Integrals and Functional Equations in Linear Topological Spaces

Thesis by

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Resumé

The classification of linear topological spaces is discussed. In particular, it is shown that locally bounded linear topological spaces are pseudo-normable and that locally compact linear topological spaces are finite dimensional. Spaces of type (F) are characterized among the class of linear topological spaces. A generalization of the Lebesgue-Fréchet-Birkhoff integral is carried out, and Riemann integrals are discussed.

By introducing suitable systems of sets, called K-systems, it is found possible to extend the method of successive approximations to functional equations in linear topological spaces. An existence theorem for K-systems is proved. Existence theorems are then obtained for the functional equation $y = f(y)$ and for first order differential equations in linear topological spaces.

~~A beginning is made on a generalization of the Riesz theory of completely continuous linear functional transformations.~~

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Introduction

A linear topological space may be characterized as being a linear space supporting a Hausdorff topology in which the operations of vector addition and scalar multiplication are continuous. A. Kolmogoroff, J. v. Neumann and A. Tychonoff were among the first* to study such spaces. Kolmogoroff in 1934, introduced the useful idea of boundedness in linear topological spaces, and was able to characterize the normed spaces among the class of linear topological spaces as those having the properties of local convexity and local boundedness. Neumann in 1930 had used several instances of linear topological spaces to good advantage in the study of functional operations in Hilbert space (cf. Neumann (1.)). In 1935, Neumann studied the notion of completeness in a linear topological space L, and in collaboration with S. Bochner, investigated almost periodic functions on an arbitrary group to L (cf. Bochner and Neumann (1.)). Tychonoff generalized the Brouwer-Schauder fixed point theorem to locally convex linear topological spaces, and gave a new way of defining the "topological product" of a collection of topological spaces.

A. D. Michal and E. W. Paxson** were the first to define a "Frechet" differential for functions with arguments and values in a linear topological space. They also developed a theory of Riemann inte-

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* Kolmogoroff (1.), Tychonoff (1.), Neumann (2.), where the numbers refer to the list of references at the end.

** Cf. Michal and Paxson (1.) and (2.)

grals for such a space.

The first chapter of this thesis, after the first four introductory sections, is concerned mainly with the classification of linear topological spaces (Begun by Kolmogoroff). The property of local boundedness is studied, and locally bounded spaces are found to be "pseudo-normable" on the one hand, and metrizable on the other. Frechet (F) spaces are completely characterized among the class of linear topological spaces. The main result is that every locally compact linear topological space is finite dimensional. In the last section of the chapter a number of examples of linear topological spaces (all non-normable) are given, illustrating various theoretical points. Chapter two is devoted to a generalization to linear topological spaces of G. Birkhoff's integral for functions with values in a Banach space, and to a brief discussion of Riemann integrals.

In the last chapter, on functional equations, certain systems of bounded sets, called K-systems, are introduced, making possible the extension of the method of successive approximations to functional equations in linear topological spaces. A method of constructing K-systems for <u>any</u> locally convex linear topological space is given. Existence theorems are proved for functional equations of the form $y = f(y)$ and for first order differential equations. ~~In the last section of the chapter completely continuous linear transformations are defined for linear topological spaces, and a beginning is made on a generalization of the Riesz theory of completely continuous linear transformations.~~

I wish to take this opportunity to express my thanks to Professor A. D. Michal for his inspiration and guidance.

Chapter 1

Linear Topological Spaces

1. The calculus of sets in a linear space

Definition 1.1 A system consisting of an abstract set T of elements $x, y, \cdots$, the set R of real numbers $\alpha, \beta, \cdots$ and operations $x+y$ and αx is called a linear space if the following postulates* are satisfied:

(i) $x+y \in T$

(ii) $\alpha x \in T$

(iii) $x+y = y+x$

(iv) $x+(y+z) = (x+y)+z$

(v) $x+y = x+z \;.\supset.\; y=z$

(vi) $\alpha(x+y) = \alpha x + \alpha y$

(vii) $(\alpha+\beta)x = \alpha x + \beta x$

(viii) $\alpha(\beta x) = (\alpha\beta)x$

(ix) $1x = x$

It is well known that these postulates imply (1) the existence of a unit θ and an inverse $-x = (-1)x$ for the operation +, (2) that $\alpha x = \beta x$ implies $\alpha = \beta$ for $x \neq \theta$ and (3) that $\alpha x = \alpha y$ implies $x = y$ for $\alpha \neq 0$. A linear space T will be called finite dimensional if every element is uniquely expressible in the form $\alpha_1 x_1 + \cdots + \alpha_\mu x_\mu$ for a fixed finite set $(x_1, \cdots, x_\mu) \subset T$, and infinite dimensional if it is not finite dimensional. The theory developed in this thesis applies equally well to

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* Cf. Banach (1.) p. 26. The sign = denotes logical identity. For an independent set of postulates for a linear space in which = is an undefined relation see Taylor and Highberg, Comptes Rendus of the Warsaw Society of Sciences, vol. 28 (1935), pp. 136-142.

finite or infinite dimensional linear spaces.

By a function on a set P to a set Q we mean a rule which assigns to each element of P one or more elements of Q. For single valued functions, where to each $x \in P$ there is ordered just one element of Q, we use a small letter such as $f(x)$ for this element. In the case of functions which are not necessarily single valued, the set of values corresponding to x will be denoted by a capital letter, viz. $F(x)$. In all cases the set P will be called the domain, and the set of correspondents of the elements of P the range of the function. If M is any subset of the domain P of a function F, the set of values $\{F(x)\}, x \in M$ will be denoted by $F(M)$.

Let S, S_1, S_2 be subsets of T, and let A be a set of real numbers. In accordance with the last paragraph we denote by $x + S$ the set of all elements $x + y$ where $y \in S$ (read y is an element of S), by $S_1 + S_2$ the set of all elements $x + y$ where $x \in S_1$ and $y \in S_2$, by αS the set of all αx with $x \in S$ and by AS the set of all αx with $\alpha \in A$ and $x \in S$. Finally we write $Co\, S$ for the "convex hull" of S, that is the set of all finite sums $x = \sum_{i=1}^{\nu} \alpha_i x_i$ where $\alpha_i > 0$, $\sum_{i=1}^{\nu} \alpha_i = 1$, $x_i \in S$, and ν runs over all the positive integers.

Definition 1.2. A subset S of T will be called convex if $\alpha S + (1-\alpha)S = S$ for $0 < \alpha < 1$. We shall make considerable use of the calculus of sets in linear spaces. For convenience we quote the following results in this calculus from a paper* by G. Birkhoff.

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* Birkhoff (1.) pp. 358-360

(a) The following properties of linear spaces hold for the vector sums of sets:

V 1 $S_1 + S_2 = S_2 + S_1$

V 2 $S_1 + (S_2 + S_3) = (S_1 + S_2) + S_3$

V 3 $\alpha(S_1 + S_2) = \alpha S_1 + \alpha S_2$

V 4 $\alpha(\beta S) = (\alpha\beta) S$

V 5 $1 S = S$

V 6 $S + \Theta = S$.

(b) S is convex if and only if $Co\, S = S$.

(c) The operation $Co\, S$ is additive and homogeneous that is

$Co(S_1 + S_2) = Co\, S_1 + Co\, S_2$ and $Co(\alpha S) = \alpha\, Co\, S$.

(d) $Co\, S$ is the smallest convex set containing S.

Besides the above vector operations on sets we shall of course make use of the usual logical operations, and these will be distinguished from the vector operations by the use of a dot. Thus $S_1 \dot{+} S_2$ will denote the <u>union</u> of S_1 and S_2, $S_1 \cdot S_2$ the <u>intersection</u> of S_1 and S_2, $S_1 \dot{-} S_2$ the set of x's for which $x \in S_1$ and $x \bar{\in} S_2$ (read x not in S_2). The empty or null set will be written $\odot$. It will be convenient to denote the linear space and the set of all elements of the space by the same letter T. For future reference we add the following properties of convex sets. If S_1 and S_2 are convex, then so are the sets $\alpha S_1 + \beta S_2$ and $S_1 \cdot S_2$, assuming that $S_1 \cdot S_2 \neq \Theta$. This statement is obvious from properties (b) and (c) and definition 1.2. Again, if $f(x)$ with domain T_1 and range T_2 is <u>distributive</u>, i.e., if

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x, y \in T_1$ and all real α, β then the set $f(S)$ of T_2 is obviously convex whenever S is convex. We may sum up these statements by saying that the property of convexity is invariant under the operations of "linear combination", "logical multiplication" and "distributive transformation".

A linear space L which is a sub-space of a linear space T (with the same operations defined in both) will be called a <u>linear manifold</u> in T. The smallest linear manifold L containing a given set S will be called the <u>linear hull</u> of S, or the linear manifold determined by S. The linear hull of a set S consists of the set of all elements of the form $x = \sum_{i=1}^{\nu} \alpha_i x_i$, where $x_i \in S$, α_i is real, and ν runs over the positive integers.

2. <u>Linear topological spaces.</u>

<u>Definition* 1.3.</u> A system $\mathcal{J}$ consisting of a linear space T and a family of point sets U of T will be called a <u>linear topological space</u> if the following postulates are satisfied:

(I a) For each point x of T there is a U_x with $x \in U_x$.

(I b) For every pair U_x, U'_x there is a U''_x such that $U''_x \subset U_x \cdot U'_x$.

(I c) If $y \in U_x$ there is a $U_y \subset U_x$.

(I d) If $x \neq y$ there is a U_x not containing y.

(II a) Given any U_{x+y} there exist U_x and U_y such that $U_x + U_y \subset U_{x+y}$.

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* This definition of a linear topological space is equivalent to that given in Kolmogoroff (1.), in which the closure $\bar{S}$ of a set S is taken as an undefined notion. See Appendix I.

(II b) Given any $U_{\alpha x}$ there is a U_x and a real open interval I_α with mid-point α such that $I_\alpha U_x \subset U_{\alpha x}$.

The sets U_x are called <u>fundamental sets</u>, or <u>neighborhoods</u>, and postulates (I) furnish a neighborhood topology for our space. (I a), (I b) and (I c) are the three neighborhood axioms of Hausdorff and (I d) is the first or Fréchet separation axiom. The linear space T will be called the <u>basis</u> of the linear topological space $\mathcal{T}$. A given linear space T may be the basis for various linear topological spaces $\mathcal{T}, \mathcal{T}', \cdots$ corresponding to different choices of the family of fundamental sets. The following definition is a statement for linear topological spaces of the Hausdorff equivalence criterion*.

<u>Definition 1.4</u>. Two linear topological spaces $\mathcal{T}$ and $\mathcal{T}'$ with the same basis T and with families $\{U_x\}$ and $\{V_x\}$ respectively of fundamental sets will be called <u>equivalent</u> if for each point x of T and each U_x there is a $V_x \subset U_x$ and for each V_x there is a $U_x \subset V_x$.

On the basis of postulate (I a) we make the following definitions. A point x is called a <u>contact point</u> (berühungspunkt) of a subset S of T if every U_x contains a point $y \in S$. If every U_x contains a point $y \in S$ with $y \neq x$ then x is called** a <u>limit point</u> (häufungspunkt) of S. The <u>closure</u> $\bar{S}$ of S is defined as the set of all the contact

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* See Alexandroff and Hopf (1.) p. 31

** In spaces satisfying (Ia) and (Id) it is clear that this definition of a limit point is equivalent to the following " x is a limit point of S if every U_x contains an infinite number of points of S."

points of S, and a set is said to be <u>closed</u> if $\bar{S} \subset S$. An open set is defined as the complement of a closed set. These definitions apply not only to linear topological spaces, but to any space E, linear or not, in which there is a neighborhood system $\{U\}$ satisfying merely (I a). The following theorem gives us an equivalent definition of an open set in any such* "umgebungsraum" E.

<u>Theorem 1.1. If E is any space with a neighborhood topology satisfying (I a), then a non-empty set S of E is open if and only if $x \in S$ implies that there exists a $U_x \subset S$.</u>

<u>Proof:</u> Let S be open and let $x \in S$. Then if there were no $U_x \subset S$ every U_x would contain an element of the complement $E \dot{-} S$, and $E \dot{-} S$ would not be closed. Conversely, let $x \in S$ imply $U_x \subset S$. Then no $x \in S$ can be a limit point of $E \dot{-} S$, since for any such x there is a neighborhood U_x containing no point of $E \dot{-} S$.

<u>Corollary: If E has a neighborhood topology satisfying both (I a) and (I c), then every neighborhood U_x is an open set. In particular every fundamental set of a linear topological space is open.</u>

From this corollary and the definition of open sets it follows at once that the three following characteristic properties of open sets hold for any umgebungsraum satisfying (I a), (I b) and (I c), and hence are valid for any linear topological space.

(i) The null set and the set of all points of the space are open.

(ii) The union of any number of open sets is open.

(iii) The intersection of two open sets is open.

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* These spaces are discussed in Alexandroff and Hopf (1.) p. 30.

We now extend the concept of "neighborhood" by the following definition (cf. Hausdorff (1.) p. 228).

Definition 1.5. A family of sets V will be called a complete neighborhood system for the space if (1.) each member of the family is open (2) for any point x and any fundamental set U_x there is a member V of the family with $x \in V \subset U_x$. A family of sets V will be called a complete neighborhood system of the point x_0 if (1.)' every V is open and contains x_0 (2.)' for every U_{x_0} there is a $V \subset U_{x_0}$.

Using the above corollary it is easy to show that any complete neighborhood system satisfies postulates (I a) - (I d), and that if any such system be taken as the family of fundamental sets of postulates (I) the resulting linear topological space is equivalent to the original space in accordance with definition 1.4. The family of all open sets of the space constitutes a complete neighborhood system for the space which is in fact the largest such system.

Now consider some consequences of postulates (II). If $f(x)$ is a function with arguments in a linear topological space $\mathcal{J}$ and values in any other linear topological space $\mathcal{J}'$, we say that $f(x)$ is continuous* at the point x if for every $U'_{f(x)}$ there is a U_x such that $f(U_x) \subset U'_{f(x)}$. A similar definition applies to functions of more than one variable. We now see that the postulates (IIa), (IIb) state merely that the functions $x + y$ and αx are continuous. Since in a linear space $-x = (-1)x$ it follows that the "inverse" $-x$ of x is a continuous function of x.

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* For other equivalent definitions of continuous functions in topological spaces see Alexandroff and Hopf (1.) pp. 52-53.

Using postulates (II) and the criterion for openess given by theorem 1.1 we shall obtain

Theorem 1.2. Let G , G_1 , G_2 be non-empty open sets of a linear topological space $\mathcal{J}$, A a non-empty open set of real numbers, β a real number $\neq 0$, and y a point of $\mathcal{J}$. Then βG , $G_1 \pm G_2$, $y \pm G$ are all open sets. Moreover if either of the conditions $0 \bar{\in} A$ or $\theta \in G$ holds the set AG is open.

Proof: Since G is open, $x \in G$ implies the existence of a $U_x \subset G$. By postulate (II b) there is a V_x with $1/\beta\, V_x \subset U_x \subset G$. Hence $V_x \subset \beta G$ and βG is open. By postulate (II a) and the fact just proved, with $\beta = -1$, it follows that given $U_x \subset G_1$ and $y \in G_2$ there exists a U_{x+y} and a U_y such that $U_{x+y} - U_y \subset U_x$. Now take $V_y \subset U_y \cdot G_2$. Then $U_{x+y} - V_y \subset U_x \subset G_1$ and hence $U_{x+y} \subset U_x + U_y \subset G_1 + G_2$ so that $G_1 + G_2$ is open. But $-G_2$ is open, so $G_1 - G_2$ is also open. Similarly $y \pm G$ is open. To prove the last statement of the theorem, first suppose $0 \bar{\in} A$. Then for any $\alpha \in A$ and any $x \in G$ there is a real interval I_α with center α such that $I_\alpha \subset A$, and a $U_x \subset G$. By hypothesis $\alpha \neq 0$, so there is a real interval $I'_{1/\alpha}$ with $1/I'_{1/\alpha} \subset I_\alpha \subset A$. By postulate (II b) there is a real interval $I''_{1/\alpha}$ and a neighborhood $U_{\alpha x}$ such that $I''_{1/\alpha} U_{\alpha x} \subset U_x$. If $I_{1/\alpha} \subset I'_{1/\alpha} \cdot I''_{1/\alpha}$ we have

$$U_{\alpha x} \subset 1/I_{1/\alpha}\, U_x \subset I_\alpha\, U_x \subset AG,$$

whence AG is open. On the other hand, without requiring $0 \bar{\in} A$, suppose that $\theta \in G$. Let $\alpha \in A$, $x \in G$. If $\alpha \neq 0$ the above proof applies and yields $U_{\alpha x} \subset AG$. If $\alpha = 0$, we must show that there exists a $U_\theta \subset AG$. Take $\beta \in A$, $\beta \neq 0$. Since $\theta \in G$, there is a $V_\theta \subset G$. But by postulate (II b), there is a U_θ with $1/\beta\, U_\theta \subset V_\theta \subset G$ or $U_\theta \subset AG$.

From this theorem and postulate (II a) it follows that the family $\{x + V\}$ where x runs over the points of $\mathcal{J}$ and V runs over any complete neighborhood system of the origin (or any other point) is a complete neighborhood system for the space $\mathcal{J}$. This fact, familiar from the theory of topological groups, is very useful in arguments involving limits.

The postulates (I) taken by themselves are weaker than the postulates for a Hausdorff space because (I d) is weaker than the second or Hausdorff separation axiom. However, postulates (I) and (II) taken together do imply not only that $\mathcal{J}$ is a Hausdorff space but indeed a regular Hausdorff space, that is, the second and third separation axioms* are both satisfied. The following theorem stating this result is proved in Kolmogoroff (1.), and is shown to hold also for topological groups.

<u>Theorem 1.3.</u> <u>A linear topological space $\mathcal{J}$ is a regular Hausdorff space. That is the following conditions are satisfied:</u>

(1.) $x \neq y$ <u>implies that there exist</u> U_x, U_y <u>with</u> $U_x \cdot U_y = 0$

(2.) <u>For any</u> U_x <u>there is a neighborhood</u> U'_x <u>whose closure is contained in</u> U_x.

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* cf. for example Alexandroff and Hopf (1.) p. 67.

Thus we may characterize a linear topological space as being (a) a regular Hausdorff space (b) a linear space in which the operations of vector addition and scalar multiplication are continuous.

The linear topological spaces which have been the most studied are the linear normed spaces, where by a linear normed space we mean a linear space T in which there is defined a function $\|x\|$ on T to the real numbers satisfying

(i) $\|x\| \geq 0$; $\|x\| = 0$ implies $x = \theta$.

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

It is easy to see that the postulates of definition 1.3 will be satisfied by any linear normed space if we take the U_x's to be the spheres $\|y - x\| < 1/\nu$, $\nu = 1, 2, \cdots$. The relation between linear topological spaces and linear normed spaces will be discussed in section 5. In section 6 a number of examples are given of linear topological spaces which are definitely not normed spaces.

3. *The postulate of convexity; the relation between the spaces $\mathcal{T}$ and Neumann's spaces L.*

Definition 1.6.* A linear topological space $\mathcal{T}$ will be said to be *locally convex* if the following postulate is satisfied:

(III) *If U is any neighborhood of the origin θ there is a set V containing θ and contained in U such that*

(1.) $x \in V$ *implies the existence of a U_x with $U_x \subset V$* (*that is V is open*).

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* Locally convex linear topological spaces are defined in Tychonoff (1.) p. 768.

(2.) $0 < \alpha < 1$ <u>implies that</u> $\alpha V + (1-\alpha)V = V$ (<u>that is</u> V <u>is convex</u>).

An equivalent statement of (III) is that there exists a complete neighborhood system $\{V\}$ of the origin with each V convex. It is this postulate of convexity which often allows us to carry over useful methods of proof from linear normed spaces to linear topological spaces. This is usually because the laws

$$\alpha V + \beta V = (\alpha + \beta) V$$

and $\alpha V \subset \beta V$ for $0 < \alpha < \beta$

hold for convex sets V containing θ, but not for arbitrary sets containing θ.

We next consider Neumann's definition of a linear topological space in relation to ours. In Neumann (2) p. 4 a linear topological space is defined as any linear space L for which there is a family $\mathcal{U}$ of sets U satisfying the following postulates

(1.) If $U \in \mathcal{U}$ then $\theta \in U$.

(2.) There is a sequence $U_1, U_2, \cdots$ of sets of $\mathcal{U}$ whose intersection is the single point θ.

(3.) If $U, V \in \mathcal{U}$ there exists $W \in \mathcal{U}$ with $W \subset U \cdot V$.

(4.) If $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $-1 \leq \alpha \leq 1$ implies $\alpha V \subset U$.

(5.) If $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ with $V + V \subset U$.

(6.) If $f \in L$, $U \in \mathcal{U}$ there is an α with $f \in \alpha U$.

If in addition the postulate

(7.) If $U \in \mathfrak{U}$, then $U + U \subset 2U$

holds, he calls L a "convex" linear topological space. Neumann has shown that a space satisfying (1.) - (6.) is a regular Hausdorff space in which the operations of addition and scalar multiplication are continuous, so that any linear space L satisfying (1.) - (6.) also satisfies our postulates (I) and (II) (choose $U_x = x + U_i$ where U_i denotes the interior of U) so that our definition of a linear topological space includes that of Neumann's. We shall now show that Neumann's "convex" linear topological spaces are likewise included in the class of what we have called locally convex linear topological spaces.

