"On Polynomials Which Have All
Their Zeros Outside the Circle
$|z|=R$ In The Complex Plane"

Thesis by

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Polynomials with Zeros Outside the Circle $|z|=R$

We shall consider polynomials with complex coefficients whose zeros are then complex numbers. In this paper we shall be interested chiefly in polynomials which have all their zeros outside the circle $|z|=R$. We prove first the elementary result:

<u>Theorem 1</u>. Let $P(z)=\sum_{i=0}^{n} b_i z^i$ be a polynomial of degree n. Then a necessary and sufficient condition for all the zeros of $P(z)$ to be outside <u>some</u> circle $|z|=R$, $R>0$, is that $b_0 \neq 0$.

Proof: <u>Necessity</u>: If all the zeros of $P(z)$ are outside $|z|=R$, $R>0$, then zero is not a "zero" of $P(z)$, i.e. $b_0 \neq 0$.

Proof: <u>Sufficiency</u>: Suppose $b_0 \neq 0$. Then zero is not a "zero" of $P(z)$. There are in all n zeros of $P(z)$. One of them, say α must be, in absolute value, closest or as close as any of the other "zeros" to the quantity zero. We can choose R as any number satisfying the inequality

$$0 < R < |\alpha|.$$

This completes the proof of the theorem.

We now seek a necessary and sufficient conditions for all the zeros of $P(z)=\sum_{i=0}^{n} b_i z^i$ to have all its zeros outside the circle $|z|=R$ where R is a <u>given</u> positive number. To this end we make use of a theorem due to I. Schur (1) The theorem is as follows:

Let $f(z)=\sum_{i=0}^{n} a_i z^i$. Then $f(z)$ has all its roots

(1) For a proof of this theorem see A. Cohn Math. Zeit. Vol. 14 (1922) P. 114-120 also see P. 110 for references to the original papers of I. Schur.

within the unit circle if and only if the $(n-1)^{st}$ degree polynomial $f_1(z) = \frac{1}{z}\{\bar{a}_n f(z) - a_0 z^n \bar{f}(\frac{1}{z})\}$ has all its zeros within the unit circle, and if also

$|a_n| > |a_0|$.

We now return to $P(z) = \sum_{i=0}^{n} b_i z^i$. We first make the change of variable $z = Ry$ giving us the polynomial $G(y) = \sum_{i=0}^{n} b_i R^i y^i$ whose zeros are $\frac{1}{R}$ times the corresponding zeros of $P(z)$.

We now consider the polynomial

$$\psi(y) = y^n \sum_{i=0}^{n} b_i R^i \frac{1}{y^i} = \sum_{i=0}^{n} b_{n-i} R^{n-i} y^i$$

The zeros of $\psi(y)$ are the reciprocals of those of $G(y)$. If $\psi(y)$ has all its zeros within the unit circle, then $G(y)$ has all its zeros outside the unit circle, and $P(z)$ has all its zeros outside the circle $|z| = R$. Conversely, if $P(z)$ has all its zeros outside $|z| = R$, then $\psi(y)$ has all its zeros inside the unit circle.

The theorem of Schur then gives us

__Theorem 2.__ Necessary and sufficient conditions for the polynomial $P(z) = \sum_{i=0}^{n} b_i z^i$ to have all its zeros outside the circle $|z| = R$ are

1° $|b_0| > |b_n R^n|$

2° $g_1(z) = \frac{1}{z}\{\bar{b}_0 g(z) - b_n R^n \bar{g}(\frac{1}{z})\}$

have all its zeros inside the unit circle, where

$$g(z) = \sum_{i=0}^{n} b_{n-i} R^{n-i} y^i$$

Theorem 2 gives us an algorithm for testing whether or not the zeros of $P(z)$ are outside $|z| = R$ for any R. This can be accomplished in at most n steps. For one can test $g_1(z)$ by

Schur's theorem and $g_1(z)$ will have all its zeros within the unit circle if a certain $(n-2)$ degree polynomial also has its zeros there, and if also a certain inequality of its coefficients is satisfied. The problem, provided this is satisfied, can then be reduced to the consideration of an $(n-2)$ degree polynomial, and so forth. Thus in a finite number of steps one can tell whether or not all the zeros of $P(z)$ are outside $|z|=R$.

The situation in which theorem 2 leaves us is thus theoretically satisfactory, but not practically so because it gives no clue as to the suse of R. We next devote ourselves to this question.

Several determinations of R are to be found in the literature amount these or the following:

(1) The zeros of $\sum_{i=0}^{n} a_i z^i$, a_i complex numbers are to be found in the annulus determined by the circles with center at origin and radii $M+1$, $\frac{1}{m+1}$ where

$$m = Max\left(\left|\frac{a_n}{a_0}\right|, \left|\frac{a_{n-1}}{a_0}\right|, \ldots, \left|\frac{a_1}{a_0}\right|\right)$$

$$M = Max\left(\left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_n}\right|, \ldots, \left|\frac{a_0}{a_n}\right|\right)$$

Thus R $= \frac{1}{m+1}$

(2) Let us write $P(z) = \sum_{i=0}^{n} a_i z^i$. Then all the zeros of $P(z)$ are in the annulus

$$S^{-\frac{1}{2}} \leq |z| \leq \frac{S^{\frac{1}{2}}}{\left|\frac{a_n}{a_0}\right|}$$

where

$$S = 1 + \left|\frac{a_1}{a_0}\right|^2 + \cdots + \left|\frac{a_n}{a_0}\right|^2.$$

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(1) See L. Koenigsberger: Uber die Abgrenzung du Losungen einer Algebraeschen Gleichung Palermo Rend. Vol. 26 P. 343-359 (1908)

(2) See S. B. Kelleher Des Limites des zeros d'un Polynome Journal de Math Ser. 7 Vol. 2 P. 169-171 (1916)

This gives us another determination of R, namely any quantity smaller than $5^{-\frac{1}{2}}$. Other determinations will be given below.

