ABSTRACT INTEGRATION

THESIS

by

Wm.A.Mersman.

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Introduction.

The subject of integration was first treated abstractly by Fréchet*. Numerous other authors have contributed to the subject, notably Saks**, Bochner***, and most recently Birkhoff†. A comprehensive discussion of the possibilities was presented by Kolmogoroff††. The purpose of this thesis is to unify some of the previous theories and to present some new ones.

Chapter I is devoted to the most obvious generalization of the classical Lebesgue theory. Chapter II extends the definition of Chapter I in such a way that no special treatment is necessary for unbounded functions or sets. Chapter III is a discussion of the theory of summation of divergent series, an application of which is made in Chapter IV to the question of divergent integrals.

Questions of relative generality of these and other integrals are usually so easily answered that no formal proofs were deemed necessary. For example, the integral of Chapter I includes the classical Lebesgue, Stieltjes, and Radon††† integrals of bounded functions. The integral of Ch Chapter II includes those of Saks and Bochner and is included in Birkhoff's. Finally those in Chapter IV are obviously generalizations of the classical Riemann- and Lebesgue-Stieltjes integrals.

Before continuing I wish to express here my appreciation to Dr. A. D. Michal for his guidance in the writing of this thesis.

Chapter I.

Integration of Measurable Functions.

1. Spaces.

In this paper the following types of spaces are considered and the indicated notation used.

B; a Banach space, normed, complete, closed under addition and under multiplication by real numbers, elements denoted by $\ell, e, f, g, I, J, F, G$, norm by $|\ldots|$.

R; the real number space, elements denoted by $\alpha, \beta, \theta, \varphi, \epsilon, \delta$ positive integers by i, j, m, n, p, q, r, M, N, absolute value by $|\ldots|$.

V; a metric space, the only property of distance being that the distance between any two points approaches zero as the distances between each of them and an arbitrary third point approach zero.

D; a metric space, elements denoted by s, t, u, v, , w, distance by $|s-t|$, with the following postulates:

(1). Equality defined, symmetric, reflexive, transitive.

(2). $|s-t| \geqq 0 . |s-t| = |t-s| : |s-t| = 0 . \equiv . s = t.$

The space D is said to be dense at the point s if, for any δ, $\exists\, t \in D,\ |t-s| < \delta$.

S; a "measurable" space, elements denoted by x, y, z, sets of elements by A, C, E, X, with the properties:

There exists at least one set X of elements of S and a family $\mathfrak{X}$ of sub-sets of X such that:

(1). $A \in \mathfrak{X} . \supset . (X - A) \in \mathfrak{X}$ (2). $A_n \in \mathfrak{X} . \supset \{\sum_1^\infty A_n\} \in \mathfrak{X}.$

(3). There exists a function, denoted by $|\ldots|$, called measure, defined throughout $\mathfrak{X}$ to R_+ such that

$$A_n \in \mathfrak{X} . \supset . \lim_{n\to\infty} |\sum_1^n A_m| \leqq \sum_1^\infty |A_n|.$$

, meaning that if the left member is infinite so is the right, while if the right member is finite so is the left, and the equality sign is to obtain if and only if the A_n are disjoint.

Such sets A will be called measurable sets, the term including not only sets A for which $\exists |A| =$ a finite number, but also sets A for which $\exists A_n, |A_n|: A = \sum_{n=1}^{\infty} A_n$, so that $|A| = \lim_{n \to \infty} |\sum_{m=1}^{n} A_m|$ and consequently $|A|$ need not be finite. Throughout it will be explicitly mentioned whenever $|A|$ is assumed to be finite.

By means of the conventions herein adopted the nature of a function will be made evident by the letters used in its expression. Also, throughout, a summation sign with no range will indicate a sum of a finite number of terms.

2. The General Principle of Convergence.

Let u be a variable over a space D. Let u' be a fixed element of D, at which D is dense. Let f(u) be a function on D to B, no requirements being made as to single-valuedness. Then the following theorem is true, the proof being exactly the same as in real number theory:

THEOREM: A necessary and sufficient condition that f(u) approach a limit as u approaches u' is that (f(u)-f(v)) approach zero as u and v approach u' independently.

This theorem will be referred to as the general principle of convergence. As a matter of reference we next state Fréchet's (1) extension of the Borel-Lebesgue theorem:

THEOREM: Let $\mathfrak{f}$ be a family of sets $I \leq A \leq V$, where A is closed and compact, such that $x \in A . \supset . x \underset{int.}{\in} I$ for some I.

Then there exists a finite number of sets $I_k \in \mathfrak{f}$ such that $x \in A . \supset . x \underset{int.}{\in} I_k$ for some k.

3. Measurable Functions. Fundamental Properties.

Definition: The function f(x) is measurable over the set A: $\equiv$:

(1). For any δ, there exists a finite number of non-overlapping, measurable sub-sets of A which almost comer A; ie., $\exists A_{n\delta}, |A_{n\delta}| : |A - \sum A_n| = 0.$

(2). For any n, δ, $x, y \in A_{n\delta}$; $|f(x) - f(y)| < \delta.$

(3). For any α, the sub-set of A containing all those and only those points for which $|f(x)| < \alpha.$ is measurable.

For any δ, the family $A_{n\delta}$ will be called a finite subdivision of A.

First of all we obtain a necessary condition that a function be measurable.

THEOREM 3.1. If f(x) is measurable over A, then f(x) is bounded almost everywhere in A.

Proof: Choose any ϵ. Then, by the above definition, there are a finite number of sets $A_{n\epsilon}$ which almost cover A, and in each of which $|f(x) - f(y)| < \epsilon$. Denote the sum of these sets by A'. Then, f(x) is bounded in A'. For, if we assume the contrary, ie., for any $\varphi > 0$, $\exists x \in A'$, $|f(x)| > \varphi$ then $\exists \{x_n\}, n=1,2,\dots; \leq A' : \{|f(x_n)|\}$ is a steadily increasing, divergent sequence. Hence, $\exists \{x'_n\} \leq \{x_n\}$ such that for any n, $|f(x'_n) - f(x'_{n+1})| > \epsilon$. Therefore, the points x'_n for distinct values of n, must lie in distinct members of the family $A_{i\epsilon}$. Therefore the range of i must be infinite if A' is to be covered by $\{A_{i\epsilon}\}$. Thus we have a contradiction.

Having obtained a necessary condition for measurability, we now obtain a sufficient one.

THEOREM 3.2. If $V \geq A \leq S$ and A is compact and closed, and if f(x) is continuous throughout A, Then f(x) satisfies the first two postulates of measurable functions (We shall see later that the third is a consequence of the first two).

Proof: In order to prove this we must make one more postulate about the measure function, namely that it exists, finite or infinite, for any hypersphere, open or closed, contained in X when S is also a V. Explicit mention will always be made when this is to be assumed. Then, since f(x) is continuous, $\exists\, \delta(\epsilon, x) : |y - x| < \delta(\epsilon, x) .\supset. |f(y) - f(x)| < \epsilon$.

Define $A(x, \epsilon) : \; y \in A(x, \epsilon) .\equiv. |y - x| < \delta(\epsilon, x)$.

Then $z \in A .\supset. z \in_{int} A(z, \epsilon)$.

Thus, $\{A(x, \epsilon)\}$, for any ϵ, is a family $\mathfrak{f}$ (see the theorem of Fréchet in section 2).

Hence, $\exists\, x_i$, range of i finite $=$ n.

$x \in A .\supset. x \in_{int} A(x_i, \frac{\epsilon}{2}) \equiv A_i(\epsilon)$ for some i.

$x \in A .\supset. |x - x_i| < \delta(\frac{\epsilon}{2}, x_i)$ for some i.

$x \in A .\supset. |f(x) - f(x_i)| < \epsilon/2$ for some i.

Thus we have $A = \sum_{1}^{n} A_i(\epsilon) : x, y \in A_i(\epsilon) \supset \cdot |f(x) - f(y)| < \epsilon$. for any ϵ.

If the A_i were disjoint, our theorem would be proved. Hence it will be sufficient ot construct disjoint, measurable sets $C_{k\epsilon}$ such that, for any k, $C_k \leq A_k : A = \sum_n C_n$.

Define $C_1 = A_1 : C_k = A_k \cdot \prod_{r=1}^{k-1} (A - C_r) = A_k(A - \sum_1^{k-1} C_r) : k = 2, \dots, n$.

Now $C_1 + C_2 = A_1 + A_2(A - A_1) = A_1 + A_2$.

Assume that for some r $\sum_1^r{}_k C_k = \sum_1^r{}_k A_k$.

Then $\sum_1^{r+1}{}_k C_k = C_{r+1} + \sum_1^r{}_k A_k = A_{r+1}(A - \sum_1^r{}_k C_k) + \sum_1^r{}_k A_k)$

$$= A_{r+1}(A - \sum_1^r{}_k A_k) + \sum_1^r{}_k A_k = \sum_1^{r+1}{}_k A_k.$$

Therefore, by induction $\sum_1^n{}_k C_k = \sum_1^n{}_k A_k = A$.

Next $C_1 C_2 = C_1 \cdot A_2 \cdot (A - C_1) = 0$.

Assume that for some r, $i \neq j \supset \cdot C_i \cdot C_j = 0, \; i, j \leq r$.

Then $C_i C_{r+1} = C_i A_{r+1} (A - C_1) \cdots (A - C_r) = 0$.

Therefore, by induction, for any i, j, $i \neq j \supset \cdot C_i C_j = 0$.

Lastly, it is obvious that, for any k, $C_k \leq A_k$.

We now show that our definition of measurability has many of the ordinary properties.

THEOREM 3.3. If f(x) and $\alpha(x)$ are measurable over A,

Then $f(x) \cdot \alpha(x)$ is measurable over A.

Proof: Let $|f(x)| < \varphi$, $|\alpha(x)| < \alpha$ almost everywhere in A.

Now $\exists A_{i\epsilon} : i \neq j \supset \cdot A_i A_j = 0 : |A - \sum A_i| = 0$.

$$x, y \in A_{i\epsilon} \supset \cdot |f(x) - f(y)| < \epsilon.$$

and $\exists B_{i\epsilon} : i \neq j \supset \cdot B_i B_j = 0 : |A - \sum B_j| = 0$.

$$x, y \in B_{i\epsilon} \supset \cdot |\alpha(x) - \alpha(y)| < \epsilon.$$

Define $A_{ij}(\epsilon) \doteq A_i(\frac{\epsilon}{2\alpha}) \cdot B_j(\epsilon/2\varphi)$.

Then $i \neq h . V . j \neq k .\supset: A_{ij} A_{hk} = 0$.

and $|A - \Sigma\Sigma A_{ij}| = |A - \Sigma A_i \cdot \Sigma B_j| = |(A - \Sigma A_i) + (A - \Sigma B_j)|$

$$\leq |A - \Sigma A_i| + |A - \Sigma B_j| = 0.$$

$$x, y \in A_{ij}(\epsilon) .\supset. |f(x)\alpha(x) - f(y)\alpha(y)| \leq$$

$$\leq |f(x)\alpha(x) - f(x)\alpha(y)| + |f(x)\alpha(y) - f(y)\alpha(y)|$$

$$< \varphi \cdot \epsilon/2\varphi + \alpha \cdot \epsilon/2\alpha = \epsilon.$$

Hence $f(x) \cdot \alpha(x)$ is measurable over A.

THEOREM 3.4. If f(x) and g(x) are measurable over A,

Then $f(x) \pm g(x)$ is measurable over A.

Proof: $\exists A_{i\epsilon} : |A - \Sigma A_i| = 0 : i \neq j .\supset. A_i A_j = 0 : x, y \in A_{i\epsilon} .\supset. |f(x) - f(y)| < \epsilon/2$.

$\exists B_{j\epsilon} : |A - \Sigma B_j| = 0 : i \neq j .\supset. B_i B_j = 0 : x, y \in B_{j\epsilon} .\supset. |g(x) - g(y)| < \epsilon/2$.

