A CLARIFICATION OF HEAVISIDE'S OPERATIONAL CALCULUS

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CONTENTS

1. INTRODUCTION

Heaviside's operational calculus[1] is fast becoming one of the most useful tools in the solution of differential equations, its power becoming generally realized and its methods expanded. However the procedures now employed, far different from those initially developed by Heaviside, involve contour integration and integral equation solution[2] rather than the much simpler algebraic manipulations of the original treatment. The supplantation was necessitated since Heaviside's treatment, making no pretense to rigor, involves unexplained rules and occasionally leads to contradictions. In effect his methods have remained merely processes of rapid evaluation which have required verification from external sources before the solutions could be deemed acceptable.[2]

Because of its remarkable simplicity, Heaviside's treatment would be of great value if it could be placed on a sound mathematical basis, despite the fact that contour integral methods can now be used to evaluate operational equations with complete mathematical rigor. It would be well to correlate the contour integral methods with Heaviside's methods in theory as well as in application.

Since with only a few exceptions Heaviside's treatment leads to correct solutions, the situation indicates that a fundamentally correct procedure underlies the work, requiring only a more careful analysis of its basic elements before rigorous operational manipulations may be discovered. Though several attempts[3] have been made to render the treatment rigorous, none of the suggested methods has succeeded in retaining the original simplicity, and our present mode of attack will be different from the above.

By but slight modification of the fundamentals of Heaviside's treatment, it will be shown possible to place the work upon a sound basis, retaining the algebraic manipulations and their accompanying simplicity. In the course of the paper the established methods will be applied to pertinent topics covered by operational calculus, permitting the treatment to be extended directly to more involved problems. The ambiguity and uncertainty of earlier operational procedures will be eliminated and the difficulties encountered by Heaviside clarified.[*]

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[*] For the sake of expedience, we shall adopt the present tendency of referring to Heaviside's treatment simply as "operational calculus" and to Carson's infinite integral equation methods and Bromwhich's and Wagner's contour integral methods as "Symbolic calculus".

2. THE UNIT FUNCTION

To represent a force which is suddenly applied at the time $t=0$, Heaviside formulated the so-called "unit function", designated by the symbol "$1(t)$", which he defined as a function such that it is zero for time before $t=0$ and unity after $t=0$. He did not specify the value of unit function at $t=0$, intimating it to be discontinuous at that instant. Such a definition of $1(t)$ is used satisfactorily in symbolic calculus because the latter, being based upon Fourier transforms, is not disturbed by a function with a single discontinuity. However, in Heaviside's calculus, where, for example, infinite derivative series operating on unit function are encountered, a discontinuous unit function presents serious mathematical difficulties, which have caused the methods to be non-rigorous. In the mode of approach which we shall adopt, the discontinuous unit function will be replaced by a continuous one, all of whose derivatives are continuous, though it will be unnecessary to designate the latter function explicitly by a specific expression.

Let us consider the electrical phenomenon which is most frequently represented by the unit function, namely that of a constant electromotive force suddenly impressed at time $t=0$ upon a circuit. In such an instance the potential established across the terminals of the circuit, though the potential appears to rise immediately, of course requires an appreciable time to build up to its full value. In actuality the exact nature of the potential rise at the terminals of the circuit cannot be determined. The problem of indeterminateness, however, is generally of no consequence, and representation of the phenomenon is considered to be approximated with sufficient accuracy by a stepped function. It is our intention now to examine whether the phenomenon can be represented also by a function which is continuous and all of whose derivatives are continuous. The investigation will be complete if it is shown that the difference in the effects of the forces which are represented by the two functions can be made as small as desired. The response in the neighborhood of $t=0$ is excluded from consideration because at that value for t the continuous and the discontinuous unit functions are not made even approximately equal, though the neighborhood can be made as small as desired.

Since only linear differential equations are considered in operational calculus, we may apply freely the superposition theorem (or the Boltzmann-Hopkinson theorem). Using electrical nomenclature after the manner of Carson[4], let $A(t)$ be the indicial admittance* of a circuit. Then

*The "indicial admittance" of a circuit is its current response to the application of a discontinuous unit function potential.

the current* response $i(t)$ due to the application of potential $e(t)$ to the circuit is given by

$$i(t) = \int_{u=-\infty}^{t} A(t-u)\, de(u) . \qquad (1)$$

When $e(t)$ is identically zero for negative values of t, the current response becomes

$$i(t) = \frac{d}{dt}\int_{0}^{t} A(t-u)\, e(u)\, du .$$

We shall proceed to compare two responses: the first, $i_1(t)$, is the current response when $e(t)$ is given by $f(t)\, 1_D(t)$, where $1_D(t)$ is the discontinuous unit function; and the second, $i_2(t)$, is the response when $e(t)$ is given by $f(t)\, 1(t)$, where $1(t)$ is the continuous unit function which is under consideration. Then the difference in the responses is

$$\Delta i(t) = i_1(t) - i_2(t) = \frac{d}{dt}\int_{0}^{t} A(t-u)\, f(u)\, du - \int_{u=-\infty}^{t} A(t-u)\, d\left(f(u)\, 1(u)\right). \qquad (2)$$

For physical problems, the difference $\Delta i(t)$ is a maximum when the indicial admittance is exponentially cumulative, $A(t) = C_1 \varepsilon^{at}$, and the force function is exponentially dissipative, $f(t) = C_2 \varepsilon^{-bt}$, where both a and b are real and positive. These conditions then represent the most exacting test which we need apply to the unit function.

As an example of a function which satisfies the necessary conditions for $1(t)$ in that it and all its derivatives

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*We shall designate current by the letter "i" and the imaginary unit, $\sqrt{-1}$, "j".

are continuous and can be made to approximate $1_D(t)$ as closely as desired, we may cite Maxwell's "discontinuous" function[5] :

$$M(t) = \frac{\varepsilon^{nt}}{1+\varepsilon^{nt}} \,, \qquad (3)$$

which may be written[6]

$$M(t) = \frac{1}{2}\left(1 + \tanh\frac{nt}{2}\right). \qquad (4)$$

The function becomes discontinuous in the limit as $n \rightarrow \infty$, so that n must be kept finite in our treatment.

It remains to determine whether Maxwell's function is such that, when it is used for $1(t)$, $\Delta i(t)$ can be made as small as desired by choosing n sufficiently large. For simplicity we shall evaluate $\Delta i(t)$ for finite negative values of t not in the neighborhood of $t=0$. Then for a given negative value of t , if n is sufficiently large, $1(t)$ acts as ε^{nt} , where $1(t)$ is identified by $M(t)$. Introducing the above most exacting conditions and the Maxwell function into Eq. (2), we obtain on evaluation that

$$|\Delta i(t)| < \left|\frac{n-b}{n-a-b} C_1 C_2 \varepsilon^{(n-b)t}\right| \;; \quad t<0 \;, \quad n \gg 0 \,.$$

Here $\Delta i(t)$ can be made as small as desired by choosing n sufficiently large. The additional difference in response when $t>0$ obviously can be made as small as desired by choosing n sufficiently large. We observe then that even for the most severe case which we may have occasion to encounter in physical problems, the difference of response to $f(t)\,1(t)$, where $1(t)$ is Maxwell's function, and to $f(t)\,1_D(t)$, where $1_D(t)$ is identically zero for $t<0$, may be made vanishingly small by the proper choice of the parameter n . The above consideration has been limited to application to physical problems, though it seems probable that the procedure may be extended directly to a rigorous mathematical treatment.

We observe that Maxwell's function would serve admirably for our proposed unit function since: it and all its derivatives are continuous; it can be made to approach the discontinuous force function as closely as desired; and the difference of response to the two functions can be made as small as desired. Of course other functions can be made to serve equally favorably as the unit function. Nevertheless, because the exact nature of the unit function need never be ascertained, it is preferred to leave the function undefined explicitly. There is another important reason for not defining the unit function by any particular expression such as the Maxwell function. Consider as an illustration the problem of a resistance R suddenly inserted at time $t=0$ across the terminals of a battery of constant electromotive force E . Then the potential across R is $E\,1(t)$ and the charge which has passed through the resistance is given by

$$q = \frac{E}{R}\int_{-\infty}^{t} 1(u)\,du\,. \qquad (5)$$

If $1(t)$ is defined by Maxwell's function, we obtain on integration that

$$q = \frac{E}{R}\,t\left[\frac{1}{2}\left(1+\frac{2}{nt}\log\cosh\frac{nt}{2}\right)\right]. \qquad (6)$$

The factor $\frac{1}{2}(1+\frac{2}{nt}\log\cosh\frac{nt}{2})$ and all its derivatives are continuous, and also, like Maxwell's function, the factor can be made to approach the discontinuous unit function as closely as desired by choosing n sufficiently large. Notwithstanding this, if $1(t)$ is defined by Maxwell's function, we cannot write

$$q = \frac{E}{R}t\ 1(t) \ . \qquad (7)$$

Though in the limit as $n \longrightarrow \infty$, the above factor and Maxwell's function are equivalent, we are prohibited from going to that limit when continuous functions are demanded. That is, since in operational equations we permit only continuous functions for operands, when q reappears in an operational equation, n must remain finite. It is only in the final solution that the limit can be taken.

These considerations lead us to define the unit function which we shall employ in operational calculus in the following general manner. The unit function, denoted by " $1(t)$ ", ~~it~~ is a function such that it and all its derivatives are continuous and one which can be made to approach the discontinuous stepped function of Heaviside as closely as desired (except in the neighborhood of the step, which neighborhood can be made as narrow as desired).

3. MULTIPLE ORDER IMPULSES

The unit function not being specified explicitly, the derivatives of the function also remain unspecified, though just as in the case of the unit function, this indeterminateness generally introduces no difficulty.

The first order impulse, namely $\frac{d}{dt}1(t)$, approximates zero for all values of t except in the neighborhood of $t=0$, where a sharp peak occurs. For our purposes the property of the impulse which is of importance is that the area bounded by the curve $\frac{d}{dt}1(t)$ and the t axis must be unity and that the area must be concentrated as closely to the instant $t=0$ as desired. Both of these conditions are obviously satisfied by the definition of $1(t)$.

In regard to multiple order impulses an analogous situation exists. It is noted that $\frac{d}{dt^n}1(t)$ approximates zero for all values of t except in the neighborhood of $t=0$, provided that n is finite. (The infinite order impulse does not exist for any value of t). The other property of the multiple order impulse which must be considered is the value of the integral $\int_{-\infty}^{t} f(u)\frac{d^n 1(u)}{du^n}du$, where $f(t)$ and all its derivatives up to and including the $(n-1)th$, exist in the neighborhood of $t=0$. The expression is readily evaluated on integration by parts. We begin with the first order impulse:

$$\int_{-\infty}^{t} f(u)\frac{d1(u)}{du}du = f(0)\,1(t) ,$$

a result which is a consequence of the fact that the area bounded by the curve $\frac{d}{dt}1(t)$ and the t axis is unity and concentrated in the neighborhood of $t=0$. Similarly on integrating by parts, we obtain

$$\int_{-\infty}^{t} f(u)\frac{d^2 1(u)}{du^2}\,du = f(u)\frac{d\,1(u)}{du}\Big|_{-\infty}^{t} - \int_{-\infty}^{t} f'(u)\frac{d\,1(u)}{du}\,du,$$

which becomes

$$= [-f'(u)]_{u=0}\,1(t), \qquad t \neq 0.$$

The method may be extended to the general case:

$$\int_{-\infty}^{t} f(u)\frac{d^n 1(u)}{du^n}\,du = (-1)^{n-1}\left[f^{(n-1)}(u)\right]_{u=0} 1(t) \quad ; \quad n \text{ finite}, \quad t \neq 0. \tag{8}$$

This identity gives the important property of the nth order impulse.

