AN INVESTIGATION OF THE EFFECTS OF EARTHQUAKES ON BUILDINGS

by

George William Housner

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INTRODUCTION

The forces induced in buildings by earthquakes are here investigated. The effects of some of the physical properties of structures are determined and from an analysis of earthquake records the general character of earthquakes is deduced. An investigation is also made of the dissipation of energy of vibration due to the propagation of elastic waves in the ground.

This thesis was written to contribute to the program of earthquake research that Professor Martel has been carrying on for the past ten years. As such, it is a continuation of the work done by M. A. Biot and M. P. White. Acknowledgment is made of the financial support of the County of Los Angeles which made possible the analysis of the earthquake records.

Sincere appreciation is expressed to Professor R. R. Martel for the advice he has given and the interest he has shown.

SUMMARY

All the earthquakes analysed are of the same general character. There are no special features distinguishing any one earthquake from another. There is no evidence that the physical properties of the ground have any effect on the character of the earthquake.

In the range of periods from 2/10 seconds to 2 seconds, all ground waves are of approximately equal importance. There are no predominating periods of ground waves.

The analysis of records provides a convenient method of measuring the intensities of earthquakes. A scale of intensities constructed in this manner is shown in the following table:-

Scale of Earthquake Intensities

El Centro, May, 1940	100
Long Beach, March, 1933, (at Vernon)	65
El Centro, December, 1934	60
Helena, Montana, October 31, 1935	55
Ferndale, September, 1938	40
Los Angeles, October, 1933	17

The analysis of the records shows that there is no so-called "dominant ground period".

For undamped structures, with periods of vibration longer than about 2/10 of a second, the maximum shearing force at the base is, for practical purposes, independent of the height of the structure and

independent of the period of vibration.

For such structures the maximum shearing forces are independent of the total mass of the structure but for ordinary construction vary in direct ratio as the mass per floor level.

When the height of a structure is increased, the shearing forces in the upper portions of the structure are increased.

Only the first few modes of vibration are of importance in producing shearing forces.

For undmaped structures flexibility and lightness of construction will reduce the magnitudes of the shearing forces. However, the flexibility of the first story alone has little effect upon the maximum shearing force at the base of the structure, although it does reduce the shearing forces in the upper portions of the structure.

Considerable elastic yielding of the ground may take place at the base of a structure without having sufficient effect on the accelerogram to be distinguished by analysis of the record.

The energy dissipated into the ground may be an important factor in reducing the shearing forces. The analysis shows that the reduction in shearing forces due to energy lost in wave propagation is greater for high frequencies than for low frequencies, and is greater for ground of low rigidity than for ground with high rigidity. Tables are constructed which indicate that structures with high frequencies may have their shearing forces appreciably reduced by this energy dissipation. It appears, however, that structures with periods of about 1 second will be little affected.

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CHAPTER I.

The analysis of the forces induced in buildings by earthquakes is hampered by certain properties of the structures to be considered. The non-uniformity of shape, mass, and rigidity of ordinary buildings would require an excessive amount of labor if a complete analysis were to be made. In addition, the vibration damping that occurs during the motion of a building is as yet undetermined. The manner in which this damping takes place, and the degree and type of damping are not known. If the damping is neglected then there are some solutions that can be easily evaluated and which represent certain types of buildings fairly well. These solutions, although not representing actual buildings, are sufficient to indicate the general behavior of undamped structures during an earthquake.

The motion* of an undamped linearly elastic structure in free vibration can be expressed as the sum of the normal coordinates. The equation of free vibration for a structure starting with zero displacement at time t = 0, is:

$$y = \sum_n A_n Y_n \sin p_n t \qquad (1)$$

where y = lateral displacement

A_n = amplitude of the nth mode

Y_n = function giving the shape of the nth mode

p_n = 2π times the frequency of the nth mode

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*The vibration theory here used is as presented in Webster's, "Partial Differential Equations of Mathematical Physics".

If these free vibrations are caused by an initial impulse; that is, if the base of the structure is given a constant acceleration a for an infinitesimal interval of time Δt, all points of the structure will be given an initial velocity, relative to the base, equal to $a\Delta t$. The coefficients A_n in Equation (1) are then to be evaluated so as to satisfy the equation:

$$\dot{y}_0 = a\Delta t = \sum_n A_n Y_n p_n$$

These coefficients can be evaluated in the Fourier manner since the types of vibrating systems here investigated have normal coordinates that are also orthogonal functions, so:

$$A_n = (a\Delta t)\frac{1}{p_n}\frac{\int_0^H m Y_n \, dx}{\int_0^H m Y_n^2 \, dx}$$

where x is measured along the height of the structure, m is the mass per unit height, and H is the total height. The equation of motion then is:

$$y = (a\Delta t)\sum_n \frac{1}{p_n}\frac{\int_0^H m Y_n \, dx}{\int_0^H m Y_n^2 \, dx} Y_n \sin p_n t \qquad (2)$$

The shearing force at any point of the structure is equal to the sum of the inertia forces above that point. In particular, the shearing force of the base is:

$$S = \int_0^H m \frac{\partial^2 y}{\partial t^2} \, dx$$

From Equation (2), this shearing force is evaluated as:

$$S = (a\Delta t)\sum_n \frac{\left[\int_0^H m Y_n \, dx\right]^2}{\int_0^H m Y_n^2 \, dx} p_n \sin p_n t \qquad (3)$$

If the base of the structure is subjected to an irregular acceleration such as caused by an earthquake, the resulting shearing forces are deduced by an application of Rayleigh's principle*. The effect of the continuous ground acceleration $\ddot{u}$ is determined by summing the effects of a succession of impulses $\ddot{u}\Delta t$; then in passing to the limit this summation becomes a definite integral giving:

$$S = \sum_n \frac{\left(\int_0^H m Y_n dx\right)^2}{\int_0^H m Y_n^2 dx} p_n \int_0^{t_1} \ddot{u} \sin p_n (t_1 - t) dt \tag{4}$$

where S is the shearing force at the time t_1. Similarly, from Equation (2), the displacements due to an arbitrary ground acceleration $\ddot{u}$ are:

$$y = \sum_n \frac{\int_0^H m Y_n dx}{\int_0^H m Y_n^2 dx} Y_n \frac{1}{p_n} \int_0^{t_1} \ddot{u} \sin p_n (t_1 - t) dt \tag{5}$$

The velocity at any point of the structure is:

$$\dot{y} = \sum_n \frac{\int_0^H m Y_n dx}{\int_0^H m Y_n^2 dx} Y_n \int_0^{t_1} \ddot{u} \cos p_n (t_1 - t) dt \tag{6}$$

Equations (5) and (6) illustrate the rule for differentiating under the integral sign. The kinetic energy of the structure is equal to:

$$K.E. = \frac{1}{2} \int_0^H m (\dot{y})^2 dx$$

Since the $\sqrt{m}\, Y_n$'s are orthogonal functions, the kinetic energy is explicitly given by:

$$K.E. = \frac{1}{2} \sum_n \frac{\left(\int_0^H m Y_n dx\right)^2}{\int_0^H m Y_n^2 dx} \left(\int_0^{t_1} \ddot{u} \cos p_n (t_1 - t) dt\right)^2 \tag{7}$$

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*Webster, "Partial Differential Equations of Mathematical Physics", or Rayleigh's "Theory of Sound", page 74.

From Equation (7), the kinetic energy of each mode of vibration can be determined. For any one mode this is:

$$KE_n = \frac{1}{2}\,\frac{\left(\int_0^H m\,Y_n\,dx\right)^2}{\int_0^H m\,Y_n^2\,dx}\left(\int_0^{t_1} \ddot{u}\cos p_n(t_1-t)\,dt\right)^2 \tag{8}$$

The maximum kinetic energy is equal to the maximum total energy attained by this mode. If the maximum values of the integrals of the type:

$$\int_0^{t_1} \ddot{u}\cos p_n(t_1-t)\,dt \tag{9}$$

are determined for an earthquake, then the maximum energy in each mode of vibration can be calculated. In determining the maximum values of such integrals as (9) , a phase factor is introduced in the cosine term in order to get the maximum possible values. It is thus seen that the maximum possible value of (9) is identical with the maximum possible value of:

$$\int_0^{t_1} \ddot{u}\sin p_n(t_1-t)\,dt \tag{10}$$

The maximum value of (10) has the dimensions of a velocity and represents the cumulative effect of the ground acceleration. The maximum values of such integrals as (10) will be designated by V_n.

From Equation (4), it is seen that the maximum shearing force in any one mode is equal to:

$$S_n = \frac{\left(\int_0^H m\,Y_n\,dx\right)^2}{\int_0^H m\,Y_n^2\,dx}\,p_n V_n \tag{11}$$

or

$$S_n = C_n V_n$$

From Equation (7), it is seen that the maximum energy in any one mode is equal to:

$$E_n = \frac{1}{2} \frac{S_n^2}{p_n C_n} \tag{12}$$

This gives a relation between the maximum shearing force and the maximum energy. Equation (12) can be written:

$$S_n = 2 \left(\frac{\pi E_n C_n}{T_n} \right)^{1/2} \tag{13}$$

where T_n is the period of vibration. Equation (13) shows that for given values of E_n and C_n the maximum shearing force varies in inverse ratio as the square root of the period of vibration.

The maximum shearing force in any one mode of vibration is thus given by the expression:

$$S_n = C_n V_n \tag{14}$$

where

$$C_n = \frac{\left(\int_0^H m Y_n \, dx \right)^2}{\int_0^H m Y_n^2 \, dx} p_n$$

The maximum shear will then be completely determined if the two factors C_n and V_n are evaluated. C_n may be evaluated from the mass, rigidity, and dimensions of the structure while V_n must be determined from the recorded ground motion of an earthquake.

The factor C_n can be conveniently evaluated for several different types of structures. There are two of these that are of particular interest. One is a structure that moves with shearing deformations, and the other is a structure that moves with bending deformations. For each of these, a structure uniform in mass and rigidity will be considered. A structure with shearing deformations and with an elastic coupling at the base will be first investigated*. Such an elastic coupling may be considered to be either a flexible first story or the elastic yielding of the ground upon which the structure rests. In the following discussion it will be taken to be a flexible first story. The value of C_n for this case is:

$$C_n = \frac{2\sin^2\lambda_n}{\lambda_n\left(1+\frac{\sin 2\lambda_n}{2\lambda_n}\right)}\sqrt{km} \tag{15}$$

where

k = modulus of rigidity of the superstructure

m = mass per unit height of superstructure

λ_n = nth root of frequency equation $\lambda \tan\lambda = \frac{kH}{Kh}$

K = modulus of rigidity of flexible first story

h = height of first story.

H = height from second floor to roof

Equation (15) may be written as:

$$C_n = D_n\sqrt{km} \tag{16}$$

where D_n has the values listed in Table I.:

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*A vibrating system of this type is discussed in Webster, "Partial Differential Equations of Mathematical Physics", page 123, and in the Ph.D. thesis of M. A. Biot, C.I.T. 1932.

TABLE I.

$\frac{kH}{Kh}$	D_1	D_2	D_3
.56	.68	.01	.003
.83	.79	.03	.008
1.11	.87	.06	.011
1.66	1.0	.09	.023
2.30	1.1	.16	.041
3.33	1.15	.21	.062
5.0	1.20	.27	.10
10.0	1.25	.36	.18
0.0	1.27	.412	.251

From Table I. it is seen that the flexibility of the first story or the elastic yielding of the ground affects the shear in the higher modes much more than the shear in the first mode. When the flexibility of the first story is so great that the ratio $\frac{kH}{Kh}$ is less than 1, the shear in the higher modes is negligible compared to the shearin the first mode. In this case the structure is in effect reduced to one degree of freedom; i.e., it behaves as a simple oscillator. Table I. clearly shows the relative importance of the modes of vibration. For ordinary structures the rigidity of the first story is approximately the same as the rigidity of the superstructure; i.e., $\frac{k}{K}=1$. For such structures an increase in height will increase the shear in each mode. The shears in the higher modes increase more rapidly than the shear in the first mode. From this it can be concluded that as the height increases the higher modes become of greater relative importance and accordingly the shears in the upper portions of the structure are increased.

The second type of structure to be considered is that having bending deformations*. The value of C_n for this case is:

$$C_n = \frac{4}{H}\left(\frac{(1-\cosh\beta_n)(1-\cos\beta_n)}{(\sinh\beta_n\cos\beta_n - \cosh\beta_n\sin\beta_n)}\right)\sqrt[4]{EIm} \tag{17}$$

where

E = Young's modulus

I = moment of inertia

m = mass per unit height

H = height of structure

β_n = nth root of frequency equation $\cosh\beta\cos\beta = -1$

Equation (17) may be written as:

$$C_n = \frac{D_n'}{H}\sqrt[4]{EIm} \tag{18}$$

where D_n' has the values listed in Table II.:

TABLE II.

D'1	D'2	D'3	D'4
5.7	4.0	4.0	4.0

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* A system of this type is discussed in Webster's, "Partial Differential Equations of Mathematical Physics", page 138, and in Timoshenko, "Vibration Problems in Engineering", page 331.

It appears from this Table that all modes of vibration are of approximately equal importance. This would indicate that structures of this type are in a very unfavorable position to resist earthquakes because of the importance of the higher modes. However, the value of V_n approaches zero as the period of vibration approaches zero and the periods of the higher modes decrease rapidly. The frequency of the nth mode is proportional to β_n^2 where β_n has the values listed in Table III.:

TABLE III.

$$\beta_1 = 1.875$$

$$\beta_2 = 4.694$$

$$\beta_3 = 7.855$$

for $n>3$

$$\beta_n = \frac{2n-1}{2}\pi$$

It is thus seen that the periods of vibration decrease very rapidly in the higher modes, being in the proportions 1; 0.16; 0.057; 0.017. Despite this decrease of period the first few modes will still be of approximately equal importance and the 2nd and 3rd modes may cause shearing forces in the upper portion of the structure comparable in magnitude to the shearing force at the base.

A surprising feature of Equation (18) is that the height H occurs in the denominator. Therefore, the shear in each mode is reduced by increasing the height. This reduction in shear is offset by the fact that as the height is increased, more modes of vibration are excited since the periods increase as H^2. This indicates that as the height is increased, the base of the structure may be somewhat relieved of stress but the upper portions will be subjected to larger stresses.

To correlate the types of structures investigated, use is made of the expressions for the periods of vibrations of the first modes.

$$T_S = 4H\sqrt{\frac{m}{K}} \qquad \text{(shear)}$$

$$T_B = 1.8\, H^2\sqrt{\frac{m}{EI}} \qquad \text{(bending)}$$

If the periods of vibration are equal, and heights and masses are equal, then:

$$\sqrt{EI} = 0.45\, H\sqrt{K}$$

The maximum shears are then:

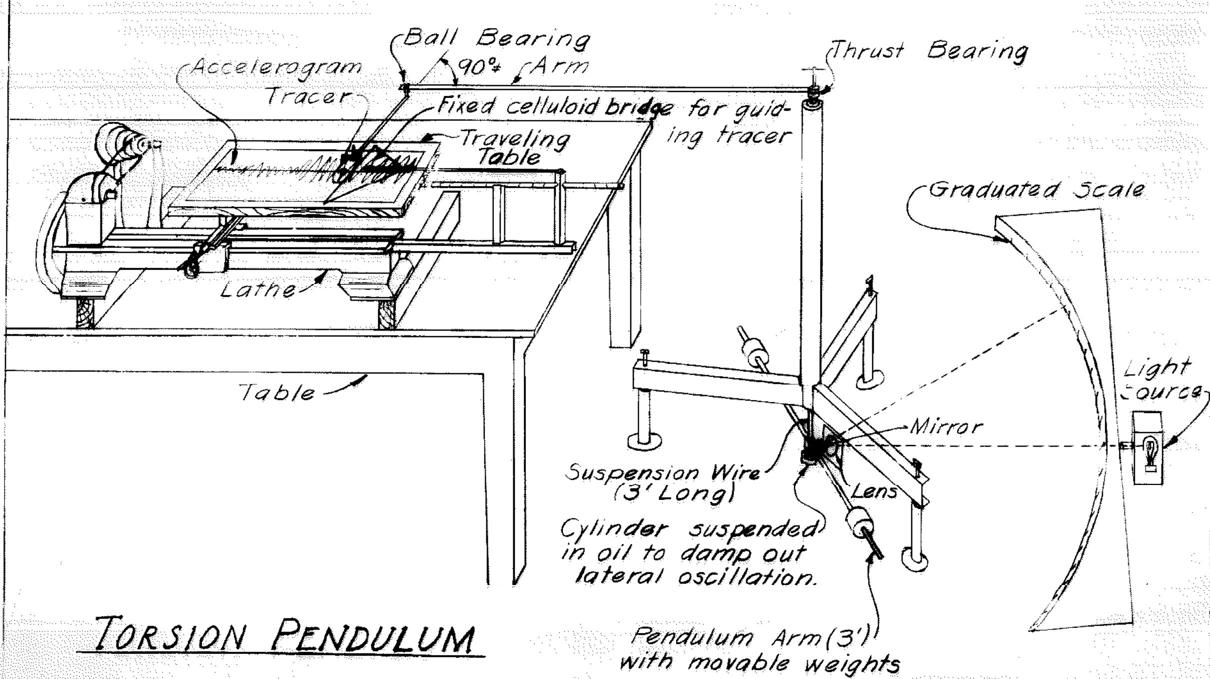
$$S_S = 1.27\sqrt{Km}\; V$$

$$S_B = \frac{5.7}{H}\sqrt{EIm}\; V = 2.55\sqrt{Km}\; V$$

For this case, then the ratio of the two shears is:

$$\frac{S_B}{S_S} = 2$$

So that if periods, mass and height are the same, the shear in a structure with bending deformations is twice that of a structure with shear deformations.