Define $A_{ij}(\epsilon) = A_{i\epsilon} \cdot B_{j\epsilon}$.

Therefore $|A - \Sigma\Sigma A_{ij}| = 0 : i \neq h . V . j \neq k .\supset: A_{ij} A_{hk} = 0$.

$$x, y \in A_{ij} .\supset. |f(x) \pm g(x) - f(y) \mp g(y)| < \epsilon.$$

Therefore $f(x) \pm g(x)$ is measurable over A.

Next we prove an extension of Egoroff's theorem:

THEOREM 3.5. If (1). For any n, $f_n(x)$ is measurable over A.

(2). $f_n(x)$ converges almost everywhere in A.

(3). $A^* \leq A : \exists |A^*| =$ a finite number.

Then, for any δ, $\exists A_\delta \leq A^* : \exists |A_\delta| > |A^*| - \delta$.

such that $f_n(x)$ converges uniformly in A_δ.

Proof: Let $A\{_j \epsilon_j\}$ be a monotonic, decreasing sequence, lim. 0.

Define $E_{n,r} \equiv A^* \cdot E\{p.q \geq n .\supset: |f_p(x) - f_q(x)| < \epsilon_r\}$.

Then for any r, $E_{n,r} \leq E_{n+1,r}$.

Also, for any n, r, $E_{n,r}$ is measurable.

For any r, $|A^*| = |\sum_1^\infty E_{n,r}|$ since $f_n(x)$ converges almost everywhere in A.

Therefore, for any r, $|A^*| = \lim_{n \to \infty} |E_{n,r}|$

Therefore, for any r, $\exists N(r,\delta): n \geq N .\supset. |A^* - E_{n,r}| < \delta/2^r$.

Also, for any r, $x \in E_{N,r} : p.q > N .\therefore\supset:. |f_p(x) - f_q(x)| < \epsilon_r$.

Define $A_\delta = \prod_{r=1}^{\infty} E_{N,r}$

Then $f_n(x)$ converges uniformly in A_δ.

and $|A^* - A_\delta| \leq \sum_1^\infty r |A^* - E_{N,r}| < \delta$.

THEOREM 3.5'. If (1). For any s in D, f(x,s) is measurable over A.

(2). D is dense at t.

(3). f(x,s) approaches a limit as s $\to$ t almost everywhere in A.

(4). $A^* \leq A : \exists |A^*| =$ a finite number.

Then, for any δ, $\exists A_\delta \leq A^* : \exists |A_\delta| > |A^*| - \delta$, such that f(x,s) approaches its limit uniformly in A_δ.

THEOREM 3.6. If f(x) is measurable over A,

Then |f(x)| is measurable over A.

Proof: $\exists A_{i\epsilon} : |A - \sum A_i| = 0 : i \neq j .\supset. A_i A_j = 0$.

$x.y \in A_{i\epsilon} .\supset: ||f(x)| - |f(y)|| \leq |f(x) - f(y)| < \epsilon$.

Therefore $|f(x)|$ is measurable over A.

THEOREM 3.7. If $f_n(x)$ is a sequence of functions, measurable over A, which converges to f(x) uniformly almost everywhere in A, Then f(x) is measurable over A.

Proof: For any n $\exists A^n_{i\epsilon} : |A - \Sigma A_i| = 0 : i \neq j .\supset. A_i A_j = 0.$

$$x, y \in A^n_{i\epsilon} :\supset: |f_n(x) - f_n(y)| < \epsilon.$$

$$\exists n_o(\epsilon) : n \geq n_o(\epsilon) .\supset. |f(x) - f_n(x)| < \epsilon.$$

Hence, define $A_{i\epsilon} = A^n_i(\epsilon/3)$ where $n = n_o(\epsilon/3)$.

Then $|A - \Sigma A_i| = 0 : i \neq j .\supset. A_i A_j = 0.$

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| - |f_n(y) - f(y)|$$

$$n = n_o(\tfrac{\epsilon}{3}) : x, y \in A_{i\epsilon} .:\supset:. |f(x) - f(y)| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3}$$

$$x, y \in A_{i\epsilon} :\supset: |f(x) - f(y)| < \epsilon.$$

ie., f(x) is measurable over A.

THEOREM 3.7'. Let f(x,s) be defined throughout AD, except when s = t. Let f(x,s) $\rightarrow$ f(x) as s $\rightarrow$ t, the convergence being uniform almost everywhere over A, For each s, let f(x,s) be measurable over A, and let D be dense at t. Then f(x) is measurable over A.

THEOREM 3.8. Let $f_n(x)$ be a sequence of functions, measurable over A, which converges to f(x) almost everywhere in A. Let $A^* \leq A \colon \ni |A^*|$ finite.

Then, for any δ, $\exists A_\delta \leq A^* \colon \ni |A_\delta| > |A^*| - \delta$.

such that f(x) is measurable over A_δ.

Proof: Theorems 3.5, 3.7.

THEOREM 3.8'. Let f(x,s) be defined throughout AD excep when s = t, measurable over A for each s. Let $f(x,s) \to f(x)$ as $s \to t$, almost every where in A. Let D be dense at t. Let $A^* \leq A \, . \ni |A^*|$ finite.

Then, for any δ, $\exists A_\delta \leq A^* \colon \ni |A_\delta| > |A^*| - \delta$,

such that f(x) is measurable over A_δ.

Proof: Theorems 3.5', 3.7'.

We now prove a theorem which shows that functions of the types considered by Saks and Bochner which are measurable according to our definition are also measurable according to their definitions.

THEOREM 3.9. If f(x) is measurable over A, then $\exists f_n(x)$ a sequence of step-functions, which $\to$ f(x) uniformly almost everywhere in A, and conversely.

Proof: Since f(x) is measurable over A, $\exists A_{i\epsilon} \colon$

$$|A - \Sigma A_i| = 0 \colon i \neq j \supset . A_i A_j = 0 \colon x, y \in A_{i\epsilon} . \supset \colon |f(x) - f(y)| < \epsilon .$$

Let ϵ_m be a monotonic decreasing sequence with limit zero, ie., $\exists m_0(\epsilon) \colon m \geq m_0(\epsilon) \supset . \epsilon_m < \epsilon$.

Now, for each i,m, define $A_{im} = A_i(\epsilon_m)$ and let $x_{im} \in A_{im}$.

For each m define $f_m(x) \colon f_m(x) = f(x_{im}) \colon \equiv \colon x \in A_{im}$

Such a function $f_m(x)$ which has only a finite number of distinct values is called a step-function.

Now, let $m > m_0(\epsilon) : \epsilon_m < \epsilon$.

For any m, $|A - \sum_i A_{im}| = 0$.

Therefore $|f(x) - f_m(x)| = |f(x) - f(x_{im})| < \epsilon_m < \epsilon$.

Therefore $f_m(x) \to f(x)$ uniformly almost everywhere in A.

Converse is obvious.

From the definition of measurability it is obvious that if f(x) is measurable over A and over C, then f(x) is measurable over A + C. This can be extended to the sum of any finite number of sets but not to the sum of a denumerable infinity of sets, since f(x) might not be bounded in the sum set. Also, if f(x) is measurable over A and if C is any measurable sub-set of A, then f(x) is measurable over C. This property has already been used.

We are now in a position to define an integral for measurable functions.

4. Integration. Definition and Fundamental Properties.

Let f(x) be measurable over A and let $|A|$ be finite.

$$\exists A_{i\epsilon} : i \neq j .\supset. A_i A_j = 0 : |A - \sum A_i| = 0.$$

$$x, y \in A_{i\epsilon} .\supset: |f(x) - f(y)| < \epsilon.$$

For any i, ϵ , let $x_{i\epsilon} \in A_{i\epsilon}$.

Definition: $\int_A f(x)\,dx = \lim_{\epsilon \to 0} \sum_i f(x_{i\epsilon}) \cdot |A_{i\epsilon}|$

if the limit exists. It is clear that if $|A| = \infty$, the sum will not exist for any ϵ.

THEOREM 4.1. If f(x) is measurable over A and if $|A|$ is finite, then $\exists \int_A f(x)\,dx$.

Proof: Define $I(\eta) \equiv \sum_i f(x_{i\eta})\cdot|A_{i\eta}|$ a function on R_+ to B.

By the general principle of convergence we need to show that $\exists\ \delta(\epsilon) : \mu.\nu \leq \delta(\epsilon) \supset : |I(\mu) - I(\nu)| < \epsilon$.

Let $\delta(\epsilon) = \epsilon/2\cdot|A|$.

$$|I(\mu) - I(\nu)| = \left|\sum_i f(x_{i\mu})\cdot|A_{i\mu}| - \sum_j f(x_{j\nu})\cdot|A_{j\nu}|\right|$$

Define $A_{ij} = A_{i\mu}\cdot A_{j\nu}$: $|A| = \sum_i\sum_j |A_{ij}|$

$$|I(\mu) - I(\nu)| = \left|\sum_i\sum_j \{f(x_{i\mu}) - f(x_{j\nu})\}\cdot|A_{ij}|\right|$$

If $A_{i\mu}, A_{j\nu}$ have no points in common, $|A_{ij}| = 0$.

If $A_{i\mu}, A_{j\nu}$ have common points, let x_{ij} be any such point, so that $x_{ij} \in A_{i\mu}, A_{j\nu}$.

Then
$$\begin{aligned}|I(\mu) - I(\nu)| &= \left|\sum_i\sum_j \{f(x_{i\mu}) - f(x_{ij}) + f(x_{ij}) - f(x_{j\nu})\}\cdot|A_{ij}|\right| \\ &\leq \sum_i\sum_j \{|f(x_{i\mu}) - f(x_{ij})| + |f(x_{ij}) - f(x_{j\nu})|\}\,|A_{ij}| \\ &\leq \sum_{i,j} \{\mu + \nu\}\cdot|A_{ij}| = (\mu+\nu)\cdot|A|\end{aligned}$$

Hence $\mu, \nu < \delta(\epsilon) \supset : |I(\mu) - I(\nu)| < \epsilon$.

THEOREM 4.2. If $\exists \int_A f(x)\,dx, \int_A g(x)\,dx$

Then $\exists \int_A \{f(x) \pm g(x)\}\,dx = \int_A f(x)\,dx \pm \int_A g(x)\,dx$.

Proof: By theorem 3.4, $f(x) \pm g(x)$ is measurable over A. By theorem 4.1, $\exists \int_A \{f(x) \pm g(x)\}\,dx$.

Using the notation of these theorems,

$$\left| \int_A \{f(x) \pm g(x)\}\, dx - \left\{ \int_A f(x)\, dx \pm \int_A g(x)\, dx \right\} \right|$$

$$\leq \left| \int_A \{f(x) \pm g(x)\}\, dx - \sum_i \sum_j \{ f(x_{ij}) \pm g(x_{ij}) \} | A_{ij\delta} | \right|$$

$$+ \left| \sum_i \sum_j \{ f(x_{ij}) \pm g(x_{ij}) \} \cdot | A_{ij\delta} | - \left\{ \int_A f(x)\, dx \pm \int_A g(x)\, dx \right\} \right|$$

$< \epsilon$ if δ is sufficiently small. But the original left member is independent of ϵ. Therefore

$$\int_A \{f(x) \pm g(x)\}\, dx = \int_A f(x)\, dx \pm \int_A g(x)\, dx.$$

THEOREM 4.3. If α is any real constant and if $\exists \int_A f(x)\, dx$,

Then $\exists \int_A \alpha \cdot f(x)\, dx = \alpha \cdot \int_A f(x)\, dx$.

Proof: It is obvious that a constant is a measurable function. Hence, by theorem 3.3, $\alpha \cdot f(x)$ is measurable. Hence, by theorem 4.1, $\exists \int_A \alpha \cdot f(x)\, dx$ which is clearly equal to $\alpha \cdot \int_A f(x)\, dx$.

