DEFORMATION OF THE FRONT OF TRAVELING WAVES ALONG POWER LINES UNDER CORONA CONDITIONS

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Deformation of the front of traveling waves along power lines under corona conditions.

Introduction.

Many experiments have been performed on actual and artificial lines in order to determine the attenuation of traveling waves, especially at their front.

Our object is to try to derive an expression, depending on the parameters of the line, which will give us the information we desire concerning the shape of the front of the wave at any point and any time.

From the knowledge of the maximum gradient of potential, the engineers set up the protective devices for the apparatus connected to the end of the line.

The telegraphists' equations, modified as we shall see, will lead us to the result. As a method of handling these equations, we have the choice between the classical method and the Heaviside operational calculus. For networks with lumped parameters, the classical method of treatment of transients is easier and does not require the symbolic operational calculus. For networks with distributed parameters, however, it is almost impossible to reach a solution with the classical method. This is due to the fact that the voltage and current distributions depend on the boundary conditions. The physical truth is thus disregarded since the waves, traveling at a finite velocity, ignore the terminal conditions until they reach the end of the line.

The operational calculus considers the waves on their trip along the line, and is therefore the natural method to use. It does not occur now to electrical engineers to use anything but the vectorial or imaginary symbolic method to solve steady a.c. problems in all cases. In a similar way, the operational calculus will become more and more the symbolic method of solution of transients, and, in fact, it is the only natural method of investigation of transients along lines.

Division of the work.

We are led to divide this work into two parts.

In the first part, we shall lay down a sound basis for the operational calculus. This will require the thorough discussion of the Fourier integral formulation, on which all the operational calculus is built.

The second part will be concerned with an application of operational calculus, i.e. the deformation of the front of traveling waves under corona conditions.

First Part.

The Fourier integral formulation.

Let f(t) be a single-valued function having a finite number of discontinuities and of maxima and minima between $t = -s$ and $t = +s$, and such that $\int_{-s}^{+s} [f(t)]^2 dt$ exists.

Then the function f(t) can be expanded in a convergent Fourier series:

$$f(t) = \frac{1}{2} b_0 + b_1 \cos \frac{\pi t}{s} + b_2 \cos \frac{2\pi t}{s} + \cdots + b_n \cos \frac{n\pi t}{s} + \cdots$$
$$+ a_1 \sin \frac{\pi t}{s} + a_2 \sin \frac{2\pi t}{s} + \cdots + a_n \sin \frac{n\pi t}{s} + \cdots$$

We find easily that:

$$b_n = \frac{\omega}{\pi} \int_{-\frac{\pi}{\omega}}^{+\frac{\pi}{\omega}} f(t) \cos(n\omega t)\, dt$$

$$a_n = \frac{\omega}{\pi} \int_{-\frac{\pi}{\omega}}^{+\frac{\pi}{\omega}} f(t) \sin(n\omega t)\, dt$$

$$b_0 = \frac{\omega}{\pi} \int_{-\frac{\pi}{\omega}}^{+\frac{\pi}{\omega}} f(t)\, dt \quad \text{where:} \quad \omega = \frac{\pi}{s}$$

Thus, for all values of t, we have:

$$f(t) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos \frac{n\pi t}{s} + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi t}{s} \qquad (1)$$

which represents the harmonic analysis of f(t).

Transformation of the Fourier series.

We know that:
$$\begin{cases} \cos(n\omega t) = \frac{1}{2}\left[\varepsilon^{jn\omega t} + \varepsilon^{-jn\omega t}\right] \\ \sin(n\omega t) = \frac{1}{2j}\left[\varepsilon^{jn\omega t} - \varepsilon^{-jn\omega t}\right] \end{cases}$$

Let:
$$\begin{cases} c_n = \frac{1}{2}\left[b_n - j a_n\right] \\ c_{-n} = \frac{1}{2}\left[b_n + j a_n\right] \end{cases}$$

Then: $$f(t) = \frac{b_0}{2} + \sum_{n=1}^{\infty} c_n \varepsilon^{jn\omega t} + \sum_{n=1}^{\infty} c_{-n} \varepsilon^{-jn\omega t}$$

where c_n and c_{-n} can be evaluated from b_n and a_n .

$$c_n = \frac{\omega}{2\pi}\int_{-\frac{\pi}{\omega}}^{+\frac{\pi}{\omega}} f(\lambda)\left[\cos(n\omega\lambda) - j\sin(n\omega\lambda)\right] d\lambda$$

or:
$$c_n = \frac{\omega}{2\pi}\int_{-\frac{\pi}{\omega}}^{+\frac{\pi}{\omega}} f(\lambda)\,\varepsilon^{-jn\omega\lambda}\,d\lambda$$

also:
$$c_{-n} = \frac{\omega}{2\pi}\int_{-\frac{\pi}{\omega}}^{+\frac{\pi}{\omega}} f(\lambda)\,\varepsilon^{jn\omega\lambda}\,d\lambda$$

The same relation holds thus for c_{-n} and c_n by changing n into (-n).

For $n=0$: $$c_0 = \frac{\omega}{2\pi}\int_{-\frac{\pi}{\omega}}^{+\frac{\pi}{\omega}} f(\lambda)\,d\lambda = \frac{b_0}{2}$$

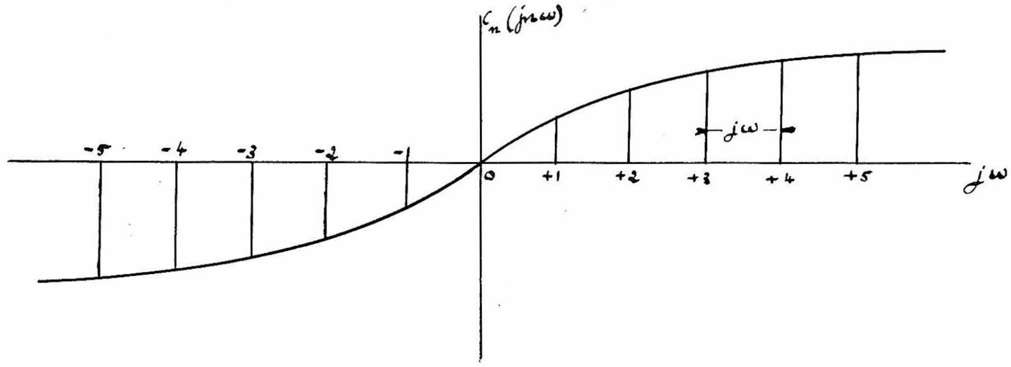
We arrive thus at the compact form:

$$f(t) = \sum_{n=-\infty}^{n=+\infty} c_n \varepsilon^{jn\omega t} \qquad (2)$$

with: $$c_n = \frac{\omega}{2\pi}\int_{-\frac{\pi}{\omega}}^{+\frac{\pi}{\omega}} f(\lambda)\,\varepsilon^{-jn\omega\lambda}\,d\lambda \qquad (3)$$

These two relations (2) and (3) are most important. They are the first form of Fourier transforms for functions having a <u>finite period</u>.

If we draw $c_n(jn\omega)$ vs $j\omega$, we obtain the <u>discontinuous frequency spectrum</u> of the function f(t) with a finite period . This " frequency spectrum " gives the relative importance of the successive harmonics.



<u>Function with infinite period, i.e. having no period whatever.</u>

This case is the limiting one of the former relations, when s tends toward infinity.

We had, for the period T : $\omega = \frac{\pi}{s} = \frac{2\pi}{T}$

When s tends toward ∞ , $\frac{\pi}{\omega}$ tends toward ∞ and so does T

Let f denote the frequency, then:

$$\lim \left(\frac{1}{T}\right)_{T=\infty} = df$$

$$\lim \left(\frac{n}{T}\right)_{T=\infty} = f \quad \text{or} \quad \lim (n\omega)_{T=\infty} = 2\pi f$$

Thus: $c_n = \lim \left[\frac{1}{T}\int_{-\infty}^{+\infty} f(\lambda)\,\varepsilon^{-j2\pi f\lambda}\,d\lambda\right]_{T=\infty}$ or $c_n = df\int_{-\infty}^{+\infty} f(\lambda)\,\varepsilon^{-j2\pi f\lambda}\,d\lambda$

and :

$$f(t) = \int_{-\infty}^{+\infty} df\,\varepsilon^{j2\pi f t}\int_{-\infty}^{+\infty} f(\lambda)\,\varepsilon^{-j2\pi f\lambda}\,d\lambda$$

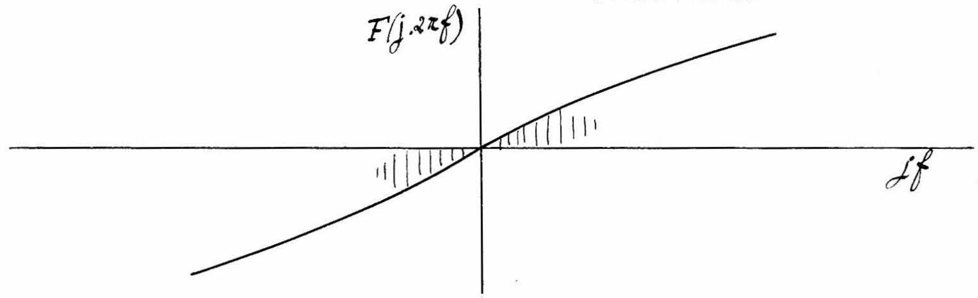
In order not to confuse this new function f(t) with the one having a finite period, let us denote G(t) a function having an infinite period.