With regard to the various determinations of R which can be given, the following theorem is of interest because it gives an upper limit to the value of R.

Theorem 3. Let $P(z) = \sum_{i=0}^{n} b_i z^i$ with $b_0 \neq 0$. Then $R < \left|\frac{b_0}{b_n}\right|^{\frac{1}{n}}$

For let $\alpha_1, \ldots, \alpha_n$ be the zeros of ~~[illegible]~~ $P(z)$. Then $|\alpha_1| \ldots |\alpha_n| = \left|\frac{b_0}{b_n}\right|$ whence either 1° $|\alpha_1| = |\alpha_2| = \cdots = |\alpha_n| = \left|\frac{b_0}{b_n}\right|^{\frac{1}{n}}$ or 2° ~~or~~ at least one of $|\alpha_i| < \left|\frac{b_0}{b_n}\right|^{\frac{1}{n}}$

This completes the proof of the theorem.

Theorem 3 can be strengthened by the finding of a necessary and sufficient condition for all the zeros of $P(z)$ to be on the circle $|z| = \left|\frac{b_0}{b_n}\right|^{\frac{1}{n}}$.

For this purpose we use a theorem of I. Schur (1) to the effect that the zeros of an n^{th} degree polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$ are all distinct and on the unit circle if and only if

$$1^\circ \quad a_0 \bar{a}_{n-\nu} = \bar{a}_n a_\nu \qquad (\nu = 1, 2, \ldots, n)$$

2° $P'(z)$ has all its zeros within the unit circle.

To apply this theorem to the problem at hand we consider the given $P(z) = \sum_{i=0}^{n} b_i z^i$

If $P(z)$ has no multiple zeros, than we apply the following:

Make the change of variable $z = Ry$, $R = \left|\frac{b_0}{b_n}\right|^{\frac{1}{n}}$.

Then $P(z) = Q(y) = \sum_{i=0}^{n} b_i R^i y^i$

and $P(z)$ has all its zeros on the circle $|z| = R$ if and only if $Q(y)$ has all its zeros on the unit circle. The zero of $Q(y)$ are all distinct since those of $P(z)$ are distinct.

(1) I. Schur Journal Fur Math. Vol. 148 P. 136 (1918)

$P(z)$ can be tested for multiple zeros. Form the G.C.D. $G(z)$ of $P(z)$ and $P'(z)$ (by Euclid's Algorithm, say). We now consider the polynomials $\frac{P(z)}{G(z)}$, $G(z)$ If $G(z)$ itself has no multiple zeros, instead of testing $P(z)$ for having all its zeros on we test $\frac{P(z)}{G(z)}$ and $G(z)$ for that property.

If $G(z)$ has multiple zeros, form $G_1(z)$ the G.C.D. of $G(z)$ and $G'(z)$. If $G_1(z)$ has no multiple zeros consider instead of $P(z)$ in our discussion above each of the polynomials

If $G_1(z)$ has multiple zeros form $G_2(z)$ the G.C.D. of $G_1(z)$ and $G_1'(z)$ and test the polynomials

in case $G_2(z)$ has no multiple zeros.

If $G_2(z)$ has multiple zeros we proceed in the way here indicated. Thus we see that there is no loss in generality if we consider our given $P(z)$ to have no multiple zeros.

This gives us the theorem:

<u>Theorem 4</u>. Let $P(z) = \sum_{i=0}^{n} b_i z^i$ be a polynomial (with distinct zeros). Then necessary and sufficient conditions that $P(z)$ have all its zeros on the circle $|z| = R = \left|\frac{b_0}{b_n}\right|^{\frac{1}{n}}$ are

$$1^\circ \quad b_n \bar{b}_{n-\nu} = \bar{b}_n b_\nu \qquad (\nu = 1, \dots, n)$$

2° $Q'(y)$ have all its zeros within $|z| = 1$, where $Q(y) = \sum_{i=0}^{n} b_i R^i y^i$

In connection with condition 2° of this theorem, necessary and sufficient conditions for 2° to be satisfied are given on page 1 above. Other necessary and sufficient conditions could also be used. (1)

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(1) I Schur: Journal Fur Math. Vol. 148 p. 134 (1918) gives one such.

Thus we have found an upper limit to possible values of R; and theorem 4 tells us when the best value for R is arbitrarily close to this upper limit.

We now return to the question of the determination of the quantity R . In addition to the formulas for the lower bound of the zeros of a polynomial given by Koenigsberger and Kelleher quoted above, there are other like formulas. Among these is an excellent one due to Chang (1).

Chang's result is the following:

Let $\alpha, \beta, \dots, \kappa, \lambda$ be positive whole numbers and $\alpha > \beta > \cdots > \kappa > \lambda > 0$

and $\nu = Min\,(\alpha-\beta, \beta-\gamma, \dots, \kappa-\lambda, \lambda)$
$\mu \quad Max\,(\alpha-\beta, \beta-\gamma, \dots, \kappa-\lambda, \lambda)$

Let $a_\alpha, a_\beta, \dots, a_\kappa, a_\lambda, a_0$ be arbitrary real or complex numbers for which each $a_\rho \neq 0$ and

$$r = Min\left(\left|\frac{a_\beta}{a_\alpha}\right|, \left|\frac{a_\gamma}{a_\beta}\right|, \dots, \left|\frac{a_\lambda}{a_\kappa}\right|, \left|\frac{a_0}{a_\lambda}\right| \right)$$

$$s = Max\left(\left|\frac{a_\beta}{a_\alpha}\right|, \left|\frac{a_\gamma}{a_\beta}\right|, \dots, \left|\frac{a_\lambda}{a_\kappa}\right|, \left|\frac{a_0}{a_\lambda}\right| \right)$$

Then each root z of the equation

$$a_\alpha x^\alpha + a_\beta x^\beta + \dots + a_\lambda x^\lambda + a_0 = 0$$

satisfies the relation

$$\left.\begin{matrix} \text{for } r \leqq 1, & \dfrac{r^{1/\nu}}{2^{1/\nu}} \\ \text{for } r \geqq 1, & \dfrac{r^{1/\mu}}{2^{1/\nu}} \end{matrix}\right\} < |z| < \left\{\begin{matrix} 2^{1/\nu}\, s^{1/\mu}, & \text{for } s \leqq 1 \\ 2^{1/\nu}\, s^{1/\nu}, & \text{for } s \geqq 1 \end{matrix}\right.$$

It is readily seen that the condition $a_\rho \neq 0$ does not limit the

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(1) T. H. Chang: Verallgemeenerung des Satzes von Kakeya Tôhoku Math. Journal. Vol. 43 Pages 79-83. (1937)

generality of this method, and that the method is applicable to every polynomial with complex coefficients.