An interesting application of a derivative series is the expansion of the operator $\varepsilon^{a\frac{d}{dt}}$, which is defined by the infinite series corresponding to an exponential. If the operand is $1(t)$ we obtain

$$\varepsilon^{a\frac{d}{dt}}1(t) \equiv \left(1 + a\frac{d}{dt} + \frac{a^2}{2!}\frac{d^2}{dt^2} + \cdots + \frac{a^n}{n!}\frac{d^n}{dt^n} + \cdots\right)1(t). \tag{9}$$

This series is recognized as an expansion about $t=0$ by Taylor's theorem, i. e.

$$\varepsilon^{a\frac{d}{dt}}1(t) \equiv 1(t+a), \tag{10}$$

where the unit function has been transferred an interval of time, a. The correlation with a Taylor's series has been made possible only by defining $1(t)$ and all its derivatives as continuous. It is observed that, though individual multiple order impulses including those of infinite order are not evaluated, the infinite series yields a result.

4. ALGEBRAIC PROPERTIES OF THE "p" OPERATOR

The customary notation wherein "p" denotes the derivative operator $\frac{d}{dt}$ and "p^n" denotes $\frac{d^n}{dt^n}$ will be used throughout. Consistent with Heaviside's methods, the "p" operator will be considered simply as abbreviatory for "$\frac{d}{dt}$" and is not to be treated as a symbolic eqivalent for "$\frac{d}{dt}$" in the manner of Carson in his infinite integral equation or Bromwich, Wagner, etc. in the case of the contour integral solution.

The ~~these~~ three necessary and sufficient conditions* which permit algebraic treatment ~~of~~ between the "p" operators can be established if the inverse operator "$\frac{1}{p}$" is appropriately defined and the operand is of appropriate form. It is first evident that combinations of the derivative operators p^a and p^b, where a and b are positive real integers satisfy the algebraic laws.

We shall define the inverse operator, namely $\frac{1}{p}$ or p^{-1},

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*The three laws are as follows:
(1) the commutative law -------------- $uv = vu$
(2) the index law ------------------ $u^a u^b = u^{a+b}$
(3) the distributive law ------------- $a(u+v) = au + av$
where a and b are constants, and u and v belong to the class of terms under examination.
Of course the "p" operators are not commutative with respect to functions of t.

by the equation

$$\frac{1}{p} f(t) = \int_{-\infty}^{t} f(u)\,du \,. \qquad (11)$$

The definition is restricted to the condition that $\int_{-\infty}^{t} |f(u)|\,du$ be finite, a condition which requires that the integrand vanish at the lower limit of Eq. (11). It is the latter result which is sought and is of utmost importance for it permits the commutation of the operators p and $\frac{1}{p}$; that is, if $\int_{-\infty}^{t} |f(u)|\,du$ is finite, we obtain the identities

$$\frac{1}{p} p f(t) \equiv p \frac{1}{p} f(t) \equiv f(t) \,,$$

and as found by ready extension of this relation,

$$p^{a} p^{b} f(t) \equiv p^{b} p^{a} f(t) \equiv p^{(a+b)} f(t) \,,$$

where a and b are any real integers, either positive or negative, and the integrands to the $\frac{1}{p}$ operation comply with the specified condition. Since the distributive law is also satisfied by the above operators, the algebraic property for such operators is established, and we may proceed with complete validity to use Heaviside's algebraic manipulations.

The condition for algebraic treatment, which we shall call the "algebrization condition", namely that the integrand $f(t)$ is such that $\int_{-\infty}^{t} |f(u)|\,du$ is finite, generally offers no difficulty in operational calculus, because all operands which are encountered in that calculus include the unit function as a factor.

5. INVERSE POLYNOMIAL OPERATORS

With the algebraic interpretation of combinations of p and $\frac{1}{p}$ operators established, other operator forms will now be examined. Consider the operational equation

$$(p+a)\, i(t) = 1(t) \quad , \qquad (12)$$

where $i(t)$ may be the current response of a series circuit, including unit inductance and resistance a, when unit function potential is impressed on the circuit. The character of a is limited only in that it must be a finite constant. An operational solution of the equation rests upon the conception of inverse operators. The inverse operator, $\frac{1}{p+a}$, we define formally as such that

$$(p+a)\frac{1}{p+a} f(t)\,1(t) \equiv f(t)\,1(t) \quad , \qquad (13)$$

where $f(t)\,1(t)$ satisfies the algebrization condition.

Let the inverse operator be expanded in negative powers of p according to the binomial theorem:

$$\frac{1}{p+a} 1(t) = \left(\frac{1}{p} - \frac{a}{p^2} + \frac{a^2}{p^3} - \cdots + (-1)^n \frac{a^n}{p^{n+1}} + \cdots\right) 1(t) \, . \qquad (14)$$

If the series as a function of t is uniformly convergent, it is obviously an inverse operator, since operation on both sides of Eq. (14) with $(p+a)$ reduces the equation to an identity. For a discontinuous unit function the series is convergent and equals $\frac{1}{a}(1-\varepsilon^{-at})1_D(t)$ as can be readily verified by performing the indicated operations. It is evident that when $1(t)$ is a continuous function, the series becomes uniformly convergent and approximates the above value as closely as desired by choosing $1(t)$ sufficiently close to $1_D(t)$. The resulting solution is in terms of a continuous function, and using the generality of the definition of $1(t)$, we obtain

$$\frac{1}{p+a}\,1(t) = \frac{1}{a}\left(1-\varepsilon^{-at}\right)1(t)\,. \qquad (15)$$

This becomes the solution of Eq. (12).* In a manner similar to the above, any operational expression may be expanded algebraically into an infinite series, provided that the resultant series as a function of t is uniformly convergent. In Eq. (14), for example, the operand $1(t)$ may be substituted by any function of t provided that the resultant series is uniformly convergent and that the algebrization condition is satisfied.

Since the $\frac{1}{p+a}$ operator is expressible in terms of simple integration, the algebraic property for combination of $(p+a)^{\pm 1}$ with $p^{\pm m}$, where m is a real integer, is established. The property can be extended further to include operators of the form $\frac{1}{(p-p_1)(p-p_2)\cdots(p-p_n)}$, where $p_1, p_2, \cdots p_n$ are constants, and, in general, to operatos such as $\frac{Y(p)}{Z(p)}$, where $Y(p)$ and $Z(p)$ are polynomials in positive integral powers of p.

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* We have by definition that $(p+a)\frac{1}{p+a}f(t)1(t) \equiv f(t)\,1(t)$, from which one may readily show that $\frac{1}{p+a}(p+a)f(t)1(t) \equiv f(t)\,1(t)$.

The inverse operator $\frac{1}{p+a}$ might have been expanded by the binomial theorem in positive powers of p were it not for the fact that the resultant series,

$$\left(\frac{1}{a} - \frac{p}{a^2} + \frac{p^2}{a^3} - \cdots + (-1)^n \frac{p^n}{a^{n+1}} + \cdots\right) 1(t) \quad , \qquad (16)$$

is not uniformly convergent. On the basis of a discontinuous unit function, one may be tempted to consider that all but the first term of the series vanish, except at $t=0$. He will then obtain the contradiction that the value of the series in positive powers of p is $\frac{1}{a} 1_D(t)$ in contrast to the result already obtained from the series in negative powers of p, namely $\frac{1}{a}(1-\epsilon^{-at}) 1_D(t)$. This contradiction has occasionally been suggested as a distinct flaw in operational calculus. However, evaluation on the basis of a discontinuous unit function when applied to series (16) is entirely without justification because, when the resultant expression is operated upon with $(p+a)$, it does not yield $1_D(t)$ for all values of t, notably at $t=0$; that is, the series operator of (16) does not represent an inverse operator. The situation may be summed up by observing that series (16) as a function of t is not uniformly convergent.

The fallacy of the above method is revealed when the inverse operator is expanded into a finite series. Having established the algebraic law for operators, we can write the following identity:

$$\frac{1}{p+a} 1(t) \equiv \left[\frac{1}{a} - \frac{p}{a^2} + \frac{p^2}{a^3} - \cdots + (-1)^n \frac{p^n}{a^{n+1}} + (-1)^{n+1} \frac{p^{n+1}}{a^{n+1}} \frac{1}{p+a}\right] 1(t) \, , \qquad (17)$$

where n is finite.. The very significant advantage of this expansion over that of Eq. (16) is that convergence need not be investigated in Eq. (17) since the number of terms therein is finite; Eq. (17) is a true identity. It can now be observed that the value of the expression $\frac{1}{p+a}\,\mathbf{1}(t)$ resides almost entirely in the first and last, or the remainder, term of the series, all other terms approximating zero.* The value of $\frac{1}{p+a}\,\mathbf{1}(t)$ has been shown to be $\frac{1}{a}(1-\varepsilon^{-at})\mathbf{1}(t)$, and substitution of this expression into Eq. (17) demonstrates the importance of the remainder term. It is interesting that, even though a may be negative, corresponding to an exponentially cummulative response, all terms of this series approximate zero except the first and the remainder terms, the latter retaining the positive order exponential. The absurdity of ignoring the remainder is evident.

It must be recognized that the insistence upon uniformity of convergence of an infinite series is required for valid algebraic manipulations of series operators, since an infinite series can be integrated term by term only if that series is uniformly convergent within the range of integration.

Generally expansion of an inverse polynomial in positive powers of p leads to a divergent series, which therefore requires evaluation of the remainder term, while

*These terms approximate zero since the unit function is a continuous one. For a finite number of such terms their aggregate value can be made as small as desired.

expansion in negative powers of p leads to a convergent series.

A very useful operational tool is the so-called Heaviside shifting theorem, given by the equation

$$F(p)\,\varepsilon^{at} f(t)\,\mathbf{1}(t) \equiv \varepsilon^{at} F(p+a)\, f(t)\,\mathbf{1}(t) \;, \qquad (18)$$

where $F(p)$ is expressible as the ratio of polynomials in p, a is a constant, and the algebrization condition is satisfied by the operands. Derivation of the theorem by the Heaviside method may be readily modified to become consistent with the treatment which we have been using, and need not be repeated here. Applying the shifting theorem to the operator $\frac{1}{p+a}$, we obtain

$$\frac{1}{p+a} f(t)\mathbf{1}(t) \equiv \frac{1}{p+a}\varepsilon^{-at}\varepsilon^{at} f(t)\mathbf{1}(t) \equiv \varepsilon^{-at}\frac{1}{p}\varepsilon^{at} f(t)\mathbf{1}(t) = \varepsilon^{-at}\int_{-\infty}^{t}\varepsilon^{au} f(u)\mathbf{1}(u)\,du \,. \qquad (19)$$

Consequently the operator $\frac{1}{p+a}$ is identical with the operation indicated by $\varepsilon^{-at}\int_{-\infty}^{t}\varepsilon^{au} \cdots du$ and may be substituted by the latter at any time. A similar procedure may be applied to a great many other operational forms.

Consider now the differential equation

$$G(p)\, i(t) = f(t) \;, \qquad (20)$$

where $f(t)$ is a known force function, $G(p)$ is a polynomial in p, and the algebrization condition is satisfied. Then the operational solution is

$$i(t) = \frac{1}{G(p)} f(t) = \frac{1}{(p-p_1)(p-p_2)\cdots(p-p_n)} f(t) \;, \qquad (21)$$

where $p_1, p_2, \cdots p_n$ are roots of $G(p)=0$. Substituting the integral identical to the inverse operator, we obtain

$$i(t) = \varepsilon^{p_1 t}\int_{-\infty}^{t}\varepsilon^{-(p_1-p_2)u_1}\int_{-\infty}^{u_1}\varepsilon^{-(p_2-p_3)u_2}\cdots\int_{-\infty}^{u_{n-2}}\varepsilon^{-(p_{n-1}-p_n)u_{n-1}}\int_{-\infty}^{u_{n-1}}\varepsilon^{-(p_n)u_n} f(u_n)\,du_1\,du_2\cdots du_n \,. \qquad (22)$$

The equation is the familiar solution of an nth order linear differential equation with constant coefficients.

