CAVITATION AND SEPARATION IN PUMPS AND TURBINES

by

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OUTLINE.

Vanes of hydraulic runners act on the surrounding fluid in a manner similar to the effect of the wing of an aeroplane. Theoretical and experimental data available about the behaviour of the latter therefore may be applied to hydraulic runners, providing the conclusions are modified according to the flow conditions in such runners, which certainly differ from those around an aeroplane wing. The present paper deals with these modifications concerning the pressure conditions in the two cases. In a straight and parallel flow, such as in the case of an aeroplane wing, the static pressures in front and behind a single aerofoil are equal, while the corresponding pressures at the intake and the discharge sections of hydraulic runners of the reaction type are different. This difference must be taken into account for all phenomena, which depend on the absolute pressure or its distribution along the vanes. The two most important phenomena of this kind are cavitation and separation, which, therefore, form the main object of our present investigations.

First we have derived simple, approximate expressions for the "basic pressure" (defined on pages 8 and 9) or average static pressure in a hydraulic runner of general characteristics, and in particular for runners of the propeller type. The latter expressions are then used for deriving theoretical criteria for the cavitation limits of runners of the propeller type, taking into account the distribution of the basic pressure. Due to the fact that for pumps the point of lowest pressure lies near the suction edges of the blades, and therefore in a region of comparatively low basic pressure and for turbines near the other end of the vanes in a region of high basic pressure, there exists an important difference in the cavitation features between pump and turbine runners (which, for instance, has not been considered in corresponding recent investigations by Spannhake).

Furthermore we have computed the relations between the cavitation limits and the specific speed for a definitely defined series of pump runners, and these relations finally have been brought into the familiar form of the chart adopted by most pump manufacturers for the "Upper limits of specific speeds for double suction, single stage centrifugal pumps".

In order to realize the effect of the change of the basic pressure on "separation" or "stalling" of airfoils or vanes, it is important to note that this form of separation is caused by a pressure increase along the low pressure side of the vane or airfoil, and we may consider the value of this pressure rise as a criterium of the commencement of separation. In order to derive from the separation or stalling limits of a single airfoil in the wind tunnel the same limits for a corresponding vane in a hydraulic runner, we add (algebraically) the change in the basic pressure to the pressure rise on the equivalent airfoil in a straight and parallel flow. If the resulting pressure risereaches a value which, if occurring on a single airfoil in the windtunnel, would lead to separation, we may assume that the same phenomenon will take place on the vanes in the runner. Based on this principle we have derived simple expressions for the separation limits of hydraulic runners. For axial flow runners the results may be expected to be fairly accurate, while for other types of runners considerably more experimental and theoretical data will have to be obtained before the given results may be used for anything beyond a preliminary estimate.

While cavitation sets an upper limit for the specific speed, separation - with respect to pump runners - determines a lower limit primarily for the unit speed ($\frac{n\,D}{\sqrt{H}}$), which also leads to a practical lower limit for the specific speed. For turbine runners such a lower limit does not exist according to our approximation. This limit of the specific speed is lower for radial flow than for axial flow pump runners, due to an important differnce between these two types of runners with respect to separation (see page 39).

Since may standard pump runners must be expected to show separation with corresponding energy losses the question arises whether these losses are large enough to warrant the considerable departures from standard designs which would be necessary to avoid separation under all circumstances.

Recent investigations have shown that the separation or stalling limits of airfoils are appreciably influenced by Reynolds' number or the scale. Hence we may expect a similar "scale influence" for hydraulic runners whenever the runner vanes are working close to the separation limit. The given formulae, therefore, allow us to estimate whether or not for a given case of model testing the scale is likely to play an important part.

CHAPTER I . ON THE APPLICATION OF AEROFOIL THEORY AND EXPERIMENTS TO HYDRAULIC RUNNERS.

Aerofoils, as used in aeronautics, and the vanes of hydraulic runners, both serve the purpose to deflect the surrounding fluid. Theoretical and experimental information obtained about the former therefore may be applied to the vanes of hydraulic machines. It is not surprising that more exact and detailed information about the behaviour of such deflecting surfaces has been obtained in aeronautics than in hydraulics, firstly, because the aeronautical engineer is mostly dealing with but one vane instead of a whole system of vanes (like a runner), so that his problem is inherently simpler, and, secondly, because of systems of vanes we can use a very simple approximation - the Eulerian theory, considering the vanes as infinitely close together - which yields fairly good overall results, while such an approximate theory is not available for an individual aerofoil; from the very beginning the aeronautical engineer, therefore, was in need of a more exact theory and more detailed experimental results. In recent years, however, the development of high speed runners has forced the hydraulic engineer to consider the vanes of hydraulic machines rather like individual aerofoils than as members of an infinitely close system of vanes, and it is quite natural that the large amount of theoretical and experimental information, accumulated in aeronautics about the behaviour of individual aerofoils, is used for the vanes of hydraulic runners. There exists, however, an important difference, which must be considered for this application: The aeronautical engineer is primarily interested in the behaviour of an aerofoil if placed into a flow which, originally, is straight, parallel, and infinitely extended, and his results are either obtained under these conditions or immediately converted to it ("Wind-tunnel corrections"). For the vane of a hydraulic runner, however, the "basic flow conditions", i.e. those flow conditions which would exist without the influence of the particular vane or aerofoil, which we consider, are greatly influenced by the other guiding surfaces (vanes, et c.), and the "basic flow" certainly is not any longer straight or parallel as assumed before.

The change of the geometrical shape of the basic flow has been considered in a very excellent paper by Betz (reference / , and appendix, section 1). This paper gives a practical method to design a vane of a parallel system so that it has the same effect as a certain given single aerofoil in a parallel, straight flow. The method applies directly to the vanes of runners of the propeller type, and to stationary vanes.

In the present paper we shall consider the change of the "basic pressure" defined as that pressure which would exist without the influence of the particular vane or aerofoil which we consider. The basic pressure of the flow conditions to which the results of the usual aerofoil theory or windtunnel tests generally apply is, of course, constant (straight, parallel, and infinitely extended basic flow), while it clear that the basic pressure in a pump will increase, and in a turbine it will decrease (in the direction of the main flow through the machine) due to the very nature of these two types of hydraulic machines. After investigating these changes of the basic pressure we shall try to express their influence on cavitation and on separation on the vanes, i.e. we shall modify the theoretical and experimental information regarding these phenomena on an individual vane in a straight, parallel flow, so that they can be applied to the vanes of hydraulic runners.

CHAPTER II. THE CHANGE OF THE BASIC PRESSURE IN HYDRAULIC RUNNERS.

The following results about the change of the basic pressure in a runner are based on the Eulerian assumption that the vane circulation or, in other words, the change of the angular momentum, is uniformly distributed along concentric circles around the axis of the runner, while it actually is concentrated at the vanes. This means, in the first place, that we consider the "basic pressure" as being equal to the static pressure in the runner. Our assumption will lead to quite accurate results in a region about half way between the suction and the pressure side of the runner. It will be less accurate with respect to the change of the basic pressure along the vanes in so far as it assumes that the total pressure change takes place in the region between the suction and pressure ends of the vanes, while, actually, this change starts in a certain distance (theoretically in an infinite distance) in front, and is completed only in a corresponding distance behind the runner. In section 2 of the appendix results for the total pressure change obtained by applying the above assumption are compared with the experimental data found in Goettingen about the pressure distribution of aerofoils arranged in a lattice (reference 2). The agreement is better than one may be inclined to expect from the comparative crudeness of our approximation, and our theoretical results are certainly accurate enough for the practical requirements of the hydraulic engineer.

For a stationary system of vanes we find that the basic pressure at any point "x" in the system

$$p(x) = p_o + \frac{V_o^2 - V_x^2}{2} \cdot \rho + w\,(\,z_o - z(x)\,) \qquad (1)$$

where the subscript o may apply to any convenient point of reference along the main flow through the system.

For a rotating system we find that :

$$p(x) = p_o + \rho \frac{v_o^2 - v_x^2}{2} + \rho \frac{u_x^2 - u_o^2}{2} + w(z_o - z(x)) \quad (2)$$

where v is the relative velocity of the fluid in the runner, and u the peripheral velocity of the runner at the point indicated by the subscript. The derivation of equation (2) is gigen in Daugherty: "Hydraulic Turbines" page 91 .

The difference in elevation $(z_o - z(x))$, which has the order of magnitude of the vertical dimensions of the runner needs to considered only for very large runners. To simplify our expersions we shall omit this term in future, and, if necessary, shall consider this pressure difference in a manner, which will be discussed later.

Generally we shall compute the change of the basic pressure with respect to the susction pressure, so that $p_o = p_s$ (see fig. 1) .

With this convention and disregarding differences in elevation within the runner, the basic pressure in a hydraulic runner of arbitrary shape may be expressed as follows:

$$p(x) - p_s = \rho \frac{v_s^2 - v_x^2}{2} + \rho \frac{u_x^2 - u_o^2}{2} \quad (2a)$$

where the first term on the right side expresses the chang e in pressure due to the change in crossection of the passages

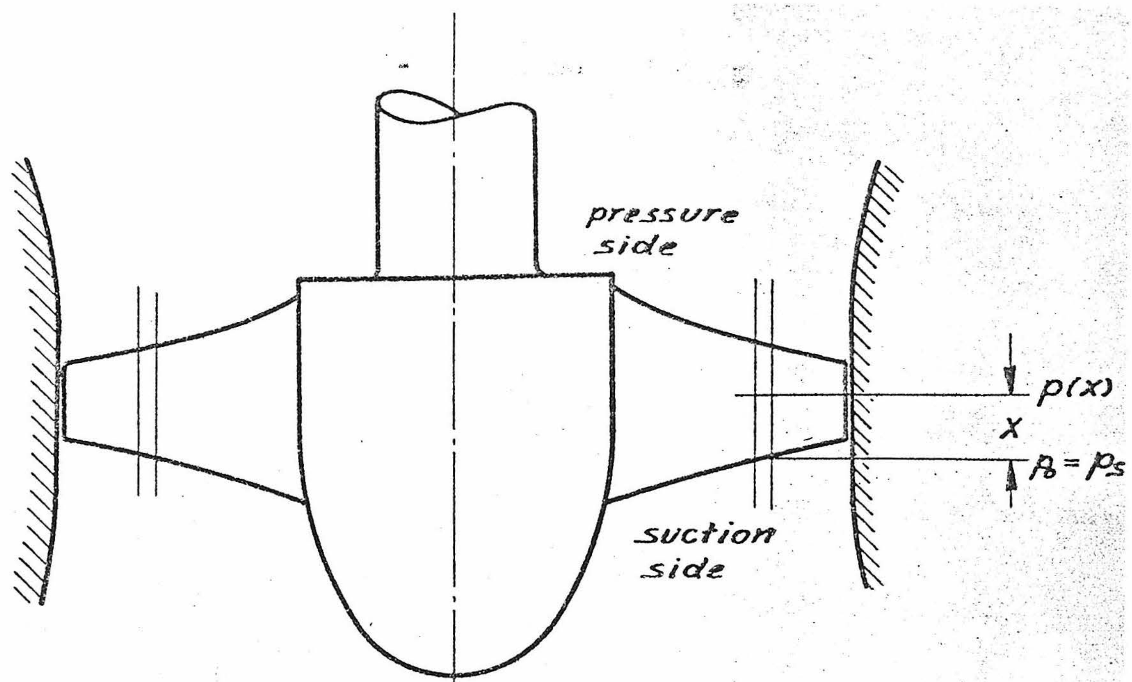


Fig. 1

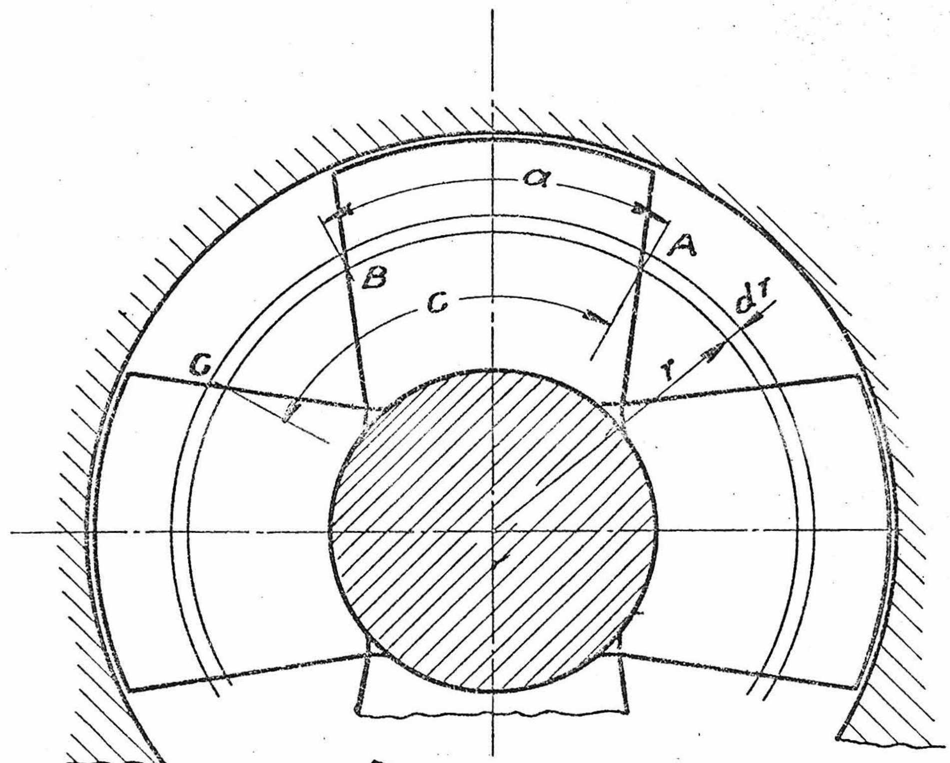


Fig. 1a

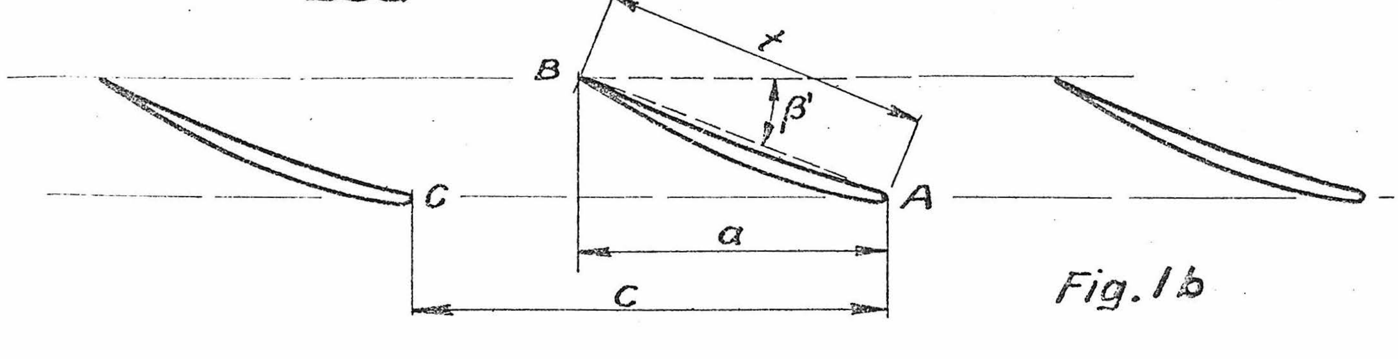


Fig. 1b

between the vanes, and the second term the change in pressure due to the centrifugal forces.