<u>Theorem 1.4</u>. <u>Any linear space satisfying (1.) - (7.) is a locally convex linear topological space</u>.

<u>Proof</u>: Let L be a linear space and $\mathfrak{U}$ a system of sets of L satisfying (1.) - (7.). Then, as we have already remarked, L satisfies (I) and (II) if we set $U_x = x + U_i$. We now show that (III) is satisfied. Neumann's theorem 12, loc. cit. p. 9, states that $U \in \mathfrak{U}$ implies that the closed hull $\overline{U_i}$ of U_i is convex. Again, given any $U \in \mathfrak{U}$ there exists a $W \in \mathfrak{U}$ such that $\overline{W_i} \subset U_i$, since L is a regular Hausdorff space by theorem 1.3. Hence for any $U \in \mathfrak{U}$ there exists a $W \in \mathfrak{U}$ such that

$$Co\, W_i \subset Co\, \overline{W_i} = \overline{W_i} \subset U_i .$$

Denote $Co\, W_i$ by V. Obviously V is convex so that condition (2.) of postulate (III) is satisfied. To verify that V also satisfies condition (1.) of the postulate, let x be any point of V. Then $x = \sum_{\iota=1}^{\nu} \alpha_\iota x_\iota$, where $\alpha_\iota > 0$, $\sum_{\iota=1}^{\nu} \alpha_\iota = 1$, $x_\iota \in W_i$.

Choose $U' \in \mathcal{U}$ so that $x_\iota + U_i' \in W_i$ and then choose W' such that $Co\, W_i' \subset U_i'$. On using the obvious equality $\sum_{\iota=1}^{\nu}{}' (\alpha_\iota\, Co\, W_i') = Co\, W_i'$ we obtain

$$x + W_i' \subset \sum_{\iota=1}^{\nu}{}' \alpha_\iota x_\iota + Co\, W_i' = \sum_{\iota=1}^{\nu}{}' \alpha_\iota (x_\iota + Co\, W_i')$$

$$\subset \sum_{\iota=1}^{\nu}{}' \alpha_\iota (x_\iota + U_i') \subset Co\, W_i \subset U_i .$$

Thus condition (1.) of postulate III may be satisfied by taking $U_x = x + W_i'$, so that any linear space for which (1.) - (7.) hold also satisfies (I) - (III).

4. Sequential convergence; the postulate of completeness.

Following the usual definition for metric spaces we shall call a sequence $\{x_\iota\}$ of points of $\mathcal{J}$ convergent if there is a point x of $\mathcal{J}$ such that corresponding to every neighborhood U_x there is an integer ν with $x_\iota \in U_x$ for all $\iota > \nu$. This statement is abbreviated by writing $\lim_{\nu \to \infty} x_\nu = x$. It follows at once from the Hausdorff separation axiom (see theorem 1.3, (1.)) that a convergent sequence has a unique limit x.

In metric spaces it is well known that any limit point of a set is the limit of at least one convergent sequence out of the set. This property however does not carry over to an arbitrary linear topological space, when the first countability axiom* is not satisfied. In fact Neumann ((1.) p. 380) has given an example of a linear topological space, namely Hilbert space with its weak topology, in which there is a denum-

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* Hausdorff (1.) p. 229, axiom (9).

erable set of points having the origin as a limit point but containing no sub-sequence which converges to the origin.

This fundamental difference between topological spaces and metric spaces has recently led several authors* to the consideration of various definitions of completeness for linear topological spaces. We shall have occasion to use only the notion of sequential completeness (see Neumann, loc. cit., p. 10) which is a direct generalization of the usual notion for linear normed spaces. It appears to be a weaker condition on a linear topological space to require sequential completeness than to demand completeness in the sense of Neumann's or Birkhoff's definitions.

Definition 1.7. A linear topological space $\mathcal{J}$ will be said to be sequentially complete if the following postulate is satisfied:

(IV.) Let $x_1, x_2, \cdots$ be a sequence of points of $\mathcal{J}$ with the property that corresponding to each neighborhood U of the origin there is a positive integer $\mu = \mu(U)$ such that $\rho, \lambda > \mu$ implies $x_\rho - x_\lambda \in U$. Then there exists a point x of $\mathcal{J}$ such that corresponding to every U_x there is an integer $\nu = \nu(U_x)$ such that $\iota > \nu$ implies $x_\iota \in U_x$.

We shall call a sequence fundamental if it satisfies the condition stated in the first sentence of postulate (IV). This postulate may now be stated in the familiar form: "Every fundamental sequence is convergent."

5. Topological boundedness and compactness.

The idea of boundedness of sets would seem to be entirely foreign

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* Neumann (2), Birkhoff (3), Graves (1).

to topological spaces in which there is no metric. However in the case of linear topological spaces the operation of scalar multiplication makes it possible to generalize the notion of boundedness from linear normed spaces in a very useful way. Three superficially different definitions* of boundedness, all using scalar multiplication, have been given recently by A. Kolmogoroff, J. v. Neumann, and A. D. Michal and E. W. Paxson.

Definition 1.8 a (Kolmogoroff) A set S of a linear topological space will be said to be bounded if α_i real, $\lim_{i \to \infty} \alpha_i = 0$, $x_i \in S$ implies $\lim_{i \to \infty} \alpha_i x_i = \theta$.

Definition 1.8 b (Neumann) A set S of a linear topological space will be said to be bounded if for any neighborhood U of the origin there is a number $\alpha = \alpha(U)$ such that $\alpha S \subset U$, $\alpha \neq 0$.

Definition 1.8 c (Michal and Paxson) A set S of a linear topological space will be said to be bounded if for any $x_0 \in S$ there is a positive number $\delta = \delta(x_0)$ such that αx_0 is not in S for any α greater than δ in absolute value.

Michal and Paxson have shown that their definition of boundedness is equivalent to Neumann's. If we can demonstrate that Kolmogoroff's definition is also equivalent to Neumann's, it will follow that all three definitions fall together.

Theorem 1.5. Definitions 1.8a, 1.8b, and 1.8c define the same class of sets.

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* Kolmogoroff (1.), Neumann(2.), Michal and Paxson (1.). The definition 1.8 a was given for the case of spaces with a metric topology by Mazur and Orlicz, Studia Math. IV (1933) p. 162.

Proof: Let* $S \subset T$ be bounded according to definition 1.8 a, and assume that S is not bounded according to 1.8 b. Then for some U and each $\alpha \neq 0$ there is an $x \in S$ with $\alpha x \in T \dot{-} U$. Let $\{\alpha_\iota\}$ be a sequence of numbers which approach 0 with $1/\iota$. Then there is a sequence $\{x_\iota\}$ with $x_\iota \in S$ such that $\alpha_\iota x_\iota \in T \dot{-} U$, so that $\lim_{\iota \to \infty} \alpha_\iota x_\iota \neq \theta$, and S is not bounded by 1.8 a. Conversely let S be bounded by 1.8 b. Let U be any neighborhood of the origin and let $\{\alpha_\iota\}$ be any real sequence converging to 0. By the continuity of αx we can choose a $\delta > 0$ and a neighborhood V of the origin such that $|\alpha| < \delta$ implies $\alpha V \subset U$. But by 1.8 b there is a real number β so small that $x_\iota \in S$ implies that $\beta x_\iota \in V$. Choose ν so large that $\left|\frac{\alpha_\iota}{\beta}\right| < \delta$ for $\iota > \nu$. Then $\iota > \nu$ implies that

$$\alpha_\iota x_\iota = \frac{\alpha_\iota}{\beta} \beta x_\iota \in \frac{\alpha_\iota}{\beta} V \subset U,$$

so that S is bounded by definition 1.8 a. This completes the proof of theorem 1.5. Besides the equivalent definitions 1.8 a - 1.8 c it will sometimes be convenient to use the following which we shall prove equivalent to the others.

Definition 1.8 d. A set S will be said to be bounded if given any neighborhood U of the origin there is an integer $\nu = \nu(U)$ such that $|\alpha| < 1/\nu$ implies $\alpha S \subset U$. This statement evidently implies the statement of definition 1.8 b. To prove the converse let S be bounded

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*As usual the letter T denotes the set of all elements of the linear topological space $\mathcal{J}$, and $T \dot{-} U$ is our notation for the complement of U.

according to Neumann's definition 1.8 b, and let U be any chosen neighborhood of θ. By the continuity of αx (postulate (II b)) there is a real interval I_0 containing 0, and a neighborhood V containing θ such that $I_0 V \subset U$. By hypothesis there exists a real β such that $\beta S \subset V$. Choose ν so large that $\alpha/\beta \in I_0$ for $|\alpha| < 1/\nu$. Then $\alpha S = (\alpha/\beta) \beta S \subset I_0 V \subset U$ for $|\alpha| < 1/\nu$. Consequently S is bounded according to 1.8 d.

In the sequel when we say that a set is bounded we shall mean according to any of the equivalent definitions 1.8 a - 1.8 d. In linear normed spaces this definition of boundedness reduces to the usual one, in which the norms of all the elements of the set are required to be less than some fixed positive number μ.

Next we consider compactness.

Definition* 1.9. A set S of a linear topological space will be called <u>compact</u> if every infinite subset of S has a limit point. If every infinite subset of S has a limit point in S we shall say that S is <u>compact in itself</u>.

It is well known that compact sets and bounded sets are identical in a finite dimensional Euclidean space. This is not the case for all linear topological spaces, since the spheres of an infinite dimensional normed linear space (for example the space of continuous functions on $[0,1]$), though bounded, are necessarily <u>not</u> compact. However we

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* Cf. for example Alexandroff and Hopf (1.) p. 84.

shall now prove the following theorem.

Theorem 1.6. Every compact set of a linear topological space is bounded.

Proof: Let $\mathcal{J}$ be a linear topological space with "basis" T, and let $S \subset T$ be compact. Assume, contrary to the theorem, that S is not bounded. Then by denying definition 1.8 a we may assert the existence of a sequence $\{x_\iota\}$, $x_\iota \in S$, a null* sequence $\{\alpha_\iota\}$ of real numbers and a neighborhood U of the origin such that for every ν there is a $\lambda > \nu$ with $\alpha_\lambda x_\lambda \in T \dot{-} U$. In other words there is a null sequence $\{\beta_\iota\}$ and a sequence $\{y_\iota\}$, $y_\iota \in S$ such that $\beta_\iota y_\iota \in T \dot{-} U$ for all ι. Now S is compact so that the sequence $\{y_\iota\}$ has a limit point x. By postulate (II b) there exists a real interval I_0 containing 0 and a neighborhood U_x with $I_0 U_x \subset U$. Since x is a limit point of $\{y_\iota\}$ there is an infinite sub-sequence $\{y_{\iota_\kappa}\}$ with $y_{\iota_\kappa} \in U_x$ for all κ. If we choose κ large enough so that $\beta_{\iota_\kappa} \in I_0$ we have $\beta_{\iota_\kappa} y_{\iota_\kappa} \in I_0 U_x \subset U$ which is a contradiction.

6. Continuous isomorphism; relations between linear topological spaces and metric spaces.

What is the relation between linear topological spaces and other linear spaces which have been more extensively studied, such as linear normed spaces, Euclidean spaces, Frechet (F) spaces? Are there simple topological conditions under which a linear topological space reduces to one of these more usual types of spaces? We shall discuss these questions in this section, making our ideas precise by using the concepts of linear transformation and continuous isomorphism.

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* We call $\{\alpha_\iota\}$ a null sequence if $\lim_{\iota \to \infty} \alpha_\iota = 0$.

Let $\mathcal{T}_1$, $\mathcal{T}_2$ be linear topological spaces whose "bases" are respectively T_1 and T_2. A function or transformation (we shall use the words interchangeably) with domain T_1 and range T_2 is called <u>additive</u> if $x, y \in T_1$ implies $f(x+y) = f(x) + f(y)$, <u>homogeneous</u> if $x \in T_1$, α real implies $f(\alpha x) = \alpha f(x)$. A function which is additive and continuous is called <u>linear</u>. By using approximating sequences of rationals it follows exactly as in the case of linear normed spaces* that any linear function is homogeneous, and thus is distributive (i.e. both additive and homogeneous). The space $\mathcal{T}_1$ will be said to be <u>continuously isomorphic</u> to the space $\mathcal{T}_2$ if there is a linear transformation $y = f(x)$ with domain T_1 and range T_2 which has a unique linear inverse $x = f^{-1}(y)$. It is clear that the relation of continuous isomorphism is reflexive, symmetric and transitive, and thus is an equivalence relation for the class of linear topological spaces. This equivalence relation is a generalization of that of definition 1.4.

<u>Definition 1.10</u>. A linear topological space will be called <u>normable</u> if it is continuously isomorphic to a linear normed space.

By a <u>metric space</u> we mean a space M for which corresponding to each pair $x, y \in M$ there is a non-negative number (x, y) with the properties

(i) $(x, y) = (y, x)$

(ii) $(x, y) = 0$ implies $x = y$

(iii) $(x, y) + (y, z) \geq (x, z)$.

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* See Banach (1.) p. 36.

Every metric space is* a regular Hausdorff space if we take the spheres $(y,x) < \alpha$ as the neighborhoods U_x. The metric is said to induce a Hausdorff topology. A linear topological space whose topology is thus induced by a metric will be called a linear metric space**.

Definition 1.11. A linear topological space will be called metrizable if it is continuously isomorphic to a linear metric space.

It is evident that in a normable space it is possible to define a norm by taking $\|x\|$ as the norm of the isomorph x' of x. Moreover, since by definition the isomorphism $x' = f(x)$ is bicontinuous and biunivocal and takes the origin θ into the origin θ' it is easy to verify that the "spheres" $\|x\| < \delta$ form a complete neighborhood system for the space. Hence in the terminology of definition 1.4, the norm $\|x\|$ generates a topology equivalent to the original Hausdorff topology. Conversely if it is possible to define a norm in $\mathcal{J}$ which gives rise to a topology equivalent to the topology of $\mathcal{J}$ then $\mathcal{J}$ is normable according to definition 1.10. A similar necessary and sufficient condition for metrizability may be stated as follows: A linear topological space is metrizable if and only if it is a linear metric space under a topology equivalent to the original Hausdorff topology. The following fundamental result is proved in Kolmogoroff (1.).

Theorem 1.7. (Kolmogoroff's normability theorem) A linear topological

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* See Alexandroff and Hopf (1.) pp. 38, 68.

** Our "linear metric spaces" are less general than the "vector metric spaces" recently considered by Adams, Trans. Am. Math. Soc. 40 (1936) pp. 421-438 since we require the operations of vector addition and scalar multiplication to be continuous.

space is normable if and only if there exists a non-vacuous sub-set of the space which is convex, open and bounded.

If we call a space locally bounded if it contains a bounded open set we have the

Corollary The necessary and sufficient condition that a linear topological space be normable is that it be locally convex and locally bounded.

We next consider the effect of requiring merely local boundedness.

Theorem 1.8. If a linear topological space $\mathcal{J}$ is locally bounded then we may define a "pseudo-norm"* $|x|$ in $\mathcal{J}$ with the following properties

(a) $|x| \geqslant 0$; $|x| = 0$ implies $x = \theta$.

(b) $|\alpha x| = |\alpha|\,|x|$

(c) The spheres $|x| < 1/\nu$, $\nu = 1, 2, \cdots$ form a complete neighborhood system of the origin.

Proof: Let G be a bounded open set containing the origin, and put

$V = G \cdot (-G)$. Then V is obviously bounded, open, and $V = -V$.

Define $|x|$ as $\inf |\alpha|$, $x \in \alpha V$. Then $|x| \geqslant 0$. If $|x| = 0$, then $x \in \alpha_\iota V$ for each α_ι of a sequence $\{\alpha_\iota\}$ converging to zero, by the definition of ~~greatest lower bound~~ inferior limit. Now since V is bounded, for any neighborhood U of the origin there is a ν such that $\iota > \nu$ implies $\alpha_\iota V \subset U$ (by def. 1.8d), i.e. the sequence $\{\alpha_\iota V\}$ is a complete neighborhood system. Hence if $|x| = 0$, x is in every neighborhood of θ, and $x = \theta$. This proves (a). To prove (b), note that

$$|\alpha x| = \inf |\beta|,\ x \in \beta/\alpha\, V = |\alpha| \left(\inf |\beta/\alpha|,\ x \in \beta/\alpha\, V\right)$$
$$= |\alpha|\,|x|.$$

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* See Appendix II for a discussion of pseudo-normed spaces.

Now consider the set $|x| < 1/\nu$. This by definition is the set of all x such that $x \in \alpha V$, $|\alpha| < 1/\nu$ and this set is open by theorem 1.2. But by definition 1.8 d, for any neighborhood U of the origin there is an integer $\nu = \nu(U)$ such that $|\alpha| < 1/\nu$ implies $\alpha V \subset U$. Hence the sphere $|x| < 1/\nu$ is contained in U for large enough ν, and (c) is demonstrated.

<u>Corollary</u>: <u>If $\mathcal{J}$ is locally bounded, then a set $S \subset T$ is bounded if and only if there is a $\mu > 0$ such that $x \in S$ implies $|x| < \mu$.</u>

On the basis of this theorem and the results of the preceding section on compact sets we shall now investigate the relation between linear topological spaces and Euclidean spaces.

<u>Theorem 1.9. The necessary and sufficient condition for a linear topological space $\mathcal{J}$ to be the continuous isomorph of a finite dimensional Euclidean space is that $\mathcal{J}$ contain a non-vacuous compact open set.</u>

<u>Proof</u>: The necessity is immediate since openness and compactness are invariant under continuous isomorphism (in fact under any homeomorphism). To prove the sufficiency, let G be a compact open set containing the origin. By theorem 1.6 of section 5 G is bounded. Hence we may define a pseudo-norm $|x|$ for $\mathcal{J}$ with the properties described in theorem 1.8. Let $S \subset T$ (where T as usual denotes the basis of $\mathcal{J}$) be bounded. Then for some $\alpha \neq 0$, $\alpha S \subset G$ and therefore αS is compact. Consequently S is compact, and the space satisfies a Weierstrass-Bolzano theorem. Moreover, since the first countability axion is evidently satisfied (the countable family $|x| < 1/\nu$ being a complete neighborhood system of θ)

every limit point is the limit of a convergent sequence. Hence by our "Weierstrass-Bolzano theorem" every bounded sequence contains a convergent sub-sequence.

Following a method of proof due to F. Riesz, ~~Acta Math~~ (cf. Riesz (1.)), we shall prove by a contradiction that the space is finite dimensional.

Lemma. If a linear topological space is locally bounded and L is a closed linear manifold which is a proper subset of a linear manifold M of the space, then corresponding to every $\eta > 0$ there is an $x_0 \in M$ such that $|x_0| = 1$ and $|x - x_0| > 1 - \eta$ for all $x \in L$.

Proof: Take $x' \in M \dot{-} L$. Then since L is closed, and since the topology according to the pseudo-norm is equivalent to the original topology of the space, if follows that there is a positive number β' such that $|x - x'| \geq \beta'$ for all $x \in L$. Let β be the least upper bound of such numbers β'. (i.e. β is the minimum "distance" of $x \in M$ from x'). Given any $\delta > 0$ there is a $y' \in L$ such that $\beta \leq |x' - y'| < \beta + \delta$ Put $x_0 = \dfrac{x'-y'}{|x'-y'|}$. Then $|x_0| = 1$. Also

$$|x - x_0| = \frac{1}{|x'-y'|}\,\big|x' - y' - |x'-y'|x\big|$$

Now if $x \in L$, $z = y' + |x'-y'|x \in L$ also, so that

$$|x - x_0| = \frac{1}{|x'-y'|}\,|x' - z| \geq \frac{\beta}{|x'-y'|} > \frac{\beta}{\beta+\delta}.$$

The lemma follows from this inequality.

Now assume that the space $\mathcal{T}$ is not finite dimensional, let y_1 be any element with $|y_1| = 1$ and let L_1 be the linear manifold αy_1. By the lemma there is a $y_2 \in T$ such that $|y_2| = 1$,

$|y_1 - y_2| > 1/2$. Let L_2 be the linear manifold determined by y_1, y_2. Then there is a $y_3 \in T$ with $|y_3| = 1$, $|y_\iota - y_3| > 1/2$ for $\iota = 1, 2$. By induction we can determine y_ν such that $|y_\nu| = 1$, $|y_\iota - y_\nu| > 1/2$ for $\iota < \nu$, and since we are supposing T to be infinite dimensional, each finite dimensional linear manifold L_ν is a proper subset of T, and we can define an infinite sequence $\{y_\iota\}$ with the properties $|y_\iota| = 1$, $|y_\mu - y_\nu| > 1/2$ for $\mu \neq \nu$. Thus the sequence $\{y_\iota\}$ is bounded (see corollary to thm. 1.8) and yet contains no convergent sub-sequence, which is a contradiction.

We have now demonstrated finite dimensionality. But in Tychonoff (1.) p. 769, it is shown that every finite dimensional linear topological space is continuously isomorphic to a Euclidean space, and the proof of the theorem is complete.

Corollary. A locally compact linear topological space is finite dimensional.

We next consider the question of metrizability. An important theorem due to G. Birkhoff* states that a Hausdorff group, that is a group with a Hausdorff topology in which the group operation and its inverse are continuous, is metrizable if and only if it satisfies the first countability axion, i.e. if there is a countable complete neighborhood system of each point. Now we are interested in linear topological spaces, which are Abelian Hausdorff groups under addition, and we shall find the following corollary of Birkhoff's theorem useful. (Although we write the

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* Birkhoff (1.)

group operation +, we do not need to suppose commutativity in proving the next theorem.)

