LAMINAR BOUNDARY LAYERS
ON SLENDER BODIES OF REVOLUTION
IN AXIAL FLOW

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ABSTRACT

An exact similar solution of the modified boundary layer equations has been obtained for the axial incompressible flow past paraboloids of revolution. It has been shown that the usual boundary layer assumptions are justified and that the local skin friction increases as the boundary layer thickness becomes large compared with the body radius.

An approximate method for obtaining the local skin friction on arbitrary slender bodies of revolution in axial incompressible flow has been developed. A comparison of the approximate results with the exact solutions for paraboloids of revolution and circular cylinders shows good agreement.

The existence of energy integrals of the modified compressible boundary layer equations is established. Similarity of the governing equations for the axial compressible flow past paraboloids of revolution has been shown; for the same bodies, a hypersonic similarity law is deduced.

An approximate method for obtaining the local skin friction on arbitrary slender insulated bodies of revolution in axial compressible flow has been developed. The results show that compressibility counterbalances the rise in local skin friction due to curvature at high Reynolds numbers (based on a characteristic length of the body) and increases the local skin friction at sufficiently low Reynolds numbers.

Velocity profiles on a slender ogive-cylinder have been obtained experimentally at a Mach number of 5.8 and at different Reynolds numbers. The results indicate a curvature effect when compared with flat plate results.

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I. INTRODUCTION

A factor of prime importance in the design of projectiles approaching the hypersonic speed range is the feasibility of using slender shapes in order to minimize the total drag. Since the total drag consists of the pressure drag and skin friction drag, it is necessary to investigate the latter only since the former decreases as the projectile becomes more slender.

Due to the complexity of the problem, present day information, both theoretical and experimental, regarding the laminar skin friction properties of slender bodies of revolution is still quite meager. As a body becomes more slender, the transverse curvature effects of the body may become more important or, equivalently, the thickness of the boundary layer may become of the same order of magnitude compared with the cross-sectional radius of the body. Any analytical solution must account for this effect.

Beginning with the Navier-Stokes equations for an incompressible fluid, C. B. Millikan (Ref. 1) obtained an approximate system of equations which are applicable in the laminar boundary layer on bodies of revolution with arbitrary shapes. Mangler (Ref. 2) showed how this system can be simplified when the boundary layer thickness is very small compared with the body radius: by a suitable set of transformations, he reduced the given system to a corresponding two-dimensional system. In a later work, the procedure was extended to the compressible case (Ref. 3). Thus Mangler's theory is not applicable to slender* bodies.

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* Unless otherwise indicated, the word "slender" as applied to a body will be used to describe the situation when the boundary layer thickness is of the same or of a larger order of magnitude compared with the body radius.

Seban and Bond (Ref. 4) made the first attempt to circumvent the restriction imposed by Mangler by solving the original system of equations derived by Millikan for a semi-infinite circular cylinder. Their results are applicable in the region near the leading edge of the cylinder. Very recently, Stewartson (Ref. 5) obtained results which are applicable in a region far from the leading edge. Both results predict an increase in the local skin friction as the cylinder becomes more slender.

It is the aim of this work to investigate the laminar skin friction properties of arbitrary slender bodies of revolution. Because of the complicated nature of the governing equations, an approximate solution will be sought. For this purpose, the von Kármán integral method will be used since it yields reasonably accurate results for the local skin friction in two-dimensions. The precise accuracy of the approximate results may then be determined through a comparison with exact solutions.

The effects of compressibility on the laminar skin friction properties of slender bodies of revolution will be examined. Again, an approximate solution will be sought.

Finally, the characteristics of the velocity profile across the laminar boundary layer on a slender ogive-cylinder in a compressible fluid will be determined experimentally.

II. LAMINAR BOUNDARY LAYER CHARACTERISTICS OF SLENDER BODIES OF REVOLUTION IN AXIAL INCOMPRESSIBLE FLOW

A. Modified Laminar Boundary Layer Equations in Cylindrical Coordinates

For a steady axial incompressible flow past a non-spinning body of revolution, the general viscous flow equations (Cf. Appendix A) reduce to the following system:

$$\frac{\partial(ur)}{\partial x} + \frac{\partial(vr)}{\partial r} = 0 \tag{1a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \nu\frac{\partial^2 u}{\partial x^2} \tag{1b}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial x\partial r} + \frac{\partial^2 v}{\partial x^2} + \frac{2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{2v}{r^2}\right] \tag{1c}$$

Together with the assumption that the thickness of the boundary layer is of the same order as the transverse radius of the body, the application of the usual limiting procedure on Eqs. (1) yields the following modified boundary layer equations (Cf. Appendix B):

$$\frac{\partial(ur)}{\partial x} + \frac{\partial(vr)}{\partial r} = 0 \tag{2a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{2b}$$

$$0 = \frac{\partial p}{\partial r} \tag{2c}$$

A similar set of equations was derived by Millikan (Ref. 1) using a slightly different coordinate system. On the additional assumption that the boundary layer thickness is small compared to the radius of the body, r_o, Eqs. (2) reduce to Mangler's boundary layer equations (Ref. 2):

$$\frac{\partial(u r_o)}{\partial x} + \frac{\partial(v r_o)}{\partial r} = 0 \tag{3a}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial r^2} \tag{3b}$$

$$0 = \frac{\partial p}{\partial r} \tag{3c}$$

The associated boundary conditions are $u = v = 0$ at $r = r_o$ and $u \to U$ as $r \to \infty$, where U is the free stream velocity.

B. Extended Mangler Transformation

Mangler has shown that Eqs. (3) can be reduced to an equivalent two-dimensional form by applying a suitable transformation. However, his transformation is no longer valid when the boundary layer thickness becomes of the same order as the body radius. Consequently, a new transformation is developed which transforms Eqs. (2) to a more amenable form although they are not strictly in the two-dimensional form.

Let

$$\bar{r} = \int_{r_o}^{r} \frac{r}{L}\, dr \quad ; \quad \bar{x} = \int_0^x \left(\frac{r_o}{L}\right)^2 dx \quad ,$$

$$\bar{u} = u \quad , \quad v = \frac{r_o^2}{rL}\bar{v} - \frac{L}{r}\frac{\partial \bar{r}}{\partial x}\bar{u} \tag{4}$$

where L is a characteristic length of the body and barred quantities are the new variables. Then, Eqs. (2) reduce to (Cf. Appendix C):

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{r}} = 0 \tag{5a}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{r}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}} + \nu\frac{\partial}{\partial \bar{r}}\left[\left(1+\frac{2L\bar{r}}{r_0^2}\right)\frac{\partial \bar{u}}{\partial \bar{r}}\right] \tag{5b}$$

$$0 = \frac{\partial \bar{p}}{\partial \bar{r}} \tag{5c}$$

The associated boundary conditions in the new variables are $\bar{u} = \bar{v} = 0$ at $\bar{r} = 0$ and $\bar{u} \to U$ as $\bar{r} \to \infty$.

Seban and Bond (Ref. 4) and Stewartson (Ref. 5) have obtained solutions to Eqs. (2) near the leading edge and far from the leading edge of a cylinder, respectively. Their governing equations could also be obtained from Eqs. (5), thus justifying to some extent the generality of Eqs. (4).

By letting

$$\frac{\partial \bar{p}}{\partial \bar{x}} = 0$$

and

$$\eta = \sqrt{\frac{U}{\nu \bar{x}}}\,\bar{r} \quad , \quad \xi = \frac{2L}{r_0^2}\sqrt{\frac{\nu \bar{x}}{U}} \quad , \quad \psi = \sqrt{U\nu\bar{x}}\, f(\xi,\eta)$$

Eqs. (5) transform into the single equation (Cf. Appendix D):

$$f\frac{\partial^2 f}{\partial \eta^2} + 2\frac{\partial}{\partial \eta}\left[(1+\xi\eta)\frac{\partial^2 f}{\partial \eta^2}\right] = \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta} - \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2}\right)$$

which is the form analyzed by Seban and Bond.

By the additional transformation

$$f_1 = \frac{2f}{\xi} \quad , \quad \eta_1 = \frac{\eta}{2\xi} + \frac{1}{2\xi^2}$$

the form of the equation analyzed by Stewartson is obtained (Cf. Appendix D):

$$f_1 \frac{\partial^2 f_1}{\partial \eta_1^2} + 2\frac{\partial}{\partial \eta_1}\left(\eta_1 \frac{\partial^2 f_1}{\partial \eta_1^2}\right) = \frac{\xi}{2}\left(\frac{\partial f_1}{\partial \eta_1}\frac{\partial^2 f_1}{\partial \xi \partial \eta_1} - \frac{\partial f_1}{\partial \xi}\frac{\partial^2 f_1}{\partial \eta_1^2}\right)$$

C. An Exact Solution of the Governing Equations for Flows Past Paraboloids of Revolution

For a paraboloid of revolution defined as

$$r_0^2 = 4L\,x = 2\sqrt{2}\,L^{3/2}\,\bar{x}^{1/2} \tag{6}$$

where L is the focal length, the inviscid solution gives for the axial pressure gradient on the body (Cf. Appendix F):

$$\frac{\partial p}{\partial x} = -\rho\,U^2 \frac{\frac{4L}{x^2}}{\left(1+\frac{L}{x}\right)^4} = -\frac{\rho U^2}{x}\frac{\left(\frac{r_0}{x}\right)^2}{\left[1+\left(\frac{r_0}{2x}\right)^2\right]^4} \tag{7}$$

Therefore, $\frac{\partial p}{\partial x}$ is small if $\left(\frac{r_0}{x}\right)^2 \ll 1$ and x not close to the leading edge. It will be shown, a posteriori, that $\frac{\partial p}{\partial x}$ is also small compared to the viscous terms in the momentum equation under the same conditions. Hence, the boundary layer characteristics for the axial flow past paraboloids of revolution will be given by the solution to the following system:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{r}} = 0 \tag{8a}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{r}} = \nu \frac{\partial}{\partial \bar{r}}\left[\left(1+\frac{2L\bar{r}}{r_0^2}\right)\frac{\partial \bar{u}}{\partial \bar{r}}\right] \tag{8b}$$

with the boundary conditions

$$\bar{u} = \bar{v} = 0 \qquad \text{at} \qquad \bar{r} = 0 \tag{9a}$$

$$\bar{u} \to U \qquad \text{as} \qquad \bar{r} \to \infty \tag{9b}$$

It is seen that this system becomes invariant under the following mapping (Cf. Appendix E):

$$\bar{x} \to k^2\bar{x} \quad , \quad \bar{r} \to k\bar{r} \quad , \quad \bar{u} \to \bar{u} \quad , \quad \bar{v} \to \frac{\bar{v}}{k} \tag{10}$$

From this invariance it follows that u and $\frac{\bar{\psi}}{\sqrt{\bar{x}}}$ must be functions of $\frac{\bar{r}}{\sqrt{\bar{x}}}$.

Now let

$$\bar{\eta} = \bar{r}\sqrt{\frac{U}{\nu\bar{x}}} \qquad \text{and} \qquad \bar{\psi} = \sqrt{U\nu\bar{x}}\ \bar{f}(\bar{\eta}) \tag{11}$$

Then

$$\begin{aligned} \bar{u} &= \frac{\partial\bar{\psi}}{\partial\bar{r}} = U\frac{\partial\bar{f}}{\partial\bar{\eta}} \quad , & \bar{v} &= -\frac{\partial\bar{\psi}}{\partial\bar{x}} = -\sqrt{U\nu}\left(\sqrt{\bar{x}}\frac{\partial\bar{f}}{\partial\bar{x}} + \frac{\bar{f}}{2\sqrt{\bar{x}}}\right) \\ \frac{\partial\bar{u}}{\partial\bar{r}} &= U\sqrt{\frac{U}{\nu\bar{x}}}\frac{\partial^2\bar{f}}{\partial\bar{\eta}^2} \quad , & \frac{\partial\bar{u}}{\partial\bar{x}} &= U\frac{\partial\bar{\eta}}{\partial\bar{x}}\frac{\partial^2\bar{f}}{\partial\bar{\eta}^2} \end{aligned} \tag{12}$$

Substituting Eqs. (12) in Eq. (8b) gives

$$\bar{f}\bar{f}'' + 2\left[(1+\beta\bar{\eta})\bar{f}''\right]' = 0 \tag{13a}$$

where

$$\beta = \frac{1}{\sqrt{2R_L}} \qquad \left(R_L = \frac{UL}{\nu}\right)$$

and primes denote differentiation with

respect to $\bar{\eta}$. The associated boundary conditions are

$$\bar{f} = \bar{f'} = 0 \qquad \text{at} \qquad \bar{\eta} = 0 \tag{13b}$$

$$\bar{f'} \to 1 \qquad \text{as} \qquad \bar{\eta} \to \infty \tag{13c}$$

Equation (13) can be solved by an electronic computer for small values of β , i.e., for β of order one. (For $\beta = 0$ the solution is known since it is the Blasius type of equation, as in the two-dimensional flat plate problem.) Some solutions for various values of β are shown in Fig. 1.* It is seen that the local skin friction increases as β increases or R_L decreases.

For $\beta \gg 1$, the solution to Eqs. (13) may be obtained analytically by a method developed by Stewartson for the cylinder problem. Before this can be done, it is necessary to perform the following transformation of the variables in Eqs. (13):

$$\bar{\eta} = \frac{1}{\beta}\left(2\beta^2 \eta - 1\right) \qquad \text{and} \qquad \bar{f} = 2\beta f \tag{14}$$

Hence, the transformed governing equation is

$$(f+1)f'' + \eta f''' = 0 \tag{15a}$$

where primes denote differentiation with respect to η . The associated boundary conditions are

$$f = f' = 0 \qquad \text{at} \qquad \eta = R_L \tag{15b}$$

$$f' \to 1 \qquad \text{as} \qquad \eta \to \infty \tag{15c}$$

Now Eq. (15a) may be written as

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* Computations were made on the REAC at the RAND Corporation.

$$\frac{f'''}{f''} = -\frac{1}{\eta} - \frac{f}{\eta}$$

which yields, after integration,

$$\ln f'' = -\ln \eta - \int_{R_L}^{\eta} \frac{f}{\eta}\, d\eta + \ln A$$

or

$$f'' = \frac{A}{\eta}\, e^{-\int_{R_L}^{\eta} \frac{f}{\eta} d\eta} \tag{16}$$

where A is a constant.

The solution to the system, Eqs. (15), now can be obtained by successive iteration. As a first approximation, take

$$f = \eta \tag{17}$$

Here, f clearly satisfies Eqs. (15a) and (15c). Substituting this on the right hand side of Eq. (16) and integrating

$$f'' = \frac{A}{\eta}\, e^{-(\eta - R_L)} \tag{18a}$$

By successive integrations this yields

$$f' = 1 + \int_{\infty}^{\eta} f''\, d\eta \tag{18b}$$

and

$$f = \eta + \int^{\eta} d\eta \int_{\infty}^{\eta} f''\, d\eta + B$$

where B is a constant. The last equation can be reduced further by an integration by parts; hence,

$$f = \eta + \eta \int_{\infty}^{\eta} f'' d\eta - \int^{\eta} \eta f'' d\eta + B$$

or

$$f = \eta - A e^{R_L} \eta \int_{\eta}^{\infty} \frac{e^{-\eta}}{\eta} d\eta + A e^{-(\eta - R_L)} + B \tag{18c}$$

The constants A and B are now determined from the boundary conditions. Hence, Eqs. (18b) and (18c) give, respectively,

$$0 = 1 - \int_{R_L}^{\infty} f'' d\eta = 1 - A \int_{R_L}^{\infty} \frac{e^{-(\eta - R_L)}}{\eta} d\eta$$

$$0 = R_L - A e^{R_L} R_L \int_{R_L}^{\infty} \frac{e^{-\eta}}{\eta} d\eta + A + B$$

The simultaneous solution of this system yields

$$A^{-1} = e^{R_L} \int_{R_L}^{\infty} \frac{e^{\eta}}{\eta} d\eta \equiv e^{R_L} ei(R_L)$$

and

$$B = -R_L - \frac{e^{-R_L}}{ei(R_L)} \left[1 - e^{R_L} R_L \, ei(R_L) \right] = - \frac{e^{-R_L}}{ei(R_L)}$$

where

$$ei(\eta) \equiv \int_{\eta}^{\infty} \frac{e^{-\eta}}{\eta} d\eta \qquad \text{(exponential integral)}$$

or, in particular,

$$ei(R_L) \equiv \int_{R_L}^{\infty} \frac{e^{-\eta}}{\eta} d\eta$$

Hence, Eqs. (18) become

$$f'' = \frac{1}{ei(R_L)} \frac{e^{-\eta}}{\eta} \tag{19a}$$

$$f' = 1 - \frac{ei(\eta)}{ei(R_L)} \tag{19b}$$

$$f = \eta - \frac{\eta\, ei(\eta)}{ei(R_L)} + \frac{e^{-\eta}}{ei(R_L)} - \frac{e^{-R_L}}{ei(R_L)} \tag{19c}$$

Equations (19) represent the second approximation to the solution of Eqs. (15). This is essentially the Oseen result for the flow past paraboloids of revolution (Cf. Appendix G).

For η small,

$$ei(\eta) = \int_{\eta}^{\infty} \frac{e^{-\eta}}{\eta}\, d\eta = \ln\frac{1}{c\eta} + \eta - \frac{\eta^2}{2 \cdot 2!} + \cdots \tag{20}$$

where $\ln c = 0.5772.$

Hence, for R_L small and η near R_L, Eqs. (19) reduce to

$$f'' = \frac{1}{\ln\frac{1}{cR_L}} \frac{e^{-\eta}}{\eta} \tag{21a}$$

$$f' = 1 - \frac{\ln\frac{1}{c\eta}}{\ln\frac{1}{cR_L}} \tag{21b}$$

$$f = \frac{1}{\ln\frac{1}{cR_L}} \left[-\eta \ln\frac{1}{c\eta} + e^{-\eta} - e^{-R_L} \right] + \eta \tag{21c}$$

Consequently, in view of the analytical forms of Eqs. (21), the solution to Eqs. (15) is assumed to be of the form

$$f = \eta + \sum_{n=1}^{\infty}{}' \frac{f_n}{(\ln \frac{1}{R_L c})^n} + O\left(\frac{R_L}{\ln R_L}\right) \tag{22}$$

This will be a unique expansion for large values of $\ln \frac{1}{R_L c}$ or small values of R_L if f satisfies Eqs. (15) identically. Thus, substituting Eq. (22) in Eq. (15a) gives

$$(1 + \eta + \sum_1^{\infty}{}' \lambda^n f_n)(\sum_1^{\infty}{}' \lambda^n f_n'') + \sum_1^{\infty}{}' \lambda^n \eta f_n''' = 0$$

or

$$\sum_1^{\infty}{}' [(1+\eta) f_n'' + \eta f_n'''] \lambda^n + \sum_1^{\infty}{}' (f_1 f_n'' + f_2 f_{n-1}'' + \cdots + f_{n-1} f_2'' + f_n f_1'') \lambda^{n+1} = 0$$

or

$$\sum_1^{\infty}{}' [(1+\eta) f_n'' + \eta f_n'''] \lambda^n + \sum_{n=1}^{\infty}{}' (\sum_{m=1}^{\infty} f_m f_{n-m+1}'') \lambda^{n+1} = 0$$

where $\lambda \equiv \frac{1}{\ln \frac{1}{R_L c}}$. Hence, equating coefficients of λ^n to zero yields

$$(1+\eta) f_n'' + \eta f_n''' = -\sum_{m=1}^{n-1}{}' f_m f_{n-m}'' \tag{23}$$

In particular, for n = 1, 2, and 3, Eq. (23) becomes

$$(1+\eta) f_1'' + \eta f_1''' = 0 \tag{24a}$$

$$(1+\eta) f_2'' + \eta f_2''' = -f_1 f_1'' \tag{24b}$$

$$(1+\eta) f_3'' + \eta f_3''' = -\sum_{m=1}^{2}{}' f_m f_{3-m}'' \tag{24c}$$

The solution to Eq. (24a) for R_L small and η near R_L is obtained directly from Eqs. (21). Hence

$$f_1'' = \frac{e^{-\eta}}{\eta} \quad ; \quad f_1' = \ln c\eta \quad ; \quad f_1 = \eta \ln c\eta \tag{25}$$

It can be proved by mathematical induction that the following expressions are valid for η near R_L and for all values of m and for R_L small:

$$f_m = D_m \eta \ln c\eta \tag{26a}$$

$$f_m' = D_m \ln c\eta + E_m \tag{26b}$$

$$f_m'' = \frac{D_m}{c\eta} \tag{26c}$$

where D_m is a constant.

For η near R_L and R_L small, Eq. (23) may be written as

$$(\eta f_n'')' = -\sum_{m=1}^{n-1} f_m f_{n-m}''$$

and if Eqs. (26) are valid for $1 \leqslant m \leqslant n-1$, then the above equation has the form

$$(\eta f_n'')' = A_n \ln c\eta$$

where A_n is a constant. Integrating this,

$$\eta f_n'' = A_n \eta (\ln c\eta - 1) + B_n$$

or

$$f_n'' = \frac{B_n}{\eta} + A_n (\ln c\eta - 1)$$

where B_n is a constant. By another integration,

$$f_n' = D_n \ln c\eta + E_n + O(\eta \ln c\eta)$$

where $B_n \equiv \frac{D_n}{c}$ and E_n is a constant. Since these expressions are valid for $n = 1$, they are valid for all values of n, thus proving the result.

By virtue of this proof, each successive term in Eq. (22) diminishes in magnitude for R_L sufficiently small. Hence, it is a suitable expansion. The first two terms together are uniformly valid across the boundary layer when R_L is sufficiently small.

Now, for η near R_L and R_L small,

$$f' = 1 + \sum_{n=1}^{\infty}{}' \frac{D_n \ln c\eta + E_n}{\left(\ln \frac{1}{R_L c}\right)^n} \tag{27}$$

and since $f' = 0$ at $\eta = R_L$,

$$0 = 1 + \sum_{n=1}^{\infty}{}' \frac{D_n \ln cR_L + E_n}{\left(\ln \frac{1}{R_L c}\right)^n}$$

or

$$\sum_{n=1}^{\infty}{}' \frac{D_n}{\left(\ln \frac{1}{R_L c}\right)^{n-1}} - \sum_{n=1}^{\infty}{}' \frac{E_n}{\left(\ln \frac{1}{R_L c}\right)^n} = 1$$

Therefore

$$D_1 = 1 \qquad \text{and} \qquad D_n = E_{n-1} \quad , \quad n \geq 2 \tag{28}$$

The procedure for determining the constants D_n and E_n will be illustrated. E_n is obtained from the boundary condition $f_n' \rightarrow 0$ as $\eta \rightarrow \infty$, and D_n is otherwise arbitrary.

From Eq. (24a)

$$f_1'' = D_1 \frac{e^{-\eta}}{\eta} \tag{29a}$$

and by subsequent integrations

$$f_1' = D_1 \int_\infty^\eta \frac{e^{-\eta}}{\eta} d\eta + E_1 \tag{29b}$$

and

$$f_1 = D_1 \eta \int_\infty^\eta \frac{e^{-\eta}}{\eta} d\eta + D_1 e^{-\eta} + E_1 \eta + F_1 \tag{29c}$$

In the limit as $\eta \to R_L$ and $R_L \to 0$, $f_1 \to 0$. Hence,

$$F_1 = -D_1$$

Since $f_1' \to 0$ as $\eta \to \infty$, then

$$E_1 = 0 \tag{30}$$

Therefore, Eqs. (29) become

$$f_1'' = \frac{e^{-\eta}}{\eta} \tag{31a}$$

$$f_1' = \int_\infty^\eta \frac{e^{-\eta}}{\eta} d\eta = -ei(\eta) \tag{31b}$$

$$f_1 = \eta \int_\infty^\eta \frac{e^{-\eta}}{\eta} d\eta + e^{-\eta} - 1 \tag{31c}$$

Now writing Eq. (24b) as

$$f_2''' + (1 + \frac{1}{\eta}) f_2'' = -\frac{f_1 f_1''}{\eta}$$

and integrating

$$\eta e^{\eta} f_2'' = -\int^{\eta} \frac{f_1}{\eta}\, d\eta + C_1 \tag{32}$$

where C_1 is a constant.

Substituting Eq. (31c) in Eq. (32) gives

$$\eta e^{\eta} f_2'' = -\int^{\eta} d\eta \int_{\infty}^{\eta} \frac{e^{-\bar{\eta}}}{\bar{\eta}}\, d\bar{\eta} - \int^{\eta} \frac{e^{-\eta}}{\eta}\, d\eta + \ln \eta + C_1$$

or

$$\eta e^{\eta} f_2'' = (1+\eta)\int_{\eta}^{\infty} \frac{e^{-\eta}}{\eta}\, d\eta - e^{-\eta} + \ln \eta + C_2 \tag{33}$$

In the limit as $\eta \to R_L$ and $R_L \to 0$, the right hand side of Eq. (33) becomes

$$-\ln C - 1 + C_2 = D_2 \equiv E_1$$

and by Eq. (30)

$$C_2 = 1 + \ln C$$

Integrating Eq. (33) gives

$$f_2' - E_2 = \int_{R_L}^{\eta} \frac{e^{-\eta}}{\eta}\left[(1+\eta)\int_{\eta}^{\infty} \frac{e^{-\bar{\eta}}}{\bar{\eta}}\, d\bar{\eta} - e^{-\eta} + \ln C\eta + 1\right] d\eta \tag{34}$$

and since $f_2' \to 0$ as $\eta \to \infty$, Eq. (34) yields in the limit as $\eta \to R_L$ and $R_L \to 0$

$$E_2 = -\int_0^{\infty} \frac{e^{-\eta}}{\eta}\left[(1+\eta)\int_{\eta}^{\infty} \frac{e^{-\bar{\eta}}}{\bar{\eta}}\, d\bar{\eta} - e^{-\eta} + \ln C\eta + 1\right] d\eta$$

$$= -2\ln 2 - \frac{\pi^2}{12}$$

(Cf. Appendix H for evaluation of the integral.)