TORSION PENDULUM

CHAPTER II.

The factors C_n which have been investigated determine how the physical properties of the structure affect the shearing forces. To determine how the earthquake affects the shearing forces, it is necessary to investigate the values of the integral:

$$V = \int_0^{t_1} \ddot{u} \sin p(t_1 - t)\,dt \Big|_{max}$$

where $\ddot{u}$ is the ground acceleration. Since the maximum possible values of this integral are to be determined, this properly requires the introduction of a phase factor in the sine term. It is thus seen that the same maximum values are given for this integral if a cosine is substituted for the sine. Using this fact and the condition that $u = \dot{u} = \ddot{u} = 0$ at $t = 0$ successive integrations by parts give:

$$V = \frac{1}{p}\left(\ddot{u} - \int_0^{t_1} \dddot{u} \sin p(t_1 - t)\,dt\right)\Big|_{max}$$
$$= \int_0^{t_1} \ddot{u} \sin p(t_1 - t)\,dt \Big|_{max}$$
$$= p\int_0^{t_1} \dot{u} \sin p(t_1 - t)\,dt \Big|_{max}$$
$$= p\left(u - p\int_0^{t_1} u \sin p(t_1 - t)\,dt\right)\Big|_{max}$$

From the first of these expressions it is seen that as p becomes large the value of V approaches $\frac{\ddot{u}}{p}$. But since there are periods on the accelerograms as low as 1/10 second $\frac{\ddot{u}}{p}$ will be the approximate value only for very short periods. The last of the above expressions shows that as the period becomes large the value of V approaches pu. These considerations establish the values of V

for extreme values of p and for intermediate values, one of the above expressions for V must be evaluated from earthquake records.

In practise these evaluations have been made either mechanically or graphically. The results of such evaluations are shown in the accompanying figures. Those marked "Stanford" were evaluated mechanically at Stanford University in the following manner. The acceleration record was integrated twice to give the ground displacements. A cam cut in the pattern of these displacements actuated a shaking table upon which a simple oscillator was placed. Since the displacement of such an oscillator is equal to $\left(u - p\int_0^{t_1} \ddot{u} \sin p(t_1-t)\,dt\right)$ the maximum recorded displacements multiplied by p give the required values of V. The figures marked "Biot" were evaluated by M. A. Biot at Columbia University using a torsion pendulum. If the point of suspension of a torsion pendulum is given angular displacements equal in numerical magnitude to the ground acceleration $\ddot{u}$, the resulting angular displacements of the pendulum are equal to $p\int_0^{t_1} \ddot{u} \sin p(t_1-t)\,dt$. The maximum values of the recorded angular displacements when multiplied by $\frac{1}{p}$ give the required values of V.

The remaining figures were evaluated at California Institute of Technology. Those marked "graphical" were done graphically. The others were evaluated by means of a torsion pendulum constructed at California Institute of Technology. A diagram of this instrument is shown in the accompanying figure. The torsion pendulum is the most rapid method and this instrument as constructed permits a skilled operator to duplicate results within a 5% variation.

Although the three methods of evaluating V should theoretically give the same results, each method has certain limitations that may affect the accuracy. The accuracy of the graphical method depends primarily upon the accuracy of the drawings made. The difficulty in making accurate drawings increases as the period decreases so that the results for short periods in general will be less reliable than for long periods. Similarly the use of the shaking table introduces inaccuracies not only in the integration of the acceleration record and construction of the cam, but also because the measurements made are of the quantities $\frac{V}{p}$. This means that for short periods the quantities measured are small compared to those measured for long periods. The reliability of the values of V will thus be less as the frequency increases. The method of the torsion pendulum is just the reverse of this, in that it measures the quantities pV. Since these quantities are very small for large periods, the reliability of the values of V decreases as the period increases.

It is thus seen that unless the operator of the particular method used is especially careful, the reliability of the results will not be uniform for all periods. This difficulty may be overcome when using the torsion pendulum. Since the displacement of the torsion pendulum is:

$$p = \int_0^{t_1} \ddot{u} \sin p(t_1 - t)\,dt$$

an integration by parts, gives:

$$p^2 \int_0^{t_1} \ddot{u} \cos p(t_1 - t)\,dt$$

If the velocity record is used instead of the acceleration record, the angular displacements of the pendulum are equal to:

$$\bar{V} = p \int_0^{t_1} \dot{u} \sin p(t_1 - t)\, dt$$

So the required values of V are equal to:

$$V = \frac{1}{p}(p\bar{V}_{max}) = \bar{V}_{max}$$

The use of the velocity record on the torsion pendulum will, therefore, give the values of V at an undistorted scale for all periods. The velocity record will thus be more satisfactory to use than the acceleration record.

When such diagrams are determined for various earthquakes, it is apparent that the method used is an important factor in interpreting the results. An inspection of the figures herein presented shows that the character of these diagrams differs according to the method used. The curves determined with the torsion pendulum at California Institute of Technology were done as carefully as possible. For each curve the values for over 100 different periods were determined. The accuracy of these results for short periods is entirely satisfactory but for periods larger than 2 seconds it was found that the curves could not be reliably determined. It has been concluded that for periods larger than 1-1/2 seconds the use of the acceleration record is no longer completely satisfactory; for these periods the velocity records should be used.

It is difficult to draw specific conclusions from these results. The diagrams indicate that an earthquake has an extremely erratic effect. A slight variation of period may mean several hundred percent variation in shear. Moreover, the magnitude of the shear forces computed from these results are much too large to be corroborated by observed earthquake damage. Since these large values are due to a pseudo-resonance, the dissipation of energy during vibration will have a marked effect. The energy may be dissipated either within the structure or into the ground. The external dissipation of the energy into the ground is investigated in Chapter IV. The investigation of the internal energy dissipation is difficult. For small displacements the dissipation is apparently small. Tests by the U. S. Coast and Geodetic Survey made on masonry structures indicate a logarithmic decrement of about 10%. This would not be a sufficient dissipation of energy to correlate computed forces with observed damage. For large displacements,such as would occur during an earthquake, the energy dissipation is undoubtedly larger but it is difficult to make any estimate of its magnitude because of the indeterminate manner of dissipation. It is of interest to note that certain steel mill-type structures have been damaged by earthquakes. This is a type of building in which energy dissipation is small even for large amplitudes. The damage to this type of structure tends to confirm the theoretical analysis which predicts that an undamped elastic structure will be subjected to large forces. In this connection it is suggested that when collecting data of earthquake damage, particular attention should be paid to structures with low damping capacity. The steel mill building and the monolithic concrete building are both structures that will have a low damping capacity until a structural failure occurs. The failures to be observed are deformations exceeding the elastic limit and the development of cracks in structural members. Both of these will cause a dissipation of strain energy. Such energy dissipation explains the observed behavior of elevated water tanks. It has been often found that the bracing rods of such structures have been broken so that the structure is at the point of collapse. The fact that such structures remained standing at the end of the earthquake has been attributed by some observers to a slight deficiency in the intensity of the earthquake or to a subsidence of ground motion immediately after such failures have occurred. But since such excessive deformations cause a dissipation of a large percentage of the energy after which the structure no longer behaves elastically, it is seen that the structure is in a favorable position to resist the remainder of the earthquake.

Before the results of this analysis can be applied to actual buildings, the effect of energy dissipation must be taken into account. Since the energy dissipation must reduce the computed forces by several hundred percent, it is evident that although the uncertainties have been removed from the earthquake, the uncertainties have in turn been placed upon the buildings.

Certain general conclusions can be drawn from these analyses of the earthquake records. The value of V/M approaches zero as the period approaches zero, and these values are in direct proportion to the period for sufficiently small periods. Any structure whose period falls in this region will be subjected to shearing forces equal in magnitude to the total mass of the structure multiplied by the maximum ground acceleration. For undamped elastic systems this will apply only when the period is very short, less than about 0.05 seconds. When damping is introduced this region may be extended to include longer periods. For structures with long periods the conclusions drawn are quite different. Undamped structures with periods from 2/10 to 2 seconds are apparently affected equally by the earthquake and the maximum shear depends primarily on the factor $\sqrt{Km}$ or $\sqrt{EIm}$. Since it is reasonable to assume that tall structures will all have internal energy dissipation of the same character, it may be concluded that even when the effects of such energy dissipation are taken into account, the shearing force will depend only on $\sqrt{Km}$ or $\sqrt{EIm}$. Except for local variations in the diagrams of V/n, these shearing forces will be practically independent of the height and period of vibration.

Certain other conclusions can be drawn from the analysis of the earthquake records. The irregular character of the diagrams of clearly indicates resonance with the ground waves. The fact that these resonance peaks are so nearly of the same general character and magnitude throughout the range from 2/10 to 2 seconds period is of particular interest. This indicates that the seismographs are functioning satisfactorily and are not influencing the record by virtue of any natural periods of the instruments. These diagrams also show the random character of the ground waves in that there are no predominant periods. These diagrams show clearly that there is no so-called "dominant ground period". These diagrams also show no characteristics that can be attributed to the physical properties of the ground in the region where the earthquake occurred. Considering the marked difference in character of the soil in El Centro, Los Angeles, and Helena, the absence of any peculiarities that could be attributed to the elastic properties of the soil is significant.

It is of interest to note that these diagrams show no marked evidence of having been influenced by the natural periods of the building in which the seismograph was housed. However, it can be shown that elastic yielding of the ground of the magnitude that could reasonably be expected would have little influence on the diagrams of

. If the amplitude of the ground yielding is A inches and the period is T seconds, the acceleration in feet per second per second is:

$$\ddot{u} = \frac{2\pi A}{12T} \sin \frac{2\pi}{T} t$$

The value of V for the condition of resonance is after n cycles equal to:

$$V = n\frac{\pi A}{12}$$

If A = 1/10 inch, T = 1 second and the motion lasts for 40 seconds, V Will equal 1 foot per second. A coupling effect of this magnitude could not be distinguished on the diagrams, since their irregularities are larger than this. It can be concluded from this that there can be considerable base motion due to coupling without noticeably affecting the diagrams of V. So, in general, it will not be possible to tell from an analysis of the earthquake records whether there has been any coupling.

These diagrams of V furnish a convenient scale of intensities of earthquakes. If the average ordinate on such a diagram is called the intensity, each earthquake may be accurately classified with respect to other earthquakes. The following table lists the intensities of the earthquakes where the El Centro earthquake of May, 1940, is taken as 100:

Vernon, March, 1933	65
El Centro, 1940	100
El Centro, 1934	60
Helena, October, 1935	55
Ferndale, September, 1938	40
Subway Terminal, October, 1933	17

This table shows that for engineering purposes neither the Mercalli scale nor the maximum ground acceleration is a satisfactory method of measuring the intensities of earthquakes.

The above conclusions can be interpreted in terms of the method of design of buildings. The maximum shearing force at the base of an undamped structure with shear deformations, in any one mode of vibration, is:

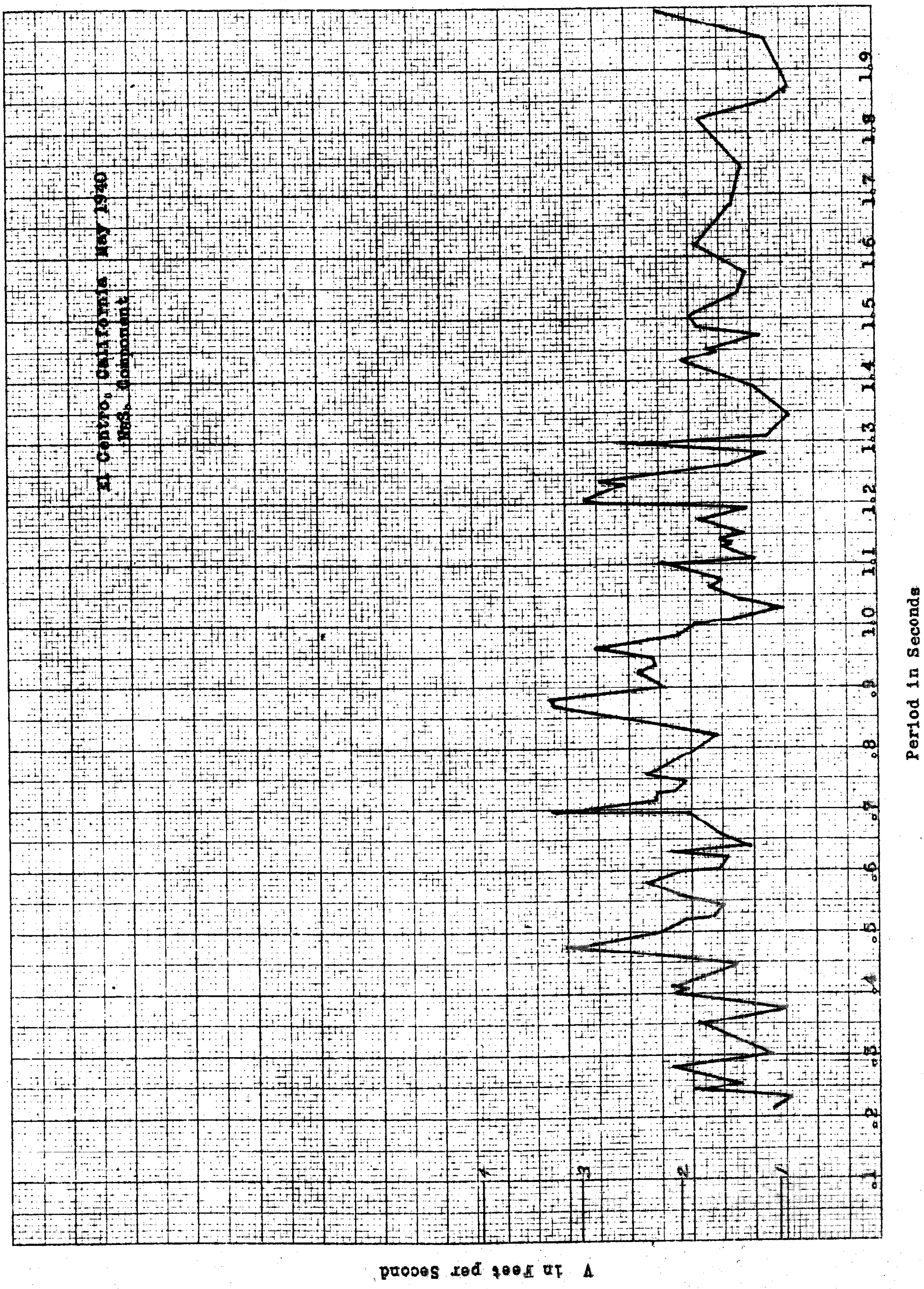
$$S_n = D_n \sqrt{Km}\ V_n \tag{19}$$

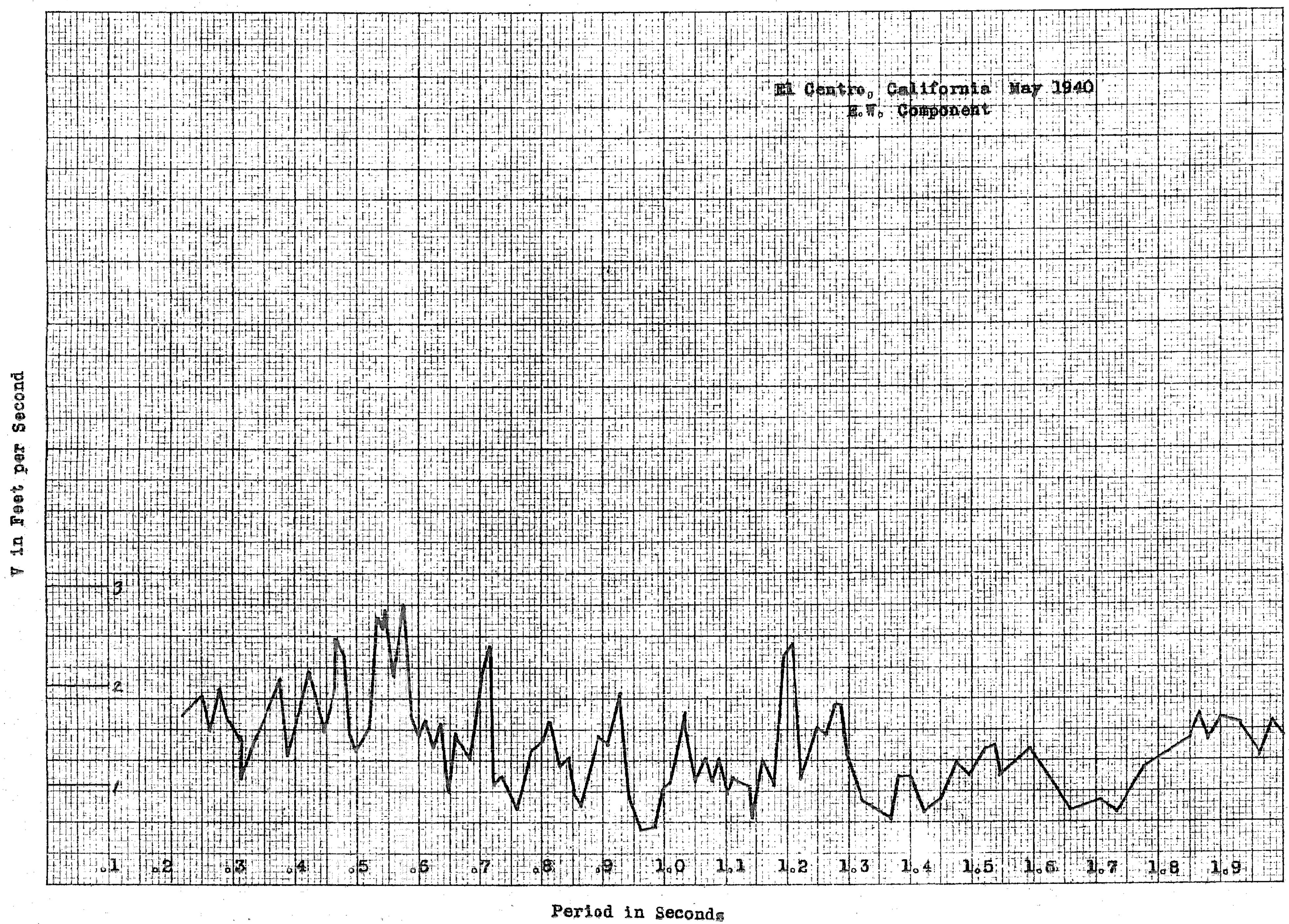
The maximum shear for a structure with bending deformations, is, for each mode:

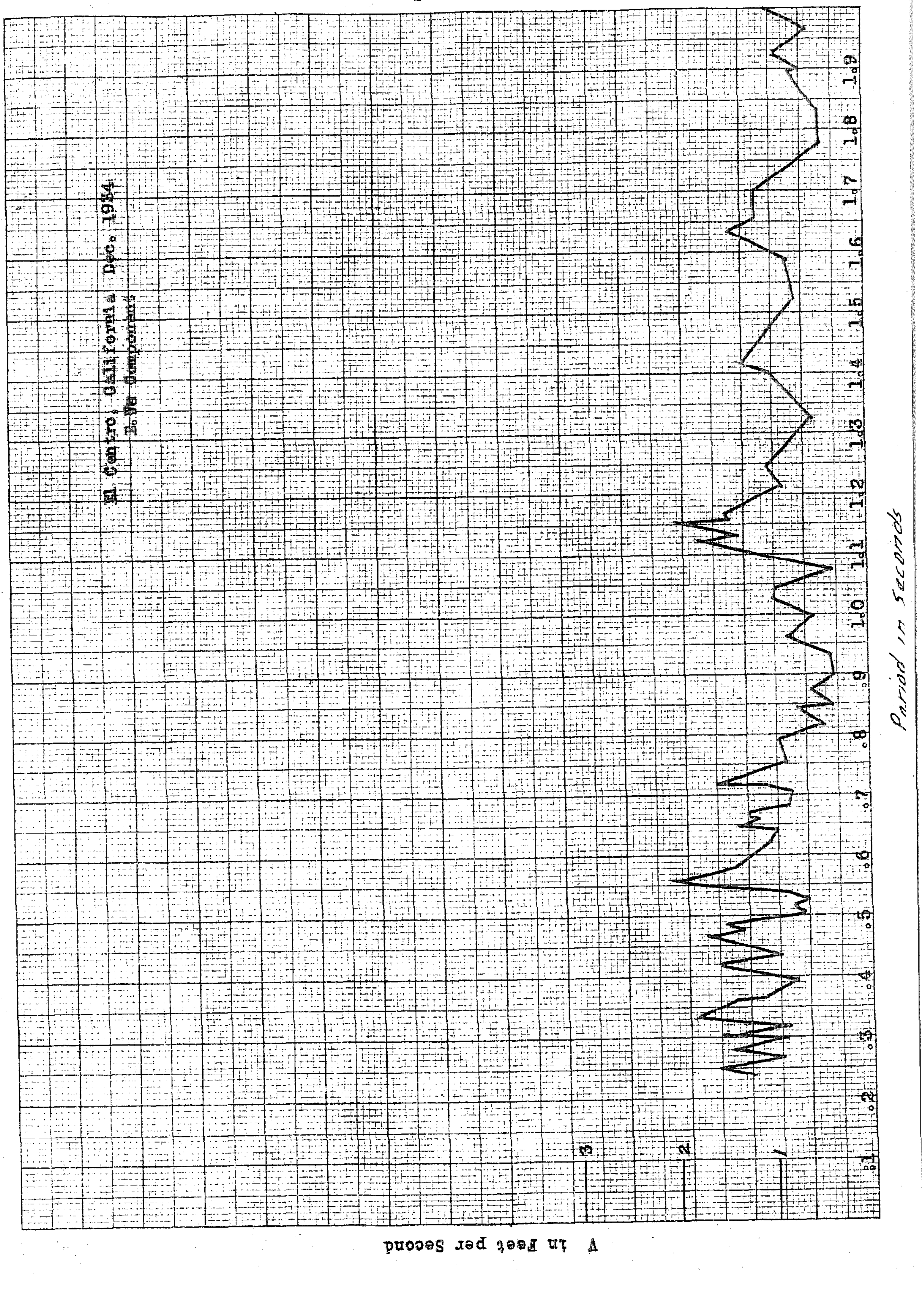
$$S_n = \frac{D_n'}{H} \sqrt{EIm}\ V_n \tag{20}$$

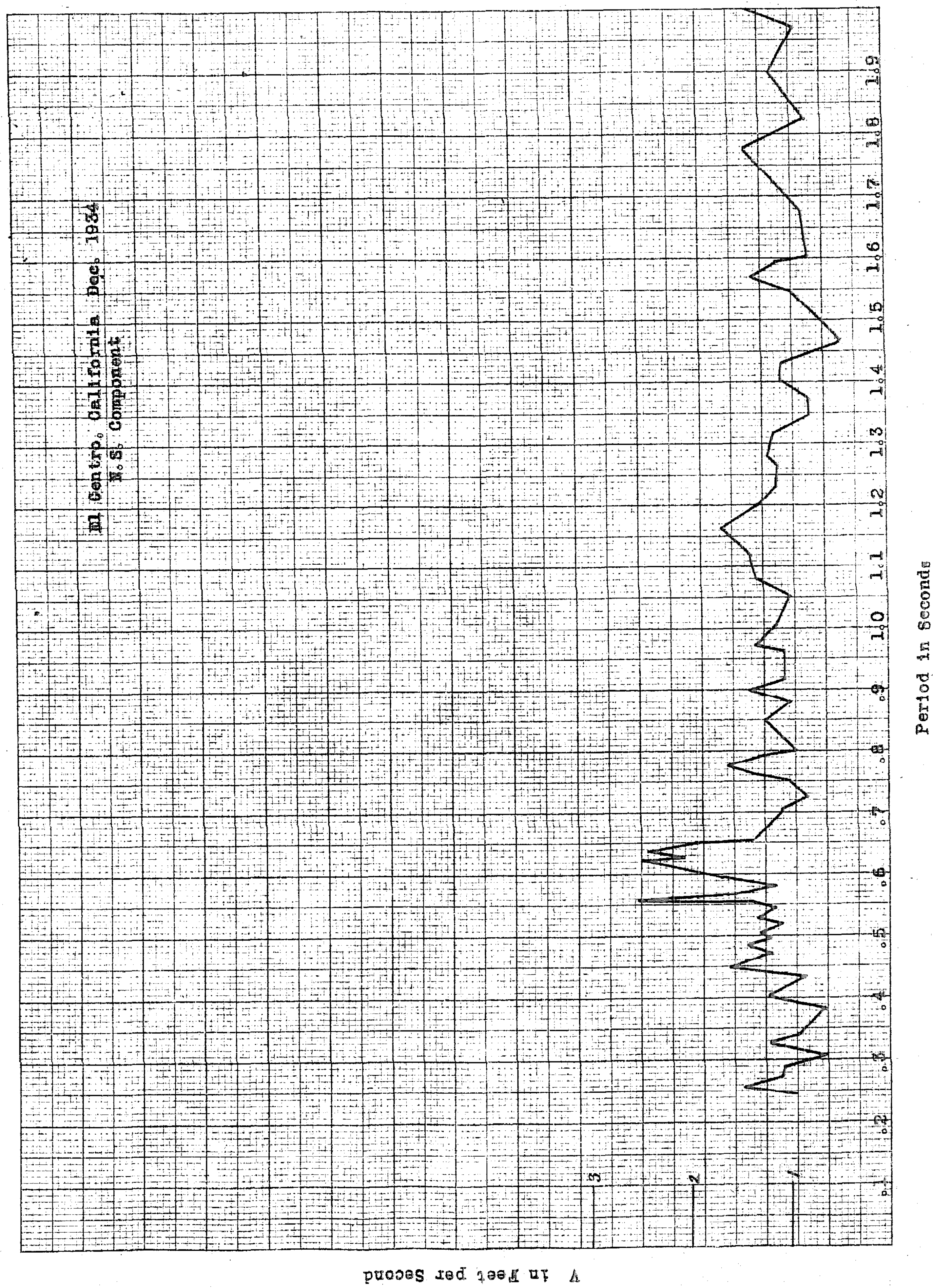
In each of these cases the so-called "percent g" method of design is the proper one to use for structures with very short periods. For tall structures, with relatively long periods, it appears from the earthquakes analysed that for design purposes the factor $1/n$ may be taken to be a constant. For such structures the maximum shearing force is independent of the period. The shearing force depends primarily upon the square root of the stiffness times the mass per unit height. Tests made on existing buildings by the U. S. Coast and Geodetic Survey show that in general the period of vibration depends only on the height. From this it can be inferred that the stiffness of ordinary buildings varies directly as the mass per unit height, and, therefore, the factor $\sqrt{km}$ is proportional to m. Since the mass per unit height is proportional to the total mass per floor, it follows that the maximum shear varies in direct proportion to the mass per floor.

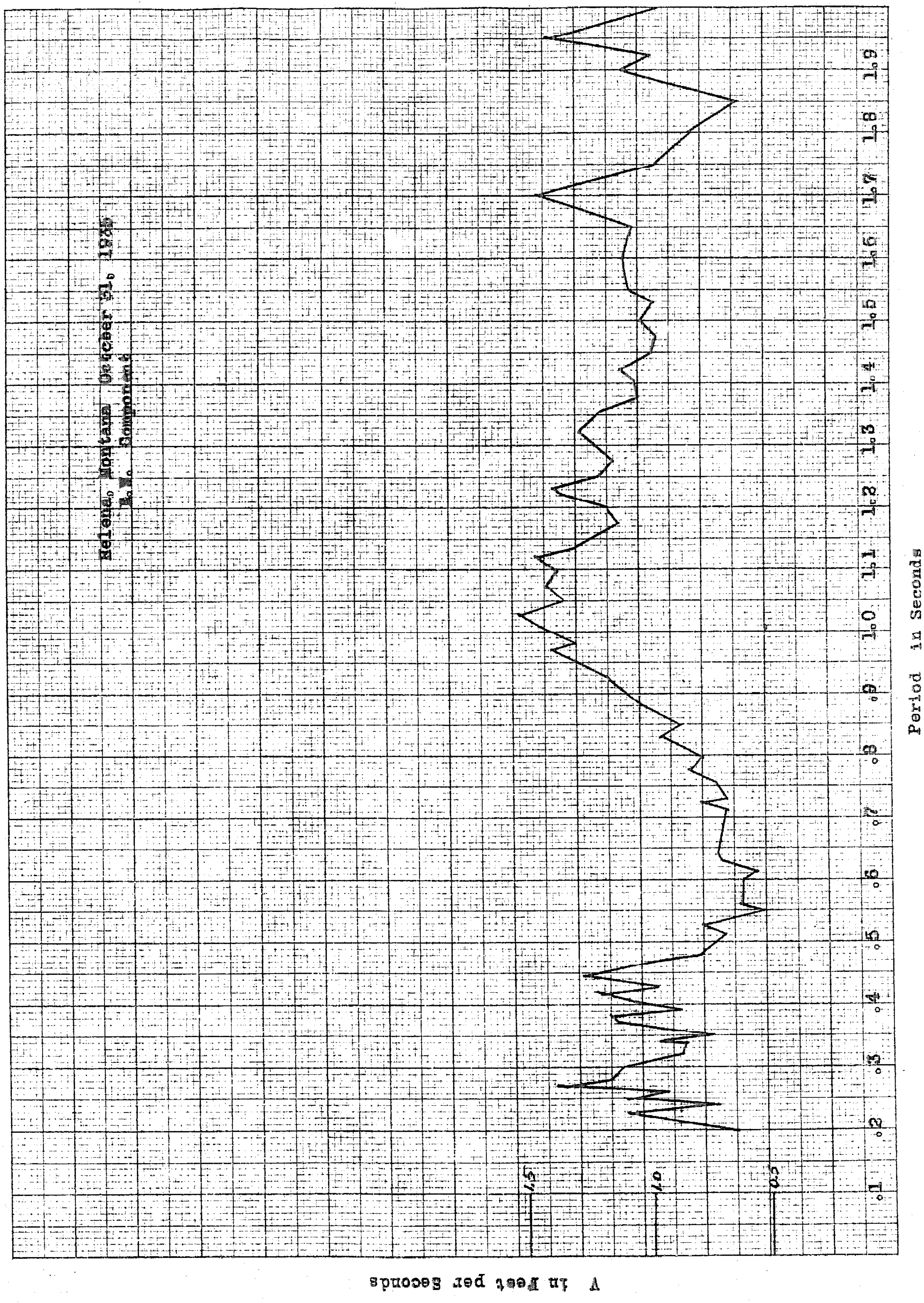
The shearing force in tall structures of the type of Equation (19) is independent of the height. For structures of the type of Equation (20), the shearing force is decreased by an increase of height. For both types of structures as the height is increased, higher modes of vibration become relatively more important. This is especially true for structures of the type of Equation (20). Therefore, as the height of a structure is increased, the strength of the upper portions should be increased.