Theorem 1.10. A Hausdorff group satisfying the first countability axiom is metrizable by means of a metric (x, y) satisfying $(x-y, \theta) = (x, y)$

Proof: Birkhoff shows that by choosing a suitable countable complete neighborhood system $\{V_K\}$ of the unit element θ, and defining an "ecart"

$$\rho(x,y) = \inf_{x-y \in V_K} (1/2)^K$$

it is possible to obtain a metric for the space such that the metric topology is equivalent to the original Hausdorff topology, the metric being given by

$$(x,y) = \inf_{\substack{u_0 = x \\ u_\nu = y}} \sum_{K=1}^{\nu} \rho(u_{K-1}, u_K)$$

It is evident that $\rho(x,y) = \rho(x-y, \theta)$

Hence $\rho(x+z, y+z) = \rho(x+z-(y+z), \theta)$

$$= \rho(x+z-z-y, \theta) = \rho(x-y, \theta) = \rho(x,y).$$

Therefore $(x-y, \theta) = \inf\limits_{\substack{u_0 = x-y \\ u_\nu = \theta}} \sum\limits_{K=1}^{\nu} \rho(u_{K-1}+y, u_K+y)$

$$= \inf_{\substack{u_0 = x \\ u_\nu = y}} \sum_{K=1}^{\nu} \rho(u_{K-1}, u_K) = (x,y).$$

By combining the results of theorems 1.8 and 1.10 we obtain

Theorem 1.11. Let $\mathcal{J}$ be a linear topological space with the basis T, and suppose that $\mathcal{J}$ is locally bounded. Then it is possible to define in T

(i) A "pseudo-norm" $|x|$ with the properties

(1.) $|x| \geq 0$; $|x| = 0$ implies $x = \theta$

(2.) $|\alpha x| = |\alpha|\,|x|$

(3.) If $|x| \to \theta$, $|y| \to \theta$ then $|x+y| \to \theta$

(ii) A metric (x, y) with the properties

(a.) $(x,y) \geq 0$; $(x,y) = 0$ implies $x = y$

(b.) $(x,y) = (y,x)$

(c.) $(x,y) + (y,z) \geq (x,z)$

(d.) $(x-y, \theta) = (x,y)$.

Moreover the topologies induced by $|x|$ and (x,y) are equivalent to the original topology of $\mathcal{J}$.

Proof: If G is a bounded open set containing θ the sets $\frac{1}{\nu} G$, $\nu = 1, 2, \cdots$ form a countable complete neighborhood system of the origin by definition 1.8 d, and (ii) follows from theorem 1.10. Property (3.) of the pseudo-norm follows immediately from theorem 1.8 when we remember that the function $x + y$ is continuous at the origin.

We are now in a position to discuss the relation between linear topological spaces and an important class of spaces known as Fréchet (F) spaces.

Definition* 1.12. A space E will be called a space of type (F) if it satisfies the following postulates

(1.) E is linear

(2.) E is a complete metric space.

(3.) The metric satisfies $(x,y) = (x-y, \theta)$

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* Cf. for example Banach (1.) p. 35.

(4.) $\lim_{\nu\to\infty} \alpha_\nu = 0$ implies $\lim_{\nu\to\infty} \alpha_\nu x = \theta$ for each $x \in E$

(5.) $\lim_{\nu\to\infty} x_\nu = \theta$ implies $\lim_{\nu\to\infty} \alpha x_\nu = \theta$ for all real α.

Our main difficulty will be in showing that a space of type (F) is a linear topological space. For, while it is an immediate result of the postulates that E is a Hausdorff space (being metric) in which the operation $x+y$ is continuous in x and y simultaneously, and in which αx is continuous on the left, and also on the right (cf. Banach (1.) p. 35 for the details), it is not at all clear that αx is a continuous function of both variables simultaneously. However we shall now prove that this is indeed the case.

Theorem 1.12. A space E of type (F) is a linear topological space, that is αx is a continuous function of α and x simultaneously.

Proof*: It will be sufficient to prove continuity at $\alpha = 0$, $x = \theta$, since we have

$$(\alpha x, \alpha_0 x_0) \leq ((\alpha - \alpha_0)(x - x_0), \theta) + (\alpha_0 (x - x_0), \theta) + ((\alpha - \alpha_0) x_0, \theta).$$

We have then to demonstrate that corresponding to any positive number η there is a positive number δ such that $(\alpha x, \theta) < \eta$ for $|\alpha| < \delta$ and $(x, \theta) < \delta$. Let H_ν denote the set of points of E such that $|\alpha| < 1/\nu$ implies $(\alpha x, \theta) \leq \eta_1 < 1/2\, \eta$. Clearly each H_ν contains θ and hence is not vacuous. Also each H_ν is closed. For let $x_\iota \in H_\nu$ be a sequence tending to a limit y. Then for any chosen α satisfying $|\alpha| < 1/\nu$ we have $(\alpha x_\iota, \theta) \leq \eta_1$ for all ι.

* The method of proof was suggested to me by reading Deane Montgomery's paper, "Continuity in Topological Groups", Bull. Am. Math. Soc., vol. 42 (1936), pp. 879-882.

But since αx is continuous on the right, and since (x,y) is a continuous functional of x for $y=\theta$ we have $(\alpha g,\theta)\leq\eta_1$, that is H_ν is closed. Since αx is also continuous on the left, every point of E is contained in some H_ν. That is

$$E=\sum_{\nu=1}^{\infty} H_\nu \qquad \text{(the union of the } H_\nu\text{'s).}$$

But since the space E is a set of the second category*, there is at least one of the H's, say H_μ of the second category. Hence H_μ is everywhere dense** in an open subset G of E. But H_μ is closed, so that $G\subset H_\mu$. Let x_0 be any point of G. Then since G is open there is a $\delta_1>0$ such that $(x,x_0)<\delta_1$ implies that $x\in G\subset H_\mu$. By the definition of H_μ we have that $(x,x_0)<\delta_1$ and $|\alpha|<1/\mu$ imply $(\alpha x,\theta)<\eta_1$. Let δ be the smaller of δ_1 and $1/\mu$. Then if $(y,\theta)<\delta$ and $|\alpha|<\delta$ we have $(\alpha y,\theta)=(\alpha(y+x_0)-\alpha x_0,\theta)$
$\leq(\alpha(y+x_0),\theta)+(\alpha x_0,\theta)\leq 2\eta_1<\eta$ since obviously $(y+x_0,x_0)<\delta_1$ and $(x,x_0)<\delta_1$. This proves the theorem.

<u>Theorem 1.13.</u> <u>A necessary and sufficient condition that a sequentially complete linear topological space $\mathfrak{T}$ be continuously isomorphic to a space of type (F) is that $\mathfrak{T}$ satisfy the first countability axiom.</u>

This theorem is an immediate consequence of theorems 1.10 and 1.12.

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* A set S of a metric space M is said to be of the second category if it cannot be given as the union of a denumerable number of non-dense sets. See Banach (1.) p. 13.

** See Banach (1.) p. 13, theorem 1.

7. Realizations of linear topological spaces.

The following instances indicate something of the scope of the theory, show the independence of various properties of linear topological spaces and serve as illustrations of the preceding theorems. None of these spaces are normable.

Example 1°. For the linear space T take the set V of all infinite dimensional vectors $x = (x_1, x_2, \cdots)$ where each x_ι is a real number, $x + y = (x_1 + y_1, x_2 + y_2, \cdots)$ and $\alpha x = (\alpha x_1, \alpha x_2$ ~~for all~~$)$. As neighborhoods of x we take the sets $U(x; \nu, \delta)$ defined as the set of all y satisfying

$$|y_\iota - x_\iota| < \delta, \quad \iota = 1, 2, \cdots, \nu.$$

The class of neighborhoods of x is obtained by letting ν vary over the positive integers while δ varies independently over the positive real numbers. We call the space thus topologized $\mathcal{V}$, and we shall now show that $\mathcal{V}$ satisfies all the postulates I - IV.

(I a) Obvious.

(I b) Given $U(x; \nu, \delta)$ and $U(x; \mu, \eta)$ it is clear that $U(x; \lambda, \beta) \subset U(x; \nu, \delta) \cdot U(x; \mu, \eta)$ providing $\lambda > \mu, \lambda > \nu$, ~~$\eta < \min_{\iota \leq \nu} \delta - |y$~~ $\beta < \delta,\ \beta < \eta$.

(I c) If $y \in U(x; \nu, \delta)$ then $U(y; \nu, \eta) \subset U(x; \nu, \delta)$ providing $\eta <$ the smaller of $\delta - |y_1 - x_1|, \cdots, \delta - |y_\nu - x_\nu|$.

(I d) If $x \neq y$, then there is an integer ν with $|x_\nu - y_\nu| > 0$. Obviously $U(x; \nu, \tfrac{1}{2}|x_\nu - y_\nu|)$ does not contain y.

(II a) Obvious from the continuity of each component.

(II b) " " " " " " "

(III) (Local convexity) If $x, y \in U(\theta; \nu, \delta)$ then $|\alpha x_\iota + (1-\alpha) y_\iota| \leq \alpha |x_\iota| + (1-\alpha)|y_\iota| < \delta$ for $0 < \alpha < 1$. That is each $U(\theta; \nu, \delta)$ is convex.

(IV) (Completeness) Let $\{x^{(\lambda)}\}$ be a fundamental sequence of vectors out of $\mathcal{V}$. Then it is clear that $|x_\iota^{(\lambda)} - x_\iota^{(\rho)}|$ tends to zero for each ι as λ and ρ tend independently to infinity. But by the Cauchy necessary and sufficient conditions for convergence of a sequence of reals, there is, for each ι, an x_ι such that $|x_\iota^{(\lambda)} - x_\iota|$ tends to zero as λ becomes infinite. But since neighborhoods are defined in terms of a finite number of components, it follows that the sequence $\{x^{(\lambda)}\}$ converges to x, where $x = (x_1, x_2, \cdots)$.

Thus the space $\mathcal{V}$ is a locally convex, sequentially complete linear topological space. Moreover, since $\mathcal{V}$ obviously satisfies the first countability axion it is metrizable. (See theorem 1.10). In fact if we examine the well known* metric

$$(x, y) = \sum_{\iota=1}^{\infty} \frac{1}{2^\iota} \frac{|x_\iota - y_\iota|}{1 + |x_\iota - y_\iota|}$$

we shall find that the metric topology is equivalent to the above neighborhood topology. To prove this let a be any point in V and consider the spheres $K(a; \alpha): (x, a) < \alpha$ where $0 < \alpha < 1$ together with the neighborhoods

$$U(a; \nu, \delta): |x_\iota - a_\iota| < \delta, \quad \iota = 1, 2, \cdots, \nu, \quad \text{where } \nu = 1, 2, \cdots$$

and $0 < \delta < +\infty$.

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* Cf. for example Banach (1) p. 10. This metric for V has been used extensively by Frechet.

Given $U(a;\nu,\delta)$ put $\alpha = \frac{\delta}{2^{\nu}(1+\delta)}$. Then $(x,a) < \alpha$ implies $\frac{|x_\iota - a_\iota|}{1+|x_\iota - a_\iota|} < \frac{\delta}{1+\delta}$ or $|x_\iota - a_\iota| < \delta$, $\iota = 1,2,\cdots,\nu$. That is, $K(a;\alpha) \subset U(a;\nu,\delta)$. On the other hand, let $K(a;\alpha)$ be given. Choose ν so large that $\sum_{\iota=\nu+1}^{\infty} \frac{1}{2^\iota} < \frac{\alpha}{2}$, and choose $\delta < \frac{\alpha}{2-\alpha}$. Then if $|x_\iota - a_\iota| < \delta$ for $\iota = 1,2,\cdots,\nu$ we have

$$\sum_{\lambda=1}^{\infty} \frac{1}{2^\lambda} \frac{|x_\lambda - a_\lambda|}{1+|x_\lambda - a_\lambda|} \leq \sum_{\lambda=1}^{\nu} \frac{1}{2^\lambda}\frac{\delta}{1+\delta} + \sum_{\lambda=\nu+1}^{\infty} \frac{1}{2^\lambda}$$

$$< \frac{\delta}{1+\delta} + \frac{\alpha}{2} < \alpha .$$

That is, $U(a;\nu,\delta) \subset K(a;\alpha)$. Hence by definition 1.4 the two topologies are equivalent. With this metric $\mathcal{V}$ is a space of type (F), and thus illustrates theorem 1.13. On the other hand $\mathcal{V}$ is not normable since it is evident that no neighborhood is bounded.

<u>Example 2°</u>. For the linear space T take the space F of all real functions $f(\xi)$ of a real variable defined on $-\infty < \xi < +\infty$, with addition of functions and multiplication by reals taken in the ordinary way. ~~and equality of functions taken as equality of the values everywhere~~. Corresponding to any finite set $(\xi_1,\cdots,\xi_\nu)$ of real numbers and any $\delta > 0$ we may define a neighborhood $U(f;\xi_1,\cdots,\xi_\nu,\delta)$ of the point f as the set of "points" g for which $|g(\xi_\iota) - f(\xi_\iota)| < \delta$, $\iota = 1,2,\cdots,\nu$. We may verify by methods similar to those used for example 1° that the space $\mathcal{T}$ with basis F and topologized by means of the neighborhoods $U(f;\xi_1,\cdots\xi_\nu,\delta)$ is a locally convex, sequentially complete linear topological space.

The space $\mathcal{F}$ provides us with an example of a linear topological space which is in fact locally convex, but which is not a Neumann space L, because Neumann's postulate (2.) requiring the existence of a countable set of neighborhoods whose intersection is the origin is not satisfied. To prove this, let $\{U(\theta; \xi_1^\iota, ---, \xi_{\nu_\iota}^\iota, \delta_\iota)\}$ be any countable sequence of neighborhoods of the origin θ (θ is the identically vanishing function). Now the union of the countable family of finite sets $(\xi_1^\iota, ---, \xi_{\nu_\iota}^\iota)$ is obviously a countable set, say $(\xi_1, \xi_2, \xi_3, ---)$. But since the real numbers are uncountable, there is a real number $\xi_0 \neq \xi_\lambda$, $\lambda = 1, 2, 3, ---$. Hence the function $g(\xi)$ defined by

$$\begin{cases} g(\xi) = 0 & \text{for } \xi \neq \xi_0 \\ g(\xi_0) = 1 \end{cases}$$

is contained in each $U(\theta; \xi_1^\iota, --- \xi_{\nu_\iota}^\iota)$ but does not vanish identically, and Neumann's second postulate does not hold for the space $\mathcal{F}$. It is now obvious that the first countability axiom cannot be satisfied by $\mathcal{F}$, so that the space $\mathcal{F}$, unlike the space $\mathcal{V}$, is not metrizable.

Example 3° (Tychonoff) As a basis take the linear space $H_{1/2}$ of all infinite dimensional vectors $x = (x_1, x_2, ---)$ for which the series $\sum_{\iota=1}^{\infty} |x_\iota|^{1/2}$ converges, and put $|x| = \left(\sum_{\iota=1}^{\infty} |x_\iota|^{1/2}\right)^2$
Now define a neighborhood $U(x;\alpha)$ as the set of points y for which

(1.) $|y - x| < \alpha$

(2.) There is a positive $\delta = \delta(y)$ such that $|z - y| < \delta$ implies $|z - x| < \alpha$.

Clearly, for all $\alpha > 0$, $x \in U(x;\alpha)$. For the proof that this space is a linear topological space see Appendix II, where the general class of "pseudo-normed" spaces is discussed. This example is due to Tychonoff (cf. Tychonoff (1.)) who has shown that the space is not locally convex. Thus an independence example is provided for the convexity postulate (III). In addition, this example taken together with say, example 1°, shows the independence of Kolmogoroff's two conditions for normability, namely local convexity and local boundedness. For it is clear from definition 1.8 a that every $U(x;\alpha)$ is bounded.

Example 4° (Neumann) As a basis take the real Hilbert space H_2 that is the set of all infinite dimensional vectors $f = (x_0, x_1, \dots)$ for which the series $\sum_{\iota=0}^{\infty} |x_\iota|^2$ converges. As is customary we put $(f,g) = \sum_{\iota=0}^{\infty} x_\iota y_\iota$ where $g = (y_0, y_1, \dots)$. Define neighborhoods $U(f_0; \varphi_1, \dots, \varphi_\nu, \delta)$ of f_0 as the sets of points g for which $|(g, \varphi_\lambda)| < \delta$ for $\lambda = 1, 2, \dots, \nu$. The set of neighborhoods of f_0 is obtained by letting $\varphi_1, \dots, \varphi_\nu$ vary over all finite sets of vectors in H_2 and letting δ range from 0 to $+\infty$. In Neumann (1.) and (2.) it is shown that this space $\mathcal{H}_w$ with the basis H_2 and the above "weak" topology is a locally convex, sequentially complete linear topological space which is not metrizable, and hence not normable. We may also prove that $\mathcal{H}_w$ is not normable by showing that no neighborhood is bounded, as follows. Given any finite set $\varphi_1, \dots, \varphi_\nu$ of vectors, there is a vector $\varphi_0 \neq \theta$ orthogonal to each φ_ι, that is $(\varphi_0, \varphi_\iota) = 0$ for $\iota = 1, 2, \dots \nu$. Put $g_\alpha = \alpha\varphi_0 + f_0$. Then $g_\alpha \in U(f_0; \varphi_1, \dots, \varphi_\nu, \delta)$ for any chosen δ, and for all real α, since $(g_\alpha - f_0, \varphi_\iota) = \alpha(\varphi_0, \varphi_\iota) = 0$ Consider the neighborhood $U(\theta; \varphi_0, 1)$. Then no matter what $\beta \neq 0$ we

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* An alternate proof will be given in the discussion of example 5°.

choose, there is a g_α such that $\beta g_\alpha \bar{\epsilon} U(\theta; \varphi_0, 1)$ as is evident from the equality $(\beta g_\alpha, \varphi_0) = \beta\alpha(\varphi_0, \varphi_0) + \beta(f_0, \varphi_0)$ and the fact that $(\varphi_0, \varphi_0) \neq 0$. Hence by definition 1.8 b, $U(f_0; \varphi_1, \cdots, \varphi_\nu, \delta)$ is an unbounded set.

<u>Example 5°</u> (Normed space with weak topology)

Let $\mathcal{E}$ be a linear normed space with basis E. Besides the ordinary "strong" topology furnished by the norm (cf. last paragraph of #1) we can define a "weak" topology for the space in terms of the linear functionals on E.

Denote by E_1 the set of all linear functions $f(x)$ on E to the real numbers. Corresponding to any finite set $(f_1, f_2, \cdots, f_\nu)$ of elements of E_1, any $\delta > 0$, and any $x \in E$ define $U(x; f_1, \cdots, f_\nu, \delta)$ as the set

$$\{y\}, \quad |f_\iota(y-x)| < \delta, \quad \iota = 1, 2, \cdots, \nu.$$

Then we can easily show that the space $\mathcal{E}_w$ with basis E and with the topology furnished by the neighborhoods $U(x; f_1, \cdots, f_\nu, \delta)$ is a locally convex linear topological space.

(I a) Obvious.

(I b) Given $U_x = U(x; f_1, \cdots, f_\nu, \delta)$, $U'_x = U(x; g_1, \cdots, g_\mu, \delta')$

Take $\delta'' < \min \delta, \delta'$. Then if $U''_x = U(x; f_1, \cdots, f_\nu, g_1, \cdots, g_\mu, \delta'')$

we have $U''_x \subset U_x \cdot U'_x$.

(I c) If $y \in U(x; f_1, \cdots, f_\nu, \delta)$, i.e. $|f_\iota(y-x)| < \delta$

for $\iota = 1, \cdots, \nu$, put $\delta_y = \min\limits_{1 \le \iota \le \nu} (\delta - |f_\iota(y-x)|)$

Then for all z satisfying $|f_\iota(z-y)| < \delta_y$

we have, since ~~for~~ each f_ι is linear, that

$$|f_\iota(z-x)| \leq |f_\iota(z-y)| + |f_\iota(y-x)|$$
$$< \delta_y + |f_\iota(y-x)| < \delta$$

That is $U(y; f_1, \cdots, f_\nu, \delta_y) \subset U(x; f_1, \cdots, f_\nu, \delta)$.

(I d) Suppose $x \neq y$, so that $\|y-x\| > 0$. By theorem 3, p. 55 of Banach (1.) there exists a linear functional $f_1(z)$ on E such that $f_1(y-x) = \|y-x\|$. Clearly $y \bar{\epsilon} U(x; f_1, \|y-x\|)$.