Then: $$G(t) = \int_{-\infty}^{+\infty} F(j.2\pi f)\, \varepsilon^{j2\pi f t}\, df \qquad (4)$$

with $$F(j2\pi f) = \int_{-\infty}^{+\infty} G(\lambda)\, \varepsilon^{-j2\pi f \lambda}\, d\lambda \qquad (5)$$

The relation (4) gives the harmonic analysis of the function G(t), whereas the relation (5) is the continuous frequency spectrum of the function G(t).

$F(j.2\pi f)$ gives the relative importance of any term of frequency f in the harmonic analysis of the given function. In other words, the equation (4) is the analysis of G(t), and the equation (5) is the synthesis to regain G(t).



The relations (4) and (5) are Fourier transforms and will serve to justify all the steps taken in the operational calculus.

We recall that the relation (4) is limited in its application to a function G(t) such that : $\int_{-\infty}^{+\infty} [G(t)]^2 dt$ exists.

This implies that G(t) goes to zero when t becomes large.

Change of variables.

Let $p = 2\pi f.j$

then $$G(t) = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} F(p)\,\varepsilon^{pt}\,dp \qquad (6)$$

with $$F(p) = \int_{-\infty}^{+\infty} G(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda \qquad (7)$$

p is a pure imaginary quantity and G(t) is altogether determined by the behavior of F(p) along the axis of imaginaries.

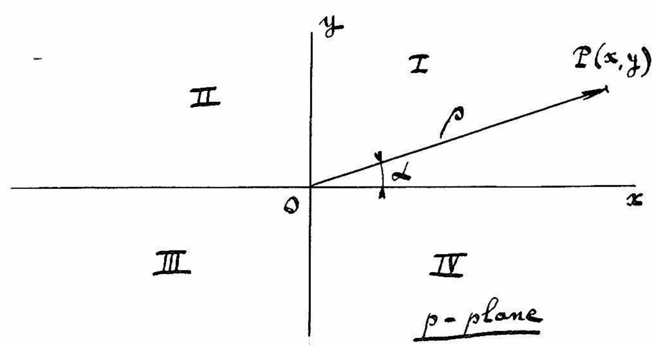
Important particular case : G(t)=0 for t<0

Then: $$G(t) = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} F(p)\,\varepsilon^{pt}\,dp \qquad (8)$$

with $$F(p) = \int_{0}^{\infty} G(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda \qquad (9)$$

The expansion (8) converges only if $\int_0^\infty [G(t)]^2 dt$ exists. If we analyze further the relation (9), we can draw some conclusions as to the nature of the function F(p) by examining what happens when the complex p is equal to infinity. Let p in the relation (9) be equal to a complex quantity :

$$p = x + jy$$
$$= \rho\,\varepsilon^{j\alpha}$$



Then, our relation (9) becomes:

$$F(p) = \int_0^\infty G(\lambda)\, \varepsilon^{-\lambda\rho\cos\alpha} \cdot \varepsilon^{-\lambda\rho j\sin\alpha}\, d\lambda \qquad (9')$$

Of course, $\varepsilon^{-j\lambda\rho\sin\alpha}$ is always less than unity.

thus: $$F(p) \leqslant \int_0^\infty G(\lambda)\, \varepsilon^{-\lambda\rho\cos\alpha}\, d\lambda \qquad (10)$$

The integration in (10) is performed along the real axis of the λ - plane, and λ is positive. If ρ goes to ∞, the integral (10) goes to zero for $\cos\alpha > 0$, i.e. for points P at infinity in the regions I and IV, or for x positive.

In order to find out what happens when $\cos\alpha < 0$ or $x < 0$, let us replace λ by $-\lambda$, then:

$$F(p) = -\int_0^{-\infty} G(-\lambda)\, \varepsilon^{\lambda\rho\cos\alpha}\, \varepsilon^{\lambda\rho j\sin\alpha}\, d\lambda \qquad (9'')$$

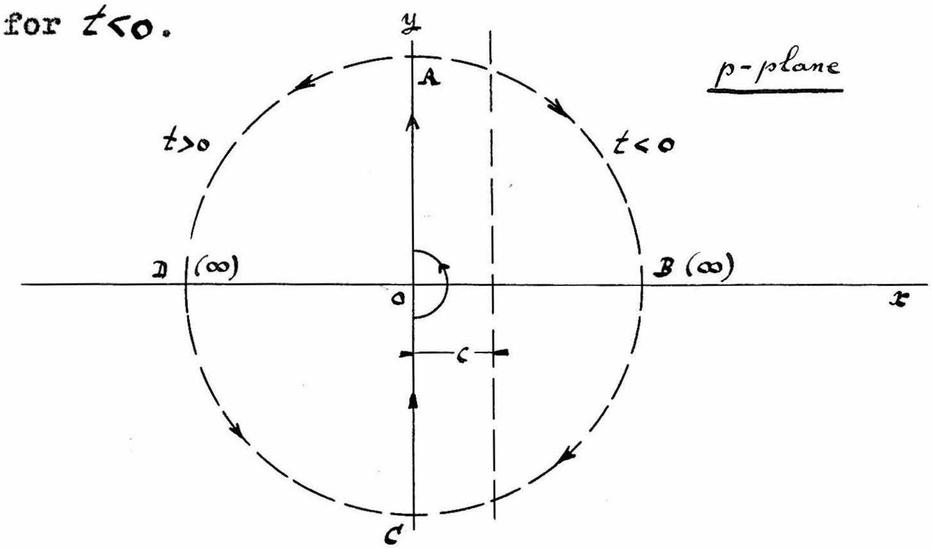
But $G(\lambda)$ is zero for negative values of the argument and $\varepsilon^{\lambda\rho\cos\alpha}$ is zero for $\rho = \infty$ and $\frac{\pi}{2} \leqslant \alpha \leqslant \frac{3\pi}{2}$ i.e. for $x < 0$.

Thus, for all points at infinity : $F(p)_{\rho=\infty} = 0$

Going back to our relation (8), in order to investigate how $F(p).\varepsilon^{pt}$ behaves at all points at infinity, we write: $F(p).\varepsilon^{pt} = F(p).\varepsilon^{xt}.\varepsilon^{jyt}$ and this will be zero if $xt < 0$, i.e. for $x > 0$ with $t < 0$

and for $x < 0$ with $t > 0$

We arrive thus at this important conclusion: the integral (8) along the axis of imaginaries in the p-plane can be replaced by a contour integral composed of the axis of imaginaries and a half circle of infinite radius to the left for $t > o$, and a half circle of infinite radius to the right for $t < o$.



Our equations (8) and (9) become now:

$$G(t) = \frac{1}{2\pi j} \oint F(p)\, \varepsilon^{pt}\, dp \qquad (11)$$

$$F(p) = \int_0^{\infty} G(\lambda)\, \varepsilon^{-p\lambda}\, d\lambda \qquad (12)$$

with $G(t) = o,\ t < o$ and the condition that $\int_0^{\infty} [G(t)]^2 dt$ exists.

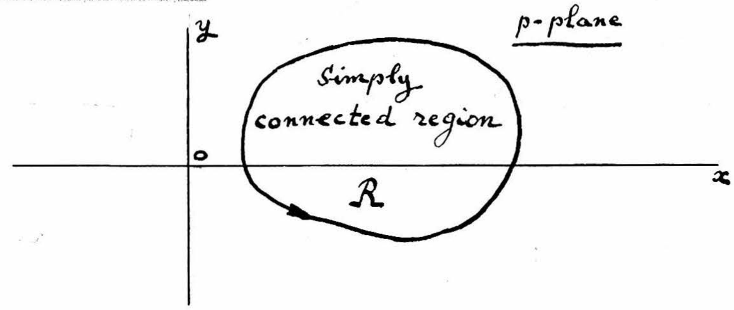
The importance of the conclusion appears here, as we are able now to evaluate the contour integral (11) with the help of the theory of residues.

If the function F(p) has a pole at the origin, i.e. if $F(o) = \int_0^{\infty} G(\lambda)\, d\lambda = \infty$, since $\int_{OABCO} F(p)\, \varepsilon^{pt}\, dp$ must be zero, we have to avoid the origin by a small half circle around it in the right half plane. This, of course, corresponds to writing:

$$G(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)\,\varepsilon^{pt}\,dp \qquad (13)$$

where c has <u>any positive value.</u>

<u>Theory of residues.</u> <u>Evaluation of the residue at a pole of any order.</u>



In order to use consistent notations, let the complex plane be spoken of as the p-plane.