In the special case of purely axial flow runners, to which most of our following considerations will apply, the change of the basic pressure can be expressed by means of the lift coefficient (c_L) and the spacing of the runner blades, which is of great advantage for the following applications. This relation follows from the simple fact that for constant axial velocity through the runner (flow between concentric cylindrical surfaces) there occurs no change of the axial momentum, so that the change in static pressure must be equal to the axial component of the pressure difference between the two sides of the vanes, or, which is the same, equal to the pressure difference acting on the axial projection of the vanes as shown on fig. 1b . We, therefore can write for any concentric ring area :

$$(p_p - p_s) . c . dr = \Delta p_{av} . a . dr$$

or

$$(p_p - p_s) = \Delta p_{av} . \frac{a}{c} \qquad (3)$$

Where Δp_{av} is the average pressure difference between the two sides of the vanes, and $c = 2 \pi r / N$, N being the number of vanes of the runner.

According to the usual aerofoil theory the average pressure difference on the vanes may be expressed as follows :

$$\Delta p_{av} = c_L \frac{\rho v^2}{2}$$

where c_L is the lift coefficient and v the "mean effective relative velocity" as defined in section 1 of the appendix. (v is approximately equal to $\frac{v_p + v_s}{2}$).
Introducing furthermore the "ratio of overlapping" :

$$j = \frac{a}{c} = \frac{N\ t\ \cos\beta'}{2\ \pi\ r}$$

we find for the rise of the static pressure in an axial flow runner, or, generally, of a straight parallel lattice:

$$p_p - p_s = \frac{\rho v^2}{2} \cdot c_L \cdot j \qquad (4)$$

In order to find the rise of the static pressure (basic pressure) up to an intermediate point or plane given by x in fig. 1 , we must consider the distribution of the pressure difference Δp along the vanes (in the direction of the relative flow). By the same reasoning as before we find :

$$(p(x) - p_s)\ c = \int_s^x \Delta p \ \ dt \cdot \frac{a}{t}$$

t being the chord of the aerofoil and dt an infinitesimal part of it.

Instead of presenting the pressure distribution of a certain aerofoil at a given velocity v directly, it is customary to plot the pressure on the vane or aerofoil divided by the stagnation pressure $\rho\, v^2/2$ against a chord of unit length. Putting :

$$\frac{\Delta p}{\varrho v^2/2} = \Delta\lambda \quad \text{and} \quad \frac{dt}{t} = d\xi$$

we can replace the above integral over the vane pressure by an integral of the customary pressure diagram as follows:

$$\int_s^x \Delta p \; dt = \frac{\varrho v^2}{2} \int_s^x \Delta\lambda \; d\xi \; . \; t$$

Consequently we can write :

$$(p(x) - p_s) \; e = \frac{\varrho v^2}{2} \cdot j \cdot \int_s^x \Delta\lambda \cdot d\xi$$

or

$$(p(x) - p_s) = \frac{\varrho v^2}{2} \cdot j \cdot \int_s^x \Delta\lambda \; d\xi \qquad (6)$$

The influence of the finite thickness of the vanes on the basic pressure will be neglected, because relatively very thin vanes are in most cases to be preferred for hydraulic runners.

While we have choosen the pressure on the suction edge of the vanes for a basis of reference, it is clear that the pressure at any other convenient crossection through the runner may be used, if x is defined correspondingly.

On fig. 2 we have shown the curve for the integral in equation (6) for an NACA profile 6406, if placed in a turbine runner, and on fig's. 3 and 4 the same curve for a Betz-Joukowsky porfile (reference 6) and for NACA profile 6406 if placed in a pump runner.

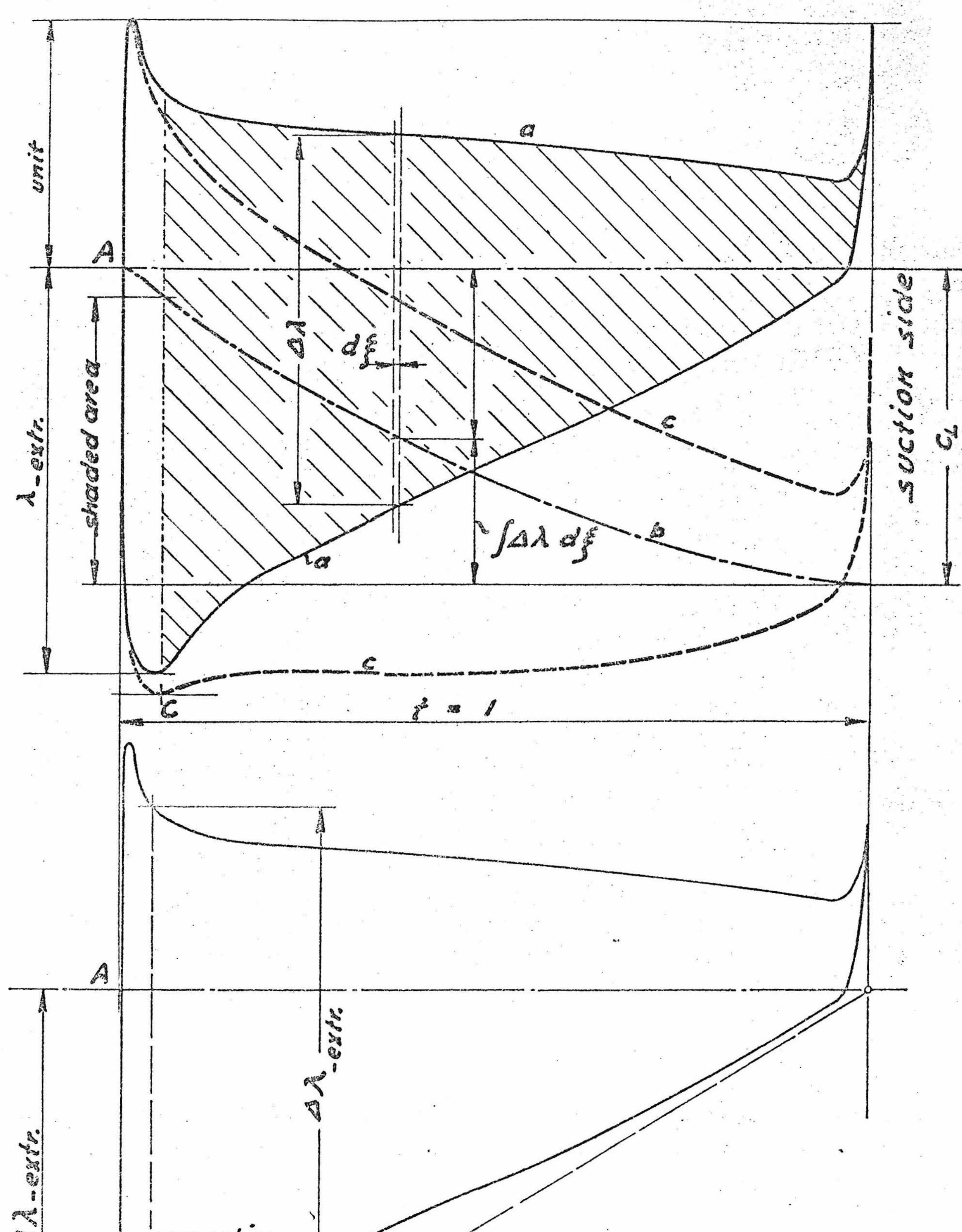


Fig.2

N A C A profile 6406 plased in a turbine rummer. c_L = 1,29

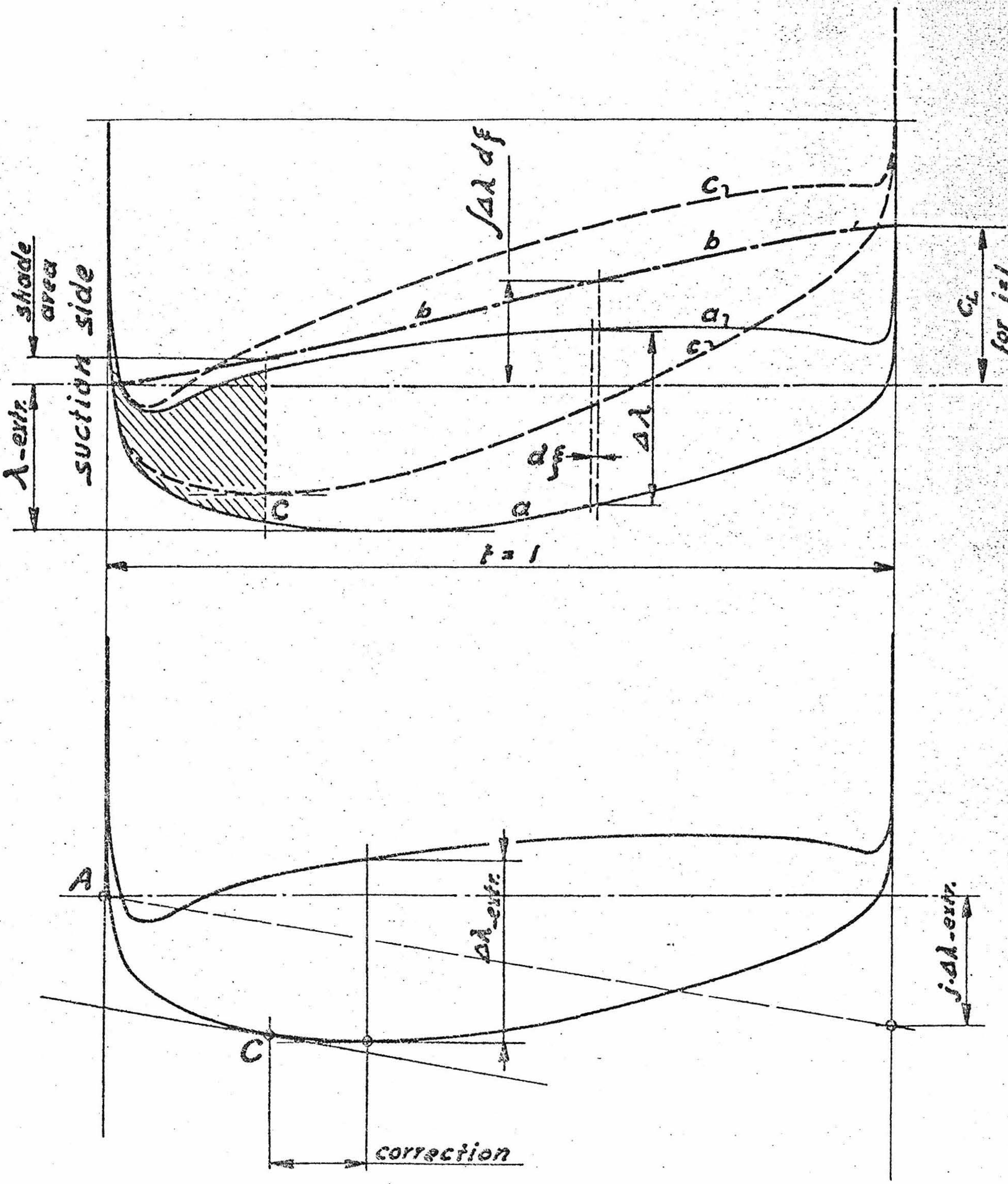


Fig. 3

Betz- Joukowsky profile 4506.8 placed in a pump runner.
(4% camber, 50% back from leading edge; 6.8% thickness.)
angle of attack = 0; c_L =

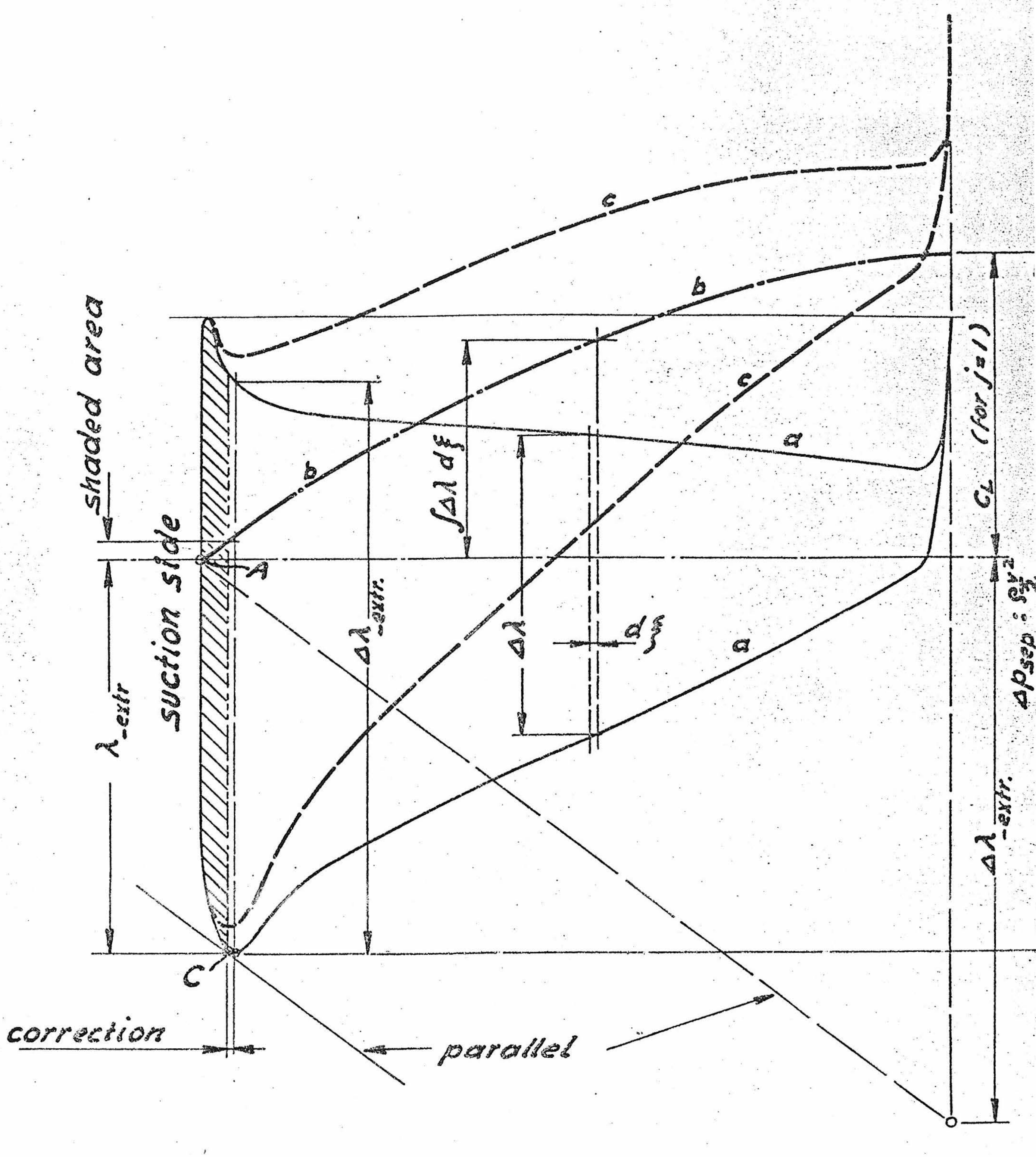


Fig. 4

N A C A profile 6406 placed in a pump runner.

$e_L = 1.29$

CHAPTER III. CAVITATION IN HYDRAULIC RUNNERS.