(II a) Given $U(x+y; f_1, \cdots, f_\nu, \delta)$, note that if $|f_\iota(x'-x)| < \delta/2$ and $|f_\iota(y'-y)| < \delta/2$ for $\iota = 1, \cdots, \nu$ then

$$|f_\iota(x'+y'-x-y)| = |f_\iota(x'-x) + f_\iota(y'-y)| < \delta/2 + \delta/2 = \delta$$

That is $U(x; f_1, \cdots, f_\nu, \delta/2) + U(y; f_1, \cdots, f_\nu, \delta/2) \subset U(x+y; f_1, \cdots f_\nu, \delta)$

(II b) Given $U(\alpha x; f_1, \cdots, f_\nu, \delta)$, choose positive numbers δ_1 and δ_2 so as to satisfy the inequalities,

$$\begin{cases} \delta_2(\delta_1 + |f_\iota(x)|) < \delta/2, & \iota = 1, 2, \cdots, \nu \\ |\alpha| \delta_1 < \delta/2 \end{cases}$$

Then if I_α denote the real open interval $|\tau - \alpha| < \delta_2$, we have for all $\tau \in I_\alpha$ and all y satisfying $|f_\iota(y-x)| < \delta_1$ for $\iota = 1, 2, \cdots, \nu$ that

$$|f_\iota(\tau y - \alpha x)| = |f_\iota((\tau-\alpha)y + \alpha(y-x))|$$
$$\leq |\tau - \alpha| \, |f_\iota(y)| + |\alpha| \, |f_\iota(y-x)|$$
$$< \delta_2(\delta_1 + |f_\iota(x)|) + |\alpha| \delta_1$$
$$< \delta/2 + \delta/2 = \delta.$$

Or, in the notation of the calculus of sets,

$$I_\alpha\, U(x; f_1, \cdots, f_\nu, \delta_1) \subset U(\alpha x; f_1, \cdots, f_\nu, \delta).$$

Chapter 2

Integrals in a Linear Topological Space

1. Introduction

A considerable number of authors, starting with Radon (1913) and Frechet* (1915), have studied integrals for functions of which either the domain or the range are suitably restricted abstract spaces. In 1935 G. Birkhoff (Cf. Birkhoff (2)) developed a very general integral of the Lebesgue type for functions with values in a Banach space, basing his work on Frechet's integral for real valued functions on an abstract "measure" domain. In defining his integral Birkhoff does not need to suppose at any time that the domain over which the integration is to extend is of finite measure, or that the function to be integrated is bounded on this domain.

The first four sections of this chapter are devoted to an extension of Birkhoff's integral to functions with values in a linear topological space, following his suggestion of (Birkhoff (2.), p.377) that his integral might be extended to linear spaces** more general than Banach spaces. We continually make use of (1.) the idea of unconditional convergence of series, due to Orlicz in the case of linear normed spaces, (2.) the calculus of convex sets, (3) the properties of convex sets.

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* Cf. Radon (1.), Frechet (1.)

** After I had obtained the results of this chapter, Birkhoff (3) was published, in which there is contained a definition of a "Lebesgue" integral for linear topological spaces. This definition does not coincide with mine.

Besides the properties of convex sets given in Chapter 1, #1 we shall need that provided by the following

<u>Theorem</u> <u>If S is convex so is its closure.</u>

<u>Proof</u>: For $x, y \in \bar{S}$ we must show that $z = \alpha x + (1-\alpha) y \in \bar{S}$ for $0 < \alpha < 1$. Given any U_z, by the continuity of vector addition and scalar multiplication there exist U_x, U_y such that $\alpha U_x + (1-\alpha) U_y \subset U_z$. Since $x, y \in \bar{S}$ there exist $x', y' \in S$ such that $x' \in U_x$, $y' \in U_y$. Therefore $z' = \alpha x' + (1-\alpha) y' \in U_z$. But since S is convex, $z' \in S$ and since U_z is arbitrary, $z \in \bar{S}$.

From this theorem it is evident that $\overline{Co\, S}$ is the smallest closed convex set containing S. In this chapter it is convenient to denote a linear topological space and its "basis" by the same letter as we shall have no occasion to consider spaces with the same basis but with different topologies. <u>Throughout this chapter T will stand for a locally convex, sequentially complete linear topological space.</u> $U, V, W, \ldots$ will denote convex neighborhoods of the origin θ.

2. <u>The domain of the independent variable.</u>

We shall be concerned with functions $F(t)$ on a space* E to T, where E satisfies the following postulates:

M 1. There exists a family $\mathfrak{D}$ of "measurable" point sets Δ of E, with $E \in \mathfrak{D}$.

M 2. The logical sum of a finite or denumerable number of sets $\Delta \in \mathfrak{D}$ is a member of $\mathfrak{D}$.

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* These postulates are a modification of those used by Birkhoff, in which the family $\mathfrak{D}$, is not explicitly mentioned.

M 3. If $\Delta \in \mathfrak{D}$ so is its complement $E \dot{-} \Delta$.

M 4. There exists a sub-family $\mathfrak{D}_1$ of $\mathfrak{D}$ such that to every Δ of $\mathfrak{D}_1$ there corresponds a non-negative real number $m(\Delta)$, called the measure of Δ.

M 5. If $\Delta = \dot{\sum}_1 \Delta_i$ is the finite or denumerable logical sum of disjoint sets Δ_i of $\mathfrak{D}_1$ and if the series $\sum_1 m(\Delta_i)$ converges, then $\Delta \in \mathfrak{D}_1$ and $m(\Delta) = \sum_1 m(\Delta_i)$.

M 6. If $\Delta \in \mathfrak{D}_1$, $\Delta' \in \mathfrak{D}$, $\Delta' \subset \Delta$ then $\Delta' \in \mathfrak{D}_1$.

M 7. Every set Δ of $\mathfrak{D}$ can be given as a finite or denumerable logical sum of sets Δ_i of $\mathfrak{D}_1$.

The consistency of the postulates $M1 - M7$ is shown by interpreting E to be the set of all real numbers, $\mathfrak{D}$ to be the set of all Lebesgue measurable linear point sets, including sets of infinite measure, and $\mathfrak{D}_1$ to be the set of all linear point sets of finite measure. From M 2 and M 3 it follows by taking complements that the logical product of a finite or denumerable number of sets $\Delta_i \in \mathfrak{D}$ is a member of $\mathfrak{D}$. If $E' \neq 0$ is a member of $\mathfrak{D}$, then E' can be made to satisfy postulates M 1 - M 7 by a process of relativisation. Simply put

$$\mathfrak{D}' = \{E' \cdot \Delta\}, \quad \Delta \in \mathfrak{D}$$

(read the set of all $E' \cdot \Delta$'s for which $\Delta \in \mathfrak{D}$)

and

$$\mathfrak{D}_1' = \{E' \cdot \Delta_1\}, \quad \Delta_1 \in \mathfrak{D}_1$$

and M 1 - M 7 are evidently satisfied by E'.

Since the logical product and the complement of sets of $\mathfrak{D}$ are sets of $\mathfrak{D}$, it follows that the logical difference

$$\Delta \dot{-} \Delta' = \Delta \cdot (E \dot{-} \Delta')$$

of Δ and Δ' is in $\mathfrak{D}$ for $\Delta, \Delta' \in \mathfrak{D}$. Now, from M 7 we have for $\Delta \in \mathfrak{D}$ that $\Delta = \dot{\sum}_1^\infty \Delta_\iota$, where $\Delta_\iota \in \mathfrak{D}_1$. Hence $\Delta = \dot{\sum}_1^\infty (\Delta_\iota \dot{-} (\dot{\sum}_{\lambda \neq \iota}^\infty \Delta_\lambda)) = \dot{\sum}_1^\infty \Delta'_\iota$, where the sets $\Delta'_\iota = \Delta_\iota \dot{-} \dot{\sum}_{\lambda \neq \iota}^\infty \Delta_\lambda$ are disjoint, and belong to $\mathfrak{D}_1$ by M 2 and M 6. If $\Delta \neq \odot$ then we can reject all the null sets Δ'_ι and obtain Δ as the sum of a denumerable number of non-vacuous disjoint sets $\Delta'_\iota \in \mathfrak{D}_1$. This result is fundamental for the theory of integrals in #4.

3. Unconditional convergence of series in T.

A series $\sum_1^\infty x_\iota = x_1 + x_2 + x_3 + \cdots$, $x_\iota \in \mathsf{T}$ will be called unconditionally convergent to an element $x \in \mathsf{T}$ if corresponding to each U there is an integer $\nu > 0$ such that $s - x \in U$, where s is any finite sum $\sum_\varsigma^1 x_{\lambda_\varsigma}$ taken from the series and including $x_1, x_2, \cdots, x_\nu$. In general it will be convenient to denote infinite series, and also their sums, when convergent, by $\sum_1^\infty x_\iota$, and finite sums, each of whose terms is selected from an infinite series by $\sum_\varsigma^1 x_{\iota_\varsigma}$.

Theorem 2.1. A necessary and sufficient condition that a series $\sum_1^\infty x_\iota$ be unconditionally convergent is that corresponding to each U there is a $\nu = \nu(U)$ such that for all finite sums in which $\overset{\iota_\rho}{\mu_\varsigma} > \nu$ we have $\sum_\varsigma^1 x_{\iota_\varsigma} \in U$.

Proof: To prove suffiency, let the condition of the theorem be satisfied. By sequential completeness the series converges to an element x . For U arbitrary take $\bar{V} - \bar{V} \subset U$ and there will exist $\nu > 0$ with $\sum_\varsigma^1 x_{\iota_\varsigma} \in V$ for $\iota_\varsigma > \nu$. If s is an arbitrary finite sum including $x_1, \cdots, x_\nu$, let $\sum_\varsigma^1 x_{\mu_\varsigma}$ be the sum including all elements not in s , but such that μ_ς is less than the greatest index in s . Clearly

$$s + \sum_\varsigma^1 x_{\mu_\varsigma} - x \in \bar{V}$$

so that $s - x \in \bar{V} - \bar{V} \subset U.$

The necessity follows in a similar way.

A countable set $\{\mathfrak{X}_\iota\}$ of point sets out of T is called unconditionally summable to the set $\mathfrak{X}$ if every series $\sum^1 x_\iota$, $x_\iota \in \mathfrak{X}_\iota$ is unconditionally convergent, and if $\mathfrak{X}$ is the set of sums x of all these series. In this case we write $\sum^1 \mathfrak{X}_\iota = \mathfrak{X}$.

Theorem 2.2. The necessary and sufficient condition that $\{\mathfrak{X}_\iota\}$ be unconditionally summable is that corresponding to every U there is a ν_0 such that ~~for all~~ $\mu_\varsigma > \nu_0$ implies $\sum_\varsigma^1 \mathfrak{X}_{\mu_\varsigma} \subset U.$

Proof: Sufficiency follows at once from theorem 2.1. To prove necessity assume that $\{\mathfrak{X}_\iota\}$ is unconditionally summable but that the condition of the theorem does not hold. Then for some U and any ν_0 , there is a set of elements $x_{\mu_\varsigma} \in \mathfrak{X}_{\mu_\varsigma}$ such that $\mu_\varsigma > \nu_0$ and $\sum^1 x_{\mu_\varsigma} \subset S \dot{-} U$. This however implies that there is a series $\sum^1 x_\mu$, $x_\mu \in \mathfrak{X}_\mu$ which is not unconditionally convergent. This contradiction proves the theorem. The following theorems are fundamental to our theory of integration.

<u>Theorem 2.3</u> <u>If the sequence of sets</u> $\{X_\iota\}$ <u>is unconditionally summable then</u> $\{\overline{Co X_\iota}\}$ <u>is also unconditionally summable and</u>

$$\sum{}' \overline{Co X_\iota} \subset \overline{Co \sum{}' X_\iota} .$$

<u>Proof</u>: Let U be any chosen neighborhood of θ. Since T is regular and locally convex, there exists a $V \subset U$ such that $\overline{V} = \overline{Co V} \subset U$. By hypothesis there exists a ν_0 such that $\iota_\varsigma > \nu_0$ implies $\sum_\varsigma{}' X_{\iota_\varsigma} \subset V$. Hence

$$\sum_\varsigma{}' \overline{Co X_{\iota_\varsigma}} \subset \overline{\sum_\varsigma{}' Co X_{\iota_\varsigma}} = \overline{Co \sum{}' X_{\iota_\varsigma}} \subset U$$

for any finite set of ι_ς's , $\iota_\varsigma > \nu_0$, and the sequence $\{\overline{Co X_\iota}\}$ is unconditionally summable.

Since $\sum_{\iota=1}^{\nu}{}' \overline{Co X_\iota} \subset \overline{Co \sum_{\iota=1}^{\nu}{}' X_\iota}$

for all integers $\nu > 0$, it follows that in the limit

$$\sum{}' \overline{Co X_\iota} \subset \overline{Co \sum{}' X_\iota} .$$

Unconditionally convergent double series may be defined by requiring that every arrangement of the double series as a single series be unconditionally convergent. From the corresponding facts for single series it is clear that a double series $\sum{}' x_{\lambda\mu}$ is unconditionally convergent to an element x if corresponding to every U there is a $\nu(U)$ such that for any finite sum s out of the rectangular array such that s ~~which~~ contains $\sum_{\lambda,\mu=1}^{\nu}{}' x_{\lambda\mu}$ we have $s - x \in U$. It follows as in theorem 2.1 that $\sum{}' x_{\lambda\mu}$ is unconditionally convergent if and only if for every U there is a $\nu(U)$ such that every finite sum $\sum_{\varsigma,\sigma}{}' x_{\lambda_\varsigma, \mu_\sigma}$ where <u>either</u> all λ_ς or all μ_σ are greater than $\nu(U)$. Since the space is regular it may easily be shown that in case $\sum x_{\lambda\mu}$ is unconditionally convergent to x, then the "sum by rows" and the "sum by columns" both exist equal to x.

Theorem 2.4 If $\mu_{\iota\lambda} > 0$, $\sum_{\lambda=1}^{\infty} \mu_{\iota\lambda} = \mu_\iota$, $\sum_{\iota=1}^{\infty} \mu_{\iota\lambda} = \mu'_\lambda$, $X_{\iota\lambda} \subset X_\iota$, $X_{\iota\lambda} \subset X'_\lambda$, $\sum_{\iota=1}^{\infty} \mu_\iota X_\iota = X$, $\sum_{\lambda=1}^{\infty} \mu'_\lambda X'_\lambda = X'$; then $\{\mu_{\iota\lambda} X_{\iota\lambda}\}$ is unconditionally summable and

$$\sum_{\iota,\lambda=1}^{\infty} \mu_{\iota\lambda} X_{\iota\lambda} \subset \overline{Co\, X}$$

$$\sum_{\iota,\lambda=1}^{\infty} \mu_{\iota\lambda} X_{\iota\lambda} \subset \overline{Co\, X'}$$

Proof: Denote by $X_\iota \dotplus \theta$ the logical sum of the set X_ι and the set consisting of the zero element θ. Then since $\sum_{\lambda=1}^{\nu} \mu_{\iota\lambda} < \mu_\iota$ we have

$$\sum_{\rho} \mu_{\iota\lambda_\rho} X_{\iota\lambda_\rho} \subset Co\, \mu_\iota (X_\iota \dotplus \theta)$$

the sum being extended over any finite set of λ_ρ's. Hence

(2.1) $$\sum_{\sigma,\rho} \mu_{\iota_\sigma,\lambda_\rho} X_{\iota_\sigma,\lambda_\rho} \subset \sum_{\sigma} Co\, \mu_{\iota_\sigma} (X_{\iota_\sigma} \dotplus \theta)$$

and similarly

(2.2) $$\sum_{\sigma,\rho} \mu_{\iota_\sigma,\lambda_\rho} X_{\iota_\sigma,\lambda_\rho} \subset \sum_{\rho} Co\, \mu'_{\lambda_\rho} (X'_{\lambda_\rho} \dotplus \theta)$$

Now $\{\mu_\iota X_\iota\}$ and $\{\mu'_\lambda X'_\lambda\}$ are unconditionally summable by hypothesis. Hence $\{\mu_\iota X_\iota \dotplus \theta\}$, $\{\mu'_\lambda X'_\lambda \dotplus \theta\}$ are also unconditionally summable, by considering unconditional remainders, and by theorem 2.3

$\{Co\,(\mu_\iota X_\iota \dotplus \theta)\}$ and $\{Co\,(\mu'_\lambda X'_\lambda \dotplus \theta)\}$

are unconditionally summable. From (2.1) and (2.2) it follows that $\{\mu_{\iota\lambda} X_{\iota\lambda}\}$ is unconditionally summable.

Now

$$\frac{\sum_{\lambda=1}^{\nu} \mu_{\iota\lambda} X_{\iota\lambda}}{\sum_{\lambda=1}^{\nu} \mu_{\iota\lambda}} \subset Co\, X_\iota$$

by the definition of $Co\, X_\iota$. Since the "sum by rows" of $\{\mu_{\iota\lambda} X_{\iota\lambda}\}$ exists equal to $\sum_{\iota,\lambda=1}^{\infty} \mu_{\iota\lambda} X_{\iota\lambda}$ we have in the limit

$$\frac{1}{\mu_\iota} \sum_{\lambda=1}^{\infty} \mu_{\iota\lambda} X_{\iota\lambda} \subset \overline{Co\, X_\iota}$$

or

$$\sum_{\lambda=1}^{\infty} \mu_{\iota\lambda} X_{\iota\lambda} \subset \overline{Co\, \mu_\iota X_\iota}$$

and

$$\sum_{\iota,\lambda=1}^{\infty} \mu_{\iota\lambda} X_{\iota\lambda} = \sum_{\iota=1}^{\infty} \sum_{\lambda=1}^{\infty} \mu_{\iota\lambda} X_{\iota\lambda} \subset \sum_{\iota=1}^{\infty} \overline{Co\, \mu_\iota X_\iota} \subset \overline{Co\, X}$$

where the last step follows from theorem 2.3. By symmetry

$$\sum_{\iota,\lambda=1}^{\infty} \mu_{\iota\lambda} X_{\iota\lambda} \subset Co\, X',$$

and the theorem is proved.

<u>Theorem 2.5</u> <u>If</u> $\mu_{\iota\lambda} > 0$, $\sum_{\lambda=1}^{\infty} \mu_{\iota\lambda} = \mu_\iota$, $X_{\iota\lambda} \subset X_\iota$ <u>every</u> X_ι <u>is bounded, and</u> $\sum \mu_\iota X_\iota = X$, <u>then</u> $\{\mu_{\iota\lambda} X_{\iota\lambda}\}$ <u>is unconditionally summable and</u>

$$\sum_{\iota,\lambda=1}^{\infty} \mu_{\iota\lambda} X_{\iota\lambda} \subset \overline{Co\, X}.$$

<u>Proof:</u> The proof differs from that of the preceding theorem only in the method used for showing that $\sum_{\iota,\lambda=1}^{\infty} \mu_{\iota\lambda} X_{\iota\lambda}$ is unconditionally convergent. We have formula (2.1) exactly as before, and we can conclude

from it that given any U there is a ν_0 such that $\iota_\sigma > \nu_0$ implies $\sum_{\sigma,\rho=1}^{\infty} \mu_{\iota_\sigma,\lambda_\rho} X_{\iota_\sigma,\lambda_\rho} \subset U$. In order to prove that $\{\mu_{\iota\lambda} X_{\iota\lambda}\}$ is unconditionally summable it remains to show that corresponding to any U there is an integer λ_0 such that $1 \leq \iota_\sigma \leq \nu_0$ and $\lambda_\rho > \lambda_0$ implies $\sum_{\sigma,\rho=1}^{\infty} \mu_{\iota_\sigma,\lambda_\rho} X_{\iota_\sigma,\lambda_\rho} \subset U$.

Let U be any chosen convex neighborhood of θ. Since for each ι the set X_ι is bounded, there is an $\alpha_\iota > 0$ such that $\alpha_\iota X_\iota \subset U$ (cf. definition 1.8 d). From the hypothesis that $\sum_{\iota,\lambda=1}^{\infty}{}' \mu_{\iota\lambda}$ converges, where $\mu_{\iota\lambda} > 0$, it follows that there is a positive integer $\lambda_\iota = \lambda_\iota(\alpha_\iota)$ such that $\nu_\rho > \lambda_\iota$ implies

$$\sum_{\rho}{}' \mu_{\iota\nu_\rho} < \frac{\alpha_\iota}{\nu_0}$$

Hence, from the elementary properties of convex sets and the hypothesis $X_{\iota\lambda} \subset X_\iota$ we have

$$\sum_{\rho}{}' \mu_{\iota\nu_\rho} X_{\iota\nu_\rho} \subset Co\left(\frac{\alpha_\iota}{\nu_0} X_\iota \dot{+} \theta\right)$$
$$\subset Co\left(\frac{1}{\nu_0} U\right) = \frac{1}{\nu_0} U$$

Consequently, if $\lambda_0 = \max_{1 \leq \iota \leq \nu_0} \lambda_\iota$, we find for $1 \leq \iota_\sigma \leq \nu_0$ and $\nu_\rho > \lambda_0$ that

$$\sum_{\sigma,\rho=1}^{\infty} \mu_{\iota_\sigma,\nu_\rho} X_{\iota_\sigma,\nu_\rho} \subset \overbrace{\left(\frac{1}{\nu_0} U + \frac{1}{\nu_0} U + \cdots + \frac{1}{\nu_0} U\right)}^{(\nu_1 \leq \nu \text{ summands})} = \frac{\nu_1}{\nu_0} U \subset U$$

where we have again made use of the convexity of U in applying the distributivity law on the left. This shows that $\sum_{\iota,\lambda=1}^{\infty} \mu_{\iota\lambda} X_{\iota\lambda}$ is unconditionally convergent and the theorem is proved.

4. Lebesque Integrals.

By a partition of the domain E of the independent variable we shall mean a set Π, finite or denumerable, of disjoint non-vacuous sets $\Delta_i \in \mathfrak{D}_1$ whose point set sum $\dot{\sum_1} \Delta_i = E$ (cf. #2). Let $F(t)$ be a function which orders to each t of E one or more points of T. If Π is any partition of E into sets $\Delta_i \in \mathfrak{D}_1$, denote by $F(\Delta_i)$ the set of function values $F(t)$ with $t \in \Delta_i$. We shall say that the function $F(t)$ is summable under the partition Π if the sequence of sets $m(\Delta_i) F(\Delta_i)$ is unconditionally summable. If F is summable under Π and each set $m(\Delta_i) F(\Delta_i)$ is bounded, we shall say that F is boundedly summable under the partition Π.