Let $\varphi(p)$ be a complex function :

$$\varphi(p) = \varphi(x+jy) = \mathcal{R}(x,y) + j\,\mathcal{I}(x,y)$$

but
$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial p} = \frac{\partial\mathcal{R}}{\partial x} + j\frac{\partial\mathcal{I}}{\partial x}$$
$$\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial p}\cdot j = \frac{\partial\mathcal{R}}{\partial y} + j\frac{\partial\mathcal{I}}{\partial y}$$

Assume that $\varphi(p)$ is <u>single-valued</u> and <u>continuous</u> in a closed region R. Then, all the derivatives of $\mathcal{R}(x,y)$ and $\mathcal{I}(x,y)$ with respect to x and y have a meaning, and we conclude at once :

$$\left[\begin{array}{l} \dfrac{\partial\mathcal{R}}{\partial x} = \dfrac{\partial\mathcal{I}}{\partial y} \qquad (14) \\[2ex] \dfrac{\partial\mathcal{R}}{\partial y} = -\dfrac{\partial\mathcal{I}}{\partial x} \end{array}\right.$$

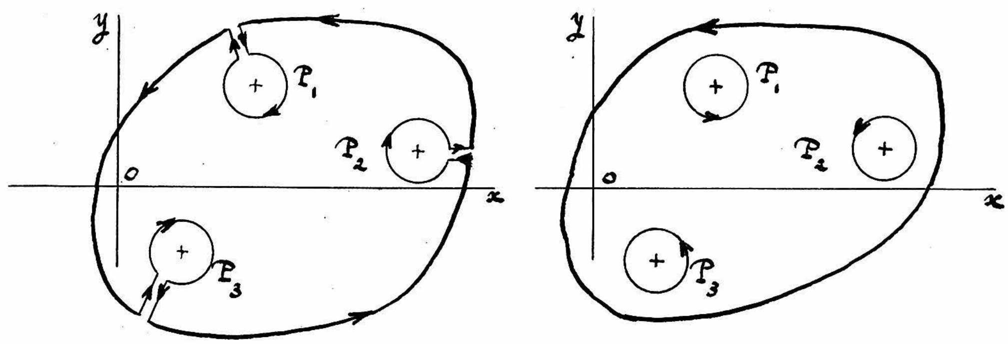
Thus : $\varphi(p) \cdot dp = (R + jI)(dx + j\,dy)$

$$= (R\,dx - I\,dy) + j(I\,dx + R\,dy)$$

and $\oint \varphi(p)\,dp = \oint (R\,dx - I\,dy) + j\oint (I\,dx + R\,dy)$

But, because of the relations (14), both expressions under the integral sign in the right hand member are <u>exact differentials</u>, thus : $\oint \varphi(p)\,dp = 0$

<u>Corollary:</u> Suppose n singularities, P_1 P_2 .....P_n



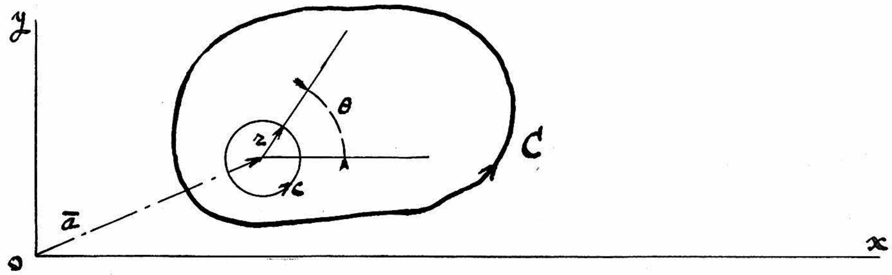
It appears at once that : The value of a line integral about a closed path in a given direction is equal to the sum of the integrals about small closed paths in the same direction, each enclosing one point of discontinuity in the original area.

Our next step is the evaluation of residues.

1. <u>Lemma</u> : <u>Cauchy's Theorem</u>.

Let $\varphi(p)$ be an analytic function (continuous and single-valued) inside of C.

Suppose $\frac{\varphi(p)}{p-a}$ analytic between C and c, c being a small circle of radius r around the point p = a.



Thus: $\oint_C \frac{\varphi(p)}{p-a}\,dp = \int_c \frac{\varphi(p)}{p-a}\,dp$

But : $p - a = r\,\varepsilon^{j\theta}$

$dp = r\,\varepsilon^{j\theta} j\,d\theta$ and $\frac{dp}{p-a} = j\,d\theta$

so that $\lim_{r=0}\left[\oint_c \frac{\varphi(p)}{p-a}\,dp\right] = \varphi(a)\int_0^{2\pi} j\,d\theta = \varphi(a)2\pi j$

Thus : $$\varphi(a) = \frac{1}{2\pi j}\oint \frac{\varphi(p)}{p-a}\,dp \qquad (15)$$

which is Cauchy's theorem.

We have also:

$$\varphi'(a) = \frac{1}{2\pi j}\oint_C \frac{\varphi(p)}{(p-a)^2}\,dp$$

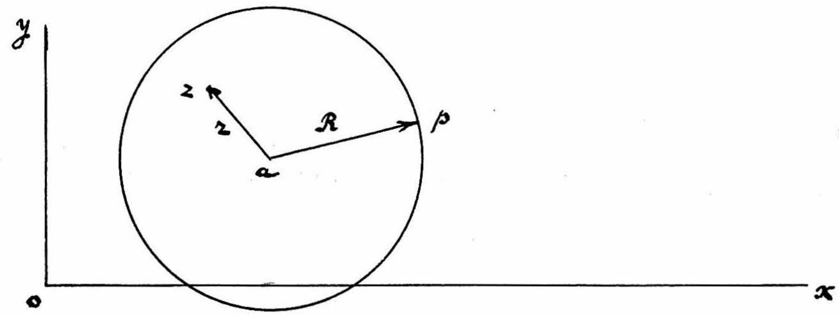
$$\varphi''(a) = \frac{1}{2\pi j}\oint_C \frac{2!\,\varphi(p)}{(p-a)^3}\,dp$$

$$\vdots$$

$$\varphi^{(n)}(a) = \frac{1}{2\pi j}\oint_C \frac{n!\,\varphi(p)}{(p-a)^{n+1}}\,dp$$

2. Theorem. Taylor's series.

Suppose $\varphi(z)$ analytic within a circle of radius R.



We know that, then: $\varphi(z) = \frac{1}{2\pi j} \oint_C \frac{\varphi(p)}{p-z}\,dp$

but: $\frac{1}{p-z} = \frac{1}{(p-a)-(z-a)} = \frac{1}{p-a} \cdot \frac{1}{1 - \frac{z-a}{p-a}}$

or $\frac{1}{p-z} = \frac{1}{p-a}\left[1 + \frac{z-a}{p-a} + \frac{(z-a)^2}{(p-a)^2} + \frac{(z-a)^3}{(p-a)^3} + \cdots\right]$

thus $\varphi(z) = \frac{1}{2\pi j}\left[\oint_C \frac{\varphi(p)}{p-a}\,dp + (z-a)\oint \frac{\varphi(p)}{(p-a)^2}\,dp + \cdots\right]$

or $\varphi(z) = \varphi(a) + (z-a)\varphi'(a) + \frac{(z-a)^2}{2!}\varphi''(a) + \cdots\cdots$

This is Taylor's series. It is quite general, as it holds for complex functions. If a function can be expanded about a point a, it is said to be regular at that point.

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3. Poles and residues.

An analytic function $\varphi(p)$ is said to have a pole of order m at a point p = a, when:

$$\varphi(p) = \frac{\psi(p)}{(p-a)^m} \qquad \text{where } \psi(a) \neq \infty$$

Thus $\psi(p)$ is a regular function at p = a, and can therefore be expanded in a Taylor's series, as:

$$\psi(p) = \psi(a) + (p-a)\psi'(a) + \frac{(p-a)^2}{2!}\psi''(a) + \cdots$$

Consequently:

$$\frac{\psi(p)}{(p-a)^m} = \overbrace{\frac{\psi(a)}{(p-a)^m} + \psi'(a)\frac{1}{(p-a)^{m-1}} + \cdots}^{\mathrm{I}} + \frac{\psi^{(m)}(a)}{m!}$$

$$\overbrace{+ \frac{\psi^{(m+1)}(a)}{(m+1)!}(p-a) + \frac{\psi^{(n+2)}(a)}{(m+2)!}(p-a)^2 + \cdots}^{\mathrm{II}}$$

The integral $\oint_C \varphi(p)\,dp$ along any contour C enclosing the pole p = a, is equal to the integral $\oint_c \varphi(p)\,dp$ along a small circle of radius r around that pole.