Hence, by Eq. (28),

$$D_3 = E_2 = -2\ln 2 - \frac{\pi^2}{12} \tag{35}$$

In a similar manner, other constants may be determined, although it is necessary to resort to numerical integration.

It is now necessary to justify the initial assumptions, and with the above analytical solution this can be readily done (Cf. Appendix I). For this purpose it is sufficient to use only the dominant terms of the expansion of the solution since the error made is small for sufficiently-small values of R_L . Consequently, the following considerations will be based on the expression

$$\frac{\bar{u}}{U} = 1 - \frac{ei(\eta)}{ei(R_L)} \tag{36}$$

which is the Oseen solution.

The following orders of magnitude are obtained from Eq. (36):

$$\frac{\delta^*}{r_0} = O\left[\frac{1}{\sqrt{R_L\, ei(R_L)}}\right] \quad , \quad \frac{\delta^*}{x} = O\left[\frac{1}{\sqrt{R_x\, ei(R_L)}}\right] \quad ,$$

$$\frac{\Delta p}{\rho U^2} = O\left[\frac{1}{R_x\, ei(R_L)}\right] \quad ,$$

$$\left.\frac{\frac{\partial p}{\partial x}}{\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)}\right|_{r_0} = O\left[\frac{R_L}{R_x}\, ei(R_L)\right] \quad , \quad \frac{\frac{\partial^2 u}{\partial x^2}}{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)} = O\left(\frac{1}{R_x}\right)$$

where δ^* is the boundary layer displacement thickness and Δp is the pressure difference across the boundary layer. Thus, it is easily seen that the initial assumptions are valid when $R_x \gg 1$ and $R_L \ll 1$ even though $\delta^*/r_o \gg 1$ for $R_L \ll 1$. Also, it is seen that $\delta^* \sim x^{1/2}$, so the effective shape of the body due to the boundary layer displacement thickness is another paraboloid of revolution. The axial pressure gradient along this effective body again diminishes with x; consequently, the pressure difference across the boundary layer will be small at large distances from the leading edge, as shown by the order of magnitude argument.

The local skin friction, C_f, is given by

$$C_f = \frac{\left(\mu \frac{\partial u}{\partial r}\right)_{r=r_o}}{\frac{1}{2}\rho U^2}$$

$$= 2\sqrt{\frac{R_L}{R_x}}\ \frac{\partial^2 f}{\partial \eta^2}\Big|_{\eta = R_L}$$

or

$$C_f\sqrt{R_x} = \frac{2}{\sqrt{R_L}} \sum_{n=1}^{\infty} \frac{D_n}{\left(\ln \frac{1}{R_L C}\right)^n}$$

$$C_f\sqrt{R_x} = \frac{2}{\sqrt{R_L}}\left[\frac{1}{\ln \frac{1}{R_L C}} - \frac{2.210}{\left(\ln\frac{1}{R_L C}\right)^3} + \cdots\right] \tag{37}$$

This result represents an analytical extension of the numerical solution to larger values of β or smaller values of R_L. Figure 2 shows the combined solutions. It is seen that Eq. (37) is valid when $R_L < .02$ approximately.

D. Approximate Solutions of the Governing Equations for Flows Past Paraboloids of Revolution by the von Kármán Integral Method

Following von Kármán (Ref. 6) an approximate solution of the governing partial differential equations for the flow past paraboloids of revolution will be obtained. The accuracy of this approximate solution can then be determined from a comparison with the exact solution of the previous section. Once sufficient accuracy is established, this approximate method may be applied to other body shapes under the same conditions.

In the preceding analysis, it was seen that the transformation to the new coordinate system $(\bar{x},\bar{r})$ played an essential part in facilitating the solution over an extensive range of R_L. Consequently, this consistency will be maintained in the application of the von Kármán integral method.

The von Kármán integral form of the governing equations will be obtained in cylindrical coordinates since it was shown previously that similarity of the velocity profiles of the flow past paraboloids of revolution stems only from the use of this system of coordinates. Hence, the governing equations will be integrated across the boundary layer from $r = r_0$ to $r = r_\delta = r_0 + \delta$, where δ is the thickness of the boundary layer.

Combining Eqs. (2a) and (2b) gives

$$\frac{\partial(u^2 r)}{\partial x} + \frac{\partial(u v r)}{\partial r} = -\frac{r}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right)$$

Hence, on integration (with x fixed),

$$\int_{r_0}^{r_\delta} \frac{\partial(u^2 r)}{\partial x}\, dr + u v r \Big|_{r_0}^{r_\delta} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\left(\frac{r_\delta^2 - r_0^2}{2}\right) + \nu r \frac{\partial u}{\partial r}\Big|_{r_0}^{r_\delta} \qquad (39)$$

By the boundary conditions: $u = v = 0$ at $r = r_0$ and $u = U$ and $\frac{\partial u}{\partial r} = 0$ at $r = r_\delta$, Eq. (39) reduces to

$$\int_{r_0}^{r_\delta} \frac{\partial (u^2 r)}{\partial x}\, dr + U(vr)_{r=r_\delta} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\left(\frac{r_\delta^2 - r_0^2}{2}\right) - \nu\left(r\frac{\partial u}{\partial r}\right)_{r=r_0} \tag{40}$$

Now integrating Eq. (2a) across the boundary layer gives

$$\int_{r_0}^{r_\delta} \frac{\partial (ur)}{\partial x}\, dr + vr\Big|_{r_0}^{r_\delta} = 0$$

or

$$(vr)_{r_\delta} = -\int_{r_0}^{r_\delta} \frac{\partial (ur)}{\partial x}\, dr$$

which, when combined with Eq. (40) yields

$$\int_{r_0}^{r_\delta}\left[\frac{\partial (U^2)}{\partial x} - U\frac{\partial u}{\partial x}\right] r\,dr = -\frac{1}{\rho}\frac{\partial p}{\partial x}\left(\frac{r_\delta^2 - r_0^2}{2}\right) - \nu\left(r\frac{\partial u}{\partial r}\right)_{r_0} \tag{41}$$

But

$$\int_{r_0}^{r_\delta}\left[\frac{\partial u^2}{\partial x} - U\frac{\partial u}{\partial x}\right] r\,dr = \int_{r_0}^{r_\delta}\left[\frac{\partial u^2}{\partial x} - \frac{\partial (Uu)}{\partial x} + u\frac{dU}{dx}\right] r\,dr$$

$$= -\frac{d}{dx}\int_{r_0}^{r_\delta} u(U-u)\,r\,dr + \frac{dU}{dx}\int_{r_0}^{r_\delta} ur\,dr$$

Hence, Eq. (41) becomes

$$\frac{d}{dx}\int_{r_0}^{r_\delta} u\,(U-u)\,r\,dr - \frac{dU}{dx}\int_{r_0}^{r_\delta} u\,r\,dr = \frac{1}{\rho}\,\frac{\partial p}{\partial x}\left(\frac{r_\delta^2-r_0^2}{2}\right) + \nu\left(r\,\frac{\partial u}{\partial r}\right)_{r_0} \qquad (42)$$

By virtue of the extended Mangler transformation, Eq. (42) reduces to the two-dimensional form ($u = \bar{u}$):

$$\frac{d}{d\bar{x}}\int_0^{\Delta} \bar{u}\,(\bar{U}-\bar{u})\,d\bar{r} - \frac{d\bar{U}}{d\bar{x}}\int_0^{\Delta} \bar{u}\,d\bar{r} = \frac{1}{\rho}\,\frac{\partial \bar{p}}{\partial \bar{x}}\,\Delta + \nu\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)_{\bar{r}=0} \qquad (43)$$

where

$$\Delta = \frac{r_\delta^2 - r_0^2}{2L}$$

In the region where the axial pressure gradient may be neglected, Eq. (43) becomes

$$\frac{d}{d\bar{x}}\int_0^{\Delta} \bar{u}\,(\bar{U}-\bar{u})\,d\bar{r} = \nu\left(\frac{\partial \bar{u}}{\partial \bar{r}}\right)_{\bar{r}=0} \qquad (44)$$

Eqs. (43) and (44) could have been obtained directly from Eq. (5).

The accuracy of this approximate method depends strongly on the choice of the analytical form of the velocity function u , and whenever possible it is convenient to choose elementary functions in order to simplify the computations. The precise form of u depends on the number of boundary conditions imposed on it. In the following considerations, several forms for the velocity function will be used, and the accuracy and range of application of each result will be discussed.

The profiles chosen will satisfy some or all of the following boundary conditions:

$$u = 0 \quad , \quad \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = 0 \qquad \text{at} \qquad r = r_o$$

$$u = u_1 \quad , \quad \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2} = 0 \qquad \text{at} \qquad r = r_\delta$$

In the transformed coordinates they become

$$\bar{u} = 0 \quad , \quad \frac{\partial}{\partial \bar{r}}\left[\left(1 + \frac{2L\bar{r}}{r_o^2}\right)\frac{\partial \bar{u}}{\partial \bar{r}}\right] = 0 \qquad \text{at} \qquad \bar{r} = 0$$

$$\bar{u} = U \quad , \quad \frac{\partial \bar{u}}{\partial \bar{r}} = \frac{\partial^2 \bar{u}}{\partial \bar{r}^2} = 0 \qquad \text{at} \qquad \bar{r} = \Delta$$

Let $F = \bar{u}/U$, $\lambda = \dfrac{\bar{r}}{\Delta}$, and $\alpha = \dfrac{2L\Delta}{r_o^2}$. The boundary conditions expressed in terms of these parameters are

(1) $F = 0$ at $\lambda = 0$

(2) $\dfrac{\partial^2 F}{\partial \lambda^2} + \alpha \dfrac{\partial F}{\partial \lambda} = 0$ at $\lambda = 0$

(3) $F = 1$ at $\lambda = 1$

(4) $\dfrac{\partial F}{\partial \lambda} = 0$ at $\lambda = 1$

(5) $\dfrac{\partial^2 F}{\partial \lambda^2} = 0$ at $\lambda = 1$

where F is, in general, a function of both λ and α.

The governing equation is thus

$$\Delta \frac{d}{d\bar{x}} [\Delta I] = \frac{\nu}{U} \left(\frac{\partial F}{\partial \lambda} \right)_{\lambda = 0}$$

or

$$\alpha \frac{d}{d\bar{x}} (r_0^2 \alpha I) = \frac{4 \nu L^2}{U r_0^2} \left(\frac{\partial F}{\partial \lambda} \right)_{\lambda = 0} \tag{45}$$

where

$$I = \int_0^1 (F - F^2)\, d\eta \equiv I(\alpha)$$

For a paraboloid of revolution

$$r_0^2 = 4 L x = 2\sqrt{2}\, L^{3/2} \bar{x}^{1/2} \tag{46}$$

which combines with Eq. (45) to give

$$\alpha \frac{d}{d\bar{x}} (\bar{x}^{1/2} \alpha I) = \frac{\nu}{2 U L \bar{x}^{1/2}} \left(\frac{\partial F}{\partial \lambda} \right)_{\lambda = 0} \tag{47}$$

Hence, it is seen that

$$\alpha = \alpha_0 = \text{constant} \tag{48}$$

is a particular solution of Eq. (47). Since

$$\alpha = \frac{2 L \Delta}{r_0^2} = \frac{r_\delta^2 - r_0^2}{r_0^2} = \left(\frac{\delta}{r_0} + 1 \right)^2 - 1 \tag{49}$$

then α = constant implies that $\delta \sim r_0 \sim x^{1/2}$ or that the boundary layer grows parabolically. This deduction agrees with the result of the exact analysis (Cf. Appendix I).

Therefore, Eq. (47) reduces to

$$\alpha_o^2 \, I(\alpha_o) = \frac{1}{R_L} \left(\frac{\partial F}{\partial \lambda} \right)_{\lambda = 0 \, , \, \alpha = \alpha_o}$$

or

$$R_L = \frac{1}{\alpha_o^2 \, I(\alpha_o)} \left(\frac{\partial F}{\partial \lambda} \right)_{\lambda = 0 \, , \, \alpha = \alpha_o} \qquad (50)$$

which gives R_L as an explicit function of α_o .

Following Pohlhausen (Ref. 7), a fourth degree polynomial will be tried of the form

$$F = a\lambda + b\lambda^2 + c\lambda^3 + d\lambda^4 \qquad (51)$$

The constants a , b , c , and d are determined from all the boundary conditions; hence,

$$a = \frac{12}{6-\alpha} \; , \quad b = \frac{6\alpha}{6-\alpha} \; , \quad c = \frac{8\alpha - 12}{6-\alpha} \; , \quad d = \frac{6-3\alpha}{6-\alpha} \qquad (51a)$$

It is seen that when $\alpha \to 0$,

$$F \to 2\lambda - 2\lambda^3 + \lambda^4$$

which agrees with the two-dimensional profile assumed by Pohlhausen under the same boundary conditions for $\alpha = 0$.

Thus, from Eq. (50),

$$R_L = \frac{12}{\alpha_o^2 \, (6 - \alpha_o) \, I(\alpha_o)} \qquad (52)$$

where

$$I(\alpha_0) = \frac{4.200 - 0.600\alpha_0}{6-\alpha_0} - \frac{20.971 - 6.315\,\alpha_0 + 0.486\,\alpha_0^2}{(6-\alpha_0)^2}$$

In general, the local skin friction is given by

$$C_f = \frac{2\mu \left.\frac{\partial u}{\partial r}\right|_{r_0}}{\rho U^2} = \frac{2\nu r_0}{U^2 L}\left.\frac{\partial u}{\partial \bar{r}}\right|_{\bar{r}=0} = \frac{2\nu r_0}{UL\Delta}\left(\frac{\partial F}{\partial \lambda}\right)_{\lambda=0}$$

$$= \frac{4\nu}{U r_0\,\alpha}\left.\frac{\partial F}{\partial \lambda}\right|_{\lambda=0} \tag{53}$$

For a paraboloid of revolution,

$$C_f\sqrt{R_x} = \frac{2}{\sqrt{R_L}}\,\frac{1}{\alpha_0}\left.\frac{\partial F}{\partial \lambda}\right|_{\substack{\lambda=0\\ \alpha=\alpha_0}} \tag{54}$$

Thus, for the fourth degree polynomial, the solution is given jointly by Eqs. (52) and (54). Figure 3 shows the comparison of this solution with the exact solution. The agreement is within 3.3% down to a value of R_L of about 7. The increase in $C_f\sqrt{R_x}$ at $R_L = 7$ over the corresponding Mangler value* is about 21%.

From the Oseen solution it was seen that for η near R_L and R_L small, the velocity behaves logarithmically in the η direction. Consequently, the following profile is assumed:

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* $\left(C_f\sqrt{R_x}\right)_{R_L\to\infty}$ gives the Mangler value.

$$F = \frac{\ln(1+\alpha\lambda)}{\ln(1+\alpha)} \tag{55}$$

By inspection, this function satisfies boundary conditions (1), (2), and (3). Since for $|\theta| < 1$

$$\ln(1+\theta) = \theta - \frac{\theta^2}{2} + \frac{\theta^3}{3} - \cdots \tag{56}$$

$F \to \lambda$ as $\alpha \to 0$, and hence, the linear profile is retained for small values of α.

Substituting Eq. (55) into Eqs. (50) and (54) yields the solution in the form

$$R_L = \frac{1}{\alpha_0 I(\alpha_0)} \frac{1}{\ln(1+\alpha_0)} \tag{57a}$$

where

$$I(\alpha_0) = \frac{1}{\ln(1+\alpha_0)} \left[1 + \frac{2}{\alpha_0} - \frac{2}{\ln(1+\alpha_0)} \right]$$

and

$$C_f \sqrt{R_x} = \frac{2}{\sqrt{R_L}} \frac{1}{\ln(1+\alpha_0)} \tag{57b}$$

From Fig. 2 it is seen that this solution applies for all values of R_L, approaching the exact solution for small values of R_L and deviating from it within 15% for large values of R_L.

Better accuracy may be obtained at large values of R_L when a profile is chosen which satisfies more boundary conditions. By assuming the function

$$F = a_1 \frac{\ln(1+\alpha\lambda)}{\ln(1+\alpha)} + a_2 \lambda \tag{57}$$

to satisfy boundary conditions (1), (3), and (4), the constants a_1 and a_2 thus determined are

$$a_1 = \frac{1}{1-A} \qquad \text{and} \qquad a_2 = \frac{A}{A-1}$$

where

$$A = \frac{\alpha}{1+\alpha} \frac{1}{\ln(1+\alpha)}$$

Thus, when $\alpha \rightarrow 0$

$$F \rightarrow 2\lambda - \lambda^2$$

which is the two-dimensional form. A comparison with the exact solution (Cf. Fig. 4) shows that the accuracy in this case for large values of R_L is within 10% for the skin friction. For $\alpha \rightarrow \infty$, $a_1 \rightarrow 1$ and $a_2 \rightarrow 0$; hence, the logarithmic form is retained for small values of R_L , and the solution approaches that given by Eqs. (57), (Cf. Appendix J).

If the form

$$F = a_1 \frac{\ln(1+\alpha\lambda)}{\ln(1+\alpha)} + a_2 \lambda + a_3 \lambda^2 \tag{59}$$

is chosen satisfying boundary conditions (1), (2), (3), and (4), then the constants determined from these conditions are

$$a_1 = \frac{2\alpha - 2}{D} \quad , \quad a_2 = 2\frac{A}{D} \qquad \text{and} \qquad a_3 = -\frac{\alpha A}{D}$$

where $A = \dfrac{\alpha}{1+\alpha} \dfrac{1}{\ln(1+\alpha)}$ and $D = (2-\alpha)A + 2\alpha - 2$

Thus, as $\alpha \rightarrow 0$

$$F \to \frac{3}{2}\lambda - \frac{1}{2}\lambda^2$$

and the two-dimensional form is retained for large values of R_L and yields an accuracy within 3% for the skin friction (Cf. Fig. 5); and as

$$\alpha \to \infty \quad , \quad a_1 \to 1 \quad , \quad a_2 \to 0 \quad , \quad a_3 \to 0 \, ,$$

the logarithmic form is retained for small values of R_L (Cf. Appendix K). Finally, if the form

$$F = a_1 \frac{\ln(1+\alpha\lambda)}{\ln(1+\alpha)} + a_2\lambda + a_3\lambda^2 + a_4\lambda^3 \tag{60}$$

is chosen satisfying all the boundary conditions, then the constants in this case are

$$a_1 = \frac{6-3\alpha}{E} \quad , \quad a_2 = -\frac{6A+3B}{E}$$

$$a_3 = \frac{\alpha}{2}\,\frac{6A+3B}{E} \quad \text{and} \quad a_4 = -\frac{A\alpha + B(1-\alpha)}{E}$$

where

$$A \equiv \frac{\alpha}{1+\alpha}\,\frac{1}{\ln(1+\alpha)} \quad , \quad B \equiv \left(\frac{\alpha}{1+\alpha}\right)^2 \frac{1}{\ln(1+\alpha)}$$

and

$$E \equiv A(2\alpha-6) + B\left(\frac{\alpha}{2}-2\right) + 6 - 3\alpha$$

Thus, as $\alpha \to 0$

$$F \to 2\eta - 2\eta^3 + \eta^4$$

and the two-dimensional form is retained, yielding an accuracy within 3.2% for the skin friction at large values of $\mathcal{R}_L$; and as $\alpha \to \infty$

$$a_1 \to 1 \quad , \quad a_2 \to 0 \quad , \quad a_3 \to 0 \quad , \quad a_4 \to 0$$

the logarithmic form is retained for small values of $\mathcal{R}_L$ (Cf. Appendix L). Figure 6 shows the comparison of this solution with the exact solution.

E. Approximate Solutions of the Governing Equations for Flows Past Circular Cylinders by the von Karman Integral Method

The results of Stewartson (Ref. 5) and Seban and Bond (Ref. 4) show that the basic parameter for flows past circular cylinders is

$$\mathcal{R} \equiv \frac{U r_0^2}{4 \nu x} \tag{61}$$

In terms of this parameter, Stewartson's result is

$$c_f \sqrt{\mathcal{R}_x} = \frac{2}{\sqrt{\mathcal{R}}} \left[\frac{1}{\ln \frac{1}{C\mathcal{R}}} - \frac{3.854}{\left(\ln \frac{1}{C\mathcal{R}}\right)^3} + \cdots \right] \tag{62}$$

where $\ln \frac{1}{C\mathcal{R}} = -\ln \mathcal{R} - 0.577$; Seban and Bond's result is

$$c_f \sqrt{\mathcal{R}_x} = 0.664 + \frac{0.698}{\sqrt{\mathcal{R}}} - \frac{0.480}{\mathcal{R}} + \cdots \tag{63}$$

where the numerical coefficients have been corrected by Kelly (Ref. 8).

Figure 8 shows that Seban and Bond's result is quite accurate down to values of about 0.7 for $\mathcal{R}$, which indicates that the convergence of Eq. (63) is quite rapid; Stewartson's result seems to be valid up to values of about 0.01 for $\mathcal{R}$.

With the facility of these exact solutions for the circular cylinders at small and large values of $\mathcal{R}$, the accuracy and the range of application of the solutions obtained from the von Kármán integral method in conjunction with some of the assumed velocity profiles of the previous section can again be determined.

For a circular cylinder let $L = r_0$; then $\bar{x} = x$ and Eq. (45) becomes

$$\alpha \frac{d}{d\bar{x}} (\alpha I) = \frac{4\nu}{U r_0^2} \left(\frac{\partial F}{\partial \lambda}\right)_{\lambda=0} \tag{64}$$

where

$$I = \int_0^1 (F - F^2)\, d\lambda = I(\alpha)$$

Eq. (64) may be written in the following form as

$$\left(\alpha^2 \frac{dI}{d\alpha} + \alpha I\right) d\alpha = \frac{4\nu}{U r_0^2} \left(\frac{\partial F}{\partial \lambda}\right)_{\lambda=0} dx$$

whence

$$\frac{1}{R} \equiv \int_0^x \frac{4\nu}{U r_0^2}\, dx = \int_0^\alpha \frac{\alpha^2 \frac{dI}{d\alpha} + \alpha I}{\left(\frac{\partial F}{\partial \lambda}\right)_{\lambda=0}}\, d\alpha \tag{65}$$

Thus, it remains to carry out the integration for any assumed velocity function F.

For the profile given by Eq. (51),

$$I(\alpha) = \frac{4.200 - 0.600\,\alpha}{6 - \alpha} - \frac{20.971 - 6.315\,\alpha + 0.486\,\alpha^2}{(6 - \alpha)^2}$$

and

$$\left.\frac{\partial F}{\partial \lambda}\right|_0 = \frac{12}{6 - \alpha}$$

Hence,

$$\frac{1}{R} = \int_0^{\alpha}\left[\frac{4.229\alpha - 2.970\alpha^2 + 0.342\alpha^3}{12(6-\alpha)} - \frac{8.458\alpha^2 - 2.970\alpha^3 + 0.228\alpha^4}{(6-\alpha)^2}\right]d\alpha$$

$$= 2.459 + 0.107(6-\alpha) - 0.028(6-\alpha)^2 + 0.003(6-\alpha)^3 - \frac{3.462}{6-\alpha} - 1.216\ln(1+\alpha).$$

The local skin friction is

$$C_f = \frac{\left(\mu\frac{\partial u}{\partial r}\right)_{r_0}}{\frac{1}{2}\rho U^2} = \frac{4\nu}{U r_0 \alpha}\left.\frac{\partial F}{\partial \lambda}\right|_0$$

or

$$C_f\sqrt{R_x} = \frac{2}{\alpha\sqrt{R}}\left.\frac{\partial F}{\partial \lambda}\right|_0 \tag{67}$$

For the above profile this gives

$$C_f\sqrt{R_x} = \frac{24}{\alpha(6-\alpha)\sqrt{R}} \tag{68}$$

Thus, the solution is given jointly by Eqs. (66) and (68) for the fourth degree polynomial. Fig. 7 shows the range of applicability of this solution and shows that the accuracy attained is within 3.3% of the exact solution of Seban and Bond.

For the profile given by Eq. (55)

$$I(\alpha) = \frac{1}{\ln(1+\alpha)} - \frac{2}{[\ln(1+\alpha)]^2} + \frac{2}{\alpha\ln(1+\alpha)}$$

and

$$\left.\frac{\partial F}{\partial \lambda}\right|_0 = \frac{\alpha}{\ln(1+\alpha)}$$

Hence,

$$\frac{1}{R} = \int_0^{\alpha} \left[1 - \frac{3}{\ln(1+\alpha)} - \frac{1}{1+\alpha}\frac{1}{\ln(1+\alpha)} + \frac{4\alpha}{1+\alpha}\frac{1}{\ln^2(1+\alpha)} \right] d\alpha$$

Although there is an apparent singularity of the integral at the lower limit, it is a removable singularity, and the integral has a finite value. Hence,

$$\frac{1}{R} = \alpha - \frac{4\alpha}{\ln(1+\alpha)} + 4 + \int_{-\infty}^{\ln(1+\alpha)} \frac{e^{\alpha}}{\alpha} d\alpha - \ln[\ln(1+\alpha)] - \gamma \tag{69a}$$

which is valid for $\alpha > 1$. The following expansion is valid for all α:

$$\frac{1}{R} = \alpha + 4 - \frac{4\alpha}{\ln(1+\alpha)} + \sum_{n=1}^{\infty} \frac{\ln^n(1+\alpha)}{n \cdot n!} \tag{69b}$$

(Cf. Appendix M.)