THEOREM 4.4. If $\exists \int_A f(x)\, dx$,

Then $\exists \int_A | f(x) |\, dx \geq \left| \int_A f(x)\, dx \right|$

Proof: By theorems 3.6, 4.1, $\exists \int_A | f(x) |\, dx$.

$$\exists\, \delta(\epsilon) \cdot \eta < \delta(\epsilon) \supset \cdot \left| \int_A f(x)\, dx - \sum_i f(x_i) | A_i | \right| < \epsilon > \left| \int_A | f(x) |\, dx - \sum_i | f(x_i) | \cdot | A_i | \right|$$

Hence $\int_A | f(x) |\, dx \geq \sum_i | f(x_i) | \cdot | A_i | - \epsilon$, $\left| \int_A f(x)\, dx \right| \leq \left| \sum_i f(x_i) | A_i | \right| + \epsilon$

Adding, we have

$$\int_A | f(x) |\, dx - \left| \int_A f(x)\, dx \right| \geq \sum_i | f(x_i) | \cdot | A_i | - \left| \sum_i f(x_i) | A_i | \right| - 2\epsilon \geq -2$$

The left member is independent of ϵ

Therefore $\int_A | f(x) |\, dx - \left| \int_A f(x)\, dx \right| \geq 0$.

THEOREM 4.5. If (1). $\exists \int_A f(x)\alpha(x)\,dx, \int_A \alpha(x)\,dx$

(2). Almost everywhere in A, $|f(x)| \leq \varphi$; $\alpha(x) \geq 0$.

Then $\left|\int_A f(x)\alpha(x)\,dx\right| \leq \varphi \int_A \alpha(x)\,dx$.

Proof: $\exists\, \delta(\epsilon) : \eta < \delta(\epsilon) : \supset : \left|\int_A f(x)\alpha(x)\,dx - \sum_i f(x_i)\cdot\alpha(x_i)\,|A_i|\right| < \epsilon$.

$$\left|\int_A \alpha(x)\,dx - \sum_i \alpha(x_i)\cdot|A_i|\right| < \epsilon.$$

Therefore $\left|\int_A f(x)\alpha(x)\,dx\right| < \epsilon + \left|\sum_i f(x_i)\alpha(x_i)|A_i|\right| \leq \epsilon + \sum |f(x_i)||\alpha(x_i)||A_i|$

$$\left|\int_A f(x)\alpha(x)\,dx\right| < \epsilon + \varphi\sum \alpha(x_i)|A_i| < \epsilon + \varphi\left\{\epsilon + \int_A \alpha(x)\,dx\right\}$$

$$\left|\int_A f(x)\alpha(x)\,dx\right| \leq \varphi\int_A \alpha(x)\,dx.$$

THEOREM 4.6. If (1). $\exists \int_A f(x)\alpha(x)\,dx, \int_A \alpha(x)\,dx$

(2). F is the smallest closed linear manifold in B containing all the values of f(x) for x in A.

Then $\exists\, f \in F : \int_A f(x)\alpha(x)\,dx = f\cdot\int_A \alpha(x)\,dx$

where f will of course depend on the function α.

Proof:
$$\frac{\int_A f(x)\alpha(x)\,dx}{\int_A \alpha(x)\,dx} = \lim_{\eta\to 0}\frac{\sum_i f(x_i)\,\alpha(x_i)\,|A_i|}{\int_A \alpha(x)\,dx}$$
$$= \lim_{\eta\to 0}\sum_i \frac{\alpha(x_i)|A_i|}{\int_A \alpha(x)\,dx}\cdot f(x_i)$$

Now $\alpha(x_i), |A_i|, \int_A \alpha(x)\,dx$ are real numbers. Since F is a closed linear manifold, the right member above is an element of F, say f.

THEOREM 4.7'. If (1). $f(x,s)$ is defined throughout AD except when $s=t$.

(2). For each s, $\exists \int_A f(x,s)\,dx$

(3). $f(x,s) \rightarrow f(x)$ as $s \rightarrow t$, uniformly almost everywhere in A.

(4). D is dense at t.

Then $\exists \int_A f(x)\,dx = \lim_{s\rightarrow t} \int_A f(x,s)\,dx.$

THEOREM 4.8. If A and C are non-overlapping sets, and if

$$\exists \int_A f(x)\,dx, \quad \int_C f(x)\,dx$$

Then

$$\exists \int_{A+C} f(x)\,dx = \int_A f(x)\,dx + \int_C f(x)\,dx.$$

Chapter II.

Integration of Summable Functions.

1. Functions having the property M (or, belonging to the class M).

As a generalization of Chapter I we now define an integral for a more extensive class of functions. In the new definition no restrictions are made regarding boundedness, either of functions or sets. However, in the case of sets the usual devices seem to be necessary for the proofs of the fundamental properties of the integral. In the case of bounded sets, on the other hand, the fundamental properties can be proved at once for unbounded as well as bounded functions, thus giving a simplification of the theory as found in Titchmarsh [2].

Definition: f(x) is of class M over A: $\equiv$:

(1). For any δ, there exists a denumerable family of disjoint, measurable, bounded sub-sets of A which almost covers A, ie.,

$$\exists A_{n\delta} : |A - \sum_1^\infty A_n| = 0 : n \neq m \supset . A_n A_m = 0.$$

(2). For any n, δ, $x, y \in A_{n\delta} : |f(x) - f(y)| < \delta.$

(3). For any α, the sub-set of A containing all those and only those points for which $|f(x)| < \alpha$ is measurable.

For any δ, such a family of sets $A_{n\delta}$ will be called a subdivision of A.

THEOREM 1.1. Any measurable function is of class M.

THEOREM 1.2. If (1). f(x) is of class M over A.

(2). B is a measurable sub-set of A.

Then f(x) is of class M over B.

THEOREM 1.3. If (1). For any n, f(x) is of class m over A_n.

(2). $n \neq m \supset. A_n A_m = 0$.

(3). $|A - \sum_{n=1}^{\infty} A_n| = 0$.

Then f(x) is of class M over A.

This depends simply on the denumerability of a denumerable set of denumerable sets.

THEOREM 1.4. If (1). $|A - \sum_1^{\infty} A_n| = 0$.

(2). $\exists |A|, |A_n|$

(3). For any n, f(x) is of class M over A_n,

Then f(x) is of class M over A.

Proof: See theorem I.3.2.

Define $A^* = \sum_1^{\infty} A_n$: Then $|A - A^*| = 0$.

Define $C_1 = A_1$: $C_n = A_n \prod_{r=1}^{n-1} (A^* - C_r) = A_n(A^* - \sum_1^{n-1} C_r)$.

For any n, $\sum_1^n C_k = \sum_1^n A_k$: $n \neq m \supset. C_n C_m = 0$.

Therefore $\sum_1^{\infty} C_n = A^*$: $|A - \sum_1^{\infty} C_n| = 0$.

By the preceeding theorem, f(x) is of class M over A.

THEOREM 1.5. If (1). $\exists |A|$, finite,

(2). For any δ, $\exists A_\delta \subseteq A . |A_\delta| > |A| - \delta$.

(3). f(x) is of class M over A_δ.

Then f(x) is of class M over A.

Proof: Let δ_n be any monotonic decreasing sequence with limit zero. Define $A_n = A_{\delta_n}$. Then $|A - \sum_{n=1}^{\infty} A_n| = 0$.

The remainder of the proof follows immediately from the preceeding theorem.

THEOREM 1.6. If (1). A is measurable.

(2). For any $A^* \leq A$ such that $\exists |A^*|$ finite, and for any δ $\exists A_\delta \leq A^* : \exists |A_\delta| > |A^*| - A_\delta$

(3). f(x) is of class M over A_δ

Then f(x) is of class M over A.

Proof: By the preceeding theorem, f(x) is of class M over A^*. Let A_n be any sequence of sets of finite measure such that $|A - \sum_{n=1}^{\infty} A_n| = 0$. Then, by theorem II.1.4, f(x) is of class M over A.

THEOREM 1.7. If A is a compact closed set in a measurable space V and if f(x) is continuous over A, Then f(x) is of class M over A.

Proof: Theorems I.3.2,and II.1.1.

THEOREM 1.8. If f(x),$\alpha(x)$, are of class M and bounded almost everywhere over A

Then $\alpha(x) \cdot f(x)$ is of class M over A.

THEOREM 1.9. If f(x),g(x) are of class M over A,

Then $f(x) \pm g(x)$ is of class M over A.

THEOREM 1.10. If f(x) is of class M over A.

Then $|f(x)|$ is of class M over A.

THEOREM 1.11. If (1). For any n, $f_n(x)$ is of class M over A.

(2). $f_n(x) \rightarrow f(x)$ uniformly almost everywhere in A.

,Then f(x) is of class M over A.

THEOREM 1.11'. If (1). D is dense at t.

(2). For any s in D, $f(x,s)$ is of class M over A.

(3) $f(x,s) \to f(x)$ as $s \to t$, uniformly almost everywhere in A.

Then $f(x)$ is of class M over A.

We also have the extensions of Egoroff's theorem analogous to theorems I. 3.5,3.5'.

THEOREM 1.12. If (1). For any n, $f_n(x)$ is of class M over A.

(2). $f_n(x)$ converges almost everywhere in A.

(3). $A^* \leq A \ni |A^*|$ finite.

Then for any δ, $\exists A_\delta \leq A^* \ni |A_\delta| > |A^*| - \delta$. such that $f_n(x)$ converges uniformly in A_δ.

THEOREM 1.12'. If (1). D is dense at t.

(2). For any s in D, $f(x,s)$ is of class M over A.

(3). $f(x,s) \to f(x)$ as $s \to t$, almost everywhere in A.

(4). $A^* \leq A \ni |A^*|$ finite.

Then, for any δ, $\exists A_\delta \leq A^* \ni |A_\delta| > |A^*| - \delta$. such that $f(x,s) \to f(x)$ uniformly in A_δ.

THEOREM 1.13. If (1). For any n, $f_n(x)$ is of class M over A.

(2). $f_n(x) \to f(x)$ almost everywhere in A.

Then $f(x)$ is of class M over A.

Proof: Theorems II.1.12,1.11,1.6.

THEOREM 1.13'. If (1). D is dense at t.

(2). For any s in D, $f(x,s)$ is of class M over A.

(3). $f(x,s) \to f(x)$ as $s \to t$. almost everywhere in A.

Then $f(x)$ is of class M over A.

Proof: Theorems II.1.12',1.11',1.6.

Thus we see that functions of class M have all the ordinary properties of measurable functions in real number theory. In particular, the class M is closed under limiting processes applied to functions or sets.

DEFINITION: $f(x)$ is an elementary function over A: $\equiv$:

$$\exists A_n, |A_n| \text{ finite}, n=1,2,\dots ; A_n \leq A : n \neq m . \supset . A_n A_m = 0 .$$

$$| A - \sum_{n=1}^{\infty} A_n | = 0 : x \in A_n . \supset . f(x) = f_n ,$$

f_n a constant for each n.

THEOREM 1.14. $f(x)$ is of class M over A if and only if $\exists f_n(x)$ a sequence of elementary functions which converges to $f(x)$ uniformly almost everywhere in A.

This shows that any function which is measurable according to Saks or Bochner is of class M.

THEOREM 1.15. In the definition of functions of the class M, postulate 3 is a consequence of the first 2.

Proof:[3]. Let $f_m(x)$ be an approximating sequence of elementary functions, as in the preceeding theorem.

Then, clearly postulate 3 is satisfied by each $f_m(x)$

Let α be any positive real number.

For any m, $\exists\, f_{n_m}(x) \therefore |f(x) - f_{n_m}(x)| < 1/2^m$.

Define $G_m = A \cdot E\{|f_{n_m}(x)| < \alpha - 1/2^m\}$.