Then : $p - a = r\,\varepsilon^{j\theta}$

$$dp = r\,\varepsilon^{j\theta} j\,d\theta$$

and $$\frac{dp}{(p-a)^m} = j\,r^{1-m}\,\varepsilon^{j(1-m)\theta}\,d\theta$$

so $$\oint_c \frac{dp}{(p-a)^m} = \frac{r^{1-m}}{1-m}\left[\varepsilon^{j(1-m)2\pi} - 1\right]$$

For $m \neq 1$, this integral is zero since r = 0 and $\varepsilon^{j(1-m)2\pi} = 1$

For m = 1 $$\oint_c \frac{dp}{p-a} = \int_0^{2\pi} j\,d\theta = 2\pi j$$

On the other hand, the terms grouped under the sign II will contribute nothing to the integral, since these terms form an analytic expression, thus :

$$\oint_C \varphi(p)\,dp = \frac{\psi^{(m-1)}(a)}{(m-1)!}\,2\pi j$$

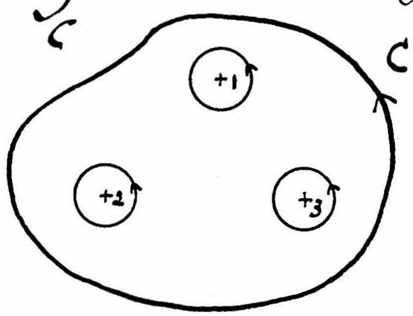
The quantity $\dfrac{\psi^{(m-1)}(a)}{(m-1)!}$ which is the only coefficient in the expansion (16) which contributes to $\oint_C \varphi(p)\,dp$ is called the residue of the pole, let us denote it R.

$$\mathcal{R} = \frac{\psi^{(m-1)}(a)}{(m-1)!} \qquad \text{but} \qquad \psi(p) = (p-a)^m \varphi(p)$$

thus, the <u>residue of a pole of mth. order</u> is :

$$\mathcal{R} = \lim_{p=a} \left\{ \frac{d^{m-1}}{dp^{m-1}} \left[\frac{(p-a)^m}{(m-1)!} \varphi(p) \right] \right\} \qquad (17)$$

and $\oint_C \varphi(p)\,dp = 2\pi j \times \mathcal{R}$ (18)



There may be several poles of any order within C, then:

$$\oint_C \varphi(p)\,dp = 2\pi j \times$$

the sum of the residues.

The expression (17) is very important: it gives a direct and always simple way to obtain the residue at a pole of any order.

For a <u>first order</u> pole: m = 1, thus,

$$\mathcal{R} = \lim_{p=a} \left[(p-a)\,\varphi(p) \right] \qquad \text{since } 0! = 1$$

which is well known relation.

<u>Evaluation of G(t) by a contour integral.</u>

We can now go back to our reciprocal relations (11) and (12) and proceed further.

As these equations stand, they give either G(t) or F(p) when the other function is given. If the complex function F(p) is given, G(t) is obtained by a contour integral.

Since G(t) is zero when t is negative, we conclude that the function $F(p) . \varepsilon^{pt}$ has <u>no poles in the right half plane</u>. When t is positive, we have :

$$\oint F(p) \varepsilon^{pt} dp = 2\pi j \times \text{ sum of the residues.}$$

Consequently: G(t) is equal to the sum of the residues at the different poles of the function $\left[F(p) . \varepsilon^{pt}\right]$.

We have, in the expression (17), the means of evaluating these residues at poles of any high order.

The integral (12) can be reduced to two real integrals. Thus, <u>from the knowledge of either F(p) or G(t), the other function can be readily obtained.</u>

<u>The operational calculus.</u>

Balth. Van Der Pol, in a publication in the Philosophical Magazine, gave in 1929 a really strong mathematical basis to the original conception of Heaviside.

The entire theory of the operational calculus is based upon a few remarkable properties of the integral equations (11) and (12).

In order to discover these properties, it is necessary to make a change of function, replacing F(p) by $\frac{H(p)}{p}$.

Our integral equations are thus:

$$G(t) = \frac{1}{2\pi j} \oint \mathcal{H}(p) \frac{\varepsilon^{pt}}{p} dp \qquad (19)$$

$$\mathcal{H}(p) = p \int_0^\infty G(\lambda) \, \varepsilon^{-p\lambda} d\lambda \qquad (20)$$

In a shorthand fashion, we write:

$$\mathcal{H}(p) \doteqdot G(t) \qquad (21)$$

which we read : H(p) is the symbolic representation or is symbolically equivalent to G(t). We may also say, together with Van Der Pol, that H(p) is the image of the original function G(t). The expression (21) is called an operational equation.

$\frac{\mathcal{H}(p)}{p}$ or F(p) is known as the Laplace transform of G(t).

Since the equations (19) and (20) are reciprocal, we shall establish a few important rules on the basis of the equation (20) :

$$\mathcal{H}(p) = p \int_0^\infty G(\lambda) \, \varepsilon^{-p\lambda} d\lambda$$

1. Let $\lambda = at$, a being a positive constant.

$$\mathcal{H}(p) = ap \int_0^\infty G(at) \, \varepsilon^{-pat} dt$$

thus : $\mathcal{H}\left(\frac{p}{a}\right) = p \int_0^\infty G(at) \, \varepsilon^{-pt} dt$

Consequently: $\mathcal{H}\left(\frac{p}{a}\right) \doteqdot G(at) \qquad (22)$

2. Let us replace $G(\lambda).\varepsilon^{-p\lambda}$ by $\frac{dG(\lambda)}{d\lambda}.\varepsilon^{-p\lambda}$

Then: $p\int_0^\infty \frac{dG(\lambda)}{d\lambda}\varepsilon^{-p\lambda}d\lambda$

integrated by parts, is equal to:

$$p\left[\varepsilon^{-p\lambda}G(\lambda) + p\int G(\lambda)\,\varepsilon^{-p\lambda}d\lambda\right]_0^\infty$$

We have shown that G(t) can not be infinite at $t=\infty$, thus :

$$p\int_0^\infty \frac{dG(\lambda)}{d\lambda}\varepsilon^{-p\lambda}d\lambda = p\left[H(p) - G(0)\right]$$

or $pH(p) = p\left[\int_0^\infty \frac{dG(\lambda)}{d\lambda}\varepsilon^{-p\lambda}d\lambda + G(0)\right] = p\int_0^\infty\left[\frac{dG(\lambda)}{d\lambda} + pG(0)\right]\frac{d\lambda}{\varepsilon^{p\lambda}}$

Consequently: $$pH(p) \doteqdot \frac{dG(t)}{dt} + pG(0) \qquad (23)$$

or $$p\left[H(p) - G(0)\right] \doteqdot \frac{dG(t)}{dt}$$

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3. Let us replace $G(\lambda)$ by $\int_0^\lambda G(\alpha)\,d\alpha$

Then: $p\int_0^\infty\left[\int_0^\lambda G(\alpha)\,d\alpha\right]\varepsilon^{-p\lambda}d\lambda = \int_0^\infty\left[\int_0^\lambda G(\alpha)d\alpha\right]d(-\varepsilon^{-p\lambda}) =$

$$= \left[-\varepsilon^{-p\lambda}\int_0^\lambda G(\alpha)\,d\alpha\right]_0^\infty + \int_0^\infty \varepsilon^{-p\lambda}d\left[\int_0^\lambda G(\alpha)\,d\alpha\right] =$$

$$= \left[\frac{-\int_0^\lambda G(\alpha)\,d\alpha}{\varepsilon^{p\lambda}}\right]_{\lambda=\infty} + \int_0^\infty \varepsilon^{-p\lambda}G(\lambda)\,d\lambda$$

We have seen that if $\frac{H(p)}{p}$ has a pole at the origin, then : $\int_0^\infty G(\lambda)\,d\lambda = \infty$

The first term has the form $\frac{\infty}{\infty}$.

Using l'Hospital's rule:

$$\lim\left[\frac{\int_0^\lambda G(\alpha)\,d\alpha}{\varepsilon^{p\lambda}}\right]_{\lambda=\infty} = \frac{G(\infty)}{p \times \infty}$$

and, as we know, $G(\infty)$ must be finite.

Thus, under all circumstances, the first term cancels and:

$$p\int_0^\infty\left[\int_0^\lambda G(\alpha)\,d\alpha\right]\varepsilon^{-p\lambda}\,d\lambda = \frac{H(p)}{p}$$

or

$$\frac{1}{p}\cdot H(p) \doteqdot \int_0^t G(t)\,dt \qquad (24)$$

---

4. In the same way, we could establish other relations, such as :

$$\frac{p}{p+a}\cdot H(p+a) \doteqdot \varepsilon^{-at}G(t) \qquad (25)$$

which is the Heaviside shifting formula.

