The Theory of Operational Equations and Differential Transformations in Kantorovitch Spaces

Thesis by

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To Maria Chapoy-Vidaurri de Villaseñor

It is my pleasure here to acknowledge my profound indebtedness and gratitude to Professor A. D. Michal for his valuable advice and encouragement and for the privilege of being associated with the group of researchers in functional analysis that have gathered about him.

Condensation

The first chapter of this thesis is devoted to a complete description of lattices and semi-ordered spaces. A general classification is carried out and is illustrated by a copious collection of specific examples of lattices.

The second chapter is devoted to an exposition of the Kantorovitch theory of semi-ordered "function-spaces". It represents a digest of the material contained in a large number of scattered papers by Kantorovitch.

The third chapter is an historical introduction to the literature on lattices. It is meant to be encyclopaedic in scope; a serious effort is made to list and discuss the important contributions and their relations with algebra, geometry, the theory of operators and the theory of combinatorial topology.

In the remainder of the thesis the theory of linear transformations between a Banach and a Kantorovitch and between two Kantorovitch spaces is developed. The Helly theory of linear equations as extended by Hahn is further extended to the cases in question.

Among other original contributions is the definition of a differential in the most general linear limit-space, having all the desired properties with regard to composition, transformation and integration (which had to be developed separately). By the aid of a real-number function intrinsic in the original space and defined over a set of points having only one limit-point, it is possible to obtain an extension of the notion of topological order of completely continuous transformations and their differentials. An extension of the Riesz theory is presented.

We consider a set S of objects or elements, forming an Abelian group under the operation of addition. That is, if we denote the elements of the set by Latin letters a,b,c,$\cdots$ x,y,z, there exists a correspondence associating to each ordered pair of objects a,b (a and b are not necessarily different) of S an object $a + b$ of S, called the sum of a and b, and subject to the following postulates, where equality is taken to mean logical identity.

I. 1.) If a and b are in S, then $a + b$ is in S.

2.) $a + b = b + a$ for every pair of elements a, b in S.

3.) $a + (b + c) = (a + b) + c$ for every three elements a,b,c (not necessarily different) of S.

4.) There exists an element, denoted by 0, in S, such that $a + 0 = a$ for every element a in S.

5.) To each element a, there exists an element -a, such that $a + (-a) = 0$.

The element 0, called the null-element in 3.) easily may be shown to be unique, as can the element -a called the inverse of a in 4.), for a fixed element a. For these and other properties of Abelian groups, see van der Waerden I 1 B, page 15. For $a + a + \cdots + a$, (n terms) we write na; for $b + (-a)$ we write $b - a$; for $-a - a - \cdots -a$ (n terms) we write -na. The following rules easily can be proved. Q is the class of negative, positive and zero integers.

I'. 1.) $n(a + b) = na + nb$ for all a,b in S and every n in Q.

2.) $(n_1 + n_2)a = n_1a + n_2a$ for all n_1, n_2 in Q, and every a in S.

3.) $n_1(n_2a) = (n_1n_2)a$ for all n_1, n_2 in Q and every a in S.

S is assumed to be a partially ordered group, that is for certain elements y in S, the relation $y > 0$ is defined subject to the following postulates:

II. 1.) If $y > 0$, then y is not equal to 0.

2.) If $y_1 > 0$ and $y_2 > 0$, then $y_1 + y_2 > 0$.

3.) To every element y in S, there exists an element $(y)_+$, such that

a.) $(y)_+ > 0$

b.) $(y)_+ - y \geqq 0$

c.) If y^* is another element having the properties of a) and b), then $y^* - (y)_+ \geqq 0$.

See p. 30 for II 4).

<u>Definition 0</u>. <u>Partially ordered group</u>. Any set of elements satisfying postulates I and II will be called a partially ordered group.

<u>Definition 1 of greater</u>. y_1 is said to be greater than y_2, that is $y_1 > y_2$, if $y_1 - y_2 > 0$.

<u>Definition 2</u>. y_1 is said to be smaller than y_2, that is $y_1 < y_2$, if $y_2 > y_1$.

Definition 3. y is said to be positive if $y > 0$, and negative if $y < 0$.

Definition 4, of positive part. $(y)_+$, see axiom II 3, is said to be the positive part of y.

Definition 5, of negative part. $(-y)_+$ is said to be the negative part of y. It is denoted by $(y)_-$.

Definition 6, of absolute value. The absolute value of y is the element $|y| = (y)_+ + (y)_-$.

It is to be noticed that, unlike the group of real numbers, it is not necessarily true that one of the relations, $y > 0$, $y = 0$, $y < 0$ must hold for every element y in S, as will be clear in examples to follow. Clearly, by definition, $(y)_+$ and $(y)_-$, and consequently $|y|$ are unique. The absolute value, $|y|$ is in this case an element of S, and not necessarily a real number. Thus $|y|$ is not a proper metric; nevertheless a topology and theory of limits very similar to the case of the real numbers may be defined by means of it.

Theorem 1

a.) If $y_1 > y_2$ and $y_2 \geqq y_3$ then $y_1 > y_3$.

Proof. $y_1 - y_2 > 0$ and $y_2 - y_3 > 0$ or else $y_2 - y_3 = 0$. Adding and using postulate II 2 and the group postulates $y_1 - y_2 + (y_2 - y_3) > 0$ or $y_1 - y_3 > 0$ or $y_1 > y_3$.

b.) If $y_1 < y_2$ and $y_2 \leqq y_3$ then $y_1 < y_3$.

Proof. Similar to a.).

c.) If $y_1 < y_2$, then $-y_1 > -y_2$.

Proof. $y_2 - y_1 > 0$ by Def. 1. Add $-y_2$ to both sides and use Def. 1 again and a.) above: $-y_1 > -y_2$.

d.) If $y_1 \geqq y_2$ then $y_1 + y \geqq y_2 + y$ for every y in S.

Proof. $y_1 - y_2 \geqq 0$ by Def. 1.

That is $y_1 + y - (y_2 + y) \geqq 0$ by adding and subtracting y and using properties of Abelian groups. But this says that

$$y_1 + y \geqq y_2 + y \text{ by Def. 1.}$$

e.) If $y_1 \geqq y_2$ and $y_3 \geqq y_4$, then $y_1 + y_3 \geqq y_2 + y_4$.

Proof. $y_1 - y_2 \geqq 0$, $y_3 - y_4 \geqq 0$ by Def. 1.

Adding and using Postulate II 2

$$y_1 + y_3 - (y_2 + y_4) \geqq 0 \text{ or } y_1 + y_3 \geqq y_2 + y_4.$$

Theorem 2

a.) $(y)_- \geqq 0$

Proof. From Def. 5 and Postulate II.

b.) $(-y)_- = (y)_+$

Proof. From Def. 5.

c.) $(y)_+ \geqq y$, $(y)_- \geqq -y$

Proof. Postulate II 3.

d.) $(y)_+ - (y)_- = y$

Proof. $y + (y)_- - y \geqq 0$, also $y + (y)_- = (-y)_+ - (-y) \geqq 0$ by Postulate II 3 and Def. 1. But these two relations state that $y + (y)_-$ satisfies the conditions of Postulates II 3 a and II 3 b. Therefore $y + (y)_- \geqq (y)_+$. Replace y by -y and get $-y + (-y)_- \geqq (-y)_+$, or $-y + (y)_+ \geqq (y)_-$, that is, $(y)_+ \geqq (y)_- + y$; this together with $y + (y)_- \geqq (y)_+$ yields $y + (y)_- = (y)_+$ by virtue of Postulate I 1, or $y = (y)_+ - (y)_-$.

e.) $(y_1 + y_2)_+ \leqq (y_1)_+ + (y_2)_+$ and $(y_1 + y_2)_- \leqq (y_1)_- + (y_2)_-$

<u>Proof</u>. $(y_1)_+ \geqq y_1$ and $(y_2)_+ \geqq y_2$. Then $(y_1)_+ + (y_2)_+ \geqq y_1 + y_2$ by Theorem 1 e. Also $(y_1)_+ + (y_2)_+ \geqq 0$. Therefore $(y_1)_+ + (y_2)_+ \geqq (y_1 + y_2)_+$ by Axiom II 3. The other result follows from $(-y)_- = (y)_+$

<u>Theorem 3</u>

a.) $|y| \geqq 0$

<u>Proof</u>. $|y| = (y)_+ + (y)_- \geqq 0$ by Postulate II 2.

b.) $|y| \geqq y$, $|y| \geqq -y$.

<u>Proof</u>. $y = (y)_+ + (y)_- \geqq (y)_+ \geqq y$.

c.) $|-y| = |y|$

<u>Proof</u>. $|-y| = (-y)_+ + (-y)_- = (y)_- + (y)_+ = |y|$

d.) If $y = 0$ then $(y)_+ = 0$ and $(y)_- = 0 = |y|$.

<u>Proof</u>. Def. 4 and Postulate II 3.

e.) If $y \neq 0$ then $|y| > 0$

Proof. Either $(y)_+ > 0$ or $(y)_- > 0$ since

$y = (y)_+ - (y)_-$. Then $y = (y)_+ + (y)_- > 0$.

f.) $|y_1 + y_2| \leqq |y_1| + |y_2|$

Proof. $|y_1 + y_2| = (y_1 + y_2)_+ + (y_1 + y_2)_- \leqq$

$(y_1)_+ + (y_2)_+ + (y_1)_- + (y_2)_- = |y_1| + |y_2|$

The results stated in Theorem 3, especially a.) $|y| \geqq 0$ c.) $|-y| = |y|$, d and e.) $|y| = 0$ if and only if $y = 0$, f.) $|y_1 + y_2| \leqq |y_1| + |y_2|$ make very striking the formal resemblance between this "absolute value", which, we recall, is an element of the space S and the ordinary absolute value of ordinary Euclidean vector analysis, which is a real positive number and not a vector, see Sierpinski IV 1 B page 75 or Kuratowski IV 1 B page 82.

Definition 7 of upper bound.

An element y^* is called an upper bound of the set of elements $y_1, y_2 \cdots y_n$ if

$$y^* \geqq y_i \quad \text{for } i = 1, \cdots n.$$

Definition 8 of lower bound.

An element y^* is called a lower bound of the set of elements $y_1, y_2 \cdots y_n$ if

$$y^* \leqq y_i \quad \text{for } i = 1, \cdots , n.$$

Theorem 4. If y is an upper bound of the set $y_1, \cdots y_n$ then $-y$ is a lower bound of the set $-y_1, \cdots -y_n$.

Proof. Theorem 1 c.

Theorem 5. Every set of elements $y_1, \cdots, y_n$, where n is a finite positive integer has at least one upper bound and at least one lower bound.

Proof. By Postulate II 3, to each element y_i there exists an element $(y_i)_+$, such that

$(y_i)_+ \geqq 0$, $(y_i)_+ \geqq y_i$. Then

$(y_1)_+ + (y_2)_+ + \ldots + (y_n)_+$ is an upper bound.

A lower bound may be found by considering the upper bound of the set $-y_1, -y_2, \cdots, -y_n$.

Definition 9 of lub and glb.

The lub (least upper bound) of any set of elements in S is an upper bound y^* such that for every other upper bound y', $y' - y^* \geqq 0$. The definition of glb (greatest lower bound) is similar.

Neither the lub nor the glb has been shown to exist for every set, but if it exists, it is necessarily unique from the definition. The following theorem guarantees for finite sets a glb and lub that are independent of the ordering of the elements.

Theorem 6. Every set with a finite number of elements $(y_1, \cdots, y_n)$ has a lub.

Proof.

a) $n = 2$. Let y_1, y_2 be any two elements in S. Put

$y^* = y_1 + (y_2 - y_1)_+$. Then y^* is the lub. For $y^* \geqq y_1$ and $(y_2 - y_1)_+ \geqq y_2 - y_1$, so that $y^* = y_1 + (y_2 - y_1)_+ \geqq y_2$. Now suppose that y' is another upper bound. Then $y' - y_1 \geqq 0$ and $y' - y_1 \geqq y_2 - y_1$, consequently $y' - y_1 \geqq (y_2 - y_1)_+$ and therefore $y' \geqq y_1 + (y_2 - y_1)_+ = y^*$. Therefore y^* is the lub.

b) Assume the theorem proved for n elements. To prove it for n + 1. Put $y^* = \text{lub}\ (\text{lub}\ (y_1, \cdots ,y_n), y_{n+1})$. Then $y^* = \text{lub}\ (y_1, \cdots ,y_n)$. For $y^* \geqq y_{n+1}$, lub $(y_1,y_2, \cdots y_n)$ by a) and if $y' \geqq y_1,y_2,\cdots,y_n,y_{n+1}$ then $y' \geqq \text{lub}\ (y_1, \cdots y_n), y_{n+1}$ consequently $y' \geqq y^*$ and y^* is the lub.

Theorem 7. Every set with a finite number of elements $y_1,y_2, \cdots,y_n$ has a glb and glb $(y_1,y_2, \cdots ,y_n) = -\ \text{lub}\ (-y_1,-y_2, \cdots ,-y_n)$.

Proof. Use Theorem 4.

Theorem 8.

a) $(y)_+ = \text{lub}\ (0,y)$; $(y)_- = \text{lub}\ (0,-y)$

Proof. See Def. 4 and Postulate II 3.

b) $\text{lub}\ (y_1,y_2, \cdots y_n,y_1) = \text{lub}\ (y_1,y_2, \cdots y_n)$

Proof. Theorem 6.

c) $\text{lub}\ (y_1+ y,y_2 + y, \cdots ,y_n + y) = y + \text{lub}\ (y_1,y_2, \cdots,y_n)$

Proof. $y + \text{lub}\ (y_1,y_2, \cdots ,y_n) \geqq y + y_i \quad (i = 1,2,\cdots,n)$

Thus α) $y + \text{lub}(y_1,y_2, \cdots ,y_n) \geqq \text{lub}(y + y_1,\cdots,y + y_n)$

Also $\text{lub}(y + y_1, \cdots ,y + y_n) \geqq y + y_i \quad (i = 1,\cdots,n)$

that is $\mathrm{lub}(y + y_1, \cdots, y + y_n) - y \geqq \mathrm{lub}(y_1, \cdots y_n)$

or β), $\mathrm{lub}(y + y_1, \cdots, y + y_n) \geqq y + \mathrm{lub}(y_1, \cdots y_n)$

From α) and β) the theorem follows.

d) $|2y| = 2\ \mathrm{lub}(y, -y)$

Proof. $|2y| = (2y)_+ + (2y)_- = \mathrm{lub}(2y, 0) + \mathrm{lub}(-2y, 0)$

$= -y + \mathrm{lub}(2y, 0) + y + \mathrm{lub}(-2y, 0)$

$= \mathrm{lub}(y, -y) + \mathrm{lub}(-y, y) = 2\ \mathrm{lub}(y, -y)$

e) If $y_i' \geqq y_i$ $(i = 1, \cdots, n)$ then

$\mathrm{lub}(y_1', \cdots, y_n') \geqq \mathrm{lub}(y_1, \cdots, y_n)$

Proof. The left side $\geqq y_i' \geqq y_i$ for all i.

Theorem 9.

a) $\mathrm{lub}\left[\mathrm{lub}(y_1, \cdots, y_n), \mathrm{lub}(y_{n+1}, \cdots, y_{n+m})\right] =$

$\mathrm{lub}\left[y_1, \cdots y_{n+m}\right]$.

b) $\left[\mathrm{lub}(y_1, y_2)\right]_+ = \mathrm{lub}\left[(y_1)_+, (y_2)_+\right]$

Proof. $\mathrm{lub}\left[(y_1)_+, (y_2)_+\right] = \mathrm{lub}\left[\mathrm{lub}(y_1, 0), \mathrm{lub}(y_2, 0)\right]$

$= \mathrm{lub}(y_1, y_2, 0) = \mathrm{lub}\left[\mathrm{lub}(y_1, y_2), 0\right]$

$= \left[\mathrm{lub}(y_1, y_2)\right]_+$

Theorem 10. $\mathrm{lub}(a,b) + \mathrm{glb}(-a,-b) = 0$

Proof. $\mathrm{lub}(a,b) + \mathrm{glb}(-a,-b) = \mathrm{lub}(a,b) - \mathrm{lub}(a,b)$

from Theorem 4.

Theorem 11. $\mathrm{lub}(a,b) + \mathrm{glb}(a,b) = a + b$

Proof. From Theorem 8 c

$\mathrm{glb}(-a,-b) + a + b = \mathrm{glb}(b,a)$ or

$a + b = \mathrm{glb}(b,a) - \mathrm{glb}(-a,-b)$. Use Theorem 10.

<u>Theorem</u> <u>12</u>. Distributive Law. $\text{glb}\left[\text{lub}(a,b),c\right] = \text{lub}\left[\text{glb}(a,c),\text{glb}(b,c)\right]$

$\text{lub}\left[\text{glb}(a,b),c\right] = \text{glb}\left[\text{lub}(a,c),\text{lub}(b,c)\right]$.

Proof 1) $\text{lub}\left[\text{lub}\{\text{glb}(a,b),c\},b\right] = \text{lub}\left[\text{glb}(a,b),b,c\right]$

$= \text{lub}\left[\text{lub}\{\text{glb}(a,b),b\},c\right] = \text{lub}\left[\text{lub}(b.c),b\right]$

$\geqq \text{lub}\left[b,\text{glb}\{a,\text{lub}(b,c)\}\right]$. Also

2) $\text{glb}\left[b,\text{lub}\{c,\text{glb}(a,b)\}\right] \geqq \text{glb}\left[b,\text{glb}(a,b)\right] =$

$\text{glb}\left[\text{glb}(a,b),\text{glb}(b,c)\right] = \text{glb}\left[b,\text{glb}\{a,\text{lub}(b,c)\}\right]$

Add inequalities 1) and 2) together and use Theorem 11.

3) $\text{lub}\left[c,\text{glb}(a,b)\right] \geqq \text{glb}\left[\text{lub}(b,c),a\right]$. Interchange

a and b and get $\text{lub}\left[c,\text{glb}(a,b)\right] \geqq \text{glb}\left[b,\text{lub}(a,c)\right]$.

$\text{lub}\left[c,\text{glb}(a,b)\right] \geqq c$, $\text{glb}\left[b,\text{lub}(a,c)\right]$, therefore

$\text{lub}\left[c,\text{glb}(a,b)\right] \geqq \text{lub}\left[c,\text{glb}(b,\text{lub}\left[a,c\right])\right] \geqq$

$\text{glb}\left[\text{lub}(a,c),\text{lub}(b,c)\right]$. This last step follows from

3) with lub (a,c) instead of a. Also

$\text{lub}\left[\text{glb}(a,b),c\right] \leqq \text{lub}\left[(a,c)\right],\text{lub}\left[(b,c)\right]$. Therefore

$\text{lub}\left[\text{glb}(a,b),c\right] \leqq \text{glb}\left[\text{lub}(a,c),\text{lub}(b,c)\right]$.

From this and the result above follows the first statement of the theorem. The second statement follows similarly.

<u>Theorem</u> <u>13</u>. $\text{lub}\left[\text{glb}(a,b),c\right] \geqq \text{glb}\left[a,\text{lub}(b,c)\right]$

<u>Proof</u>. See 3) in Theorem 12.

<u>Theorem</u> <u>14</u>. $\text{lub}(a+c,b+c) \geqq \text{lub}(a,c) + \text{glb}(b,c)$ or

$\text{lub}(a,b) \geqq (a)_+ - (b)_-$

<u>Proof</u>. $a + c \geqq a + \text{glb}(b,c)$; $a + c - \text{glb}(b,c) \geqq a$

$\text{lub}\left[a + c - \text{glb}(b,c), \text{lub}(b,c)\right] \geqq \text{lub}(a,c)$. Add glb(b,c) to both sides, use Theorem 8 c and

$\text{lub}\left[a + c, \text{lub}(b,c) + \text{glb}(b,c)\right] \geqq \text{lub}(a,c) + \text{glb}(b,c)$

Use Theorem 11 and the left hand side is seen to be lub(a + c,b + c). The second statement of Theorem 14 follows if c = 0 in the first statement.

Up to now, we have been concerned with an Abelian group with a partial-ordering relation. Now we assume that there exists an operation of multiplication of the elements of this group by real numbers, obeying the usual laws of algebra. This is stated in the form of the following set of postulates.

III

1.) If y is in S and λ is in the space R of real numbers, then λy is in S.

2.) $(\lambda_1 + \lambda_2)y = \lambda_1 y + \lambda_2 y$ for all λ_1, λ_2 (not necessarily different) in R, and every y in S.

3.) $(\lambda_1 \lambda_2)y = \lambda_1(\lambda_2 y)$ for all λ_1, λ_2 (not necessarily different) in R and every y in S.

4.) $\lambda(y_1 + y_2) = \lambda y_1 + \lambda y_2$ for every in R and all y_1, y_2 (not necessarily different) in S.

5.) $1 \cdot y = y$ for every y in S.

6.) If $y > 0$, $\lambda > 0$, then $\lambda y > 0$, where y is in S and λ is in R.

We have already explicitly defined a multiplication of the elements by integers, positive, negative or zero. The operation of multiplication by real numbers is taken as an undefined notion. The significance of this operation will vary with the particular example of space considered. All that we require of it is that it satisfy the postulates III (1 - 6). The postulates III (1 - 5) assure that it will be consistent with the defined operation governed by the rules I' (1 - 3). That is, by an obvious application of the postulates III (1 - 5) we can prove that if λ is an integer, positive, negative or zero, the defined multiplication and the undefined multiplication by λ must coincide.

Definition 10 of partially ordered system. A set of elements satisfying postulates I - III will be called a partially ordered system.

Theorem 15.

a) If $\lambda < 0$ and $y > 0$, then $\lambda y < 0$.

Proof. $-(\lambda y) = (-\lambda)y > 0$. Therefore $\lambda y < 0$ from Theorem 1 c.

b) If $\lambda > 0$ and $y < 0$, then $\lambda y < 0$. For $-(\lambda y) = \lambda(-1\cdot y) = \lambda(-y) > 0$ since if $y < 0$, $-y > 0$. Therefore $-\lambda y > 0$ and $\lambda y < 0$.

c) If $\lambda < 0$ and $y < 0$, then $\lambda y > 0$. $-(\lambda y) = (-\lambda)y < 0$ from b). Therefore $\lambda y > 0$.

Theorem 16.

a) If $\lambda > 0$, then $\text{lub}(\lambda y_1, \lambda y_2, \cdots, \lambda y_n) = \lambda\, \text{lub}(y_1, \cdots y_n)$.

<u>Proof</u>. $\text{lub}(y_1, y_2, \cdots, y_n) \geqq y_i \quad (i = 1, 2, \cdots, n)$

Multiply both sides of the inequality by $\lambda > 0$ and use postulate III 6.

$\lambda\,\text{lub}(y_1, \cdots, y_n) \geqq \lambda y_i, \ (i = 1, \cdots, n).$

Therefore (1) $\lambda\,\text{lub}(y_1, \cdots, y_n) \geqq \text{lub}(\lambda y_1, \cdots, \lambda y_n)$

Substitute $1/\lambda$ for λ and λy_i for y_i, then

$\frac{1}{\lambda}\,\text{lub}(\lambda y_1, \cdots, \lambda y_n) \geqq \text{lub}(y_1, \cdots, y_n)$. Multiply both sides by λ and get

(2) $\text{lub}(\lambda y_1, \cdots, \lambda y_n) \geqq \lambda\,\text{lub}(y_1, \cdots, y_n)$. This together with (1), gives the equality.

b) If $\lambda < 0$, then $\text{lub}(\lambda y_1, \cdots, \lambda y_n) = \lambda\,\text{glb}(y_1, \cdots y_n)$

<u>Proof</u>. $\text{lub}(\lambda y_1, \cdots, \lambda y_n) = \text{lub}(-|\lambda| y_1, \cdots, -|\lambda| y_n) =$

$-\text{glb}(|\lambda| y_1, \cdots, |\lambda| y_n)$ by Theorem 4 $= -|\lambda|\,\text{glb}(y_1, \cdots y_n) =$

$\lambda\,\text{glb}(y_1, \cdots, y_n)$ by extension of a).

c) If $\lambda > 0$ then $(\lambda y)_+ = \lambda (y)_+$ and $(\lambda y)_- = \lambda (y)_-$

<u>Proof</u>. $(\lambda y)_+ = \text{lub}(0, \lambda y) = \lambda\,\text{lub}(0, y) = \lambda (y)_+$

$(\lambda y)_- = \text{lub}(0, -\lambda y) = \lambda\,\text{lub}(0, -y) = \lambda (y)_-$ by Theorem 8 a)

d) $|\lambda y| = |\lambda|\,|y|$

<u>Proof</u>. If $\lambda > 0$, then $|\lambda y| = (\lambda y)_+ + (\lambda y)_- =$

$\lambda (y)_+ + \lambda (y)_- = \lambda[(y)_+ + (y)_-] = |\lambda|\,|y|$

If $\lambda < 0$ then

$|\lambda y| = |-\lambda y| = |-\lambda| \cdot |y| = |\lambda| \cdot |y|$ by Theorem 3 c) and above.

e) $y = \text{lub}(y_1, -y)$

Proof. $|2y| = 2\ \text{lub}(y,-y)$ by Theorem 8 d.

$= 2\ |y|$ by d) above.

Multiply both sides by 1/2.

A useful concept is that of a "structure", see Hausdorff IV 1 B, p. 139 or Ore, I, 2 which we define as a set of elements y, with an ordering relation $\geqq$ existing between some pairs of elements x,y, subject to the following postulates, where $x \geqq y$ means $y \leqq x$.

IV. 1) $x \geqq x$

2) If $x \geqq y$ and $y \geqq z$ then $x \geqq z$.

3) If x and y are any two elements there is an element $x \cap y$ such that $x \cap y \leqq y$; $x \cap y \leqq x$ and $x \cap y \geqq$ any element z such that $z \leqq y, x$. $x \cap y$ is called the cross-cut of x and y.

4) If x and y are any two elements, there is an element $x \cup y$ such that $x \cup y \geqq x,y$ and $x \cup y \leqq$ any z such that $z \geqq x,y$. $x \cup y$ is called the union of x and y.

This set need not be an Abelian group. We see, however, that any partially ordered group satisfies axioms IV. Some examples of partially ordered sets that are especially important are as follows. If $x \geqq y$ but $x \neq y$ then we write $x > y$.

Example 1. Sub-classes of a class.

Consider any class of well-defined elements. Consider now the class whose <u>elements</u> are <u>sets of elements</u> of the given class and in fact is made up of the totality of such sets (the given class is now considered as fixed). If the given class consists of the numbers 1, 2, 3, then the class of sub-sets is composed of 0, (1), (2), (3), (1,2), (1,3), (2,3), (1,2,3). Each one of these brackets, regarded as entity is an element of the new class. The number 0, written at the first of the above sequence represents the null-class, a mathematical fiction which has been created for convenience, and means the class without any members. The symbol (1) means the class with one member, namely the integer 1. The symbol (1,2) is the class with the two members 1 and 2. Remember that the numbers 1, 2 are <u>elements</u> of the given class. Notice that we include the given class itself as a sub-set of itself (called an improper sub-set). Also notice that no attention is paid to order in sets. This is an example wherein the initial set is finite. In the case of an infinite set, the procedure is similar.

<u>Example I</u>. Our structure will consist of the set of sub-sets A, B, etc., where $A \geqq B$, read, "A includes B", if every element of the original set occurring in B also occurs in A. $A \cap B$ is the set of elements of the original set occurring in both A and B. $A \cup B$ is the set of elements of the original set occurring in either A or B, or both. In most treatises on the theory of point sets, $A \supseteqq B$ is written for $A \geqq B$ and $A \supset B$ is written for $A > B$.

Example 2. Positive integers.

The set is that of the positive integers, where $a \leqq b$ means that $\frac{b}{a}$ is an integer, i.e. a divides b. $a \cup b$ is the least common multiple of a and b, called the l.c.m. and is the smallest (in the ordinary sense) integer that is a multiple of both a and b. $a \cap b$ is the greatest common divisor called the g.c.m. and is by definition the greatest (in the ordinary sense) integer that divides both a and b. Both these numbers exist, see the algorithm of division, Dickson I 1B, pp. 1-2.

Example 3. Set of polynomials.

The set is that of all polynomials such as $a_n x^n + \cdots + a_o$ in a single unknown x, and real numbers as coefficients. $A \cup B$ is the least common multiple of the two polynomials, that is the polynomial of smallest (in the ordinary sense) degree such that it = fA and = gB, where f and g are polynomials in x. $A \cap B$ is the greatest common divisor of the two polynomials, defined as the polynomial of greatest degree (in the ordinary sense) that will yield a polynomial quotient when A and B are divided by it. These always can be found, see van der Waerden I 1 B vol. I pp. 52-53 and König I 1 B. $A > B$ if A/B is a polynomial, that is if B divides A with no remainder.

Example 4. Linear sub-spaces of a linear space.

A linear space is an Abelian group with number multipliers satisfying postulates I (1 - 5) and III (1 - 5). A linear sub-space of a space is a set of elements of the given space that themselves form a linear space. One element is considered as being a linear space, degenerately. If we consider the set of all linear sub-spaces of a linear space, these form a partially ordered set, where $A > B$ is defined as in example 1. The cross-cut of two linear-sub spaces is automatically a linear sub-space. However $A \cup B$ is defined as the smallest linear space containing A and B, and is got by taking all possible linear combinations, with real number coefficients, of elements in A and B, see van der Waerden I 1 B.

A finite dimensional linear sub-space is one composed of all possible linear combinations, with real number multipliers, of elements of the space selected from a certain fixed set of elements, finite in number $y_1, y_2, \cdots y_n$. Notice now that the totality of finite-dimensional linear sub-spaces of a given linear space form a structure, which is a sub-structure of the first-mentioned structure.

Example 5. Ordered sets.

An ordered set is defined as a set of elements a, b, etc. with an ordering relation $>$ such that for every pair a,b of elements either $a > b$, $b < a$, or $a = b$, where $a = b$ is taken as identity, and there is no fourth possibility. Moreover, the three possibilities are considered as mutually exclusive. Obviously, given the pair a,b, the union of a and b is either a or b according as $a > b$ or $a < b$; in case $a = b$ then $a \cup b = a = b$. $a \cap b$ is either a or b according as b $>$ a or a $>$ b, etc. Examples of ordered sets are: the set of real numbers; the set of positive integers; any set of cardinal numbers (for a definition of cardinal numbers and a proof of this statement, see Hausdorff IV 2 B, pp. 25-77 or Kamke IV 1 B, or Pierpont IV 1 B, vol. II, pp. 276-323); a set of ordinal numbers; the set of all circles with centers at the origin and lying in the Euclidean plane, ordered according to length of radius.

Example 6. Hereditary and additive families.

Consider a class P of sets X,Y, ··· , of objects. This class, or family is called additive if, for any two sets X,Y in the family P, X + Y is a set in P, (X + Y is defined as the point-set sum, see example 1). It is called hereditary if X in P and Y $<$ X, in the sense of example 1, together imply that Y is in P (is a member of the family P). This structure can be considered a generalization and abstraction of example 1. See Kuratowski, IV 1 B, pp. 29-32. Examples 7 - 11 are special cases of this type of family.

Example 7. Sub-groups of a group.

A group is a set of objects a,b, ··· , such that to each ordered pair a,b of objects (not necessarily distinct) there is associated an object of the group designated by ab, and called the product ab. The products ab and ba may or may not happen to be distinct. The product aa exists and is again an object of the group.

When an operation (such as this product) always yields an object of a given set when applied to every possible pair of objects (or elements) in the set, we say that the set is closed under this operation. We require further of a group G, that the objects satisfy the following axioms, see van der Waerden I, 1 B, p. 15

V. 1.) To each order pair a,b of objects in G there is an object ab in G.

2.) a(bc) = (ab)c for every triple a,b,c (not necessarily different) in G.

3.) There exists an element 1 in G, such that for every element a in G

$$a \cdot 1 = 1 \cdot a = a$$

4.) To each element a, there exists an element a^{-1} in G such that

$$a^{-1}a = 1 = aa^{-1}.$$

A sub-group H of G is any set of elements of G that themselves form a group. The class of all sub-groups (themselves considered as elements) A,B,··· of a given group is a structure if we define $A \cap B$ as the cross-section (in the point-set sense) of A and B and $A \cup B$ as the smallest sub-group of G containing both A and B. $A \succ B$ is defined as point-set inclusion, example 1. See van der Waerden I 1 B, Chap. 2, vol. I.

Example 8. Sub-rings of a ring.

A ring is defined as a set of elements with two different operations, sum and product, each of which assigns to each ordered pair of elements another element, a + b and ab resp. The ring is an Abelian group with respect to the sum operation, see postulates I and further, the product operation obeys the associative law:

$$a(bc) = (ab)c$$

for all elements (not necessarily different) in G. Also we have the distributive laws:

VI. 1.) $a(b + c) = ab + ac$

2.) $(b + c)a = ba + ca$

If in addition, ab = ba for all elements then the set is known as a commutative ring. A sub-ring is any sub-set of elements of a ring that themselves, as a totality, form a ring.

The class of all sub-rings A,B,··· of a ring is a structure if by $A \cap B$ we mean the cross-cut, in the point-set sense, of A and B. Obviously $A \cap B$ is a sub-ring. $A \cup B$ is the smallest ring K that includes A and B, in the point-set sense. When we say the smallest ring, we mean that if we omit a single element a from K, then either a is in A or B or both; or in case a is not in A or B, the set resulting from the omission of the element a is not a ring.

Example 9. Ideals of a ring, left or right.

A left (right) ideal of a ring R, is a sub-ring H such that rh,(hr) is in the sub-ring H whenever h is in H and r is any element of R. The class of all left-ideals of a given ring R forms a structure, as does the class of all right ideals. $A \cap B$ is taken to be the cross-cut in the point-set sense. $A \cup B$ is the smallest ideal containing the union in the point-set sense; $A > B$ if A includes B in the point-set sense.

Example 10. Sub-fields of a field.

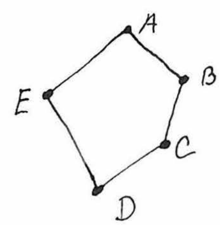
A field is a set of objects with two operations sum and product; a + b and ab resp., such that it is an Abelian group with respect to the sum and a group with respect to the product and it satisfies the distributive laws VI. Thus a field is simply a ring whose product is uniquely solvable and that contains a unit element. A sub-field is a set of elements of the given field, that form a field. The class of all sub-fields A,B,... of a given field is a structure if we define $A > B$ as point-set inclusion and $A \cap B$, $A \cup B$ similarly to examples 7 and 8.

Example 11. The ideals of a field.

See example 9.

Example 12. Hasse diagrams

The Hasse diagram is often used by algebraists to represent structures with a finite number of elements. They study these diagrams and work out certain theorems by a synthetic graphical method and then write up their results using proofs that are simply analytical verifications of the theorems and are non-informative as to the origin of the idea. Then they pretend that it was all an inspiration sent from heaven. In the Hasse diagram, each element of the structure is represented by a point in the plane $A < B$ if the y-coordinate of B is greater than that of A. If $A < B$ and there is no element X such that $A < X < B$ and X is different from A and B, then A and B are to be joined by a straight line segment. Every connected plane complex of lines and points is to be regarded as a structure if it obeys the postulates IV.



Example 13. Boolean algebras.

A Boolean algebra, in the classical sense, is an algebra of logic in which the elements are propositions A,B ··· . $A < B$ means, the proposition B implies the proposition A. $A \cup B$ is the disjunction and means either A is true or B is true or both are true. $A \cap B$ means both A and B are true. A' is the negation of A (to be used later). N is the identically false proposition, O is the identically true proposition.

Example 14. The structure of equivalence relations. See Birkhoff I 8.

Let C be any class of objects and let x and y be equivalence relations on C. An equivalence relation is a relation that is either true or not true for each pair of elements a, b of C.

1.) a x a

2.) If a x b, then b x a

3.) If a x b and b x c then a x c.

We define the cross-cut (or meet) $x \cap y$ of the equivalence relation x and y as the relation w on C such that:

a w b if and only if a x b and a y b.

By the union (join) $x \cup y$ of x and y is the equivalence relation z with the property that a x b or a y b implies a z b. The union and cross-cut are obviously equivalence relations. $x \supseteqq y$ if $x \cap y = y$ and $x \subseteqq y$ if $x \cup y = y$.

We shall consider a few specific examples of spaces satisfying postulates I & II. Postulate III can be shown to be included in I.

Example 15. n-dimensional vector space, $R^{(n)}$.

Its elements are ordered sets $a = (a_1, \cdots a_n)$ of n real numbers. It is composed of the totality of such sets. The sum, a + b of the vectors $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ is defined as the vector $a + b = (a_1 + b_1, \cdots, a_n + b_n)$. The element $y = (y_1, \cdots, y_n)$ is > 0 if $y_i \geqq 0$ for $i = 1, \cdots, n$, and at least one $y_i > 0$. $(y)_+ = \text{lub}(0,y)$ is defined as the vector $(\max(y_1,0), \cdots, \max(y_n,0))$, where max(l,m) means the greater of the two numbers l,m. Thus $|y|$ is the vector $(|y_1|, |y_2|, \cdots, |y_n|)$ and lub(y,z) is the vector $(\max(y_1,z_1), \cdots, \max(y_n,z_n))$. The vector λy, where λ is a real number and y is a vector is defined as $(\lambda y_1, \cdots, \lambda y_n)$. Thus for n = 1 we have the space of real numbers. For n = 2 we have the space of complex numbers.

Example 16. Space of n-by-m rectangular matrices. Real numbers.

This space is composed of the totality of rectangular matrices with m rows and n columns, the components, a_{ij}, of the matrix being real numbers, and m and n being fixed for the particular space. A rectangular matrix is an array of elements arranged in the following form of m rows and n columns.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1m} \\ a_{21} & a_{22} & \cdots\cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{pmatrix} = A$$

This is one natural generalization of a vector. The sum A + B of two matrices is defined as the matrix, each of whose components is the sum of the two similarly placed components in the matrices A,B. That is, A + B is the matrix

$$\begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots\cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots\cdots & a_{2n} + b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots\cdots & a_{mn} + b_{mn} \end{pmatrix} = A + B$$

The matrix A is said to be > 0 if every one of its components is $\geqq 0$ and at least one of its components is > 0. The matrix $(A)_+$ is the matrix

$$\begin{pmatrix} \max(a_{11},0) & \max(a_{12},0) & \cdots & \max(a_{1n},0) \\ \max(a_{21},0) & \cdots\cdots & & \max(a_{2n},0) \\ \cdots & \cdots & \cdots & \cdots \\ \max(a_{m1},0) & \cdots\cdots & & \max(a_{m,n},0) \end{pmatrix}$$

The matrix λA, where λ is a real number, is the matrix, each of whose elements is equal to the product of λ and the similarly placed element in A.

<u>Example 17</u>. Matrices with components not real numbers.