Then, in G_m, $|f(x)| < \alpha$.

Conversely, if $|f(x)| < \alpha$ let $|f(x)| = \alpha - \epsilon$.

Let $\frac{1}{2^m} < \frac{\epsilon}{2}$; $-\epsilon < -2/2^m$

Then $|f_{n_m}(x)| < |f(x)| + \frac{1}{2^m} = \alpha - \epsilon + \frac{1}{2^m} < \alpha - 1/2^m$

Therefore $A \cdot E\{|f(x)| < \alpha\} = A \cdot \sum_1^\infty G_m$ except for a set of measure zero. Hence postulate 3 is satisfied.

2. Summable Functions.

Definition: f(x) is summable over A: $\equiv$:

(1). f(x) is of class M over A.

(2). For any δ, any subdivision $A_{n\delta}$, any $x_n \in A_n$ the series $\sum_{n=1}^\infty |f(x_n)||A_n|$ converges.

Hence, the series $\sum_{n=1}^\infty f(x_n)|A_n|$ converges.

We begin by finding sufficient conditions that a function be summable.

THEOREM 2.1. If (1). f(x) is of class M over A.

(2). f(x) is bounded over A.

(3). $|A|$ is finite.

Then f(x) is summable over A.

Proof: Let $\varphi \geq |f(x)|$

Then $\sum_{n=1}^\infty |f(x_n)||A_n| \leq \varphi \cdot |A|$

But, any bounded series of positive terms is convergent.

THEOREM 2.2. If (1). $f(x)$ is of class M over A.

(2). For any δ, some subdivision $A_{n\delta}$ any $x_n \in A_n$; $\exists \sum_{n=1}^{\infty} |f(x_n)||A_n|$

(3). For any n, δ, A_{nm} is a subdivision of A_n, $x_{nm} \in A_{nm}$

Then $\exists \sum_{n,m=1}^{\infty} |f(x_{nm})||A_{nm}|$, the manner of summing being immaterial since all the terms are positive.

Proof: Lemma. $\exists \sum_{1}^{\infty} \varphi_n |A_n|$ where φ_n is the least upper bound of $f(x)$ in A_n. From the definition of the class M of functions it is clear that $\exists \varphi_n$, any n, δ.

Also, by the definition of a least upper bound, for any n, δ, ϵ, $\exists x_n \in A_n$ such that $0 \le \varphi_n - |f(x_n)| \le \epsilon$.

Choose $\epsilon = \varphi_n/2$, Then $\varphi_n \le 2|f(x_n)|$.

Therefore $\sum_{n=1}^{\infty} \varphi_n |A_n| \le 2\sum_{n=1}^{\infty} |f(x_n)||A_n|$

and, by hypothesis, the right member of the inequality exists.

Now, $\sum_{n,m=1}^{\infty} |f(x_{nm})||A_{nm}| \le \sum_{n=1}^{\infty} \varphi_n \sum_{m=1}^{\infty} |A_{nm}| = \sum_{n=1}^{\infty} \varphi_n |A_n|$

and, by the lemma, the extreme right member exists.

THEOREM 2.3. If (1). $|A|$ is finite.

(2). $f(x)$ is of class M over A.

(3). For any δ, some subdivision $A_{n\delta}$ any $x_n \in A_n$, $\exists \sum_{n=1}^{\infty} |f(x_n)||A_n|$

Then $f(x)$ is summable over A.

Proof: By the definition of summability it must be shown that if, for any δ, $B_{n\delta}$ is any subdivision of A, $y_n \in B_n$ then $\exists \sum_{n=1}^{\infty} |f(y_n)||B_n|$

Define $A_{nm} = A_n B_m$. Let $x_{nm} \in A_{nm}$.

Then

$$\sum_{m=1}^{\infty} |f(y_m)| \cdot |B_m| = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(y_m)||A_{nm}|$$
$$\leq \sum_{m}{}_1^{\infty} \sum_{n}{}_1^{\infty} |f(y_m) - f(x_{nm})||A_{nm}| + \sum_{m}{}_1^{\infty} \sum_{n}{}_1^{\infty} |f(x_{nm})||A_{nm}|$$
$$< \delta \cdot |A| + \sum_{m}{}_1^{\infty} \sum_{n}{}_1^{\infty} |f(x_{nm})||A_{nm}|$$

By the preceeding theorem, the right member is, for any δ finite.

THEOREM 2.4. If (1). $|A|$ is finite.

(2). f(x),g(x) are summable over A.

Then $f(x) \pm g(x)$ is summable over A.

Proof: Let A_n, B_n be the respective $\delta/2$ subdivisions of A for f,g.

Define $A_{nm} = A_n B_m$.

Then A_{nm} is a δ subdivision of A for $f(x) \pm g(x)$

Therefore, by hypothesis 2, $\exists \sum_{n,m}{}_1^{\infty} |f(x_{nm}) \pm g(x_{nm})||A_{nm}|$

Therefore, by theorem II.2.3, $f(x) \pm g(x)$ is summable over A.

THEOREM 2.5. If $|A|$ is finite, then any measurable function is summable.

THEOREM 2.6. If f(x) is summable over A, so is $|f(x)|$ and conversely.

THEOREM 2.7. If f(x) is summable over A, and if B is a measurable sub-set of A, then f(x) is summable over B.

THEOREM 2.8. If f(x) is summable over A and over B, Then f(x) is summable over A+B, AB.

THEOREM 2.9. If f(x) is summable over A, and if α is any real number, then $\alpha \cdot f(x)$ is summable over A.

THEOREM 2.10. If (1). $\beta(x) \geqslant 0$, summable.

(2). $\alpha(x) \geqslant 0$, of class M.

(3). $\alpha(x) \leq \beta(x)$.

(4). $|A|$ is finite.

Then $\alpha(x)$ is summable.

Proof: Let A_n, B_n be δ subdivisions of A for α, β respectively. Then $A_{nm} \equiv A_n B_m$ is a δ subdivision for α and for β. $\sum_{n,m=1}^{\infty} |\alpha(x_{nm})||A_{nm}| \leq \sum_{n,m=1}^{\infty} |\beta(x_{nm})||A_{nm}|$

and, by hypothesis (1), the right member exists.

Therefore, by theorem II.2.3, $\alpha(x)$ is summable.

THEOREM 2.11. If (1). $|A|$ is finite.

(2). For any n, $f_n(x)$ is summable over A.

(3). $f_n(x) \rightarrow f(x)$ uniformly almost everywhere in A.

Then f(x) is summable over A.

Proof: Let A_n^m be a $\delta/3$ subdivision of A for $f_m(x)$

Let m be such that $|f(x) - f_m(x)| < \delta/3$

almost everywhere in A.

Then A_n^m is a δ subdivision of A for f(x).

$$\sum_{n=1}^{\infty} |f(x_n)||A_n^m| \leq \sum_{n=1}^{\infty} |f(x_n) - f_m(x_n)||A_n^m| + \sum_{n=1}^{\infty} |f_m(x_n)||A_n^m|$$

$$< \frac{\delta \cdot |A|}{3} + \sum_{n=1}^{\infty} |f_m(x_n)||A_n^m|$$

and, by hypothesis 2, the right member exists.

THEOREM 2.12. If (1). $|A|$ is finite.

(2). For any n, $f_n(x)$ is summable over A.

(3). For every n, $|f_n(x)| < \alpha(x)$ where $\alpha(x)$ is summable over A.

(4). $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ almost everywhere in A.

Then f(x) is summable over A.

Proof: By theorem II.1.13, f(x) is of class M over A. Hence, by theorem II.2.10, f(x) is summable over A.

3. Integration of Summable Functions.

Definition: $F(\epsilon) = \sum_{n=1}^{\infty} f(x_n)\cdot|A_n|$ where A_n is any ϵ subdivision of A for f(x), $x_n \in A_n$.

Definition: $\int_A f(x)dx = \lim_{\epsilon \to 0} F(\epsilon)$ if the limit exists uniquely.

We begin with existence theorems.

THEOREM 3.1. If (1). f(x) is summable over A.

(2). $|A|$ is finite.

Then $\exists \int_A f(x)\,dx$.

Proof: It is sufficient to show that $F(\epsilon)$ satisfies the Cauchy condition.

$$|F(\epsilon) - F(\delta)| = \left|\sum_{1}^{\infty}{}_n f(x_{n\epsilon})|A_{n\epsilon}| - \sum_{1}^{\infty}{}_m f(x_{m\delta})\,|A_{m\delta}|\right|$$

As in the proof of Theorem I.4.1,

Define $A_{nm} = A_{n\epsilon}\cdot A_{m\delta}$ Then $|A| = \sum_{n,m\,1}^{\infty} |A_{nm}|$ since A_{nm} are disjoint. Let $x_{nm} \in A_{nm}$

Then $|F(\epsilon) - F(\delta)| \leq \left|\sum_{n,m\,1}^{\infty}\{f(x_{n\epsilon}) - f(x_{nm})\}|A_{nm}|\right| + \left|\sum_{m,n\,1}^{\infty}\{f(x_{nm}) - f(x_{m\delta})\}|A_{nm}|\right|$

$$\leq \sum_{n,m\,1}^{\infty} |f(x_{n\epsilon}) - f(x_{nm})||A_{nm}| + \sum_{m,n\,1}^{\infty} |f(x_{nm}) - f(x_{m\delta})||A_{nm}|$$

$$< (\epsilon + \delta)\cdot|A|.$$

and the right member can be made arbitrarily small by making ϵ, δ sufficiently small.

The above manipulations of series are permissible since each is absolutely convergent.

THEOREM 3.2. If (1). $f(x)$ is measurable over A.

(2). $|A|$ is finite.

Then the two definitions of its integral are equivalent.

Proof: Theorems I.3.1,4.1,II.2.1,3.1, and the general principle of convergence.

Thus if a function of class M happens to be measurable, there will be no ambiguity in the expression: $\int_A f(x)\,dx$.

THEOREM 3.3. If $|A|$ is finite, then a necessary and sufficient condition that $\exists \int f(x)\,dx$ is that $\exists \int |f(x)|\,dx$; then $\left|\int_A f(x)\,dx\right| \le \int_A |f(x)|\,dx$.

Proof: By theorems II.2.6, 3.1, the equivalence is evident. The remainder of the proof is exactly the same as that of theorem I.4.4.

THEOREM 3.4. If $\exists \int_A f(x)\,dx, \int_A g(x)\,dx, |A|$ finite,

Then $\exists \int_A \{f(x) \pm g(x)\}\,dx = \int_A f(x)\,dx \pm \int_A g(x)\,dx$.

Proof: Theorems II.2.4,3.1, remainder of proof as in theorem I.4.2.

THEOREM 3.5. If $\exists \int_A f(x)\,dx$ and α is any real number,

Then $\exists \int_A \alpha f(x)\,dx = \alpha \int_A f(x)\,dx$.

Proof: Theorem II.2.9.

THEOREM 3.6. If (1). $\exists \int_A \alpha(x) f(x)\,dx$, $\int_A \alpha(x)\,dx$

(2). $|f(x)| \leq \varphi$, $\alpha(x) \geq 0$ almost everywhere in A.

Then $\left|\int_A \alpha(x) f(x)\,dx\right| \leq \varphi \int_A \alpha(x)\,dx$.

Proof: Same as that of theorem I.4.5.

THEOREM 3.7. If (1). $\exists \int_A \alpha(x) f(x)\,dx$, $\int_A \alpha(x)\,dx$

(2). F is the smallest closed linear manifold in B containing all the values of f(x) for x in A.

Then $\exists f \in F$ such that $\int_A \alpha(x) f(x)\,dx = f \cdot \int_A \alpha(x)\,dx$.

where f will of course depend on the function α.

Proof: Same as that of theorem I.4.6.

THEOREM 3.8. If (1). $|A|$ is finite.

