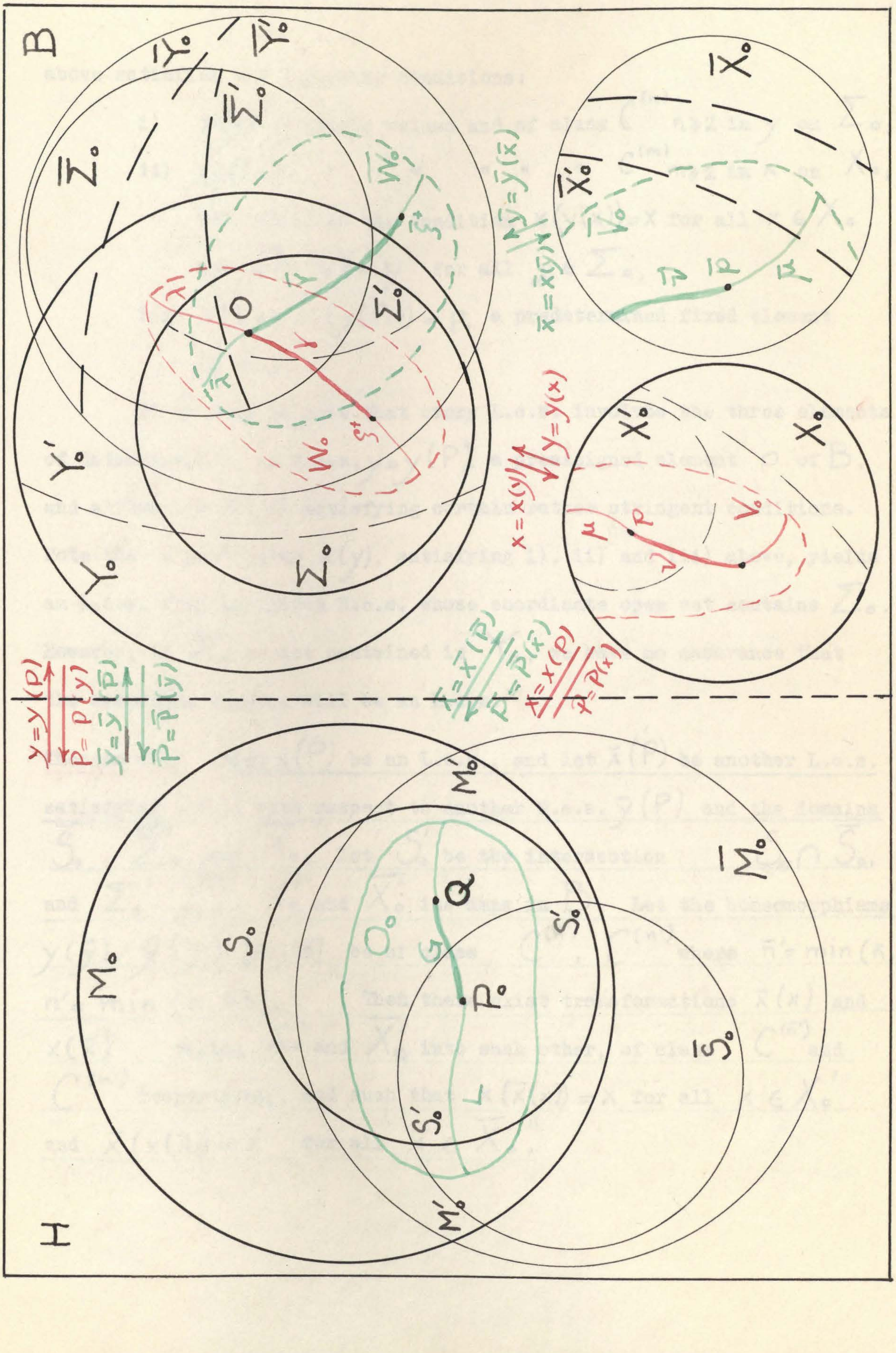

Figure 2.

above satisfies the following conditions:

i) $x(y)$ is single valued and of class $C^{(n)}$ $n \geq 2$ in y on Σ_0,

ii) $y(x)$ " " " " " " $C^{(m)}$ $m \geq 2$ in x on X_0,

and satisfies the condition $x(y(x)) = x$ for all $x \in X_0$ and $y(x(y)) = y$ for all $y \in \Sigma_0$,

iii) $x(0) = x(y(P_0)) = p$, a predetermined fixed element of X_0.

It is well to note that every L.c.s. involves the three elements of determination: an N.c.s. $y = y(P)$, a preassigned element p of B, and a function $x(y)$ satisfying certain rather stringent conditions. Note that a particular $x(y)$, satisfying i), ii) and iii) above, yields an L.c.s. from any given N.c.s. whose coordinate open set contains Σ_0. However, if Σ_0 is not contained in Y_0, we have no assurance that the resulting mapping will be an L.c.s.

Theorem 22.2 "Let $x(P)$ be an L.c.s., and let $\bar{x}(P)$ be another L.c.s. satisfying (22.1) with respect to another N.c.s. $\bar{y}(P)$ and the domains $\bar{S}_0$, $\bar{\Sigma}_0$, and $\bar{X}_0$. Let S_0' be the intersection $S_0 \cap \bar{S}_0$, and Σ_0', $\bar{\Sigma}_0'$, X_0' and $\bar{X}_0'$ its maps in B. Let the homeomorphisms $y(\bar{y})$, $\bar{y}(y)$ [21.24] be of class $C^{(\bar{n}')}$, $C^{(n')}$ where $\bar{n}' = \min(\bar{n}, m)$ $n' = \min(n, \bar{m})$. Then there exist transformations $\bar{x}(x)$ and $x(\bar{x})$ taking X_0' and $\bar{X}_0'$ into each other, of class $C^{(\bar{n}')}$ and $C^{(n')}$ respectively, and such that $x(\bar{x}(x)) = x$ for all $x \in X_0'$ and $\bar{x}(x(\bar{x})) = \bar{x}$ for all $\bar{x} \in \bar{X}_0'$."

Proof: - 1) $y=y(P)$ maps S_0' on the set $\Sigma_0' \subset \Sigma_0 \subset Y_0$,
$P=P(y)$ " Σ_0' on " " S_0', and similarly for
$\bar{y}=\bar{y}(\bar{P})$, $\bar{P}=\bar{P}(\bar{y})$.

2) An argument analogous to that used in the proof of theorem [21.25] will show Σ_0', $\bar{\Sigma}_0'$ are open sets. {S_0' is not necessarily a neighborhood, however.}

3) Then the homeomorphism $y=y(\bar{y})$, $\bar{y}=\bar{y}(y)$ [21.24] maps the open sets Σ_0', $\bar{\Sigma}_0'$ on each other.

4) By hypothesis and by (22.1) $\bar{x}(\bar{y})$ is of class $C^{(\bar{n})}$, $\bar{y}(y)$ of class $C^{(\bar{n}')}$, $y(x)$ of class $C^{(m)}$.
Hence $\bar{x}=\bar{x}(\bar{y}(y(x)))=\bar{x}(x)$ exists and is of class $C^{(\bar{n}')}$ ($\bar{n}'=\min(\bar{n},m)\geqslant 2$, as before)

5) Likewise $x=x(y(\bar{y}(\bar{x})))=x(\bar{x})$ exists and is of class $C^{(n')}$, $(n'\geqslant 2)$.

6) Clearly $x=x(y(\bar{y}(\bar{x}(\bar{y}(y(x))))))=x(\bar{x}(x))$
and $\bar{x}=\bar{x}(\bar{y}(y(x(y(\bar{y}(\bar{x}))))))=\bar{x}(x(\bar{x}))$
for every $x\in X_0'$ and $\bar{x}\in\bar{X}_0'$ as required.

(Q.E.D.)

Definition 22.3 Contravariant vector associated with P_1. - A geometric object having components ξ, $\bar{\xi}$ in every pair of L.c.s. $x(P)$ and $\bar{x}(\bar{P})$ satisfying (22.1) and the hypotheses of [22.2] {or coordinate systems satisfying all these conditions but $n,m,\bar{n},\bar{m}\geqslant 2$, this being replaced by $n,m,\bar{n},\bar{m}\geqslant 1$} and $P_1\in S_0'$. And further, such that these components satisfy the linear relations,

(2.7) $$\bar{\xi} = \bar{x}(x_1; \xi) \qquad \xi = x(\bar{x}_1; \bar{\xi})$$

where $x_1 = x(P_1)$ $\qquad \bar{x}_1 = \bar{x}(P_1)$.

Definition 22.4 Contravariant Vector field - A set of contravariant vectors (c.v.) such that one c.v. of the set is associated with each point P_1 of the set considered.

We now establish the existence of a "connection" in the space of paths associated with P_0 and use this to express the local differential properties of this space.

Definition 22.5 Linear connection - A geometric object with the component

(2.8) $$\Gamma(x, \xi_1, \xi_2) = -x(y; \xi_1^+; \xi_2) = -d^{yy}_{\xi_1^+\xi_2^+} x(y)$$

in the L.c.s. $x(P)$ dependent on the N.c.s. $y(P)$. The increments ξ_1^+, ξ_2^+ are assumed arbitrary and free to take any value in B and are related to the arguments ξ_1, ξ_2 by

(2.9) $$\xi_i = x(y; \xi_i^+) \qquad \xi_i^+ = y(x; \xi_i) \qquad i = 1, 2.$$

It is clear that if both relations (2.9) are to be satisfied simultaneously for arbitrary ξ_i^+, then these relations are solvable linear in the sense that

$$\xi_i = x(y; y(x; \xi_i)) \qquad \xi_i^+ = y(x; x(y; \xi_i^+)) \qquad i = 1, 2.$$

Hence it follows that the set of values $\xi_i = x(y; \xi_i^+)$, $\xi_i^+ \in B$ is a closed linear manifold in B and is either B itself or a sub-Banach space of B.

Theorem 22.6 "The linear connection (2.8) is symmetric and bilinear in its two arguments ξ_1, ξ_2."

Proof: Since, from [22.1], the second Fréchet differential exists and is continuous in y for $y \in Y_0$ the theorem follows from the definition of Fréchet differential and a well known theorem on the symmetry of the second Fréchet differential {Kerner 1933-1}.

The theory of normal coordinates as developed in Michal-Hyers {1938-1} leads to a case in which the homeomorphism $\bar{y}(y)$ is linear. The assumptions made in the present study are not sufficient to assure this, nor even to assure the differentiability of $\bar{y}(y)$.

Suppose two L.c.s. $x(P)$ and $\bar{x}(P)$ (22.1) have been defined and $\bar{y}(y)$ satisfies the hypothesis of [22.2] so that $\bar{x}(x)$, $x(\bar{x})$ are of class $C^{(n')}$. Let $\bar{\bar{y}}(P)$ be a third N.c.s. such that $\bar{\bar{y}}(y)$ $y(\bar{\bar{y}})$ is not linear or differentiable (see example II §2.4). Hence we have a function $x(\bar{\bar{y}}) = x(y(\bar{\bar{y}}))$, not necessarily of class $C^{(n')}$, which defines the same L.c.s. $x(P)$ as before but from an N.c.s. $\bar{\bar{y}}(P)$. However, this does no violence to definition (22.1), for nothing was said there about uniqueness of defining function nor about differentiability of all such possible defining functions.

Under these conditions then, we would have no assurance that a given c.v. with components in x, $\bar{x}$, y, $\bar{y}$ etc. would have any component defined in $\bar{\bar{y}}$. From these concepts we are able to distinguish two further subclasses of the general class Ω of coordinate systems.

Definition 22.7 $C^{(n)}$ N.c.s. - Given any N.c.s. $y(P)$ then all other N.c.s. $\bar{y}(P)$ whose homeomorphisms $y(\bar{y}), \bar{y}(y)$ are of class $C^{(n)}$ will be said to be $C^{(n)}$ N.c.s. with respect to $y(P)$, or $y(P)$ and $\bar{y}(P)$ will be said to be a pair of $C^{(n)}$ N.c.s.

Definition 22.8 Irrational L.c.s. (I.L.c.s.) - Any L.c.s. $x(P)$ satisfying (22.1) and the further condition that $\xi_0 = x(0;\xi^+) = \left(\frac{\partial x}{\partial s}\right)_{s=0} = \xi^+$ for all $\xi^+ \in B$.

A pair of L.c.s. satisfying both these conditions with respect to their defining N.c.s. will be termed a pair of $I.C^{(n)}L.c.s.$

Theorem 22.9 "The linear connection (2.8) undergoes a transformation of the form

$$\bar{\Gamma}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \bar{x}(x; \Gamma(x, \xi_1, \xi_2)) - \bar{x}(x; \xi_1; \xi_2) \tag{2.10}$$

$$= \bar{x}(x; \Gamma(x, \xi_1, \xi_2) + x(\bar{x}; \bar{\xi}_1; \bar{\xi}_2))$$

under the transformation $\bar{x} = \bar{x}(x)$ of L.c.s. on V_0 to $\bar{V}_0$."