Let us consider the space of all matrices with m rows and n columns, m and n fixed, whose components are themselves elements of some partially-ordered space satisfying postulates I - III. All the necessary definitions are simply paraphrases of those in example 2. This represents a very considerable generalization of examples 1 and 2.

<u>Example 18</u>. Integer functions.

The space S is here the totality of functions $f(t)$ with real-number values, where t varies in the whole closed interval of real numbers $(0,1)$. The sum and difference of two functions are defined as usual. The element (of S), y is defined as > 0 if the corresponding function $y(t)$ is $\geqq 0$ for all t in $(0,1)$ and $y(t_1) > 0$ for at least one t_1, in $(0,1)$. Multiplication by real numbers is defined as usual.

Example 19. Functions on abstract sets.

The space S is here the set of all numerically-valued functions defined on an abstract set, T, of elements, t. That is, a correspondence is set up between each member t of the class T of abstract elements, and a real number, so that a unique real number is assigned to each such member. Such a correspondence is known as a function f(t) defined on T. Our space is simply the totality of all possible such correspondences defined over a fixed set T. The various definitions are analogous to those of example 4.

Example 20. Polynomials according to Bernstein.

The space S is the totality of polynomials y(t) defined in the interval (0,1), of degree up to and including n. We define $y > 0$ if in the corresponding polynomial y(t), expressed in the Bernstein normal form

$$y(t) = \sum_{k=0}^{n} c_n^k t^k (1 - t)^{n-k} S_{k,n},$$

the coefficients $S_{k,n} \geqq 0$ for all $k \leqq n$ and, in addition $y(t) \not\equiv 0$. This space is isomorphic to the space $R^{(n+1)}$ see Bernstein II 1.

Example 21. Classical Hilbert space.

The space consists of the totality of infinite vectors $y = (y_1, \cdots y_n, \cdots)$ where the "components", $y_1, \cdots y_n, \cdots$, are complex numbers. The various notions are defined just as in example 1, with slight modifications. Formally, this is a slight generalization. Actually it is considerable.

<u>Example 22</u>. Positive linear functionals.

The space S is taken to be the totality of linear functionals F_f defined on the class of continuous functions f(t) where $0 \leqq t \leqq 1$. We define a functional over a certain class of functions f(t), as a correspondence that assigns to each member f(t) of the class, a unique real number F, called the value. It is a positive functional if $F_f \geqq 0$ for every f of the given class such that $f(t) \geqq 0$, $0 \leqq t \leqq 1$, and $F_f > 0$ for at least one function of the class; that is, the value of the functional is positive or zero for every non-negative function in the given class. It is an additive functional if

$$F_{f+g} = F_f + F_g.$$

It is a continuous functional if for any sequence of functions $f_n(t)$ in the given class such that $\lim_{n \to \infty} f_n(t) = f(t)$, a function also in the class, we have

$$\lim_{n \to \infty} F_{f_n} = F_f$$

In the definition we do not state what $\lim_{n \to \infty} f_n(t)$ means. It depends to some extent on choice and on the particular class of functions f(t). Ordinarily, for convenience, in the case of the class of continuous functions we say that it has the meaning of uniform convergence; that is if $\lim_{n \to \infty} f_n(t) = f(t)$ then to any $\epsilon > 0$ we can find a positive number $N(\epsilon)$ such that $|f(t) - f_n(t)| < \epsilon$ for $n > N(\epsilon)$ regardless of t; in other words, an N can be found that will make the inequality valid for any t whatsoever. The N will depend only on the ϵ, and in general will be larger the smaller is ϵ. A functional that is additive and continuous over a particular class of functions is called linear over that class. See Banach IV 1 B, pp. 16 and 23.

Now we define the functional F to be > 0 if it is positive in the above defined sense. F + G is an element defined by the equation

$$(F + G)_f = F_f + G_f.$$

The null-element is the functional that is identically zero: it assigns the value 0 to every function of the class. The element λ F is defined by the equation

$$(\lambda F)_f = \lambda (F)_f.$$

We shall have occasion to consider this example latter in greater detail. An example of a linear functional is the Riemann integral of a function i.e. $F_f = \int_0^1 f(t)dt$. That this is continuous for the set of continuous functions considered above is a theorem of the elementary theory of real variables, see Titchmarsh IV 1 B, p. 36 or Townsend IV 1 B, p. 373. This is obviously a positive functional and linear by very elementary theorems.

We shall develop the theory of convergence by means of the notion of upper and lower bounds of sequences. A postulate will be added assuring the existence of the exact upper and lower bounds for certain infinite sets not necessarily denumerable. (A set is called denumerable if there exists a one to one correspondence between the members of the set and the positive integers). After that a topology of "Cauchy sequences" (see Theorem 32) will be developed in a form sufficient for our later needs. A "topology" is a theory of limits and convergence developed in connexion with the theory of transformations defined on the given set of elements.

Definition 11 of bounded above. A set E (infinite or otherwise) of elements in S is said to be bounded above if there exists an element y of S such that $y \geqq z$ for every z in E. y is called an upper bound.

Definition 12 of bounded below.

The dual of definition 11.

Definition 13 of bounded. A set E of elements in S is said to be bounded if it is bounded above and bounded below.

Definition 14 of lub and glb. If an element y of S exists so that for every z in $E \subset S$

1.) $y \geqq z$

2.) for every upper bound w of E $w \geqq y$:

then y is said to be the lub E. A similar definition holds for glb.

Note $\{y\}$ is a short notation for the set of all elements of which a typical one is y.

Theorem 16a. If the set $E = \{y\}$ is bounded, then so is also the set

$E_1 = \{|y|\}$, the elements of which are the absolute values of the corresponding elements of E.

Proof. By Definition 13 there exist y_1 and y_2, such that

$y_1 \leqq y \leqq y_2$ for all y in E.

Therefore

$(y)_+ \leqq (y_2)_+$, $(y)_- \leqq (y_1)_-$ by Definitions 4 and 5 and postulate II 3.

Put $y_0 = (y_2)_+ + (y_1)_-$ then $0 = |y| = y_0$ for all y in E. That is, $\{|y|\}$ is bounded.

Notation. If the set E is provided with indices of any sort (not necessarily integers)

$E = \{y_\xi\}_{\xi \text{ in } \Xi}$ then the lub and glb of this set

will be written

$$\text{lub } E = \underset{\xi \text{ in } \Xi}{\text{lub}}\; y_\xi$$

$$\text{glb } E = \underset{\xi \text{ in } \Xi}{\text{glb}}\; y_\xi$$

In case the set E is a sequence y ,y_2, ... y , ... then lub E = $\underset{n}{\text{lub}} \{y_n\}$= lub ($y_1$,$y_2$, ... y_n, ...) etc.

We now introduce an axiom that enables us to obtain a development of the theory of sequences similar to those of Cantor and Dedekind.

Postulate II

II. 4.) Every set bounded above has an lub. The set may be infinite or otherwise.

Theorem 17. The postulates I, II, III are not independent. II 3 follows from the rest.

Proof. See Kantorovitch II 13, p. 129, Satz 13.

Theorem 18. If the set $E = \{y_\xi\}_{\xi \text{ in } \Xi}$ is bounded below, then the glb E exists, and

$$\text{glb } E = - \underset{\xi \text{ in } \Xi}{\text{lub}} \{- y_\xi\}$$

Proof. Let y_0 be a lower bound of E, $y_0 \leqq y_\xi$ for ξ in Ξ. Then $-y_0 \geqq -y_\xi$; consequently every lower bound y_0 of $\{y_\xi\}$ is such that $-y_0$ is upper bound of $\{- y_\xi\}$. By postulate II 4, the set $\{- y_\xi\}$, being bounded above by $-y_0$ has a lub $\{-y_\xi\} = -y_0$. Therefore $y_0 \geqq -\text{lub}\{-y_\xi\}$ for every lower bound y_0 and $-\text{lub}\{-y_\xi\} = \text{glb } \{y_\xi\}$

Theorem 19. The space S has no greatest and no least element.

Proof. Suppose y is the greatest element and y_1 is an element $\neq 0$. Then $y + |y_1| > y$. Contradiction.

Definition 15 of $+\infty$, $-\infty$.

$+\infty$ and $-\infty$ are two fictitious ideal elements with the properties

a) $-\infty < y < +\infty$ for all y in S.

b) $+\infty + y = +\infty$ " " " " "

$-\infty + y = -\infty$ " " " " "

c) $+\infty + \infty = +\infty$

$-\infty + (-\infty) = -\infty$

$-\infty + \infty$ has no meaning.

d) $-(+\infty) = -\infty$; $-(-\infty) = +\infty$

e) If $\lambda > 0$ then $\lambda(+\infty) = +\infty$; $\lambda(-\infty) = -\infty$

$\lambda < 0$ then $\lambda(+\infty) = -\infty$; $\lambda(-\infty) = +\infty$

Notation lub $E = +\infty$ means E is not bounded above.

glb $E = -\infty$ means E is not bounded below.

Theorem 20. Every set is bounded (bounds include $\pm\infty$ as possibilities).

Theorem 21.

a.) If $E \neq 0$, (the null-set) then lub $E \geqq$ glb E

b.) If $E_1 \supseteq E_2$ then lub $E_1 \geqq$ lub E_2

c.) $\underset{\xi \text{ in } \Xi}{\text{lub}} \{y_\xi + y\} = y + \underset{\xi \text{ in } \Xi}{\text{lub}} \{y_\xi\}$

Proof. See Theorem 8 c.

d.) If $E = \sum_{\eta \text{ in } H} E_\eta$, where the point-set sum is meant, then

$$\text{lub } E = \underset{\eta \text{ in } H}{\text{lub}} \{\text{lub } E_\eta\}$$

Proof. $E = \sum_{\eta \text{ in } H} E_\eta$. Then lub $E \geqq$ lub E_η, η in H, by b)

(1) Therefore lub $E \geqq \underset{\eta \text{ in } H}{\text{lub}} \{\text{lub } E_\eta\}$

Also $\underset{\eta \text{ in } H}{\text{lub}} \{\text{lub } E_\eta\} \geqq$ lub E_η, η in H

or $\underset{\eta \text{ in } H}{\text{lub}} \{\text{lub } E_\eta\} \geqq y$, for y in any E_η.

Therefore $\underset{\eta \text{ in } H}{\text{lub}} \{\text{lub } E_\eta\} \geqq y$, for y in E.

(2) or $\geqq$ lub E. (1) and (2) together yield the result.

e.) Suppose two sets E_1 and E_2 such that for every y_1 in E_1 and every y_2 in E_2 $y_1 \leqq y_2$. Then lub E_1 and glb E_2 both exist and lub $E_1 \leqq$ glb E_2.

Proof. Under the hypotheses, any y_2 is an upper bound for E_1 and $y_2 \geqq$ lub E_1 for every y_2 in E_2. Therefore lub E is a lower bound for E_2 and therefore

$$\text{glb } E_2 \geqq \text{lub } E_1.$$

f.) If $y \neq 0$, then the set $\{n\, y\}$, $n = 1,2,3, \cdots$ is not bounded.

Proof. $y \neq 0$. Therefore either $(y)_+$ or $(y)_-$ is $\neq 0$ by Theorem 3 e. Assume $(y)_+ \neq 0$. Now assume $ny \leqq y_0$ for $n = 1,2, \cdots$ lub $\{ny\} = (n + 1)y$ for all n lub $(ny) - y = ny$. Therefore lub $\{ny\} - y =$ lub $\{ny\}$ and $-y = 0$. Or $y \leqq 0$ and by Definition 4 $(y)_+ = 0$.

Note. This last sub-theorem corresponds to a kind of "Archimedean" axiom. The Archimedean axiom is, that to any two elements $a,b, > 0$ there exists an integer n such that $na > b$. The result f) above is definitely weaker than this. Actually the Archimedean axiom does not hold in the space S, as the simple example shows:

Using the notation of Example 15 and the space $R^{(2)}$, $(1,0) > 0$ and $(1,1) > 0$, but there is no integer n, such that

$$(n,0) > (n,n)$$

Theorem 22. Let E_1 and E_2 be two sets, such that E_1 consists of all the elements of S that are smaller than all the elements of the set E_2, and E_2 consists of all the elements of S that are larger than all the elements of E_1. Then there exists an element y_0, giving the cut. It may belong either to E_1 or to E_2. We have:

E_1 = the set of $y < y_0$, E_2 = the set of $y \geqq y_0$ if y_0 in E_2

E_1 = the set of $y \leqq y_0$, E_2 = the set of $y > y_0$ if y_0 not in E_2

Proof. Put y_0 = lub E_1. Then there are two cases.

1) y_0 in E_2. Then

$y_0 > y$ for all y in E_1

$y_0 \leqq y$ for all y in E_2

because y_0 = lub E_1 and every y in E_2 is an upper bound for E_1. In this case

E_1 = the set of $y < y_0$

E_2 = the set of $y \geqq y_0$

2) y_0 not in E_2. Then $y_0 < y$ for all elements y in E_2 and therefore y_0 is in E_1. In this case

E_1 = the set of $y \leqq y_0$,

E_2 = the set of $y > y_0$.

This last theorem gives as a theorem, the meat of the Dedekind theory of cuts as applied to ordinary real numbers, see Bohnenblust IV 1 B. p. 4, or Osgood 1 B, pp. 34 - 61. The element y_0 is sometimes known as the cut determined by the two classes E_1 and E_2. The class E_1 is the L-class of Dedekind and the class E_2 is the R-class. We are now able to introduce a notion of limit of a sequence of elements and all the consequent apparatus of convergence, almost precisely analogous to the case of real numbers.

<u>Definition 16</u> of upper limit and lower limit of sequences.

Let y_n be a sequence of elements of S.

We define

$$\overline{\lim_{n \to \infty}}\, y_n = \operatorname{glb}_n \left\{ \operatorname{lub} (y_n, y_{n+1}, \dots) \right\},$$ called the upper limit of the sequence $\{y_n\}$.

$$\underline{\lim_{n \to \infty}}\, y_n = \operatorname{lub}_n \; \operatorname{glb}(y_n, y_{n+1}, \dots),$$ called the lower limit of the sequence $\{y_n\}$. $\pm\infty$ is included as a possible value for either the upper or the lower limit.

<u>Definition 17</u> of limit of a sequence.

If $\overline{\lim_{n\to\infty}}\, y_n = \underline{\lim_{n\to\infty}}\, y_n$ then we say that the sequence has a limit and we write $\lim_{n \to \infty} y_n$ for the common value of the left and right-hand sides. See Pringsheim [N8]. $+\infty$ and $-\infty$ are included as possible values in the above equality.

<u>Definition 18</u> of upper and lower limits of a double sequence.

If we have a double sequence y_{nm}, that is, a set of elements of S such that to each ordered pair of positive integers n,m there corresponds a unique element $y_{n,m}$, then we define

$$\overline{\lim_{m,n\to\infty}}\ y_{mn} = \operatorname{glb}_s\left[\operatorname{lub}_{m,n \geqq s} (y_{mn})\right]$$

$$\underline{\lim_{m,n\to\infty}}\ y_{mn} = \operatorname{lub}_s\left[\operatorname{glb}_{m,n \geqq s} (y_{mn})\right]$$

<u>Definition 19</u> of limit of a double sequence y_{mn}

In case

$$\overline{\lim_{m,n\to\infty}}\left(y_{mn}\right) = \underline{\lim_{m,n\to\infty}}\left(y_{mn}\right)$$

we call the common value $\lim_{m,m\to\infty}\left(y_{mn}\right)$ and we say that the limit of the double sequence exists.

<u>Convention</u>. When we write $\lim y_n$, we mean that the limit exists.

<u>Theorem 23</u>

a) $\operatorname{lub}_n\left\{y_n\right\} \geqq \overline{\lim_{n\to\infty}}\left\{y_n\right\} \geqq \lim_{n\to\infty}\left\{y_n\right\} \geqq \underline{\lim_{n\ \infty}}\, y_n \geqq \operatorname{glb}_n\left\{y_n\right\}$

Even in case $\lim y_n$ does not exist, we have the rest of the inequality satisfied if those members exist.

<u>Proof</u>.

$$\operatorname{lub}(y_1,y_2,\ldots) \geqq \operatorname{lub}(y_n,y_{n+1},\ldots) \geqq \operatorname{glb}(y_n,y_{n+1},\ldots) \geqq \operatorname{glb}(y_1,y_2,\ldots).$$

Therefore, from Theorem 21 e we have the result.

b) $\overline{\lim_{n\to\infty}}\ (y_n + y) = y + \overline{\lim_{n\to\infty}}\ y_n$

Proof. $\overline{\lim\limits_{n \to \infty}} (y_n + y) = \operatorname{glb}\limits_{n} \left[\operatorname{lub}(y_n + y, y_{n+1} + y, \ldots) \right]$

$= \operatorname{glb}\limits_{n} \left[y + \operatorname{lub}(y_n, y_{n+1}, \ldots) \right]$ by Theorem 21 c,

$= y + \operatorname{glb}\limits_{n} \left[\operatorname{lub}(y_n, y_{n+1}, \ldots) \right] = y + \overline{\lim\limits_{n \to \infty}} y_n.$

c) $\underline{\lim\limits_{n \to \infty}} y_n = - \overline{\lim\limits_{n \to \infty}} (-y_n)$

Proof. From Theorem 18,

$$\underline{\lim\limits_{n \to \infty}} y_n = \operatorname{lub}\limits_{n} \left[\operatorname{glb}(y_n, y_{n+1}, \ldots) \right] = \operatorname{lub}\limits_{n} \left[-\operatorname{lub}(-y_n, -y_{n+1}, \ldots) \right]$$

$$= -\operatorname{glb} \left[\operatorname{lub}(-y_n, -y_{n+1}, \ldots) \right] = - \overline{\lim\limits_{n \to \infty}} (-y_n)$$

Theorem 24.

a) If $y_1 \geqq y_2 \geqq \ldots$ then $\lim\limits_{n \to \infty} y_n = \operatorname{glb}(y_1, y_2 \ldots)$

b) $y_1 \leqq y_2 \leqq \ldots$ then

$\lim\limits_{n \to \infty} y_n = \operatorname{lub}(y_1, y_2, \ldots)$ exists in the sense that there is either a finite element y with the properties of Definition 17 and the above equality holds or else $+\infty$ has the properties of Definition 17. Similarly for a)

Proof. We demonstrate b). Put

$\operatorname{lub}(y_1, y_2, \ldots) = y$, where y is a proper element of S if y_n is bounded above, and is $+\infty$ otherwise. Then by definition and the hypothesis:

$$\text{lub}(y_n, y_{n+1}, \ldots) = y; \ \text{glb}(y_n, y_{n+1}, \ldots) = y_n$$

$$\overline{\lim_{n \to \infty}} \ y_n = \underset{n}{\text{glb}} \left[\text{lub}(y_n, y_{n+1}, \ldots)\right] = y$$

$$\underline{\lim_{n \to \infty}} \ y_n = \underset{n}{\text{lub}} \left[\text{glb}(y_n, y_{n+1}, \ldots)\right] = \underset{n}{\text{lub}} \{y_n\} = y$$

Therefore $\overline{\lim_{n \to \infty}} \ y_n = \underline{\lim_{n \to \infty}} \ y_n = y = \lim_{n \to \infty} y_n$

Theorem 25. For any sequence y_n

$$\overline{\lim_{n \to \infty}} \ y_n = \lim_{n \to \infty} \left[\text{lub}(y_n, y_{n+1}, \ldots)\right]$$

Proof. Put $\text{lub}(y_n, y_{n+1}, \ldots) = \mathcal{Y}_n$, then $\{\mathcal{Y}_n\}$ is a non-increasing sequence. Apply Theorem 24 and obtain

$\lim_{n \to \infty} \mathcal{Y}_n = \text{glb}(\mathcal{Y}_1, \mathcal{Y}_2, \ldots.)$ or, substituting back

$$\lim_{n \to \infty} \left[\text{lub}(y_n, y_{n+1}, \ldots)\right] = \underset{n}{\text{glb}} \left[\text{lub}(y_n, y_{n+1}, \ldots)\right]$$

Theorem 26. If $\overline{y}_n \geqq y_n \geqq y'_n$ and $\lim_{n \to \infty} \overline{y}_n = \lim_{n \ \infty} y'_n = y$ then $\lim_{n \to \infty} y_n = y$ exists.

Proof. Use Theorem 21 e.

Theorem 27. If $\lim y_n = y$ and $\lim y'_n = y'$ then $\lim(y_n + y'_n) = \lim y_n + \lim y'_n$. Use Theorem 23 b) and thus consider only sequences for which $y' = 0 = y$. To prove that $\lim(y_n + y'_n) = 0$ if $\lim y_n = 0 = \lim y'_n$.

Proof. Put $Y_n = \text{lub}(y_n, y_{n+1}, \ldots)$

$Y_n' = \text{lub}(y_n', y_{n+1}', \ldots)$

The sequences Y_n and Y_n' are non-increasing and so, by Theorem 24,

$$\lim Y_n = 0 = \lim Y_n' = \underset{n}{\text{glb}}\, Y_n = \text{glb}\, Y_n' .$$

Also the sequence $Y_n + Y_n'$ is non-decreasing and

$$\lim(Y_n + Y_n') = \text{glb}(Y_n + Y_n', Y_{n+1} + Y_{n+1}', \ldots)$$

$$\leqq \text{glb}(Y_n + Y_n', Y_n + Y_{n+1}', \ldots)$$

$$\leqq Y_n + \underset{n}{\text{glb}}(Y_n', Y_{n+1}', \quad) = Y_n$$

But this is true for every n, thus

$\lim(Y_n + Y_n') \leqq \underset{n}{\text{glb}}\, y_n = 0$. Transcribing this back we have $\overline{\lim_{n \to \infty}} (y_n + y_n') \leqq \lim_{n\ \infty} (Y_n + Y_n') = 0.$

Now substitute $-y_n$ and $-y_n'$

$$\overline{\lim}(-y_n - y_n') \leqq 0, \text{ that is } \overline{\lim}(y_n + y_n') \geqq 0.$$

This together with the above yields $\lim(y_n + y_n') = 0$.

Theorem 28. If $\lim y_n = y$ then $\lim(y_n)_+ = y_+$ and $\lim(y_n)_- = (y)_-$.

Proof. To prove $\lim(y_n)_+ = \lim \left[\text{lub}(0, y_n)\right]$

$= \text{lub}(0, y)$

Put $Y_n = \text{lub}(y_n, y_{n+1}, \ldots)$ and $\overline{Y}_n = \text{glb}(y_n, y_{n+1}, \ldots)$. Then

$Y_n \geqq y_n \geqq \overline{Y_n}$ and $\lim Y = \lim \overline{Y_n} = y$ by Theorem 24, since Y_n is a non-increasing function of integer n and $\overline{Y_n}$ is a non-decreasing function of integer n.

The sequence $(\overline{Y_n})_+$ is non decreasing and so $\lim (\overline{Y_n})_+ = \overline{Y}$ exists. $\overline{Y} \geqq (\overline{Y_n})_+$, therefore $\overline{Y} \geqq \overline{Y_n}$ and so $\overline{Y} \geqq \underset{n}{\text{lub}}\, \overline{Y_n} = \lim \overline{Y_n} = y$. But $\overline{Y} \geqq 0$. Therefore $\overline{Y} \geqq (y)_+$. Moreover $y \geqq \overline{Y_n}$ therefore $(y)_+ \geqq (\overline{Y_n})_+$ and $(y)_+ \geqq \text{lub}(Y_n)_+ =$ $= \lim (Y_n)_+ = \overline{Y}$. And so finally, $(y)_+ = \overline{Y} = \lim (\overline{Y_n})_+$. The sequence $-Y_n$ is non-decreasing, therefore $\lim(Y_n)_- =$ $\lim (-Y_n)_+ = (-y)_+ = y_-$. Then from Theorem 28,

$$\lim (Y_n)_+ = \lim_{n\to\infty} \left[Y_n + (Y_n)_-\right] = y + (y)_- = (y)_+.$$

But $(\overline{Y_n})_+ \leqq (y_n)_+ \leqq (\overline{Y_n})_+$, so that $\lim (y_n)_+ = (y)_+$ and from this $\lim (y_n)_- = \lim (-y_n)_+ = (-y)_+ = y_-$

<u>Theorem 29</u>. $\lim |y_n| = |y|$

<u>Proof</u>. $|y_n| = (y_n)_+ + (y_n)_-$. Use Theorems 27 and 28.

$$\lim |y_n| = \lim \left[(y_n)_+ + (y_n)_-\right] = (y)_+ + (y)_- = |y|$$

<u>Theorem 30</u>. $\lim (\text{lub}\,[y_n, y_n'])= \text{lub}\,(y,y')$

$\lim \left[\text{glb}\,(y_n, y_n')\right] = \text{glb}\,(y,y')$, if $\lim y_n = y$, $\lim y_n' = y'$.

<u>Proof</u>. $\text{lub}\,(y_n,y_n') = y_n + \text{lub}\,(0, y_n' - y_n)$

$\lim (y_n' - y_n) = y' - y$. Use Theorem 28 and 29 and get

$$\lim \left[\text{lub}\,(y_n,y_n')\right] = y + \text{lub}\,(0, y' - y) = \text{lub}\,(y,y')$$

$$\lim \left[\text{glb}(y_n,y_n')\right] = -\lim \left[\text{lub}(-y_n,-y_n')\right] = -\text{lub}(-y,-y')$$
$$= \text{glb}\,(y,y').$$

Theorem 31. The necessary and sufficient condition that $\lim y_n = 0$ is

$$\lim_{m,n \to \infty} \left[\text{lub}\,(|y_n|, \cdots |y_m|)\right) = 0$$

Proof.

a) Necessity.

Use Theorem 29 and $\lim |y_n| = 0$. Therefore

$\lim_{n \to \infty} \left[\text{lub}(|y_n|, |y_{n+1}|, \ldots)\right] = 0$, using Theorem 25.

Therefore $\lim_{m,n \to \infty} \left[\text{lub}\ |y_n|, |y_{n+1}|, \cdots, |y_m|\right] = 0$

For $\overline{\lim}_{m,n \to \infty} \left[\text{lub}(|y_n|, |y_{n+1}|, \ldots, |y_m|)\right] =$

$\lim_{s \to \infty} \left[\text{lub}_{m,n \geqq s}\ \text{lub}(|y_n|, |y_{n+1}|, \ldots, |y_m|)\right] =$

$\lim_{s \to \infty} \left[\text{lub}\,(|y_s|, |y_{s+1}|, \ldots)\right] = 0$. This next to the last relation is proved just as in Theorem 25.

Also, by definition

$$\lim_{m,n \to \infty} \left[\sup\,(|y_n|, |y_{n+1}|, \ldots, |y_m|)\right] = 0.$$

Therefore the limit exists and is equal to zero.

b) Sufficiency.

Assume $\lim_{m,n \to \infty} \left[\text{lub}(|y_n|, |y_{n+1}|, \cdots, |y_m|)\right] = 0$.

Put $m = n$ in particular so that $\lim_{n \to \infty} |y_n| = 0$ and so $\lim (-|y_n|) = 0$. But $-|y_n| = y_n = |y_n|$, therefore by Theorem 26.

Theorem 32. Fundamental Theorem. Generalized Cauchy Theorem.

The necessary and sufficient condition that the sequence $\{y_n\}$ have a finite limit is $\lim_{m,n \to \infty} |y_n - y_m| = 0$.

Proof.

a) Necessity. Assume $\lim y_n = y$.

$$|y_n - y_m| \leqq |y_n - y| + |y_m - y| \text{, so}$$

$$\overline{\lim_{m,n \to \infty}} |y_n - y_m| \leqq \overline{\lim_{n \to \infty}} |y_n - y| + \overline{\lim_{m \to \infty}} |y_m - y| = 0.$$

b) Sufficiency. Assume $\lim_{m,n \to \infty} |y_n - y_m| = 0$.

Put $Y_s = \operatorname{lub}_{m,n \geqq s} |y_n - y_m|$. Then $\lim Y_s = 0$.

$\operatorname{lub}(y_n, y_{n+1}, \cdots, y_m) - \operatorname{glb}(y_m, y_{m+1}, \cdots, y_m) =$

$\operatorname{lub}(y_n, y_{n+1}, \cdots, y_m) + \operatorname{lub}(-y_n, -y_{n+1}, \cdots -y_m) =$

$\operatorname{lub}\left[y_n + \operatorname{lub}(-y_n, -y_{n+1}, \cdots, -y_m), y_{m+1} + \operatorname{lub}(-y_n, \cdots, -y_m), \cdots, y_m + \operatorname{lub}(-y_n, \cdots, -y_m)\right]$. This last step is got by putting $\operatorname{lub}(-y_n, -y_{n+1}, \cdots) = y$ in Theorem 21 c. Then the equation continues

$= \operatorname{lub}\left[\operatorname{lub}(0, y_n - y_{n+1}, \cdots, y_n - y_m, \operatorname{lub}(y_{n+1}, -y_n, 0, \cdots y_{n+1} - y_m), \cdots, \operatorname{lub}(y_m - y_n, y_m - y_{n+1}, \cdots, 0)\right] \leqq$

$\operatorname{lub}_{m \geqq i,k \geqq n} |y_i - y_k| \leqq Y_n$. This is true for all m, thus true for limit. Thus

$$\operatorname{lub}(y_n, y_{n+1}, \cdots) - \operatorname{glb}(y_n, y_{n+1}, \cdots) \leqq Y_n.$$

Therefore $0 \leqq \overline{\lim_{n\to\infty}}\, y_n - \underline{\lim_{n\to\infty}}\, y_n = \lim_{n\to\infty} \Big[\text{lub}(y_n, y_{n+1}, \cdots)$

$- \text{glb}\,(y_n, y_{n+1}, \cdots)\Big] \leqq \lim Y_n = 0$

Therefore $\lim y_n = 0$ and so exists.

<u>Theorem 33</u> a) $\lambda < 0$ implies $\text{lub}\{\lambda\, y_\xi\} = \lambda\, \text{glb}\,\{y_\xi\}$ $\quad \xi$ in Ξ

b) $\lambda > 0$ implies $\underset{\xi \text{ in } \Xi}{\text{lub}}\{\lambda\, y_\xi\} = \lambda\, \text{lub}\,\{y_\xi\}$

<u>Proof</u>. $\text{lub}\,\{y_\xi\} \geqq y_\xi$ $\quad \xi$ in Ξ . And

$\lambda\, \text{lub}\,\{y_\xi\} \geqq \lambda\, y_\xi$ $\quad \xi$ in Ξ , that is

$\lambda\, \text{lub}\,\{y_\xi\} \geqq \text{lub}\,\{\lambda y_\xi\}$. Replace λ by $\frac{1}{\lambda}$

and y_ξ by $\lambda\, y_\xi$ and get the inequality reading the other way. If $\lambda < 0$ then by the above and Theorem 18 we have

$$\text{lub}\,\{\lambda\, y_\xi\} = \text{lub}\,\{-|\lambda|\, y_\xi\} = |\lambda|\, \text{lub}\,\{-y_\xi\} =$$
$$-|\lambda|\, \text{glb}\,\{y_\xi\} = \lambda\, \text{glb}\,\{y_\xi\}.$$

<u>Theorem 34</u>. If $\{\lambda_\xi\}$, ξ in Ξ is a transfinite sequence of real numbers and $y > 0$ then

$\text{lub}\,\{\lambda_\xi\, y\} = y\, \text{lub}\, \lambda_\xi$, where now $\text{lub}\,\{\lambda_\xi\}$ is taken in the ordinary sense (see Bohnenblust IV B; or simply use the foregoing theory on the certainly (partially) ordered set of real numbers).

<u>Proof</u>. Put $\text{lub}\, \lambda_\xi = \lambda$ $\quad \xi$ in Ξ. Then

$\lambda_\xi \leqq \lambda$ $\quad \xi$ in Ξ, $\quad \lambda_\xi\, y \leqq \lambda\, y$ and

therefore $y_1 = \text{lub}\{\lambda_\xi\, y\} \leqq \lambda y$. Then for any real number $\epsilon > 0$ $(\lambda - \epsilon)y \leqq y_1 \leqq \lambda y_n$

But, using Theorem 21 f), this implies $y_1 = \lambda y$.

Theorem 35. a) If $\lambda > 0$, then $\overline{\lim}_{n \to \infty} \lambda y_n = \lambda \overline{\lim}_{n \to \infty} y_n$

b) If $\lambda < 0$, then $\overline{\lim}_{n \to \infty} \lambda y_n = \lambda \underline{\lim}_{n \to \infty} y_n$

c) For any real λ, $\lim_{n \to \infty} \lambda y_n = \lim_{n \to \infty} y_n$

Proof. $\lambda > 0$. $\overline{\lim}_{n \to \infty} \lambda y_n = \underset{n}{\text{glb}} \left[\text{lub}(\lambda y_n, \lambda y_{n+1}, \ldots)\right] =$

$= \underset{n}{\text{glb}} \left[\lambda \, \text{lub}(y_n, y_{n+1}, \ldots)\right] =$

$= \lambda \underset{n}{\text{glb}} \left[\text{lub}(y_n, y_{n+1}, \ldots)\right] = \lambda \lim_{n \to \infty} y_n.$

If $\lambda < 0$, have an analogous proof. If $\lim_{n \to \infty} y_n$ exists, then by application of the two taken together we have c).

Theorem 36. a) If y_n is a bounded sequence and $\lim_{n \to \infty} \lambda_n = 0$, where $\{\lambda_n\}$ is an ordinary real number sequence, then $\lim_{n \to \infty} \lambda_n y_n = 0$.

Proof. By hypothesis, there is a y such that $|y_n| \leqq y$ from Theorem 16. Then

$\overline{\lim}_{n \to \infty} \lambda_n y_n \leqq \overline{\lim}_{n \to \infty} |\lambda_n| y = y \, \overline{\lim}_{n \to \infty} |\lambda_n| = 0$

$\underline{\lim}_{n \to \infty} \lambda_n y_n \geqq \underline{\lim} \, (-|\lambda_n| y) = -\overline{\lim}_{n \to \infty} |\lambda_n| y = 0$

Also, by Theorem 23 $\overline{\lim} \{\lambda_n y_n\} \geqq \underline{\lim} \{\lambda_n y_n\} = 0$

Therefore $\overline{\lim}\{\lambda_n y_n\} = 0$ and by a similar argument, $\underline{\lim}\{\lambda_n y_n\} = 0$ and so $\lim\{\lambda_n y_n\}$ exists and = 0.

b) If $\lim_{n\to\infty} \lambda_n = \lambda$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} \lambda_n y_n = \lambda y$.

<u>Proof</u>. $\lambda_n y_n - \lambda y = (\lambda_n - \lambda) y_n + \lambda(y_n - y)$

The first term has zero as limit by a) above. The second term has zero as limit by Theorem 35 c). The sum, as a consequence, has zero as limit by Theorem 27.

c) If y_1 is an element with the property that $y > y_1$ implies $y > 0$, then $y_1 \geqq 0$.

<u>Proof</u>. Assume the contrary: that $(y_1)_- \neq 0$ (that y_1 has a non-zero negative part, see Postulate II 3, and Definition 5, of negative part). From Theorem 21 f we can find a positive integer n such that $n(y_1)_- \nless (y_1)_+$, (where $\nless$ is the negation of $<$). Also we can choose n large enough so that

(1) $(y_1)_+ - (1 - \frac{1}{n})(y_1)_- \ngtr 0$, for otherwise $(y_1)_+ - (1 - \frac{1}{n})(y_1)_- > 0$ for all n, and by Theorem 23 and the fact that the limits of both sides of the inequality exist, by Theorem 36 b) we should have

$y_1 = (y_1)_+ - (y_1)_- \geqq 0$ contrary to the hypothesis. Now put $y = (y_1)_+ - (1 - \frac{1}{n})(y_1)_-$ then $y - y_1 = \frac{1}{n} y_1 > 0$ by the assumption. Therefore y is a suitable y for the hypothesis of the theorem and it should follow that y 0 according to the same hypothesis (above just after c)). This contradiction proves the theorem.

The Postulates of Regularity II 5

a) Let E_n be a sequence of subsets of S (each one bounded above) such that a finite or infinite limit $\lim_{n \to \infty} \left[\text{lub } E_n\right] = y$ exists. Then from each set E_n, a finite subset $E_n' \subset E_n$ of elements of S can be chosen so that

$$\lim_{n \to \infty} \left[\text{lub } E_n'\right] = y$$

b) The postulate will be said to hold in the "broad sense" if the following condition also holds. If E_n are sets for which lub $E_n = +\infty$, then there are finite subsets $E_n' \subset E$ such that

$$\text{lub}_n \left[\text{lub } E_n'\right] = +\infty$$

Definition 20 of regular space.

A space S that satisfies axioms I 1-5, II 1-5, III 1-6 shall be called regular.

Theorem 37. If $E_n \subset S$ and the limit $\lim_{n \to \infty} \left[\text{glb } E_n\right]$ exists, then there exist finite subsets $E'_n \subset E_n$ such that

$$\lim_{n \to \infty} \left[\text{glb } E'_n \right] = \lim_{n \to \infty} \left[\text{glb } E_n \right] .$$

Proof. Let E^*_n be the set of elements y, for which -y is in E . Then lub E^*_n = -glb E_n by Theorem 18 and the limit exists:

$$- \lim_{n \to \infty} \left[\text{glb } E_n \right] = \lim_{n \to \infty} \left[\text{lub } E^*_n \right]$$

Now by Postulate II 5, there exist finite subsets E'^*_n such that

$$\lim_{n \to \infty} \left[\text{lub } E'^*_n \right] = \lim_{n \to \infty} \left[\text{lub } E^*_n \right] = - \lim_{n \to \infty} \left[\text{glb } E_n \right]$$

Let E'_n be the set of elements y, such that -y is in E'^*. Then E'_n is a finite subset of E_n and also glb E'_n = - lub E'^*_n and finally

$$\lim_{n \to \infty} \left[\text{glb } E'_n \right] = - \lim_{n \to \infty} \left[\text{lub } E'^*_n \right] = \lim_{n \to \infty} \left[\text{lub } E_n \right]$$

Note. The theory of transfinite induction will be used in the following two theorems, both in the enunciation and the proof. For an account in English sufficient for our needs, see Pierpont IV B, vol. II, or Sierpinski IV B, appendix. Kamke IV B has probably the clearest succinct account (in German) to be found in a book.