The following is a method for determining R in a very special case, but is rather interesting. (1)

If $|c_0| > |c_1| > \cdots > |c_n| > 0$

and $R = Max \left\{ \dfrac{|c_{v-1} - c_v|}{|c_{v-1}| - |c_v|} \right\}$ $\qquad (v = 1, 2, \ldots, n)$

Then all the zeros of the polynomial

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n$$

are outside the circle $|z| = \frac{1}{R}$.

Another determination of R is contained in a paper by Angheluțza (2) which contains the result that the absolute values of the zeros of the polynomial $\sum_{i=0}^{n} a_i z^i$ are all contained in the internal (x_1, x_2) where x_1 and x_2 are the single positive zeros of the polynomials

$$|a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x - |a_0|$$

$$|a_n| x^n - |a_{n-1}| x^{n-1} - \cdots - |a_1| x - |a_0|$$

respectively.

The question of the upper bound of zeros of a polynomial has been investigated much more carefully and extensively than the question of lower bound which we have considered. It is often possible to utilize this information in the determination of the lower bound R.

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(1) See K. Kimura: Über die Nullstellen von den Polynome and der Potenzreihen. Tohoku Math. Journ. Vol. 35. (1932) P. 233-236

(2) Angheluțza Th. Sur une extension d'un Theoreme de Hurwitz Bull. Sect. Sci. Acad Roum. Vol 16. P. 119-121 (1934)

These upper bounds are to be found in many and various places in the literature. If we restrict ourselves to polynomials with real coefficients the number of upper bounds found in the literature increases enormously. Such upper bounds are even found in elementary texts on the theory of equations. (1)

However we shall consider here the general case where the coefficients of the polynomial may be complex numbers.

We enumerate some upper bounds:

1. $M+1$ where M is given by Koenigsberger as cited above. (P. 3)

2. $\dfrac{S^{-1/2}}{\left|\frac{a_n}{a_0}\right|}$ where

$$S = 1 + \left|\frac{a_1}{a_0}\right|^2 + \cdots + \left|\frac{a_n}{a_0}\right|^2$$

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(1) See Dickson, First Course in the Theory of Equations Pages 21, 22.

given by Kelleher as cited above. (P. 3).

3. The criterion of Chang as cited above.

4. The criterion of Anghelutza as cited above.

5. (1) $Z^n + a_1 Z^{n-1} + \dots + a_n$ has its zeros satisfying the relation $|Z| < \sqrt{1 + |a_1 - 1|^2 + |a_2 - a_1|^2 + \dots + |a_n - a_{n-1}|^2 + |a_n|^2}$

6. The zeros of $Z^n + a_1 Z^{n-1} + \dots + a_n$ all lie on or within the circle $|Z| = |a_1| + |a_2|^{\frac{1}{2}} + \dots + |a_n|^{\frac{1}{n}}$ (2)

7. The absolute value of the zeros of $Z^n + a_1 Z^{n-1} + \dots + a_n$ has the upper limit

$$\left\{ 1 + \left(|a_1|^{1+p} + |a_2|^{1+p} + \dots + |a_n|^{1+p} \right)^{\frac{1}{p}} \right\}^{\frac{p}{1+p}}$$

where $p > 0$. (3)

8. The polynomial in 7. has as the upper limit of the absolute value of its zeros the maximum of the numbers

$$\left(\frac{\lambda_k}{a_k} \right)^{\frac{1}{k}} \qquad (k = 1, 2, \dots, n)$$

where $\lambda_k > 0$, $\lambda_1^{-1} + \lambda_2^{-2} + \dots + \lambda_n^{-1} \leqq 1$ (4); and also the

9. maximum of the numbers

$$|a_1| + \rho_1, \quad |a_2| \rho_1^{-1} + \rho_2, \quad \dots, \quad |a_{n-1}| (\rho_1 \rho_2 \cdots \rho_{n-2})^{-1} + \rho_{n-1},$$
$$|a_n| (\rho_1 \rho_2 \cdots \rho_{n-1})^{-1}$$

where $\rho_1 > 0$, $\rho_2 > 0$, $\dots$, $\rho_{n-1} > 0$ (5)

10. The number $\left[1 + |a_1|^m + \dots + |a_n|^m \right]^{\frac{1}{m}}$, $(1 < m \leqq 2)$ is an upper limit also. (6)

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(1) Carmichael & Mason, Bull. Am. Math. Soc. Vol. 21 (1914) P. 21
(2) J. L. Walsh, Annals of Math. Seras 2 Vol. 25, P. 286 (1924)
(3) Kuneyedo Tohoku Math. Journal Vol. 10 P. 187-188 (1916)
(4) Fujiwara Tohoku Math. Journal Vol. 10 P. 167-171 (1916)
(5) Kojima Tohoku Math. Journal Vol. 11 P. 119-127 (1917)
(6) Montel Comptes Renduss Vol. 193 P. 974-976 (1931)

11. Designate by $A_0, \dots, A_{n+1}$ the points $1, a_1, \dots, a_n, 0$ then designate by L the length of the broken line obtained by joining two consecutive points. The absolute values of the zeros are less than L. (1)

12. Let $A_0, \dots, A_{n+1}$ have the same meaning as in 11. Then let $l_0, l_1, \dots, l_n$ be the lengths of $A_0A_1, A_1A_2, \dots, A_n0$ and let $L_m^n = l_0^m + l_1^m + \dots + l_n^m$

then $M_m = \left[1 + L_m^{\frac{m}{m-1}} \right]^{\frac{m}{m-1}} \qquad (m > 1)$

is an upper limit of the absolute values of the zeros. (2)

Some of these upper limits are special cases of others in the list. For example: 1 is a special case of 10, where m approaches unity, and 11 is a special case of 12. However it is not our purpose to investigate the relations between these upper limits.