Proof: - 1) For $x \in V_0$ we may write $x(y) = x(s\xi^+)$

and $\dfrac{\partial x(y)}{\partial s} = x(s\xi^+; \xi^+) = x(y; \xi^+) = \xi$ by (2.9)

2) Then $\bar{\xi} = \bar{x}(\bar{y}; \bar{\xi}^+) = \bar{x}(\bar{y}(y); \bar{y}(y; \xi^+))$

3) Now we know from [22.2] that $\bar{x}(x) = \bar{x}(\bar{y}(y(x)))$,

so

$$\bar{\xi} = \frac{\partial \bar{x}}{\partial s} = \bar{x}(\bar{y}(y(x)); \bar{y}(y(x); y(x;\xi))).$$

4) But, using the notation of Michal and Elconin {1937-2},

$$\bar{x}(x;\xi) = d^{x}_{\xi}\,\bar{x}(\sigma) = d^{x}_{\xi}\,\bar{x}(\bar{y}(y(\sigma))) =$$

$$= \bar{x}(\bar{y}(y(x)); d^{x}_{\xi}\,\bar{y}(y(\sigma)) = \bar{x}(\bar{y}(y(X); \bar{y}(x;\xi))).$$

5) Hence $\bar{\xi} = \bar{x}(x;\xi) = \bar{x}(x(y); x(y;\xi^{+})) = \bar{x}(\bar{y}(y); \bar{y}(y;\xi^{+}))$

where the last equality must be an identity for $y \in W_0 \cdot \Sigma_0$

6) Note now that $\bar{\xi}^{+}_{1}$ is not a function of y for any path - hence $d^{y}_{\xi^{+}_{2}}\,\bar{\xi}^{+}_{1} = 0$. So differentiate the identity just obtained and get

$$\bar{x}(\bar{y}(y); \bar{y}(y;\xi^{+}_{1}); \bar{y}(y;\xi^{+}_{2})) + 0 =$$

$$= \bar{x}(x(y); x(y;\xi^{+}_{1}); x(y;\xi^{+}_{2})) + \bar{x}(x(y); x(y;\xi^{+}_{1};\xi^{+}_{2})).$$

7) Which gives the first form of (2.10) on substituting from (2.8), (2.9) $\bar{\xi}^{+}_{1} = \bar{y}(y;\xi^{+}_{1})$ obtained as in steps 4) and 5).

8) The second form of (2.10) follows from a basic identity proved by Michal {1937-1}. (Q.E.D.)

One should notice that the usual restriction that ξ_1, ξ_2 of (2.10) should be components of contravariant vectors is not made here as this property is established in the course of the proof from the property (2.9) of (22.5).

We shall need several properties of the second partial derivative $\frac{\partial^2 x}{\partial s^2}$ in our further study of the linear connection, and these we now establish.

Theorem 22.10 "The function $\frac{\partial^2 x}{\partial s^2} = x\left(s\,y(x;\frac{\partial x}{\partial s});\, y(x;\frac{\partial x}{\partial s});\, y(x;\frac{\partial x}{\partial s})\right)$ is homogeneous of degree two in $\frac{\partial x}{\partial s}$."

Proof: - 1) Let $x = x(s\,\xi^+)$ undergo an affine transformation $s = at + b$ (a, b fixed and real) of the parameter s.

2) $$x = x((at+b)\,\xi^+) = f(at+b, \xi^+)$$
$$\xi = \frac{\partial x}{\partial s} = \frac{dt}{ds}\frac{\partial x}{\partial t} = x(s\,\xi^+; \xi^+) = \frac{1}{a}\, x((at+b)\xi^+; a\,\xi^+)$$

3) $$\xi = \frac{1}{a}\frac{\partial f}{\partial t} = \frac{1}{a} f_t(at+b, \xi^+).$$

4) Similarly $$\frac{\partial \xi}{\partial s} = \frac{dt}{ds}\frac{\partial \xi}{\partial s} = \frac{1}{a^2} f_{tt}(at+b, \xi^+).$$

5) Now replace ξ by $\lambda\xi$ in 3) and get $\xi = \frac{1}{\lambda a} f_t(at+b, \xi^+)$ and note that this replacement, with $a, at+b, \xi^+$ invariant, has the same effect on 3) as the replacement of a by $a\lambda$, with $\xi, at+b, \xi^+$ invariant, for all λ not zero.

6) Hence in 4) by the replacement $a \to a\lambda$ we get

$$\frac{\partial^2 x}{\partial s^2} = \frac{1}{(a\lambda)^2} f_{tt}(at+b, \xi^+)$$

or $\lambda^2 \frac{\partial^2 x}{\partial s^2} = \frac{1}{a^2} f_{tt}\ (at+b, \xi^+)$.

7) But, by the equivalence established in 5), this implies

$$\left[\frac{\partial^2 f(s,\sigma)}{\partial s^2}\right]_{\sigma=\lambda\xi^+} = \lambda^2 \left[\frac{\partial^2 f(s,\sigma)}{\partial s^2}\right]_{\sigma=\xi^+}.$$

(Q.E.D.)

Theorem 22.11 "The function $F(x, s, \xi_1, \xi_2) =$ $x(sy(x;\xi_1); y(x;\xi_2); y(x;\xi_2))$ is a homogeneous polynomial of degree zero in ξ_1 and is a homogeneous polynomial of degree two in ξ_1."

Proof: - 1) $\lambda^2 \frac{\partial^2 x}{\partial s^2} = x\ (sy(x;\lambda\xi); y(x;\lambda\xi); y(x;\lambda\xi))$ by [22.10]

2) $\frac{\partial^2 x}{\partial s^2} = x\ (sy(x;\lambda\xi); y(x;\xi); y(x;\xi))$ by linearity

$= x\,(sy(x;\xi); y(x;\xi); y(x;\xi))$.

3) Hence F is a homogeneous function of ξ_1 of degree zero.

4) Now in all previous considerations of $y = s\xi^+$ we have excluded $\xi^+ = 0$, and $\xi_1^+ = y(x;\xi_1)$ is zero at $\xi_1 = 0$ so we must permit this value to have $F(x, s, \xi_1, \xi_2)$ defined and continuous at $\xi_1 = 0$

5) So, given $\epsilon > 0$, there exists $\delta > 0$ such that

$\|\lambda\xi_3\| < \delta \ :\supset:\ \|F(x,s,0,\xi_2) - F(x,s,\lambda\xi_3,\xi_2)\| =$

$= \|F(x,s,0,\xi_2) - F(x,s,\xi_3,\xi_2)\| < \epsilon$.

6) But for <u>any</u> $\xi_3 \in B$ there exists a $\lambda > 0$ such that $\|\lambda \xi_3\| < \delta$ hence

$$F(x, s, 0, \xi_2) = F(x, s, \xi_3, \xi_2) \quad \text{for } \underline{\text{all}} \quad \xi_3 \in B,$$

- that is $F(x, s, \xi_1, \xi_2)$ is independent of ξ_1.

7) Then $F(x, s, \xi_1 + \lambda \xi_3, \xi_2) = F(x, s, \xi_1, \xi_2)$ which is of degree zero in ξ_1. This satisfies the definition for a polynomial in ξ_1 of degree zero {Martin 1932-1}

8) Since F is twice <u>linear</u> in ξ_2, and the rest of the theorem follows. (Q.E.D.)

<u>Theorem 22.12</u> "$\Gamma(x, \xi_1, \xi_2)$ becomes $F(x, s, \xi, \frac{\partial x}{\partial s}) = \frac{\partial^2 x}{\partial s^2}$ in the case $\xi_1 = \xi_2 = \frac{\partial x}{\partial s}$. If the closed linear manifold of values of $\partial x/\partial s$ is B itself, then $\Gamma(x, \xi_1, \xi_2)$ is the unique polar of $\partial^2 x/\partial s^2$ considered as a homogeneous polynomial of degree two in $\partial x/\partial s$."

<u>Proof</u> by a direct application of theorem 13.64 in Martin {1932-1} and [22.11].

<u>Theorem 22.13</u> "The equations $x = x(s)$ obtained by fixing the value of ξ^+ in $x = x(s\, \xi^+)$ are solutions of the second order differential equation -

$$\frac{d^2 x}{d s^2} + \Gamma\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right) = 0. \text{"} \tag{2.11}$$

Proof: - $\frac{d^2x}{ds^2} = x(y;\xi^+;\xi^+) = x(y;\frac{dx}{ds};\frac{dx}{ds})$ (ξ^+ fixed)

$$= x(y(x);y(x;\frac{dx}{ds});y(x;\frac{dx}{ds})) = -\Gamma(x,\frac{dx}{ds},\frac{dx}{ds}).$$ (Q.E.D.)

Theorem 22.14 "The N.c.s. component $^+\Gamma(y,\xi_1^+,\xi_2^+)$ of the affine connection vanishes for every $y \in W_0$."

Proof I: - 1) From definition (22.5) equation (2.8) we have

$$^+\Gamma(y,\xi_1^+,\xi_2^+) = -d^{y\ y}_{\sigma\ \sigma\ \xi_1^+\xi_2^+}\, y(\sigma)$$ where we are to

interpret $y(y)$ as $y(x(y))=y$, the identity transformation of W_0 into W_0.

2) Hence, since the arbitrary increment ξ_1^+ is independent of y, $^+\Gamma(y,\xi_1^+,\xi_2^+) = -d^{y}_{\xi_2^+}\,\xi_1^+ = 0.$

(Q.E.D.)

Proof II: - By theorem [22.9] equation (2.10)

$$^+\Gamma(y,\xi_1^+,\xi_2^+) = y(x;\Gamma(x,\xi_1,\xi_2) + x(y;\xi_1^+;\xi_2^+)),$$

and by equation (2.8) the linear argument on the right is zero. (Q.E.D.)

Theorem 22.15 "The components $\Gamma(x,\xi_1,\xi_2)$ of linear connection in any L.c.s. $x = x(P)$ dependent on the N.c.s. $y = y(P)$ vanish for all $x \in V_0$."

Proof: - 1) By equation (2.10)

$$\Gamma(x, \xi_1, \xi_2) = x(y\,;\, {}^{+}\Gamma(y, \xi_1^+, \xi_2^+) + y(x\,;\,\xi_1\,;\,\xi_2))$$

for all $y \in W_0 \cdot \Sigma_0$.

2) Now $y(x\,;\,\xi_1\,;\,\xi_2) = d^{x}_{\xi_2} y(x\,;\,\xi_1) = d^{x}_{\xi_2}\xi_1^+$

where, as before, ξ_1^+ is independent of x, hence

$y(x\,;\,\xi_1\,;\,\xi_2) = 0$ for all $x \in V_0$.

3) The theorem then follows from theorem [22.14] and the linearity of the Fréchet differential. (Q.E.D.)

Definition 22.16 The curvature form $B(x, \xi_1, \xi_2, \xi_3)$ based on the linear connection $\Gamma(x, \xi_1, \xi_2)$.- A geometric object whose component in the coordinate system $x = x(P)$ is

$$B(x, \xi_1, \xi_2, \xi_3) = \Gamma(x, \xi_1, \xi_2\,;\,\xi_3) - \Gamma(x, \xi_1, \xi_3\,;\,\xi_2) + \\ + \Gamma(x, \Gamma(x, \xi_1, \xi_2), \xi_3) - \Gamma(x, \Gamma(x, \xi_1, \xi_3), \xi_2). \tag{2.12}$$

The above definition is valid for all linear connections which we may consider which have components in $x = x(P)$ which are of class at least $C^{(1)}$. In the present case it is clear that, since the components of linear connection in all N.c.s. and L.c.s. are zero throughout their domains of definition, these components possess successive differentials which are themselves all zero. This yields at once the following.

Theorem 22.17 "The components ${}^{+}B(y, \xi_1^+, \xi_2^+, \xi_3^+)$ and $B(x, \xi_1, \xi_2, \xi_3)$

in N.c.s. $y = y(P)$ and L.c.s. $x = x(P)$, respectively exist and are zero throughout their range of definition.

Definition 22.18 Locally flat space - A geometric space H satisfying the following condition. Given any $P \in H$ and any coordinate system $x = x(P)$ whose geometric domain M contains P, there exists a neighborhood of P, N_p, contained in M such that $Q \in N_p$ implies that the component $B(x, \xi_1, \xi_2, \xi_3)$ of the curvature form is defined at $x = q = x(Q)$, and $B(q, \xi_1, \xi_2, \xi_3) = 0$ for all ξ_1, ξ_2, ξ_3 in the coordinate space B.

Theorem 22.19 "The space H is locally flat under the definition (22.16) of curvature form."