Theorem 38. For any set $E \subset S$, one can find a countable sub-set (that is, a sub-set that can be put into one-to-one correspondence with the positive integers) $E^* \subset E$, such that

$$\text{lub } E^* = \text{lub } E; \text{ glb } E^* = \text{glb } E.$$

<u>Proof</u>. In Postulate II 5, let every $E_n = E$ for every n. Then we can find, by the same postulate, finite subsets $E^*_n \subset E$ such that

$$\lim_{n \to \infty} \left[\text{lub } E^*_n\right] = \lim_{n \to \infty} \left[\text{lub } E_n\right] = \text{lub } E.$$

By Theorem 37, we can find finite subsets $E^{**}_n \subset E_n = E$, such that

$$\lim_{n \to \infty} \left[\text{glb } E^{**}_n\right] = \text{glb } E$$

Put $E' = \sum_{n=1}^{\infty} (E^*_n + E^{**}_n)$

Then this is a suitable E' to satisfy the conditions of the theorem. For $E' \subset E$ and E' is countable this follows from the well-known theorem that a countable set of countable sets of elements is itself countable, when considered as made up of the last-mentioned elements. Also

$$\text{lub } E \geqq \text{lub } E' \geqq \text{lub } \left(\sum_{n=1}^{\infty} E^*_n\right) \geqq \text{lub}_n(\text{lub } E^*_n) \geqq$$

$$\geqq \lim_{n \to \infty} \left[\text{lub } E^*_n\right] = \text{lub } E.$$ This implies lub E' = lub E. Similarly, glb E' = glb E. In case lub E = + ∞,

(or glb $E = -\infty$) the proof can be carried through by means of Postulate II 5 in the broad sense.

Theorem 39. A well-ordered monotonically increasing sequence of elements of S is unconditionally countable (see Sierpinski IV B 2). That is if

$$y_1 \leqq y_2 \leqq y_3 \cdots \leqq y_\omega \leqq y_{\omega+1} \leqq \quad \cdots \quad \leqq y_\alpha \leqq \cdots$$

then there is an α of the first or second number-class such that beginning with α, we have

$$y_\alpha = y_{\alpha+1} = \cdots$$

Proof. Let E be the set of all y_α $(1 \leqq \alpha \leqq \Omega)$. From a) there exists a countable subset E', such that lub E' = lub E. Let α' be the number, of the first or second number-class, that comes immediately after all those α's for which y_α in E'. Then $y \leqq y_{\alpha'}$ for every y in E', and so lub E' $\leqq y_\alpha =$ lub E. But lub E' = lub E. Therefore lub E = $y_{\alpha'}$ and $y_{\alpha'} = y_{\alpha'+1} = \cdots$.

Theorem 40. If $\lim_{k \to \infty} y_i^{(k)} = y_i$, for $i = 1, 2,$ and $\lim_{i \to \infty} y_i = y$, then there exists an infinite sequence of indices $k_1, k_2, \cdots$ $(k_1 < k_2 < \cdots)$ such that

$$\lim_{i \to \infty} y_i^{(k_i)} = y$$

<u>Proof</u>. $\text{lub}\,(y_i^{(k)}, y_i^{k+1}, \ldots) \geqq y_i \geqq \text{glb}(y_i^{(k)}, y_i^{(k+1)}, \ldots)$

by Theorem 23; and

$$\underset{k}{\text{glb}}\left[\text{lub}(y_i^{(k)}, y_i^{k+1}, \ldots)\right] = \underset{k}{\text{lub}}\left[\text{glb}(y_i^{(k)}, y_i^{(k+1)}, \ldots)\right] = y_i$$

Now write

$$Z_i^{(k)} \equiv \text{lub}\,(y_i^{(k)}, y_i^{(k+1)}, \ldots)$$

$$w_i^{(k)} \equiv \text{glb}\,(y_i^{(k)}, y_i^{(k+1)}, \ldots)$$

Then $\underset{k}{\text{glb}}\left(Z_i^{(k)}\right) = \underset{k}{\text{lub}}\left(w_i^{(k)}\right) = y_i$ and

$$(1)\quad \lim_{i\to\infty} y_i = \lim_{i\to\infty}\left[\text{glb}\ Z_i^{(k)}\right] = \lim_{i\to\infty}\left[\text{lub}\ w_i^{(k)}\right]$$

Before proceeding with the proof, let us remark that if we pick any finite set of elements out of the sequence $Z_i^1, Z_i^2, \cdots$ that the glb of this set will actually lie in the set, for, as in Theorem 18 b $Z_i^{(k)}$ is a monotonically decreasing sequence (considering i fixed) and so the element, in the above sequence, whose upper index is greatest will be the glb. Similar considerations apply to lub of a finite number of $w_i^{(k)}$, (i fixed).

By Theorem 37 and Postulate II 5, we can (from (1)) pick a finite set E_i of the $Z_i^{(k)}$ for each i such that

$$y = \lim_{i\to\infty}\left[\text{glb}\ E_i\right] = \lim_{i\to\infty}\left[\text{lub}\ E_i^*\right],$$

where E_i^* is a set of w_i^k with similar properties. Using the above remark, we can pick <u>one</u> element $Z_i^{\,i}$ out of E_i and one element w_i out of E_i^* such that $y = \lim_{i \to \infty} \left[\text{glb } Z_i^{\ i}\right] = \lim_{i \to \infty} \left[\text{lub } w_i^{\ i}\right]$. Now pick a $k_i > l_i, m_i$ so that the k_i form an increasing sequence. Then we have

$$Z_i^{\ i} \geqq Z_i^{k_i} \geqq w_i^{k_i} \geqq w_i^{m_i}$$

and hence, by Theorem 18 c

$$\lim_{i \to \infty} Z_i^{k_i} = \lim w_i^{k_i} = y \quad \text{and thus}$$

$$\overline{\lim_{i \to \infty}}\, y_i^{k_i} = \lim_{i \to \infty} \left[\text{lub } (y_i^{k_i}, y_i^{k_i+1}, \quad)\right] =$$

$$\lim_{i \to \infty} \text{glb } (y_i^{k_i}, y_i^{k_i+1}, \cdots) = \underline{\lim}_{i \to \infty}\, y_i^{k_i} \text{ , by Theorem 18 b.}$$

Finally then $\lim_{i \to \infty} y_i^{k_i} = y.$

This last theorem enables us to enunciate a principle that is to be found in most topologies of point-sets. If first, we define a "limit-point" of a set E of elements to be an element of S (not necessarily of E) that can be expressed as the limit, in our sense, of a convergent sequence of elements y_i (all different) in E. The set of limit points of E is denoted by "E'". Now we can consider the limit-points of E', that is, E". The above theorem states that all the points in E" can be expressed as limits of points in <u>E</u>. That is, are automatically points of E'. If a set contains <u>all</u> its limit-points, we say it is "closed under the limiting-process" or simply "closed". It is seen that E' is closed for any set E of points in S. For spaces in which this is not always true, see Frechet IV.

Theorem 41. A necessary and sufficient condition that the set $E \subset S$ be bounded is that the following condition hold.

For every sequence of real numbers $\{\lambda_n\}$ such that $\lim \lambda_n = 0$, and every sequence $\{y_n\}$ of elements in E we shall have

$$\lim \lambda_n y_n = 0$$

Proof. a) Necessity. This follows from Theorem 18 d).

b) Sufficiency. Assume the contrary, that E is not bounded above. Then lub E = + ∞. Let E_n be the set of elements y for which ny in E. Then lub E_n = + ∞ and from Postulate II 5, in the broad sense, there exist finite subsets $E_n' \subset E_n$ such that $\underset{n}{\text{lub}}\,(\text{lub}\, E_n') = \overline{\lim_{n \to \infty}}\,(\text{lub}\, E_n') = +\infty$. Let the elements (finite in number) of E_k' be

$y_1^k, y_2^k, \cdots y_{n_k}^k$. Now put

$$y_{n_1+n_2+ \cdots +n_{k-1}+ s} = k\, y_s^k \quad \text{and}$$

$$\lambda_{n_1+n_2+ \cdots +n_{k-1} + s} = \frac{1}{k} \text{ for } 1 \leqq s \leqq n_k.$$

Then y_n and λ_n are determined for every n. Since y_s^k in E_k, every y_n in E. Obviously $\lim_{n \to \infty} \lambda_n = 0$. But it is not true that $\lim_{n \to \infty} \lambda_n y_n = 0$. For if it were we should have

$$\overline{\lim_{n\to\infty}} \lambda_n y_n = \lim_{n\to\infty} (\text{lub}\ (\lambda_n y_n,\ \lambda_{n+1} y_{n+1}, \dots)) \geqq$$

$$= \overline{\lim_{k\to\infty}} (\text{lub}\ (\tfrac{1}{k} y_{n_1+n_2+\ ..\ +n_{k-1}+1} \cdots\ y_{n_1+n_2+\ ..\ +n_k}))=$$

$$= \overline{\lim_{k\to\infty}} (\text{lub}\ E_k') = +\infty \text{ and a contradiction.}$$

Theorem 42. If $\{y_n\}$ is a sequence of elements for which $\lim y_n = 0$, then there exists a sequence of real numbers λ_n, such that $\lim_{n\to\infty} \lambda_n = +\infty$ and $\lim_{n\to\infty} \lambda_n y_n = 0$.

Proof. Put $Z_n = \text{lub}\ (|y_n|, |y_{n+1}|, \dots)$, then $\lim_{n\ \infty} Z_n = 0$ from Theorems 19 c) and 18 b). The Z_n form a monotonically decreasing sequence, $(Z_1 \geqq Z_2 \geqq \dots)$. Put $E_k = \{k Z_n\}$, $n = 1, 2, \cdots$ Then $\text{glb}\ (E_k) = 0$ and so $\lim_{k\to\infty} (\text{glb}\ E_k) = 0$. Now apply Theorem 37. There is then a finite set $E_k' \subset E_k$ for each k such that $\lim_k (\text{lub}\ E_k') = \lim_k (\text{lub}\ E_k)$. Since Z_n is monotonically decreasing, lub E_k' is a member y_{n_k} of E'. Therefore

$$\lim_{k\to\infty} (kZ_{N_k}) = 0 \text{ where without loss of}$$

generality, N_k is an increasing infinite sequence of suffixes. Now put $\lambda_n = k$ in case $N_k \leqq N < N_{k+1}$. Then $\lim \lambda_n = +\infty$ and $\lim_{n\to\infty} \lambda_n Z_n = \lim k\, y_{N_k} = 0$ and therefore $\lim \lambda_n y_n = 0$.

Theorem 43. The necessary and sufficient condition that we shall have $\lim_{n \to \infty} y_n = y$ for a sequence $\{y_n\}$, is that there exist an element Z in S with the property that to any positive real number e we can find a positive integer N_e such that

$$|y_n - y| < e\,Z \quad \text{for every } n \geqq N_e.$$

Proof. Necessity. Assume $\lim |y_n - y| = 0$. Then from Theorem 42 we can find a sequence of real positive numbers l_n such that

$\lim l_n = +\infty$ and $\lim_{n \to \infty} l_n |y_n - y| = 0$. This limit is finite, so the sequence must be bounded, by the definition of limit. Therefore the sequence has a lub, Z, and this is the required Z. For let e be a positive number, then we can find a N_e, such that for $n \geqq N_e$,

$$\frac{1}{l_n} < e \quad \text{and consequently}$$

$$|y_n - y| < \frac{Z}{l_n} < e\,Z.$$

Sufficiency. Let e_n be the smallest real number for which $|y_n - y| \leqq e_n Z$. e_n exists for each n, because the class of real numbers e_n satisfying the inequality is bounded below. Now according to the hypothesis, $\lim e_n = 0$. Then

$$0 \leqq \underline{\lim}_{n\to\infty} |y_n - y| \leqq \overline{\lim}_{n\to\infty} |y_n - y| \leqq \overline{\lim}_{n\to\infty} e_n Z = 0.$$

Therefore

$$\underline{\lim}_{n\to\infty} y_n - y = \overline{\lim}_{n\to\infty} |y_n - y| = 0 = \lim_{n\to\infty} |y_n - y|.$$

Therefore, by the fundamental theorem $\lim y_n = y$.

<u>Theorem 44</u>. If $\lim_{K\to\infty} y_i^k = y_i$ then a Z in S can be found such that to any positive real number e we can find a K_e^i such that

$$|y_i^k - y_i| \leqq e Z \quad \text{for } k \geqq K_e^i.$$

<u>Proof</u>. From Theorem 43, we can find a Z' to each i, such that

$|y_i^k - y_i| \leqq e Z^i$ for $k \geqq K(e,i)$. But

$\lim_{n\to\infty} \frac{1}{n} Z^i = 0$ for each i, so we can choose n_i

so that $\lim_{i\to\infty} \frac{1}{n_i} Z' = 0$, by the following lemma

1. Thus, by Theorem 41, the elements $\frac{1}{n_i} Z^i$ form

a bounded sequence. Take its lub and it will

be the required Z.

<u>Lemma 1</u>. If $\lim_{n\to\infty} \frac{1}{n} y^i = 0$ for every i then we can find a sequence of integers n_i, such that $\lim_{i\to\infty} \frac{1}{n_i} y^i = 0$.

<u>Proof</u>. Let $Y_n^i = \frac{1}{n} y^i$. Then $\lim_{n\to\infty} Y_n^i = Y^i = 0$ and so

$\lim_{i\to\infty} Y^i = 0$. Therefore, by Theorem 40, we can

pick out a sequence $Y_{i_n}^i$ such that

$$\lim_{i\to\infty} Y_{i_n}^i = 0 = \lim \frac{1}{i}$$

Note to Chapter III.

It will be found convenient and in fact almost indispensable, to write $\lambda \cdot x$, where λ is a real number and x an element of the Kantorovitch space S, in the form $x \cdot \lambda$ or simply $x\lambda$. The striking analogies between these theories and the classical ones can be brought out more clearly by the use of this notation. It need cause no confusion, since Greek letters are used exclusively for real numbers with the exceptions of positive integer suffixes, when n, m, j, k are used. As before latin letters mean elements of abstract spaces.

Chapter II

Introduction

In this chapter we shall examine the consequences of the postulates of a partially-ordered set (not necessarily an Abelian group), as we shall need something of this theory in the later work on transformations. The branch of mathematics dealing with partially-ordered sets is comparatively new, so that there are no systematic developments available in textbooks. The following historical sketch is necessarily brief. For completeness and convenience mention is made of certain applications in finite algebra, theory of groups and mathematical logic, subjects that will not concern us directly in what is to follow, but which actually may influence the turn that the research shall take.

Dedekind seems to have been the first to study abstractly the properties of the partially-ordered set as such. He used the term "dual group" to describe a set of objects with the following postulates, see Dedekind I B, p. 493, I 1, 2,3. We call this set a "structure". M.

VI. 1.) To every ordered pair of elements a, b (not necessarily different) there corresponds an uniquely determined element $a \cup b$ of M, called the union of a and b.

2.) To every ordered pair of elements a,b (not necessarily different) there corresponds an uniquely determined element $a \cap b$ of M, called the cross-cut of a and b.

For all a,b,c (not necessarily different) in M we have:

3.) $a \cup b = b \cup a \qquad a \cap b = b \cap a$

4.) $a \cup (b \cup c) = (a \cup b) \cup c, \qquad a \cap (b \cap c) = (a \cap b) \cap c$

5.) $a \cup (a \cap b) = a, \qquad a \cap (a \cup b) = a$

Definition 21 of null element

A null-element, if it exists, is an element E_0 such that $E_0 \cap A = E_0$ for every element A of the structure. A null-element does not necessarily exist for every structure.

Definition 22 of all element

An all-element, if it exists, is an element 0_o such that $0_o \cup A = 0_o$ for every element A of the structure. An all-element does not necessarily exist for every structure.

These axioms are Dedekind's. Ore I 3 showed that they are equivalent to our axioms V if we define $a \leqq b$ when $a \cup b = b$ or $a \cap b = a$ (these two conditions are equivalent as can be seen by putting the first into 5_1 above). Ore I 3 speaks of a "structure"; Birkhoff I 1,2 of a "lattice"; Kantorovitch II 13 of a "halbgeordneter Raum" or of an "espace semi-ordonné"; Hausdorff IV 1 B of a "teilweise geordnete Menge"; Glivenko II 1 of a "chose"; Tucker IV 1 of a "cell-space"; Weisner I 1 of an "hierarchy"; Caratheodory II 1 of a "soma". A name for structure sometimes encountered in German literature is "Ding". The term "partially-ordered set" is sometimes used in a broader sense; it is required only that if $a \leqq b$ and $b \leqq c$

then $a \leqq c$. Birkhoff IV 5 has used the idea of a "directed set"; a directed set of points x_a where x_a are in some topological space X is a set of points that are ordered by means of the subscripts a of a class A (not necessarily in X). The class A is partially ordered, and in addition to each pair of elements a,b of A there is an element $c > a,b$. This c need not be unique. The set A is distinctly more general than the set satisfying the axioms VI.

Bennett I 1 has given a postulational treatment of what he calls "semi-serial order", which in our notation is essentially the partial order characterizing a structure.

E. Foradori II 1 has considered a theory of the abstract domain, a member of which need not be a point-set in the classical sense, but may be any one of a very much wider collection of entities. He takes as fundamental a certain abstraction of a "sub-set" relation, $A \subset B$. He postulates that either $A \subset B$ or else A is not $\subset B$, for each pair of "members" A,B. Further $A \subset A$ for every A and $A \subset B$, $B \subset C$ together imply $A \subset C$. Neither the union nor the cross cut is postulated, but if they both exist, they are of course unique and the laws hold:

$$(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$$

$$(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C).$$

He defines the "residue" of B with respect to A. First let A and B be elements such that $A \subset B$. Then A_B will represent any elements of the domain that satisfy (for A fixed):

$$B \subset A \cup A_B.$$

The elements A_B are called complements of A in B. Now if there exists an A_B that is contained in every A_B, it is called the residue of B with respect to A. The residue may not exist. It is designated by B-A. It can be proved to be unique and that $B-A \subset B$. Also $A \cup (B-A) = B$ where $X = Y$ simply means $X \subset Y$ and $Y \subset X$, and no more. He proves, moreover, that if $A \cup B$, $(A \cup B) - C$, $A - C$, $B - C$, $(A - C) \cup (B - C)$ exist, then $(A \cup B) - C = (A - C) \cup (B - C)$. It is seen that this "residue" anticipates to in a certain measure the generalized residual of Ward and Dilworth, see below. Their residual is in one way more general, but it occurs in a structure, which is a less general system than the domain of Foradori.

Finally he defines a "Schachtelung" or "nested domain", a domain S such that for each pair of elements A,B, at least one of the relations holds

$$A \subset B, \quad B \subset A.$$

We shall return to this theory later in the section on topology.

The classfication of the recent researches in structure theory is necessarily based on the classification of the subjects to which they are applied. Even this is difficult, owing to the multifold inter-dependence of subjects that at first glance seem quite unrelated. Roughly we can divide structure theory into the purely algebraic, the topological, the analytical, that branch dealing with probability and the geometrical. It is difficult to make sharp distinctions, but we shall follow this classification throughout.

Section 1. Structures in Connexion with Algebra.

Perhaps the most striking examples of structures occur in group theory, number theory and ideal theory. We may study a group by directly examining the relations between the individual elements and tabulate properties according to the behavior of one element with respect to another. Thus we may study the properties of a left-unit element, that is an e, such that e a = a for all elements a of the group. Or we may study the properties of the left-inverse or right-inverse of an element. (In the case of a group, each of these is unique). On the other hand, we may study the properties of subsets of a group, regarding these subsets as entities and never mentioning their elements directly. We can speak of the class of subgroups of a given group and can define operations defined over this class. Usually the groups of any certain class of subgroups are, in the final analysis, groups that are characterized by the behavior of their individual elements; an example is the class of all subgroups each of which is composed only of powers of a single element (a cyclic group). However, in many cases the characterizing property may be abstracted and re-expressed in terms of the behavior of the subgroups themselves with respect to certain operations. As was pointed out in the last chapter (see example 7) the class of all subgroups of a given group forms a structure under the operations of union and cross-cut. It is this structure and these operations that are so important in modern algebra.

<u>Definition 23 of modular structure.</u>

A structure is said to be modular if $A \leqq C$ implies $A \cup (B \cap C) = (A \cup B) \cap C$.

The modular structure is sometimes known as a Dedekind structure. Dedekind I 3 used the condition in the form, for every triple a,b,c:

$$\left[A \cap (B \cup C)\right] \cup \ (B \cap C) = \left[A \cup (B \cap C)\right] \cap \ B \cup C\ .$$ Another equivalent condition is (see Ore I 2)

$$C \geqq \overline{C},\ C \cup D = \overline{C} \cup D,\ C \cap D = \overline{C} \cap D \quad \text{imply } C = \overline{C}.$$

If we draw this same conclusion from just the last two of the relations then we have a more special structure called an arithmetic structure. Dedekind has shown that for any structure $G \geqq H$ implies

$$(H \cup C) \cap G \geqq H \cup (G \cap C)$$

see Dedekind I 3.

<u>Definition 24 of isomorphism.</u>

Two structures are said to be isomorphic if there is a one-to-one correspondence between their elements, preserving union and cross-cut of every pair of elements.

<u>Definition 25 of quotient of elements.</u>

By the quotient A/B of two elements A,B such that A   B, is meant the totality of elements X such that $A \geqq X \geqq B$.

Definition 26 of prime quotient.

A quotient A/B is called prime if there is no element X such that $A > X > B$.

One of the most important theorems of group theory is what is called the first theorem of isomorphism, see van der Waerden I 1 B. A normal subgroup N of a group G is a group such that the group product gn can be put in the form $n_1 g$ whenever g is in G and n is in N, and where n_1 is in N. Two groups G and $\overline{G}$ are said to be isomorphic if there is a one-to-one correspondence between their elements such that ab and $\overline{a}\,\overline{b}$ correspond to one another, where a and b represent in turn every pair in one group and $\overline{a}$ and $\overline{b}$ in turn every pair in the other group.

If we have a correspondence $a \to \overline{a}$ between two groups (single-valued in one direction, but not necessarily so in the other) that preserves all products, then it can be proved that all the elements of the group G that correspond to the unit of $\overline{G}$ form a normal subgroup N in G. Furthermore if we examine the class of sets gN, where gN means the set of products gn (g fixed and n representing in turn every element of N), we can prove that each element of G lies in exactly one of them and that if we consider each of these sets as an entity, the class will form a group, where the product (aN)(bN) is (ab)N. Thus this group, which we call the factor group G/N of G with respect to N can be shown to be isomorphic to $\overline{G}$. The first theorem of isomorphism states that if N is a normal subgroup in G and H is a normal subgroup of G, then the set of elements contained in both H and N, i.e. H N, form a normal subgroup and

$$H\,N/N \text{ is isomorphic to } H/H \cap N,$$

where H N is defined as the smallest subgroup of G containing H and N. So far we actually have been operating with properties of <u>individual</u> elements. This theorem need not be stated thus.

Let us consider a modular structure, and the quotient $H/(H \cap N)$, that is, the totality of X such that

$$H \geqq X \geqq H \cap N$$

and the quotient $(N \cup H)/N$ composed of the X' such that

$$N \cup H \geqq X' \geqq N$$

Now let us try $X' = N \cup X$, then $N = N \cup (H \cap N) \leqq N \cup X \leqq N \cup H$, from the above, and because $A \leqq B$ implies $A \cup C \leqq B \cup C$ and $A \cap C \leqq B \leqq C$. This X' is indeed in $(N \cup H)/N$. Moreover

$X' \cap H = (X \cup N) \cap H = X \cup (N \cap H) = X$, so the ordering between X and X' is one-to-one. Also $X'_1 \cup X'_2 = (N \cup X\ ) \cup (N \cup X_2) =$ $(N \cup N) \cup (X_1 \cup X_2) = N \cup (X_1 \cup X_2)$ and

$X_1 \cap X_2 = (H \cap X'_1) \cap (H \cap X'_2) = H \cap (X'_1 \cap X'_2)$, so that the correspondence is a structure isomorphism. Now if we interpret N, H, X, X' as normal subgroups of a group G and $\geqq$, $\cup$, $\cap$ as in the example on group theory at the end of Chapter I, then we have simply the first theorem of isomorphism.

Definition 27 of similar quotients.

The quotients C/D and C'/D are called similar if there exist relations expressing two of these elements say K,L in terms of the other two say A,B, by means of the operations $\cup$ and $\cap$, such that $(A \cup B)/A$ is isomorphic to $B/(A \cap B)$.

A composition series is a sequence $G \supset G_1 \supset G_2 \supset \cdots \supset C$ of subgroups of G (C is the group composed of the unit element of G) such that G_{i+1} is a normal subgroup of G_i for all i, and such that for no value of i can we find a subgroup H between G_{i+1} and G_i with the property that H is normal in G_i and G_{i+1} is normal in H.

The principal theorem on composition series is that of Jordan-Holder (see van Waerden I 1 B); it states that between each pair of composition series of a given group there exists a one-to-one correspondence such that the factor group G_{i+1}/G_i is isomorphic to the factor group $\overline{G}_{j+1}/\overline{G}_j$ for all i and j. Expressed in the language of modular structures this is stated as follows: If between two elements A,B of a modular structure there is an ordered chain

$$A_0 = A < A_1 \cdots < A_{n-1} < A_n = B$$ of elements A_i

such that all the quotients A_{i+1}/A_i are prime and the A_i are finite in number, then every other similarly defined chain between A and B has the same length and its prime quotients are similar, in some order, to the prime quotients A_{i+1}/A_i.

Ore has, in a series of papers, developed the theory of structures with the object of abstracting and generalizing the theorem of Jordan-Hölder to other than series of normal subgroups. He considers structures in which the elements are themselves quotients of a given structure. $(A/B) \cap (A_1/B_1)$ is defined as $(A \cap A_1)/(B \cap B_1)$ and $(A/B) \cup (A_1/B_1)$ as $(A \cup A_1)/(B \cup B_1)$. A/B is defined $\geqq A_1/B_1$ if $A \geqq A_1$ and $B \geqq B_1$. If the original structure has a null element E_0 such that $A \cap E_0 = E_0$ then the substructure (of the associated quotient-structure) composed of the elements A/E_0 is obviously isomorphic to the original structure. A quotient A/B is called prime over a quotient C/D if the element A is prime over C and $B = D$, or if $A = C$ and B is prime over D. The quotients A/A are called unit quotients. Two quotients A/B and C/D are called relatively prime if $(A/B) \cap (C/D)$ is a unit quotient. A product $(A/B) \times (B/C)$ is defined as A/C. Let us denote quotients by small letters. Instead of $a \times b = c$ we write $b = a^{-1} \times c$ and $a = c \times b^{-1}$. The transformed of $a = A/B$ by $c = C/B$ (they must have the same denominators) is defined as the element $c\, a\, c^{-1} = (a \cup c) \times c = (A \cup C)/C$. If a and c are relatively prime, then $a' = c\, a\, c^{-1}$ is called similar to a. Now we are ready to state the theorem of Jordan-Hölder in a still more suggestive form. We state the hypothesis as follows: if $A/B = p_1 \times p_2 \times \cdot\cdot \times p_r$ where p are all prime quotients, then every other representation of A/B as a product of prime quotients has the same number of factors, which are similar to the given factors taken in some order. Moreover (in a modular structure) two equal products

$$a_1 \times a_2 \times \cdots \times a_r = b_1 \times \cdots \times b_s$$

can be further factored so that the same number of factors are on both sides and they are similar, taken in some order. This is a generalization of the lemma of Schreier on the refinements of normal series, see van der Waerden I 1 B, p. 139 or Zassenhaus I 1. By defining a semi-normal element Ore, I 5 has been able to prove that chains of elements, of which each is semi-normal in the preceeding satisfy a theorem analogous to that of Jordan-Schreier.

Kurosch I 2 and Birkhoff I have each considered generalizations of the Jordan-Holder theorem to the case of transfinite well-ordered chains. Kurosch considers that the members of the chains are each normal subgroups in the member immediately following. By suitably defining a composition series, he shows that each two composition series are isomorphic. Birkhoff considers chains of T-invariant subgroups, where T is any automorphism group containing all the inner automorphisms of the given group. For definitions see van der Waerden I 1 B.

Closely connected with the above theory is that of the decomposition of groups and structures into direct factors (with respect to union or cross-cut). Kurosch I 1 has shown that in modular structures there is a factorization theory <u>for the elements</u>. If the element a can be represented as the cross-cut of the different elements A_i, $i = 1, \cdots, m$ and as the cross-cut of the different elements B_j, $j = 1, \cdots, m$ and if these representations are minimal (that is, A is not contained in

$A_1 \cap A_2 \quad \cdots \quad A_{i-1} \cap A_{i+1} \cap \cdots \cap A_n$, got by omitting an A_i from the sequence, and similarly for the b_j) then $n = m$, and every A_i can be replaced by a proper b_j in a one-to-one manner. The dual theorem (with respect to union) can be proved easily from this result. Klein I 14 has defined an "independent set" of elements of a modular structure. A set of element $X_1, \cdots, X_m$ is called independent if and only if $(X_1 \cup \cdots \cup X_{i-1}) \cap X_i = 0$ for $i = 2, \quad , n$. He proves that the definition is symmetric in the X_π and that if $Y_1, \cdots, Y_k$ are unions of disjoint subsets of the X_i, then the Y_i are independent (thus any subset of an independent set is independent). For further abstract treatments of linear dependence, see Menger II 1, Bergmann II 1, Menger II 5, Mac Lane II 1, Nakasawa II 1 (also under the section on projective geometry) and Whitney I 1.

Ore I 3,4,7 has built up a very pretentious theory of the decomposition of quotients. Let $a = a_1 \cup a_2 \cup \cdots \cup a_r$ be a quotient expressed as the union of a number of quotients with the same denominators; a is said to be "factored". The factorization is said to be proper if no factor a_i is contained (is $\leqq$) in its complement $\bar{a}_i = a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cdots \cup a_r$. The component with respect to a_i of any factor b of a is $a_i^b \equiv a_i \cap (b \cup \bar{a}_i)$. He proves that $a_i^b \cup a^c = a_i^{b \cup c}$ see Ore I 3. From this he deduces that if $a = a_1 \cup \cdots \cup a_r = b_1 \cup \cdots \cup b_s$ and if $a_{ij} = a_i^{b_j}$, $b_{ji} = b_j^{a_i}$ then $a_i = a_{i1} \cup \cdots \cup a_{is}$, $b_j = b_{j1} \cup \cdots \cup b_{jr}$ and in $a = a_{11} \cup \cdots \cup a_{rs}$, a_{ij} can be replaced by b_{ji}.

The quotient $a = A/A_0$ is called reducible if there exist two elements B, C such that $A_0 < B < A$, $A_0 < C < A$ and $A = B \cup C$, and a is then the union of B/A_0 and C/A_0. Otherwise a is called irreducible. The quotient is called of finite length if between A and A_0 there is a composition series, that is a series such as is used in the generalized theorem of Jordan-Hölder. It can be shown that if A/A_0 is irreducible, then every composition series $A_0 < A_1 \cdots < A_{n-1} < A$ possesses the same A_{n-1} and A/A_0 is said to belong to the prime quotient A/A_{n-1}. Ore proves for quotients a theorem similar to the one above of Kurosch: if A/A_0 is a quotient of finite length of a modular structure, then it may be represented as a minimal union of a finite number of irreducible quotients. Two such representations have the same number of factors, which belong to similar prime quotients taken in proper order, that is, there is a c_i that is relatively prime to a_i and b_i (the factors in the two respective factorizations) such that $c_i a_i c_i^{-1} = c_i b_i c_i^{-1}$. Each a_i in the first factorization can be replaced by a suitable b_j from the second one. He further investigates the possibility of carrying over the theorem to the case of quotients not necessarily of finite length. He shows that it will be true on the conditions that the descending chain condition holds in the structure, (the descending chain condition requires that every chain $A_1 > A_2 \cdots$ of decreasing elements be finite) and that from two direct decompositions $k = a \cup b = c \cup d$ (that is when a,b and c,d are respectively relatively prime) one can deduce the two, $k = c \cup b = a \cup d$. He proves a number of results on "completely reducible" structures. A completely reducible structure is one with a null element E_0 and such that $A > B > E_0$ implies the existence of at least one $C \neq A$ such that $A = B \cup C$. Further, he defines an n-fold component and finds certain invariants of this process that will not concern us here.

Klein has examined what he calls "Birkhoff structures", which are those in which one of the following "one directional" forms of the modular condition holds, (see Birkhoff I 1 p. 445),

a) If A and B cover C and $A \neq B$ then $A \cup B$ covers A and B.

b) If C covers A and B, and $A \neq B$, then A and B cover $A \cap B$.

An element T is said to "cover" an element Q if $T > Q$, $T \neq Q$ and there is no element P such that $T > P > Q$, $P \neq T, Q$. He shows that when both Birkhoff conditions hold, the modular condition holds. Klein also shows that any structure with more than 6 elements has a substructure with six elements.

<u>Definition 28 of distributive structure.</u>

A structure is called distributive if for every triple of elements (not necessarily different) A,B,C we have

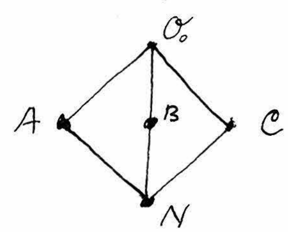
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

and

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

It can be proved that either of these conditions is a consequence of the other, see Dedekind I 2.

It may be proved that every distributive structure is modular see Birkhoff I 1. It is not true that every modular structure is distributive, for examine the following diagram (for explanation see the Hasse diagrams of Chapter I).



$A \cup (B \cap C) = A \cup N = A$, but $(A \cup B) \cap (A \cup C) = O_o \cap O_o = O_o$. This structure is easily seen to be modular. However, Birkhoff I 2 and Klein I 7 have shown that every modular non-distributive structure has a sub-structure that is distributive.

Furthermore, Klein I 7, Birkhoff I 1 and Ore I 3 have shown that for any structure

$$(A \cup B) \cap (A \cup C) \geqq A \cup (B \cap C)$$

$$A \cap (B \cup C) \geqq (A \cap B) \cup (A \cap C)$$ for any three elements A, B, C of the structure. Dedekind has priority see p. 61.

The structure of the ideals of a finite algebraic number field (that is, a field with a finite number of elements, composed of sums quotients, products and differences of algebraic numbers) is distributive. This follows from the theorem on the unique representation by prime ideals see van der Waerden I 1 B vol. II, pp. 36-38. The structure formed of the ideals of a commutative ring is modular. A divisorless ideal is one that is contained in no other ideal then the complete ring itself. A prime ideal is one that is different from the ring itself and such that whenever it contains the product ab, it contains at least one of the factors a, b. As an illustration of structure theory we shall prove that a divisorless prime ideal that contains (is greater than) $A \cap B$, contains at least one factor. For, by hypothesis $P \cup (A \cap B) = P$ and so, using the distributivity,

$(P \cup A) \cap (P \cup B) = P$. But P is divisorless and so $P \cup A$ and $P \cup B$ can be only equal to P or to O (the ring itself). Thus at least one must be equal to P.

The set-structures see example 1 last chapter are distributive. Birkhoff I 1 has proved the converse, that every distributive structure is isomorphic, in the structure sense, to a set-structure. We shall discuss this example more in detail in the sections on logic and topology. It might be mentioned that a finite distributive structure with a composition series of maximal length n is isomorphic to a structure of the sub-sets of a class of n elements and so he proves that the number of structurally different possible such structures as there are ways of partially ordering a set of n elements, see Birkhoff II 6. By means of set-representations he shows that a modular structure with the finite chain condition (ascending) is distributive if and only if each one of its elements can be uniquely represented as a minimal cross-cut (see above under Kurosch's work). He proves also that there are as many inequivalent point-set representations of a structure with a null-element and all-element as there are functions whose arguments describe all the prime-ideals of R and which possess arbitrary cardinal numbers as values. Furthermore, by these methods, he proves that if R is a finite distributive structure and $n < a_1 \quad \ldots < a_r = o$ is a composition series, then the totality of x with $(a_{i-1} \cup x) \cap a_i = a_{i-1}$ form a prime ideal in R and all prime ideals can be got thus.

Klein I 9 has shown that the axiom of distributivity may be expressed as $(A \cup B) \cap (B \cup C) \cap (C \cup A) = (A \cap B) \cup (B \cap C) \cup (C \cap A)$. He also proves that a structure of higher order than 5 is modular or distributive respectively if and only if all substructures of order 5 are modular or distributive respectively.

Klein I 4 has proved that in a distributive structure with a null element and such that every element contains only a finite number of elements there exists for each element A at most one representation of the form

$$A = P_1 \cup P_2 \cup \ldots \cup P_n \quad ,$$

where the P_i are "primary elements" over certain "prime elements". A "prime" element is one that contains no other elements with the possible exception of the null element of the structure (if it exist). The element Q is called "primary" over P if P is the only prime element contained in Q. Furthermore if it is assumed that every not-primary element $A \neq E_0$ factors (with respect to $\cup$) into two different elements then he has proved that there exists at least one representation of the above form for every element of the structure. Klein I 12 has shown that the elements of a distributive structure have unique representations as unions of prime elements. Klein I 11 examines the solutions X of the system of equations

$$A = B \cup X, \qquad C = B \cap X. \qquad\qquad A > B > C.$$

in various structures. The existence of at most one solution is shown to be characteristic of distributive structures. Klein I 3 defines a "characteristic mapping", which is a one-to-one correspondence between the elements $A, B, \cdots$ of the structure and the elements $a, b, \cdots$ of an Abelian group or a subset of an Abelian group, satisfying the following condition:

$\overline{A \cup B} + \overline{A \cap B} = \overline{A} + \overline{B}$, where $\overline{A}$ is the correspondent (the group element) of A (the structure element). The + is the group operation. Examples of such a mapping are: $\overline{A}$ is the number of elements in the set A; $\overline{A}$ is the sum of the divisors of the integer A, where $\cap$ and $\cup$ are the g.c.d. and the l.c.m.; $\overline{A}$ is the number of divisors of the integer A with $\cup$ and $\cap$ defined as above. In both the last examples multiplication is taken as the group operation. A generalized condition may be deduced from the one given above for a characteristic mapping, and in this form gives the "sieve of Eratosthenes" see also H. J. S. Smith I 1 and Brun I 1. Klein's formula is

$$\overline{A_1 \cup \cdots \cup A_n} = \underset{n}{\nabla} \overline{A_1} + \underset{n}{\nabla} \overline{A_1 \cap A_2 \cap A_3} + \cdots - \underset{n}{\nabla} \overline{A_1 \cap A_2}$$

$$- \underset{n}{\nabla} \overline{A_1 \cap A_2 \cap A_3 \cap A_4} - \cdots$$

where $\underset{n}{\nabla} \overline{A_1 \cap \cdots \cap A_i}$ means $(A_1 \cup A_2 \cup \cdots \cup A_i) + A_1 \cap A_2 \cap \cdots \cap A_{i+1} + \cdots$ etc. that is where $A_1, \cdots, A_n$ are taken i at a time and all different.

Birkhoff I 1 defined a "rank function" for finite structures. If $\overline{L}$ is any finite structure satisfying either of the one-directional forms of the modular condition and O_0 is the all-element and E_0 is the null-element (these both exist in finite structures) than it can be proved that a maximal chain connecting O_0 and E_0 exists and that any other such chain has the same length n. This number is called the rank $\rho(\overline{L})$ of the structure $\overline{L}$. The length of a maximal chain connecting E_0 with a given element A is the rank of A, $\rho(A)$. He proves that the one-directional modular conditions a) and b) imply

$$\rho(A \cup B) - \rho(B) \leqq \rho(A) - \rho(A \cap B)$$

$$\rho(A) - \rho(A \cap B) \leqq \rho(A \cup B) - \rho(B)$$

respectively and if both a) and b) hold then

$$\rho(A) + \rho(B) = \rho(A \cup B) + \rho(A \cap B).$$

This rank function is simply an extension of the dimension function, see Menger II 1, and the section on projective geometry and continuous geometry.

