A DETERMINATION OF THE DEVIATION OF THE CALIBRATION CONSTANT OF A FORCE VOLTMETER FROM ITS THEORETICAL VALUE FOR ISOLATED SPHERES DUE TO THE PRESENCE OF OTHER BODIES

Thesis by

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SUMMARY OF RESULTS

The quantitative results of this investigation are:

1) The effect of shanks, the diameter of which is 5% of the diameter of the spheres of a Force-Voltmeter, is to decrease the calibration factor k, in $V = k\sqrt{F}$, by 1.5% ± 1% for spacings between 70% and 110% of the diameter of the spheres.

2) The effect of a grounded, open-ended enclosing cubical cage, each face of which is 6 diameters from the spheres, is to increase k by 10% ± 2% for spacings between 40% and 100% of the sphere diameter.

The qualitative conclusions are:

1) The method of calibrating a Force-Voltmeter by making capacitance measurements appears feasible.

2) The presence of high potential bodies other than the sphere and nearby portions of its shank, is to decrease k by important and unexpected amounts (up to 15% at 100% spacing).

LIST OF ILLUSTRATIONS

INTRODUCTION.

The force voltmeter experiments, made in the High Tension Laboratory as a part of the high voltage measurements program proposed by Professor Sorensen,[1] consist of measurements with two one-meter spheres. These spheres are so mounted that the one connected to the unknown potential is stationary, while the other, which is at ground potential, is free to swing along the line between them. When a voltage is applied, the attractive force moves the grounded sphere toward the other one. If this attraction is now balanced by a mechanically applied force which returns the sphere to its original position, the attractive force can be measured.

To determine the value of the applied potential when a knowledge of the attractive force and of all physical dimensions is complete, it is necessary to know the form of the function relating these quantities. From three fundamental concepts:

1) that the force between point charges is $F = \frac{q_1 q_2}{r^2}$,

2) that the charge on each electrode is $q = CV$,[2]

and 3) that the relative distribution of charges over the surfaces of the electrodes is independent of the voltage;

it is clear that the function relating force and voltage, when all physical dimensions are held constant, is of the form $F = KV^2$ or $V = k\sqrt{F}$. In the latter equation k is a function of the physical dimensions. Primarily, it is a function of the ratio of spacing (minimum distance between the spheres) to diameter, but it also depends in some measure on the position of other surfaces at ground potential and at the high potential.

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1. J. E. Hobson Ph. D. thesis 1935.
"The Sparkless Sphere Gap Voltmeter", by Sorensen, Hobson, and Ramo; Electrical Engineering, June 1935.

2. It is recognized that the charges on the two electrodes are not equal. The charge on the high potential sphere will be V times C_{11} and that on the grounded sphere will be V times C_{12}, where C_{11} and C_{12} are Maxwellian coefficients of capacity and induction. The important thing here is that charge is proportional to voltage, and that the factor of proportionality depends only on physical dimensions.

Lord Kelvin has published tables giving values of k as a function of $\frac{\text{spacing}}{\text{diameter}}$ for two isolated spheres.[3] The values given are easily checked by the use of a few terms of the infinite series of image charges in the region $\frac{\text{spacing}}{\text{diameter}} < \frac{1}{2}$, where convergence is rapid.

In any actual arrangement it is impossible to isolate the two spheres for there are, in the vicinity, other surfaces at ground potential and at the high potential. Hence the use of Kelvin's values of k, with measured values of force will give only approximate values of voltage. These other surfaces are primarily, 1) the shanks of the two spheres, and 2) the walls, floor, and ceiling of the laboratory. This classification is made because analysis of a qualitative type indicates that the effect of both shanks is to decrease k while the effect of all grounded bodies is to increase k.[4]

The work to be described was undertaken to investigate the deviation of the constant k in $V = k\sqrt{F}$ from its value for isolated spheres, when the effect of shanks and grounded bodies is included.

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3. Kelvin, "Papers on Electricity and Magnetism", p. 96. Kelvin does not give "k" itself, but rather $\frac{1}{k^2}$, (which he calls "A"). It should be noted also that he uses the ratio of center-to-center distance to radius, instead of the ratio of spacing to diameter.

4. The Effect of Shanks:
Any increase in the size of the high potential electrode (with V constant) occasions more charge on it, and hence more force. Similarly, any increase in the size of the grounded electrode increases the charge on it and also the force.
The Effect of Ground Planes:
Any ground plane parallel to the line of centers of the spheres may be replaced by image charges behind it. When these image charges are re-imaged back into the two spheres, it is found that the charge on the high potential sphere has been slightly increased while that on the grounded sphere has been decreased by exactly the same amount. Since the charge on the grounded sphere was originally the smaller, the product of the charges after this addition and subtraction is always less than the original product. That is $(A + \alpha)\cdot(B - \alpha) < AB$ if $B < A$. If the ground plane is perpendicular to the line of centers, the re-image charges will not be exactly the same, but for all positions of practical importance (say not nearer than 3 diameters from the nearest sphere), the difference in these re-image charges will not be sufficient to affect the result already stated, namely that the force will be decreased.

CHRONOLOGICAL OUTLINE.

The work reported in this thesis, as a part of the California Institute high voltage measurements program, was started in June, 1935 in the Institute high voltage laboratory and continued there until September of that year. From September until April, 1936, the work herein reported was performed at Iowa State College, Ames, Iowa.

After checking one of Kelvin's values for isolated spheres, a similar computation was made in which the effect of a ground plane parallel to the line joining the sphere centers was included. From this computation it appeared that the calculation of the simultaneous effect of four or more ground planes would be vastly more complicated but not out of the question. Leaving this computation for a more auspicious time, experimental work on the same problem was carried on during the summer of 1935 in the High Tension Laboratory. The results indicated that it would be impossible to determine experimentally the effect of ground planes without some knowledge of the effect of shanks, and vice-versa. As an incidental result, some doubt was raised regarding the applicability of Peek's flashover values for 12.5 cm. spheres to the tests made in 1927 and repeated in 1936. (See page 47.)

During the fall of 1935 the calculation of the simultaneous effect of four ground planes was resumed. A double infinite series was developed, which was proven convergent but which was found hopeless from the standpoint of computation. Approximately 10,000 terms would be required for 1% accuracy. (See page 31 for details of the method.)

Attention was then directed toward an experimental determination of the effect of shank size. In considering methods, several possibilities presented themselves. First there was the direct method of measuring force and voltage with different sized shanks. A setup was made following this method, but unusual difficulties in measuring both force and voltage were encountered and it was abandoned. Another possibility was the use of an electrolytic setup. Conduction currents in such an arrangement are equivalent to capacitance currents in the electrostatic case. The electrolytic experiment has an important advantage in that perfect insulators are available. It has also, however, the serious disadvantage that the specific conductivity of solutions varies extraordinarily with temperature.[5] For this reason, the electrolytic experiment was not attempted. A third possibility lay in the measurement of capacitance. This may be shown as follows:

In any arrangement, the energy in the electrostatic field is

$$E_f = \frac{1}{2} CV^2.$$

If, now, the capacitance is increased by dC (by reducing the spacing) while the voltage is kept constant, the increase in the energy of the electrostatic field will be

$$dE_f = \frac{1}{2} V^2 dC.$$

However, the source must have supplied energy amounting to

$$dE_s = Vdq = V^2 dC.$$

Since only half the energy supplied appears in the electrostatic field, the remaining half must appear as work done by the system on the external world in changing the capacitance. Thus

$$dE_m = dE_s - dE_f = V^2 dC - \frac{1}{2} V^2 dC = \frac{1}{2} V^2 dC.$$

If this change of capacitance occurred by the change of a linear dimension, x, then

$$F_x = -\frac{dE_m}{dx} = -\frac{1}{2} V^2 \frac{dC}{dx} \quad \text{Or, in } V = k\sqrt{F},\ k = \frac{1}{\sqrt{\frac{1}{2}\frac{dC}{dx}}}$$

Hence, if, for a given shank size, the curve of capacitance against spacing can be determined experimentally, the value of the constant relating voltage to the square root of force is determined. Unusually good equipment for such capacitance measurements was available and a complete set of tests was run.

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5. Approximately 10% per 0.1°C.

DESCRIPTION OF THE WORK

The following detailed description of the work will be divided into three parts: first, the determination of the effect of shanks by capacitance measurements, second, the determination of the effect of ground planes by measurement of force and voltage, and third, the analysis of the effect of ground planes by the method of images.

PART I

The Determination of the Effect of Shank Size by Capacitance Measurement.

It is felt that the importance of the method far transcends the results so far attained by its use. By means of it, the final calibration of any force-voltmeter may be obtained, with no regard for size, position, or uniformity of ground surfaces or shanks (except that the formation of corona must be avoided). Too much emphasis cannot be laid on the point that a calibration, determined by measurements on the actual meter in its permanent place, is vastly superior to one which is calculated from figures for an "ideal" arrangement modified by correction factors determined by any method.

Description of the Apparatus

Spherical, copper-plated, pressure floats, 3" in diameter, were used with shanks of $\frac{1}{4}"$, $\frac{1}{2}"$, and 1". The arrangement is shown in Figure 1, page 7. The floats were tapped for the $\frac{1}{4}"$ shanks, and these shanks, both 30" long, were used to support the spheres during all tests. When measurements of the larger shanks were to be made, sleeves of $\frac{1}{2}"$ (or 1") outside diameter and 12" long, were slipped over the $\frac{1}{4}"$ permanent shanks and centered by means of accurately turned washers. Between the two 4" glazed porcelain insulators supporting one shank, a wooden engineer's-scale was fastened so that one edge closely paralleled the shank. On this shank a vernier scale was scribed so that the position of the sphere could be determined to 0.005 inches.

The circuit used in measuring the capacitance is shown in Figure 2, page 7. The general scheme was to place a vernier condenser in parallel with the spheres, of which the capacitance change was to be measured, in one arm of a capacitance bridge, and to place a fixed capacitance in the other arm. Any change in the capacitance of the spheres must then be equalled by an opposite change in the capacitance of the vernier condenser before the bridge is restored to balance.

The fixed capacitance was a General Radio No. 222 precision variable condenser. Its setting was kept unchanged for the duration of the tests.

The vernier condenser was made by the Iowa State College Instrument Shop. It consisted of a stationary vertical outer cylinder and a concentric inner cylinder mounted on a micrometer lead screw having a pitch of .5 mm. per revolution. The circumference of the head was divided into 25 parts and the vertical travel was 50 mm. so that the total range of capacitance was divided into 2500 divisions. The total range was found to be about 220 Micro-Micro-farads so that each division was approximately 0.1 m.m.f..[6] Further, each division on the condenser covered a space of about 3 mm. so that tenths could be easily estimated. Hence values of capacitance were recorded down to 0.01 m.m.f.. The fact that the curves obtained are extremely smooth is good evidence that the figures are significant to this precision.

The capacitance bridge, a General Radio No. 216 precision type was completely shielded and had shielded transformers in both the oscillator circuit and the output circuit. The fixed resistance arms had 5,000 ohms in each and the variable resistance (decade type, from 10000 ohms

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6. The abbreviation "m.m.f." will be used for micro-micro-farad hereafter.