The Heaviside expansion theorem solution of the equation

$$i(t) = E\,\frac{Y(p)}{Z(p)}\,1(t)\,, \qquad (23)$$

where $Y(p)$ and $Z(p)$ are polynomials in p, requires little elucidation, the proof following directly from the algebraic property of the p operator. The expansion of $\frac{Y(p)}{Z(p)}$ into partial fractions is permitted, of course, only when the degree of $Z(p)$ is higher than that of $Y(p)$, a condition which is demanded by a similar expansion when p is considered purely algebraic. A failure to abide by this limitation results at worse in the loss of impulsive terms.

6. LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

It has been suggested by Bush[7] that Heaviside's operational methods may be profitably applied to the case of linear differential equations with variable coefficients, the justification for such treatment being placed by Bush upon the fact that the results in general agree with those obtained by the classical methods. However, we are now in position to establish directly the rigor of operational calculus.

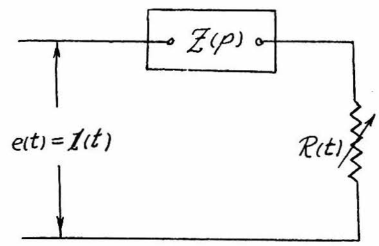

Fig. 1

Let us consider the simple case, Fig. 1, of a unit function potential applied to a series circuit including an impedance unit, whose impedance operator* is $Z(p)$, and a resistance $R(t)$, the latter being the only circuit element which varies with time. Applying Kirchhoff's potential law to the circuit we obtain

$$[Z(p) + R(t)]\, i(t) = 1(t) \quad , \qquad (24)$$

where $i(t)$ as usual is the current response. Though the inverse operator $\frac{1}{Z(p) + R(t)}$ is a function of both p and t, it is defined in the customary manner, i. e. by the identity

$$[Z(p) + R(t)]\, \frac{1}{Z(p) + R(t)}\, 1(t) \equiv 1(t) \quad . \qquad (25)$$

Using this definition we may write

$$i(t) = \frac{1}{Z(p) + R(t)}\, 1(t) \, ,$$

or

$$= \frac{1}{Z(p)\left\{1 + \frac{1}{Z(p)} R(t)\right]}\, 1(t) \, . \qquad (26)$$

To expand the inverse operator according to the binomial theorem would lead to error since the resulting series would not satisfy the property of an inverse operator defined by Eq. (25). However expansion following the reversal of the

*An impedance operator of a circuit element is the ratio of potential across the element and current flowing through it, the operator being expressed as a function of p.

position of $\frac{1}{Z(p)}$ and $R(t)$ in the denominator does satisfy the inverse property. The solution becomes:

$$i(t) = \frac{1}{Z(p)} \left\{ 1 - R(t)\frac{1}{Z(p)} + \left[R(t)\frac{1}{Z(p)}\right]^2 - \cdots + (-1)^n \left[R(t)\frac{1}{Z(p)}\right]^n + \cdots \right\} I(t) , \qquad (27)$$

where we define, e. g.

$$\left[R(t)\frac{1}{Z(p)}\right]^3 \equiv R(t)\frac{1}{Z(p)}\, R(t)\frac{1}{Z(p)}\, R(t)\frac{1}{Z(p)} .$$

The expansion of Eq. (27) is valid only if the resultant series is uniformly convergent as a function of time.

To take a simple example, let $Z(p) = pL$ and $R(t) = R_0 t\, I(t)$, where L and R_0 are constants. Then the current response is given by Eq. (27); on performing the indicated operations, we obtain

$$i(t) = \frac{t}{L}\left[1 - \frac{R_0 t^2}{3L} + \frac{R_0^2 t^4}{3\cdot 5 L^2} - \cdots + (-1)^n \frac{2^n n!\, R_0^n t^{2n}}{(2n+1)!\, L^n} + \cdots \right] I(t) . \qquad (28)$$

The method can be readily extended to more involved circuits.* One must be reminded, however, that, in general, expansion of the inverse operator must be in terms of negative powers of to obtain a convergent series solution.

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*By analogy with the successful treatment of linear differential equations with variable coefficients by Heaviside's operational methods, it has been attempted to extend the same method to non-linear equations. Unfortunately this cannot be done. Since the functional operator is not distributive with respect to its operand, the functional operator cannot be expanded into a series in a manner which is requisite to the usual Heaviside treatment. For a solution of the problem see J. R. Carson, Phys. Rev. 17 p 129 (1921).

7. FRACTIONAL ORDER DERIVATIVES

The function of Heaviside's operational calculus is to simplify operational expressions into forms which can be directly evaluated. The simplification may proceed either to a point where ready integration or differentiation alone remains to be performed, or else it may stop with expressions whose values are known. Operational calculus in itself, however, has no provision for the evaluation of its basic operational forms, such as the derivative or the integral, so that, when half order derivatives are encountered, the basic form, namely $\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} 1(t)$ or $p^{\frac{1}{2}} 1(t)$, must be evaluated by other than operational methods.

In the study of physical problems, the only fractional order derivatives which have been investigated to any great extent are those of half-integer order. The half order differentiation is defined as an operation such that when it is repeated twice, the resultant operation is a complete differentiation. The higher order derivatives, $p^{n+\frac{1}{2}}$, where n is a positive integer, can be treated as $p^n p^{\frac{1}{2}}$. In this way the higher half-integer order derivatives may be obtained readily from $p^{\frac{1}{2}}$, as also can half-integer order integrations. Similarly to the extent of physical problems, the general expression $p^{\frac{1}{2}} f(t) 1(t)$ may be derived by operational manipulations from $p^{\frac{1}{2}} 1(t)$. We observe then that it is required to evaluate only the expression $p^{\frac{1}{2}} 1(t)$ by other than operational methods before Heaviside's treatment may be applied to half-integer order differentiation and integration operations.

There are several ways of evaluating $p^{\frac{1}{2}} 1(t)$, the application of symbolic calculus[2] perhaps being the most satisfactory. Using the discontinuous unit function $1_D(t)$, it is found that $p^{\frac{1}{2}} 1_D(t) = \frac{1}{\sqrt{\pi t}} 1_D(t)$. To evaluate $p^{\frac{1}{2}} 1(t)$ where $1(t)$ is the continuous function, we use the superposition theorem equations, i. e.

$$p^{\frac{1}{2}} 1(t) = \int_{u=-\infty}^{t} \frac{d\,1(u)}{\sqrt{\pi(t-u)}} , \qquad (29)$$

where, consistent with previous notation, $A(t) = \frac{1}{\sqrt{\pi t}}$, $e(t) = 1(t)$. Here $1(t)$ has been considered to be made up of infinitesimal steps whose tread lengths approach zero in the limit as the summation of effects of each incremental rise is expressed as an integral. The resulting function $p^{\frac{1}{2}} 1(t)$ and all its derivatives are continuous, and the function approaches* $\frac{1}{\sqrt{\pi t}} 1_D(t)$ as closely as desired by choosing $1(t)$ sufficiently close to $1_D(t)$. To write the integral of Eq. (29) every time $p^{\frac{1}{2}} 1(t)$ is evaluated would be awkward, and therefore a new notation must be used to designate such a function. It is evident that the function cannot strictly be written simply

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*We have that $\lim_{1(t) \to 1_D(t)} \int_{u=-\infty}^{t} f(u)\, d\,1(u) = f(0)\, 1_D(t)$.

Therefore $\lim_{1(t) \to 1_D(t)} \int_{u=-\infty}^{t} \frac{d\,1(u)}{\sqrt{\pi(t-u)}} = \frac{1_D(t)}{\sqrt{\pi t}}$.

$\frac{1}{\sqrt{\pi t}} 1(t)$ because the latter is both discontinuous at $t=0$ and imaginary for negative values of t. We could denote the integral of Eq. (29) by a symbol such as $\left[\frac{1}{\sqrt{\pi t}} 1_D(t)\right]_c$, where " c " designates that the function and all its derivatives are continuous and that the function approaches $\frac{1}{\sqrt{\pi t}} 1_D(t)$ as closely as desired. However, it is unnecessary to complicate notation by this symbol and we shall indeed employ the customary expression $\frac{1}{\sqrt{\pi t}} 1(t)$ with the understanding that the term expresses a continuous function.

The expression $p^{\frac{3}{2}} 1(t)$ is obtained by differentiation of Eq. (29). Since direct differentiation of the right hand side as it stands is not possible, it is first integrated by parts, i. e.

$$p^{\frac{1}{2}} 1(t) = 2\int_{-\infty}^{t} \sqrt{\frac{t-u}{\pi}} \, \frac{d^2 1(u)}{du^2} \, du .$$

Now differentiating the equation, we obtain

$$p^{\frac{3}{2}} 1(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-u)}} \, \frac{d^2 1(u)}{du^2} \, du .$$

Recalling the property of the second order impulse and employing the above established notation, we could write

$$p^{\frac{3}{2}} 1(t) = \left[\frac{1_D(t)}{-2\sqrt{\pi}\, t^{3/2}}\right]_c + \left[\text{Impulsive term at } t=0\right] ,$$

since $p^{\frac{3}{2}} 1(t)$, as well as all its derivatives, is continuous and approaches $\frac{1_D(t)}{-2\sqrt{\pi}\, t^{3/2}}$ as closely as desired when t is not in the neighborhood of $t=0$. Similarly we obtain for higher order derivatives

$$p^{n+\frac{1}{2}} 1(t) = \left[\frac{1_D(t)}{\Gamma(-n+\frac{1}{2})\, t^{n+\frac{1}{2}}}\right]_c + \left[\text{Impulsive terms at } t=0\right] , \qquad (30)$$

where n is a positive integer and " Γ " denotes the Gamma function. Again to simplify notation, we shall return to the more customary notation, omitting the brackets.

For integrations, i. e. $p^{-\frac{1}{2}}1(t) = \frac{1}{p}p^{\frac{1}{2}}1(t)$, we observe that

$$p^{\frac{1}{2}}1(t) = \frac{d}{dt}\int_{u=-\infty}^{t} 2\sqrt{\frac{t-u}{\pi}}\, d\,1(u) ,$$

for which we obtain

$$p^{-\frac{1}{2}}1(t) = \int_{u=-\infty}^{t} 2\sqrt{\frac{t-u}{\pi}}\, d\,1(u) .$$

Therefore, we write as before $p^{-\frac{1}{2}}1(t) = [2\sqrt{\frac{t}{\pi}}\,1_D(t)]_c$,

and in general

$$p^{-\frac{n}{2}}1(t) = [\frac{t^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\,1_D(t)]_c \quad , \qquad (31)$$

brackets to be omitted in the usual notation.

The definition of the half order derivative permits it to be incorporated into the realm of operators which obey algebraic properties, so that any operator which is a function of integral and half-integral order differentiations and integrations may be treated algebraically. The Heaviside shifting theorem also applies to such operators.

8. INFINITE NON-INDUCTIVE CABLE

The familiar partial differential equation for the potential, $e(x,t)$, along a non-inductive cable is given by

$$\frac{\partial^2 e(x,t)}{\partial x^2} = RC\,\frac{\partial e(x,t)}{\partial t} ,$$

where R is the distributed series resistance of the cable and C is its distributed capacitance to ground, while the leakage conductance and series inductance are considered negligible. All circuit parameters are assumed constant. It is easily verified by substitution into the partial differential equation that a solution of the equation is

$$e(x,t) = \varepsilon^{-\sqrt{pRC}\,x} f_1(t) + \varepsilon^{\sqrt{pRC}\,x} f_2(t) \,,$$

where in the customary manner the exponential operator is defined by the infinite series corresponding to the exponential, e. g.

$$\varepsilon^{-\sqrt{pRC}\,x} f_1(t) = \left[1 - \sqrt{pRC}\,x + \frac{pRC}{2!}x^2 - \dots + (-1)^n \frac{(pRC)^{\frac{n}{2}}}{n!} x^n + \dots\right] f_1(t) \,.$$

The solution follows from consideration of the algebraic property of the p operator raised to half-integer powers.