Definition 2.1 Let Π be a partition under which $F(t)$ is summable. The set $\overline{Co \sum_1 m(\Delta_i) F(\Delta_i)}$ is called the integral range of F relative to Π and will be denoted by $J_\pi(F)$. To emphasize the fact that the set Δ_i belongs to Π we shall occasionally write Δ_i^π for Δ_i. We may now prove the following fundamental theorems on the integral range of a partition "consecutive" to a given partition.

Theorem 2.6 If Π_1 and Π_2 are any two partitions of E under which F is summable and if $\Pi_1 \cdot \Pi_2$ denotes the partition of E consisting of the non-vacuous sets in $\{\Delta_i^{\pi_1} \cdot \Delta_j^{\pi_2}\}$, then F is summable under $\Pi_1 \cdot \Pi_2$ and

$$J_{\pi_1 \cdot \pi_2}(F) \subset J_{\pi_1}(F) \cdot J_{\pi_2}(F).$$

Proof: Note that theorem 2.4 applies here if we take

$$\mu_{\iota\lambda} = m(\Delta_\iota^{\pi_1}\cdot\Delta_\lambda^{\pi_2}), \quad \mu_\iota = m(\Delta_\iota^{\pi_1}),$$
$$\mu'_\lambda = m(\Delta_\lambda^{\pi_2}), \quad X_{\iota\lambda} = F(\Delta_\iota^{\pi_1}\cdot\Delta_\lambda^{\pi_2})$$
$$X_\iota = F(\Delta_\iota^{\pi_1}), \quad X'_\lambda = F(\Delta_\lambda^{\pi_2}).$$

Hence $\sum_{\iota,\lambda=1}^{\infty} m(\Delta_\iota^{\pi_1}\cdot\Delta_\lambda^{\pi_2})F(\Delta_\iota^{\pi_1}\cdot\Delta_\lambda^{\pi_2})$ is unconditionally convergent and is contained in both $\overline{\sum_\iota \mathrm{Co}\, m(\Delta_\iota^{\pi_1})F(\Delta_\iota^{\pi_1})}$ and in $\overline{\sum_\lambda \mathrm{Co}\, m(\Delta_\lambda^{\pi_2})F(\Delta_\lambda^{\pi_2})}$. Consequently, on taking convex hulls we obtain

$$J_{\pi_1\cdot\pi_2}(F) \subset J_{\pi_1}(F)\cdot J_{\pi_2}(F).$$

<u>Theorem 2.7</u> <u>Let $\mathcal{P}: (\Delta_1, \Delta_2, \cdots)$, $\Delta_\iota\cdot\Delta_\kappa = 0$, be a decomposition of E into sets Δ_ι of $\mathfrak{D}$ (Δ_ι not necessarily of finite measure) and let Π be a partition of E into sets Δ'_ι of $\mathfrak{D}_1$.</u>
<u>Let $\Pi\cdot\mathcal{P}$ denote the partition of E consisting of the non-vacuous sets in $\{\Delta'_\iota\cdot\Delta_\lambda\}$. If $F(t)$ is boundedly summable under Π then $F(t)$ is also boundedly summable under $\Pi\cdot\mathcal{P}$ and</u>

$$J_{\Pi\cdot\mathcal{P}}(F) \subset J_\Pi(F).$$

<u>Proof:</u> Theorem 2.5 applies here if we take $\mu_{\iota\lambda} = m(\Delta'_\iota\cdot\Delta_\lambda)$, $\mu_\iota = m(\Delta'_\iota)$, $X_{\iota\lambda} = F(\Delta'_\iota\cdot\Delta_\lambda)$, $X_\iota = F(\Delta'_\iota)$, and we obtain

$$\sum_{\iota,\lambda=1}^{\infty} m(\Delta'_\iota\cdot\Delta_\lambda)F(\Delta'_\iota\cdot\Delta_\lambda) \subset \overline{\mathrm{Co}\sum_\iota m(\Delta'_\iota)F(\Delta'_\iota)}$$

Taking closed convex hulls of both sides, we have

$$J_{\pi\cdot p}(F) \subset J_{\pi_\nu}(F).$$

Definition 2.2 A function $F(t)$ (single valued or not) on E to T is said to be integrable over E if there exists an element $j(F)$ of T and a sequence of partitions Π_ν with the property that for every U there corresponds an integer ν_0 such that $\nu > \nu_0$ implies that $J_{\Pi_\nu}(F) - j(F) \subset U.$

The element $j(F)$ is called the integral of $F(t)$ over E.

Definition 2.2a If $F(t)$ is integrable over E, and if F is not only summable, but boundedly summable over each of the above partitions Π_ν then we shall say that F is boundedly integrable over E.

Theorem 2.8 The integral of definition 2.2, when it exists, is unique.

Proof: Let $\{\Pi_\nu\}$ and $\{\Pi'_\nu\}$ be sequences of partitions defining integrals $i(F)$ and $j(F)$ respectively. Then by theorem 2.6

$$J_{\pi_\nu\cdot\pi'_\lambda} \subset J_{\pi_\nu}\cdot J_{\pi'_\lambda}$$

for any ν, λ. Let U be any neighborhood of the origin, and let V be such that $V - V \subset U$. Then by definition 2.2 we have for $\nu > \nu_0(V)$, $\lambda > \lambda_0(V)$ that

$$J_{\pi_\nu\cdot\pi'_\lambda} - j(F) \subset V$$

$$J_{\pi_\nu\cdot\pi'_\lambda} - i(F) \subset V$$

Hence $J_{\pi_\nu\cdot\pi'_\lambda} - J_{\pi_\nu\cdot\pi'_\lambda} + i(F) - j(F) \subset V - V.$

But since $\theta \in J_{\pi_\nu\cdot\pi'_\lambda} - J_{\pi_\nu\cdot\pi'_\lambda}$ this implies

$$i(F) - j(F) \in V - V \subset U,$$

whence $i(F) = j(F)$, since T is a Hausdorff space, and $i(F) - j(F)$ is contained in every neighborhood of θ . From definition 2.2 and the fact that T is both regular and locally convex we have the following theorem.

Theorem 2.9 A necessary and sufficient condition for $F(t)$ to be integrable over T is that there exist a sequence of partitions Π_ν of E and an element $j(F)$ of T such that, given U there is a $\lambda = \lambda(U)$ such that $\nu > \lambda$ implies

$$\sum_{\Delta_i \in \Pi_\nu}' m(\Delta_i)\, F(\Delta_i) \subset j(F) + U.$$

The following "Cauchy criterion" for the existence of an integral is often useful.

Theorem 2.10 In order that $F(t)$ be integrable it is necessary and sufficient that there exist a sequence of partitions Π_ν with the property that, given U there is an integer $\lambda = \lambda(U)$ such that $\nu > \lambda$ implies

$$J_{\pi_\nu} - J_{\pi_\nu} \subset U.$$

Proof: Necessity is obvious. To prove sufficiency let U be chosen and let ν_0 be the integer corresponding to U in accordance with definition 2.2. From theorem 2.6 the sequence of closed sets

has the properties

(1.) Each set contains the following

(2.) $J_{\pi_1 \cdots \pi_\nu} - J_{\pi_1 \cdots \pi_\nu} \subset J_{\pi_\nu} - J_{\pi_\nu} \subset U$

for $\nu > \nu_0$

By the axiom of choice (applied here to only a countable number of sets) there exists a sequence $\{x_\nu\}$ such that $x_\nu \in J_{\pi_1 \cdots \pi_\nu}$. Then $\mu, \nu > \nu_0$ implies $x_\mu - x_\nu \in U$, so that by sequential completeness, the sequence $\{x_\nu\}$ converges to a point $x \in T$. Each of the closed sets $J_{\pi_1 \cdots \pi_\nu}$ contains x so we have $x \in J_{\pi_\nu}$ for each ν. Hence $J_{\pi_\nu} - x \subset U$, and F is integrable to $j(F) = x$.

<u>Corollary</u>: <u>In theorem 2.10 we may replace J_{π_ν} by $\sum_{\Delta_\iota \in \pi_\nu} m(\Delta_\iota) F(\Delta_\iota)$ and still have a true statement.</u>

By comparison with G. Birkhoff's definition (cf. Birkhoff (2.) definitions 1 and 3, pp. 366, 367) of an integral for Banach spaces, we see that our integral reduces to Birkhoff's in case T is a Banach space and the function $F(t)$ is <u>boundedly</u> integrable in accordance with definition 2.2 a.

<u>Theorem 2.11</u> <u>If $F(t)$ is integrable over E it is integrable over every set Δ of $\mathfrak{D}$ to an element $j(F, \Delta)$, and we have</u>

$$j(F, \Delta_1 \dotplus \Delta_2) = j(F, \Delta_1) + j(F, \Delta_2).$$

<u>Proof</u>: Let $F(t)$ be summable under a partition π of E into sets $\Delta_\iota \in \mathfrak{D}$, and suppose $\Delta \in \mathfrak{D}$. Then

$$F(\Delta'_\iota) \subset F(\Delta_\iota)$$
$$F(\Delta^2_\iota) \subset F(\Delta_\iota)$$

$$m(\Delta'_\iota) + m(\Delta^2_\iota) = m(\Delta_\iota)$$

where $\Delta_i^1 = \Delta_i \cdot \Delta$, $\Delta_i^2 = (E \dot{-} \Delta) \cdot \Delta_i$.

Therefore

$$m(\Delta_i^1) F(\Delta_i^1) + m(\Delta_i^2) F(\Delta_i^2) \subset Co\, m(\Delta_i) F(\Delta_i) .$$

Evidently $m(\Delta_i^1) < m(\Delta_i)$ so that

$$\begin{aligned} m(\Delta_i^1)\, Co\, F(\Delta_i^1) &\subset m(\Delta_i^1)\, Co\,(F(\Delta_i^1) \dot{+} \theta) \\ &\subset Co\,(m(\Delta_i) F(\Delta_i) \dot{+} \theta) \end{aligned}$$

From theorem 2.2 the sequence $\{m(\Delta_i) F(\Delta_i) \dot{+} \theta\}$ is easily found to be unconditionally summable, since by hypothesis $\{m(\Delta_i) F(\Delta_i)\}$ is unconditionally summable, and hence by theorem 2.3 we may conclude that the sequence of sets $m(\Delta_i^1)\, Co\, F(\Delta_i^1) = Co\, m(\Delta_i^1) F(\Delta_i)$ is unconditionally summable. A similar agrument shows that $\{Co\, m(\Delta_i^2) F(\Delta_i^2)\}$ is unconditionally summable. From the distributivity of the "Co" operation and the fact that $\overline{M} + \overline{N} \subset \overline{M + N}$ for any two sets M, N in $\mathcal{T}$ we obtain

$$\overline{Co \sum_{i=1}^{\infty} m(\Delta_i^1) F(\Delta_i^1)} + \overline{Co \sum_{i=1}^{\infty} m(\Delta_i^2) F(\Delta_i^2)}$$
$$\subset \overline{Co \sum_{i=1}^{\infty} m(\Delta_i) F(\Delta_i)}$$

That is,

(2.3) $J_{\pi^1}(F,\Delta) + J_{\pi^2}(F, E \dot{-} \Delta) \subset J_{\pi}(F, E)$,

where $J_{\pi^1}(F,\Delta)$ for instance is the integral range of F over Δ with respect to the partition $\pi^1 : \{\Delta_i \cdot \Delta\}$ of Δ . Now consider a sequence Π_ν of partitions of E for which $J_{\Pi_\nu} - J_{\Pi_\nu} \to \theta$ in accordance with theorem 2.10. Given U we have for all $\nu > \lambda(U)$ on using the inequality (2.3) for $\Pi = \Pi_\nu$,

$$J_{\pi^1_\nu}(F,\Delta) + J_{\pi^2_\nu}(F, E \dot{-} \Delta) - J_{\pi^1_\nu}(F,\Delta) - J_{\pi^2_\nu}(F, E - \Delta)$$
$$\subset J_{\pi_\nu}(F,E) - J_{\pi_\nu}(F,E) \subset U.$$

But since the sets $J_{\pi^1_\nu}(F,\Delta) - J_{\pi^1_\nu}(F,\Delta)$ and $J_{\pi^2_\nu}(F,E \dot{-}\Delta) - J_{\pi^2_\nu}(F, E\dot{-}\Delta)$ each contain θ, this implies that

$$J_{\pi^2_\nu}(F, E \dot{-} \Delta) - J_{\pi^2_\nu}(F, E \dot{-} \Delta) \subset U$$

and
$$J_{\pi^1_\nu}(F,E) - J_{\pi^1_\nu}(F,E) \subset U$$

for all $\nu > \lambda(U)$. It follows (theorem 2.10) that $F(t)$ is integrable over Δ to some $j(F,\Delta)$ and over $E \dot{-} \Delta$ to some $j(F, E \dot{-} \Delta)$. As in the proof of theorem 2.10 we have

$$j(F,\Delta) + j(F, E \dot{-} \Delta) \subset J_{\pi^1_\nu}(F,\Delta) + J_{\pi^2_\nu}(F, E \dot{-} \Delta)$$
$$\subset J_{\pi_\nu}(F,E) \subset j(F,E) + U$$

for $\nu > \lambda(U)$, for arbitrary U. Hence, ~~by the uniqueness theorem 2.8~~,

$$j(F,\Delta) + j(F, E \dot{-} \Delta) = j(F,E).$$

The theorem now follows, if we replace E by the "sub-space" E'.

Theorem 2.12 If $F(t)$ is boundedly integrable over E then the set function $j(F,\Delta)$ is completely additive.

Proof: Since any member E' of $\mathfrak{D}$ by relativisation satisfies the postulates for an M-space, it will be sufficient to show that if $\mathcal{P}$ is any denumerable collection of non-vacuous sets $\Delta_\iota \in \mathfrak{D}$ with

$$\dot{\sum} \Delta_\iota = E, \quad \Delta_\iota \cdot \Delta_\lambda = \odot \qquad (\iota \neq \lambda)$$

then $\sum_{\iota=1}^{\infty} j(F, \Delta_\iota)$ converges and is equal to $j(F, E)$.

To prove this, let $\Pi_\nu : (\cdots, \Delta_\iota^\nu, \cdots)$, where $\Delta_\iota^\nu \in \mathfrak{D}_1$ be a sequence of partitions of E such that

$$J_{\pi_\nu}(F, E) \to j(F, E).$$

Having chosen U let ν_0 be such that $\nu > \nu_0$ implies $J_{\pi_\nu}(F, E) - j(F, E) \subset U$. Consider the partition $\Pi_\nu \cdot \mathcal{P}$, for any $\nu > \nu_0$. By hypothesis each set $F(\Delta_\iota^\nu)$ is bounded, so we can apply theorem 2.5, putting

$$\Pi = \Pi_\nu, \quad \mu_{\iota\lambda} = m(\Delta_\iota^\nu \cdot \Delta_\lambda), \quad \mu_\iota = m(\Delta_\iota^\nu),$$

$$X_{\iota\lambda} = F(\Delta_\iota^\nu \cdot \Delta_\lambda), \quad X_\iota = F(\Delta_\iota^\nu),$$

and conclude that $\{m(\Delta_\iota^\nu \cdot \Delta_\lambda) F(\Delta_\iota^\nu \cdot \Delta_\lambda)\}$ is unconditionally summable and that the following inequality is satisfied.

$$(2.4) \quad \sum_{\iota,\lambda=1}^{\infty} m(\Delta_\iota^\nu \cdot \Delta_\lambda) F(\Delta_\iota^\nu \cdot \Delta_\lambda) \subset \overline{Co \sum_{\iota=1}^{\infty} m(\Delta_\iota^\nu) F(\Delta_\iota^\nu)} = J_{\pi_\nu}(F, E).$$

Since by theorem 2.11 $F(t)$ is integrable over Δ_λ, we have

$$j(F, \Delta_\lambda) \subset J_{\pi_\nu \cdot \Delta_\lambda}(F, \Delta_\lambda) = \overline{Co \sum_{\iota=1}^{\infty} m(\Delta_\iota^\nu \cdot \Delta_\lambda) F(\Delta_\iota^\nu \cdot \Delta_\lambda)}$$

Summing on λ from 1 to ∞, we have

$$\sum_{\lambda=1}^{\infty} j(F, \Delta_\lambda) \subset \sum_{\lambda=1}^{\infty} \overline{Co \sum_{\iota=1}^{\infty} m(\Delta_\iota^\nu \cdot \Delta_\lambda) F(\Delta_\iota^\nu \cdot \Delta_\lambda)}$$

But from theorem 2.3 this implies that

$$\sum_{\lambda=1}^{\infty} j(F,\Delta_\lambda) \subset \overline{Co \sum_{\iota,\lambda=1}^{\infty} m(\Delta_\iota^\nu\cdot\Delta_\lambda)\, F(\Delta_\iota^\nu\cdot\Delta_\lambda)}$$

From inequality (2.4)

$$\sum_{\lambda=1}^{\infty} j(F,\Delta_\lambda) \subset J_{\pi_\nu}(F,E)$$

for all $\nu > \nu_0$,

so that $$\sum_{\lambda=1}^{\infty} j(F,\Delta_\lambda) = j(F,E).$$

Theorem 2.13 If $F(t)$ and $G(t)$ are boundedly integrable over E then $\alpha F(t)$ and $F(t)+G(t)$ are integrable over E and

$$j(\alpha F) = \alpha\, j(F), \quad j(F+G) = j(F) + j(G).$$

Proof: The statements about $\alpha F(t)$ are evident. To prove those about $F+G$, let $\{\pi_F^\nu\}$ and $\{\pi_G^\lambda\}$ be sequences of partitions of E into sets of $\mathcal{D}$, such that

$$J_{\pi_F^\nu}(F) \to j(F)$$

$$J_{\pi_G^\lambda}(G) \to j(G)$$

as ν and λ respectively $\to \infty$. Given U, choose V so that $\overline{V+V} \subset U$. By theorem 2.7,

$$J_{\pi_F^\nu\cdot\pi_G^\lambda}(F) \subset J_{\pi_F^\nu}(F)$$

$$J_{\pi_F^\nu\cdot\pi_G^\lambda}(G) \subset J_{\pi_G^\lambda}(G)$$

But from the properties of convex hulls,

$$\overline{Co\,(A+B)} = \overline{\overline{Co A} + \overline{Co B}}$$

for any $A, B \subset T$.

so that

$$J_{\pi_F^\nu \cdot \pi_G^\lambda}(F+G) = \overline{J_{\pi_F^\nu \cdot \pi_G^\lambda}(F) + J_{\pi_F^\nu \cdot \pi_G^\lambda}(G)}$$
$$\subset \overline{J_{\pi_F^\nu}(F) + J_{\pi_G^\lambda}(G)}.$$

Consequently for large enough ν, λ we have

$$J_{\pi_F^\nu \cdot \pi_G^\lambda}(F+G) - j(F) - j(G) \subset \overline{J_{\pi_F^\nu} - j(F) + J_{\pi_G^\lambda}(G) - j(G)}$$

so that $F+G$ is integrable and

$$j(F+G) = j(F) + j(G).$$

<u>Theorem 2.14</u> If Δ is of finite measure, that is if $\Delta \in \mathfrak{D}_1$, then we have the following law of the mean for L-integrable functions:

$$j(F, \Delta) \subset m(\Delta)\, \overline{Co\, F(\Delta)}.$$

<u>Proof</u>: $$\frac{1}{m(\Delta)} j(F,\Delta) \in \frac{1}{m(\Delta)} J_\pi(F,\Delta) = \overline{Co \sum_{i=1}^{\infty} \frac{m(\Delta_i \cdot \Delta)}{m(\Delta)} F(\Delta_i \cdot \Delta)}$$
$$\subset \overline{Co\ \overline{Co\ F(\Delta)}}$$
$$= \overline{Co\ F(\Delta)}.$$

5. <u>Riemann integrals and continuous functions</u>

Riemann integrals of functions on a closed* real interval to a Neumann linear topological space** are considered in Michal and

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* The interval $\alpha < \tau < \beta$ will be denoted by (α, β), and the intervals $\alpha \leq \tau < \beta$, $\alpha < \tau \leq \beta$, $\alpha \leq \tau \leq \beta$ will be denoted respectively by $[\alpha, \beta)$, $(\alpha, \beta]$, $[\alpha, \beta]$.

** Such a space is slightly less general than T. See Chap. 1,#3 and #7, Ex. 2°. My results on Riemann integrals were obtained independently from those of Michal and Paxson.

Paxson (1.) and (2.) and are studied in detail in Paxson (1.). However in view of our later applications, and in order to correlate the "Lébesgue" integral of the preceding sections with the Riemann integral, a brief discussion of the latter will be given in this section.