Klein I 8 studied "ausgeglichene" structures. These are structures in which all chains $A > A_1 > \cdots > A_n > B$ connecting every given pair of elements A,B, are finite and of the same length. He defined a rank function $[A,B]$, which for two dependent elements A,B (these are elements such that either $A > B$ or $A < B$) is simply the length of the chain connecting them. He defines a bridge as a sequence of elements $A_1, \cdots, A_m$ such that for each pair A_{i+1}, A_i at least one of the relations $A_{i+1} > A_i$, $A_{i+1} < A_i$ holds. He proves that for any such bridge

$$\sum_{i=1}^{m-1} A_i, A_{i+1} = [A_1, A_1 \cup A_m] + [A_1 \cup A_m, A_m]$$
$$= [A_1, A_1 \cap A_m] + [A_1 \cap A_m, A_m]$$

He defines the symbol $[A,B]$ for any pair of elements A,B.

$$[A,B] = \sum_{i=1}^{m-1} [A_i, A_{i+1}]$$ where $A_1, \cdots, A_m$ is a bridge

connecting A and B. He proves that for any three elements A,B,C

$$[A,B] + [B,C] = [A,C].$$

He designates K(A) as the set of all X for which $[X,A] = [A,X] = 0$ and proves that K(A) and K(B) are either identical or without common elements, so that the structure is the point-set union of all the classes K(A), K(B), etc.

A host of scattered results on finite structures have been found by Klein and Birkhoff. Klein I 7 showed that there are 1,1,2,5,15 kinds respectively of structures with 2,3,4,5,6 elements respectively. All the structures of order up to and including 4 are distributive, one is modular but not distributive and one is not modular. He shows that a structure of order n contains a substructure of order n - 1 for $2 \leqq n \leqq 7$, but not for n = 8. A structure of higher than order k contains a substructure of order k for $k = 5$, but in general not for k = 7. Most of the proofs are dependent on the representations as Hasse diagrams.

Birkhoff has treated of the "free" structure which is a structure in which every element can be represented as the result of applying the operations of cross-cut and union a finite number of times to elements selected from a certain fixed finite set of elements. One can also say that the structure is "generated" by a certain fixed "basis" of elements, under the operations of union and cross-cut. Birkhoff I 8 proves that the free structure generated by 3 elements is infinite (if not modular, distributive, etc). The most general modular structure generated by 3 elements possesses 28 elements and was mentioned by Dedekind I 3. Birkhoff I 1 proves that the most general modular structure generated by 4 elements is infinite; on the contrary that every distributive structure generated by a finite number of elements is finite.

Skolem I 1 claims to have discovered distributive structures but the date (1913) to which he attaches his claim is later than that of Dedekind. He defined canonical forms for free distributive structures and defined direct products of structures. The direct product of two structures with elements $A_1, B_1, \cdots$ and $A_2, B_2, \cdots$ respectively is the structure of ordered pairs (A_1, A_2) where union and cross-cut are defined by:

$$(A_1, A_2) \cup (B_1, B_2) = (A_1 \cup B_1,\ A_2 \cup B_2)$$

$$(A_1, A_2) \cap (B_1, B_2) = (A_1 \cap B_1,\ A_2 \cap B_2).$$

Ward I 1 and 2 has introduced a new operation, corresponding roughly to a quotient, called the residual. It is an abstraction of the residual, or ideal quotient used in ordinary ideal theory see van der Waerden I 1 B vol. II p. 29. By the ideal quotient A:B where A is an ideal and B is not necessarily such (A and B are both sets of elements of a commutative ring O) we understand the totality of elements c of the ring O for which c·b is in the ideal A whenever the element b is in the set B. If B and A are ideals then $B\cdot(A:B) \subset A$, by definition, (the ideal-product A·B is the ideal that consists of all the sums of the form $\sum_{i=1}^{n} a_i b_i$ where a_i is in A and b_i is in B). Ward I 2 considers an abstract structure in which there is defined a commutative and associative multiplication A·B, such that the distributive law $A\cdot(B \cup C) = (A\cdot B) \cup (A\cdot C)$ holds. If O is the all element of the structure then it is assumed that $O\cdot A = A$ for every element A of the structure. He then proves the existence of the residual A:B (a sort of inverse of multiplication) and deduces many of the usual properties of the ideal quotient. He shows that the structure is necessarily distributive in the usual sense if the condition is added:

if $A \supset B$ then there exists a P for which $A = B:P$. (The relation

$$M:(A \cap B) = (M:A) \cup (M:B)$$

is equivalent to the above added condition).

He proves that this relation implies that

$$(A:B) \cup (B:A) = 0$$

$$(A \cup B):M = (A:M) \cup (B:M)$$

$$A \cdot B = (A \cup B) \cdot (A \cap B).$$

Dilworth I 1 has carried through a postulational treatment of the residual defined over a structure. His postulates concerning the residual itself are

VII 1.) $A:A = 0$

2.) $(A:B):C = (A:C):B$

3.) $A:(B \cap C) = (A:B) \cap (A:C)$

4.) $(A \cup B):C = (A:C) \cup (B:C)$

5.) If $A:B = B:A = 0$, then $A = B$.

He considers a separate set of postulates that are got from the above by interchanging union and cross-cut and interchanging 0 and E_0.

Ward and Dilworth I 3 give the following sufficient condition that the ideal theory of E. Noether I 3, (see also Noether and Schmeidler I 1) apply to a residuated structure

1.) The structure is modular

2.) $A^m \cap B^n \leqq A\ B$ for every A and B and suitable exponents for each pair.

There are a few systems that have been developed from time to time under other names than structures, but that are actually structures. Whitney I 1 has considered sets of elements called "matroids", which are equivalent to the Birkhoff structures of finite order admitting a "rank" function, see Klein I 10. Whitney indicates four equivalent ways in which to construct a "matroid", using in turn the ideas of rank of a subset of the given set M; of a set of independent elements; of a basis and of a circuit. He uses the investigation of various systems of axioms to obtain results on matrices the elements of which are integers (mod 2).

Weisner I 1,2 has investigated a system of elements with a relation x/y called "hierarchy". There are six postulates

1.) Reflexivity x/x.

2.) Asymmetry x/y and y/x imply x = y.

3.) x/y and y/z imply x/z.

4.) Existence of a greatest common divisor of two elements (if x/y is read x divides y).

5.) Existence of the least common multiple of every two elements.

6.) There is only a finite number of elements x such that a/x/b for a fixed pair of elements a,b.

It is seen without difficulty that this system is a certain kind of structure. He uses the system to obtain an excellent generalization of the Dedekind inversion formula for finite series, see Bell I 1 B. In a paper following Weisner I 2 applies this theory of inversion to the theory of prime-power groups. This reference and its significance were pointed out to me by Mr. R. P. Dilworth.

We have given here only the barest sketch of the theory of structures as connected with algebra and have stressed only the most obvious connexions. The entire development of modern, and indeed present-day algebra has been dominated implicitly or explicitly, by the notions that are classed under the heading of structure theory. Many closely related results may be found in the papers of Fitting I 1-4, Remak I, P. Hall I 1, M. Hall I 1, Noether, etc. Results of a somewhat different nature were secured quite early by Daniell II (1917) (see the section on topology), and Menger 1922 (see geometry). A comparatively early paper of Grell I1 (1926) deals with algebraic problems in an exceedingly structure-theoretic manner. Krull I 2 has developed a dimension-theory in special types of rings.

Section 2 The structures connected with logic

Definition 29 of complemented structure.

A structure with an all element and a null element is called complemented if to each A there is at least one A' such that $A \cap A' = E_0$ and $A \cup A' = O$.

It can be proved that $(A')' = A$ and that $A \leqq B$ implies $B' \leqq A'$ for distributive complemented structures.

Definition 30 of Boolean algebra

A distributive complemented structure is known as a Boolean algebra see Boole I 1 B, 2 B. The significance of this algebra in logic is described in example (13) of Chapter I.

We shall give a comparatively full account of the researches of Stone on the connexions of Boolean algebras, structures, Boolean rings and Boolean spaces. A mention of some of the outstanding contributions of other men will follow.

Stone I 2 introduces a multiplication and addition into a Boolean algebra by means of the equation

$$A + B = (A \cap B') \cup (A' \cap B)$$

$$AB = A \cap B$$

and thus there results a ring with a principal unit element O - an element such that AO = A for all A of the ring - , and the null element E_0 - an element for which $A + E_0 = A$ for all A in the ring. Every element A is idempotent —, $A \cdot A = A$. Also $A + A = E_0$ for every A.

Definition 31 of Boolean ring, (or generalized Boolean ring)

A Boolean ring is a ring in which every element is idempotent and which does not necessarily contain a unit element.

Definition 32 of Boolean ring with unit.

A Boolean ring with unit is a Boolean ring in which a principal unit element E_0 exists and in which $A + A = 0_0$ for all A.

We may go from a Boolean ring with unit to a Boolean algebra by defining union and cross-cut in terms of the addition and multiplication as follows

$$A \cup B = A + B + AB$$

$$A \cap B = AB$$

Thus Boolean algebras are identified with Boolean rings with units. This idea of going from a ring to a Boolean algebra is not new, see Gegalkine I 1, and Daniell II 1, but Stone has made it the starting-point of his extensive theory of generalized Boolean algebras. In the theory of sets of points, $(A \cap B') \cup (A' \cap B)$ is sometimes known as the symmetric difference of A and B and has also been used frequently in combinatorial topology. Stone I 3 extends the theories of sub-algebras, congruence relations, and ideals to Boolean algebras (Boolean rings, on transformation).

He defines the subsystems of a generalized Boolean ring $\bar{A}$, with elements a, b, $\cdots$, as the subclasses of $\bar{A}$ that contain ab and a + b whenever they contain a and b. He proves that the subsystems of $\bar{A}$

are actually generalized Boolean rings. He defines the sum $B \cup C$ of two subsystems B, C of $\bar{A}$ as simply the subsystem generated by the point set sum of B and C, and the product $B \cap C$ or BC as simply the point-set product. An ideal of $\bar{A}$ is a non-void sub-class that contains a + b whenever it contains a and b, and that contains c whenever $c < a$ (c is said to be less than a if ac = c). It is shown that the subsystems form a structure, a substructure of which is formed by the class (J) of ideals of the given generalized Boolean ring. A(a) is defined as the class of elements b of $\bar{A}$ such that ab = b, (a fixed). A(a) is shown to be an ideal. If A is a subset of $\bar{A}$ then A' is defined as the set of elements a' such that $a' \cdot a = e$ for every element a in A. A' is called the orthocomplement of A in $\bar{A}$, and it is proved that A' is an ideal (regardless of whether A is or not) and $A' \cap A = E_o$ and A' contains every subclass P of $\bar{A}$ such that $A \cap P = E_o$ (A and P are said to be orthogonal). It is shown that $A \subset A''$, if A is an ideal. An ideal is said to principal (P) if A = A(a) for some element a; semi-principal (P*) if A = A(a) or A = A'(a) for some element a; simple (S*) if $A \cup A' = \bar{A}$; normal (N) if A = A". He shows that the respective classes of elements (P), (P*), (S*), (N) satisfy the inclusion relations $(P) \subset (P^*) \subset (S^*) \subset (N) \subset (J)$, where the elements of the class (N), for example, are those ideals satisfying the condition of normality. He proves that (N) = (S*) if and only if products of arbitrarily many elements are defined in $\bar{A}$; (S*) = (P) if and only if $\bar{A}$ has a "unit" o for which $o \cdot x = x \cdot o = x$ for all x; (N) = (P) if and only if $\overline{A}$ corresponds to a complete Boolean algebra, see Lewis-Langmuir I 1 B, p. 350 ; (J) = (P) if and only if the algebra is finite. This last follows from the fact that either the ascending or descending chain condition implies finiteness see Birkhoff I 1.

Tarski I 1 has set forth a notion of atomic element in a Boolean ring. In our notation an atomic element a is a non-null element such that $a \succ b$ implies b = a or else b = e, or equivalently, ab = 0 or ab = a for every b in the Boolean ring. It is actually a prime element with respect to the operation of the Boolean ring. He shows that the assumption that every element contains an atomic element is equivalent to the proposition that every element is equal to the sum of all its atomic elements. A class $\overline{S}$ of atomic elements is said to form a complete atomic system if b = 0 is the only element such that ba = 0 for every a in $\overline{S}$. Stone I 4 has investigated atomic elements and "atomic bases" quite exhaustively. Stone I 4 defines a barrier ideal as an ideal $B \neq \overline{A}$ such that $B' = E_0$ and such that there exist normal ideals C and D for which $A = C \cup D$ and $C \cap D = CD = E_0$. He proves that in a countable Boolean algebra every prime ideal is normal or else a barrier ideal. He defines a prime ideal just as in ordinary ideal theory.

Stone was the first to make a complete and systematic study of the relation between rings and Boolean algebras, but others, notably B. A. Bernstein, had previously announced many scattered results bearing on this subject. For example B. A. Bernstein I 2 showed that a ring with a finite number of elements, all of which are idempotent, can be represented as the direct sum of a finite number of prime fields of characteristic 2. Radokovic I 1, in 1929, considered a logical system of "basic" propositions A, B, C, ··· such that each of these is either true or false. The system contains the tautology T (the identically true proposition) and the contradiction K (the identically false proposition). Now a set of "generalized" propositions is constructed from these by applying the operation of union A + B (the proposition that is false whenever A and B are false, otherwise true) and cross-cut A B (true when A and B are both true, otherwise false). A numerically-valued dimension function d(A) is defined as having the value k for all the basic true propositions and the value t for all the basic false propositions. It obeys the law $d(A) + d(B) = d(A + B) + d(A \cdot B)$. Both the distributive laws hold i.e.,

$$A + (B \cdot C) = (A + B) \cdot (A + C)$$

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

and the union and cross-cut are each associative and commutative. Also

$$A + R = A$$

$$A \cdot T = A$$

He shows that his system satisfies the axiom of Menger-Bergmann for n-dimensional projective geometries. Even in the case of a finite set of propositions, he is able to define a non-trivial dimension function.

Stone systematically applies ring-theory to Boolean algebras, and wherever possible, group-theory. B. A. Bernstein I 1 had long before shown that a Boolean algebra could be represented as a group. Moisil has given an extended treatment somewhat resembling that of Stone. He considers a system of elements with two operations, each associative and commutative separately and such that

$$a + a = a, \quad a \cdot a = a$$

He examines the case in which the two distributive laws hold, the case in which the two laws

$$a + a \cdot b = a, \quad a \cdot (b + a) = a$$

hold (the case of semi-serial logic, or structure,) and finally, the case in which they both hold (the case of a Boolean algebra). He finds that, of the systems generated by two elements, the distributive system contains 6 elements and the semi-serial one 4 elements, while if there are 3 generators, there result 33 elements and 31 elements, respectively. He defines a subset M to be a subsystem of the system L if it is closed under the operations + and $\cdot$; M is an invariant subsystem if, in addition, for each a in M and p in L, a + p is in M; it is normal if for each a in M and x in L, a + x in M implies x in M; it is an ideal if it is normal and invariant. He proves the result found by Birkhoff I 1 that a distributive system with null and all elements is semi-serial (that is, a structure). He considers systems in which an operation (not relation) of implication $\supset$ is defined, with the properties

$$(a + b) \supset c = (a \supset c)\cdot(b \supset c)$$

$$a + (a \supset b) = a\cdot b$$

He proves, if p is a fixed element of a distributive logic L, then the transformation $a \rightarrow \bar{\bar{a}}$ is an automorphism, where $\bar{a} = a \supset p$; furthermore the system consisting of the residue-classes with respect to a family of automorphisms $T, T^2, T^3, \ldots$ is a structure.

Moisil I 2 in a later paper makes use of systems in which an "equivalence operation" and a symmetric difference operation are defined. He defines, for "ideals" in this system a "deducibility", b is said to be deducible from a with respect to an ideal M of a distributive system if and only if b belongs to every ideal that includes M and a. He states a well-known result of Gentzen in the form: if c is deducible from a and from b (both with respect to the same ideal M) and a + b is in M, then c is in M.

R. Vaidyanathaswamy I 1 has also investigated representations of Boolean algebras as groups. Duthie I 1 has made use of the symmetric difference of two elements of a Boolean algebra to define a (Boolean) function of bounded variation. He proves that the complement of a function of bounded variation is a function of bounded variation; that the complement of a function not of bounded variation is a function not of bounded variation; if f(x) is a function of bounded variation then so is f(x'). He extends the definition and some results to functions of two variables.

Birkhoff and von Neumann, see Birkhoff I 1, have put forth a logic that they consider meets the needs of the quantum mechanics. The basic ideas are a unary operation of priming, yielding <u>a</u>' from <u>a</u>, and a relation $a \subset b$. The axioms are in substance as follows

1.) $a \subset b$ and $b \subset c$ together imply $a \subset c$.

2.) $a \subset b$ implies $b' \subset a'$.

3.) $a \subset (a')'$ and $(a')' \subset a$.

4.) To every pair of propositions x and y there is a proposition $x \cap y$ such that $(x \cap y) \subset x$, $(x \cap y) \subset y$ and $z \subset x$ and $z \subset y$ together imply $z \subset (x \cap y)$.

5.) $a \subset a'$ implies $a \subset x$ for every <u>x</u>.

$a = b$ is defined as meaning the two relations $a \supset b$ and $b \supset a$ taken together. From 5) it follows that there is a proposition <u>e</u> such that $e \subset x$ for every x. From 1) and 3) follows $a \subset a$. The proposition e' has the property that $x \subset e'$ for every <u>x</u>. It can be proved that there exists a proposition $x \cup y$, $= (x' \cap y')'$, such that $x \subset x \cup y$ and $y \subset x \cup y$ and $x \subset z$, $y \subset z$ taken together imply $x \cup y \subset z$. From this we see that this logic is a structure. It can be proved to be modular if a numerically valued "dimension function" d(a) is introduced, with the following properties

a.) $a \subset b$ and $a \neq b$ implies $d(a) < d(b)$.

b.) $d(a) + d(b) = d(a \cap b) + d(a \cup b)$.

Now if we simply assume axioms 1 - 5 and distributivity and introduce the following two conditions

c.) There is a finite upper bound for the length n of any chain connecting the null element with the all element

$$e_{0} = a_{0} < a_{1} \quad \cdots \quad < a_{n} = o$$

d.) The structure is irreducible, that is, there is no "neutral" element $x \neq e_{0}$, o, such that

$(a \cap x) \cup (a \cap x') = a$ for all a and x fixed.

we can say that the structure is isomorphic to a projective geometry over a non-commutative field. For definitions of projective geometries see the section below on geometries.

This logic arose from the opinion that "one can reasonably expect to find a calculus of propositions that is formally indistinguisable from the calculus of linear subspaces with respect to set products, linear sums and orthogonal complements, and resembles the usual calculus of propositions with respect to <u>and</u>, <u>or</u>, and <u>not</u>." (See Birkhoff and von Neumann I 4). A phase-space in the quantum mechanics is a mathematical idea defined as follows. Any physical system S is at each instant hypothetically associated with a "point" <u>p</u> in a fixed "phase-space" P. The point <u>p</u> is supposed to represent mathematically the "state" of S and the "state" of S is supposed to be ascertainable by "maximal observations". The reader unfamiliar with these notions can find ample discussion in von Neumann's book on the quantum mechanics, see von Neumann I B 1. Moreover, every point p_{o} associated with S at a time t_{o}

together with a prescribed mathematical "law of propogation", fix the point p_t associated with S at any later time t.

In electrodynamics the points of P can only be specified after certain functions - such as the electrostatic and electromagnetic potential, are known; P is in this case a space with an infinitude of dimensions. The law of propagation is contained in Maxwell's equations. In quantum theory, the points of P correspond to the "wave function" and so P is a function-space usually assumed to be Hilbert space. In this case the law of propagation is contained in Schrodinger's equations.

To establish the correspondence to the experimental, or observation space S, we let the mathematical representative of a subset S_1 of the observation space (determined by compatible observations $\alpha_1, \ldots, \alpha_n$) for a quantum-mechanical system, be the set of all points of the phase-space that are linearly determined by proper functions f_k satisfying $\alpha_1 f_k = \phi_1 f_k, \ldots, \alpha_n f_k = \phi_n f_k$, where $(\phi_1, \ldots, \phi_n)$ is a point of S_1. It can be shown by quantum-mechanical methods that there exists, then, a set of mutually orthogonal closed linear subspaces P_i of P (which corresponds to the families of proper function f, satisfying $\alpha_1 f = \phi_{i1} f, \ldots \alpha_n f = \phi_{in} f$) such that every point (or function) f of P can be represented uniquely in the form $f = c_1 f_1 + c_2 f_2 + \ldots$, ($f_i$ in P_i). It can be shown that the mathematical representative of any experimental proposition is a closed linear subspace of Hilbert space. From the fact that all operators of quantum mechanics are Hermitian it can be deduced that the mathematical representative of the negative of any experimental proposition is the orthogonal complement of the mathematical representative of the proposition itself. If the following postulate is added to the usual ones of the quantum mechanics, then a suitable correspondence will result.

(A) The set-theoretical product of any two mathematical representatives of experimental propositions concerning a quantum-mechanical system is itself the mathematical representative of an experimental proposition.

From this postulate, together with the other deductions, it follows that the set-theoretical product and closed linear sum of any two, and the orthogonal complement of any one closed linear subspace of Hilbert space representing mathematically an experimental proposition concerning a quantum-mechanical system S, itself represents an experimental proposition concerning S.

In classical physics with this sort of logic of propositions, the distributive law holds. In ordinary Hilbert space the modular axiom holds only when the modul-sum of two subspaces representing propositions, is always closed.

It is not my intention to even sketch the full subject of mathematical logic. I have attempted only to indicate a few of the interdependences between the mathematical logics and the abstract structures. An entertaining and elementary treatment of some of the chief theories of mathematical logic is to be found in Frink I 1.

Section 3. Structures connected with probability.

In the theory of probability there is an ample field for the application of structure theory. The application comes usually by way of Boolean algebras. We shall pause here long enough to indicate a few of the best treatises. Boole I 1,2,3,4 and Poretski I 3 were among the first to treat probability by means of logic. Venn in his "Logic of Chance" presented one of the first extended treatments. Keynes in "A Treatise on Probability" and Reichenbach "Wahrscheislichkeitslehre", treat the modern problems fully, and give complete bibliographies.

Those familiar with ergodic theory will probably be acquainted with the possibilities associated with the introduction of structure theory. Birkhoff I 9 has begun a study of probability from the modern point of view especially as connected with ergodic theory. We shall not go into this further, but shall consider a most instructive example due to Markoff I 1.

Markoff considers the space of space-time points x,y, etc and defines between some pairs x,y the relation "precedes". $x \subsetneq y$ means x precedes y. The axioms concerning this relation are similar to those for structures.

A. If $x \subsetneq y$, then y does not $\subsetneq$ x.

B. If $x \subsetneq y$ and $y \subset z$ then $x \subsetneq z$.

C. A chain $x = z_0 \subsetneq z_1 \quad \ldots \subsetneq z_n = y$ between two given points x and y is limited in the number of terms.

D. To each pair of points x,y there is a point u and a point v, for which

$$x,y \lessdot u; \quad v \lessdot x,y.$$

Axiom C is a sort of "finite chain condition". Axiom D is considerably weaker than the axioms that guarantee the existence of a lub and a glb see Postulates VI 1,2. By the point x, we shall mean the ordered set (x_0, x_1, x_2, x_3). Now the relation $x \lessdot y$ can be established by $x_0 < y_0$, $(y_0 - x_0)^2 - \sum_{i=1}^{3} (x_i - y_i)^2 \geqq 1$, and it satisfies all the above axioms. Conclusions about relations of the metric defined by these conditions and the usual metrics of relativity are deduced by Markoff.

Section 4. Structures connected with projective geometries.

Many of the methods used in projective geometry since its origin have been structure-theoretic in essence, but the connexions with structure-theory were greatly obscured by the presence of many kinds of elements - points, lines, planes, etc. - each of which seemed to have its own separate identity. Although projective geometry had long been known as the geometry of projection and section (union and cross-cut, respectively) because of the commanding position these two ideas assumed, still there were few efforts to try to develop abstractly the general theory of n-dimensional projective geometry using the ideas of union and cross-cut as the fundamental primary notion.

In his theory of casts, von Staudt II 1, II 2 B, touched the idea. Veblen and Young II 1 came very near and moreover, without abstracting the idea of a dimension function they proved that if dim L represents the dimension of the linear sub-spaced L then

$$\dim L + \dim M = \dim (L \cup M) + \dim (L \cap M)$$

(see pp. 32-33, vol. I, theorems S_n2 and S_n3).

Menger II 4 formally stated the postulates of the ordinary n-dimensional projective geometry in terms of union and cross-cut. He started with a structure with all - and null-elements etc., and first postulated a transitive, symmetric and reflexive equivalence relation, "=", such that A = B or A ≠ B for every pair of objects. To these postulates he added a "subtraction" postulate: if A + S = S then there exists a B such that A + B = S and AB = null element; and a "dimension" axiom: to each element A there is associated an integer dim A such that

a.) if every element properly contained in A has dimension $\leqq n - 1$ then dim $A \leqq n$.

b.) if a proper element of A has dimension $\geqq n - 1$, then dim $A \geqq n$.

c.) dim A + dim B = dim $(A \cup B)$ + dim $(A \cap B)$.

Points are defined as elements of dimension zero, lines as elements of dimension 1, etc. If we take the structure as an n-dimensional number-space, A and B as linear sub-spaces, $A \cap B$ as the greatest linear sub-space contained in A and B, $A \cup B$ as the least linear sub-space containing A and B, then we have the postulates of n-dimensional projective geometry.

Bergmann II 1, took a slightly altered set of postulates in which both distributive laws hold. Then both operations were proved to be uniquely reversible. Instead of the "dimension postulate" he took further conditions on the operations and proved the existence of a proper dimension-function. He showed that if we take dim of the null-element = -1 and the dimension postulate of Menger with the exception of dim A + dim B = dim $A \cup B$ + dim $A \cap B$, we can prove dim A + dim B $\geqq$ dim $A \cup B$ + dim $A \cap B$. He II 2 made further critical studies of systems of postulates, some involving the formation of complements as the fundamental idea. Menger II 7 showed that if the subtraction axiom be further sharpened to require uniqueness, then his system of postulates describe a structure isomorphic with the structure of all subsets of a set, this having the power of the dimension of the all-element.

Wald (see Menger II 8) investigated the effect of requiring the unique reversibility of the operation of union. Schreiber II 1 investigated the effect on the structure of substituting various axioms on betweenness and on congruence relations. He bases his investigations on the observation that in the plane the group of similarity transformations may be defined as a certain cross-cut of the group of the projective transformation with that of the transformations of the form $x' + iy' = \frac{a}{x + iy}$ $\underline{x'}, \underline{y'}, \underline{x}, \underline{y}, \underline{a}$ all real. Schreiber has studied the effect of substituting postulates concerning the relation of an element to three other given elements.

Garrett Birkhoff II 3 shows that if we start from any n-dimensional vector space with coordinates in a number field the set of vector subspaces of the given space will form a complemented modular structure. If these elements are taken as the "elements" of a projective geometry, taking an "element" of dimension 1 as a point, an element of dimension 2 as a line, etc. then according to classical n-dimensional projective geometry,

P_1 Two distinct points are contained in one and only one line.

P_2 If A,B,C are points not all on the same line, and D,E ($D \neq E$) are points such that B,C,D are on a line, then there is a point F such that A,B,F are on a line and also D,E,F are on a line.

P_3 Every line contains at least three points.

P_4 The points on lines through an K-dimensional element and a fixed point not on the element are a (k + 1)-dimensional element, and every (k + 1)-dimensional element can be defined in this way.

He proves that any set of elements satisfying P_1 to P_4 with a dimension function, defined for every element and such that there is a finite upper bound to the dimensions of the elements, can be defined as a projective geometry. Birkhoff proves that any projective geometry is a complemented modular structure, if intersections are defined as cross-cuts and conjunctions as unions. This structure also has the finite chain conditions and has of course an all element and a null element. Conversely he proves that any modular complemented structure with the condition that every sequence of decreasing elements $a_1 > a_2 > a_3 \cdots$ has at most n terms, satisfies postulates P_1 to P_4, where union and cross-cut are defined as before, and the dimension of an element is the length of the decreasing chain that starts with it. The dimension of the structure is the dimension of its all element. He proves that every complemented modular structure of finite dimensions is isomorphic with the direct product of a finite Boolean algebra and a finite number of projective geometries and conversely that any direct product of a finite Boolean algebra and a finite number of projective geometries is a complemented modular structure of finite dimensions. Menger II 10 (see also below) has proved a similar result in somewhat different form.

Birkhoff and von Neumann II 4 have studied the problem of determining which projective geometries may be complemented in such a fashion that to any element there corresponds an A' such that (A')' = A and $A \leqq B$ implies $B' \leqq A'$. They express the result by saying that the number-field used in forming the projective geometry must have an involutory anti-isomorphism (that is, a transformation $a \to \bar{a}$ such that $\bar{a} \to a$ and $ab \to \overline{ba}$) to which there is associated a definite Hermitean diagonal form $\sum_{1}^{n} \bar{x}_i g_i x_i$ with $g_i = \bar{g}_i$ (which vanishes only for $x_1 = x_2 = \cdots = x_n = 0$). Some of the theory of such an isomorphism may be got from Albert I 1 B.

We have seen (see the section on algebra, Whitney) how the idea of linear dependence may arise in connexion with the theory of structures. Birkhoff II 1 has considered a matroid M (in the sense of Whitney) in which all sets consisting of at most two elements are independent (in the matroid sense). Now let L(M) be the set of all linearly closed sets, that is, all subsets A of M such that every element of M that depends linearly on A belongs to A. Then L(M) forms a structure with respect to the union and cross-cut defined in L(M). The nature of the matroid M is uniquely determined by the nature of the structure L(M), since the rank of a linearly closed set A is exactly the length of the longest ascending chain of linearly closed subsets of A. For further work of Birkhoff II 2, showing the connexion between the "discrete spaces" of Alexandroff and n-dimensional projective geometries see the discussion of Alexandroff in the section below on topology.

Nakasawa II 1 has treated axiomatically linear dependence in n-dimensional projective spaces, using a calculus due to G. Thomsen II 1: certain sequences of element of elements of an abstract space are designated as "cycles" with a product defined and an equivalence relation known as "to be valid", $a_1a_2 \cdots a_s = 0$ or for short $a_1a_2 \cdots a_s$; or not to be valid $a_1a_2 \cdots a_s \neq 0$. There are associated the following axioms

a.) aa

b.) $a_1 \cdots a_s$ implies $a_1 \cdots a_s x$; $s = 1,2,\cdots$

c.) $a_1 \cdots a_i \cdots a_s$ implies $a_i \cdots a_1 \cdots a_s$; $s = 2,3,$
$i = 2, \cdots s.$

d.) $a_1 \neq 0, \cdots a_s \neq 0, xa_1 \cdots a_s, a_1 \cdots a_s y$ imply
$xa_1 \cdots a_s y$; $s = 1,2, \cdots$

"To be valid" is the same as "to be linearly dependent". Elements can be taken as points. The totality of elements x that satisfy the relation

$a_1 \cdots a_n x$ where $a_1, a_2, \cdots, a_n$ is a fixed set of elements and $a_1 a_2 \cdots a_n \neq 0$, is defined as a linear space R^n generated by the "basis" $a_1 \cdots a_n$, and n is the rank of the space. The notion of basis and rank are then extended to any set M of elements of the given space.

Menger II 10 examines the notion of linear dependence in a generalized projective geometry. He defines m points $P_1, P_2, \cdots, P_m$ of a projective geometry as independent if

$$P_k \cap (P_1 \cup \cdots \cup P_{k-1} \cup P_{k+1} \quad \cdots \quad \cup P_m) = \text{null element}$$

for $k = 1, 2, \cdots, m$.

An hyperplane is defined as an element that is contained only in itself and the all-element. m hyperplanes $H_1, H_2, \cdots, H_m$ are called independent if $H_k \cup (H_1 \cap H_2 \cap \cdots \cap H_{k-1} \cap H_{k+1} \cap \cdots \cap H_m)$ = all element for $k = 1, 2, \cdots, m$. Certain necessary and sufficient conditions for linear dependence are stated. The dimension of an element is defined as the number of independent points whose union is equal to the element. If the finite chain condition holds, then every element will have a finite dimension dim V = -1 where V is the null element. dim A = 1 + max (dim A') for all A' contained in A. As before dim A + dim B = dim $(A \cup B)$ + dim $(A \cap B)$. He proves the theorem (see Birkhoff above) on the decomposition of a projective geometry into a direct product, by means of the notion of generalized simplex. He defines parallelism in terms of union and cross-cut and linear dependence.

MacLane II 1 has studied abstract linear dependence in projective geometry by means of a "k dimensional schematic figure", and has shown the relation to the matroids of Whitney. He defines a schematic plane figure as a system of a finite number of "points" and certain sets of these points, called "lines" and satisfying the axioms:

1.) Any pair of points belongs to one and only one line.

2.) Every line contains at least two points.

3.) No line contains all the points.

4.) There are at least two points.

It is not necessary that this schematic figure correspond to any actual figure in the plane. A schematic three-dimensional figure is a set of "points", "lines", planes satisfying

1.) Every triple of points belong to no one line belongs to one and only one plane.

2.) Every plane contains three points not on a line.

3.) No plane contains all the points.

4.) If a plane contains two points of a line, it contains all the points of that line.

The definition of a k-dimensional schematic figure is similar. He sets up a one-to-one correspondence between the schematic n-dimensional figures and the matroids of rank n + 1, where the rank of a set of points A is defined as the smallest r such that all points of A are contained in some (r - 1) plane. He proves that a schematic n-dimensional figure is completely determined if its (n - 1) planes are given, moreover, if a set of "points" and certain subsets of this set are given, these subsets will be (n - 1) planes of some figure if and only if there are circuit complements of a matroid M^*, that is, if these subsets satisfy the axioms C_1^*, C_2^*, while their complements satisfy Whitney's axioms C_1 and C_2 for circuits.

C_1^*. Every element is omitted from at least one circuit complement.

C_2^*. For every pair of elements e_1, e_2 there is a circuit complement containing e_1 but not e_2.

C_1. No proper subset of a circuit is a circuit.

C_2. If P_1 and P_2 are circuits, if e_1 is in both P_1 and P_2, and if e_2 is in P_1 but not in P_2, then there is a circuit P_3 in $P_1 + P_2$ containing e_2 but not e_1.

Further he gives examples of matroids that are not representable as the set of columns of a matrix, but proves that if a matroid M is representable in this way by a matrix of complex numbers, then M can also be represented by a matrix with elements from an algebraic field of finite degree. He indicates the relation, in certain cases, between the non-representability of a matroid by a matrix, and the degeneracy of the associated schematic figure.

Menger II 12 and II 13 has defined parallelism and the generalized parallelism and non-parallelism of Lobatschewsky and Bolyai by means of union and cross-cut, and has developed the theory of non-Euclidean geometry axiomatically from the structure-theoretic point of view.

Section 5. Structures connected with continuous geometries

Von Neumann II 2 has invented a most remarkable kind of geometry, one in which there are no points; that is a geometry in which there are no non-vacuuous elements of least dimension. A continuous geometry is defined as a structure L, defined by means of a partial-ordering relation, such that for the two postulates on the existence of a greatest lower bound and a least upper bound of any pair of elements (the cross-cut and the union) we have instead the postulates

(U) To each set $\overline{S}$ of elements S, there exists an element M_S such that $S \leqq M_S$ for all S in $\overline{S}$ and for any element M' with the same property, $M' \geqq M$.

(P) To each set $\overline{S}$ of elements S there exists an element D such that $D \leqq S$ for all S in $\overline{S}$ and such that for any element D' with the same property, $D' \leqq D$.

D is designated by $\underset{S \text{ in } \overline{S}}{P(S)}$ and M is designated by $\underset{S \text{ in } \overline{S}}{U(S)}$. Moreover, let L be modular, complemented and irreducible, and let the operations of union and cross cut be continuous, that is

(C) Let Ω be any aleph, A_α be a transfinite sequence defined for all ordinal numbers $\alpha < \Omega$. Then

a.) If $A_\alpha \leqq B_\beta$ for $\alpha < \beta < \Omega$, then

$$U_\alpha (A_\alpha \cup B) = \left[U_\alpha (A_\alpha)\right] \cup B$$

b.) If $A_\beta \leqq A_\alpha$ for $\alpha < \beta < \Omega$ then

$$P_\alpha (A_\alpha \cap B) = \left[P_\alpha (A_\alpha)\right] \cap B.$$

Now he has shown that a dimension function can be defined over the structure L such that

1.) $0 \leqq d(A) \leqq 1$

2.) $d(N) = 0$, $d(O_o) = 1$ where N and O_o are the null-element and the all-element respectively.

3.) $d(A \cup B) + d(A \cap B) = d(A) + d(B)$.

Further he shows that for the set of values of d(A) as A goes over the structure, there are only two cases possible. In the first case this set of values consists of the numbers $0, \frac{1}{n}, \frac{2}{n}, \cdots 1$, for some fixed n, characteristic of the particular structure. In this case, the structure L can be shown to be an abstract projective geometry in the sense of Birkhoff (see above). The only other possible case is that in which the set of values of the dimension-function consists of all the real numbers between 0 and 1. This is the proper continuous geometry and it may be considered as a limiting case of a projective geometry of dimension n (in the sense of Birkhoff), as n approaches infinity.

Von Neumann II 3 defines a metric, a generalized distance between two elements as $(A,B) = d(A \cup B) - d(A \cap B)$. This metric satisfies the conditions

1.) $(A,A) = 0$; $(A,B) > 0$ for $A \neq B$.

2.) $(A,B) = (B,A)$.

3.) $(A,C) \leqq (A,B) + (B,C)$

Since the set of values for the metric are, in the case of a projective geometry, a finite discrete set, no significant topology arises there by its introduction. In the case of a continuous geometry proper, there results an important non-trivial topology. The geometry is complete and connected with respect to this metric.

He (see II 4) seeks to algebraize the continuous geometry, and shows that it is structure-isomorphic, with respect to union and cross-cut (defined in the usual manner for ideals), with the structure of all principal right ideals of an unique (within ring-isomorphism) irreducible regular ring. A ring is defined as regular if it has an unit and if to each element A in the ring, there is an element Y, such that A Y A = A (this condition can also be stated that in the structure of the ideals of the ring, every principal ideal has a complement). It is shown that this ideal-structure need not be a continuous geometry and is indeed merely a complemented modular structure, in general. He (see II 1) proves that the centrum of the ring, as a consequence of the irreducibility, is a commutative division algebra. This division algebra is not itself always sufficient to characterize the original continuous geometry. In the special case that the continuous geometry is improper, that is, a projective geometry, the examples can be given in which two distinct projective geometries are associated with the same division algebra.