For the specific example of a unit function electromotive force impressed at $x=0$ upon a non-inductive cable which extends to infinity in the positive x direction, the potential equation becomes

$$e(x,t) = \varepsilon^{-\sqrt{pRC}\,x} \, 1(t) \,,$$

which can be readily shown to be uniformly convergent for all values of x and t. Now if $i(x,t)$ is the current along the cable, we have that $\frac{\partial e(x,t)}{\partial x} = -R\,i(x,t)$, and substituting for $e(x,t)$ we obtain

$$i(x,t) = \sqrt{\frac{pC}{R}}\,\varepsilon^{-\sqrt{pRC}\,x} \, 1(t) \,.$$

At the head of the cable, i. e. $x=0$, the current is simply $i(0,t)=\sqrt{\frac{pC}{R}}\,1(t)$ so that the impedance operator of the infinite cable is $\sqrt{\frac{R}{pC}}$.

Consider now the problem of determining the response of the infinite cable when the latter is terminated at its source by a resistance R_0 .

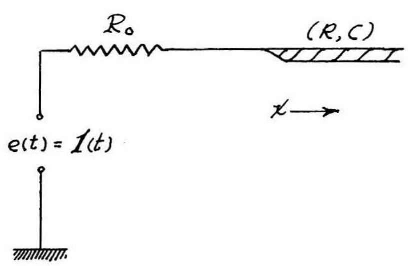

Fig. 2

The impedance operator $Z(p)$ of the complete circuit becomes $Z(p)=R_0+\sqrt{\frac{R}{pC}}$, and the current response $i(t)$ to unit function electromotive force is

$$i(t)=\frac{\sqrt{pC}}{R_0\sqrt{pC}+\sqrt{R}}\,1(t)\,. \qquad (32)$$

We shall treat an operator of this form, namely

$$I(t)=\frac{\sqrt{p}}{\sqrt{p}+\sqrt{a}}\,1(t) \quad , \qquad (33)$$

where a is a positive real constant. The solution of this equation will be obtained in two forms: namely, as an infinite convergent series and as an asymptotic series. Our treatment will differ from others in that the complete remainder terms for both solutions will be obtained, terms which Heaviside failed to consider *.

* Though remainder terms have previously been evaluated by putting the original operational expression into an integral which is integrated by parts, such a method lacks the flexibility of the complete operational solution since the integral form is preferably introduced only for the final remainder term in the manner now to be described.

The convergent series is obtained by expansion of the operator in negative powers of p , according to the binomial theorem:

$$\frac{\sqrt{p}}{\sqrt{p}+\sqrt{a}}\,1(t) = \left[1-\left(\frac{a}{p}\right)^{\frac{1}{2}}+\left(\frac{a}{p}\right)-\left(\frac{a}{p}\right)^{\frac{3}{2}}+\left(\frac{a}{p}\right)^{2}-\cdots+(-1)^{n}\left(\frac{a}{p}\right)^{\frac{n}{2}}\right]1(t)+(-1)^{n+1}\left(\frac{a}{p}\right)^{\frac{n+1}{2}}\frac{\sqrt{p}}{\sqrt{p}+\sqrt{a}}\,1(t)\,. \quad (34)$$

The expansion is exact since the remainder term is included. To evaluate the remainder, first the half-powers of its denominator are eliminated:

$$R = (-1)^{n+1}\left(\frac{a}{p}\right)^{\frac{n+1}{2}}\frac{p-\sqrt{ap}}{p-a}\,1(t)\ ,$$

and, using the shifting theorem for the denominator factor $(p-a)$, we obtain

$$R = (-1)^{n+1}\,\varepsilon^{at}\int_{-\infty}^{t}\varepsilon^{-au}\left(\frac{a}{p}\right)^{\frac{n+1}{2}}(p-\sqrt{ap})\,1(u)\,du\ .$$

Since ε^{-at} is greatest in the range $t \geqslant 0$ at $t=0$, a dominant to the absolute value of R is obtained by removing that exponential from the equation, i. e.

$$|R| \leqslant \left|\varepsilon^{at}\frac{1}{p}\left(\frac{a}{p}\right)^{\frac{n+1}{2}}(p-\sqrt{ap})\,1(t)\right| = \left|\frac{(at)^{\frac{n+1}{2}}\varepsilon^{at}}{\Gamma(\frac{n+3}{2})}-\frac{(at)^{\frac{n}{2}+1}\varepsilon^{at}}{\Gamma(\frac{n}{2}+2)}\right|1(t)\ ,$$

or

$$R = (-1)^{n+1}\,\theta\,(at)^{\frac{n+1}{2}}\left(\frac{1}{\Gamma(\frac{n+3}{2})}-\frac{\sqrt{at}}{\Gamma(\frac{n}{2}+2)}\right)1(t)\ , \qquad (35)$$

where θ is a real constant such that $0 \leqslant \theta \leqslant 1$.

The complete solution is now obtained by evaluating Eq. (34) term by term using Eq. (31):

$$I(t) = \left[1-\frac{(at)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}+\frac{at}{1!}-\frac{(at)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}+\cdots+\frac{(-1)^{n}(at)^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}+(-1)^{n+1}\theta\,(at)^{\frac{n+1}{2}}\varepsilon^{at}\left(\frac{1}{\Gamma(\frac{n+3}{2})}-\frac{\sqrt{at}}{\Gamma(\frac{n}{2}+2)}\right)\right]1(t). \quad (36$$

The presence of the positive order exponential in the remainder term renders the remainder rapidly more uncertain with increasing values of t so that an asymptotic form of solution is required and is next obtained.

Expanding the original operator, Eq. (33), in positive powers of p according to the binomial theorem, we have

$$I(t) = \left[\left(\frac{p}{a}\right)^{\frac{1}{2}} - \frac{p}{a} + \left(\frac{p}{a}\right)^{\frac{3}{2}} - \cdots + (-1)^n \left(\frac{p}{a}\right)^{\frac{n+1}{2}}\right] 1(t) + (-1)^{n+1}\left(\frac{p}{a}\right)^{\frac{n+1}{2}} \frac{\sqrt{p}}{\sqrt{p}+\sqrt{a}}\, 1(t) \,. \qquad (37)$$

The expansion again is exact. On the elimination of half-order derivatives from the denominator of the remainder, the latter becomes

$$R = (-1)^{n+1}\left(\frac{p}{a}\right)^{\frac{n+1}{2}} \frac{p - \sqrt{ap}}{p-a}\, 1(t) \,.$$

Let $\frac{n+1}{2} = m$, an integer, and apply the shifting theorem:

$$R = \left(\frac{p}{a}\right)^m \frac{p}{p-a}\, 1(t) - \left(\frac{p}{a}\right)^m \varepsilon^{at}\int_{-\infty}^{t} \varepsilon^{-au}\sqrt{ap}\; 1(u)\, du \,.$$

The first term R_1 may be reduced directly, since $\frac{p}{p-a}1(t) = \varepsilon^{at}1(t)$ to

$$R_1 = \left(\frac{p}{a}\right)^m \varepsilon^{at} 1(t) = \varepsilon^{at} 1(t) \quad , \qquad (38)$$

where the impulsive terms are ignored because the asymptotic solution is not applied to the neighborhood of $t=0$.

For the second term R_2 of the remainder, the integral is separated according to

$$\int_{-\infty}^{t} = \int_{-\infty}^{\infty} + \int_{\infty}^{t} .$$

Then

$$R_2 = -\left(\frac{p}{a}\right)^m \varepsilon^{at}\int_{-\infty}^{\infty} \varepsilon^{-au}\sqrt{\frac{a}{\pi u}}\, I(u)\,du - \left(\frac{p}{a}\right)^m \varepsilon^{at}\int_{\infty}^{t}\varepsilon^{-au}\sqrt{ap}\, I(u)\,du .$$

But

$$\int_{0}^{\infty} \varepsilon^{-au}\frac{1}{\sqrt{u}}\,du = \sqrt{\frac{\pi}{a}} ,$$

so that

$$R_2 = -\left(\frac{p}{a}\right)^m \varepsilon^{at} I(t) - \left(\frac{p}{a}\right)^m \varepsilon^{at}\int_{\infty}^{t}\varepsilon^{-au}\sqrt{ap}\, I(u)\,du ,$$

$$= -\varepsilon^{at} I(t) - \left(\frac{p}{a}\right)^m \varepsilon^{at}\int_{\infty}^{t}\varepsilon^{-au}\sqrt{ap}\, I(u)\,du .$$

Therefore the complete remainder is

$$R = R_1 + R_2 = -\left(\frac{p}{a}\right)^m \varepsilon^{at}\int_{\infty}^{t}\varepsilon^{-au}\sqrt{ap}\, I(u)\,du . \qquad (39)$$

To evaluate this expression we recall that integration and differentiation operations are commutative provided that the operand vanishes at the definite limit of the integral. In Eq. (39) the operand under consideration is $\varepsilon^{-at}\sqrt{ap}\, I(t)$, which indeed vanishes in the limit as t approaches infinity. Then using the commutative property, we may "shift" the multiple derivative operator $(ap)^m$ past the first exponential, bring it into the integrand, "shift" it past the second exponential, and complete its journey by operation on $\sqrt{ap}\, I(t)$. The important aspect of the integral is that the impulsive terms are not present in the range of t over which the integral extends. We find for $t > 0$

$$\mathcal{R} = -a\varepsilon^{at}\int_{\infty}^{t}\varepsilon^{-au}\left(\frac{p}{a}\right)^{m+\frac{1}{2}} 1(u)\,du \quad ,$$

$$= -a\varepsilon^{at}\int_{\infty}^{t}\varepsilon^{-au}\frac{du}{\Gamma(-m+\frac{1}{2})\,(au)^{m+\frac{1}{2}}} \quad .$$

A dominant to the absolute value of $\mathcal{R}$ is obtained if the exponential is removed from the integrand past the integral sign; then

$$\mathcal{R} = \frac{-\theta}{\Gamma(-m+\frac{3}{2})\,(at)^{m-\frac{1}{2}}} \quad ; \quad 0 < \theta < 1 \quad , \quad t > 0 \quad . \qquad (40)$$

The complete solution obtained on term by term evaluation of Eq. (37) becomes

$$I(t) = \frac{1}{\sqrt{at}}\left[\frac{1}{\Gamma(\frac{1}{2})} + \frac{1}{\Gamma(\frac{-1}{2})at} + \frac{1}{\Gamma(\frac{-3}{2})(at)^2} + \cdots + \frac{1}{\Gamma(-m+\frac{3}{2})(at)^{m-1}} - \frac{\theta}{\Gamma(-m+\frac{3}{2})(at)^{m-1}}\right] \; ; \; 0 < \theta < 1 \; , \; t > 0 \quad . \qquad (41)$$

The impulsive terms are dropped since $t > 0$. It will be noted that the error incurred by stopping the series with the mth term is less than the magnitude of the mth term.