Proof: - If we take N_p to be M the conditions of (22.18) are satisfied, since we have shown that although $x(y)$ was only assumed of class $C^{(n)}$ $n \geqslant 2$, the second Fréchet differential is zero, and hence all higher differentials are defined and zero - hence $B(x, \xi_1, \xi_2, \xi_3)$ is defined and zero is required. (Q.E.D.)

§2.3 A More General Case

A theory of the space of paths which is more inclusive in its scope will be considered now. A weakening of some of our hypotheses leads to a somewhat more general concept of differential than the Fréchet concept used up to this point. This, in turn, permits definitions of connection and curvature for cases to which the previous definitions do not apply. Further, we shall show that the space of paths meeting our new hypotheses is not necessarily locally flat.

For this purpose, let us adapt for present use certain ideas concerning sets and dimensions which are already well known.

Definition 23.1 n-dimensional subset E_n - Any subset of a real Banach space B having the three properties: -

i) E_n contains at least $n+1$ elements, (n finite),

ii) corresponding to every set of $n+1$ elements $\xi_0, \xi_1, \xi_2, \cdots, \xi_n$ of E_n there exists a set of n real numbers $a^1, a^2, \cdots, a^n$ such that

$$\xi_0 = a^1\xi_1 + a^2\xi_2 + \cdots + a^n\xi_n,$$

iii) there exists in E_n at least one set of n elements $\eta_1, \eta_2, \cdots, \eta_n$ such that $\sum_{j=1}^{n} b^j\eta_j = 0 :\supset: b^j = 0,\ j = 1, 2, \cdots, n.$

Definition 23.2 Open n-dimensional subset E_n^o - Any n-dimensional subset E_n which is open in its own linear manifold.

Definition (23.2) implies that E_n^o is an E_n such that if $x_0 \in E_n^o$, then there exists some $\delta(x_0) > 0$ such that $\|x_0 - x\| < \delta(x_0)$, where $x = \sum_{j=1}^{n} a^j\eta_j$ for some set of real numbers $a^1, \cdots, a^n$ and some set of $\eta_1, \eta_2, \cdots, \eta_n$ satisfying (23.1 iii), implies that $x \in E_n^o$.

Definition 23.3 Linear n-dimensional subset E_n^L - Any subset of a real Banach space B which is an E_n and also a linear manifold under the operations of the original Banach space.

Definition 23.4 n-dimensional sub-Banach space C_n - Any subset of B which is an E^L_n and is also complete.

The following properties are almost evident from the above definitions and will be stated without formal proof.

Theorem 23.5 i) Every E^L_n is automatically an E^o_n.

ii) Every E_n determines uniquely an E^L_n which contains E_n

iii) Every E^L_n contains at least one set of elements $v_1, v_2, \cdots, v_n$ with the three properties:

a) $\|v_1\| = \|v_2\| = \cdots = \|v_n\| = 1.$

b) Every $\xi \in E^L_n$ can be expressed in the form $\xi = \sum_{j=1}^{n} a^j v_j.$

c) $\sum_{j=1}^{n} b^j v_j = 0 \;:\supset:\; b^j = 0 \qquad j = 1, 2, \cdots, n.$

Definition 23.6 Basis of an $E^L_n (E_n)$ - Any set $v_1, v_2, \cdots, v_n$ satisfying the conditions [23.5 iii)] (for the E^L_n determined by E_n).

Theorem 23.7 "Every E^L_1 is automatically a C_1."

Proof: 1) If $\xi \in E^L_1$, then $v = \frac{1}{\|\xi\|}\xi \in E^L_1$ is a basis of E^L_1, and any $x \in E^L_1$ can be expressed as $x = a v$.

2) Let $\{a_i = a_i v\}$ be a regular (or Cauchy) sequence of elements of E^L_1.

3) Hence, given $\epsilon > 0$ there exists an $m(\epsilon)$ such that $n > m$ and $p = 1, 2, \cdots \;:\supset:$

$$\| x_{n+p} - x_n \| = \| a_{n+p} v - a_n v \| = | a_{n+p} - a_n | < \epsilon$$

4) However, since the real number space is complete, there exists a real number $a = \lim_{j \to \infty} a_j$ by step 3).

5) But $av \in E_1^L$ by [23.3], hence E_1^L is complete.

(Q.E.D.)

Definition 23.8 Sets $E_{(n)}^{(o)}$ open relatively to n-dimensional subsets – Any subset of a real Banach space B such that if E_n^o is any open n-dimensional subset (23.2) contained in $E_{(n)}^{(o)}$ (and $E_{(n)}^{(o)}$ contains at least one such E_n^o), then there corresponds to every $x_o \in E_n^o$ a number $\delta(x_o) > 0$ such that $\|x_o - x\| < \delta(x_o)$ implies $x_o \in E_{(n)}^{(o)}$.

The parenthetic restriction is inserted to prevent this definition from being satisfied vacuously. It will be noticed that this definition imposes no restriction on the dimensionality of $E_{(n)}^{(o)}$ other than the natural restriction of the dimensionality of the containing Banach space.

Theorem 23.9 "Any set $E_{(n)}^{(o)}$ of dimensionality d (provided $E_{(n)}^{(o)}$ contains at least one E_m^o) is necessarily a set $E_{(m)}^{(o)}$, $d \leq \infty$ and $n < m < \infty$ but is not necessarily a set $E_{(1)}^{(o)}$, $1 < n$ (even though $E_{(n)}^{(o)}$ necessarily contains at least one E_1^o). A set $E_{(o)}^{(o)}$ is an open set in B, and conversely any open set in B is a set $E_{(n)}^{(o)}$ for all n, $o \leq n \leq d$."

Proof: - 1) Let E_m^o be an open m-dimensional subset of $E_{(n)}^{(o)}$, $m < \infty$ and $n < m \leq d$, and x_o an arbitrary element of E_m^o

2) We need only show now that there exists at least one E_n^o containing x_o and contained in $E_{(n)}^{(o)}$ and the theorem will follow from our hypothesis and (23.8).

3) Let $x_o = \sum_{j=1}^{m} a^j v_j$, and $\delta(x_o) > 0$ satisfy (23.2) for E_m^o.

4) Consider then the set of elements x of the form

$$x = \sum_{j=1}^{n} (a^j + \Delta a^j) v_j + \sum_{j=n+1}^{m} a^j v_j \quad ; \text{ where}$$

$$|\Delta a^j| < \frac{\delta(x_o)}{n}, \quad j = 1, 2, \cdots, n.$$

5) By (23.2) this is a set E_n^o which is contained in E_m^o and hence in $E_{(n)}^{(o)}$ and it contains x_o. (Q.E.D.)$_1$

6) We prove the second part of the theorem by an example:- let $E_{(3)}^{(o)}$ consist of the open unit sphere about the origin in rectangular cartesian 3-space plus the open unit circle in the xy-plane with center (3,0,0).

7) In this example the only E_3^o are in the sphere, but there exists an E_2^o (that part of $E_{(3)}^{(o)}$ in $z = 0$) which does not satisfy our condition. (Q.E.D.)$_2$

8) Every E_o^o consists of one element only (vacuously open). Conversely, every element of $E_{(o)}^{(o)}$ is an E_o^o, and hence any $E_{(o)}^{(o)}$ satisfies our definition as an open set in B. (Q.E.D.)$_3$

In particular, one may assume that an $E_{(n)}^{(o)}$ is not an $E_{(n-1)}^{(o)}$, though we shall not make this restriction consistently.

Definition 23.10 L_1F-differential - A function $F_1(x;\delta x)$ is the L_1F-differential of $F_1(x)$ at $x = x_0 \in E_1^\circ \subset E_{(1)}^{(0)}$ with respect to E_1° if it satisfies the following conditions:-

i) $F_1(x_0;\delta x)$ is defined for all $\delta x \in E_1^L = C_1$ corresponding to E_1° and has values in B,

ii) $F_1(x_0;\delta x)$ is homogeneous of degree 1 in δx on C_1,

iii) given $\epsilon > 0$, there exists $\delta > 0$ such that $\|\delta x\| < \delta$,

$$\delta x \in C_1 \quad :\supset: \|F_1(x_0+\delta x) - F_1(x_0) - F_1(x_0;\delta x)\| \leq \epsilon \|\delta x\|$$

Theorem 23.11 "The L_1F-differential of $F_1(x)$ is additive in δx, continuous in δx in some neighborhood of zero, and hence is linear in δx."

Proof: - 1) Let $x_0 \in C_1$ and let V be a basis of C_1, then

$$F_1(x_0;\delta x) = a\, F_1(x_0;V),$$

2) $F_1(x_0;\delta_1 x + \delta_2 x) = (a_1 + a_2)\, F_1(x_0;V) =$

$$= a_1 F_1(x_0;V) + a_2 F_1(x_0;V) = F(x_0;\delta_1 x) + F_1(x_0;\delta_2 x).$$

3) $F_1(x_0;0) = 0$ and $\|F_1(x_0;\delta x)\| = |a| \cdot \|F_1(x_0;V)\|$.

4) Hence, given $\epsilon > 0$ there exists $\delta = \dfrac{\epsilon}{\|F_1(x_0;V)\|} > 0$ such that $\|\delta x\| = |a| < \delta \;:\supset:$

$$|a| \cdot \|F_1(x_0;V)\| = \|F_1(x_0;\delta x)\| < \epsilon.$$

(Q.E.D.)

Theorem 23.12 "A necessary and sufficient condition that $F_1(x)$ be L_1F-differentiable throughout $E_1^\circ \subset E$ is that the derivative of

$F_1(s\bar{x})$ with respect to s exist throughout E_1°, where $\bar{x}$ is an arbitrary element of E_1°."

<u>Proof</u>: - <u>Nec</u>. - 1) $\epsilon > 0$ there exists $\delta > 0$ such that

$$\| dx \| = \| \tau \bar{x} \| < \delta' = \frac{\delta}{\|\bar{x}\|} \quad :\supset:$$

$$\| F_1(x+dx) - F_1(x) - F_1(x;dx) \leqslant \epsilon' \| \tau\bar{x} \| = \frac{\epsilon}{\|\bar{x}\|} \| \tau \bar{x} \|$$

$$:\supset: \| F_1((s+\tau)\bar{x}) - F_1(s\bar{x}) - F_1(s\bar{x};\tau\bar{x}) \| \leqslant \epsilon |\tau|$$

$$:\supset: \left\| \frac{F_1((s+\tau)\bar{x}) - F_1(s\bar{x})}{\tau} - F_1(s\bar{x};\bar{x}) \right\| \leqslant \epsilon$$

2) But this is just the condition that

$$\lim_{|\tau| \to 0} \frac{F_1((s+\tau)\bar{x}) - F_1(s\bar{x})}{\tau} = \frac{\partial F_1(s\bar{x})}{\partial s}$$

exist and equal $F_1(s\bar{x};\bar{x})$. <u>(Q.E.D.) Nec.</u>

3) The steps of this proof may be reversed in order in an obvious manner to prove <u>sufficiency</u>. <u>(Q.E.D.)</u>

<u>Definition 23.13</u> <u>$L_n F$-differential</u> - A function $F_n(x;dx)$ is the $L_n F$-differential of $F_n(x)$, $n \geqslant 2$, at $x = x_0 \in E_n^\circ \subset E_{(n)}^{(o)}$ with respect to E_n° if it satisfies the following conditions:

i) $F_n(x_0;dx)$ is defined for all $dx \in E_n^L$ corresponding to E_n° and is valued in B,

ii) $F_n(x_0;dx)$ is linear in dx on E_n^L,

iii) given $\epsilon > 0$ there exists $\delta > 0$ such that $\|\delta x\| < \delta$ and $dx \in E_n^L$ $:\supset:$

$$\| F_n(x_0+dx) - F_n(x_0) - F_n(x_0;dx) \| \leqslant \epsilon \|\delta x\|.$$

Attention is called to the need for assuming linearity in (23.13 ii), as theorem [23.11] is not necessarily true in subspaces of dimensionality greater than one.

Theorem 23.14 "If the Fréchet differential $F_n(x_0; dx)$ of $F_n(x)$, $n \geqslant 1$, exists at $x = x_0$, then the $L_n F$-differentials with respect to any E_n^o containing x_0 and contained in $E_{(n)}^{(o)}$ exist at $x = x_0$ and are equal to $F_n(x_0; dx)$ for all $dx \in E_n^L$."

Proof: - The theorem is an immediate consequence of the definition of Fréchet differential and (23.10) and (23.13).

Definition 23.15 Partial $L_n F$-differential - The function $F_n(x, \lambda_1, \cdots \lambda_s; \delta x)$, $n \geqslant 1$ $s \geqslant 1$, is the partial $L_n F$-differential of $F_n(x, \lambda_1, \cdots, \lambda_s)$ with respect to x and E_n^o at $x_0 \in E_n^o \subset E_{(n)}^{(o)}$ if, for any set $\lambda_1, \cdots, \lambda_s$ it satisfies conditions i), ii) and iii) of (23.10) and (23.13) for $n = 1$ and $n \geqslant 2$ respectively.