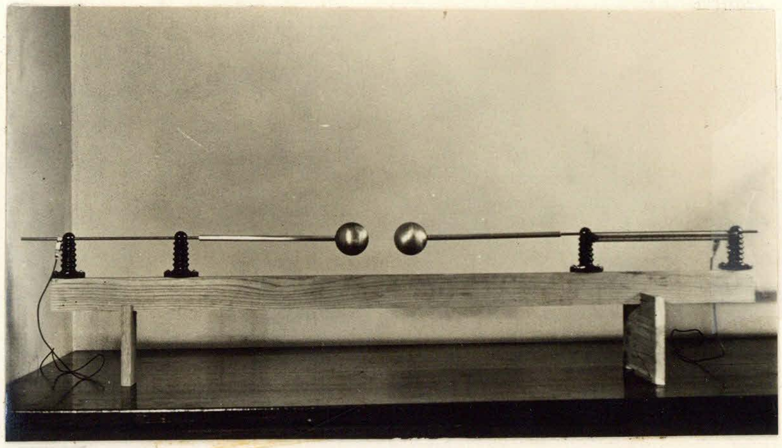


FIGURE 1
Arrangement of Spheres, $\frac{1}{2}$" Shanks in Place

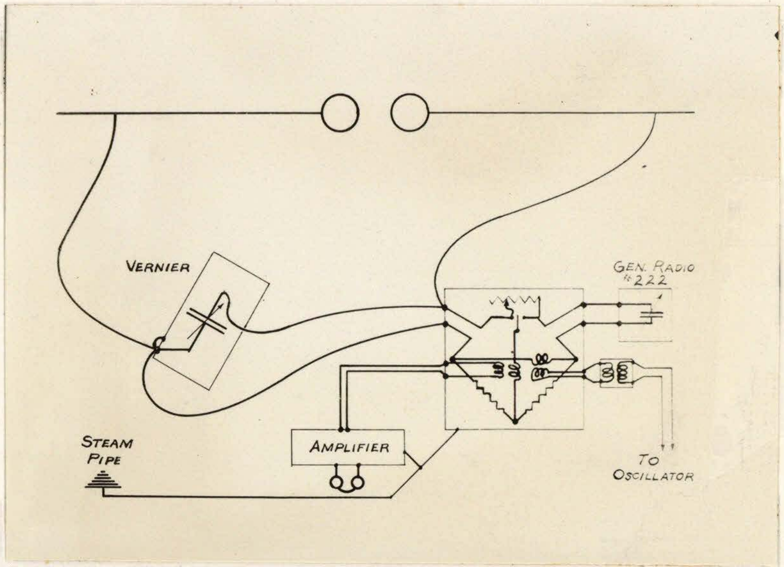


FIGURE 2
Wiring Diagram

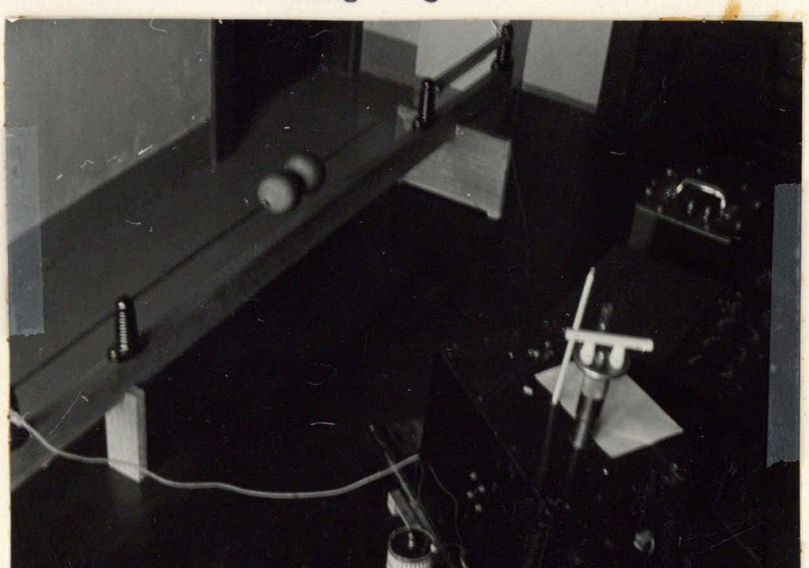


FIGURE 3
Arrangement of Apparatus

to 1 ohm) could be switched into either condenser arm.

The oscillator was a Western Electric type 8A, audio-frequency oscillator. It was about 40 feet from the bridge, in a separate room and on the floor above. A frequency of 1000 cycles was used thruout the tests. Power from the oscillator was led to the bridge over a twisted pair. It was found experimentally that a sharper balance could be obtained if a shielded transformer, with ungrounded case, was placed in the oscillator leads, close to the bridge. The transformer used was an "Amertran" Mixing Transformer. A similar one was also tried in the output circuit but since no improvement was observed, it was removed.

The amplifier was a Leeds and Northrup No. 210301 two stage audio-frequency amplifier which could be broadly tuned for 500, 1000 or 2000 cycles. The tubes used were one 201A and one 222, which were so extremely microphonic that the whole amplifier was placed on sponge rubber pads on a separate table. The amplifier was completely shielded, and this shield, together with the shield in the bridge, was grounded to the steam pipe. Early in the work it was found necessary to bridge a 0.02 m.f. condenser across the input to the amplifier in order to eliminate interference from the broadcast transmitter located in the same building. This condenser, having an impedance of 8000 ohms at 1000 cycles, undoubtedly reduced the sensitivity of the bridge but the amplifier more than made up the difference. As a matter of fact, it may well have been that the condenser reduced the sensitivity greatly when the bridge was far out of balance and had progressively less effect as the bridge was brought to balance. It would thus act somewhat as an automatic volume control. In any case, a much closer and more nearly reproducable balance could be obtained with the use of the amplifier and condenser than without them. The phones were Western Electric No. 509-W.

A photograph showing all the equipment with the exception of the amplifier, which is just out of the picture in the right foreground, is included as Figure 3, page 7.

Description of the Tests

The calibration of the vernier condenser was considered of paramount importance and hence it constituted the first task. For this work the General Radio No. 222 variable condenser was placed in parallel with the vernier condenser in one arm of the bridge, and a third condenser, the setting of which was kept constant thruout the calibration, was placed in the other arm. After a balance had been obtained, the No. 222 condenser setting was changed and the bridge was rebalanced by adjusting the vernier condenser. This process was repeated for about 50 points during which the complete range of the vernier was covered several times. It was found that the vernier condenser deviated measurably from a straight line for the first 20% of its range, but for the middle 60% out of a total of 43 points taken, 2 deviated from the best straight line by 1.5m.m.f. and all the others were within 0.7 m.m.f.. The range covered by this straight line portion was from 183.3 m.m.f. at setting 2000 to 325.6 m.m.f. at setting 500. Hence 1 division on the vernier is

$$\frac{325.6 - 183.3}{2000 - 500} = 0.09495 \text{ m.m.f.}.$$

From the beginning of the calibration work it was apparent that the General Radio No. 222 condenser had a noticeable blacklash. Backlash is, of course, undesirable in any precision equipment but it is particularly unfortunate and annoying when a null method is employed. To eliminate the effect of this backlash, at each point the bridge was unbalanced by moving the No. 222 condenser in a specified direction, and the rebalance was effected with the vernier condenser, which had no detectable backlash. To increase the accuracy further, the No. 222 condenser was used only within the range where its calibration was an exact straight line. Temperatures were recorded, and varied somewhat from day to day. No correlation between temperature and calibration was found.

The procedure in conducting the tests was as follows. (For wiring diagram refer to figure 2, page 7.) 1) The position of the left-hand sphere was adjusted so that the desired range of spacing could be attained by motion of the other sphere alone. 2) The right-hand sphere was then adjusted roughly to any convenient spacing (say 0.1") and its position, as read on the engineer's scale and the vernier scribed on the shank, was recorded. 3) The bridge was balanced with the vernier condenser and its reading recorded. 4) The right-hand sphere was moved to increase the spacing approximately one half inch, its position recorded and the bridge rebalanced with the vernier condenser. This process was repeated until the desired range of spacing had been covered, from 3" to 4.5", (that is, from 100% to 150% of the sphere diameter). The right-hand sphere was then returned to an initial spacing of approximately 0.2" and readings were taken at half inch intervals as before. The sphere was then returned to 0.3", 0.4", and 0.5" initial spacing and the process repeated. It should be noted that while the <u>adjustment</u> of the sphere at each point was only approximate, its <u>position</u> was known and recorded to 0.005". At the end of a complete series, the amplifier was turned off and the right-hand sphere was carefully moved toward the left. When it touched the other sphere, the bridge was violently unbalanced and the 1000 cycle note could be heard. The position of the right-hand sphere was recorded, and this reading was subtracted from all points to get the spacing. The results were plotted as a curve of vernier setting against sphere position on which the points were approximately 0.1" apart, though the range had been covered, not once, but five times. This process was followed to eliminate any changes in condenser calibration or, in fact, any error which might be changing continuously thruout the run. When such changes occurred, as they did in many of the early tests, they were immediately detected by the appearance of "waves" having a wavelength of 5 points, in the final curve. It was found that keeping the room temperature constant within, say, 2 degree C. for 6 hours preceding each test and during it, eliminated the difficulty. This indicates that there was a variation of some capacitance with temperature, notwithstanding the contrary conclusion reached during the calibration, in regard to the vernier condenser. Rough calculations immediately eliminate the possibility of the expansion of the shanks changing the spacing of the spheres in sufficient amount, and while both the vernier condenser and the No. 222 condenser in the other arm are open to question, it is considered more probable that the vernier condenser was at fault. The effect was so small as to be negligible as far as calibration was concerned while still of importance in making the position of the capacitance-spacing curve doubtful. The data of a typical run is included in Appendix I.

To find $\frac{dC}{dx}$ from curves of vernier condenser setting versus sphere position, it was necessary to construct a curve with ordinates the slopes of the original curve, and then to apply the calibration factor of the vernier condenser. These slopes were found by means of a plane metallic mirror about 10" long, which was placed across the curve so that its reflecting face was normal to the plane of the paper. The horizontal angle between the mirror and the curve was then adjusted by rotating the mirror, until the curve on the paper and its image in the mirror had no discontinuity of slope at the mirror edge. If the observer now views the curve and its reflection at an angle of 5 or 10 degrees to the paper, the position of the mirror can be determined with considerable precision, and a line can be scribed using it as a straightedge. From this line, which is normal to the curve, the slope of the tangent can be calculated. The values of the tangents found by this process were plotted on the same sheet with the original curve of vernier condenser setting versus sphere position.

The final sets of data, taken after several weeks of trials, consisted of two complete runs for each size of shank, $\frac{1}{4}$", $\frac{1}{2}$", and 1". Each run consisted of points at 0.1" intervals over a 4.5" range of spacing (150% of sphere diameter). The values of slope from the two curves for one shank size were plotted on the same sheet, and a smooth curve was drawn thru them. From this curve values of k were computed as follows:

In $V = k\sqrt{F}$, k should preferably be of such dimensions that with F in grams, V will be in kilovolts. When, on page 4, the equation $E = \frac{1}{2} CV^2$ was written, the units implied were

E = energy in ergs

C = capacitance in stat-farads (= centimeters)

V = voltage in stat-volts.

When the derivative with respect to distance is taken,

$F = -\frac{dE}{dx} = -\frac{1}{2} V^2\frac{dC}{dx}$, where F is in dynes if x is in centimeters.

Dimensionally, this equation is, then,

$$\text{Dynes} = \frac{1}{2}\,(\text{stat-volts})^2 \cdot \frac{\text{stat-farads}}{\text{cm.}}$$

But it is desired to have:

F in grams

V in kilovolts

$\frac{dC}{dx}$ in micro-micro-farads per cm.

If the symbols F, V, C, and x are re-defined in these units, the dimensional equation above must still be satisfied and there results

$$F \cdot 980.6 = \frac{1}{2}\left(\frac{1}{.2998}\right)^2 \cdot 10(.2998)\,\frac{dC}{dx}$$

From which

$$k = \frac{V}{\sqrt{F}} = \sqrt{\frac{980.6 \times 2 \times (.2998)^2}{10(2998)^2\,\frac{dC}{dx}}} = \frac{14.00}{\sqrt{\frac{dC}{dx}}}$$

This is the constant of general applicability but one further step will be required to find k in terms of the ordinates of the last curve plotted from experimental data. The ordinate of this curve (call it "N") is in vernier divisions per inch. It was shown on page 9 that 1 division = 0.09495 m.m.f., and 1 inch = 2.54 cm.; hence

$$\frac{dC}{dx} = \frac{N \times (0.09495)}{2.54}$$

$$\text{And } k = \frac{14.0}{\sqrt{N}\left(\frac{0.09495}{2.54}\right)^{\frac{1}{2}}} = \frac{72.4}{\sqrt{N}}$$

Values of k found in this manner for the 3 shank sizes are plotted against $\frac{\text{Spacing}}{\text{Diameter}}$ in curves C, D, and E of Figure 4.

