POLYNOMIALS IN ABSTRACT SPACES

Thesis
by

Ivar E. Highberg

In Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy

Pasadena, California

1936

INTRODUCTION

This thesis really had its inception in some work of Fréchet. In 1909, Fréchet, (1), gave a functional definition of a polynomial, where the polynomial is a real, continuous function of a real variable. His definition was essentially: $f(x)$ is a polynomial, if it is continuous, and if for some integer n, $\Delta^{n+1}_{u_1, \cdots, u_{n+1}} f(x) \equiv 0$. He then showed that such a function could be represented in the usual form. Gateaux (1), next gave a definition of polynomials for functionals. His definition was that $f(x)$ is a polynomial if,

$$f(\mu \cdot x + \lambda \cdot y) = \mu^n \cdot k_0(x,y) + \mu^{n-1}\lambda \cdot k_1(x,y) + \cdots + \lambda^n \cdot k_n(x,y) ,$$

where λ, μ are complex numbers, and $k_j(x,y)$ is a functional of x and y. In 1929, Fréchet, (2), extended his definition of a polynomial to a class of abstract spaces called "espaces algébrophiles", essentially linear spaces with a topology. He practically showed the equivalence of his definition and the Gateaux definition, and did in fact in the case of functions on the real numbers to the abstract space.

He, however, did not consider "espaces algébrophiles" in which multiplication by complex numbers is allowed. R.S. Martin, in his 1932 thesis, considered extensively the properties of polynomials from the Gateaux standpoint in a complex Banach space. Some of this has been published. See Michal and Martin (1), and Michal and Clifford (1). Michal ,(1), has also considered the definition of polynomials by means of polars. Martin conjectured that if $f(x)$, besides satisfying

Frêchet's conditions, were also Frêchet differentiable at $x = 0$, then $f(x)$ would be a polynomial in the Gateaux sense. That this is not enough is shown by the example $f(z) = \bar{z}^2$, which possesses a Frêchet differential at the origin.

To make Frêchet's definition equivalent to Gateaux' definition, it has been found sufficient to add the requirement of Gateaux differentiability to Frêchet's conditions. See Chapter II. An algebraic condition has also been found, see Chapter III.

It was thought profitable to investigate the nature of a function which satisfies Frêchet's conditions in a "complex espace algébrophile", and this led to the definition of pseudo-polynomials and the consideration of their properties. Pseudo-polynomials are the functions with which this thesis is mainly concerned, and in order to develop the theory, the work is arranged in the following manner. Chapter I contains the postulates for the space, and as much of the theory as is needed in the sequel. For a more extended account, see Frêchet (2).

Chapter II develops the theory of polynomials in both the Gateaux and Frêchet sense, and consists mainly of rewriting theorems from either Martin or Frêchet. Section 4, in which the equivalence of the definitions for complex spaces is proved, is, however, new.

In Chapter III pseudo-polynomials are defined from what might be termed a generalized Gateaux sense. The theory is developed in much the same manner as polynomial theory. For every theorem on polynomials, there is a corresponding theorem for pseudo-polynomials, with one or two exceptions. The main result of the chapter is section 3, the equivalence theorems.

Chapter IV is an application of the theory to the problem of generating a norm by means of a Hermitian bilinear form, and Chapter V is an example concerning the relation between the modulus of a polynomial and its polar. Appendix A contains a theorem needed in Chapter III, and Appendix B contains a simpler derivation of the equivalence theorem of Chapter III, but which was not found until the other had been done.

It seems as if most of the results proved in this thesis could be established for a linear topological space. It was thought best, however, to retain the "espace algébrophile" at this time. Modular properties of polynomials are not discussed, and the theory of polars has been barely touched. There is more variety of possibility in polars of pseudo-polynomials. For example, $\lambda\bar{\lambda}$ may have the polar $\lambda_1 \bar{\lambda}_2$ or $\bar{\lambda}_1 \lambda_2$.

In general the notation is that of Martin's thesis, and we do not give long explanations which may be found there. The functional notation is that of E.H. Moore, and we use some point-set theoretical notation also. The numbers after an author's name refer to the bibliography at the end.

In conclusion I wish to express my appreciation to Professor A.D. Michal, who first suggested to me the problem of polynomials in complex spaces, for his criticisms and suggestions.

TABLE OF CONTENTS

CHAPTER I

THE ESPACE ALGÉBROPHILE

In this chapter we give the postulates for an "espace algébrophile" and develop as much of the general theory as is used throughout the body of the work.

§1 The Postulates

A complex "espace algébrophile" E(C), is a system which consists of:-

A set $\mathcal{E}$, of elements of any nature whatever, and which we may call abstract points;

Families $\{V_x\}$, of sets, V_x, of abstract points associated with each abstract point x; $\{V_x\}$ constituting the family of "neighborhoods", V_x, of x;

Operations concerning the abstract points, or the abstract points and the complex number system C, and denoted by the symbols, "$+$", "$\cdot$", "$\|x\|$";

A system of relations between the set $\mathcal{E}$, the sets of neighborhoods $\{V_x\}$, and the operations denoted above.

In the following we shall assume that equality has the properties of an equivalence relation.

I The First Group of Conditions.

1. If $(x,y) \in E$, then $(x + y) \in E$.

2. $$x + y = y + x .$$

3. $$(x + y) + z = x + (y + z) .$$

4. If $a \in C$, and $x \in E$, then $(a \cdot x) \in E$.

5. $$a \cdot (x + y) = a \cdot x + a \cdot y .$$

6. $$(a + b) \cdot x = a \cdot x + b \cdot x .$$

7. $$a \cdot (b \cdot x) = ab \cdot x .$$

8. $$1 \cdot x = x .$$

9. If $x \in E$, then $\|x\|$ is a real number $\geqslant 0$.

10. There exists one and only one element 0 in E, such that

$$\|0\| = 0 .$$

11. $$\|a \cdot x\| = |a| \, \|x\| .$$

We write $x - y$ for $x + -1\, y$, and I will now state some definitions which are needed for the second group of conditions.

Abstract Line. The set of points z , where

$$z = x + t \cdot (y - x) \qquad t \in C ,$$

will be called the abstract line xy . It is obvious that the points x and y "lie on the line". That the line xy coincides with the line yx , may be seen by setting $t = 1 - t'$.

Translation. A transformation of E into itself by an equation of the form,

$$y = x + x_0 \qquad x_0 \in E ,$$

will be called a translation.

Homothetie. A transformation of E into itself by an equation of the form,

$$y = x_0 + a \cdot (x - x_0) ,$$

where x_0 is a fixed point in E , and a is a fixed point in C , will be called a homothetic transformation.

Continuity. If $z = f(x,y)$ is a function defined on E_1E_2 to E_3, and $z_0 = f(x_0,y_0)$, then $z = f(x,y)$ is said to be continuous at (x_0,y_0), if for every neighborhood V_{z_0} of z_0, there exists a neighborhood V_{x_0} of x_0, and a neighborhood V_{y_0} of y_0, such that when $x \in V_{x_0}$ and $y \in V_{y_0}$, then $z \in V_{z_0}$.

This definition of continuity has been phrased in terms of two variables for convenience. For functions of one variable, we drop one variable from the definition.

Point of Accumulation. A point x will be called a point of accumulation of a set U, if each neighborhood of x contains at least one point of U (different from x).

We now proceed with the second group of conditions.

II The Second Group of Conditions.

12. If U_x and V_x are any two neighborhoods of x, then there exists a neighborhood W_x of x, such that $W_x \subset U_x \cdot V_x$.

13. The logical product of all the neighborhoods of a point x is the point x alone.

14. The transformation $z = x + y$ is continuous in the pair (x,y) at all points (x,y) of E.

15. Homothetic transformations are continuous at all points of E.

16. The points of accumulation of an abstract line lie on the line.

17. A necessary and sufficient condition that a point x_0 of an abstract line be a point of accumulation of a set of points U on the line, is that the lower bound of $\|x - x_0\|$ be zero as x ranges over U (remaining distinct from x_0).

§2 Some Theorems on Continuity

Theorem 1. If $z = f(y)$ and $y = g(x)$ are functions defined in "espaces algébrophiles", and z is continuous at $y = y_0$, and y is continuous at $x = x_0$, then $z = f(g(x))$ is continuous at $x = x_0$.

Theorem 2. If $z = f(x,y)$ is continuous in the pair (x,y) at (x_0,y_0), and $x = g(t)$, $y = h(t)$ are continuous at $x_0 = g(t_0)$, $y_0 = h(t_0)$, then $z = f(g(t),h(t))$ is continuous at $t = t_0$.

Theorem 3. A linear combination $z = a\cdot g(u) + b\cdot h(u)$ is continuous if $g(u)$ and $h(u)$ are continuous.

In the above three theorems I have not mentioned the domains of definition in order to avoid prolixity. It is clear, however, that the domains may be situated in the same or different spaces. Theorem 1 follows from the definition of continuity. Theorem 2 requires postulate 12 in addition. The proof of theorem 3 requires postulates 15 and 14, and theorem 1.

Theorem 4. If $f(\lambda)$ is a continuous complex function of a complex variable, and u_0 is a fixed point in E, then $y = f(\lambda)\cdot u_0$ is a continuous function of λ.

Proof: Let $y_0 = f(\lambda_0)\cdot u_0$, and let V_{y_0} be a neighborhood of y_0. Let us assume that the theorem is false. Then there must exist at least one sequence $\lambda_n \to \lambda_0$, such that $y_n = f(\lambda_n)\cdot u_0$ is not in V_{y_0} for any n. However,

$$\|y_n - y_0\| = |f(\lambda_n) - f(\lambda_0)|\ \|u_0\| \to 0 ,$$

and hence y_0 is a point of accumulation of the set $\{y_n\}$. We thus arrive at a contradiction with the hypothesis and hence the theorem is true.

Theorem 5. If u_i are fixed points in E, and $f_i(\lambda)$ are continuous functions of a complex variable, then,

(1) $$y = f_0(\lambda)\cdot u_0 + \cdots + f_n(\lambda)\cdot u_n$$

is a continuous function of λ.

Proof: Use theorems 3 and 4, and mathematical induction.

§3 The Gateaux Differential

Definition 1. Let f(x) be defined on an "espace algĕbrophile" E_1, to a like space E_2. Then f(x) will be said to possess a Gateaux differential at x_0, if for any z in E_1,

$$\lim_{t \to 0} \frac{f(x_0 + t\cdot z) - f(x_0)}{t} \qquad t \in G$$

exists, independent of the way in which $t \to 0$. We denote the Gateaux differential by $f(x_0;z)$.

Theorem 6. Let $F(\lambda) = f(\lambda)\cdot u_0$, where $f(\lambda)$ is a function on C to C having a derivative everywhere. Then $F(\lambda)$ is Gateaux differentiable everywhere.

Proof: $$\frac{F(\lambda + t\mu) - F(\lambda)}{t} = \mu \frac{f(\lambda + t\mu) - f(\lambda)}{t\mu} \cdot u_0$$

Since $\lim_{t \to 0} \frac{f(\lambda + t\mu) - f(\lambda)}{t\mu} = f'(\lambda)$, and since $g(\mu)\cdot u_0$ is a continuous function of μ, we conclude that the Gateaux differential exists, and $F(\lambda;\mu) = \mu f'(\lambda)\cdot u_0$.

Theorem 7. A function of the form (1) is Gateaux differentiable if all the $f_i(\lambda)$ are derivable.

Proof: This is clear from theorems 5 and 6.

§4 Examples of Espaces Algébrophiles

It is obvious that the complex number system C , is itself an instance of an "espace algébrophile". We may take for the family of neighborhoods attached to the point z , the interiors of circles having z as a center and rational radii. A complex Banach space is also an instance, with spheres about a point as neighborhoods. For a formal proof and for other examples see Fréchet (2).

CHAPTER II

POLYNOMIALS IN E

The object of this chapter is to give a definition of polynomials, lay down some of their fundamental properties, and to consider the equivalence of two different definitions which have been given. The results of this chapter appear in a paper which has not yet been published, Highberg, (1).

§1 Polynomials on C to E(C)

We begin by considering functions on C to E(C) , and the natural definition of a polynomial on C to E(C) is as follows.

Definition 1. A function $p(\lambda)$ defined on C to E(C) , which can be expressed in the form, valid for all λ ,

$$(1) \qquad p(\lambda) = u_0 + \lambda \cdot u_1 + \cdots + \lambda^n \cdot u_n ,$$

where the u_i are fixed elements in E , will be called a C-polynomial. The index of the non-vanishing u_i with highest index will be called the degree of the polynomial. A polynomial of degree 0 is a constant. A polynomial which is identically zero may be regarded as having any degree whatever.

Theorem 1. A C-polynomial is continuous.

Proof: This is a special case of I, §2, theorem 5, with $f_i(\lambda) = \lambda^i$.

Theorem 2. A linear combination of C-polynomials of degree $\leqslant m$, is a C-polynomial of degree $\leqslant m$.