The phenomenon of cavitation has been discussed extensively with respect to hydraulic machines, so that we shall restrict our investigations to the problem of finding theoretical criteria for the commencement of cavitation in hydraulic runners. It is exceedingly difficult to solve this problem theoretically with respect to radial flow and mixed flow runners of a general form, which are common in a region of medium and low specific speeds. Fortunately there exists a sufficient supply of test data on the cavitation limits of such runners, so that, to some extent, the problem can be solved empirically. An outstanding example for such a solution is offered by the well known chart for the "Upper limits of specific speeds for double suction single stage centrifugal pumps", adopted by most pump manufacturers. On the other hand we find that in the field of exceedingly high speed runners, namely for those of the propeller type, where test data are not so plentyful as yet, the theoretical solution of the problem does not seem to offer appreciable difficulties. The latest attack on this problem has been made in a recent paper by Spannhake (reference 4). The present investigations, although carried out independently, naturally had to follow similar lines, as those by Spannhake, they differ, however , from the latter in the following points:

1) Spannhake considers but one very special vane profile, for which the minimum pressure always occurs in the center of the vane, and , consequently, the value of the basic pressure had to be determined for this point only. Due to this symmetry of the pressure distribution the results by Spannhake show but little difference between pumps and turbines regarding the cavitation limit. In accordance with the general tendency of the present paper we shall consider the influence of the basic pressure in a more general manner, so that the results may be applied to vane profiles of arbitrary shapes. Thereby also a marked difference between the cavitation properties of pumps and turbines will become apparent.

2) We have carried our computations so far that for a particular example the cavitation limits could be shown in their familiar relation to the specific speed. Spannhake carefully avoids such a special representation in order not to create the impression that there might be but <u>one</u> definite relation between the specific speed and the cavitation limit, while, actually, this relation must depend on special choices for the design of the runner. Although the scientific justification of omitting therefore such a final representation cannot be doubted, we believe that the value of these theoretical investigations will hardly be appreciated, unless the results - though not quite general — are represented in a practical, useful form. Such a definite evaluation

seems the more permissible, as changes of the designing features within reasonnable limits will not change the results very radiacally.

3) Spannhake expresses the specific speed primarily by flow characteristics, while we have used form characteristics of the runner. The representation by flow characteristics is scientifically more correct, the representation by form characteristics, on the other hand, corresponds to the practical procedure of assigning to one particular type of geometrically similar runners but one specific speed, while actually the specific speed will, of course, vary as the operating point is shifted along the characteristic curve. Our specific speed formula, therefore, must be understood as applying to the operating point for which the runner has been designed.

4) In order to avoid complications we have disregarded friction losses. This seems justified in so far as for instance the resistance, which a profile would show in the windtunnel will hardly apply to the vane of the runner, where the end effects (wall friction) will play a deciding part. In this connection it is important to keep in minde that the profile drag of the vanes in may cases tends rather to increase than to decrease the desired effect of the vanes, while the friction on all stationary walls will always decrease this effect. The influence of friction, therefore, partly cancels out, which tends to make all calculations regarding friction in high speed runners the more uncertain.

5) Spannhake has carried out very valuable investigations on the influence of certain changes in the flow conditions, considering especially a rotation of the fluid on the suction side of the runner. In order not to distract attention from the main object of our investigations, we have considered the simplest flow conditions only, especially purely axial and constant fluid velocity at the suction side of the runner.

Before presenting the main part of our investigations it will be necessary to define what we mean by the "cavitation limit". Theoretical investigations, naturally, can detect only the very beginning of cavitation, when at the point of absolutely lowest pressure the vapor or gas pressure of the liquid has just been reached. Considering a typical cavitation-test diagram of a screw pump, we may say that we consider as "cavitation limit" always the point where the characteristic curve starts to deviate from its normal course, not the point of complete cavitation, where the characteristic curve suddenly drops to zero. It is quite possible that the cavitation limit, which we compute, will lie still in front of the point where the characteristic curve shows first indications of a drop below its normal course, which would mean that locally restricted cavitation phenomena do not appreciably influence the behaviour of the machine.

In accordance with the ususal procedure of the theory for runners of the propeller type we assume that the flow through the runner proceeds along cylindrical, coaxial surfaces (fig. 1, 1a,) and we consider the flow conditions in one of these stream surfaces. Furthermore we assume that the change in angular momentum is constant for different distances from the axis of the runner.

As mentioned before we must investigate conditions at the point of absolutely lowest pressure in the runner, and this point will be near to, but not exactly identical with the point of lowest pressure on the vane or aerofoil if investigated in a straight, parallel, and infinite flow; the exact location of the point of lowest pressure in the runner will be changed by the deviation of the basic pressure from its constant value in the undistorted flow. We shall determine this change in the location of the point of lowest pressure, assuming, however, that at the new low pressure point the pressure drop below the basic pressure is the same as the maximum pressure drop below the constant basic pressure for the vane in the straight, parallel flow. Designating this extreme pressure drop as " $p_{-extr.}$", so that :

$$p_{-extr.} = p_{basic} - p_{minimum} \qquad (7)$$

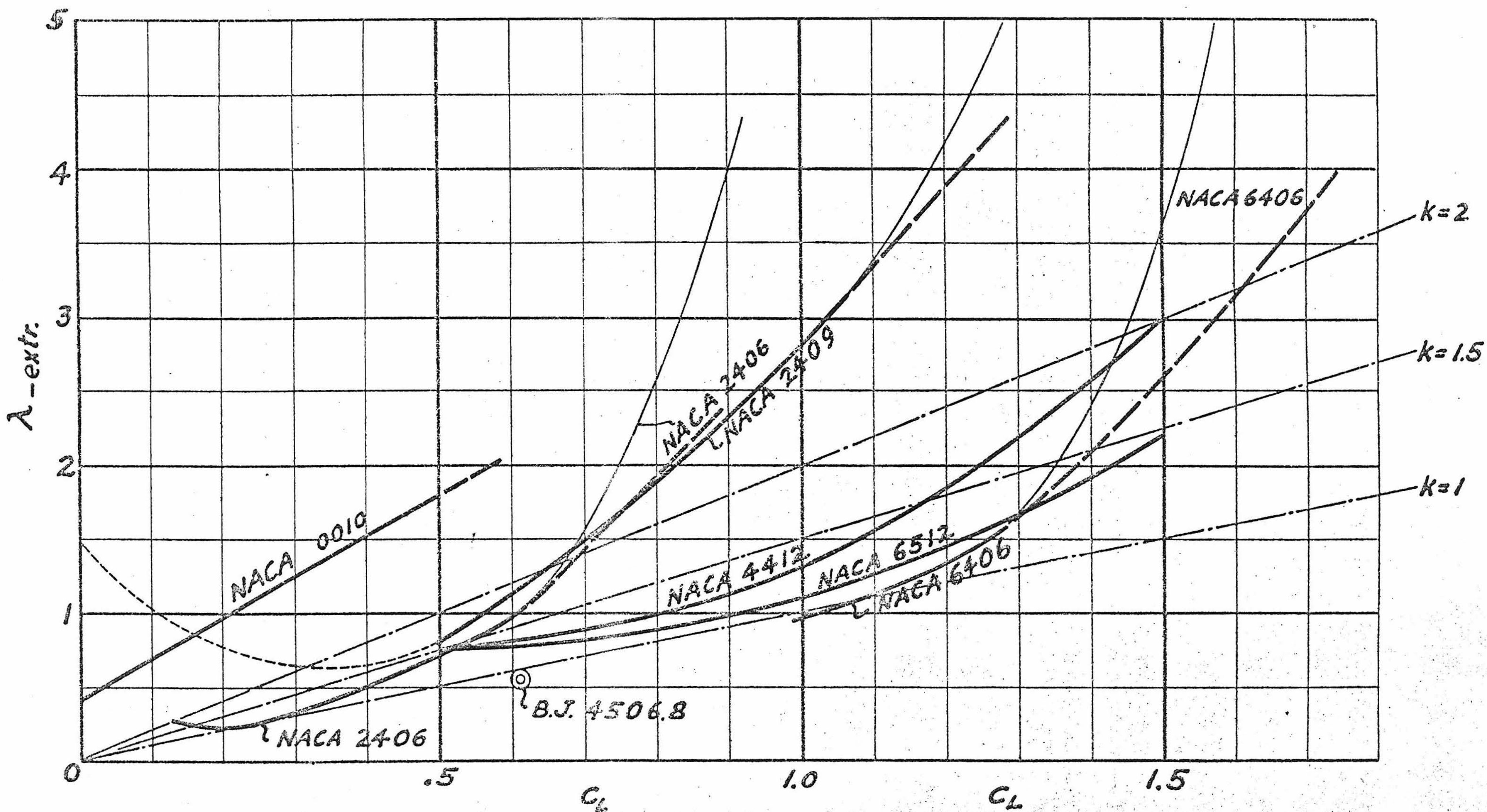
we find with reference to the usual pressure distribution diagram (see fig's. 2 to 4) :

$$\frac{p_{-extr.}}{\frac{\rho v^2}{2}} = \lambda_{-extr.} \qquad (7a)$$

We are particularly interested in the ratio between this extreme pressure drop and the average pressure difference on the vanes, for which we may write :

$$\frac{p_{-extr.}}{p\ av.} = \frac{\lambda_{-extr.}}{c_L} = k \qquad (8)$$

It is clear that low values of k are desirable with respect to the danger of cavitation. On the diagram on fig. 5 we present k values for a number of NACA aerofoils (reference 5) taken from theoretical pressure distribution curves. The dotted extensions indicate k values which may actually be expected in a region where the theoretical curves show very great peaks for the underpressure, the k values pertaining to the latters being indicated by thin, solid lines. The isolated point marked: B J 4506.8 gives the k value for the Betz-Joukowsky profile for which the princîl form characteristics and the theoretical pressure distribution are given on fig. 3 , the angle of attack being equal to zero. This position approximately yields what - in the language of the hydraulic engineer -- is called the condition of "shockless approach", and therefore is favorable for obtaining low values for k . It is easy to anticipate that this condition is particularly important whenever the danger of cavitation becomes the deciding factor.



5
4
3
2
1
0
λ-extr.
NACA 0010
NACA 2406
NACA 2409
NACA 4412
NACA 6512
NACA 6406
NACA 6406
NACA 2406
B.J. 4506.8
k=2
k=1.5
k=1
.5
1.0
1.5
c_L
c_L

Fig. 5

We have still to determine the position of the point of the absolute pressure minimum. This means that we have to find a point, where the curve of the sum of the differential aerofoil pressure (curve a on fig's. 2 to 4) and the basic pressure (curve b) , i.e. the curve c on fig's. 2 to 4 has a horizontal tangent. This point, however, has of course the same position (along the aerofoil) as a point, where the slope of the differential aerofoil pressure curve (a) has the negative value of the slope of the curve (b) of the basic pressure. The latter, however, is, according to equation 6, if written in the form :

$$\frac{d\,p(x)}{d\xi} = j \cdot \Delta\lambda \cdot \frac{\rho v^2}{2} \qquad (6a)$$

equal to the total difference in pressure on the aerofoil times the ratio of overlapping (j), if the chord of the aerofoil or vane (t) is put equal to unity. The point satisfying the previous requirement, therefore, can readily be determined by a process of iteration, starting from the point where the normal λ- curve has its minimum; generally the first step of this process, which has been indicated in fig's. 2 to 4, is sufficient.

We are now ready to formulate the condition for the commencement of cavitation as follows :

Denoting by p_c that underpressure (pressure below atmospheric) at which cavitation will just begin, we can say that the relation :

$$p_{basic} - p_{-extr.} = p_c \qquad (9)$$

gives us the limit regarding the pressure conditions in the runner, at which cavitation will start.

According to equation (6), the basic pressure is given by :

$$p(x) = \frac{\rho v^2}{2} \cdot j \cdot \int_s^x \Delta\lambda \, d\xi + p_s$$

where the point x should be that of the absolute pressure minimum as determined before, according to fig's. 2 to 4 . On the suction side of the runner the pressure above atmospheric pressure is :

$$p_s = w \cdot h_s - \frac{\rho V_s^2}{2}$$

h_s being the static head on the suction side of the runner (measured in feet), and V_s the absolute velocity of the fluid at the same place. As mentioned before we shall assume that, for our further calculations, $V_s = V_{axial}$, i.e. that the fluid has no rotation on the suction side of the runner.

So far we have neglected the difference in elevation from the suction side of the runner to the point of lowest pressure (see chapter II). Whenever the dimensions of the runner warrant it , this difference in elevation can easily be taken into account by measuring h_s not to the suction side of the runner but to the point, where the absolute pressure minimum is expected, i.e. we put :

h_s = difference in elevation between the free water level on the suction side and the point of absolutely lowest pressure in the runner. (10)

h_s is positive is positive, if the the free su tion level lies above the point of lowest pressure in the runner.

(Strictly speaking, for vertical-shaft runners the point of lowest pressure will slightly differ from that determined above, due to the additional pressure gradient from the differences in elevation. This influence, however, can be neglected as long as the axial dimensions of the runner are not a very considerable portion of the total head.)

If one wishes to consider friction losses in the suction line, there value (in feet) should be added to h_s in the case of turbines, and subtracted from h_s in the case of pumps.

Putting $p_e = w\, h_e$, we obtain from equation (9) :

$$w\,(h_s + h_e) = \frac{\rho V^2_{axial}}{2} + p_{-extr.} - \frac{\rho v^2}{2}\, j \int_s^x \Delta\lambda \; d\xi$$

and after substituting for $p_{-extr.}$ the value given by the equations (7a) and (8), we find :

$$h_s + h_e = \frac{V^2_{axial}}{2\,g} + \frac{v^2}{2g} \cdot c_L \cdot k - \frac{v^2}{2g}\; j \int_s^x \Delta\lambda \; d\xi$$

or

$$h_s + h_e = \frac{v^2}{2g} \left(\frac{V^2_{axial}}{v^2} + c_L \cdot k - j \cdot \int_s^x \Delta\lambda \; d\xi \right) \quad (11)$$

Since all pressures in the runner are, for otherwise similar conditions, proportional to the total head H consumed or produced by the runner, it is advisable to represent the values of $h_s + h_c$ in a dimensionless form by dividing by H. In section 1 of the appendix we have derived for the total head of any cylindrical section through an axial flow runner the expression :

$$H = \frac{u \cdot v}{2g} \cdot \frac{c_L\, j}{\cos\beta'} \qquad (12)$$

By means of this equation and a few simple transformations, given in section 3 of the appendix, we obtain :

$$\frac{h_s + h_c}{H} = \left(1 - \frac{c_L\, j}{4 \cos\beta'} \right) \left(\frac{\sin^2\beta_s}{c_L\, j} + \frac{k}{j} - \frac{\int_s^x \Delta\lambda \, d\xi}{c_L} \right) \qquad (13)$$

<u>This equation is fundamental for our method of computing the cavitation limit.</u> It is important to note that the right side of this equation is entirely determined by the form of the runner and by the direction of the relative fluid velocity on the suction side of the runner, which may be considered as the "initial condition".