It will be readily seen that by appropriate changes in wording of the statements and proofs of theorems [23.11], [23.12] and [23.14] equivalent theorems for partial $L_n F$-differentials may be obtained.

Definition 23.16 Higher order $L_n F$-differentials - If the $(m-1)^{th}$ $L_n F$-differential $F_n(x, \lambda_1, \cdots, \lambda_s; \delta_1 x; \cdots; \delta_{m-1} x)$ of $F_n(x, \lambda_1, \cdots, \lambda_s)$ with respect to x and E_n^o exists at $x = x_0 \in E_n^o$ and itself has a (partial) $L_n F$-differential $F_n(x, \lambda_1, \cdots, \lambda_s; \delta_1 x; \cdots; \delta_{m-1} x; \delta_m x)$ with respect to x

and the same E_n^o at $x = x_o \in E_n^o$, then $F_n(x,\lambda_1,\cdots,\lambda_s;\delta_1 x;\cdots;\delta_m x)$ will be called the mth L_nF-differential of $F_n(x,\lambda_1,\cdots,\lambda_s)$ with respect to x and E_n^o at $x = x_o$.

Note particularly in the above inductive definition that all successive definitions are to be taken with respect to the same open n-dimensional subset E_n^o containing x_o. It is possible that a function may be continuous in a restricted sense while not satisfying the general criterion for continuity. The importance of the set E_n^o increases if the function and its differential are defined at all points of E_n^o.

Definition 23.17 Continuity with respect to an E_n^o - A function $f(x)$ is continuous at $x_o \in E_n^o$ with respect to E_n^o if, given $\epsilon > 0$, there exists a $\delta(x_o,\epsilon) > 0$ such that $\delta x \in E_n^o$ and $\|\delta x\| < \delta(x_o,\epsilon)$ $:\supset:$

$$\|f(x_o+\delta x) - f(x_o)\| < \epsilon.$$

We can readily see that the associated ideas of uniform continuity, locally uniform continuity and so forth may also be considered with respect to some particular E_n^o.

Theorem 23.18 "If $F_n(x)$ has an L_nF-differential with respect to x and E_n^o at $x = x_o \in E_n^o$, then $F_n(x)$ is continuous in x with respect to E_n^o at $x = x_o$."

Proof: - 1) $F_n(x_o;\delta x)$ is linear in δx, hence there exists a modulus $M(x_0)$ such that

$$\| F_n(x_0; \delta x) \| \leqslant M(x_0) \| \delta x \|$$

for all $\delta x \in E_n^o$.

2) Given $\epsilon > 0$, there exists a $\delta(x_0, \epsilon) > 0$ such that $\| \delta x \| < \delta(x_0, \epsilon)$ and $\delta x \in E_n^o$ $:\supset:$

$$\| F_n(x_0 + \delta x) - F(x_0) \| \leqslant \epsilon \| \delta x \| + M(x_0) \| \delta x \|.$$

3) Now let $\delta = \min\left(\delta(x_0, \epsilon), \dfrac{\epsilon}{\epsilon + M(x_0)}\right)$ and $\| \delta x \| < \delta$.

(Q.E.D.)

Corollary 23.19 "If $F_n(x, y_1, \cdots, y_t)$ has a partial $L_n F$-differential with respect to x and E_n^o at $x = x_0 \in E_n^o$, then $F_n(x, y_1, \cdots, y_t)$ is continuous in x with respect to E_n^o at $x = x_0$.

If we restrict the increment in the usual definition of Gateaux differential to lie in E_n^o, then the existence of the $L_n F$-differential implies the existence of this Gateaux differential. It then becomes merely a matter of rephrasing the proof of theorem 1.1 in Michal {1936-1} to establish the following.

Theorem 23.20 "Let $F_n(x, y)$ be linear in y and have a first partial $L_n F$-differential, $F(x, y; \delta x)$ with respect to x and E_n^o. Then the total $L_n F$-differential exists and has the form

$$F_n(x_0, y; \delta x, \delta y) = F_n(x_0, y; \delta x) + F_n(x_0, \delta y)$$

at $x = x_0$. Furthermore if $F_n(x, y; \delta x)$ is continuous in x at $x = x_0$ with respect to E_n^o, then $F_n(x, y; \delta x, \delta y)$ is continuous in x as well as in y, δx, and δy with respect to E_n^o."

Corollary 23.21 "If y in theorem [23.20] is a function $y = y(x)$ of x having an $L_n F$-differential on the same E_n^o, then the total differential with respect to x and E_n^o exists and has the form

$$F_n(x, y(x); \delta x, \delta y) = F_n(x, y(x); \delta x) + F_n(x, y(x; \delta x))."$$

Definition 23.22 Class $C^{(m)}_{L_n}$ - A function $F_n(x, y_1, \cdots, y_s)$, $n \geq 1, s \geq 0$, will be said to be of class $C^{(m)}_{L_n}$ if its $L_n F$-differentials of the first m orders with respect to x and E_n^o exist and are continuous in x (with respect to E_n^o) for all x in E_n^o and all $\| \delta_i x \| < 1$, $i = 1, 2, \cdots, m$.

In the present section we shall again use the notation of figures 1 and 2 with the meanings given in (21.23) except that L.c.s., and all notation pertaining to them, will be replaced by the following P.L.c.s.

Definition 23.23 P.L. coordinate system (P.L.c.s.) - Any coordinate system $x(P) = x(y(P), x_0)$ defined in terms of an N.c.s. by a function $x = x(y, x_0)$ satisfying the following conditions:-

i) $x(y, x_0)$ is defined, single valued and continuous in the ordinary sense for all $y \in \Sigma_0$, and it is of class $C^{(m)}_{L_1}$ with respect to y and any E_1^0 of the form $y = s\,\mathfrak{f}^+$ contained in $\Sigma_0' = W_0 \cdot \Sigma_0$, $(m \geq 2)$.

ii) x_0 is a preassigned element of B and $x(0, x_0) = x_0$, hence $x_0 \in X_0$.

iii) There exists a function $y(x, x_0)$ which is defined, single valued and continuous in the ordinary sense for all $x \in X_0$ and which satisfies the conditions

$$x(y(x, x_0), x_0) = x \qquad \text{for all } x \in X_0$$

$$y(x(y, x_0), x_0) = y \qquad \text{for all } y \in \Sigma_0 .$$

iv) $y(x, x_0)$ is of class $C_{L_1}^{(m)}$ with respect to x and any $E_1^0 \subset V_0$ for all $x \in V_0$, where V_0 is the map of $\Sigma_0' = W_0 \cdot \Sigma_0$ by $x = x(y, x_0)$.

<u>Theorem 23.24</u> "Let $x(P)$ and $\bar{x}(P)$ be P.L.c.s. obtained from N.c.s. $y(P)$ and $\bar{y}(P)$ respectively. Then there exists a pair of mutually inverse transformations $x(\bar{x})$ and $\bar{x}(x)$ which constitute a homeomorphism between X_0' and $\bar{X}_0'$.

<u>Proof:</u> - 1) By [21.24] there exists a homeomorphism $y(\bar{y}), \bar{y}(y)$ between Σ_0' and $\bar{\Sigma}_0'$.

2) By the transitivity of homeomorphisms and (21.1)

$$x(\bar{x}) = x(y(\bar{y}(\bar{x}))) \qquad \bar{x}(x) = \bar{x}(\bar{y}(y(x)))$$

exist and satisfy the theorem. (Q.E.D.)

<u>Theorem 23.25</u> "If $x(P)$ and $\bar{x}(P)$ are two P.L.c.s. defined in

terms of a pair $y(P), \bar{y}(P)$ of $C^{(n')}_{L_1}$ N.c.s.* $n \geq 2$ then the homeomorphism $x(\bar{x}), \bar{x}(x)$ is of class $C^{(r)}_{L_1}$, $r \geq 2$."

Proof of this theorem exactly parallels that of [22.2] with the same range of validity.

Definition 23.26 L_1c.v. associated with P_1 - Let $x(\bar{x})$ and $\bar{x}(x)$ satisfy [23.24] and $P_1 \in S_0'$. Then the geometric object whose components $\xi, \bar{\xi}$ in all such $C^{(r)}_{L_1}$ P.L.c.s. are related by

$$\bar{\xi} = \bar{x}\,(x\,(P_1);\, x\,(y\,(P_1),\, x_0;\, \xi^+)) = \bar{x}\,(x;\, \xi) \tag{2.13}$$

$$\xi = x\,(\bar{x}\,(P_1);\, \bar{x}\,(\bar{y}\,(P_1),\, \bar{x}_0;\, \bar{\xi}^+)) = x\,(\bar{x};\, \bar{\xi})$$

will be termed an L_1c.v. associated with P_1.

From this definition it follows that this L_1c.v. has components $\xi^+, \bar{\xi}^+$ in the $C^{(r)}_{L_1}$ N.c.s., which satisfy

$$\xi^+ = y\,(x_1;\, \xi) \qquad \bar{\xi}^+ = y\,(\bar{y}_1;\, \bar{\xi}^+). \tag{2.14}$$

Definition 23.27 P-Linear Connection - A geometric object whose components in P.L.c.s. $x(P)$ is

$$\Gamma_P\,(x, \xi_1, \xi_2) = -\, x\,(y, x_0;\, \xi_1^+;\, \xi_2^+) \tag{2.15}$$

$$= -\, x\,(s\,y\,(x, x_0;\, \xi);\, y(x, x_0;\, s_1\,\xi);\, y(x, x_0;\, s_2\,\xi))$$

where ξ^+, ξ_1^+, ξ_2^+ are in the same $E_1^0\,(\lambda)$ with respect to which

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* The definition of $C^{(n')}_{L_1}$ N.c.s. is analogous to (22.7).

the L.F.-differentials were taken.

Theorem 23.28 "The P-linear connection $\Gamma_P(x, \xi_1, \xi_2)$ exists and is symmetric and bilinear in ξ_1, ξ_2."

Theorem 23.29 "If a linear connection $\Gamma(x, \xi_1, \xi_2)$ (22.5) exists, then the P-linear connection exists and the two are equal over their common range of definition."

Proof of these two theorems involves only a direct application of definitions (22.14) (23.10) (23.16) (23.23) and theorems [23.11] [23.14].

Theorem 23.30 "The P-linear connection undergoes the transformation

$$\overline{\Gamma}_P(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \bar{x}(x; \Gamma_P(x, \xi_1, \xi_2)) - \bar{x}(x; \xi_1; \xi_2) \tag{2.16}$$

on any $E_1^{\circ}(\mu)$ to $\bar{E}_1^{\circ}(\bar{\mu})$ contained in X_0' and $\bar{X}_0'$ respectively of the pair $x(P), \bar{x}(P)$ of $C_{L,}^{(u)}$L.c.s. $n \geq 2$, and ξ of (2.15) is an L.c.v. at all points of O_0."

Proof: - 1) By our hypothesis

$$\bar{\xi} = \bar{x}(x(y, x_0); x(y, x_0; \xi^+)) \qquad y \in W_0'$$

$$\bar{\xi} = \bar{x}(\bar{y}(y); \bar{y}(y; \xi^+)).$$

2) Equating these yields an identity in y which by [23.21] may be differentiated with respect to y and E_1° to give

$$\bar{x}(x(y,x_0);x(y,x_0;\xi_1^+);x(y,x_0;\xi_2^+))+$$
$$+\bar{x}(x(y,x_0);x(y,x_0;\xi_1^+;\xi_2^+))=$$
$$=\bar{x}(\bar{y}(y);\bar{y}(y;\xi_1^+);\bar{y}(y;\xi_2^+))+\bar{x}(\bar{y}(y);\bar{y}(y;\xi_1^+;\xi_2^+)).$$

3) Now note that ξ_1^+ is a fixed element of E_1^L determined by $E_1^\circ(\lambda)$, hence

$$\lambda: \quad y = s\xi_1^+ \qquad s \in J$$
$$\bar{\lambda}: \quad \bar{y} = \bar{s}\bar{\xi}_1^+ \qquad \bar{s} \in \bar{J},$$

where $\bar{\xi}_1^+ = \bar{y}(y;\xi_1^+)$ is independent of s.

4) Hence $\bar{y}(y;\xi_1^+;\xi_2^+) = d\bar{\xi}_1^+$ with respect to y and $E_1^\circ(\lambda)$ is zero.

5) ∴ The last term on the right is zero by the linearity [23.11] of the L.F.-differential, and we obtain (2.16) directly upon substitution from (2.15) (Q.E.D.)