Theorem 3. If $p(\lambda)$ is a C-polynomial of degree m, and $g(\lambda)$ is a polynomial on C to C of degree n, then $g(\lambda)\cdot p(\lambda)$ is a C-polynomial of degree $m + n$.

Theorem 4. If $p(\lambda)$ is a C-polynomial of degree m, and $q(\lambda)$ is a C-polynomial of degree n, and $p(\lambda) \equiv q(\lambda)$; then $m = n$ and $u_i = v_i$, where u_i and v_i are the coefficients in $p(\lambda)$ and $q(\lambda)$ respectively.

Theorem 5. The representation of a C-polynomial in the form (1) is unique.

The proofs of theorems 2-5 have been omitted, for the details are no different from the method in Martin's thesis, Martin (1), pp. 19-21.

Theorem 6. A C-polynomial is Gateaux differentiable everywhere.
Proof: This is a special case of I, §3, theorem 7.

§2 Polynomials on E_1 to E_2

In this section we discuss functions defined on one "espace algébrophile E_1, to a like space E_2.

Definition 2. Let $p(x)$ be a function on E_1 to E_2. Then $p(x)$ will be said to be an E-polynomial, if

1° $p(x)$ is continuous,

2° for every pair (x,y), $p(x + \lambda\cdot y)$ is a C-polynomial in λ.

It will be said to be of degree n, if for some (x,y), $p(x + \lambda\cdot y)$ is a C-polynomial in λ of degree n, and for all (x,y) is a C-polynomial in λ of degree $\leqslant n$.

An E-polynomial can therefore be represented in the form,

$$(2) \qquad p(x+\lambda\cdot y) = k_0(x,y) + \lambda\cdot k_1(x,y) + \cdots + \lambda^n\cdot k_n(x,y),$$

where the $k_i(x,y)$ are functions on E_1^2 to E_2. When $p(x)$ is represented in the form (2), the functions $k_i(x,y)$ are called the <u>associated functions</u> to $p(x)$.

<u>Theorem 7.</u> A linear combination of E-polynomials of degree $\leqslant n$, is an E-polynomial of degree $\leqslant n$.

Proof: This follows immediately from I, §2, theorem 3, and theorem 2 above.

<u>Theorem 8.</u> If $p(x)$ is an E-polynomial of degree $\leqslant n$, and $c \in C$, then $p(c\cdot x + x_0)$ is an E-polynomial in x of degree $\leqslant n$.

Proof: First, $p(c\cdot x + x_0)$ is a continuous function of x by I, §2, theorems 3 and 1. Secondly, using (2),

$$p(c\cdot x + c\lambda\cdot y + x_0) = p(c\cdot x + x_0 + \lambda c\cdot y)$$

$$= k_0(c\cdot x + x_0, c\cdot y) + \lambda\cdot k_1(c\cdot x + x_0, c\cdot y) + \cdots + \lambda^n\cdot k_n(c\cdot x + x_0, c\cdot y)$$

and this is a C-polynomial in λ of degree $\leqslant n$.

<u>Theorem 9.</u> If $p(x)$ is an E-polynomial, and $k_r(x,y)$ is an associated function to it, then $k_r(x,y)$ is homogeneous in y of degree r, that is,

$$k_r(x, \mu\cdot y) = \mu^r\cdot k_r(x,y).$$

<u>Theorem 10.</u> If $p(x)$ is an E-polynomial of degree $\leqslant n$, and $k_r(x,y)$

is an associated function to it, then for fixed x , $k_r(x,y)$ is an E-polynomial in y of degree $\leqslant n$, and for fixed y , $k_r(x,y)$ is an E-polynomial in x of degree $\leqslant n$.

Theorem 11. If $h(x)$ is an E-polynomial of degree n , and if it is homogeneous of degree r , then $n = r$. *

Theorem 12. If $h(x)$ is an E-polynomial homogeneous of degree m , and $k_r(x,y)$ is an associated function to it, then $k_r(x,y)$ is homogeneous in x of degree $m - r$, that is,

$$k_r(\mu \cdot x,y) = \mu^{m-r} \cdot k_r(x,y) .$$

The proofs of the last four theorems depend only on the linearity of the space, the definition, and theorems 7 and 8 above. Similar theorems may be found in Martin (1), pp.28-31.

By setting $x = 0$, $\lambda = 1$, $h_r(y) = k_r(0,y)$ in (2), we get the theorem,

Theorem 13. If $p(x)$ is an E-polynomial of degree n , it can be uniquely represented as a sum of homogeneous polynomials in the form,

(3) $$p(x) = h_0(x) + h_1(x) + \cdots + h_n(x) .$$

Canonical Representation The representation of an E-polynomial in the form (3), will be called the canonical representation.

---

*Note: A function which is homogeneous is not necessarily a polynomial. For example, in a real Banach space, $\|t \cdot x\|^2 = t^2 \|x\|^2$, but $\|x\|^2$ is not a polynomial. If in fact it were a polynomial, the norm would be of Hilbert type, that is, generated by a quadratic form.

§ 3 Differences

Let $F(x)$ be a function defined on one "espace algébrophile" E_1, to a like space E_2. We define successive differences inductively. The first difference is written

$$\Delta_u^1 F(x) = F(x + u) - F(x) ,$$

and the nth difference with independent increments is defined as

$$\Delta_{u_1,\cdots,u_n}^n F(x) = \Delta_{u_n}^1 \left[\Delta_{u_1,\cdots,u_{n-1}}^{n-1} F(x) \right] .$$

In particular, $\Delta_{u_1,\cdots,u_n}^n F(0)$ means $\left[\Delta_{u_1,\cdots,u_n}^n F(x) \right]_{x=0}$. It can easily be proved by induction that the nth difference of a function is symmetric in the increments.

If we apply this to a function of the form (1), we see that

$$\Delta_\mu^1 p(\lambda) = \mu \cdot u_1 + \left[(\lambda+\mu)^2 - \lambda^2 \right] \cdot u_2 + \cdots + \left[(\lambda+\mu)^n - \lambda^n \right] \cdot u_n ,$$

so that we may assert,

Theorem 14. If $p(x)$ is a C-polynomial of degree n, then $\Delta_\mu^1 p(\lambda)$ is a C-polynomial of degree $\leqslant n - 1$, and for some choice of μ is of degree exactly $n - 1$.

Proof: This is evident from the equation above.

Theorem 15. If $p(\lambda)$ is a C-polynomial of degree n, then $\Delta_{\mu_1,\cdots,\mu_{n+1}}^{n+1} p(\lambda) \equiv 0$, and for some choice of the increments, $\Delta_{\mu_1,\cdots,\mu_n}^n p(\lambda) \not\equiv 0$.

Proof: Use theorem 14 successively.

Theorem 16. If $p(x)$ is a homogeneous E-polynomial of degree n, then $\Delta_u^1 p(x)$ is an E-polynomial of degree $\leqslant n - 1$, and for some u is of degree exactly $n - 1$.

Proof: Using the representation (2) and theorem 12,

$$p(\mu\cdot x + \lambda\cdot y) = \mu^n\cdot k_0(x,y) + \mu^{n-1}\lambda\cdot k_1(x,y) + \cdots + \lambda^n\cdot k_n(x,y) .$$

Putting $\mu = 0$, we see that $k_n(x,y)$ is independent of the first argument. Further, setting $\mu = 0$ and $\lambda = 1$, we get $p(y) = k_n(x,y)$. From theorem 12, $k_i(0,y) = 0$ if $i \neq n$, so if in the equation above, we set $\lambda = \mu = 1$, replace x by u and y by x , we get

$$\underset{u}{\Delta'} p(x) = k_0(u,x) + k_1(u,x) + \cdots + k_{n-1}(u,x) .$$

By theorem 9 this is a sum of polynomials of degree $\leqslant n - 1$ in x , and hence is of degree $\leqslant n - 1$.

To show the second part of the theorem choose $u = \lambda\cdot x$. Then

$$\underset{\lambda\cdot x}{\Delta'} p(x) = p(x + \lambda\cdot x) - p(x) = [(1 + \lambda)^n - 1]\cdot p(x)$$

$$= \lambda^n\cdot k_0(x,x) + \lambda^{n-1}\cdot k_1(x,x) + \cdots + \lambda\cdot k_{n-1}(x,x) .$$

Equating the coefficients of λ on both sides of the equation we get,

$$n\cdot p(x) = k_{n-1}(x,x) .$$

Since p(x) is of degree n , there is at least one value of x , say x_1, for which $p(x_1) \neq 0$. Choose $u = x_1$. Then $k_{n-1}(x_1,x) \not\equiv 0$, and hence is of degree exactly $n - 1$, and from this, $\underset{x_1}{\Delta'} p(x)$ is of degree exactly $n - 1$.

<u>Theorem 17</u>. If p(x) is an E-polynomial of degree n , then $\underset{u}{\Delta'} p(x)$ is an E-polynomial in x of degree $\leqslant n - 1$, and for some choice of u is an E-polynomial of degree $n - 1$.

Proof: Use the representation of p(x) in the canonical form (3) and then employ theorem 16.

Theorem 18. If p(x) is an E-polynomial of degree n, then

$\Delta^{n+1}_{u_1,\cdots,u_{n+1}} p(x) \equiv 0$, and for some choice of the increments, $\Delta^{n}_{u_1,\cdots,u_n} p(x) \not\equiv 0$

Proof: Use theorem 17 successively.

Theorem 19. An E-polynomial is Gateaux differentiable.

Proof: Using the representation (2) we have,

$$p(x+\lambda\cdot y) - p(x) = \lambda\cdot k_1(x,y) + \cdots + \lambda^n\cdot k_n(x,y) .$$

Dividing by λ and letting λ tend to zero, it is evident that the limit exists, and that

$$p(x;y) = k_1(x,y) .$$

§ 4 A Second Definition of E-Polynomials and the Equivalence of the Definitions

In this section we shall give a definition of polynomials, which for real "espaces algébrophiles" is due to Fréchet. In the real spaces, condition 3° of definition 2′ is unnecessary. The equivalence theorems for complex "espaces algébrophiles" are new results.

Definition 2′. Let p(x) be a function on an E_1 to an E_2. Then p(x) will be said to be an E-polynomial, if

1° p(x) is continuous,

2° for some integer n, $\Delta^{n+1}_{u_1,\cdots,u_{n+1}} p(x) \equiv 0$,

3° p(x) possesses a Gateaux differential everywhere. *

It will be said to be of degree n, if $\Delta^{n}_{u_1,\cdots,u_n} p(x) \not\equiv 0$, for some $u_1, \cdots, u_n$.

*Note: The condition of Gateaux differentiability may be replaced by a purely algebraic condition. See Chapter III.

This definition includes the case of polynomials defined on C to E, and in discussing the equivalence of definitions 2 and 2', it is convenient to consider the C-polynomials first.

Theorem 20. In definition 2' let E_1 be C. Then definition 2' and definition 1 are equivalent.

Proof: That a function which is a C-polynomial of degree n by definition 1, is a C-polynomial of degree n by definition 2', is clear from theorems 1, 6, and 15. To show the converse we must show that a function on C to E_2, which satisfies conditions 1°, 2°, 3° above, has the form (1). We make the proof by induction.

Case I. $$\Delta^1_{\mu} f(\lambda) \equiv 0 .$$

In this case, $f(\lambda) = u_0$, u_0 a constant, and is of form (1).

Case II. $$\Delta^2_{\mu_1, \mu_2} f(\lambda) \equiv 0 .$$

Setting $\lambda = 0$ in this equation we have,

(4) $$f(\mu_1 + \mu_2) - f(\mu_1) - f(\mu_2) + f(0) = 0 ;$$

Let us define $\phi(\mu) = f(\mu) - f(0)$. Clearly $\phi(\mu)$ is continuous and Gateaux differentiable everywhere. Also, using (4),

(5) $$\phi(\mu + \nu) = \phi(\mu) + \phi(\nu) .$$

Then by a well known limiting process,

$$\phi(t\mu) = t \cdot \phi(\mu) , \qquad t \text{ real.}$$

Hence if $\lambda = \lambda_1 + i\lambda_2$,

$$\phi(\lambda) = \lambda_1 \cdot \phi(1) + \lambda_2 \cdot \phi(i) ,$$

$$= \frac{\lambda + \bar{\lambda}}{2} \phi(1) + \frac{\lambda - \bar{\lambda}}{2i} \phi(i) ,$$

$$\phi(\lambda) = \lambda \cdot u_1 + \bar{\lambda} \cdot u_2 , \tag{6}$$

where $\bar{\lambda}$ is the complex conjugate of λ. Since $\phi(\lambda)$ is Gateaux differentiable, and the first term on the right of (6) is likewise, we conclude that $\bar{\lambda} \cdot u_2$ is Gateaux differentiable. This is a contradiction since $\bar{\lambda}$ does not possess a derivative, and hence u_2 must be zero. We have therefore proved that

$$f(\lambda) = f(0) + \lambda \cdot u_1 ,$$

and is therefore of the form (1).