Also:

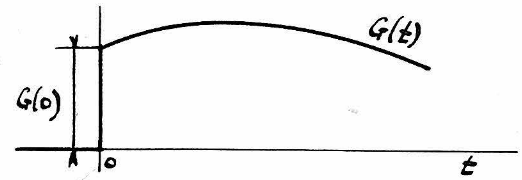
$$\varepsilon^{-ap}H(p) \doteqdot \begin{cases} 0 & t < a \\ G(t-a) & t > a \end{cases} \qquad (26)$$

where ε^{-ap} is the Heaviside transfer operator.

Conclusion. By looking at the operational equations (23) and (24), we can draw important conclusions.

Suppose first that $G(0) = 0$, then p is equivalent to taking the derivative $\frac{d}{dt}$, whereas $\frac{1}{p}$ corresponds to the integral sign $\int_0^t dt$. Of course, if G(0) is not zero, then there is an additional term in the derivative:

$$pH(p) \doteq \frac{dG(t)}{dt} + p\,G(0)$$



The second term pG(0) corresponding to the discontinuity is called an impulse.

In the same way:

$$p^2 H(p) \doteq \frac{d^2G(t)}{dt^2} + p\left[\frac{dG(t)}{dt}\right]_{t=0} + p^2 G(0)$$

so, that: p^n represents $\frac{d^n}{dt^n}$, plus the impulse terms in the function and the (n-1) first derivatives at the origin.

p^{-n} represents $\int_0^t \int_0^t - - - - - \int_0^t dt^n$

Consequently, p which is the independent variable in the image, becomes a symbolic operator in the original function. This property, which is the clue of the operational method, has given the name: operational calculus.

Operational equations.

The fundamental problem in operational calculus is to find the original function of a given image.

An operational equation is thus of the type:

$$G(t) \doteq H(p)$$

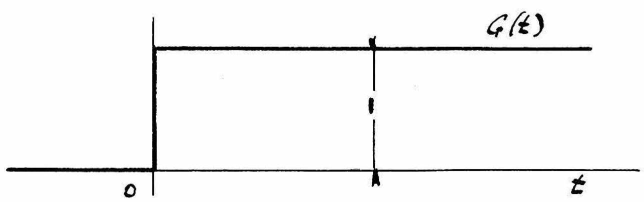
where H(p) is a given function of p, and G(t) an unknown function of t. The solution of an operational equation reduces itself to the solution of the integral equation (20), and can be obtained by the theory of residues.

However, this solution can be obtained in most cases by reducing it to the solution of a certain number of fundamental equations which we shall now consider. We have already found the meaning of p^n and p^{-n}.

Fundamental equations, obtained from :

$$H(p) = p\int_0^\infty G(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda$$

1. Let G(t) be equal to unity.



This is the well-known unit function of Heaviside.

We notice at once that $\int_0^\infty [G(t)]^2\,dt$ does not exist, i.e. G(t) is not of summable squares. Such a function G(t) is not representable by the Fourier integral as established in the equations (8) and (9). We have there:

$$G(t) = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty}\varepsilon^{pt}\,dp\int_0^\infty G(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda \qquad (27)$$

This is not true any more.

However, the function $\varepsilon^{-ct}\cdot G(t)$, where c is a real positive constant, is of summable squares, i.e. the integral

$$\int_0^\infty \varepsilon^{-2ct} \cdot \overline{G(t)}^2 \, dt$$ exists.

Thus, our equation (27) holds for $\varepsilon^{-ct} \cdot G(t)$:

$$\varepsilon^{-ct} \cdot G(t) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \varepsilon^{pt} dp \int_0^\infty \varepsilon^{-c\lambda} G(\lambda) \, \varepsilon^{-p\lambda} d\lambda$$

Let $u = c + p \qquad du = dp$

$$\varepsilon^{-ct} \cdot G(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \varepsilon^{ut} \cdot \varepsilon^{-ct} du \int_0^\infty G(\lambda) \, \varepsilon^{-u\lambda} d\lambda$$

or $$G(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{\varepsilon^{ut}}{u} H(u) \, du \qquad (28)$$

where $$H(u) = u \int_0^\infty G(\lambda) \, \varepsilon^{-u\lambda} d\lambda \qquad (29)$$

These relations hold for $\int_0^\infty [G(t)]^2 dt = \infty$

The equations (28) and (29) are the generalized Fourier transforms. We notice that the equation (28) only has changed. The equation (28) merely means that, when we integrate along the axis of imaginaries, we must avoid the origin by a small half circle in the right half plane.

Thus, if $G(t) = 1$, the generalized equation (29) gives $H(u) = 1$. The unit function is thus expressed by the contour integral: $1 = \frac{1}{2\pi j} \oint \frac{\varepsilon^{pt}}{p} dp$ and $1 \doteqdot 1 \quad (30)$

Consequently, a constant is its own image.

---

2. Since $1 \doteqdot 1$ and $\frac{1}{p^n} \doteqdot \int_0^t \cdots \int_0^t dt^n$

we derive : $$\frac{1}{p^n} \doteqdot \frac{t^n}{n!} \qquad (31)$$

Many more relations can be found; it is the object of textbooks on operational calculus to establish them.

Application of the operational calculus to electric networks.

So far, we have built a logical ensemble of deductions from the Fourier integral formulation. We have been led to the operational calculus. Now that the basis of the operational calculus is firmly established, we pass from the purely mathematical point of view to the practical applications of operational calculus to electrical engineering problems. The link between the former part and its application to electric networks rests on the following property. Let us consider the differential equation of any high order with constant coefficients:

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = E(t) \quad (32)$$

with the conditions: $E(t) = 0$ for $t < 0$

$E(t) \equiv E(t)$ for $t > 0$

and $y = \dfrac{dy}{dt} = \cdots = \dfrac{d^{n-1} y}{dt^{n-1}} = 0$ for $t = 0$

where $\dfrac{d^n y}{dt^n}$ at $t = 0$ is not necessarily zero.

These conditions express that at $t = 0$, a disturbing force E(t) is suddenly applied to a linear system previously in equilibrium.

Let us replace the equation (32) by an integral equation.

We know that: $G(t) \doteq H(p) = p\int_0^\infty G(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda$

Let : $y \doteq h(p)$

$$a_n y \doteq a_n\, p\int_0^\infty y(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda$$

then $a_{n-1}\left[\frac{dy}{dt} + p\,y(0)\right] \doteq a_{n-1}\,p^2\int_0^\infty y(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda$

Since the system is in equilibrium: $y(0) = 0$, i.e. the impulsive term disappears, and:

$$a_{n-1}\cdot\frac{dy}{dt} \doteq a_{n-1}\,p^2\int_0^\infty y(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda$$

$$\vdots \qquad\qquad \vdots$$

$$a_0\cdot\frac{d^n y}{dt^n} \doteq a_0\,p^{n+1}\int_0^\infty y(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda$$

since the (n-1) derivatives are zero at $t=0$.

The image of the left hand member is:

$$p\left(a_0 p^n + a_1 p^{n-1} + \cdots + a_{n-1}\,p + a_n\right)\int_0^\infty y(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda \qquad (33)$$

The image of the right hand member is:

$$p\int_0^\infty E(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda \qquad (34)$$

Images (33) and (34) must be equal:

$$\left(a_0 p^n + a_1 p^{n-1} + \cdots + a_n\right)\int_0^\infty y(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda = \int_0^\infty E(\lambda)\,\varepsilon^{-p\lambda}\,d\lambda \qquad (35)$$

The integral equation (35) takes the place of the differential equation (32).

Let $E(t) \doteqdot M(p)$

and $Z(p) = a_0 p^n + a_1 p^{n-1} + \cdots + a_n$

Then, the integral equation (35) can be written as:

$$Z(p) \cdot h(p) = M(p) \doteqdot E(t)$$

Thus: $h(p) = \dfrac{M(p)}{Z(p)}$ (36)

or $y(t) \doteqdot \dfrac{M(p)}{Z(p)}$ (36) or $y(t) \doteqdot \dfrac{1}{Z(p)} E(t)$

We have also $\dfrac{dy}{dt} \doteqdot p \dfrac{M(p)}{Z(p)}$, and so on.

$\displaystyle\int_0^t y\,dt \doteqdot \frac{1}{p} \frac{M(p)}{Z(p)}$, and so on.

Consequently, we have reduced the solution of a linear differential equation (32) with equilibrium time boundary conditions, to the solution of an operational equation (36).

This is the justification for our developed mathematical background on operational calculus.

<u>Case of an applied unit force.</u>

Suppose E(t) = 1.

Since : $1 \doteqdot 1$, M(p) = 1

and : $y(t) \doteqdot \dfrac{1}{Z(p)}$ (37)

which means that: $\dfrac{1}{Z(p)} = p \displaystyle\int_0^\infty y(\lambda) \cdot \varepsilon^{-p\lambda} d\lambda$ (38)

Let us now consider what the symbolic equation (37) means. If the function y(t) that we are trying to determine is a current, and if the unit function E(t) is a suddenly applied unit voltage, then Z(p) plays the role of an impedance. Thus Z(p) is the symbolic or the operational impedance of the circuit, for a suddenly applied unit voltage. This symbolic impedance is not a disturbing new concept; it is very much like the complex impedance $Z(j\omega)$ in the case of sinusoidal a.c. currents. In the same way, Z(p) gives the means to treat problems in the non-steady state by means of equations having the form of the Ohm's law. The inverse of Z(p), $\frac{1}{Z(p)}$, is called the generating function, and $\frac{1}{pZ(p)}$ is called the Laplacian transform of A(t).