We now proceed to give a method whereby these results may be applied to the determination of R.

Let us suppose that a value for R is determined by one of the formulae above, (Page 3, 6, for example.) Let us call this value ρ_1. We have then that all zeros of are outside $|z| = \rho_1$. We make the change variable

$$z = \rho_1 y$$

Then $$Q(y) = \sum_{i=0}^{n} b_i \rho_1^i y^i$$

$Q(y)$ has all its zeros outside the unit circle

Form $$S(y) = y^n Q\left(\frac{1}{y}\right) = \sum_{i=0}^{n} b_{n-i} \rho_1^{n-i} y^i$$

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(1) Montel Comptes Rendues Vol. 193 P. 974-976 (1931)
(2) Montel Tohoku Math. Journal Vol. 41 P. 311-316 (1936)

$S(y)$ has as its zeros the reciprocals of the zeros of $\Phi(y)$ and all the zeros of $S(y)$ are within the unit circle.

We apply the methods 1. to 12. above to the polynomial $S(y)$ and thereby get a quantity, B_1 say, which is an upper limit to the absolute values of the zeros of $S(y)$. We take the least such quantity B_1 that these methods give us. Now $\beta_1 \leqq 1$. Suppose $\beta_1 < 1$. Then the zeros of $\Phi(y)$ are outside $|z| = \frac{1}{\beta_1}$ and hence the zeros of $P(z)$ are outside $|z| = \frac{\rho_1}{\beta_1}$. Thus if $\beta_1 < 1$ we have a better determination of R, namely $\rho_2 = \frac{\rho_1}{\beta_1}$. If we obtained $\beta_1 = 1$ the method we give here does not enable us to improve on our value for R. We can now proceed as before using ρ_2 in place of ρ_1. And if β_2 (obtained as was β_1) is less than unity, we obtain a value ρ_3 which is a better determination of R than either ρ_1 or ρ_2. We proceed in this manner as far as possible, that is, as long as the β's obtained are less than unity.

We now turn our attention to other questions, to properties of polynomials which have all their zeros outside the circle Our first result is the

<u>Theorem 5</u>. Let $P(z) = \sum_{i=0}^{n} b_i z^i$ be a polynomial of degree n with all its zeros outside the circle $|z| = R$. Let $n > 1$, $b_{n-1} \neq 0$. Then the polynomial of the $(n-1)^{st}$ degree

$$S(z) = \sum_{i=0}^{n-1} (n-i) b_i z^i$$

also has all its zeros outside the circle $|z| = R$

<u>Proof</u>: We divide our proof into two cases; case 1, $R \geqq 1$ case 2 $R < 1$*.

*This is unnecessary as the proof for case 1 is equally valid for case 2.

Case 1. $R \geq 1$.

$$Q(y) = y^n \sum_{i=0}^{n} b_i \frac{1}{y^i} = \sum_{i=0}^{n} b_{n-i} y^i$$ has all its zeros within the circle $|z| = \frac{1}{R}$ since the zeros of $Q(y)$ are the reciprocals of the zeros of $P(z)$ and $R \geq 1$.

There is a theorem due to Walsh (1) which states that if the zeros of a polynomial all lie within a circle in the z-plane, then all the zeros of the derivative of that polynomial also lie within that circle. Applying this result we have that all the zeros of

$$Q'(y) = \sum_{i=1}^{n} i b_{n-i} y^{i-1} = \sum_{i=0}^{n-1} (i+1) b_{n-i-1} y^i$$

all lie within the circle $|z| = \frac{1}{R}$.

If $b_{n-1} \neq 0$, the polynomial

$$S(y) = y^{n-1} \sum_{i=0}^{n-1} (i+1) b_{n-i-1} \frac{1}{y^i} = \sum_{i=0}^{n-1} (i+1) b_{n-i-1} y^{n-i-1}$$

has as its zeros the reciprocals of the zeros of $Q'(y)$. Hence all the zeros of $S(y)$ are outside $|z| = R$. We rewrite $S(y)$ in the following manner. As i ranges from 0 to $n-1$, $n-i-1$ ranges from $n-1$ to 0.

Thus $S(y) = \sum_{i=0}^{n-1} (n-i) b_i y^i$ has all its zeros outside $|z| = R$.

Case 2. $R < 1$.

We make the change of variable $z = Ry$, giving us that $P(z) = Q(y)$ where $Q(y) = \sum_{i=0}^{n} b_i R^i y^i$ has zeros $\frac{1}{R}$ times the corresponding zeros of $P(z)$. Thus $Q(y)$ has all its zeros outside the unit circle.

Let $Q(y) = \sum_{i=0}^{n} c_i y^i$ where $c_i = b_i R^i$, we apply the result of case one and have that

$$S(y) = \sum_{i=0}^{n-1} (n-i) c_i y^i = \sum_{i=0}^{n} (n-i) b_i R^i y^i$$

(1) J. L. Walsh: Trans. Am. Math. Soc. Vol. 22 P. 115 (1921)

has all its zeros outside the unit circle.

We now make the change of variable $w = Ry$ obtaining the polynomial $T(w)$ which has its zeros R times the corresponding zeros of $s(y)$. All the zeros of $T(w)$ are then outside $|z| = R$.

$$T(w) = \sum_{i=0}^{n-1} (n-i) b_i R^i \frac{w^i}{R^i} = \sum_{i=0}^{n-1} (n-i) b_i w^i$$

The proof of theorem 5 is now complete.