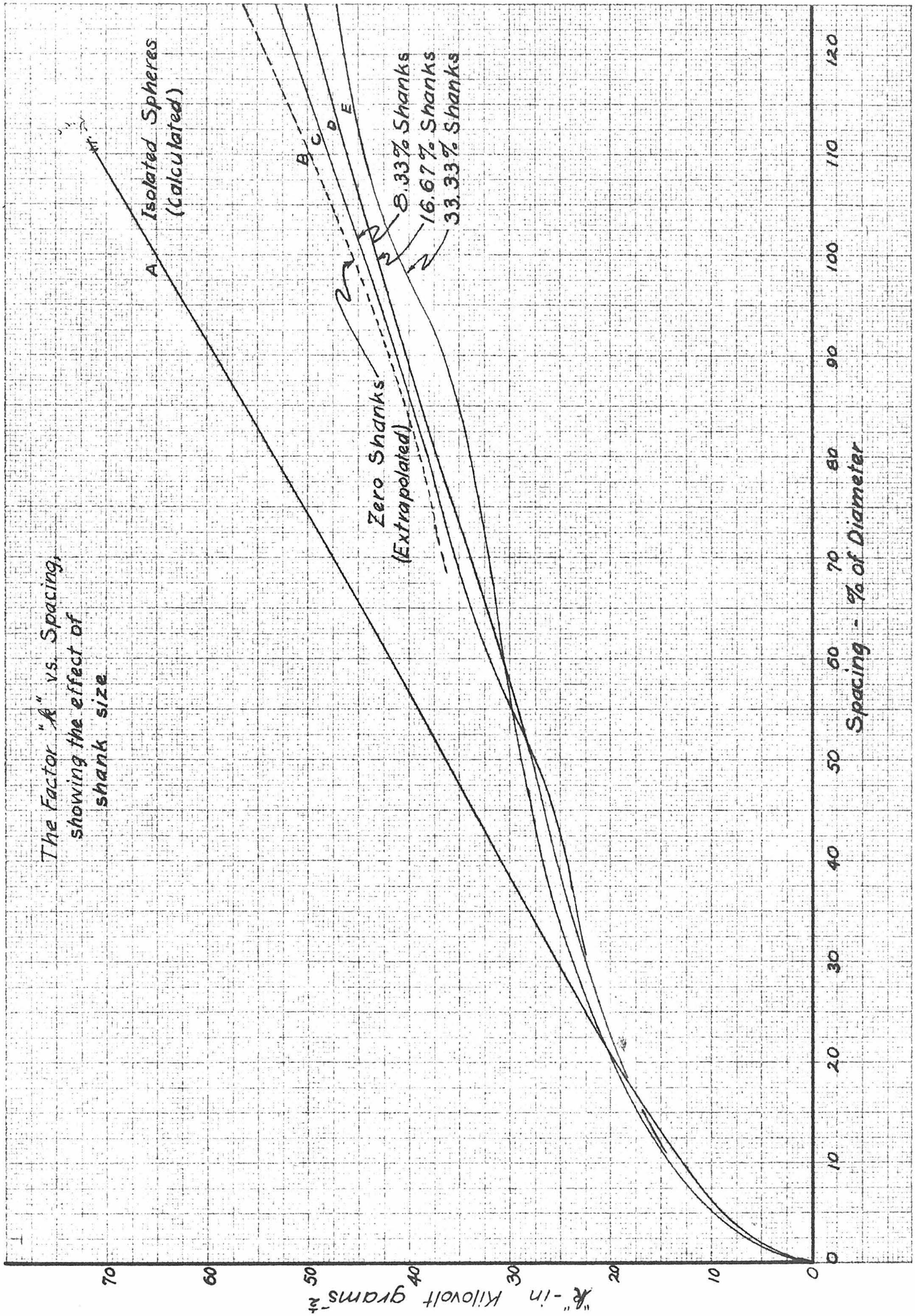
Results of Capacitance Measurements to Determine

The Effect of Shank Size.

The results of the tests described are shown graphically in Figure 4, page 14, on which values of k, in $V = k\sqrt{F}$, are plotted against the ratio of Spacing to Diameter. Curves C, D, and E represent the experimental results for shanks of $\frac{1}{4}$", $\frac{1}{2}$", and 1" respectively. (Since the spheres were 3" diameter these shank sizes correspond to 8.33%, 16.67% and 33.33% of the sphere diameter.) Curve A is the theoretical curve for isolated spheres (from Lord Kelvin's data, footnote 2, page 2). Curve B is the extrapolation of curves C, D, and E to zero shank size. The comparison of curves C, D, and E with curve B is the essential result. From them it is seen that a shank size of 5% of the sphere diameter (which is 2" for 1 meter spheres) results in a reduction of k of approximately 2%, for spacings from 70% to 115% of the sphere diameter.

If there had been no equipotential surfaces other than the spheres and shanks in the field, curve B should have coincided with curve A. That they do not coincide results from the proximity of the vernier condenser to the movable sphere. The case of this condenser was connected to the stationary (left-hand) sphere. Hence when the right-hand sphere was moved, the resultant capacitance change had two parts, 1) the change in capacitance between the spheres, and 2) the change in capacitance between the right-hand sphere and the vernier condenser case.

FIGURE 4


The Factor "k" vs. Spacing, showing the effect of shank size
Isolated Spheres (Calculated)
A
B
C
D
E
8.33% Shanks
16.67% Shanks
33.33% Shanks
Zero Shanks (Extrapolated)
Spacing - % of Diameter
"k"-in Kilovolt grams
0
10
20
30
40
50
60
70
80
90
100
110
120

Since the right-hand sphere corresponds to the grounded one in an actual voltmeter, connecting the vernier condenser case to the other sphere corresponds to the addition of a large high-voltage body in the actual meter. Its effect, in this instance, well illustrates that bringing in any body at high voltage, other than the one sphere and its shank, changes the calibration of the voltmeter.

The fact that the curves cross at about 55% spacing has not been explained. Nevertheless it is not believed to indicate solely lack of precision because the two curves run for each shank are in agreement to somewhat smaller values. It is estimated that these curves are reproducable to within 1%.

PART II

THE DETERMINATION OF THE EFFECT OF GROUND PLANES BY MEASUREMENT OF FORCE AND VOLTAGE.

Description of the Apparatus

The use of a model arrangement rather than the use of the 1 meter spheres was decided upon primarily because of the comparative ease with which different arrangements of ground planes could be set up.

A pair of 12.5 cm. copper spheres which had been in use as a sphere gap were chosen. These spheres were set up inside a wooden cage 8' high by 10' long by 6' wide as shown in Figure 5, page 17. The high voltage one was mounted so that the spacing could be varied, but in addition, so that when it was clamped in place its position was fixed. The $\frac{5}{8}$" threaded and calibrated shank about 14" long, which was attached to it in the sphere gap, was retained. The metal shank of the other sphere was replaced by one of wood about 5' long. This sphere and shank were mechanically suspended by a double "V" suspension exactly similar to the large voltmeter, and the sphere was grounded by a No. 20 wire running along the wood shank and connecting to a coiled pigtail at the rear "V" support. Close to the rear end of the wood shank a brass ring was mounted, across the vertical diameter of which a 0.5 mil tungsten wire was stretched. This wire was observed with the cathetometer to detect motion of the sphere. A fluid damper proved to be of great aid in speeding observations. The vane was attached to the shank, and the container, a cocoa can, rested on the frame. The details will be made clear by a glance at Figure 6.

In measuring the force it was considered better, from the standpoint of accuracy and speed, to read the sphere deflection in the cathetometer and to calculate the force from these deflections by use of a calibration curve. For this calibration, and also to prevent too

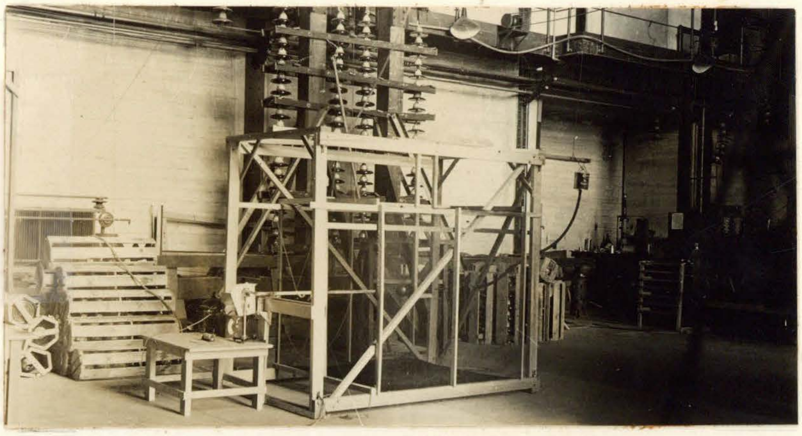


FIGURE 5

Model Voltmeter

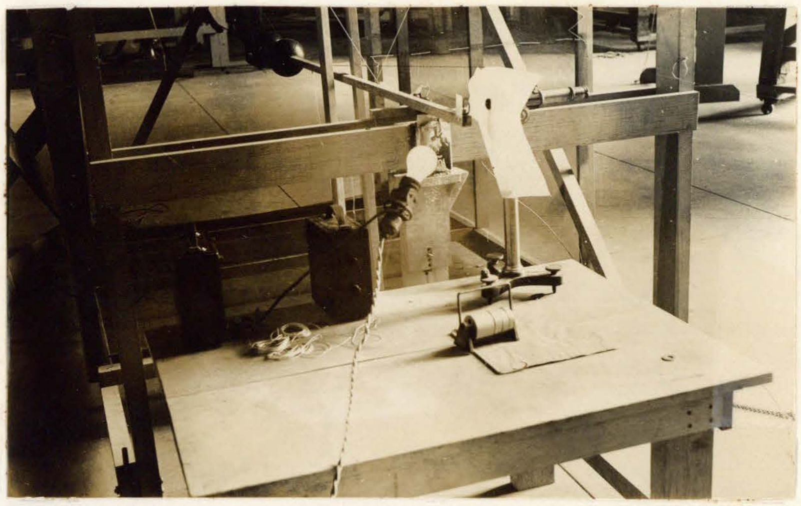


FIGURE 6

Details of Arrangement

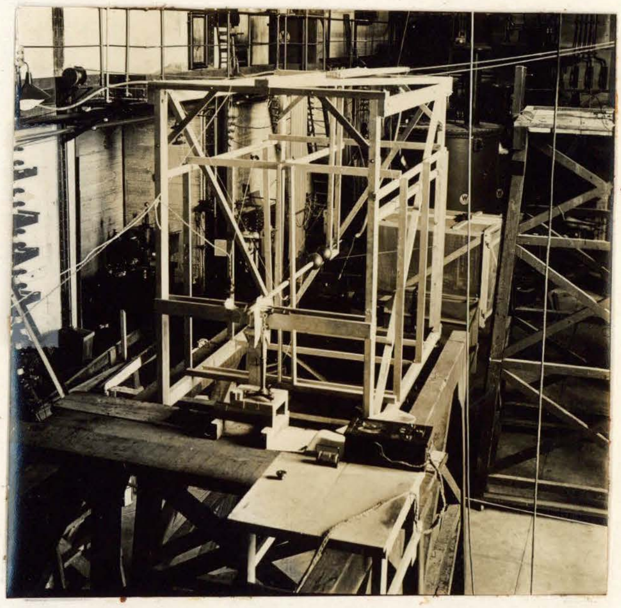


FIGURE 7

Model Voltmeter on Tower

large deflections, a means of applying a horizontal force was required. The method used was to suspend a weight from two equal threads, of which one was attached to the wooden shank close to the front "V" support, and the other to the stationary frame at the rear. Care was taken to place the two points of support on the same level and to make the distance between them approximately equal to the length of each thread. From measurements of the three distances, the horizontal force due to any weight could be calculated.

With this arrangement, horizontal forces from 1 to 5 grams were applied to the grounded sphere and the deflections in the cathetometer were observed. The calibration curve so obtained was accurately linear. Though the weights were within only about 3% of their marked values, each one was carefully weighed on a sensitive balance to the nearest milligram, and this value was used in calculating the horizontal force. The maximum actual deflection of the sphere, if the cross wire was to remain in the cathetometer field of view, was 2 mm..