The local skin friction is

$$C_f \sqrt{R_x} = \frac{2}{\sqrt{R}} \frac{1}{\ln(1+\alpha)} \tag{70}$$

Thus, the solution is given jointly by Eqs. (69) and (70). The comparison of this solution with the exact solution is shown in Fig. 8, from which it is seen that the accuracy is almost the same as that for a paraboloid with the same profile.

For other profiles, it is necessary to resort to numerical integration, but sufficient analysis has been given to demonstrate the general procedure for any particular case.

III. LAMINAR BOUNDARY LAYER CHARACTERISTICS OF SLENDER BODIES OF REVOLUTION IN AXIAL COMPRESSIBLE FLOW

A. Governing Equations in Cylindrical Coordinates

If the following assumptions are made

$$p = R\rho T$$

$$h = c_p T \quad ; \quad c_p = \text{constant}$$

$$\sigma = \mu c_p / k = \text{constant}$$

$$\gamma = c_p / c_v = \text{constant}$$

$$\lambda = -\tfrac{2}{3}\mu$$

$$F^i = Q = 0$$

then the governing equations for a steady axial compressible flow past a non-spinning body of revolution are

$$\frac{\partial(\rho u r)}{\partial x} + \frac{\partial(\rho v r)}{\partial r} = 0 \tag{71a}$$

$$\rho r\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r}\right) = 2\frac{\partial}{\partial x}\left(\mu r\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial r}\left[\mu r\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r}\right)\right] - r\frac{\partial p}{\partial x}$$
$$- \frac{2}{3} r\frac{\partial}{\partial x}\left\{\frac{\mu}{r}\left[\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r}\right]\right\} \tag{71b}$$

$$\rho r\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r}\right) = -r\frac{\partial p}{\partial r} + \frac{\partial}{\partial x}\left[\mu r\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right)\right] + 2\frac{\partial}{\partial r}\left(\mu r\frac{\partial v}{\partial r}\right) - \frac{2\mu v}{r}$$
$$- \frac{2}{3} r\frac{\partial}{\partial r}\left\{\frac{\mu}{r}\left[\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r}\right]\right\} \tag{71c}$$

$$\rho\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial r}\right) - u\frac{\partial p}{\partial x} - v\frac{\partial p}{\partial r} = \Phi + \frac{1}{r}\frac{\partial}{\partial x}\left(\frac{r\mu}{\sigma}\frac{\partial h}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu}{\sigma} r\frac{\partial h}{\partial r}\right) \tag{71d}$$

where

$$\Phi = 2\mu\left(\frac{\partial u}{\partial x}\right)^2 + \mu\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r}\right) + \mu\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right) + 2\mu\frac{\partial v}{\partial r}$$
$$+ 2\mu\left(\frac{v}{r}\right)^2 - \frac{2}{3}\mu\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r}\right]^2$$

By the assumptions

$$v \ll u \qquad \text{and} \qquad \frac{\partial}{\partial x} \ll \frac{\partial}{\partial r} \tag{72}$$

Eqs. (71) can be simplified, as in the incompressible case, to

$$\frac{\partial(\varrho u r)}{\partial x} + \frac{\partial(\varrho v r)}{\partial r} = 0 \tag{73a}$$

$$\varrho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r}\right) = -\frac{\partial p}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial u}{\partial r}\right) \tag{73b}$$

$$0 = \frac{\partial p}{\partial r} \tag{73c}$$

$$\varrho u\frac{\partial h}{\partial x} + \varrho v\frac{\partial h}{\partial r} = u\frac{\partial p}{\partial x} + \mu\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu}{\sigma}r\frac{\partial h}{\partial r}\right) \tag{73d}$$

These equations will be called the modified laminar boundary layer equations for a compressible fluid; they are valid only within the above assumptions.

B. Existence of the Energy Integral

For $\sigma = 1$, an energy integral may be deduced from Eqs. (73). If Eq. (73b) is multiplied by u and combined with Eq. (73d), the following form results:

$$\left(\rho u \frac{\partial}{\partial x} + \rho v \frac{\partial}{\partial r}\right)\left(h + \frac{u^2}{2}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\mu \frac{\partial}{\partial r}\left(h + \frac{u^2}{2}\right)\right] \tag{74}$$

Hence, it is seen that

$$h + \frac{u^2}{2} = a = \text{constant} \tag{75}$$

is a particular integral of Eq. (74).

If $\frac{\partial p}{\partial x} = 0$, a particular integral of Eq. (74) is

$$h + \frac{u^2}{2} = a + bu \tag{76}$$

since it reduces Eq. (74) identically to the momentum equation in the x direction. Hence, the energy integrals given by Eqs. (75) and (76) exist in axi-symmetric flow as well as in two-dimensional flow.

The constants a and b may be determined from the following boundary conditions:

$$h = h_1 \quad \text{at} \quad u = u_1$$

$$h = h_o \quad \text{at} \quad u = 0$$

$$\frac{\partial h}{\partial r} = q_o \quad \text{at} \quad u = 0$$

(Subscripts "1" and "o" denote conditions in the free stream and at the wall, respectively.) Under the first two conditions,

$$b = \frac{1}{u_1}\left(h_1 - h_o + \frac{u_1^2}{2}\right) \quad \text{and} \quad a = h_o$$

Hence, Eq. (76) becomes

$$h = h_o + \left(h_1 - h_o + \frac{u_1^2}{2}\right)\frac{u}{u_1} - \frac{1}{2}u^2 \tag{77}$$

If the body surface is insulated, then

$$\frac{\partial h}{\partial r} = \left(h_1 - h_0 + \frac{u_1^2}{2}\right)\frac{\partial}{\partial r}\left(\frac{u}{u_1}\right) - u\frac{\partial u}{\partial r} = 0 \qquad \text{at} \qquad u = 0$$

or

$$\left(h_1 - h_0 + \frac{u_1^2}{2}\right)\frac{\partial}{\partial r}\left(\frac{u}{u_1}\right)\bigg|_{u=0} = 0$$

Since $\left.\frac{\partial u}{\partial r}\right|_{u=0}$, or the local skin friction, is not zero, then

$$h - h_0 + \frac{u_1^2}{2} = 0$$

or

$$h_1 + \frac{u_1^2}{2} = h_0$$

Thus, for an insulated surface, Eq. (77) may be rewritten in the form

$$\frac{h}{h_1} = \frac{h_0}{h_1} - \frac{u^2}{2h_1}$$

or

$$\frac{h}{h_1} = 1 + \frac{\gamma-1}{2}M_1^2 - \frac{\gamma-1}{2}M_1^2\left(\frac{u}{u_1}\right)^2 \tag{78}$$

since $\frac{u_1^2}{2h_1} = \frac{\gamma-1}{2}M_1^2$, where M_1 is the free stream Mach number.

C. Simplification of the Modified Compressible Boundary Layer Equations for Flows Past Paraboloids of Revolution

Eqs. (73) can be simplified by an analysis similar to that used previously. If the extended Mangler transformation is applied to this system, it reduces to (Cf. Appendix N)

$$\frac{\partial(\bar{\rho}\bar{u})}{\partial\bar{x}} + \frac{\partial(\bar{\rho}\bar{v})}{\partial\bar{r}} = 0 \tag{79a}$$

$$\bar{\rho}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{\rho}\bar{v}\frac{\partial \bar{u}}{\partial \bar{r}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial}{\partial \bar{r}}\left[\left(1+\frac{2L\bar{r}}{r_0^2}\right)\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{r}}\right] \tag{79b}$$

$$0 = \frac{\partial \bar{p}}{\partial \bar{r}} \tag{79c}$$

The energy integrals are invariant under the transformation since h is a function of u only and $u = \bar{u}$.

By assuming that the axial pressure gradient along paraboloids of revolution may be neglected, the pressure is then constant everywhere. Hence, from the equation of state,

$$\frac{\rho}{\rho_1} = \frac{h_1}{h} \tag{80}$$

By the power law for the viscosity,

$$\frac{\mu}{\mu_1} = \left(\frac{h}{h_1}\right)^{\omega}$$

For the present purpose, ω will be taken to be unity; hence

$$\frac{\mu}{\mu_1} = \frac{h}{h_1} \tag{81}$$

Also, an insulated surface will be considered; thus, Eq. (78) will be applicable.

If it is assumed that

$$\bar{u} = u_1\, g(\bar{\eta}) \tag{82}$$

where

$$\bar{\eta} = \frac{2L\bar{r}}{\beta_c\, r_0^2} \qquad \left(\beta_c = \sqrt{\frac{\nu_1}{2u_1 L}}\right)$$

then Eq. (79b) gives

$$\bar{\rho}\bar{v} = \sqrt{\frac{u_1}{\nu_1 x}}\,\frac{1}{g'}\left\{\left[(1+\beta_c\bar{\eta})\bar{\mu}g'\right]' + \nu_1\bar{\rho}\bar{\eta}gg'\right\}$$

which, when combined with Eq. (73a), yields

$$2\left\{\frac{1}{g'}\left[(1+\beta_c\bar{\eta})\frac{\bar{\mu}}{\mu_1}g'\right]'\right\}' + \frac{\bar{\rho}}{\rho_1}g = 0 \tag{83}$$

where $\bar{\rho}$ and $\bar{\mu}$ are functions of $\bar{u}$ or g only. (Primes denote differentiation with respect to $\bar{\eta}$.)

Therefore, the assumption that $\bar{u}$ is a function of $\bar{\eta}$ only is justified since Eq. (83) is an ordinary differential equation. By letting $g = \bar{f}'$ and letting $\bar{\rho}$ and $\bar{\mu}$ be constants, Eq. (83) reduces to the incompressible form:

$$2\frac{d}{d\bar{\eta}}\left[(1+\beta_c\bar{\eta})\bar{f}''\right] + \bar{f}\bar{f}'' = 0$$

The boundary conditions associated with Eq. (83) are

$$g = 0 \quad \text{at} \quad \bar{\eta} = 0 \tag{84a}$$

$$\left[(1+\beta_c\bar{\eta})\bar{\mu}g'\right]' = 0 \quad \text{at} \quad \bar{\eta} = 0 \tag{84b}$$

$$g \to 1 \quad \text{as} \quad \bar{\eta} \to \infty \tag{84c}$$

D. A Similarity Law for Hypersonic Flows Past Paraboloids of Revolution

Let $m = \frac{\gamma-1}{2}M_1^2$; then, for $M_1 \gg 1$,

$$m \gg 1$$

and Eq. (78) simplifies to

$$\bar{\rho}\bar{v} = \sqrt{\frac{u_1}{\nu_1 \bar{x}}}\,\frac{1}{g'}\left\{\left[(1+\beta_c\bar{\eta})\bar{\mu}g'\right]' + \nu_1\bar{\rho}\bar{\eta}gg'\right\}$$

which, when combined with Eq. (73a), yields

$$2\left\{\frac{1}{g'}\left[(1+\beta_c\bar{\eta})\frac{\bar{\mu}}{\mu_1}g'\right]'\right\}' + \frac{\bar{\rho}}{\rho_1}g = 0 \tag{83}$$

where $\bar{\rho}$ and $\bar{\mu}$ are functions of $\bar{u}$ or g only. (Primes denote differentiation with respect to $\bar{\eta}$.)

Therefore, the assumption that $\bar{u}$ is a function of $\bar{\eta}$ only is justified since Eq. (83) is an ordinary differential equation. By letting $g = \bar{f}'$ and letting $\bar{\rho}$ and $\bar{\mu}$ be constants, Eq. (83) reduces to the incompressible form:

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The boundary conditions associated with Eq. (83) are

$$g = 0 \quad \text{at} \quad \bar{\eta} = 0 \tag{84a}$$

$$\left[(1+\beta_c\bar{\eta})\bar{\mu}g'\right]' = 0 \quad \text{at} \quad \bar{\eta} = 0 \tag{84b}$$

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$$m \gg 1$$

and Eq. (78) simplifies to

$$\frac{h}{h_1} = m\left(1 - \frac{\bar{u}^2}{u_1^2}\right) \tag{85}$$

Combining Eq. (85) with Eq. (83) gives

$$2m^2\left\{\frac{1}{g'}\left[(1+\beta_c\bar{\eta})(1-g^2)g'\right]'\right\}' + \frac{g}{1-g^2} = 0 \tag{86}$$

Now let

$$1+\beta_c\bar{\eta} = (m\beta_c)^2\eta_1$$

then Eq. (86) becomes

$$2\frac{d}{d\eta_1}\left\{\frac{1}{\frac{dg}{d\eta_1}}\frac{d}{d\eta_1}\left[\eta_1(1-g^2)\frac{dg}{d\eta_1}\right]\right\} + \frac{g}{1-g^2} = 0 \tag{87}$$

The associated boundary conditions are

$$g = 0 \quad \text{at} \quad \eta_1 = \frac{1}{(m\beta_c)^2} = \frac{2R_{L_c}}{\left(\frac{\gamma-1}{2}M_1^2\right)^2} \qquad \left(R_{L_c} = \frac{u_1 L}{\nu_1}\right) \tag{88a}$$

$$\frac{d}{d\eta_1}\left[\eta_1(1-g^2)\frac{dg}{d\eta_1}\right] = 0 \quad \text{at} \quad \eta_1 = \frac{2R_{L_c}}{\left(\frac{\gamma-1}{2}M_1^2\right)^2} \tag{88b}$$

$$g \rightarrow 1 \quad \text{as} \quad \eta_1 \rightarrow \infty \tag{88c}$$

Therefore, the similarity law for $M_1 \gg 1$ is

$$\frac{R_{L_c}}{M_1^4} = K \qquad \left(R_{L_c} = \frac{u_1 L}{\nu_1}\right) \tag{89}$$

where K is a constant.

The local skin friction is now a function of K and R_x or

$$C_f\sqrt{R_{x_c}} = G(K) \qquad \left(R_{x_c} = \frac{u_1 x}{\nu_1}\right)$$

E. Approximate Solution of the Modified Compressible Boundary Layer Equations for Flows Past Insulated Paraboloids of Revolution by the von Kármán Integral Method

The following transformations

$$\left.\begin{aligned} dr_c &= \frac{\rho r}{\rho_0 L}\, dr \quad , & d\bar{x} &= \left(\frac{r_0}{L}\right)^2 dx \\ u &= \bar{u} \quad , & v &= \frac{\rho_0 L}{\rho r}\left[\left(\frac{r_0}{L}\right)^2 v_c - \frac{\partial r}{\partial x} u\right] \end{aligned}\right\} \qquad (9)$$

reduce Eqs. (73), with $\frac{\partial p}{\partial x} = 0$, to

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial v_c}{\partial r_c} = 0 \qquad (91a)$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + v_c \frac{\partial \bar{u}}{\partial r_c} = \nu_0 \frac{\partial}{\partial r_c}\left[\left(1 + \frac{2L}{r_0^2}\int_0^{r_c} \frac{\rho_0}{\rho}\, dr_c\right)\frac{\rho \mu}{\rho_0 \mu_0}\frac{\partial \bar{u}}{\partial r_c}\right] \qquad (91b)$$

(Cf. Appendix 0)

The boundary layer thicknesses are related by

$$\left(\frac{\delta}{r_0}\right)^2 = 1 + \frac{2L}{r_0^2}\int_0^{\Delta_c} \frac{\rho_0}{\rho}\, dr_c \qquad (92)$$

where Δ_c is the transformed boundary layer thickness.

Now, integrating Eqs. (91) across the boundary layer from $r_c = 0$ to $r_c = \Delta_c$, the momentum integral equation in the transformed coordinates is

$$\frac{d}{d\bar{x}}\int_0^{\Delta_c} \bar{u}\,(u_1 - \bar{u})\, dr_c = \nu_0\left(\frac{\partial \bar{u}}{\partial r_c}\right)_{r_c = 0} \qquad (93)$$

The velocity profiles chosen will satisfy some or all of the following boundary conditions:

$$u = 0 \quad , \quad \frac{\partial}{\partial r}\left(\mu r \frac{\partial u}{\partial r}\right) = 0 \qquad \text{at} \qquad r = r_o$$

$$u = U \quad , \quad \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2} = 0 \qquad \text{at} \qquad r = r_\delta$$

Also, an insulated surface will be considered. In the transformed coordinates, the boundary conditions become

$$\bar{u} = 0 \quad , \quad \frac{\partial^2 \bar{u}}{\partial r_c^2} + \frac{2L}{r_o^2}\frac{\partial \bar{u}}{\partial r_c} = 0 \qquad \text{at} \qquad r_c = 0$$

$$\bar{u} = u_1 \quad , \quad \frac{\partial \bar{u}}{\partial r_c} = \frac{\partial^2 \bar{u}}{\partial r_c^2} = 0 \qquad \text{at} \qquad r_c = \Delta_c$$

Let $F = \frac{\bar{u}}{u_1}$, $\lambda_c = \frac{r_c}{\Delta_c}$ and $\alpha_c = \frac{2L\Delta_c}{r_o^2}$.

Then Eq. (93) becomes

$$\Delta_c \frac{d}{d\bar{x}}\left[\Delta_c I\right] = \frac{\nu_o}{u_1}\left.\frac{\partial F}{\partial \lambda_c}\right|_{\lambda_c = 0}$$

or

$$\alpha_c \frac{d}{d\bar{x}}\left(r_o^2 \alpha_c I\right) = \frac{4\nu_o L^2}{u_1 r_o^2}\left(\frac{\partial F}{\partial \lambda_c}\right)_{\lambda_c = 0} \tag{94}$$

where $I(\alpha_c) = \int_0^1 (F - F^2)\, d\lambda_c$. The associated boundary conditions are:

(1) $F = 0$ at $\lambda_c = 0$

(2) $\frac{\partial^2 F}{\partial \lambda_c^2} + \alpha_c \frac{\partial F}{\partial \lambda_c} = 0$ at $\lambda_c = 0$

(3) $F = 1$ at $\lambda_c = 1$

(4) $\dfrac{\partial F}{\partial \lambda_c} = 0$ at $\lambda_c = 1$

(5) $\dfrac{\partial^2 F}{\partial \lambda_c^2} = 0$ at $\lambda_c = 1$

Henceforth, a solution in the transformed plane is equivalent to a solution in the physical plane under the transformations.

For a paraboloid of revolution and for a viscosity power law, Eq. (94) reduces to

$$\alpha_c \frac{d}{d\bar{x}}\left(\bar{x}^{1/2} \alpha_c I\right) = \frac{\nu_1 \left(\frac{T_0}{T_1}\right)^{\omega+1}}{2 u_1 L \bar{x}^{1/2}} \left(\frac{\partial F}{\partial \lambda_c}\right)_{\lambda_c = 0} \tag{95}$$

It is seen from Eq. (95) that

$$\alpha_c = \alpha_{c_0} = \text{constant}$$

is a particular solution. Since

$$\alpha_c = \frac{\left(\frac{\delta}{r_0} + 1\right)^2 - 1}{\int_0^1 \frac{\rho_0}{\rho}\, d\lambda_c}$$

then α_c = constant implies that the boundary layer grows parabolically as in the incompressible case.

Eq. (95) now reduces to

$$R_{L_c} = \frac{\left(\frac{T_0}{T_1}\right)^{\omega+1}}{\alpha_{c_0}^2 I(\alpha_{c_0})} \left(\frac{\partial F}{\partial \lambda_c}\right)_{\lambda_c = 0}, \quad \alpha_c = \alpha_{c_0}$$

or, since the surface is insulated,

$$R_{L_c} = \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\omega+1} \left[\frac{\frac{\partial F}{\partial \lambda_c}}{\alpha_c^2 I}\right]_{\lambda_c = 0,\ \alpha_c = \alpha_{c_0}} \tag{96}$$

The local skin friction is given by

$$C_f \sqrt{R_{x_c}} = \left(1 + \frac{\gamma-1}{2} M_1^2\right)^{\omega} \left[\frac{2}{\sqrt{R_{L_c}}} \frac{1}{\alpha_c} \frac{\partial F}{\partial \lambda_c}\right]_{\substack{\lambda_c = 0 \\ \alpha_c = \alpha_{c_0}}} \tag{97}$$

The solution for the compressible flow past paraboloids of revolution is thus given jointly by Eqs. (96) and (97); in view of their forms, the solution may be obtained directly from the incompressible results.

For $M_1 \gg 1$ and $\omega = 1$, Eq. (96) gives

$$\frac{R_{L_c}}{M_1^4} = \left(\frac{\gamma-1}{2}\right)^2 \left[\frac{\frac{\partial F}{\partial \lambda_c}}{\alpha_c^2 I}\right]_{\substack{\lambda_c = 0 \\ \alpha_c = \alpha_{c_0}}}$$

This relation then establishes a significance for the constant K in the hypersonic similarity law as given by Eq. (89).

An example will now be given to show the effect of compressibility. Let ω = 0.768, γ = 1.4, and

$$F = \frac{\ln(1 + \alpha_c \lambda_c)}{\ln(1+\alpha_c)}$$

For small values of α_c or large values of R_{L_c} , $F \rightarrow \lambda_c$, which is the linear profile in the Howarth coordinates (a good approximation for hypersonic Mach numbers).

Eqs. (96) and)97) give

$$R_{L_c} = \left(\frac{T_0}{T_1}\right)^{\omega+1} \left[\frac{1}{\alpha_c I} \frac{1}{\ln(1+\alpha_c)}\right]_{\alpha_c = \alpha_{c_0}}$$

where

$$I(\alpha_{c_0}) = \frac{1}{\ln(1+\alpha_{c_0})} \left[1 + \frac{2}{\alpha_{c_0}} - \frac{2}{\ln(1+\alpha_{c_0})}\right]$$

and

$$c_f \sqrt{R_{x_c}} = \left(\frac{T_0}{T_1}\right)^{\omega} \left[\frac{1}{\sqrt{R_{x_c}}} \frac{1}{\ln(1+\alpha_{c_0})}\right]$$

These results are shown in Fig. 9 for various values of the free stream Mach number M_1 .

F. Approximate Solution of the Modified Compressible Boundary Layer Equations for Flows Past Insulated Circular Cylinders by the von Kármán Integral Method

For a circular cylinder, $L \equiv r_0$, and Eq. (94) reduces to

$$\alpha_c \frac{d}{dx} (\alpha_c I_c) = \frac{4\nu_1}{u_1 r_0^2} \left(\frac{T_0}{T_1}\right)^{\omega+1} \left(\frac{\partial F}{\partial \lambda_c}\right)_{\lambda_c = 0}$$

where

$$I_c = \int_0^1 (F - F^2)\, d\lambda_c$$

By a slight manipulation, this can be written as

$$\left(\frac{T_0}{T_1}\right)^{\omega+1} \frac{1}{R_c} = \int_0^x \frac{4\nu_1}{U r_0^2} \left(\frac{T_0}{T_1}\right)^{\omega+1} dx = \int_0^{\alpha_c} \frac{\alpha_c^2 \frac{dI_c}{d\alpha_c} + \alpha_c I_c}{\left(\frac{\partial F}{\partial \lambda_c}\right)_{\lambda_c = 0}} d\alpha_c \tag{98}$$

where

$$\frac{1}{R_c} \equiv \frac{4\nu_1 x}{U r_0^2}$$

The local skin friction is given by

$$c_f \sqrt{R_{x_c}} = \frac{2}{\alpha_c \sqrt{R_c}} \left(\frac{T_0}{T_1}\right)^{\omega} \frac{\partial F}{\partial \lambda_c}\bigg|_0 \tag{99}$$

Eqs. (98) and (99) comprise the solution for the circular cylinder in a compressible fluid for any particular profile.

An example will be given for $\omega = 0.768$, and $\gamma = 1.4$ and for a logarithmic profile

$$F = \frac{\ln(1+\alpha_c \lambda_c)}{\ln(1+\alpha_c)}$$

This profile again reduces to the linear profile in the Howarth plane for small values of α_c. From the incompressible results, the solution is given by

$$\frac{\left(\frac{T_0}{T_1}\right)^{\omega+1}}{R_c} = \begin{cases} \alpha_c + 4 - \dfrac{4\alpha_c}{\ln(1+\alpha_c)} + \displaystyle\int_{-\infty}^{\ln(1+\alpha_c)} \frac{e^{\alpha_c}}{\alpha_c}\, d\alpha_c - \ln\left[\ln(1+\alpha_c)\right] - \gamma & (\alpha_c > 1) \\[2ex] \alpha_c + 4 - \dfrac{4\alpha_c}{\ln(1+\alpha_c)} \displaystyle\sum_{n=1}^{\infty}{}' \frac{\ln^n(1+\alpha_c)}{n \cdot n!} & (\text{all } \alpha_c) \end{cases}$$

and

$$C_f \sqrt{R_{x_c}} = \frac{\left(\frac{T_0}{T_1}\right)^{\omega}}{\sqrt{R_c}} \frac{1}{\ln(1+\alpha_c)}$$

These results are shown in Fig. 10 for various values of M_1.

IV. AN EXPERIMENTAL DETERMINATION OF THE VELOCITY PROFILE ACROSS THE LAMINAR BOUNDARY LAYER ON SLENDER BODIES OF REVOLUTION IN AXIAL COMPRESSIBLE FLOW

A. Description of Test Apparatus

The investigation was conducted in Leg No. 1 of the GALCIT 5 x 5 inch Hypersonic Wind Tunnel. The reservoir pressure and temperature controls, the manometer system for the measurement of static and impact pressure, and the details of the tunnel installation are described in Ref. 12.