We can, of course, extend this result to the polynomial

$$Q(z) = \sum_{i=0}^{n-1} (n-i) b_i z^i$$

if $b_{n-2} \neq 0$, $n > 2$, for $Q(z)$ has all its zeros outside $|z| = R$ we obtain the result that, in this case,

$$M(z) = \sum_{i=0}^{n-2} (n-1-i)(n-i) b_i z^i$$

has its zeros all outside $|z| = R$.

Further if $b_{n-3} \neq 0$, $n > 3$ we have that

$$N(z) = \sum_{i=0}^{n-3} (n-2-i)(n-1-i)(n-i) b_i z^i$$

has all its zeros outside $|z| = R$.

Proceeding in this manner we have the

Theorem 6. Let $P(z) = \sum_{i=0}^{n} b_i z^i$ be a polynomial of degree $n > 1$ and let $b_i \neq 0$ $(i = 0, 1, \ldots, n)$. Then $R < \left| \frac{n b_0}{b_1} \right|$. where R is any quantity such that all the zeros of $P(z)$ are outside the circle $|z| = R$.

Proof: By theorem 1 there exists such an R, since $b_0 \neq 0$. Now $b_i \neq 0$ for each i. Hence by theorem 5 and the discussion above the polynomial

$$L(z) = \sum_{i=0}^{1} (2-i)(3-i) \cdots (n-i) b_i z^i$$

has all its zeros outside $|z| = R$.

But $L(z)$ has only one zero, namely $z = -\frac{n b_0}{b_1}$.

But this zero must lie outside $|z| = R$, that is

$$R < \left|\frac{n b_0}{b_1}\right|$$

In a paper in the Tohoku Mathematical Journal Obrechkoff gives some interesting theorems. Among these is the following.(1)

Let the zeros of the polynomial $P(z)$ of degree n be all outside the circle $|z| = R$, and let the zeros of the polynomial $Q(z)$ of degree m be in the circle $|z| \leqq r$, $r < R$. Let $\lambda > 0$ be an arbitrary (positive) number. Then the polynomial $2z[P'(z)Q(z) - \lambda P(z)Q'(z)] + (m\lambda - n)P(z)Q(z)$ has no zeros in the annulus $r < |z| < R$.

Now by the theorem of Walsh used in the proof of theorem 5, the zeros of $Q'(z), Q''(z), \ldots, Q^{(m-1)}(z)$ are all in the circle $|z| \leqq r$. Let $P(z) = \sum_{i=0}^{n} b_i z^i$, $Q(z) = \sum_{i=0}^{m} a_i z^i$

We have that

$$2z[P'(z)Q'(z) - \lambda P(z)Q''(z)] + (m\lambda - n)P(z)Q'(z)$$

has no zeros in $r < |z| < R$ and likewise

$$2z[P'(z)Q^{(m-1)}(z) - \lambda P(z)Q^{(m)}(z)] + (m\lambda - n)P(z)Q^{(m-1)}(z).$$

has no zeros in the annulus $r < |z| < R$.

But $Q^{(m-1)}(z) = m!\,a_m z + (m-1)!\,a_{m-1}$, $Q^{(m)}(z) = m!\,a_m$

Then $2z\left[\left(\sum_{i=0}^{n} i b_i z^{i-1}\right)(m!\,a_m z + (m-1)!\,a_{m-1}) - \lambda\left(\sum_{i=0}^{n} b_i z^i\right)(m!\,a_m)\right]$

$$+ (m\lambda - n)\left(\sum_{i=0}^{n} b_i z^i\right)(m!\,a_m z + (m-1)!\,a_{m-1})$$

has no zeros in the annulus $r < |z| < R$.

Factoring out an $(m-1)!$ and rewriting we have that

$$S(z) = (m\lambda - n)a_{m-1} b_0 + \sum_{i=0}^{n}\{2ma_m(i-1)b_{i-1} + 2a_{m-1} i b_i - 2\lambda m a_m b_{i-1}$$
$$+ ma_m(m\lambda - n)b_{i-1} + (m\lambda - n)a_{m-1} b_i\} z^i$$
$$+ \{2ma_m n b_n - 2\lambda m a_m b_n + ma_m(m\lambda - n)b_n\} z^{n+1}$$

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(1) N. Obrechkoff: Tôhoku Mathematical Journal: Sur les Racines des E'quations Algebriques Vol. 38 P. 93--100. See theorem VIII P. 100 (1933)

has no zeros within $r < |z| < R$ giving us the

Theorem 7. If $P(z) = \sum_{i=0}^{n} b_i z^i$ has all its zeros outside $|z| = R$, $Q(z) = \sum_{i=0}^{m} a_i z^i$ has all its zeros within $|z| \leq r$, $r < R$, then $S(z)$ has no zeros in the annulus $r < |z| < R$.

We note the result is independent of the numbers $a_{m-2}, \ldots, a_0$ and hence we obtain the same result by taking $a_{m-2} = a_{m-3} = \cdots = a_0 = 0$ and $Q(z) = a_m z^m + a_{m-1} z^{m-1} = z^{m-1}[a_m z + a_{m-1}]$

Note $Q(z)$ has an $(m-1)$ fold zero at $z = 0$ and a zero at $-\frac{a_{m-1}}{a_m}$. This gives us the

Theorem 8. Let $P(z) = \sum_{i=0}^{n} b_i z^i$ have all its zeros outside $|z| = R$. Let λ be any positive number, m any positive integer, α and β any two numbers such that $\left|\frac{\beta}{\alpha}\right| < R$. Then the polynomial

$$(\lambda m - n) b_0 \beta + \sum_{i=1}^{n} \{ m\alpha [2(i-\lambda-1) + m - n] b_{i-1} + \beta [2i + m^2\lambda - n] b_i \} z^i$$
$$+ m \alpha b_n [n + \lambda(m-2)] z^{n+1}$$

has no zeros in the annulus $\left|\frac{\beta}{\alpha}\right| < |z| < R$.