Corollary 23.31 - "The $C_{L_1}^{(n)}$ N.c.s. component of P-linear connection satisfies

(2.17) $$^+\Gamma_P(y,y_1,y_2) = \frac{s_1 s_2}{s^2}\,{}^+\Gamma_P(y,y,y) = 0$$

for all s, s_1, s_2 such that $y, y_1, y_2 \in E_1^\circ \subset W_0$."

Corollary 23.32 "The $C_{L_1}^{(n)}$N.c.s. component of the curvature form based on the P-linear connection (22.13) satisfies

$$^+B_P(y,y_1,y_2,y_3) = \frac{s_1 s_2 s_3}{s^3}\,{}^+B_P(y,y,y,y) = 0.$$

for all s, s_1, s_2, s_3 such that $y, y_1, y_2, y_3 \in E_1^0 \subset W_0$."

We now find, however, that our definition (22.18) of a locally flat space is not in general satisfied under our hypotheses. This is seen from the fact that the curvature form (22.16) is not necessarily defined throughout any neighborhood of a given point in H. This result, moreover, is still in perfect harmony with [23.31] and [23.32] as we shall show in § 3.3 that the normal coordinate component of linear connection and hence the curvature form based on it vanish for values of the increments in the same linear set as y (i.e. on the same path).

§ 2.4. Examples and Illustrations

The first of the two following examples relates the present theory to the well known theory developed in spaces of finite dimensionality. The second exhibits a case in a dimensionless space of the modified differential of (23.10).

Example I. Consider the case in which B is the n-dimensional arithmetic space. Here under (21.4) equation (2.1) is expressed as

$$(2.18) \qquad \lambda: \quad y^i = s\,\xi^i \qquad s \in J \qquad i = 1, 2, \cdots, n$$

where $\xi = (\xi^1, \xi^2, \cdots, \xi^n)$ is fixed and $\sum_{i=1}^{n} (\xi^i)^2 \neq 0$ in the normal coordinate system $y^i = y^i(P)$ and as

$$(2.19) \qquad \bar{\lambda}: \quad \bar{y}^i = \bar{s}\,\bar{\xi}^i \qquad \bar{s} \in J \qquad i = 1, 2, \cdots, n$$

where again $\bar{\xi} = (\bar{\xi}^1, \bar{\xi}^2, \ldots, \bar{\xi}^n)$ is fixed and $\sum_{i=1}^{n} (\bar{\xi}^i)^2 \neq 0$ in the $\bar{y}^i(P)$ N.c.s.

In previous treatments of this case the assumption has been made that $\bar{s} = s$, throughout their common range $\bar{J} = J$. We shall here only insist that $\bar{s}(s)$, and $s(\bar{s})$ shall be (1 - 1) continuous, and mutually inverse over their ranges of definition. Hence we obtain

$$\frac{d\bar{y}^i}{d\bar{s}} = \bar{\xi}^i = \frac{d\bar{y}^i}{dy^j}\frac{dy^j}{d\bar{s}} = \frac{ds}{d\bar{s}}\frac{d\bar{y}^i}{dy^j}\xi^j \tag{2.20}$$

which reduces to the usual form if we let $s = \bar{s}$. Multiply both sides by $\bar{s}$ and we have

$$\bar{y}^i = \bar{s}\bar{\xi}^i = \bar{s}\frac{ds}{d\bar{s}}\frac{d\bar{y}^i}{dy^j}\xi^j = \alpha^i_j y^j \tag{2.21}$$

where $\alpha^i_j = \frac{\bar{s}}{s}\frac{ds}{d\bar{s}}\frac{d\bar{y}^i}{dy^j}$ which may or may not be a function of s. In case it is not (as for $s = \bar{s}$) then the transformation of normal coordinates is linear, and the ξ are components of a contravariant vector.

Let us point out that it is perfectly possible to impose the condition that a certain finite set of points (or even an infinite set) on every path shall have equal values of the parameters s and $\bar{s}$ in the two N.c.s., without thereby necessarily making $s = \bar{s}$ for all points of the paths. In particular consider

(A) $\bar{s} = s^3 \qquad s = \sqrt[3]{\bar{s}}, \qquad |s| \leqslant 1 \qquad |\bar{s}| \leqslant 1;$

$\bar{s} = s \qquad\qquad |s| > 1 \qquad |\bar{s}| > 1;$

$$\bar{y}^i(P) = \{s(P)\}^2 y^i(P). \qquad (\text{and } \bar{\xi} = \xi)$$

(B) $\bar{s} = \sqrt{2} \sin \tfrac{1}{4}\pi s \qquad s = \tfrac{4}{\pi}$ (principal value) $\arcsin \dfrac{\bar{s}}{\sqrt{2}},$

$|s| \leqslant 1 \qquad |\bar{s}| \leqslant 1;$

$\bar{s} = s, \qquad |s| > 1 \qquad |\bar{s}| > 1;$

$$\bar{y}^i(P) = \sqrt{2}\, \frac{\sin \frac{1}{4}\pi s(P)}{s(P)}\, y^i(P). \qquad (\text{and } \bar{\xi} = \xi)$$

It can be shown without difficulty that in each case the mapping $\bar{y}(P)$ has the same geometric domain as $y(P)$ and is continuous and solvable with continuous inverse, and that every path $L \in M_0$ will have a map $\bar{\lambda}$ of form $\bar{y}^i = \bar{s}\,\bar{\xi}^i$ in $\bar{y}(P)$. Hence, from our definitions, $\bar{y}(P)$ is an N.c.s. associated with P_0. Yet the transformations $\bar{y}(y), y(\bar{y})$ are not everywhere differentiable, and hence the ξ is not a component of a c.v. throughout O_0. It is also possible to construct a similar example in which the first derivative exists and is continuous but the second does not exist throughout O_0.

On the other hand, if we do assume that $\bar{y}(y)$ is <u>linear</u>, the above development leads to precisely the usual definitions and laws of transformation for the components of linear connection and curvature form $\{$e.g. Veblen 1927-1$\}$.

Example II Here we take the coordinate space to be a general Banach space and consider the properties of certain functions. As these depend on the norm (which is not Fréchet differentiable), the uniform continuity of the norm is first established. Ultimately, a function yielding a P.L.c.s. but not an L.c.s. is studied.

Theorem 24.1 "The function $r(y) = \|y\|$ on B to the real numbers is uniformly continuous on B."

Proof: 1) $\|y_0\| + \|\delta y\| \geqslant \|y_0 + \delta y\|$,

2) $\|y_0\| - \|\delta y\| \leqslant \|y_0 + \delta y\|$.

3) Given $\epsilon > 0$ there exists $\delta(\epsilon) = \epsilon > 0$ such that

i) if $\|y_0 + \delta y\| \geqslant \|y_0\|$, $\|\delta y\| < \delta(\epsilon)$ $:\supset:$

$\big|\|y_0 + \delta y\| - \|y_0\|\big| = \|y_0 + \delta y\| - \|y_0\| \leqslant \|\delta y\| < \epsilon$, step 1),

ii) if $\|y_0 + \delta y\| \leqslant \|y_0\|$, $\|\delta y\| < \delta(\epsilon)$ $:\supset:$

$\big|\|y_0 + \delta y\| - \|y_0\|\big| = \|y_0\| - \|y_0 + \delta y\| \leqslant \|\delta y\| < \epsilon$, step 2).

4) Hence, given $\epsilon > 0$, there exists $\delta > 0$ independent of y such that

$\|\delta y\| < \delta$ $:\supset:$ $|r(y + \delta y) - r(y)| < \epsilon$. (Q.E.D.)

Theorem 24.2 "The function $x = f(y) = \|y\|^2 y$ on B to B is solvable and has the inverse $y = g(x) = \frac{1}{\|x\|^{2/3}} x$ $\{g(0) = 0\}$ and the two functions have L, F-differentials with respect to all E_1^0 not containing the zero element of B, which are continuous throughout their range of definition. The two differentials are

$dx = 3\|y\|^2 dy$ and $dy = \frac{1}{\|x\|^{2/3}} dx$ * and are solvable linear (i.e. they are mutually inverse) in dx and dy.

Proof: - 1) First we show $g(x)$ is continuous at $x = 0$. Given $\epsilon > 0$, there exists a $\delta(\epsilon) = \epsilon^3 > 0$ such that

$$\|\delta x\| < \delta(\epsilon) \;:\supset:\; \|g(0+\delta x) - g(0)\| =$$
$$= \|g(\delta x)\| = \|\delta x\|^{1/3} < \epsilon \qquad *$$

2) To show solvability, we have by direct substitution

$$x = \left\| \frac{x}{\|x\|^{2/3}} \right\|^2 \frac{1}{\|x\|^{2/3}} x = \frac{\|x\|^2}{|\|x\|^{2/3}|^2 \|x\|^{2/3}} x = x \text{ and}$$

$$y = \frac{\|y\|^2}{\|\|y\|^2 y\|^{2/3}} y = \frac{\|y\|^2}{|\|y\|^2|^{2/3} \|y\|^{2/3}} y = y \text{ for all } x, y \in B.$$

3) E_1° of B will be of the form $y = s\xi$ where ξ is fixed and s satisfies (say) $0 < s < \infty$. Then to show $x = \|y\|^2 y$ has the L.F-differential

$$dx = f(y_0; \delta y) = 3\|y_0\|^2 \delta y$$

with respect to y and E_1° at $y = y_0 = s_0 \xi$ we have

$$\|\Delta\| = \|f(y_0 + \delta y) - f(y_0) - f(y_0; \delta y)\| =$$
$$= \left| |s_0 + \delta s|^2 (s_0 + \delta s) - |s_0|^2 s_0 - 3|s_0|^2 \delta s \right| \cdot \|\xi\|^3 =$$
$$= \left| s_0^3 + 3s_0^2 \delta s + 3s_0(\delta s)^2 + (\delta s)^3 - s_0^3 - 3s_0^2 \delta s \right| \cdot \|\xi\|^3 =$$
$$= |\delta s|^2 \cdot |3s_0 + \delta s| \cdot \|\xi\|^3 = \|\delta y\|^2 \|3y_0 + \delta y\|.$$

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* Throughout the following proofs, fractional exponents will be used to denote the positive real roots of positive real numbers.

4) Hence, given $\epsilon > 0$ there exists $\delta = \min\left(\|y_0\|, \frac{\epsilon}{4\|y_0\|}\right) > 0$ such that $\|\delta y\| < \delta$ implies

$$\|\Delta\| \leqslant 4\|\delta y\|^2 \cdot \|y_0\| \leqslant \epsilon\|\delta y\|.$$

(Q.E.D.).

5) The uniform continuity of $f(y;\delta y)$ in y follows from theorem [24.1].

6) To show that $g(x_0;\delta x) = \frac{1}{3\|x_0\|^{2/3}}\delta x$ is the L, F- differential of $g(x) = \|x\|^{-2/3}x$ with respect to X and E_1^0 of the form $x = t\rho$, $0 < t < \infty$ at $x = x_0 = t_0\rho$, we have

$$\|\Delta\| = \|g(x_0+\delta x) - g(x_0) - g(x_0;\delta x)\| =$$

$$= \left|\frac{t_0+\delta t}{|t_0+\delta t|^{2/3}} - \frac{t_0}{|t_0|^{2/3}} - \frac{\delta t}{3|t_0|^{2/3}}\right| \cdot \|\rho\|^{1/3} =$$

$$= \left|\left(1+\frac{\delta t}{t_0}\right)^{1/3} - 1 - \frac{1}{3}\frac{\delta t}{t_0}\right| \cdot \|x_0\|^{1/3}.$$

7) Now by Taylor's theorem it can be shown that

$$\left(1+\frac{\delta t}{t_0}\right)^{1/3} = 1 + \frac{1}{3}\frac{\delta t}{t_0} + \left(\frac{\delta t}{t_0}\right)R,$$

where $|R| < 2$ $\qquad |\delta t| < \frac{1}{2}|t_0|$

8) Hence, given $\epsilon > 0$, there exists $\delta = \min\left(\frac{\|x_0\|}{2}, \frac{\epsilon\|x_0\|^{5/3}}{2}\right) > 0$ such that $\|\delta x\| < \delta$ implies

$$\|\Delta\| = \left|\frac{\delta t}{t_0}\right|^2 |R| \cdot \|x_0\|^{1/3} = \|\delta x\|^2 \frac{|R|}{\|x_0\|^{5/3}} \leqslant \|\delta x\|^2 \frac{2}{\|x_0\|^{5/3}} \leqslant \epsilon\|\delta x\|.$$

9) Again by theorem [24.1], $g'(x;\delta x)$ is continuous in x at all points of B except zero. However, we cannot establish uniform continuity over regions containing zero in their closure.