(2). $\beta(x) \geq 0 \leq \alpha(x) \leq \beta(x)$ almost everywhere in A.

(3). $\beta(x)$ is summable over A.

Then $\exists \int_A \alpha(x)\,dx$, $\int_A \beta(x)\,dx \geq \int_A \alpha(x)\,dx$.

Proof: By theorem II.2.10, $\alpha(x)$ is summable.

By theorem II.3.1, $\exists \int_A \alpha(x)\,dx$, $\exists \int_A \beta(x)\,dx$.

Inequality is obvious.

THEOREM 3.9. If $\exists \int_A f(x)\,dx$, $\int_B f(x)\,dx$, $|AB| = 0$,

Then $\exists \int_{A+B} f(x)\,dx = \int_A f(x)\,dx + \int_B f(x)\,dx$.

Proof: Theorem II.2.8.

THEOREM 3.10. If $\exists \int_A f(x)\,dx$, B is a measurable sub-set of A,

Then $\exists \int_B f(x)\,dx$

Proof: Theorem II.2.7.

THEOREM 3.11. If (1). $|A|$ is finite.

(2). For any n, $\exists \int_A f_n(x)\,dx$.

(3). $f_n(x) \to f(x)$ uniformly almost everywhere in A.

Then $\exists \int_A f(x)\,dx = \lim_{n\to\infty} \int_A f_n(x)\,dx$.

Proof: By theorem II.2.11, f(x) is summable over A.
By theorem II.3.1, $\exists \int_A f(x)\,dx$.
Remainder of proof is the same as that of theorem I.4.7.

THEOREM 3.12. If f(x) is of class M over A, where $|A|$ is finite, then a necessary and sufficient condition that $\exists \int_A f(x)\,dx$ is that the series $\sum_{m=0}^{\infty} m|A_m|$ converge, where for any m,

$$A_m = A \cdot E\{ m \le |f(x)| < m+1 \}.$$

Proof: For any ϵ, let $A_{n\epsilon}$ be any ϵ subdivision of A.
By theorems II.2.2, 2.3, 3.1, we must prove that the condition stated in the theorem is a necessary and sufficient condition for the convergence of the series $\sum_{n,m\ 1,0}^{\infty} |f(x_{nm\epsilon})||A_{n\epsilon} \cdot A_m|$
where $x_{nm\epsilon} \in A_{n\epsilon} \cdot A_m$.
Now $|A| + \sum_{m\ 0}^{\infty} m \cdot |A_m| \geq \sum_{n,m\ 1,0}^{\infty} |f(x_{nm\epsilon})||A_{n\epsilon} A_m| \geq \sum_{m\ 0}^{\infty} m \cdot |A_m|$
from which inequality the result is evident.

COROLLARY 1. If $|A|$ is finite, and if $\exists \int_A f(x)\,dx$.

Then $\lim_{m\to\infty} m \cdot |B_m| = 0$.

where $B_m = A \cdot E\{ m \le |f(x)| \}$.

Proof: Since $\sum_{m\ 1}^{\infty} m \cdot |A_m|$ converges,

$\exists M(\epsilon): \sum_{m\ M}^{\infty} m \cdot |A_m| < \epsilon$.

But $M \cdot |B_M| = M \cdot \sum_{M}^{\infty} m\, |A_m| \leq \sum_{M}^{\infty} m\, |A_m| < \epsilon.$

COROLLARY 2. The necessity does not depend on the finiteness of $|A|$.

Therefore $\sum_{1}^{\infty} m\, |A_m|$ must converge.

Therefore $|B_1|$ is finite.

COROLLARY 3. Similarly, if for any positive number α,

$$B_\alpha \equiv A \cdot E\{\alpha \leq |f(x)|\}$$

and if $\exists \int_A f(x)\,dx$.

Then $|B_\alpha|$ is finite.

THEOREM 3.13. Define $f_n(x) = f(x) \;:\equiv:\; |f(x)| < n.$

$= 0 \;:\equiv:\; |f(x)| \geq n.$

If $|A|$ is finite and if $\exists \int_A f(x)\,dx$.

Then $\int_A f(x)\,dx = \lim_{n \to \infty} \int_A f_n(x)\,dx.$

Proof: Using the notation of the preceeding theorem and its corollaries, $\left|\int_A f(x)\,dx - \int_A f_n(x)\,dx\right| = \left|\int_{B_n} f(x)\,dx\right|$

For any δ, let $A_{n\delta}$ be a δ subdivision of A.

Define $A_{n\delta m} = A_{n\delta} \cdot A_m$ Let $x_{n\delta m} \in A_{n\delta m}$.

Then $\left|\int_{B_n} f(x)\,dx\right| \leq \left|\int_{B_n} f(x)\,dx - \sum_{1}^{\infty} n \sum_{h}^{\infty} m\, f(x_{n\delta m})|A_{n\delta m}|\right|$

$+ \sum_{1}^{\infty} n \sum_{h}^{\infty} m\, |f(x_{n\delta m})||A_{n\delta m}| \leq \quad '' \quad + \sum_{1}^{\infty} n \sum_{h}^{\infty} m\, (m+1)|A_{n\delta m}|$

$\leq \quad '' \quad + \sum_{h}^{\infty} m\, |A_m| + \sum_{h}^{\infty} m\, m \cdot |A_m|$

By the preceeding theorem, the third term can be made arbitrarily small by a suitable choice of h. Similarly for the second term. After h has been so chosen, the first term can be made arbitrarily small by a suitable choice of δ, from the definition of the integral.

COROLLARY . If $|A|$ is finite, $\exists \int_A f(x)\,dx$.

Then $\exists \int_A |f(x)|\,dx = \lim_{n\to\infty} \int_A |f_n(x)|\,dx$.

We now prove a quasi-converse of the above theorem:

THEOREM 3.14. If, for any n, $\exists \int_A |f_n(x)|\,dx$

$\exists \lim_{n\to\infty} \int_A |f_n(x)|\,dx$, $|A|$ finite,

using the notation of the previous theorems,

Then $\exists \int_A f(x)\,dx = \lim_{n\to\infty} \int_A f_n(x)\,dx$.

Proof: By theorems II.3.1,3.13, it is sufficient to show that f(x) is summable over A. Hence, by theorem II.3.12, it is sufficient to show that the series $\sum_1^\infty m\cdot|A_m|$ converges. Now, $\exists\, n_0(\epsilon) : N \geq M \geq n_0 \,.\supset.\, \left|\int_A |f_N(x)|\,dx - \int_A |f_M(x)|\,dx\right| < \epsilon$

$$\int_{A_M+\cdots+A_N} |f(x)|\,dx < \epsilon.$$

Let $A_{n\delta}$ be any δ subdivision of A, $A_{nm} = A_{n\delta}A_m$, $x_{nm} \in A_{nm}$

Then $\sum_M^N m|A_m| = \sum_{n=1}^\infty \sum_M^N m|A_{nm}| \leq \sum_M^N \sum_1^\infty |f(x_{nm})|\cdot|A_{nm}|$

$$\leq \left|\sum_M^N \sum_1^\infty |f(x_{nm})||A_{nm}| - \int_{A_M+\cdots+A_N} |f(x)|\,dx\right|$$

$$+ \int_{A_M+\cdots+A_N} |f(x)|\,dx.$$

The last term can be made small by choosing M sufficiently large. For any choice of M,N, the first term is small with δ which does not appear in the left member of the inequality.

THEOREM 3.15. If (1). $|A|$ is finite.

(2). $\exists \int_A f(x)\,dx.$

(3). $A = \sum_1^\infty {}_m B_m$ where the B_m are disjoint, measurable.

Then $\int_A f(x)\,dx = \sum_1^\infty {}_n \int_{B_n} f(x)\,dx$.

Proof: Let $A_{n\delta}$ be any δ subdivision of A.

Let A_p be defined as in theorem 3.12.

Define $A_{nmp} = A_{n\delta} B_m A_p$.

Then $\left| \int_A f(x)\,dx - \sum_1^{M-1} {}_n \int_{B_n} f(x)\,dx \right| = \left| \int_{\sum_M^\infty {}_n B_n} f(x)\,dx \right|$

$$\leq \left| \int_{\sum_{n=M}^\infty B_n} f(x)\,dx - \sum_M^\infty {}_m \sum_1^\infty {}_{n,p} f(x_{nmp}) |A_{nmp}| \right| + \sum_M^\infty {}_m \sum_1^\infty {}_{n,p} (p+1) |A_{nmp}|$$

$$\leq \quad " \quad + \sum_M^\infty {}_m \sum_1^\infty {}_p (p+1) |B_m A_p|$$

Now the series $\sum_1^\infty {}_p (p+1)|A_p|$ converges and is equal to

$$\sum_1^\infty {}_m \sum_1^\infty {}_p (p+1) |B_m A_p|$$

Therefore the second term of the right member of the above inequality approaches zero with 1/M. For any M, the first term approaches zero with δ, which does not appear in the left member.

COROLLARY . If (1). $|A|$ is finite.

(2). $\exists \int_A f(x)\,dx$.

(3). $A = \sum_{m=1}^\infty B_m$ where the B_m are disjoint, measurable.

Then $\int_A |f(x)|\,dx = \sum_1^\infty {}_n \int_{B_n} |f(x)|\,dx$.

THEOREM 3.16. If (1). $|A|$ is finite.

(2). $\exists \int_A f(x)\,dx$.

(3). $B_n \leq A : \exists |B_n| : \lim_{n\to\infty} |B_n| = 0$.

Then

$$\lim_{n\to\infty} \int_{B_n} f(x)\,dx = 0 : \lim_{n\to\infty} \int_{B_n} |f(x)|\,dx = 0.$$

Proof: It is sufficient to prove the second conclusion.

Consider the construction.

$$A_1 = A - \sum_1^\infty {}_n B_n \; ; \quad A_2 = B_1 - \sum_2^\infty {}_n B_n .$$

$$A_3 = B_2 - B_2\{B_1 + \sum_3^\infty {}_n B_n\} \; ; \quad A_4 = B_1 B_2 - B_1 B_2 \{\sum_3^\infty {}_n B_n\} .$$

$$A_5 = B_3 - \text{etc.} \qquad A_6 = B_1 B_3 - \text{etc.}$$

$$A_7 = B_2 B_3 - \text{etc.} \qquad A_8 = B_4 - \text{etc.}$$

etc.

$$A_{K_n} = B_n - B_n\{\sum_1^{n-1} {}_r B_r + \sum_{n+1}^\infty {}_r B_r\} .$$

$$A_{K_n+1} = B_1 B_n - \text{etc.} \qquad \text{etc.}$$

Then, evidently for any n, there is an integer $K_n (= 2 + \frac{n(n-1)}{2})$ such that $B_n \le \sum_{K_n}^\infty {}_r A_r$ and $K_n \to \infty$ as $n \to \infty$.

Also $A = \sum_1^\infty {}_n A_n$: $n \ne m . \supset . A_n A_m = 0$.

Hence, by theorem II.3.15,, $\lim\limits_{n\to\infty} \int_{\sum_{K_n}^\infty {}_r A_r} |f(x)| dx = 0$.

Therefore $\lim\limits_{n\to\infty} \int_{B_n} |f(x)| dx = 0$.

THEOREM 3.17. Analog of Lebesgue's convergence theorem.

If (1). $|A|$ is finite.

(2). For any n, $\exists \int_A f_n(x)\, dx$.

(3). $|f_n(x)| \le \alpha(x)$ almost everywhere in A.

(4). $\exists \int_A \alpha(x)\, dx$.

(5). $f(x) = \lim\limits_{n\to\infty} f_n(x)$ almost everywhere in A.

Then $\exists \int_A f(x)\, dx = \lim\limits_{n\to\infty} \int_A f_n(x)\, dx$.

Proof: Same as that of Titchmarsh, with the use of theorem II.2.10.