If in the theorem of Obreschkoff we take $Q(z) = z^m$ we have that the polynomial

$$2z[P'(z) z^m - \lambda P(z) m z^{m-1}] + (m\lambda - n) P(z) z^m$$

has no zeros within $|z| = R$ except $z = 0$. Factoring out z^m we have that the polynomial

$$2zP'(z) - 2\lambda m P(z) + (m\lambda - n) P(z) = 2zP'(z) - (\lambda m + n) P(z)$$

has no zeros within the circle $|z| = R$ except possibly $z = 0$. Moreover $z = 0$ cannot be a zero for if it were the constant term of the polynomial in question would vanish, i.e.

$$-(\lambda m + n) b_0 = 0.$$

But $b_0 \neq 0$, for $P(z) = \sum_{i=0}^{n} b_i z^i$ has all its zeros outside the circle $|z| = R$ and we apply theorem 1. Moreover $\lambda > 0$, m and n are positive integers, hence $\lambda m + n > 0$ and

$$-(\lambda m + n) b_0 \neq 0$$

and hence $z = 0$ is not a zero of our polynomial.

Moreover
$$2zP'(z) - (\lambda m + n) P(z) = \sum_{i=1}^{n} 2i b_i z^i - (\lambda m + n) \sum_{i=0}^{n} b_i z^i$$
$$= \sum_{i=0}^{n} (2i - \lambda m - n) b_i z^i$$

Moreover since $\lambda > 0$ is an arbitrary positive number, m an arbitrary positive integer then λm is an arbitrary positive number.

We sum all these results in the

<u>Theorem 9.</u> Let $P(z) = \sum_{i=0}^{n} b_i z^i$ be a polynomial of degree n with all its zeros outside the circle $|z| = R$. Let $\mu > 0$ be an arbitrary positive number. Then the polynomial

$$2zP'(z) - (\mu + n) P(z) = \sum_{i=0}^{n} (2i - \mu - n) b_i z^i$$

has no zeros within the circle $|z| = R$

We notice that the theorem leaves open the possibility of $\sum_{i=0}^{n} (2i - \mu - n) b_i z^i$ having zeros on the circle $|z| = R$ * At any rate, all the zeros of $\sum_{i=0}^{n} (2i - \mu - n) b_i z^i$ are outside the circle $|z| = R - \epsilon_1$, where ϵ_1 is an arbitrarily small positive number. Theorem 9 can now be applied the polynomial giving us the result that the polynomial

$$\sum_{i=0}^{n} (2i - \mu_1 - n)(2i - \mu - n) b_i z^i$$

where μ_1 is an arbitrary positive number has no zeros within the circle $|z| = R - \epsilon_1$

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* The polynomial cannot have any zeros on $|z| = R$ as shown in the Appendix. A similar statement can be made for the polynomials of theorem 10.

We can apply theorem 9 to this new polynomial and obtain the result that

$$\sum_{i=0}^{n} (2i-\mu_2-n)(2i-\mu_1-n)(2i-\mu-n)\, b_i z^i$$

the circle $|z| = R-\epsilon_1-\epsilon_2$ where μ_2 is an arbitrary positive number and ϵ_2 is an arbitrarily small positive number.

Proceeding in this way it is easy to show that the polynomials

$$R_s(z) = \sum_{i=0}^{n} \prod_{n=1}^{s} (2i-\mu_n-n)\, b_i z^i \qquad (s=1,2,\ldots)$$

where $\mu_1,\ldots,\mu_n,\ldots$ are arbitrary positive numbers, have no zeros within the circle $|z| = R-(\epsilon_1+\cdots+\epsilon_s)$ where $\epsilon_1,\ldots,\epsilon_s$ are arbitrarily small positive numbers. The proof of this statement is an easy induction on s. But we can make a stronger statement, the

Theorem 10.* Let $P(z) = \sum_{i=0}^{n} b_i z^i$ be a polynomial of degree n with all its zeros outside the circle $|z| = R$. Let $\mu_1,\mu_2,\ldots,\mu_n,\ldots$ be a sequence of arbitrary positive numbers. Then the polynomials

$$R_s(z) = \sum_{i=0}^{n} \prod_{n=1}^{s} (2i-\mu_n-n)\, b_i z^i \qquad (s=1,2,\ldots)$$

have no zeros within the circle $|z| = R$.

For suppose that $R_s(z)$ had the zero $z = z_1$ within the circle $|z| = R$, $|z_1| < R$, $R - |z_1| = \delta$, say. It was shown above that $R_s(z)$ has no zeros within the circle $|z| = R-(\epsilon_1+\cdots+\epsilon_s)$ where $\epsilon_1,\ldots,\epsilon_s$ are arbitrarily small positive numbers. We choose $\epsilon_1,\ldots,\epsilon_s$ so small that

$$\epsilon_1+\cdots+\epsilon_s = h < \delta$$

Then no zeros of $R_s(z)$ lie within the circle $|z| = R-h$. But z_1 lies within the circle $|z| = R-h$. Hence we have a

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* $R_s(z)$ cannot have zeros on or within $|z| = R$. See Appendix for this and for a simplified proof of theorem 10.

contradiction, and the theorem is proved.

Results different from, but similar to, theorems 7, 8, 9, 10, may be obtained from another theorem of Obrechkoff. (1)

The theorem states that if the zeros of $P(z)$, degree n are outside the circle $|z| = R$ and that if those of $Q(z)$ degree m are within the circle $|z| \leqq r$, $r < R$ and if α, α_1 are arbitrary, (non-null) complex numbers such that

$(1-|\alpha|)(1-|\alpha_1|) > 0$ and γ is a complex number such that $|\gamma| = 1$ then the polynomial $R(z) = P(\alpha z)\, Q\left(\frac{z}{\alpha_1}\right) + \gamma P\left(\frac{z}{\bar{\alpha}}\right) Q(\bar{\alpha}_1 z) \frac{\alpha^n}{\bar{\alpha}_1^{\,m}}$

has no zeros in the annulus $r \leqq |z| \leqq R$.