In seeking to characterize the continuous rings, which are those regular rings for which the principal right ideal structure is a continuous geometry, von Neumann II 5 is led to the notion of a "rank ring".

A rank ring is a irreducible regular ring in which a numerically-valued rank function, $\overline{R}(a)$, is defined for all elements a of the ring such that

1.) $a \leqq \overline{R}(a) \leqq 1$

2.) $\overline{R}(a) = 0$ if and only if $a = 0$

3.) $\overline{R}(1) = 1$

4.) $\overline{R}(ab) \leqq \overline{R}(a)$; $\overline{R}(ab) \leqq \overline{R}(b)$

5.) For $e^2 = e$, $f^2 = f$, $ef = fe = 0$, we have

$\overline{R}(e + f) = \overline{R}(e) + \overline{R}(f)$

He shows that $\overline{R}(a + b) \leqq R(a) + R(b)$ and sets up $\overline{R}(a - b)$ as a metric distance, calls it the "rank metric". Now the principal right-ideal structure of a regular irreducible ring is proved to be a continuous ring only if it is a rank ring, complete according to the topology induced by the rank metric. Conversely if it is a complete rank ring then it is either an n^{th} order matrix algebra over a suitable division algebra, or else it is a proper continuous ring. In an extension and consolidation of algebraic results concerning rank rings he considers algebraicity of ring elements with respect to the centrum of the ring, and algebraic numbers, defined as limits of sequences of "algebraic integers" (in a certain generalized sense).

Mac Lane I has worked with "continuous structures" (structures in which postulates (U) and (P) are valid) and has defined an "exchange structure" L satisfying the conditions

(E) The exchange postulate.

If a is in L and p and q are elements prime over zero, then $a < a \cup p \leqq a \cup q$ implies $q \leqq a \quad p$.

(Z) Existence of prime elements.

If $b < a$ are in L then there exists a prime element p in L such that $b < b \cup p \leqq a$.

(F) Finiteness of dependence.

If Q is a set of elements prime over zero and p is an element prime over zero such that $p \leq U(Q)$, then there exists a finite set of elements prime over zero $q_1, q_2, \ldots q_n$ of Q with $p \leqq q_1 \cup q_2 \cup \ldots \cup q_n$.

He has used these structures to investigate transcendence degrees and p-bases. It might be mentioned in passing that as early as 1887 Kempe (see II 1,2,3) had attempted to correlate geometry and the calculus of propositions by introducing a ternary function that could be interpreted in terms of betweenness, in special cases.

Section 6. Structures connected with topology

There are, roughly, two ways in which structures may be used in connexion with topology. We may study a given topological space by means of structure-theoretical methods, or we may start with a structure and observe the results of endowing it with a topology. The first way is commonly found in combinational topology, and the second is found in the general theory of sets of points.

Borel II 1 B (see p. 18) started with a set of points in which the idea of subset was defined, and then introduced the notion of the limit superior ("ensemble limite complet") of a sequence of such subsets, and the limit inferior of a similar sequence ("ensemble limite restreinte"). The first is defined as the class of elements such that each one is in an infinite number of the sets of the given sequence. The second is defined as the class of elements such that each one is in all but a finite number. He derives the formulas,

$$\overline{\lim}\, E_n = (E_1 + E_2 + \cdots)(E_2 + E_3 + \cdots)(E_3 + E_4 + \cdots)$$

$$\underline{\lim}\, E_n = E_1 \cdot E_2 \cdot E_3 \cdots + E_2 \cdot E_3 \cdots + E_3 \cdot E_4 \cdots + \cdots$$

where + and · are just the union and cross-cut for set theory (see Chapter I, example 1). C. de la Vallee Poussin IV 1 and IV 1 B treated the same question by means of characteristic functions, see also Hausdorff IV 2 B. The sequence is said to have a limit if $\overline{\lim}\, E_n = \underline{\lim}\, E_n$.

Daniell II 1 investigated the limit sets by means of the symmetric difference. He puts $|A - B| = A'B + AB'$ and calls it the "modular difference". Some of the properties of this modular difference are

$$|A - B| = |B - A|$$
$$|A - B| = |\bar{A} - \bar{B}|$$
$$\text{If} \quad |A - B| = 0 \quad \text{then } A = B \text{ and conversely}$$
$$|A - B| \subset A + B$$
$$|A - C| \subset |A - B| + |B - C|$$
$$|A - C| \cdot |B - C| \subset |A - C| + |B - C|$$
$$|A - B| + |A - C| + \cdots + |A - N| = (A + B + C + \cdots + N) - ABC\cdots N$$

Incidentally he notes that the standard type of equation in symbolic logic, having a unique solution, is of the form $XA' + X'A = B$ and that the solution may be expressed $X = |B - A|$.

He defines a sequence of classes $A_1, A_2, \cdots, A_n, \cdots$ as having a limit A if a set A exists with the property $|A - A_n| \subset S_n$ for all n, where S_n is the n^{th} term of a sequence such that $S_n \supset S_{n-1}$ and there is no point common to all the S_n (i.e. $\lim S_n = 0$). He shows that this definition is equivalent to that of Borel and furthermore that the Cauchy condition for sequences holds, that the necessary and sufficient condition that the sequence A_n have a limit is that

$$|A_n - A_{n+p}| \subset S_n \quad \text{for some decreasing null-sequence}$$

S_n .

Mac Neille starts simply with a set of elements a,b,c, partially ordered by a transitive and reflexive but not necessarily symmetric relation $\subset$. If a partially ordered set K is closed under certain operations, then a set L is defined as an extension of K if L is a partially ordered set closed under a certain class of operations, such that we can set up an one-to-one correspondence between the operations of K and a sub-class of those of L, and such that there exists an one-to-one correspondence between the elements of K and a sub-set of L, preserving the partial order, and preserving corresponding operations. Furthermore L is to be the smallest such set. He defines the cross-cut and union for each subset of K and postulates their existence for a complete K. He defines several kinds of extensions. The first extension is an adjunction of the units with respect to cross-cut and union. The second extension is the embedding of an arbitrary partially-ordered set K with units in a complete structure L, by means of "cuts". A cut (A,B) of K consists of two subsets A and B of K such that $a \subset b$ for every a in A and every b in B, $a \subset x$ for every a in A implies x in B, $y \subset b$ for every b in B implies y in A. He partially orders the cuts by putting $(A,B) \subset (C,D)$ if every element of A is an element of C. The system L of the cuts will be a structure and will be complete (in terms of union and cross-cut). A third extension embeds an arbitrary multiplicative partially-ordered system K in a distributive structure L. The elements of L are the subsets A,B, etc. of K. $A \subset B$ by definition if, for every a_i in A, $a_i = \sum_j a_i b_j$ where the sum is taken over every b_j in B, and $\sum_j a_i b_j$ is distributive. $\sum p_j$ is said to be distributive if to every element x such that $\sum p_j \supset x$ there exists a sum $\sum q_i = x$ where for each q_i there is a p_i with $p_i \supset x$. Then L is a distributive structure such that each of its subsets has an union in L, which union is distributive in L. Moreover any such structure that has a subset isomorphic to K, has a subset isomorphic to a subset of L. A fourth extension embeds an arbitrary distributive structure K with units in a Boolean algebra L. K' is the set of ordered pairs (a_i,a_j) of elements in K. $(a_i,a_j) \subset (b_i,b_j)$ if $a_i \subset a_j + b$ and $a_i b_j \subset a_j$. Now apply the third extension to K' and we get L.

Foradori, in his theory of inclusion (see the introduction to Chapter II for the notation) has set up a very general theory of limits and limiting sets by means of his essentially structure-theoretic methods. He calls a "domain" K an F-domain ("Gefüge"), if every pair of elements contain a common element of K. A domain G is called an R-domain ("Gerüst") if there exists a "sub-domain" A with the property that to every element h of H there is an F-domain K_h, whose cross-cut is h. H is called a basis for G, its elements are called basiselements and the F-domains K_h the basis domains. If a,b,c are elements of a domain B, and K is an F-domain in B then c is said to connect a and b by means of K if c is the cross-cut of the elements of K and every member of K contains at least one element contained in a and at least one element contained in b. If G is an R-domain in the domain B then an element s of B is said to be connected in G if to each pair of elements p,q in B such that $p \cup q = s$ there is at least one basis-element g of G that is contained in s and connects p and q by means of G. Let, as before, s be an element of B, g a basiselement of G then g is a cluster-element of s if g is the cross-cut of all elements of K_g (the F-domain associated with g) that properly contain g, and if every member of K_g that contains g properly always contains a sub-element of s different from g. By a subelement of s we mean an element contained in s. s is said to be closed in the R-domain G if it contains all its cluster elements. s is said to be continuous ("stetig") in G if it is connected and closed in G. A "nested domain" (see Chapter II introduction) S is said to be "unbroken" ("kontinuierlich") if every member i of S is the cross-cut of all the members of S that contain i properly, if there is contained in S the cross-cut of the elements of each nested subdomain T of S and finally if the cross-cut of all the elements of S is in S. Now an element k of the domain B is said to be a continuum if every nested domain all of whose elements are contained in k is a nested subdomain of an "unbroken" nested domain all of whose elements are contained in k.

Let a and b be two elements of a domain B; then a and b are said to be isomorphic or of similar structure, if the elements contained in a can be made to correspond in an one-to-one manner to the elements contained in b so that that the inclusion or partial order is preserved. Now he proves that if k and l are two isomorphic elements of a domain, and if k is a continuum, then l is a continuum. The property of being a continuum is thus a structural property. It can be shown, however, that the property of being a continuous set is not a structural property.

He applies these results to work out an enormously more general theory of measure than any previous one, see II 3. He is able to reduce Lebesque and Caratheodory measure to special cases of his measure. He works out a general Borel-Lebesgue theorem.

Let us pause a moment to notice a few of the connexions between this theory and the commoner topologies. A very good representative of the classical topologies is that of Hausdorff. For convenience in the later work of this thesis we shall note down the postulates of Hausdorff (see IV 1 B and 2 B). In a set H of elements we label certain sub-sets neighborhoods and impose on them the following postulates:

VII 1.) To each element a corresponds at least one of its neighborhoods; every neighborhood contains a.

2.) If V_1 and V_2 are two neighborhoods of a, there exists a neighborhood V of a such that $V \subset V_1 \cap V_2$.

3.) To each element b contained in a neighborhood V of an element a, there exists a neighborhood W of b such that $W \subset V$.

4.) To each pair of different elements a,b there exists a neighborhood V_1 of a and a neighborhood V_2 of b without common elements.

An element (or point) a is said to be a cluster point of a given subset S if every neighborhood of a contains elements of the set. The set S is said to be closed if it contains all its cluster-points. It is open if there is a disjoint closed set such that the union of the two is the whole space. The closure of a set is defined as the union of the set with the set of its cluster-points. It is said to be connected if it is impossible to express it as the point-set sum of two sets without common points and without common cluster-points. Now it can be shown that if we take the inclusion relation and the cross-cut and union just as in the classical point-set theory (see example 1, Chapter I), then we get a domain of Foradori, and it can be shown easily that the ordinary topological notions of connected set, closed set, etc. are identical in this case, with the corresponding ones of Foradori. However, his definitions are not simple generalizations of the point-set definitions as he shows quite clearly. Moreover, a set of sets can be partially ordered in an indefinite number of ways, none of which is equivalent in any respect to the ordinary point-set manner. His theory includes all of these, and produces some striking new results.

Perhaps the most highly developed theory of structures with regard to their direct connexion with Hausdorff topological spaces has been set forth by Stone II 1,2,3. For the nomenclature involved in the following description, the reader is referred to the first part of the section on logic. As the ideas of homomorphism and isomorphism are important in the sequel, we shall recall the fundamental theorem of homomorphism, which applies, of course, to a Boolean ring. In order that a Boolean ring $\overline{B}$ be homomorphic to a given Boolean ring $\overline{A}$, it is necessary and sufficient that there exist an ideal A in $\overline{A}$ such that the quotient ring $\overline{A}/A$ be isomorphic to B. Now regarding the symbols $\cup$, + and $\cdot$, ($\cap$ is regarded as identical with $\cdot$ according to the equation at the first of the section on logic) if an algebraic system B is homomorphic to a Boolean ring A with respect to the pair of operations +, $\cdot$ or with respect to the pair $\cup$ and $\cdot$, then B is homomorphic to A with respect to all three of the operations, and B is a Boolean ring. If the algebraic system B is homomorphic to a Boolean ring A with unit with respect to the pair of operations $\cup$ and $'$, or with respect to the pair of operations $\cup$ and $\cdot$, then B is homomorphic to A with respect to all four of the operations +, $\cup$, $\cdot$, and $'$, and B is a Boolean ring with a unit.

As a first step in establishing the connexions between a Boolean ring and a Hausdorff topological space Stone II 1,2 proves that every algebra of classes with more than one element is isomorphic to a reduced algebra of classes by virtue of an element-to-element correspondence of the basic classes. A reduced algebra of classes is a Boolean ring $\bar{A}$ with elements that are sub-classes of a fixed class E_A with elements e_π such that every element e_π in E_A is contained in some element of A and is the only element of E common to all the elements of $\bar{A}$ containing it. Moreover it can be proved that if $\bar{A}$ is actually an algebra of sub classes of a class E, A is an arbitrary ideal in $\bar{A}$ and E(A) is the union of all those subclasses of E that are elements of the ideal A, then the correspondence $A \rightarrow E(A)$ defines an homomorphism of the system of all ideals in $\bar{A}$, with unrestricted addition and finite multiplication as operations, to the system of all classes E(A), with the operations of forming arbitrary unions and finite intersections in accordance with the rules

$$\sum_{A \text{ in } B} E(A) = E(\underset{A \text{ in } \bar{B}}{S} A), \qquad \prod_{A \text{ in } B} E(A) \supset E(\underset{A \text{ in } \bar{B}}{P} A)$$

for A in a non-vacuous class of ideals in $\bar{A}$. $\sum$ and $\prod$ are union and cross-cut in $\bar{A}$, S and P are union and cross-cut in the ideal structure associated with $\bar{A}$. The algebra $\bar{A}$ of subclasses of the class E is said to be perfect if this homorphism is an isomorphism. From this result he proves that the necessary and sufficient condition for an abstract Boolean ring B to be isomorphic to the algebra of all sub-classes of some class E, is that every normal ideal in B be principal and that B contain a complete atomic system. The definitions of these terms have been given in the section on logic.

Now let $\bar{A}$ be a Boolean ring with elements a,b,c, C(a) the class of all prime ideals of $\bar{A}$ that are not divisors (do not contain in the point-set sense) of the principal ideal A(a), and B(A) the algebraic system with the classes C(a) as elements and the operations of forming finite unions symmetric differences and finite intersections. Then B(A) is a Boolean ring or algebra of classes and is isomorphic to $\bar{A}$ in the following way

$$C(a) \longleftrightarrow a$$

$$C(a + b) = C(a) + C(b) \quad \text{(symmetric difference)}$$

$$C(a \cup b) = C(a) \cup C(b) \quad \text{(union)}$$

$$C(ab) = C(a)\ C(b) \quad \text{(cross-cut)}$$

Moreover this is a perfect representation. In justification of the correspondence between a and C(a) it can be shown by transfinite induction that this does indeed exist and is one-to-one.

Now if in connexion with the prime ideal A of $\bar{A}$ we consider the class C(A) of all prime ideals of $\bar{A}$ that are not divisors (in the above sense) of A, then (see Stone I 3) the class C of all prime ideals A in $\bar{A}$ will be a topological space if we consider the ideals A as points of that space, and each C(A) as a neighborhood of every element it contains (in the point-set sense). An equivalent topology is yielded by taking the neighborhoods as the C(a). C then is a totally disconnected, locally bicompact Hausdorff space. The classes C(A) are characterized as the open sets in C. The classes C(a) are characterized as the bicompact open sets in C. The space is actually bicompact if and only if $\overline{A}$ is a Boolean ring with an unit. We recall that an Hausdorff space is bicompact by definition if from any class of open sets such that every point in the space belongs to at least one of the sets, we can select a finite class of them with the same property. It is said to be totally disconnected if every pair of points of it can be contained in two disjoint closed sets having the entire space as their union. It is locally bicompact if for each point there exists a neighborhood of this point whose closure is bicompact.

The converse is provable, that if S is a totally-disconnected, locally bicompact Hausdorff space, then the bicompact open subsets of of S constitute a Boolean ring A. Now if we proceed to topologize this A by the preceding paragraph, it can be shown that we arrive back to a space topologically equivalent to S. Moreover, it can be shown that a necessary and sufficient condition that a set be both open and closed is that it be a set C(a) described in the preceding paragraphs.

Now certain of these results can be generalized by establishing a certain type of mapping function of an Hausdorff space into a Boolean space and a complete mathematical equivalence is established between the theory of Hausdorff spaces and the theory of Boolean algebras by the result that the theory of arbitrary spaces is equivalent to the theory of densely distributed classes of mutually disjunct closed sets in Boolean spaces, see Stone II 1,3.

This theory produces definitely new results in the theory of approximation of polynomials and bounded continuous functionals. Applications are also made to the Brouwer-Menger-Urysohn theory of dimension.

Wallman has extended the connexions of the above type to the case of general bicompact Hausdorff spaces and a type of distributive structure. Let us take a distributive structure L with elements a,b,··· and null- and all-elements. A collection G of elements of L is called a "point" if the cross-cut of any finite number of the elements of the collection is not zero and G is a proper subset of no such collection. With each element a of L is associated a "basic a-set" of "point"s consisting of all the points that have the given element as one of their coordinates. The coordinates of a "point" are simply the elements that go to form the point. A closed set of "points" is defined as the "point"-set intersection of a finite, or infinite number of basic sets. Now it can be shown that the set of all "points" with this definition of closed set forms a bicompact T_1-space. A T_1-space is a topological space satisfying the axiom of Hausdorff except for the fourth, separation axiom (see above). This axiom is replaced by Frechet's axiom:

for every pair of points, each point possesses a neighborhood not containing the other. Moreover, there is a basis for the closed sets of S that is a structure-homomorphic image of L. This homomorphism is an isomorphism if and only if L has the property that if a and b are different elements of L, there exists an element C of L such that one of ac and bc is null and the other is not null. If L is complemented, then we simply get Stone's theory.

If we start with L as the distributive structure (with null-element and all-element) of the closed sets of a T_1-space R where the union and cross-cut are simply those of ordinary point-set theory and if we define an "ordinary point" of the associated space S as a "point" of S determined by the collection of all closed sets of R containing a given point of R, then the correspondence between the original space R and the space of "ordinary points" of S is an homeomorphism. An homeomorphism between two topological spaces is an one-to-one transformation that transforms open sets into open sets and such that open sets are the transforms of open sets. Furthermore, if R is bicompact, then every "point" of S is ordinary. He proves that the homology theory of Čech is identical in the cases of R and S, and dim R = dim S.

Kline II 1 has studied set-structures, both complemented and otherwise, with regard to limiting relations and their transforms.

Extending the idea, first enunciated carefully by Menger, of considering spaces wherein the elements are of various types, within the same space, and of different dimensions, Tucker as early as 1933 see II 1 formulated an abstract approach to arbitrary topological manifolds. A cell x of a combinatorial complex (for these definitions, see Seifert and Threlfall IV 1 B) K is said to include a cell y if and only if the closure of x contains y. The cells of K form then, a partially ordered set in which incidence relations, closure and boundary can be defined in terms of inclusion, such that the boundary of any boundary is void. Generalizing this idea, he calls certain partially ordered sets "cell-spaces" see II 3.

Alexandroff II 1 has considered discrete spaces ("espaces discrets"). A set E of any elements form a topological space if certain subsets (including always the vacuous set and E) are called closed, and the cross-cut and union (point-set) of any number of closed sets are themselves closed. The complements of closed sets are called open sets and are the neighborhoods of their elements. (This is essentially reversing the procedure above under Hausdorff spaces). Now E is called discrete D if the union as well as the cross-cut of a finite or infinite number of closed sets is closed. Every element p of D is in a smallest closed set $\bar{p}$ and in a smallest open set Op. The sets $\bar{p}$, Op, the vacuous set, and D are called elementary sets. By means of the introduction of a dimension function and certain basic axioms for the corners, or elements of dimension 0, he studies the homeomorphy of a discrete space composed of a finite number of elements with a discrete space consisting of polyhedra of an Euclidean space.

Garrett Birkhoff II 2 has studied Alexandroff discrete spaces. He shows that the discrete spaces with the multiplication axiom "the cross-cut of a finite or infinite number of elementary sets is an elementary set", is identical with a structure. He shows the relation between a finite dimensional discrete space and the direct sum of a finite number of projective geometries and a finite number of isolated points. He proves that a structure is n-dimensional and distributive if and only if it is isomorphic to the structure of the closed point-sets of a discrete space consisting of n points.

Kodaira II 1 has also written on the relation of cell-spaces with combinatorial analysis situs.

H. Cartan has considered a system, closely connected with structures, called a "filter". A filter of sets of a class J is a family F that includes all oversets of any one, and the product of any two of its members. A "basis" of F is a subfamily B such that any two subsets of B have a common subset in B, and such that the oversets include all sets in F. The system is essentially an abstraction of a topological space I, where F is the class of all sets whose interiors contain a fixed point p and B is a neighborhood system equivalent topologically to F. A neighborhood system is said to be equivalent to another if every neighborhood of one include, in the point-set sense, at least one neighborhood of the other and conversely.

Kurepa II 3 has applied generalizations and extensions of the Cantor theory of transfinite numbers to the study of families of sets. He defines a ramified class as a system of non-vacuous sets X such that for each pair of such sets, either they are disjoint, or else one is a subset of the other. A well-ramified system F, or ramified table is one such that to every subsystem $F' \subseteqq F$ there is an initial system $R_0(F') \subseteqq F'$ such that every set of F' is a subset of a set of $R_0(F')$. He applies these notions to an analysis and some novel restatements of the problem of Souslin: is a linearly ordered set without gaps or jumps in which every set of intervals without common inner points necessarily the linear continuum?

He examines correspondences between partially ordered spaces E_1 to E_2 and defines such a correspondence as an increasing function, if it is single-valued and $a_1 < b_1$ implies $f(a_1) < f(b_1)$ where a, b are in E_1. It is called a real increasing function if E_2 is similar, in the sense of Cantor, to a linear set. He proves that every non-denumerable ramified table in which there is defined a real increasing function has the same transfinite power as one of its subsets composed of elements that are pairwise independent, that is neither of $a < b$, $a > b$ holds for each pair. Also that for any non-denumerable ramified table T of open sets and closed sets taken from a space V that is perfectly separable, T has the same transfinite power as one of its sub-families of pairwise disjoint closed sets (see II 10). Further, in II 11 he proves that

$$\max (p_e E, p_d E, p_s E) \leqq pE \leqq 2p_s E^{\max(p_e E, p_d E)}$$

where pE is the power of the partially-ordered set E, $p_e E$ is the upper bound for the powers of subsets of E, $p_d E$ is the upper bound for the powers of subsets well ordered in both directions and $p_s E$ is the upper bound for the powers of subsets consisting of pairwise independent elements. A set is called partially well-ordered if every one of its ordered subsets is well-ordered. He uses this idea and the hypothesis of the continuum to obtain an hypothesis equivalent to that of Souslin.

The idea of pseudo-set is due to Shirai II 1. Let certain relations $\bar{e}$, $\underline{e}$ and $\parallel$ be defined between the elements of a set X and certain objects of any form than a given object M called a pseudo-set with reference to X if for all elements x of X we have $x \,\bar{e}\, M$, $x \,\underline{e}\, M$ and $x \parallel M$. Equality, inequality, sum and product of pseudo-sets can be introduced.

Section 7. Structures connected with function-spaces.

We start with a discussion of certain questions that are usually regarded as algebraic, but which are actually connected with analysis in their applications. Loewy II 1, as early as 1906, considered differential equations from the algebraic point of view. He considers a linear homogeneous differential expression of order n,

$$A_0(x)\,\frac{d^ny}{dx^n} + A_1\,\frac{d^{n-1}y}{dn^{n-1}} + \cdots + y$$

where the A's are rational expressions in x, with coefficients in some field. The associated differential equation is got by equating the expression to zero. This equation is called irreducible if it has no integral in common with a similar linear homogeneous differential equation of lower order, otherwise it is called reducible. Now according to a theorem of Kronecker, a linear homogeneous differential equation, like those above, that is satisfied by one solution of an irreducible linear homogeneous differential equation, is satisfied by all the solutions of that irreducible equation. Now a linear homogeneous differential equation is called completely reducible if one can find a finite number of different irreducible linear homogeneous differential equations $J_1 = 0$, $J_2 = 0, \cdots, J_g = 0$, such that the order of the given equation is equal to the sum of the orders of the g preceding equations and the given equation is the equation of lowest order among all those that are satisfied by the totality of the integrals of $J_1 = 0$, $J_2 = 0$, etc. The given equation is called the least common multiple of the J's. This can be shown to exist. He proves without use of the Picard-Vessiot theory of the rationality-group that every completely reducible differential equation is representable either uniquely, or in an infinite number of ways as the least common mutliple of a set of irreducible differential equations. The necessary and sufficient condition for the second possibility is that at least two of the set have simultaneously the same property as the given equation. Moreover the necessary and sufficient condition that a linear homogeneous differential equation be satisfied by the integrals of an infinite number of different irreducible linear homogeneous differential equations is that the given equation be representable in an infinitude of ways as the least common multiple of sets of the irreducible linear homogeneous differential equations. The works of Mc Coy II 1 and Raudenbush II 1 have carried this on further, but from the point of view of ring theory, rather than structure-theory, except insofar as they considered ideals. One of Ore's first contributions to the theory of structures was his paper on non-commutative polynomials, essentially a theory of differential expressions. The "multiplication" of two differential expressions $J_1(y)$ and $J_2(y)$ is defined as $J_1(J_2(y))$ or $J_2(J_1(y))$ respectively. This "multiplication" is not commutative. For the intricacies of notation and concept, the reader is referred to Ore's papers II 1 - 4.

Perhaps the first systematic use of a partial ordering of real functions of an abstract variable was used by P. J. Daniell (1917) see II 2. He defines the cross-cut of two functions $f_1(x)$, $f_2(x)$ just as we have in example 22, Chapter I, and similarly for the union. The sum of $f_1(x)$ and $f_2(x)$ is the function with the real values $f_1(x) + f_2(x)$. He studied various classes of numerically-valued functional-operators U(f) on the given function-space to itself (see the last example of Chapter I). Among these were the class of linear operations and the class (M) of operations M(f) such that when $f \leqq g$, then $M(f) \leqq M(g)$ and $U(f) \leqq M(f)$ for all operations U of the class (M). He defined an I integral (inferior integral) as any operation that is linear and positive (see example 22). He defined an S-integral (superior integral) as any operation that is linear and belongs to the class (M). A principal integral is an operation satisfying the condition of an I-integral and an S-integral simultaneously. It can be extended to the case of limit-functions of the first class if they are all bounded by a common upper bounding number. He gives examples in a later paper II 3.

Riesz II 1 made use of partial ordering of functions to secure theorems on the decomposition of linear numerically-valued functionals. First we know that for any such functional defined over the space of functions of one variable continuous on the closed interval (a,b),

$|A_f| \leqq M \cdot \max |f(x)|$ where M is a positive real number not depending on f(x). According to a theorem of Riesz III 1 every linear functional operation can be expressed as a Stieltjes integral

$$A_f = \int_a^b f(x)\, d\alpha(x),$$ where $\alpha(x)$ is a function of bounded variation depending only on the operation A and determined by it, essentially. Frechet III 1 studied a decomposition of (x) into three parts

$$\alpha(x) = \alpha_1(x) + \alpha_2(x) + \alpha_3(x),$$ where $\alpha_1(x)$ is the absolutely continuous part (this is the indefinite integral of $\alpha'(x)$); $\alpha_2(x)$ is the function of singularities, a continuous function of bounded variation whose derivative vanishes almost everywhere; $\alpha_3(x)$ is the function of jumps. Now Riesz was able to secure this decomposition by a method involving the theory of partial ordering of operations. The operation A is said to be greater than the operation B if the operation A - B is a positive operation, that is, $A_f - B_f \geqq 0$ for every non-negative function f(x). He proves that a set of operations that is bounded in the above sense, has a least upper bound, by decomposing f(x) into any number of functions of the same type continuous and non-negative, and applying to each one of them any one of the operations A belonging to the given set and finally adding the values. Then the lub of the set is defined as the least upper bound of all these sums.

A and B are called disjoint if their cross-cut (greatest lower bound) is the zero-operator (called simply zero). The operation I_f is defined as $\int_a^b f(x)dx$. A singular operation is one disjoint with I.

Three classes of operations are defined and it is shown that in each of them a finite or infinite number of operations possess an lub in the respective class. These classes are (1) the regular operations, those operators that are disjoint with all singular operations; (2) the continuous operations, consisting of the singular operations that are disjoint to all operations of the type $A_f = f(x_0)$; (3) the purely discontinuous operations consisting of the singular operations that are disjoint with all continuous operations. He proves that every linear operation can be decomposed into a sum of three operations, one from each of the above classes and that moreover this decomposition yields exactly the decomposition of $\alpha(x)$ due to Frechet. The decomposition of the operation is accomplished by purely lattice theoretic methods.

Kantorovitch has given the first complete theory of partially ordered function-spaces. The theory given in Chapter I is a direct exposition of his work see II 13.

Freudenthal II 1, 2, 3 has considered partially ordered function spaces much after the manner of Kantorovitch. He has considered elements of an arbitrary field as multipliers for the elements, instead of real numbers, as we have. He represents certain elements as Lebesque-Stieltjes integrals $\int \gamma \, de_\gamma$ where the γ is a real variable and e_γ is a montonic function of γ, with values in a certain Boolean algebra $E \subseteqq L$, where L is the partially-ordered space. For more details, see below under Stein. Furthermore, he characterizes his partially ordered space as the direct sum of spaces of two types. The first space is an L^2 space composed of functions on the interval (0,1) such that $|f(x)|^2$ is Lebesque-integrable, where $f \geqq 0$ if $f(x) \geqq 0$ almost everywhere. The second type is a Hilbert space (or in special case, Euclidean space) with the partial-ordering given above.

Glivenko II 1, 2 has considered an ordinary structure with a dimension function such that $d(0) = 0$, where the zero on the left is the null-element of the structure, and such that $a < b$ implies $d(a) < d(b)$. He proves that $|x - y| \equiv d(x \cup y) - d(x \cap y)$ is a metric, following the work of von Neumann see the section on continuous geometry. Conversely, the dimension function and the choice of null-element can be made to determine the algebra of the structure. He considers a general metric space with a distance function (a,b) and defines the element c to be between the elements a and b if $(a,b) = (a,c) + (c,b)$. He gives a necessary and sufficient condition, in terms of partial ordering that a partially ordered metric space be completely ordered.

Steen II 1 has used partially-ordered operator-rings in connexion with an axiomatisation of spectraltheory . With respect to the operation of addition, the elements of the space behave just as do the elements of the space considered in detail in Chapter I. With respect to the ring-multiplication he has $A > 0$ and $B > 0$ together imply $AB \geqq 0$. If $A \neq 0$ then $A^2 > 0$; there exists an element C such that $CA = 0$ implies

$A = 0$ for every A. Also he has the axiom $A \not\equiv B$ implies $A + D \not\equiv B + D$. Also A_+ and A_- are such that $A_+ \cdot A_- = 0$. For every element $P > 0$ there is defined an n^{th} root $P^{1/n}$. It is the lub $(\bar{K})$ where $\bar{K}$ is the set of all $B > 0$ such that $B^n \leqq P$. Now $\lim_{n \to \infty} A^{1/n} = E$ is proved where E has the properties $A_+ E = 0$ and $A_- E = A$. It is called a projection element. If we consider the class of all projection elements F with $A_+ F = 0$ for a fixed A, then lub $(F) = E_-$ exists and is itself a projection element with $A_+ E_- = 0$, $A_- E_- = A_-$. It can be proved that there exists an element I such that $A = I \cdot A$ for all A. Obviously this has the properties of the "C" above. The spectral family E_λ associated with A is obtained by equating E_{λ_0} to the particular E_- associated with $A - \lambda_0 I$ by the process above. Suppose A_x is defined for $a \leqq x \leqq b$, $-\infty \leqq a < b \leqq \infty$; that $A_{x_1} \leqq A_{x_2}$ for $x_1 < x_2$; that B_x is defined for $a \leqq x \leqq b$; that $H_k \leqq B_x \leqq K_k$ for x in the interval (x_k, x_{k+1}), $x_k < x_{k+1}$ where $H_k = \underset{x \text{ in } (x_k, x_{k+1})}{\text{glb}} (B_x)$, $K_k = \underset{x \text{ in } (x_k, x_{k+1})}{\text{lub}} (B_x)$; that $H_\ell = \sum_{k=-\infty}^{+\infty} H_k(x_{k+1} - x_k)A_x$ and $K_\ell = \sum_{k=-\infty}^{+\infty} K_k(x_{k+1} - x_k)A_x$ both exist; that $\lim_{\ell \to 0} H_\ell = \lim_{\ell \to 0} K_\ell$, where ℓ = the greatest of the intervals $x_{k+1} - x_k$. Then he calls their common value $\int_a^b B_x \, dA_x$ and if $B = f(x) \cdot I$ we write $\int_a^b f(x) \, dA_x$. He proves only by means of this abstract theory the fundamental theorem of Hilbert spaces and spectral theory, that $A = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda$. If these elements are interpreted as permutable self-adjoint transformations in

Hilbert space, then we have the usual theory. $A > 0$ means that A is positive definite, that is $(Af,f) \geqq 0$ for all f, and not always equal to zero. (,) is the inner product function. A + B, AB mean closed linear extensions of the transformations ordinarily represented as sum and product, respectively. A = B means that A and B are identical. $A^2 > 0$ is true if A is self adjoint. Now in the above fundamental theorem we can add conditions on E_λ, namely $E_\lambda^2 = E_\lambda$; $E_\lambda E_\mu = E_\mu$ if $\mu < \lambda$; $E_\lambda \to I$ as $\lambda \to +\infty$; $E_\lambda \to 0$ as $\lambda \to -\infty$, $E_{\lambda+\epsilon} \longrightarrow E$ as $\epsilon \to 0_+$. This theory is very much similar to that of Freudenthal on the same subject. He proves further II 2 that for all functions F(A) that can be constructed from the operation A and other operators B,C, ..., by finite addition and multiplication of operators and proceeding to limits by means of uniformly convergent sequences, there is a representation as an integral.

$$F(A) = \int_{-\infty}^{+\infty} F(\lambda)\, dE_\lambda .$$ If $F_n(\lambda) \to F(\lambda)$ as n ∞ for each λ in the interval $\lambda_1 \leqq \lambda \leqq \lambda_2$ then $F_n(\lambda) \to F(\lambda)$ uniformly in $\lambda_1 \leqq \lambda \leqq \lambda_2$ of given $H > 0$, there is an operator K with $0 < K \leqq H$ such that

$$F(\lambda)\, K - K^2 \leqq F_n(\lambda)\, K \leqq F(\lambda)\, K + K^2 \quad \text{for } n > n_0(H),$$

where $n_0(H)$ is independent of λ

Maeda II 1 has also written on structure-theory as connected with the theory of orthogonal systems in Hilbert space.

Chapter III

The Theory of Derivatives and Differentials

Section 1 Functions of real variables

We return now to the space S of Kantorovitch exposed in Chapter I, especially from page 28 onward to the end of that chapter.

Definition 1 of function of real variable.

$f(\phi)$ is said to be a function of the real variable ϕ in the interval $\phi_1 \leqq \phi \leqq \phi_2$, where ϕ_1 and ϕ_2 are two fixed real numbers, if to each value of ϕ in that interval, there is assigned one and only one element $y = f(\phi)$ in the space S. The function is defined similarly if the domain of values of ϕ is not an interval.

Definition 2 of limit of function.

If the sequence of elements $f(\phi_1)$, $f(\phi_2), \cdots, f(\phi_n), \cdots$ approaches the limit (see definition 17, Chapter I) Z (a fixed element throughout), for each sequence $\phi_1, \phi_2, \cdots \phi_n, \cdots$, approaching the same fixed number ϕ_0, the $f(\phi)$ is said to approach the limit Z as ϕ approaches ϕ_0 and one writes

$$\lim_{\phi \to \phi_0} f(\phi) = Z.$$

Theorem 1 If $\lim_{\phi \to \phi_0} f(\phi) = Z$ exists, then there is an element

y_o with the property that to each real number $\theta > 0$, there exists a real number $\pi > 0$ such that

(1) $|f(\phi) - Z| < \theta y_o$ for all ϕ such that $0 < |\phi - \phi_o| < \pi$

Proof By contradiction. Suppose that to each possible y_o there is a $\theta > 0$ and a point ϕ such that (1) is not satisfied. Let $\{\lambda\}$ be all such points (of the real numbers). Let the glb $|\phi_o - \lambda| = \mu$. Now if $\mu > 0$, take $0 < \pi < \mu$ and (1) is satisfied. If $\mu = 0$, let λ_i be a monotonic sequence extracted from the set $\{\lambda\}$, and approaching ϕ_o as a limit (this is easily possible by the properties of the glb). Then, the sequence $f(\lambda_1)$, $f(\lambda_2)$, ... $f(\lambda_n)$, ... does not converge to Z by Theorem 43, Chapter I. But from the definition of the limit of a function and the hypothesis of the present theorem, $f(\lambda_n)$ does approach a limit. Hence a contradiction.

Definition 3 of limit of function. (2nd definition).

The abstract function $f(\phi)$ of the real variable ϕ is said to approach the limit Z as ϕ approaches ϕ_o if there is an element y_o with the property that to any real number

$\theta > 0$, there is a $\pi_1 > 0$ such that

$$|f(\phi) - Z| < \theta y_0 \text{ whenever } 0 < |\phi - \phi_0| < \pi_1$$

This definition is justified by Theorem 1.

Definition 4 of left-limit of function (right limit)

The abstract function $f(\phi)$ of the real variable ϕ is said to approach the left-limit ℓ as ϕ approaches ϕ_0 if there is a fixed y_0 such that to any real number $\theta > 0$ there is a real number $\pi_1 > 0$, with the property that

$$|f(\phi) - \ell| < \theta y_0 \text{ whenever } 0 < |\phi_0 - \phi| < \pi_1 \text{ and } 0 < \phi_0 - \phi$$

A similar definition may be made for the right limit. In this case we restrict ϕ by the inequality

$$0 < \phi - \phi_0 < \pi'$$

Definition 2 of continuous function at a point.