Case III. $$\Delta^{n+1}_{\mu_1, \cdots, \mu_{n+1}} f(\lambda) \equiv 0 .$$

Setting $\mu_{n+1} = \mu$, $\Delta^{n}_{\mu_1, \cdots, \mu_n} [f(\lambda + \mu) - f(\lambda)] \equiv 0$. Now, $f(\lambda + \mu) - f(\lambda)$, considered as a function of λ, is continuous and Gateaux differentiable everywhere.* Moreover, if $f(\lambda)$ is of degree n by definition 2', then $f(\lambda + \mu) - f(\lambda)$ is of degree $n - 1$ by definition 2'. Hence under the induction hypothesis we shall suppose that $f(\lambda + \mu) - f(\lambda)$ is a C-polynomial of degree $n - 1$ by definition 1. Let us set,

$$\psi(\lambda, \mu) = f(\lambda + \mu) - f(\lambda) - f(\mu) . \tag{7}$$

Then $\psi(\lambda, \mu)$ is a C-polynomial in λ of degree $n - 1$, and since it is symmetric in λ, μ, it is also a C-polynomial in μ of degree $n - 1$.

---

* Note: It is essential that $f(\lambda)$ be Gateaux differentiable everywhere, for otherwise $f(\lambda + \mu) - f(\lambda)$ may not be differentiable anywhere. Differentiability at one point is not sufficient. For example $\phi(\lambda) = \bar{\lambda}^2$ is differentiable at $\lambda = 0$, but nowhere else.

Then by a purely manipulative process, the details of which may be found in Fréchet (2), pp.86-88, we find

$$(8) \qquad \psi(\lambda,\mu) = g(\lambda+\mu) - g(\lambda) - g(\mu) \quad ,$$

where

$$g(\lambda) = u_0 + \sum_{1}^{r}{}_j \lambda^j \cdot u_j \quad ,$$

and where the u_j are fixed points in E. We set

$$\phi(\lambda) = f(\lambda) - g(\lambda) \quad .$$

Then, using (8),

$$\phi(\lambda+\mu) = \phi(\lambda) + \phi(\mu) \quad .$$

Now $f(\lambda)$ is continuous and Gateaux differentiable by hypothesis, and $g(\lambda)$ is likewise since it is a C-polynomial by definition 1. Hence $\phi(\lambda)$ is continuous and Gateaux differentiable, and therefore by Case II, $\phi(\lambda) = \lambda\cdot\phi(1)$. Hence,

$$f(\lambda) = u_0 + \lambda\cdot\phi(1) + \sum_{1}^{r}{}_j \lambda^j \cdot u_j ,$$

and this is of the form (1). To show that the degrees are the same, we observe that the left hand side of (8) is of degree n-1 in λ, while the right hand side is of degree r-1. Hence r = n, and the equivalence of the two definitions is completely established.

The equivalence theorem for E-polynomials is now readily established, and we shall give it as,

Theorem 21. Definitions 2 and 2' are equivalent.

Proof: That a function which is an E-polynomial of degree n by definition 2, is an E-polynomial of degree n by definition 2', is clear from definition 2, theorem 18, and theorem 19. To show the converse, we will suppose that $F(x)$ satisfies the conditions of definition 2' and is of degree n. Then $f(\lambda) = F(x + \lambda \cdot y)$ is continuous in λ, and is Gateaux differentiable everywhere. Moreover, $\Delta^{n+1}_{\mu_1, \cdots, \mu_{n+1}} f(\lambda) \equiv 0$. Hence by theorem 20, $f(\lambda)$ is a C-polynomial in λ of degree $\leqslant n$. Therefore, $F(x)$ is an E-polynomial of degree $\leqslant n$ by definition 2. To show that $F(x)$ must be of degree exactly n, we use theorem 18, and with that we have proved that the definitions are equivalent.

CHAPTER III

PSEUDO POLYNOMIALS

In order to preserve the equivalence of the Fréchet and Gateaux definitions of polynomials in complex "espaces algébrophiles", it was found necessary to add the condition of Gateaux differentiability to Fréchet's conditions. It is of some interest to examine the nature of the functions which arise when th differentiability condition is suppressed. It is proved in this chapter that such functions are pseudo-polynomials, that is, $\bar{C}$- or $\bar{E}$-polynomials. The plan of this chapter is to develop the properties of pseudo-polynomials which correspond to the properties of ordinary polynomials, and finally, to show that a function satisfying Fréchet's original conditions is a pseudo-polynomial.

§1 $\bar{C}$-polynomials

Definition 1. A function $p(\lambda)$ defined on C to E, which can be expressed in the form,

$$(1) \qquad p(\lambda) = u_{00} + \lambda\cdot u_{10} + \bar{\lambda}\cdot u_{11} + \cdots + \lambda^n\cdot u_{n0} + \lambda^{n-1}\bar{\lambda}\cdot u_{n1} + \cdots + \bar{\lambda}^n\cdot u_{nn},$$

where the u_{rs} are fixed elements in E, will be called a $\bar{C}$-polynomial. It will be said to be of degree n, if not all the u_{nj} are 0. A $\bar{C}$-polynomial of degree 0 is a constant, and a $\bar{C}$-polynomial which vanishes identically may be regarded as having any degree whatever.

Theorem 1. A $\bar{C}$-polynomial is continuous.

Theorem 2. A linear combination of $\bar{C}$-polynomials of degree $\leq m$, is a $\bar{C}$-polynomial of degree $\leq m$.

Theorem 3. If $p(\lambda)$ is a $\bar{C}$-polynomial of degree n, and $g(\lambda)$ is a $\bar{C}$-polynomial on C to C of degree m, then $g(\lambda)\cdot p(\lambda)$ is a $\bar{C}$-polynomial of degree $m + n$.

The proof of the three above theorems is similar to the proofs for C-polynomials.

The next theorem which should be proved is that the representation of a $\bar{C}$-polynomial in the form (1) is unique. To do this we must be able to solve for the u_{rs} in terms of values of $p(\lambda)$. In order to do this it must be shown that we can choose $\frac{1}{2}(n+1)(n+2)$ values of λ, say $\lambda_{00}, \lambda_{10}, \lambda_{11}, \cdots \lambda_{n0}, \cdots \lambda_{nn}$, such that the determinant

$$\begin{vmatrix} 1, \lambda_{00}, \bar{\lambda}_{00}, \cdots, \lambda_{00}^{n}, \lambda_{00}^{n-1}\bar{\lambda}_{00}, \cdots, \bar{\lambda}_{00}^{n} \\ 1, \lambda_{10}, \bar{\lambda}_{10}, \cdots, \lambda_{10}^{n}, \lambda_{10}^{n-1}\bar{\lambda}_{10}, \cdots, \bar{\lambda}_{10}^{n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 1, \lambda_{nn}, \bar{\lambda}_{nn}, \cdots, \lambda_{nn}^{n}, \lambda_{nn}^{n-1}\bar{\lambda}, \cdots, \bar{\lambda}_{nn}^{n} \end{vmatrix}$$

does not vanish. That this can be done is proved in the appendix. If we denote by A_{rs}^{mj} the normalized cofactors of the above determinant, (1) can be solved for the constants,

$$u_{rs} = \sum_{0}^{n}{}_{m} \sum_{0}^{m}{}_{j} A_{rs}^{mj}\cdot p(\lambda_{mj}) \quad . \tag{2}$$

Using this result we are enabled to prove

Theorem 4. If $p(\lambda)$ and $q(\lambda)$ are $\bar{C}$-polynomials of degree m and n, and $p(\lambda) \equiv q(\lambda)$, then $m = n$ and $u_{rs} = v_{rs}$, where u_{rs} and v_{rs} are the coefficients in $p(\lambda)$ and $q(\lambda)$ respectively.

Lemma 1. Let $p(\lambda) = \lambda^{r}\bar{\lambda}^{s}\cdot u_{0}$. Then $\underset{\mu}{\Delta'} p(\lambda)$ is of degree $r + s - 1$

in λ , and

$$\Delta^{r+s}_{\mu_1,\cdots,\mu_{r+s}} p(\lambda) = r!\,s!\,\sum_\kappa \Pi_\kappa\, \mu_{\kappa_1}\cdots\mu_{\kappa_r}\,\bar\mu_{\kappa_{r+1}}\cdots\bar\mu_{\kappa_{r+s}}\cdot u_0 ,$$

where the summation is extended over all permutations of the $r + s$ increments, r unbarred, and s barred.

Proof:

$$\Delta'_\mu p(\lambda) = [(\lambda+\mu)^r(\bar\lambda+\bar\mu)^s - \lambda^r\bar\lambda^s]\cdot u_0$$

which is of degree $r + s - 1$ in λ . The two terms of highest degree are,

$$\{\lambda^r\cdot s\cdot\bar\lambda^{s-1}\bar\mu + r\cdot\lambda^{r-1}\mu\,\bar\lambda^s\}\cdot u_0 .$$

Repeating this process successively we get the result in the lemma.

Theorem 5. If $p(\lambda)$ is a $\bar{C}$-polynomial of degree n , then $\Delta'_\mu p(\lambda)$ is a $\bar{C}$-polynomial of degree $\leqslant n - 1$, and for some μ is of degree $n - 1$.
Proof: This follows immediately from the lemma and the uniqueness theorem.

Theorem 6. If $p(\lambda)$ is a $\bar{C}$-polynomial of degree n , then $\Delta^{n+1}_{\mu_1,\cdots,\mu_{n+1}} p(\lambda) \equiv 0$, and for some choice of $\mu_1,\cdots,\mu_n$ $\Delta^n_{\mu_1,\cdots,\mu_n} p(\lambda) \not\equiv 0$.
Proof: If $p(\lambda)$ is expressed in the form (1), then using lemma 1,

$$(3)\quad \Delta^n_{\mu_1,\cdots,\mu_n} p(\lambda) = n!\,\mu_1\cdots\mu_n\cdot u_{n0} + (n-1)!\,1!\,\sum_\kappa \Pi_\kappa\,\mu_{\kappa_1}\cdots\mu_{\kappa_{n-1}}\bar\mu_{\kappa_n}\cdot u_{n1}$$
$$+ (n-2)!\,2!\,\sum_\kappa \Pi_\kappa\,\mu_{\kappa_1}\cdots\mu_{\kappa_{n-2}}\bar\mu_{\kappa_{n-1}}\bar\mu_{\kappa_n}\cdot u_{n2} + \cdots + n!\,\bar\mu_1\cdots\bar\mu_n\cdot u_{nn} .$$

That this cannot vanish for all choices of the increments follows from the fact that

$$(4)\quad \Delta^n_{\mu,\cdots,\mu} p(\lambda) = n!\,\{\mu^n\cdot u_{n0} + \mu^{n-1}\bar\mu\cdot u_{n1} + \cdots + \bar\mu^n\cdot u_{nn}\} ,$$

and by the uniqueness theorem, if this vanishes identically, $u_{ni} = 0$ for all i , and the original polynomial could not have been of the nth degree. That the $n + 1$ st difference vanishes is evident.

§ 2 $\bar{E}$-Polynomials

In this section we discuss functions defined on one "espace algébrophile" E_1, to a like space E_2.

Definition 2. Let $p(x)$ be a function on an E_1 to an E_2. Then $p(x)$ will be said to be an $\bar{E}$-polynomial, if

1° $p(x)$ is continuous,

2° for each pair x,y, $p(x + \lambda\cdot y)$ is a $\bar{C}$-polynomial in λ.

It will be said to be of degree n, if for all x,y, $p(x + \lambda\cdot y)$ is a $\bar{C}$-polynomial of degree $\leqslant n$, and for some x,y is a $\bar{C}$-polynomial of degree exactly n.

An $\bar{E}$-polynomial can therefore be represented in the form,

$$(5) \quad p(x + \lambda\cdot y) = k_{00}(x,y) + \lambda\cdot k_{10}(x,y) + \bar{\lambda}\cdot k_{11}(x,y) + \cdots + \lambda^n\cdot k_{n0}(x,y) + \lambda^{n-1}\bar{\lambda}\cdot k_{n1}(x,y) + \cdots + \bar{\lambda}^n\cdot k_{nn}(x,y),$$

where the functions $k_{rs}(x,y)$ are on E_1 to E_2. When $p(x)$ is represented in the form (5), the functions $k_{rs}(x,y)$ are called the associated functions to $p(x)$.

Theorem 7. A linear combination of $\bar{E}$-polynomials of degree $\leqslant n$ is an $\bar{E}$-polynomial of degree $\leqslant n$.

Theorem 8. If $p(x)$ is an $\bar{E}$-polynomial of degree $\leqslant n$, and $c \in C$, then $p(c\cdot x + x_0)$ is an $\bar{E}$-polynomial in x of degree $\leqslant n$.

The proofs of these two theorems is similar to that of the corresponding theorems in Chapter II, and are therefore omitted.