B. Description of Model and Instrumentation

A slender cylindrical rod with an ogival frontal section, rigidly supported by a long plate, was used (Fig. 11). Three static pressure orifices spaced 2.5" apart were located on the cylinder. A single impact pressure probe with an 0.004" thick opening and an 0.030" width was mounted on the vertical actuator system located at the top of the nozzle block.

C. Test Procedure

All the runs were made at a fixed reservoir temperature of 225°F and at reservoir pressures of 24.3, 46.9, and 94.4 psia, respectively. The impact pressure measurements were all taken above the third static pressure orifice located 8.5" from the nose where the axial static pressure gradient is quite small. Sufficient time was allowed for each pressure to reach equilibrium before a reading was taken. The measurements show good agreement when repeated.

D. Reduction of Data

The following quantities were measured:

T_o - reservoir temperature (°F)

P_o - reservoir pressure (psia)

P_o' - boundary layer impact pressure (cm. Silicone)

P - cylinder surface static pressure (cm. Silicone)

The local Mach number, M , in the boundary layer can be obtained from Rayleigh's pitot formula:

$$\frac{P}{P_o'} = \left[\frac{2\gamma}{\gamma+1} M^2 - \frac{\gamma-1}{\gamma+1} \right]^{\frac{1}{\gamma-1}} \left[\frac{\gamma+1}{2} M \right]^{-\frac{\gamma}{\gamma-1}}$$

The static pressure, P , is assumed constant across the boundary layer.

By assuming that the Prandtl number is unity and that the cylinder surface is insulated, the velocity distribution can be obtained from the energy integral

$$\frac{u}{u_1} = \frac{\sqrt{1 + \frac{\gamma-1}{2} M_1^2}}{M_1} \frac{M}{\sqrt{1 + \frac{\gamma-1}{2} M^2}}$$

where u_1 is the free stream velocity and M_1 is the free stream Mach number.

E. Discussion of Results

Fig. 12 shows the variation of impact pressures across the boundary layer at a constant $M_1 = 5.8$.

In order to bring out the transverse curvature effect, the velocity is plotted against the actual height from the surface for P_o = 24.3 and 94.4 psia (Cf. Fig. 13). From this figure, it is seen that the ratios of boundary layer thickness to the body radius are approximately equal to 1 and 2 for P_o = 94.4 and 24.3 psia, respectively. In Fig. 14, the velocity is plotted against the normalized actual height -- normalized with respect to the boundary layer thickness (defined as the height where the velocity is 0.995 times the free stream velocity). The latter figure appears to indicate a transverse curvature effect when compared with the flat plate data obtained by Korkegi* (corresponding to a circular cylinder of infinite radius).

As far as local properties are concerned, the laminar boundary layer velocity profiles of other slender bodies of revolution will be similar to that investigated herein.

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* To be published in a forthcoming report.

V. SUMMARY

It has been shown in this work that certain basic questions could be answered regarding the laminar boundary layer on slender semi-infinite bodies of revolution in axial flow, and several new developments in this regard were introduced. They are summarized as follows:

(1) The use of the modified boundary layer equations for the treatment of slender bodies of revolution in axial incompressible flow has been justified by an a posteriori argument.

(2) By adopting the usual cylindrical coordinate system and by a suitable transformation (of which Mangler's transformation is a special case), it was shown that the modified boundary layer equations exhibit a similarity for paraboloids of revolution in axial incompressible flow. By virtue of this similarity property, an exact solution of the modified boundary layer equations was obtained which is valid over the complete range of R_L (Reynolds number based on the focal length of the paraboloid).

(3) It was seen that this exact solution reduces to known solutions at the extreme ends of R_L, i.e., to the Oseen solution for paraboloids of revolution for small values of R_L and to the Mangler value for large values of R_L.

(4) The following deductions were also made from the exact solution for paraboloids of revolution:

(a) For $R_L \ll 1$, as for slender paraboloids, and for $R_L \gg 1$, the axial pressure gradient has a negligible effect on the local skin friction. Also, for $R_L \ll 1$ and $R_x \gg 1$, Prandtl's assumption of a thin boundary layer together with the neglect of the normal pressure gradient is still justified.

(b) The local skin friction increases as R_L decreases or $\frac{\delta}{r_0}$ increases (Cf. Fig. 1a).

(5) Since the exact solution is valid for the complete range of R_L or δ/r_0, the precise range of applicability of Mangler's theory, valid only for $\frac{\delta}{r_0} \ll 1$, can be easily bracketed.

(6) On the basis of the exact solution for paraboloids of revolution, several approximate forms of the velocity profile across the boundary layer were chosen, which, in conjunction with the von Kármán integral method, yield results in good agreement with the exact solution.

(7) The generality of this approximate method was then tested on circular cylinders. The results again yield good agreement with known exact solutions.

(8) A comparison of the exact solutions for paraboloids of revolution and circular cylinders shows that the shape effect on the local skin friction is negligible when $\frac{u_1 r_0^2}{4\nu x} \ll 1$ or when $\frac{\delta}{r_0} \gg 1$ (Cf. Fig. 8a).

(9) The modified boundary layer equations, applicable for a compressible fluid, are obtained. For this system, an energy integral was shown to exist which agrees identically with that deduced by Crocco in two dimensions.

(10) When the axial pressure gradient is neglected, the compressible modified boundary layer equations exhibit similarity for paraboloids of revolution, as in the incompressible case.

(11) A similarity law was deduced from the compressible modified boundary layer equations for the hypersonic flow ($M_1 \gg 1$) past

paraboloids of revolution:

$$\frac{M_1^4}{R_{L_c}} = K = \text{constant}$$

(12) By introducing a new set of transformations, the von Kármán integral equation for a compressible fluid may be reduced to an equivalent incompressible form. As a consequence, the approximate results obtained for paraboloids of revolution and circular cylinders in the incompressible case can be transformed directly into the compressible case. For sufficiently large values of $\frac{u_1 r_0^2}{4\nu_1 x}$, compressibility counteracts the rise in $c_f \sqrt{R_{x_c}}$ and, below a certain value of $\frac{u_1 r_0^2}{4\nu_1 x}$, compressibility tends to increase the value of $c_f \sqrt{R_{x_c}}$ (Cf. Figs. 9 and 10).

(13) The laminar boundary layer characteristics on slender ogive-cylinders in a compressible fluid were investigated experimentally. Velocity profiles across the laminar boundary layer were obtained on the cylindrical portion of the body at different Reynolds numbers. Measurements were made at a fixed position where the effects of self-induced pressure gradients are small. The results show clearly that the thickness of the boundary layer can become of the same order of magnitude compared with the body radius and that the velocity profiles exhibit a behavior different from that on a flat plate (cylinder of infinite radius). Both of these deductions serve to indicate that the transverse curvature effects become more important as δ/r_0 increases.

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APPENDIX A

EXPANSION OF THE COMPRESSIBLE VISCOUS FLOW EQUATIONS IN CYLINDRICAL COORDINATES

In an unpublished set of notes, Lagerstrom has derived the compressible viscous flow equations in generalized coordinates. They are expanded below in cylindrical coordinates.

Let ξ^i , $i = 1, 2, 3$ be the cylindrical coordinates and x_i , $i = 1, 2, 3$ be the rectangular coordinates. They are related by the following transformation:

$$x_1 = \xi^1 \quad , \quad x_2 = \xi^2 \sin \xi^3 \quad , \quad x_3 = \xi^2 \cos \xi^3$$

The components of the metric tensor $g_{ij} = \dfrac{\partial x_k}{\partial \xi^i} \dfrac{\partial x_k}{\partial \xi^j}$ are thus:

$$g_{11} = 1 \quad , \quad g_{22} = 1 \quad , \quad g_{33} = (\xi^2)^2 \quad \text{and} \quad g_{ij} = 0 \quad (i \neq j) .$$

Now

$$g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\xi^2)^2 \end{vmatrix} = (\xi^2)^2$$

and by definition,

$$g^{ij} = \frac{\text{cofactor of } g_{ij} \text{ in } g}{g}$$

Hence, $g^{11} = 1$, $g^{22} = 1$, $g^{33} = \dfrac{1}{(\xi^2)^2}$, and $g^{ij} = 0 \quad (i \neq j)$.

The Christoffel symbols, $\Gamma^h_{ik} = \frac{1}{2}\sum_{\ell} g^{h\ell}\left(\frac{\partial g_{i\ell}}{\partial \xi^k} + \frac{\partial g_{k\ell}}{\partial \xi^i} - \frac{\partial g_{ik}}{\partial \xi^\ell}\right)$, reduce, in cylindrical coordinates, to

$$\Gamma^2_{33} = -\xi^2 \ , \quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{\xi^2} \ , \quad \text{and all other} \quad \Gamma^i_{jk} = 0 \ .$$

If q^i are the contravariant velocity components, then

$$(\text{true velocity})^2 = (q^1)^2 + (q^2)^2 + (\xi^2)^2(q^3)^2$$

and denoting the components of the true velocity in the (ξ^1, ξ^2, ξ^3) directions by u , v , and w , respectively, then

$$u = q^1 \quad , \quad v = q^2 \quad , \quad w = \xi^2 q^3$$

Denoting the coordinates (ξ^1 , ξ^2 , ξ^3) by x , r , and θ , the compressible viscous flow equations are:

<u>Continuity Equation</u>

$$\frac{\partial(\rho\sqrt{|g|})}{\partial t} + \frac{\partial(\rho\sqrt{|g|}\,q^i)}{\partial \xi^i} = 0$$

or

$$\frac{\partial(\rho r)}{\partial t} + \frac{\partial(\rho u r)}{\partial x} + \frac{\partial(\rho v r)}{\partial r} + \frac{\partial(\rho w)}{\partial \theta} = 0 \qquad \text{(A-1)}$$

<u>Momentum Equations</u>

$$\rho\sqrt{|g|}\left(\frac{\partial q^i}{\partial t} + q^j q^i_{,j}\right) = \frac{\partial H^{ij}}{\partial \xi^j} + H^{jk}\Gamma^i_{jk} + \sqrt{|g|}\,g^{ij}(k-p)_{,j} + \rho\sqrt{|g|}\,F^i$$

where

$$H^{ij} = \mu\sqrt{|g|}\left(g^{ik}q^j_{,k} + g^{i\ell}q^i_{,\ell}\right) \quad , \quad k = \frac{\lambda}{\sqrt{|g|}}\frac{\partial}{\partial \xi^\ell}\left(\sqrt{|g|}\,q^\ell\right) \quad ,$$

or

$$\rho r\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}+\frac{w}{r}\frac{\partial u}{\partial \theta}\right)=2\frac{\partial}{\partial x}\left(\mu r\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial r}\left[\mu r\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial r}\right)\right]$$
$$+\frac{\partial}{\partial \theta}\left[\mu r\left(\frac{1}{r}\frac{\partial w}{\partial x}+\frac{1}{r^2}\frac{\partial u}{\partial \theta}\right)\right]-r\frac{\partial p}{\partial x}$$
$$+r\frac{\partial}{\partial x}\left\{\frac{\lambda}{r}\left[\frac{\partial(ru)}{\partial x}+\frac{\partial(rv)}{\partial r}+\frac{\partial w}{\partial \theta}\right]\right\}+\rho r F_x \qquad \textbf{(A-2)}$$

$$\rho r\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}+\frac{w}{r}\frac{\partial v}{\partial \theta}-\frac{w^2}{r}\right)=\frac{\partial}{\partial x}\left[\mu r\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right)\right]+2\frac{\partial}{\partial r}\left(\mu r\frac{\partial v}{\partial r}\right)$$
$$+\frac{\partial}{\partial \theta}\left[\mu r\left(\frac{1}{r}\frac{\partial w}{\partial r}-\frac{w}{r^2}+\frac{1}{r^2}\frac{\partial v}{\partial \theta}\right)\right]-\frac{2\mu}{r}\left(\frac{\partial w}{\partial \theta}+v\right)$$
$$+r\frac{\partial}{\partial r}\left\{\frac{\lambda}{r}\left[\frac{\partial(ru)}{\partial x}+\frac{\partial(rv)}{\partial r}+\frac{\partial w}{\partial \theta}\right]\right\}-r\frac{\partial p}{\partial r}+\rho r F_r \qquad \textbf{(A-3)}$$

$$\rho r\left(\frac{1}{r}\frac{\partial w}{\partial t}+\frac{u}{r}\frac{\partial w}{\partial x}+\frac{v}{r}\frac{\partial w}{\partial r}+\frac{w}{r^2}\frac{\partial w}{\partial \theta}+\frac{vw}{r^2}\right)=\frac{\partial}{\partial x}\left[\mu\left(\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial r}\left[\mu\left(\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial r}-\frac{w}{r}\right)\right]$$
$$+\frac{2}{r^2}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{\partial w}{\partial \theta}+v\right)\right]+2\mu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial v}{\partial \theta}-\frac{w}{r^2}\right)$$
$$+\frac{1}{r}\frac{\partial}{\partial \theta}\left\{\frac{\lambda}{r}\left[\frac{\partial(ru)}{\partial x}+\frac{\partial(rv)}{\partial r}+\frac{\partial w}{\partial \theta}\right]\right\}-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho r F_w$$

(A-4)

Energy Equation

$$\rho\left(\frac{\partial h}{\partial t} + q^j h_{,j}\right) - \left(\frac{\partial p}{\partial t} + q^j p_{,j}\right) = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial \xi^i}\left(\sqrt{|g|}\, k\, g^{ij} T_{,j}\right) + \Phi + \rho Q$$

where

$$\Phi = \mu q^j_{,i}\left(g_{jk}\, g^{il} q^k_{,l} + q^i_{,j}\right) + \lambda\left(q^l_{,l}\right)\left(q^i_{,i}\right)$$

or

$$\rho\left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial r} + \frac{w}{r}\frac{\partial h}{\partial \theta}\right) - \left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial r} + \frac{w}{r}\frac{\partial p}{\partial \theta}\right) = \Phi + \rho Q$$
$$+ \frac{1}{r}\frac{\partial}{\partial x}\left(rk\frac{\partial T}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) \qquad \text{(A-5)}$$

where

$$\Phi = 2\mu\left(\frac{\partial u}{\partial x}\right)^2 + \mu\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r}\right) + \mu\frac{\partial}{\partial x}\left(\frac{w}{r}\right)\left[r\frac{\partial w}{\partial x} + \frac{\partial u}{\partial \theta}\right] + \mu\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right) + 2\mu\frac{\partial v}{\partial r}$$
$$+ \mu\left[\frac{\partial}{\partial r}\left(\frac{w}{r}\right) + \frac{w}{r^2}\right]\left\{r^2\left[\frac{\partial}{\partial r}\left(\frac{w}{r}\right) + \frac{w}{r^2}\right] + \frac{\partial v}{\partial \theta} - w\right\} + \mu\frac{\partial u}{\partial \theta}\left(\frac{1}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial w}{\partial x}\right)$$
$$+ \mu\left(\frac{\partial v}{\partial \theta} - w\right)\left[\frac{1}{r^2}\left(\frac{\partial v}{\partial \theta} - w\right) + \frac{\partial}{\partial r}\left(\frac{w}{r}\right) + \frac{w}{r^2}\right] + 2\mu\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{v}{r}\right)^2$$
$$+ \lambda\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{v}{r}\right]^2$$

Equation of State

$$p = \rho R T \qquad \text{(A-6)}$$

(F^i and Q represent external sources of momentum and energy, respectively. p , λ , μ , and k are functions of the thermodynamical variables.)

APPENDIX B

MODIFIED BOUNDARY LAYER EQUATIONS

The governing equations for a steady axial incompressible flow past a non-spinning body of revolution are as follows:

$$\frac{\partial(ur)}{\partial x} + \frac{\partial(vr)}{\partial r} = 0 \tag{B-1a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial x^2}\right] \tag{B-1b}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial x\partial r} + \frac{\partial^2 v}{\partial x^2} + \frac{2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{2v}{r^2}\right] \tag{B-1c}$$

Now, assuming that the thickness of the boundary layer is of the same order as the body radius, that is, $\delta = O(r_0)$, letting $r = \sqrt{\nu}\, r^*$, $v = \sqrt{\nu}\, v^*$, and leaving the other variables unchanged, Eqs. (B-1) become

$$\frac{\partial(u^* r^*)}{\partial x^*} + \frac{\partial(v^* r^*)}{\partial r^*} = 0$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial r^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u^*}{\partial r^*}\right) + \nu\frac{\partial^2 u^*}{\partial x^{*2}}$$

$$\sqrt{\nu}\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial r^*}\right) = -\frac{1}{\rho\sqrt{\nu}}\frac{\partial p^*}{\partial r^*} + \nu\left[\frac{1}{\sqrt{\nu}}\frac{\partial^2 u^*}{\partial x^*\partial r^*} + \sqrt{\nu}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{2}{\sqrt{\nu}}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial v^*}{\partial r^*}\right) - \frac{1}{\sqrt{\nu}}\frac{2v^*}{r^{*2}}\right]$$

Holding the asterisk variables fixed and taking the limit as yields

$$\frac{\partial(u^* r^*)}{\partial x^*} + \frac{\partial(v^* r^*)}{\partial r^*} = 0$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial r^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial u^*}{\partial r^*}\right)$$

$$0 = \frac{\partial p^*}{\partial r^*}$$

which, in terms of the original variables, becomes

$$\frac{\partial(ur)}{\partial x} + \frac{\partial(vr)}{\partial r} = 0 \qquad \text{(B-2a)}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) \qquad \text{(B-2b)}$$

$$0 = \frac{\partial p}{\partial r} \qquad \text{(B-2c)}$$

This system of equations will be called the modified boundary layer equations.

APPENDIX C

EXTENDED MANGLER TRANSFORMATION

The modified boundary layer equations of Appendix B can be transfomed into the following system:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{r}} = 0 \tag{C-1a}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{r}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}} + \nu\frac{\partial}{\partial \bar{r}}\left[\left(1 + \frac{2L\bar{r}}{r_0^2}\right)\frac{\partial \bar{u}}{\partial \bar{r}}\right] \tag{C-1b}$$

$$0 = \frac{\partial \bar{p}}{\partial \bar{r}} \tag{C-1c}$$

<u>Proof</u> Let

$$\bar{r} = \int_{r_0}^{r} \frac{r}{L}\, dr \quad , \quad \bar{x} = \int_0^x \left(\frac{r_0}{L}\right)^2 dx$$
$$\bar{\psi} = \frac{1}{L}\psi \quad , \quad \bar{p} = p \tag{C-2}$$

Hence, the transfomation operators are

$$\frac{\partial}{\partial x} = \left(\frac{r_0}{L}\right)^2\frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{r}}{\partial x}\frac{\partial}{\partial \bar{r}} \qquad \text{and} \qquad \frac{\partial}{\partial r} = \frac{r}{L}\frac{\partial}{\partial \bar{r}}$$

From the continuity equation,

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r} = \frac{\partial \bar{\psi}}{\partial \bar{r}} = \bar{u} \tag{C-3a}$$

$$v = -\frac{1}{r}\frac{\partial \psi}{\partial x} = -\frac{1}{r}\left[\left(\frac{r_0}{L}\right)^2\frac{\partial \bar{\psi}}{\partial \bar{x}} + \frac{\partial \bar{r}}{\partial x}\frac{\partial \bar{\psi}}{\partial \bar{r}}\right] = \frac{r_0^2}{rL}\bar{v} - \frac{L}{r}\frac{\partial \bar{r}}{\partial x}\bar{u} \tag{C-3b}$$

Hence, the x momentum equation becomes

$$\bar{u}\left[\left(\frac{r_0}{L}\right)^2 \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{r}}{\partial x} \frac{\partial \bar{u}}{\partial \bar{r}}\right] + \frac{r}{L} \frac{\partial \bar{u}}{\partial \bar{r}}\left[\frac{r_0^2}{rL} \bar{v} - \frac{L}{r} \frac{\partial \bar{r}}{\partial x} \bar{u}\right] = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial \bar{x}} \left(\frac{r_0}{L}\right)^2 + \frac{1}{L} \frac{\partial}{\partial \bar{r}}\left(\frac{r^2}{L} \frac{\partial \bar{u}}{\partial \bar{r}}\right)$$

which simplifies to

$$\left(\frac{r_0}{L}\right)^2\left(\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{r}}\right) = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial \bar{x}} \left(\frac{r_0}{L}\right)^2 + \frac{\partial}{\partial \bar{r}} \left(\frac{r^2}{L^2} \frac{\partial \bar{u}}{\partial \bar{r}}\right)$$

and division by $\left(\frac{r_0}{L}\right)^2$ yields

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{r}} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial}{\partial \bar{r}} \left(\frac{r^2}{r_0^2} \frac{\partial \bar{u}}{\partial \bar{r}}\right) \tag{C-4}$$

where

$$\frac{r^2}{r_0^2} = 1 + \frac{2 L \bar{r}}{r_0^2}$$

APPENDIX D

REDUCTION OF MODIFIED BOUNDARY LAYER EQUATIONS FOR FLOWS PAST CIRCULAR CYLINDERS

The modified boundary layer equations for a cylinder are

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{r}} = 0 \tag{D-1a}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{r}} = \nu \frac{\partial}{\partial \bar{r}}\left[\left(1 + \frac{2L\bar{r}}{r_0^2}\right)\frac{\partial \bar{u}}{\partial \bar{r}}\right] \tag{D-1b}$$

Let $\quad \eta = \sqrt{\frac{U}{\nu \bar{x}}}\,\bar{r} \quad , \quad \xi = \frac{2L}{r_0^2}\sqrt{\frac{\nu \bar{x}}{U}} \quad$ and $\quad \bar{\psi} = \sqrt{U\nu\bar{x}}\, f(\xi, \eta)$

Then,

$$\frac{2L\bar{r}}{r_0^2} = \xi\eta \quad , \quad \frac{\partial}{\partial \bar{r}} = \sqrt{\frac{U}{\nu \bar{x}}}\frac{\partial}{\partial \eta} \quad , \quad \frac{\partial}{\partial \bar{x}} = \frac{\xi}{2x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial \bar{x}}\frac{\partial}{\partial \eta}$$

and

$$u = \frac{\partial \bar{\psi}}{\partial \bar{r}} = U\frac{\partial f}{\partial \eta} \tag{D-2a}$$

$$v = -\frac{\partial \bar{\psi}}{\partial \bar{x}} = -\sqrt{U\nu\bar{x}}\left(\frac{\xi}{2\bar{x}}\frac{\partial f}{\partial \xi} + \frac{\partial \eta}{\partial \bar{x}}\frac{\partial f}{\partial \eta}\right) - \frac{1}{2}\sqrt{\frac{U\nu}{\bar{x}}}\, f \tag{D-2b}$$

Hence, the above equations reduce to

$$f\frac{\partial^2 f}{\partial \eta^2} + 2\frac{\partial}{\partial \eta}\left[(1+\xi\eta)\frac{\partial^2 f}{\partial \eta^2}\right] = \xi\left(\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \xi \partial \eta} - \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2}\right) \tag{D-3}$$

which is the analytical form of Seban and Bond.

Now let $f_1 = \frac{2f}{\xi}$ and $\eta_1 = \frac{\eta}{2\xi} + \frac{1}{2\xi^2}$.

Then,

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi} + \frac{\partial \eta_1}{\partial \xi}\frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial \eta} = \frac{1}{2\xi}\frac{\partial}{\partial \eta_1} \quad ; \text{ and}$$

$$\frac{\partial f}{\partial \eta} = \frac{1}{4}\frac{\partial f_1}{\partial \eta} \quad , \quad \frac{\partial^2 f}{\partial \eta^2} = \frac{1}{8\xi}\frac{\partial^2 f_1}{\partial \eta_1^2}$$

$$\frac{\partial^2 f}{\partial \xi \partial \eta} = \frac{1}{4}\left(\frac{\partial^2 f_1}{\partial \xi \partial \eta_1} + \frac{\partial \eta_1}{\partial \xi}\frac{\partial^2 f_1}{\partial \eta_1^2}\right) \quad , \quad \frac{\partial f}{\partial \xi} = \frac{1}{2}\left(\xi \frac{\partial f_1}{\partial \xi} + f_1 + \xi \frac{\partial \eta_1}{\partial \xi}\frac{\partial f_1}{\partial \eta_1}\right)$$

Consequently, the analytical form of the Seban and Bond result transforms into the analytical form of Stewartson:

$$f_1 \frac{\partial^2 f_1}{\partial \eta_1^2} + 2\frac{\partial}{\partial \eta_1}\left(\eta_1 \frac{\partial^2 f_1}{\partial \eta_1^2}\right) = \frac{\xi}{2}\left(\frac{\partial f_1}{\partial \eta_1}\frac{\partial^2 f_1}{\partial \xi \partial \eta_1} - \frac{\partial f_1}{\partial \xi}\frac{\partial^2 f_1}{\partial \eta_1^2}\right) \qquad \text{(D-4)}$$

APPENDIX E

SIMILARITY OF THE MODIFIED BOUNDARY LAYER EQUATIONS FOR FLOWS PAST PARABOLOIDS OF REVOLUTION

As will be shown in Appendix I the axial pressure gradient, $\frac{\partial p}{\partial x}$, for flows past paraboloids of revolution is small under certain conditions and, hence, may be neglected in the transformed modified boundary layer equations. Thus,

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{r}} = 0 \qquad \text{(E-1a)}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{r}} = \nu \frac{\partial}{\partial \bar{r}}\left[\left(1 + \frac{2L\bar{r}}{r_0^2}\right)\frac{\partial \bar{u}}{\partial \bar{r}}\right] \qquad \text{(E-1b)}$$

with the boundary conditions $\bar{u} = \bar{v} = 0$ at $r_0 = \bar{\alpha}\,\bar{x}^{\frac{1}{4}}$ and $\bar{u} \rightarrow U$ as $\bar{r} \rightarrow \infty$. Now consider the mapping $\bar{r} \rightarrow k\bar{r}$, $\bar{x} \rightarrow a\bar{x}$, $\bar{u} \rightarrow b\bar{u}$, $\bar{v} \rightarrow c\bar{v}$. Eqs. (E-1) thus become

$$\frac{b}{a}\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{c}{k}\frac{\partial \bar{v}}{\partial \bar{r}} = 0$$

$$\frac{b^2}{a}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{bc}{k}v\frac{\partial \bar{u}}{\partial \bar{r}} = \frac{b}{k^2}\nu\frac{\partial}{\partial \bar{r}}\left[\left(1 + \frac{k}{\sqrt{a}}\,\frac{2L\bar{r}}{\bar{\alpha}\sqrt{\bar{x}}}\right)\frac{\partial \bar{u}}{\partial \bar{r}}\right]$$

and the boundary conditions become

$\bar{u} = \bar{v} = 0$ at $k r_0 = \bar{\alpha}\,(a\bar{x})^{1/4}$ and $b\bar{u} \rightarrow U$ as $\bar{r} \rightarrow \infty$.