The source of voltage was Unit No. 1 of the One Million Volt Cascade transformers. This unit has a maximum secondary voltage of 250 KV. It contains a tertiary (voltmeter) winding having a turn ratio with respect to the secondary of 1 : 833; hence 300 volts on the tertiary coil represented 250 KV on the secondary. Since these transformers were designed to have very low reactance (1.24% impedance by test), a series resistance to limit flashover current was an absolute necessity. The resistance used was five feet of $\frac{3}{4}$" garden hose filled with tap water (about 8,000 ohms per foot) which can be seen at the rear of the cage in Figure 5. Considerable care was used to be sure that flashover did not occur, particularly after the first time, when it was discovered that the water column acted more like an expulsion cutout than like a protective resistance.

The electrical cage consisted of enameled iron fly screen in the form of an open-ended cube. Each face was 29.5" (or 6 diameters) from the centers of the two spheres. The end into which the high voltage lead entered was omitted.

Description of the Tests.

At the outset of the tests it was recognized that if there was any uncertainty regarding the transformer's secondary voltage in terms of its tertiary coil voltage, the results of the tests could be only comparative; that is, only the percent change in k (in $V = k\sqrt{F}$) when the grounded cage was brought up, could be determined. On the other hand, if the true voltage applied to the spheres could be determined with satisfactory accuracy, the numerical value of k under each condition could be found.

The unusually low reactance of the transformer should indicate the maintenance of the nominal ratio between tertiary and secondary over the full load range within a fraction of 1 percent, and since the transformer load during these tests (simply charging current) was less than 100 m.a., compared to the rated current of 1 ampere, the ratio should have been maintained within even closer limits. In addition, since the leakage reactance is essentially independent of the flux density and since the charging current is directly proportional to the voltage (up to the beginning of corona), the IX drop should be exactly proportional to the voltage. Hence if the calibration curve of secondary KV against tertiary coil volts deviates from the turn ratio at all, it should remain a straight line with merely a slightly different slope. Since the charging current leads the voltage, the IX drop adds to the induced voltage and the final conclusion is, then, that if the ratio of secondary KV to tertiary volts, which is constant in any case, differs from the turn ratio at all, then it must be higher by not more than 1%.

The conclusion from theoretical considerations, that for any maintained arrangement the force between two bodies is exactly proportional to the square of the voltage, seems to require no qualifying limitations to its practical application, except, perhaps, the statement that the formation of corona is electrostatically equivalent to an increase in size of the conductor. Thus, with no knowledge of the numerical value of k, it was possible to use the model spheres to check the linearity of the ratio of secondary KV to tertiary volts.

The first tests indicated exact linearity between F and v^2, up to v = 100 (V = 83.3 KV) approximately.[7] At about this value a slight but definite change in slope occurred corresponding to a reduction in k of 3 or 4 %. After this break was observed it was noted that audible and visible corona occurred on the high voltage lead at v = 58.5 volts (V = 49 KV), and on the shank at v = 123.5 volts (V = 103 KV). All tests run after this discovery were carried only up to v = 100 volts.

The procedure on a typical run was as follows. With the cathetometer adjusted so that the wire was near the center of the field and its reading taken, the sphere to be connected to the high potential was moved over until it touched the suspended one. The calibrated shank was read and the sphere was moved away to approximately the desired spacing, when the shank was read again. After rechecking the zero cathetometer reading, the voltage was applied, the cathetometer again read, and the tertiary coil voltmeter read. The voltage was increased and readings of cathetometer and voltmeter recorded again. When the wire approached the limit of the cathetometer field, a 5 gram weight was hung from the double thread support described on page 18, and the cathetometer read again before any further change in voltage was made. The voltage was then increased in steps until the wire had again moved to the edge of the cathetometer field when the 5 gram weight was replaced by one of 10

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7. The lower case "v" will be used for tertiary coil voltage, in contradistinction to the upper case "V" which will continue to be used for kilovolts applied to the meter (that is, secondary KV).

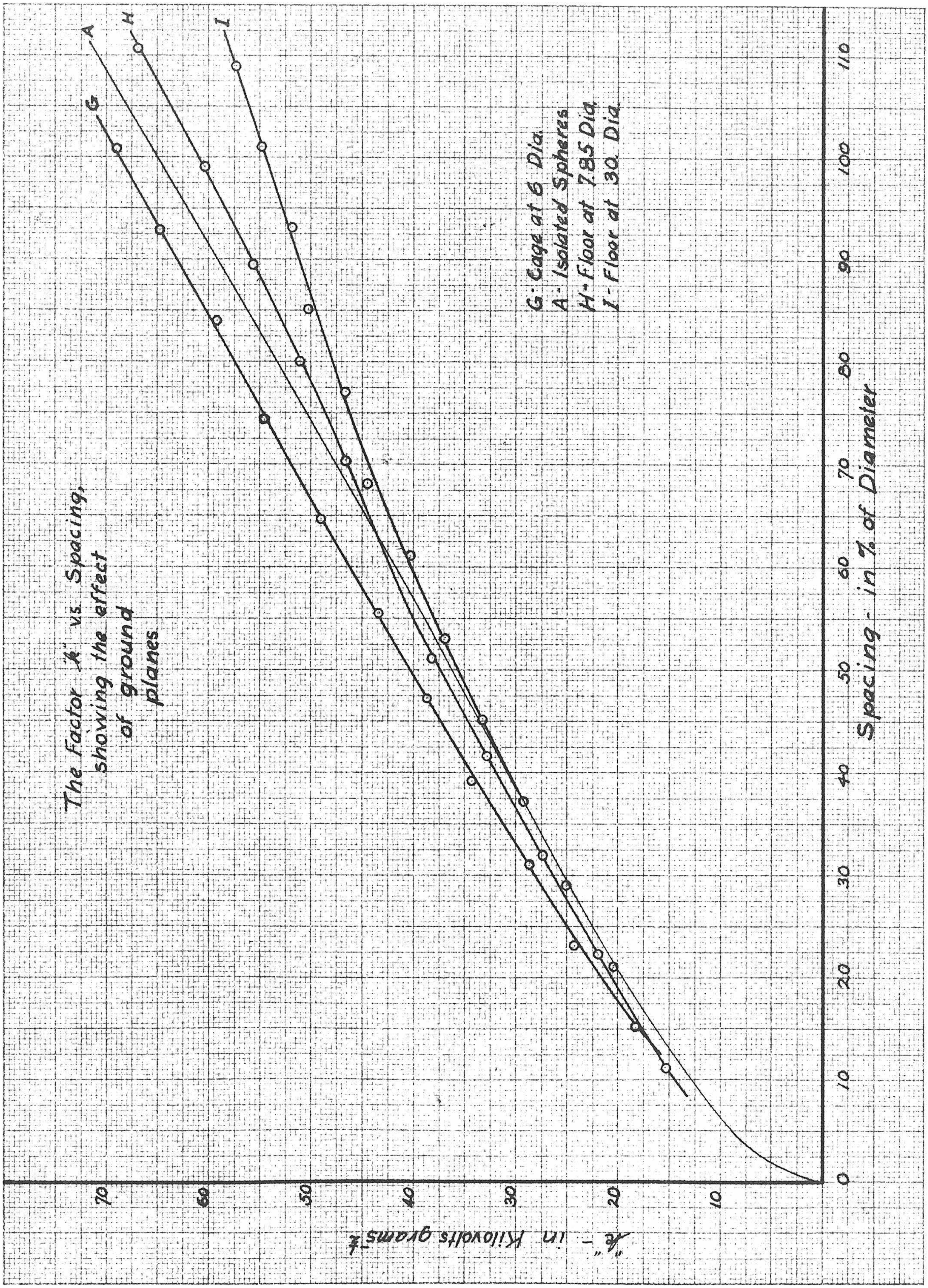
grams. For short spacings it was sometimes necessary to make a third traverse across the cathetometer field with a total weight of 15 grams. (The ratio of horizontal force applied to the sphere, to weight hung from the thread, was 0.3 to 1, hence the horizontal force due to a 5 gram weight was 1.5 grams.) Since no break in the curve of v^2 against F was noted when the 5 grams weight was added (thus changing the spacing slightly), it was assumed that no correction for this change in spacing need be made. When the curve of v^2 with respect to F was plotted, the slope, with the assumption that the turn ratio between v and V was valid, gave a single value of k for the spacing used, and for that particular arrangement of ground planes.

The spacing was then changed, while keeping the arrangement otherwise constant, and a new value of k determined. Eleven runs were made, from which eleven values of k at different spacings could be determined, for each arrangement of ground planes. Three general arrangements were investigated: 1) no cage, the only close ground plane being the floor which was distant 7.85 diameters, 2) grounded cubical cage (except for one end), each side being distant 6 diameters, 3) no cage, apparatus being on a wooden tower of such height that the floor was distant 30 diameters. On this last setup, it was noted only after the data had been taken that one of the 1 meter spheres, which was about 15 diameters to one side and about 6 diameters behind the model voltmeter, had been connected to the high voltage during this test only.

Results of the Determination of the Effect of Ground Planes By Measurements of Force and Voltage.

The results are shown graphically in Figure 8 on page 22. On the hypothesis that the use of the turn ratio of 833 : 1 is a valid means of determining secondary voltage from tertiary coil voltmeter readings, numerical values of k are significant, as is also the comparison of the experimental curves G, H, and I with the theoretical curve A, for

FIGURE 8


The Factor 'k' vs. Spacing, showing the effect of ground planes
G: Cage at 6 Dia.
A - Isolated Spheres
H - Floor at 7.85 Dia
I - Floor at 30 Dia.
G
A
H
I
Spacing - in % of Diameter
0
10
20
30
40
50
60
70
80
90
100
110
"k" - in Kilovolts grams
10
20
30
40
50
60
70

isolated spheres. Evidence that the use of the turn ratio does not introduce very serious error is that a sphere gap calibration curve run by W. A. Lewis shows a deviation from the turn ratio of 1.5% at 70 KV. A similar calibration by sphere gap (described in Appendix II) which was run at the conclusion of the force and voltage measurements shows a deviation of 3.5% at 70 KV. When it is recalled that each value of k is determined by the average slope of a Voltage-Force curve on which nearly all the points lie below 70 KV, it is seen that these deviations would cause errors of much smaller amounts.

The most interesting of the curves of Figure 8 is I, for the tests made on the tower. During these tests the nearest ground plane, the floor of the laboratory, was at a distance of 30 diameters. The all-important question is, why are the values of k so far below curve A for isolated spheres (that is, why is the force for a given voltage so much higher than would be expected)? The explanation offered is that the effective high voltage electrode includes much more than the one sphere. It appears that the effect of the shank, the lead, the hose resistance, and the one-gallon can at its upper end, as well as the 1 meter sphere mentioned earlier, was more important than had been supposed. This conclusion is fully born out by the similar results of the capacitance measurements, discussed on page 13.

Curve H, in which the disturbing effect of the 1 meter sphere is absent, and in which the ground plane (floor) at 7.85 diameters tends to increase k, is still below the curve for isolated spheres. This indicates definitely that the increased effective size of the high potential electrode was a factor which cannot be neglected.

Curve G, for the open cube at 6 diameters indicates an increase in k over curve H, of 9 to 12% between spacings of 40 to 100% of diameter. These values are of particular interest because the cage at 6 diameters roughly approximates the High Tension Laboratory enclosing the 1 meter spheres (6 diameters, for 1 meter spheres, is 20 feet).

PART III

Analytical Treatment of the Effect of Grounded Bodies.

From qualitative arguments, it is clear that the disturbing effect of all ground planes and shanks will become of greater importance as the spacing of the spheres is increased. Since other and quite satisfactory means are available for the measurement of voltages up to 300 KV, no great interest is attached to a knowledge of these disturbing effects below 20 cm. spacing for 1 meter spheres. On the other hand, the maximum force for voltages just below flashover decreases rapidly for spacings larger than one diameter. The disturbing effects are therefore calculated for a spacing of one diameter, with the understanding that the results will be maximum values which will never be exceeded in practice. A further reason for choosing so large a spacing is that the infinite series of images converges rapidly and hence only a few terms need be considered.

The effect of a single infinite ground plane, parallel to the line of centers of the two spheres and distant from it 5 diameters will be calculated, the infinite series for the effect of four planes forming an infinitely long prism of square cross-section will be developed, and the effect of enclosing spherical cages of ten and twenty times the diameter of the voltmeter spheres will be found. To make the method clear it will be applied first to the case of two isolated spheres of unit radius (8) at a spacing of 2(= 1 diameter), one sphere being at potential 1 and the other at potential zero.