<u>Definition 3.</u> A function $f(x)$ will be said to be pseudo-homogeneous of degree r and s, if

$$f(\mu\cdot x) = \mu^r \bar{\mu}^s \cdot f(x) \qquad \mu \in C .$$

<u>Theorem 9.</u> If $p(x)$ is an $\bar{E}$-polynomial of degree n, then the associated functions $k_{rs}(x,y)$ are pseudo-homogeneous in y of order $r-s$ and s.

Proof:

$$\begin{aligned} p(x+\lambda\mu\cdot y) &= k_{oo}(x,\mu\cdot y) + \cdots + \lambda^{r-s}\bar{\lambda}^s\cdot k_{rs}(x,\mu y) + \cdots \\ &= k_{oo}(x,y) + \cdots + (\lambda\mu)^{r-s}(\bar{\lambda}\bar{\mu})^s k_{rs}(x,y) + \cdots . \end{aligned}$$

By theorem 4 we can equate coefficients of $\lambda^{r-s}\bar{\lambda}^s$ in the two equations, thus getting,

$$k_{rs}(x,\mu\cdot y) = \mu^{r-s}\bar{\mu}^s\cdot k_{rs}(x,y) .$$

<u>Theorem 10.</u> If $p(x)$ is an $\bar{E}$-polynomial of degree n, and $k_{rs}(x,y)$ is an associated function, then for fixed x, $k_{rs}(x,y)$ is an $\bar{E}$-polynomial in y of degree $\leqslant n$, and for fixed y is an $\bar{E}$-polynomial in x of degree $\leqslant n$.

Proof: Using the numbers A^{mj}_{rs} of § 1, and the method given there, we can solve for $k_{rs}(x,y)$ in (5) as,

$$(6) \qquad k_{rs}(x,y) = \sum_o^n{}_m \sum_o^m{}_j A^{mj}_{rs}\cdot p(x+\lambda_{mj}\cdot y) .$$

Theorems 7 and 8 now assure us of our assertion, since on the right we have the sum of functions which are $\bar{E}$-polynomials in either x or y.

<u>Theorem 11.</u> If $h(x)$ is an $\bar{E}$-polynomial of degree n, and if it is pseudo-homogeneous of degree r and s, then $n = r+s$.

Proof: Setting $x=0$ in (5) we have,

$$h(\lambda\cdot y) = \lambda^r\bar{\lambda}^s\cdot h(y) = \sum_o^n{}_m \sum_o^m{}_j \lambda^{m-j}\bar{\lambda}^j\cdot k_{mj}(0,y) .$$

Hence by theorem 4, $r + s \leqslant n$. On the other hand,

$$h(x + \lambda\cdot y) = h(\lambda(\tfrac{1}{\lambda}\cdot x + y)) = \lambda^r \bar{\lambda}^s \cdot h(\tfrac{1}{\lambda}x + y)$$

$$= \sum_{0}^{n}{}_m \sum_{0}^{m}{}_j \lambda^{r+j-m} \bar{\lambda}^{s-j} \cdot K_{mj}(y, x) .$$

In order for this to be a pseudo-polynomial, we must have for any m,

(7) $\qquad r + j - m \geqslant 0 \quad , \quad s - j \geqslant 0 .$

Applying (7) for $m = n$, we get $r + s \geqslant n$, and hence $r + s = n$.

Corollary 1. If $p(x)$ is an $\bar{E}$-polynomial and $p(\lambda\cdot x) = \lambda^n \cdot p(x)$, then $p(x)$ has the form,

$$p(x + \lambda\cdot y) = k_{00}(x,y) + \lambda\cdot k_{10}(x,y) + \lambda^2\cdot k_{20}(x,y) + \cdots + \lambda^n\cdot k_{n0}(x,y).$$

Corollary 2. If $p(x)$ is an $\bar{E}$-polynomial and $p(\lambda\cdot x) = \lambda\bar{\lambda}\cdot p(x)$, then $p(x)$ has the form,

$$p(x + \lambda\cdot y) = k_{00}(x,y) + \lambda\cdot k_{10}(x,y) + \bar{\lambda}\cdot k_{11}(x,y) + \lambda\bar{\lambda}\cdot k_{21}(x,y) .$$

These corollaries are immediate applications of (7).

Theorem 12. If $p(x)$ is an $\bar{E}$-polynomial which is pseudo-homogeneous of degree r and s, and $k_{mj}(x,y)$ is an associated function, then $k_{mj}(x,y)$ is pseudo-homogeneous in x of degree $r + j - m$ and $s - j$, and is pseudo-homogeneous in y of degree $m - j$ and j.

Proof: The latter part of the theorem has already been proved. To prove the former we have,

$$p(\mu\cdot x + \lambda\cdot y) = \sum_{0}^{r+s}{}_m \sum_{0}^{m}{}_j \lambda^{m-j} \bar{\lambda}^j \cdot K_{mj}(\mu\cdot x, y)$$

$$p(\mu\cdot x + \lambda\cdot y) = \mu^r \bar{\mu}^s \cdot p(x + \tfrac{\lambda}{\mu} y) = \sum_{0}^{r+s}{}_m \sum_{0}^{m}{}_j \lambda^{m-j} \bar{\lambda}^j \mu^{r+j-m} \bar{\mu}^{s-j} \cdot K_{mj}(x, y) .$$

Equating like powers of $\lambda^{m-j}\bar{\lambda}^{j}$ in these two equations we obtain,

$$k_{mj}(\mu\cdot x,y) = \mu^{r+j-m}\bar{\mu}^{s-j}\cdot k_{mj}(x,y) \quad .$$

If in equation (5) we replace x, λ, y by $0,1,x$ and set $h_{rs}(x) = k_{rs}(0,x)$, we get

<u>Theorem 13.</u> If $p(x)$ is an $\bar{E}$-polynomial of degree n, it can be uniquely represented as a sum of pseudo-homogeneous polynomials in the form,

$$(8) \qquad p(x) = h_{00}(x) + h_{10}(x) + h_{11}(x) + \cdots + h_{n0}(x) + \cdots + h_{nn}(x) \quad ,$$

where $h_{mj}(\mu\cdot x) = \mu^{m-j}\bar{\mu}^{j}\cdot h_{mj}(x)$.

<u>Canonical Representation</u> The representation of an $\bar{E}$-polynomial in the form (8) will be termed canonical.

<u>Theorem 14.</u> If $h(x)$ is an $\bar{E}$-polynomial which is pseudo-homogeneous of degree r and s, then $\underset{u}{\Delta}' h(x)$ is an $\bar{E}$-polynomial in x of degree $\leqslant r+s-1$, and for some choice of u is an $\bar{E}$-polynomial of degree exactly $r+s-1$.

Proof: Let $n = r+s$, and let $h(x)$ be represented by (5). Using (7) we have $k_{nj}(x,y) = 0$ unless $j = s$. From theorem 12 we deduce that $k_{mj}(0,y) = 0$ for $m < n$, and $k_{ns}(x,y) = h(y)$. In equation (5) replace x by u, and y by x, and set $\lambda = 1$. Then,

$$\underset{u}{\Delta}' h(x) = \sum_{0}^{n-1}{}_{m}\sum_{0}^{m}{}_{j}\, k_{mj}(u,x) \quad ,$$

and by theorem 7, this is an $\bar{E}$-polynomial in x of degree $\leqslant r+s-1$. Using (7) we find that the terms of highest degree in the above equation are

$$k_{n-1,s-1}(u,x) + k_{n-1,s}(u,x) \quad .$$

To show the second part of the theorem we must show that u can be chosen so that the above expression is not identically zero. To do this, set $u = \lambda \cdot x$. Then,

$$\Delta'_{\lambda \cdot x} h(x) = h(x + \lambda \cdot x) - h(x) = \left[(1+\lambda)^r(1+\bar{\lambda})^s - 1\right] \cdot h(x)$$

$$= \sum_{0}^{n-1}{}_m \sum_{0}^{m}{}_j \lambda^{r+j-m} \bar{\lambda}^{s-j} \cdot K_{mj}(x, x) \quad .$$

Equating coefficients of λ and $\bar{\lambda}$ in these two equations we have,

$$k_{n-1,s-1}(x,x) = s \cdot h(x) \quad , \quad k_{n-1,s}(x,x) = r \cdot h(x) \quad .$$

Hence if we choose x_1 so that $h(x_1) \neq 0$, which is clearly possible since $h(x)$ is of degree n, we get

$$k_{n-1,s-1}(x_1, x_1) + k_{n-1,s}(x_1, x_1) = (s + r) \cdot h(x_1) \neq 0 \; .$$

For this value of x, $\Delta'_{x_1} h(x)$ is of degree exactly $r + s - 1$.

<u>Theorem 15</u>. If $p(x)$ is an $\bar{E}$-polynomial of degree n, then $\Delta'_u p(x)$ is an $\bar{E}$-polynomial in x of degree $\leqslant n - 1$, and for some choice of u is an $\bar{E}$-polynomial of degree exactly $n - 1$.

Proof: The first part of the theorem follows by expressing $p(x)$ in canonical form and then using theorem 14. To show the second part of the theorem we will take only the highest degree terms in the canonical representation,

$$p(x) = + \cdot \cdot \cdot \cdot \cdot + h_{n0}(x) + h_{n1}(x) + \cdot \cdot \cdot + h_{nn}(x) \; ,$$

$$\Delta'_u p(x) = + \cdot \cdot \cdot + \Delta'_u h_{n0}(x) + \cdot \cdot \cdot \cdot \cdot \cdot + \Delta'_u h_{nn}(x) \; .$$

Now consider $h_{nj}(x)$. It is pseudo-homogeneous of degree $n - j$ and j. Hence by the last part of theorem 14, the highest degree terms in $\Delta'_x h_{nj}(x)$

consist of, $$k^{[n,j]}_{n-1,j-1}(x,x) \quad + \quad k^{[nj]}_{n-1,j}(x,x) \quad = \quad (n - j + j)\cdot h_{nj}(x) \ .$$

Adding these terms together, we get

$$\Delta'_{x} p(x) \quad = \quad + \cdot \cdot \cdot + \quad n\cdot\left[h_{n0}(x) + h_{n1}(x) + \cdot\cdot\cdot + h_{nn}(x)\right] \ .$$

Now if $p(x)$ is of degree n, there must be at least one value x_1 for which

$$h_{n0}(x_1) + \cdot \cdot \cdot + h_{nn}(x_1) \quad \neq \quad 0 \ ,$$

and for that value of x, $\Delta'_{x_1} p(x)$ is of degree exactly $n - 1$ and hence the theorem.

<u>Theorem 16.</u> If $p(x)$ is an $\bar{E}$-polynomial of degree n, then $\Delta^{n+1}_{u_1,\cdots,u_{n+1}} p(x) \equiv 0$, and for some $u_1, \cdots, u_n$, $\Delta^{n}_{u_1,\cdots,u_n} p(x) \not\equiv 0$.

Proof: Use theorem 15 successively.

§ 3 A Second Definition of Pseudo-Polynomials and the Equivalence of the Definitions

<u>Definition 2'.</u> Let $p(x)$ be a function defined on an E_1 to an E_2. Then $p(x)$ will be said to be an $\bar{E}$-polynomial if,

1° $p(x)$ is continuous,

2° for some integer n, $\Delta^{n+1}_{u_1,\cdots,u_{n+1}} p(x) \equiv 0$.

It will be said to be of degree n, if for some choice of $u_1, \cdots, u_n$

$$\Delta^{n}_{u_1,\cdots,u_n} p(x) \not\equiv 0 \ .$$

It is to be remarked that this definition is the same as the definition 2' of chapter II, except that the requirement of Gateaux differentiability has been dropped. This definition, as in chapter II, includes the case of $\bar{C}$-polynomials, and in proving the equivalence of the definitions it is convenient to consider that case first.

Theorem 17. In definition 2', let $E_1 = \mathcal{C}$, Then definition 2' and definition 1 are equivalent.

Proof: That a function which is a $\bar{C}$-polynomial of degree n by definition 1, is a $\bar{C}$-polynomial of degree n by definition 2', is clear from theorems 1 and 6. To prove the converse, we must show that a function on C to E_2 which satisfies conditions 1° and 2° of definition 2' has the form (1). We make the proof by induction.

Case I $\quad \Delta^1_{\mu} f(\lambda) \equiv 0.$

In this case $f(\lambda) = u_{00}$, a constant, and it is trivial.

Case II $\quad \Delta^2_{\mu_1,\mu_2} f(\lambda) \equiv 0.$

In this case, using equation (6) of II, §4 , we have,

$$f(\lambda) = f(0) + \lambda\cdot u_{10} + \bar{\lambda}\cdot u_{11} ,$$

which is of the form (1).