A function $f(\phi)$ of a real variable ϕ is said to be continuous at the point ϕ_0 if $f(\phi)$ is defined throughout some interval of which ϕ is an interior point, and if

$$\lim_{\phi \to \phi_0} f(\phi) = f(\phi_0).$$

It is said to be left-continuous at ϕ_0 if $f(\phi)$ is defined throughout some interval of which ϕ_0 is the right end-point and

$$\underset{\phi \to \phi_0}{\text{left limit}}\, f(\phi) = f(\phi_0).$$

Definition 3 of continuous function in interval.

A function $f(\phi)$ of a real variable ϕ is said to be continuous in the interval $\phi_1 \leqq \phi \leqq \phi_2$ if it is continuous at every interior point of the interval and at the end-points the variable ϕ of definition 2 is assumed to take only values in the closed interval.

Definition 4 of hypercontinuous function in the interval.

$f(\phi)$ is said to be hypercontinuous on the interval $\phi_1 \leqq \phi \leqq \phi_2$ if there exists a y_0, such that to any real number $\epsilon > 0$, there is a real number $\delta > 0$ with

$$\left| f(\phi_1) - f(\phi_2) \right| < \epsilon \, y_0 \quad \text{whenever}$$

$$\left| \phi_1 - \phi_2 \right| < \delta$$

Definition 5 of bounded function on the interval.

$f(\phi)$ is said to be a bounded function on the interval $\phi_1 \leqq \phi \leqq \phi_2$, if the set of its values is bounded (above and below).

We shall develop a short theory of "Riemann integration" for bounded functions of a real variable. The development will be pursued just as far as is needed for the later theory of functional differentiation.

Let the bounded function $f(\phi)$ be defined on the interval $\phi_1 \leqq \phi \leqq \phi_2$. Let the symbol π denote a partition of (ϕ_1, ϕ_2)

into closed subintervals Δ_i of lengths Δ_i. The norm $N\pi$ of the partition is defined as the greatest of the quantities Δ_i, which are taken to be positive. Let λ_i be an arbitrary point of the closed subinterval Δ_i. Then if $\lim_{N\pi \to 0} \sum_{\pi} f(\lambda_i)\Delta_i$ exists (in a sense to be made precise in the next sentence) we say that $F(\phi)$ is integrable on (ϕ_1, ϕ_2) and denote it by $\int_{\phi_1}^{\phi_2} f(\phi)d\phi$.

To any sequence of partitions $\pi^{(n)}$ such that $N\pi^{(n)} \to 0$ there is associated the corresponding sequence of sums $\sum = \sum_{i_n} f(\lambda_{i_n}^{(n)})\Delta_{i_n}^{(n)}$. Every possible such sequence must approach the same limit, namely the integral.

<u>Theorem 2</u>

An hypercontinuous function has a definite integral

$$\int_{\lambda}^{\mu} f(\phi)d\phi$$

where (λ, μ) is the interval of hypercontinuity.

<u>Proof</u>. First let us notice that if π is a partition such that when λ' and λ'' belong to the same subinterval of π we have

$$\left| f(\lambda') - f(\lambda'') \right| < \epsilon y_0$$ then upon the

insertion of new points we have for the resulting

Σ':
$|\Sigma - \Sigma'| < \epsilon\ (\mu - \lambda) y_0$. For if in any given interval there be inserted one point we have the term $f(\lambda_i)\Delta_i = f(\lambda_i)\ (\Delta_i' + \Delta_i^2)$ before, and after, we have $f(\lambda_i')\Delta_i' + f(\lambda_i^2)\Delta_i^2$ and the difference is

$$\left[f(\lambda_i) - f(\lambda_i')\right]\Delta_i' + \left[f(\lambda_i) - f(\lambda_i^2)\right]\Delta_i \leqq$$
$$\left|f(\lambda_i) - f(\lambda_i')\right| \Delta_i' + \left|f(\lambda_i) - f(\lambda_i^2)\right|\Delta_i^2 <$$

$\epsilon\, y_0 \Delta_i$. The case of the insertion of more than one point follows by induction and the entire theorem by the addition of contributions from all the subintervals.

Now let $\pi^{(n)}$ be any sequence of partitions such that $N\pi^{(n)} \to 0$. Now we can find a δ for which

$$\left|f(\lambda') - f(\lambda'')\right| < \frac{y_0}{2(b-a)} \quad \text{when}$$

$|\lambda' - \lambda''| < \delta$ Let n_0 be a number for which $N\pi^{(n)} < \delta$ when $n > n_0$. Then if $n_1, n_2 > n_0$ and λ', λ'' are in the same subinterval of either $\pi^{(n_1)}$ or $\pi^{(n_2)}$

$$\left|f(\lambda') - f(\lambda'')\right| < \frac{y_0}{2(b-a)}.$$ Now form the partition π formed by the subdivisions of $\pi^{(n_1)}$

and $\pi^{(n_2)}$. We have

$$\left|\Sigma^{n_1} - \Sigma\right| < \frac{\epsilon y_o}{2(b - a)} (b - a) = \frac{y_o}{2}$$

$$\left|\Sigma^{n_2} - \Sigma\right| < \frac{\epsilon y_o}{2(b - a)} (b - a) = \frac{y_o}{2}$$

And so, by the triangular inequality,

$\left|\Sigma^{n_1} - \Sigma^{n_2}\right| < \epsilon\ y_o$ for $n_1, n_2 > n_o$. This, however, by Theorem 32 , guarantees the existence of a limit Σ_o for the sequence $\Sigma^{(n)}$. A sequence $\Sigma'^{(n)}$ would have the limit Σ_o', corresponding to a sequence of partitions $\pi'^{(n)}$. We have then elements Y_o, Z_o and numbers n_3, n_4, n_o, n'_o for which

$$\left|\Sigma^{n_1} - \Sigma^{n_2}\right| < \epsilon\ y_o \quad \text{for } n_1, n_2 > n_o$$

$$\left|\Sigma'^{n'_1} - \Sigma'^{n'_2}\right| < \epsilon\ y_o \quad \text{for } n'_1, n'_2 > n'_o$$

$$\left|\Sigma'^{n'} - \Sigma_o'\right| < \epsilon\ Z_o \qquad n' > n_4$$

$$\left|\Sigma^{n} - \Sigma_o\right| < \epsilon\ Y_o \qquad n > n_3$$

Consequently, by the remark at the first of the proof,

$$\left|\Sigma^{n_1} - \Sigma'^{n'_1}\right| < \epsilon\ y_o \quad \text{for } n'_1, n_1 \quad n_o, n'_o.$$

And $\left|\Sigma'^{n_1'} - \Sigma_0\right| \leqq \left|\Sigma'^{n_1'} - \Sigma^{n_1}\right| + \left|\Sigma^{n_1} - \Sigma_0\right| <$

$\epsilon\,(y_0 + Y_0)$

Therefore $\Sigma'^{n_1'}$ approaches Σ_0 as a limit, $\Sigma_0 = \Sigma_0'$ and the integral is unique.

Notice that the y_0 in the above depended only on the function to be integrated, that ϵ depended only on δ and that neither depended directly on the particular sequence of partitions. Also that, but the last equation above, $y_0 + Y_0$ serves as a suitable Z_0 and hence does not depend on the particular sequence of partitions.

<u>Theorem</u> 3 If $\psi(\lambda)$ is a real function of a real variable λ on (λ_1, λ_2) that doesn't change sign on that interval and if $\psi(\lambda)\cdot f(\lambda)$, $\psi(\lambda)\cdot\left|f(\lambda)\right|$ are integrable on (λ_1, λ_2) then

$$\left|\int_{\lambda_1}^{\lambda_2} \psi(\lambda) f(\lambda)\, d\lambda\right| \leqq \left|\int_{\lambda_1}^{\lambda_2} \psi(\lambda) \left|f(\lambda)\right| d\lambda\right|$$

<u>Proof</u>. Assume $\psi(\lambda) > 0$

$$\left|\Sigma_i\, \Delta_i\, \psi(\lambda_i)\, f(\lambda_i)\right| \leqq \Sigma_i\, \Delta_i\, \psi(\lambda_i) \left|f(\lambda_i)\right|$$

Now proceed to the limits. In case $\psi(\lambda) \leqq 0$ for all λ in (λ_1, λ_2), then use $\psi(\lambda)$ and note the following lemma.

<u>Lemma</u> $$\int_{\lambda_1}^{\lambda_2} k\,\psi(\lambda)f(\lambda)d\lambda = k\int_{\lambda_1}^{\lambda_2}\psi(\lambda)f(\lambda)d\lambda,$$

for any real k.

<u>Proof</u> Immediate from properties of the limit and the definition of integral.

<u>Theorem 4</u> If $\psi(\lambda)$ is a real function of a real variable on (λ_1, λ_2), $\psi(\lambda)\,f(\lambda)$, $\psi(\lambda)$ are integrable on (λ_1, λ_2) and $f(\lambda)$ is bounded. I.e.,

$$|f(\lambda)| \leqq y \text{ for all } \lambda, \text{ then}$$

$$\left|\int_{\lambda_1}^{\lambda_2}\psi(\lambda)f(\lambda)d\lambda\right| \leqq y\int_{\lambda_1}^{\lambda_2}\psi(\lambda)d$$

<u>Proof</u>. Similar to that of Theorem 3.

<u>Corollary</u>

$$\left|\int_{\lambda_1}^{\lambda_2} f(\lambda)d\lambda\right| \leqq y\,|\lambda_1 - \lambda_2|$$

<u>Theorem</u> 5 Let not $\psi(\lambda)$ change sign in (λ_1, λ_2), let $\psi(\lambda)f(\lambda)$, $\psi(\lambda)$ be integrable in (λ_1, λ_2) and M any closed convex set containing the image of (λ_1, λ_2) by means of the function $f(\lambda)$. Then there is in M an element X, such that

$$\int_{\lambda_1}^{\lambda_2}\psi(\lambda)f(\lambda)d\lambda = X\int_{\lambda_1}^{\lambda_2}\psi(\lambda)d\lambda$$

<u>Proof</u>

$$\frac{\sum \Delta_i\,\psi(\lambda_i)f(\lambda_i)}{\sum \Delta_i\,\psi(\lambda_i)}$$ belongs to the set M,

owing to its convexity. Also, owing to the fact that M is closed, the limit

$$X = \frac{\int_{\lambda_1}^{\lambda_2} \psi(\lambda) f(\lambda)\, d\lambda}{\int_{\lambda_1}^{\lambda_2} \psi(\lambda)\, d\lambda} \quad \text{belongs to M.}$$

Corollary

$$\int_{\lambda_1}^{\lambda_2} f(\lambda)\, d\lambda = X(b - a) \quad \text{for some X in M.}$$

Definition 6 of derivative

A function $f(\lambda)$ on (λ_1, λ_2) to S is said to have a derivative $\left.\frac{df}{d\lambda}\right]_{\lambda = \lambda_3} \equiv f'(\lambda_3)$ where λ_3 is an interior point of (λ_1, λ_2) if

$$\lim_{\mu \to 0} \frac{f(\lambda_3 + \mu) - f(\lambda_3)}{\mu}$$ exists. This limit is called the derivative. Left-hand and right-hand derivatives are defined in an obvious manner.

Definition 7 of derivable over an interval.

$f(\lambda)$ is said to be derivable over an interval, if it has a derivative at every point of the interval.

Definition 8 of hyperderivable function (over an interval).

$f(\lambda)$ is said to be hyperderivable over the interval (λ_1, λ_2) if it is derivable and if there exists a y_0 such that

$$\left| \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} - f'(\lambda_0) \right| < \epsilon\, y_0 \quad \text{for} \quad |\lambda - \lambda_0| < \delta_{\epsilon, \lambda_0}$$

Theorem 6

If $f(\lambda)$ is derivable at λ_0, it is continuous at λ_0.

Proof

$$\left|\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} - f'(\lambda_0)\right| < \epsilon\, y_{\lambda_0} \quad \text{for} \quad |\lambda - \lambda_0| < \delta_{\epsilon,\lambda_0}$$

or $$|f(\lambda) - f(\lambda_0)| < |f'(\lambda_0)| \cdot |\lambda - \lambda_0| + \epsilon |\lambda - \lambda_0|\, y_{\lambda_0}$$

$$= |\lambda - \lambda_0| \cdot Y_0$$ where Y_0

is fixed for λ_0 fixed, and λ variable.

Theorem 7

If $f(\lambda)$ is hyperderivable on the interval and its derivative is bounded over the interval, then it is hypercontinuous.

Proof. Similarly to Theorem 6 we get

$$|f(\lambda) - f(\lambda_0)| < |\lambda - \lambda_0|\, Y, \text{ for } |\lambda - \lambda_0| < \delta_{\epsilon,\lambda_0}$$

but where now Y doesn't depend on λ_0. Now the result follows by classical arguments following the Heine-Borel theorem.

Theorem 8

If the function $f(\lambda)$ on (λ_1, λ_2) to S is hyperderivable on (λ_1, λ_2) and that hyperderivative has an integral, then

$$f(\lambda_2) - f(\lambda_1) = \int_{\lambda_1}^{\lambda_2} f'(\lambda)\, d\lambda. \quad \lambda_1 < \lambda_2$$

<u>Proof</u> Let $\epsilon > 0$ be arbitrary. Since $f'(\lambda)$ is integrable, there exists $\delta > 0$ such that for every partition π with $N\pi < \delta$, and every choice of the λ_i in the closed intervals Δ_i of π

$$(1) \quad \left| \sum_{\pi} f'(\lambda_i)\Delta_i - \int_a^b f'(\lambda)d\lambda \right| < \epsilon y_0$$

where y_0 doesn't depend on the particular partition. From the definition of hyperderivation there is, for each point r of (λ_1, λ_2) a positive number $\alpha_i \leqq \delta$ with the property that for every point λ' of (λ_1, λ_2) subject to $|\lambda' - \lambda| \leqq \alpha_i$ and a fixed element y_1 of S we have

$$(2) \quad \left| f(\lambda') - f(\lambda) - f'(\lambda)(\lambda - \lambda') \right| \leqq \epsilon\, y_1 |\lambda' - \lambda|$$

The set of open intervals $I_r \equiv (\lambda - \alpha_i < \lambda' < \lambda + \alpha_i)$ cover the closed interval (λ_1, λ_2). Since (λ_1, λ_2) is an internal (closed) of real numbers, we may use the Borel-Lebesque theorem and find that a finite aggregate of them $I_1, \cdots, I_n$, with centers at $\rho_1 < \rho_2 < \cdots < \rho_n$, respectively, cover it. Assume that none of the I_j completely contains any other. Now applying (2), above, to the interval I_j

$$(3) \qquad \left| f(\lambda) - f(\rho_j) - f'(\rho_j)(\lambda - \rho_j) \right| < \epsilon \left| \lambda - \rho_j \right| y_1$$

(λ is assumed to be in I_j).

In the interval $\rho_j \leqq \lambda = \rho_{j+1}$, delete all end-points of intervals I_k such that $k \neq j$. The remaining end-points, together with the ρ_j determine a partition π of (λ_1, λ_2) of norm $\leqq \delta$. Each interval Δ_i of π has at least one of the points ρ_k as an end point. If there are two, we choose the left-hand one and denote the one chosen from Δ_i by ρ_i. By (3), we have

$$(4) \qquad \left| f(r_{i+1}) - f(r_i) - f'(\rho_i)\Delta_i \right| < \epsilon \Delta_i y_1$$

Adding together all equations of type (4) and using (1) we have

$$(5) \qquad \left| f(\lambda_2) - f(\lambda_1) - \int_{\lambda_1}^{\lambda_2} f'(\lambda)d\lambda \right| < \Sigma \epsilon \Delta_i y_1 + \epsilon y_o$$

$$= \epsilon \left| \lambda_2 - \lambda_1 \right| y_1 + \epsilon y_o$$

The left side, however doesn't depend on δ , while the right side can be made "arbitrarily small" for δ small enough. Thus the theorem is proved.

Theorem 9

A functions continuous on a closed interval is bounded on that interval.

<u>Proof</u> By contradiction. Suppose a sequence of values of $f(\lambda)$ that are unbounded. Then the corresponding set of values for the λ 's have a limit-point. Extract a sequence from this set approaching this limit. Then the values of <u>f</u> for these λ 's will form an unbounded sequence and hence a divergent one. But the continuity implies the converse. Hence a contradiction.

<u>Theorem</u> 10 If $f(\lambda)$ is continuous at λ and integrable, and

$$F(\lambda) = \int_{\lambda_1}^{\lambda} f(\lambda)d\lambda, \text{ then } \frac{dF}{d\lambda} = f(\lambda).$$

<u>Proof</u> $F(\lambda + \mu) - F(\lambda) = \int_{\lambda}^{\lambda+\mu} f(\phi)d\phi$ for μ small enough.

Also $F(\lambda + \mu) - F(\lambda) - \mu f(\lambda) = \int_{\lambda}^{\lambda+\mu} \left[f(\lambda) - f(\phi)\right] d\phi$

Then $\left|F(\lambda + \mu) - F(\lambda) - \mu f(\lambda)\right| \leqq y\left|\mu\right|$ by Theorem 4 above, where $y = \max\left|f(\lambda) - f(\phi)\right|$. (See Theorem 9) $\left|\frac{F(\lambda + \mu) - F(\lambda)}{\mu} - \mu f(\lambda)\right| = y$

Since $f(\phi)$ is continuous, $y \rightarrow 0$ as $\phi \rightarrow \lambda$ (i.e. as $\mu \rightarrow 0$).

Theorem 11 The hyperderivative equation

$\frac{dF}{d\lambda} = f(\lambda)$ where $f(\lambda)$ is hypercontinuous has an unique solution taking the value zero for $\lambda = \lambda_1$.

It is $F(\lambda) = \int_{\lambda_1}^{\lambda} f(\lambda)d\lambda$

<u>Proof</u> Assume another $F_1(\lambda)$ such that

$F(\lambda_1) = 0$, $\frac{dF_1}{d\lambda} = f(\lambda)$. Then

$$F_1(\lambda_1) - F(\lambda_1) = 0$$

$$\frac{d\left[F_1(\lambda) - F(\lambda)\right]}{d\lambda} = 0$$ and using Theoerm 8,

$$F_1(\lambda) - F(\lambda) = 0$$

<u>Theorem 12</u>

If $f(\lambda)$ has an integrable continuous hyperderivative $f'(\lambda)$, then

$$\left|f(\lambda_2) - f(\lambda_1)\right| \leqq \left|f'(\quad)\right|_{max} |\lambda_2 - \lambda_1|$$

The usual theorem on integration by parts can be stated with respect to hyperderivatives, as can Taylor theorems and existence theorems on derivative equations. We shall return later to such considerations.

We shall press on now to the theory of abstract differentials and their applications.

Section 2. Differentials and Functions of abstract variables.

We shall frame the definition of the abstract differential in terms of any L*-space of elements x,y,z, etc., satisfying postulates I, III 1 - 5 of Chapter I. In addition, to each one of a certain class of preferred sequences, called "convergent sequences", there is associated an unique element of the space, called the "limit" of the sequence and subject to the following rules.

(L*) 1. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$, then $x = y$.

2. If $\lim_{n\to\infty} x_n = x$ and $\{n_i\}$ is any infinite subsequence of $\{n\}$, then $\lim_{i\to\infty} x_{n_i} = x$.

3. If $x_n = x$ for all n then $\lim_{n\to\infty} x_n = x$.

4. If $\{x_n\}$ is not a convergent sequence then it contains (in the point-set sense) a subsequence $\{x_{n_i}\}$ such that no subsequence of this latter is convergent.

5. If $\lim \lambda_n = \lambda$ for a sequence of real numbers $\{\lambda_n\}$ and $\lim x_n = x$ for a sequence of elements, then $\lim \lambda_n x_n = \lambda x$

6. If $\lim x_n = x$, $\lim y_n = y$, (†) then $\lim (x_n + y_n) = x + y$.

Let F(x) be a function on one L* space, L_1 to another L* space, L_2. Then this function is said to have a differential at a point x_0 in L_1 if there exists a function $F(x_0;y)$ of two

† Note: F(x) defined "on" L^* means defined throughout

variables on L_1^2 to L_2 and a function $E(x\ ,y,z)$ of three variables on L_1^3 to L_2 with properties as follows.

1.) a) $F(x_0;y)$ is linear in y, that is:

a) $F(x_0;y + z) = F(x_0;y) + F(x_0;z)$ for any two elements y,z of L_1. (It is additive).

b) $\lim_{n\to\infty} F(x_0;y_n) = 0$ for every sequence $\{y_n\}$ such that $\lim_{n\to\infty} y_n = 0$. (It is continuous in y at y = 0.)

2.) a) $\lim_{n\to\infty} E(x_0,y_n,y_n) = 0$ for every sequence $\{y_n\}$ such that $\lim_{n\to\infty} y_n = 0$.

b) $\lim_{n\to\infty} E(x_0,y_n,z) = 0$ for every sequence $\{y_n\}$ such that $\lim_{n\to\infty} y_n = 0$.

c) $E(x_0,y,\lambda z) = \lambda E(x_0,y,z)$ for every real number λ.

3.) $F(x_0 + y) - F(x_0) - F(x_0;y) = E(x_0,y,y)$.

From 1.) a) and b), it may be proved by classical methods that $F(x_0; \lambda y) = \lambda F(x_0;y)$ for every real number λ. If this last equation is satisfied, $F(x\ ;y)$ is said to be homogeneous in y. We note that $E(x\ ,y,z)$ is homogeneous in Z, by 2) c).

Theorem 1. If $F(x)$ possesses a differential at x_0, then it is unique.

Proof $F(x_0 + \lambda_n \delta x) - F(x_0) - F(x_0; \lambda_n \delta x) =$

$$E(x_0, \lambda_n \delta x, \lambda_n \delta x),$$

where $\{\lambda_n\}$ is a sequence of real numbers, $\lambda_n \to 0$ as $n \to \infty$, and $\lambda_n \neq 0$ for any n.

Now divide both sides by λ_n

$$\frac{F(x_0 + \lambda_n \delta x) - F(x_0)}{\lambda_n} - F(x_0; \delta x) =$$

$$E(x_0, \lambda_n \delta x, \delta x)$$

by property 2 c. Now lim $\lambda_n \delta x = 0$, by the postulated property of the limit, and therefore $\lim_{n \to \infty} E(x_0, \lambda_n \delta x, \delta x) = 0$, by property 2 b of the differential.

Therefore $\lim_{n \to \infty} \frac{F(x_0 + \lambda_n \delta x) - F(x_0)}{\lambda_n} - F(x_0; \delta x) = 0$

or $\lim_{n \to \infty} \frac{F(x_0 + \lambda_n \delta x) - F(x_0)}{\lambda_n} = F(x_0; \delta x)$.

Now for a particular sequence, the limit on the left is unique. Thus for a particular sequence $F(x\ ; \delta x)$ is unique. But $F(x_0; \delta x)$ evidently exhibits no explicit or implicit dependence on $\{\lambda_n\}$, therefore the limit of the left-hand side doesn't depend on the particular sequence chosen, and $F(x_0; \delta x)$ is unique.

<u>Theorem</u> 2 Let $w = f(x)$ be a function on L_1 to L_2 and

$y = g(w)$ be a function on L_2 to L_3

where L_1, L_2, L_3 are all $L*$ spaces of the type described at the head of the chapter. Let further $f(x)$ be differentiable at x_0 and $g(w)$ be differentiable at $w_0 = f(x_0)$. Then $g(f(x))$ is differentiable at x_0, and its differential is

$$g(f(x_0); f(x_0; \delta x)).$$

<u>Proof</u> (1) $g(w_0 + \delta w) - g(w_0) - g(w_0; \delta w) = E_{23}(w_0, \delta w, \delta w)$

where $w_0 + \delta w = f(x_0 + \delta x)$

$$w_0 = f(x_0)$$

Then $f(x_0 + \delta x) - f(x_0) - f(x_0; \delta x) = E_{12}(x_0, \delta x, \delta x)$

and the left-hand side of (1) is

$$g(f(x_0 + \delta x)) - g(f(x_0)) - g(f(x_0); E_{12}(x_0, \delta x, \delta x) + f(x_0; \delta x)) = E_{23}(w_0, \delta w, \delta w).$$

(2) $$g(f(x_0 + \delta x)) - g(f(x_0)) - g(f(x_0); f(x_0; \delta x)) = E_{23}(w_0, \delta w, \delta w) + g(f(x_0); E_{12}(x_0, \delta x, \delta x)).$$

Now the right-hand side of (2) has two terms. Call

$g(f(m); E_{12}(m,n,p)) = E(m,n,p)$ and

$$E_{23}(w_0, \delta w, \delta w) \equiv E_{23}(f(x_0), f(x_0; \delta x) + E_{12}(x_0, \delta x, \delta x), f(x_0; \delta x) + E_{12}(x_0, \delta x, \delta x)).$$

Call $E_{23}(f(m), f(m;n) + E_{12}(m,n,n,), f(m;p) + E_{12}(m,n,p)) \equiv F(m,n,p)$.

(A) Then $E(m,n,p)$ is homogeneous in p.

$E(m,n,p) \to 0$ as $n \to 0$, p,m fixed.

$E(m,n,n) \to 0$ as $n \to 0$

(B) $F(m,n,p)$ is homogeneous in p

$F(m,n,p) \to 0$ as $n \to 0$, m and p fixed

$F(m,n,n) \to 0$ as $n \to 0$ m fixed.

Now, put $E_3(m,n,p) \equiv E(m,n,p) + F(m,n,p)$ and it will have the desired properties; wherewith the theorem is proved.

Definition 1 of continuous function

A function $F_2(x_1)$ on L_1 to L_2 is said to be continuous at x_1 if for every sequence $\{x_1^{(n)}\}$ such that $\lim_{n \to \infty} x_1^{(n)} = x_1$, we have $\lim_{n \to \infty} F_2(x_1^{(n)}) = F_2(x_1)$.

Theorem 3 If $F(x)$ on L_1 to L_2 has a differential at x_0, then it is continuous at x_0.

Proof $F(x_0 + \delta x) - F(xo) = F(x_0; \delta x) + E_{12}(x_0, \delta x, \delta x)$

But the right-hand side has limit zero for $\lim \delta x = 0$ (no matter what sequence is used).

Therefore $\lim_{\delta x \to 0} f(x_0 + \delta x) = f(x_0)$. But $x_0 + \delta x$ can be any typical sequence with limit x_0.

<u>Theorem</u> 4 If $f(x)$ on L_1 to L_2 has a differential $f(x_0; \delta x)$ at x_0, then $\alpha f(x)$ has the differential $\alpha f(x_0; \delta x)$ at x_0.

<u>Proof</u> $f(x_0 + \delta x) - f(x_0) - f(x_0; \delta x) = E_{12}(x_0, \delta x, \delta x)$

Multiply through by α and αE_{12} has the desired properties.

<u>Theorem</u> 5 If $g(x)$ on L_1 to L_2 has a differential $g(x_0; \delta x)$ and $f(x)$ on L_1 to L_2 a differential $f(x_0; \delta x)$ both at the point x_0, then $\phi(x) \equiv f(x) + g(x)$ has a differential $\phi(x_0; \delta x) \equiv f(x_0; \delta x) + g(x_0; \delta x)$.

<u>Proof</u> $f(x_0 + \delta x) - f(x_0) - f(x_0; \delta x) = E_{12}(x_0, \delta x, \delta x)$

$g(x_0 + \delta x) - g(x_0) - g(x_0; \delta x) = F_{12}(x_0, \delta x, \delta x)$

Add and call $E_{12}(m,n,p) + F_{12}(m,n,p) \equiv C_{12}(m,n,p)$.

$$f(x_0 + \delta x) + g(x_0 + \delta x) - \left[f(x_0) + g(x_0)\right] - \left[f(x_0; \delta x) + g(x_0; \delta x)\right] = C_{12}(x_0, \delta x, \delta x)$$

<u>Definition</u> 2 of derivative of a function $f(\lambda)$.

$f(\lambda)$, a function on the real number segment (λ_1, λ_2) to L_2 is said to have a derivative at λ_0 if there exists a function $f'(\lambda)$ such that

$$\lim_{n \to \infty} \frac{f(\lambda_0 + \lambda_n) - f(\lambda_0)}{\lambda_n} = f'(\lambda_0) \text{ for every sequence}$$

$\{\lambda_n\}$ in (λ_1, λ_2), and that $\lambda_n \to 0$, where $f'(\lambda_0)$ doesn't depend on the particular sequence.

Theorem 6 If $f'(\lambda_0)$ exists, it is unique.

Proof. Obvious from the definition of a limit.

Definition 3 of convex set.

A set is said to be convex if for every pair of points x_1, x_2 in the set, it is true that $x_1 + \alpha(x_2 - x_1)$ is in the set for every real α in the interval $(0,1)$.

Theorem 7 If $f(x)$ on L_1 to L_2 has a differential at all points x of a convex set $C \subset L_1$, then $\phi(\lambda) \equiv f(x_1 + \lambda(x_2 - x_1))$, where x_1, x_2 are in C, has a derivative $\phi'(\lambda)$ for all λ such that $0 \leqq \lambda \leqq 1$.

Proof. Let $\{\lambda_n\}$ be a sequence of real numbers such that $\lambda_n \to 0$ as $n \to \infty$.

$$\frac{\phi(\lambda + \lambda_n) - \phi(\lambda)}{\lambda_n} = \frac{f(x_1 + (\lambda + \lambda_n)(x_2 - x_1)) - f(x_1 + \lambda(x_2 - x_1))}{\lambda_n}$$

$$= \frac{f(x_1 + \lambda(x_2 - x_1) + \lambda_n(x_2 - x_1)) - f(x_1 + \lambda(x_2 - x_1))}{\lambda_n}$$

$f(x_1 + \lambda(x_2 - x_1); x_2 - x_1)$. That is, the last quantity is a limit that exists if $x_1 + \lambda(x_2 - x_1)$ is in C, by hypothesis. But $0 \leqq \lambda \leqq 1$ and C is convex. The theorem follows immediately.

Section 3 Mappings from Banach to semi-ordered spaces.

Notation a,b,c, etc. denote elements from the space B; x,y,z etc. those from the space S.

<u>Definition 1 of function.</u>

A function $F(a)$ "on" B "to" S is a correspondence between a Banach space B and a semi-ordered space S, such that to each element of B corresponds exactly one element of S.

<u>Definition 2 of linear function on B to S.</u>

$F(a)$ on B to S is said to be linear if it is

1) Additive, i.e. $F(a + b) = F(a) + F(b)$

2) Continuous, i.e. to every sequence $\{a_n\}$ from B such that $a_n \to a_0$ we shall have $F(a_n) \to F(a_0)$ for a_0 in B. The two limits are to be taken in the respective spaces.

<u>Theorem 1</u> If a function $F(a)$ on B to S is linear, then it is homogeneous, i.e. $F(\lambda a) = \lambda F(a)$ for real.

<u>Proof.</u> $F(a_1 + a_2) = F(a_1) + F(a_2)$ in particular

$F(a + a) = 2F(a)$ and by induction

$F(na) = nF(a)$ Put $\lambda a = b$, $a = b/\lambda$ for λ an integer.

$$\frac{1}{\lambda} F(b) = F\left(\frac{b}{\lambda}\right)$$

$$F\left(\mu \frac{b}{\lambda}\right) = \mu F\left(\frac{b}{\lambda}\right)$$

$$= \frac{\mu}{\lambda} F(b) \quad , \mu, \lambda \text{ integers.}$$

Now let a given irrational number λ be the limit of a sequence of rationals $\{\lambda_n\}$ then

$F(\lambda_n a) = \lambda_n F(a) \longrightarrow \lambda F(a)$

$F(\lambda_n a) \to F(\lambda a)$ owing to the continuity.

$\lambda F(a) = F(\lambda a)$.

The following three theorems have been indicated without proof by Kantorovitch II 3.

Theorem 2. In order that an additive operation be continuous, it is necessary and sufficient that it be bounded in the sphere $\|a\| = 1$.

Proof. Necessity, assume F(a) continuous.

If $\{a_n\}$ be any sequence such that $\|a_n\| = 1$ for all n, then

$\left|F(\frac{a_n}{p_n})\right| \to 0$ as $n \to \infty$, where p_n is any infinite subsequence of the natural numbers. Now suppose $F(a_n)$ unbounded. That means that to any element y of S, we can find a subsequence $\{a_{n_p}\}$ of $\{a_n\}$ such that

$|F(a_{n_p})| \nless n\,y$. Divide both sides by n.

$\left|F(\frac{a_{n_p}}{n})\right| \nless y$. But $F(\frac{a_{n_p}}{n}) \to 0$. Therefore by Theorem 43, page 54, there exists a y such that

$$\left|F(\frac{a_{n_p}}{n})\right| < e\,y \quad \text{for } n > N_e.$$

Contradiction.

Sufficiency. Retrace the steps above, remembering that the last inequality is necessary and sufficient for convergence and that by Theorem 4 below, if F(a) is discontinuous anywhere, it will be discontinuous at zero.

Theorem 3. If y = F(a) on B to S is linear, then there exists an element $|F|_B$ of S such that $|F(a)| \leqq |F|_B \cdot \|a\|$ for all a in B.

$$|F|_B = \operatorname{lub}_{\|a\|=1} \{F(a)\}$$

Proof. By contradiction. Suppose $\|a\| > 1$ and

$\left|F(a)\right| \nleqq \left|F\right|_B \cdot \|a\|$ Divide by $\|a\|$ Then

$\left|F\left(\frac{a}{\|a\|}\right)\right| \nleqq \left|F\right|_B \cdot 1 = U\left\|\frac{a}{\|a\|}\right\|$, since $\left\|\frac{a}{\|a\|}\right\| = 1$

But in the above hypothesis $\left|F\right|_B$ was defined so that $\left|F(b)\right| \leqq \left|F\right|_B$ when $\|b\| = 1$. Contradiction.

Definition 3 of norm. The norm of F(a) is the above $\left|F\right|_B$.

Theorem 4. If F(a) is an additive function of a and it is continuous at the point a_o (i.e. $F(a_n) \to F(a_o)$ for every sequence $\{a_n\}$ such that $a_n \to a_o$), then it is continuous at every other point of the space B.

Proof. Let $b_n \to b$ where b_n, b are all in B. Then

$(b_n - b + a_o) \to a_o$ and $F(b_n - b + a_o) \to F(a\ )$.

Rewriting, $F(b_n) - F(b) + F(a_o) \to F(a\ )$, owing to the additivity or $E(b_n) \to F(b)$, subtracting.

Theorem 5. Let G be a linear subspace of B and b_o an element of B not in G. Let G' be the set composed of the totality of elements of the form $a + \alpha b_o$ (a in G, α real). Let f(a) be a linear functional defined on G to S. Then there exists a linear function F(a) on G' to S such that

a) $F(a) = f(a)$ a in G

b) $\left|F\right|_{G'} = \left|f\right|_G$

Proof For a_1 and a_2 in G,

$$f(a_2 - a_1) \leqq |f(a_2 - a_1)| \leqq |f|_G \cdot \|a_2 - a_1\| =$$

$$|f|_G \cdot \|a_2 + b_o - b_o - a_1\| \leqq$$

$$\{\|a_2 + b_o\| + \|a_1 + b_o\|\} |f|_G$$

and so

(1) $$-\|a_1 + b_o\| \cdot |f|_G - f(a_1) \leqq \|a_2 + b_o\| \; |f|_G - f(a_2)$$

Now the elements (of S), $-\|a_1 + b_o\| \cdot |f|_G - f(a_1)$ are bounded above (by any one of the elements on the left side of inequality (1)) and so by Postulate II 4, (p. 30), there exists an exact upper bound of the set, say $\underline{u}$, and similarly there exists an exact lower bound, say $\underline{w}$, of the set of elements (of S) $\|a_2 + b_o\| \cdot |f|_G - f(a_2)$. Now let $\underline{v}$ be any element (of S) such that

$$u \leqq v \leqq w.$$

Now put

(2) $$F(a + \alpha b_o) \equiv f(a) + \alpha v; \text{ a in G, } \alpha \text{ real.}$$

If $\alpha \neq 0$,

$$-\left\|\frac{a}{\alpha} + b_o\right\| \cdot |f|_G - f\left(\frac{a}{\alpha}\right) \leqq v \leqq \left\|\frac{a}{\alpha} + b_o\right\| \cdot |f|_G - f\left(\frac{a}{\alpha}\right)$$

and so

$$-\left\|\frac{a}{\alpha} + b_0\right\| \cdot |f|_G \leqq v + f\left(\frac{a}{\alpha}\right) \leqq \left\|\frac{a}{\alpha} + b_0\right\| \cdot |f|_G$$

and finally

$$\left|f\left(\frac{a}{\alpha}\right) + v\right| = \left\|\frac{a}{\alpha} + b_0\right\| \cdot |f|_G$$

or

$$\left|f(a) + \alpha v\right| \leqq \left\|a + \alpha b_0\right\| \cdot |f|_G \text{ . But from (2)}$$

above, this means that

$$|F(C)| \leqq |f|_G \, \|C\|, \; C \text{ in } G'.$$

This last equation says that $|F|_{G'}$ is at the greatest equal to $|f|_G$; but $F(x) = f(x)$ when x is in G, therefore $|F|_{G'}$ is at least equal to $|f|_G$.

Theorem 6. If G is a linear subspace of B and f(a) is a linear function defined on G, there exists a linear function F(a) defined in B, such that

$$F(a) = f(a) \quad \text{and} \quad |F|_B = |f|_G \quad \text{for a in G.}$$

Proof. By transfinite induction. Suppose (which is legitimate) that the set E - G is well ordered (see Kamke IV B). Now we apply the previous theorem, immediately above, to the successive elements of E - G, in order. Assume the above F(a) to be defined for all ordinal numbers $\xi < \xi_0$. Then if ξ_0 is not a limit-ordinal, the

F(a) can easily be defined in one step by the above theorem for the set with suffixes up to and including ξ_0. If ξ_0 is a limit-ordinal, we see from the equation

$$\left| F(C_\xi) \right| = |f|_G \cdot \| C_\xi \| \qquad C_\xi \text{ in } G''$$

(where G" is the set of elements of G plus those of the set E - G with suffixes less than ξ_0) that

$$\lim_{\xi \to \xi_0} F(C_\xi) \text{ exists and that}$$

$$\lim F(C_\xi) = \left| \lim F(C_\xi) \right| \leqq |f|_G \; \| C_{\xi_0} \|$$

This completes the proof by transfinite induction.

<u>Theorem</u> 7. A linear function from B to S may be defined that does not vanish identically.

<u>Proof</u>. Let a_0 be an element of B not equal to 0. The elements αa_0 compose a linear subspace G of B. Put $f(\alpha a_0) = (f)(\alpha) \| a_0 \|$, where f is any non-zero element of S. This function is obviously linear. Now apply the theorem immediately above.