Force Between Two Isolated Spheres at a Spacing of 1 Diameter.

The basis on which the theory of images rests is that if the real conductors are removed and an artificial system of charges is set up such

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8. That is, the radius will be taken as the unit of length. Hence the spacing is 2, and the center-to-center distance is 4.

that equipotential surfaces are created which exactly replace the real conductors, then the field outside the equipotentials will be identically the same as the field outside the conductors. It is necessary, then, to remove the two spheres and to replace them with a set of charges which will produce identical equipotential surfaces.

A unit charge, placed at the center of the sphere at potential 1, will produce a spherical equipotential of unit potential at unit radius, but it produces an unwanted field at the position of the grounded sphere. It is easily shown that a spherical equipotential of zero value and radius a will be produced by any charge e at radius r and a charge $- e \cdot \frac{a}{r}$ at radius $\frac{a^2}{r}$ on the straight line connecting the center to charge e. Hence a charge $- 1(-\frac{1}{4})$ (= .250) placed at a radius $\frac{1^2}{4}$ (= .250) from the center of the grounded sphere will combine with the original charge to produce an equipotential which exactly replaces the grounded sphere. But this charge has distorted the equipotential which replaced the sphere at unit potential. To correct this difficulty a third charge must be added, whose value is $- (-.250) \frac{1}{(4 - .250)}$ (= + .06667) at a radius $\frac{1}{4 - .250}$ (= .2667). The addition of this charge has distorted the grounded sphere and hence another charge must be added inside it. A systematic tabulation will be necessary for clearness; let the original unit charge be labelled q_o, the first image be q_1, its image be q_2, etc., and let the distance from the center of the nearest sphere to charge q_n be r_n. The following tabulation results:

	Sphere at Unit Potential		Sphere at Zero Potential	
Charge number (n in q_n)	Value of Charge	Radius	Value of Charge	Radius
0	1.00000	0.0		
1			-.25000	.25000
2	.06667	.26667		
3			-.01786	.26786
4	.00478	.26794		
5,7,9,11			-.00138	.26795
6,8,10,12	.00037	.26795		
	1.07182		-.26924	

$$q_n = - q_{n-1} \frac{1}{(4 - r_{n-1})}$$

$$r_n = \frac{1^2}{(4 - r_{n-1})}$$

It is seen that some of the charges have been combined. This has been done because they are so close together as to be indistinguishable in the computation of forces.

To find the force between the spheres, it is only necessary to find the sum of the forces between all the charges within one sphere and all the charges within the other. The attractive force between two charges is of course, minus their product divided by the square of the distance between them. Thus:

$$f_{nm} = - \frac{q_n \cdot q_m}{(d_{nm})^2} = - \frac{q_n \cdot q_m}{(4 - r_n - r_m)^2}$$

For example, the force between q_2 and q_3 is

$$f_{23} = - \frac{(.06667)\cdot(-.017857)}{(4 - .26667 - .26786)^2}$$

$$= .000099$$

The results of this and all similar calculations are tabulated:

Charge Numbers	Force
0 - 1	.017778
0 - 3	.001282
0 - 5,7,9,11	.000099
2 - 1	.001374
2 - 3	.000099
2 - 5,7,9,11	.000008
4 - 1	.000099
4 - 3	.000007

Total force = .020746, which may be rounded off to .02075.

In regard to units, if the potential of the one sphere is one stat-volt, and its radius is one centimeter, then q_o must be one stat-coulomb, and all forces are in dynes. It should further be noted that, if the radius of the spheres is C centimeters, then the n^{th} charge becomes Cq_n, and distance d_n becomes Cd_n.

And $f = -\frac{(Cq_n)\cdot(Cq_m)}{(Cd_{nm})^2} = -\frac{q_n q_m}{(d_{nm})^2}$ in dynes, just as before.

The force is thus independent of the radius of the spheres (as long as the relative spacing, and the potential is maintained). If the potential is E stat-volts, charge q_o becomes Eq_o, and the force becomes E^2f. The force in dynes between two spheres at any spacing, S, one at potential E and the other at zero, becomes, then,

$$f = E^2 f_o(S)$$

where $f_o(S)$ is the force on two spheres at spacing S, with one at unit potential and the other at zero; that is, $f_o(S)$ is the value for spacing S corresponding to the figure .02075 which has just been calculated for spacing 2.

If the force is expressed as F grams, and the potential is V kilovolts, these changes in units will require this equation to be rewritten

$$f_{(dynes)} = F\ (980.6) = E^2 f_o(S) = \left[\frac{V}{0.2998}\right]^2 f_o(S)$$

$$\text{or } V_{(kv)} = \sqrt{\frac{F\ (980.6)\cdot(.2998)^2}{f_o(S)}}$$

$$V = \frac{9.389}{\sqrt{f_o(S)}}\cdot\sqrt{F}$$

$$\text{Or in } V = k\sqrt{F}, \quad k = \frac{9.389}{\sqrt{f_o(S)}}$$

For the example chosen, isolated spheres at 1 diameter spacing,

$$k = \frac{9.389}{.02075} = 65.1 \text{ KV. gm.}^{\frac{1}{2}}$$

The Effect of One Infinite Ground Plane.

The value of f_o will now be calculated for the case of two spheres at 1 diameter spacing with the addition of a ground plane parallel to the line of centers and 5 diameters distant from it. As before, let the radius of the spheres be the unit of length.

Following the same argument, if the spheres and plane are removed, and replaced by the set of charges found before, there will be equipotentials of the proper values to replace the spheres, but the plane will not be so replaced. To produce the desired plane equipotential, for each charge in front of it, an equal and opposite charge must be placed behind it. Each such image charge must be on the extension of the perpendicular from the original charge to the plane, and as far behind the plane as the original charge is in front of it. If a set of charges identical to those used to produce the spherical equipotentials except of opposite sign, is so placed behind the plane, then the plane becomes an equipotential, but the spheres are now distorted. This second set must now be re-imaged inside each sphere, and each new charge in one sphere must then be imaged in the other sphere. All these charges will have distorted the plane, and the whole process must be repeated.

The computation is greatly simplified if advantage is taken of some approximations which introduce only a very small error in the calculation of the force. The charges to be placed behind the plane are 20 units from the spheres (plane is at 5 dia. = 10 radii) and hence the re-image of each of these charges within the sphere is at $\frac{1^2}{20} = .05$ units from the center. Since, furthermore, this distance is nearly perpendicular to the line of centers, the effect on the other sphere of an accurately placed charge will be indistinguishable from the effect of a charge of equal value, but placed at the center. If all the re-images in each sphere are to be placed at its center, then the charges behind the plane may as well be lumped together at once. One further approximation should be made. It is almost obvious that charges whose positions differ only 1 or 2 in the third significant digit will have an effect indistinguishable from that of a single charge of their lumped value at their mean position. Thus in the table on page 25, charges 2 and 4 should be lumped with 6, 8, 10, 12; and charge 3 with 5, 7, 9, 11. When these combinations are made, that table becomes:

	Sphere at Unit Potential		Sphere at Zero Potential	
Charge number	Value of Charge	Radius	Value of Charge	Radius
0	+ 1.00000	0.0		
1			-.25000	.250
2 - 12	+ .07182	.267		
3 - 11			-.01924	.268
	+ 1.07182		-.26924	

The net charge in the two spheres is + 1.07182 - .26924 = +.80258. Hence the lumped charge behind the plane will be -.80258, and the re-image of this in each sphere is $- (-.80258)\,\frac{1}{20}$ = +.04013. It now appears necessary to repeat the calculations for the two isolated spheres, starting first with q_o = .04013 at the center of the grounded sphere. If these calculations are made, it will be discovered that the net charge on the two spheres is no longer .80258 but some larger value. Hence the value .04013 is not correct and another series of calculations must be carried thru. There is a short-cut, however, by which much of this labor can be avoided. It consists in noting that the effect of starting with a charge + q_o at the center of one sphere results finally (in the case of isolated spheres) in a total charge on that sphere of 107.2% of q_o, and in a total charge on the other sphere of - 26.9% of q_o. Now let q_o = 1.0 and let α be the re-image charge (.04013 in the first approximation above). Superimposing charge α at the center of the unit potential sphere will add 1.072α to its final charge and will add -.269α to the charge on the grounded sphere. Similarly, placing charge α at the center of the grounded sphere will add + 1.072α to its charge, and -.269α to the charge of the other one. The total charge on the unit potential sphere is now 1.072 + 1.072α - .269α, and that on the grounded sphere is -.269 + 1.072α - .269α.

The net charge on both spheres is $(1.072 - .269)(1 + 2\alpha)$. But α is to be $-(-\frac{1}{20})$ times this net charge,

Hence

$$\alpha = \frac{1}{20}(1.072 - .269)(1 + 2\alpha)$$

$$= .04013 + .08026$$

$$\alpha = \frac{.04013}{.9197} = .0436$$

Some detailed calculations must be made to find the force, but they need be done only once and in these calculations the figures of the table on page 29 may be used as percentages so that much of the tedious part can be avoided. The start will be made with $q_o = (1 + \alpha) = 1.0436$, and all charges closer than .001 will be combined.

	Sphere at Unit Potential		Sphere at Ground Potential	
Charge Number	Value of Charge	Radius	Value of Charge	Radius
0	+ 1.0436	0.0		
1			-.2609	.250
2 to 12	+ .0749	.267		
3 to 11			-.0201	.268
0'			+.0436	0.0
1'	- .0109	.250		
2' to 12'			+.0031	.267
3' to 11'	- .0008	.268		
	+ 1.1068		-.2343	

The net charge is 1.1068 - .2343 = .8725; hence the image behind the plane should be -.8725 and the re-image within each sphere(α) should be $-(-.8725 \frac{1}{20}) = .0436$, check. The forces may be calculated just as before, except that the forces due to the charges behind the plane need to be considered now. For this purpose, the lumping of the charges of opposite signs will not be allowable because the images of those in the grounded sphere produce no component of force (on the grounded sphere) in the direction of the line of centers. The desired component of the force due to the charges behind the plane is

$$f_i = -\frac{(-.2343)(-1.1068)}{4^2 + (20)^2} \cdot \frac{4}{\sqrt{4^2 + (20)^2}}$$

$$= -.00012$$

A further saving of computation time may be made by noting that each of charges 0 to 12 are increased over those in the table on page 29 by 4.36%; the total force between them must then be that found before, (.02075), times $(1.0436)^2$. Similarly, charges 0' to 12' are just 4.36% of these values and the forces due to them are therefore $(.0436)^2$ times the original force. Thus the only forces that need to be calculated by $\frac{q_n q_m}{(d_{nm})^2}$ are the forces between the unprimed charges and the primed ones. The forces as tabulated below, are for convenience separated into attractive (+) and repulsive (-).

Charge numbers	Force: Attractive	Force: Repulsive
(0 to 12) - (1 to 11)	.02259	
(0' to 12') - (1' to 11')	.00004	
1 to 11 and 0' to 12' - Image		.00012
0 - 0'		.00285
0 - (2' to 12')		.00023
1 - 1'		.00023
0 - (3' to 11')		.00002
2 - 0'		.00024
	+ .02263	-.00369

Net attractive force with ground plane .02263 - .00369 = .01894 dynes

Net attractive force with spheres isolated = .02075

Net reduction of force due to the ground plane = .00181 dynes

= 8.7%

The Effect of Four Ground Planes.

The next problem is to find the force by the method of images when the two spheres, still at 1 diameter spacing, are enclosed by four ground planes parallel to the line of centers, forming a rectangular prism of square cross-section. This case is of some importance because real planes of finite width may be treated mathematically as if they were of infinite extent. The mathematical difficulties of dealing with the "edge effects" of finite planes are thus avoided while at the same time the inaccuracies ordinarily accompanying the change from theory, involving infinite planes, to practice, involving finite ones, are also avoided.