In order that this system be invariant under the above mapping, it is necessary that $a = k^2$, $b = 1$ and $c = 1/k$. Consequently, u must be a function of $\frac{\bar{r}}{\sqrt{\bar{x}}}$.

In terms of the stream function, $\bar{\psi}$, Eqs. (E-1) reduce to

$$\frac{\partial \bar{\psi}}{\partial \bar{r}} \frac{\partial^2 \bar{\psi}}{\partial \bar{x} \partial \bar{r}} - \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial^2 \bar{\psi}}{\partial \bar{r}} = \nu \frac{\partial}{\partial \bar{r}} \left[\left(1 + \frac{2L\bar{r}}{r_o^2} \right) \frac{\partial^2 \bar{\psi}}{\partial \bar{r}^2} \right] \tag{E-2}$$

with the boundary conditions $\bar{\psi} = 0$ at $r_o = \bar{\alpha} x^{-\frac{1}{4}}$ and $\bar{\psi} \to U \bar{r}$ as $\bar{r} \to \infty$.

Now consider the mapping $\bar{r} \to k\bar{r}$, $\bar{x} \to a\bar{x}$, $\bar{\psi} \to c\bar{\psi}$. Eq. (E-2) and the boundary conditions thus become

$$\frac{c}{a k^2} \frac{\partial \bar{\psi}}{\partial \bar{r}} \frac{\partial^2 \bar{\psi}}{\partial \bar{x} \partial \bar{r}} - \frac{c^2}{a k^2} \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial^2 \bar{\psi}}{\partial \bar{r}^2} = \frac{c}{k^3} \nu \frac{\partial}{\partial \bar{r}} \left[\left(1 + \frac{k}{\sqrt{a}} \frac{2L\bar{r}}{\bar{\alpha}\sqrt{\bar{x}}} \right) \frac{\partial^2 \bar{\psi}}{\partial \bar{r}^2} \right]$$

and $\bar{\psi} = 0$ at $K r_o = \bar{\alpha} (a\bar{x})^{1/4}$, $c\bar{\psi} \to KU\bar{r}$ as $r \to \infty$. In order that this system be invariant under the mapping, it is necessary that $a = k^2$, $c = k$. Consequently, ψ must be a function of $\sqrt{\bar{x}}\, fcn\left(\frac{\bar{r}}{\sqrt{\bar{x}}}\right)$.

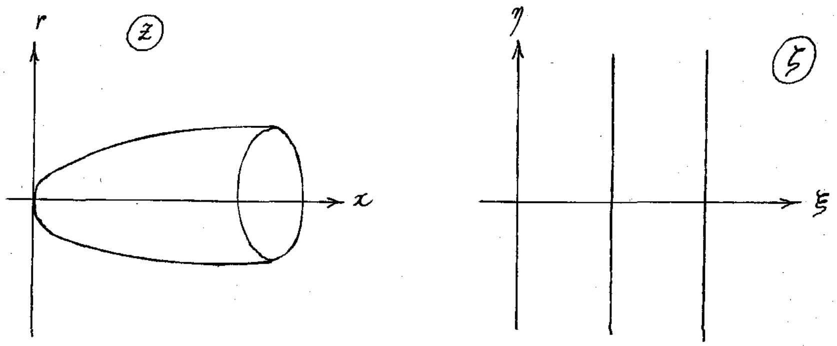
APPENDIX F

INVISCID FLOW PAST PARABOLOIDS OF REVOLUTION

The following transformation between the $z = x + ir$ plane and the $\zeta = \xi + i\eta$ plane

$$z = L - c\zeta^2 \qquad \text{(F-1)}$$

maps the surfaces ξ = constant into paraboloids of revolution.



Let $\xi = \xi_0$ be a particular paraboloid and L its focus length. It is shown in Ref. 9 that the stream function for the flow over this body is given by

$$\psi = 2c^2 U (\xi^2 - \xi_0^2)\eta^2 \qquad \text{(F-2)}$$

In the (x, r) coordinates, this may be written as

$$\psi = \frac{Ur^2}{2}\left[1 - \frac{2L}{L - x + \sqrt{(x-L)^2 + r^2}}\right] \qquad \text{(F-3)}$$

which identically satisfies the following inviscid equation of motion:

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = 0 \qquad \text{(F-4)}$$

Eq. (F-3) also satisfies the two boundary conditions: (1) $\psi = 0$ at $r_0^2 = 4Lx$

and (2) $\frac{1}{r}\frac{\partial\psi}{\partial r} = U$ at infinity.

The velocity components are

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r} = U\left[1 - L\,\frac{2(L - x + \sqrt{}) - \frac{r^2}{\sqrt{}}}{(L - x + \sqrt{})^2}\right] \tag{F-5a}$$

and

$$v = -\frac{1}{r}\frac{\partial\psi}{\partial x} = \frac{U L r}{\sqrt{}}\,\frac{1}{L - x + \sqrt{}} \tag{F-5b}$$

where

$$\sqrt{} = \sqrt{(x-L)^2 + r^2}$$

On the body, $r_0^2 = 4Lx$, the components reduce to

$$u_B = \frac{U}{1 + \frac{L}{x}} \qquad \text{and} \qquad v_B = \frac{\left(\frac{L}{x}\right)^{1/2}}{1 + \frac{L}{x}}\,U \tag{F-6}$$

Hence, $u_B \to U$ and $v_B \to 0$ as $\frac{L}{x} \to 0$. In other words, $v_B \ll u_B$ when $L \ll x$.

The axial pressure gradient on the body is

$$\frac{1}{\rho U^2}\frac{\partial p}{\partial x}\bigg|_B = -\frac{\partial}{\partial x}\left(\frac{u}{U}\right)^2\bigg|_B = -\frac{\frac{4L}{x^2}}{\left(1 + \frac{L}{x}\right)^4}$$

Hence, the pressure gradient is favorable and approaches zero when either $L \to 0$ or $x \to \infty$.

APPENDIX G

SOLUTION TO THE OSEEN BOUNDARY LAYER EQUATIONS FOR FLOW PAST PARABOLOIDS OF REVOLUTION

For sufficiently small values of R_L , Eqs. (2) can be simplified by the Oseen approximation of linearization about free stream conditions. Thus, replacing the operator $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial r}$ by $U\frac{\partial}{\partial x}$ in Eq. (2b) gives

$$U\frac{\partial u}{\partial x} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{G-1}$$

where the axial pressure gradient has been assumed to be small in comparison to the other terms when R_L is small.

By the transformation operators of Appendix C, Eq. (G-1) becomes

$$U\left[\left(\frac{r_o}{L}\right)^2\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{r}}{\partial x}\frac{\partial \bar{u}}{\partial \bar{r}}\right] = \nu\frac{\partial}{\partial \bar{r}}\left[\left(\frac{r}{L}\right)^2\frac{\partial \bar{u}}{\partial \bar{r}}\right] \tag{G-2}$$

Since $r_o^2 = 4Lx$ and $\bar{r} = \frac{r^2 - r_o^2}{2L}$, then $\frac{\partial \bar{r}}{\partial x} = -2$. Also $\frac{r_o^2}{L^2} = \sqrt{\frac{8\bar{x}}{L}}$ and $\frac{r^2}{r_o^2} = 1 + \frac{\bar{r}}{\sqrt{2L\bar{x}}}$. Hence, Eq. (G-2) reduces to

$$U\left[\frac{\partial \bar{u}}{\partial \bar{x}} - \sqrt{\frac{L}{2\bar{x}}}\frac{\partial \bar{u}}{\partial \bar{r}}\right] = \nu\frac{\partial}{\partial \bar{r}}\left[\left(1 + \frac{\bar{r}}{\sqrt{2L\bar{x}}}\right)\frac{\partial \bar{u}}{\partial \bar{r}}\right] \tag{G-3}$$

Now let

$$\bar{\eta} = \bar{r}\sqrt{\frac{U}{\nu\bar{x}}} \qquad \text{and} \qquad \bar{u} = U\frac{\partial \bar{f}}{\partial \eta}$$

and

$$\frac{\partial \bar{u}}{\partial \bar{x}} = U \bar{f}''\left(-\frac{\eta}{2\bar{x}}\right)$$

$$\frac{\partial \bar{u}}{\partial \bar{r}} = U\sqrt{\frac{U}{\nu \bar{x}}}\,\bar{f}''$$

Hence, Eq. (G-3) becomes

$$U\left[-\frac{U}{2}\frac{\bar{\eta}}{\bar{x}}\bar{f}'' - \frac{U}{\bar{x}}\sqrt{\frac{UL}{2\nu}}\,\bar{f}''\right] = \frac{U^2}{\bar{x}}\frac{d}{d\bar{\eta}}\left[(1+\beta\bar{\eta})\bar{f}''\right]$$

or

$$\eta\bar{f}'' + \frac{1}{\beta}\bar{f}'' + 2\left[(1+\beta\bar{\eta})\bar{f}''\right]' = 0$$

By Eqs. (14), this equation becomes

$$(\eta+1)f'' + \eta f''' = 0 \qquad \text{(G-4)}$$

with the boundary condition $f' = 0$ at $\eta = R_L$ and $f' \to 1$ as $\eta \to \infty$.

Eq. (G-4) is a linear differential equation and can be solved. Write it as

$$\frac{f'''}{f''} = -1 - \frac{1}{\eta}$$

and integrate

$$\ln f'' = -\eta - \ln \eta + \ln A$$

or

$$f'' = \frac{A}{\eta}e^{-\eta} \qquad \text{(G-5)}$$

where A is a constant. By a second integration,

$$\int_{\infty}^{\eta} f'' d\eta = A \int_{\infty}^{\eta} \frac{e^{-\eta}}{\eta} d\eta$$

or

$$f' = 1 + A \int_{\infty}^{\eta} \frac{e^{-\eta}}{\eta} d\eta \tag{G-6}$$

by virtue of the boundary condition at infinity. Now, since $f' = 0$ at $\eta = k_L$, the constant A is given by

$$A^{-1} = \int_{k_L}^{\infty} \frac{e^{-\eta}}{\eta} d\eta = ei(k_L)$$

Hence, Eq. (G-6) becomes

$$f' = 1 - \frac{ei(\eta)}{ei(k_L)}$$

and this is precisely Eq. (19b).

APPENDIX H

EVALUATION OF THE CONSTANT E_2

$$E_2 = -\int_0^\infty \frac{e^{-\eta}}{\eta}\left[(1+\eta)\int_\eta^\infty \frac{e^{-\bar{\eta}}}{\bar{\eta}}\,d\bar{\eta} - e^{-\eta} + \ln c\eta + 1\right] d\eta$$

The detail evaluation of the constant E_2 occurring in Eq. (27) will now be carried out in detail. From Ref. 10,

$$\ln c\eta = \int_0^\eta (1-e^{-\eta})\frac{d\eta}{\eta} - \int_\eta^\infty \frac{e^{-\eta}}{\eta}\,d\eta \tag{H-1}$$

so that

$$E_2 = -\int_0^\infty \frac{e^{-\eta}}{\eta}\left[\eta\int_\eta^\infty \frac{e^{-\bar{\eta}}}{\bar{\eta}}\,d\bar{\eta} - e^{-\eta} + 1 + \int_0^\eta (1-e^{-\bar{\eta}})\frac{d\bar{\eta}}{\bar{\eta}}\right] d\eta$$

$$= \int_0^\infty e^{-\eta}\,d\eta \int_\eta^\infty \frac{e^{-\bar{\eta}}}{\bar{\eta}}\,d\bar{\eta} - \int_0^\infty (1-e^{-\eta})\frac{e^{-\eta}}{\eta}\,d\eta - \int_0^\infty \frac{e^{-\eta}}{\eta}\,d\eta \int_0^\eta (1-e^{-\bar{\eta}})\frac{d\bar{\eta}}{\bar{\eta}}$$

$$= -\int_0^\infty \frac{e^{-\bar{\eta}}}{\bar{\eta}}\,d\bar{\eta}\int_0^{\bar{\eta}} e^{-\eta}\,d\eta - \int_0^\infty (1-e^{-\eta})\frac{e^{-\eta}}{\eta}\,d\eta - \int_0^\infty \frac{e^{-\eta}}{\eta}\,d\eta \int_0^\eta (1-e^{-\bar{\eta}})\frac{d\bar{\eta}}{\bar{\eta}}$$

$$= 2\int_0^\infty \frac{e^{-\bar{\eta}}}{\bar{\eta}}(e^{-\bar{\eta}}-1)\,d\bar{\eta} + \int_0^\infty \frac{e^{-\eta}}{\eta}\,d\eta\,(e^{-\bar{\eta}}-1)\frac{d\bar{\eta}}{\bar{\eta}} \tag{H-2}$$

Substituting

$$e^{-\bar{\eta}} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n \bar{\eta}^n}{n!}$$

in Eq. (H-2) gives

$$E_2 = 2\int_0^{\infty} \frac{e^{-\bar{\eta}}}{\bar{\eta}} \sum_{n=1}^{\infty} \frac{(-1)^n \bar{\eta}^n}{n!} \, d\bar{\eta} + \int_0^{\infty} \frac{e^{-\eta}}{\eta} \, d\eta \int_0^{\eta} \sum_{n=1}^{\infty} \frac{(-1)^n \bar{\eta}^n}{n!} \frac{d\bar{\eta}}{\bar{\eta}}$$

$$= 2\sum_{n=1}^{\infty} \frac{(-)^n}{n!} \int_0^{\infty} \bar{\eta}^{n-1} e^{-\bar{\eta}} d\bar{\eta} + \int_0^{\infty} \frac{e^{-\eta}}{\eta} d\eta \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n.\, n!} \eta^n \right]$$

$$= \sum_{n=1}^{\infty} \left[2\, \frac{(-1)^n}{n!} + \frac{(-1)^n}{n.\,n!} \right] \int_0^{\infty} \eta^{n-1} e^{-\eta} \, d\eta$$

From Ref. 11,

$$\int_0^{\infty} \eta^{n-1} e^{-\eta} \, d\eta = (n-1)!$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln 2 \qquad , \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

Hence

$$E_2 = -2 \ln 2 - \frac{\pi^2}{12}$$

APPENDIX I

JUSTIFICATION OF INITIAL ASSUMPTIONS

It is first necessary to define a displacement thickness, δ^*, for an axially-symmetric boundary layer. The mass flux for an inviscid fluid at any station along the body is

$$\int_{r_0}^{\infty} \rho U \,(2\pi r\, dr)$$

while for a viscous fluid it is

$$\int_{r_0}^{\infty} \rho u \,(2\pi r\, dr)$$

Hence, δ^* may be defined as

$$\pi \rho U \left[(r_0 + \delta^*)^2 - r_0^2 \right] = 2\pi \int_{r_0}^{\infty} (\rho U - \rho u)\, r\, dr$$

which reduces for an incompressible fluid to

$$(r_0 + \delta^*)^2 = r_0^2 + 2 \int_{r_0}^{\infty} \left(1 - \frac{u}{U}\right) r\, dr \qquad \text{(I-1)}$$

By Eqs. (4), (11), and (14),

$$r\, dr = L\, d\bar{r} = \frac{r_0^2}{2} \beta\, d\bar{\eta} = \beta^2 r_0^2\, d\eta = \frac{r_0^2}{2R_L}\, d\eta \qquad \text{(I-2)}$$

Hence, Eq. (I-1) reduces to

$$\left(1 + \frac{\delta^*}{r_0}\right)^2 = 1 + \frac{1}{R_L} \int_{R_L}^{\infty} \left(1 - \frac{u}{U}\right) d\eta$$

or

$$\frac{\delta^*}{r_o} = \left[1 + \frac{1}{R_L}\int_{R_L}^{\infty}\left(1 - \frac{u}{U}\right) d\eta\right]^{1/2} - 1 \tag{I-3}$$

From the Oseen solution

$$\frac{u}{U} = 1 - \frac{ei(\eta)}{ei(R_L)} \equiv f' \tag{I-4}$$

Hence,

$$\frac{\delta^*}{r_o} = \left[1 + \frac{1}{R_L}\int_{R_L}^{\infty} \frac{ei(\eta)}{ei(R_L)}\, d\eta\right]^{\frac{1}{2}} - 1 \tag{I-5}$$

but

$$\int_{R_L}^{\infty} ei(\eta)\, d\eta = \int_{R_L}^{\infty} d\eta \int_{\eta}^{\infty} \frac{e^{-\bar{\eta}}}{\bar{\eta}}\, d\bar{\eta} = \int_{R_L}^{\infty} \frac{e^{-\bar{\eta}}}{\bar{\eta}}\, d\bar{\eta} \int_{R_L}^{\bar{\eta}} d\eta = e^{-R_L} - R_L\, ei(R_L)$$

Therefore,

$$\frac{\delta^*}{r_o} = \left[\frac{e^{-R_L}}{R_L\, ei(R_L)}\right] - 1 = O\left(\frac{1}{\sqrt{R_L\, ei(R_L)}}\right)$$

for R_L sufficiently small.

Now

$$\frac{\delta^*}{x} = \left(\frac{\delta^*}{r_o}\right)\left(\frac{r_o}{x}\right) = 2\,\frac{\delta^*}{r_o}\sqrt{\frac{R_L}{R_x}}$$

and hence,

$$\frac{\delta^*}{x} = O\left[\frac{1}{\sqrt{R_x\, ei(R_L)}}\right] \tag{I-6}$$

Since the pressure gradient across the boundary layer must balance the centrifugal force, then

$$\frac{\partial p}{\partial r} = O\left(\rho U^2 \frac{\delta^*}{x^2}\right)$$

and the pressure difference is thus

$$\frac{\Delta p}{\rho U^2} = O\left(\frac{\delta^{*^2}}{x^2}\right)$$

Hence, by Eq. (I-6)

$$\frac{\Delta p}{\rho U^2} = O\left[\frac{1}{R_x \, ei(R_L)}\right] \tag{I-7}$$

which is small for $R_x \gg 1$ and $R_L \ll 1$. Consequently, it is sufficient to establish the order of the following ratio on the body surface:

$$R_1 = \left. \frac{\dfrac{\partial p}{\partial x}}{\dfrac{\mu}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)} \right|_{r=r_0}$$

From Appendix F

$$\frac{\partial p}{\partial x} = O\left(\frac{\rho U^2 L}{x^2}\right)$$

when $\frac{L}{x} \ll 1$. By Eqs. (4), (12), and (14)

$$\frac{\partial u}{\partial r} = \frac{r}{L}\frac{\partial u}{\partial \bar{r}} = \frac{2r}{\beta r_0^2}\frac{\partial u}{\partial \eta} = \frac{2rU}{\beta r_0^2}\frac{\partial^2 f}{\partial \eta^2}$$

$$r\frac{\partial u}{\partial r} = 2U\eta f''$$

Hence,

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{2\mu U}{\beta^2 r_0^2}\left(\eta f''\right)' \tag{I-8}$$

By Eq. (I-4)

$$(\eta f'')' = -\frac{e^{-\eta}}{ei\,(R_L)}$$

Therefore,

$$R_1 = O\left[\frac{R_L}{R_X}\; ei\,(R_L)\right] \tag{I-9}$$

Finally, the following ratio must be examined:

$$R_2 = \frac{\dfrac{\partial^2 u}{\partial x^2}}{\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)}$$

Now by Eqs. (4), (12), and (14)

$$\frac{\partial u}{\partial x} = \left(\frac{r_0}{L}\right)^2 \frac{\partial u}{\partial \bar{x}} = \left(\frac{r_0}{L}\right)^2 \left(-\frac{\eta}{2\bar{x}}\right) \frac{\partial u}{\partial \eta} = -\frac{U}{2\bar{x}}\left(\frac{r_0}{L}\right)^2 \eta f''$$

so that

$$\frac{\partial^2 u}{\partial x^2} = \frac{U}{4\bar{x}^2}\left(\frac{r_0}{L}\right)^4 (\eta f'')'$$

but by Eq. (4)

$$\bar{x}^2 = \frac{4x^4}{L^2}$$

Hence,

$$\frac{\partial^2 u}{\partial x^2} = \frac{U}{x^2}\,(\eta f'')'$$

and combining with Eq. (I-8) yields

$$R_2 = \frac{1}{R_X}$$

APPENDIX J

SOLUTION WITH $F = a_1 \dfrac{\ln(1+\alpha\lambda)}{\ln(1+\alpha)} + a_2\lambda$

The von Kármán integral with the above profile will now be evaluated for paraboloids of revolution.

$$\int_0^1 F\,d\lambda = \frac{a_1}{\ln(1+\alpha)} \int_0^1 \ln(1+\alpha\lambda)\,d\lambda + \frac{a_2}{2}$$

$$= \frac{a_1}{\ln(1+\alpha)} \left[\frac{1+\alpha}{\alpha} \ln(1+\alpha) - 1 \right] + \frac{a_2}{2}$$

$$\int_0^1 F^2\,d\lambda = \left[\frac{a_1}{\ln(1+\alpha)}\right]^2 \int_0^1 \ln^2(1+\alpha\lambda)\,d\lambda + \frac{2a_1a_2}{\ln(1+\alpha)} \int_0^1 \lambda \ln(1+\alpha\lambda)\,d\lambda + \frac{1}{3}a_2^2$$

$$= \frac{1+\alpha}{\alpha}\left[\frac{a_1}{\ln(1+\alpha)}\right]^2 \left\{ \left[\ln(1+\alpha) - 1\right]^2 + 1 - \frac{2}{1+\alpha} \right\}$$

$$+ \frac{2a_1a_2}{\ln(1+\alpha)} \left\{ \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)\left[\frac{1+\alpha}{\alpha}\ln(1+\alpha) - 1\right] + \frac{1}{4} \right\} + \frac{a_2^2}{3}$$

where

$$a_1 = \frac{1}{1-A}\,, \qquad a_2 = \frac{A}{A-1}\,, \text{ and } \quad A = \frac{\alpha}{1+\alpha}\,\frac{1}{\ln(1+\alpha)}\,.$$

The solution is given jointly by

$$\mathcal{R}_L = \left\{ \frac{1}{\alpha I} \left[\frac{a_1}{\ln(1+\alpha_o)} + \frac{a_2}{\alpha_o} \right] \right\}_{\alpha=\alpha_o} \tag{J-1}$$

and

$$C_f \sqrt{R_x} = \frac{2}{\sqrt{R_L}} \left[\frac{a_1}{\ln(1+\alpha_o)} + \frac{a_2}{\alpha_o} \right]_{\alpha=\alpha_o} = 2\sqrt{R_L}\, \alpha\, I(\alpha_o) \tag{J-2}$$

$$F = \frac{\ln(1+\alpha\lambda) - \frac{2}{1+\alpha}\lambda}{\ln(1+\alpha) - \frac{\alpha}{1+\alpha}}$$

$$= \frac{\alpha\left(\lambda - \frac{\alpha}{2}\lambda^2 + \cdots\right) - \frac{\alpha}{1+\alpha}\lambda}{\alpha\left(1 - \frac{\alpha}{2} + \cdots\right) - \frac{\alpha}{1+\alpha}}$$

$$= \frac{\lambda - \frac{\alpha}{2}\lambda^2 - \lambda(1 - \alpha + \alpha^2 - \cdots)}{1 - \frac{1}{2}\alpha - (1 - \alpha + \alpha^2 - \cdots)} = 2\lambda - \lambda^2 + O(\alpha)$$

Hence,

$$F \to 2\lambda - \lambda^2 \quad \text{as} \quad \alpha \to 0$$

Since $A = \dfrac{1}{1+\frac{1}{\alpha}} \dfrac{1}{\ln(1+\alpha)} \to 0$ as $\alpha \to \infty$, then

$a_1 \to 1$ and $a_2 \to 0$ as $\alpha \to \infty$.