4. Relations of this to other recently defined integrals.

THEOREM 4.1. If f(x) is summable over A, then a necessary and sufficient condition that f(x) be integrable over A is that if $x_n, y_n \in A_{n\epsilon}$
$$\left|\sum_{1}^{\infty}{}_n \{f(x_n) - f(y_n)\}|A_n|\right| < \epsilon.$$

Proof: This is equivalent to the Cauchy condition, since
$$|F(\epsilon) - F(\eta)| \leq \left|\sum_{1}^{\infty}{}_n f(x_{n\epsilon})|A_{n\epsilon}| - \sum_{1}{}_{n,m} f(x_{nm})|A_{n\epsilon}A_{m\eta}|\right|$$
$$+ \left|\sum_{1}^{\infty}{}_{n,m} f(x_{nm})|A_{n\epsilon}A_{m\eta}| - \sum_{1}^{\infty}{}_m f(x_{m\eta})|A_{m\eta}|\right|$$

$< \delta$ if ϵ, η sufficiently small.

We might, with Birkhoff (4), take simply this condition as our definition of integrability. Taht is, f(x) is integrable if and only if the series $\sum_{n=1}^{\infty} |f(x_n)||A_n|$ converges and this Cauchy condition is satisfied. However, this definition does not seem to yield Egoroff's theorem. For, consider the simple real function $f(x) = x^{-1/2}$

Let A be the interval $[0, 1]$.

For any ϵ, the interval $B_\epsilon \equiv [\frac{\epsilon^2}{4}, 1]$ can be subdivided into a <u>finite</u> number of sub-intervals in each of which the oscillation of f(x) is less than $\epsilon/4$, so that

$$\sum_B O_n|B_n| < \frac{\epsilon}{4}|B| < \frac{\epsilon}{4} \quad : O_n = \underset{x,y \in B_n}{\text{l.u.b.}} |f(x) - f(y)|$$

Hence, define $A_{n\epsilon} = [\frac{\epsilon^2}{2^{2n+1}}, \frac{\epsilon^2}{2^{2n}}], \quad n \geq 1.$

so that $|A_n| = \frac{3\epsilon^2}{2^{2n+2}}, \quad O_n = 2^n/\epsilon.$

Then $\sum_A |f(x_n)||A_n| = \sum_{n=1}^{\infty} |f(x_n)||A_n| + \sum_B$ ''

and the sum over B contains only a finite number of terms, while, $\sum_{n=1}^{\infty} \leq \sum_{n=1}^{\infty} \frac{2^{n+1}}{\epsilon} \cdot \frac{3\epsilon^2}{2^{2n+2}} = \frac{3\epsilon}{2} \sum_{1}^{\infty} \frac{1}{2^n} = \frac{3\epsilon}{2}$

so that f(x) is summable over the given subdivision.

Furthermore, $\sum_A O_n |A_n| = \sum_1^{\infty} O_n |A_n| + \sum_B$ "

$$\leq \sum_1^{\infty} \frac{2^n}{\epsilon} \cdot \frac{3\epsilon^2}{2^{2n+2}} + \frac{\epsilon}{4} = \epsilon.$$

so that this Cauchy condition is satisfied. But, $O_n = 2^n/\epsilon$ which, for any ϵ, is not uniformly bounded in n. That is, the function is not of class M with this choice of subdivisions, and the proof of Egoroff's theorem rests directly on the properties of functions of the class M.

However, in such a simple example the difficulty is removable. There do exist subdivisions satisfying the postulates of the class M. The question, then, of the sufficiency of the condition of theorem 4.1, for the existence of such subdivisions still remains an open one.

It is evident from the definitions that this integral includes those of Saks and Bochner, and Radon.

Also, Birkhoff's integral is more general than this only in so far as "unconditional" convergence is more general than absolute. If these are equivalent, then so are the integrals.

Chapter III.

Theory of Divergent Series.

Before proceeding to the next and last integral concept, we need some elements of the theory of divergent series.

1. The Cesàro Theory.[5]

Consider the series $\sum_{0}^{\infty} {}_k a_k$: $A_m^0 = \sum_{0}^{m} {}_k a_k$.

Define $(n, k) = n! / k!(n-k)!$: ≡ : $n \geq k \geq 0$.

$(n, -1) = 0.$ $n \geq 0.$

$(n, m) = 0,$ $n < m.$

$A_n^r = \sum_{0}^{n} {}_m (r+m-1, m) A_{n-m}^0$

$\alpha_n^r = A_n^r / (r+n, n)$ "(C, r)". $\sum_{k=0}^{\infty} a_k = \lim_{n \to \infty} \alpha_n^r.$

If $A = \sum_{n=0}^{\infty} a_n, B = \sum_{n=0}^{\infty} b_n$ are summable (C, r), then the following properties are well known.[5]:

1). $a_n \to 0 \ (C, r).$

2). $(A - A_n^0) \to 0 \ (C, r).$

3). $\sum_{0}^{\infty} {}_n (a_n - a_{n+1}) = a_0 \ (C, r).$

4). A change in one term of A will affect its sum in the obvious manner.

5). The addition of a constant other than zero to each term of A will destroy its summability.

6). If $a_n \geq 0$, then $\sum_{n=0}^{\infty} a_n$ is convergent.

7). $\sum_{0}^{\infty} {}_n (a_n + b_n) = A + B \quad (C, r).$

8). $a_n \leq b_n \ .\supset. \ A \leq B.$

9). $a_n \leq b_n .\supset. \sum_{n=0}^{\infty} b_n - a_n$ converges to (B-A).

We also need the following two more important theorems:

a). If $A \equiv \sum_0^\infty a_n$ is summable (C,r),

Then $A' \equiv \sum_0^\infty \frac{a_n}{n+1}$ is summable $(C, r-1)$ and

$\sum_{n=0}^{\infty} \alpha_n$ is summable $(C,r-1)$ to A, where

$$\alpha_n = \sum_n^\infty \frac{a_m}{m+1} \qquad (C, r-1).$$

Also, $a_n = (n+1)(\alpha_n - \alpha_{n+1})$. $\quad r \geq 1$.

b). If $A \equiv \sum_0^\infty \alpha_n$ is summable $(C,r-1)$,

Then $\sum_0^\infty a_n$ is summable (C,r) to A, where

$$a_n = (n+1)(\alpha_n - \alpha_{n+1}).$$

We shall adopt the following notation: $A_n \equiv \sum_0^n \frac{a_m}{m+1}$

so that $A_n = \alpha_0 - \alpha_{n+1}$

The purpose of this section is to prove the

THEOREM. If (1). $a_n \leq d_n \leq b_n$ all n.

(2). $A \equiv \sum_0^\infty a_n$, $B \equiv \sum_0^\infty b_n$ are summable (C,r),

Then $D \equiv \sum_0^\infty d_n$ is summable (C,r) and $A \leq D \leq B$.

Proof: The theorem is obviously true for $r = 0$. We proceed by induction on r, which is a positive integral variable.

Let α_n, β_n be defined as in theorem (a). We have the following results immediately:

Lemma 1. $\alpha_n \leq \beta_n$ all n.

Proof: Theorems 8 and (a).

Lemma 2. $\sum_{n=0}^{\infty} (\beta_n - \alpha_n)$ converges to (B-A).

Proof. Theorems 9 and (a), lemma 1.

Hence $\lim_{n \to \infty} (\beta_n - \alpha_n) = 0$.

Lemma 3. $\alpha_0 + D_n - A_n \leq \beta_0 - (B_n - D_n)$ all n.

Proof: Since $\alpha_0 - A_n = \alpha_{n+1}$, $\beta_0 - B_n = \beta_{n+1}$

it follows from lemma 1.

Hence it is permissible to speak of the interval.

$I_n \equiv [\alpha_0 + D_n - A_n, \beta_0 - (B_n - D_n)]$. Clearly, $I_n < [\alpha_0, \beta_0]$ all n.

Lemma 4. The sequence of intervals I_n defines a real number.

Proof: We must show that

1). $I_n \geq I_{n+1}$ all n.

2). $\lim_{n \to \infty}(\text{length of } I_n) = 0$.

Clearly, $D_n - A_n \geq 0 \leq B_n - D_n$ since $a_n \leq d_n \leq b_n$.

Thus to prove (1), we need only show that

$D_{n+1} - A_{n+1} \geq D_n - A_n$ and $B_{n+1} - D_{n+1} \geq B_n - D_n$.

This is true since $D_{n+1} - A_{n+1} = D_n - A_n + \dfrac{d_{n+1} - a_{n+1}}{n+2}$

and $B_{n+1} - D_{n+1} = B_n - D_n + \dfrac{b_{n+1} - d_{n+1}}{n+2}$.

As for (2), the (length of I_n) $= \beta_0 - \alpha_0 - B_n + A_n = \beta_{n+1} - \alpha_{n+1}$.

and the result follows from lemma 2.

Let δ_0 be the number defined by the sequence of intervals I_n. Then $\alpha_0 \leq \delta_0 \leq \beta_0$; $(\alpha_0 + D_n - A_n) \leq \delta_0 \leq (\beta_0 - [B_n - D_n])$ ie., δ_0 lies in I_n, all n.

Lemma 5. If δ_n be defined by $d_n = (n+1)(\delta_n - \delta_{n+1})$ $n \geq 0$.

Then $\alpha_n \leq \delta_n \leq \beta_n$ all n.

Proof: $\delta_{n+1} = \delta_0 - D_n$; $\alpha_{n+1} = \alpha_0 - A_n$; $\beta_{n+1} = \beta_0 - B_n$.

But, since $\alpha_0 + D_n - A_n \leq \delta_0 \leq \beta_0 - (B_n - D_n)$

it follows that $\alpha_0 - A_n \leq \delta_0 - D_n \leq \beta_0 - B_n$

and hence the result.

We are now able to complete the proof of the theorem stated at the beginning of the paragraph.

Proceeding by induction, assume the truth of the theorem for (r-1). By theorem (a), $\sum_0^\infty \alpha_n$, $\sum_0^\infty \beta_n$ are summable (C,r-1).

By the induction hypothesis and lemma 5,

$$\sum_0^\infty \delta_n \quad \text{is summable } (C, r-1).$$

Hence, by theorem (b), $\sum_0^\infty d_n$ is summable (C,r).

Then $A \leq D \leq B$. by theorem 8.

Since we are discussing the subject of divergent series, it seems convenient to introduce here the following new method of summation.

2. New Summation Method for Divergent Series.

The method to be given here is a modification of that due to Euler-Knopp[6]. For the weighted means of the partial sums we use the binomial coefficients, but instead of beginning with the first we begin with the central one,ie., with the greatest. Thus the initial terms always receive the greatest weight, as in the Cesàro-Hölder method.

Therefore let us consider the series $\sum_{k,0}^\infty a_k$.

with the partial sums. $A_n = \sum_{k,0}^n a_k$

Define $S_n^1 = \sum_{k,0}^n (2n+1, n-k) \cdot A_k / 2^{2n}$.

and $S_n^{r+1} = \sum_{k,0}^n (2n+1, n-k)\, S_k^r / 2^{2n} \qquad r \geq 1$

If there exists $\lim_{n \to \infty} S_n^r = S^r$ then S^r is called the "sum" of the series.

Our first problem is to find the relation of this method of summation to that of Cesàro. To do this we recall the following well-known equality[7]:

$$(1-x)^r \sum_{n,0}^\infty A_n^r x^n = \sum_{n,0}^\infty A_n x^n$$

both series converging absolutely in the open interval (-1,1).