Analysing equation (13) we find that the first factor comes from using on the left side the <u>total</u> head H instead of the change in <u>static</u> pressure, only, i.e. if we would replace H by the change in static pressure, this first factor would become equal to one. In the second factor the first term comes from the pressure drop due to the axial velocity (which is the larger the steeper the vanes, i.e. the larger β_s), the second term from the pressure difference on the vanes, and the third term from the pressure increase between the suction side of the runner and the point of lowest pressure. The different origins appear somewhat obscured since they various terms have been divided by the total head H. The last term is simply the ratio between the shaded areas in fig's. 2 to 4 and the total area of the pressure diagram. It is clear from equation (13) that one will try to make this ratio as large as possible, which means physically that one will try to place the point of lowest pressure as far as possible into a region of high basic pressure. Since for standard aerofoils the pressure minimum always lies nearer to the leading edge than to the trailing edge, the former being in a turbine the high pressure end, and in a pump the low pressure end of the vanes, we find that it is easy to make the last term of equation (13) comparatively large for a turbine (about .9) while for a pump it will hardly be possible to increase its value beyond about $^1/_3$, and generally we may have to figure with values around $^1/_5$ (compare fig. 2 with fig's. 3 and 4). <u>This fact appears to be the main difference between pumps and turbines regarding cavitation.</u> It is partly counterbalanced by the other fact that for vertical shaft runners the leading edges for turbine vanes lie at a higher elevation than for pumps (for the same position of the runner), but the former effect will in most cases be the dominating one. (On the other hand turbines usually need longer draft tubes to recover the velocity head in the latter; this requires a higher position for the runner above the lowest point of the excavations. h_s, therefore, will be algebraically smaller (mostly negative) for turbines. For our present investigations, however, which concern the runner only, this difference will not appear explicitly.)

Equation (13) so far applies to all coaxial, cylindrical stream surfaces through the runner, and, therefore, should be evaluated for a number of such surfaces to find that surface, for which the danger of cavitation, i.e. the value of $\frac{h_s + h_c}{H}$ reaches a maximum. It is possible, however, to estimate in advance where this maximum will occur by using equation (3) of chapter II in the following form :

$$p_p - p_s = \Delta p_{av} \cdot j$$

For axial flow runners of high specific speeds (if the velocity on the suction side is purely axial) the change in static pressure ($p_p - p_s$) mostly constitutes the greater part of the total head H ; and, therefore, is nearly constant across the whole runner. Hence :

$$\Delta p_{av} \cdot j \approx \text{constant}$$

or

$$\Delta p_{av} \approx \frac{\text{constant}}{j} \qquad (14)$$

The region where j is a minimum, therefore, has in first approximation the greatest pressure difference on the vanes, and under otherwise constant conditions thereby the greatest danger of cavitation.

Further considerations reveal that the most important deviations from our first approximation - namely the change in $(p_p - p_s)$, which becomes smaller towards the center of the runner, and the change in the distribution of the vane pressure, which usually is less uniform towards the center so that k increases - have opposite effects, and, therefore, cancel each other to some extent. The first approximation, therefore, can be considered as sufficient.

From the following formula for the specific speed we shall see that for a given value for n_{sp} the product $c_L j$ must increase towards the hub of the runner. We always can satisfy this condition for a given specific speed by increasing j towards the center of the runner, so that the average vane pressure difference and thereby the danger of cavitation becomes greatest at the outer periphery of the runner. It is, therefore, sufficient to investigate the cavitation limit for the outer periphery, only, because it is always possible to design the rest of the runner for a given specific speed safer with respect to cavitation than the outer tips of the vanes.

As our next and last problem in the chapter on cavitation we shall compute the cavitation limits for a certain series of axial flow pump runners as a function of their specific speeds.

For the outer periphery of the impeller the specific speed of axial flow pumps can be expressed by the equation :

$$n_{sp} = C \frac{\sqrt{\sin\beta_s} \ \cos^{3/4}\beta'}{(c_L\ j)^{3/4}} \sqrt{1 - \frac{D_i^2}{D_o^2}} \left(\cos\beta' + \frac{c_L\ j}{4 \cos\beta'} \right)^{1/4} \quad (15$$

where $C = 8160$ if n_{sp} is defined by the relation :

$$n_{sp} = \frac{n \sqrt{GPM}}{H^{3/4}}$$

Equation (15) is derived in section 4 of the appendix. It expresses the specific speed completely by form characteristics of the impeller, if β_s ; i.e. the direction of the initial flow in front of the runner, is given. On the other hand we remember that the characteristic figure for cavitation:

$$\frac{h_s + h_a}{H}$$

(see equation 13) also was completely determined by the form of the runner and by β_s. We therefore can represent $\frac{h_s + h_a}{H}$ as a function of the specific speed, if we make the form of the runner a definite function of n_{sp}, which, of course, is possible only, if one introduces definite assumptions about the way in which the form characteristics of the runner shall change with the specific speed, because different types of runners may have the same specific speed, or, in other words, different values for c_L, j, β', and D_i/D_o, if substituted into equation (15) may lead to the same value for n_{sp}.

The most influential factor is the product $c_L . j$, and therefore we have introduced at first definite assumptions about j, c_L, and D_i/D_o as functions of $c_L.j$. These assumed relations are given by the diagram on fig. 6

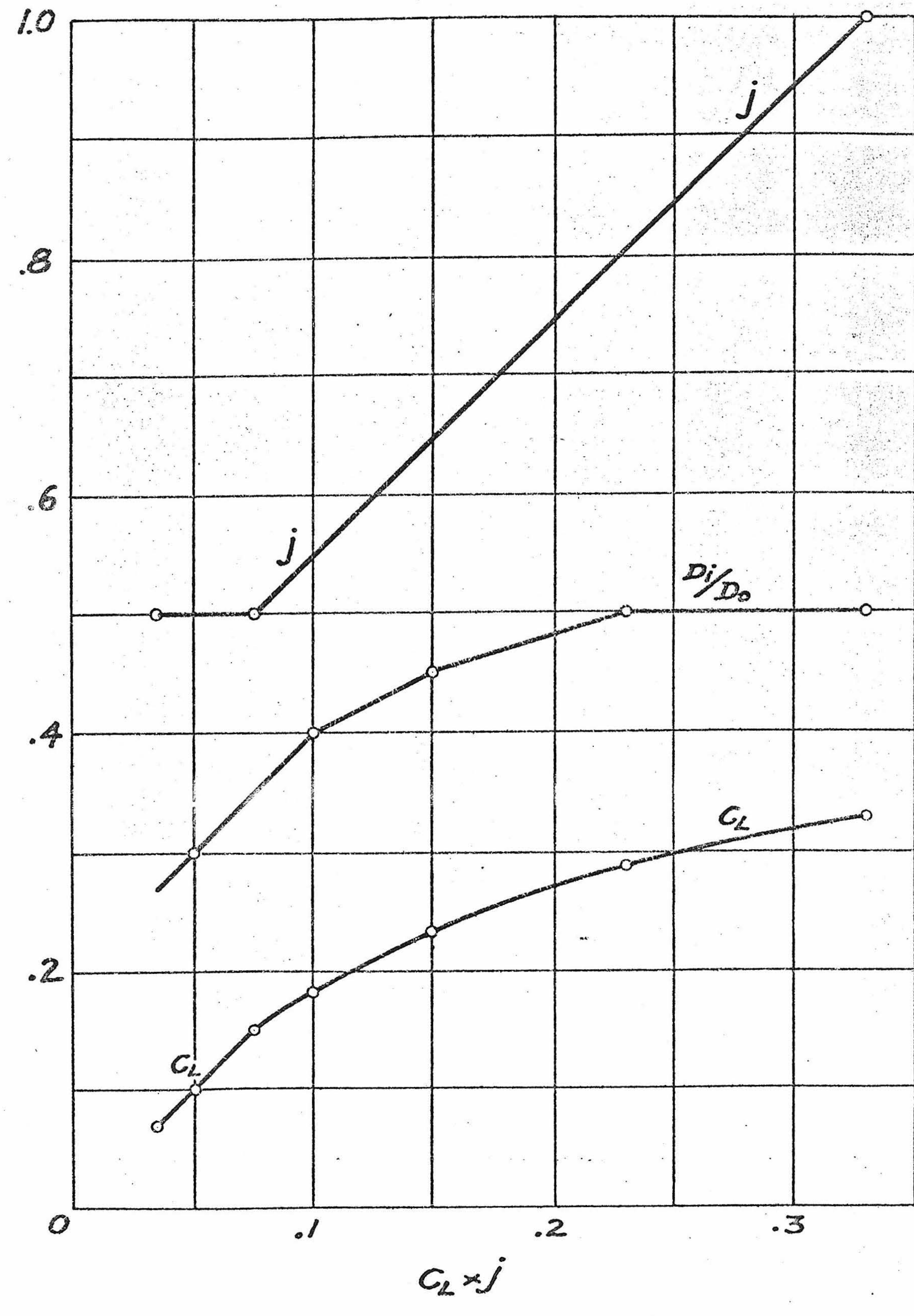
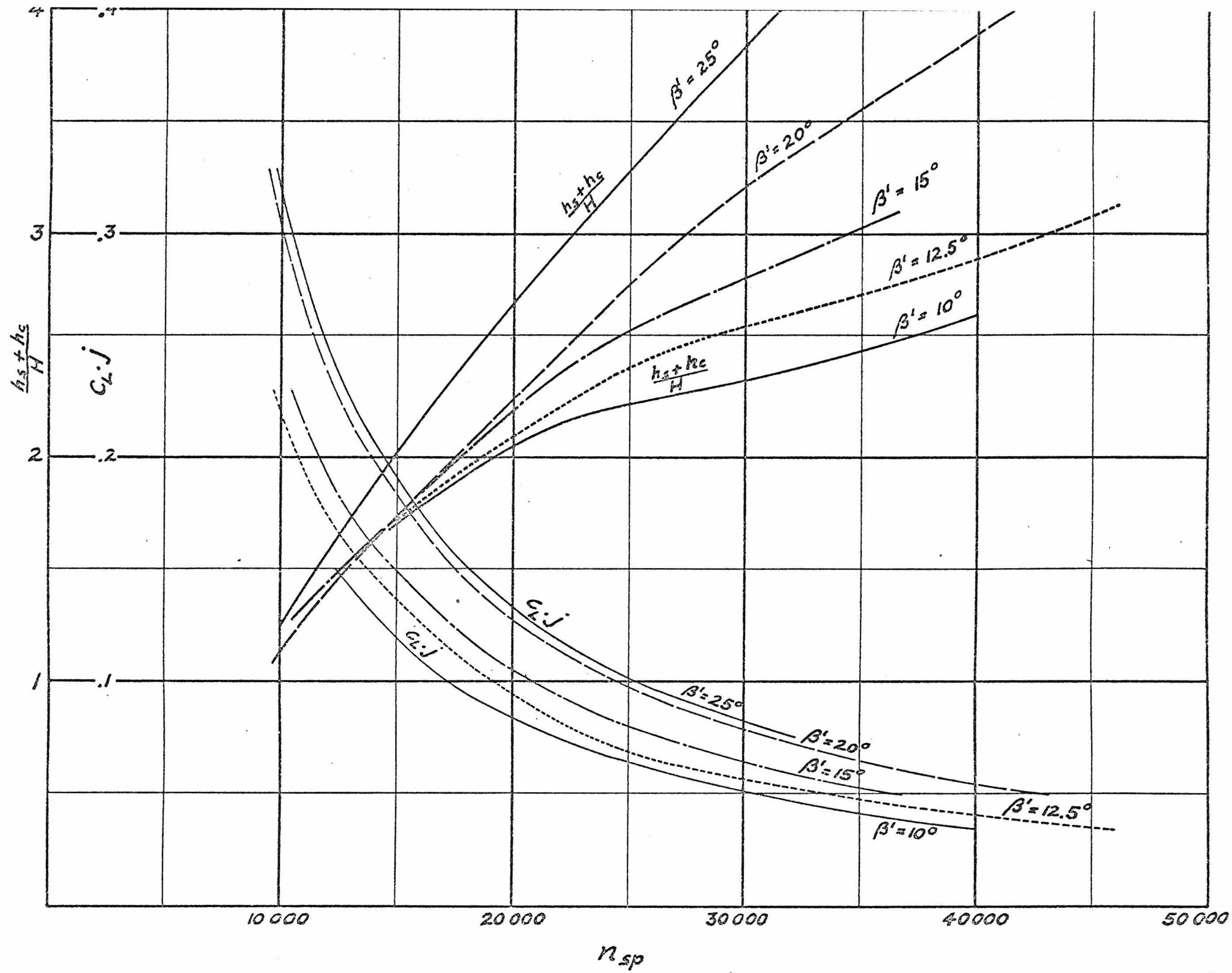


Fig. 6

For our computations we have put $\beta_s = \beta'$, which seems permissible, since the difference between the two angles, which is equal to half the deflection of the fluid plus the angle of attack, is always very small at the outer periphery of screw pump impellers.

With these assumptions we can now compute n_{sp} as a function of $(c_L \cdot j)$ for different, constant values of β', and on the diagram on fig. 7 we have plotted the inverse functions, namely $c_L \cdot j$ as a function of n_{sp} for various values of the vane angle β'.

These last curves give us for every choice for β' definite form characteristics as a function of n_{sp}, which can be substituted into the right side of equation (13), leading finally for every value of β' to definite values of $\frac{h_s + h_c}{H}$ as a function of the specific speed. Since c_L does not reach very high values, we may assume that the angle of attack will be zero, which leads to pressure distributions along the vanes similar to that shown on fig. 3 ; we, therefore, have chosen for the evaluation of equation (13) $k = 1$ and $\int_s^x \Delta\lambda \, d\xi / c_L = .2$. The corresponding results for $\frac{h_s + h_c}{H}$ also have been plotted on the diagram on fig. 7 . This diagram represents the cavitation limits for different types of propeller pumps as far as the steepness of the vanes (β') is concerned; it is, however, not by any means generally valid, since other form characteristics remain restricted by the choice represented on fig. 6 . For other assumptions



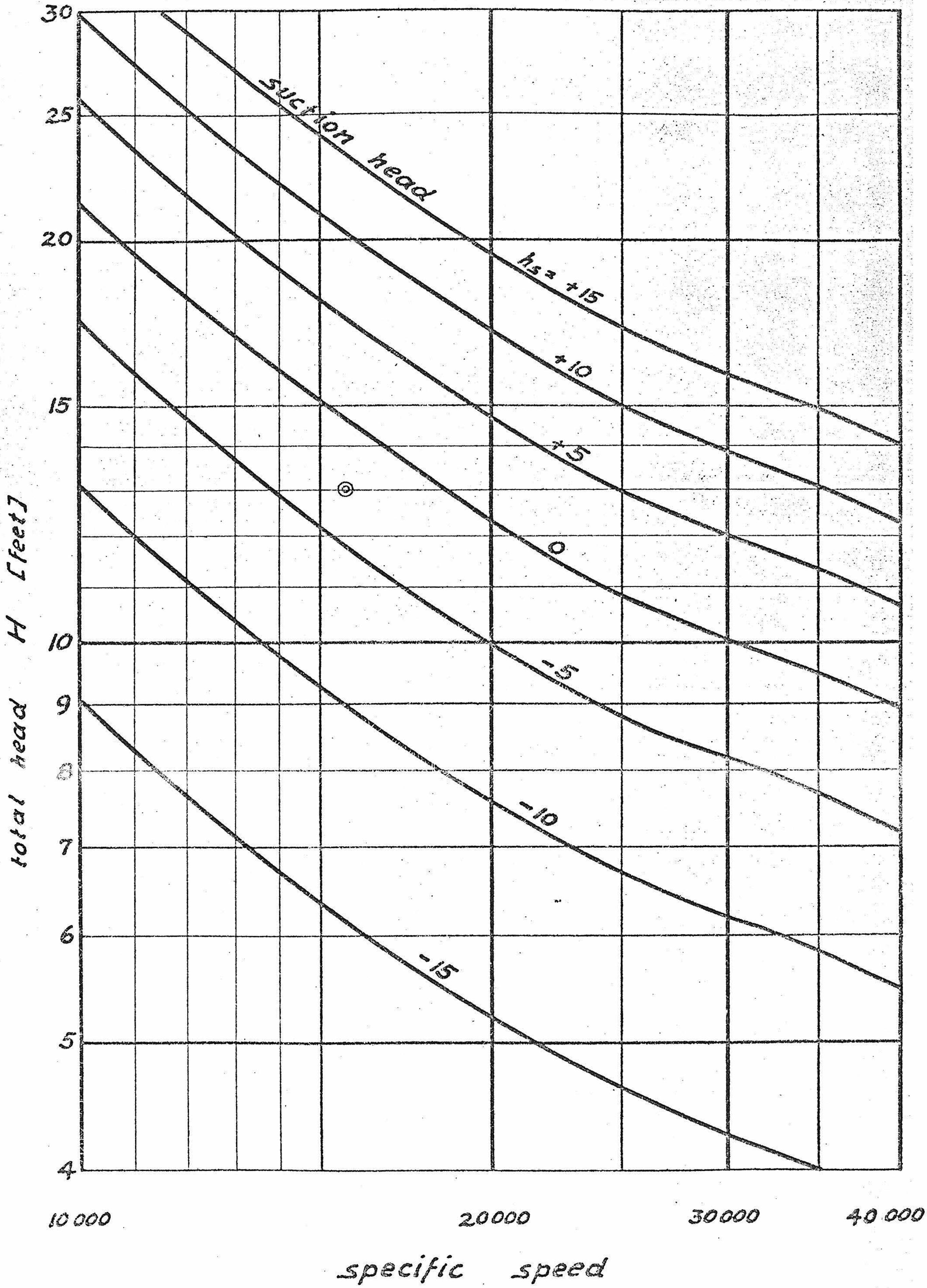
(hs+hc)/H
CL·j
β' = 25°
β' = 20°
β' = 15°
β' = 12.5°
β' = 10°
10000
20000
30000
40000
50000
n sp

Fig. 7

about these form characteristics the curves on fig. 7 would be altered, these alterations, however, will not become very radical as long as the form characteristics are changed within reasonnable, practical limits only. The diagram on fig. 7 therefore has some general significance also.