Usually the divergent series obtained by expansion of an operator in positive powers of p is asymptotic and Heaviside generally evaluated such a series by stopping it with its smallest term, presuming that the uncertainty of the remainder is less than that smallest term. The above example satisfies this condition. However the procedure occasionally fails. Consider the problem of a sinusoidal electromotive force impressed suddenly on an infinite non-inductive cable. The current response may be written:

$$i(t) = \frac{1}{Z(p)} e(t) = E\sqrt{\frac{pC}{R}}\ \varepsilon^{j\omega t} 1(t),$$

where the real part of the solution is the response to a potential $E \cos \omega t\ 1(t)$. Converting the exponential term to its operational equivalent,

$$i(t) = E\sqrt{\frac{C}{R}}\ \frac{p^{\frac{3}{2}}}{p - j\omega}\ 1(t) . \qquad (42)$$

Expanding in positive powers of p according to the binomial theorem, we obtain

$$i(t) = E\sqrt{\frac{C}{R}}\left[-\frac{p^{\frac{3}{2}}}{j\omega} + \frac{p^{\frac{5}{2}}}{\omega^{4}} + \frac{p^{\frac{7}{2}}}{j\omega^{3}} - \cdots + \left(\frac{1}{j}\right)^{n+2}\frac{p^{n+\frac{1}{2}}}{\omega^{n}}\right] 1(t) + \left(\frac{1}{j}\right)^{n}\frac{p^{n+\frac{3}{2}}}{\omega^{n}(p-j\omega)}\ 1(t) . \qquad (43)$$

The remainder term is

$$R = \left(\frac{1}{j}\right)^{n} E\sqrt{\frac{C}{R}}\ \frac{p^{n+1}}{\omega^{n}}\ \varepsilon^{j\omega t}\int_{-\infty}^{t}\varepsilon^{-j\omega u}\sqrt{p}\ 1(u)\, du .$$

Separating the upper limit and evaluating as in the preceding example, we find

$$R = E\sqrt{\frac{j\omega C}{R}}\ \varepsilon^{j\omega t} + \frac{\theta}{\Gamma(-n+\frac{1}{2})\,\omega^{n} t^{n+\frac{1}{2}}},$$

where $0 < |\theta| < 1$. The complete solution is $(t > 0)$:

$$i(t) = E\sqrt{\frac{j\omega C}{R}}\ \varepsilon^{j\omega t} + E\sqrt{\frac{C}{Rt}}\left[\frac{1}{\Gamma(-\frac{3}{2})(\omega t)^{2}} - \frac{1}{\Gamma(-\frac{7}{2})(\omega t)^{4}} + \cdots + \frac{(-1)^{n+2}}{\Gamma(-n+\frac{1}{2})(\omega t)^{n}} + \frac{\theta}{\Gamma(-n+\frac{1}{2})(\omega t)^{n}}\right]$$

$$+ jE\sqrt{\frac{C}{Rt}}\left[\frac{1}{\Gamma(-\frac{1}{2})(\omega t)} - \frac{1}{\Gamma(-\frac{5}{2})(\omega t)^{3}} + \cdots + \frac{(-1)^{n+1}}{\Gamma(-n+\frac{3}{2})(\omega t)^{n-1}} + \frac{\theta_2}{\Gamma(-n+\frac{1}{2})(\omega t)^{n}}\right] . \qquad (44)$$

where n is made even and θ_1, θ_2 are real quantities such that $0 < |\theta_1| < 1$; $0 < |\theta_2| < 1$. The steady state term evidently is $E\sqrt{\frac{j\omega C}{R}}\,\epsilon^{j\omega t}$, which, we note, was obtained from the remainder term of the expansion. Therefore a failure to take the remainder into account would lead to an absurd result. The example illustrates the necessity for the development of an asymptotic series in a manner which permits of the evaluation of the uncertainty consequent to stopping short the series.

Sometimes the error resulting from the ommission of the remainder term is not as obvious as above, for example when the remainder contains a negative order exponential. Since an asymptotic series cannot reveal the presence of a negative order exponential, it is to be expected that if the solution contains such exponentials, the latter will persist in the remainder term when the result is expressed in asymptotic form. Heaviside's usual methods would fail in these instances and he has used intuitive methods to handle the situation, methods which are not rigorous and not explicitly defined. A conspicuous example of such a problem is that of an infinite non-inductive cable terminated by inductance. By the application of the methods herein established, the problem is treated in detail in the appendix of this paper. The solution is made cumbersome at one point due to the fact that the remainder of the asymptotic expansion of $\frac{1}{p+(\alpha+j\beta)}\,p^{\frac{1}{2}}1(t)$ does not lend itself to ready evaluation when α is real and positive and β is real.

We have examined the method of treating inverse polynomials in integral and half-integral powers of p. An analogous method, though more cumbersome, may be applied to the more advanced operational expressions. For example, in the case of operators pertaining to the inductive transmission line, the expansion may be made to proceed not as a power series in t but in terms of Bessel functions, as used by Heaviside, permitting the evaluation of a dominant to the remainder of the series. Space does not permit a detailed exposition, and the latter procedure can be derived by utilizing Heaviside's own methods, with the slight modifications which have been discussed in the course of this paper.

9. SYSTEM INITIALLY NOT IN EQUILIBRIUM

Thus far we have been dealing with solutions of differential equations under the simplified boundary conditions pertaining to a circuit undisturbed prior to application of the electromotive force which is under consideration. The methods already developed are readily extended to a treatment of general boundary conditions.

We have observed that operational equations for lumped parameter circuits state explicitly the entire history of the dependent variable, but in the case of systems which are initially $(t=0)$ not in equilibrium, the values of the variable are both unknown and of no consequence prior to $t=0$. The obvious procedure in setting up the operational equation is to assume that the system is in equilibrium before $t=0$ and that at this instant an impulsive force is introduced such as to create a state identical with the boundary conditions.

Consider an nth order differential equation:

$$(a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0)\, i(t) = 0 \, . \qquad (45)$$

We are to determine the dissipation of the variable $i(t)$ following an initial excitation, specified by $i(0)$, $[i'(t)]_{t=0}$, $\ldots$, $[i^{(n-1)}(t)]_{t=0}$. The operational equation assumes the following form:

$$(a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0)\, i(t) = (b_n p^n + b_{n-1} p^{n-1} + \cdots + b_1 p)\, I(t) \, , \qquad (46)$$

where the right-hand side represents the impulsive force which is necessary to produce the initial conditions. There remains then only to identify the b coefficients with these initial conditions. Divide the equation by the left-hand polynomial in p ,

$$i(t) = \Big\{ \frac{b_n}{a_n} + \frac{1}{a_n} p^{-1} [b_{n-1} - a_{n-1} K_0] + \frac{1}{a_n} p^{-2} [b_{n-2} - a_{n-2} K_0 - a_{n-1} K_1]$$

$$+ \cdots + \frac{1}{a_n} p^{-n+1} [b_1 - a_1 K_0 - a_2 K_1 - \cdots - a_{n-1} K_{n-2}] + \frac{c_1 + c_2 p^{-1} + \cdots + c_n p^{-n+1}}{a_n p^n + \cdots + a_1 p + a_0} \Big\} I(t) \, , \qquad (47)$$

where K_m is the coefficient of p^{-m}, i. e. $K_0 = \frac{b_n}{a_n}$; $K_1 = \frac{1}{a_n}(b_{n-1} - a_{n-1} K_0)$; etc., and $c_1, c_2, \cdots c_n$ are constants.

As we approach $t=0$ from the side of positive values of t, we find that, since the integration terms approach zero,

$$i(t) \longrightarrow \frac{b_n}{a_n} I(t) = K_0 I(t) \quad ; \quad t \to 0 .$$

But this approach is exactly what we mean by the first boundary condition, namely $i(0)$, so that $K_0 = i(0)$.

Now differentiate both sides of Eq. (47) with respect to t :

$$p\,i(t) = \left[K_0 p + K_1 + K_2 p^{-1} + \cdots + K_{n-1} p^{-n+2} + \frac{c_1 p + c_2 + \cdots + c_n p^{-n+2}}{a_n p^n + \cdots + a_1 p + a_0} \right] I(t) .$$

Approaching $t=0$ as before, we have

$$p\,i(t) \longrightarrow K_0 p I(t) + K_1 I(t) \quad ; \quad t \longrightarrow 0 .$$

The impulsive term on the right hand side vanishes except in the neighborhood of $t=0$ and the neighborhood is as small as desired, so that we obtain in $K_1 I(t)$ exactly what we mean by the second boundary condition, or $[i'(t)]_{t=0} = K_1$. The method is extended by successive differentiations, giving for the general term:

$$[p^{(m)} i(t)]_{t=0} = K_m = \frac{1}{a_n}\left[b_{n-m} - a_{n-m} K_0 - a_{n-m+1} K_1 - \cdots - a_{n-1} K_{m-1} \right] .$$

The b coefficients can now be evaluated in terms of the specified initial conditions:

$$b_n = a_n\, i(0)$$

$$b_{n-1} = \left[a_n i'(t) + a_{n-1}\, i(t)\right]_{t=0}$$

$$b_{n-2} = \left[a_n i''(t) + a_{n-1}\, i'(t) + a_{n-2}\, i(t)\right]_{t=0}$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$b_1 = \left[a_n i^{(n-1)}(t) + a_{n-1}\, i^{(n-2)}(t) + \cdots + a_2 i'(t) + a_1 i(t)\right]_{t=0}$$

The operational equation becomes:

$$\begin{aligned}(a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0)\, i(t) = \Big\{ & p\left[a_1 i(t) + a_2 i'(t) + \cdots + a_{n-1} i^{(n-2)}(t) + a_n i^{(n-1)}(t)\right]_{t=0} \\ & + p^2\left[a_2 i(t) + a_3 i'(t) + \cdots + a_n i^{(n-2)}(t)\right]_{t=0} \\ & + \ldots \qquad \ldots \qquad \ldots \\ & + p^{n-1}\left[a_{n-1} i(t) + a_n i'(t)\right]_{t=0} \\ & + p^n a_n\, i(0) \Big\} 1(t)\,. \end{aligned} \qquad (48)$$

Immediately after the impulsive force is applied, the force vanishes and the proper initial conditions are assumed; therefore the validity of the equation is established.* The solution of the above equation proceeds in the usual operational manner.

If an additional force is impressed on the system, a function representing the force is added to the right-hand side of the equation according to the principle of superposition.

In electrical circuit problems, it is a series of simultaneous differential equations rather than an nth order differential equation which is involved. For a

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*H. Jeffreys introduces the above general boundary conditions, though for a more restricted treatment of the fundamental operator $\frac{1}{p}$, "Operational Methods in Mathematical Physics" Cambridge Tract, 1927.

passive network of n branches under initial excitation, the first current equation of a series of simultaneous equations is

$$(L_{11}p^2 + R_{11}p + \frac{1}{C_{11}})\, i_1(t) + \cdots \cdots + (L_{1n}p^2 + R_{1n}p + \frac{1}{C_{1n}})\, i_n(t)$$

$$= \left\{ p^2 L_{11}\, i_1(0) + p\left[L_{11} i_1'(t) + R_{11} i_1(t)\right]_{t=0} + \cdots \cdots + p^2 L_{1n}\, i_n(0) + p\left[L_{1n} i_n'(t) + R_{1n} i_n(t)\right]_{t=0} \right\} 1(t) \,,$$

where e. g. L_{rs} indicates the mutual inductance between the branches r and s. That the boundary conditions specified by

$$i_1(0)\,,\ [i_1'(t)]_{t=0}\ ;\ \cdots\ \cdots\ ;\ i_n(0)\,,\ [i_n'(t)]_{t=0}$$

are satisfied by this equation is evident by comparison with Eq. (48).

In setting up the current equation it is more convenient to express the terms in dimensions of potential, to which form the above equation may be turned by operating on the equation with $\frac{1}{p}$. Furthermore, to consider a more general problem, the presence of sources of potential within the branches will be taken into account. Since inital conditions have already been introduced into the equations, potential functions are considered to vanish for negative values of t and consequently include the unit function as a factor. The first of the system of equations becomes:

$$(L_{11}p + R_{11} + \frac{1}{pC_{11}})\, i_1(t) + \cdots \cdots + (L_{1n}p + R_{1n} + \frac{1}{pC_{1n}})\, i_n(t)$$

$$= \left\{ E_1 + pL_{11} i_1(0) + \left[L_{11} i_1'(t) + R_{11} i_1(t)\right]_{t=0} + \cdots \cdots + pL_{1n} i_n(0) + \left[L_{1n} i_n'(t) + R_{1n} i_n(t)\right]_{t=0} \right\} 1(t)\,.$$

Again the solution proceeds in the customary operational manner.