It will be assumed, as in the previous case, that the planes are at sufficient distance from the spheres that the sets of charges in the two spheres may be combined into one in the first image behind each plane. If a cross-section thru the center be taken, it is seen that the whole plane must be sprinkled with images to make the planes equipotentials. (See Figure 9, page 34.) If α is the distance between adjacent images (in terms of sphere radius as a unit) and + q is the net charge on the two spheres, then the re-image within either sphere of any one of the first four images is $-(-q\frac{1}{\alpha}) = \frac{q}{\alpha}$. Since there is symmetry about both the horizontal and the vertical axis, the effect of all the images will be 4 times the effect of all those in one quadrant. The lower right-hand quadrant will be used. The re-image of the change in row 0, column n, is

$$-\left[(-1)^n q\right]\frac{1}{n\alpha} = (-1)^{n+1}\frac{q}{n\alpha}$$

The sum of the re-images of all the charges in row 0 is

$$Q_0 = \frac{q}{\alpha}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

The re-image of the charge in row 1, column n is

$$-\left[(-1)^{n+1} q\right]\frac{1}{\sqrt{\alpha^2+n^2\alpha^2}} = (-1)^n\frac{q}{\alpha}\frac{1}{\sqrt{1+n^2}}$$

and the sum of the re-images of all the charges in row 1 is

$$Q_1 = \frac{q}{\alpha}\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{1+n^2}}$$

The sum of the re-images of all the charges in row s is

$$Q_s = \frac{q}{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^{s+n+1}}{\sqrt{s^2+n^2}}$$

Combining the re-images for all the rows:

$$Q = \sum_{s=0}^{\infty} Q_s = \frac{q}{\alpha} \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{s+n+1}}{\sqrt{s^2+n^2}}$$

It is plain that this series has diagonal symmetry with respect to the two parameters s and n, by taking advantage of which the labor of computation can be cut almost in half. That is, the term n = 55, s = 72 has identically the same value as the term n = 72, s = 55. It is only for charges on the diagonal (n = s) and in the first row (s = 0) that there are no similar ones with which they can be paired.(9)

Before this pairing is attempted, it should be noted that the double series is convergent. First, for row s the series Q_s can be compared term by term with the convergent series $\sum_{m=1}^{\infty} \frac{(-1)^m}{m}$

For m = n = k

$$\left| \frac{1}{\sqrt{s^2+k^2}} \right| < \left| \frac{1}{k} \right|$$

Therefore the series Q_s is convergent. Furthermore,

$$|Q_s| < \left| \frac{q}{\alpha} \frac{1}{\sqrt{s^2+1}} \right| < \left| \frac{q}{\alpha} \frac{1}{s} \right|$$

and the sign of Q_s will be that of its first term.

Similarly it can be shown that Q_{s+1} is convergent, and it is also true that

$$|Q_{s+1}| < \left| \frac{q}{\alpha} \frac{1}{\sqrt{(s+1)^2+1}} \right| < \left| \frac{q}{\alpha} \frac{1}{s+1} \right|$$

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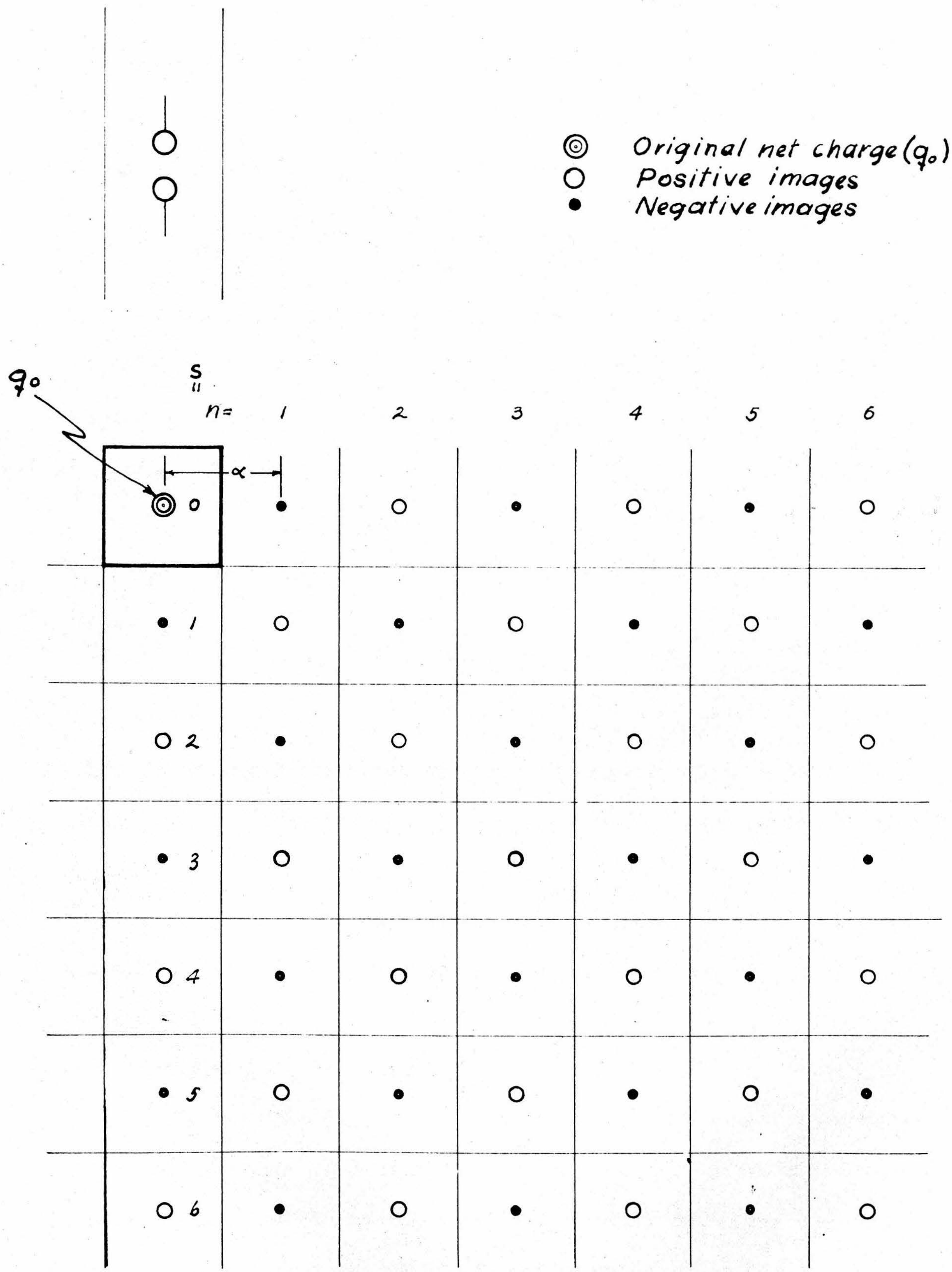
9. The values in row s = 0 could be paired with the values in column n = 0 but if this were done, more than one quarter of the total number of charges would have been included.

FIGURE 9



Original net charge (q_0)
Positive images
Negative images
q_0
S
n= 1 2 3 4 5 6
α
0
1
2
3
4
5
6

and that Q_{s+1} takes the sign of its first term. Now since the sign of the first term of the several series oscillates, the sign of the sums oscillates, and the comparison with the series $\frac{(-1)^m}{m}$ demonstrates the convergence of the series of the sums.

To return to the simplification of computation by the use of the symmetry, let the row s = 0 be separated first:

$$Q = \frac{8}{\alpha}\left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \sum_{s=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{s+n+1}}{\sqrt{s^2+n^2}} \right\}$$

The first series is recognized as one whose sum is $\log_e 2$.

Now let the terms for which n = s be separated. The remaining terms can then be paired.

$$Q = \frac{8}{\alpha}\left\{ \log 2 + \sum_{s=1}^{\infty} \frac{(-1)^{2s+1}}{\sqrt{2s^2}} + 2\sum_{s=1}^{\infty}\sum_{n=s+1}^{\infty} \frac{(-1)^{s+n+1}}{\sqrt{s^2+n^2}} \right\}$$

This step has had the unfortunate result of introducing a divergent series,

$$\sum_{s=1}^{\infty} \frac{(-1)^{2s+1}}{\sqrt{2s^2}} = -\frac{1}{\sqrt{2}} \sum_{s=1}^{\infty} \frac{1}{s}$$

Since the original sum is convergent, it is presumed that this means simply that the reamining double series is also divergent and of opposite sign.

By physical intuition, it seems safe to make the calculation for a square array, that is, out to n = s = k rather than out to $n^2 + s^2 = k^2$ which would be a circular array.

In any case, since the double series is oscillating, it seems safe to say that the maximum error will be limited by the largest term neglected. Now the first term, log 2 is .693, and it would be interesting to see how far the calculations must be carried before the individual terms are as small as 1% of this value.

Take s = 1: to find n such that

$$\frac{1}{\sqrt{s^2+n^2}} \leqq .00693$$

$$1 + n^2 \doteqdot \left(\frac{1}{.00693}\right)^2 = (144.4)^2$$

$$n \doteqdot 144$$

If the upper limit is finite, the double series can be written

$$\sum_{s=1}^{k} \sum_{n=s+1}^{k+1} \frac{(-1)^{s+n+1}}{\sqrt{s^2+n^2}}$$

in which the number of terms is $N = \frac{k\,(k+1)}{2}$

If n must be carried to 144, then k + 1 = 144 and

$$N = \frac{(143)(144)}{2} = 10{,}296.$$

In addition, there are 144 terms in the single series $\sum_{s=1}^{\infty} \frac{1}{s}$,

making a total of 10,340 terms.

The Effect of a Spherical Enclosing Cage of Diameter Ten Times That of the Voltmeter Spheres.

(Voltmeter Spheres at 1 Diameter Spacing.)

The method of computation will be 1) to assume the total charge within each voltmeter sphere, 2) to calculate the image of each of these thru the enclosing cage, and 3) to check the accuracy of the assumed values. This check is made by comparing the sum of the images of the other three charges into one voltmeter sphere with the total charge assumed in that sphere. It should be noted that a unit charge at the center in addition to the images is required to raise the one sphere to unit potential.

Let the radius of the voltmeter spheres again be taken as the unit of length. To simplify the computation, all the charges in the high potential sphere will be assumed to be concentrated at its center and, for the present, all charges in the grounded sphere at .27 units from its center. (The figure .27 is the radius of the center of gravity of these charges for isolated spheres.)

For the first trial, the values for isolated spheres will be used: + 1.07 in the high potential sphere and - .27 in the grounded one.[10] The images of these charges outside the cage, and their distances from its center are:

$$-(1.07)\,\frac{10}{2} = -\,5.35 \text{ at } \frac{(10)^2}{2} = 50.0$$

$$-(-.27)\,\frac{10}{2-.27} = +\,1.56 \text{ at } \frac{(10)^2}{2-.27} = 57.8$$

Under the influence of these four charges, the three spheres should be equipotentials. If they are, then the total charge within each should be the sum of the images of the other three.

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10. Charges are in electrostatic units and forces are in dynes. Refer to page 26 for complete discussion.

The images within the high potential sphere are:

$$-\frac{-5.38}{48.0} = +.111$$

$$-\frac{-.27}{3.73} = +.072$$

$$-\frac{+1.56}{59.8} = -.026$$

	+ .157
The unit charge is	1.000
Total	1.157

which is to be compared

with the assumed value of 1.070

The images within the grounded sphere are:

$$-\frac{-5.35}{52.0} = +.103$$

$$-\frac{+1.07}{4.0} = -.268$$

$$-\frac{+1.56}{55.8} = -.028$$

Total - .193

which is to be compared

with the assumed value of - .270

For the second trial, the charges assumed will be + 1.16 and - .190. The images outside the cage then become - 5.8 at 50 and + 1.10 at 57.8.

The images within the high potential sphere are:

$$-\frac{-5.8}{48.0} = +.121$$

$$-\frac{-.19}{3.73} = +.051$$

$$-\frac{+1.10}{59.8} = -.018$$

	+ .154
The unit charge is	1.000
Total	1.154

Compared with 1.16 assumed.

The images within the grounded sphere are:

$$- \frac{-5.8}{52.0} = + .112$$

$$- \frac{+1.16}{4.0} = - .290$$

$$- \frac{+1.10}{55.8} = - .020$$

Total $- .198$

Compared with $- .190$ assumed.

For the third trial assume charges + 1.154 and - .198. Their images outside the cage are - 5.77 and + 1.145.