Case III $\quad \Delta^{n+1}_{\mu_1,\cdots,\mu_{n+1}} f(\lambda) \equiv 0 .$

Then $\Delta^n_{\mu_1,\cdots,\mu_n}\left[f(\lambda+\mu) - f(\lambda)\right] \equiv 0$. Now $f(\lambda+\mu) - f(\lambda)$ is a continuous function of λ , and moreover its nth difference vanishes identically. If $f(\lambda)$ is of degree n, then for some choice of the increments, the (n - 1)th difference of $f(\lambda+\mu) - f(\lambda)$ does not vanish. Hence under the induction hypothesis, we will assume that $f(\lambda+\mu) - f(\lambda)$ is a $\bar{C}$-polynomial of degree n - 1 by definition 1, and will show that this leads to the conclusion that $f(\lambda)$ is a $\bar{C}$-polynomial of degree n by definition 1. Let us set

$$\text{(9)} \qquad \psi(\lambda,\mu) = f(\lambda+\mu) - f(\lambda) - f(\mu) .$$

Since subtracting a constant cannot affect the degree of a $\bar{C}$-polynomial,

$\Psi(\lambda, \mu)$ is a $\bar{C}$-polynomial in λ of degree exactly $n - 1$. Moreover, since it is symmetric in λ, μ it is also a $\bar{C}$-polynomial in μ of degree exactly $n - 1$. By using (9) we may verify a result which is of considerable use later, namely,

$$(10) \qquad \Psi(\lambda + \mu, \nu) + \Psi(\lambda, \mu) = f(\lambda + \mu + \nu) - f(\lambda) - f(\mu) - f(\nu)$$

The expression on the left is completely symmetric in the three variables λ, μ, ν.

Since $\Psi(\lambda, \mu)$ is a $\bar{C}$-polynomial of degree $n - 1$ in λ and in μ, we may write it as,

$$(11) \qquad \begin{aligned} \Psi(\lambda, \mu) &= u_{00}(\mu) + \lambda \cdot u_{10}(\mu) + \bar{\lambda} \cdot u_{11}(\mu) + \cdots \\ &= u_{00}(\lambda) + \mu \cdot u_{10}(\lambda) + \bar{\mu} \cdot u_{11}(\lambda) + \cdots \end{aligned}$$

Using the numbers A_{rs}^{mj}, λ_{mj} of §1, we may solve the latter of equations (11) for $u_{rs}(\lambda)$ as,

$$u_{rs}(\lambda) = \sum_{0}^{n-1}{}_m \sum_{0}^{m}{}_j A_{rs}^{mj} \cdot \Psi(\lambda, \lambda_{mj}) .$$

If we calculate the value of $\Psi(\lambda, \lambda_{mj})$ from the first equation, introduce this into $u_{rs}(\lambda)$, and introduce this into the latter of equations (11), we get an expression for $\Psi(\lambda, \mu)$ in the form,

$$(12) \qquad \Psi(\lambda, \mu) = \sum \lambda^p \bar{\lambda}^q \mu^r \bar{\mu}^s \cdot A_{p,q,r,s} ,$$

where the $A_{p,q,r,s}$ are constant elements of E_2. This expression is of degree $n - 1$ in $\lambda, \bar{\lambda}$ together, and of degree $n - 1$ in $\mu, \bar{\mu}$ together. Hence it is of degree $2n - 2$ at most in the four variables $\lambda, \bar{\lambda}, \mu, \bar{\mu}$. If we group together the terms which are of the same degree in the four variables, we may arrange it in the form,

(13) $$\psi(\lambda,\mu) = \psi_0(\lambda,\bar\lambda,\mu,\bar\mu) + \cdots + \psi_r(\lambda,\bar\lambda,\mu,\bar\mu) ,$$

where $r \leqslant 2n-2$, and

$$\psi_\kappa(t\lambda, t\bar\lambda, t\mu, t\bar\mu) = t^\kappa \cdot \psi_\kappa(\lambda,\bar\lambda,\mu,\bar\mu)$$

for t real. Therefore,

(14) $$\psi(t\lambda,t\mu) = \psi_0 + t\cdot\psi_1(\lambda,\bar\lambda,\mu,\bar\mu) + \cdots + t^r\cdot\psi_r(\lambda,\bar\lambda,\mu,\bar\mu).$$

We may therefore solve (14) for $\psi_s(\lambda,\bar\lambda,\mu,\bar\mu)$ as a linear combination of terms of the form $\psi(L\lambda,L\mu)$, where L is an integer say. From this, using (9) and (10), it is easy to show that ψ_s is symmetric in λ,μ , and $\psi_s(\lambda+\mu,\bar\lambda+\bar\mu,\nu,\bar\nu) + \psi_s(\lambda,\bar\lambda,\mu,\bar\mu)$ is completely symmetric in λ,μ,ν. We shall now attempt to find the special form of $\psi_s(\lambda,\bar\lambda,\mu,\bar\mu)$. We write it in the following form,

(15) $$\begin{aligned}\psi_s(\lambda,\bar\lambda,\mu,\bar\mu) = {} & \sum_0^s{}_j \lambda^{s-j}\bar\lambda^j\cdot a_j + \sum_0^s{}_j \mu^{s-j}\bar\mu^j\cdot b_j + \sum_1^{s-1}{}_j \lambda^{s-j}\mu^j\cdot c_j \\ & + \sum_1^{s-1}{}_j \bar\lambda^{s-j}\bar\mu^j\cdot d_j + \sum_1^{s-1}{}_j \lambda^{s-j}\bar\mu^j + \sum_1^{s-1}{}_j \bar\lambda^{s-j}\mu^j\cdot f_j \\ & + \sum \lambda^p\bar\lambda^q\mu^r\cdot g_{p,q,r} + \sum \bar\lambda^p\mu^q\bar\mu^r\cdot k_{p,q,r} \\ & + \sum \lambda^p\bar\lambda^q\bar\mu^r\cdot h_{p,q,r} + \sum \lambda^p\mu^q\bar\mu^r\cdot j_{p,q,r} \\ & + \sum \lambda^u\mu^v\bar\lambda^w\bar\mu^x\cdot l_{u,v,w,x} ,\end{aligned}$$

where $\quad p+q+r = s \qquad 1 \leqslant p,q,r \leqslant s-2$

$\qquad u+v+w+x = s \qquad 1 \leqslant u,v,w,x \leqslant s-3$.

Because of the symmetry of $\psi_s(\lambda,\bar\lambda,\mu,\bar\mu)$ in λ,μ , we have

$$b_j = a_j , \qquad c_{s-j} = c_j , \qquad d_{s-j} = d_j ,$$

$$f_j = e_j , \qquad j_{p,q,r} = g_{q,r,p} , \qquad k_{p,q,r} = h_{q,r,p} ,$$

$$l_{u,v,w,x} = l_{v,u,x,w} ,$$

The expression $\Psi_s(\lambda+\mu, \bar\lambda+\bar\mu, \nu, \bar\nu) + \Psi_s(\lambda, \bar\lambda, \mu, \bar\mu)$ is completely symmetric in λ, μ, ν, so we now consider it.

(16) $$\Psi_s(\lambda+\mu, \bar\lambda+\bar\mu, \nu, \bar\nu) + \Psi_s(\lambda, \bar\lambda, \mu, \bar\mu) =$$

$$\sum_{0}^{s}{}_j (\lambda+\mu)^{s-j}(\bar\lambda+\bar\mu)^j \cdot a_j + \sum_{0}^{s}{}_j \nu^{s-j}\bar\nu^j \cdot a_j + \sum_{1}^{s-1}{}_j (\lambda+\mu)^{s-j}\nu^j \cdot c_j$$

$$+ \sum_{1}^{s-1}{}_j (\bar\lambda+\bar\mu)^{s-j}\bar\nu^j \cdot d_j + \sum_{1}^{s-1}{}_j (\lambda+\mu)^{s-j}\bar\nu^j \cdot e_j + \sum_{1}^{s-1}{}_j (\bar\lambda+\bar\mu)^{s-j}\nu^j \cdot e_{s-j}$$

$$+ \sum (\lambda+\mu)^p(\bar\lambda+\bar\mu)^q \nu^r \cdot g_{p,q,r} + \sum (\lambda+\mu)^p \nu^q \bar\nu^r \cdot g_{q,r,p}$$

$$+ \sum (\lambda+\mu)^p(\bar\lambda+\bar\mu)^q \bar\nu^r \cdot h_{p,q,r} + \sum (\bar\lambda+\bar\mu)^p \nu^q \bar\nu^r \cdot h_{q,r,p}$$

$$+ \sum (\lambda+\mu)^u \nu^v (\bar\lambda+\bar\mu)^w \bar\nu^x \cdot l_{u,v,w,x}$$

$$+ \sum_{0}^{s}{}_j \lambda^{s-j}\bar\lambda^j \cdot a_j + \sum_{0}^{s}{}_j \mu^{s-j}\bar\mu^j \cdot a_j + \sum_{1}^{s-1}{}_j \lambda^{s-j}\mu^j \cdot c_j$$

$$+ \sum_{1}^{s-1}{}_j \bar\lambda^{s-j}\bar\mu^j \cdot d_j + \sum_{1}^{s-1}{}_j \lambda^{s-j}\bar\mu^j \cdot e_j + \sum_{1}^{s-1}{}_j \bar\lambda^{s-j}\mu^j \cdot e_{s-j}$$

$$+ \sum \lambda^p\bar\lambda^q\mu^r \cdot g_{p,q,r} + \sum \lambda^p\mu^q\bar\mu^r \cdot g_{q,r,p}$$

$$+ \sum \lambda^p\bar\lambda^q\bar\mu^r \cdot h_{p,q,r} + \sum \bar\lambda^p\mu^q\bar\mu^r \cdot h_{q,r,p}$$

$$+ \sum \lambda^u\mu^v\bar\lambda^w\bar\mu^x \cdot l_{u,v,w,x}$$

In equation (16) consider the coefficients of $\lambda^{s-j}\bar{\lambda}^{j}$ and $\nu^{s-j}\bar{\nu}^{j}$.
They are respectively $2a_j$ and a_j . Because of the symmetry we must have

$$2a_j = a_j \quad \text{or} \quad a_j = 0 \ .$$

Now consider the coefficients of $\lambda^{s-1-j}\mu^{j}\nu$ and $\lambda^{s-1-j}\mu\nu^{j}$. The condition of equality yields,

$$(17) \qquad \binom{s-1}{j}\cdot c_1 = (s-j)\cdot c_j \quad \text{or} \quad c_j = \frac{1}{s}\binom{s}{j}\cdot c_1 \ .$$

On considering the complex conjugates of the above terms, we are led in a similar manner to,

$$(18) \qquad d_j = \frac{1}{s}\binom{s}{j}\cdot d_1 \ .$$

We next consider the coefficients of $\lambda^{s-i-j}\mu^{i}\bar{\nu}^{j}$ and $\bar{\lambda}^{j}\mu^{i}\nu^{s-i-j}$, where we have permuted λ and ν , and where $i,j \geqslant 1$. From this we get,

$$(19) \qquad g_{i,j,s-i-j} = \binom{s-j}{i}\cdot e_j \ .$$

In a similar fashion we prove,

$$(20) \qquad h_{i,j,s-i-j} = \binom{s-i}{j}\cdot e_{s-i} \ .$$

If we consider the equality of the coefficients of $\lambda^{i}\nu^{j}\bar{\mu}^{k}\bar{\nu}^{s-i-j-k}$ and $\lambda^{i}\mu^{j}\bar{\nu}^{k}\bar{\mu}^{s-i-j-k}$, where $i,j,k \geqslant 1$, we get

$$l_{i,j,k,s-i-j-k} = \binom{i+j}{k}\cdot h_{i+j,s-i-j-k,k} \ .$$

On substituting the value of h from (20), we get

$$(21) \qquad l_{i,j,k,s-i-j-k} = \binom{i+j}{j}\binom{s-i-j}{k}\cdot e_{s-i-j} \ .$$

We have now evaluated all the coefficients of $\Psi_s(\lambda,\bar{\lambda},\mu,\bar{\mu})$ in terms of c_1, d_1, and the e_j . It remains to find its special form. First we will find all the terms which have the coefficient c_1 .