To obtain finally the greatest permissible head H as a function of the specific speed and the suction head, which corresponds to the well known cavitation chart adopted for double suction pumps, it is necessary to make a final choice regarding the vane angle β', and to determine the value of h_c. the latter should be done experimentally, since there exists a multitude of influences for the actual commencement of cavitation, which can hardly be covered by theoretical investigations. The figure chosen for h_c also should include a certain margin of safety to account for deviations from the exact desired shape of the vanes. The curves on fig. 8 have been drawn for $\beta' = 12.5^\circ$ and $h_c = 26$ feet of water. For specific speeds not exceeding about 16000 fig. 7 shows that the influence of changing β' between 10° and 20° is small, so that the diagram on fig. 8 should be valid for all vane angles of this range, if $n_{sp} < 16\,000$. The isolated point marks the condition of the propeller pump developed by Pfleiderer (reference 11 Page 319), which agrees well with the statements of his text.

The reason why our diagrams have not been extented to lower values of n_{sp} will be given in the following chapter.



suction head
hs = +15
+10
+5
0
-5
-10
-15
30
25
20
15
10
9
8
7
6
5
4
total head H [feet]
10 000
20000
30000
40 000
specific speed

Fig. 8

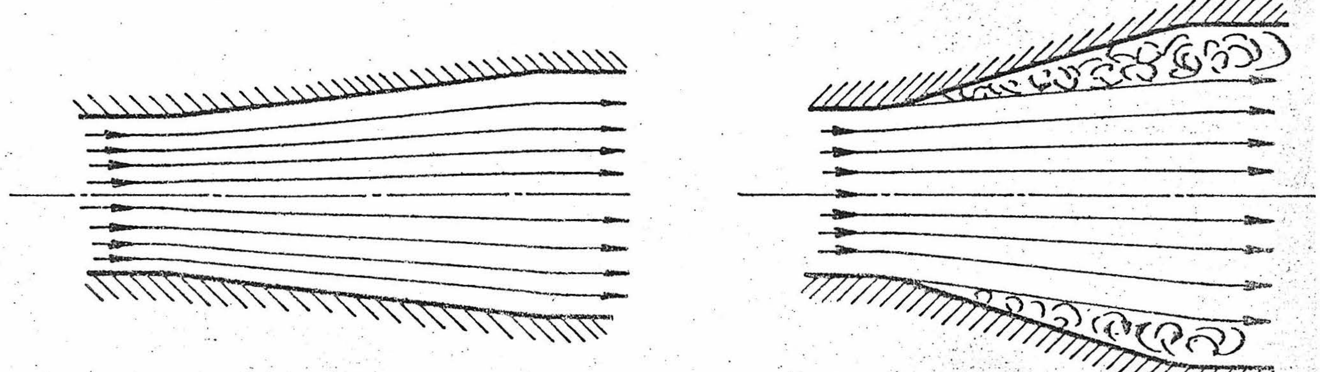
CHAPTER IV. SEPARATION IN HYDRAULIC RUNNERS.

By the term "Separation" we mean the phenomenon that the stream lines of a flow along a rigid body cease to follow the surface of this body, forming between them and the body a region of highly disturbed, whirling fluid motions, which do not show any preferred direction ("turbulent wake"). Separation is observed chiefly under the following three conditions :

1) In a diffusor, if the angle between the sides exceeds certain limits (see fig's. 9a and 9b). In this form the phenomenon has been considered extensively by hydraulic engineers.

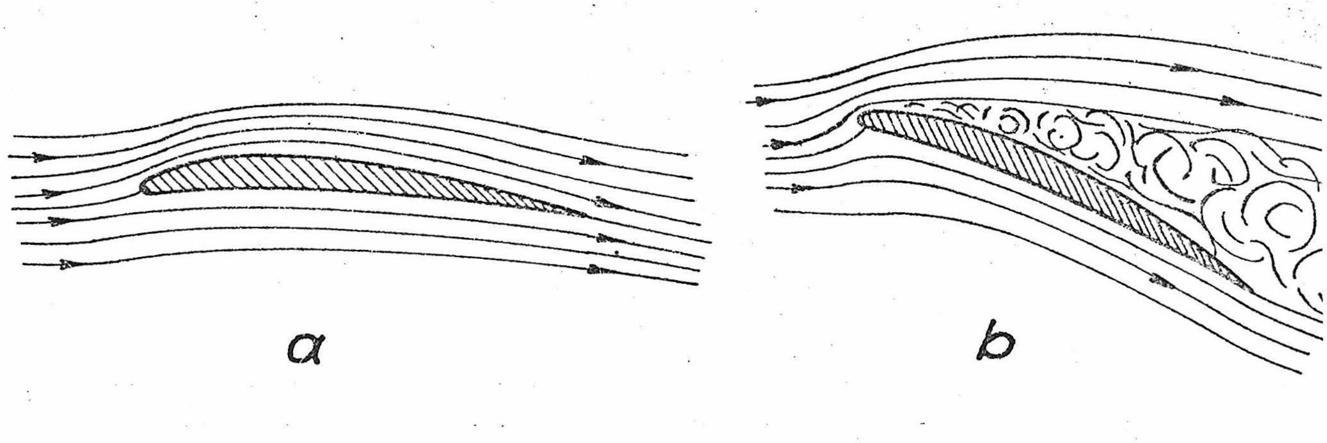
2) In the case of the flow around a sharp corner or along a wall which is curved rather abruptly in comparison with the size of the crossection of the flow (like in a short-radius elbow), especially if the flow is deflected by a large angle.

3) On the suction side of an aerofoil or any vane, if the angle of attack is increased beyond certain limits. (see fig's. 10a and 10b). This form of the phenomenon, in which we are chiefly interested here, has been the object of many investigations in aeronautics, where it is referred to as the "stalling" of aeroplane wings. With respect to hydraulic



a b

Fig. 9



a b

Fig. 10

machines the formation of a turbulent wake on the suction side of the vanes has been observed in the hydraulic laboratory of Thoma in Munich (reference 3), and it s effect on the behaviour of the runner has been considered in a chiefly qualitative manner by Pfleiderer (reference 11 page 106). Numerical criteria for the commencement of separation in hydraulic runners, however, have not yet been derived. To establish such criteria, therefore, will be the main object of this chapter.

No attempt will be made to reach this goal by an exact, mathematical procedure, since not even for a single aerofoil this phenomenon can be described completely by theoretical means. Instead of entering into rigorous theoretical investigations, we shall try to cover only the most fundamental facts by means of simple and plausible approximations.

The following consideration offers itself as first step: For an individual aerofoil in a straight and parallel flow the limit to which this aerofoil can be used without showing separation phenomena ("stalling") may be expressed by the greatest lift coefficient (c_L), which can be reached before separation occurs. (The phenomenon of separation or stalling is experimentally indicated by the fact that from a certain value of the angle of attack and the corresponding lift coefficient on the latter suddenly does not increase any more - in many cases it even decreases - as the angle of attack is further increased.) On the other hand it is possible to define for the vanes of a hydraulic machine a "lift coefficient" in the same manner as for a single aerofoil (the only difficulty being the suitable definition for an average relative velocity of the fluid) . We could then say that <u>separation will occur, if the lift coefficient of the vane is increased to a value, for which separation would occur on a single aerofoil of similar hydrodynamic features.</u> In this manner the separation limit has so far been considered with respect to runners of the propeller type as far as they have been computed by means of the aerofoil theory. For runners of a more general shape the computation of a "lift coefficient" for the vanes and the determination of a corresponding separation limit apparently has not been attempted as yet.

The forgoing approximation is incomplete for the following reason :

We have no right to assume that a vane as part of a whole system of vanes (like a runner) will "stall" (i.e. experience separation) at the same lift coefficient, for which a "corresponding" single aerofoil (see appendix, section 1) in a straight, parallel flow will stall. This possible influence of the arrangement of aerofoils as vanes in a system, which was the main object of our previous investigations about cavitation, will again be in the forground of our present investigations on separation. In order to attack this problem it will be necessary to consider the cause of separation , because only then we will be in the position to decide, which changes in the general flow conditions may have an effect on the phenomenon of separation.

The fundamental idea of the present explanation for separation on aerofoils follows :

The fluid particles near to the surface of a rigid body will be retarded by the friction on the surface. If these particles come into a region where the pressure along the wall increases in the direction of the flow, they will, due to their initially lower velocity be retarded by the pressure gradiant much quicker than the faster moving particles in a greater distance from the wall. The layer of the fluid nearest to the wall, therefore, will eventually by brought to rest and even accelerated in the reversed direction, if the pressure increase is sufficiently long and steep, while the fluid in a greater distance from the wall will still retain its orginal direction of motion. Separation starts at a point, where the layers near to the wall have come to a stop or have reversed their motion. The cause of this form of separation, therefore, is a pressure increase along a fixed boundary (like on the suction side of an aerofoil after the point of lowest pressure).

It is interesting to observe that according to this theory the reason for separation in a diffusor is exactly the same as for separation on an aerofoil. On the other hand it can probably be said that the separation phnomena in the case of abrupt changes in the di rection of the flow cannot be explained in this manner, and are very likely due to a certain instability of the force distribution in a curved flow. This form of separation is not very completely explained as yet, so that it does not seem possible to consider it with respect to hydraulic runners theoretically, except in a purely qualitative manner in so far, as the tendency for separation probably increases as the curvature of the absolute flow through the runner is increased. Especially in the case of high speed runners, however, the curvature of the flow (considering as usual the development of the cylindrical stream surface as shown in fig's. 1, 1a, and 1b) is not very much greater as for a single aerofoil in a straight flow, and we are therefore justified to assume in this case that the phenomenon of separation is controlled by the pressure increase along the vanes, only.

Separation in Runners of the Propeller Type:

We shall demonstrate our computations about separation at first on runners of the propeller type, where these investigations can be carried through in a straight-forward and simple manner.

We have seen before that the pressure rise on the suction side of an aerofoil is responsible for the "stalling" i.e. separation. This pressure rise is considered as equal to p_{-extr} as defined by equation (7). (Theoretically one would have to add to this amount the stagnation pressure, but the pressure rise from the "basic pressure", i.e. the pressure at infinity for a single aerofoil in a straight and parallel flow, to the stagnation pressure is hardly ever realized at the trailing edge of the aerofoil.)

According to the equation (7a) and (8) we have:

$$p_{-extr.} = \frac{\rho v^2}{2} \lambda_{-extr.} = \frac{\rho v^2}{2} \cdot k \cdot c_L \qquad (16)$$

For the further computations we introduce the following fundamental assumption :

The pressure rise, which determines the danger of separation on a vane in a hydraulic runner is equal to the pressure rise on the "corresponding aerofoil"(see appendix, section 1) in a straight, parallel, and infinite flow PLUS the total change of the basic pressure in the runner, the latter being positive in the case of a pump,and negative in the case of a turbine.

By this assumption we, therefore, consider that part of the change in basic pressure, which occurs before the point of $p_{-extr.}$ on the vane is reached (i.e. between A and C on fig's. 2 to 4) as also contributing towards or against separation in the fillowing region of increasing vane pressure.. This seems justified, since we are considering all the time the same "boundary layer" (see references 7 and 8), whose future behaviour may well be expected to be influenced by the pressure changes in its very first part.

We shall denote the total pressure rise on the vanes, which controlls separation, by Δp_{sep}, and we put, according to our above assumption :

$$\Delta p_{sep} = p_{-extr.} \pm (p_p - p_s) \qquad (17)$$

where the positive sign applies to pumps, and the negative sign to turbines.

Substituting for $p_p - p_s$ the value given by equation (4) and for $p_{-extr.}$ that from the equation (7a) and (8), we obtain :

$$\Delta p_{sep} = \frac{\rho v^2}{2} \cdot k \cdot c_L \pm \frac{\rho v^2}{2} \cdot j \cdot c_L$$

$$= \frac{\rho v^2}{2} \cdot c_L (k \pm j) \qquad (18)$$

Since we are used to judge the danger of separation on an aerofoil by the value of the lift coefficient, we shall introduce a "coefficient of separation" c_s, which has the same relation to Δp_{sep} in a system of vanes as c_L has to $p_{-extr.}$ for an aerofoil in a straight and parallel flow. By analogy with equation (16) we therefore define the coefficient of separation by the relation :

$$\Delta p_{sep} = \frac{\rho v^2}{2} \cdot k' \cdot c_s \qquad (19)$$

or

$$c_s = \frac{1}{\frac{\rho v^2}{2}} \; \frac{\Delta p_{sep}}{k'} \qquad (19a)$$

where k' is a new k value taken from the diagram on fig. 5 corresponding to the value of c_s in the place of c_L.

Substituting into equation (19a) for Δp_{sep} the value given by equation (18), we obtain :

$$c_s = c_L \left(\frac{k}{k'} \pm \frac{i}{k'} \right) \tag{20}$$

Since the boundary layer theory as yet does not give us sufficient information about the influence of the shape of the pressure curve on separation to decide which value we should choose for k', we shall use for our further consideration the simplifying assumption

$$k' = k \tag{20a}$$

which leads to the equation :

$$c_s = c_L \left(1 \pm \frac{i}{k} \right) \tag{21}$$

where we should keep in minde that we have to pick our value for k with some respect to the value which we expect for c_s.