The images within the high potential sphere are:

	+ .120
	+ .053
	- .019
	+ .154
The unit charge is	1.000
Total	1.154

Compared with 1.154 assumed.

The images within the grounded sphere are:

	+ .111
	- .288
	- .021
Total	- .198

Compared with - .198 assumed.

In the computation of the force on the grounded sphere the accuracy will be improved by separating the charge within it into two parts: 1) - .288 at .27 from its center and 2) + .111 - .021 = + .090 at the center. The force on each of these due to the other three charges is:

$$- (- .288)\left[\frac{1.154}{(3.73)^2} - \frac{5.77}{(51.73)^2} - \frac{1.145}{(56.1)^2}\right]$$

$$- (+ .090)\left[\frac{1.154}{4^2} - \frac{5.77}{(52)^2} - \frac{1.145}{(55.8)^2}\right]$$

$$f = .288\ (.08294 - .00215 - .00036) - .090(.07212 - .00214 - .00037)$$

$$= .01693 \text{ dynes.}$$

This figure is to be compared to .02075 dynes for isolated spheres

$$\frac{.02075 - .01693}{.02075} = .184 = 18.4\%.$$

Conclusion:

The presence of a spherical grounded cage of 10 times the diameter of the spheres results in a reduction of force of 18.4% when the voltmeter spacing is 1 diameter.

The Effect of a Spherical Enclosing Cage of Diameter Twenty Times that of the Voltmeter Spheres.

(Voltmeter Spheres at 1 Diameter Spacing.)

To make a similar calculation for an enclosing sphere of 20 times the diameter of the voltmeter spheres, assume charges of value midway between those for the 10 diameter enclosure and those for isolated spheres: assume + 1.12 and - .230. The images outside the cage are

$$-(+1.12)\frac{20}{2} = -11.20 \text{ at } \frac{(20)^2}{2} = 200$$

$$-(-.230)\frac{20}{1.73} = +2.66 \text{ at } \frac{(20)^2}{1.73} = 231$$

The images within the high potential sphere are:

$$-\frac{-11.20}{198.0} = +.0565$$

$$-\frac{-.23}{3.73} = +.0617$$

$$-\frac{+2.66}{233.0} = -.0114$$

	+ .1068
The unit charge is	1.0000
Total	1.107
Compared with	1.120 assumed.

The images within the grounded sphere are:

$$-\frac{-11.20}{202.0} = +.0555$$

$$-\frac{+1.12}{4.0} = -.2800$$

$$-\frac{+2.66}{229.0} = -.0116$$

Total	- .2361
Compared with	- .230 assumed.

For the second trial assume + 1.107 and - .236. The images of these charges are - 11.07 and + 2.73.

The images within the high potential sphere are:

	+ .0559
	+ .0633
	- .0117
	+ .1075
The unit charge is	1.0000
Total	1.1075
Compared with	1.107 assumed.

The images within the grounded sphere are:

	+ .0548
	- .2768
	- .0119
Total	- .2339
Compared with	- .2360 assumed.

As in the previous case, the charge within the grounded sphere will be divided into - .2768 at .27, and + .0548 - .0119 = + .0429 at the center. The forces are:

$$f = -(-.2768)\left[\frac{1.107}{(3.73)^2} - \frac{11.07}{(201.7)^2} - \frac{2.73}{(229.3)^2}\right]$$

$$-(+.0429)\left[\frac{1.107}{4^2} - \frac{11.07}{(202)^2} - \frac{2.73}{(229)^2}\right]$$

$$= + .2768\ (.07956 - .00027 - .0005) - .0429(.06919 - .00027 - .00005)$$

$$= .2768\ (.0792) - .0429\ (.0689)$$

$$= .01898 \text{ dynes.}$$

$$\frac{.02075 - .01898}{.02075} = .085 = 8.5\%$$

Conclusion:

A grounded spherical cage of 20 times the voltmeter sphere diameter decreases the force 8.5%.

Compilation of Results of Calculations of the Effect of Grounded Bodies.

Since $F = K \cdot V^2$, or $V = k\sqrt{F}$, the percentage reduction in the force at a given voltage is approximately twice the increase in voltage for a given force.

For spheres at a spacing of 1 diameter, the presence of the grounded body in column 1 results in the changes from values for isolated sphere stated in columns 2 and 3.

(1) Grounded Body	(2) Reduction in Force at a Given Voltage	(3) Increase in Voltage at a Given Force (Increase in k)
One infinite plane parallel to the centerline and 5 diameters distant	8.7%	4.3%
Sphere of ten times the diameter of the voltmeter spheres	18.4%	8.8%
Sphere of twenty times the diameter of voltmeter spheres	8.5%	4.2%

The values in column (3) are values by which a voltage, obtained by force measurement and calculation on the assumption of isolation, must be increased to get the true value.

APPENDIX I

Typical Data for the Determination of the Effect of Shank Size By Capacitance Measurement.

The data for a typical capacitance measurement run is given on page 45 and a photograph of the original curve sheet plotted from these data appears on page 46. The two curves marked "Capacitance - Spacing" are plotted from the data given, to the same "Spacing" scale, but to different "Capacitance" scales. The one on the right is plotted on the right-hand "Vernier" scale, and that on the left on the left-hand scale. The straight lines are those traced along the mirror as described on page 11. From their slopes, the slopes of the tangents to the curves were computed and plotted as the curve marked "Slope - Spacing". This curve was combined with another for the same ($\frac{1}{4}$") shanks taken on a different day, the ordinates changed from Vernier Divisions per Inch to Kilovolt grams $^{-\frac{1}{2}}$ (k) and the result plotted as curve C, Figure 4, page 14.

TYPICAL DATA SHEET

$\frac{1}{4}$" Shanks

Time	Spacing	Vernier	Resistance	Room Temp.
10:57	0.5	Touching		17.8°
	0.7	1485.4	9880	
	1.2	1473.7	9880	
	1.7	1468.95	9880	
	2.2	1464.8	9880	
	2.7	1462.65	9880	
	3.2	1460.95	9885	
	3.7	1459.5	9885	
	4.2	1458.45	9885	
	4.7	1457.5	9885	
	0.6	1493.7	9885	
	1.1	1475.15	9885	
	1.6	1469.65	9875	
	2.1	1465.5	9875	
	2.6	1462.95	9875	
	3.1	1461.2	9875	
	3.6	1459.7	9875	
	4.1	1458.55	9875	
	4.6	1457.6	9875	
	0.65	1488.45	9875	
	1.0	1476.85	9875	
	1.5	1470.8	9875	
	2.0	1466.55	9875	
	2.5	1463.6	9875	
	3.0	1461.55	9875	
	3.5	1460.05	9875	
	4.0	1458.85	9875	
	4.5	1457.9	9875	
	5.0:	1457.0	9875	
	0.9	1478.9	9895	
	1.4	1471.5	9885	
	1.9	1469.9	9885	
	2.4	1464.1	9885	
	2.9	1461.9	9885	
	3.4	1460.3	9885	
	3.9	1458.95	9885	
	4.4	1458.0	9885	
	4.9	1457.15	9885	
	0.8	1481.3	9885	
	1.31	1472.5	9885	
	1.8	1467.9	9885	
	2.3	1464.35	9880	
	2.8	1462.1	9880	
	3.3	1460.45	9880	
	3.8	1459.1	9880	
	4.3	1458.05	9880	
	4.8	1457.15	9880	
11:29	0.5	Touching		17.8°

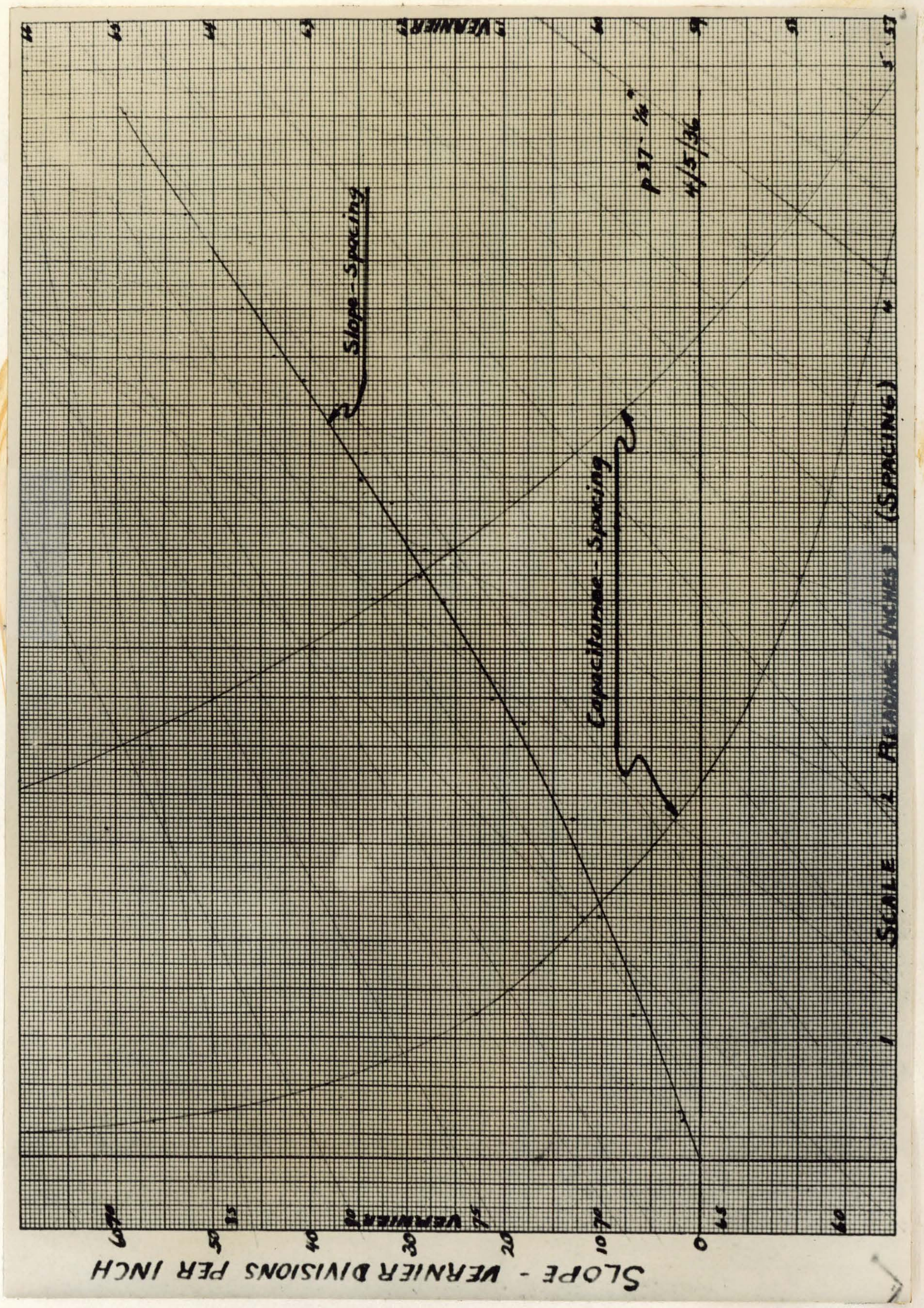


FIGURE 10
Working Graph of Capacitance Measurements.

APPENDIX II

Calibration of Transformer by Sphere Gap Flashover.

Since considerable confidence was placed in the validity of the model voltmeter measurements indicating rather exact linearity between tertiary voltage-v, and secondary KV - V, for the transformer up to 100 KV, and since this conclusion was at variance with the sphere-gap calibration curve run by Lewis in 1927, it was decided to re-run the sphere-gap curve. For this purpose, while the setup was still on the tower, the grounded sphere was clamped and flashover measurements were made. The hose resistor referred to was replaced by 25' of $\frac{3''}{4}$ garden hose thru which tap water was run continuously.