If $F(\tau)$ is a function (possibly many valued) on $[\alpha,\beta]$ to $\mathcal{T}$, we can obtain a definition for the Riemann integral of $F(\tau)$ over $[\alpha,\beta]$ from the general definition of #3 by requiring the partitions π to be the usual subdivisions of $[\alpha,\beta]$ into a finite number of intervals. More specifically, for $\alpha = \tau_0 < \tau_1 < \cdots < \tau_\nu = \beta$ let $\mathcal{S}_\nu$ denote the collection of sub-intervals $\Delta_1 = [\tau_0,\tau_1]$, $\Delta_2 = (\tau_1,\tau_2]$, $\cdots$, $\Delta_\nu = (\tau_{\nu-1},\tau_\nu]$. Put $\ell(\Delta_\iota) = \tau_\iota - \tau_{\iota-1}$, $\delta(\mathcal{S}_\nu) = \max\limits_{0 \le \iota \le \nu} \ell(\Delta_\iota)$, and define the Riemann integral as follows:

<u>Definition 2.3</u> A function $F(\tau)$ on $[\alpha,\beta]$ to $\mathcal{T}$ will be said to be <u>Riemann</u> <u>integrable</u> to $j(F) = \int_\alpha^\beta F(\sigma)\,d\sigma$ if, corresponding to each U and each $\eta > 0$, there is a sub-division $\mathcal{S}_\nu$ such that $\delta(\mathcal{S}_\nu) < \eta$ and

$$\sum_{\iota=1}^{\nu} \ell(\Delta_\iota) F(\Delta_\iota) - j(F) \subset U.$$

Since we are always supposing that the space $\mathcal{T}$ is locally convex, and since any linear topological space is regular, it is clear that the content of the definition is unchanged if we replace $\sum_{\iota=1}^{\nu} \ell(\Delta_\iota) F(\Delta_\iota)$ by $\overline{Co \sum_1^\nu \ell(\Delta_\iota) F(\Delta_\iota)}$. As in the case of the L-integral of #3, it is convenient to define the (Riemann) integral range $J_{\mathcal{S}_\nu}(F)$ of $F(\tau)$ relative to $\mathcal{S}_\nu$ by

$$J_{\mathcal{S}_\nu}(F) = \overline{Co \sum_{\iota=1}^{\nu} \ell(\Delta_\iota) F(\Delta_\iota)}.$$

Thus definition 2.3 could have been phrased in terms of integral ranges. (Compare with definition 2.2 and theorem 2.9).

If we interpret the M-space E of #1 to be the real interval $[\alpha, \beta]$ where $\mathcal{S}$ and $\mathcal{S}_1$ are identical, and each consists of the set of all measurable sub-sets of $[\alpha, \beta]$ then definition 2.2 furnishes us with the definition of a Lebesgue integral of a function $F(\tau)$ on $[\alpha, \beta]$.

Theorem 2.15 *If $F(\tau)$ on $[\alpha, \beta]$ to T is Riemann integrable over $[\alpha, \beta]$ then it is Lebesgue integrable over $[\alpha, \beta]$ and the two integrals are equal.*

Proof: The theorem is clear from the uniqueness theorem 2.8 when we notice that (1) every sub-division $\mathcal{S}_\nu$ is a partition Π of $[\alpha, \beta]$ into measurable sets, (2) every "Riemann" integral range is a "Lebesque" integral range, (3) by taking a sequence of η's $\to 0$ in definition 2.3, we obtain a sequence of integral ranges satisfying the conditions of definition 2.2.

In the following theorems on Riemann integrals and continuous functions we shall, for the most part, merely outline the proofs, for the reason given at the beginning of this section.

Theorem 2.16 *Every (single valued) continuous function on $[\alpha, \beta]$ to T is uniformly continuous.*

Proof: As in the case of real functions, use Heine-Borel theorem.

Theorem 2.17 (Cauchy condition) *A necessary and sufficient condition for $F(\tau)$ on $[\alpha, \beta]$ to be Riemann integrable is that corresponding to every U there is a sub-division $\mathcal{S}_\nu : \{\Delta_\iota\}$ of $[\alpha, \beta]$ such that*

$\delta(\mathcal{D}_\nu) < \eta$ and

$$\sum_{\iota=1}^{\nu}{}' \, l(\Delta_\iota) F(\Delta_\iota) - \sum_{\iota=1}^{\nu}{}' \, l(\Delta_\iota) F(\Delta_\iota) \subset U.$$

Proof: See proof of theorem 2.10, and corollary.

Theorem 2.18 If $f(\tau)$ is a (single valued) continuous function on $[\alpha, \beta]$ to T, then $f(\tau)$ is Riemann integrable.

Proof: Use theorems 2.16, 2.17 and the postulate of local convexity.

It is convenient to define $\int_\gamma^\eta F(\sigma)d\sigma$ as $-\int_\eta^\gamma F(\sigma)d\sigma$ in case $\gamma \geq \eta$.

Theorem 2.19 Let the Riemann integral of the functions $F(\tau)$, $G(\tau)$ exist over $[\alpha, \beta]$. Then we have

(i) The integral of $F(\tau)$ over any sub-interval exists and

$$\int_\gamma^\eta F(\sigma)d\sigma = \int_\gamma^\varsigma F(\sigma)d\sigma + \int_\varsigma^\eta F(\sigma)d\sigma \quad \text{for } \gamma, \eta, \varsigma \in [\alpha, \beta]$$

(ii) $\int_\alpha^\beta (\lambda F(\sigma) + \mu G(\sigma))d\sigma$ exists, $= \lambda \int_\alpha^\beta F(\sigma)d\sigma + \mu \int_\alpha^\beta G(\sigma)d\sigma$

(iii) $\int_\alpha^\beta F(\sigma)\,d\sigma \in (\beta - \alpha)\, \overline{Co\, F([\alpha, \beta])}$

In the law of the mean (iii), $F([\alpha, \beta])$ denotes the range of , i.e. the set $\{F(\tau)\}$, $\tau \in [\alpha, \beta]$. The proofs of (i), (ii), (iii) are similar to those of theorems 2.11, 2.13, and 2.14, respectively. Here the proofs are greatly simplified, since all series $\sum' l(\Delta_\iota) F(\Delta_\iota)$ are finite, and hence automatically unconditionally convergent. Consequently, it is not necessarily to bring in the notion of "boundedly integrable" functions.

Definition 2.4 A function $f(\tau)$ on the open interval (α, β) to T

will be said to possess a derivative $f'(\tau_0)$ at the point $\tau_0 \in (\alpha, \beta)$ if for every U there is a $\delta > 0$ such that $|\tau - \tau_0| < \delta$ implies

$$\frac{f(\tau - \tau_0) - f(\tau_0)}{\tau - \tau_0} - f'(\tau_0) \subset U.$$

Since T is a Hausdorff space, a function $f(\tau)$ can have at most one derivative at τ_0.

Theorem 2.20 *If $f(\tau)$ is continuous on $[\alpha, \beta]$ to T and*

$$g(\tau) = \int_\alpha^\tau f(\sigma)\, d\sigma,$$

then $g(\tau)$ has the derivative $g'(\tau) = f(\tau)$ at each point of (α, β).

Proof: As for real functions, using theorem 2.19.

Corollary: *Under the conditions of the theorem $g(\tau)$ is continuous on $[\alpha, \beta]$.*

Theorem 2.21 *If a function $f(\tau)$ on $[\alpha, \beta]$ to T has a derivative at each point of (α, β) which vanishes identically on (α, β) then $f(\tau)$ is a constant element of T.*

Proof: Since $f'(\tau) = \theta$ for all $\tau \in (\alpha, \beta)$, there exists for each U (we may suppose U convex) and each $\tau \in (\alpha, \beta)$, a positive number $\delta = \delta(\tau)$ such that

$$(2.5) \quad \frac{f(\sigma) - f(\tau)}{\sigma - \tau} \in U, \quad \frac{f(\tau) - f(\varrho)}{\tau - \varrho} \in U \quad \text{for } \tau - \delta < \varrho < \tau < \sigma < \tau + \delta.$$

Hence since U is convex, we have

$$(2.5)\text{a} \quad \frac{f(\sigma) - f(\varrho)}{\sigma - \varrho} \in \frac{\sigma - \tau}{\sigma - \varrho} U + \frac{\tau - \varrho}{\sigma - \varrho} U = U,$$

$$\text{for } \tau - \delta(\tau) < \varrho < \tau < \sigma < \tau + \delta(\tau).$$

Let $[\gamma, \eta]$ be any closed sub-interval of (α, β). We wish to show that

$f(\gamma) = f(\eta)$. By employing the Heine-Borel theorem for linear point sets we may assert the existence of a finite covering of $[\gamma, \eta]$ by open intervals $\Delta_\iota = (\tau_\iota - \delta_\iota, \tau_\iota + \delta_\iota)$ with the following properties:

(1.) $\delta_\iota < \delta(\tau_\iota)$ (so that we can apply inequality (2.5)) and $\delta_\iota < \frac{1}{2}\delta(\eta)$

(2.) $\tau_{\iota+1} > \tau_\iota$

(3.) Δ_0 has γ as its mid-point.

(4.) The intersection of the left open half of Δ_ι and the right open half of $\Delta_{\iota-1}$ contains at least one point σ_ι.

(5.) Some Δ_ι, say Δ_ν, contains η.

If η is not the mid-point of Δ_ν, put $\sigma_0 = \gamma$, $\sigma_{\nu+1} = \eta$, select σ_ι in accordance with (4.), for $1 \leq \iota \leq \nu$, and we may clearly suppose $\sigma_{\iota-1} < \sigma_\iota$, $\iota = 1, \cdots, \nu+1$. Then we have by inequalities (2.5) and (2.5)a that

$$\frac{f(\sigma_\iota) - f(\sigma_{\iota-1})}{\sigma_\iota - \sigma_{\iota-1}} \subset U \qquad \text{for } \iota = 1, \cdots, \nu+1.$$

If η is the mid-point of Δ_ν, we can stop ι at ν instead of $\nu+1$.

In any case we have

$$\frac{f(\eta) - f(\gamma)}{\eta - \gamma} = \frac{1}{\eta - \gamma} \sum\nolimits_1 (\sigma_\iota - \sigma_{\iota-1}) \frac{f(\sigma_\iota) - f(\sigma_{\iota-1})}{\sigma_\iota - \sigma_{\iota-1}} \subset \mathcal{C}o\, U = U.$$

But since U is an arbitrary neighborhood of θ and $\eta - \gamma \neq 0$ we have $f(\eta) - f(\gamma) = \theta$, q.e.d.

<u>Theorem 2.22</u> <u>If $f'(\tau)$ exists and is continuous on (α, β), then, for $\gamma, \eta \in (\alpha, \beta)$ we have</u>

$$f(\eta) - f(\gamma) = \int_\gamma^\eta f'(\sigma)\, d\sigma.$$

Proof: See Theorems 2.20, 2.21.

Theorem 2.23 If $\{f_\nu(\tau)\}$ is a sequence of continuous functions on $[\alpha,\beta]$ to any linear topological space, converging uniformly on $[\alpha,\beta]$ to a function $f(\tau)$, then $f(\tau)$ is continuous on $[\alpha,\beta]$.

Proof: As for real functions, In this theorem the postulates of local convexity and sequential completeness are not used.

Theorem 2.24 If $\{f_\nu(\tau)\}$ is a sequence of continuous functions on $[\alpha,\beta]$ to T converging uniformly on this interval to $f(\tau)$ then

$$\int_\alpha^\beta f(\sigma) = \lim_{\nu\to\infty} \int_\alpha^\beta f_\nu(\sigma)\,d\sigma.$$

Proof: Use theorem 2.23, the law of the mean, and the local convexity and regularity of T.

Theorem 2.25 If $f(\tau)$ is a continuous function on $[\alpha,\beta]$ to any linear topological space then its range $f([\alpha,\beta])$ is compact in itself.

Proof: Let $f_1, f_2, ---$ be an infinite sequence of (distinct) values of the function $f(\tau)$, and $A_1, A_2, ---$ the original sets corresponding to $f_1, f_2, ---$ i.e. $A_\iota = \{\tau\}$, $\tau \in [\alpha,\beta]$, $f(\tau) = f_\iota$. By the axiom of Zermelo (used here for only a countable number of sets) we may choose $\tau_\iota \in A_\iota$, obtaining an infinite sequence $\{\tau_\iota\}$ of distinct points of $[\alpha,\beta]$, such that $f(\tau_\iota) = f_\iota$. By the Weierstrass-Bolzano theorem there is a limit point τ_0 of the set $\{\tau_\iota\}$. Evidently $\tau_0 \in [\alpha,\beta]$ and since $f(\tau)$ is continuous, the corresponding point $f(\tau_0)$ of T is a limit point of the set $\{f_\iota\}$.

<u>Corollary</u> <u>The range of a continuous function on $[\alpha,\beta]$ to a linear topological space is a bounded set.</u>

<u>Proof</u>: Cf. theorem 1.6.

Chapter 3

Existence Theorems for Functional Equations in Linear Topological Spaces

1. K-systems

Let there be ordered to each element x of a linear topological* space T a set $K(x)$ subject to the following conditions.

K 1. Each $K(x)$ is convex.

K 2. $x \in K(x)$; $\theta \in K(x)$ If $x \neq \theta, \alpha > 1$ then $\alpha x \bar{\in} K(x)$.

K 3. $K(\alpha x) = \alpha K(x)$ for $\alpha \geq 0$.

K 4. $y \in K(x)$ implies $K(y) \subset K(x)$.

K 5. Each $K(x)$ is bounded.

Then we shall call the family of sets $K(x)$ a K-system for the space. If instead of K 5 the condition

K 5a. Given any neighborhood U of θ there exists a neighborhood of θ such that

$$\sum_{x \in V}{}^{\bullet} K(x) \subset U$$

is satisfied, we shall call $\{K(x)\}$ a continuous K-system. It follows readily from K 3 and the continuity of scalar multiplcation that K 5a implies K 5, so that a continuous K-system is a K-system. When each of the sets $K(x)$ is closed, we shall call the family a closed K-system.

In the case of a linear normed space, the spheres $\|y\| \leq \|x\|$

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* As in chapter 2, we shall denote both the basis and the space by the same letter when there is no chance for confusion. It will be explicitly stated when postulates (II.) and (III.) are required.

form a closed, continuous $\mathcal{K}$-system for the space. In any linear topological space the closed line segments joining the origin to the point form a closed K-system $\mathcal{K}_0$ which is "minimal" in the sense that each set $K(x)$ of an arbitrary K-system contains the corresponding member $K_0(x)$ of the system $\mathcal{K}_0$. ($K_0(x)$ being merely the line segment αx, $0 \leq \alpha \leq 1$.)

For locally convex linear topological spaces K-systems have a special significance. In particular, the minimal K-system $\mathcal{K}_0$ is continuous for such spaces. We now state a fundamental existence theorem for K-systems in locally convex spaces.

<u>Theorem 3.1</u> <u>In any locally convex linear topological space T there always exist complete neighborhood systems $\mathcal{U} : (U, V, W, \cdots)$ of the origin with the properties</u>

(i) If $U \in \mathcal{U}$ then U is convex

(ii) If $U \in \mathcal{U}$ then $\alpha U \in \mathcal{U}$.

<u>For any such neighborhood system $\mathcal{U}$ define</u>*

$$K(x) = \dot{\prod_{x \in U,\, U \in \mathcal{U}}} (U)$$

<u>Then the sets $K(x)$ form a closed continuous K-system for the space T, and we have</u> $K(x) = \dot{\prod_{x \in \bar{U},\, U \in \mathcal{U}}} (\bar{U})$.

<u>Proof:</u> If $\mathcal{W}$ is any complete neighborhood system of the origin in the locally convex space T, there is ordered to each $W \in \mathcal{W}$ a convex open set $V \subset W$ with $\theta \in V$, by the definition of local convexity. Then the family $\{V\}$, $V \in W$, $W \in \mathcal{W}$ forms a complete neighborhood system of the origin, with each V convex. Now add to $\{V\}$ the sets $U = \alpha V$,

*i.e. $K(x)$ = intersection of all $U \in \mathcal{U}$ for which $x \in U$.

where α ranges over the positive real numbers, and V ranges over $\{V\}$. By theorem 1.2 of #1, Chapter 1, each U is open, and clearly each U is convex. The system $\mathcal{U}:\{U\}, \substack{U=\alpha V,\\ V\subset W\in\mathcal{W}}$ satisfied (i) and (ii), and the first statement of the theorem is true.

To prove the second statement note that, due to the continuity of scalar multiplication, for each $x\in T$ and each $U\in\mathcal{U}$ there is an $\alpha>0$ with $\alpha x\in U$, that is $x\in 1/\alpha\, U = U'\in\mathcal{U}$

Hence $K(x)=\dot{\prod\limits_{x\in U\in\mathcal{U}}}(U)$ is not empty and in fact contains x, θ.

We now verify that properties K 1 - K 5a hold for the set $K(x)$.

For convenience we denote by $U(\theta,x)$ any $U\in\mathcal{U}$ with $x\in U$.

Ad K 1: $K(x)$ is convex, since the intersection of any family of convex sets is convex.

Ad K 2: Obviously $x,\theta\in K(x)$. Take $x\neq\theta$, $\alpha>1$, and $U\in\mathcal{U}$ such that $x\bar{\in} U$. Then put $\gamma=\sup\beta,\ \beta x\in U$. Choose γ' so as to satisfy $1/\alpha\,\gamma<\gamma'<\gamma$, and put $V=1/\gamma'\,U$. Then $x\in V$ since $\gamma' x\in U$, but $\alpha x\bar{\in} V$ since $\gamma'\alpha>\gamma$ and therefore $(\gamma'\alpha)x\bar{\in} U$. But since $K(x)=\dot{\prod}\{U(\theta,x)\}\subset V$, $\alpha x\bar{\in} K(x)$.

Ad K 3: $\alpha K(x)=\alpha\dot{\prod}\{U(\theta,x)\}=\dot{\prod}\{\alpha U(\theta,x)\}=\dot{\prod}(U(\theta,\alpha x))$

Ad K 4: If $y\in K(x)$, then $y\in U(\theta,x)$ for all $U(\theta,x)\in\mathcal{U}$. Hence every $U(\theta,x)$ is a $U(\theta,y)$, and

$$\dot{\prod}\{U(\theta,y)\}\subset\dot{\prod}\{U(\theta,x)\}.$$

Ad K 5a. Given any neighborhood W of the origin, there exists a $U\in\mathcal{U}$

such that $U \subset W$. Then if $x \in U$, $\dot{\Pi}\{U(\theta,x)\} \subset U \subset W$.

That is $\dot{\sum}_{x \in U} K(x) = \dot{\sum}_{x \in U} \dot{\Pi}\{U(\theta,x)\} \subset W,$ q.e.d.

In order to show that $K(x)$ is closed we first establish the following lemmas which are of interest in themselves and should perhaps be included in Chapter 1.

Lemma 3.1 <u>If T is a linear topological space and G is a convex open set of T which contains the origin, then $\alpha \bar{G} \subset G$ for all α,</u> $0 \leq \alpha < 1$.

Proof: If $0 < \alpha < 1$, then by hypothesis

$$\alpha G + (1-\alpha) G = G$$

Let x be any point of $\overline{(\alpha G)} = \alpha \bar{G}$. Since αG and $(1-\alpha)G$ are open sets containing θ, there is a neighborhood U_x of x such that $U_x - U_x \subset (\alpha G) \cdot (1-\alpha) G)$, by the continuity of subtraction. Since x is a ~~limit~~ contact point of αG there exists a point $y \in (\alpha G) \cdot U_x$. Hence we have

$$x = y + x - y \subset \alpha G + U_x - U_x$$
$$\subset \alpha G + (1-\alpha) G = G.$$

That is $\alpha \bar{G} \subset G$.

Lemma 3.2 <u>If $U(\theta,x)$ is any convex neighborhood of the origin containing x, and α is any positive number less than unity, there is a number β, $\alpha < \beta < 1$ such that</u> $x \in \beta U(\theta,x)$.

Proof: Either the lemma holds, or there is a number γ, $0 < \gamma < 1$ such that $x \,\bar{\in}\, \gamma U(\theta,x)$. Put $\eta = \sup \gamma$, $x \,\bar{\in}\, \gamma U(\theta,x)$ so that $\eta \leq 1$. Then x is in the frontier of the open set $\eta U(\theta,x)$.

For if V_x is any neighborhood of x, by continuity there is a $\delta > 0$ such that $\lambda x \in V_x$ for all λ satisfying $|\lambda - 1| < \delta$. But $\lambda > 1$ implies $x \bar{\in} \eta/\lambda\, U(\theta, x)$ or $\lambda x \bar{\in} \eta\, U(\theta,x)$, while $\lambda > 1$ implies $x \in \eta/\lambda\, U(\theta,x)$ or $\lambda x \in \eta\, U(\theta,x)$. Hence each V_x contains points of $\eta\, U(\theta,x)$ and also of its complement, and X must lie on the frontier of $\eta\, U(\theta,x)$. But since $x \in U(\theta,x)$ where $U(\theta,x)$ is open, x is not in the frontier of $U(\theta,x)$, whence $\eta < 1$. By the definition of η, $x \in \beta\, U(\theta,x)$ for $\beta > \eta$, and the lemma 3.2 is proven.