The upper limit of the coefficient of separation c_s is in first approximation the highest lift coefficient, which can be reached in the windtunnel forsimilar aerofoils (and for the same Reynolds number). To obtain an exact criterium for the commencement of separation it would be necessary to determine the values for c_s and for k experimentally.

We have seen before that for the same velocity conditions the head converted in the runner is chiefly determined by the product $c_L \cdot j$ (see equation (12) in chapter III and section 1 of the appendix). In order to judge the limits of this value with respect to separation we rewrite equation (21) in the form :

$$c_L \cdot j = \frac{c_s}{\frac{1}{j} \mp \frac{1}{k}} \qquad (22)$$

with respect to turbines (negative sign) this equation expresses the fact that for $j = k$ the value $c_L \cdot j$, and thereby the head which can be converted in the runner without separation, is unlimited. Physically this means that under these conditions the drop of the basic pressure always cancels the usual pressure rise on the suction side of the vanes to such an extent, that no separation can take place, regardless how far c_L may be increased (see fig. 2). (In the case $j > k$ equation (22) looses its physical meaning.)

With respect to pump runners (positive sign) we find that $c_L \cdot j$ remains definitely bounded if we increase j arbitrarily, i.e. we find that

$$c_L \cdot j < c_s \cdot k \qquad (23)$$

In the diagram on fig. 11 we have plotted the values of

$$\frac{c_L \cdot j}{c_s} = \frac{1}{\frac{1}{j} + \frac{1}{k}} \qquad (22a)$$

as a function of the ratio of overlapping j, showing that

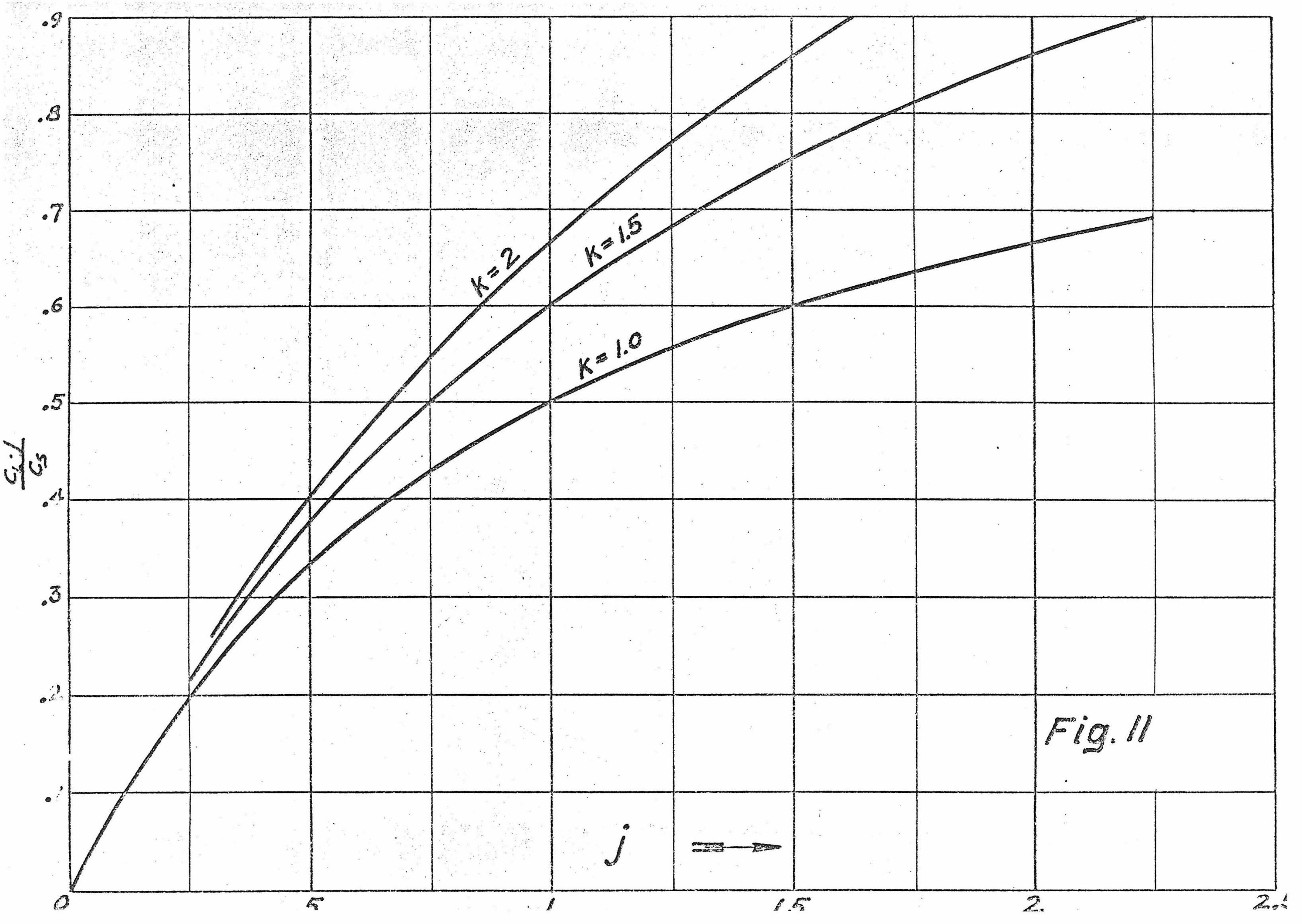


Fig. II

an increase in j, i.e. an increase of the length of the vanes compared with their distance, contributes less and less towards an increase of the highest possible $c_L \cdot j$ as j is increased, which is plausible, because as we increase j, we increase the rise in basic pressure, and therefore must decrease c_L in order to keep the total pressure rise on the suction side of the vanes within permissible limits.

These considerations also yield definite lower limits for the specific speed of pumps of the propeller type.

Corresponding to equation (15) presented in chapter III, we can express the specific speed also as a function of the form characteristics at the innermost cylindrical stream surface of the runner. This expression also has been derived in section 4 of the appendix :

$$n_{sp} = C \cdot \frac{\sqrt{\sin\beta_s}\;\cos^{3/4}\beta}{(c_L \cdot j)^{3/4}} \sqrt{\frac{D_o^2}{D_i^2} - 1}\;\left(\cos\beta' + \frac{c_L j}{4\cos\beta'}\right)^{1/4} \tag{24}$$

where we have to take the angles and other form characteristics from a cylindrical section with the inner diameter D_i. C has the same value as before.

The only difference between equation (24) and the corresponding equation (15) for the outer surface is the fact, that the value $D_o^2/D_i^2 - 1$ has been substituted in place of $1 - D_i^2/D_o^2$. Since the new value is always larger than the old one, we must, in order to obtain the same value for the specific speed, make the value of $c_L \cdot j$ considerably larger at the inside than at the outer periphery. This is physically very easy to comprehend, considering that near the hub we have to convert the same total head at a considerably lower vane velocity or relative velocity (see also equation (12)). Hence we find that separation on the vanes (stalling) is more difficult to avoid at the inside than at the outer periphery of an axial flow runner, and that the lower limit for the specific speed has to be determined from the conditions near the hub.

In accordance with the form characteristics given in the previous chapter for the series of rpopeller pumps, which we considered, we choose for the form characteristics of the inner stream surface the following values : $\beta' = 40^\circ$ $D_i/D_o = .5$.

With $c_L \cdot j = 1$ we obtain according to equation (24) for the specific speed $n_{sp} = 8000$. Since $c_L \cdot j = 1$ requires, according to the diagram on fig. 11 already a ratio of overlapping of about 1.25, which is rather large for runners of the propeller type, we have, for our previously derived cavitation charts, considered $n_{sp} = 10\ 000$ as practical lower limit for the specific speed of such runners.

For turbines such a lower limit for the specific speed does nor seem to exist (from the point of view of separation and within the limits of our present approximations), since we have seen that for such runners $c_L \cdot j$ can be increased indefinitely without danger of separation.

Separation in radial flow runners.

With respect to radial flow (and mixed flow) runners we can only hope to mke rather crude estimates as to whether we should expect separation to take place in a certain runner or not.

Regarding (reaction-) turbines of the above type it is well known that the relative motion of the water is always so highly accelerated, i.e. accompanied by a rapid pressure drop, that separation can always be avoided, and we shall, therefore, restrict our following investigations to pump runners only.

Although for radial flow runners we do not have such a simple relation between runner vanes and a "corresponding" aerofoil as in the case of axial flow runners, we are yet in the position to compute for the vanes of such runners a "lift coefficient" defined in the same manner as for a single aerofoil. Thereby we obtain a primitive idea regarding the danger of separation for the particular vane.

In section 5 of the appendix we have derived the following formula for the "lift coefficient" of the vanes of a radial flow runner under the assumption that the vanes are curved as logarithmic spirals (always assuming no rotation of the fluid at the suction side of the runner).

$$c_L = \frac{2}{j} \quad \frac{v_{u_p}}{v_p} \quad \cos \beta' \qquad (25)$$

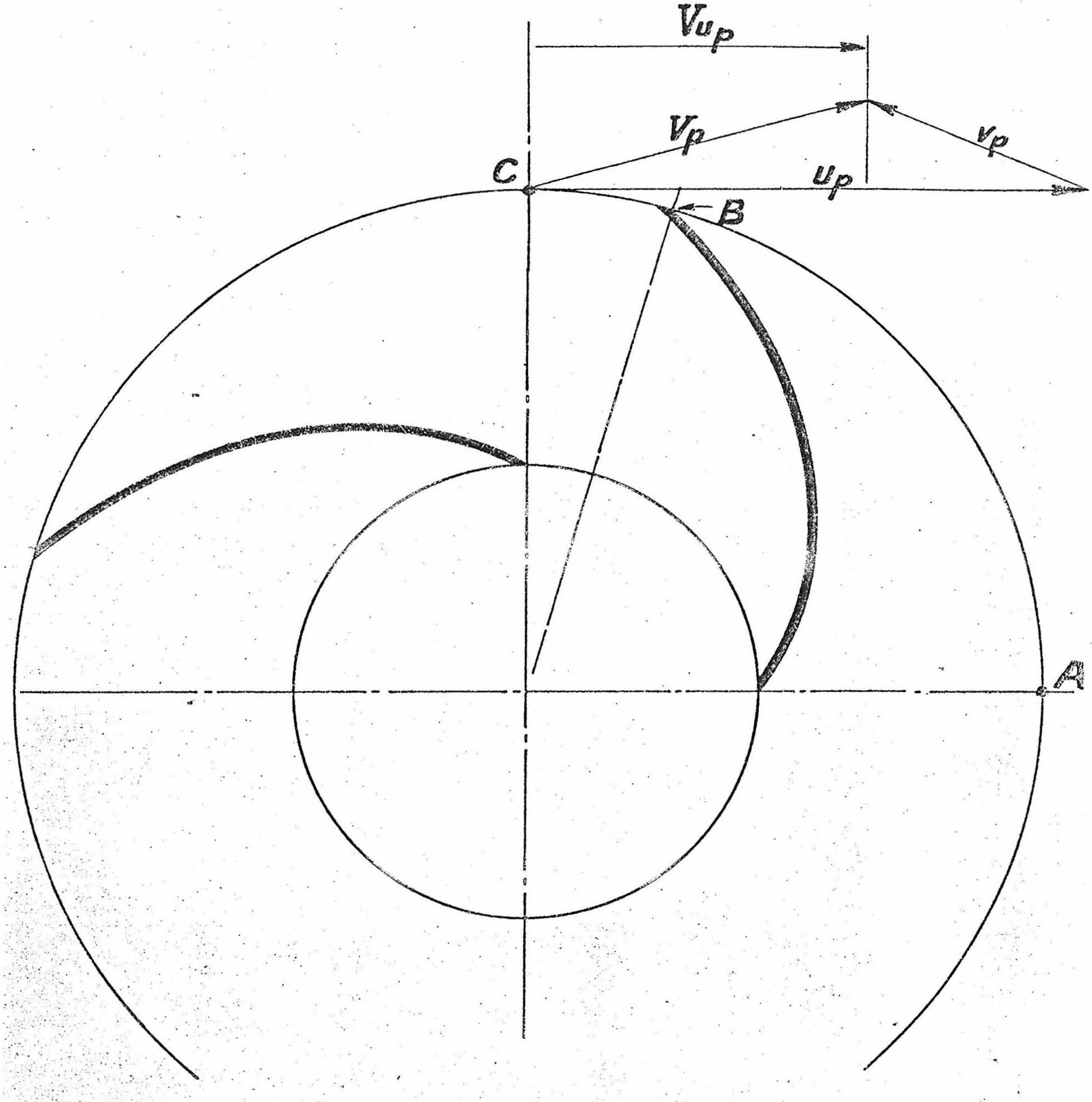


Fig. 12

where $j = \frac{AB}{AC}$ as defined by fig. 12 , AB and AC being measure along the outer periphery of the impeller.
(It is interesting to note that equation (25) is identical with the corresponding equation for axial flow runners, except that j and the velocities have to be defined in a different manner.)

For the investigation about the influence of the basic pressure on the danger of separation it is of greatest importance to note, that that part of the increase in the basic pressure, which is due to the centrifugal forces, does not contribute towards separation. This fact was first pointed out by von Kàrmàn on ground of the perfect equivalence between a field of centrifugal forces and a gravitational field, since the pressure increase due to gravitation (for instance in a vertical pipe) does not lead to separation. A further investigation immediately reveals the fact that the pressure increase due to centrifugal forces cannot contribute towards the retardation of a certain particle of fluid, which we consider, since it is balanced by - and numerically identical to - the centrifugal force acting on the same particle. Having seen already that the pressure changes due to gravitation (differences in elevation) do not have any effect on the possibility of separation either, we find that according to equation (2a) (chapter II) only the pressure changes due to the changes in relative vleocity of the fluid, i.e. the diffuser effect of the channel between the vanes has to be taken into account.

For the pressure increase along the vanes which has an influence on the commencement of separation we, therefore, may write :

$$\Delta p_{sep} = \frac{\rho v^2}{2} \cdot k \cdot e_L + \rho \frac{v_s^2 - v_p^2}{2} \qquad (26)$$

where v is again the "mean effective velocity" as defined in section 1 of the appendix, if we extend this definition to the conditions of radial flow runners.

By analogy with our previous procedure we compute now a "coefficient of separation" as defined by equation (19), leading to the following expressions :

$$e_s = \frac{\Delta p_{sep}}{k} \frac{1}{\frac{\rho v^2}{2}} = e_L + \frac{v_s^2 - v_p^2}{2 k v^2} \qquad (27)$$

and if we assume that : $v = \frac{v_s + v_p}{2}$:

$$e_s = e_L + \frac{2}{k} \frac{v_s^2 - v_p^2}{(v_s + v_p)^2} = e_L + \frac{2}{k} \frac{v_s - v_p}{v_s + v_p}$$

or finally :

$$e_s = e_L + \frac{2}{k} \frac{1 - v_p/v_s}{1 + v_p/v_s} \qquad (27a)$$

Substituting for e_L the value given by equation (25) we obtain :

$$e_s = 2 \left\{ \frac{\cos\beta'}{j} \frac{V_{u_p}}{v_p} + \frac{1}{k} \frac{1 - v_p/v_s}{1 + v_p/v_s} \right\} \qquad (28)$$

This formula, on account of the derivation for c_L applies only to runners with vanes curved as logarithmic spirals. We shall, however, use equation (28) in connection with radial flow and mixed flow impellers of quite general form characteristics, considering expression (25) simply as definition for the lift coefficient c_L of runner vanes of arbitrary shapes. For β' we may choose in this case the average inclination of the vane, i.e. the inclination of a logarithmic spiral drawn through the two ends of the vane. (Note that equation (28) is equally valid for axial flow runners if j is interpreted correspondingly, and v_p in the first term is replaced by v.)