From the nature of the setup, it was impossible to change the gap spacing while the voltage was being applied. The procedure was 1) to read the scale on the shank, 2) to bring up the voltage to about 90% of the expected flashover value, maintain it for at least 45 seconds, and then to raise it approximately 1% every 15 seconds. This process was repeated at least five times for every gap spacing. It was found that when the voltage was raised at a rapid rate, flashover occurred at a lower value than when it was raised slowly. The cause of this effect may be the "conditioning" of the gap as suggested by Peek. In any case, for each spacing the first readings are erratic and are all below the later ones. These data are summarized in the data sheet on page 49 in which the last 5 values of tertiary coil voltage at each spacing are given. The flashover voltage for each spacing was taken from Peek's data, correction (amounting to 1.3%) being made for air density in accord with his recommendation. The results are plotted in Figure 11, page 50. Each point on this curve is the average of the three highest readings. Since the reactance of the transformer is so very low (1.24%) and since the current just before flashover must be at a considerable leading power-factor, no possible reason can be suggested for the calibration curve to fall below the ratio line. It is therefore concluded that the sphere-gap test was run under test conditions essentially different from Peek's and that results obtained by use of his data are not reliable in this case.

DATA SHEET FOR FLASHOVER TESTS

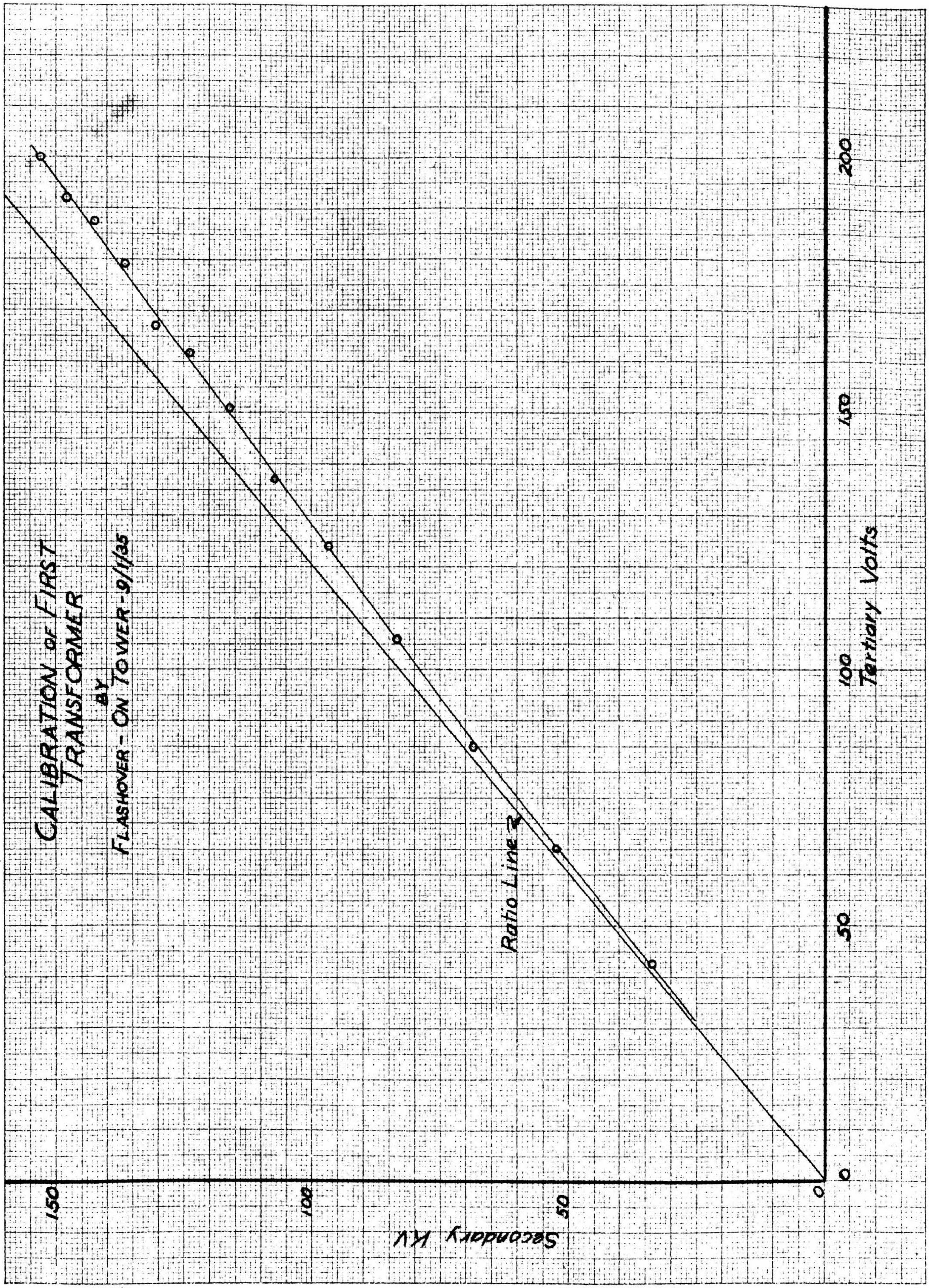
Time	Shank Reading	Tertiary Volts at Flashover
7:30 A.M.	14	189
		190
		193
		193
		190
7:55	12	178
		178.5
		176
		179.5
		179
8:15	10	159
		160
		161
		161
		161
8:37	11	167
		168
		167
		163
		167
8:52	9	150.5
		137.5
		137
		137
		137
9:20	6	105.3
		105.8
		106
		106
		106
9:30	4	65
		65
		65
		65
10:20	7	124
		123
		123.2
		123

Time	Shank Reading	Tertiary Volts at Flashover
10:34	5	85
		85
		84.5
		85
10:47	4	65
		65
		65
		65.3
10:55	3	42.8
		42.5
		42.5
		42.4
		42.5
11:08	10	163
		162.5
		160.2
		159
		163
11:24	15	200
		199.5
		199.6
		200
12:35	13	188
		187
		187
		187
	1.35	Contact

12:54

Barometric Pressure	742.3 mm.
Temperature	21.5° C.
Relative Humidity	76.0%

FIGURE 11


CALIBRATION OF FIRST TRANSFORMER
BY
FLASHOVER - ON TOWER - 9/1/35
Ratio Line
Tertiary Volts
0
50
100
150
200
Secondary KV
0
50
100
150

APPENDIX III

Typical Data for the Determination of the Effect of Ground Planes By Measurement of Force and Voltage.

The following table is a typical set of data taken with the 12.5 cm. spheres as a force voltmeter. From column 1, the shank reading, and the reading at contact, 1.2cms., the spacing can be found. In column 2, weight, the nominal value of the weight hung from the double thread support described on page 18 is tabulated. Columns 3 and 4 give the tertiary coil voltages as read on a 75 volt and a 150 volt voltmeter, respectively. These voltmeters contained spring pushbuttons so that they were never both connected to the circuit at the same time. At 75 volts, the additional load of the 150 volt voltmeter (connected as a special trial case) reduced the reading of the 75 volt voltmeter from 75.0 to 74.9. Column 5 gives the Cathetometer reading as read, and column 6 gives the values of deflection obtained by subtracting the reading 4.60, for zero voltage, from each reading. The horizontal force due to the deflection determined by the cathetometer, calculated from a calibration curve, is tabulated in column 7. The horizontal force due to the weight noted in column 2, and calculated from the trigonometry of the suspension triangle is given in column 8. Column 9 is the sum of columns 7 and 8, and is the abscissa of the curves of Figure 12, page 53. Column 10 is (Tertiary Coil Volts) squared and divided by 1000. It is the ordinate of the curves on Figure 12. On this figure the set of data is plotted as the curve next to the top one, that is, for Spacing = 99.1% of diameter.

TYPICAL DATA SHEET

NO CAGE

	1 Shank Reading	2 Weight	3 Voltage 75 Scale	4 150 Scale	5 Cathetometer Reading	6 Deflection	7 Force	8 Weight Force	9 Total Force	10 V^2
2:45	13.6	0	0		4.60					
			22.8		4.81	.21	.13			.52
			31.7		5.00	.40	.24			1.00
			40.4	41.3	5.21	.61	.37			1.65
			49.4	50.1	5.51	.91	.55			2.45
			59.3	57.4	5.84	1.24	.76			3.28
			65.7	66.0	6.26	1.66	1.01			4.32
		5	65.8		3.71	- .89	- .54	1.50	.96	
			73.9	74.0	4.17	- .43	- .20	1.50	1.24	5.47
				84.2	4.80	.20	.12		1.62	7.10
				91.2	5.28	.68	.41		1.91	8.30
				97.4	5.73	1.13	.69		2.19	9.48
		0		0	4.60					
									10.98	47.89

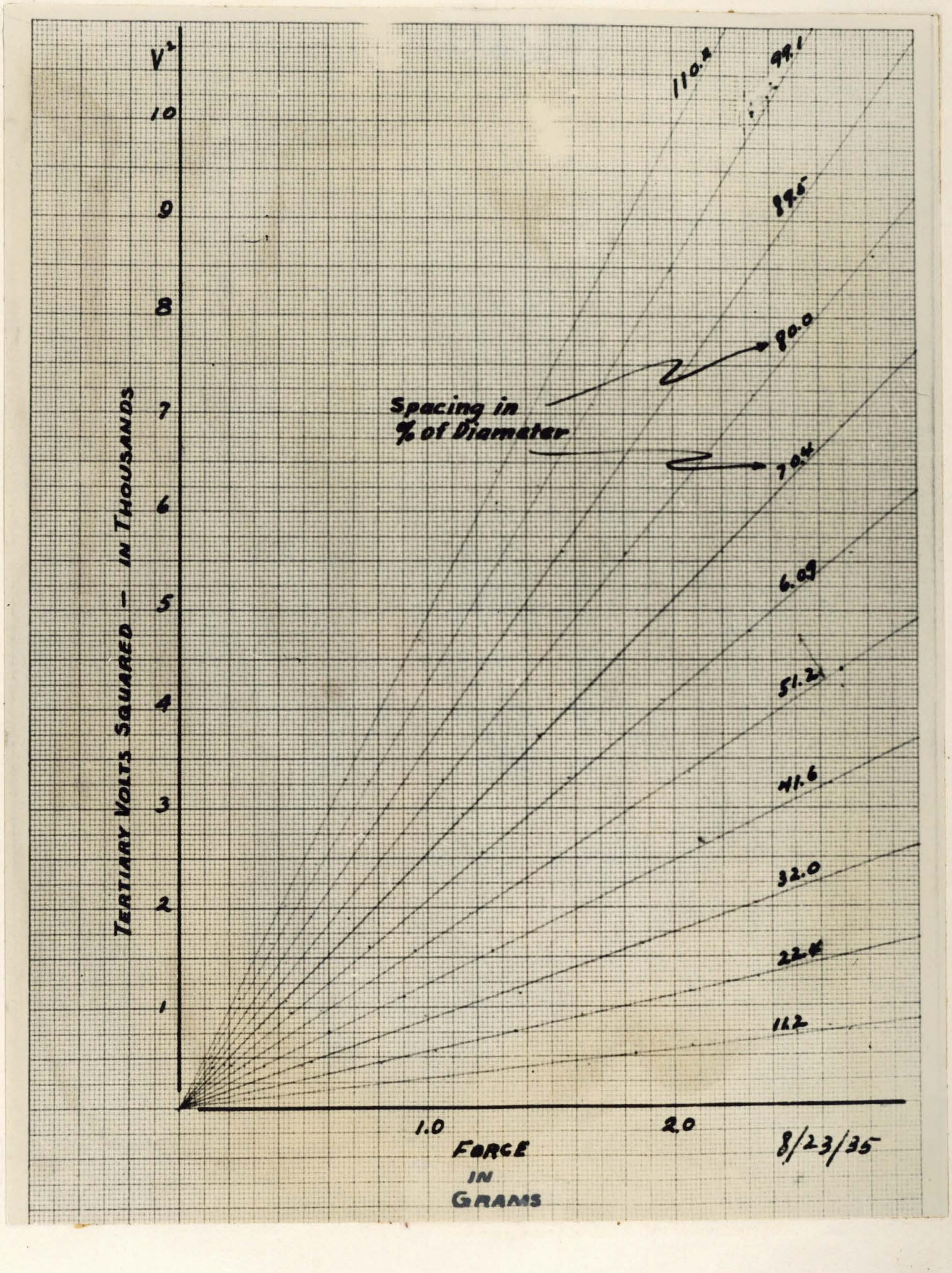


FIGURE 12

Working Graph of Force and Voltage Measurements