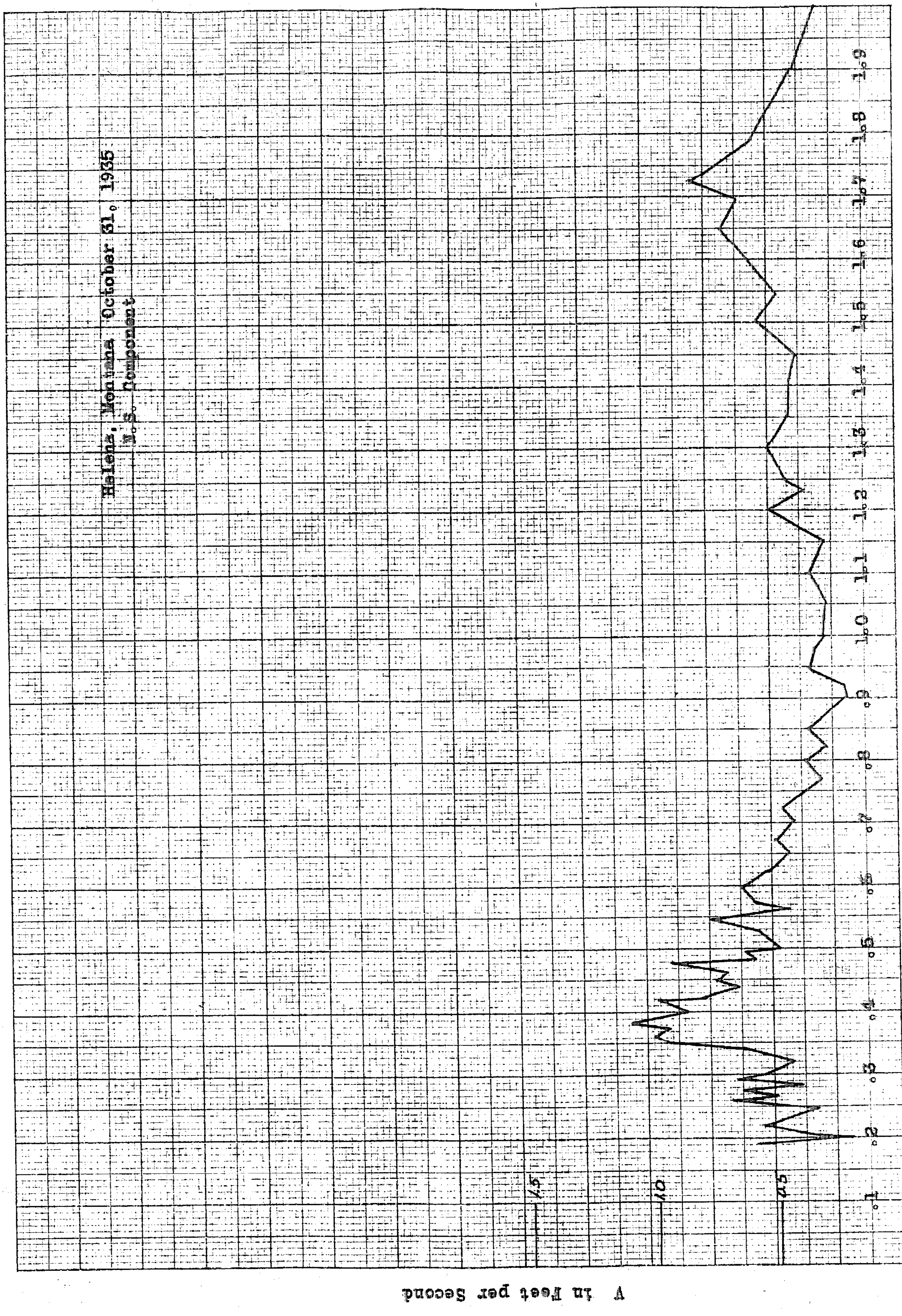

El Centro, California May 1940
N-S. Component
Period in Seconds
V in Feet per Second
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
1
2
3
4

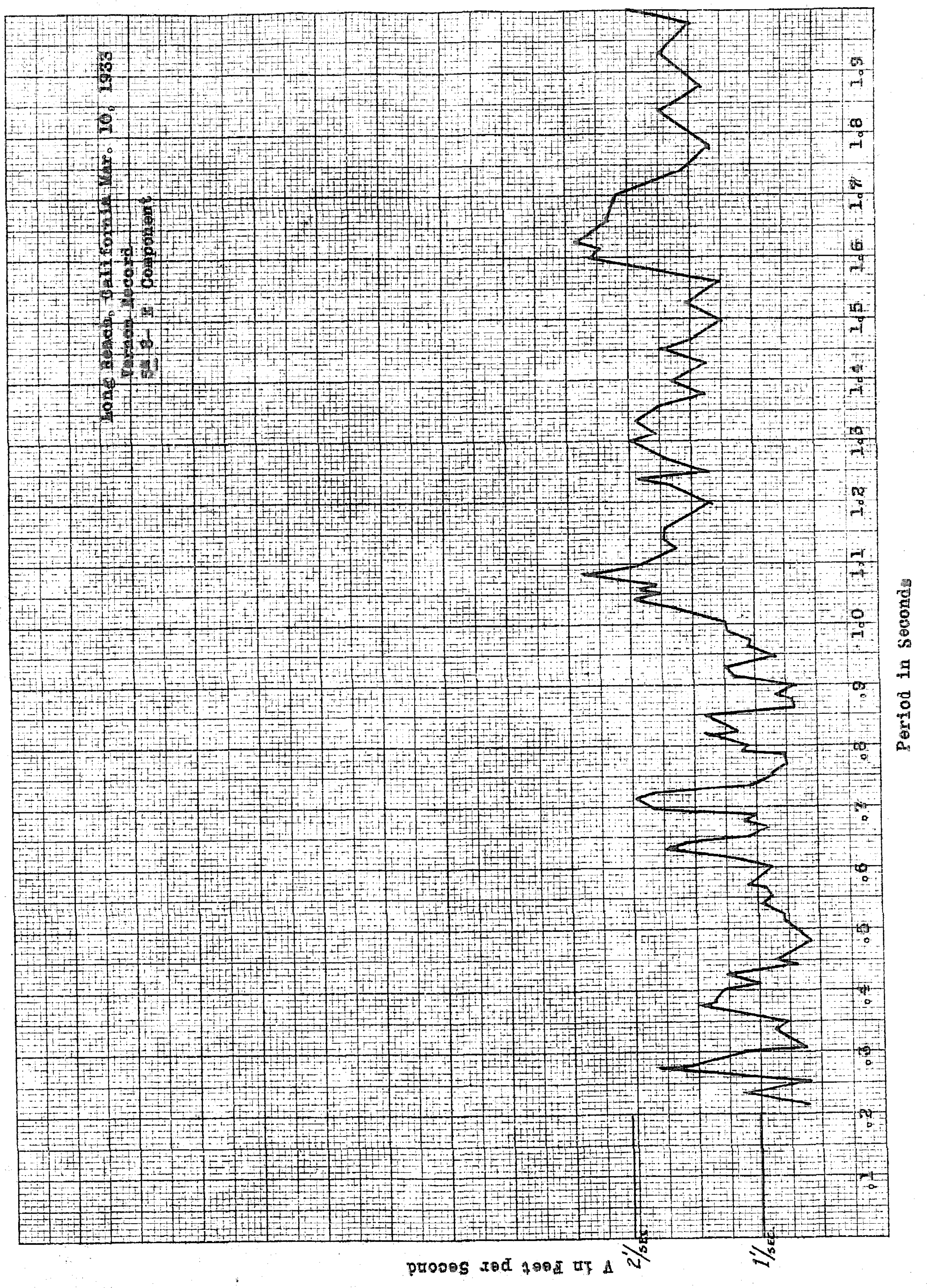

El Centro, California May 1940
E.W. Component
V in Feet per Second
3
2
1
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
Period in Seconds

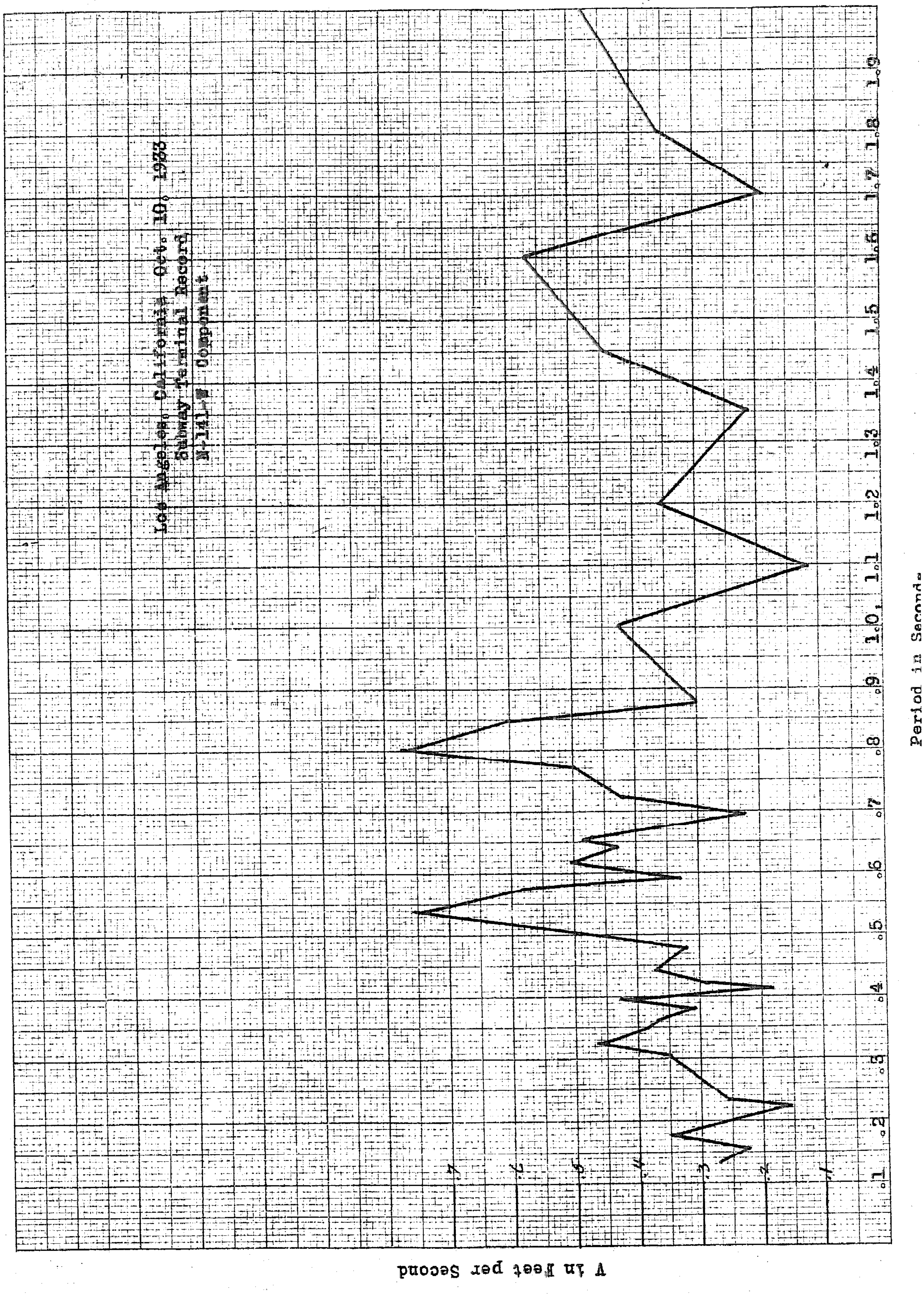

El Centro, California Dec. 1934
E. W. Component
Period in Seconds
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9
1 2 3
V in Feet per Second

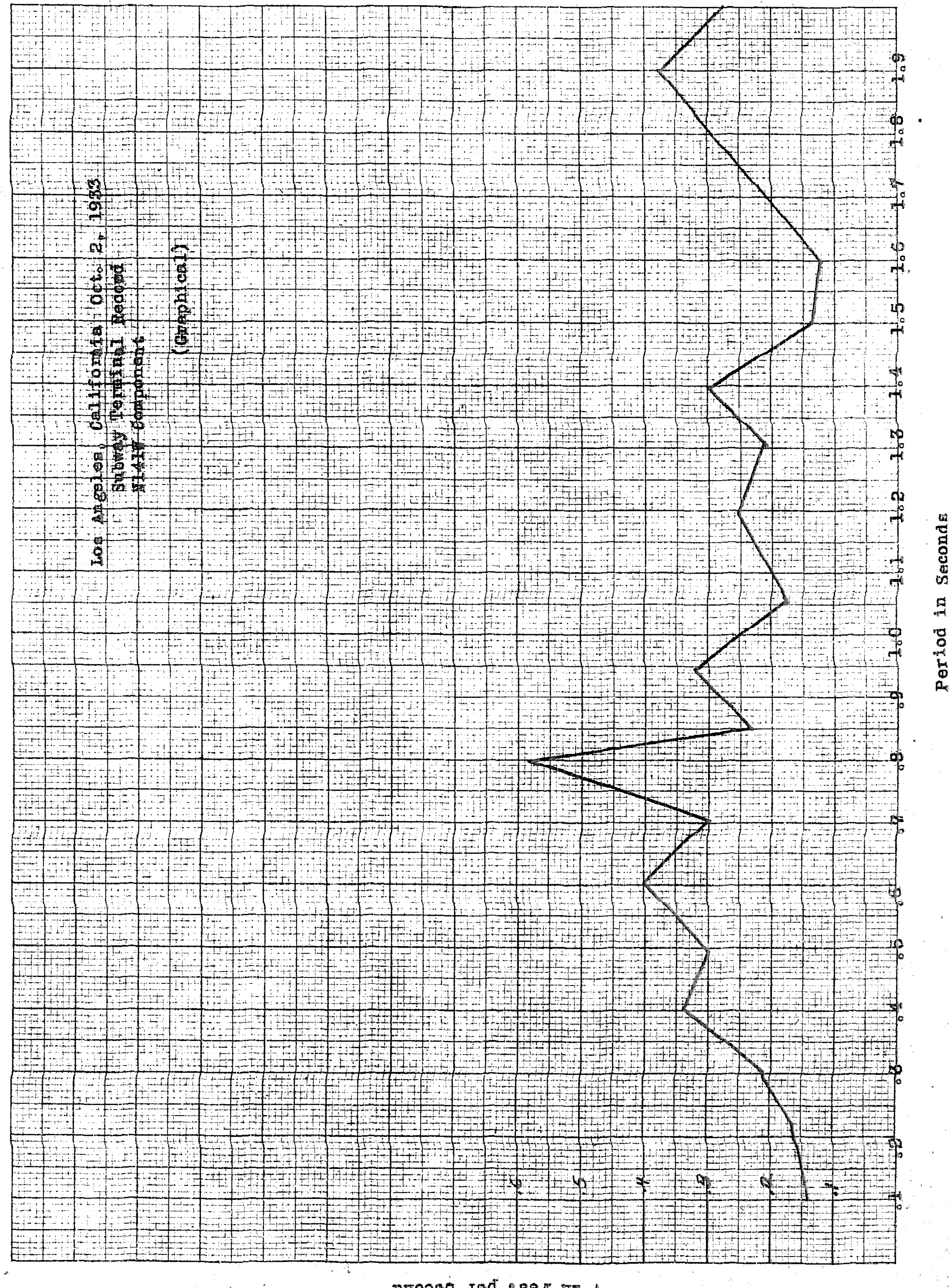

El Centro, California Dec. 1934
N.S. Component
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
1
2
3
Period in Seconds
V in Feet per Second