To demonstrate that $K(x) = \dot{\prod}_{x \in \bar{U}, U \in \mathcal{U}} (\bar{U})$ and therefore that $K(x)$ is closed we first prove that $\dot{\prod}_{x \in \bar{U}, U \in \mathcal{U}} (\bar{U}) \subset K(x)$. Suppose that $y \in \bar{U}$, for all $U \in \mathcal{U}$ such that $x \in \bar{U}$. Let U' be any member of $\mathcal{U}$ containing x. By lemma 3.2 there exists an α, $0 < \alpha < 1$ such that $x \in \alpha U'$. By hypothesis (i), $\alpha U'$ is a member of $\mathcal{U}$, call it $U''(\theta, x)$. Using lemma 3.1, we obtain

$$\overline{U''(\theta,x)} = \alpha \bar{U}' \subset U'$$

Now by hypothesis, $y \in \overline{U''(\theta,x)}$, so that $y \in U'$, and it follows that

$$\dot{\prod}_{x \in \bar{U}, U \in \mathcal{U}} (\bar{U}) \subset \dot{\prod}_{x \in U, U \in \mathcal{U}} (U) = K(x).$$

To demonstrate that $K(x) \subset \dot{\prod}_{x \in \bar{U}, U \in \mathcal{U}} (\bar{U})$ suppose that $y \in K(x) = \dot{\prod}_{x \in U \in \mathcal{U}} (U)$, and let U be any member of $\mathcal{U}$ such that $x \in \bar{U}$. We must prove that $y \in \bar{U}$. By lemma 3.1 $\bar{U} \subset 1/\alpha\, U$ for $0 < \alpha < 1$, hence $x \in 1/\alpha\, U \in \mathcal{U}$. Since $y \in K(x)$, we also have $y \in 1/\alpha\, U$ or $\alpha y \in U$ for all $\alpha,\ 0<\alpha<1$. Since scalar multiplication is continuous, y is a limit point of U, i.e. $y \in \bar{U}$. Combining our results we have that $K(x) = \prod_{x\in \bar{U},\, U \in \mathcal{U}} (\bar{U})$. Also, since the intersection of a family of closed sets is closed, $K(x)$ is closed.

In view of the later applications to functional equations it will be of interest to examine some examples of K-systems in a few of the important instances of non-normable linear topological spaces (cf. #7, chap. 1).

(A) In the space $\mathcal{V}$ of infinite dimensional vectors the sets $K(x)$: $\{y\},\ |y_i| \leq |x_i|,\ i=1,2,\cdots$, form a continuous, closed K-system.

(B)* In the space $\mathcal{F}$ of all real functions the sets $K(f)$: $\{g\},\ |g(\xi)| \leq |f(\xi)|$ for $-\infty < \xi < +\infty$, form a continuous, closed K-system.

(C)* In the weakly topologized Hilbert space $\mathcal{H}_w$, the sets $K(f)$: $\{g\},\ |(g,\varphi)| \leq |(f,\varphi)|$ for all φ, form a continuous, closed K-system.

(D) In a weakly topologized linear normed space $\mathcal{E}_w$, the sets $K(x)$: $\{y\},\ |f(y)| \leq |f(x)|$ for all linear functionals f, form a continuous, closed K-system. In each case, the space considered is locally convex, and the set $K(x)$ is the intersection of all those neighborhoods containing x which belong to a complete neighborhood system of the origin having properties (i), (ii) of theorem 1. For example in (A) $K(x)$ is the intersection of all the neighborhoods $U(\theta;\delta_1,\cdots,\delta_\nu)$:

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* The spaces $\mathcal{F}$ and $\mathcal{H}_w$ are non-metrizable as well as non-normable. Cf. #7 of Chap. 1.

$\{y\}, \ |y_\iota| < \delta_\iota, \quad \iota = 1, 2, \cdots, \nu,$ as ν varies over the positive integers, and δ_ι varies over the real numbers greater than $|x_\iota|$. From the general existence theorem 3.1 it follows that the families of sets $K(x)$ given in (A) - (D) are continuous closed K-systems.

2. The equation $y = f(y)$.

Theorem 3.2 Let $f(y)$ be a continuous function on Y to Y where Y is a sequentially complete sub-set of a linear topological space T . Let $\{K(y)\}$ be any K-system with respect to which the function $f(y)$ satisfies the "Lipschitz" condition

$$f(y) - f(z) \subset \mu K(y-z), \qquad 0 < \mu < 1,$$

for all pairs $y, z \in Y$.

Then the transformation $z = f(y)$ of Y into a part of itself has a unique fixed point.

Proof: We employ successive approximations. Having chosen $y_0 \in Y$, put

$$y_{\iota+1} = f(y_\iota), \quad \iota = 0, 1, 2, \cdots .$$

Clearly each y_ι is well defined. Using K 3 and K 4 we get

$$\begin{aligned} y_2 - y_1 &= f(y_1) - f(y_0) \subset \mu K(y_1 - y_0) = K(\mu(y_1 - y_0)) \\ y_3 - y_2 &= f(y_2) - f(y_1) \subset \mu K(y_2 - y_1) \\ &\subset \mu K(\mu(y_1 - y_0)) \\ &= \mu^2 K(y_1 - y_0), \end{aligned}$$

and by induction we obtain

$$y_{\iota+1} - y_\iota \subset \mu^\iota K(y_1 - y_0) \qquad \text{for } \iota = 1, 2, 3, \cdots$$

Consider the series $u_0 + u_1 + u_2 + \cdots$ where $u_0 = y_0$, $u_{\iota+1} = y_{\iota+1} - y_\iota$ for $\iota = 0, 1, 2, \cdots$. A Cauchy remainder $r_{\nu,\rho} = u_{\nu+1} + \cdots + u_{\nu+\rho}$

satisfies

$$r_{\nu,\rho} \subset \mu^{\nu} K(y_1 - y_0) + \mu^{\nu+1} K(y_1 - y_0) + \cdots + \mu^{\nu+\rho-1} K(y_1 - y_0)$$
$$= (\mu^{\nu} + \cdots + \mu^{\nu+\rho-1}) \, K(y_1 - y_0)$$

since $K(y_1 - y_0)$ is convex by K 1.

But since by K 5 the set $K(y_1 - y_0)$ is bounded there is a $\delta > 0$ corresponding to any neighborhood U of the origin such that $0 < \beta < \delta$ implies $\beta K(y_1 - y_0) \subset U$. By hypothesis $0 < \mu < 1$, so that $\sum_{i=1}^{\infty} \mu^i$ converges, and so for all sufficiently large ν, $r_{\nu,\rho} \subset \beta \, K(y_1 - y_0) \subset U$ for all ρ. Since Υ is supposed complete, the series $u_0 + u_1 + \cdots$ and therefore the sequence $y_0, y_1, \cdots$ converge to an element y' of Υ. But $y_i = f(y_{i-1})$ and f is continuous, so that $f(y_i) \to y'$. Hence $y' = f(y')$ q.e.d.

To demonstrate the uniqueness of the fixed point, let z and w be two solutions of $y = f(y)$ with $z \in \Upsilon$, $w \in \Upsilon$. Then

$$z - w = f(z) - f(w) \subset \mu \, K(z - w).$$

Now $\mu < 1$ by hypothesis. Putting $\alpha = 1/\mu$ we have $\alpha(z - w) \subset K(z - w)$ where $\alpha > 1$. Hence $z - w = \theta$ by K 2.

Remark: The property $\alpha x \bar{\in} K(x)$ for $\alpha > 1, x \neq \theta$ is not used in establishing the existence of a solution, but only in the uniqueness proof.

3. An existence theorem for a differential equation

Theorem 3.3. Let T be a locally convex, sequentially complete linear topological space and let $\mathcal{U}$ be a complete neighborhood system of the origin with the properties

(i) Each $U \in \mathcal{U}$ is convex

(ii) $U \in \mathcal{U}$ implies $\alpha U \in \mathcal{U}$.

Let I be a real open interval, G an open set of T. If $f(\tau, y)$ is continuous on IG to T and satisfies the "Lipschitz" condition

$$(3.1)\quad f(\tau, y) - f(\tau, z) \subset \mu \dot{\prod_{y-z \in U \in \mathcal{U}}} (U) \qquad (\mu \text{ a constant} > 0)$$

for all $\tau \in I$ and $y, z \in G$, then for any chosen $\alpha \in I$ and $a \in G$ we may assert that

(1) The differential system

$$(3.2)\quad \frac{dy}{d\tau} = f(\tau, y), \quad y(\alpha) = a$$

has at most one continuous solution with values in G.

(2) There exists a $\delta > 0$ such that the differential system (3.2) has a continuous solution $y = y(\tau)$ on the closed interval $[\alpha - \delta, \alpha + \delta]$ to G.

Existence proof: We seek for a continuous solution of the integral equation

$$(3.2)\text{a}\quad y(\tau) = a + \int_\alpha^\tau f(\sigma, y(\sigma))\, d\sigma$$

where the integral is of Riemann type. Since $f(\tau, y)$ is a continuous function of τ, y, $f(\tau, y(\tau))$ is a continuous function of τ for a continuous $y(\tau)$. (Cf. Alexandroff and Hopf (1.), p. 53) Hence by the corollary to theroem 2.20 the functional transformation

$$(3.3)\quad z(\tau) = a + \int_\alpha^\tau f(\sigma, y(\sigma))\, d\sigma$$

takes a continuous function $y(\tau)$ on I to G into a continuous function $z(\tau)$ on I.

Since $a \in G$ where G is open, and since the space T is regular and locally convex there is a convex neighborhood V of Θ such that $a + \bar{V} \subset G$. Choose $\beta \in I$ so as to satisfy simultaneously the inequalities

(3.4) (a.) $0 < \beta - \alpha < 1/2$ (b.) $(\beta - \alpha)a \subset 1/2 V$ (c.) $\beta - \alpha < 1/\mu$

and keep β fixed. Now if $y(\sigma) - a \in \bar{V}$ for $\sigma \in [\alpha, \beta]$ we have, on using the law of the mean for Riemann integrals (Theorem 2.19, (iii)), that

$$z(\tau) - a = \int_\alpha^\tau f(\sigma, y(\sigma))\, d\sigma \subset (\tau - \alpha)\, \overline{Co(a + \bar{V})}$$

Therefore, since V, and hence $\bar{V}$, is convex, we have by inequalities (3.4) that for $\tau \in [\alpha, \beta]$,

$$\begin{aligned} z(\tau) - a &\subset (\tau - \alpha)a + (\tau - \alpha)\bar{V} \\ &\subset \frac{(\tau - \alpha)}{(\beta - \alpha)} \tfrac{1}{2}\bar{V} + \tfrac{1}{2}\bar{V} \subset \tfrac{1}{2}\bar{V} + \tfrac{1}{2}\bar{V} = \bar{V}. \end{aligned}$$

That is the functional transformation (3.3) carries the set Y of continuous functions $y(\sigma)$ satisfying $y(\sigma) \in a + \bar{V}$ for $\sigma \in [\alpha, \beta]$ into a part of itself. It remains to show that the functional transformation (3.3), carrying Y into a part of itself has a "fixed point", i.e. that the integral equation (3.2)a has a solution. We shall prove this by the method of successive approximations, after a few necessary preliminaries.

Put $K(x) = \dot{\prod\limits_{x \in U \in \mathcal{U}}} (U)$. Then by theorem 3.1 the sets $K(x)$ form

a K-system for the space T, and we have $K(x) = \prod_{x \in \bar{U}, U \in \mathcal{U}}^{\cdot} (\bar{U})$
Hence for any $U \in \mathcal{U}$ we have $K(x) \subset \bar{U}$ for each $x \in \bar{U}$.
Let us denote by $K(M)$ the set $\sum_{x \in M}^{\cdot} K(x)$, where M is any subset of our space T. Then, from what was just shown, $K(\bar{U}) \subset \bar{U}$
But since $x \in K(x)$, we may also write $\bar{U} \subset K(\bar{U})$, so that $K(\bar{U}) = \bar{U}$ for each $U \in \mathcal{U}$.

Define $y_\iota(\tau)$ inductively by

$$(3.5) \quad \begin{cases} y_0(\tau) = a \\ y_{\iota+1}(\tau) = a + \int_\alpha^\tau f(\sigma, y_\iota(\sigma))\, d\sigma, \quad \iota = 0, 1, 2, \cdots \end{cases}$$

for $\tau \in [\alpha, \beta]$.

Clearly $y_\iota(\tau) \in Y$ for each ι, so that for each $\tau \in [\alpha, \beta]$ we have $y_\iota(\tau) \in a + \bar{V}$. In order to prove that the sequence $y_0(\tau)$, $y_1(\tau), y_2(\tau), \cdots$ is uniformly convergent for $\tau \in [\alpha, \beta]$, let W be an arbitrary neighborhood of the origin. Since $\mathcal{U}$ is a complete neighborhood system of θ, and since T is a regular Hausdorff space, there exists a $U \in \mathcal{U}$ with $\bar{U} \in W$. Clearly $y_1(\tau) - y_0(\tau)$ is a continuous function on $[\alpha, \beta]$ to T, and by the corollary to theorem 2.25, its range is a bounded set. That is there exists a positive constant γ such that $y_1(\tau) - y_0(\tau) \in \gamma U$ for all $\tau \in [\alpha, \beta]$. By an induction we shall show that $y_{\nu+1}(\tau) - y_\nu(\tau) \in \gamma \mu^\nu (\beta - \alpha)^\nu \bar{U}$ for $\nu = 1, 2, \cdots$.
This statement is evidently true for $\nu = 0$, so let us suppose that it is true for $\nu = \iota - 1$. On using the law of the mean for integrals and introducing the Lipschitz condition (3.1) we find that

$$y_{\iota+1}(\tau) - y_\iota(\tau) = \int_\alpha^\tau \big(f(\sigma, y_\iota(\sigma)) - f(\sigma, y_{\iota-1}(\sigma)) \big)\, d\sigma$$

$$\in (\beta - \alpha)\ \overline{Co \sum_{\sigma \in [\alpha,\beta]}^{\cdot} \big(f(\sigma, y_\iota(\sigma)) - f(\sigma, y_{\iota-1}(\sigma)) \big)}$$

$$\subset (\beta - \alpha)\ Co \sum_{\sigma \in [\alpha,\beta]}^{\cdot} \mu\, K\big(y_\iota(\sigma) - y_{\iota-1}(\sigma)\big)$$

where, of course, $K(x) = \dot{\prod}_{x \in U \in \mathcal{U}} (U)$.

The induction hypothesis states that

$$\sum_{\sigma \in [\alpha,\beta]}^{\cdot} \big(y_\iota(\sigma) - y_{\iota-1}(\sigma) \big) \subset \gamma \mu^{\iota-1} (\beta-\alpha)^{\iota-1}\, \overline{U}.$$

Hence, since the set operations $Co(M)$ and $K(M) = \sum_{x \in M}^{\cdot} K(x)$ are both homogeneous with respect to positive scalar multipliers we obtain

$$y_{\iota+1}(\tau) - y_\iota(\tau) \in (\beta - \alpha)\, \mu\ \overline{Co\ K\big(\gamma \mu^{\iota-1} (\beta-\alpha)^{\iota-1}\, \overline{U}\big)}$$

$$\subset \gamma (\beta-\alpha)^\iota \mu^\iota\ \overline{Co\ K(\overline{U})}$$

$$= \gamma (\beta-\alpha)^\iota \mu^\iota\ \overline{Co\ \overline{U}}$$

$$= \gamma (\beta-\alpha)^\iota \mu^\iota\ \overline{U}$$

The last equality is true because the closure of the convex set U is convex. The induction step is now established, and it follows that

$y_{\nu+1}(\tau) - y_\nu(\tau) \in \gamma (\beta-\alpha)^\nu \mu^\nu\, U$ for $\nu = 0, 1, 2, \cdots$

Since β satisfies the inequality (3.4), (c), the series $\sum_{\iota=1}^{\infty} (\beta-\alpha)^\iota \mu^\iota$ is convergent. Let κ be an integer so large that $\nu > \kappa$ implies

$\sum_{\iota=\nu}^{\infty} (\beta-\alpha)^{\iota} \mu^{\iota} < 1/\gamma$. Then we have for any integer $\nu > \kappa$ any positive integer ρ and for all $\tau \in [\alpha, \beta]$ that

$$
\begin{aligned}
y_{\nu+\rho}(\tau) - y_{\nu}(\tau) &= \sum_{\iota=0}^{\rho-1} (y_{\nu+\iota+1} - y_{\nu+\iota}) \\
&\in \gamma \sum_{\iota=0}^{\rho-1} (\beta-\alpha)^{\nu+\iota} \mu^{\nu+\iota}\, \bar{U} \\
&= \left(\gamma \sum_{\iota=0}^{\rho-1} (\beta-\alpha)^{\nu+\iota} \mu^{\nu+\iota}\right) \bar{U} \\
&\subset \bar{U} \subset W.
\end{aligned}
$$

The last two steps are justified since $\bar{U}$ is convex and contains θ . But since W was an arbitrarily chosen neighborhood of the origin, $y_0(\tau), y_1(\tau), \cdots$ is a fundamental sequence for each $\tau \in [\alpha, \beta]$. Since the space is complete, and the choice of κ was independent of $\tau \in [\alpha, \beta]$ it follows that the $y_\nu(\tau)$ converge uniformly to a function $w(\tau)$, which is therefore (cf. theorem 2.23) continuous on $[\alpha, \beta]$. On taking limits in both sides of (3.5), recognizing that $f(\tau, y_\nu(\tau)) \rightarrow f(\tau, w(\tau))$ uniformly, we see from theorem 2.24 that

$$w(\tau) = a + \int_{\alpha}^{\tau} f(\sigma, w(\sigma))\, d\sigma .$$

Differentiating both sides, making use of theorem 2.20, we have

$$\frac{d w(\tau)}{d\tau} = f(\tau, w(\tau)) \qquad \text{for each} \quad \tau \in [\alpha, \beta]$$

where $w(\alpha) = a$.

A similar proof shows that there exists a $\beta' \in I$, $\beta' < \alpha$, and a continuous function $z(\tau)$ on $\tau \in [\beta', \alpha]$ with

$$\frac{dz(\tau)}{d\tau} = f(\tau, z(\tau)) \qquad \text{for each } \tau \in [\beta', \alpha] \text{ , where}$$

$z(\alpha) = a$.

The conclusion (2.) of the theorem now follows on putting
$\delta = \min(|\beta-\alpha|, |\beta'-\alpha|)$, $y(\tau) = z(\tau)$ on $[\alpha-\delta, \alpha]$
and $y(\tau) = w(\tau)$ on $[\alpha, \alpha+\delta]$.

<u>Uniqueness Proof</u>: To establish conclusion (1.) of the theorem let $y_1(\tau)$ and $y_2(\tau)$ be two continuous solutions of the differential system (3.2) on I to G. Either $y_1(\tau) = y_2(\tau)$ on I or $y_1(\tau_1) \neq y_2(\tau_1)$ at some point $\tau_1 \in I$. We may suppose $\tau_1 > \alpha$, for if $\tau_1 < \alpha$, a similar proof applies. Put $\beta = \sup \sigma$, $y_1(\tau) = y_2(\tau)$ on $[\alpha, \sigma]$ so that $\alpha \leq \beta < \tau_1$, Then there is a γ, $0 < \gamma - \beta < 1/\mu$, such that
$y_1(\tau_0) - y_2(\tau_0) \neq \theta$ at some point $\tau_0 \in [\beta, \gamma]$. By hypothesis the functions $y_1(\tau)$ and $y_2(\tau)$ are continuous, so that $f(\tau, y_i(\tau))$, $i = 1, 2$ are continuous*. Hence by theorem 2.22, $y_i(\tau)$ satisfy the equation

$$y(\tau) = a + \int_\alpha^\tau f(\sigma, y(\sigma))\, d\sigma \qquad \text{for} \quad \tau \in I.$$

Thus, by our selection of β,

$$y_1(\tau) - y_2(\tau) = \int_\beta^\tau f(\sigma, y(\sigma))\, d\sigma \qquad \text{for} \quad \tau \in I,\ \tau > \beta.$$

Since $y_1(\tau_0) - y_2(\tau_0) \neq \theta$, there is a neighborhood U of the complete neighborhood system $\mathcal{U}$ of the origin such that

$$y_1(\tau_0) - y_2(\tau_0) \bar{\in} \bar{U}.$$

Now the function $y_1(\tau) - y_2(\tau)$ is continuous on the closed interval

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* Cf. Alexandroff and Hopf (1.), Satz III, p. 53.

$[\beta,\gamma]$, so its range for this interval is bounded (cor. to thm. 2.25) and there is a constant $\eta > 0$ such that

$$y_1(\tau) - y_2(\tau) \in \eta U \quad \text{for all} \quad \tau \in [\beta,\gamma].$$

Using the law of the mean and the Lipschitz condition, exactly as in the existence proof, we obtain for $\tau \in [\beta,\gamma]$,

$$(3.6)\quad \begin{cases} y_1(\tau) - y_2(\tau) = \displaystyle\int_\beta^\tau \big(f(\sigma, y_1(\sigma)) - f(\sigma, y_2(\sigma))\big)\,d\sigma \\ \qquad \in (\gamma-\beta)\,\overline{Co \displaystyle\sum_{\sigma\in[\beta,\gamma]}^{\cdot} \mu K(y_1(\sigma) - y_2(\sigma))} \end{cases}$$

Hence, since $\bar{U}$ is convex, and $K(\bar{U}) = \bar{U}$,

$$y_1(\tau) - y_2(\tau) \in (\gamma-\beta)\mu\eta\,\bar{U} \qquad \text{for all } \tau \in [\beta,\gamma].$$

Substituting this result in the right member yields

$$y_1(\tau) - y_2(\tau) \in (\gamma-\beta)^2\mu^2\eta\,\bar{U} \qquad \text{for } \tau \in [\beta,\gamma],$$

and by induction, it is easy to establish that $y_1(\tau) - y_2(\tau) \in (\gamma-\beta)^\iota\mu^\iota\eta\,\bar{U}$ for all $\tau \in [\alpha,\beta]$ and any given positive integer ι . But $0 < (\gamma-\beta)\mu < 1$, so that for sufficiently large ι , $(\gamma-\beta)^\iota\mu^\iota\eta < 1$. Thus for sufficiently large ι ,

$$y_1(\tau_0) - y_2(\tau_0) \in (\gamma-\beta)^\iota\mu^\iota\eta\,\bar{U} \subset \bar{U},$$

contrary to the hypothesis that $y_1(\tau_1) \neq y_2(\tau_1)$. Hence $y_1(\tau) = y_2(\tau)$ on I , and the uniqueness proof is complete.