In the field of partial differential equations, an analogous method may be employed in the treatment of boundary conditions. As an example, we shall consider the wave equation:

$$\frac{\partial^2 e(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 e(x,t)}{\partial t^2} = 0 \quad ,$$

governed by the initial conditions:

$$e(x,0) = E_1(x) ,$$
$$\left[\frac{\partial e(x,t)}{\partial t}\right]_{t=0} = E_2(x) . \qquad (49)$$

In the manner of the previous problem, the equation is written operationally

$$\frac{\partial^2 e(x,t)}{\partial x^2} - \frac{1}{c^2} p^2 e(x,t) = p^2 f_1(x)\, 1(t) + p\, f_2(x)\, 1(t) , \qquad (50)$$

where the right-hand side corresponds to the force function necessary to produce the required initial conditions. To ascertain the function $f_1(x)$ and $f_2(x)$, we first operate on both sides of the equation with $\frac{1}{p^2}$:

$$\frac{1}{p^2}\frac{\partial^2 e(x,t)}{\partial x^2} - \frac{1}{c^2} e(x,t) = f_1(x)\, 1(t) + f_2(x)\frac{1}{p}\, 1(t) .$$

Since $e(x,t)$ includes $1(t)$ as a factor, as we approach $t=0$, the first and last terms of the equation tend to zero, leaving

$$-\frac{1}{c^2} e(x,t) \longrightarrow f_1(x)\, 1(t) \qquad \text{as } t \longrightarrow 0 .$$

Therefore, comparing with the first boundary condition Eq. (49), we obtain

$$f_1(x) = -\frac{1}{c^2} E_1(x) .$$

Similarly, operation on Eq. (50) with $\frac{1}{p}$ leads to the second boundary condition

$$f_2(x) = -\frac{1}{c^2} E_2(x) \ .$$

The operational equation becomes

$$\frac{\partial^2 e(x,t)}{\partial x^2} - \frac{1}{c^2} p^2 e(x,t) = -\frac{1}{c^2} E_1(x) p^2 1(t) - \frac{1}{c^2} E_2(x) p 1(t) \ . \qquad (51)$$

In the customary manner the equation is integrated with respect to x , treating p as a constant, the justification for such a procedure resting upon the algebraic property of the p operator. The solution may be expressed in terms of the particular integral, $I(x,t)$, and the solution of the complimentary equation:

$$e(x,t) = \varepsilon^{-\frac{p}{c}x} F_1(t) + \varepsilon^{\frac{p}{c}x} F_2(t) + I(x,t) \ ,$$

or

$$= F_1(t - \tfrac{x}{c}) + F_2(t + \tfrac{x}{c}) + I(x,t) \ . \qquad (52)$$

The familiar method of determining the particular integral by operational methods, letting $D = \frac{\partial}{\partial x}$, gives

$$I(x,t) = \frac{-1}{D^2 - \frac{p^2}{c^2}} \left(\frac{1}{c^2} E_1(x) p^2 1(t) + \frac{1}{c^2} E_2(x) p 1(t) \right). \qquad (53)$$

The operator is expressed in partial fractions,

$$\frac{-1}{D^2 - \frac{p^2}{c^2}} \equiv \frac{c}{2p} \left(\frac{1}{D + \frac{p}{c}} - \frac{1}{D - \frac{p}{c}} \right) ,$$

and each fraction may be substituted by its equivalent, e. g.

$$\frac{1}{D + \frac{p}{c}} \equiv \varepsilon^{-\frac{p}{c}x} \int_{-\infty}^{x} \varepsilon^{\frac{p}{c}u} \cdots du \ ,$$

where the lower limit is chosen at $-\infty$ in order to simplify the derivation.

Evaluating the first term $I_1(x,t)$ of the particular integral:

$$I_1(x,t) = \frac{1}{D+\frac{p}{c}} E_1(x)\frac{p}{2c}\,\mathbf{1}(t) = \frac{1}{2c}\,\varepsilon^{-\frac{p}{c}x}\int_{-\infty}^{x}\varepsilon^{\frac{p}{c}u}E_1(u)\,p\,\mathbf{1}(t)\,du$$

or

$$= \frac{1}{2c}\,\varepsilon^{-\frac{p}{c}x}\int_{-\infty}^{x}E_1(u)\,p\,\mathbf{1}(t+\tfrac{u}{c})\,du\,.$$

But the function $p\,\mathbf{1}(t+\frac{u}{c})$, as a function of u , represents a peak concentrated at $u=-ct$ which encompasses an area of magnitude c with the u axis. Therefore the expression becomes

$$I_1(x,t) = \tfrac{1}{2}\,\varepsilon^{-\frac{p}{c}x}E_1(-ct)\,\mathbf{1}(t+\tfrac{x}{c}),$$

or

$$= \tfrac{1}{2}E_1(x-ct)\,\mathbf{1}(t)\,. \qquad (54)$$

In a similar manner,we find

$$I_2(x,t) = \frac{-1}{D-\frac{p}{c}}E_1(x)\frac{p}{2c}\,\mathbf{1}(t) = \tfrac{1}{2}E_1(x+ct)\,\mathbf{1}(t)\,.$$

Since the remaining terms $I_3(x,t)$ of the particular integral are of the same form as those already derived except for the presence of a p factor, we have

$$I_3(x,t) = \frac{1}{2p}\left[E_2(x+ct)+E_2(x-ct)\right]\mathbf{1}(t)\,.$$

Proceeding to the limit as $\mathbf{1}(t)$ is made to approach the discontinuous unit function, the particular integral becomes:

$$I(x,t) = \tfrac{1}{2}\left\{E_1(x+ct)+E_1(x-ct)\right\}+\tfrac{1}{2}\int_0^t\left\{E_2(x+cu)+E_2(x-cu)\right\}du \quad ;\ t>0\,. \qquad (55)$$

By change of variable the integral may be written in a more familiar form:

$$I(x,t) = \frac{1}{2}\left\{E_1(x+ct) + E_1(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} E_2(u)\,du\right., \quad t > 0 . \tag{56}$$

The complete solution of the wave equation becomes:

$$e(x,t) = F_1\left(t-\frac{x}{c}\right) + F_2\left(t+\frac{x}{c}\right) + \frac{1}{2}\left\{E_1(x+ct) + E_1(x-ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} E_2(u)\,du . \tag{57}$$

The two functions, F_1 and F_2, are determined by the particular conditions specified at the terminals of the line, the functions representing reflected waves. If an electromotive force is impressed at the terminals, the corresponding force function is also involved in the two boundary functions.

As in the field of lumped parameter circuits, the operational equations pertaining to distributed parameter circuits describe an entire past history of the equations, or at least a history which simulates the actual in regard to the production of the initial conditions. In contrast to the differential equation, which is true for only particular intervals of its independent variables, the operational equation is true for all values of the time variable. It is this completeness which renders operational calculus so useful, for the initial conditions are inherently stated in the equations and do not require further attention.

10. SUMMARY

A mathematically sound treatment of operational calculus has been shown to be possible following several modifications of Heaviside's methods. The basic difficulty in Heaviside's presentation, namely that which accompanies differentiations of a discontinuous function, has been eliminated by the restriction that operands ~~of~~ of operational equations, as well as all successive derivatives of the operands, must always be continuous. As a consequence, the unit function $1(t)$ has been defined as a function such that it and all its derivatives are continuous and which function can be made to approach the Heaviside's discontinuous stepped function as closely as desired.

To render the operators p and $\frac{1}{p}$ commutative, $\frac{1}{p}$ is defined by the operation $\int_{-\infty}^{t} \cdots du$, where the integrand $f(t)$ is required to satisfy the condition that $\int_{-\infty}^{t} |f(u)| du$ be finite.

Linear differential equations with variable coefficients are shown to be amenable to solution by operational calculus.

The half order differentiation, $p^{\frac{1}{2}}$, is defined as an operation, such that when it is repeated twice, it performs the same operation as a complete differentiation.

Its basic evaluation is found from other than operational calculus methods to be $p^{\frac{1}{2}} 1_D(t) = \frac{1}{\sqrt{\pi t}} 1_D(t)$, where $1_D(t)$ is the discontinuous unit function. For our purposes, using the continuous unit function, we have by the superposition theorem that

$$p^{\frac{1}{2}} 1(t) = \int_{u=-\infty}^{t} \frac{d\,1(u)}{\sqrt{\pi(t-u)}} ,$$

which, as well as all its derivatives, is continuous and which approaches $\frac{1}{\sqrt{\pi t}} 1_D(t)$ in the limit as $1(t) \to 1_D(t)$. The expression is abbreviated by $\frac{1}{\sqrt{\pi t}} 1(t)$. The algebraic law can be readily shown to extend to operators which are functions of integral and half-integral powers of p .

In the treatment of asymptotic series, the remainder terms must be determined before the series can be considered evaluated. When inverse polynomials in p and $p^{\frac{1}{2}}$ are expanded into either the convergent or asymptotic form of series, the remainder terms may be obtained if the finite series expansions are used, where the finite operational series are true identities with respect to the original operators. Examples of the procedure are presented.

The operational methods herein established are applied to the problems involving systems which are initially not in equilibrium, where both ordinary and partial differential equations are considered.

A derivation of an asymptotic expansion, which includes the remainder term, of $\varepsilon^{-(\alpha+j\beta)t}\int_0^t \varepsilon^{(\alpha+j\beta)u}\frac{1}{\sqrt{u}}\,du$, where α and β are real and α is positive, is given in the appendix in the course of treatment of the non-inductive cable terminated by inductance. The derivation supplies the only deficiency which has been present thus far to the determination of an asymptotic solution with remainder term for any operational equation involving polynomials in half-integer powers of p.

The writer takes pleasure in acknowledging his indebtedness to Dr. Morgan Ward and to Dr. D. H. Weinstein of California Institute of Technology who have kindly assisted in the preparation of this thesis.

APPENDIX

NON-INDUCTIVE CABLE TERMINATED BY INDUCTANCE

A problem which has been the center of extensive investigation is that of determining the asymptotic solution for the response to unit function electromotive force of a non-inductive cable terminated at its source by inductance. It has been mentioned, p 32, that the difficulty lies in the presence of negative order exponential terms which appear in the remainder of the asymptotic expansion and which are not readily evaluated[10].

Consider the circuit, Fig. 3; the impedance operator of the cable has been found to be $\sqrt{\frac{R}{pC}}$, so that the impedance operator $Z(p)$ of the complete circuit becomes

$$Z(p) = pL_0 + \sqrt{\frac{R}{pC}}.$$

The current response $i(t)$ to unit function electromotive force is

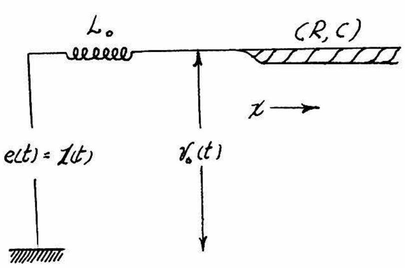

Fig. 3

$$i(t) = \frac{\sqrt{pC}}{L_0\sqrt{C}\,p^{3/2} + \sqrt{R}}\,1(t). \tag{1}$$

Then the potential $v_0(t)$ at the head of the line, Fig. (3), is

$$v_0(t) = 1(t) - pL_0\,i(t) = \frac{\sqrt{R}}{L_0\sqrt{C}\,p^{3/2} + \sqrt{R}}\,1(t). \tag{2}$$

We shall treat an operator of this form, namely

$$V_0(t) = \frac{1}{(ap)^{3/2} + 1}\,1(t). \tag{3}$$

Let us expand the operator of Eq.(3) in positive powers of p according to the binomial theorem.

$$V_0(t) = \frac{1}{(ap)^{3/2}+1}\,1(t) = \left[1-(ap)^{\frac{3}{2}}+(ap)^{3}-(ap)^{\frac{9}{2}}+\dots+(-1)^{n-1}(ap)^{\frac{3}{2}(n-1)}+(-1)^{n}\frac{(ap)^{\frac{3}{2}n}}{(ap)^{3/2}+1}\right]1(t). \quad (4)$$

We shall evaluate first the remainder term, $R(t) = (-1)^n \dfrac{(ap)^{\frac{3}{2}n}}{(ap)^{3/2}+1} 1(t)$. Clearing the fractional order derivative from the denominator by operating on $R(t)$ with $\dfrac{(ap)^{\frac{3}{2}}-1}{(ap)^{3/2}-1}$, we obtain

$$R(t) = (-1)^n (ap)^{\frac{3}{2}n}\,\frac{(ap)^{\frac{3}{2}}-1}{(ap)^3-1}\,1(t). \quad (5)$$

Now we expand the operator into partial fractions, and let n be even so that $\frac{3}{2}n = m-1$, where m is $4, 7, 10, \cdots$:

$$R(t) = (ap)^{m-1}\frac{1}{3a}\left[\frac{1}{p-\frac{1}{a}}+\frac{-1+j\sqrt{3}}{2(p-\frac{-1+j\sqrt{3}}{2a})}+\frac{-1-j\sqrt{3}}{2(p-\frac{-1-j\sqrt{3}}{2a})}\right]\left[(ap)^{\frac{3}{2}}-1\right]1(t). \quad (6)$$

By $R_1(t)$ we denote the terms of the remainder which involve only integral powers of p. Using the relation $\frac{p}{p-b}1(t) = \varepsilon^{bt}1(t)$ already evaluated, we have that

$$R_1(t) = -\frac{(ap)^{m-2}}{3}\left(\varepsilon^{\frac{t}{a}}+\frac{-1+j\sqrt{3}}{2}\varepsilon^{\frac{-1+j\sqrt{3}}{2}\frac{t}{a}}+\frac{-1-j\sqrt{3}}{2}\varepsilon^{\frac{-1-j\sqrt{3}}{2}\frac{t}{a}}\right)1(t). \quad (7)$$

The multiple differentiation is now performed and the terms combined:

$$R_1(t) = -\frac{1}{3}\varepsilon^{\frac{t}{a}} - \frac{2}{3}\varepsilon^{-\frac{t}{2a}}\cos\frac{\sqrt{3}\,t}{2a}, \quad t > 0. \quad (8)$$

The neighborhood of $t=0$ is excluded from consideration since the asymptotic series is not applicable to this region.