The current response to a unit voltage has been called the indicial admittance by Carson, and denoted A(t), thus:

$$A(t) \doteq \frac{1}{Z(p)} \qquad (39)$$

which means:

$$\frac{1}{Z(p)} = p\int_0^\infty A(\lambda)\, \varepsilon^{-p\lambda}\, d\lambda \qquad (40)$$

and:

$$A(t) = \frac{1}{2\pi j} \oint \frac{1}{pZ(p)}\, \varepsilon^{pt}\, dp \qquad (41)$$

These equations have the same limitations as before:

1. $A(\infty)$ must be finite, which is physically the case for dissipative networks (all the roots of Z(p) are thus in the left half plane, with, eventually, the exception of the origin).

2. $\int_0^\infty [A(t)]^2 dt$ exists.

The equation (40) is known in electrical engineering as the <u>infinite integral theorem</u>. It is the definition that was used by Balth. Van Der Pol for operators. Carson, in his " Theory of transient oscillations of electrical networks" (Trans. A.I.E.E.38,1919,p.345), makes of this theorem the central feature of operational calculus. He thus agrees with Balth. Van Der Pol.

The equation (41) is known as the <u>Bromwich-Wagner theorem</u>: its advantage is to give the indicial admittance through a complex integration in the complex plane.

Concerning the infinite integral theorem, we should point out that while Van Der Pol used it to solve differential equations, Carson considered it rather as a check on operational equations; in other words, Carson did not use it to solve operational equations.

<u>The Heaviside expansion theorem.</u>

It is deduced quite naturally from the Bromwich-Wagner theorem, by the use of the theory of residues.

The poles of $\dfrac{\varepsilon^{pt}}{p\,Z(p)}$ are given by $p = 0$

and $Z(p) = 0$

$Z(p) = 0$ is called the <u>determinantal equation.</u>

Heaviside established his theorem in the case where all the roots are different. In the case where there are coincident roots, Wagner gave a general form of the expansion theorem, but it is simpler then to keep the integral (41) combined with the theory of residues.

Suppose that Z(p)= 0 gives the single roots:

$$p = p_1 \; \cdots \; p_K \; \cdots \; p_n$$

Residue at the origin.

$$\mathcal{R}_o = \lim_{p=0} \left[p \cdot \frac{\varepsilon^{pt}}{p\,Z(p)} \right] = \frac{1}{Z(o)}$$

Residue at the root p_K.

$$\mathcal{R}_K = \lim_{p=p_K} \left[(p-p_K) \frac{\varepsilon^{pt}}{p\,Z(p)} \right]$$

but $Z(p) = a(p-p_1) \cdots (p-p_K) \cdots (p-p_n)$

thus $\dfrac{Z(p)}{p-p_K} = a(p-p_1)\cdots(p-p_{K-1})(p-p_{K+1})\cdots(p-p_n)$

and $\lim \left[\dfrac{Z(p)}{p-p_K} \right]_{p=p_K} = Z'(p_K)$

Thus: $\mathcal{R}_K = \dfrac{\varepsilon^{p_K t}}{p_K Z'(p_K)}$

and since $A(t) = \mathcal{R}_o + \mathcal{R}_1 + \cdots + \mathcal{R}_n$

we have $$A(t) = \frac{1}{Z(o)} + \sum_{K=1}^{K=n} \frac{\varepsilon^{p_K t}}{p_K Z'(p_K)} \qquad (42)$$

Since A(t) must be real, we deduce that the complex imaginary roots come out in pairs of conjugate complex quantities.

In general: $p_K = -\alpha_K + j\beta_K$

α_K is the natural decrement

β_K is the natural angular velocity

p_K is the generalized natural angular velocity

Heaviside established his theorem twice; in his " Electromagnetic Theory ", Vol.II, p.127, and in his " Electrical Papers ", Vol.II, p.226, as a footnote.

Vallarta first presented an algebraic indiscutable proof.

Bromwich studied the case of an infinite number of roots.

Carson developed a form of expansion theorem for use when the applied voltage is of the form $\varepsilon^{j\omega t}$; let us treat this case.

Our equation (36) gives the solution: $i(t) \doteq \dfrac{M(p)}{Z(p)}$

where $\quad M(p) \doteq \varepsilon^{j\omega t}$

or $\quad \dfrac{p}{p-j\omega} \doteq \varepsilon^{j\omega t}$

thus: $\quad i(t) = \dfrac{1}{2\pi j} \oint \dfrac{\varepsilon^{pt}}{p} \cdot \dfrac{p}{p-j\omega} \cdot \dfrac{1}{Z(p)}\, dp$

or $\quad i(t) = \dfrac{1}{2\pi j} \oint \dfrac{\varepsilon^{pt}}{p-j\omega} \cdot \dfrac{1}{Z(p)} \cdot dp$

We must find the poles of $\quad \dfrac{\varepsilon^{pt}}{p-j\omega} \cdot \dfrac{1}{Z(p)}$

i.e. for $\quad p - j\omega = 0 \quad$ and $\quad Z(p) = 0$

<u>Pole at</u> $p = j\omega$

$$R = \lim_{p=j\omega} \left[(p-j\omega) \cdot \frac{\varepsilon^{pt}}{p-j\omega} \cdot \frac{1}{Z(p)} \right] = \frac{\varepsilon^{j\omega t}}{Z(j\omega)}$$

<u>Pole at</u> $p = p_K$

$$R = \lim_{p=p_K} \left[(p-p_K) \frac{\varepsilon^{pt}}{p-j\omega} \cdot \frac{1}{Z(p)} \right] = \frac{\varepsilon^{p_K t}}{(p_K - j\omega) Z'(p_K)}$$

thus: $$i(t) = \frac{\varepsilon^{j\omega t}}{Z(j\omega)} + \sum_{K=1}^{n} \frac{\varepsilon^{p_K t}}{(p_K - j\omega) Z'(p_K)} \qquad (43)$$

Discussion of the equations (42) and (43).

After a time $t = \infty$, the exponential terms disappear. The steady-state term is therefore $\frac{1}{Z(0)}$ for a unit applied voltage.

The steady state term for a voltage $\varepsilon^{j\omega t}$ is $\frac{1}{Z(j\omega)}$, in which we see that the symbolic impedance Z(p) becomes the complex imaginary impedance Z $(j\omega)$; and the whole operational calculus reduces to the complex imaginary calculus. The examples (42) and (43) are sufficient to show the wide use of application af the Carson and Bromwich-Wagner theorems.

General case.

The disturbing e.m.f. is a function E(t) of any shape whatever. We want the current solution i(t).

First method. We replace E(t) by its image:

$$M(p) \doteq E(t) \quad \text{where} \quad M(p) = p\int_0^\infty E(\lambda)\,\varepsilon^{-p\lambda}d\lambda$$

Then, it follows at once:

$$i(t) = \frac{1}{2\pi j}\oint \frac{M(p)}{p\,Z(p)}\,\varepsilon^{pt}dp$$

As a check: $$\frac{M(p)}{Z(p)} = p\int_0^\infty i(\lambda)\,\varepsilon^{-p\lambda}d\lambda$$

Second method. The solution is reached in two steps. We first determine the indicial admittance:

$$A(t) = \frac{1}{2\pi j} \oint \frac{1}{p\,Z(p)} \varepsilon^{pt} dp$$

Then, by the use of the superposition theorem (Boltzmann-Duhamel integral), we have :

$$i(t) = \frac{d}{dt} \int_0^t A(\lambda)\, E(t-\lambda)\, d\lambda$$

Conclusion: This first part indicates the compact form into which the development of the whole of the operational calculus may be put.

On the basis of the Fourier integral theorem, we have drawn conclusions on operational equations. By showing then that the solution of a differential equation can be reduced to the solution of an operational equation, we have applied our conclusions on operational equations to electrical engineering practice.

All the well known theorems of Carson, Bromwich-Wagner, and of Heaviside are readily obtained, with their right meaning and their limitations.

This entire procedure is opposite to the historical development of the operational calculus.

Heaviside established his theorem, and also the meaning of the operator p, but he proceeded by intuition, and went wrong often, because the whole matter seemed mysterious. Mainly, the impulse p1 was a most indefinite matter.

My opinion is that operational calculus should be taught by finishing with the Heaviside expansion theorem. The actual didactic method in which this theorem is only the very start can only lead into misconceptions.

Second Part.

Deformation of the front of traveling waves along power lines under corona conditions.

Introduction.