APPENDIX K

SOLUTION WITH $F = a_1 \dfrac{\ln(1+\alpha\lambda)}{\ln(1+\alpha)} + a_2\lambda + a_3\lambda^2$

The von Kármán integral with the above profile will now be evaluated for paraboloids of revolution:

$$\int_0^1 F\,d\lambda = \frac{a_1}{\ln(1+\alpha)} I_0 + \frac{a_2}{2} + \frac{a_3}{3}$$

$$\int_0^1 F^2\,d\lambda = \frac{a_1}{\ln(1+\alpha)}\left[\frac{a_1}{\ln(1+\alpha)} I_1 + 2a_2 I_2 + 2a_3 I_3\right] + \frac{a_2^2}{3} + \frac{a_3^2}{5} + \frac{1}{2} a_2 a_3$$

where

$$I_0 = \frac{1+\alpha}{\alpha}\ln(1+\alpha) - 1$$

$$I_1 = \frac{1+\alpha}{\alpha}\left\{\left[\ln(1+\alpha) - 1\right]^2 + 1 - \frac{2}{1+\alpha}\right\}$$

$$I_2 = \frac{1}{2}\left(1 - \frac{1}{\alpha}\right) I_0 + \frac{1}{4}$$

$$I_3 = \frac{1}{3} I_0 - \frac{2}{3\alpha} I_2 + \frac{2}{9}$$

and $a_1 = \dfrac{2\alpha - 2}{D}$, $a_2 = 2\dfrac{A}{D}$, $a_3 = \dfrac{-\alpha A}{D}$, $A = \dfrac{\alpha}{1+\alpha}\,\dfrac{1}{\ln(1+\alpha)}$,

$$D = (2-\alpha)A + 2\alpha - 2$$

The solution is given jointly by

$$R_L = \left\{ \frac{1}{\alpha I} \left[\frac{a_1}{\ln(1+\alpha)} + \frac{a_2}{\alpha} \right] \right\}_{\alpha=\alpha_0} \tag{K-1}$$

and

$$C_f \sqrt{R_L} = 2 \sqrt{R_L} \, (\alpha I)_{\alpha=\alpha_0} \tag{K-2}$$

$$F = \frac{(2\alpha - 2)\ln(1+\alpha\lambda) + (2A\lambda - \alpha A \lambda^2)\ln(1+\alpha)}{D \ln(1+\alpha)}$$

$$= \frac{\alpha(2\alpha-2)\left[\lambda - \frac{\alpha}{2}\lambda^2 + \frac{\alpha^3}{3}\lambda^3 + \cdots\right] + \frac{\alpha}{1+\alpha}(2\lambda - \alpha\lambda^2)}{\alpha(2-\alpha) + (2\alpha-2)(1+\alpha)\ln(1+\alpha)}$$

$$= \frac{(2\alpha^2\lambda - \frac{2}{3}\alpha^2\lambda^3 + \cdots)(1+\alpha)}{\frac{4}{3}\alpha^2 + \cdots}$$

$$= \frac{3}{2}\lambda - \frac{1}{2}\lambda^3 + O(\alpha)$$

Hence,

$$F \to \frac{3}{2}\lambda - \frac{1}{2}\lambda^3 \qquad \text{as} \qquad \alpha \to 0$$

Now

$$a_1 = \frac{1}{\frac{\alpha(2-\alpha)}{(1+\alpha)(2\alpha-2)} \frac{1}{\ln(1+\alpha)} + 1} \to 1 \qquad \text{as} \qquad \alpha \to \infty$$

Also,

$$a_2 \quad \text{and} \quad a_3 \to 0 \qquad \text{as} \qquad \alpha \to \infty$$

APPENDIX L

SOLUTION WITH $$F = a_1 \frac{\ln(1+\alpha\lambda)}{\ln(1+\alpha)} + a_2\lambda + a_3\lambda^2 + a_4\lambda^3$$

The von Kármán integral with the above profile will now be evaluated for paraboloids of revolution:

$$\int_0^1 F\,d\lambda = \frac{a_1}{\ln(1+\alpha)} I_0 + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4}$$

$$\int_0^1 F^2\,d\lambda = \frac{a_1}{\ln(1+\alpha)}\left\{\frac{a_1}{\ln(1+\alpha)} I_1 + 2a_2 I_2 + 2a_3 I_3 + 2a_4 I_4\right\}$$

$$+\frac{1}{3}a_2^2 + \frac{1}{5}a_3^2 + \frac{1}{7}a_4^2 + \frac{1}{2}a_2 a_3 + \frac{2}{5}a_2 a_4 + \frac{1}{3}a_3 a_4 .$$

where

$$I_0 = \frac{1+\alpha}{\alpha}\ln(1+\alpha) - 1$$

$$I_1 = \frac{1+\alpha}{\alpha}\left\{\left[\ln(1+\alpha) - 1\right]^2 + 1 - \frac{2}{1+\alpha}\right\}$$

$$I_2 = \frac{1}{2}\left(1 - \frac{1}{\alpha}\right) I_0 + \frac{1}{4}$$

$$I_3 = \frac{1}{3} I_0 - \frac{2}{3\alpha} I_2 + \frac{2}{9}$$

$$I_4 = \frac{1}{4} I_0 - \frac{3}{4\alpha} I_3 + \frac{3}{16}$$

and $$a_1 = \frac{6-3\alpha}{D}, \quad a_2 = -\frac{6A-3B}{D}, \quad a_3 = \frac{\alpha}{2}\,\frac{6A+3B}{D}, \quad a_4 = -\frac{A\alpha + B(1-\alpha)}{D}$$

$$A = \frac{\alpha}{1+\alpha}\,\frac{1}{\ln(1+\alpha)} \quad , \quad B = \left(\frac{\alpha}{1+\alpha}\right)^2 \frac{1}{\ln(1+\alpha)} \quad , \quad D = A\,(2\alpha-6) + B\left(\frac{\alpha}{2}-2\right) + 6 - 3\alpha \, .$$

The solution is given jointly by

$$R_L = \left\{ \frac{1}{\alpha I}\left[\frac{a_1}{\ln(1+\alpha)} + \frac{a_2}{\alpha}\right]\right\}_{\alpha=\alpha_0} \tag{L-1}$$

and

$$C_f \sqrt{R_x} = 2\sqrt{R_L}\,(\alpha I)_{\alpha=\alpha_0} \tag{L-2}$$

$$F = \frac{1}{D\ln(1+\alpha)}\left\{ (6-3\alpha)\ln(1+\alpha\lambda) - [\ln(1+\alpha)](6A+3B)\left(\lambda - \frac{\alpha}{2}\lambda^2\right) + (-A\alpha + B - B\alpha)\lambda^3 \ln(1+\alpha)\right\}$$

$$= \frac{(6+9\alpha-3\alpha^3)\left(\alpha\lambda - \frac{\alpha^2}{2}\lambda^2 + \frac{\alpha^3}{3}\lambda^3 - \frac{\alpha^4}{4}\lambda^4\right) - \left(6\alpha\lambda - 3\alpha^2\lambda^2 + 9\alpha^2\lambda - \frac{9}{2}\alpha^3\lambda^2\right) - 2\alpha^3\lambda^3}{\alpha\,(2\alpha - 6 + 2\alpha^2 - 6\alpha) + \frac{\alpha^3}{2} - 2\alpha^2 + (6+9\alpha-3\alpha^3)\left(\alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{3} - \frac{\alpha^4}{4} + \cdots\right)}$$

$$= \frac{-3\alpha^4\lambda + 3\alpha^4\lambda^3 - \frac{3}{2}\alpha^4\lambda^4 + O(\alpha^5)}{-\frac{3}{2}\alpha^4 + O(\alpha^5)}$$

$$= 2\lambda - 2\lambda^3 + \lambda^4 + O(\alpha)$$

Hence,

$$F \longrightarrow 2\lambda - 2\lambda^3 + \lambda^4 \qquad \text{as} \qquad \alpha \to 0$$

Since A and $B \to 0$ and $D \to \infty$ as $\alpha \to \infty$, then

$$a_1 \to 1 \quad ; \quad a_2 \, , \, a_3 \, , \, a_4 \longrightarrow 0 \qquad \text{as} \qquad \alpha \to \infty$$

APPENDIX M

SOLUTION OF THE VON KÁRMÁN INTEGRAL RELATION WITH $F = \frac{\ln(1+\alpha\lambda)}{\ln(1+\alpha)}$ FOR CIRCULAR CYLINDER

The evaluation of the following integral will now be carried out:

$$\frac{1}{R} = \int_0^\alpha \left[1 - \frac{3}{\ln(1+\alpha)} - \frac{1}{1+\alpha}\,\frac{1}{\ln(1+\alpha)} + \frac{4\alpha}{1+\alpha}\,\frac{1}{\ln^2(1+\alpha)}\right] d\alpha \qquad \text{(M-1)}$$

Let $\theta = 1+\alpha$, then

$$\frac{1}{R} = \int_1^\theta \left[1 - \frac{3}{\ln\theta} - \frac{1}{\theta\ln\theta} + \frac{4(\theta-1)}{\theta\ln^2\theta}\right] d\theta \qquad \text{(M-2)}$$

Let $\omega = \ln\theta$, then

$$\frac{1}{R} = \int_0^\omega \left(e^\omega - \frac{3e^\omega+1}{\omega} + 4\,\frac{e^\omega-1}{\omega^2}\right) d\omega$$

$$= (e^\omega-1) - \int_0^\omega \frac{3e^\omega+1}{\omega}\,d\omega + 4\left[(e^\omega-1)\frac{-1}{\omega}\Big|_0^\omega + \int_0^\omega \frac{e^\omega}{\omega}\,d\omega\right]$$

$$= (e^\omega-1) - 4\left(\frac{e^\omega-1}{\omega} - 1\right) + \int_0^\omega \frac{e^\omega-1}{\omega}\,d\omega$$

$$= (e^\omega-1) - 4\,\frac{e^\omega-1}{\omega} + 4 + Ei(\omega) - \ln\omega - \gamma$$

$$= \alpha + 4 - \frac{4\alpha}{\ln(1+\alpha)} + Ei\left[\ln(1+\alpha)\right] - \ln\left[\ln(1+\alpha)\right] - \gamma \qquad (\alpha>1)$$

(M-3)

since

$$\int_0^{\omega} \frac{e^{\omega}-1}{\omega}\, d\omega = Ei(\omega) - \int_1^{\infty} (1-e^{-\omega}) \frac{d\omega}{\omega} - \int_1^{\omega} \frac{d\omega}{\omega}$$

and

$$\gamma = \int_1^{\infty} (1-e^{-\omega}) \frac{d\omega}{\omega} - \int_1^{\infty} \frac{e^{-\omega}}{\omega}\, d\omega = 0.5772$$

where

$$Ei(\omega) = \int_{-\infty}^{\omega} \frac{e^{\omega}}{\omega}\, d\omega = \gamma + \ln \omega + \sum_{n=1}^{\infty} \frac{\omega^n}{n\, n!} \qquad \text{(M-4)}$$

By an expansion of the above result the following series is applicable for all α :

$$\frac{1}{R} = \alpha + 4 - \frac{4\alpha}{\ln(1+\alpha)} + \sum_{n=1}^{\infty} \frac{\ln^n(1+\alpha)}{n \cdot n!} \qquad \text{(M-5)}$$

APPENDIX N

APPLICATION OF THE EXTENDED MANGLER TRANSFORMATION TO THE MODIFIED COMPRESSIBLE BOUNDARY LAYER EQUATIONS

The transformation equations are

$$\bar{r} = \int_{r_0}^{r} \frac{r}{L}\,dr \quad , \quad \bar{x} = \int_{0}^{x} \left(\frac{r_0}{L}\right)^2 dx \quad , \quad \bar{\psi} = \frac{\psi}{L}$$
$$p = \bar{p} \quad , \quad \rho = \bar{\rho} \quad , \quad \mu = \bar{\mu} \tag{N-1}$$

The transformation operators are

$$\frac{\partial}{\partial x} = \left(\frac{r_0}{L}\right)^2 \frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{r}}{\partial x}\frac{\partial}{\partial \bar{r}} \quad \text{and} \quad \frac{\partial}{\partial r} = \frac{r}{L}\frac{\partial}{\partial \bar{r}}$$

From the continuity equation,

$$\rho u = \frac{1}{r}\frac{\partial \psi}{\partial r} = \frac{\partial \bar{\psi}}{\partial \bar{r}} = \bar{\rho}\,\bar{u} \quad \text{or} \quad u = \bar{u} \tag{N-2a}$$

$$\rho v = -\frac{1}{r}\frac{\partial \psi}{\partial x} = -\frac{1}{r}\left[\left(\frac{r_0}{L}\right)^2 \frac{\partial \bar{\psi}}{\partial \bar{x}} + \frac{\partial \bar{r}}{\partial x}\frac{\partial \bar{\psi}}{\partial \bar{r}}\right] = \frac{r_0^2}{rL}\bar{\rho}\bar{v} - \frac{L}{r}\frac{\partial \bar{r}}{\partial x}\bar{\rho}\bar{u} \tag{N-2b}$$

Substituting these quantities in the x - momentum equation and simplifying yields

$$\bar{\rho}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{r}}\right) = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial}{\partial \bar{r}}\left[\left(1 + \frac{2L\bar{r}}{r_0^2}\right)\bar{\mu}\frac{\partial \bar{u}}{\partial \bar{r}}\right] \tag{N-3}$$

APPENDIX O

TRANSFORMATION OF THE MODIFIED COMPRESSIBLE BOUNDARY LAYER EQUATIONS WITH $\frac{\partial p}{\partial x} = 0$

The governing equations are

$$\frac{\partial(\rho u r)}{\partial x} + \frac{\partial(\rho v r)}{\partial r} = 0 \tag{O-1}$$

$$\rho u r \frac{\partial u}{\partial x} + \rho v r \frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(\mu r \frac{\partial u}{\partial r}\right) \tag{O-2}$$

Let

$$dr_c = \frac{\rho r}{\rho_0 L}\, dr \quad , \quad d\bar{x} = \left(\frac{r_0}{L}\right)^2 dx \quad , \quad \bar{\psi} = \frac{\psi}{\rho_0 L}$$

$$\frac{\partial}{\partial x} = \left(\frac{r_0}{L}\right)^2 \frac{\partial}{\partial \bar{x}} + \frac{\partial r_c}{\partial x}\frac{\partial}{\partial r_c} \quad , \quad \frac{\partial}{\partial r} = \frac{\rho r}{\rho_0 L}\frac{\partial}{\partial r_c}$$

Then

$$u = \frac{1}{\rho r}\frac{\partial \psi}{\partial r} = \frac{1}{\rho_0 L}\frac{\partial \psi}{\partial r_c} = \frac{\partial \bar{\psi}}{\partial r_c} = \bar{u} \tag{O-3a}$$

$$v = -\frac{1}{\rho r}\frac{\partial \psi}{\partial x} = \frac{\rho_0 L}{\rho r}\left[\left(\frac{r_0}{L}\right)^2 v_c - \frac{\partial r_c}{\partial x}\, u\right] \tag{O-3b}$$

and substituting these quantities in Eq. (O-2) yields

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + v_c \frac{\partial \bar{u}}{\partial r_c} = \nu_0 \frac{\partial}{\partial r_c}\left[\frac{r^2}{r_0^2}\frac{\rho\mu}{\rho_0\mu_0}\frac{\partial \bar{u}}{\partial r_c}\right] \tag{O-4a}$$

where

$$\frac{r^2}{r_0^2} = 1 + \frac{2L}{r_0^2}\int_0^{r_c} \frac{\rho_0}{\rho}\, dr_c \tag{O-4b}$$

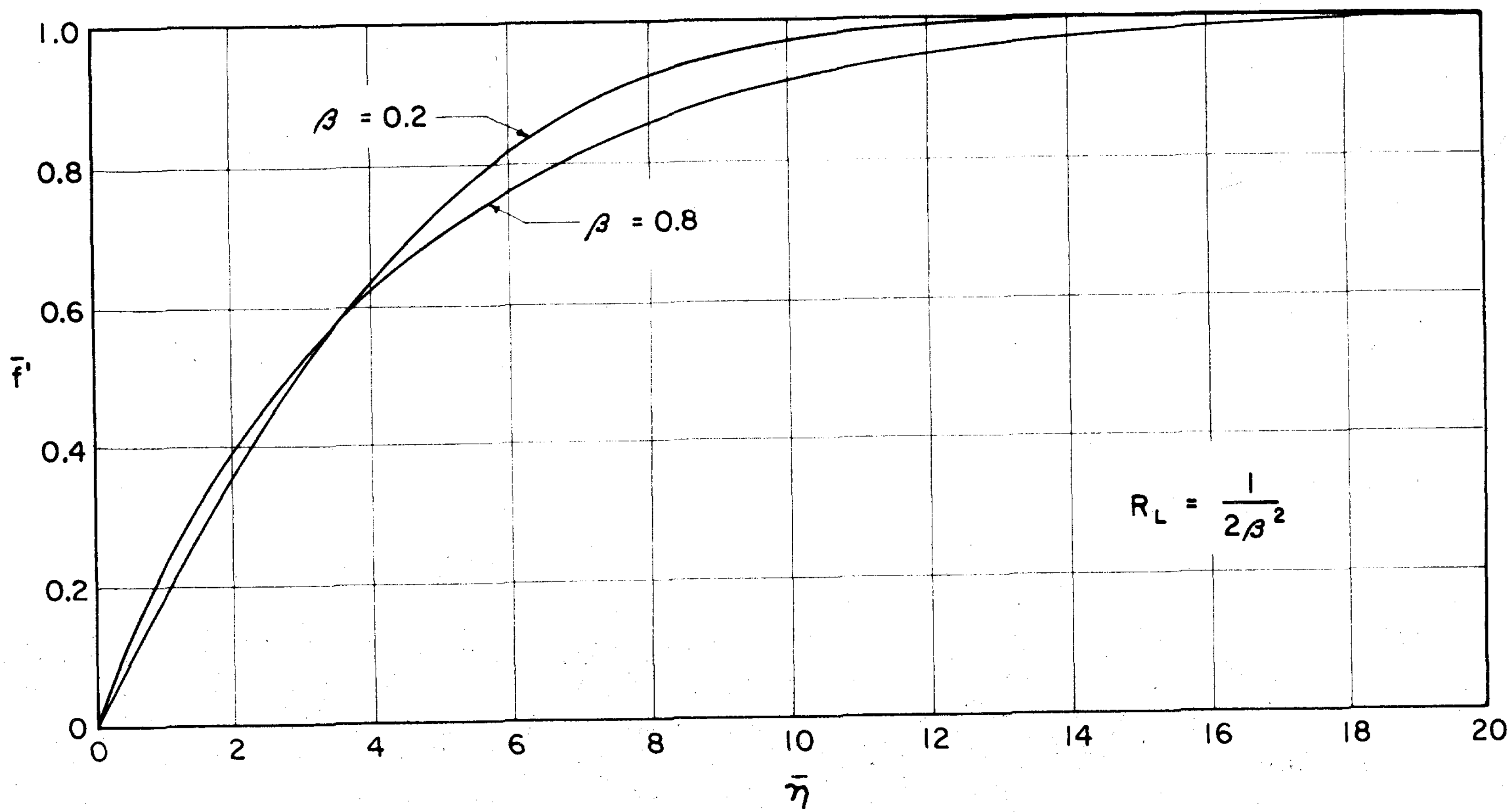


FIG. 1 VELOCITY PROFILES ON PARABOLOID OF REVOLUTION AT DIFFERENT R_L VALUES

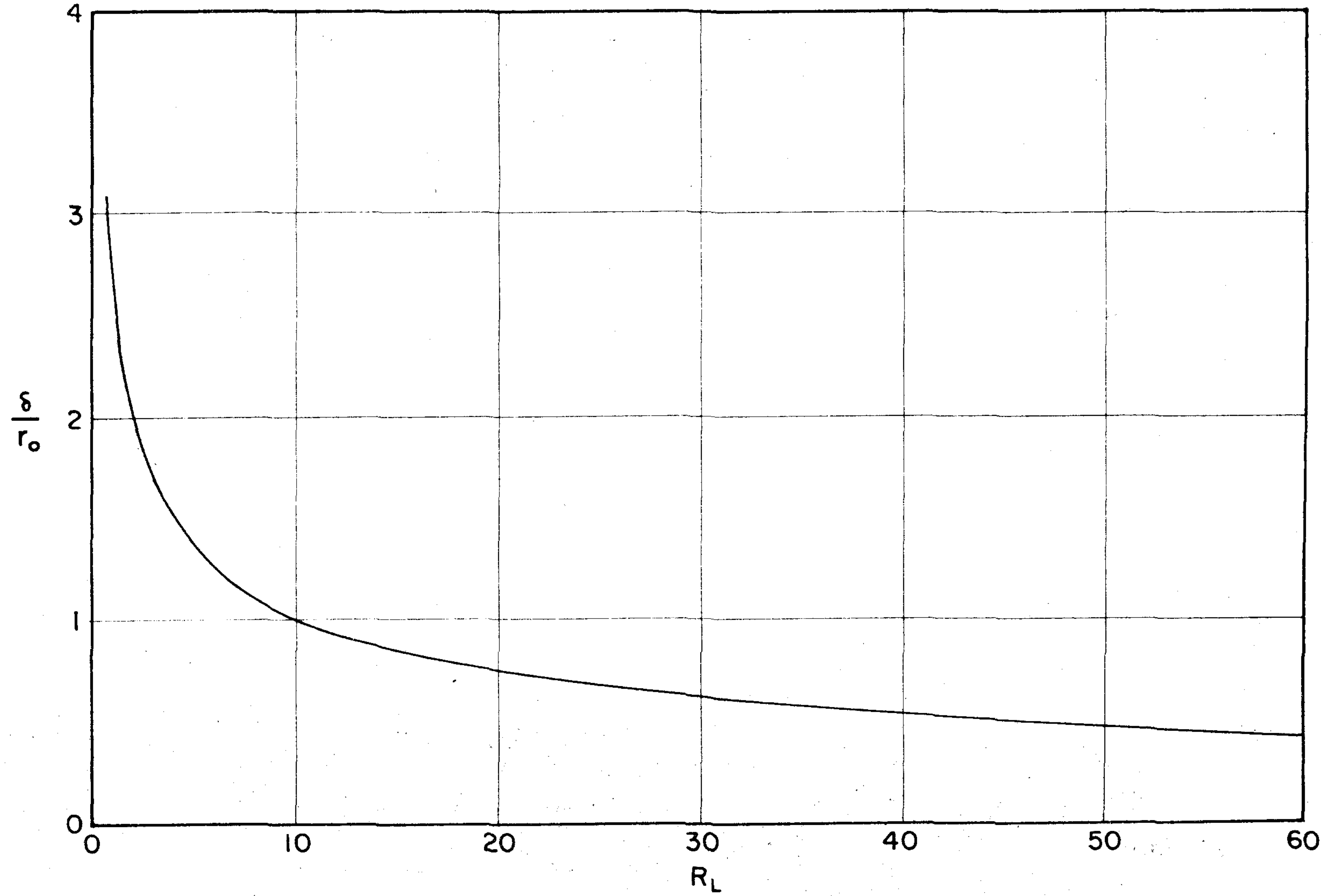


FIG. 1a RATIOS OF BOUNDARY LAYER THICKNESS TO THE TRANSVERSE RADIUS OF PARABOLOIDS OF REVOLUTION FOR DIFFERENT R_L VALUES