They are from (17),

$$\frac{1}{s}\sum_{1}^{s-1}{}_j \binom{s}{j} \lambda^{s-j}\mu^{j}\cdot c_1 \quad = \quad \frac{1}{s}\left[(\lambda+\mu)^s - \lambda^s - \mu^s\right]\cdot c_1 \quad . \tag{22}$$

Next we will find the terms involving e_1 . They consist of the coefficients of

$$g_{i,1,s-1-i} \quad = \quad \binom{s-1}{i}\cdot e_1 \quad ,$$

$$h_{s-1,j,1-j} \quad = \quad \binom{1}{j}\cdot e_1 \quad ,$$

but do not involve any terms from l_{uvwx} . Since the subscripts on the h term must be at least one, we see that the h term contributes nothing. The terms in Ψ_s which involve e_1 , are then

$$\lambda^{s-1}\bar{\mu}\cdot e_1 \quad + \quad \sum_{1}^{s-2}{}_j \binom{s-1}{j} \lambda^{j}\bar{\lambda}\mu^{s-1-j}\cdot e_1 \quad + \quad \sum_{1}^{s-2}{}_j \binom{s-1}{j} \lambda^{s-1-j}\mu^{j}\bar{\mu}\cdot e_1 \quad + \quad \bar{\lambda}\mu^{s-1}\cdot e_1$$

$$= \quad \left[(\lambda+\mu)^{s-1}(\bar{\lambda}+\bar{\mu}) \; - \lambda^{s-1}\bar{\lambda} \; - \mu^{s-1}\bar{\mu}\right]\cdot e_1 \quad . \tag{23}$$

We shall now calculate the coefficients of e_t , where $2 \leqslant t \leqslant s - 2$. Such coefficients arise from (19), (20), and (21), and are

$$g_{j,t,s-t-j} \quad = \quad \binom{s-t}{j}\cdot e_t \quad ,$$

$$h_{s-t,j,t-j} \quad = \quad \binom{t}{j}\cdot e_t \quad ,$$

$$l_{j,s-t-j,k,t-k} \quad = \quad \binom{s-t}{j}\binom{t}{k} e_t \quad .$$

The coefficient of e_t is then, from (15),

$$\lambda^{s-t}\bar{\mu}^{t}\cdot e_t \quad + \quad \sum_{1}^{s-1-t}{}_j \binom{s-t}{j} \lambda^{j}\bar{\lambda}^{t}\mu^{s-t-j}\cdot e_t \quad + \quad \sum_{1}^{s-1-t}{}_j \binom{s-t}{j} \lambda^{s-t-j}\mu^{j}\bar{\mu}^{t}\cdot e_t$$

$$+ \quad \sum_{1}^{t-1}{}_j \binom{t}{j} \lambda^{s-t}\bar{\lambda}^{j}\bar{\mu}^{t-j}\cdot e_t \quad + \quad \sum_{1}^{t-1}{}_j \binom{t}{j} \bar{\lambda}^{t-j}\mu^{s-t}\bar{\mu}^{j}\cdot e_t$$

$$+ \quad \sum_{1}^{s-t-1}{}_j \sum_{1}^{t-1}{}_k \binom{s-t}{j}\binom{t}{k} \lambda^{j}\mu^{s-t-j}\bar{\lambda}^{k}\bar{\mu}^{t-k}\cdot e_t \quad + \quad \bar{\lambda}^{t}\mu^{s-t}\cdot e_t$$

$$= \lambda^{s-t}\bar\mu^{t}\cdot e_t + \bar\lambda^{t}[(\lambda+\mu)^{s-t} - \lambda^{s-t} - \mu^{s-t}]\cdot e_t + \bar\mu^{t}[(\lambda+\mu)^{s-t} - \lambda^{s-t} - \mu^{s-t}]\cdot e_t$$

$$+ \lambda^{s-t}[(\bar\lambda+\bar\mu)^{t} - \bar\lambda^{t} - \bar\mu^{t}]\cdot e_t + \mu^{s-t}[(\bar\lambda+\bar\mu)^{t} - \bar\lambda^{t} - \bar\mu^{t}]\cdot e_t$$

$$+ [(\lambda+\mu)^{s-t} - \lambda^{s-t} - \mu^{s-t}][(\bar\lambda+\bar\mu)^{t} - \bar\lambda^{t} - \bar\mu^{t}]\cdot e_t + \bar\lambda^{t}\mu^{s-t}\cdot e_t$$

(24) $$= [(\lambda+\mu)^{s-t}(\bar\lambda+\bar\mu)^{t} - \lambda^{s-t}\bar\lambda^{t} - \mu^{s-t}\bar\mu^{t}]\cdot e_t \quad .$$

In a manner similar to that in which we got (22) and (23), we find that the coefficients of e_{s-1} and d_1 are,

(25) $$[(\lambda+\mu)(\bar\lambda+\bar\mu)^{s-1} - \lambda\bar\lambda^{s-1} - \mu\bar\mu^{s-1}]\cdot e_{s-1} \quad ,$$

(26) $$\frac{1}{s}[(\bar\lambda+\bar\mu)^{s} - \bar\lambda^{s} - \bar\mu^{s}]\cdot d_1 \quad .$$

We finally get then, using equations (22) to (26), the result that Ψ_s can be expressed in the form,

$$\Psi_s(\lambda,\bar\lambda,\mu,\bar\mu) = \sum_{0}^{s}{}_j\, [(\lambda+\mu)^{s-j}(\bar\lambda+\bar\mu)^{j} - \lambda^{s-j}\bar\lambda^{j} - \mu^{s-j}\bar\mu^{j}]\cdot e_{sj}$$

where $e_{s0} = \frac{1}{s}c_1$, $e_{sr} = e_r$, $e_{ss} = \frac{1}{s}d_1$. Let

$$g_s(\lambda) = \sum_{0}^{s}{}_j\, \lambda^{s-j}\bar\lambda^{j}\cdot e_{sj} \qquad \text{and} \qquad g(\lambda) = \sum_{1}^{r}{}_s\, g_s(\lambda) \quad .$$

Then $\Psi_s(\lambda,\bar\lambda,\mu,\bar\mu) = g_s(\lambda+\mu) - g_s(\lambda) - g_s(\mu)$,

and using (13),

(27) $$\Psi(\lambda,\mu) = \Psi_0 + g(\lambda+\mu) - g(\lambda) - g(\mu) \quad .$$

Let $H(\lambda)$ be defined by, $H(\lambda) = \Psi_0 + f(\lambda) - g(\lambda)$. Using (9) and (27) it may easily be verified that, $H(\lambda+\mu) = H(\lambda) + H(\mu)$. Now from the definition of $H(\lambda)$ it is clear that it is a continuous function. Hence, using the result of Case II, we have,

$$H(\lambda) = \lambda\cdot u_{10} + \bar\lambda\cdot u_{11} \quad .$$

Substituting this result in the definition of $H(\lambda)$, we get

$$f(\lambda) = -\psi_0 + \lambda \cdot u_{10} + \bar{\lambda} \cdot u_{11} + \sum_{1}^{r}{}_s \sum_{0}^{s}{}_j \lambda^{s-j} \bar{\lambda}^j \cdot e_{sj} ,$$

an expression for $f(\lambda)$ as a pseudo-polynomial by the first definition. The degree of $\psi(\lambda, \mu)$ is exactly $n - 1$ in λ by our induction hypothesis, and the degree of the right hand side of (27) is exactly $r - 1$. Hence $r = n$, and we have completely proved the equivalence of the two definitions of $\bar{C}$-polynomials.

The equivalence theorem for $\bar{E}$-polynomials is now readily established. We state it as

Theorem 18. Definitions 2 and 2' are equivalent.

Proof: That a function which is an $\bar{E}$-polynomial of degree n by definition 2, is an $\bar{E}$-polynomial of degree n by definition 2' is clear from theorem 16 and the definition. To show the converse, let $p(x)$ satisfy the conditions of definition 2'. Then $f(\lambda) = p(x + \lambda \cdot y)$ is continuous in λ , and $\Delta^{n+1}_{\mu_1, \cdots, \mu_{n+1}} f(\lambda) \equiv 0$. Hence $f(\lambda)$ is a $\bar{C}$-polynomial of degree $\leqslant n$ by definition 2. That its degree is exactly n, is assured by theorem 16, and thereby the equivalence is completely established.

Note: In definition 2' of chapter II, the condition of Gateaux differentiability may be replaced by an algebraic one. If the condition is left out we get a pseudo-polynomial by the results of this chapter. Hence if in equation (2), we equate $u_{rs} = 0$, if $s \neq 0$, we get an algebraic condition on $p(\lambda)$ equivalent to the differentiability condition.

CHAPTER IV

CONDITION THAT A BANACH SPACE BE A HILBERT SPACE

This chapter concerns itself with the problem of generating an arbitrary norm in a vector space by means of a Hermitian bilinear form. The definition of a vector space will be assumed as will also the notion of a Hilbert space. For the former see Martin (1), and the latter, Stone (1).

In a Hilbert space H , notions of continuity and distance are derived from a real valued function $\|f\|_H$, which is generated from a function of two variables $((f, g))$, which has the following properties,

$$
\begin{array}{lll}
 & 1^\circ & ((f, g)) \in C \\
 & 2^\circ & ((a\cdot f, g)) = a\cdot((f, g)) \qquad a \in C \\
(1) & 3^\circ & ((f + g, h)) = ((f, h)) + ((g, h)) \\
 & 4^\circ & ((f, g)) = \overline{((g, f))} \\
 & 5^\circ & ((f, f)) \geqslant 0 \text{, and } ((f, f)) = 0 \text{ implies } f = 0 .
\end{array}
$$

We then define $\|f\|_H = ((f, f))^{\frac{1}{2}}$, and prove that $\|f\|_H$ has the usual properties of a norm, namely, that it is real valued, vanishes only for the zero element of the space, is triangular, and $\|a\cdot x\|_H = |a|\cdot\|x\|$.

In a Banach space B , notions of continuity and distance are derived from a postulated norm, $\|x\|$. Fréchet (3) has recently given criteria which $\|x\|$ in a Banach space must satisfy in order that there exist a function $((x, y))$ in the Banach space, which enjoys the properties (1), and for which $\|x\| = ((x, x))^{\frac{1}{2}}$. These conditions have been simplified by J. von Neumann and P. Jordan (1), and may be stated as,

<u>Theorem 1.</u> A necessary and sufficient condition that there exist a function $((x, y))$ defined in a Banach space B , having properties (1), and the further property that $((x, x))^{\frac{1}{2}} = \|x\|$ for all x , is that,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 , \tag{2}$$

for every pair of elements x, y in B . Furthermore, the function $((x, y))$ is defined by,

$$((x, y)) = \frac{1}{4}\left\{\|x + y\|^2 - \|x - y\|^2\right\} - \frac{1}{4}i\left\{\|ix + y\|^2 - \|ix - y\|^2\right\}$$

The proof of this theorem may be found in the reference above, and is omitted because we prefer to give our own.

*<u>Theorem 1'.</u> A necessary and sufficient condition that there exist a function $((x, y))$ defined in a Banach space B , having properties (1), and the further property that $\|x\| = ((x, x))^{\frac{1}{2}}$ for all x , is that,

$$\Delta^3_{u_1, u_2, u_3} \|x\|^2 \equiv 0 . \tag{3}$$

Proof: The necessity is evident, for $\|x\|$ is a priori continuous, and if $\|x\| = ((x, x))^{\frac{1}{2}}$, then

$$\|x + \lambda \cdot y\|^2 = ((x, x)) + \lambda((y, x)) + \bar{\lambda}((x, y)) + \lambda\bar{\lambda}((y, y))$$

so that $\|x\|^2$ is a pseudo-polynomial of degree 2, and hence $\Delta^3_{u_1, u_2, u_3}\|x\|^2 \equiv 0$. Now to prove the sufficiency. Suppose therefore (3) holds. Then by III,

---

* The corresponding simpler theorem in real Hilbert space was pointed out by Prof. Michal in his seminar on abstract spaces, 1935.

§ 3, theorem 17, $\|x\|^2$ is a pseudo-polynomial of degree 2. But

$$\|\lambda\cdot x\|^2 = \lambda\bar{\lambda}\|x\|^2 \quad ,$$

so that $\|x\|^2$ is a pseudo-polynomial of degree 2, pseudo-homogeneous of degree 1 and 1, and hence by III, § 2, corollary to theorem 11, $\|x + \lambda\cdot y\|^2$ has the form,

$$(4) \qquad \|x + \lambda\cdot y\|^2 = k_{00}(x,y) + \lambda\, k_{10}(x,y) + \bar{\lambda}\, k_{11}(x,y) + \lambda\bar{\lambda}\, k_{21}(x,y) \ ,$$

where the functions $k_{rs}(x,y)$ are complex numbers. We shall show that the function $k_{11}(x,y)$ has the properties (1). Setting $y = x$, and equating coefficients of λ on both sides of (4), we obtain

$$\|x\|^2 = k_{00}(x,x) = k_{10}(x,x) = k_{11}(x,x) = k_{21}(x,x) \ ,$$

so that $k_{11}(x,y)$ has properties 1° and 5° of (1). Further, because of the pseudo-homogeneity (see III, § 2, theorem 12), we have, $k_{11}(\mu\cdot x,y) = \mu\, k_{11}(x,y)$, so that it has property 2°. Now,

$$\|x + \lambda y\|^2 = \|\lambda\cdot(\tfrac{1}{\lambda}\cdot x + y)\|^2 = \lambda\bar{\lambda}\,\|\tfrac{1}{\lambda}x + y\|^2$$

$$(5) \qquad = \lambda\bar{\lambda}\, k_{00}(y,x) + \bar{\lambda}\, k_{10}(y,x) + \lambda\, k_{11}(y,x) + k_{21}(y,x) \ .$$

Equating coefficients in (4) and (5), we arrive at

$$(6) \qquad k_{00}(y,x) = k_{21}(x,y) \quad , \quad k_{10}(y,x) = k_{11}(x,y) \ .$$