Since in the majority of practical cases we do not know the pressure distribution along the vanes, which determines the value permissible for c_s, this value as well as k should be found by experimental observations. The determination of c_s, in particular, would require methods for the observation of separation on the vanes for a given machine of general design. Such methods are not available as yet, although promising attempts have been made in this direction.

Notwithstanding this present restriction, equation (28) permits us to estimate for a given case, whether we can reasonnable expect separation, because we know that the lift coefficient of a single aerofoil can hardly be expected to exceed 1.6 very much, while k will probably lie between 1 and 2. Due to the comparatively long passages enclosed between the vanes and the sharp curvature of the flow in the case of lower specific speeds, $c_{s_{max}}$ for impellers must be expected to lie somewhat below the corresponding values $c_{L_{max}}$ for individual aerofoils.

There are a few general remarks to be made with respect to equation (27a) and (28) :

1) If we examine standard centrifugal pumps as built for rather high head we find that in many cases c_s will reach values which are considerably higher than the highest lift coefficients reached with the best aerofoils, so that one must conclude that separation does actually take place on the vanes of such runners. (This agrees with the observations made by Thoma, see reference 3). Separation on pump runners, therefore, does not seem to have such a radiacal effect as for individual aerofoils (see normal lift coefficient curves, for instance in reference 5), which is not surprising, since in the case of separation from the suction side of a particular vane, the following vane will prevent the wake from becoming very wide, and the flow will not be able to deviate very radiacally from its ideal direction. However, one would expect that the formation of any wake on the vanes must cause some energy losses, so that separation should be avoided where very high efficiencies are essential.

The most influential factor in equation (28) is the ratio V_{up}/v_p, especially since an increase in j (length of the vanes) can hardly be expected to effectively prevent separation due to the corresponding increase in wall friction. Hence, in order to keep e_s below certain limits, we must keep V_{u_p}/v_p below corresponding limits. With reference to the velocity diagram on fig. 12 we may put v_p approximately equal to $u - V_{u_p}$ thereby obtaining for the ratio V_{u_p}/v_p the approximate expression :

$$\frac{V_{u_p}}{v_p} \approx \frac{1}{u/V_{u_p} - 1} \tag{29}$$

The ratio u/V_{u_p}, however, is proportional to the square of the unit speed, the latter being defined as

$$n_I = \frac{n\,D}{\sqrt{H}}$$

Summing up the previous steps we therefore arrive at the conclusion that in order to keep e_s below certain limits (i.e. to avoid separation), the unit speed n_I must lie above a corresponding lower limit. The unit speed, however, cannot be increased very far without increasing the specific speed, since otherwise the impeller would assume a hydraulically unfavorable shape. One, therefore, obtains the result that, as a general rule, also the specific speed must be kept above a certain lower limit in order to prevent separation. This conclusion is the more probably, since an increase in specific speed usually is accompanied by a decrease in friction surface (wider crossections), which must be beneficial for preventing separation, since we have seen that this phenomenon is caused primarily by friction on a rigid wall. Due to the lack of general information, however, regarding the values to be used for e_s and k it is at present impossible to give definite figures for the lowest specific speed at which separation can be avoided, which would be the more difficult since from equation (28) we only could derive an approximate lower limit for the unit speed, So that one would have to make additional assumptions about the remaining characteristics (unit capacity). From the present state of information one merely can estimate that the lower limit for the specific speed, at which separation can be avoided, may lie in the vicinity of 2500 , unless one uses special means for preventing separation, corresponding to "high lift devices" on aeroplane wings, or perhaps by subdividing the impeller into two or more successive parts. Especially for large pumps such means are well within the reach of practical methods of impeller construction, and their applicability should be seriously studied, provided it can be shown that the energy losses due to separation on vanes of standard runners of low specific speeds are large enough to warrant such departures from the conventional designs. By this remark the writer wishes to suggest that the energy conditions in high head pump runners may have to be considered in a very different manner from those of the flow about an aerofoil, so that the above lower limits for unit and specific speed merely indicates the limit to which aerofoil conceptions are applicable to hydraulic runners.

3) The fact that the pressure increase, which is due to the centrifugal force, does not contribute towards separation is expressed once more by equation (27a) in so far as a radial flow runner for which the second term on the right side of equation (27a) would vanish or become negative (corresponding to constant or increasing relative velocity) still creates static head due to the centrifugal forces. Such a condition, of course, is impossible for purely axial flow runners. For the computation of the runner this means that for radial or mixed flow runners e_L under certain conditions can be taken as equal or even larger than e_S, which is impossible for purely axial flow pump runners, because here the decrease in relative velocity is necessary for any increase in static head within the runner, so that the blades of axial flow pump runners always work under relatively more unfavorable pressure conditions than a single aerofoil in a straight and parallel flow, which is not necessarily the case for radial or mixed flow runners.

This difference between radial and axial flow pump runners is one of the most fundamental reasons why radial flow impellers are to be preferred for pumps of low specific speeds.

The Influence of Separation on the Scale Effect.

Since the turbulence of the flow in hydraulic runners can be assumed to be very high, a change in the scale of the flow, or, more exactly, a change in Reynolds' number, cannot be expected to have a great effect on the behaviour of the pump or turbine. This fact has so far been verified in most cases where the scale effect could be observed experimentally, and all model testing of hydraulic runners is generally based on this assumption.

Recent experiments, however, carried out at the California Institute of Technology (see references 9 and) have shown that the limits for separation on aerofoils are appreciablg influenced by the value of Reynolds' number (and by the degree of turbulence). This means that one may expect a greater scale effect than usual in such cases, where the impeller blades of a particular pump, for instance, work close to the limit of separation, and our previous formulae may be used to estimate whether or not such an effect should be expected in a given case.

CHAPTER V. SUMMARY AND RESULTS.

1) We define the "basic pressure" of a flow about a vane or aerofoil as that pressure, which would exist without the effect of the particular vane or aerofoil which we consider.

2) Most information about aerofoils pertains to conditions with constant basic pressure. The main object of this paper is the modification of such results regarding cavitation and separation on aerofoils, so that they may be applied to hydraulic runners, where the basic pressure is not constant.

3) In chapter II we derive simple formulae for the basic pressure in a runner, considering the basic pressure as equal to the static pressure in the runner as computed according to the Eulerian theory for hydraulic runners (infinite number of vanes).

4) The change in basic pressure influences the lowest pressure as found on a single aerofoil regarding its value as well as its location along the vane. Both changes are determined in chapter III with respect to axial flow runners, and the results are used for computing cavitation charts (fig's. 7 and 8) for propeller pumps. Since in pump runners the point of lowest pressure lies in a region of low basic pressure, and in turbine runners in a region of high basic pressure, there exists an important difference between the cavitation limits of these two types of runners.

5) Since separation on aerofoils begins if the pressure rise along the suction side of the aerofoil exceeds certain limits, we assume that separation on runner vanes will commence, if the pressure rise, which a vane with the same lift coefficient as the runner vane would have in a straight and parallel flow, PLUS the rise in basic pressure exceed the same limits. (The lift coefficient for runner vanes, therefore, had to be derived for runners of arbitrary shape, also.) In chapter IV we present formulae for the separation limits for pump runners of the propeller type and of general shape. The separation limits practically determine lower limits for the unit speed of pump runners. For turbine runners such separation limits mostly do not exist.

6) Since the beginning of separation is considerably influenced by Reynolds' number, our formulae allow to estimate, in which cases such an influence must be expected for hydraulic runners (scale effect).

List of Notations.

While the "American Standard Symbols for Hydraulics" were used whereever possible, the subject required the introduction of a number of special notations. Dimensions are given only if essential in the place where the particular notations are used.

c_L	lift coefficient (appendix, page III)
D	diameter
D_a	outer diameter of an axial flow runner
D_i	inner diameter of an axial flow runner (diam. of hub)
F	force (total pressure)
g	graviatational acceleration
H	total head of a runner (feet)
h_s	"suction head" as defined on page 23 (feet)
h_c	head drop below atmospheric pressure, at which cavitation will just begin (feet)
j	"ratio of overlapping" (page 13)
k	$\frac{\text{pressure drop on an aerofoil below pressure at infinity}}{\text{average pressure difference between the two sides of the vane}}$
N	number of vanes in a runner
n	number of R P M
n_{sp}	specific speed $= \frac{n \sqrt{GPM}}{H^{\frac{3}{4}}}$
n_I	unit speed $= \frac{n\ D}{H^{\frac{1}{2}}}$
p	pressure (lbs/inch^2)
Δp	pressure difference on the vanes

Q	rate of flow ($feet^3/sec$)
r	radius
t	chord or length of an aerofoil or vane
u	peripheral velocity of the runner (feet/sec)
V	absolute fluid velocity (feet/sec)
v	relative fluid velocity (feet/sec) (without subscript: "mean effective relative velocity" as defined in the appendix page II fig. A2.)
w	density (weight per unit of volume)
x	coordinate in the axial direction of an axial flow runner
z	elevation (feet)
β	angle between the relative velocity and the peripheral direction (fig. A 1 of the appendix) (without subscript it refers to the "mean effective relative velocity" as defined in the appendix, p. II)
β'	angle between the vane and the peripheral direction (fig. 1 b)
Γ	circulation, defined as the line integral of the velocity component pointing in the direction of the closed path of integration (page I of the appendix).
λ	vane pressure divide by the stagnation pressure of the "mean effective relative velocity".
$\Delta\lambda$	difference between the pressures on the two sides of the vane divided by the same stagnation pressure as before.
ε	angle of attack
ξ	coordinate along the vane or aerofoil divided by t
ρ	w/g (specific mass)
ω	angular velocity

Subscripts:

a refers to the axial direction of an axial flow runner (except section VI of the appendix)

p refers to the high pressure side of the runner

s refers to the low pressure side of the runner (except in the connection e_s which denotes the coefficient of separation)

u indicates peripheral component of a velocity

A few other notations are explained in the text.

List of References.

Papers, Reports, et c. :

1) Betz - Diagramme zur Berechnung von Fluegelreihen - Ingenieurarchiv, vol. II, 1931, page 359

2) Prandtl - Betz - Ergebnisse der Aerodynamischen Versuchsanstalt zu Goettingen - III. Lieferung. 16 Untersuchungen ueber Druckverteilung an gestaffelten Fluegelgittern. page 132.

3) Fischer - Untersuchung der Stroemung in einer Zentrifugalpumpe. - in D. Thoma - Mitteilungen des hydraulischen Instituts der technischen Hochschule Muenchen, Heft 4, page 7 .

4) Spannhake - Problems of Modern Pump and Turbine Design. - (HYD-56-1) Transactions of the A.S.M.E. April 1934, vol.56, No.4.

5) Jacobs - Ward - Pinkerton - Report 460 of the National Advisory Committee for Aeronautics.(N.A.C.A.)

6) Betz - Verallgemeinerung der Joukowsky Profile - Zeitschrift fuer Motorluftschiffahrt (Z.F.M.) 1924 .

7) von Kàrmàn - Turbulence and Skin Friction - Journal of Aeronautical Sciences - vol.1, No.1, January 1934.

8) von Kàrmàn - Theorie des Reibungswiderstandes - in : Hydraulische Probleme des Schiffsantriebs - Hamburg 1932.

9) C.B.Millikan and A.L.Klein - The Effect of Turbulence - Aircraft Engineering - August 1933.

Text Books and Hand Books :

10) Daugherty - Hydraulic Turbines - 3d edition

11) Pfleiderer - Die Kreiselpumpen - 2nd edition

There exists, of course, a large number of other valuable reference books; we have quoted, however, only those which were used directly in the present paper.

APPENDIX TO THESIS

Section 1. Some theoretical facts about axial flow runners.

The flow is assumed to proceed in the runner along cylindrical, coaxial stream surfaces. The change in angular momentum, or the "vane circulation" is assumed to have the same value for every one of these stream surfaces. The flow is investigated in the developments of the latter, and the resulting flow pictures are treated as two-dimensional, so that the hydraulic problem is reduced to that of an initially parallel flow through a straight, parallel, and infinitely long lattice. The relative motion is irrotational, i.e. the "Bernoulli constant" or total energy of the flow has the same value for every point in the field (not merely along the same stream line). We are therefore considering primarily the relative motions of the fluid.

The blades of the lattice are computed like aerofoils exposed to the "mean effective velocity" v, defined by the velocity diagrams on fig's. A1 and A2. These diagrams show relative velocities only.

If the fluid has no absolute rotation on the suction side of the runner :

$$V_{up} = |v_{us} - v_{u_2}| \text{ (for pumps and turbines)} \qquad (A1)$$

With reference to the contour A B C D on fig. A 1 we find for the circulation around one vane the expression:

$$\Gamma = \oint_{ABCD} \bar{V} \cdot d\bar{s} = \frac{2\pi r}{N}(V_{u_s} - V_{u_p}) = \frac{2\pi r}{N} V_{up} \qquad (A2)$$