4. <u>An application of the general theorems</u>

By specializing the abstract space T we can obtain existence theorems for various functional equations from the general theorems 3.2 and 3.3. We shall consider below an existence theorem for an infinite system of differential equations, obtained by interpreting T as the space* $\mathcal{V}$ of infinite dimensional vectors $x = (x_1, x_2, \cdots)$.

Since $\mathcal{V}$ is metrizable, a function $g(\tau, x)$, where τ is a real number, x a vector in $\mathcal{V}$ and the values of the function are also in $\mathcal{V}$, is continuous** if and only if $g(\tau_\nu, x^\nu) \to g(\tau, x)$ for any real sequence $\tau_\nu \to \tau$ and any sequence of vectors $x^\nu \to x$.
In terms of the components of the vectors g and x, this criterion for continuity reduces to: $g(\tau, x)$ is continuous if and only if, for each ι, $\tau' \to \tau$, $x'_\lambda \to x_\lambda$ for $\lambda = 1, 2, \cdots$, implies $g_\iota(\tau', x') \to g_\iota(\tau, x)$

In the space $\mathcal{V}$, consider the complete neighborhood system $\mathcal{U}$ of the origin consisting of all sets $U(\delta_1, \cdots, \delta_\nu) : \{y\}, |y_\lambda| < \delta_\lambda$, $\lambda = 1, 2, \cdots, \nu$, obtained by letting the δ_ι's run over the positive reals, the ν's over the positive integers. As we have already remarked in #1, each $U(\delta_1, \cdots, \delta_\nu)$ is convex, and we also have $\alpha U(\delta_1, \cdots, \delta_\nu)$ $= U(\alpha\delta_1, \cdots, \alpha\delta_\nu) \in \mathcal{U}$ for $\alpha > 0$, so that conditions (i) and (ii) of theorem 3.1 are satisfied by the neighborhood system $\mathcal{U}$. Evidently the set $K(x) = \dot{\prod\limits_{x \in U \in \mathcal{U}}} (U)$ is identical with the set

$$\{y\}, \quad |y_\iota| \leq |x_\iota|, \quad \iota = 1, 2, 3, \cdots$$

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*Cf. example 1°, Ch. 1, #7

** This is the "Heine condition" for continuity in metric spaces. See Alexandroff and Hopf (1.), p. 58.

Theorem 3.4 Let G_λ be a sequence of open sets of real numbers, all but a finite number of the G_λ's consisting of the real axis $(-\infty, +\infty)$ and let I be a real open interval. Let $f_\iota(\tau, x_1, x_2, \cdots)$ be an infinite set of real functions of an infinite number of real variables, and for each ι, suppose that $f_\iota(\tau, x_1, x_2, \cdots)$ is defined for $\tau \in I$, $x_\lambda \in G_\lambda$ and is continuous in the sense that $\tau' \to \tau$, $x'_\lambda \to x_\lambda$ implies $f_\iota(\tau', x'_1, x'_2, \cdots) \to f_\iota(\tau, x_1, x_2, \cdots)$ for all $\tau \in I$, $x_\lambda \in G_\lambda$. Finally, we suppose that for each ι, f_ι satisfies the Lipschitz condition

$$|f_\iota(\tau, y_1, y_2, \cdots) - f_\iota(\tau, z_1, z_2, \cdots)| \leq \mu |y_\iota - z_\iota|$$

for $\tau \in I$, $y_\lambda, z_\lambda \in G_\lambda$ where μ is a positive constant independent of ι. Then

(1.) for any chosen $\alpha \in I$, $a_\lambda \in G_\lambda$ the differential system

$$\begin{cases} \dfrac{dy_\iota}{d\tau} = f_\iota(\tau, y_1, y_2, \cdots) \\ y_\iota(\alpha) = a_\iota \end{cases}$$

has at most one set of continuous solutions $y_\iota = y_\iota(\tau)$ with $y_\iota(\tau) \in G_\iota$ for $\tau \in I$.

(2.) there exists a $\delta > 0$ such that this differential system has a set of continuous solutions $y_\iota = y_\iota(\tau)$ for $|\tau - \alpha| < \delta$, where $y_\iota(\tau) \in G_\iota$ for $|\tau - \alpha| < \delta$.

Proof: Immediate from Theorem 3.3, the above considerations, and the fact that $\mathfrak{V}$ is sequentially complete (cf. discussion of ex. 1^o, ch. 1, #7).

5. Completely continuous linear transformations*

Let $y = f(x)$ be a linear transformation of a linear topological space T into a sub-set L of T. If there is a neighborhood U of the origin whose transform $f(U)$ is a compact set, then we shall say that f is a completely continuous linear transformation (function). It is clear that a completely continuous linear transformation f takes bounded sets into compact sets, for if S is bounded, then for some real α, $S \subset \alpha U$ and

$$f(S) \subset f(\alpha U) = \alpha f(U),$$

whence $f(S)$ is compact. Thus, when T is a linear normed space, the above definition of complete continuity reduces to the usual one*.

We have proved in theorem 1.9, chap. 1, #6 that every locally compact linear topological space is finite dimensional. This suggests the possibility of extending the Riesz theory of completely continuous linear functional transformations to linear topological spaces, with the above definition of complete continuity for such spaces. I shall show below that several of the simpler theorems of Riesz concerning the equation $y = x - f(x)$ where $f(x)$ is completely continuous are immediately extensible. The following theorem illustrates the power of topological methods and will often permit simplification of Riesz's proofs. What is more important for us, it will make it possible to eliminate the role of the norm.

~~* Cf. Riesz (1.), p. 74.~~

* Author's Note:

The proof of the key theorem 3.5 is incorrect. Therefore section 5 should be omitted. For a correct treatment of the Riesz theory in the case of locally bounded spaces see the author's paper in Revista de Ciencias, vol. 41 (1939) pp. 555-574.

<u>Theorem 3.5: If $y=f(x)$ is a completely continuous linear transformation carrying a linear topological space T into a subset L of T, then L is a finite dimensional linear manifold.</u>

Proof: L is obviously a linear manifold, for if $y_1 \in L$, $y_2 \in L$, where $y_1 = f(x_1)$, $y_2 = f(x_2)$, then $\alpha y_1 + \alpha y_2 = f(\alpha x_1 + \alpha x_2) \in L$.

We wish to find a topology for L which will make L a locally compact linear topological space.

Let U be any neighborhood of the origin. For any $y \in L$ put $\tilde{U}_y = y + f(U)$. Then with the sets $\tilde{U}_y$ as fundamental sets, L forms a linear topological space.

<u>Verification of the postulates</u>

(I a) Obvious, since $\theta \in U$

(I b) Given $\tilde{U}_y = y + f(U)$, $\tilde{U}'_y = y + f(U')$, choose $U'' \subset U \cdot U'$. Then clearly $f(U'') \subset f(U)$ and $f(U'') \subset f(U')$.

Hence $\tilde{U}''_y = y + f(U'') \subset (y + f(U)) \cdot (y + f(U'))$.

That is $\tilde{U}''_y \subset \tilde{U}_y \cdot \tilde{U}'_y$.

(I c) If $z \in \tilde{U}_y = y + f(U)$ then $z = y + f(x_1)$, where $x_1 \in U$. Take U' so that $x_1 + U' \subset U$. Then, since f is linear, $f(x_1) + f(U') = f(x_1 + U') \subset f(U)$, whence $\tilde{U}'_z = z + f(U') = y + f(x_1) + f(U') \subset y + f(U)$

i.e., $\tilde{U}'_z \subset \tilde{U}_y$.

(I d) If $y \neq z$, $y, z \in L \subset T$, there is a neighborhood V of the origin such that $z - y \bar{\in} V$. Since

$f(x)$ is continuous at θ there is a U such that $f(U) \subset V$.
Hence $z - y \,\bar{\epsilon}\, f(U)$, or, putting $\tilde{U}_y = y + f(U)$
we have $z \,\bar{\epsilon}\, \tilde{U}_y$.

(II a) Given $\tilde{U}_{y+z} = y + z + f(U)$, choose V so that $V + V \subset U$
Then, since f is linear, we have

$$y + f(V) + z + f(V) = y + z + f(V+V) \subset y + z + f(U)$$

That is, $\tilde{V}_y + \tilde{V}_z \subset \tilde{U}_{y+z}$

(II b) Given $\tilde{U}_{\alpha y} = \alpha y + f(U)$, let the real interval I_α and the neighborhood U' of θ be such that $\beta \in I_\alpha$ implies
$\beta(x + U') \in \alpha x + U$, where $y = f(x)$. Then, since f is linear, we obtain

$$\beta(y + f(U')) = \beta f(x + U') = f(\beta(x + U')) \subset f(\alpha x + U) = \alpha y + f(U),$$

for all $\beta \in I_\alpha$. That is, on putting $\tilde{U}'_y = y + f(U')$, $I_\alpha \tilde{U}'_y \subset \tilde{U}_{\alpha y}$
Thus L forms a linear topological space $\mathcal{L}$ if we take the sets $\tilde{U}_y = y + f(U)$ as fundamental sets. But since $f(x)$ is supposed completely continuous, there is a neighborhood U whose transform $f(U)$ is compact. Hence $\mathcal{L}$ is locally compact, and by theorem 1.9, chapter 1, #6, the "basis" L of $\mathcal{L}$ is finite dimensional. ?

<u>Theorem 3.6</u> <u>If $y = f(x)$ is a completely continuous linear transformation of a linear topological space T into a sub-set L of T then the equation $x - f(x) = \theta$ has only a finite number of linearly independent solutions.</u>

<u>Proof</u>: The set of points x for which $x - f(x) = \theta$ forms a linear

sub-space M of the finite dimensional linear space L which is the range of the transformation $y = f(x)$.

Theorem* 3.7 If $f(x)$ is a completely continuous linear function, and if the equation $y = x - f(x)$ has a solution for each y, then the homogeneous equation $x - f(x) = \theta$ has the unique solution $x = \theta$.

Proof: Put $t'(x) = t(x) = x - f(x)$

$$t^{(\nu)}(x) = t(t^{(\nu-1)}(x)), \quad \nu = 2, 3, \cdots$$

Let M_ν be the linear manifold consisting of all the elements $x \in T$ for which $t^{(\nu)}(x) = \theta$. Suppose that there is an element $x_1 \neq \theta$ for which $t(x_1) = \theta$. By hypothesis there is a solution $x = x_2$ of $t(x) = x_1$, a solution $x = x_3$ of $t(x) = x_2$, and a solution $x = x_\nu$ of $t(x) = x_{\nu-1}$, for any chosen ν, i.e. $x_{\nu-1} = t(x_\nu)$, $\nu = 2, 3, \cdots$. Then $t^{(\nu)}(x_{\nu+1}) = x_1 \neq \theta$, while $t^{(\nu+1)}(x_{\nu+1}) = \theta$.
That is $x_{\nu+1} \in M_{\nu+1} \dot{-} M_\nu$. But obviously $M_\nu \subset M_{\nu+1}$ so that M_ν is a proper subset of $M_{\nu+1}$, for $\nu = 1, 2, \cdots$.
Using operator notation, where "e" denotes the identity operator, we have

$$t^{(\nu)}(x) = (e - f)^\nu x = (e - g_\nu)x = x - g_\nu(x)$$

where $g_\nu = -\sum_{\iota=1}^{\nu} \binom{\nu}{\iota}(-f)^\iota = f\left(\sum_{\iota=1}^{\nu} \binom{\nu}{\iota}(-f)^{\iota-1}\right)$

and where $\binom{\nu}{\iota} = \nu!/\iota!(\nu-\iota)!$. Clearly then, the range $g_\nu(T)$ of the completely continuous transformation g_ν is

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* Cf. Riesz (1.), p. 82 and Banach (1.), p. 153.

contained in the range $f(T)$ of the completely continuous transformation f, and therefore, since $M_\nu \subset g_\nu(T)$, we have $M_\nu \subset f(T)$ for $\nu = 1, 2, 3, \cdots$. Now by theorem 3.5, the linear manifold L is finite dimensional, so that there must be an integer μ such that $M_\mu = M_{\mu+1}$, since a finite dimensional linear space cannot contain an infinite sequence of linear manifolds, each a proper subset of the following one. Thus by supposing $x_1 \neq \theta$ we have arrived at a contradiction, and the theorem follows. We have incidentally proved the

Corollary*: There is a non-negative integer ν such that for $\mu > \nu$ every solution of the homogeneous equation $t^{(\mu)}(x) = \theta$ also satisfies $t^{(\nu)}(x) = \theta$, while for $0 \leq \mu < \nu$, the equation $t^{(\mu+1)}(x) = \theta$ has a solution not satisfying $t^{(\mu)}(x) = \theta$.

Proof: Take ν to be the first integer such that $M_\nu = M_{\nu+1}$.

Theorem 3.8 Put $t(x) = x - f(x)$, and denote by L_μ the range $t^{(\mu)}(T)$ of the linear transformation $t^{(\mu)}(x)$. Then there exists an integer ν such that for $\mu > \nu$, $L_\mu = L_\nu$, but for $0 \leq \mu < \nu$, $L_{\nu+1}$ is a proper subset of L_ν. This integer ν is the same as that of the preceding corollary.

Proof: Clearly L_μ, being the range of a linear transformation is a linear manifold, and since $t^{(\mu+1)}(T) = t^{(\mu)}(t(T))$, we have $L_{\mu+1} \subset L_\mu$. Also, if $L_{\mu+1} = L_\mu$, then $L_\rho = L_\mu$ for all $\rho > \mu$. Thus, if the theorem does not hold, then every $L_{\mu+1}$ is a proper subset of L_μ. Therefore there exists an element x_μ in

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* Cf. Riesz (1.), p. 81

L_μ, but not in $L_{\mu+1}$. Now we have

$$f(x_\mu) - f(x_{\mu+1}) = x_\mu - (x_{\mu+1} + t(x_\mu) - t(x_{\mu+1}))$$
$$= x_\mu - x$$

where $x = x_{\mu+1} + t(x_\mu) - t(x_{\mu+1}) \in L_{\mu+1}$ since

$x_{\mu+1},\ t(x_\mu),\ t(x_{\mu+1}) \in L_{\mu+1}$

Since $x \in L_{\mu+1}$, while $x_\mu \bar{\in} L_{\mu+1}$, we see that $x_\mu - x \bar{\in} L_{\mu+1}$. Hence $f(x_\mu - x_{\mu+1}) \in L_\mu \dot{-} L_{\mu+1}$. But if $L_{\mu+1}$ is a proper sub-set of L_μ for all μ, then we are led to an infinite sequence of linearly independent elements $y_\mu = f(x_\mu - x_{\mu+1})$, with $y_\mu \in L = f(T)$, contrary to the fact that L is finite dimensional (cf. theorem 3.5). Therefore the first conclusion of the theorem holds, and we have $L_\mu \neq L_{\mu+1}$ for $\mu < \nu$, but $L_\nu = L_{\nu+1}$. That is $t(x)$ takes L_ν into itself. Hence, for any $y \in L_\nu$, there exists a solution $x = z \in L_\nu$ of the equation $t(x) = y$. The rest of the argument may be carried over word for word from Riesz's proof, see Riesz (1.) p.85.

From theroem 3.7 and corrollary, and from theorem 3.8 we obtain at once the following criterion for the solution of the equation $t(x) = y$

<u>Theorem 3.9</u> <u>If $f(x)$ is a completely continuous linear function on T to T then the equation $x - f(x) = y$ has a solution for all $y \in T$ if and only if the homogeneous equation $x - f(x) = \theta$ has no solution other than $x = \theta$.</u>

<u>Proof</u>: If $x = \theta$ is the unique solution of $t(x) = \theta$ then we clearly have the case $\nu = 0$ of theorem 3.8. But then $t(x)$ takes

$L_0 = T$ into itself, and for every $y \epsilon T$, the equation $t(x) = y$ has a solution. The necessity was proved in theorem 3.7.

Appendix I: The Kuratowski Postulates for a Topological Space.

Let $\overline{}$ be an undefined set operation which orders to each subset M of a "space" E, a subset $\overline{M}$ of E, subject to the following postulates (cf. Kuratowski (1))

(i) $\overline{M \dotplus N} = \overline{M} \dotplus \overline{N}$

(ii) $p \in E \;.\!\supset\!.\; (\overline{p}) = (p)$ where (p) is the set consisting of a single element p.

(iii) $\overline{\overline{M}} \subset \overline{M}$

(iv) $\overline{0} = 0$ where 0 is the null set.

A space E satisfying (i) - (iv) also satisfies postulates (Ia) - (Id), and conversely, as is shown for example in Alexandroff and Hopf (1.) p. 43 and p. 59. If T is a linear space satisfying (i) - (iv), and if in addition the following postulates hold for T.

(v) $\overline{M} + \overline{N} \subset \overline{M + N}$ for all sub-sets M, N of T

(vi) $\overline{A}\,\overline{M} \subset \overline{AM}$ for all sets A of real numbers and all sub-sets M of T, where $\overline{A}$ denotes the closure of A, then is the "basis" for a linear topological space $\mathcal{J}$, where neighborhoods U_x are open sets containing x, a set U being defined as open if $\overline{T \dot{-} U} = T \dot{-} U$. The postulates (v) and (vi) are simply another way of stating the continuity of the operations of vector addition and scalar multiplication, making use of the following definition of continuous functions:

Let E and E' be spaces satisfying the Kuratowski postulates (i) - (iv). Then a function f on E to E' will be said to be continuous on E if $f(\bar{M}) \subset \overline{f(M)}$ for any subset M of E. For the proof of the equivalence of this definition of continuity to the definition in terms of neighborhoods used in Chapter I, see Alexandroff and Hopf (1.) pp. 52 - 54 (Satz IV). The postulates (i) - (iv) are essentially those used in Kolmogoroff (1.) and Tychonoff (1.).

Appendix II: Pseudo-normed spaces.

A linear space T will be said to be pseudo-normed* if to each of T there corresponds a real number $|x|$ satisfying the conditions

(1.) $|x| \geqslant 0$; $|x| = 0 \;.\supset.\; x = \theta$

(2.) $|\alpha x| = |\alpha|\,|x|$

(3.) If $|x| \to 0$, $|y| \to 0$ then $|x+y| \to 0$.

We shall show that a linear pseudo-normed space is a linear topological space in the following way: If S is any sub-set of T we denote by $\bar{S}$ the set of "contact" points of S, where by a contact point of S we mean a point x such that for every $\delta > 0$ there is a $y \in S$ with $|y - x| < \delta$. Then we may demonstrate that the postulates (i) - (iv) of Kuratowski are verified. (See Appendix I)

Ad (i) It is clear that $\bar{M} \dot{+} \bar{N} \subset \overline{M \dot{+} N}$. To prove that $\overline{M \dot{+} N} \subset \bar{M} \dot{+} \bar{N}$, take $x \in \overline{M \dot{+} N}$. If x is not a contact point of N then there is a δ such that no point y of N satisfies $|x - y| < \delta$. Therefore for each $\eta > \delta$ there is a $z \in M$ such that $|x - z| < \eta$, i.e. $x \in \bar{M}$. Similarly if $x \bar{\in} \bar{M}$ then $x \in \bar{N}$. Postulates (ii) and (iv) are evidently satisfied.

Ad (iii) Given $\eta > 0$, by property (3.) of the pseudo-norm there exists a $\delta > 0$ such that $|x+y| < \eta$ for $|x|, |y| < \delta$. Given $x_0 \in \bar{\bar{M}}$, there exists an $x \in \bar{M}$ such that $|x - x_0| < \delta$, and similarly there exists a $y \in M$ such that $|y - x| < \delta$. Hence $|y - x_0| < \eta$, where $y \in M$, i.e. $\bar{\bar{M}} \subset \bar{M}$.

Ad (v) If $x \in \bar{M}$, $y \in \bar{N}$, there exists sequences $m_\nu \in M$, $n_\nu \in N$, with $|m_\nu - x| \to 0$, $|n_\nu - y| \to 0$. Hence by (3.),

*This pseudo-norm is not to be confused with Neumann's pseudo-metric for linear topological spaces.

$|m_\gamma + n_\gamma - x - y| \to 0$ i.e. $x + y \in \overline{M+N}$.

Ad (vi): Follows from the fact that

$$|\alpha x - \alpha_0 x_0| = |\alpha x - \alpha x_0 + \alpha x_0 - \alpha_0 x_0| \to 0$$

as $|\alpha x - \alpha x_0| = |\alpha|\,|x - x_0| \to 0$ and $|\alpha x_0 - \alpha_0 x_0| = |\alpha - \alpha_0|\,|x_0| \to 0$.

From Ch. 1, #6 we now have the

<u>Theorem</u> <u>The necessary and sufficient condition for a linear topological space to be pseudo-normable is that it contain a bounded open set.</u>

It is clear from Tychonoff's example $H_{1/2}$, and the results of ch. 1, #6 that a pseudo-normed space is metrizable, but not necessarily normable.

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