Taking complex conjugates of both sides of equation (4) and equating coefficients, we get

$$(7) \qquad \begin{aligned} k_{00}(x,y) &= \overline{k_{00}(x,y)} \\ k_{21}(x,y) &= \overline{k_{21}(x,y)} \\ k_{10}(x,y) &= \overline{k_{11}(x,y)} \ . \end{aligned}$$

From the latter of equations (6) and (7), we find

(8) $$k_{11}(x,y) = \overline{k_{11}(y,x)} \quad .$$

We have thereby proved that property 4° is satisfied, and there remains only the additivity. Now $\Delta^2_{u_1,u_2} \| \lambda \cdot x \|^2 \not\equiv 0$, and in fact is a constant. Calculating it we find that

$$\| \lambda \cdot x + (u_1 + u_2) \|^2 = \lambda \bar{\lambda} K_{00}(x, u_1 + u_2) + \bar{\lambda} K_{10}(x, u_1 + u_2) + \lambda K_{11}(x, u_1 + u_2) + K_{21}(x, u_1 + u_2)$$

$$- \| \lambda \cdot x + u_1 \|^2 = - \lambda \bar{\lambda} K_{00}(x, u_1) - \bar{\lambda} K_{10}(x, u_1) - \lambda K_{11}(x, u_1) - K_{21}(x, u_1)$$

$$- \| \lambda \cdot x + u_2 \|^2 = - \lambda \bar{\lambda} K_{00}(x, u_2) - \bar{\lambda} K_{10}(x, u_2) - \lambda K_{11}(x, u_2) - K_{21}(x, u_2)$$

$$\| \lambda \cdot x \|^2 = \lambda \bar{\lambda} K_{00}(x, 0)$$

If we add the four equations above, since $k_{00}(x,y)$ is independent of y , we get

$$\Delta^2_{u_1,u_2} \| \lambda \cdot x \|^2 = \begin{aligned} & \bar{\lambda} \cdot \{ k_{10}(x, u_1 + u_2) - k_{10}(x, u_1) - k_{10}(x, u_2) \} \\ & + \lambda \cdot \{ k_{11}(x, u_1 + u_2) - k_{11}(x, u_1) - k_{11}(x, u_2) \} \\ & + \{ k_{21}(x, u_1 + u_2) - k_{21}(x, u_1) - k_{21}(x, u_2) \} \quad . \end{aligned}$$

If this is to be constant, we must have in particular,

(9) $$k_{11}(x, u_1 + u_2) = k_{11}(x, u_1) + k_{11}(x, u_2) \quad .$$

Equation (9) together with (8) enables us to show that $k_{11}(x,y)$ satisfies property 3° , and with that we have demonstrated the theorem.

This proof might have been simplified right from the beginning, for from (3) it is easy to show that (2) holds. In fact (2) can be written as, $\Delta^3_{x,y,-y} \| 0 \|^2 = 0$, which is a special case of (3). We preferred, however, to bring in the polar from the expansion.

CHAPTER V

AN EXAMPLE OF A MODULAR PROPERTY

For the notation of this chapter see Martin (1), p. 45. Martin proved that if $h(x)$ is a homogeneous polynomial of degree n in a Banach space, and if its modulus is denoted by M, and the modulus of its polar is denoted by M_ρ, then the moduli satisfy the inequality,

$$M \leqslant M_\rho \leqslant \frac{n^n}{n!} M .$$

However, it was not known whether this bound could be reduced, that is, do there exist homogeneous polynomials, the modulus of whose polar comes arbitrarily close to $\frac{n^n}{n!} M$. That the answer is affirmative in the case $n = 2$, is shown by the following example.

We shall be concerned with C, the space of continuous functions defined in the interval $a \leqslant t \leqslant b$. This is a Banach space with, $\|x\| = \max_{a \leqslant t \leqslant b} |x(t)|$. Now consider the following function of x,

$$H(x) = \{x(t)\}^2 + \int_a^b K(s) \{x(s)\}^2 ds ,$$

where $K(s)$ is a continuous function. $H(x)$ is a homogeneous polynomial of degree 2, for

1° it is continuous,

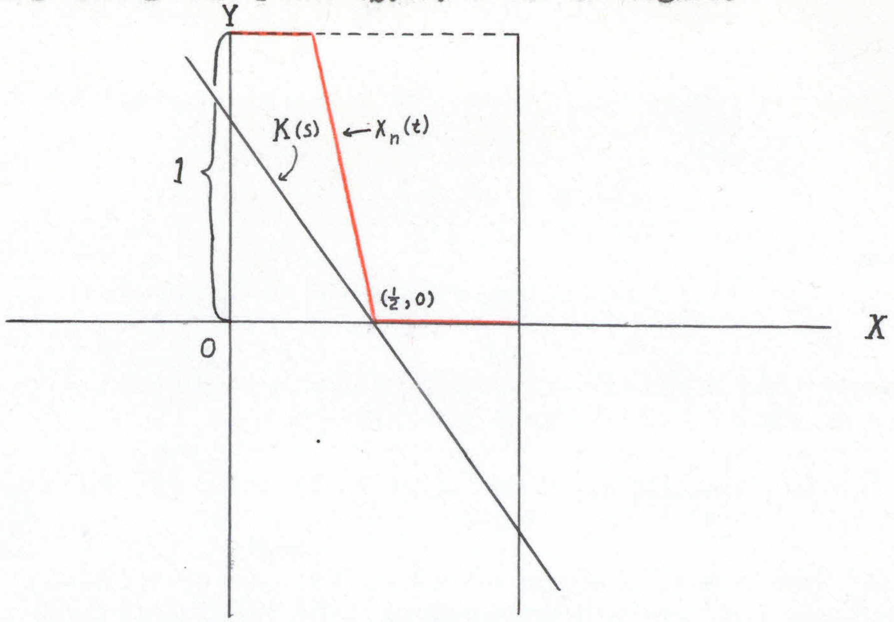
2° $H(x + \lambda \cdot y)$ is a polynomial in λ,

3° $H(\lambda \cdot x) = \lambda^2 \cdot H(x)$.

As such it has a modulus M, satisfying the relation,

$$\|H(x)\| \leqslant M \|x\|^2 .$$

Let us choose $a = 0$, $b = 1$, and $K(s)$ a linear function passing through the point $(\frac{1}{2},0)$, as in the figure.



In order to calculate the modulus of $H(x)$, we must get the least upper bound of the expression,

$$\left| \{x(t)\}^2 + \int_0^1 K(s) \{x(s)\}^2 \, ds \right| ,$$

while the maximum of $|x(s)|$ is unity. Now $\{x(t)\}^2$ is positive, but $K(s)$ is negative between $\frac{1}{2}$ and 1. To get the largest possible value from the integral then, we choose a sequence of functions $\{x_n(t)\}$, defined by,

$$x_n(t) = 1 , \quad 0 \leqslant t \leqslant \tfrac{1}{2} - \tfrac{1}{n}$$

$$x_n(t) = n(\tfrac{1}{2} - t) , \quad \tfrac{1}{2} - \tfrac{1}{n} < t \leqslant \tfrac{1}{2}$$

$$x_n(t) = 0 , \quad \tfrac{1}{2} < t \leqslant 1 .$$

These functions are continuous, but of course do not have a continuous limit. One representation is shown in the figure. The upper bound of the integral, under this sequence of functions, is $\frac{K}{2}$, where

$$K = \int_0^1 |K(s)| \, ds \quad .$$

Since the integral is added to $\{x(t)\}^2$, clearly the modulus of H is

$$M = 1 + \frac{K}{2} \quad .$$

Now consider the expression,

$$H_2(x,y) = x(t)\,y(t) + \int_0^1 K(s)\,x(s)\,y(s)\,ds \quad .$$

$H_2(x,y)$ is the polar of $H(x)$, for it is bilinear, symmetric, and $H_2(x,x) = H(x)$. Then,

$$\| H_2(x,y) \| \leqslant M_p \, \|x\| \, \|y\| \quad .$$

To calculate M_p, we must get the least upper bound of $\| H_2(x,y) \|$, while $\|x\| \, \|y\| = 1$. It can easily be proved that it is sufficient to take $\|x\| = \|y\| = 1$. Much as in the first part, we choose

$$y(t) = 1 \quad ,$$

$$x_n(t) = 1 \quad , \quad 0 \leqslant t \leqslant \tfrac{1}{2} - \tfrac{1}{n}$$

$$x_n(t) = n\left(\tfrac{1}{2} - t\right) \quad , \quad \tfrac{1}{2} - \tfrac{1}{n} < t < \tfrac{1}{2} + \tfrac{1}{n}$$

$$x_n(t) = -1 \quad , \quad \tfrac{1}{2} + \tfrac{1}{n} \leqslant t \leqslant 1 \quad .$$

In this case, the least upper bound of the integral is K, and as before, we find

$$M_p = 1 + K \quad .$$

We have therefore given an example in which,

$$M < M_p \quad \text{and} \quad M_p < 2M \quad .$$

It is also obvious that we can make $\frac{M_p}{M}$ approach arbitrarily near to 2 , for K can be increased by multiplication of K(s) by a real number. That is,

$$\frac{M_p}{M} = \frac{1+K}{1+\frac{K}{2}} \longrightarrow 2 \text{, as } K \to \infty \quad .$$

APPENDIX A

In order to solve the problems of Chapter III, it is necessary to be able to solve the equation,

$$(1) \qquad f(\lambda) = a_{00} + \lambda\cdot a_{10} + \bar{\lambda}\cdot a_{11} + \cdots + \lambda^{n}\cdot a_{n0} + \cdot + \lambda^{n-j}\bar{\lambda}^{j}\cdot a_{nj} \quad ,$$

by means of determinants. In other words we wish to show that if

$f(\lambda) \equiv 0$, then $a_{sj} = 0$, for $s = 0,\cdots,n$, $j = 0,\cdots,s$.

In order to do this we must show that there exist complex numbers

$$\lambda_1 , \cdots , \lambda_r \qquad\qquad r = \tfrac{1}{2}\, n(n+1) + j + 1 \quad ,$$

such that the determinant

$$D = \begin{vmatrix} 1, \lambda_1, \bar{\lambda}_1, \cdots, \lambda_1^{n}, \cdots, \lambda_1^{n-j}\bar{\lambda}_1^{j} \\ 1, \lambda_2, \bar{\lambda}_2, \cdots, \lambda_2^{n}, \cdots, \lambda_2^{n-j}\bar{\lambda}_2^{j} \\ \cdots\cdots\cdots\cdots\cdots \\ 1, \lambda_r, \bar{\lambda}_r, \cdots, \lambda_r^{n}, \cdots, \lambda_r^{n-j}\bar{\lambda}_r^{j} \end{vmatrix}$$

does not vanish. We shall make a proof by induction. Therefore we assume that there exist values $\lambda_1, \cdots, \lambda_r$ such that $D \neq 0$, and we now wish to show that we can choose a value of λ , so that the determinant, which follows, does not vanish.

$$\begin{vmatrix} 1, \lambda_1, \bar{\lambda}_1, \ldots, \lambda_1^n, \ldots, \lambda_1^{n-j}\bar{\lambda}_1^j, \lambda_1^{n-j-1}\bar{\lambda}_1^{j+1} \\ 1, \lambda_2, \bar{\lambda}_2, \ldots, \lambda_2^n, \ldots, \lambda_2^{n-j}\bar{\lambda}_2^j, \lambda_2^{n-j-1}\bar{\lambda}_2^{j+1} \\ \cdots\cdots\cdots\cdots\cdots \\ 1, \lambda_r, \bar{\lambda}_r, \ldots, \lambda_r^n, \ldots, \lambda_r^{n-j}\bar{\lambda}_r^j, \lambda_r^{n-j-1}\bar{\lambda}_r^{j+1} \\ 1, \lambda, \bar{\lambda}, \ldots, \lambda^n, \ldots, \lambda^{n-j}\bar{\lambda}^j, \lambda^{n-j-1}\bar{\lambda}^{j+1} \end{vmatrix}$$

Expanding this determinant by elements of the last row, we get

$$(2) \qquad \phi(\lambda) = b_{00} + \lambda \cdot b_{10} + \bar{\lambda} \cdot b_{11} + \cdots + \lambda^n \cdot b_{n0} + \cdots + \lambda^{n-j}\bar{\lambda}^j \cdot b_{nj} + \lambda^{n-j-1}\bar{\lambda}^{j+1} \cdot b_{n,j+1}$$

where $b_{n,j+1} = D \neq 0$.