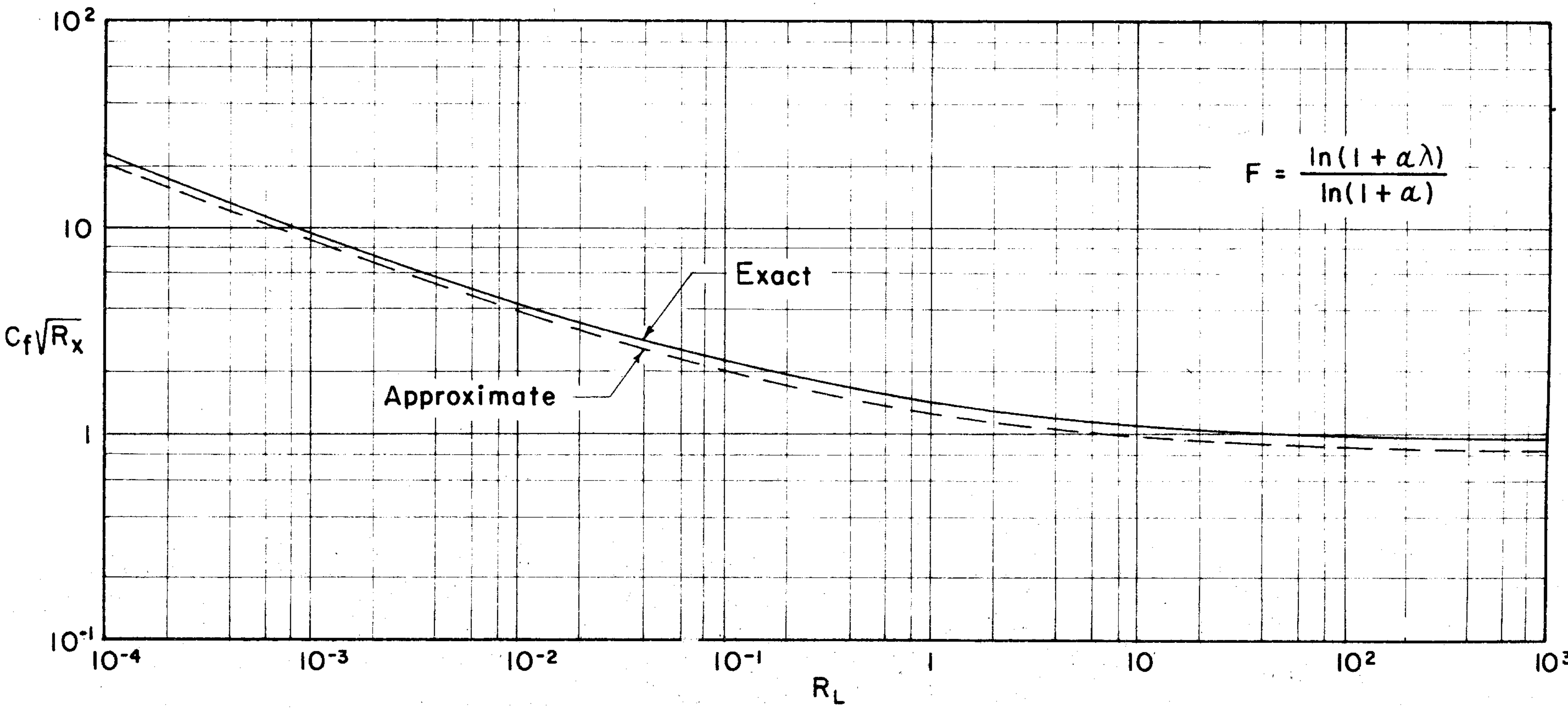


FIG. 2 COMPARISON OF EXACT AND APPROXIMATE LOCAL SKIN FRICTION OF PARABOLOIDS OF REVOLUTION

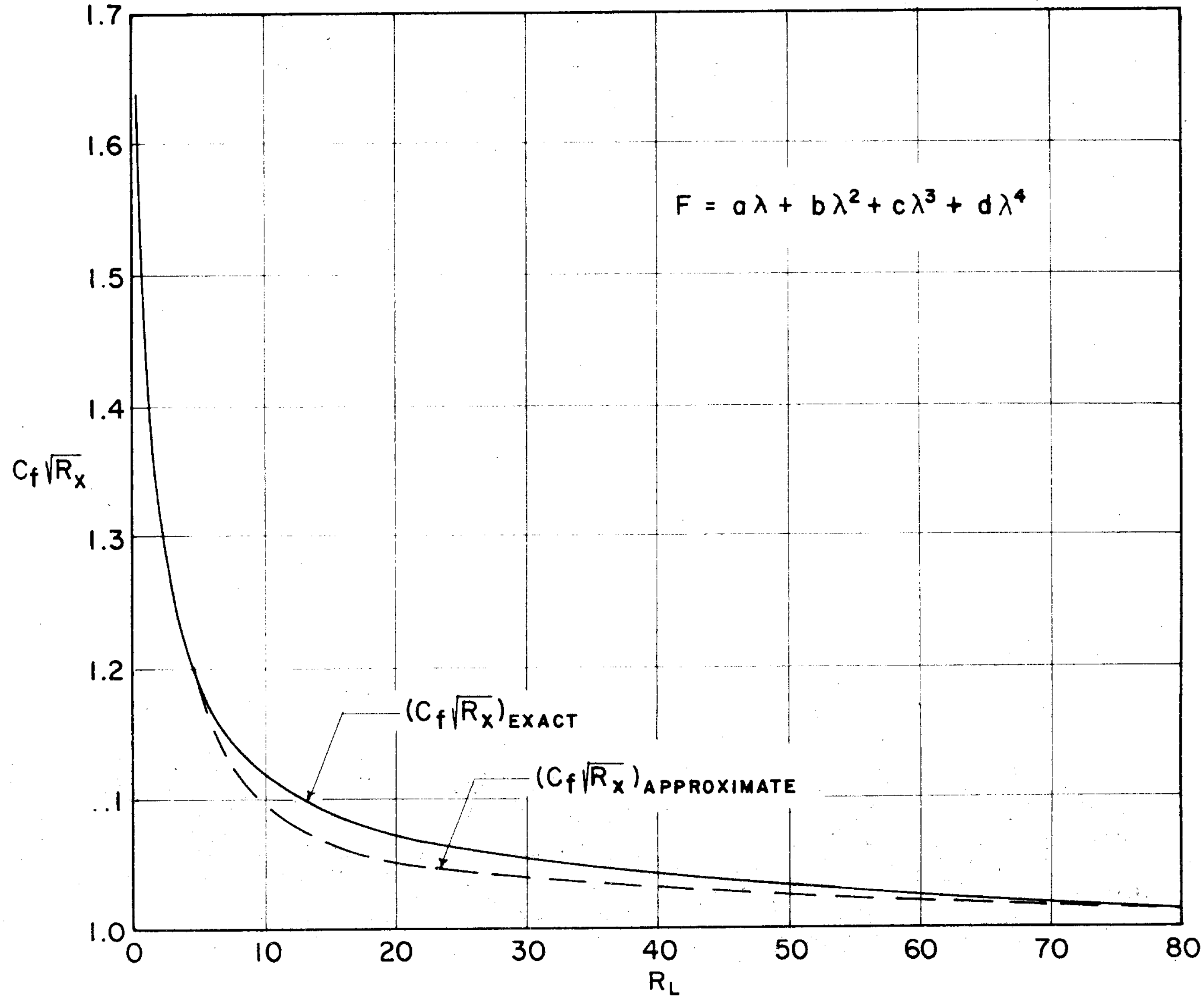


FIG. 3 COMPARISON OF EXACT AND APPROXIMATE LOCAL SKIN FRICTION OF PARABOLOIDS OF REVOLUTION

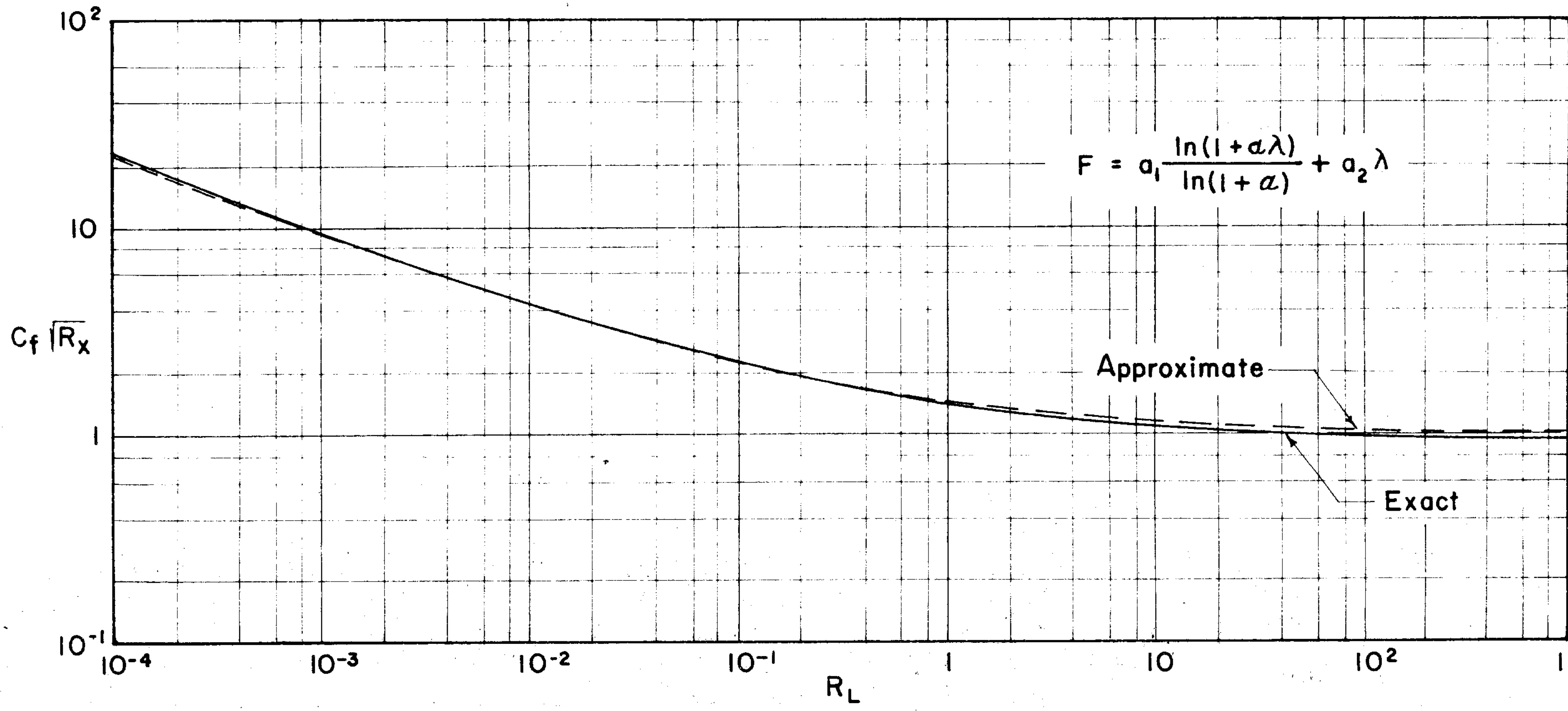


FIG. 4 COMPARISON OF EXACT AND APPROXIMATE LOCAL SKIN FRICTION OF PARABOLOIDS OF REVOLUTION

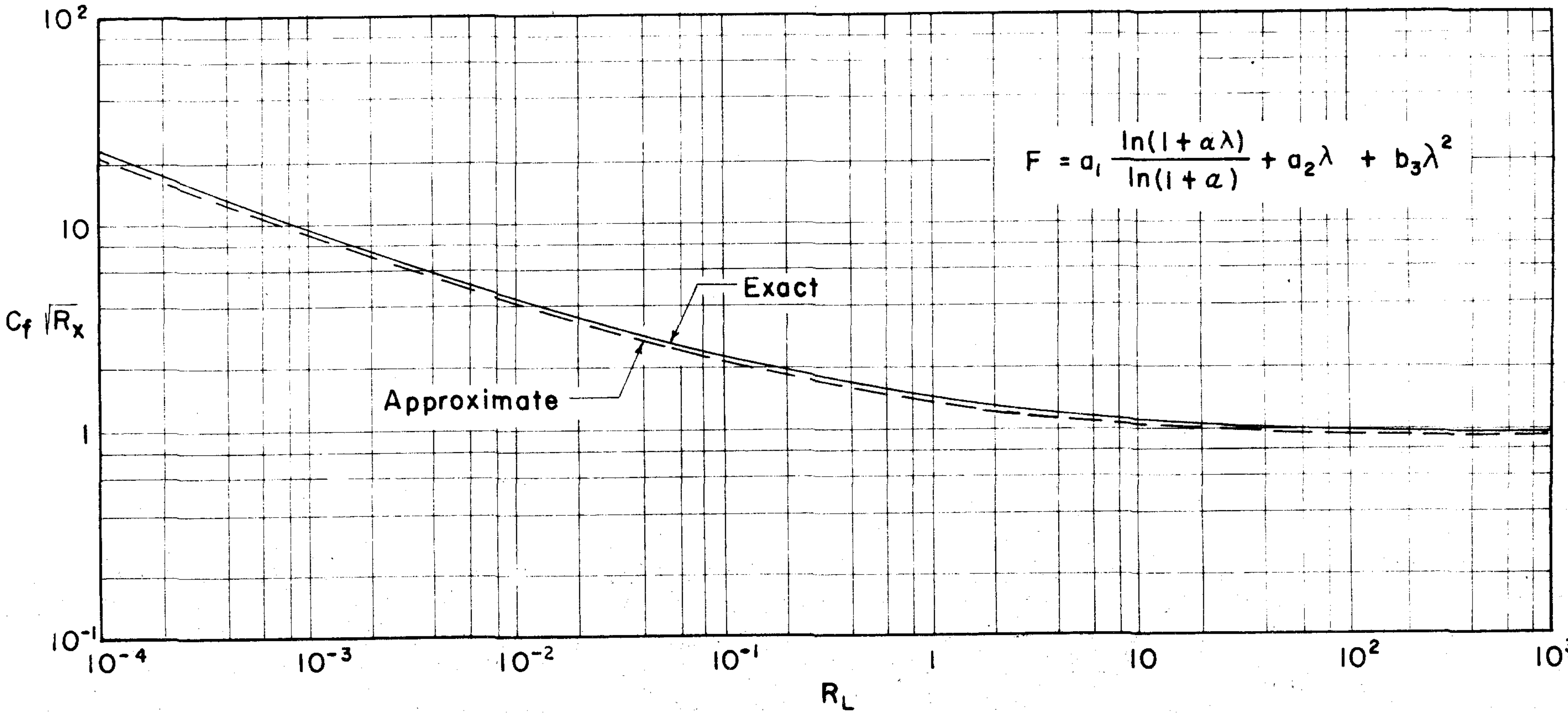


FIG. 5 COMPARISON OF EXACT AND APPROXIMATE LOCAL SKIN FRICTION OF PARABOLOIDS OF REVOLUTION

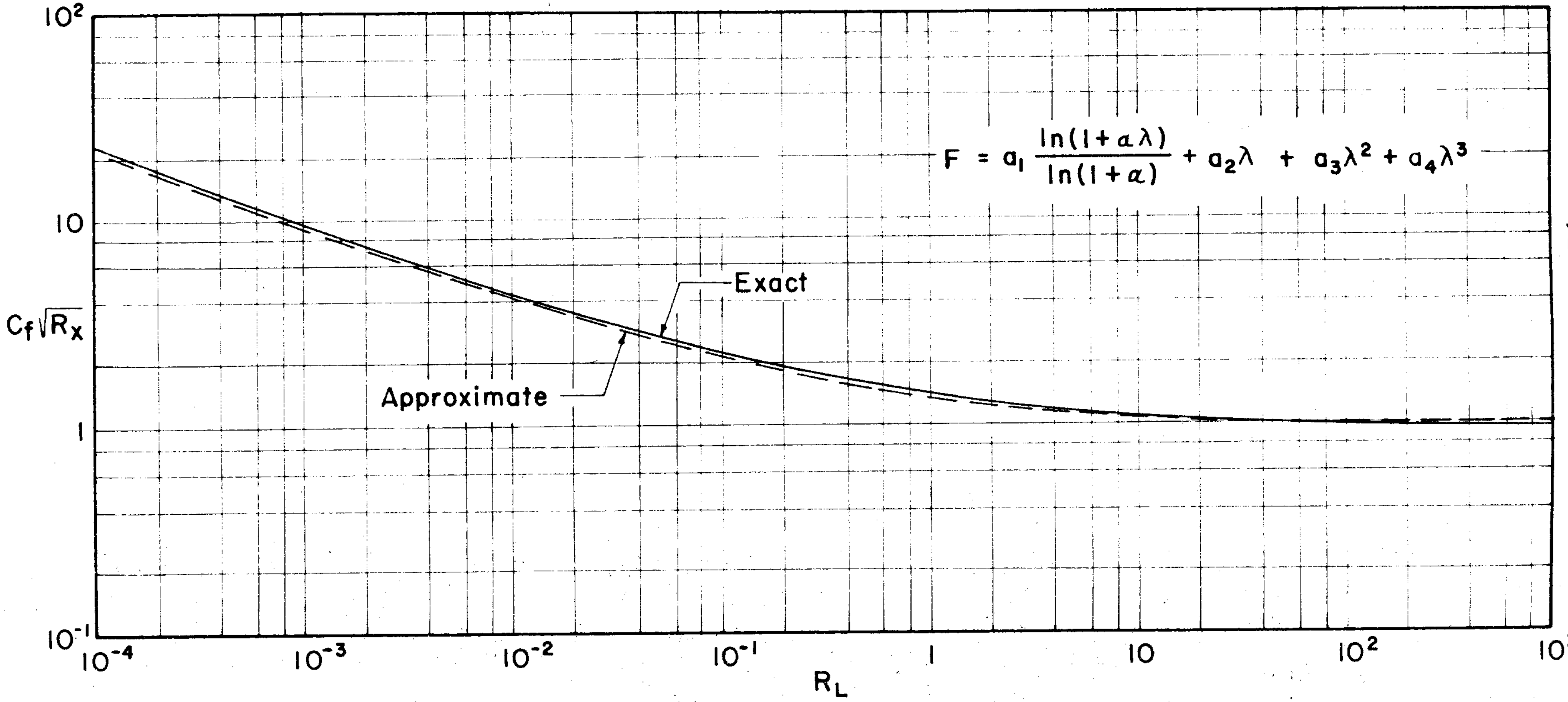


FIG. 6 COMPARISON OF EXACT AND APPROXIMATE LOCAL SKIN FRICTION OF PARABOLOIDS OF REVOLUTION

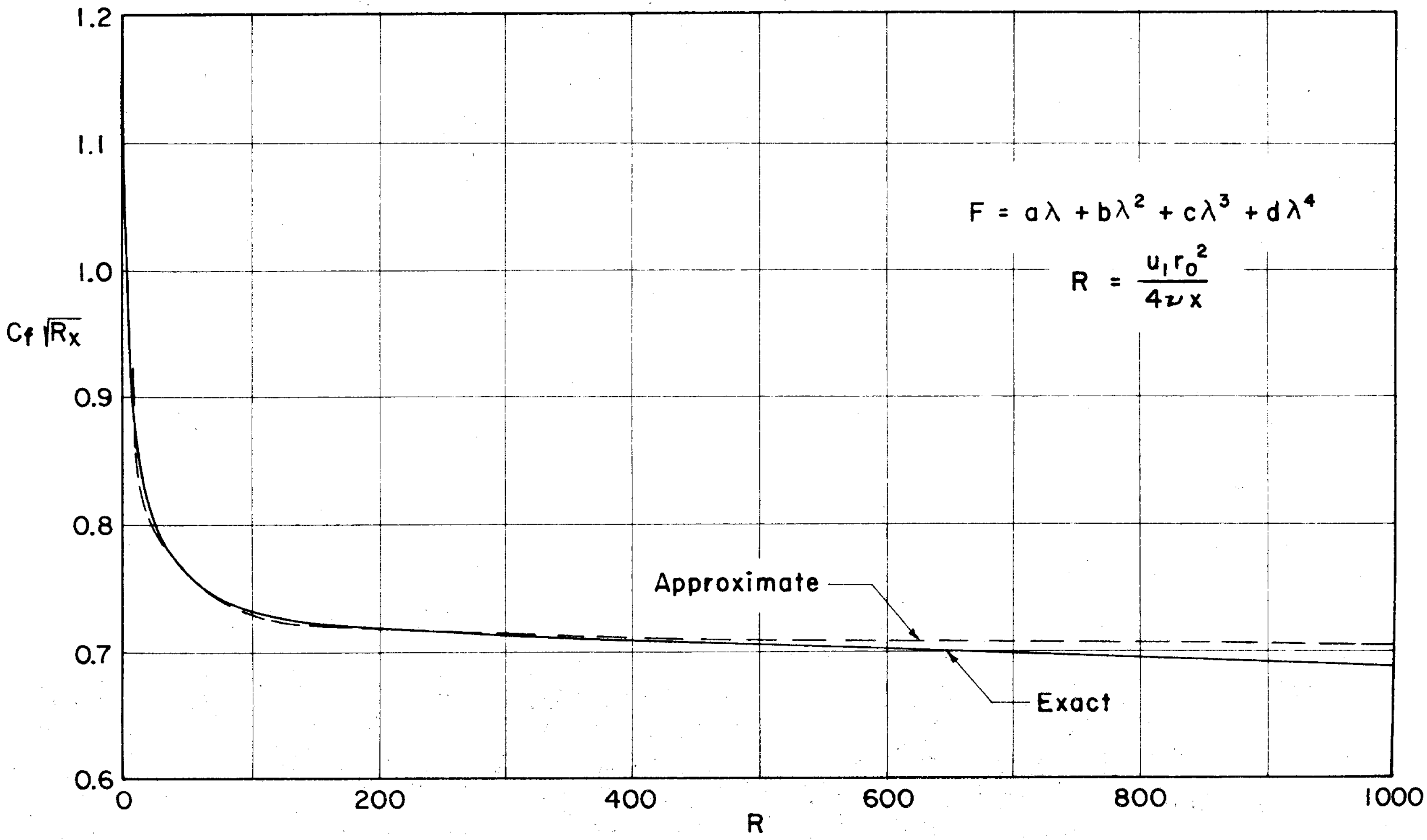


FIG. 7 COMPARISON OF EXACT AND APPROXIMATE LOCAL SKIN FRICTION OF A CIRCULAR CYLINDER

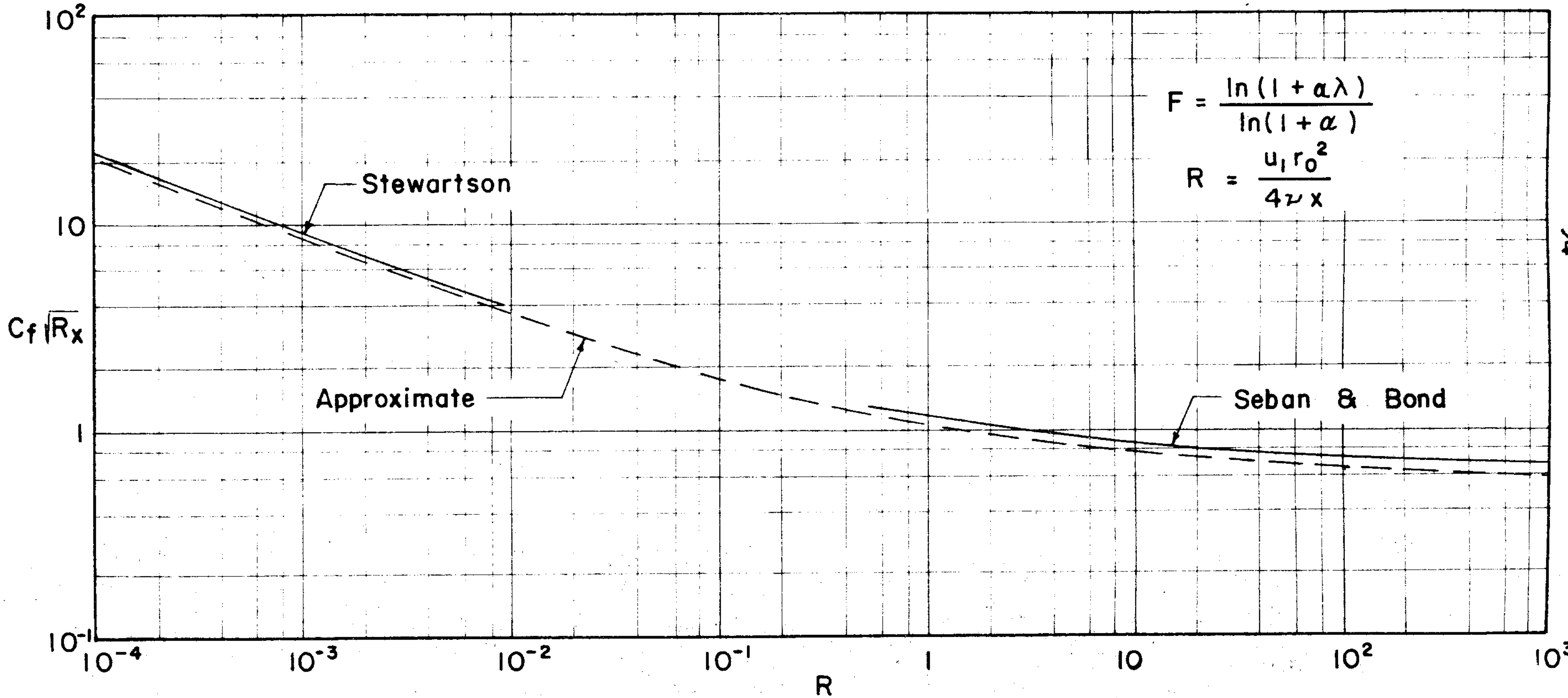


FIG. 8 COMPARISON OF EXACT AND APPROXIMATE LOCAL SKIN FRICTION OF A CIRCULAR CYLINDER