<u>Theorem</u> 8. Given $\{a_n\}$ a sequence of B, $\{x_n\}$ a sequence of elements of S and $X > 0$ an element of S. The necessary and sufficient condition for the existence of a linear function f(a) such that

$f(a_n) = x_n$ for all n, and $|f|_B \leqq X$ is that

$$\left|\sum_{i=1}^{n} \lambda_i x_i\right| \leqq X \cdot \left\|\sum_{i=1}^{n} \lambda_i a_i\right\|$$

Proof. Necessity. If f(x) exists, then

$$\left| f\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right| = \left|\sum_{i=1}^{n} \lambda_i f(a_i)\right| = \left| f\left(\sum_{i=1}^{n} \lambda_i a_i\right)\right| \leqq$$

$$|f|_B \left\|\sum_{i=1}^{n} \lambda_i a_i\right\|$$ and the X is in this case just $|f|_B$.

Sufficiency. Let G be the linear subspace of B composed of the totality of elements of the form $\sum_{i=1}^{n} \lambda_i a_i$. Define f(a) as follows: if $a = \sum_{i=1}^{n} \lambda_i a_i$ then put $f(a) = \sum_{i=1}^{n} \lambda_i x_i$. Then $|f(a)| \leqq X \|a\|$ by hypothesis and by the above theorems the required linear function may be defined over B.

Definition 4 of distance from point to set.

If E is a set of points of B and a_o is an element (point) of B, then the g.l.b. of all the numbers of the form $\|a - a_o\|$ where a is in E is called the distance of the point a_o from the set E.

Theorem 9. Given G, a linear subspace of B and a_o an element of B whose distance from G is $\delta > 0$ and an element $X > 0$ of S, then there exists a linear function F(a) on B to S such that

1. $F(a) = 0$ a in G
2. $F(a_0) = X$
3. $|F|_B = \frac{X}{\delta}$

Proof. Call G' the linear subspace formed by the elements $b = a + \alpha a_0$ where $\underline{a}$ is in G, and define

$f(b) \equiv \alpha X$. Then for $\alpha \neq 0$

$\|b\| = \|a + \alpha a_0\| = \alpha \left\| \frac{1}{\alpha} a + a_0 \right\|$. But the element $\frac{-1}{\alpha}$ a belongs to G since it is a linear subspace and $\left\| \frac{a}{\alpha} + a_0 \right\|$ is the distance between a_0 and this element; this distance is less than or equal to δ.

Therefore $\|b\| \geqq |\alpha| \cdot \delta$ or $|\alpha| \leqq \frac{\|b\|}{\delta}$ and $|f(b)| = |\alpha X| \leqq \frac{\|b\|}{\delta} X$ since $X = |X|$ (owing to the fact that $X > 0$). Now it follows immediately, by the definition of the norm of a function, that

$$|f|_{G'} \leqq \frac{X}{\delta}.$$ Now for a in G,

$X \leqq |f|_{G'} \cdot \|a + a_0\| \leqq \frac{X}{\delta} \|a + a_0\|$. The last inequality follows immediately from the above by multiplication with the factor (numerical) $\|x + y\|$.

The first inequality follows from the fact that $f(a + y_o) = X$ (from the definition of $f(b)$). Since the distance from a_o to G is , there exists a sequence of elements $\{ a_n \}$ such that

$\| a_n + a_o \| \to \delta$ Finding the limits in accordance with this, in the above double inequalities, we get

$$X = |f|_{G'} \cdot \delta \leqq X. \text{ Therefore}$$

$$|f|_{G'} = \frac{X}{\delta}. \text{ Now we may apply the}$$

Theorem 5 on extensions of linear functions, and extend the values throughout the space B.

Corollary If a linear subspace G has the property that every linear function F(a) taking values zero over G necessarily is zero for all a, then G is identical with B.

Theorem 10. Given a function p(a) on B to S such that

$$p(a + b) \leqq p(a) + p(b)$$

$$p(\lambda a) = \lambda p(a) \qquad \lambda \text{ real and } \geqq 0.$$

Then there exists an additive function f(a) on B to S such that

$$-p(-a) \leqq f(a) \leqq p(a) \qquad a \text{ in B}.$$

Proof. (Inductive). Let f(a) be an additive function defined on a linear subspace G of B and such that $f(a) \leqq p(a)$.

Now consider an element $\underline{a_0}$ of B not in G. For $\underline{a}$ and $\underline{b}$ in G we have

$$f(a + b) \leqq p(a + b) = p(a + a_0 + b - a_0) \leqq$$

$$p(a + a_0) + p(b - a_0) \text{ and so}$$

(1) $$-p(b - a_0) + f(b) \leqq p(a + a_0) - f(a).$$ Now let x be the lub of the bounded (above) set $-p(b - a_0) + f(b)$ for all b in G. Let X be the glb of the set $p(a + a_0) - f(a)$ for all $\underline{a}$ in G. The above inequality shews that $x \leqq X$.

Now take any element (of S) Z, such that $x \leqq Z \leqq X$. Let C be any element of the form $a + \lambda a_0$, ($\underline{a}$ in G, $\underline{\text{any}}$ real number). Define F(C) by:

$$F(C) \equiv F(a + \lambda a_0) = f(a) + \lambda Z.$$

Obviously F(C) = f(C) if C is in G, and furthermore F(C) is additive.

Now, multiplying inequality (1) by $\lambda \geqq 0$

$$\lambda Z \leqq p(\lambda a + \lambda a_0) - f(\lambda a), \text{ x in G.}$$

Define $b \equiv \lambda a$ and $b + \lambda a_0 \equiv C$ and get

$$f(b) + \lambda Z \leqq p(b + \lambda a_0), \text{ or}$$

$F(C) \leqq p(C)$. But if $\lambda \leqq 0$, multiply through by $-\lambda$ and get

$-p(-\lambda b + \lambda a_o) + f(-\lambda b) \leqq -\lambda Z$. Define $-\lambda b = a$ and $a + \lambda Z = C$, then the last inequality yields

$f(a) + \lambda Z \leqq p(C)$, that is

$F(C) \leqq p(C)$. Thus we have been able to extend the domain of definition of the function, retaining its additivity, and maintaining the desired inequalities. This assumed it possible in the first place to define a function $f(a)$. We now shew it is possible to define such a function on some linear subspace of B. Let the space B be supposed well-ordered, and in such a way that $a_1 = 0$ is the first element. Put $f(0) = 0$, so that in this case $f(a_1) \leqq p(a_1)$; in fact $f(a_1) = p(a_1) = 0$. Now extend the domain of definition of $f(a)$ by the above inductive process, utilizing transfinite induction in this case.

<u>Definition</u> 5 of upper limit of transfinite sequence.

Let ϑ be a transfinite limit-ordinal number. Let $\{x_\xi\}$ be a transfinite sequence (of ordinal type ϑ) of elements of S, and let them be bounded, as a totality, that is, let $|x_\xi|$ L for all ξ such that $1 \leqq \xi \leqq \vartheta$. Now let K be the g.l.b. of all the elements of S such that $C_\xi \leqq K_\alpha$ for $\xi \geqq \alpha$. Now we define $\overline{\lim}_{\xi \to \vartheta} C_\xi$ as the g.l.b. of all elements K_α .

$$-\overline{\lim_{\xi\to\vartheta}}(-C_\xi) \equiv \underline{\lim_{\xi\to\vartheta}}(C_\xi)$$

<u>Theorem</u> 11 $\overline{\lim_{\xi\to\vartheta}}(C_\xi + d_\xi) \leqq \overline{\lim_{\xi\to\vartheta}}(C\quad) + \overline{\lim_{\xi\to\vartheta}}\, d_\xi$

<u>Proof</u>. Obvious from the definition of $\overline{\lim}$.

<u>Theorem</u> 11[a] If $\lambda \geqq 0$ then

$$\overline{\lim_{\xi\to\vartheta}}(\lambda C_\xi) = \lambda\,\overline{\lim}(C_\xi)$$

<u>Proof</u>. Follows directly from definition of $\overline{\lim}$.

<u>Theorem</u> 11[b] Let $\{f_\xi(a)\}$ be a transfinite sequence of linear functions of ordinal type ϑ, such that $|f_\xi| \leqq M$ for $1 \leqq \xi < \vartheta$. Then there exists a fixed linear function f(a) such that for every <u>a</u>,

$$\underline{\lim_{\xi\to\vartheta}}\, f_\xi(a) \leqq f(a) \leqq \overline{\lim_{\xi\to\vartheta}}\, f_\xi(a)$$

<u>Proof</u>. Put $p(a) \equiv \overline{\lim_{\xi\to\vartheta}}\, f_\xi(a)$, then $p(-x) = -\underline{\lim_{\xi\to\vartheta}}\, f_\xi(a)$

Now

$p(a + b) \leqq p(a) + p(b)$, since from Theorem 11

above, $\overline{\lim}\ f_\xi(a + b) = \overline{\lim}\left[f_\xi(a) + f_\xi(b)\right] \leqq$

$\leqq \overline{\lim}\ f_\xi(a) + \overline{\lim}\ f_\xi(b)$.

Therefore, by Theorem 10, there exists an additive function f(a) such that

$-p(-a) \leqq f(a) \leqq p(a)$. Also $|p(a)| \leqq M \cdot \|a\|$

so $|f(a)| \leqq M \cdot \|a\|$ and f(a) is linear, obviously.

<u>Definition</u> 6 of hyperlinear function on S to S.

A function F(x) defined on the Kantorovitch space S to the space S is said to be hyperlinear if it is linear and if there exists a real number $_F$ such that

$$|F(x)| \leqq \lambda_F |x|.$$

$$\left| F(x) \right| \leqq \lambda_F |x| .$$

Definition 7 of norm of hyperlinear function.

The norm of F(x) is defined as the greatest lower bound of the λ_F's in definition 6.

Definition 8 of "to span".

A set (A) of elements is said to "span" a linear subspace L_o of the given linear space L if L_o is composed solely of linear combinations of finite sets of elements of (A) and limits of such linear combinations.

Theorem 12. Given (A), a set of elements of a Kantorovitch space L and L_o, the linear subspace spanned by (A). Given a function $f_a(x)$ defined on (A) to L. The necessary and sufficient condition that there exist an hyperlinear function f(x) on L_o to L, with norm μ, taking values in (A) identical with the corresponding ones of $f_a(x)$, is that for every finite linear combination $\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n$ of elements of (A) we shall have:

(1), $$\left| \lambda_1 f_a(x_1) + \lambda_2 f_a(x_2) + \dots + \lambda_n f_a(x_n) \right| \leqq \mu \left| \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n \right|$$

Proof. Sufficiency. Assume the above inequality. Then if $\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = 0$, we have

$$\lambda_1 f\ (x_1) + \lambda_2 f_a(x_2) + \cdots + \lambda_n f_a(x_n) = 0.$$

It follows, then, that if

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \lambda'_1 x'_1 + \lambda'_2 x'_2 + \cdots + \lambda'_p x'_p,$$ we have

$$\lambda_1 f(x_1) + \lambda_2 f_a(x_2) + \cdots + \lambda_n f_a(x_n) =$$

$$\lambda'_1 f_a(x'_1) + \lambda'_2 f_a(x'_2) + \cdots + \lambda'_p f_a(x'_p)$$

Therefore, we may define $f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)$ as $\lambda_1 f_a(x_1) + \lambda_2 f_a(x_2) + \ldots + \lambda_n f_a(x_n)$ for any point of L_0 that may be represented as a linear combination of a finite number of elements of (A). Consider now any point Z of L_0, and $\{x_\nu\}$ a sequence of such linear combinations of elements of (A) such that $x_\nu \to Z$. Now from

$$\left| f(x_\nu) - f(x_{\nu'}) \right| = \left| f(x \quad - x \quad) \right| \leqq \mu \left| x_\nu - x_{\nu'} \right|$$

But $x_\nu \to Z$, therefore the elements $w_\nu \equiv f(x_\nu)$ of L form a Cauchy sequence and so have a limit, by Theorem 32, Chapter I (p. 42). We define f(Z) as $\lim f(x_\nu)$. Obviously, from the fact that $\lim (w_\nu + w'_\nu) = \lim w_\nu + \lim w'_\nu$ if the two latter exist we see that f(Z) is additive. From the fact that $\lambda \lim w_\nu = \lim (\lambda w_\nu)$ for λ a real number, we see easily that $f(\lambda Z) = \lambda f(Z)$. That f(x) is hyperlinear follows from the fact that

$$|f(x_\nu)| \leqq \mu |x_\nu|$$ and so in the limit,

$$|f(Z)| \leqq \mu |Z|$$

Definition 9^a of complete space

L_o, a linear subspace of a Kantorovitch space is said to be complete if Theorem 32, Chapter I (p. 52) is valid in it, considered as a semi-ordered space.

Theorem 13. Given L_o a complete linear subspace of L and $f_o(x)$ an hyperlinear function on L_o to L, with norm μ. Then there exists an hyperlinear function on L to L , with norm μ, and taking values identical with those of $f_o(x)$ in the corresponding points of L_o.

Proof. Consider any well-ordered set L_ξ of linear subspaces of L, starting with L_o and ending with L, and such that each one include all the preceding ones. Assume that for an ordinal number ξ_o we have defined all those preceding. If ξ_o is not a limit number then considering L_{ξ_o-1} and any point x not in L_{ξ_o-1} we shall define L_{ξ_o} as the space spanned by x together with the elements of L_{ξ_o-1} . In case ξ_o is a limit number, we simply define L_{ξ_o} as the space spanned by the union of all the L_ξ such that $\xi < \xi_o$. Since, by the well-ordering hypothesis, L can be regarded as well-ordered the foregoing inductive process can not go beyond a certain ordinal.

Now let us prove the given theorem by an inductive process. Assume that for all of the subspaces L_ξ , $\xi < \xi_0$ $f(x)$ has been defined. In case ξ_0 is not a limit-ordinal, consider the linear subspace $L_{\xi_0 - 1}$ and the point Z of L_{ξ_0} , used above. Now

$$f_0(\bar{x} - x) \leq \left| f_0(\bar{x} - x) \right| \leq \mu \left| \bar{x} - x \right| = \mu \left| \bar{x} - Z - (x - Z) \right| \leq \mu \left| x - Z \right| + \mu \left| \bar{x} - Z \right|$$

and so

(2) $f_0(\bar{x}) - \mu|\bar{x} - Z| \leq \mu|x - Z| + f_0(x)$ for every pair of elements $x, \bar{x}$ of $L_{\xi_0 - 1}$. Take the least upper bound $\underline{u}$ of the set $f_0(\bar{x}) - \mu|\bar{x} - Z|$, (this exists by axiom II 4, p. 30, Chapter I, since the set $f_0(\bar{x}) - \mu|\bar{x} - Z|$ is bounded above, by inequality (2)). Take the greatest lower bound ℓ of the set $+f_0(x) - \mu|x - Z|$, (this exists by similar considerations). Then

$$u \leq \ell$$

Choose any element e of L such that

(3) $$u \leq e \leq \ell$$

Now any element x' of L can be represented, by definition, as a linear combination $x + \lambda e$ of the element $\underline{e}$ and an element $\underline{x}$ of $L_{\xi_0 - 1}$. We define

$$f(x') \equiv f(x + \lambda Z) \equiv f_0(x) + \lambda e, \quad \text{that is } f_0(x)$$

has been extended to an additive homogeneous function $f(x)$

and $f(Z) = e$. To see that it is hyperlinear with norm μ, we notice that $-\frac{x}{\lambda}$ is a point of $L_{\xi_0 - 1}$ and so by inequalities (2) and (3):

$$-f\left(\frac{x}{\lambda}\right) - \mu\left|\frac{-x}{\lambda} - Z\right| \leqq f(Z) \leqq -f\left(\frac{x}{\lambda}\right) + \mu\left|\frac{-x}{\lambda} - Z\right|$$

Therefore

$$f(Z) + f_0\left(\frac{x}{\lambda}\right) \leqq \mu\left|\frac{x}{\lambda} + Z\right|$$

$$-\mu\left|\frac{x}{\lambda} + Z\right| \leqq f(Z) + f_0\left(\frac{x}{\lambda}\right) \quad \text{therefore}$$

$$\left|f(Z) + f_0\left(\frac{x}{\lambda}\right)\right| \leqq \mu\left|\frac{x}{\lambda} + Z\right|.$$ Multiply through by $|\lambda|$ and get

$$\left|\lambda f(Z) + f_0(x)\right| \leqq \mu\left|x + \lambda Z\right|.$$ Or, by the definition above,

$$\left|f(x')\right| = \left|f(x + \lambda Z)\right| \leqq \mu\left|x + \lambda Z\right| = \mu\left|x\right| .$$

In case ξ_0 is a limit-ordinal, L_{ξ_0} was defined as space spanned by the union (A) of all spaces L_ξ for which $\xi < \xi_0$. Assume that $f(x)$ is defined for all L , . If $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$ is a finite linear combination of points of (A), then each x_i belongs to some L_{ξ_i} ($\xi_i < \xi_0$). Let the largest of the ordinals ξ_i be ξ', then all the x_i will belong to $L_{\xi'}$. But, by the inductive assumption $f(x)$ is defined on $L_{\xi'}$ and has a norm μ. Therefore

$$\left|\lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n)\right| =$$

$$\left|f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)\right| \leqq$$

$$\mu\left|\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n\right|.$$ Now the

hypotheses of Theorem 12 are satisfied. Therefore Theorem 12 is true and f(x) may be defined, with the desired properties, over L_{f_0} .

Consider now the set $\overline{S}$ of all linear functions f(a) on B to S.

Definition 9b of absolute value of a linear function.

The absolute value of a function f(a) on B to S is defined as the norm $|f|_B$ of the function. This exists by Theorem 3 and is in S. The suffix B will be omitted if no ambiguity results.

Theorem 14. The absolute value $|f|_B$ of f(a) on B to S has the properties

1. $|f|_B \geqq 0$ and $|f|_B = 0$ if and only if f(a) is identically zero.
2. $|\lambda f| = \lambda |f|$
3. $|f + g| \leqq |f| + |g|$

Proof.

1. Obvious
2. Obvious
3. $|f(a)| \leqq |f| \cdot \|a\|$; $|g(a)| \leqq |g| \, \|a\|$

   $|g(a) + f(a)| \leqq |g(a)| + |f(a)| = \{|g| + |f|\} \cdot \|a\|$

Let us, for convenience, designate the value of the linear function u(a) at the point a by B(u,a). Then

$|B(u,a)| \leqq |u| \cdot \|a\|$.

Definition 9^c of $F_+(a)$, $F_-(a)$.

If $F(a)$ is a function, linear or otherwise, on B to S, we define the values of $F_+(a)$ and $F_-(a)$ for each value of a by the equations

$$F_+(a) = \text{lub } (0, F(a)$$

$$F_-(a) = \text{lub } (0, -F(a))$$

Theorem 14^a

$$F_+(a) - F_-(a) = F(a)$$

$$F_+(a) + F_-(a) = |F(a)|$$

$$F_+(a) \geqq 0; \ F_-(a) \geqq 0.$$

Proof. Obvious.

Theorem 15. If $F(a)$ is linear on B to S, there exist "moduli" F_+ and F_- such that

$$F_+(a) \leqq F_+ \ \|a\|$$

$$F_-(a) \leqq F_- \ \|a\|$$

and

$$F_+ \leqq |F| \ ; \ F_- \leqq |F|$$

$$|F| \leqq F_+ + F_- \leqq 2|F|$$

Proof. We see that

$$F_+(a) \leqq F(a)_+ + F_-(a) = |F(a)| \leqq |F| \cdot \|a\|$$

$$F_-(a) \leqq F_+(a) + F_-(a) = |F(a)| \leqq |F| \cdot \|a\|$$

and so (1) $F_+(a) \leqq |F| \ \|a\|$

(2) $F_-(a) \leqq |F| \ \|a\|$.

Let F_+ be the glb of all the elements having the property of the F in equation (1). Similarly, let F_- be the glb of all the elements having the property of $|F|$ in 2. Thus

$$F_+(a) \leqq F_+ \|a\| \quad \text{and} \quad F_+ \leqq |F|$$

$$F_- \leqq |F|$$

$$F_-(a) \leqq F_- \|a\| \quad \text{and so adding,}$$

$$F_+(a) + F_-(a) = |F(a)| \leqq \{F_+ + F_-\} \|a\|$$

$$\text{Also} \quad F_+ + F_- \leqq 2|F|$$

It is then evident that $F_- + F_+ \geqq |F|$, from the definition of F and the last equation.

<u>Definition 10^a of star-modulus.</u>

$$F_+ + F_- \equiv |F|^*$$

<u>Theorem 15^a</u> a) $|F|^* = 0$ if and only if $F(a) \equiv 0$

b) $|F + G|^* \leqq |F|^* + |G|^*$

c) $|F - G|^* \geqq \big| |F|^* - |G|^* \big|$

d) $|\lambda F|^* = |\lambda| \cdot |F|^*$

Proof. a) Obvious.

b) From Theorem 2e, Chapter I, p. 5

$$\{F(a) + G(a)\}_+ \leqq \{F(a)\}_+ + \{G(a)\}_+ \leqq \{F_+ + G_+\} \cdot \|a\|$$

$(F + G)_+$, being the glb of all elements having the property of $F_+ + G_+$ above satisfies the inequality

$(F + G)_+ \leqq F_+ + G_+$. Similarly

$(F + G)_- \leqq F_- + G_-$. Adding

$|F + G|^* \leqq |F|^* + |G|^*$, by definition of the star modulus.

c) In b) put $F + G = H$, $F = G - H$

$$|H|^* \leqq |G - H|^* + |G|^*.$$

(1) $|G - H|^* \geqq |H|^* - |G|^*$

(2) $|H - G|^* \geqq |H|^* - |G|^*.$

But $\{-K(a)\}_- = K_+(a)$ has modulus K_+

$\{-K(a)\}_+ = K_-(a)$ has modulus K_-

and so adding $-K(a)$ has modulus $|K|^*$ thus

$|-K|^* = |K|$. Hence from (1) and (2)

$$|G - H|^* \geqq \big| |H|^* - |G|^* \big|.$$

d) If $\lambda > 0$ from Theorem 16c, Chapter I, p. 13,

$$\{\lambda F(a)\}_+ = \lambda F_+(a)$$

$$\{\lambda F(a)\}_- = \lambda F_-(a)$$

It follows easily that $\lambda F_+(a)$ has modulus λF_+ and $\lambda F_-(a)$ has modulus λF_-, so that adding, we see that $|\lambda F|^* = \lambda |F|^*$ for $\lambda > 0$. To extend the result to the case of $\lambda < 0$, notice the remark at the end of the proof of c). We shall have three simple sorts of convergence :

defined for sequences of functions, one will be called modular convergence, the other star-modular convergence and the third, weak convergence.

Definition 10^c of modular convergence.

The sequence of linear functions F(a) is said to converge modularly to the linear function F(a) if $|F_n - F|$ converges to 0.

Definition 10^d of star-modular convergence.

The sequence of linear functions $F_n(a)$ is said to converge to the linear function F(a) star-modularly if $|F_n - F|^*$ converges to 0.

Definition 10^e of weak convergence.

A sequence of functions $F_n(a)$ is said to converge weakly to the linear function F(a) if $F_n(a) \longrightarrow F(a)$ for each a in B.

Theorem 15^b. If a sequence of linear functions converges modularly to a linear function it converges star-modularly and vice versa.

Proof. If $F_n(a) \longrightarrow F(a)$ star modularly,

$$|F_n - F| \leqq (F_n - F)_+ + (F_n - F)_- = |F_n - F|^* < \epsilon y_0$$

for $n > N_e$

Thus $|F_n - F| < \epsilon y_0$ for $n > N_\epsilon$ and $F_n(a) \longrightarrow F(a)$ modularly. Similarly for the converse, use the inequality

$$|F_n - F|^* \leqq 2|F_n - F|.$$

Theorem 15^c. If a sequence of linear functions converges to a linear function modularly, it converges weakly.

Proof. Suppose $F_n(a) \longrightarrow F(a)$ modularly. Then

$$|F_n - F| < \epsilon y_0, \quad n > N_\epsilon$$

$$|F_n(a) - F(a)| \leqq |F_n - F| \cdot \|a\| < \epsilon \|a\| y_0$$

and so for each $\underline{a}$, $F_n(a)$ converges to $F(a)$.

Theorem 15[d]. The space $\overline{S}$ of functions on B to S is a semi-ordered space of Kantorovitch (not necessarily regular) satisfying Postulates I 1 - 5, II 1 - 4, III 1 - 6. It is complete with the ~~additional~~ topology imposed by the modular convergence.

Proof. By definition, $F_1 > F_2$, where F_1 and F_2 represent the functions $F_1(a)$ and $F_2(a)$, considered as elements of the sets $\overline{S}$, if

$$F_1(a) \geqq F_2(a) \text{ for all a and}$$

$$F_1(a_0) > F_2(a_0) \text{ for at least one } a_0, \text{ or,}$$

what is equivalent

$$\{F_1 - F_2\}_+ \neq 0$$

$$\{F_1 - F_2\}_- = 0$$

The latter part of the theorem follows from Theorem 15[c].

Theorem 16. $R(u,a_0)$, $\underline{a_0}$ a fixed element of R) is an hyperlinear function with norm $\|a_0\|$.

Proof. Since

$|R(u,a_0)| \leqq \|a_0\| \cdot |u|$, it follows that there is a least number (real) μ such that $|R(u,a_0)| \leqq \mu |u|$

μ is certainly less than or equal to $\|a_0\|$. We now proceed to shew that we can find a function $\overline{u}$ (a) that actually takes the value $|\overline{u}| \cdot \|a_0\|$ for $a = a_0$. Now by Theorem 9, we can find a function $\overline{u}(a)$ that takes the value k in a_0 and such that $u = \frac{k}{\|a_0\|}$. Then

$$\overline{u}(a_0) = k = \frac{k}{\|a_0\|} \|a_0\| = \overline{u} \quad \|a_0\|$$

Thus $\|a_0\|$ is the smallest possible value for μ.

Theorem 17. The totality of linear transformations R(u,a) where a is a parameter on B lies in a Banach space.

Proof. The norm of an element is simply $\|a\|$, etc.

Let us now consider a family T of linear transformations belonging to $\overline{S}$. Let a typical one be $u_y(a)$, where y varies over a set with the same cardinal number as T. Let C_y be elements of S. We shall examine the set of equations

$$U_y(a) = C_y.$$

Definition 11 of regular system of equations.

If each linear function f(u) on T_0, the space spanned by the totality of the above u_y, is of the form $R(u,a_0)$ for a_0 in B, the system above is said to be regular.

Definition 12 of regular Banach space B.

The Banach space B is said to be regular if every linear function f(u) defined on the associated space $\overline{S}$ is of the form

$B(u,a_o)$.

Theorem 18. The necessary and sufficient condition that there exists a point a_o of B satisfying $U_y(a_o) = C_y$ is that there exist a number μ such that for every finite linear combination $\lambda_1 U_{y_1} + \lambda_2 U_{y_2} + \cdots + \lambda_n U_{y_n}$ of the above U'S, the following inequality shall exist.

$$\left|\lambda_1 C_{y_1} + \lambda_2 C_{y_n} + \cdots + \lambda_n C_{y_n}\right| \leqq \mu \left|\lambda_1 U_{y_1} + \lambda_2 U_{y_2} + \cdots + \lambda_n U_{y_n}\right|$$

Proof. Necessity. Let a_o be a solution. Then

$$\lambda_1 U_{y_1}(a_o) + \lambda_2 U_{y_2}(a_o) + \cdots + \lambda_n U_{y_n}(a_o) = \lambda_1 C_{y_n} + \cdots + \lambda_n C_{y_n}$$

and

$$\left|\lambda_1 C_{y_1} + \cdots + \lambda_n C_{y_n}\right| = \left|\lambda_1 U_{y_1}(a_o) + \cdots + \lambda_n U_{y_n}(a_o)\right| \leqq \|a_o\| \cdot \left|\lambda_1 U_{y_1} + \cdots + \lambda_n U_{y_n}\right|$$

This last follows from the regularity of the space and the fact that $\lambda_1 U_{y_1}(a\ ) + \cdots + \lambda_n U_{y_n}(a)$ is a linear function.

Sufficiency. Assume the inequality of the hypothesis. Then by Theorem 12 there is an hyperlinear function $f(u)$ on T_o, with norm u, and for which $f(Uy) = Cy$. Owing to the regularity there is a point a_o, such that $f(u) = B(u,a_o)$. So

$$U_y(a_o) = B(U_y,a_o) = f(U_y) = C_y \text{ for all } U_y \text{ in } T_o.$$

<u>Definition</u> 12 of minimal solution.

a_o is said to be a minimal solution of $U_y(a) = C_y$ if for any solution a',

$$\|a_o\| \leqq \|a'\|$$

<u>Theorem</u> <u>19</u>. If B is regular and if there is a μ such that

$$\left|\lambda_1 C_{y_1} + \cdots + \lambda_n C_{y_n}\right| \leqq \mu \quad \left|\lambda_1 U_{y_1} + \cdots + \lambda_n U_{y_n}\right|$$

for every finite linear combination of the U'S, then the system

$U_y(a) = C_y$ has a minimal solution $\underline{a}_o$.

Proof. Let μ be the smallest possible. Then certainly every linear function f(a) on T_o, taking the value C_y has a norm at least equal to μ. By the preceding theorem, there exists at least one $\underline{a}_o$ for which $R(U_y, a_o) = C_y$. The norm of this hyperlinear function is $\|a_o\|$ and so $\|a_o\| \geqq \mu$. But by Theorem 12 there exists an hyperlinear function with norm μ, taking the value C_y for U_y at the point a'_o. Owing to the regularity it is of the form $R(u, a'_o)$. So $\|a'_o\| = \mu$ and this together with the inequality $\|a_o\| \geqq \mu$ for every solution of the system gives the result that a'_o is a minimal solution.

Now it is possible to extend the original Banach space B to a Banach space B_1 in such a way that the system of equations of Definition 11 is regular in B_1. Let the totality of hyperlinear functions $W = f(u)$ on $\overline{S}$ to S be the space S_1. From Theorem 16 it follows that the set of functions of the form

$R(u, a_0)$, is a sub-set of S_1.

Theorem 20. The space S_1 is a complete Banach space using the hyper-norm as the Banach norm. It is also a semi-ordered space.

Proof. Suppose $W_n = f_n(u)$ is a sequence of hyperlinear functions such that

$$\| f_n - f_{n'} \| < \epsilon \qquad \text{for } n, n' > N_\epsilon$$

this means that

$$| f_n(u) - f_{n'}(u) | < \epsilon \cdot u$$ Thus for each u, the limit exists:

$\lim_{n \to \infty} f_n(u) = f(u)$. Owing to the additive homogeneous properties of the limit operation, f(u) is additive and homogeneous. But owing to the fact that the f_n form a Cauchy sequence, they are bounded in the norm from the ordinary theory of Banach spaces, see Banach IV[b], 1, p. 137 Theorem 5. Thus

$$\| f_n \| < \lambda \text{, all } n \text{, and so}$$

$$| f_n(u) | \leqq \lambda |u| . \text{ Thus}$$

$|f(u)| \leqq \lambda |u|$ and the hyperlinearity of f(u) follows easily by taking as its hypernorm the glb of all λ's satisfying the last inequality. The remainder of the axioms for a Banach space are easily verified. The semi-ordering is accomplished by a process similar to that of Theorem 15[d].

Now if $W = f(u)$ is an hyperlinear function of the type $R(u,a)$ where $\underline{a}$ is a suitable element of B then we shall define $\|W\|$ as $\|a\|$. Thus we have an one-to-one correspondence between the elements of B and the elements of S_1 of the form $R(u,a)$. Now we shall identify $R(u,a)$, considered as an element of S with $\underline{a}$ of B. Further, we shall add to B all the other hyperlinear functions $W = f(u)$ as new points. The extended space we shall call B_1. $\|W\|$ is simply defined as the hypernorm of the function $W = f(u)$.

If $U(a)$ is a linear function on B to S, define $U(a)$ at the point b_0 of B by the convention: $U(b_0)$ is the value that $b_0 = f_0(u)$ (hyperlinear function) takes at the point U of $\overline{S}$.

Theorem 21 The modulus $|U|_{B_1}$ of the linear function $U(b)$ in B_1 is the same as $|U|_B$ in B.

Proof. Since $B \subseteq B_1$, we have

$$|U|_{B_1} \geqq |U|_B. \text{ But}$$

$$U(b_0) = f_0(U(b)). \text{ The modulus of } f_0(u) \text{ is } \|b_0\|$$

$$|U(b_0)| \leqq |U| \cdot \|b_0\| \text{ . Therefore in } B_1,$$

$$|U|_{B_1} \leqq |U|_B. \text{ And so finally,}$$

$$|U|_{B_1} = |U|_B$$

Let now $U(b)$ be a linear function on B_1 to S and let $R(u,b)$ be the value that $U(b)$ takes at the point b of B_1. Then every linear function $f_0(U)$ on $\overline{S}$ to S is got from $R(u,b)$ by choosing a suitable b from B_1. For, $f_0(u)$

defines a point b_0 of B and $f_0(u) = U(b_0) = R(u, b_0)$. Thus every system of linear equations of the type of Definition 11 is regular and the space S_1 is identified with the space B_1.

Chapter IV

We now examine the consequences of the additional postulate, which I shall call postulate IV. Remembering that if a sequence y_n approaches a limit (necessarily unique) then the sequence is bounded above and below, we frame it as follows:

(IV) If $y_n \to 0$ then for any y such that $|y_n| \leqq y$, all n, it is true that $|y_n| < \epsilon y$ for $n > N_\epsilon$.

Theorem 1. If $y_n \to z$ and $|y_n| \leqq y$, then $|y_n - z| \leqq \epsilon y$ for all $n > N_\epsilon$.

Proof. The first part is just a restatement of Theorem 36 b and Theorem 41. The second part follows immediately from the fact that $(y_n - z) \to 0$ and that if $|y_n| \leqq y$ for all n, then $|\lim y_n| \leqq y$ and so $|y_n - z| \leqq |y_n| + |z| \leqq 2y$. This postulate is valid in all the common applications, such as in the space of functions bounded almost everywhere, as we shall see later. It enables us to introduce a numerically-valued function which we shall call the weak metric on a sequence.

Definition 1 of the weak metric on a sequence.

When $y_n \to 0$, we fix a y as in postulate (IV) and then define $\|y_n\|_y$ as the smallest ϵ_n such that $|y_n| \leqq \epsilon_n y$. We can define the norm $\|x\|$ of any element x in the space, with regard to this fixed y, as the smallest positive real number μ such that

$$|x| \leqq \mu y, \text{ if it exists.}$$

$\|x\| = \text{glb}\, \mu$ such that $|x| \leqq \mu y$. It must be pointed out that the norm of an arbitrary element does not always exist.

It follows easily that $\|\lambda x\| = |\lambda| \cdot \|x\|$

<u>Theorem 2</u>. If $\|x\|_y$ exists and $\|z\|_y$ exists then $\|x + z\|_y$ exists and

$$\|x + z\|_y \leqq \|x\|_y + \|z\|_y$$

$$\Big| \|x\|_y - \|z\|_y \Big| \leqq \|x - z\|_y$$

<u>Proof</u>.

$$|x| \leqq \|x\|_y \cdot y$$

$$|z| \leqq \|z\|_y \cdot y$$

$$|x + z| \leqq |x| + |z| \leqq \left\{\|x\|_y + \|z\|_y\right\} \cdot y$$

Considering the first and last terms of this inequality we see that $\|x + z\|_y$ exists and is less than or equal to $\|x\|_y + \|z\|_y$. By writing $x - z$ for x in this formula, we get the second formula.

<u>Theorem 3</u>. If $\|x\|_y$ exists, then $\|x\|_y = 0$ if and only if $x = 0$, where y is arbitrary but fixed.

<u>Proof</u>. Obvious from definition.

<u>Theorem 4</u>. If $x_n \rightarrow x$ then if $\|x_n\|_y$ all exist except for a finite number of n, $\|x\|_y$ will exist and $\|x_n\|_y \longrightarrow \|x\|_y$.

<u>Proof</u>. The first statement follows from Theorem 1. The second statement follows from the fact that

$$|x_n - x| \leqq \epsilon\, y \quad \text{for } n > N_\epsilon$$

and $\| x_n - x \| \leqq \epsilon$ for $n > N_\epsilon$.

But $\left| \|x_n\| - \|x\| \right| \leqq \| x_n - x \| \leqq \epsilon \qquad n > N_\epsilon$

so that $\| x_n \| \rightarrow \| x \|$.

Theorem 5. Any bounded set A can be metrized with respect to the bounding element.

Proof. The preceding four theorems.

Definition 2 of weakly compact and weakly compact in itself.

A set of elements is said to be compact if every infinite sub-set of different elements possesses a sub-sequence that converges to a limit element in S not necessarily. The original set is said to be compact in itself if it always contains the mentioned limit-element.

Definition 3 of strongly compact and strongly compact in itself.

A compact set is a bounded set such that every infinite sequence of elements in the set contains a sub-sequence converging to an element in S. This last element need not be in the given set. If it is without exception, then the given set is said to be compact in itself.

Theorem 6. If a set A is strongly compact then the set of all elements of A together with the limit-elements of all convergent sequences of such elements will form a complete metric space.

Proof. Let G be any element such that $|x| \leqq G$ for all x in A. Now metrize according to Theorem 5. The limit-elements are of course metrizable by Theorem 4.

If we consider any element x_0 not belonging to the closed, strongly compact set A of the above Theorem, then it will be at a distance $\delta > 0$ from the set, since $y_0 = \text{lub} \{|x_0|, g\}$ will be a bound for all elements concerned, and using the metric with respect to y_0, which obviously by Theorem 5 is equivalent to the metric with respect to g, we can apply the well-known metric theorems concerning the minimum distance of an exterior element from a closed compact set. See Sierpinski IV B p. 82.

Theorem 7. If A is a closed, strongly compact set then to each $\epsilon > 0$ there is a set of elements $x_1, x_2, \cdots, x_n$ of A, finite in number such that to each point x of A at least one of the elements satisfies the inequality

$$\| x_i - x \| \leq \frac{\epsilon}{2} \quad \text{or}$$

$$| x_i - x | \leq \frac{\epsilon}{2} g, \text{ where } g \text{ is a fixed upper.}$$

bound of all x in A.

Proof. By the preceding theorem the set A is metric and by the corresponding theorem in metric spaces (see Sierpinski IV B p. 81 Theorem 48) this theorem follows.

Theorem 8. If A is a closed, strongly-compact set and $\epsilon > 0$, there exists a linear transformation T(x) of the set into an n-dimensional set such that

$$| T(x) - x | < \epsilon \, g \quad \text{for all } x \text{ in } A.$$

g is an upper bound of the x in A. The n and the ϵ are associated as in the preceding theorem.

Proof. Put

$$T(x) = \frac{\sum_i \mu_i(x)\, x_i}{\sum_i \mu_i(x)}$$ where x_i are the elements of the set of the preceding theorem and

$$\mu_i(x) = \epsilon - \varpi_i \quad \text{if } |x - x_i| < \epsilon g$$

$$\mu_i(x) = 0 \quad \text{if } |x - x_i| \nless \epsilon g$$

and $\varpi_i = \text{glb}\ \xi$ such that

$|x - x_i| < \xi y$. Then

$$|T(x) - x| = \left| \frac{\sum_i \mu_i(x)\, x_i}{\sum_i \mu_i(x)} - \frac{\sum_i \mu_i(x)\, x}{\sum_i \mu_i(x)} \right|$$

$$\leqq \frac{|\sum_i \mu_i(x - x_i)|}{|\sum_i \mu_i(x)|} \leqq \frac{\sum \mu_i |x - x_i|}{\sum u_i}$$

$$\leqq \epsilon g.$$

Definition 4 of completely continuous function.