We wish to show that $\phi(\lambda)$ has at least one non-zero value, or in other words, that it cannot vanish identically. In order to prove this, let us make the supposition that $\phi(\lambda) \equiv 0$, and show that this leads to a contradiction. Let us set

$$\lambda = x + iy \;, \quad \bar{\lambda} = x - iy \quad \text{and} \quad b_{rs} = b'_{rs} + ib''_{rs} \;.$$

Then by substituting in (2),

$$\phi(\lambda) = \phi_1(x,y) + i \cdot \phi_2(x,y)$$

where ϕ_1 and ϕ_2 are real functions of the real variables x,y. Hence if $\phi(\lambda) \equiv 0$, then $\phi_1(x,y) \equiv 0$, $\phi_2(x,y) \equiv 0$. Now since $\phi_1(x,y)$ and $\phi_2(x,y)$ are polynomials in x,y which vanish identically, all the coefficients in the representation must vanish. In particular all terms whose total degree is m must vanish identically. Clearly, these terms arise only from the terms in (2) which are of total degree m, say

(3) $$h_m(\lambda, \bar{\lambda}) = \sum_{0}^{m}{}_j \lambda^{m-j} \bar{\lambda}^{j} \cdot b_{mj} \quad .$$

Consider the term $\lambda^{m-s}\bar{\lambda}^{s}$. Replacing λ by $x + iy$,

$$\lambda^{m-s}\bar{\lambda}^{s} = (x+iy)^{m-s}(x-iy)^{s} = \sum_{0}^{m-s}{}_{\alpha} \binom{m-s}{\alpha} x^{m-s-\alpha}(iy)^{\alpha} \sum_{0}^{s}{}_{\beta} (-)^{\beta} \binom{s}{\beta} x^{s-\beta}(iy)^{\beta} \quad .$$

In the above expression the coefficient of $x^{m-\mu}(iy)^{\mu}$ is

(4) $$p^{m}_{\mu,s} = \sum_{\alpha+\beta=\mu} (-)^{\beta} \binom{m-s}{\alpha}\binom{s}{\beta} \quad .$$

Or rewriting,

$$\lambda^{m-s}\bar{\lambda}^{s} = \sum_{0}^{m}{}_{\mu} p^{m}_{\mu,s} x^{m-\mu}(iy)^{\mu}$$

yields,

$$h_m(\lambda,\bar{\lambda}) = \sum_{0}^{m}{}_j b_{mj} \sum_{0}^{m}{}_{\mu} p^{m}_{\mu,j} x^{m-\mu}(iy)^{\mu}$$

$$= \sum_{0}^{m}{}_{\mu} x^{m-\mu} y^{\mu} \sum_{0}^{m}{}_j (i)^{\mu} p^{m}_{\mu,j} \cdot b'_{mj} + \sum_{0}^{m}{}_{\mu} x^{m-\mu} y^{\mu} \sum_{0}^{m}{}_j (i)^{\mu+1} p^{m}_{\mu,j} \cdot b''_{m,j}$$

Since the coefficient of $x^{m-\mu} y^{\mu}$ is real in the first of the above sums when the coefficient of $x^{m-\mu} y^{\mu}$ in the second sum is imaginary, we may assert that if $h_m(\lambda, \bar{\lambda}) \equiv 0$, then

$$\sum_{0}^{m}{}_j p^{m}_{\mu,j} \cdot b'_{mj} = 0 \quad , \qquad \sum_{0}^{m}{}_j p^{m}_{\mu,j} \cdot b''_{mj} = 0 \quad .$$

These equations have one solution $b'_{mj} = b''_{mj} = 0$, and we wish to prove that that is the unique solution. To do that, it is sufficient to show that

$$\det_{\mu,j=0,\cdots,m} \left| p^{m}_{\mu,j} \right| \neq 0 \quad .$$

Now consider $(1 + x)^{m-\nu}(1 - x)^{\nu}$. This is equal to

$$\sum_{0}^{m}{}_{\mu} x^{\mu} \sum_{\alpha+\beta=\mu} (-)^{\alpha} \binom{\nu}{\alpha}\binom{m-\nu}{\beta} = \sum_{0}^{m}{}_{\mu} p^{m}_{\mu,\nu} x^{\mu} \quad .$$

If we set $x = -1$, we find

$$(5) \qquad \sum_{0}^{m}{}_{\mu} (-)^{\mu} p^{m}_{\mu,\nu} = 0 \quad , \text{ for } \nu = 0, \cdots, m-1$$

$$= 2^{m} \quad , \text{ for } \nu = m \ .$$

Hence,

$$(6) \qquad \begin{vmatrix} p^{m}_{0,0} , \cdots , p^{m}_{0,m} \\ \cdots\cdots\cdots \\ p^{m}_{m,0} , \cdots , p^{m}_{m,m} \end{vmatrix} = (-2)^{m} \begin{vmatrix} p^{m}_{0,0} , \cdots , p^{m}_{0,m-1} \\ \cdots\cdots\cdots \\ p^{m}_{m-1,0} , \cdots , p^{m}_{m-1,m-1} \end{vmatrix} \ .$$

This is evident on multiplying every other row in the determinant on the left by -1 , adding to the last row, and using (5).

Now consider the determinant on the right of (6). I shall now show that it is det $|p^{m-1}_{\mu,\nu}|$. By writing out the expressions and using (4) it is easy to verify that

$$(7) \qquad p^{m}_{\mu,\nu} - p^{m-1}_{\mu-1,\nu} = p^{m-1}_{\mu,\nu} \ .$$

Also, from the definition, $p^{m}_{0,\nu} = p^{m-1}_{0,\nu} = 1$. Hence, if in the determinant on the right of (6), we replace $p^{m}_{0,\nu}$ by $p^{m-1}_{0,\nu}$, and subtract the first row from the second, we find that the elements of the second row become $p^{m-1}_{1,\nu}$. Subtracting this row from the third, and so on, we finally reduce the determinant to $|p^{m-1}_{\mu,\nu}|$, $\mu , \nu = 0, \cdots, m-1$. Hence,

$$|p^{m}_{\mu,\nu}| = (-2)^{m} \ |p^{m-1}_{\mu,\nu}| \quad ,$$

and therefore,

$$|p^{m}_{\mu,\nu}| = (-2)^{\frac{1}{2}m(m+1)} \neq 0 \ .$$

Hence in (3), $b_{mj} = 0$, for all m and j , which leads to a contradiction since $b_{n,j+1} \neq 0$. Therefore we have proved, that for any r , there exist numbers $\lambda_1, \cdots, \lambda_r$ such that the determinant D does not vanish, and hence that the equation (1) can be solved for the coefficients a_{mj} .

APPENDIX B

In a paper which has recently appeared in Studia Mathematica, S. Mazur and W. Orlicz, (1), have made some contributions to the study of polynomials which are very fundamental and beautiful, and which can be used to simplify some of the proofs in this thesis. First we must have some definitions; these hold in any linear space.

K-additivity: A function $U_k^*(x_1, x_2, \cdots, x_k)$ will be said to be k-additive if it is completely symmetric and additive in each place.

Rational homogeneity: A function $U_k(x)$ will be said to be rationally homogeneous of degree k, if it can be expressed as

$$U_k(x) = U_k^*(x, \cdots, x) \quad ,$$

where $U_k^*(x_1, \cdots, x_k)$ is k-additive. In this case,

$$U_k(t \cdot x) = t^k \cdot U_k(x) \quad , \quad t \text{ a rational real.}$$

$U_k^*(x_1, \cdots, x_k)$ is called the generating function of $U_k(x)$.

They are then able to prove the following fundamental theorem.

Theorem If $U(x)$ is defined on one linear space E_1 to another, E_2 , and if $\Delta^{n+1}_{u, \cdots, u} U(x) \equiv 0$, (where the increments are all the same), then $U(x)$ is uniquely representable as the sum of rationally homogeneous functions in the form,

$$U(x) = U_0(x) + U_1(x) + \cdots + U_n(x) \quad ,$$

and where moreover,

$$U^*_k(x_1, \cdots, x_k) = \frac{1}{k!} \Delta^k_{x_1,\cdots,x_k} U(0) .$$

The proof of this theorem is rather complicated and the reader is referred to the Studia article for details.

Let us now assume that λ is a complex rational, that is, $\lambda = \lambda_1 + i\lambda_2$, where λ_1, λ_2 are rational reals. Then if $U_k(x)$ is rationally homogeneous of degree k,

$$U_k(\lambda\cdot x) = U_k(\lambda_1\cdot x + \lambda_2\cdot ix)$$

$$= \sum_{s=0}^{k} \lambda_1^s \lambda_2^{k-s} \frac{k!}{s!\,(k-s)!} U^*_k(\underbrace{x,\cdots,x}_{s}\ \underbrace{ix,\cdots,ix}_{k-s})$$

$$= \sum_{s=0}^{k} \left(\frac{\lambda+\bar\lambda}{2}\right)^s \left(\frac{\bar\lambda-\bar\lambda}{2i}\right)^{k-s} \binom{k}{s} U^*_k(\underbrace{x,\cdots,x}_{s}\ \underbrace{ix,\cdots,ix}_{k-s})$$

$$= \sum_{s=0}^{k} \sum_{r=0}^{s} \binom{s}{r} \lambda^{s-r}\bar\lambda^{r} \sum_{t=0}^{k-s} \binom{k-s}{t} \lambda^{k-s-t}\bar\lambda^{t}(-)^t(-i)^{k-s} \frac{1}{2^k}\binom{k}{s} U^*_k$$

$$= \sum_{s=0}^{k} \sum_{m=0}^{k} \sum_{r+t=m} \lambda^{k-m}\bar\lambda^{m}(-i)^{k-s}(-)^t \binom{s}{r}\binom{k}{s}\binom{k-s}{t} \frac{1}{2^k} U^*_k(\underbrace{x,\cdots,x}_{s}\ \underbrace{ix,\cdots,ix}_{k-s}) .$$

If we now define $H_{km}(x)$ as

$$H_{km}(x) = \sum_{s=0}^{k} \frac{1}{2^k}\binom{k}{s}(-i)^{k-s} \sum_{r+t=m} (-)^t \binom{s}{r}\binom{k-s}{t} U^*_k(\underbrace{x,\cdots,x}_{s}\ \underbrace{ix,\cdots,ix}_{k-s}) ,$$

we have

$$U_k(\lambda\cdot x) = \lambda^k\cdot H_{k0}(x) + \lambda^{k-1}\bar\lambda\cdot H_{k1}(x) + \cdots + \bar\lambda^k\cdot H_{kk}(x) .$$

Using this result we see that $U(x)$ is uniquely representable in the form

$$U(x) = H_{00}(x) + H_{10}(x) + H_{11}(x) + \cdots + H_{n0}(x) + \cdots + H_{nn}(x),$$

where

$$H_{rs}(\lambda\cdot x) = \lambda^{r-s}\bar\lambda^{s}\cdot H_{rs}(x)$$

for λ a rational complex number.

If the spaces are, besides being linear, also "espaces algébrophiles", then it is easily seen by taking a sequence $\{\lambda_n\}$ of complex rationals tending to λ, that

$$U(\lambda \cdot x) = H_{00}(x) + \lambda \cdot H_{10}(x) + \bar{\lambda} \cdot H_{11}(x) + \cdots + \lambda^n \cdot H_{n0}(x) + \cdots + \bar{\lambda}^n \cdot H_{nn}(x) ,$$

where λ is any complex number, provided of course that $U(x)$ is a continuous function.

REFERENCES TO THE LITERATURE

Fréchet, Maurice

(1) Une Définition Fonctionelle de Polynomes.

Nouvelles Annales de Mathématiques, Series 4, Tome 9, 1909, p.145.

(2) Les Polynomes Abstraits.

Journal de Mathématiques, Series 9, Tome 8, pp. 71-92 (1929)

(3) Sur la définition axiomatiques d'une classe d'espaces vectoriels distanciés applicable vectoriellement sur l'espace de Hilbert.

Annals of Mathematics, Series II, Vol.36, pp. 705-718 (1935)

Gateaux, R.

(1) Sur Diverses Questions du Calcul Fonctionnel.

Bulletin de la Société Mathématique de France, Tome 50, pp.1-57, (1922)

Highberg, I.E.

(1) A Note on Abstract Polynomials in Complex Spaces.

(To be published in the Journal de Mathématiques)

von Neumann, J. and P. Jordan

(1) On Inner Products in Linear Metric Spaces.

Annals of Mathematics, Series II, Vol.36, pp.719-23 (1935)

Mazur, S. and W. Orlicz

(1) Grundlegende Eigenschaften der Polynomischen Operationen

Studia Mathematica, Tome V, pp.50-68, (1934)

Michal, A.D. and R.S. Martin

(1) Some Expansions in Vector Space.

Journal de Mathématiques, Series 4, Tome 13, pp.69-91 (1934)

Michal, A.D. and A.H. Clifford

(1) Fonctions Analytiques Implicites dans des Espaces Vectoriels Abstraits.

Comptes Rendus, Tome 197, p.735 (1933).

Michal, A.D.

(1) Notes on Abstract Vector Spaces.

I Definition of Abstract Polynomials.

(Unpublished)

Stone, M.H.

(1) Linear Transformations in Hilbert Space and their Applications to Analysis.

Colloquium Publications of the American Mathematical Society, No. 15, (1932)