The problem of the deformation of the front of traveling waves or surges along power lines is a very important one. It is known experimentally, that, as a wave travels, its front flattens out, reducing thus the dangerous maximum potential gradient which is the direct cause of flashover in transformers and a.c. machinery.

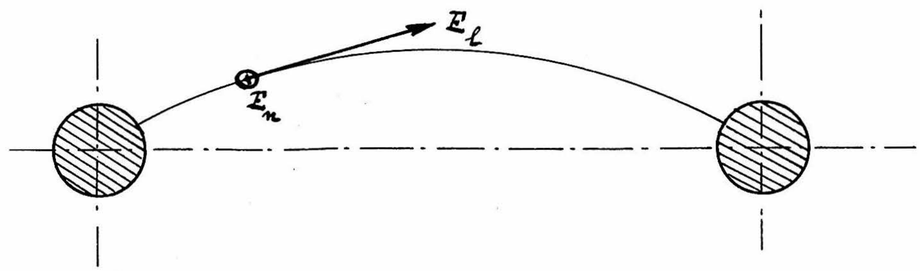
The attenuation of the waves at their front is much higher for cables than for aerial lines and is in fact inversely proportional to the square of the surge impedance of the line, as will come out of the theory.

The factors influencing these attenuation and distortion are skin-effect and corona.

Our object is to introduce these factors in the theory and to predetermine the shape of the waves at their front, as they travel along the line.

We shall take corona into account by the introduction of a leakage conductivity which will be treated as a constant. The actual variation of that coefficient with the voltage, according to experimental data, will be taken into account afterwards in the solution.

General considerations.



Let us consider a double line of cylindrical circular wires. In a transversal plane, we have a series of lines of force flowing from one conductor to the other. The component of the electric field in a transversal plane will be denoted by E_l . In fact, the total vector electric field $\overline{E}$ has also a component E_n in the direction of the wires. The magnetic field is in the transversal plane and is perpendicular to the transversal lines of force.

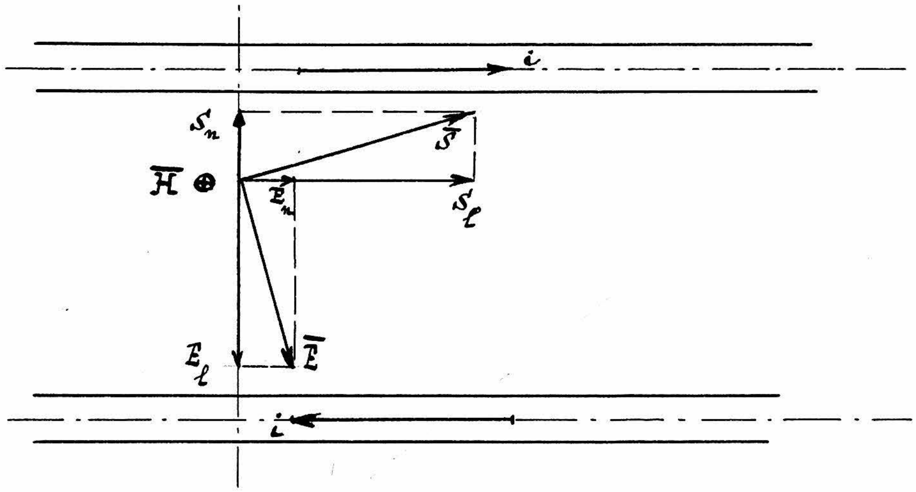
Consequently, the Poynting vector is not parallel to the wires: this means that, in the successive transversal planes, at the front of the wave for instance, the flow of energy, traveling at the velocity of propagation, is not a constant, but damps out.

If we express the electrical quantities in the c.g.s. electrostatic system and the magnetic quantities in the c.g.s. electromagnetic system of units, then we must introduce the ratio of the e.m.u. to the e.s.u. of charge, i.e. the velocity c of light, as was first theoretically shown by Maxwell.

Then: $\overline{S} = \frac{c}{4\pi}\left[\overline{E} \times \overline{H}\right]$

$\overline{S}$: Poynting vector

$\overline{E}x\overline{H}$: vectorial product.



We shall be interested in the dase of a double wire line. We suppose that the conductors are so remote from each other that the current flowing in each wire has no effect whatever upon the distribution of the current in the other. In other words, the skin effect in each conductor will be the same as if it were isolated; this approximation is excellent.

The <u>vectorial Ohm's law</u> is: $\overline{\lambda} = \sigma \overline{E}$

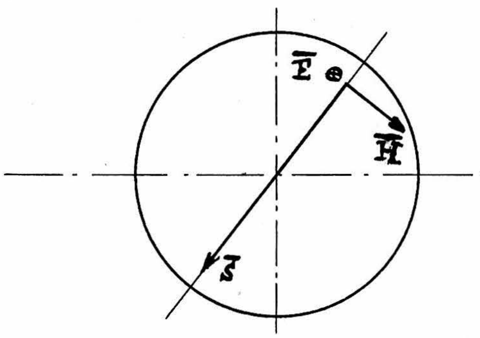
where $\overline{\lambda} = \sigma \overline{E}$ - current density vector

σ - conductivity

$\overline{E}$ - electric field

Since we assume that there are no eddy currents, and therefore $\overline{E}$, are parallel to the wires. The electric field $\overline{E}$, <u>inside of the wires</u>, is therefore purely <u>longitudinal.</u> The Poynting vector, inside of the wires, is therefore purely radial too. This means that there is no longitudinal propagation of the electromagnetic field inside of the wire.

This, by the way, is a very good proof of the fact that the energy is propagated along a power line by the electromagnetic field outside of the wires, and not by the current inside of them. We may also, as another consequence, expect to have for $\overline{\lambda}$ or $\overline{E}$ a diffusion equation. This corresponds to neglecting the displacement current inside of the wire. This last assumption is excellent, because the transversal dimensions of the wires are small, and, on the other hand, the velocity of propagation is very close to that of light, i.e. 3×10^{10} cms. per second.



The Poynting vector, inside of the wire, is radial and directed toward the center, as can be seen at once.

Therefore, the energy supplying the losses in the wires, flows from the outside to the inside.

C.H.Manneback, in his publication " Radiation from Transmission Lines " (n.35, March 1923- M.I.T. Bulletin), developed that point very clearly. He showed, together with Carson, that the electromagnetic field was propagated from the outside toward the inside of the wire. Manneback, in that way, corrected Steinmetz's theory which was that the field was propagated from the inside toward the outside of the wire.

Outside of the wires, the component of the Poynting's vector, parallel to the wires, corresponds to the transmitted power.

We shall now establish the Maxwell's equations which are at the basis of our study, and to which we shall apply the operational calculus method.

The Maxwell's equations for the electromagnetic field.

We shall denote:

$\bar{E}$	electric field
$\bar{H}$	magnetic field
ε	specific inductivity
μ	magnetic permeability
$\bar{D} = \varepsilon . \bar{E}$	eledtric displacement
$\bar{B} = \mu . \bar{H}$	magnetic induction
σ	conductivity
$\Phi = \int B_n \, dS$	magnetic flux
$N = \int D_n \, dS$	electric flux
∇	gradient of
$\nabla \times$	curl of
$\nabla .$	divergence of
∇^2	Laplacian of
$\bar{a} . \bar{b}$	scalar product

$\bar{a} \times \bar{b}$: vectorial product

Note: Jeans uses the polarisation vector $\bar{P}$ instead of the electric displacement vector $\bar{D}$, where $\bar{P} = \frac{\epsilon}{4\pi}\bar{E}$. In my opinion, it is better to work with $\bar{D}$, as it is quite analogous to $\bar{B}$.

First law. The first Maxwell equation, written for a finite closed circuit, is:

$$-\frac{d\Phi}{dt} = \oint E_l \, dl$$

or $$-\frac{d\Phi}{dt} = \oint \bar{E}.\overline{dl} = \int E \, dl \cos(\bar{E}, \overline{dl})$$

For an elementary circuit, by Stokes theorem:

$$-\nabla \times \bar{E} = \frac{\partial \bar{B}}{\partial t} + \nabla \times [\bar{B} \times \bar{u}]$$

where $\bar{u}$ is the velocity of the elementary loop.

However, this law is true only if $\bar{u}$ is much less than the velocity of propagation of the electromagnetic field. But we are interested only in fixed circuits: $\bar{u} = 0$

and $$-\nabla \times \bar{E} = \frac{\partial \bar{B}}{\partial t}$$

If $\bar{E}$ is expressed in c.g.s. electrostatic units and $\bar{B}$ in c.g.s. electromagnetic units:

$$-\nabla \times \bar{E} = \frac{1}{c} \cdot \frac{\partial \bar{B}}{\partial t} \qquad 1)$$

where c is the ratio between an e.m.unit and an e.s.unit of charge.