The other terms of the remainder, to be grouped under $R_2(t)$ are

$$R_2(t) = \frac{(ap)^m}{3a}\left[\frac{1}{p-\frac{1}{a}} + \frac{-1+j\sqrt{3}}{2(p-\frac{-1+j\sqrt{3}}{2a})} + \frac{-1-j\sqrt{3}}{2(p-\frac{-1-j\sqrt{3}}{2a})}\right]\sqrt{ap}\ 1(t)\,. \qquad (9)$$

For the first of these fractions, $R_{2,1}(t)$, we transform the inverse operator to its equivalent, i.e.

$$R_{2,1}(t) = \frac{(ap)^m}{3a}\varepsilon^{\frac{t}{a}}\int_{-\infty}^{t}\varepsilon^{-\frac{u}{a}}\sqrt{ap}\ 1(u)\,du\,.$$

Then separating the integral according to $\int_{-\infty}^{t} = \int_{-\infty}^{\infty} + \int_{\infty}^{t}$, and evaluating $\sqrt{p}\ 1(t)$, we have

$$R_{2,1}(t) = \frac{(ap)^m}{3a}\varepsilon^{\frac{t}{a}}\int_{-\infty}^{\infty}\varepsilon^{-\frac{u}{a}}\sqrt{\frac{a}{\pi u}}\ 1(u)\,du + \frac{(ap)^m}{3a}\varepsilon^{\frac{t}{a}}\int_{\infty}^{t}\varepsilon^{-\frac{u}{a}}\sqrt{ap}\ 1(u)\,du\,. \qquad (10)$$

The value of the definite integral approximates a as $1(t)$ is made to approach the discontinuous unit function. To evaluate the second term, we shift the derivative operator $(ap)^m$ within the integral in the manner already discussed on p 29. We find for $t > 0$

$$R_{2,1}(t) = \frac{\varepsilon^{\frac{t}{a}}}{3} + \frac{\varepsilon^{\frac{t}{a}}}{3a}\int_{\infty}^{t}\varepsilon^{-\frac{u}{a}}(ap)^{m+\frac{1}{2}}\,1(u)\,du\,. \qquad (11)$$

A dominant to the absolute value of $R_{2,1}(t)$ is obtained if the exponential is removed from the integrand past the integral sign; then

$$R_{2,1}(t) = \frac{\varepsilon^{\frac{t}{a}}}{3} + \frac{\theta_1}{3\Gamma(-m+\frac{3}{2})}\left(\frac{a}{t}\right)^{m-\frac{1}{2}}\ ; \quad 0 < \theta_1 < 1\ , \quad t > 0\,. \qquad (12)$$

The positive order exponential cancels that of Eq. (8) when the complete remainder $R(t)$ is derived.

The second and third terms, $R_{2,2}(t)$ and $R_{2,3}(t)$ respectively, of $R_2(t)$ Eq. (9), present certain difficulties in their evaluation. Let us consider a general expression of which $R_{2,2}(t)$ and $R_{2,3}(t)$ are examples, namely

$$R_I(t) = \left(\frac{-p}{\alpha+j\beta}\right)^m \epsilon^{-(\alpha+j\beta)t}\int_{-\infty}^{t}\epsilon^{(\alpha+j\beta)u}\,p^{\frac{1}{2}}1(u)\,du \quad , \qquad (13)$$

where α is real and positive, and β is real. This expression is the remainder after the m^{th} term of the asymptotic expansion of

$$I(t) = \frac{1}{p+(\alpha+j\beta)}\,p^{\frac{1}{2}}1(t) = \epsilon^{-(\alpha+j\beta)t}\int_{-\infty}^{t}\epsilon^{(\alpha+j\beta)u}\frac{1}{\sqrt{\pi u}}\,1(u)\,du \qquad (14)$$

To the writer's knowledge no rigorous method of evaluating a useful dominant of the remainder has yet appeared in literature *. For the remainder of the particular operator series in our problem Heaviside obtained an exponential †, with the uncertainty less than the smallest term of the series when the series is stopped with that term, but no proof was furnished by Heaviside though his conclusions are correct,as we shall observe presently.

The following treatment of $I(t)$, Eq.(14), obtaining a complete and useful asymptotic expansion, was outlined to the writer by Dr. Morgan Ward. The expression is first expanded in convergent form, either by integration by parts or by familiar operational procedure:

$$I(t) = 2\sqrt{\frac{t}{\pi}}\left\{1 - \frac{2at}{3} + \frac{(2at)^2}{3\cdot5} - \frac{(2at)^3}{3\cdot5\cdot7} + \cdots + (-1)^m\frac{(2at)^m}{3\cdot5\cdots(2m-1)} + \cdots\right\} 1(t) \ ,$$

*For the specific case of $\beta=0$, T.J.Stieltjes, Acta Math.IX p 167 (1886), has shown that the remainder is less than the smallest term of the asymptotic series, when the series is stopped with that term.

†Treating the problem by contour integration T.J.I'A Bromwich Proc.Phys.Soc. 41 p 420 (1929) (discussion of an article written by W. E. Sumpner), derives the negative order exponential but does not evaluate the uncertainty of the remainder.

where $\lambda = \alpha + j\beta$, and we let $|\lambda| = 1$. The series is a hypergeometric one, namely

$$I(t) = 2\sqrt{\frac{t}{\pi}}\ {}_1F_1\left(1; \tfrac{3}{2}; -\lambda t\right) 1(t) .$$

An asymptotic expansion is obtained by employing the Kummer function

$$\Phi(\alpha, \gamma, z) = \frac{j}{2\pi} z^{-\alpha} \int_c (-w)^{\alpha-1} \left(1 + \frac{w}{z}\right)^{\gamma-\alpha-1} \varepsilon^{-w} dw ,$$

where $|\arg z| < \pi$; the contour is designated by Fig.4, and requires that the singularity at $w = -z$ be outside the contour, as shown. The Kummer function is identified with the hypergeometric function by:

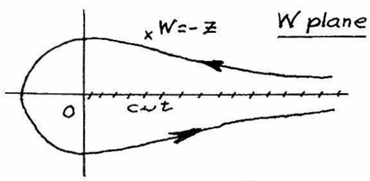

FIG. 4.

$$\Phi(\alpha, \gamma, z) = \frac{\Gamma(1-\gamma)}{\Gamma(1-\alpha)\Gamma(\alpha+1-\gamma)}\ {}_1F_1(\alpha; \gamma; z) + \frac{\sin \pi\alpha}{\pi} \Gamma(\gamma-1) z^{1-\gamma}\ {}_1F_1(\alpha-\gamma+1; -\gamma+2; z) .$$

For the problem under consideration, we shall let $\alpha = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $z = -\lambda t$ Then

$$\Phi\left(\tfrac{1}{2}, \tfrac{1}{2}, -\lambda t\right) = {}_1F_1\left(\tfrac{1}{2}; \tfrac{1}{2}; -\lambda t\right) + \frac{1}{\pi}\Gamma\left(-\tfrac{1}{2}\right) z^{\frac{1}{2}}\ {}_1F_1\left(1; \tfrac{3}{2}; -\lambda t\right) .$$

But

$$\begin{aligned} {}_1F_1\left(\tfrac{1}{2}; \tfrac{1}{2}; -\lambda t\right) &= 1 - \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} - \cdots + (-1)^n \frac{(\lambda t)^n}{n!} \\ &= \varepsilon^{-\lambda t} . \end{aligned}$$

Therefore, we obtain on substitution

$$I(t) = \left[\frac{\varepsilon^{-\lambda t}}{\sqrt{-\lambda}} - \frac{1}{\sqrt{-\lambda}} \Phi\left(\tfrac{1}{2}, \tfrac{1}{2}, -\lambda t\right)\right] 1(t) . \qquad (15)$$

To evaluate the contour integral

$$\Phi\left(\tfrac{1}{2}, \tfrac{1}{2}, -\lambda t\right) = \frac{j}{2\pi} \frac{1}{\sqrt{-\lambda t}} \int_c (-w)^{-\frac{1}{2}} \left(1 - \frac{w}{\lambda t}\right)^{-1} \varepsilon^{-w} dw ,$$

we first expand by the binomial theorem

$$\left(1 - \frac{w}{\lambda t}\right)^{-1} = 1 + \frac{w}{\lambda t} + \left(\frac{w}{\lambda t}\right)^2 + \cdots + \left(\frac{w}{\lambda t}\right)^m + \left(\frac{w}{\lambda t}\right)^{m+1} \left(1 - \frac{w}{\lambda t}\right)^{-1} ,$$

and substitute into the integrand:

$$\Phi\left(\tfrac{1}{2}, \tfrac{1}{2}, -\lambda t\right) = \frac{j}{2\pi} \frac{1}{\sqrt{-\lambda t}} \int_c \varepsilon^{-w} \left\{ (-w)^{-\frac{1}{2}} - \frac{(-w)^{\frac{1}{2}}}{\lambda t} + \frac{(-w)^{\frac{3}{2}}}{(\lambda t)^2} - \cdots + \frac{(-1)^m (-w)^{m-\frac{1}{2}}}{(\lambda t)^m} + \frac{(-1)^{m+1} (-w)^{m+\frac{1}{2}}}{(\lambda t)^{m+1}} \left(1 - \frac{w}{\lambda t}\right)^{-1} \right\} dw .$$

But it is known that

$$\frac{1}{\Gamma(z)} = \frac{j}{2\pi} \int_c (-w)^{-z} \varepsilon^{-w} dw ,$$

where the contour is identical with that of Fig. 4. Then

$$\Phi(\tfrac{1}{2},\tfrac{1}{2},-\alpha t) = \frac{1}{\sqrt{-\alpha t}}\left[\frac{1}{\Gamma(\frac{1}{2})} - \frac{1}{\Gamma(-\frac{1}{2})\alpha t} + \frac{1}{\Gamma(-\frac{3}{2})(\alpha t)^2} + \cdots + (-1)^m \frac{1}{\Gamma(-m+\frac{1}{2})(\alpha t)^m}\right]$$

$$+ \frac{j}{2\pi}\frac{(-1)^{m+1}}{\sqrt{-\alpha t}\,(\alpha t)^{m+1}}\int_C \varepsilon^{-w}(-w)^{m+\frac{1}{2}}\left(1-\frac{w}{\alpha t}\right)^{-1}dw. \qquad (16)$$