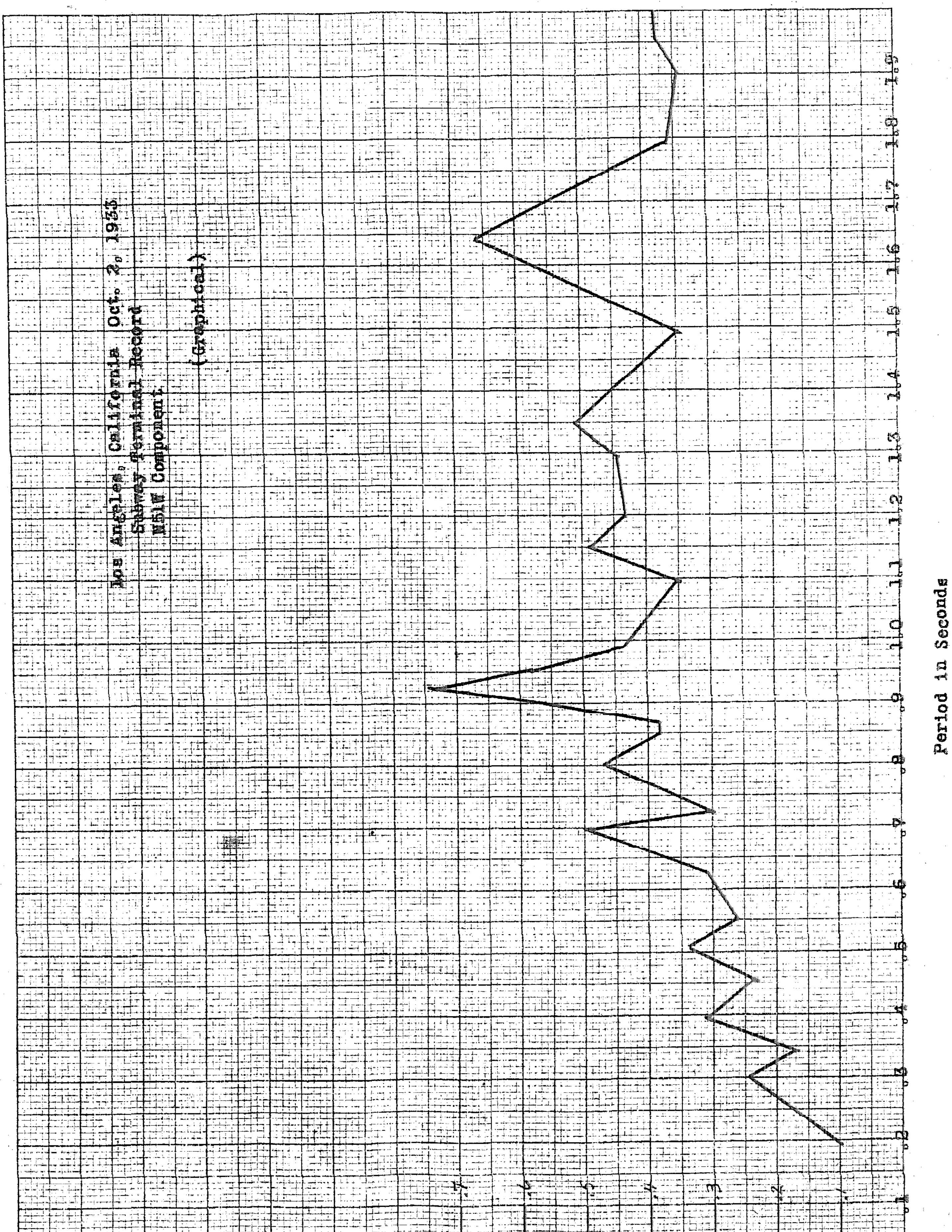

Helena, Montana October 31, 1935
E. W. Component
Period in Seconds
V in Feet per Seconds
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
0.5
1.0
1.5

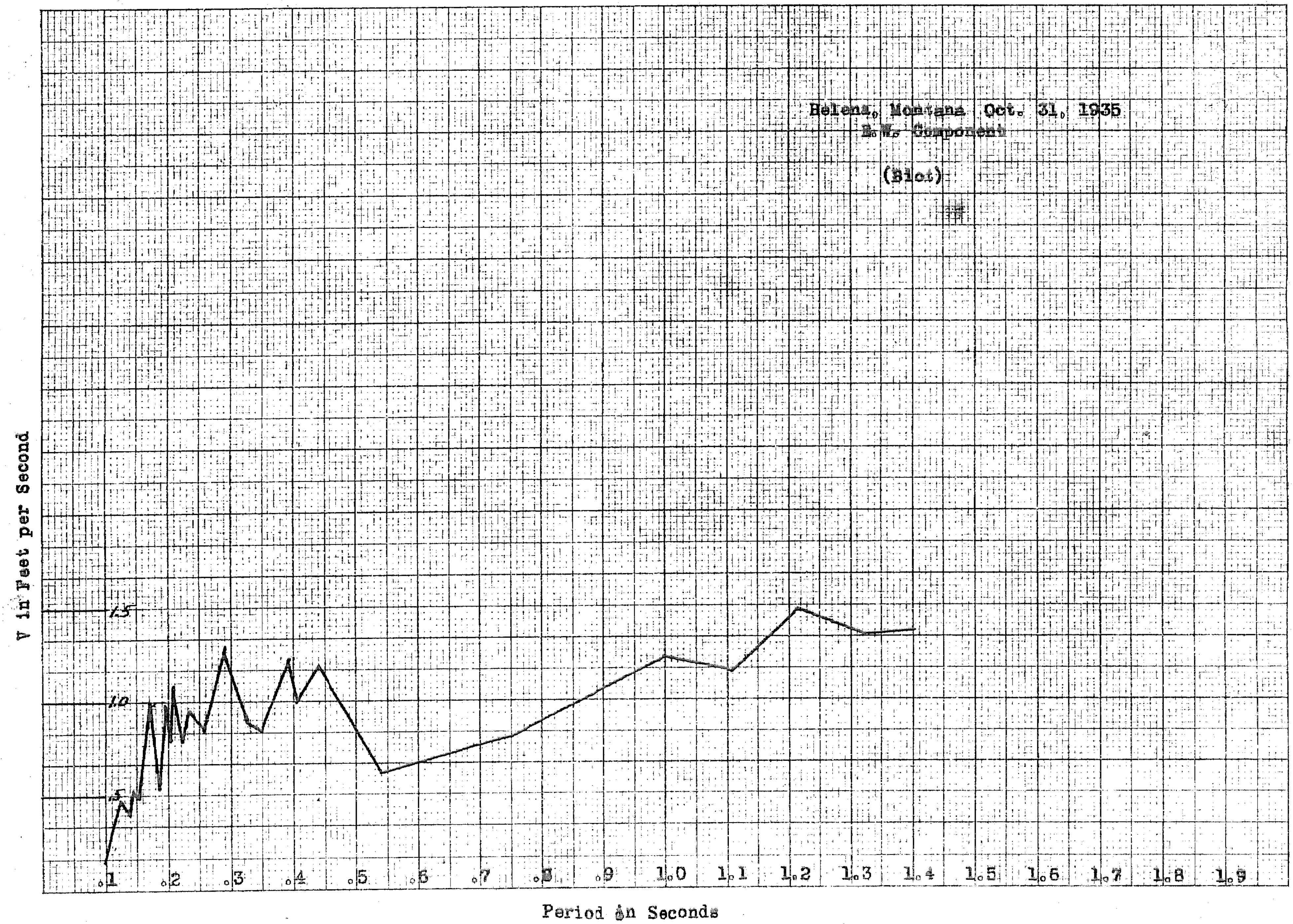

Helena, Montana October 31, 1935
N.S. Component
Period in Seconds
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
0.5
1.0
1.5
V in Feet per Second

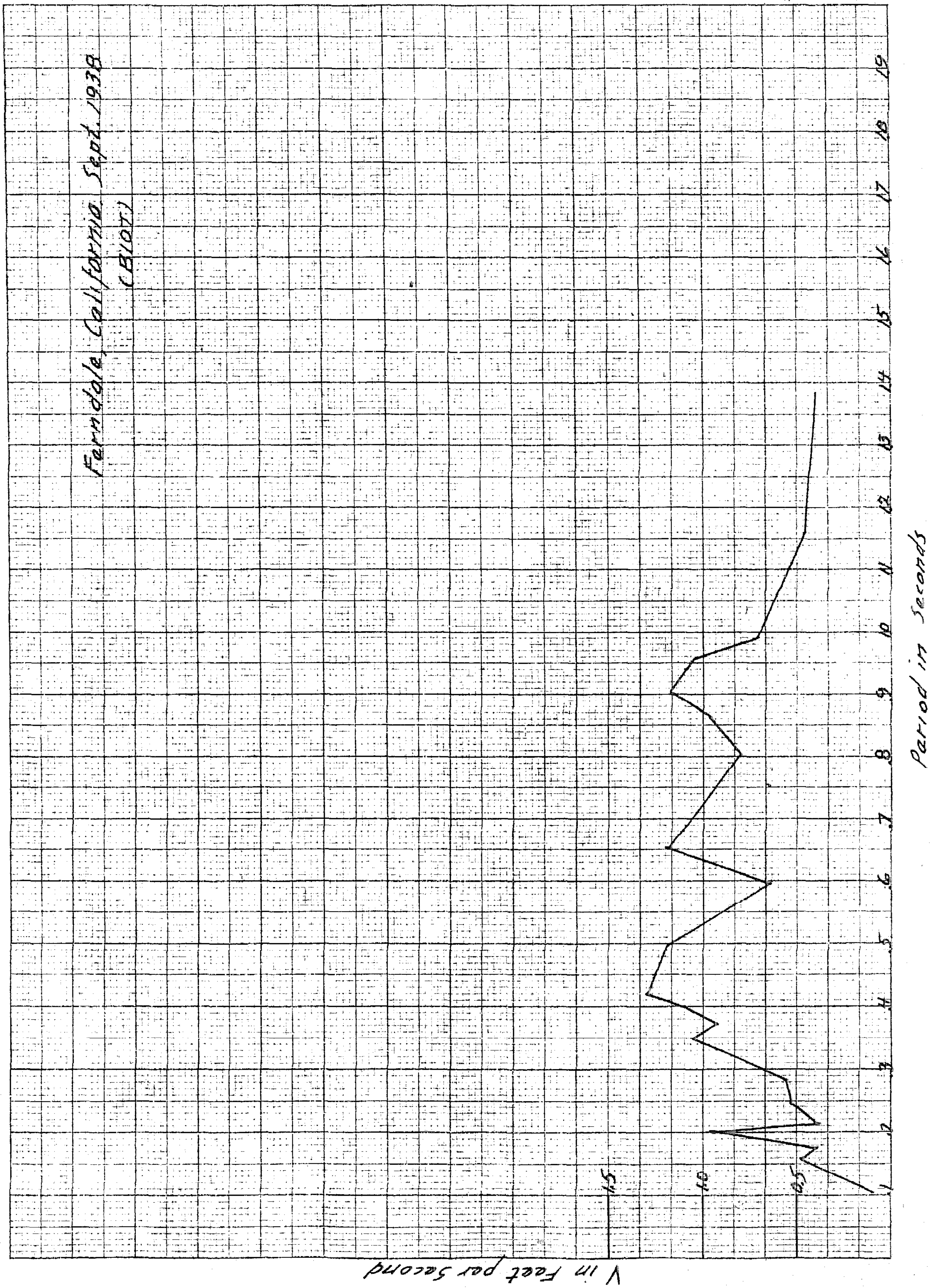

Long Beach, California Mar. 10, 1933
Vernon Record
S. 8 E. Component
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
Period in Seconds
2'/sec.
1'/sec.
V in Feet per Second

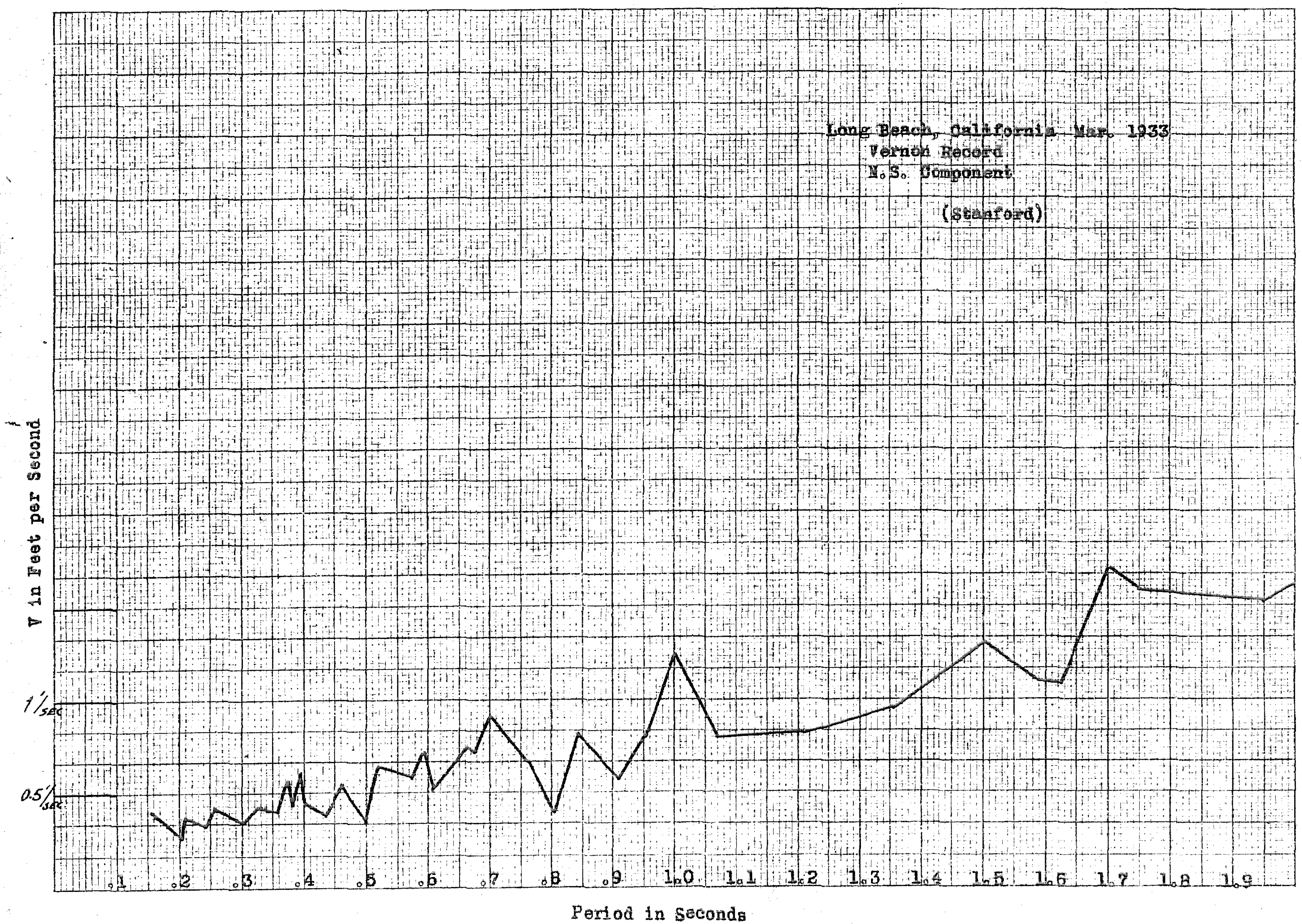

Los Angeles, California Oct. 10, 1933
Subway Terminal Record
N-141-W Component
Period in Seconds
V in Feet per Second
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9
1 2 3 4 5 6 7