10) Lastly, by a direct substitution, it is verified that the two $L.F$-differentials are mutually inverse in their increments

$$dx = 3\|y\|^2 \frac{1}{3\|x\|^{2/3}}\, dx = \frac{\|y\|^2}{\|\,\|y\|^2 y\|^{2/3}}\, dx = dx$$

$$dy = \frac{1}{3\|x\|^{2/3}}\, 3\|y\|^2 dy = dy.$$

(Q.E.D.)$_3$

It will be readily seen that the functions in the above theorem do not possess higher order $L.F$-differentials, hence the following case is of greater importance. The existence of a linear functional L, not identically zero, in any normed vectorial space has been established by Banach {Theorem 3, p. 55, 1932-2} and we shall assume that the function $L(x)$ on B to R in the following theorem is such a function.

Theorem 24.3 "Define

(2.22) $\quad x = f(y, x_0) = \|y\|^2 L(y)\,y + x_0$

(2.23) $\quad y = g(x, x_0) = \|x - x_0\|^{-\frac{1}{2}} \,|L(x - x_0)|^{\frac{1}{4}}\,(x - x_0)$

Then - i) (2.22) and (2.23) are mutually inverse provided we take x_0, the set of values of (2.22) as the domain of definition of $g(x, x_0)$ and $g(x_0, x_0) = 0$.

ii) (2.22) and (2.23) are each of class $C^{(2)}_{L_1}$ on subsets E_1° $(y = s\xi,\ s \in J)$ and $E_1^{\circ\prime}$ $(x = t\rho,\ t \neq 0\ t \in J')$ and the L_1F-differentials are of the following form

(2.24) $$f(y, x_0; \delta y) = 4\|y\|^2 L(\delta y) y = 4\|y\|^2 L(y)\delta y$$

(2.25) $$f(y, x_0; \delta_1 y; \delta_2 y) = 12\|y\|^2 L(\delta_1 y)\delta_2 y$$

(2.26) $$g(x, x_0; \delta x) = \frac{1}{4}\|x - x_0\|^{-\frac{1}{2}} |L(x - x_0)|^{-\frac{1}{4}} \delta_1 x$$

(2.27) $$g(x, x_0; \delta_1 x; \delta_2 x) = -\frac{3}{16}\|x - x_0\|^{-\frac{1}{2}} |L(x - x_0)|^{-\frac{5}{4}} L(\delta_1 x)\delta_2 x.$$

iii) $f(0, x_0) = x_0$ $\qquad g(x_0, x_0) = 0$.

iv) $\delta_1 x = d^{y}_{\delta_1 y} f(y, x_0)$ and $\delta_1 y = d^{x}_{\delta_1 x}\, g(x, x_0)$

are mutually inverse linear functions of their increments."

In the proof we shall take $x_0 = 0 \in B$ which simplifies the notation and does not lessen the generality.

<u>Proof:</u> - 1) To prove (2.24). Given any E_1° as in ii), $y_0 \in E_1^\circ$, and $\epsilon > 0$, there exists a

$$\delta_1 = \min\left(\|y_0\|, \frac{\epsilon}{11}\|y_0\|^{-2} |L(y_0)|^{-1}\right) > 0$$

such that $\|\delta y\| < \delta_1$ and $\delta y \in E_1^0$ implies

$$\|f(y_0+\delta y) - f(y_0) - f(y_0;\delta y)\| =$$

$$= \| \|y_0 + \delta y\|^2 L(y_0+\delta y)(y_0+\delta y) - \|y_0\|^2 L(y_0)y_0 -$$

$$- 4\|y_0\|^2 L(\delta y) y_0 \| =$$

$$= \left| |s_0+\delta s|^2 (s_0+\delta s)^2 - |s_0|^2 s_0^2 - 4|s_0|^2 s_0 \delta s \right| \cdot \|\xi\|^3 |L(\xi)| =$$

$$= \left| 6 s_0^2 (\delta s)^2 + 4 s_0 (\delta s)^3 + (\delta s)^4 \right| \cdot \|\xi\|^3 |L(\xi)| =$$

$$= |\delta s|^2 \left| 6 s_0^2 + 4 s_0 \delta s + (\delta s)^2 \right| \cdot \|\xi\|^3 |L(\xi)| \leq \epsilon \|\delta y\|.$$

2) Remembering that $\delta_1 y$ is independent of y (2.24) follows from theorem [24.2].

3) To prove (2.25), we have for any $E_1^{0\prime}$ as in ii), $x_1 \in E_1^{0\prime}$ and $\epsilon > 0$, there exists a

$$\delta_2 = \min\left(\frac{\|x_1\|}{k_1}, \frac{\epsilon}{m_1}\|x_1\|^{3/4} |L(x_1)|^{\frac{1}{4}}\right) > 0$$

where we understand that k_1 is so chosen that, in the Taylor's expansion $\left(1+\frac{\delta_1 t}{t_1}\right)^{1/4} = 1 + \frac{1}{4}\frac{\delta t}{t_1} + \left(\frac{\delta t}{t_1}\right)^2 R_1$,

$$|\delta t| < \frac{|t_1|}{k_1} \quad \text{implies} \quad |R_1| < m_1$$

$$\|g(x_1+\delta x) - g(x_1) - g(x_1;\delta x)\| =$$

$$= \| \|x_1+\delta x\|^{-\frac{1}{2}} |L(x_1+\delta x)|^{-\frac{1}{4}} (x_1+\delta x) - \|x_1\|^{-\frac{1}{2}} |L(x_1)|^{-\frac{1}{4}} x_1 -$$

$$- \frac{1}{4}\|x_1\|^{-\frac{1}{2}} |L(x_1)|^{-\frac{1}{4}} \delta_1 x \| =$$

$$= \left\| |\pm(t_1+\delta t)|^{-3/4}(t_1+\delta t) - |\pm t_1|^{-3/4} t_1 - \right.$$

$$\left. - \tfrac{1}{4}|\pm t_1|^{-3/4}\delta t \right| \cdot \|\mp \rho\|^{-\frac{1}{2}} |L(\mp\rho)|^{-\frac{1}{4}} \|\rho\|$$

where the ambiguous sign is to be chosen so that $\pm t_1 > 0$ and hence $|\pm t_1| = \pm t_1$.

It follows then that $\|\delta x\| < \delta_2$ implies

$$\|g(x_1+\delta x) - g(x_1) - g(x_1;\delta x)\| =$$

$$= |\pm t_1|^{\frac{1}{4}} \left|\frac{\delta t}{\pm t_1}\right|^2 |R_1| \cdot \|\mp\rho\|^{-\frac{1}{2}} |L(\mp\rho)|^{-\frac{1}{4}} \|\rho\| =$$

$$= \|\delta x\|^2 \frac{m_1}{\|x_1\|^{3/4}|L(x_1)|^{1/4}} \leqslant \epsilon \|\delta x\| .$$

4) To prove (2.27) - given $E^{o\prime}_1$ as above and $\epsilon > 0$, there exists

$$\delta_3 = \min\left(\frac{\|x_1\|}{k_2}, \frac{\epsilon}{m_2}\|x_1\|^{5/4}|L(x_1)|^{5/4}|L(\delta_1 x)|^{-1}\right) > 0$$

such that $\|\delta_2 x\| < \delta_2$ implies (ambiguous sign as above) -

$$\|g(x_1+\delta_2 x;\delta_1 x) - g(x_1;\delta_1 x) - g(x_1;\delta_1 x;\delta_2 x)\| =$$

$$= \| \tfrac{1}{4}\|x_1+\delta_2 x\|^{-\frac{1}{2}} |L(x_1+\delta_2 x)|^{-\frac{1}{4}}\delta_1 x - \tfrac{1}{4}\|x_1\|^{-\frac{1}{2}} |L(x_1)|^{-\frac{1}{4}}\delta_1 x +$$

$$+ \tfrac{3}{16}\|x_1\|^{-\frac{1}{2}} |L(x_1)|^{-\frac{5}{4}} L(\delta_1 x)\, \delta_2 x \| =$$

$$= \left| |\pm(t_1+\delta_2 t)|^{-\frac{3}{4}} - |\pm t_1|^{-\frac{3}{4}} + \tfrac{3}{4}|\pm t_1|^{-\frac{7}{4}}\delta_2 t \right| \cdot$$

$$\cdot \|\tfrac{1}{4}\delta_1 x\| \cdot \|\mp\rho\|^{-\frac{3}{4}} |L(\rho)| \cdot |L(\mp\rho)|^{-\frac{5}{4}} =$$

$$= |\pm t_1|^{-\frac{3}{4}} \left| \left(1+\frac{\delta t}{\pm t_1}\right)^{-\frac{3}{4}} - 1 + \tfrac{3}{4}\left(\frac{\delta t}{\pm t_1}\right) \right| \cdot \|\tfrac{1}{4}\delta_1 x\| \cdot \|\mp\rho\|^{-\frac{3}{4}} \cdot |L(\rho)| \cdot$$

$$\cdot |L(\mp\rho)|^{-\frac{5}{4}} = \tfrac{1}{4}\|\delta_2 x\|^2 \cdot \|x_1\|^{-\frac{5}{4}} |L(x_1)|^{-\frac{5}{4}} \cdot |R_2| \cdot |L(\delta_1 x)| \leqslant \epsilon \|\delta_2 x\|$$

5) A direct substitution will verify iii) and iv) (Q.E.D.)

It is well known that the norm of a Banach variable is not Fréchet differentiable, hence, though the functions (2.22) and (2.23) will permit the definition of a P.L.c.s. from a given N.c.s., they will not define an L.c.s.

Several other functions similar to the above have been studied in this connection with the following results which will be presented without proof.

Theorem 24.4 "$x = f(y) = \|y\| L(y) y$ has a first L, F-differential $f(y;\delta y) = 3\|y\| L(\delta y) y = 3\|y\| L(y)\delta y$ with respect to y and E_1^0 but has no inverse."

Theorem 24.5 "The general problem of finding real exponents α and β such that $x = \|y\|^{\alpha} L(y) y$ $\quad y = \|x\|^{\beta} (L(x))^{\pm 1} x$ will be mutually inverse has no solution."

Theorem 24.6 "$x = f(y) = \|y\| y$ has a first L, F-differential $f(y;\delta y) = 2\|y\| y$ and an inverse $y = g(x) = \frac{1}{2}\|x\|^{-\frac{1}{2}} x$ also with first L, F-differential $g(x;\delta x) = \frac{1}{2}\|x\|^{-\frac{1}{2}}\delta x$."

Theorem 24.7 "$S = \|y\|^{n}$ as a real valued function, is not L, F-differentiable."

III

ADDITIONAL RESULTS

§ 3.1 Abstract Normal Coordinates

The first of the following two theorems relating to the paper on abstract normal coordinates by Michal and Hyers {1938-1} extends the corollary on p. 166 (see also theorem 17.2 {1939-1}) by showing that the normal coordinate component of linear connection still vanishes if the second two arguments are not even maps of points on a path, but merely are points in the coordinate space which are scalar multiples of the first argument. The second theorem applies the methods of {1938-1} to proving the validity of an identity in the curvature form analogous to one due to Veblen {1922-1}. This is proved both for spaces in which an abstract covariant differential {1936-1} is defined and for spaces in which a normal coordinate system and hence a kth extension can be defined. Equations mentioned in the latter theorem are specific references to equations in {1938-1}

Theorem 31.1 "In any normal coordinate system with coordinate domain Y there exists a number $\rho > 0$ such that

(3.1) $$\overset{+}{\Gamma}(s\lambda, t\lambda, u\lambda) = 0.$$

for any $\lambda \in B$, $|t|<\infty$, $|u|<\infty$, $|s| \leq \frac{\rho}{\|\lambda\|}$. Further, if $Y \neq B$, then there exists a greatest such $\rho < \infty$."

Proof: - 1) For any $\xi \in Y$ and $0 \leq s \leq 1$, $y = s\xi$ satisfies

$$\frac{d^2y}{ds^2} + {}^{+}\Gamma\left(y, \frac{dy}{ds}, \frac{dy}{ds}\right) = 0.$$

2) Hence ${}^{+}\Gamma(s\xi, \xi, \xi) = 0$.

3) Since Y is an open set containing $0 \in B$, there exists a $\rho > 0$ such that $\|y\| < \rho$:$\supset$: $y \in Y$. Furthermore if $Y \neq B$ there exists a maximum such $\rho < \infty$. (Compare theorem [21.12]).

4) Let $\lambda \in B$ be arbitrary and $\xi' = \frac{\theta\rho}{\|\lambda\|}\lambda$, $0 < \theta < 1$.

5) Then $\|\xi'\| = \theta\rho < \rho$ and by step 3) $\xi' \in Y$, and by step 2)

$${}^{+}\Gamma(s'\xi', \xi', \xi') = 0 \qquad 0 \leq s' \leq 1.$$

6) By the bilinearity and hence homogeneity of ${}^{+}\Gamma$ we have on multiplying through by $\frac{\|\lambda\|^2}{\rho^2} t u$ where t and u are any finite numbers,

$${}^{+}\Gamma(s'\xi', t\lambda, u\lambda) = 0.$$

7) Let $s\lambda = s'\lambda' = s'\frac{\theta\rho}{\|\lambda\|}\lambda$, then on substitution we obtain (3.1) from steps 5) and 6) and the corresponding restriction on the range of the parameter s. (Q.E.D.)