To evaluate the remainder, we shall deform the contour as shown in Fig. 5, where the radius of the circle about the origin is made vanishingly small and the lines Γ_1 and Γ_2 are made to approach the real axis. A dominant to the absolute value of the remainder may be obtained by permitting the factor $(1-\frac{w}{\alpha t})^{-1}$ to assume its greatest absolute value in the range of integration, i.e. from Fig. 5

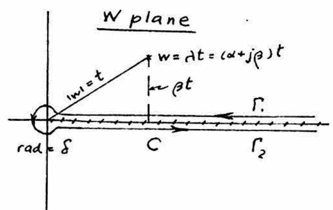

FIG. 5

$|(1-\frac{w}{\alpha t})^{-1}| \leq \frac{1}{\beta}$, since $|\alpha| = 1$. The remainder term of Eq. (16) therefore becomes:

$$|R_m| < \left|\frac{1}{2\pi\beta}\frac{1}{(\alpha t)^{m+\frac{3}{2}}}\int_C \varepsilon^{-w}(-w)^{m+\frac{1}{2}}dw\right|$$

$$= \frac{\theta}{\Gamma(-m-\frac{1}{2})\beta(\alpha t)^{m+\frac{3}{2}}}, \quad \text{where} \quad |\theta| < 1 .$$

On substituting into Eq. (15), we obtain the final solution*:

$$\varepsilon^{-\alpha t}\int_{-\infty}^{t}\varepsilon^{\alpha u}p^{\frac{1}{2}}I(u)du = \left[\frac{\varepsilon^{-\alpha t}}{\sqrt{-\alpha}} + \frac{1}{\alpha t}\left\{\frac{1}{\Gamma(\frac{1}{2})} - \frac{1}{\Gamma(-\frac{1}{2})(\alpha t)} + \frac{1}{\Gamma(-\frac{3}{2})(\alpha t)^2} + \cdots + (-1)^m\frac{1}{\Gamma(-m+\frac{1}{2})(\alpha t)^m}\right.\right. \qquad (17)$$

$$\left.\left. + \frac{\theta}{\beta}\frac{1}{\Gamma(-m-\frac{1}{2})(\alpha t)^{m+1}}\right\}\right] I(t) \quad ; \ |\theta| < 1 \ ; \alpha = \alpha + j\beta \ , |\alpha| = 1 \ ; \ \alpha, \beta \text{ real} \ , \alpha > 0 \ , \ \beta \neq 0 .$$

The remainder term, it will be noted, is multiplied by the term of the asymptotic series following the last considered. The remainder $R_I(t)$, Eq. (13), is

$$R_I(t) = \left\{\frac{\varepsilon^{-(\alpha+j\beta)t}}{\sqrt{-(\alpha+j\beta)}} + \frac{\theta}{\beta}\frac{1}{\Gamma(-m-\frac{1}{2})(\alpha+j\beta)^{m+1}t^{m+\frac{3}{2}}}\right\} I(t) . \qquad (18)$$

---

*The uncertainty factor θ is different in the various equations where it appears, subject only to the condition that $|\theta| < 1$.

The expansion is extremely useful as it now permits us to obtain complete asymptotic solutions for any operational equation involving polynomials in half integer powers of p .

We now apply the above results to the problem on hand, namely the evaluation of $R_{2,2}(t)$ and $R_{2,3}(t)$,Eq.(9). Comparing Eq.(9) with the remainder, $R_I(t)$, Eq.(13), already evaluated in Eq.(18), we have that

$$R_{2,2}(t) = \frac{1}{3}(\alpha+j\beta)^m \frac{\varepsilon^{-(\alpha+j\beta)\frac{t}{a}}}{\sqrt{-(\alpha+j\beta)}} + \frac{\theta}{3\sqrt{3}} \frac{1}{\Gamma(-m-\frac{1}{2})}\left(\frac{a}{t}\right)^{m+\frac{3}{2}}; \quad |\theta|<1 \quad, t>0 .$$

And substituting* $-(\alpha+j\beta) = -\frac{1}{2}+j\frac{\sqrt{3}}{2} = \varepsilon^{j\frac{2\pi}{3}}$; $m = 4, 7, 10, \cdots$,

$$R_{2,2}(t) = -\frac{1}{3}\varepsilon^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})\frac{t}{a}} + \frac{2\theta}{3\sqrt{3}} \frac{1}{\Gamma(-m-\frac{1}{2})}\left(\frac{a}{t}\right)^{m+\frac{3}{2}} ; \quad |\theta|<1 \quad, t>0. \tag{19}$$

In a similar manner, we obtain the conjugate remainder term:

$$R_{2,3}(t) = -\frac{1}{3}\varepsilon^{(-\frac{1}{2}-j\frac{\sqrt{3}}{2})\frac{t}{a}} + \frac{2\theta}{3\sqrt{3}} \frac{1}{\Gamma(-m-\frac{1}{2})}\left(\frac{a}{t}\right)^{m+\frac{3}{2}} ; \quad |\theta|<1 \quad, t>0 \tag{20}$$

The complete remainder is evaluated by the combination of the four constituents: $R = R_1 + R_{2,1} + R_{2,2} + R_{2,3}$ Eq's (8,12,19,20);

$$R(t) = -\frac{4}{3}\varepsilon^{-\frac{t}{2a}}\cos\frac{\sqrt{3}\,t}{2a} + \frac{4\theta}{3\sqrt{3}}\frac{1}{\Gamma(-m-\frac{1}{2})}\left(\frac{a}{t}\right)^{m+\frac{3}{2}} + \frac{\theta}{3}\frac{1}{\Gamma(-m+\frac{3}{2})}\left(\frac{a}{t}\right)^{m-\frac{1}{2}} ; \quad |\theta|<1, \ t>0.$$

The final potential equation therefore becomes:

$$V_0(t) = 1 - \frac{4}{3}\varepsilon^{-\frac{t}{2a}}\cos\frac{\sqrt{3}\,t}{2a} - \left(\frac{a}{t}\right)^{\frac{3}{2}}\Big\{\frac{1}{\Gamma(-\frac{1}{2})} + \frac{1}{\Gamma(-\frac{7}{2})}\left(\frac{a}{t}\right)^3 + \frac{1}{\Gamma(-\frac{13}{2})}\left(\frac{a}{t}\right)^6 + \cdots + \frac{1}{\Gamma(-m+\frac{7}{2})}\left(\frac{a}{t}\right)^{m-4}$$

$$+ \frac{4\theta}{3\sqrt{3}}\left(\frac{a}{t}\right)^m + \frac{\theta}{3}\frac{1}{\Gamma(-m+\frac{3}{2})}\left(\frac{a}{t}\right)^{m-2}\Big\} ; \quad |\theta|<1 \quad, t>0 \quad, \quad m = 4, 7, 10, \cdots \quad . \tag{21}$$

The exponential term which we have obtained agrees with that given by Heaviside. It is unnecessary, however, despite Heaviside's statement, to stop the series with its smallest

*In the evaluation of $\sqrt{-(\alpha+j\beta)}$, it must be recalled that $|\arg -(\alpha+j\beta)| < \pi$.

term in order to ascertain the uncertainty of the remainder.

A treatment by convergent series of the non-inductive cable terminated by inductance permits ready evaluation of the uncertainty of the remainder after the nth term. Expanding the operator in negative powers of p , we obtain

$$V_o(t) = \frac{1}{(ap)^{3/2}+1}\, 1(t)$$

$$= \left[\frac{1}{(ap)^{3/2}} - \frac{1}{(ap)^3} + \frac{1}{(ap)^{9/2}} - \cdots + (-1)^{n-1}\frac{1}{(ap)^{\frac{3}{2}n}} + (-1)^n \frac{1}{(ap)^{\frac{3}{2}n}}\,\frac{1}{(ap)^{\frac{3}{2}}+1}\right] 1(t) \,. \quad (22)$$

The remainder term, $R(t)$, is expanded into partial fractions, following the clearing of fractional order derivatives from the denominator, as for the asymptotic series:

$$R(t) = \frac{(-1)^n}{3a}\left[\frac{1}{p-\frac{1}{a}} + \frac{-1+j\sqrt{3}}{2(p-\frac{-1+j\sqrt{3}}{2a})} + \frac{-1-j\sqrt{3}}{2(p-\frac{-1-j\sqrt{3}}{2a})}\right]\frac{(ap)^{\frac{3}{2}}-1}{(ap)^{\frac{3}{2}n}}\, 1(t) .$$

Applying the shifting theorem to each fraction and combining terms, we obtain

$$R(t) = (-1)^n \frac{1}{3a}\, \varepsilon^{\frac{t}{a}} \int_{-\infty}^{t} \varepsilon^{-\frac{u}{a}} \left[(ap)^{-\frac{3}{2}(n-1)} - (ap)^{-\frac{3}{2}n}\right] 1(u)\, du \quad (23)$$

$$+ (-1)^n \frac{2}{3a}\, \varepsilon^{-\frac{t}{2a}} \cos\left(\frac{\sqrt{3}t}{2a} + \frac{2\pi}{3}\right) \int_{-\infty}^{t} \varepsilon^{\frac{u}{2a}} \cos\frac{\sqrt{3}u}{2a} \left[(ap)^{-\frac{3}{2}(n-1)} - (ap)^{-\frac{3}{2}n}\right] 1(u)\, du$$

$$+ (-1)^n \frac{2}{3a}\, \varepsilon^{-\frac{t}{2a}} \sin\left(\frac{\sqrt{3}t}{2a} + \frac{2\pi}{3}\right) \int_{-\infty}^{t} \varepsilon^{\frac{u}{2a}} \sin\frac{\sqrt{3}u}{2a} \left[(ap)^{-\frac{3}{2}(n-1)} - (ap)^{-\frac{3}{2}n}\right] 1(u)\, du ,$$

It is evident that a dominant to $R(t)$ may be obtained by proper removal of exponentials and trigonometric functions from the integrand, and one form of the remainder is certainly given by

$$R(t) = \frac{\theta}{3}\left(\varepsilon^{\frac{t}{a}} + 2\right)\left[(ap)^{-\frac{3}{2}n+\frac{1}{2}} - (ap)^{-\frac{3}{2}n-1}\right] I(t) \quad , 0 \leq |\theta| \leq 1 .$$

On integrating and taking the limit as the continuous unit function is made to approach the discontinuous one,

$$R(t) = \frac{\theta}{3}\left(\varepsilon^{\frac{t}{a}} + 2\right)\left[\frac{1}{\Gamma(\frac{3}{2}n+\frac{1}{2})}\left(\frac{t}{a}\right)^{\frac{3}{2}n-\frac{1}{2}} - \frac{1}{\Gamma(\frac{3}{2}n+2)}\left(\frac{t}{a}\right)^{\frac{3}{2}n+1}\right]; \; 0 \leq |\theta| \leq 1 , \; t \geq 0. \qquad (24)$$

Finally the potential equation is obtained by performing the indicated operations of Eq. (21), i.e.

$$V_0(t) = \frac{1}{\Gamma(\frac{5}{2})}\left(\frac{t}{a}\right)^{\frac{3}{2}} - \frac{1}{3!}\left(\frac{t}{a}\right)^{3} + \cdots + (-1)^{n-1}\frac{1}{\Gamma(\frac{3}{2}n+1)}\left(\frac{t}{a}\right)^{\frac{3}{2}n}$$

$$+ \frac{\theta}{3}\left(\varepsilon^{\frac{t}{a}} + 2\right)\left[\frac{1}{\Gamma(\frac{3}{2}n+\frac{1}{2})}\left(\frac{t}{a}\right)^{\frac{3}{2}n-\frac{1}{2}} - \frac{1}{\Gamma(\frac{3}{2}n+2)}\left(\frac{t}{a}\right)^{\frac{3}{2}n+1}\right]; \; 0 \leq |\theta| \leq 1 , \; t \geq 0 . \qquad (25)$$

The expansion is useful for small values of $\frac{t}{a}$; as $\frac{t}{a}$ increases, greatly more terms of the series are required to maintain accuracy of computations.

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