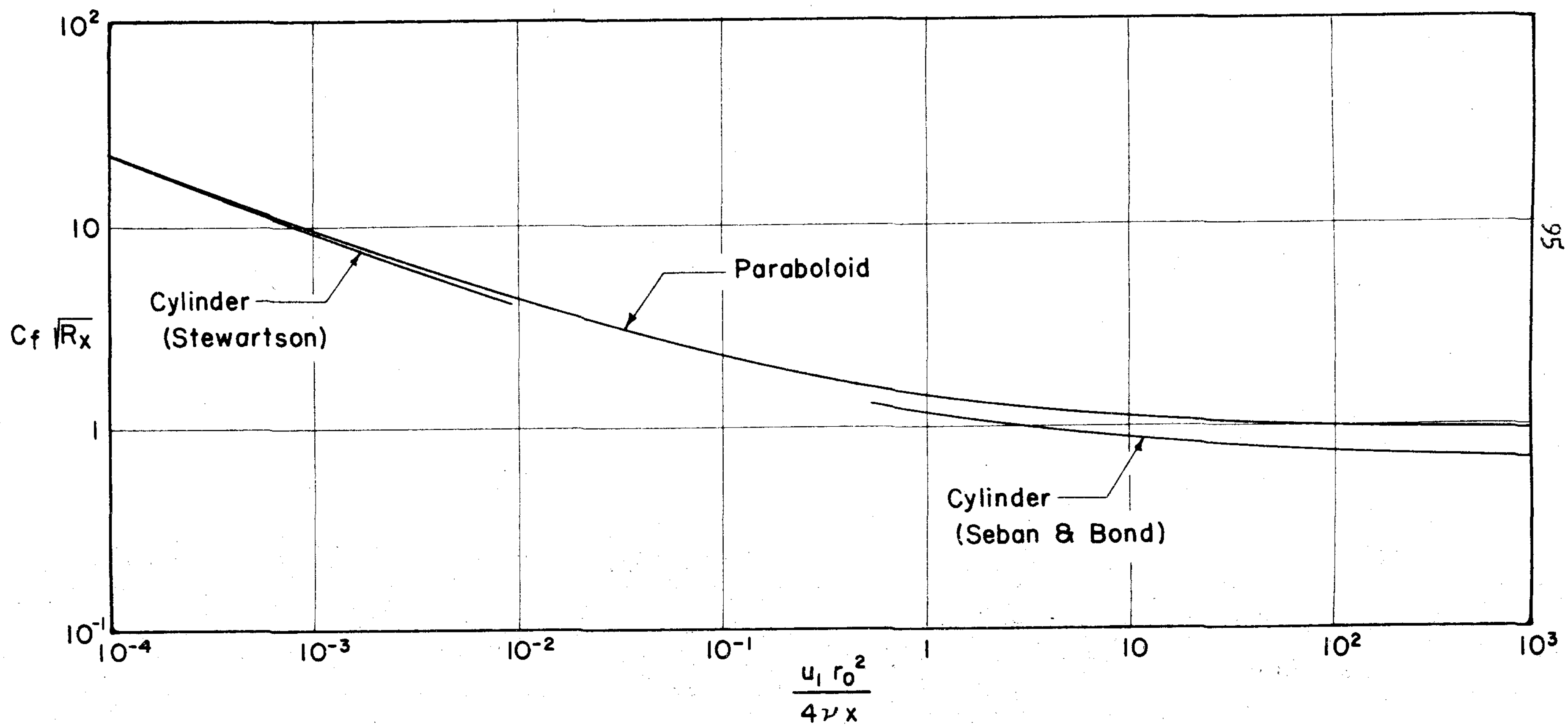


FIG. 8a COMPARISON OF EXACT LOCAL SKIN FRICTION OF PARABOLOIDS AND CIRCULAR CYLINDERS IN AXIAL INCOMPRESSIBLE FLOW

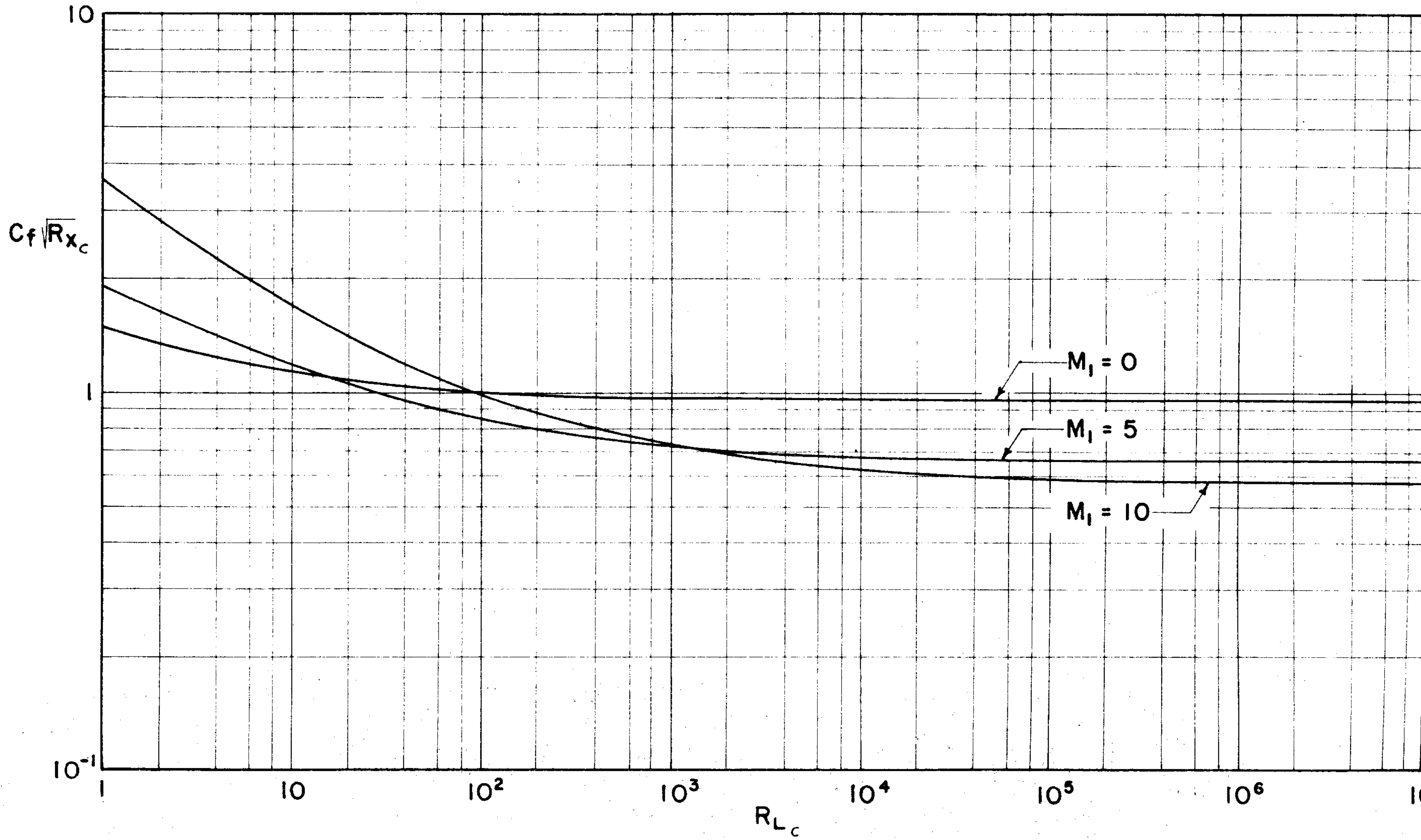


FIG. 9 APPROXIMATE LOCAL SKIN FRICTION OF PARABOLOIDS OF REVOLUTION AT DIFFERENT FREE STREAM MACH NUMBERS

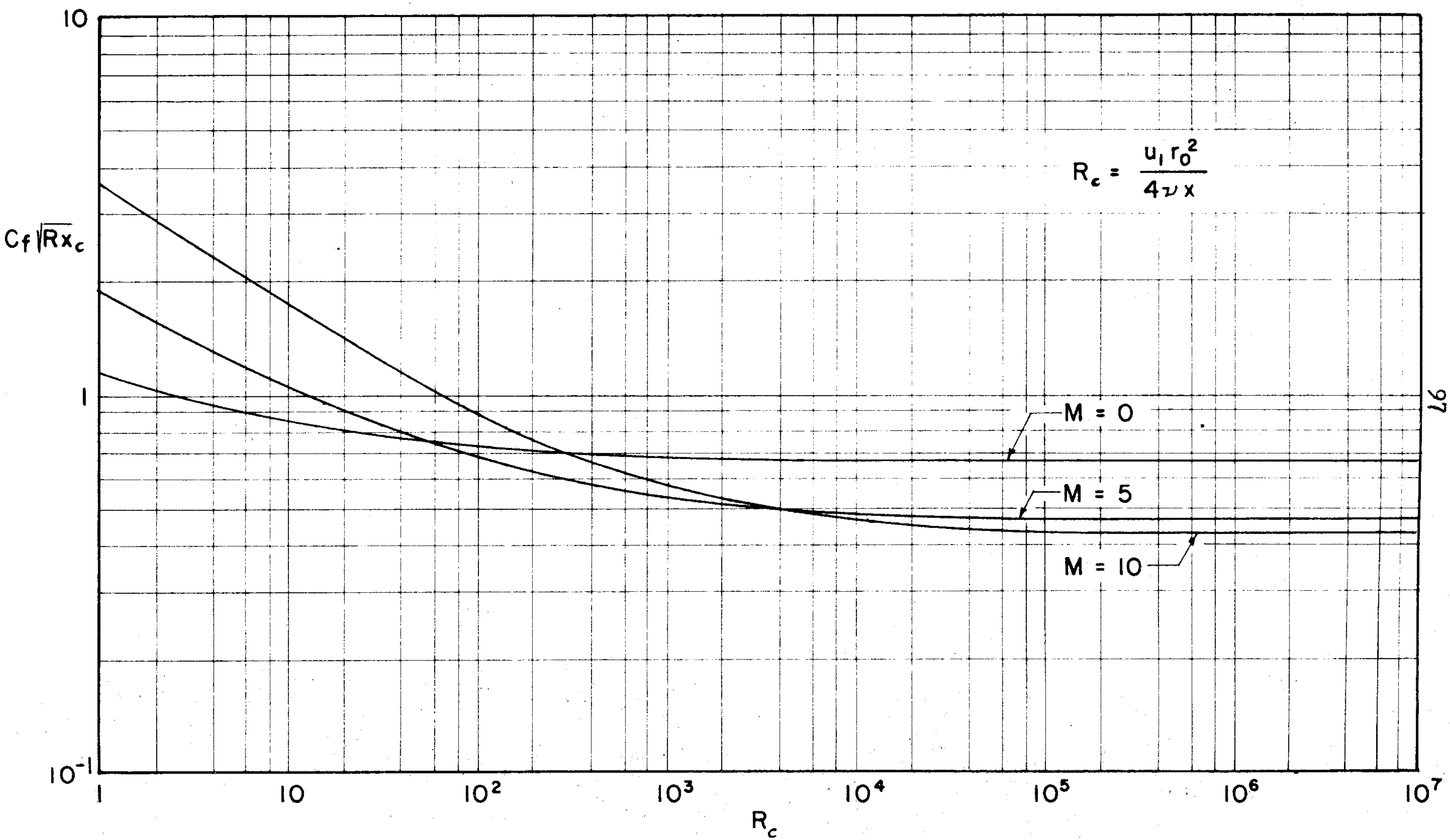


FIG. 10 APPROXIMATE LOCAL SKIN FRICTION OF CIRCULAR CYLINDERS AT DIFFERENT FREE STREAM MACH NUMBERS

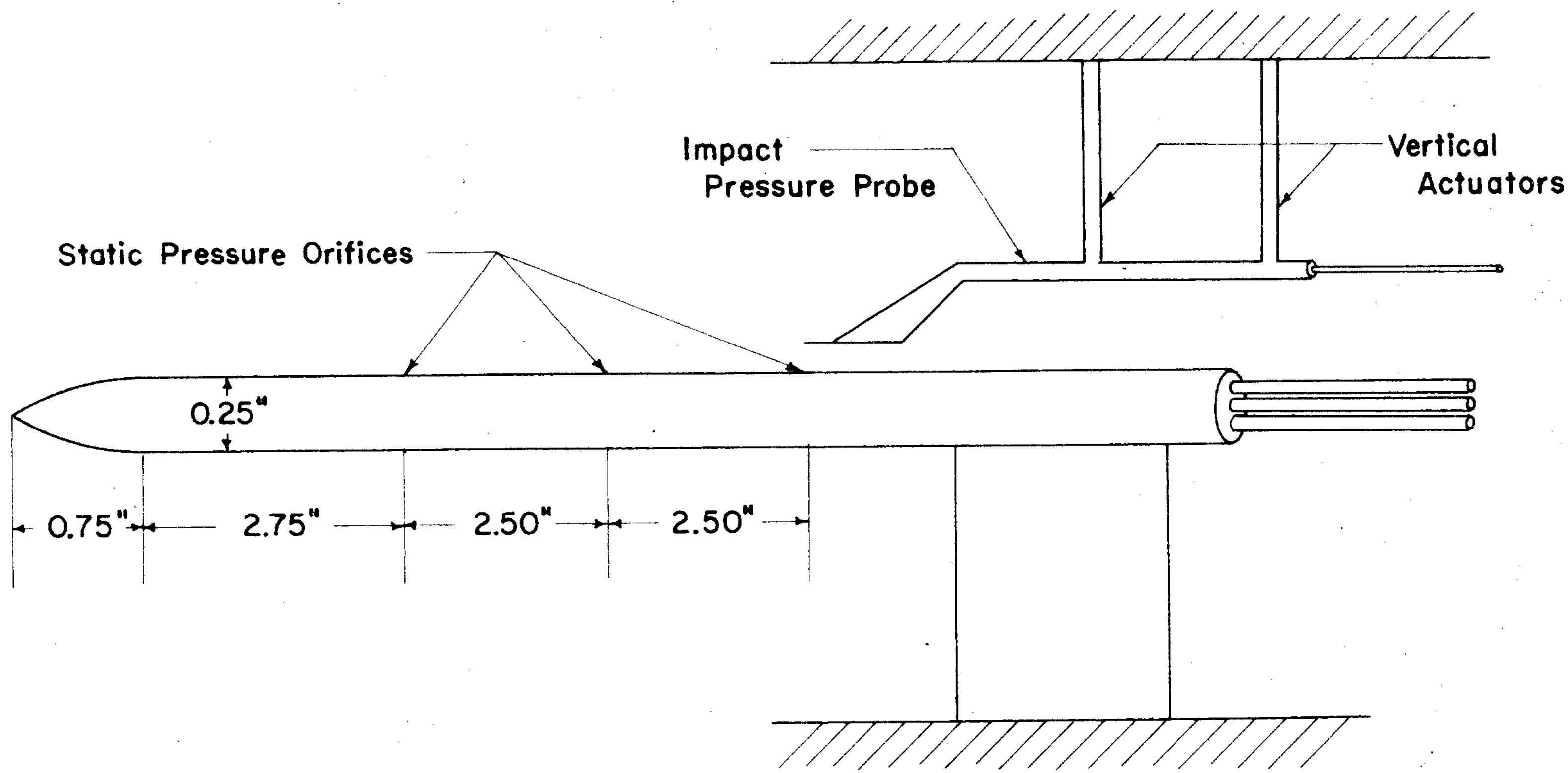


FIG. 11 SCHEMATIC DIAGRAM OF MODEL AND INSTRUMENTATION

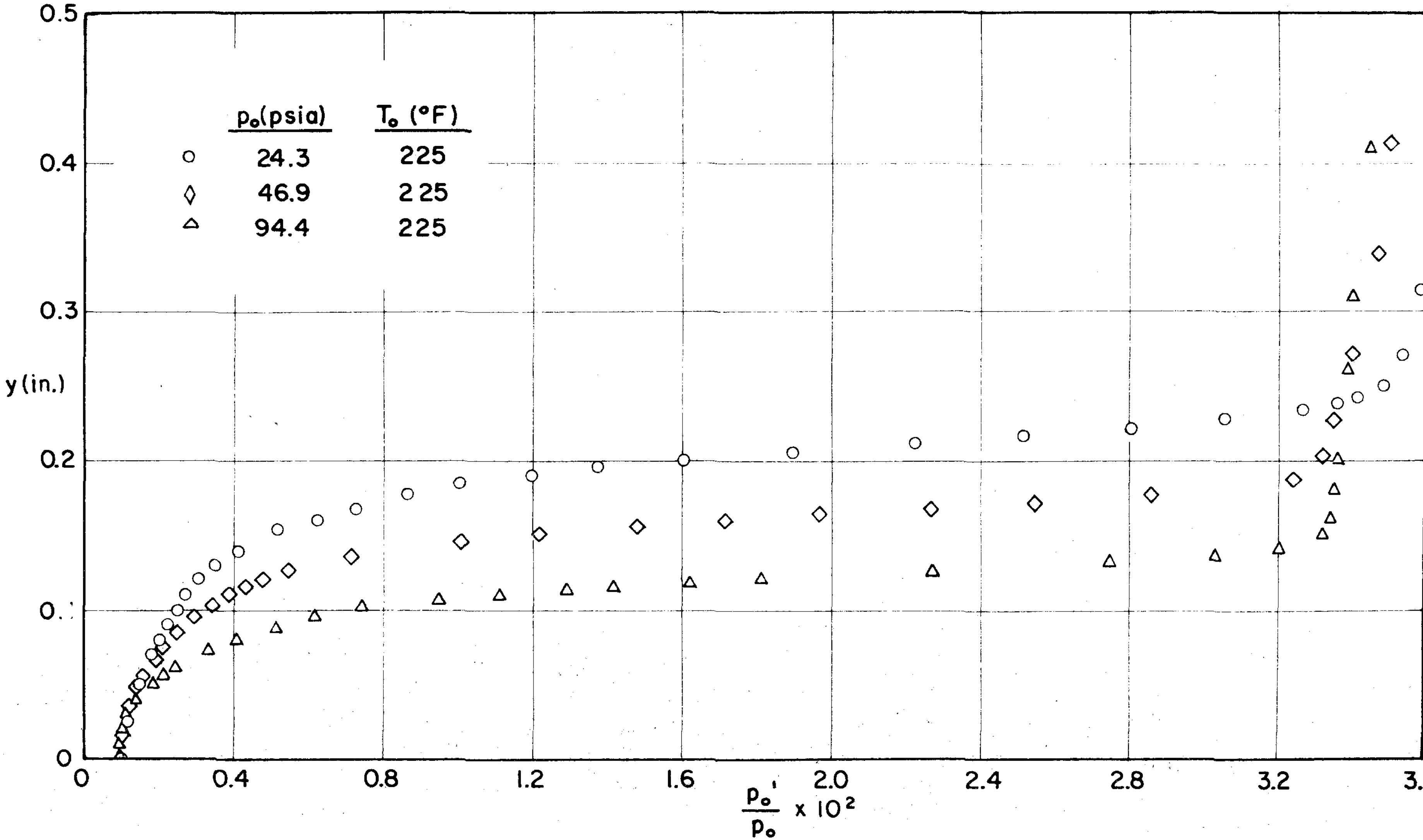


FIG. 12 IMPACT PRESSURE MEASUREMENTS FOR DIFFERENT STAGNATION PRESSURES AT $M_1 = 5.8$

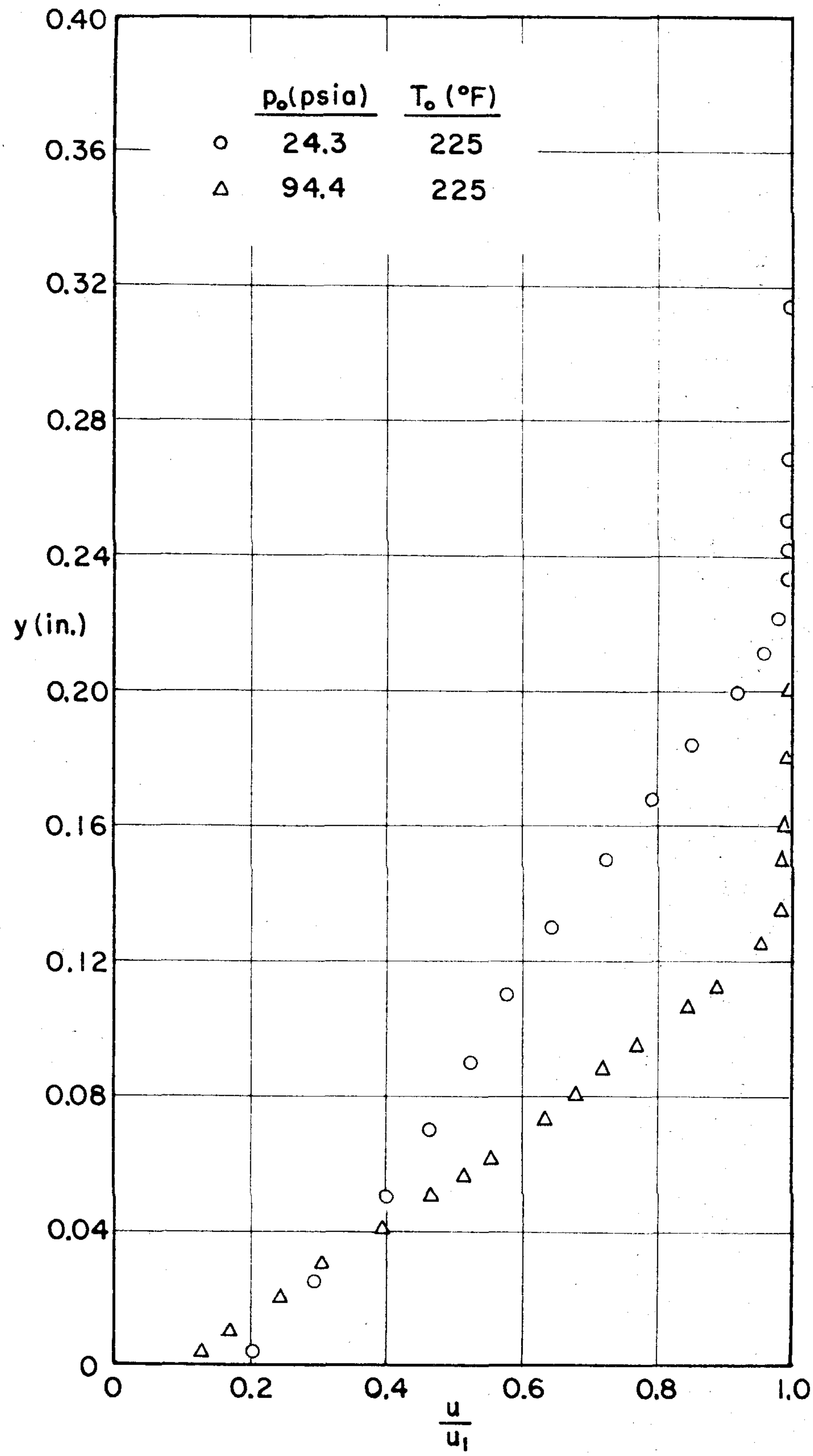


FIG. 13 VELOCITY PROFILES ON OGIVE-CYLINDER AT $M_1 = 5.8$ AT DIFFERENT STAGNATION PRESSURES

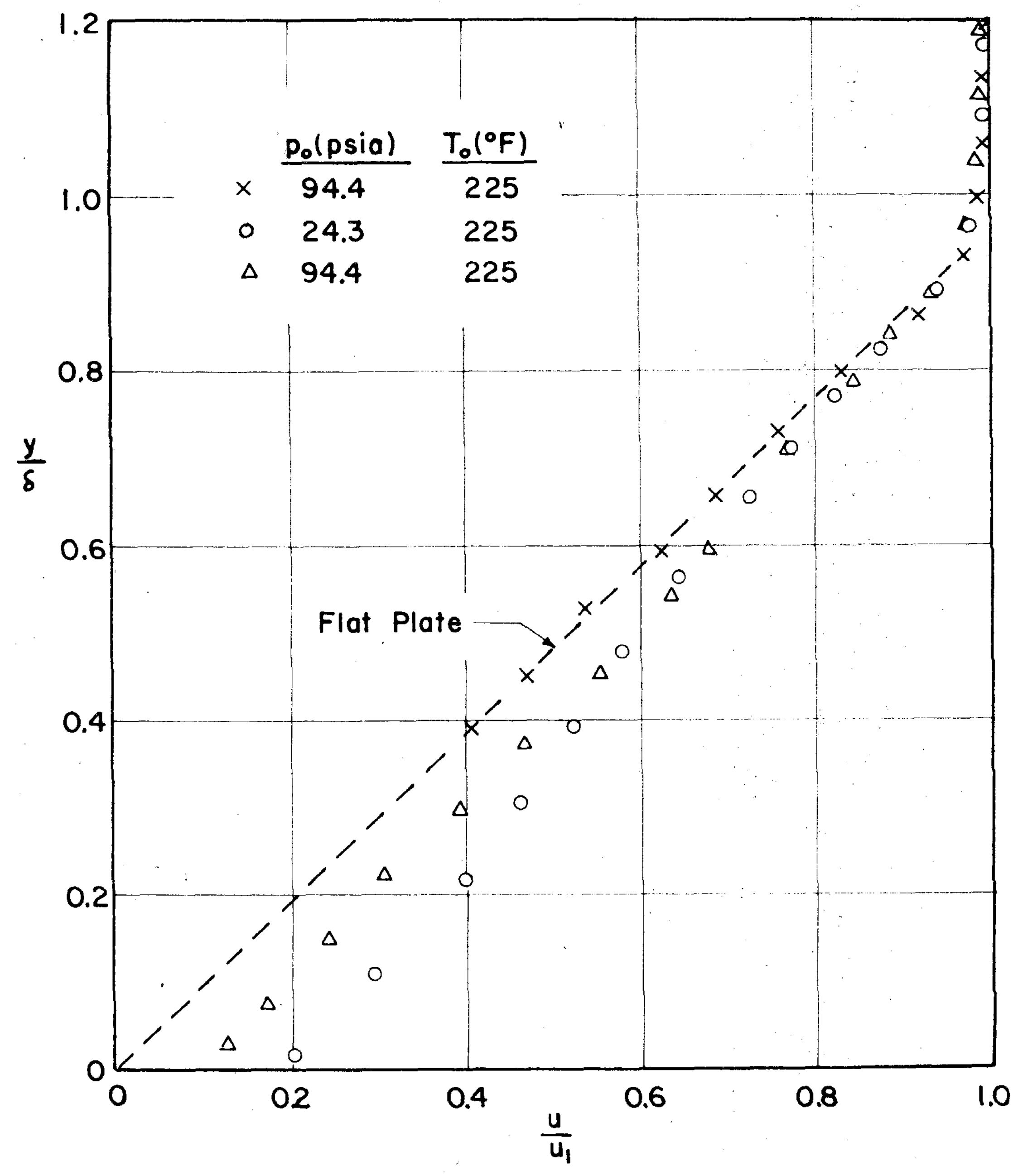


FIG. 14 COMPARISON OF VELOCITY PROFILES ON OGIVE-CYLINDER AND FLAT PLATE AT $M_1 = 5.8$