Let $P(z) = \sum_{i=0}^{n} b_i z^i$, $Q(z) = \sum_{i=0}^{m} a_i z^i$

By Walsh's theorem cited above $Q^{(m-1)}(z)$ also has all its zeros within the circle $|z| \leqq r$.

$$Q^{(m-1)}(z) = (m-1)!\, a_{m-1} + m!\, a_m z$$

$$Q^{(m-1)}\left(\frac{z}{\alpha_1}\right) = (m-1)!\, a_{m-1} + \frac{m!\, a_m}{\alpha_1}\,, \quad Q^{(m-1)}(\bar{\alpha}_1 z) = (m-1)!\, a_{m-1} + m!\, a_m \bar{\alpha}_1$$

Then we have that

$$R(z) = P(\alpha z)\left\{(m-1)!\, a_{m-1} + \frac{m!\, a_m z}{\alpha_1}\right\} + \gamma P\left(\frac{z}{\bar{\alpha}}\right)\left\{(m-1)!\, a_{m-1} + m!\, \bar{\alpha}_1 a_m z\right\} \frac{\alpha^n}{\bar{\alpha}_1^{\,m}}$$

has no zeros in the annulus $r \leqq |z| \leqq R$.

Using the same argument as in the proof of theorem 8 we have the

<u>Theorem 11.</u> Let $P(z)$ be a polynomial of degree n with all its zeros outside the circle $|z| = r$, let α and α_1, be non-null complex numbers such that

$$(1-|\alpha|)(1-|\alpha_1|) > 0$$

Let γ be a complex number such that $|\gamma| = 1$, and let c, d

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(1) N. Obrechkoff, Loc. Cit. P. 99 Theorem VII

be any two complex numbers such that $\left|\frac{c}{d}\right| < R$. Then the polynomial

$$R(z) = P(\alpha z)\left\{c + \frac{m d z}{\alpha_1}\right\} + \gamma P\left(\frac{z}{\bar{\alpha}}\right)\left\{c + m d \bar{\alpha}_1 z\right\}\frac{\alpha^n}{\bar{\alpha}_1^{\,m}}$$

where m is any positive integer, has no zeros in the annulus

$$\left|\frac{c}{d}\right| \leq |z| \leq R.$$

We note that theorem 11 is valid of we change the hypotheses on α, α_1, γ to read $|\alpha| > 1$, $|\alpha_1| > 1$, $|\gamma| \leq 1$.

If we take $Q(z) = z^m$, we get new results. Here

$$Q^{(m-1)}(z) = m!\, z$$

This gives us the

<u>Theorem 12.</u> Under the hypotheses (on the α, α_1, γ) of theorem 11, m, any positive integer, then the polynomial

$$R(z) = P(\alpha z)\left(\frac{m!\, z}{\alpha_1}\right) + \gamma P\left(\frac{z}{\bar{\alpha}}\right)(m!\, \bar{\alpha}_1 z)\frac{\alpha^n}{\bar{\alpha}_1^{\,m}}$$

has no zeros within or on the circle $|z| = R$ except $z = 0$. To show this we merely take the λ in the theorem of Obrechkoff to be equal to ϵ, an arbitrarily small number. Let us factor out the $m!\,z$ yielding us that

$$\mathcal{R}(z) = \frac{P(\alpha z)}{\alpha_1} + \gamma P\left(\frac{z}{\bar{\alpha}}\right)\bar{\alpha}_1 \frac{\alpha^n}{\bar{\alpha}_1^{\,m}}$$

has no zeros within or on $|z| = R$ except possibly $z = 0$.

If we write $P(z) = \sum_{i=0}^{n} b_i z^i$, $(b_0 \neq 0)$, we have that $z = 0$ is a zero of $\mathcal{R}(z)$ if and only if

$$\frac{b_0}{\alpha_1} + \gamma b_0 \bar{\alpha}_1 \frac{\alpha^n}{\bar{\alpha}_1^{\,m}} = 0$$

or if and only if $\frac{1}{\alpha_1} + \gamma \bar{\alpha}_1 \frac{\alpha^n}{\bar{\alpha}_1^{\,m}} = 0$

or if and only if $\bar{\alpha}_1^{\,m} + \gamma |\alpha_1|^2 \alpha^n = 0$. This we state as a <u>Corellary</u>. Under the hypotheses of theorem 12 the polynomial $R(z) = \frac{P(\alpha z)}{\alpha_1} + \gamma P\left(\frac{z}{\bar{\alpha}}\right)\frac{\bar{\alpha}_1 \alpha^n}{\bar{\alpha}_1^{\,m}}$ has

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(3) N. Obrechkoff loc. cit. Page 100

no zeros within or on the circle $|z|=R$ except when $\bar{\alpha}_1{}^m + \gamma|\alpha_1|^2\alpha^n = 0$. Then $z=0$ is a zero of $R(z)$.

Thus if α, α_1, γ satisfy the hypotheses of theorem 12 and we add the additional hypothesis that $\bar{\alpha}_1{}^m + \gamma|\alpha_1|^2\alpha^n \neq 0$ we have that $R(z)$ has all its zeros outside the circle

Multiplying by $\bar{\alpha}_1{}^m \alpha_1$ we have that

$$R(z) = \sum_{i=0}^{n} \left(\bar{\alpha}_1{}^m \alpha^i + \frac{\gamma|\alpha_1|^2\alpha^n}{\alpha^i} \right) b_i z^i$$

has all its zeros outside $|z|=R$ under the additional hypothesis.