It might be noted in passing that (4.10) of Michal and Hyers {1938-1} can be written as

$$H_3(\xi) = \Gamma_{r-1}(x, \xi, \cdots, \xi; \xi) - (r-1)\,\Gamma_{r-1}(x, \Gamma(x, \xi, \xi)\xi, \cdots, \xi)$$

by using the symmetry of Γ_{r-1}.

Theorem 31.2 "The curvature form $B(x, \xi_1, \xi_2, \xi_3)$ of (5.12) satisfies the identity

$$(3.2) \quad B(x, \xi_1, \xi_2, \xi_3 | \xi_4) + B(x, \xi_3, \xi_1, \xi_4 | \xi_2) + \\ + B(x, \xi_4, \xi_3, \xi_2 | \xi_1) + B(x, \xi_2, \xi_4, \xi_1 | \xi_3) = 0."$$

Proof: - 1) Whether the above terms be regarded as covariant differentials or as first extensions (6.1) the expression (6.2) gives the following expansion for the first term above

$$B(x, \xi_1, \xi_2, \xi_3 | \xi_4) = B(x, \xi_1, \xi_2, \xi_3; \xi_4) - \\ - B(x, \Gamma_2(x, \xi_1, \xi_4), \xi_2, \xi_3) - B(x, \xi_1, \Gamma_2(x, \xi_2, \xi_4), \xi_3) - \\ - B(x, \xi_1, \xi_2, \Gamma(x, \xi_3, \xi_4)) + \Gamma_2(x, B(x, \xi_1, \xi_2, \xi_3), \xi_4).$$

2) Direct application of definition (5.12) to the right side of the above expansion gives a set of 22 terms each in five, six, or seven arguments, hence the verification of (3.2) will involve combination of 88 such terms. However, these terms fall into six types which we shall denote as

follows: -

i) $\Gamma_2(x, \xi_1, \xi_2; \xi_3; \xi_4)$ by 12,3,4

ii) $\Gamma_2(x, \Gamma_2(x, \xi_1, \xi_4), \xi_2; \xi_3)$

$= \Gamma_2(x, \xi_2, \Gamma_2(x, \xi_1, \xi_4); \xi_3)$ by 14|2,3

iii) $\Gamma_2(x, \Gamma_2(x, \xi_1, \xi_2; \xi_4), \xi_3)$ by 12,4|3

iv) $\Gamma_2(x, \Gamma_2(x, \Gamma_2(x, \xi_1, \xi_4), \xi_2), \xi_3)$

$= \Gamma_2(x, \Gamma_2(x, \xi_2, \Gamma_2(x, \xi_1, \xi_4), \xi_3)$ by 14|2|3

v) $\Gamma_2(x, \xi_1, \xi_2; \Gamma_2(x, \xi_3, \xi_4))$ by 12,|34

vi) $\Gamma_2(x, \Gamma_2(x, \xi_1, \xi_2), \Gamma_2(x, \xi_3, \xi_4))$ by 12||34

where terms of types i), v) and vi) are symmetric in the first pair and in the second pair, but terms of types ii), iii) and iv) are symmetric only in the first pair.

3) Under the above notation we have -

$$B(x, \xi_1, \xi_2, \xi_3 | \xi_4) = 12,3,4 - 13,2,4 + 12|3,4 + 12,4|3 - 13|2,4 - 13,4|2 - \{14|2,3 - 14|3,2 + 14|2|3 - 14|3|2 + 24|1,3 - 13,|24 + 24|1|3 - 13||24 + 12,|34 - 34|1,2 + 12||34 - 34|1|2\} + 12,3|4 - 13,2|4 + 12|3|4 - 13|2|4.$$

4) Since we cannot have combinations between these types of terms, we must have the terms of each type cancelling among themselves as the subscripts are

permuted (1342). This takes place as follows -

5) There are eight terms each of types i) v) and vi) which combine as follows -

$$\begin{array}{l|l|l}
12,3,4^{1} - 13,2,4^{2} & 12,|34^{1} - 13,|24^{2} & 12||34^{1} - 13||24^{2} \\
+31,4,2^{2} - 34,1,2^{3} & +31,|42^{2} - 34,|12^{3} & +21||42^{2} - 34||12^{3} \\
+43,2,1^{3} - 42,3,1^{4} & +43,|21^{3} - 42,|31^{4} & +43||21^{3} - 42||31^{4} \\
+24,1,3^{4} - 21,4,3^{1} & +24,|13^{4} - 21,|43^{1} & +24||13^{4} - 21||43^{1}
\end{array}$$

6) There are sixteen terms of type iii) which combine as follows -

$$\begin{array}{l}
12,4|3^{1} - 13,4|2^{5} + 12,3|4^{8} - 13,2|4^{2} \\
+31,2|4^{2} - 34,2|1^{6} + 31,4|2^{5} - 34,1|2^{3} \\
+42,1|2^{3} - 42,1|3^{7} + 43,2|1^{6} - 42,3|1^{4} \\
+24,3|1^{4} - 21,3|4^{8} + 24,1|3^{7} - 21,4|3^{1}.
\end{array}$$

7) There are twenty-four terms of types ii) and iv) which combine as follows -

$$\begin{array}{l}
12|3,4^{1} - 13|2,4^{5} - 14|2,3^{9} + 14|3,2^{11} - 24|1,3^{4} + 34|1,2^{6} \\
+31|4,2^{2} - 34|1,2^{6} - 32|1,4^{10} + 32|4,1^{12} - 12|3,4^{1} + 42|3,1^{7} \\
+43|2,1^{3} - 42|3,1^{7} - 41|3,2^{11} + 41|2,3^{9} - 31|4,2^{2} + 21|4,3^{8} \\
+24|1,3^{4} - 21|4,3^{8} - 23|4,1^{12} + 23|1,4^{10} - 43|2,1^{3} + 13|2,4^{5}.
\end{array}$$

(Terms are indexed in red in pairs which cancel). (Q.E.D.)

Note: - A briefer proof of this theorem consists in showing that (1.2) holds in normal coordinates as follows -

1) All terms of types ii)$\cdots$ vi) vanish at the origin of normal coordinates.

2) By step 5) above all terms of type i) cancel out in normal coordinates evaluated at the origin of coordinates.

3) Since $B(x, \xi_1, \xi_2, \xi_3 | \xi_4)$ and hence any sum of such is c.v.f. valued it vanishes in all coordinates at a point, if it vanishes in one coordinate system.

4) Since $x_0 \leftrightarrow y_0 = 0$ was any arbitrary point of the space, the theorem holds in general.

(Q.E.D.)

§3.2 The δ-property

In a recent paper on the projective geometry of paths {1939-2, 1940-1} special use was made of the following restriction on the Fréchet differential.

Definition 32.1 The δ-property - Let $f(x,y)$ have arguments and values in Banach spaces. The Fréchet differential $f(x_0, y; \delta x)$ of $f(x,y)$ at $x = x_0$ is said to have the δ-property with respect to y if for every $\epsilon > 0$, $a > 0$ there exists a $\delta(\epsilon, a, x_0) > 0$ independent of y such that

(3.3) $\quad \|\delta x\| < \delta(\epsilon, a, x_0)$ and $\|y\| < a$

$$:\supset: \|f(x_0 + \delta x, y) - f(x_0, y) - f(x_0, y; \delta x)\| \leq \epsilon \|\delta x\|,$$

This definition may be restated in terms of the following modified concept of uniform continuity

Definition 32.2 Relative uniform continuity at a point*- Let $f(x,y)$ be on E_1 H to E_2 where E_1, E_2 are Banach spaces and H is a topologic space. Then $f(x,y)$ will be said to be continuous at $x = x_0$ uniformly with respect to y on the open set $S \subset H$ if, given $\epsilon > 0$, there exists a $\delta(\epsilon, x_0) > 0$ independent of y such that

(3.4) $$\|\delta x\| < \delta(\epsilon, x_0), \qquad y \in S \quad :\supset: \\ \|f(x_0 + \delta x, y) - f(x_0, y)\| \leqq \epsilon.$$

Definition 32.3 Relative locally uniform continuity at a point* - Let $f(x,y)$ be as in (32.2). Then $f(x,y)$ will be said to be continuous at $x = x_0$ locally uniformly with respect to y on the open set $S \subset H$ if, given $\epsilon > 0$ and $y_0 \in S$, there exists a neighborhood $U_{y_0} \subset S$ and a $\delta(\epsilon, x_0, y_0) > 0$ independent of y such that

(3.5) $$\|\delta x\| < \delta(\epsilon, x_0, y_0), \quad y \in U_{y_0} \quad :\supset: \\ \|f(x_0 + \delta x, y) - f(x_0, y)\| \leqq \epsilon.$$

In many instances we will take the space H to be also a Banach space topologized by taking spheres as neighborhoods.

Definition 32.4 The δ-property - Let $f(x,y)$ be as in (32.2) and be Fréchet differentiable in x at $x = x_0$. The Fréchet differential $f(x_0, y; \delta x)$ will be said to have the δ-property

* These definitions differ from the use made of the same term by Hildebrandt and Graves 1927-2 .

with respect to y on S if

$$\epsilon(x_0, y, \delta x) = \frac{f(x_0+\delta x, y) - f(x_0, y) - f(x_0, y; \delta x)}{\|\delta x\|}$$

has the value $\epsilon(x_0, y, 0) = 0$ at $\delta x = 0$ and is continuous in δx at $\delta x = 0$ uniformly with respect to y for all y in a given open set $S \epsilon H$.

Theorem 32.5 "The definitions (32.1) and (32.4) are equivalent when H is a Banach space and S contains $0 \epsilon H$."

Proof: - 1) Let (32.4) be satisfied, then by the definition of open set S, there exists a $\rho > 0$ such that $\|y\| < \rho \quad :\supset: \quad y \epsilon S$.

2) Given any $\epsilon > 0$ and the above $\rho > 0$, there exists a $\delta(\epsilon, x_0) > 0$ such that

$$\|\delta x\| < \delta(\epsilon, x_0), \qquad \|y\| < \rho \quad :\supset:$$
$$\|\epsilon(x_0, y, \delta x) - \epsilon(x_0, y, 0)\| =$$

$$= \left\| \frac{f(x_0+\delta x, y) - f(x_0, y) - f(x_0, y; \delta x)}{\|\delta y\|} \right\| \leq \epsilon$$

3) or $\|f(x_0+\delta x, y) - f(x_0, y) - f(x_0, y; \delta x)\| \leq \epsilon \|\delta x\|$.

4) Let (32.1) be satisfied, then for any $\epsilon > 0$, $\|y\| < a$, $\|\epsilon(x_0, y, \delta x)\| \leq \epsilon$ for all $\|\delta x\| < \delta(\epsilon, a, x_0)$.

5) But this is the condition of (32.2) that $\epsilon(x_0, y, \delta x)$

be continuous at $\delta x = 0$ uniformly with respect to y on the sphere S_a of radius a about the origin, and have the value $\epsilon(x_0, y, 0) = 0$. (Q.E.D.)

Theorem 32.6 "Let $f_1(x,y)$ and $f_2(x,y)$ be two functions satisfying (32.4). Then any linear combination

$$F(x,y) = k_1 f_1(x,y) + k_2 f_2(x,y)$$

with constant real coefficients k_1 and k_2 will satisfy (32.4)."

Proof: - 1) Let $\epsilon_i(x_0, y, \delta x) =$

$$= \frac{1}{\|\delta x\|}\{f_i(x_0+\delta x, y) - f_i(x_0, y) - f_i(x_0, y; \delta x)\} \quad i = 1, 2.$$

2) By (32.2) there exists $\delta_i(\epsilon', x_0) > 0$ such that

$$\|\delta x\| < \delta_i\left(\frac{\epsilon}{2k_i}, x_0\right) \quad y \in S :\supset: \|\epsilon_i(x_0, y, \delta x)\| \leq \frac{\epsilon}{2k_i}$$

3) $\therefore \|\delta x\| < \delta = \min_{i=1,2} \delta_i\left(\frac{\epsilon}{2k_i}, x_0\right), \quad y \in S :\supset:$

$$\|\epsilon(x_0, y, \delta x)\| = \left\|\frac{1}{\|\delta x\|}\{F(x_0+\delta x, y) - F(x_0, y) - F(x_0, y; \delta x)\}\right\| =$$

$$= \left\|\sum_{i=1}^{2} k_i \epsilon_i(x_0, y, \delta x)\right\| < \epsilon.$$

(Q.E.D.)