Hence, equating coefficients we obtain the partial sums of the original series as linear functions of the Cesàro means.

$$A_n = \sum_{0}^{n}{}_k (-1)^k (r,k)(r+n-k,r)\,\alpha^r_{n-k} \qquad : n \le r.$$

$$A_n = \sum_{0}^{r}{}_k \quad '' \qquad n \ge r.$$

Thus we can express our means as linear functions of Cesàro's.

$$S'_m = \sum_{0}^{m}{}_n \sum_{0}^{n}{}_k (-1)^k (2m+1,m-n)(r,k)(r+n-k,r)\,\alpha^r_{n-k} / 2^{2m} \qquad m \le r$$

and $S'_m = \sum_{0}^{r}{}_n \sum_{0}^{n}{}_k \; '' \; + \sum_{r+1}^{m}{}_n \sum_{0}^{r}{}_k \; ''$. $\qquad m \ge r+1.$

or, collecting coefficients and rearranging

$$S'_m = \sum_{0}^{m}{}_j \alpha^r_j \left\{ \sum_{j}^{m}{}_n [(-1)^{n-j}(2m+1,m-n)(r,n-j)(r+j,j) / 2^{2m}] \right\} \quad m \le r.$$

and $S'_m = \sum_{0}^{m-r}{}_j \alpha^r_j \{ \sum_{j}^{j+r}{}_n [\,..\,]\} + \sum_{m-r+1}^{m}{}_j \alpha^r_j \{ \sum_{j}^{m}{}_n [\,..\,]\}, \qquad m \ge r+1.$

Hence we may write $\quad S'_m = \sum_{0}^{m}{}_j a^r_{mj}\,\alpha^r_j$.

by defining $\quad a^r_{mj} = \sum_{j}^{j+r}{}_n (-1)^{n-j}(2m+1,m-n)(r,n-j)(r+j,j) / 2^{2m} : \; m \ge r+1.$

Now we make use of the following

Toeplit's theorem: Consider the transformation:

$$t_m = \sum_{0}^{q(m)}{}_j a_{mj}\, s_j$$

In order that $\quad \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n$

whenever the latter exists, it is necessary and sufficient that

1). $\lim_{m\to\infty} \sum_{0}^{q(m)}{}_j a_{mj} = 1.$

2). $\lim_{m\to\infty} a_{mj} = 0 \quad$ all j.

3). $\sum_{0}^{q(m)}{}_j |a_{mj}| < K \qquad$ all m, K independent of m.

Thus we shall not need the values of the coefficients

a^r_{mj} for $m \le r$, since all three of the conditions of the theorem depend on the behavior of the a^r_{mj} as $m \longrightarrow \infty$.

Now let us consider the first condition, which is that

$$\lim_{m\to\infty} \sum_0^m {}_j\, a^r_{mj} = 1.$$

We have $\sum_0^m {}_j\, a^r_{mj} = \sum_0^m {}_j \sum_j^{j+r} {}_n (-1)^{n-j}(r+j,j)(r,n-j)(2m+1,m-n)/2^{2m}$.

Interchanging the order of summing, we have

$$\sum_0^m {}_j\, a^r_{mj} = \sum_0^r {}_n \left(\frac{1}{2^{2m}}\right)\cdot(2m+1,m-n)\cdot\sum_0^n {}_j (-1)^{n-j}(r+j,j)(r,n-j)$$

$$+ \sum_{r+1}^m {}_n \{\,..\,\}\cdot\sum_{n-r}^n {}_j \{\,..\,\}.$$

We can write this as $\sum_0^m {}_j\, a^r_{mj} = \sum_0^m {}_n (-1)^n (2m+1,m-n)\, T[r,n]/2^{2m}$

where $T[r,n] = \sum_0^n {}_j (-1)^j (r+j,j)(r,n-j) \qquad n \le r$

and $T[r,n] = \sum_{n-r}^n {}_j \quad " \qquad n \ge r.$

By means of the recursion relations $(j,k) = (j-1,k-1)+(j-1,k)$ we find that $T[r,n] = -T[r,n-1]+T[r-1,n-1]+T[r-1,n]$; $n,r \ge 2$.

while $T[r,1] = -1$; $T[1,n] = (-1)^n$, $T[r,0] = 1$, $r,n \ge 0$.

by actual calculation. Thus, by induction, $T[r,n] = (-1)^n$ for every n, r. Hence, $\sum_0^m {}_j\, a^r_{mj} = 1$ for every r, m. Therefore Toeplitz's first condition is satisfied.

Now let us consider the second condition, namely

$$\lim_{m\to\infty} a^r_{mj} = 0 \qquad \text{any } r, j.$$

We have $a^r_{mj} = \sum_j^{j+r} {}_n (-1)^{n-j}(2m+1,m-n)(r,n-j)(r+j,j)/2^{2m} \qquad m \ge r+1.$

$$= \{(r+j,j)/2^{2m}\}\cdot\sum_j^{j+r} {}_n (-1)^{n-j}(r,n-j)(2m+1)!/(m-n)!\,(m+n+1)!$$

$$= \{(r+j,j)(2m+1)!/2^{2m}(m-j)!(m+j+r+1)!\} \sum_{n=j}^{j+r} (-1)^{n-j}(r,n-j)\{(m-n+1)\cdots(m-j)\}\{(m+n+2)\cdots(m+j+r+1)\}$$

where there are n-j factors in the first bracket and j + r-m in the second, or r in all. That is, each term in the sum is a polynomial in m of degree r. The leading coefficient in each term is $(-1)^{n-j}(r,n-j)$; hence, for the entire sum the leading coefficient is $\sum_{n=j}^{j+r} (-1)^{n-j}(r,n-j) = (1-1)^r = 0.$

Thus the entire sum is a polynomial in m of degree r-1 at most. Denote it by P.

$$a^r_{mj} = \frac{(r+j,j)\cdot 1\cdot 3\cdot 5\cdots(2m-2j-1)\cdot\{(2m-2j+1)\cdots(2m+1)\}\cdot P}{2^j\cdot 2\cdot 4\cdot 6\cdots(2m-2j)\cdot\{(m+1)\cdots(m+j+r+1)\}}$$

where the bracket in the numerator has j + 1 factors and that in the denominator j + r + 1.

Therefore $|a^r_{mj}| \le (r+j,j)\cdot O(1/m)$

Hence

$$\lim_{m\to\infty} a^r_{mj} = 0.$$

Now for the third condition, which is:

$$\sum_{j=0}^{m} |a^r_{mj}| < K(r) \qquad \text{K independent of m.}$$

$$a^r_{mj} = \sum_{n=j}^{j+r} (-1)^{n-j}(r,n-j)(r+j,j)(2m+1,m-n)/2^{2m}$$

substitute h for n - j , yielding

$$a^r_{mj} = \sum_{h=0}^{r} (-1)^h (r,h)(r+j,j)(2m+1,m-j-h)/2^{2m}$$

Now, in the statement of Abel's lemma[9], we have

$$f_h = (2m+1,m-j-h)(r+j,j)/2^{2m} \;;\; f_i \ge f_{i+1} > 0.$$

and[10]: $a_h = (-1)^h(r,h) \;;\; S_k \equiv \sum_{h=0}^{k} a_h = \sum_{h=0}^{k} (-1)^h (r,h) = (-1)^k (r-1,k).$

and hence, again by footnote [9],

$$\sum_{0}^{r}{}_h\, a_h f_h = \sum_{0}^{r-1}{}_K\, S_K (f_K - f_{K+1}) + S_r f_r$$

or, since $S_r = 0$

$$a^r_{mj} = \sum_{0}^{r-1}{}_K\, (-1)^K (r-1, K)(r+j, j)\{(2m+1, m-j-K) - (2m+1, m-j-K-1)\}/2^{2m}$$

Now, in the special case $r = 2$,

$$a^2_{mj} = (j+2, j)\{(2m+1, m-j) - 2(2m+1, m-j-1) + (2m+1, m-j-2)\}/2^{2m}$$

$$= (2m+1)!\,(j+1)(j+2)(-m+2j^2+6j+3)\,/\,2^{2m}(m-j)!\,(m+j+3)!$$

$$a^2_{mj} \leq 0 \quad \text{if} \quad j \leq \sqrt{\frac{m}{2} + \frac{3}{4}} - \frac{3}{2} \equiv \varkappa.$$

while $a^2_{mj} \geq 0$ if $j > \varkappa$.

$$\therefore \sum_{0}^{m}{}_j\, |a^2_{mj}| = -\sum_{0}^{\varkappa}{}_j\, a^2_{mj} + \sum_{\varkappa+1}^{m}{}_j\, a^2_{mj} = \sum_{0}^{m}{}_j\, a^2_{mj} - 2\sum_{0}^{\varkappa}{}_j\, a^2_{mj}$$

$$= 1 - 2\sum_{0}^{\varkappa}{}_j\, a^2_{mj}.$$

Therefore $\displaystyle \sum_{0}^{m}{}_j\, |a^2_{mj}| = 1 + \frac{4(2m+1)!}{2^{2m+1}} \sum_{0}^{\varkappa}{}_j\, \frac{(j+1)(j+2)(m-2j^2-6j-3)}{(m-j)!\,(m+j+3)!}$

$$\leq 1 + \frac{4(2m+1)!}{2^{2m+1}\, m!\,(m+1)!} \sum_{0}^{\varkappa}{}_j\, \frac{(j+1)(j+2)(m-2j^2-6j-3)}{(m+2)(m+3)}$$

Now [11] : $\displaystyle \sum_{0}^{\varkappa}{}_j\, j^4 = O(\varkappa^5) = O(m^{5/2})$

$$\therefore \sum_{0}^{m}{}_j\, |a^2_{mj}| \leq 1 + \frac{4(2m+1)!}{2^{2m+1}\, m!\,(m+1)!}\, O(\sqrt{m}).$$

Thus, we need only show that

$$f_m \equiv \frac{\sqrt{m}\,(2m+1)!}{2^{2m+1}\, m!\,(m+1)!}$$ is a bounded function of m.

$$f_m = \frac{\sqrt{m}\,\Gamma(2m+2)}{2^{2m+1}\,\Gamma(m+1)\,\Gamma(m+2)}$$ and, using Stirling's theorem

we have $f_m \longrightarrow \dfrac{\sqrt{m}\,(2m+2)^{2m+1+1/2}\sqrt{2\pi}\; e^{m+1}\, e^{m+2}}{e^{2m+2}\,(m+1)^{m+1/2}\sqrt{2\pi}\,(m+2)^{m+1+1/2}\sqrt{2\pi}\;\; 2^{2m+1}}$

$$f_m \longrightarrow \frac{e}{\sqrt{\pi}}\;\frac{(2m+1)^{2m+1}\left(1+\frac{1}{2m+1}\right)^{2m+1}}{m^m\left(1+\frac{1}{m}\right)^m (m+1)^{m+1}\left(1+\frac{1}{m+1}\right)^{m+1}}\cdot\frac{1}{2^{2m+1}}$$

$$f_m \longrightarrow \frac{1}{\sqrt{\pi}}\cdot\frac{1+1/2m}{1+1/m} \longrightarrow \frac{1}{\sqrt{\pi}}.$$

Hence we have proved the

THEOREM 1. Any series which is summable (C,2) is summable by this method to the same sum.

Next we investigate the relation of this method of summation to that of Euler-Knopp. Using Rey-Pastor's [13] result

$$A_n = \sum_0^n{}_r (-1)^{n-r}(n,r)\,2^r\cdot E_r \;:\; E_r = \frac{1}{2^r}\sum_0^r{}_n (r,n)\,A_n.$$

we find that $S'_m = \sum_0^m{}_r\, a_{mr}\, E_r$

where $a_{mr} = \sum_r^m{}_n (-1)^{n-r}(n,r)(2m+1,m-n)\,2^r/2^{2m}$

We wish to see if the matrix $\|a_{mr}\|$ is a Toeplitz matrix.

Is it true that $\sum_{r=0}^m a_{mr} \to 1$ as $m \to \infty$?