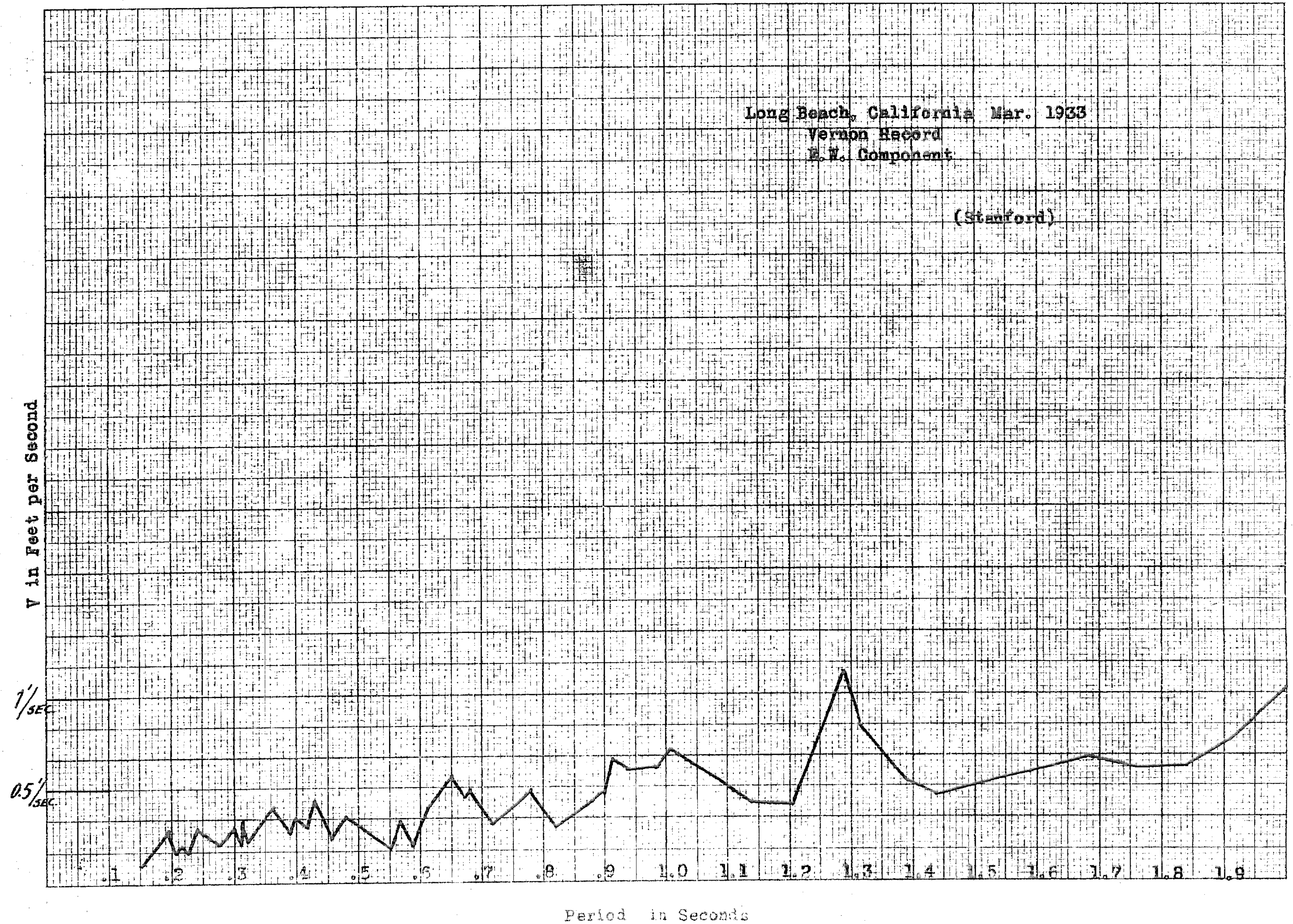

Los Angeles, California Oct. 2, 1933
Subway Terminal Record
N14°W Component
(Graphical)
Period in Seconds
V in Feet per Second
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9
.1 .2 .3 .4 .5 .6


Los Angeles, California Oct. 2, 1933
Subway Terminal Record
N51W Component
(Graphical)
Period in Seconds
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
V in Feet per Second


Helena, Montana Oct. 31, 1935
E.W. Component
(Blot)
V in Feet per Second
1.5
1.0
.5
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
Period in Seconds


Ferndale, California Sept. 1938
(B107)
Period in Seconds
V in Feet per Second
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
0.5
1.0
1.5


Long Beach, California Mar. 1933
Vernon Record
N.S. Component
(Stanford)
V in Feet per Second
1'/sec
0.5'/sec
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
Period in Seconds


Long Beach, California Mar. 1933
Vernon Record
E.W. Component
(Stanford)
V in Feet per Second
1'/sec
0.5'/sec
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
Period in Seconds

CHAPTER III.

Derivation of Equations of Motion for Structure with Shear Deformations

The following notation is used:-

x = coordinate measured along height from top downward.

y = lateral displacement.

t = time.

K = modulus of rigidity of superstructure.

k = modulus of rigidity of first story.

m = mass per unit height (constant).

H = height from second story to roof.

h = height of first story.

λ = root of frequency equation.

The fundamental equation of motion for a structure with shear deformation is:-

$$m \frac{\partial^2 y}{\partial t^2} = K \frac{\partial^2 y}{\partial x^2}$$

This expresses the fact that at each point the rate of change of slope is proportional to the inertia force. Substituting $y = XT$, where X is a function of x only and T a function of t only, reduces the above equation to:-

$$\frac{m}{K} \frac{1}{T} \frac{d^2T}{dt^2} = \frac{d^2X}{dx^2} \frac{1}{X}$$

Since the left side of this equation is independent of x and the right side is independent of t, the equation can hold true only if each side is constant. This equation reduces then to the two equations:-

$$\frac{d^2T}{dt^2} = -\frac{K}{m} p^2 T$$

and

$$\frac{d^2X}{dx^2} = -p^2 X$$

where $-p^2$ is an undetermined constant. The solutions of these equations are:-

$$T = A_1 \sin p\sqrt{\tfrac{K}{m}}\, t + A_2 \cos p\sqrt{\tfrac{K}{m}}\, t$$
$$X = B_1 \sin px + B_2 \cos px$$

The shear at the top of the structure must be zero and at the base it is equal to $\frac{k}{h}$ times the displacement of the second floor. Also only those solutions are desired that will give zero displacement at zero time. Therefore, the following conditions must be satisfied:-

$$\frac{dX}{dx} = 0 \quad at \quad x = 0$$
$$K\frac{dX}{dx} + \frac{k}{h} X = 0 \quad at \; x = H$$
$$T = 0 \quad at \quad t = 0$$

The first and third of these conditions requires the arbitrary constants A_2 and B_1 to be zero. The second condition requires:-

$$-Kp \sin pH + \frac{k}{h} \cos pH = 0$$

or setting $pH = \lambda$

$$\lambda \tan \lambda = \frac{kH}{Kh}$$

The roots of this equation determine the frequencies of vibration. Calling the nth root of the equation λ_n , the displacement of the structure is written:-

$$y = \sum_n C_n \cos\frac{\lambda_n}{H}x \sin\frac{\lambda_n}{H}\sqrt{\frac{K}{m}}\, t \tag{1a}$$

where the arbitrary constants C_n are determined from the initial velocity.

If the base of the structure is given a constant acceleration "a" for an infinitesimal interval of time Δt , this will give all points of the structure an initial relative velocity equal to $a\Delta t$. For this case the constants C_n must be determined to satisfy the equation:-

$$(a\Delta t) = \sum_n C_n \left(\cos\frac{\lambda_n}{H}x\right)\left(\frac{\lambda_n}{H}\sqrt{\frac{K}{m}}\right)$$

Since the terms $\cos\frac{\lambda_n}{H}x$ form a set of orthogonal functions*, the constants C_n may be evaluated by multiplying both sides of the equation by $\cos\frac{\lambda_m}{H}x$ and integrating. Since:-

$$\int_0^H \cos\frac{\lambda_n}{H}x \cos\lambda_m\frac{x}{H}\,dx = 0 \quad \text{if } m \neq n$$

$$C_n = \frac{a\Delta t}{\frac{\lambda_n}{H}\sqrt{\frac{K}{m}}}\;\frac{\int_0^H \cos\frac{\lambda_n}{H}x\,dx}{\int_0^H \cos^2\frac{\lambda_n}{H}x\,dx}$$

and

$$C_n = (a\Delta t)\,\frac{2\sin\lambda_n}{\lambda_n^2\left(1+\frac{\sin 2\lambda_n}{2\lambda_n}\right)}\,\frac{H}{\sqrt{\frac{K}{m}}}$$

The displacements of the structure are then:-

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*Webster, "Partial Differential Equations of Mathematical Physics".

(2a)

When the height of the flexible story is zero, this reduces to the case of a uniform structure anchored at its base. The value of λ_n then is $\frac{2n-1}{2}\pi$ and the displacements are:-

$$y = (a\Delta t)\frac{8H}{\pi^2\sqrt{\frac{K}{m}}}\sum_n \frac{1}{(2n-1)^2}\cos\frac{2n-1}{2H}\pi x \sin\frac{2n-1}{2H}\pi\sqrt{\frac{K}{m}}\,t$$

The shearing force at the base of the structure is given by:-

$$S = \int_0^H m\frac{\partial^2 y}{\partial t^2}dx$$

or

$$S = m(a\Delta t)\sum_n \frac{\lambda_n}{H}\sqrt{\frac{K}{m}}\;\frac{\left(\int_0^H \cos\lambda_n\frac{x}{H}dx\right)^2}{\int_0^H \cos^2\lambda_n\frac{x}{H}dx}\sin\frac{\lambda_n}{H}\sqrt{\frac{K}{m}}\,t$$

which gives:-

$$S = (a\Delta t)\sum_n \frac{2\sin^2\lambda_n}{\lambda_n\left(1+\frac{\sin 2\lambda_n}{2\lambda_n}\right)}\sqrt{Km}\;\sin\frac{\lambda_n}{H}\sqrt{\frac{K}{m}}\,t \qquad (3a)$$

Equations (2a) and (3a) give the response of the structure to a single sharp shock. If the base of the structure is given an arbitrary acceleration $\ddot{u}$, the response of the structure is given by applying "Rayleigh's principle" *. The displacement and shear are then:-

$$y = \sum_n \frac{2\sin\lambda_n}{\lambda_n^2\left(1+\frac{\sin 2\lambda_n}{2\lambda_n}\right)}\frac{H}{\sqrt{\frac{K}{m}}}\cos\frac{\lambda_n}{H}x\int_0^{t_1}\ddot{u}\sin\frac{\lambda_n}{H}\sqrt{\frac{K}{m}}\,t\,dt \qquad (4a)$$

$$S = \sum_n \frac{2\sin^2\lambda_n}{\lambda_n\left(1+\frac{\sin 2\lambda_n}{2\lambda_n}\right)}\sqrt{Km}\int_0^{t_1}\ddot{u}\sin\frac{\lambda_n}{H}\sqrt{\frac{K}{m}}\,t\,dt \qquad (5a)$$

*Rayleigh, "Theory of Sound", Vol. 1, page 74.

Derivation of the Equation of Motion for a Structure with Bending Deformations

The following notation is used:-

x = coordinate measured along height.

y = lateral displacement.

H = height.

E = Young's modulus.

I = moment of inertia.

t = time.

β_n = nth root of frequency equation.

m = mass per unit height (constant).

The fundamental equation of motion is:-

$$EI \frac{\partial^4 y}{\partial x^4} = -m \frac{\partial^2 y}{\partial t^2}$$

This equation states that the structure is deformed as if acted upon by a load equal to the inertia forces. Substituting in this equation $y = XT$, where X is independent of t and T is independent of x, gives:-

$$\frac{EI}{m} \frac{1}{X} \frac{d^4 X}{dx^4} = -\frac{1}{T} \frac{d^2 T}{dt^2}$$

Since the left side of this equation is independent of t and the right side is independent of x, it can be satisfied only if each side is equal to a constant. This reduces then to the two equations:-

$$\frac{d^2 T}{dt^2} = -p^2 T$$

$$\frac{d^4X}{dx^4} = \frac{mp^2}{EI}X$$

where p^2 is an undetermined constant. The solution of the first equation is:-

$$T = A_1 \sin pt + A_2 \cos pt$$

The solution of the second equation is:-

$$X = B_1 \cos kx + B_2 \sin kx + B_3 \cosh kx + B_4 \sinh kx$$

where
$$k = \sqrt[4]{\frac{mp^2}{EI}}$$

For determining the arbitrary constants the following conditions are available. The shear and bending moment are zero at the top and the displacement and slope are zero at the bottom. Also only such solutions are desired that will give zero displacement when $t=0$. Therefore:-

$$\frac{d^3X}{dx^3} = 0; \quad \frac{d^2X}{dx^2} = 0 \qquad at\ x = H$$

$$\frac{dX}{dx} = 0; \quad X = 0 \qquad at\ x = 0$$

$$T = 0 \qquad at\ t = 0$$

The last of these requires $A_2 = 0$. The second set of conditions requires $B_1 = -B_3$ and $B_2 = -B_4$. Therefore:-

$$X = B_1(\cos kx - \cosh kx) + B_2(\sin kx - \sinh kx) \tag{0a}$$

Substituting this in the first set of conditions, gives the equations:-

$$B_1 k^2(-\cos kH - \cosh kH) + B_2 k^2(-\sin kH - \sinh kH) = 0$$

$$B_1 K^3(\sin KH - \sinh KH) + B_2 K^3(-\cos KH - \cosh KH) = 0$$

Solving each of these for B_2 gives:-

$$B_2 = -B_1 \frac{\cos KH + \cosh KH}{\sin KH + \sinh KH}$$

$$B_2 = B_1 \frac{\sin KH - \sinh KH}{\cos KH + \cosh KH}$$

The constant K must be so determined that both of these give the same value of B_2. This requires that:-

$$\frac{\sin KH + \sinh KH}{\cos KH + \cosh KH} = -\frac{\cos KH + \cosh KH}{\sin KH - \sinh KH}$$

Clearing of fractions and using the relation $\cosh^2 KH - \sinh^2 KH = +1$ results in the equation:-

$$\cos\beta \cosh\beta = -1$$

where $\beta = KH$. This is the frequency equation whose roots determine the proper values of β. These values of β are listed in the following table:-

β_1	β_2	β_3	for $n > 3$
1.875	4.694	7.855	$\beta \cong \frac{2n-1}{2}\pi$

Since $K = \sqrt[4]{\frac{p^2 m}{EI}}$ these values of β determine the frequencies of vibration of the normal modes by determining p_n. i.e.:-

$$p_n = K_n^2 \sqrt{\frac{EI}{m}} = \beta_n^2 \sqrt{\frac{EI}{mH^4}}$$

The displacements of the structure are:-

$$y = \sum_n X_n T_n$$

or

$$y = \sum_n C_n \left[\left(\cosh\beta_n \frac{x}{H} - \cos\beta_n \frac{x}{H} \right) - \frac{\cosh\beta_n + \cos\beta_n}{\sinh\beta_n + \sin\beta_n} \left(\sinh\beta_n \frac{x}{H} - \sin\beta_n \frac{x}{H} \right) \right] \sin p_n t \tag{6a}$$

For convenience, this equation may be written:-

$$y = \sum_n C_n W_n \sin p_n t \tag{7a}$$

where the constants C_n are to be determined from the initial velocity. If the base of the structure is given a constant acceleration "a" for an infinitesimal interval of time Δt, each point of the structure will be given an initial relative velocity equal to $a\Delta t$. The values of the constants C_n must be determined to satisfy the equation:-

$$(a\Delta t) = \sum_n C_n W_n p_n$$

It can be shown that the functions W_n form a set of orthogonal functions so that:- *

$$\int_0^H W_n W_m \, dx = 0 \quad \text{if } n \neq m$$

Multiplying both sides of the above equation by W_n and integrating gives:-

$$C_n = (a\Delta t) \frac{\int_0^H W_n \, dx}{\int_0^H W_n^2 \, dx} \frac{1}{p_n}$$

---

* Webster, "Partial Differential Equations of Mathematical Physics".