This new polynomial itself has all its zeros outside and hence the corollary may be applied to it. If we proceed in this way we obtain finally the

<u>Theorem 13</u>. Let $P(z) = \sum_{i=0}^{n} b_i z^i$ be a polynomial of degree n, with all its zeros outside the circle $|z|=R$. Let $[\alpha_r, \beta_r, \gamma_r], (r=1,2,\ldots)$ be sets of complex numbers satisfying the relations

$$(1-|\alpha_r|)(1-|\beta_r|) > 0, \quad |\gamma_r| > 1, \quad \alpha_r \neq 0, \quad \beta_r \neq 0$$
$$\bar{\beta}_r{}^m + \gamma_r|\beta_r|^2\alpha_r^n \neq 0 \qquad (r=1,2,\ldots)$$

and let m be any positive integer then the polynomials

$$R_s(z) = \sum_{i=0}^{n} \prod_{r=1}^{s} \left(\bar{\beta}_r{}^m \alpha_r{}^i + \frac{\gamma_r|\beta_r|^2\alpha_r^n}{\alpha_r{}^i} \right) b_i z^i$$

have all their zeros outside the circle

An easy induction on s is all that is necessary to complete the proof.

If we take $\bar{\beta}_r = \alpha_r$, $m=n$ we have that the condition $\bar{\beta}_r{}^m + \gamma_r|\beta_r|^2\alpha_r^n \neq 0$ is automatically satisfied for if it were not $1 + \gamma_n |\beta_n|^2 = 0$, $|\beta_n| = \frac{1}{\gamma_n}$, $|\beta_n| = 1$ which contradicts the hypothesis $(1 - |\alpha_n|)(1 - |\beta_n|) > 0$. Moreover $|\beta_n| = |\bar{\beta}_n|$ so that theorem 13 now can be made to read <u>Theorem 14</u>. Let $P(z) = \sum_{i=0}^{n} b_i z^i$ be a polynomial of degree n with all its zeros outside the circle $|z| = R$, let $\alpha_1, \dots, \alpha_n, \dots$ be any sequence of non-null complex numbers, let $\gamma_1, \dots \gamma_n, \dots$ be complex numbers whose modulii are unity, then the polynomials

$$R_s(z) = \sum_{i=0}^{n} \prod_{n=1}^{s} \left(\alpha_n^i + \frac{\gamma_n |\alpha_n|^2}{\bar{\alpha}_n^i} \right) b_i z^i \quad (s = 1, 2, \dots)$$

have all their zeros outside the circle $|z| = R$.

APPENDIX

Theorem 9 can be written to read:

Let $P(z) = \sum_{i=0}^{n} b_i z^i$ be a polynomial of degree n with all its zeros outside $|z| = R$. Let μ be an arbitrary positive number.

Then the polynomial

$$Q(z) = \sum_{i=0}^{n} (2i - \mu - n) b_i z^i$$

has all its zeros outside $|z| = R$

The proof given above shows that $P(z)$ has no zeros within $|z| = R$. Let $\alpha = \mathrm{Min}(|\alpha_1|, \ldots, |\alpha_n|)$ where $\alpha_1, \ldots, \alpha_n$ are the zeros of $P(z)$. Then $\alpha - \rho = \epsilon > 0$ and we can find a ρ such that $R < \rho < \alpha$.

Then by the proof of theorem 9 given above since $P(z)$ has its zeros all outside $|z| = \rho$ then $Q(z)$ has no zeros within $|z| = \rho$, and hence cannot have any zeros on $|z| = R$. Hence $Q(z)$ must have all its zeros outside $|z| = R$.

Theorem 10 can be rewritten to read:

Let $P(z) = \sum_{i=0}^{n} b_i z^i$ be a polynomial of degree n with all its zeros outside $|z| = R$. Let be a sequence of arbitrary positive numbers . Then the polynomials

$$R_s(z) = \sum_{i=0}^{n} \prod_{r=1}^{s} (2i - \mu_r - n) b_i z^i \qquad (s = 1, 2, \ldots)$$

have all their zeros outside $|z| = R$.

The proof is an easy induction on s .

SUMMARY

~~The material of the last ten pages is entirely new and original, that of the first twelve pages is not all entirely new by any means but the treatment is original~~

Summary of first part: (Theorems 1 to 5)

Necessary and sufficient conditions for all the zeros of a polynomial to be outside the circle $|z|=R$ are derived. The problem of the determination of the best value of R for a given polynomial with a non-zero constant term is next taken up. An upper limit for possible values of R is derived, and necessary and sufficient conditions are given for that upper limit to be approached arbitrarily closely. Various formulae which estimate R are cited from the literature. Upper limits for the modulis of zeros of polynomials are cited and their use is enlisted in a method for the determination of the best value for R.

Summary of second part: (Theorems 5 through 14)

Among other results, it is shown that if $P(z)=\sum_{i=0}^{n} b_i z^i$ $b_{n-1}\neq 0, n>1$ has all its zeros outside $|z|=R$, then so has

$$\sum_{i=0}^{n-1}(n-i)\,b_i z^i$$

If $\mu_1, \dots, \mu_s, \dots$ is a sequence of positive (non-null) numbers then the polynomials

$$R_s(z)=\sum_{i=0}^{n}\prod_{r=1}^{s}(2i-\mu_r-n)\,b_i z^i \qquad (s=1,2,\dots)$$

have ~~no~~ all their zeros ~~within~~ outside $|z|=R$ if $P(z)$ has all its zeros outside $|z|=R$. Also if the α's, β's, γ's satisfy certain conditions then the polynomials (where m is a positive integer)

$$R_s(z)=\sum_{i=0}^{n}\prod_{r=1}^{s}\left(\bar{\beta}_r^{\,m}\alpha_r^{\,i}+\frac{\gamma_r|\beta_r|^2\alpha_r^{\,m}}{\bar{\alpha}_r^{\,i}}\right)b_i z^i \qquad (s=1,2,\dots)$$

have all their zeros outside $|z|=R$.

Also if $\alpha_1, \ldots, \alpha_r, \ldots$ is any sequence of complex numbers and $\gamma_1, \ldots, \gamma_r, \ldots$ are any complex numbers situated on the unit circle then the polynomials

$$R_s(z) = \sum_{i=0}^{m} \prod_{r=1}^{s} \left(\alpha_r^{\,i} + \frac{\gamma_r |\alpha_r|^2}{\alpha_r^{\,i}} \right) b_i z^i \qquad (s=1,2,\ldots)$$

have all their zeros outside the circle $|z|=R$.