We have $\sum_0^m{}_r\, a_{mr} = \sum_0^m{}_r \sum_r^m{}_n (-1)^{n-r}(n,r)(2m+1,m-n)\,2^r/2^{2m}$

$$= \sum_0^m{}_n \{(2m+1,m-n)/2^{2m}\}\sum_0{}_r (-1)^{n-r}(n,r)\,2^r$$

$$= \sum_0^m{}_n \{(2m+1,m-n)/2^{2m}\}(2-1)^n$$

$$= 1$$

Thus the first condition is satisfied. We prove that the third condition is also satisfied by showing that $a_{mr} \geq 0$, for all m, r. $\quad a_{mr} = \dfrac{2^r}{2^{2m}} \sum_{n=r}^{m} (-1)^{n-r} (n,r)(2m+1, m-n)$

$$a_{mr} = \frac{2^r (-1)^{m-r}}{2^{2m}} \sum_{K=0}^{m-r} (-1)^K (2m+1, K)(m-K, r)$$

Now, in the statement of Abel's lemma we have (9) :

$$f_K = (m-K, r) > f_{K+1} > 0.$$

$$f_K - f_{K+1} = (m-K, r) - (m-K-1, r) = (m-K-1, r-1)$$

$$a_K = (-1)^K (2m+1, K)$$

(10) $\quad \therefore S_K = \sum_{h=0}^{K} a_h = (-1)^K (2m, K).$

and $\quad \sum_{K=0}^{t} a_K f_K = \sum_{K=0}^{t-1} S_K (f_K - f_{K+1}) + S_t f_t .$

Therefore $\quad a_{mr} = \dfrac{2^r (-1)^{m-r}}{2^{2m}} \cdot \left\{ \sum_{K=0}^{m-r-1} (-1)^K (2m, K)(m-K-1, r-1) + (-1)^{m-r} (2m, m-r) \right\}$

$$= \frac{2^r (-1)^{m-r}}{2^{2m}} \sum_{K=0}^{m-r} (-1)^K (2m, K)(m-K-1, r-1)$$

Now let $\quad g(s) \equiv \sum_{K=0}^{m-r} (-1)^K (2m+1-s, K)(m-K-s, r-s) \qquad 0 \leq s \leq r.$

The same method shows that $\quad g(s) = g(s+1).$

Therefore, by induction $\quad g(0) = g(r).$

But, $\quad g(r) = \sum_{K=0}^{m-r} (-1)^K (2m+1-r, K) = (-1)^{m-r} (2m-r, m-r)$

Also $\quad a_{mr} = \dfrac{2^r (-1)^{m-r}}{2^{2m}} g(0) = \dfrac{2^r (-1)^{m-r}}{2^{2m}} g(r)$

Therefore $\quad a_{mr} = \dfrac{2^r (2m-r, m-r)}{2^{2m}} \geq 0.$

Hence $\quad \sum_{r=0}^{m} |a_{mr}| = \sum_{r=0}^{m} a_{mr} = 1.$

To prove condition 2, we have $\quad a_{mr} = \dfrac{2^r (2m-r, m-r)}{2^{2m}}$

$$a_{mr} = \frac{2^r (2m-r)!}{2^{2m} m! (m-r)!} = \frac{2^r \cdot 1\cdot 2\cdot 3 \cdots (2m-2r)\cdot \{(2m-2r+1)\cdots(2m-r)\}}{2^{m+r} \cdot 2\cdot 4\cdot 6 \cdots (2m-2r) \cdot m!}$$

where the bracket in the numerator contains r factors.

Therefore $$a_{mr} = \frac{[1\cdot 3\cdot 5 \cdots (2m-2r-1)]\cdot[(2m-2r+1)\cdots(2m-r)]}{2^m \cdot m!}$$

where the brackets in the numerator have respectively m-r and r factors.

Thus $$a_{mr} = \frac{1\cdot 3\cdot 5 \cdots (2m-2r-1)}{2\cdot 4\cdot 6 \cdots (2m-2r)} \cdot \frac{1}{2^r} \cdot \left[\frac{2m-2r+1}{m-r+1} \cdots \frac{2m-r}{m}\right]$$

where the bracket cpntains r factors.

Therefore $$a_{mr} = (1-\tfrac{1}{2})(1-\tfrac{1}{4})\cdots(1-\frac{1}{2m-2r})\cdot\left[\frac{2m-2r+1}{2m-2r+2} \cdots \frac{2m-2r}{2m}\right]$$

and hence $a_{mr} \to 0$ as $m \to \infty$.

Thus we have proved the

THEOREM 2. Any series summable (E,1) is summable by this method to the same sum.

Chapter IV.

Divergent Integrals.

The possible methods of generalizing the integral concept have been well discussed by Kolmogoroff [††]. He suggests that we may

(1). Take infinite instead of finite subdivisions.

(2). Consider abstract functions.

(3). Generalize the measure [†††].

He overlooks, however, the possibility of generalizing the sum, which is presented by (1). We propose to make this generalization, with particular reference to the Cesàro summation method. Since, however, the sum of a divergent series does not remain invariant under commutation, we must always be able to order the subdivision sets according to a definite rule. For this reason we find it convenient to consider only real functions of a real variable in this chapter. We present generalizations of both the Riemann and Lebesgue integrals.

1. Riemann Integration.

Let A be a semi-open interval $[a, b)$ (b may be infinite). Let $\varphi(\alpha)$ be a real function of a real variable defined throughout A except possibly at b, and have the following properties.

(1). For any $\epsilon > 0$, A can be covered by a denumerable infinity of closed intervals $A_{n\epsilon}$, any two having at most one point in common, such that $\alpha \leq \beta$ if $\alpha \in A_n, \beta \in A_{n+1}$ and $\varphi(\alpha)$ is bounded in each interval.

(2). $|A_{n\epsilon}| \leq \epsilon$.

Definition. $\varphi(\alpha)$ is summable in A: $\equiv$:

For any ϵ, the series

$$\Phi(\epsilon) \equiv \sum_1^{\infty}{}_n \varphi(\alpha_n) \cdot |A_{n\epsilon}|$$

is summable by a regular method (always the same), where $\alpha_n \in A_{n\epsilon}$.

Definition. $\int_A \varphi(\alpha)\, d\alpha = \lim_{\epsilon \to 0} \Phi(\epsilon)$.

if the right member exists.

By a "regular" summation method is meant one with the following properties:

a). $\sum a + \sum b = \sum (a+b)$.

b). If the ordinary sum exists, both exist and are equal.

c). Changing a finite number of terms affects the sum in the obvious manner.

d). $a_n \leq d_n \leq b_n : \exists \sum a_n, \sum b_n :\supset:$

$$\exists \sum d_n : \quad A \leq D \leq B.$$

As was shown in III.1, the Cesàro methods of integral order are thus regular.

Now define $\overline{\varphi}_{n\epsilon} = \text{l.u.b. } \varphi(\alpha), \ \alpha \in A_{n\epsilon}$

$$\underline{\varphi}_{n\epsilon} = \text{g.l.b.} \quad ''$$

Define $\overline{\Phi(\epsilon)} = \sum_1^{\infty}{}_n \overline{\varphi}_{n\epsilon} |A_{n\epsilon}|$.

$$\underline{\Phi(\epsilon)} = \sum_1^{\infty}{}_n \underline{\varphi}_{n\epsilon} |A_{n\epsilon}|.$$

Then $\overline{\Phi(\epsilon)} \geq \Phi(\epsilon) \geq \underline{\Phi(\epsilon)}$ for any ϵ, from property d.

Thus we obtain the first existence

THEOREM 1. A necessary and sufficient condition for the existence of $\int_A \varphi(\alpha)\,d\alpha$ is the existence of the common limit

$$\lim_{\epsilon \to 0} \overline{\Phi(\epsilon)} = \lim_{\epsilon \to 0} \underline{\Phi(\epsilon)}.$$

Now, for any ϵ, any subdivision $A_{n\epsilon}$, define

$$\overline{\Phi'(\epsilon)} = \sum \overline{\varphi}_{n\epsilon} |A_{n\epsilon}|$$

where in the $\sum$ the terms are associated in any manner such that all the $A_{n\epsilon}$ in any group have a total length $\leq \epsilon$. Similarly for $\underline{\Phi'(\epsilon)}$.

Then we can prove the more powerful existende

THEOREM 2. Let $A_{n\epsilon}$ be any ϵ subdivision, and $B_{n\epsilon}$ any resubdivision of $A_{n\epsilon}$ (in particular, B may be A itself).

Let $\overline{\Phi_A(\epsilon)} = \sum \overline{\varphi}_{An\epsilon} |A_{n\epsilon}|$

Let $\overline{\Phi_B(\epsilon)} = \sum \overline{\varphi}_{Bn\epsilon} |B_{n\epsilon}|$

the terms being associated as above. Similarly, define $\underline{\Phi_A(\epsilon)}$, $\underline{\Phi_B(\epsilon)}$

Then, a necessary and sufficient condition for the existence of $\int_A \varphi(\alpha)\,d\alpha$ is that it be possible to enclose $\overline{\Phi}_A, \underline{\Phi}_A, \overline{\Phi}_B, \underline{\Phi}_B$ in an interval whose length $\rightarrow$ 0 with ϵ.

Proof: Let $A_{n\epsilon}, A_{n\delta}$ be two subdivisions, B_n the product subdivision, so that B satisfies the hypothesis ralative to both A's. Let Φ' refer to the $\sum$ with B in it, associated so that the nth term contains precisely those B's which add up to $A_{n\epsilon}$. Similarly, in Φ'' those in the nth group add up to $A_{n\delta}$. Let Φ''' be the simple unassociated series of the B's.

Then, firstly, all the series considered are summable, Φ', Φ'' by d), and Φ''' because φ is summable.

Now, to show integrability, we need only show that $\overline{\Phi(\epsilon)}$, $\underline{\Phi(\epsilon)}$, $\overline{\Phi(\delta)}$, $\underline{\Phi(\delta)}$ lie in an interval of arbitrarily small length.

Well, $$\left|\overline{\Phi(\epsilon)} - \overline{\Phi(\delta)}\right| \le \left|\overline{\Phi(\epsilon)} - \overline{\Phi'}\right| + \left|\overline{\Phi'} - \overline{\Phi'''}\right| + \left|\overline{\Phi'''} - \overline{\Phi''}\right| + \left|\overline{\Phi''} - \overline{\Phi(\delta)}\right|$$

and the result follows from the hypotheses.

It is clear that $\int_A \varphi(\alpha)\,d\alpha$ is additive with respect to φ and with respect to A.

We might have allowed the values of the function to be abstract, but then we should not have had these existence theorems.

2. Lebesgue Integration.

Let A be a measurable set of real numbers, $\varphi(\alpha)$ a real function defined and measurable in A. Then, as in classical theory, let $\cdots < y_{-2} < y_{-1} < y_0 = 0 < y_1 < y_2 < \cdots$ and $y_n - y_{n-1} \le \epsilon$ for all n. Define $A_{n\epsilon} = A \cdot \sum_\alpha \{y_{n-1} \le \varphi(\alpha) < y_n\}$.

Definition. $\varphi(\alpha)$ is summable in A: $\equiv$:

The series $\Phi(\epsilon) \equiv \sum_1^\infty {}_n \{\varphi(\alpha_n)|A_n| + \varphi(\alpha_{-n})|A_{-n}|\}$

is summable for any ϵ , with $\alpha_n \in A_n$.

Definition : $\int_A \varphi(\alpha)\, d\alpha = \lim_{\epsilon \to 0} \Phi(\epsilon)$.

Then the analogs of the two preceeding theorems are valid.

Here also we might have generalized to the case of an abstract independent variable, since both the ordering of the sets and the above existence theorems depend only on the realness of the values of the function.

Finally, it is evident that these integrals are true generalizations of the classical Riemann and Lebesgue integrals.

3. Examples.

The function $f(x) = (-2)^n$, $0 \le n \le x < n+1$, all n, is integrable in either of these senses, using the Euler-Knopp summation method, the value of the integral being simply 1/3, although this function is not integrable according to any other definition with which we are familiar.

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