The displacements of the structure thus are:-

$$y = (a\Delta t)\sum_n \frac{\int_0^H W_n\,dx}{\int_0^H W_n^2\,dx} W_n \frac{1}{p_n} \sin p_n t \tag{8a}$$

The shear at the base is:-

$$S = \int_0^H m \frac{\partial^2 y}{\partial t^2}\,dx$$

or

$$S = m(a\Delta t)\sum_n \frac{\left(\int_0^H W_n\,dx\right)^2}{\int_0^H W_n^2\,dx} p_n \sin p_n t \tag{9a}$$

Applying "Rayleigh's principle", gives for the response of the structure to an arbitrary ground acceleration $\ddot{u}$:-

$$y = \sum_n \frac{\int_0^H W_n\,dx}{\int_0^H W_n^2\,dx} W_n \frac{1}{p_n} \int_0^{t_1} \ddot{u} \sin p_n (t_1 - t)\,dt \tag{10a}$$

$$S = m\sum_n \frac{\left(\int_0^H W_n\,dx\right)^2}{\int_0^H W_n^2\,dx} p_n \int_0^{t_1} \ddot{u} \sin p_n (t_1 - t)\,dt \tag{11a}$$

To complete the solution, it is necessary to evaluate the integrals:-

$$\int_0^H W_n\,dx$$

$$\int_0^H W_n^2\,dx$$

where

$$W_n = \left(\cosh\beta_n \frac{x}{H} - \cos\beta_n \frac{x}{H}\right) - \frac{\cosh\beta_n + \cos\beta_n}{\sinh\beta_n + \sin\beta_n}\left(\sinh\beta_n \frac{x}{H} - \sin\beta_n \frac{x}{H}\right)$$

The first of the above integrals can be evaluated directly and the second is evaluated in the following manner. Consider two solutions W_m and W_n which satisfy the equations:-

$$\frac{d^4 W_m}{dx^4} = \frac{p_m^2}{a^2} W_m$$

$$\frac{d^4 W_n}{dx^4} = \frac{p_n^2}{a^2} W_n \qquad \text{where } a^2 = \frac{EI}{m}$$

Multiplying the first of these equations by W_n and the second by W_m, subtracting and integrating gives:-

$$\left(\frac{p_n^2 - p_m^2}{a^2}\right)\int_0^H W_m W_n \, dx = \int_0^H \left(W_m \frac{d^4 W_n}{dx^4} - W_n \frac{d^4 W_m}{dx^4}\right) dx$$

integrating the right side by parts:

$$\left(\frac{p_n^2 - p_m^2}{a^2}\right)\int_0^H W_m W_n \, dx = \left| W_m \frac{d^3 W_n}{dx^3} - W_n \frac{d^3 W_m}{dx^3} + \frac{dW_n}{dx}\frac{d^2 W_m}{dx^2} - \frac{dW_m}{dx}\frac{d^2 W_n}{dx^2} \right|_0^H \qquad (12a)$$

If $m \neq n$ the right side of this equation is equal to zero and when $m = n$ the left side of the equation is equal to zero. This difficulty is avoided by the following artifice. Let:-

$$W_n = W_m + \frac{dW_m}{dk}\delta k$$

where $k = \sqrt{\frac{p_m}{a}}$ as in the original solution of the differential equation. This gives then:-

$$k + \delta k = \sqrt[4]{\frac{p_n^2}{a^2}}$$

or

$$\frac{p_n^2}{a^2} = (k + \delta k)^4 = k^4 + 4k^3 \delta k$$

and $$\frac{p_n^2 - p_m^2}{a^2} = 4k^3 \delta k$$

Equation (12a) then becomes:-

$$4k^3 \int_0^H W_m^2 \, dx = \left| W_m \frac{d}{dk}\frac{d^3 W_m}{dx^3} - \frac{dW_m}{dk}\frac{d^3 W_m}{dx^3} + \frac{d}{dk}\left(\frac{dW_m}{dx}\right)\frac{d^2 W_m}{dx^2} \right.$$

$$\left. - \frac{dW_m}{dx}\frac{d}{dk}\left(\frac{d^2 W_m}{dx^2}\right)\right|_0^H \qquad (13a)$$

where infinitesimals of higher order are neglected. Now considering the original differential equation:-

$$\frac{d^4W}{dx^4} = K^4W$$

and changing the independent variable to:

$$x = \frac{z}{K} \qquad dz = K\,dx$$

this equation becomes:-

$$\frac{d^4W}{dx^4} = \frac{d^4W}{dz^4}\left(\frac{dz}{dx}\right)^4 = \frac{d^4W}{dz^4}K^4 = K^4W$$

or $W'''' = W$

where the primes indicate derivatives with respect to z. Also:-

$$\frac{dW}{dx} = \frac{dW}{dz}\frac{dz}{dx} = W'K$$

and

$$\frac{dW}{dK} = \frac{dW}{dz}\frac{dz}{dK} = W'x$$

Equation (13a) then becomes:-

$$4K\int_0^H W_m^2\,dx = \Big/ 3W_mW_m''' + Kx\,W_m^2 - 2Kx\,W_m'W_m''' + Kx(W_m'')^2 \Big/_0^H$$

From the end conditions specified all terms on the right side are zero except the second, so:-

$$\int_0^H W_m^2\,dx = \frac{1}{4}\Big/ x\,W_m^2 \Big/_0^H = \frac{H}{4}W_m^2\Big/_{x=H}$$

The coefficients in Equation (11a) are then evaluated as follows:-

$$\frac{\left(\int_0^H W_n\,dx\right)^2}{\int_0^H W_n^2\,dx} = \frac{\left(\int_0^H W_n\,dx\right)^2}{\frac{H}{4}W_n^2\Big/_{x=H}}$$

$$= \frac{4H}{\beta_n^2}\left(\frac{(\sinh\beta_n - \sin\beta_n) - \dfrac{\cosh\beta_n + \cos\beta_n}{\sinh\beta_n + \sin\beta_n}(\cosh\beta_n + \cos\beta_n)}{(\cosh\beta_n - \cos\beta_n) - \dfrac{\cosh\beta_n + \cos\beta_n}{\sinh\beta_n + \sin\beta_n}(\sinh\beta_n - \sin\beta_n)}\right)^2$$

This reduces to:-

$$\frac{4H}{\beta_n^2}\,\frac{(1-\cosh\beta_n)(1-\cos\beta_n)}{(\sinh\beta_n\cos\beta_n - \cosh\beta_n\sin\beta_n)}$$

Equation (11a), which gives the shearing force at the base of the structure, becomes then:-

$$S = m\sum_n \frac{4H}{\beta_n^2} F_n\, p_n \int_0^{t_1} \ddot{u}\,\sin p_n(t_1 - t)\,dt \qquad (14a)$$

where $F_n = \dfrac{(1-\cosh\beta_n)(1-\cos\beta_n)}{(\sinh\beta_n\cos\beta_n - \cosh\beta_n\sin\beta_n)}$

Substituting:-

$$p_n = \frac{\beta_n^2}{H^2}\sqrt{\frac{EI}{m}}$$

gives:-

$$S = \frac{4\sqrt{EIm}}{H}\sum_n F_n \int_0^{t_1} \ddot{u}\,\sin p_n(t_1 - t)\,dt \qquad (15a)$$

The numerical values of F_n are listed in the table below:-

F_1	F_2	F_3	F_4
1.44	1.00	1.00	1.00

CHAPTER IV.

Energy Loss Due to Wave Propagation

The motion of structures during an earthquake may be considerably affected by the dissipation of energy of vibration. Such energy dissipation may take place in two ways. There may be an internal dissipation through the various elements of the structure, or there may be an external dissipation into the ground. This external dissipation of energy may be due either to inelastic deformation of the ground immediately underlying the structure, or it may be due to the propagation of elastic waves. The latter is the one here investigated.

The problem of propagation of elastic waves is exceedingly complex. It is only because of certain special features of the case here considered and because certain simplifications can be made that the following approximate solution is possible. The special features are as follows. During an earthquake a structure builds up energy of vibration and for ordinary structures at least 80% of this energy is in the first mode of vibration. The force exerted on the ground by this mode is equal to the shear at the base of the structure. As was shown, this shear force is proportional to $\sin p(t_1 - t)$. The problem reduces then to that of finding the energy dissipated by the action of a sinusoidal force acting on the surface of a semi-infinite solid. This force is actually distributed over the area covered by the base of the structure. This distributed force may be replaced by a concentrated force and the magnitude of error involved by so doing can be shown to be small. Another simplifying feature is that for ordinary structures the velocity of vibration is very small compared to the velocity of the propagated waves. It is assumed that no surface waves will be produced by the action of this force on the ground.

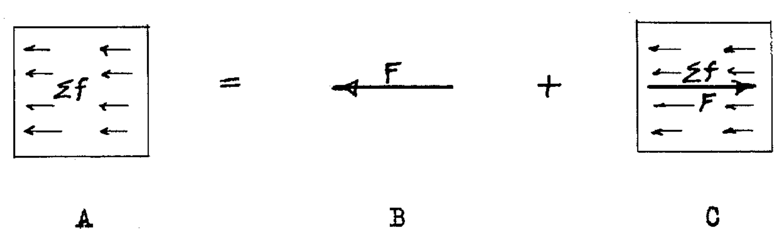
In the following discussion the ordinary elasticity theory is used with the system of notation used in Love's "Theory of Elasticity". All theorems and special solutions here used will be found in Love's book, except one which will be worked out in detail.

To investigate the assumption of a single concentrated force acting, the following theorems are used.

Theorem I. When a single static force acts at a point in an infinite or semi-infinite elastic solid the resulting displacements vary inversely as the distance from this point.

Theorem II. When in a limited region a system of forces with zero resultant acts the displacements vary inversely as the square of the distance from this region.

If a structure exerts a static system of forces on the ground it may be considered to be composed of two systems of forces as follows, where :



If r is the distance from the point of application of the force the displacements of case B are proportional to $\frac{1}{r}$. The displacements of case C are proportional to $\frac{1}{r^2}$ except in the immediate vicinity of the points of application of the forces. Therefore, the displacements of case A will be satisfactorily represented by B except in a local region near the base of the structure.

If a static force F acts at a point in an infinite or semi-infinite solid the displacements in the x, y, z, directions are:-

$$
\begin{aligned}
u &= F\frac{\Gamma_u}{r} \\
v &= F\frac{\Gamma_v}{r} \\
w &= F\frac{\Gamma_w}{r}
\end{aligned}
\tag{1b}
$$

where r, θ, ϕ, are polar coordinates with origin at the point of application of the force and Γ_u, Γ_v, Γ_w, are functions of θ, ϕ, and the elastic constants. The assumption is now made that if F varies as a function of the time, the displacement at any point is given by Equations (1b) with the proper phase factor added. This is equivalent to assuming that the velocity of propagation of strain is very large compared to the velocity of the displacement at the origin. For the case of a building on the ground these velocities are in the approximate ratio of $\frac{1}{10,000}$. In accordance with this assumption the velocity at any point is:-

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\Gamma_u}{r}\frac{\partial F}{\partial t} \\
\frac{\partial v}{\partial t} &= \frac{\Gamma_v}{r}\frac{\partial F}{\partial t} \\
\frac{\partial w}{\partial t} &= \frac{\Gamma_w}{r}\frac{\partial F}{\partial t}
\end{aligned}
$$

The kinetic energy of an element of volume is then equal to:-

$$dT = \frac{1}{2}\rho\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]r^2\sin\theta\, d\theta\, d\phi$$

where ρ is the mass density.

The kinetic energy of a spherical shell of thickness dr at a distance r from the origin is:-

$$\iint_{\theta\,\phi} dT = \frac{1}{2}\,\Theta\left(\frac{\partial F}{\partial t}\right)^2 dr \qquad (3b)$$

where Θ is a function of the elastic constants. If F is directed parallel to the x-axis, then:-

$$\frac{\partial F}{\partial t} = \frac{\partial u'}{\partial t}\,\frac{1}{\Gamma_{u'}}$$

where u' is a point on the x-axis which will be taken at a distance R . Equation (3b) may be written:-

$$T = \frac{1}{2}\,\frac{\Theta}{(\Gamma_{u'})^2}\left(\frac{\partial u'}{\partial t}\right)^2 dr$$

The apparent mass of a spherical shell is therefore equal to:-

$$m = \frac{\Theta}{\Gamma_{u'}^2}\,dr$$

which is a constant independent of the distance from the origin. The apparent mass of the infinite solid is therefore the same as the mass of a heavy, infinitely long, stretched string. Moreover, since the displacements of the infinite solid are propagated with a velocity "c" the analogy is complete. For energy considerations, then, the case of such an elastic solid is similar to a stretched string of mass "m" per unit length and with velocity of propagation "c".

The propagation of motion along the string is given by the expression:-

$$u = \phi(x - ct)$$

where

u = lateral displacement.

x = coordinate measured along string.

c = velocity of propagation.

ϕ = arbitrary function.

If the end of the string is given a sinusoidal displacement:-

$$u_0 = A' \sin(-\omega t)$$

then

$$u = A' \sin \frac{\omega}{c}(x - ct)$$

The force required at the end of the string to produce this displacement is:-

$$f = -mc^2\left(\frac{\partial y}{\partial x}\right)_{x=0}$$

$$= -A' mc\omega \cos \omega t$$

The velocity at the end of the string is:-

$$\dot{u}_0 = -A'\omega \cos \omega t$$

so

$$f = mc\dot{u}_0$$

The force f is thus proportional to the velocity so the energyloss due to wave propagation has the same effect as a viscous damping. The rate of dissipation is $mc(\dot{u}_0)^2$. When the rigidity of the elastic solid is small, the velocity of propagation is relatively small and when the rigidity is large, the velocity $\dot{u}_0$ is small. The energy dissipation, therefore, is a maximum when the rigidity is neither extremely small nor extremely large.

To investigate the effect of this energy loss on the displacement, a force F is considered to act at the origin where:-

$$F = A \sin pt$$

so

$$mc\dot{u} + \frac{1}{\Gamma} u = A \sin pt$$

where u is now taken to be the displacement at the point R, on the x-axis. The solution of this equation is:-

$$u = \frac{A\Gamma}{(1+(mc\Gamma p)^2)} (\sin pt - mc\Gamma p \cos pt)$$

The magnitude of the term $(mc\Gamma p)$ represents the effect of the energy dissipation. To illustrate the effect of this term in a specific case the terms m and Γ will be evaluated for the case of a force acting at a point in an infinite solid. The exact solution will then be worked out so that a comparison may be made.

The static displacement caused by such a concentrated force acting parallel to the x-axis is:-

$$u_r = \frac{F}{4\pi\mu} \frac{\cos\theta}{r}$$

$$u_\theta = \frac{1}{2} \frac{a^2+b^2}{a^2} \frac{F}{4\pi\mu} \frac{\sin\theta}{r}$$

$$u_\phi = 0$$

where μ is Lame's constant and "a" and "b" are the velocity of propagations of compression and shear waves, respectively. The term θ appearing in the expression for the apparent mass is:-

$$\theta = \rho \int_0^{2\pi}\int_0^{2\pi} \left(\frac{1}{4\pi\mu}\right)^2 \left(\cos^2\theta - \left(\frac{a^2+b^2}{2a^2}\right)^2 \sin^2\theta\right) d\theta\, d\phi$$

$$= \rho \left(\frac{1}{4\pi\mu}\right)^2 \left(\pi\left(1 - \left(\frac{a^2+b^2}{2a^2}\right)^2\right)\right) 2\pi$$

also Γ for this case is:-

$$\Gamma = \frac{1}{4\pi\mu R}$$

so
$$m\Gamma = \frac{\pi\rho R}{2\mu}\left(1 - \left(\frac{a^2+b^2}{2a^2}\right)^2\right)$$

If Poisson's ratio is taken equal to 1/4 then $a^2 = 3b^2$

and $$m\Gamma = \frac{5}{18}\pi\frac{\rho R}{\mu}$$

now, $\mu = \rho b^2$ and if c is taken equal to b:-

$$m\Gamma c p = \frac{pR}{1.14b}$$

The displacement of Equation (3b) then becomes:-

$$u = \frac{A}{4\pi\mu R}\left(\frac{1}{1+\left(\frac{pR}{1.14b}\right)^2}\right)\left(\sin pt - \frac{pR}{1.14b}\cos pt\right) \qquad (4b)$$

or rewriting this equation and neglecting small quantities of higher order:-

$$u = \frac{A}{4\pi\mu R}\left[\left\{1-\left(\frac{pR}{1.14b}\right)^2\right\}\sin pt - \frac{pR}{1.14b}\cos pt\right] \qquad (5b)$$

If the loss of energyis completely neglected, the displacement is:-

$$u = \frac{A}{4\pi\mu R}\sin pt$$

For purposes of comparison, the exact solution will be derived for the case of a sinusoidal force acting at a point in an infinite solid. The notation used is:-

ρ = mass density.

λ, μ = Lame's elastic constants.

a, b = velocities of compression and shear waves, respectively,

so $a^2 = \frac{\lambda + 2\mu}{\rho}$, $b^2 = \mu/\rho$

u, v, w = displacement in x, y, z direction.

Δ = dilation = div. $(u,v,w) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

ϕ = scalar potential, F, G, H = vector potential, so (u, v, w) =

$-\text{grad}\,\phi + \text{curl}\,(F, G, H)$ where div. (F, G, H) = 0

$\nabla^2 \phi = -\Delta$ and , etc..