Corollary 32.7 "Let $f_i(x,y)$, $i = 1, 2, \cdots, t$ satisfy (32.4).

Then

$$F(x,y) = \sum_{i=1}^{t} k_i f_i(x,y)$$

with constant coefficients $k_i \in R$ satisfies (32.4)."

Definition 32.8 δ-property (locally) uniformly in y - Let $f(x,y)$ and $f(x_0,y;\delta x)$ satisfy (32.4). Then $f(x_0,y;\delta x)$ will be said to have the δ-property (locally) uniformly in y if $f(x_0,y;\delta x)$ is continuous in δx at $\delta x = 0$ (locally) uniformly in y on the open set $S \subset H$ according to (32.2) $\{$or (32.3)$\}$.

This property is not automatically satisfied by the linearity of $f(x_0,y;\delta x)$ in δx since the modulus M in

$$\|f(x_0,y;\delta x)\| \leq M\|\delta x\|$$

is itself a function of x_0 and y. It is clear, however, that if (32.8) is satisfied we will have for $y \in S$ $(y \in U_{y_0})$ a modulus M which depends on x_0 alone.

Theorem 32.9 "A necessary and sufficient condition that $f(x,y)$ have a Fréchet differential with the δ-property (locally) uniformly in y on $S \subset H$ is that $f(x_0,y;\delta x)$ exist with the δ-property and $f(x,y)$ be continuous in x at $x = x_0$ (locally) uniformly in y on $S \subset H$."

Proof: - Necessity 1) Let $f(x_0,y;\delta x)$ satisfy (32.8), then, given $\epsilon > 0$ and $S \subset H$, there exists $\delta_1(\frac{\epsilon}{2}, x_0) > 0$ such that

$$\|\delta x\| < \delta_1(\tfrac{\epsilon}{2}, x_0) \ :\supset: \ \|\epsilon(x_0,y,\delta x)\| \leq \tfrac{\epsilon}{2}.$$

2) By (32.2) $\{$or (32.3)$\}$ there exists $\delta_2(\frac{\epsilon}{2}, x_0) > 0$ (and

such that

$$\|\delta x\| < \delta_2\left(\tfrac{\epsilon}{2}, x_0\right) \;:\supset:\; \|f(x_0, y; \delta x)\| \leq \tfrac{\epsilon}{2}.$$

3) Now step 1) implies

$$\|f(x_0+\delta x, y) - f(x_0, y)\| - \|f(x_0, y; \delta x)\| \leq \tfrac{\epsilon}{2}\|\delta x\|$$

4) hence there exists $\delta = \min\left(1, \delta_1\left(\tfrac{\epsilon}{2}, x_0\right), \delta_2\left(\tfrac{\epsilon}{2}, x_0\right), \epsilon\right) > 0$

independent of y such that

$$\|\delta x\| < \delta, \quad y \in S \quad (y \in U_{y_0}) \;:\supset:$$

$$\|f(x_0+\delta x, y) - f(x_0, y)\| \leq \epsilon.$$

(Q.E.D.)

Sufficiency 5) Let $f(x, y)$ satisfy (32.2) (or (32.3)),

then there exists $\delta_3\left(\tfrac{\epsilon}{2}, x_0\right) > 0$ (and U_{y_0})

such that

$$\|\delta x\| < \delta_3\left(\tfrac{\epsilon}{2}, x_0\right), \quad y \in S \quad (y \in U_{y_0})$$

implies

$$\|f(x_0+\delta x, y) - f(x_0, y)\| < \tfrac{\epsilon}{2}.$$

6) By (32.4) there exists $\delta_4\left(\tfrac{\epsilon}{2}, x_0\right) > 0$ such that

$$\|\delta x\| < \delta_4\left(\tfrac{\epsilon}{2}, x_0\right) \qquad y \in S. \;:\supset:$$

$$\|f(x_0, y; \delta x)\| - \|f(x_0+\delta x, y) - f(x_0, y)\| \leq$$

$$\leq \|\delta x\| \cdot \|\epsilon(x_0, y, \delta x)\| \leq \tfrac{\epsilon}{2}\|\delta x\|.$$

7) But this implies that there exists $\delta = \min(1, \delta_3(\frac{\epsilon}{2}, x_0),$
$\delta_4(\frac{\epsilon}{2}, x_0)) > 0$ independent of y such that
$\|\delta x\| < \delta \qquad y \in S \quad (y \in U_{y_0})$
$: \supset: \quad \|f(x_0, y; \delta x)\| \leqslant \epsilon.$

(Q.E.D.)

In the following theorems H will be a Banach space. Hence we shall use definition (32.1) of the δ-property as being more convenient. It will, of course, be assumed that the wording of other definitions and theorems hsve been modified to conform to this.

Theorem 32.10 "Let $f_1(x,y)$ be on $E_1(x_0)_a\, E_2$ to E_3, linear in y, continuous in x uniformly in y on $E_2(0)_b$ and such that the Fréchet differential $f_1(x_0, y; \delta x)$ exists and has the δ-property at $x = x_0$.

Let $f_2(x,z)$ be on $E_1(x_0)_a\, E_3$ to E_4 linear in z and such that the Fréchet differential $f_2(x_0, z; \delta x)$ exists and has the δ-property at $x = x_0$.

E_1, E_2, E_3, E_4 are Banach spaces and $E_1(x_0)_a$ and $E_2(0)_b$ are the open spheres $\|x_0 - x\| < a$ and $\|y\| < b$.

Then - i) $F(x,y) = f_2(x, f_1(x,y))$ is on $E_1(x_0)_a\, E_2$ to E_4 linear in y, and ii) $F(x,y)$ has the Fréchet differential

$$F(x_0,y;\delta x) = f_2(x_0, f_1(x_0,y);\delta x) + f_2(x_0, f_1(x_0,y;\delta x))$$

with the δ-property uniformly in y on $E_2(0)_b$ at $x = x_0$."

Proof: - i) follows at once from the hypotheses and the definition of linearity.

ii) 1) Since $f_1(x,y)$ is linear in y, it follows that $f_1(x_0,y;\delta x)$ is bilinear in y and δx and hence there exists a modulus $M_1(x_0) > 0$ such that

$$\|f_2(x_0, \tau; \delta x)\| \leq M_1(x_0)\,\|\tau\|\cdot\|\delta x\| \qquad \tau \in E_3,$$

2) and also a modulus $M_2(x_0) > 0$ such that

$$\|f_2(x_0,\sigma)\| \leq M_2(x_0)\,\|\sigma\|.$$

3) Let $\epsilon > 0$, $b > 0$ be given and remain fixed.

4) By the δ-property of df_1, there exists a constant

$$\delta_1 = \delta_1\left(\frac{\epsilon}{3M_2(x_0)}, b, x_0\right) > 0 \text{ independent of } y$$

such that $\|\delta x\| < \delta_1 \quad \|y\| < b \ :\supset:$

$$\|\sigma\| = \|f_1(x_0+\delta x, y) - f_1(x_0,y) - f_1(x_0,y;\delta x)\| \leq \frac{\epsilon}{3M_2(x_0)}$$

5) Steps 2) and 4) further imply

(3.6) $$\|f_2(x_0, f_1(x_0+\delta x, y)) - f_2(x_0, f_1(x_0,y)) - f_2(x_0, f_1(x_0,y;\delta x))\| \leq \frac{\epsilon}{3}$$

6) Since $f_1(x,y)$ is continuous in x uniformly in y, there exists a constant $\delta_2 = \delta_2\left(\frac{\epsilon}{3M_1(x_0)}, b, x_0\right) > 0$

independent of y such that $\|\delta x\| < \delta_2 \; :\supset:$

$$\|T\| = \|f_1(x_0 + \delta x, y) - f_1(x_0, y)\| \leq \frac{\epsilon}{3 M_1(x_0)}.$$

7) If also $\|\delta x\| < 1$, steps 1) and 6) further imply

$$\|f_2(x_0, f_1(x_0 + \delta x, y); \delta x) - f_2(x_0, f_1(x_0, y); \delta x)\| \leq \frac{\epsilon}{3}. \tag{3.7}$$

8) Since $f_1(x, y)$ is linear in y, there exists a modulus $M_3(x_0) > 0$ such that $\|f_1(x_0, y)\| \leq M_3(x_0)\|y\|$ $y \in E_2$ and so by step 6) and the triangle inequality

$$\|\omega\| = \|f_1(x_0 + \delta x, y)\| \leq \frac{\epsilon}{3 M_1(x_0)} + \|f_1(x_0, y)\| \leq$$

$$< \frac{\epsilon}{3 M_1(x_0)} + M_3(x_0) b = c \quad \text{for} \quad \|y\| < b.$$

9) Given c as in step 8), we have by the δ-property of df_2 that there exists a constant $\delta_3 = \delta_3(\frac{\epsilon}{3}, c, x_0) > 0$ independent of ω such that

$$\|\delta x\| < \delta_3 \quad \text{and} \quad \|\omega\| < c \quad :\supset:$$

$$\|f_2(x_0 + \delta x, \omega) - f_2(x_0, \omega) - f_2(x_0, \omega; \delta x)\| \leq \frac{\epsilon}{3}\|\delta x\|. \tag{3.8}$$

10) Now let $\delta = \delta(\epsilon, b, x_0, M_1(x_0), M_2(x_0), M_3(x_0)) =$

$$= \min(1, \delta_1, \delta_2, \delta_3) > 0$$

which is a positive constant independent of y such that $\|\delta x\| < \delta$, $\|y\| < b \; :\supset:$ (3.6) (3.7) (3.8).

11) Add these three equations and make use of the triangle inequality and additivity and we have -

$$\| f_2(x_0, f_1(x_0+\delta x, y)) - f_2(x_0, f_1(x_0,y)) - \tag{3.9}$$
$$- f_2(x_0, f_1(x_0,y;\delta x)) + f_2(x_0, f_1(x_0+\delta x, y);\delta x) -$$
$$- f_2(x_0, f_1(x_0,y);\delta x) + f_2(x_0+\delta x, f_1(x_0+\delta x, y)) -$$
$$- f_2(x_0, f_1(x_0+\delta x, y)) - f_2(x_0, f_1(x_0+\delta x, y);\delta x)\| \leqslant \epsilon.$$

12) Hence, by the linearity of the Fréchet differentials the hypothesis on the linearity of $f_2(x, z)$ in z and steps 10) and 11)

$$F(x_0, y; \delta x) = f_2(x_0, f_1(x_0,y); \delta x) + f_2(x_0, f_1(x_0, y; \delta x)) \tag{3.10}$$

is linear in δx and satisfies definition (32.1).

13) By theorem (32.9) and the expression (3.10) it follows that $F(x,y)$ is continuous in x uniformly in y and hence $F(x_0, y; \delta x)$ has the δ-property uniformly in y. (Q.E.D.)

Corollary 32.11 "Theorem [32.10] holds if the words 'locally uniformly' be substituted for 'uniformly' wherever they occur".

Theorem 32.12 "The set Φ of all functions $f(x,y)$ on $E_1(x_0)_a E_2$ to E_2 linear and solvable in y, continuous in x uniformly in y on E_2 and with differentials having the δ-property plus the function $\ominus(x,y) = 0 \in E_2$, under suitable definitions of $\oplus$, $\odot$ and $\|\cdots\|$, form a Banach ring."

Proof: - 1) By theorem [32.6] and corollary [32.7] this is linear set and $\oplus$ is the plus operation of E_2.

2) i) $\oplus$ is commutative,

ii) $\ominus(x,y)$ is the identity under $\oplus$,

iii) if $f(x,y) \in \Phi$ then $-f(x,y)$ is also linear and solvable, etc. and hence is in Φ.

3) The operation $\odot$ in Φ will be defined as iteration, and by theorem [32.10] the set is closed under $\odot$.

4) The identity function $I(x,y) = y$ for all $x \in E_1(x_0)_a$ is in Φ.

5) To show that the inverse $g(x,y)$ of $f(x,y) \in \Phi$ is in Φ we need only show its Fréchet differential $g(x,y;\delta x)$ has the δ-property uniformly in y at $x = x_0$ This follows from

$$I(x,y;\delta x) = 0 = f(x, g(x,y); \delta x) + f(x, g(x,y;\delta x)).$$

6) Associativity is readily verified.

7) Distributivity follows from the linearity of every $f(x,y) \in \Phi$ in y.

8) The norm will be defined as

$$\|f(x,y)\| = \max_{x \in E_1(x_0)_a} M(x)$$

where $M(x)$ is the modulus of $f(x,y)$, and this can be shown to have the norm properties.

(Q.E.D.)

Corollary 32.13 "The Banach ring Φ is a subring of the Banach ring R_1 of {1940-1}."

Bibliography

This list does not pretend in any sense to be complete, as many such bibliographies are available (an excellent one is to be found in {1939-1}). Hence, only papers and books to which special reference has been made in the thesis will be found listed here.

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