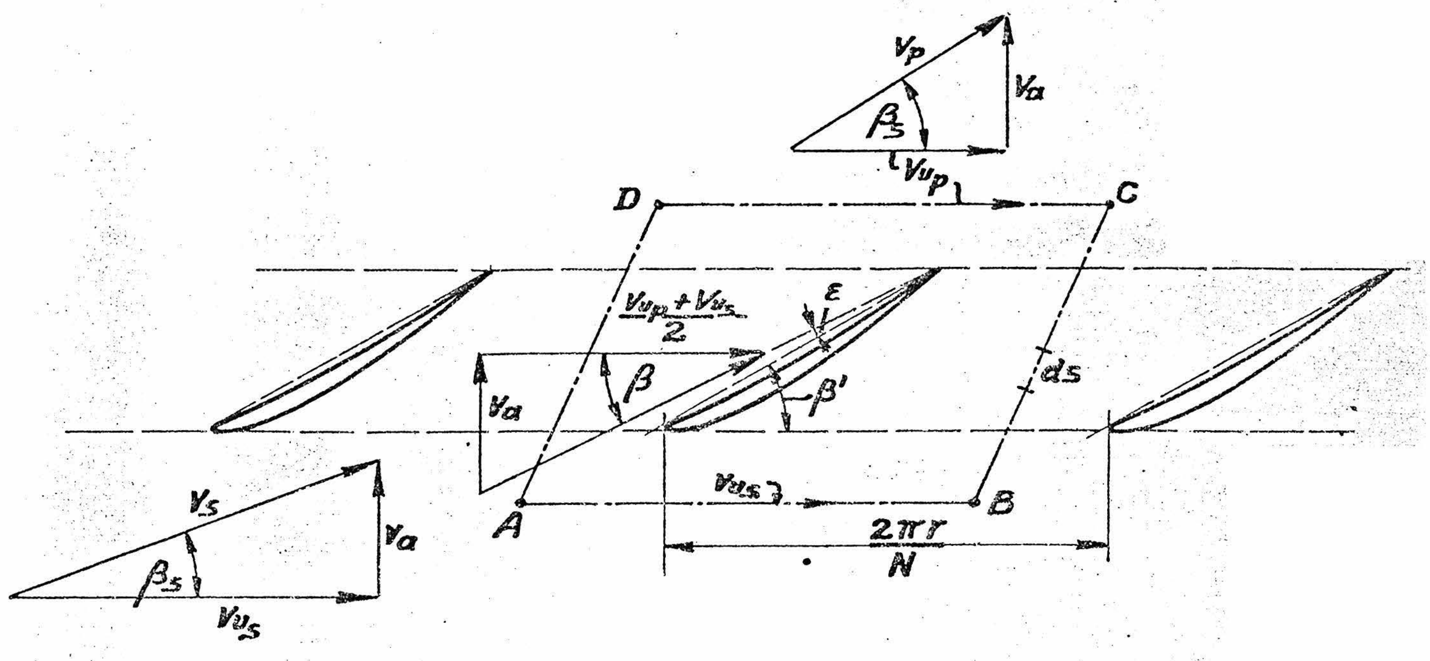


Fig. A1

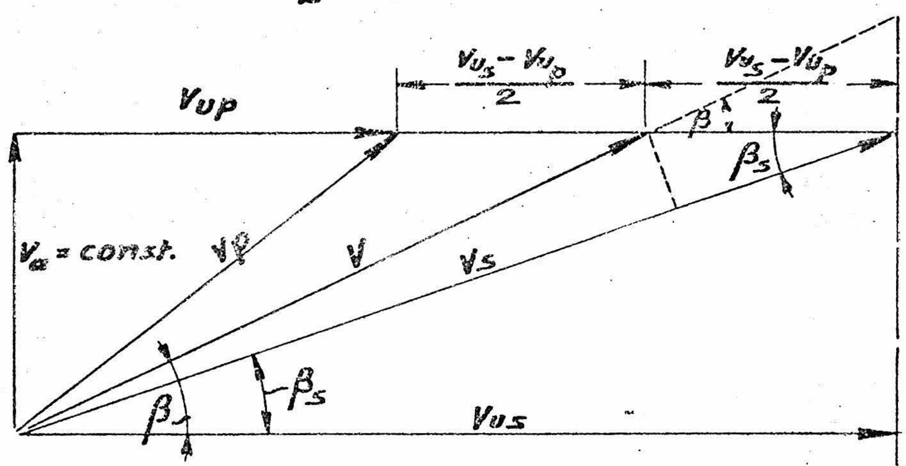


Fig. A 2

Fig's A 1 and A 2 show conditions pertaining to pump runners. The corresponding pictures for turbine runners are obtained by reversing the directions of all velocities, and by interchanging leading and trailing on the vanes.

The circulation about an aerofoil, in general, has the following relation to the "lift coefficient" c_L ;

$$\Gamma = \frac{c_L}{2} \cdot v \cdot t$$

or

$$c_L = \frac{2\Gamma}{t v} \qquad \text{(A3)}$$

(the lift coefficient c_L is defined by the following relation :

$$\text{Lift} = c_L \cdot \frac{\rho v^2}{2} \cdot t \cdot \text{width}$$

the "lift" being the force action between the fluid and the vane perpendicular to the (relative) velocity v .)

From equations (A2) and (A3) we obtain :

$$c_L = \frac{4\pi r}{N t} \frac{V_{up}}{v} = \frac{2}{j} \frac{V_{up}}{v} \cos\beta' \qquad \text{(A4)}$$

(about the definition of j see chapter II.)

The head converted in the runner (neglecting losses and assuming again that the fluid has no absolute rotation on the suction side) is expressed by the following momentum equation:

$$H = \frac{u \cdot V_{up}}{g} \qquad \text{(A5)}$$

Substituting for V_{up} the value expressed implicitly by equation (A4) leads to the following relation:

$$H = \frac{u v}{2g} \frac{c_L \cdot j}{\cos\beta'} \qquad \text{(A6)}$$

In order to express the "mean effective velocity" (v) and its direction β as a function of the initial flow conditions given by v_s and β_s we derive from fig. A2 and equations (A1) and (A4) the relation :

$$v = v_s - \frac{V_{up}}{2} \cos\beta_s = v_s - \frac{c_L j \, v \cos\beta_s}{4 \cos\beta'}$$

Using the approximation : $\cos\beta_s = \cos\beta'$, we obtain:

$$V = \frac{v_s}{1 + \frac{c_L i}{4}} \qquad \text{(A7)}$$

For turbines one may prefer to express v as a function of v_p. By the same approximation as before we find:

$$V = \frac{v_p}{1 - \frac{c_L i}{4}} \qquad \text{(A8)}$$

For the direction of the "mean effective velocity" we find in the same manner:

$$\begin{aligned} \sin\beta &= \sin\beta_s \cdot \left(1 + \frac{c_L i}{4}\right) \\ &= \sin\beta_p \cdot \left(1 - \frac{c_L i}{4}\right) \end{aligned} \qquad \text{(A9)}$$

So far we have considered the whole vane in the system as being exposed to a constant and straight flow with the velocity v and the direction β. Actually the relative velocity changes steadily along the vane. We shall assume that for all our computations this change is taken into account by the method developed by Betz (reference 1), which is based on the following idea:

We start from an aerofoil which in a straight and parallel flow would have the desired lift coefficient, pressure distribution, et c. This aerofoil is then <u>deformed</u> in such a manner that under the changing relative flow conditions in the system it will have the same distribution of the total forces (not of the specific pressures) as the original aerofoil in a straight and parallel flow. In our text we refer to the latter as the "corresponding aerofoil".

All our data about the vanes of axial flow runners, like the chord (t), the direction of the chord (β'), the ratio of overlapping (j), The vane pressure ($\Delta p, \Delta\lambda, \lambda$), and the lift coefficient (c_L) pertain to the "corresponding aerofoil" which would have the desired hydraulic properties if exposed to a flow with constant velocity v and constant direction β; they do NOT pertain to the actual vane shape in the styem after the Betz deformation has been applied. It is the "corresponding aerofoil", not the final vane shape, for which we select standard profiles according to their known properties. Just this possibility, namely to carry out all considerations as if the "mean effective relative velocity" were constant over the whole length of the vanes, and to apply the necessary corrections afterwards when designing the actual vane, this appears to be the greatest practical and theoretical advantage of the Betz correction method. It may be mentioned that the Betz corrections become negligible for small values of j (about $\frac{1}{2}$), while they are quite important for large j values (1 or more) especially if β' is small.

The contradiction, that we consider the change of the "basic pressure" along the vanes, while considering the relative velocity (basic velocity) as constant in the same range, vanishes, if the Betz correction is applied afterwards. Both, the velocity and the pressure corrections appear to have approximately the same order of importance.

Section 2. Comparison of given pressure formulae with experimental results.

The results of equation (4) in chapter II have been compared with the experimental results given in reference 2 pages 132 to 138. The investigated system corresponds to a turbine runner (accelerated relative motion). The given pressure diagrams are drawn with the stagnation pressure of the initial flow (v_p) not the "mean effective flow" (v) as unit, which complicates the comparison. Another difficulty lies in the fact that the experiment had, of course, to be carried out with one shape for the profile placed into different positions, while our formulae assume that the vanes are corrected by the Betz method (section 1), which for a fixed "corresponding aerófoil" shape would call for different actual vane shapes in different positions in the system. In order to minimize this effect we have considered only those results which were obtained with the steepest vanes (in the reference : $\delta = 29°$, $\alpha = 4.2°$). The comparisons gave the following results :

Test series ("Versuchsreihe")	change in basic pressure	
	theoretical $c_L \cdot j$	from the experiment
I	.24	.19
II	.34	.26
III	.44	.30
IV	.56	.37
V	.71	.60
VI	.76	.76

Considering the uncertainty of the comparison in itself (the diagrams are so small that values taken from them can hardly be better than ± 5% to 10% accurate, no direct readings for the leading or trailing edges being given) The results are sufficiently consistent to justify the use of the given formulae for practical designing work and similar computations. If the influence of the neighbouring vanes would be taken into account more accurately, the results may well be expected to approach, as j increases, greater accuracy in a more consistent manner.

Section 3. Some transformations for the cavitation formula (13).

From equations (11) and (13) in chapter III we obtain at first :

$$\frac{h_s + h_c}{H} = \frac{V}{U}\,\frac{\cos\beta'}{c_L j}\left\{ \frac{V_a^2}{V^2} + c_L \cdot K - j \cdot \int_0^x \Delta\lambda \, d\xi \right\}$$

$$= \frac{V}{U}\cos\beta'\left\{ \left(\frac{V_a}{V}\right)^2 \frac{1}{c_L j} + \frac{K}{j} - \frac{\int_0^x \Delta\lambda \, d\xi}{c_L} \right\} \qquad \text{(A10)}$$

According to fig. A1 in section 1 we derive the relation :

$$\frac{v_a}{v} = \sin\beta$$

From the same diagram we find that :

$$\cos\beta = \frac{u}{V + \frac{V_{up}}{2\cos\beta}}$$

because $v_{u_s} = u$ if the fluid has no absolute rotation on the suction side of the runner.

Hence :

$$\cos\beta = \frac{u}{V}\,\frac{1}{1 + \frac{V_{up}}{V}\,\frac{1}{2\cos\beta}}$$

or

$$\cos\beta + \frac{V_{up}}{V} = \frac{u}{V}$$

By equation (A4) in section 1 :

$$\frac{V_{up}}{V} = \frac{c_L \cdot j}{2\cos\beta'}$$

whence :

$$\cos\beta + \frac{c_L j}{4\cos\beta'} = \frac{U}{V} \tag{A11}$$

or

$$\frac{V}{U}\cos\beta = \frac{1}{1+\frac{c_L j}{4\cos^2\beta'}} \approx 1 - \frac{c_L j}{4\cos^2\beta'} \tag{A12}$$

putting $\cos\beta' = \cos\beta$

(Note that $c_L \cdot j$ always has a low value at the outer periphery of the runner, to which the following equation generally applies.)

Substituting the above values into equation (A10) we obtain:

$$\frac{h_s + h_c}{H} = \left(1 - \frac{c_L \cdot j}{4\cos^2\beta'}\right)\left(\frac{\sin^2\beta}{c_L \cdot j} + \frac{K}{j} - \frac{\int_0^x \Delta\lambda \, df}{c_L}\right) \tag{A13}$$

q. e. d.

Section 4. Specific Speed of an axial flow runner expressed by form characteristics only.

Definition : $$n_{sp} = \frac{RPM \sqrt{GPM}}{H^{\frac{3}{4}}} \tag{A14}$$

$$= \frac{n \sqrt{Q[ft^3/s]}}{H^{3/4}} \quad \frac{\sqrt{GPM}}{\sqrt{ft^3/s}} = \frac{n\sqrt{Q}}{H^{3/4}} \times 21.2$$

where $Q = V_a \cdot \frac{\pi}{4} \cdot \left(1 - \frac{D_i^2}{D_o^2}\right) D_o^2 = V_a \frac{\pi}{4} \cdot \left(\frac{D_o^2}{D_i^2} - 1\right) D_i^2$

and $n\,(RPM) = \frac{30}{\pi} \cdot \omega$

Substituting these expressions as well as expression (A6) for H into the specific speed formula, we obtain (using the first one of the two expressions for Q) :

$$n_{sp} = \frac{\frac{30}{\pi}\sqrt{\pi}\cdot(2g)^{3/4}\cdot 21.2\cdot\sqrt{V_a}\,(\omega\cdot\frac{D_o}{2})\sqrt{1-(\frac{D_i}{D_o})^2}\cdot\cos^{3/4}\beta'}{U^{3/4}\;V^{3/4}\cdot(c_L\cdot j)^{3/4}}$$

Since $(\omega\cdot\frac{D_o}{2}) = U$, we have to investigate the form :

$$\frac{\sqrt{V_a}\cdot U}{V^{3/4}\;U^{3/4}} = \frac{\sqrt{V_a}}{\sqrt{V}}\cdot\frac{U^{1/4}}{V^{1/4}} = \sqrt{\sin\beta_s}\cdot\left(\frac{U}{V}\right)^{1/4}$$

From equation (A11) section 3 we put :

$$\frac{U}{V} = \cos\beta' + \frac{c_L\cdot j}{4\cos\beta'}$$

using again the simplification $\cos\beta = \cos\beta'$ in order to obtain an expression containing form characteristics only. After these transformations we finally obtain :

$$n_{sp} = C\,\frac{\sqrt{\sin\beta_s}\cdot\cos^{3/4}\beta'}{(c_L\cdot j)^{3/4}}\sqrt{1-\left(\frac{D_i}{D_o}\right)^2}\left(\cos\beta'+\frac{c_L j}{4\cos\beta'}\right)^{1/4} \qquad \text{(A15)}$$

where : $C = \frac{30}{\sqrt{\pi}}(2g)^{3/4}\times 21.2 = 8160$

Using instead of the first the second of the above expressions for Q we find:

$$n_{sp} = C\,\frac{\sqrt{\sin\beta_s}\cdot\cos^{3/4}\beta'}{(c_L j)^{3/4}}\sqrt{\left(\frac{D_o}{D_i}\right)^2-1}\left(\cos\beta'+\frac{c_L j}{4\cos\beta'}\right)^{1/4} \qquad \text{(A16)}$$

Equation (A15) applies to the outer, and equation (A16) to the innermost stream surface of axial flow pump runners. For turbine runners the same formulae will hold with the exception that the constant C will have a different value.

<u>Section VII. Derivation of a formula for the lift coefficient for the vanes of a radial flow runner curved as logarithmic spirals.</u>

We shall not attempt to give a rigorous derivation. Instead of such a procedure we shall justify the application of a formula, which primarily was derived for a straight, parallel system (equation (A4), section 1), to a radial flow system by means of the following simple, physical conside- rations :

If we compare the radial flow system, with the vanes as well as the relative flow lines considered as logarithmic spirals, with a system of straight and parallel vanes (corresponding to fig. A 1), we find :

1) That equal divisions on the parallel vanes correspond to divisions along the spirals which are proportional (in lengt to their distance from the center of the system, providing radial lines in the radial flow system correspond to parallel lines normal to the general direction of the straight system.

2) The relative (basic) velocity, which is constant in the straight, parallel system, is inversely proportional to the distance from the center of the radial flow system.

Hence: The products of the lengths of the divisions, which we compare, with their respective relative velocities are equal in the two systems. As the divisions become smaller towards the center of the radial flow system the basic relative velocity increases correspondingly.

Assuming that the difference in the actual relative velocity on the two sides of the vanes (vorticity on the vanes) is the same at corresponding points of the two sytems, we find that the force distribution is also the same in the two systems because it can be proven that the energy equation :

$$\text{Force} = \rho \cdot \Delta t \cdot \frac{V_a^2 - V_b^2}{2}$$

$$= \rho \cdot \Delta t \cdot \frac{V_a + V_b}{2} \cdot (V_a - V_b)$$

(v_a and v_b being the relative velocities at the two sides of the vanes, and Δt one of the divisions of the vane) holds for the relative motion in the radial flow system also. Hence with

$$\frac{V_a + V_b}{2} = V \quad = \text{basic relative velocity}$$

and $v \cdot \Delta t$ as well as $v_a - v_b$ being the same at corresponding points of the two systems, the above statement about the force distribution follows.

With the force distribution (and the vorticity) along the vanes being the same in the two systems, we also may call the "lift Coefficient" the same for the two systems, which leads to the application of equation (A4), section 1 to radial flow systems, q. e. d. The new definition for j follows from the above condition regarding the relation between "corresponding" divisions along the vanes in the two systems, if starting the consideration from the outside of the radial flow runner; this is also the reason why v in equation (A4) has to be replaced by v_p .