F(x) defined on a closed bounded set A in S to a set B in S is said to be completely continuous with respect to A, if **F(A)** = B is strongly compact and F(x) is continuous. Usually if the set A is fixed for a particular problem we simply say that F(x) is completely continuous.

Theorem 9. If F(x) defined on a closed bounded set A is completely continuous, then to any $\epsilon > 0$ we can find a corresponding transformation $F_\epsilon(x)$ such that the range of values of $F_\epsilon(x)$ is contained in a linear finite-dimensional space E_n and such that $F_\epsilon(x)$ approximates F(x) to an error of ϵ, that is

$$\left| F_\epsilon(x) - F(x) \right| < \epsilon g \quad \text{for } x \text{ in } \bar{A}$$

$$F_\epsilon(\bar{A}) \subseteqq E_n.$$

Proof. The elements F(x) are contained in a strongly compact set B and so by Theorem 8, we can find a transformation, T(w) (linear) of those elements into the E_n, i.e.

$$F_\epsilon(x) \equiv T(F(x)).$$

We devote a space to a review of the definition and properties of the degree of a single-valued function F(x) defined on a bounded closed sub-set A of an n-dimensional Euclidean E_n space to a sub-set of the same space. For complete definitions of the terms used, see Alexandroff and Hopt IV B. If x_1 is an element not on F(fr A) where fr(A) is the frontier of A, then degree of the function F(x) at the point x_1 is defined by approximating to F(x) by the aid of a simplicial function $F_\epsilon(x)$. If the simplicial approximations are fine enough, then the number of positive simplexes containing x_1 minus the number of negative ones containing x_1 will be the same for all these latter functions $F_\epsilon(x)$. The degree, $\gamma(F,A,x_1)$ of the function F(x) at the point x_1 is defined as this number. The properties of the degree (Cf. Leray and Schauder IV[d]) are as follows

1. If A_1 and A_2 are two closed bounded sets in E_n, without common points and x_1 is not contained in $F(fr(A_1))$, $F(fr(A_2))$, then $\gamma(F,A_1,x_1) + \gamma(F,A_2,x_1) = \gamma(F,A_1 \cup A_2,x_1)$.

2. If $\gamma(F,A,x_1) \neq 0$ then x_1 is in $F(\text{int } A)$ where int A is the set A exclusive of its frontier points.

3. $\gamma(F,A,x_1)$ is constant if the transformation $F(x)$, the set A and the point x_1 vary continuously, A, x_1 and F being subject to the same restrictions as before.

Let us state a lemma of Leray and Schauder (IV^d 1, p. 49) in a form suitable for our purposes. Consider a linear metric space E_{n+p} of n + p dimensions and a bounded closed set A_{n+p} contained therein. If we consider a linear metric subspace E_n containing points of A_{n+p} and some given point x_1 of E_{n+p}, then the common part A_n of the space E_n and the set A_{n+p}, being the intersection of two closed sets is necessarily closed and of course, bounded. Certainly $fr(A_n) \subseteq fr(A_{n+p})$. Now consider a function $F_{n+p}(x)$ on A_{n+p} to a subset of E_{n+p} and a point x_1 not contained in $F_{n+p}(fr\ A)$ and such that $F(x) - x$ is an element in E_n whenever x is in A_{n+p} (in this case the function $F(x)$ is known as an n-translation). Then $F_{n+p}(x)$ will transform A_n into a set belonging to E_n. Let $F_n(x)$ be defined only for x in A_n and let $F_n(x) \equiv F_{n+p}(x)$ when x is in A_n. The first lemma of Leray and Schauder then states that

$$\gamma(F_n,A_n,x_1) = \gamma(F_{n+p},A_{n+p},x_1)$$

We shall now be occupied with functions of the form $x - F(x)$ in which x is contained in a fundamental closed bounded set, $F(x)$ is a completely continuous function defined on that set. There follow some preliminary theorems and lemmas. The proof of the following theorem does not depend on Postulate IV. A simpler proof can be given if that postulate is assumed.

Theorem 10. A linear semi-ordered space of Kantorovitch, with a finite number n of dimensions is homeomorphic to an Euclidean space of n dimensions.

Proof. According to the hypothesis there exists a linearly independent set of elements $x_1, x_2, \ldots, x_n$ such that every element in the given space can be expressed as a linear combination of those elements (with real number coefficients), i.e. for x arbitrary there exists a set of real numbers $\alpha_1, \cdots, \alpha_n$ for which $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$. Let $x \leftrightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be the correspondence between the two spaces. Then to prove that if $x^{(i)} = \sum_{j=1} \alpha_j^{(i)} x_j$ and if $\alpha_j^{(i)} \to \alpha_j$ as $i \to \infty$, then $x^{(i)} \to \sum_{j=1}^{n} \alpha_j x_j$ and the converse, that if $x^{(i)} \to \sum \alpha_j x_j$, then $\alpha_j^{(i)} \to \alpha_j$ for each j. The first part follows easily from the theorems on the limit of sums, etc. Prove the second part for the special case $x^{(i)} = \sum \alpha_j^{(i)} x_j \to 0$. Suppose by contradiction that $\beta^{(i)} \equiv \max\left\{|\alpha_1^{(i)}|, |\alpha_2^{(i)}|, \ldots, |\alpha_n^{(i)}|\right\}$

does not converge to zero. Then pick a sub-sequence from $x^{(i)}$ (by re-numbering we can treat it as if it were the original sequence) with the property that $\frac{x^{(i)}}{\beta^{(i)}} \to 0$ and furthermore the sequences $\frac{\alpha_j^{(i)}}{\beta^{(i)}}$ converge to finite limits λ_j, for each j. This last is accomplished by noting that the sequences $\frac{\alpha_j^{(i)}}{\beta^{(i)}}$ are each bounded. From the first sequence $\frac{\alpha_1^{(i)}}{\beta^{(i)}}$ can be picked a convergent subsequence say $\frac{\alpha_1^{(i_n)}}{\beta^{(i_n)}}$, with indices i_n. Now from the sequence $\frac{\alpha_2^{(i_n)}}{\beta^{(i_n)}}$, with indices i_n. Now from the sequence $\frac{\alpha_2^{(i_n)}}{\beta^{(i_n)}}$ can be picked a convergent sequence with indices i_{n_p}, a subset of the indices i_n; if we continue this process to $\frac{\alpha_n^{(i)}}{\beta^{(i)}}$ we arrive with an infinite sequence of indices which we shall, without loss of generality call $\{i\}$.

Thus

$$\sum_{j=1}^{n} \lambda_j x_j = \lim_{i \to \infty} \sum_{j=1}^{n} \frac{\alpha_j^{(i)} x_j}{\beta^{(i)}} = 0 \text{ follows}$$

that $\lambda_j = 0$, since the x_j are linearly independent. But since $\max\left\{|\alpha_1^{(i)}|, |\alpha_2^{(i)}|, \cdots, |\alpha_n^{(i)}|\right\} \geqq |\alpha_1^{(i)}|, |\alpha_2^{i}|, \cdots, |\alpha_n^{(i)}|$, $\frac{|\alpha_j^{(i)}|}{\beta^{(i)}} \leqq 1$ and so $\lambda_j \leqq 1$.

Then there is at least one $\lambda_k = 1$. For if they were all 1, we should have for $i > N$

$\frac{|\alpha_j^{(i)}|}{\beta^{(i)}} < 1$ for all j, but this gives a contradiction since at least one $\frac{\alpha_j^{(i)}}{\beta^{(i)}}$ must be equal to

1 for every i. The proof for a non-zero limit follows now by subtraction.

Theorem 11. If L_n is a linear manifold of dimension n contained in S, then L_n is a closed linear manifold.

Proof. Suppose that $\left| \sum_{j=1}^{n} \alpha_j^{(i)} \cdot x_j - x \right| \to 0$ where x is in S. Then by contradiction assume that x cannot be expressed as a linear combination of the $x_1, \cdots, x_n$, that is if

$$\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + \beta x = 0 \quad \text{then}$$

$\beta_y = 0 = \beta$ for all y. Then all possible linear combinations of $x_1, \cdots, x_n$, x will form a linear manifold L_{n+1} in S and by the preceding theorem if

$$\sum_{y=1}^{n} \alpha_y^{(i)} x_y - x \to 0 \text{ then } \alpha_y^{(i)} \to 0 \text{ as } i \to \infty \text{ and}$$

$-1, -1, \cdots -1 \cdots$; as a sequence has a zero limit, which is false. This completes the proof.

Lemma 1. If A is a closed, bounded set, then fr A, the frontier of A, is closed.

Proof. The frontier of A is defined as $\overline{A} \cap \overline{S - A}$, where $\overline{A}$ means the set A together with all its limit-points. S - A is the set of elements of S not in A and $\overline{S - A}$ means the set of all limit points of S - A together with the points of S - A. But the part common to two closed sets is closed, obviously.

Theorem 12. If $f(x) = x - F(x)$ where $F(x)$ is completely continuous defined on a closed bounded set A and y is not on f(fr A) then there is no sequence $\{x_n\}$ in fr A such that

$$x_n - F(x_n) \to y.$$

Proof. By contradiction. Suppose there does exist such a sequence. Then since the set $\{F(x_n)\}$ is strongly compact, we can pick a convergent subsequence out of $\{F(x_n)\}$, which without loss of generality can be assumed to be the same as $\{F(x_n)\}$ itself. Let $F(x_n) \to x_o$. Then

$$x_n = [x_n - F(x_n)] + F(x_n) \to y + x_o$$

And so $z = y + x_o$ will belong to fr A and so

$$\lim_{n \to \infty} x_n - F(x_n) = y = z - F(z).$$ But the last equality says that y is on f(fr A). Contradiction.

Theorem 13. Under the same hypothesis as the above theorem, y is at a distance $\delta > 0$ from f(fr A), where the δ is simply the distance according to the metric with respect to a certain element (arbitrary but fixed in a particular case) bounding y and the set A.

Proof. The set A is bounded, as is the set F(A) and so is then the set x - F(x) where x is in A. The proof is now immediate in view of the remarks immediately preceding Theorem 7.

We shall now define the degree at a point y of a function $f(x) = x - F(x)$ where F(x) is completely continuous on a closed bounded set A, and y is not on f(fr A). We first find a linear function $F_\delta(x)$ on A to an n-dimensional sub-space of S such that

$\| F(x) - F_\delta(x) \| < \delta$ where δ is the distance of y from

f(fr A) reckoned with the metric with regard to an element bounding y, the set F(A), and the set f(A). Now let E_{n_δ} be a space containing the (n-dimensional) set $F_\delta(A)$, the point y, and at least one point of A. Now the intersection, A_{n_δ}, of E_{n_δ} with A will be non-vacuous, closed and bounded and will be, moreover, n_δ-dimensional. Also $fr(A_{n_\delta}) \subseteq fr(A)$. Thus y is at a distance $\geqq \delta$ from $fr(A_n\ )$. The function f_δ on the set A_{n_δ} is then a function on one set of a finite dimensional space to another set of the same space. The degree of this function is well-defined at y and has the properties 1. - 3. on page 189 We now define

$$\gamma[f(x), A, y] \equiv \gamma[f_\delta(x), A_n\ , y].$$

This definition is justified by the following theorems.

Theorem 14. Consider $f(x) = x - F(x)$, $F(x)$ completely continuous and defined on a bounded closed set A in an n_1 dimensional sub-space of S and

$$f_{\delta_1}(x) = x - F_{\delta_1}(x)$$
$$f_{\delta_2}(x) = x - F_{\delta_2}(x)$$ be two functions in

which $F_{\delta_1}(x)$, $F_{\delta_2}(x)$ are both linear functions defined on the same closed bounded set A and whose ranges of values are both contained in the same linear sub-space E_{n_δ} and such that $0 < \delta_1, \delta_2 < \delta$ and let y be an element in E_{n_δ} and at a distance greater than δ from F(fr A). $F_{\delta_1}(x)$, $F_{\delta_2}(x)$ satisfy the inequalities

$$\| F(x) - F_{\delta_1}(x) \| < \delta$$
$$\| F(x) - F_{\delta_2}(x) \| < \delta, \quad x \text{ in } A.$$

Then the degrees of the functions f_{δ_1} and f_{δ_2} at y are the same.

Proof. Consider the function $\eta f_{\delta_1}(x) + (1 - \eta) f_{\delta_2}(x)$ where varies from 0 to 1. Then

$$\left| \eta f_{\delta_1}(x) + (1 - \eta) f_{\delta_2}(x) - f(x) \right| =$$
$$\left| \eta F_{\delta_1}(x) + (1 - \eta) F_{\delta_2}(x) - F(x) \right| =$$
$$\left| \eta (F(x) - F_{\delta_1}(x)) + (1 - \eta)(F(x) - F_{\delta_2}(x)) \right| \leqq$$
$$\eta \left| F(x) - F_{\delta_1}(x) \right| + (1 - \eta) \left| F(x) - F_{\delta_2}(x) \right| \leqq (1 - \eta + \eta)\delta = \delta$$

Thus by property 3 (see p. 190) of the degree of a function on one n-dimensional metric space to another, the theorem follows.

This theorem justifies the definition of the degree of a function to the extent of shewing that it doesn't depend on the choice of the approximating linear function if the $E_{n_{\delta_1}}$, $E_{n_{\delta_2}}$ are the same for two linear functions $f_{\delta_1}(x)$, $f_{\delta_2}(x)$ for which

$$\| f_{\delta_i}(x) - f(x) \| = \| F(x) - F_{\delta_i}(x) \| < \delta \qquad i = 1,2,$$

The next theorem shews that these associated linear spaces need not be identical.

Theorem 15. Using the above notation, if $E_{n_{\delta_1}}$, $E_{n_{\delta_2}}$ are different the respective degrees of the transformation $f_{\delta_1}(x)$, $f_{\delta_2}(x)$ are the same.

Proof. This reduces to the previous theorem if we consider E_n a finite-dimensional linear space containing the finite-dimensional spaces $E_{n\delta_1}$ and $E_{n\delta_2}$. Let A be the point-set intersection of E_n and A. The two linear functions are both defined on the same set A_n and their values lie in the same linear n-dimensional space E_n. Also by the lemma of Leray and Schauder, see above, the degree of $f_{\delta_1}(x)$ on A_n is the same as that on $A_{n\delta_1}$.

Now we shall prove that the three characteristic properties of the degree of a are valid in the case of the functions f(x) in Kantorovitch spaces.

Theorem 16. If A and B are two closed bounded sets without common interior points, then $\gamma[f,A,y] + \gamma[f,B,y] = \gamma[f,A \cup B,y]$ where y is a point not on f(fr A) or f(fr B).

Proof. In determining the degree the linear sub-space $E_{n\delta}$ is made to contain points from both A and B.

Theorem 17. If $\gamma[f,A,y] \neq 0$ then y is in f(A).

Proof. Immediate.

Definition 5 of continuous deformation of f(x).

The function g(x) is said to be continuously deformable into the f(x), where

$$g(x) = x - G(x)$$

$f(x) = x - F(x)$, G(x) and F(x) being completely continuous and defined on the same closed bounded set A, if there exists a function $F(x, \lambda)$ on $S \times R$ where S is a Kantorovitch space and R is the real number segment $0 \leqq \lambda \leqq 1$ with the properties:

1. $F(x,0) = G(x)$, $F(x,1) = F(x)$.
2. There is a y in S such that to any $\epsilon > 0$, there exists a $\delta > 0$ independent of x, for which
$$|F(x, \lambda_1) - F(x, \lambda_2)| < \epsilon \; y \quad \text{for} \quad |\lambda_1 - \lambda_2| < \delta$$
3. $F(x, \lambda_0)$, λ_0 fixed, is completely continuous on A.

The following theorem is a generalization of one stated by Leray and Schauder and proved by Rothe IV 2 p. 301, Hilfssatz 2.

Theorem 18. The totality of values of the function $F(x, \lambda)$ x in A, λ in $[0,1]$ form a compact set.

Proof. $F(x, \lambda)$ is a function continuous in the variables combined, that is there exists a Z such that to any $\epsilon > 0$ there exists a $\delta_1 > 0$ and $\delta_2 > 0$ for which
$$|F(x, \lambda) - F(x_0, \lambda_0)| \leqq \epsilon \; Z$$
when
$$|x - x_0| < \delta_1 \; g$$
$$|\lambda - \lambda_0| < \delta_2,$$
where g is a bounding metrizing element of A. This follows from the inequality
$$|F(x, \lambda) - F(x_0, \lambda_0)| = |F(x, \lambda) - F(x, \lambda_0) + F(x, \lambda_0) - F(x_0, \lambda_0)| \leqq$$
$$\leqq |F(x, \lambda) - F(x, \lambda_0)| + |F(x, \lambda_0) - F(x_0, \lambda_0)|$$
$$\frac{\epsilon}{2} y + \frac{\epsilon}{2} y_{\lambda_0} \leqq \epsilon \, \text{lub} \, (y, y_{\lambda_0}).$$

Now to prove that from any sequence of pairs (x_n, λ_n) can be extracted a sequence (x_{n_i}, λ_{n_i}) for which $\lim_{i \to \infty} F(x_{n_i}, \lambda_{n_i})$ exists. This may be proved if we first prove that there exists a fixed w such that to any $\epsilon > 0$ we can find a subsequence of pairs (x_{n_j}, λ_{n_j}) for which

$$\left| F(x_{n_j}, \lambda_{n_j}) - F(x_{n_k}, \lambda_{n_k}) \right| < \epsilon w.$$

The proof will then be completed by the diagonal method.

Since the original sequence $\{\lambda_n\}$ is bounded, we can by the Bolzano-Weierstrass theorem extract a convergent subsequence. Thus for simplicity that $\{\lambda_n\}$ already converges. From property 2 of Definition 5, we can choose N_1 so that

$$\left| F(x, \lambda_n) - F(x, \lambda_{n'}) \right| \leqq \frac{\epsilon}{3} y, \quad \text{for } n, n' \geqq N_1$$

and a suitable fixed y. Now pick a subsequence x_{n_i} from x_n so that the sequence $F(x_{n_i}, \lambda_{N_1})$ converges. Then there is a number N_2 for which

$$\left| F(x_{n_i}, \lambda_{N_1}) - F(x_{n_i'}, \lambda_{N_1}) \right| \leqq \frac{\epsilon}{3} y_{N_1}$$

whenever $n_i, n_i' \geqq N_2$. Now delete from the sequence of integers n_i all those less than $N = \max(N_1, N_2)$. Now for the resulting sequence

$$(1) \quad \left| F(x_{n_i}, \lambda_{n_i}) - F(x_{n_i'}, \lambda_{n_i'}) \right| \leqq$$

$$\left| F(x_{n_i}, \lambda_{n_i}) - F(x_{n_i}, \lambda_N) \right| + \left| F(x_{n_i}, \lambda_N) - F(x_{n_i'}, \lambda_N) \right| +$$

$$\left| F(x_{n_i'}, \lambda_N) - F(x_{n_i'}, \lambda_{n_i'}) \right| \leqq \frac{\epsilon}{3} y + \frac{\epsilon}{3} y_{\lambda_N} + \frac{\epsilon}{3} y \leqq$$

$$\epsilon \, \mathrm{lub}(y, y_{\lambda_N})$$

But $\left|F(x_{n_i}, \lambda_k)\right| \leqq \left|F(x_{n_i}, \lambda_k) - F(x_{n_i}, \lambda_N)\right| + \left|F(x_{n_i}, \lambda_N)\right|$

$$\frac{\epsilon}{3} y + y_1 ; n_i, k > N$$

y_1 is the upper bound of the set $F(x_{n_i}, \lambda_N)$, n variable, which exists, owing to the complete continuity of $F(x, \lambda)$ for fixed λ, $(= \lambda_N)$.

Now let $w = \text{lub}\,(y, y_1, y_{\lambda_N})$. Thus:

$$(1') \qquad \left|F(x_{n_i}, \lambda_{n_i}) - F(x_{n'}, \lambda_{n_i'})\right| < \epsilon w$$

Inequality (1') says what we started out to prove as a first step. Now if we select a null-sequence $\{\epsilon_n\}$, then we can pick a sub-sequence $(x_n^{(1)}, \lambda_n^{(1)})$ such that

$$\left|F(x_n^{(1)}, \lambda_n^{(1)}) - F(x_{n'}^{(1)}, \lambda_{n'}^{(1)})\right| < \epsilon_1 w$$

Now consider this sub-sequence. We can fix a real integer P so large that $\left|F(x, \lambda_n^{(1)}) - F(x, \lambda_m^{(1)})\right| < \frac{\epsilon_2}{3} w$ for $n, m \geqq P$ and all x in the sequence $\left\{x_n^{(1)}\right\}$. Now we can pick a sub-sequence $\left\{x_{n_i}^{(1)}\right\}$ of $\left\{x_n^{(1)}\right\}$ such that $F(x_{n_i}^{(1)}, \lambda_P^{(1)})$ is convergent, i.e. so that $\left|F(x_{n_i}^{(1)}, \lambda_P^{(1)}) - F(x_{n_i}^{(1)}, \lambda_P^{(1)})\right| < \frac{\epsilon_2}{3} w$ for $n_i > P_{\epsilon_2/3}$ etc.

Finally by steps similar to those above, we arrive at a subsequence of pairs $\left\{x_n^{(2)}, \lambda_n^{(2)}\right\}$ for which

$$\left|F(x_n^{(2)}, \lambda_n^{(2)}) - F(x_{n'}^{(2)}, \lambda_{n'}^{(2)})\right| < \epsilon_2 w$$

Notice that this is the same w. Now this process can be continued indefinitely so that we have the sequence $F(x_n^{(p)}, \lambda_n^{(p)})$ for which

$$\left| F(x_n^{(p)}, \lambda_n^{(p)}) - F(x_n^{(p)}, \lambda_n^{(p)}) \right| < \epsilon_p \, w.$$ Now pick out the diagonal sequence $\left\{ F(x_n^{(n)}, \lambda_n^{(n)}) \right\}$. This sequence will then have the property that

$$\left| F(x_n^{(n)}, \lambda_n^{(n)}) - F(x_{n'}^{(n')}, \lambda_{n'}^{(n')}) \right| < \epsilon_N \, W$$

for all $n, n' \geqq N$. But this says that the sequence $\left\{ F(x_n^{(n)}, \lambda_n^{(n)}) \right\}$ is convergent.

Theorem 19. Under the same hypotheses as those of the preceding theorem the set $f(x, \lambda)$ is bounded.

Proof. Consider $f(x, \lambda)$ for any fixed x_0 then

$$|f(x, \lambda)| \leqq |f(x, \lambda) - f(x,0)| + |f(x,0)|$$

The set $f(x,0)$ is bounded for x in A, since $f(x,0) = x - F(x,0)$ and $F(x,0)$ is completely continuous. Let its bound be Z. Now to shew that $|f(x, \lambda) - f(x,0)|$ is bounded for λ and x variable. To a number $\epsilon > 0$ we can find a $\delta > 0$ such that

$$|F(x, \lambda) - F(x, \lambda')| = |f(x, \lambda) - f(x, \lambda')| < \epsilon \, y$$ for $|\lambda - \lambda'| < \delta$. Now in the interval $0 \leqq \lambda \leqq 1$, lay off lengths $\delta_1 < \delta$ (δ_1 fixed) with the first interval starting at zero and the last interval or a fraction of it ending at 1. There will be k of these, say, with end-points $0 = \lambda_1, \lambda_2, \dots, \lambda_{k+1} = 1$.

Now if $\lambda_{r+1} \geqq \lambda_k \geqq \lambda_r$

$$\left|f(x,\lambda) - f(x,0)\right| = \left|f(x,\lambda) - f(x,\lambda_r) + f(x,\lambda_r) - f(x,\lambda_{r-1}) + \cdots + f(x,\lambda_2) - f(x,\lambda_1)\right| \leqq$$

$$\left|f(x,\lambda) - f(x,\lambda_r)\right| + \left|f(x,\lambda_r) - f(x,\lambda_{r-1})\right| + \cdots + \left|f(x,\lambda_2) - f(x,\lambda_1)\right| \leqq r\epsilon y \leqq k\epsilon y,$$ where k is fixed for the time. Thus putting $k\epsilon = \ell$

$\left|f(x,\lambda)\right| \leqq z + k\epsilon y \leqq \text{lub}(z, \ell y) = w_0$, and w_0 is independent of x and λ.

<u>Theorem 20</u>. The set $x - F(x,\lambda)$, where F, x, λ are the same as in Definition 5, is closed.

<u>Proof</u>. Let g be an element such that $x_n - F(x_n,\lambda_n) \rightarrow g$. Now we can assume that $F(x_n,\lambda_n)$ is convergent by Theorem 18. (If it is not, renumber a suitable subsequence). Then $F(x_n,\lambda_n) \rightarrow b$. $x_n = \left[x_n - F(x_n,\lambda_n)\right] + F(x_m,\lambda_n) \rightarrow b + g$. Thus x_n is convergent. This means that the sequence of pairs $\{x_n,\lambda_n\}$ is convergent to $\{b + g, \lambda_0\}$, since $\{\lambda_n\}$ can be assumed convergent by Theorem 18. Now $F(x_m,\lambda_m)$, Theorem 18 is continuous in the variables x, λ combined. Therefore $b = F(b + g, \lambda_0)$ and

$$g = b + g - b = \lim\left[x_n - F(x\ ,\lambda_n)\right]$$
$$= (b + g) - F(b + g, \lambda_0)$$

is the value of $f(x,\lambda)$ for x in A(= b + g) and λ in $[0,1]$.

<u>Theorem 21</u>. If g is an element not on the frontier of the set $f(x,\lambda) = x - F(x,\lambda)$, then it is at a distance $\delta > 0$ from that set.

<u>Proof</u>. The set in question, together with the element g can be metrized. Then Theorems 18-21 together with the remarks preceding Theorem 7 lead to the result.

<u>Theorem 22</u>. If $f_1(x), f_2(x)$ are defined on A as above and $\|f_1(x) - f_2(x)\| < \mu < \delta_1$, where $\|g - f_1(x)\| > \delta_1$ then $\gamma(f_1,A,g) = \gamma(f_2,A,g)$. Let the metric be as above.

<u>Proof</u>. Define an approximation function (linear n-dimensional) f_ϵ such that

(1) $\|f_2(x) - f_\epsilon(x)\| < \epsilon < \delta_1 - \mu$

Then

$$\|f_\epsilon(x) - f_1(x)\| \leqq \|f_\epsilon(x) - f_1(x)\| + \|f_1(x) - f_2(x)\| \leqq \delta_1 - \mu + \mu = \delta_1$$

$$\leqq \delta_1 - \mu + \mu = \delta_1$$

which says $\gamma(f_1,A,g) = \gamma(f_\epsilon,A,g)$. But also

$$\|g - f_2(x)\| \geqq \|g - f_1(x)\| - \|f_1(x) - f_2(x)\| > \delta_1 - \mu$$

which together with inequality (1) says that $\gamma(f_\epsilon,A,g) = \gamma(f_2,A,g)$. Thus finally from the last two equations,

$$\gamma(f_1,A,g) = \gamma(f_2,A,g).$$

<u>Theorem 23</u>. (Property 3). If $F(x,\lambda)$ varies by continuous deformation so that for each value of λ, g is not on the set $F(A,\lambda)$, then $\gamma(f(x,\lambda),A,g)$ is independent of λ.

<u>Proof</u>. By Theorem 13, g is at a distance $\delta > 0$ from the frontier of the set of elements $x - F(x,\lambda)$ x in A and 0 $0 \leqq \lambda \leqq 1$. Now if, as is possible, we choose η so small that

$$\| F(x, \lambda_1) - F(x, \lambda_2) \| < \frac{\delta}{2} \quad \text{for} \quad |\lambda_1 - \lambda_2| \leqq \eta$$

Now divide the interval $[0,1]$ into sub-intervals each of length $\eta/2$, starting at 0 and such that the last interval or fraction of interval ends at 1. Then by the preceding theorem the degrees of the set of functions for values of λ in any sub-interval are the same. Also by the same theorem so are the degrees of functions corresponding to values of λ in two adjacent sub-intervals. Thus step by step we may shew the degree independent of λ.

We now return to a consideration of the abstract differential and its connexion with the theory of the degree of the corresponding function, (see p. 149 for the definition and properties of the differential). We shall assume now in addition to the properties there listed that the function $E(x_0,y,z)$ has the property

4. If the sequence $\{z_n\}$ is bounded then $E(x_0,x_n,z_n) \to 0$ uniformly as $x_n \to 0$. That is, there is a $\eta_{y,h}$ and a $g_{y,h}$ such that $|E(x_0,x_n,z_n)| < \epsilon\, g_{y,h}$ when $|x_n| < \delta_\epsilon\, y$ and $|z_n| < h$.

<u>Theorem 24</u>. If A is a bounded set in B and F(a) is linear function defined on B to S, then F(A) is a bounded set, where F(A) means the set of all F(a) with <u>a</u> in A.

<u>Proof</u>. The necessary and sufficient condition that the set A be bounded is that for each countable set of elements $\{a_n\}$

in A, and every null-sequence $\{\lambda_n\}$ of real numbers (i.e. $\lambda_n \to 0$), we have $\lambda_n a_n \to 0$, see Banach IV . The necessary and sufficient condition that a set X in S be bounded is by Theorem 41 p 52 the same as the above condition, simply replacing a_n by x_n, an element in X.

Now suppose by contradiction that we can pick a sequence (not necessarily convergent) $\{a_n\}$ such that the sequence $F(a_n)$ is unbounded. Now

$$\lambda_n F(a_n) = F(\lambda_n a_n) \to 0 \text{ for each null-sequence } \{\lambda_n\}.$$

But this simply means that $F(a_n)$ is bounded and we have a contradiction.

<u>Definition</u> of completely continuous differential.

F(x) on S to S is said to have a completely continuous differential at the point <u>x_0</u> if it has a differential $F(x_0;dx)$ at x_0 that is completely continuous, that is, such that if the <u>dx</u> are in a bounded set, the set $F(x_0;dx)$, (x_0 fixed and dx variable) will form a compact set.

<u>Definition</u> of non-singular differential at a point.

F(x) on S to S is said to have a non-singular differential at the point x_0 if it has a differential $F(x_0;dx)$ with the property that for x_0 fixed,

$$F(x_0;dx) = 0 \text{ if and only if } dx = 0.$$

Definition of isolated solution.

y_o is said to be an isolated solution of the equation

$G(x) = G(y_o)$, where $G(x)$ is defined over all of the space S, if there is no sequence $\{x_n\}$ such that $x_n \to y_o$ and

$$G(x_n) = G(y_o) \text{ for all } n.$$

Theorem 25. If the function $F(x)$ on S to S has a completely continuous differential at x_o and the function $x - F(x)$ has a non singular differential at x_o then x_o is an isolated solution of the equation:

$$x - F(x) = x_o - F(x_o).$$

Proof. By contradiction. Suppose a sequence $\{x_n\}$, $x_n \to x_o$ such that $x_n - F(x_n) = x_o - F(x_o)$, or

$x_n - x_o - [F(x_n) - F(x_o)] = 0$. But

$F(x_n) - F(x_o) = F(x_o;x_n - x_o) + E(x_o,x_n - x_o,x_n - x_o)$,

owing to the existence of a differential. Thus

(1) $x_n - x_o - F(x_o;x_n - x_o) = E(x_o,x_n - x_o,x_n - x_o)$

Now since $x_n - x_o \to 0$ there is a $y > 0$ for which

$|x_n - x_o| < \epsilon\, y$. Now metrize the sequence

$x_n - x_o$ with respect to y and divide equation (1) by

$\|x_n - x_o\|$

$$\frac{x_n - x_o}{\|x_n - x_o\|} - F(x\ ;\frac{x_n - x_o}{\|x_n - x_o\|}) = E(x\ ,x_n - x\ ,\frac{x_n - x_o}{\|x_n - x_o\|})$$

Now as $x_n \to x_o$, the $\frac{x_n - x_o}{\|x_n - x_o\|}$ are bounded (are all less than or equal to y except for a finite number of preliminary terms) and so $E(x_o, x_n - x_o, \frac{x_n - x_o}{\|x_n - x_o\|}) \to 0$ owing to the restrictions on E. Thus

(2) $\frac{x_n - x_o}{\|x_n - x_o\|} - F\left(x ; \frac{x_n - x_o}{\|x_n - x_o\|}\right) \to 0$. But since $\frac{x_n - x_o}{\|x_n - x_o\|}$ are bounded, the set $F(x_o; \frac{x_n - x_o}{\|x_n - x_o\|})$ is compact. Extract a sequence approaching a limit-point g. Without loss of generality we can preserve the notation and take this subsequence as the original sequence. Thus

$\frac{x_n - x_o}{\|x_n - x_o\|} \to g$ from equation (2), and $g \neq 0$ from Theorem 2, since its norm exists and is 1. But owing to the continuity of $F(x_o;z)$ in z,

$$\lim_{n \to \infty} \left\{ \frac{x_n - x_o}{\|x_n - x_o\|} - F\left(x_o; \frac{x_n - x_o}{\|x_n - x_o\|}\right) \right\} = 0 =$$

$$\lim_{n \to \infty} \frac{x_n - x_o}{\|x_n - x_o\|} - \lim_{n \to \infty} F(x_o; \frac{x_n - x}{\|x_n - x_o\|}) =$$

(3) $g - F(x_o;g) = 0$, and $g \neq 0$. But the differential of $x - F(x)$ at x_o is $x - x_o - F(x_o;x - x_o)$ or, since x in the latter is arbitrary, $u - F(x_o;u)$. Equation (3) then says that this differential is singular, contrary to hypothesis.

Now if the point x_o is an isolated point of the equation

$$f(x) \equiv x - F(x) = x_o - F(x_o),$$ then the function $f(x) - f(x_o)$ carries the point x_o into 0, and any "sphere" $|x - x_o| \leqq \epsilon_g$ g for a given g and small enough corresponding ϵ, will not contain further points x_1 such that

$f(x_1) - f(x_o) = 0$. The degree of the function $f(x) - f(x_o)$ at the point 0 will be defined and furthermore for all μ such that $0 < \mu \leqq \epsilon_g$ the degree will be the same. This last can be proved by the use of approximating functions in finite-dimensional spaces. But, since the function $F(x)$ has a completely continuous differential we have

$$f(x) - f(x_o) = x - F(x) - \left[x_o - F(x_o)\right] = x - x_o - \left[F(x) - F(x_o)\right] =$$

$$x - x_o - \left[dF(x_o;x - x_o) + E(x_o,x - x_o,x - x_o)\right]$$

or

$$\left|f(x) - f(x_o) - \{x - x_o - dF(x_o;x - x_o)\}\right| = \left|E(x_o,x - x_o,x - x_o)\right|$$

or

$$\left|f(x) - f(x_o) - df(x_o;x - x_o)\right| = \left|E(x_o,x - x_o,x - x_o)\right|$$

Now let Z_g be the element bounding the sets $f(x)$ and $df(x_o;x - x_o)$ when x is in the "sphere" $|x - x_o| \leqq \epsilon_g\, g$. Then we can write the equality

$$\left\|f(x) - f(x_o) - df(x_o;x - x_o)\right\|_{Z_g} = \left\|E(x_o,x - x_o,x - x_o)\right\|_{Z_g}$$

Now for $\|x - x_o\|$ small enough, we have

$$\left\|df(x_o;x - x_o)\right\| > \left\|E(x_o,x - x_o,x - x_o)\right\|$$

for by contradiction if

$$\left\|df(x_o;x_n - x_o)\right\| \leqq \left\|E(x_o,x_n - x_o,x_n - x_o)\right\|$$

for a sequence $\{x_n - x_o\}$ of different elements approaching zero, then metrize this sequence according to the method on p . Divide the above inequality by $\|x_n - x_o\|$.

$$\left\| df\left(x_o; \frac{x_n - x_o}{\|x_n - x_o\|}\right) \right\| \leqq \left\| E\left(x_o, x_n - x_o, \frac{x_n - x_o}{\|x_n - x_o\|}\right) \right\|$$

Now as $x_n \rightarrow x_o$ the left-hand side does not approach zero, by the proof of Theorem 25, (using now the additional restriction that x_n be in the "sphere"), while the right-hand side does.

But ~~if~~ we consider the function $(1 - \lambda)\{f(x) - f(x_o)\} + \lambda\, df(x_o; x - x_o)$, which for $\lambda = 0$ is just $f(x) - f(x_o)$ and for $\lambda = 1$ is $df(x_o; x - x_o)$. Now apply Theorem 23 to this function and we see that the degree of the point $\underline{0}$ with respect to the function $f(x) - f(x_o)$ on the set $|x - x_o| \leqq \epsilon_g g$ is the same as it is with respect to the function $df(x_o; x - x_o)$ on the same set.

To actually determine the degree of the function (linear), $x - x_o - dF(x_o; x - x_o) = df(x_o; x - x_o)$ we shall have need of the Riesz theory of linear completely continuous functions see Riesz IV 1. Suffice it to say that if we impose further restrictions on the space we can use his theory. Sufficient such restrictions are

1) It shall be possible to norm any given linear

manifold consisting of the point set sum of an infinite increasing sequence of finite dimensional linear manifolds. The topology of this norm shall be equivalent to the original one.

2) If the linear manifolds L_1 and L_2 have only the element 0 in common and if at least one is finite dimensional then they are simultaneously metrizable equivalently to the original topology so that for each element x_1 in L_1 and each element x_2 in L_2

$\phi \| x_1 + x_2 \| \geqq \| x_1 \| + \| x_2 \|$ for a ϕ determined only by the two spaces and not dependent on the particular choice of x_1, x_2.

The degree is then determined by considering the function

$$x - x_0 - \lambda \, dF(x_0; x - x_0), \text{ where } 0 \leqq \lambda \leqq 1.$$

The reader may refer to Leray and Schauder IV, p. 58 and to Goursat, Cours d'Analyse III, Chap. 31, part II.

Appendix

Bibliography

The bibliography is divided into four parts. The first part deals with the theory of structures as connected with algebraic questions such as the theory of numbers, the theory of groups and rings, mathematical logic, probability. These respective subdivisional classifications are indicated by (N), (A), (L), (P) placed at the right of the reference. The second part deals with the theory of structures as connected with questions of geometry (projective and continuous), topology (point-set and combinatorial), function-spaces. These are indicated respectively by (G), (T), (F). The third part is devoted to the subject of abstract derivatives, differentials and integrals. The fourth part deals with pure topology and with theorems on fixed points of transformations. In each part the books are listed separately from the papers that appeared in periodicals, and a B is attached to the respective Roman numeral to indicate this.

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