∇^2 = La Placian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

The equations of equilibrium are:-

$$(\lambda+2\mu)\frac{\partial \Delta}{\partial x} + \mu \nabla^2 u + \rho X = \rho \frac{\partial^2 u}{\partial t^2}$$

$$(\lambda+2\mu)\frac{\partial \Delta}{\partial y} + \mu \nabla^2 v + \rho Y = \rho \frac{\partial^2 v}{\partial t^2} \qquad (1c)$$

$$(\lambda+2\mu)\frac{\partial \Delta}{\partial z} + \mu \nabla^2 w + \rho Z = \rho \frac{\partial^2 w}{\partial t^2}$$

Expressing the body forces in terms of the scalar potential Φ and vector potentials (L, M, N):-

$$(X, Y, Z) = -\text{grad}\,\Phi + \text{curl}\,(M, N, L)$$

Writing Equations (1c) in terms of the potentials, gives:-

$$-(\lambda+2\mu)\frac{\partial}{\partial x}\nabla^2\phi + \mu\left(\frac{\partial}{\partial y}\nabla^2 H - \frac{\partial}{\partial z}\nabla^2 G\right) - \rho\left(\frac{\partial \Phi}{\partial x} - \frac{\partial N}{\partial y} + \frac{\partial M}{\partial z}\right)$$

$$= -\rho\left(\frac{\partial^2}{\partial t^2}\left(\frac{\partial \phi}{\partial x} - \frac{\partial H}{\partial y} + \frac{\partial G}{\partial z}\right)\right.$$

Collecting terms:-

$$\frac{\partial}{\partial x}\left(\rho\frac{\partial^2\phi}{\partial t^2}-(\lambda+2\mu)\nabla^2\phi-\rho\Phi\right)-\frac{\partial}{\partial y}\left(\rho\frac{\partial^2 H}{\partial t^2}-\mu\nabla^2 H-\rho N\right)$$
$$+\frac{\partial}{\partial z}\left(\rho\frac{\partial^2 G}{\partial t^2}-\mu\nabla^2 G-\rho M\right)=0$$

with similar equations for v and w. Particular solutions will be obtained if solutions are found for the equations:-

$$\frac{\partial^2\phi}{\partial t^2}-a^2\nabla^2\phi=\Phi\ ;\ \frac{\partial^2 F}{\partial t^2}-b^2\nabla^2 F=L\ ;\ \text{etc.} \qquad (2c)$$

These equations are of the form called Lorenz's equation*, the solution of which is given by the "retarded potential":-

$$\phi=\frac{1}{4\pi a^2}\iiint\frac{1}{r}\,\Phi'(t-r/a)\,dx'dy'dz'$$
$$F=\frac{1}{4\pi b^2}\iiint\frac{1}{r}\,L'(t-r/b)\,dx'dy'dz' \qquad (3c)$$

The values of the functions Φ', L', etc., are to be calculated at the point x', y', z' at the time $(t-r/a)$ etc.. The integration is over the volume in which Φ', L', etc., are not zero.

As in Love's Theory of Elasticity Φ, L, M, etc., can be expressed by:-

$$\Phi=\frac{1}{4\pi}\iiint\left(X'\frac{\partial r^{-1}}{\partial x}+Y'\frac{\partial r^{-1}}{\partial y}+Z'\frac{\partial r^{-1}}{\partial z}\right)dx'dy'dz'$$

$$L=\frac{1}{4\pi}\iiint\left(Z'\frac{\partial r^{-1}}{\partial y}-Y'\frac{\partial r^{-1}}{\partial z}\right)dx'dy'dz' \quad \text{etc.}$$

where X', Y', Z' are the values of X, Y, Z at the point (x', y', z'), r is the distance from the point (x, y, z). In the limit when the body

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* Webster, "Partial Differential Equations in Mathematical Physics," p. 216.

forces reduce to a single force in the x-direction, these integrals become:-

$$\iiint \rho X' dx' dy' dz' = f(t)$$

$$\iiint \rho Y' dx' dy' dz' = 0 \quad etc.$$

The potentials reduce to:-

$$\Phi = \frac{1}{4\pi\rho} f(t) \frac{\partial r^{-1}}{\partial x} ; \; L = 0 ; \; M = \frac{1}{4\pi\rho} f(t) \frac{\partial r^{-1}}{\partial z}$$

$$N = -\frac{1}{4\pi\rho} f(t) \frac{\partial r^{-1}}{\partial y} \qquad (4c)$$

From Equations (3c) and (4c) the potentials are evaluated giving:-

$$\phi = \frac{1}{4\pi\rho} \frac{\partial r^{-1}}{\partial x} \int_0^{r/a} t \, f(t_1 - t) \, dt$$

$$F = 0$$

$$G = \frac{1}{4\pi\rho} \frac{\partial r^{-1}}{\partial z} \int_0^{r/b} t \, f(t_1 - t) \, dt \qquad (5c)$$

$$H = -\frac{1}{4\pi\rho} \frac{\partial r^{-1}}{\partial y} \int_0^{r/a} t \, f(t_1 - t) \, dt$$

From the above expressions the displacements are calculated. The integrals in these expressions are differentiated in the following manner:-

$$\frac{\partial}{\partial x} \int_0^{r/a} t (f(t_1 - t) \, dt = r/a \, f(t - r/a) \frac{1}{a} \frac{\partial r}{\partial x}$$

The displacements are given by the expressions:-

$$u = \frac{1}{4\pi\rho} \frac{\partial^2 r^{-1}}{\partial x^2} \int_{r/a}^{r/b} t \, f(t_1 - t) \, dt + \frac{1}{4\pi\rho b^2 r} f(t_1 - r/b) + \frac{1}{4\pi\rho r} \left(\frac{\partial r}{\partial x}\right)^2 \left\{ \frac{1}{a^2} f(t_1 - r/a) - \frac{1}{b^2} f(t_1 - r/b) \right\}$$

$$v = \frac{1}{4\pi\rho} \frac{\partial^2 r^{-1}}{\partial x \partial y} \int_{r/a}^{r/b} t \, f(t_1 - t) \, dt + \frac{1}{4\pi\rho r} \frac{\partial r}{\partial x} \frac{\partial r}{\partial y} \left\{ \frac{1}{a^2} f(t_1 - r/a) - \frac{1}{b^2} f(t_1 - r/b) \right\}$$

$$w = \frac{1}{4\pi\rho}\frac{\partial^2 r^{-1}}{\partial x \partial z}\int_{r/a}^{r/b} t\, f(t_1 - t)\, dt + \frac{1}{4\pi\rho r}\frac{\partial r}{\partial x}\frac{\partial r}{\partial z}\left\{\frac{1}{a^2} f(t_1 - r/a)\right.$$

$$\left. - \frac{1}{b^2} f(t_1 - r/b)\right\} \qquad (6c)$$

When the applied force is:-

$$f(t) = A \sin pt,$$

the displacements in the direction of the forces, at points on the x-axis are:-

$$u = \frac{1}{4\pi\rho}\left[\frac{2}{r^2 p}\left(\frac{\cos p(t_1 - r/b)}{b} - \frac{\cos p(t_1 - r/a)}{a}\right)\right.$$

$$\left. + \frac{2}{r^3 p^2}\left(\sin p(t_1 - r/b) - \sin p(t_1 - r/a)\right) + \frac{1}{r a^2}\sin p(t_1 - r/a)\right] \qquad (7c)$$

This equation is in the wave form, but it is desired to reduce this to an equation giving the displacement at a fixed point. This is done by expanding each term of the above equation. Since r is to be very small compared to "a" and "b", all terms of high order in 1/a or 1/b are neglected. The first term in the brackets is expanded as follows:-

$$\frac{2}{r^2 p}\left(\frac{(\cos pt \cos \frac{pr}{b} + \sin pt \sin pr/b)}{b} - \frac{(\cos pt \cdot \cos pr/a + \sin pt \sin pr/a)}{a}\right)$$

$$= \frac{2}{r^2 p}\left(\cos pt\left(\frac{\cos pr/b}{b} - \frac{\cos pr/a}{a}\right) + \sin pt\left(\frac{\sin pr/b}{b} - \frac{\sin pr/a}{a}\right)\right)$$

The second term expanded thus gives:-

$$\frac{2}{r^3 p^2}\left(\cos pt(\sin pr/a - \sin pr/b) + \sin pt(\cos pr/b - \cos pr/a)\right)$$

Expanding in series gives for the sum of these two:-

$$\cos pt \left| \begin{array}{l} \frac{2}{r^2 p}\left(\frac{1}{b}\left[1-\left(\frac{pr}{b}\right)^2\frac{1}{2!}+\cdots\right] - \frac{1}{a}\left[1-\left(\frac{pr}{a}\right)^2\frac{1}{2!}+\cdots\right]\right) \\ + \frac{2}{r^3p^2}\left(\left[pr/a-\left(\frac{pr}{a}\right)^3\frac{1}{3!}+\cdots\right]-\left[\frac{pr}{b}-\left(\frac{pr}{b}\right)^3\frac{1}{3!}+\cdots\right]\right) \end{array} \right|$$

with a similar expansion for the term containing $\sin pt$. Combining terms and neglecting those of higher order, gives:-

$$-\frac{2}{3}p\left(\frac{1}{b^3}-\frac{1}{a^3}\right)\cos pt + \left[\frac{1}{r}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)-\frac{p^2r}{4}\left(+\frac{1}{b^4}-\frac{1}{a^4}\right)\right]\sin pt$$

The third term in Equation (7c) gives:-

$$\frac{1}{ra^2}\left(1-\frac{p^2r^2}{2a^2}\right)\sin pt - \frac{p}{a^3}\cos pt$$

Combining these and using the relation $\mu=\rho b^2$, gives for the displacement:-

$$u=\frac{A}{4\pi\mu R}\left\{\left[1-\left(\frac{pr}{2b}\right)^2\left(\frac{a^4+b^4}{a^4}\right)\right]\sin pt - \frac{r}{3b}\left(2+\frac{b^3}{a^3}\right)\cos pt\right\}$$

For comparison with the approximate solution previously derived, the value $a^2=3b^2$, corresponding to Poisson's ratio of 1/4, is substituted giving:-

$$u=\frac{A}{4\pi\mu r}\left\{\left[1-\left(\frac{pr}{1.9b}\right)^2\right]\sin pt - \frac{pr}{1.4b}\cos pt\right\} \qquad (8c)$$

Comparing this with Equation (5b), it is seen that they differ only in the magnitude of the numerical factors. The approximate solution indicates a larger energy dissipation than the exact solution. This is consistent with the assumptions made. It can be inferred that the assumptions underlying the approximate solution do not seriously affect the validity of the results.

The approximate solution for the case of a horizontal force applied to the surface of a semi-infinite solid is worked out in the same manner as for a force in an infinite solid. For this case the static displacements are:- *

$$u_r = \frac{F}{2\pi(\lambda+\mu)r}\left(1 - \frac{\lambda+3\mu}{2\mu}\cos\theta - \frac{1}{4}\sec^2\frac{\theta}{2}\right)\cos\phi$$

$$u_\theta = \frac{F}{2\pi(\lambda+\mu)r}\left(\frac{\lambda+3\mu}{2\mu} - 1 + \frac{1}{4}\sec^2\frac{\theta}{2}\right)\sin\phi \qquad (9c)$$

$$u_\phi = \frac{F}{2\pi(\lambda+\mu)r}\left(\tan\frac{\theta}{2} - \frac{\lambda+2\mu}{\mu}\sin\theta\right)\cos\phi$$

The displacement of a point in the direction of the force $F = A\sin pt$ is:-

$$u = \frac{A}{3\pi(\lambda+2\mu)R}\left(\left(1 - \left(\frac{11.2\,pR}{b}\right)^2\right)\sin pt - \frac{11.2\,pR}{b}\cos pt\right) \qquad (10c)$$

It is seen that in this case the energy dissipation has a much larger effect than in the case of the force acting at a point in an infinite solid.

As was shown, the energy dissipated due to the propagation of elastic waves is of the same character as a viscous damping, so since the displacement due to a force $F = A\sin pt$ is of the form:

$$u = \Gamma A \sin(pt+\alpha)$$

the rate of dissipation is:-

$$mc\left(\Gamma^2 A^2 p^2 \cos^2(pt+\alpha)\right)$$

---

* J. H. Michell, "Proc. London Math. Soc.", 1900, vol. 32, page 34.

Since m, c, and Γ are functions of the elastic properties of the ground, let $\epsilon = mc\Gamma^2$ and the rate of dissipation then is:-

$$\epsilon A^2 p^2 \cos^2(pt+\alpha)$$

The total energy dissipated in one cycle is:-

$$\pi \epsilon A^2 p$$

The average rate of dissipation is:-

$$\frac{\epsilon A^2 p_n^2}{2}$$

The energy of vibration of the structure, from Equation (12), Chapter I, is equal to:-

$$E = \frac{1}{2} \frac{A^2}{p C_n}$$

The average rate of dissipation can, therefore, be written:-

$$\epsilon p^3 C_n E_n = \epsilon \frac{(2\pi)^3}{T_n^3} D_n \sqrt{km}\, E_n = \epsilon \frac{(2\pi)^4}{T_n^4} \left(\frac{D_n}{\lambda_n}\right) M E_n$$

where the notation is that of Chapter I. and M is the total mass of the structure. If ϵ and M are constant, then the rate of dissipation varies directly as the energy E and inversely as the period to the fourth power.

This dissipation of energy will reduce the magnitude of the shearing forces computed for an undamped structure. To illustrate this, let it be assumed that the energy of an undamped structure increases linearly with the time to a maximum value of E' at the time t_1. The energyin the corresponding damped structure at any time t is:-

$$E = \frac{E' t}{t_1} - \int_0^{t_0} \bar{a} E \, dt$$

where $\bar{a}E$ is the rate of dissipation. The solution of this equation is:-

$$E = \frac{E'}{\bar{a}t_1}(1 - e^{-\bar{a}t_1})$$

The maximum value of E occurs at the time t_1 and the ratio of the energies at the time t_1 is:-

$$\frac{E}{E'} = \frac{(1 - e^{-\bar{a}t_1})}{\bar{a}t_1}$$

The following table gives the values of this ratio for different values of "$\bar{a}$" and t_1:-

TABLE III.

Ratios of $\frac{E}{E'}$

$\bar{a}$ \ t_1	5	10	15
.01	.98	.95	.93
.05	.88	.78	.71
.10	.78	.63	.52
.15	.70	.52	.40
.20	.63	.43	.32
.25	.57	.36	.26
.50	.40	.20	.13

If the maximum shearing force in the undamped structure is (S') and in the damped structure (S), then:-

$$\frac{S}{S'} = \left(\frac{E}{E'}\right)^{1/2}$$

Table IV. shows how the damping affects the magnitude of the maximum shearing force.

TABLE IV.

Ratios of $\frac{S}{S'}$

$\bar{a}$ \ t_1	5	10	15
.05	.94	.88	.84
.10	.88	.79	.72
.15	.84	.72	.63
.20	.79	.65	.57
.25	.75	.60	.51
.50	.63	.45	.36
.75	.52	.36	.30
1.00	.45	.32	.26
1.50	.37	.26	.21

From this table it is seen that if the energy is not pumped in too rapidly and if "ā" is relatively large, there is a considerable reduction in the magnitude of the maximum shearing force.

The factor "ā" will now be evaluated for the case of Equation (10c), where R = 1:-

$$\bar{a} = \frac{98\, D_n \sqrt{km}}{\rho\, b^3\, T^3} = \frac{98 \left(\frac{D_n}{\lambda_n}\right) 2\pi}{\rho\, b^3\, T^4} \quad M$$

Substituting the following typical values for a 10 story building:-

$$\sqrt{Km} = 3 \times 10^5$$
$$\mathcal{T} = 1$$
$$\rho = 3$$
$$b = 2000$$
$$D_n = 1.25$$

gives $\bar{a}$ = 0.00154 and if the period T is taken equal to 1/10, then $\bar{a}$ = 1.5. This value of b corresponds to a shearing modulus of elasticity of 84,000 pounds per inch. The probable range of the moduli of various soils is from 5000 to 100,000 pounds per inch. The following table lists the values of "$\bar{a}$" corresponding to given values of T and shearing modulus G:-

TABLE V.

Values of "$\bar{a}$"

G \ T	1.0	.75	.50	.25	.10
100,000	.0012	.0028	.0096	.077	1.2
75,000	.0018	.0042	.014	.115	1.8
50,000	.0033	.0078	.0265	.211	3.3
25,000	.0092	.0215	.073	.590	
10,000	.038	.0890	.305	2.43	
5,000	.105	.245	.805		

Tables IV. and V. indicate that for long periods the reduction in maximum shearing force is small. The short periods show considerable reduction particularly for the less rigid soils.

The numerical values of Table V. are, of course, not accurate, because of the assumptions underlying Equation (10c) and because the ground is not a homogeneous elastic solid. This table does, however, give an indication of the rate of energy dissipation due to wave propagation. Considering other forms of energy dissipation, such as internal damping in the ground immediately underlying the structure, it is seen that the ground may have a considerable effect in reducing the shearing forces in buildings with short periods.