<u>Second law.</u> The second law of Maxwell for a finite closed circuit is:

$$4\pi \sum I + \frac{dN}{dt} = \oint H_l \, dl$$

or $$\oint H_l \, dl = \oint \overline{H} . \overline{dl} = \oint H \, dl \cos(\overline{H}, \overline{dl})$$

For an elementary circuit, by Stokes theorem:

$$\nabla \times \overline{H} = 4\pi\sigma \overline{E} + \varepsilon \frac{\partial \overline{E}}{\partial t}$$

If $\overline{E}$ is expressed in c.g.s. electrostatic units, and $\overline{H}$ and σ in c.g.s. electromagnetic units:

$$\nabla \times \overline{H} \text{ e.m.u.} = c . \nabla \times \overline{H} \text{ e.s.u.}$$

$$\sigma \text{ e.m.u.} = c^2 . \sigma \text{ e.s.u.}$$

and $$\nabla \times \overline{H} = 4\pi\sigma c \overline{E} + \frac{1}{c} \varepsilon \frac{\partial \overline{E}}{\partial t}$$

or $$\nabla \times \overline{H} = 4\pi \left[\sigma c \overline{E} + \frac{\varepsilon}{4\pi c} \frac{\partial \overline{E}}{\partial t} \right] \qquad (2)$$

$\sigma c \overline{E}$ is the conduction current density

$\frac{\varepsilon}{4\pi c} \frac{\partial \overline{E}}{\partial t}$ is the displacement current density

We shall stick to the following rule : electric quantities such as $\varepsilon, \overline{E}, \overline{D}$ will be expressed in c.g.s. electrostatic units; magnetic quantities such as $\mu, \overline{H}, \overline{B}$

σ, Φ will be expressed in c.g.s. electromagnetic units.

Continuity equations.

$$\nabla . \overline{D} = 4\pi\rho \quad (3)$$

where ρ is the density of electric charges. This is a Poisson-type equation.

$$\nabla . \overline{B} = 0 \quad (4)$$

, since there are no free magnetic charges. This is a Laplace-type equation.

We are now in possession of four important relations. It was desirable to establish them for the sake of the units as used.

Fundamental relations concerning L and C.

We consider a double line and make the fundamental assumption that the transversal section of each wire is equipotential.

Furthermore, we recall that each conductor is treated as if it were at an infinite distance from the other. In other words, the action of the exterior flux is the same on all the elements of current inside the wire, and may thus be neglected, at least as far as the skin effect is concerned.

An integral equation for skin effect in parallel conductors of any shape of cross section has been developed by C.H.Manneback, but it only applies to the steady state and could not be of any help in our problem.

Although we know that L and C are not really constant, we shall see that the assumption of considering them as such is very good. We shall follow Abraham and Becker in their " Theorie der Elektrizität", and consider a double line of infinite conductivity σ .

Since $\bar{\lambda} = \sigma \bar{E}$, $\bar{E} \equiv 0$ inside of the wire,
or, if x is the axis direction: $E_x = 0$ also $H_x = 0$

From equation (4): $\bar{B} = \nabla \times \bar{A}$
with $\bar{A} = A \cdot \bar{r}_x$ $A_y = A_z = 0$

where $\bar{A}$ is the magnetic vector potential

Then, from equation (1):

$$\bar{E} = -\frac{1}{c} \cdot \frac{\partial \bar{A}}{\partial t} + \nabla \varphi$$

where φ is the scalar potential

In quasi-stationary fields, $\frac{\partial \bar{A}}{\partial t} = 0$, thus:

$$\begin{cases} E_y = -\dfrac{\partial \varphi}{\partial y} \\ E_z = -\dfrac{\partial \varphi}{\partial z} \end{cases} \qquad \begin{cases} H_y = \dfrac{\partial A}{\partial z} \\ H_z = -\dfrac{\partial A}{\partial y} \end{cases}$$

But, since there are no static charges:

$$\nabla . \bar{D} = \nabla . \bar{E} = 0 \quad \text{or} \quad \nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \qquad (5)$$

Also $\nabla \times \mathcal{H} = 0$ thus $\nabla^2 A = \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2} = 0$ (6)

These two conditions (5) and (6) are sufficient to define L and C as constants having a geometrical dimension (Abraham and Becker, Tome I. p.200).

In the actual problem, however:

1. $\sigma < \infty$, though very large for copper

2. The electromagnetic field is not stationary; the velocity of light is, however, so large, and we are interested in a region so near to the wires, that we have there a double justification in assuming a stationary field.

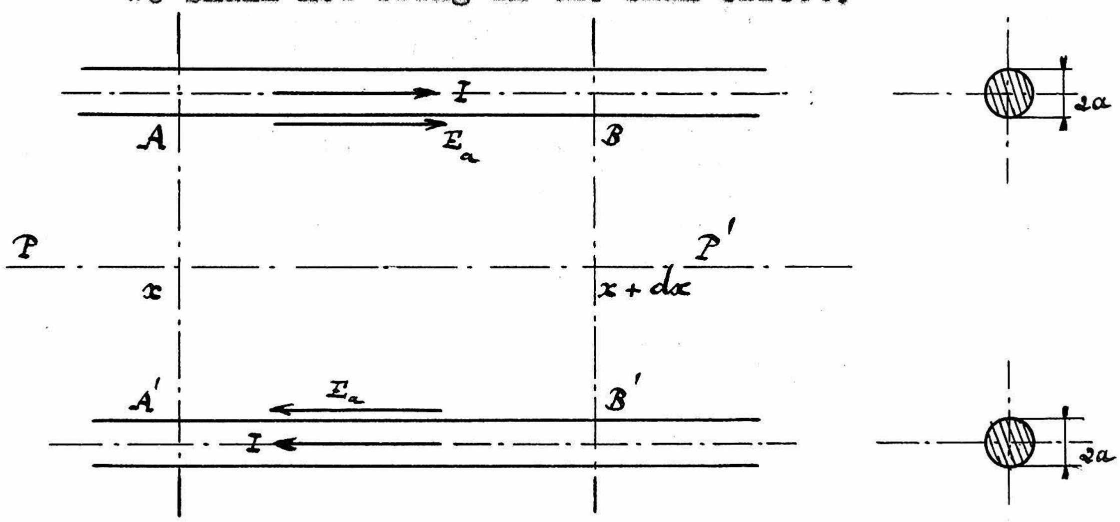
We conclude, therefore, that <u>it is possible with sufficient accuracy, to handle L and C as constants</u>, exactly as for the steady state.

<u>The Telegraphists' equations</u>.

As we know, the theory of propagation of electrical disturbances does not give a correct picture of what happens near to the wave front. This theory was developed by Heaviside and Poincaré. Heaviside was the first to reach a solution for the transient voltage and current distribution in the case of an initial rectangular voltage wave. His theory, entirely operational, supposes an instantaneous penetration of the current in the wires, i.e. a uniform distribution throughout their cross section.

On the same hypothesis, Poincaré developed the classical solution of the telegraphists' equations. The classical solution introduces constants which can only be determined by the boundary conditions. This fact does not correspond to the physical truth, since the waves ignore the terminal conditions until they reach the end of the line. Therefore, the operational method, in which the waves are considered on their trip along the line is the only natural method of investigation of disturbances.

We shall now bring in the skin effect.



Let V be the difference of potential between A and A'.

$$V = \int_A^{A'} E_l \, dl$$

also $$V + \frac{\partial V}{\partial x} dx = \int_B^{B'} E_l \, dl$$

We now apply the first Maxwell law to the loop ABB'A'A, outside of the wires:

$$-L \, dx \frac{\partial I}{\partial t} = \left(V + \frac{\partial V}{\partial x} dx\right) - V + 2 E_a \, dx$$

I being the total current in the section

E_a the electric field at the periphery of the wires of radius a

L^e the self inductance for the pair of conductors

The term $2E_a dx$ replaces the known term $R dx . I$, since we do not know anything about R, due to skin effect.

Thus:
$$-\frac{\partial V}{\partial x} = 2E_a + L\frac{\partial I}{\partial t} \qquad (A)$$

We have also the continuity equation:

$$I_x dt = I_{x+dx} . dt + C dx \frac{\partial V}{\partial t} dt + G dx . V dt$$

or
$$-\frac{\partial I}{\partial x} = GV + C\frac{\partial V}{\partial t} \qquad (B)$$

We have a system of two equations (A) and (B), with three unknowns: V, I and E_a.

The additional equation will be given by the study of the electromagnetic field, by the use of Maxwell's equations.

We shall establish an operational relation between the tangential electric field and the total current in the section; it will be the third equation.

The electric and magnetic field equations.

1. ε and μ are supposed constant.
2. There are no free electric charges.

Thus: $$-\nabla\times\overline{E} = \frac{\mu}{c}\frac{\partial\overline{H}}{\partial t} \qquad (5)$$

$$\nabla\times\overline{H} = 4\pi\left[\sigma c\,\overline{E} + \frac{\varepsilon}{4\pi c}\frac{\partial\overline{E}}{\partial t}\right] \qquad (6)$$

$$\nabla.\overline{E} = 0 \qquad (7)$$

$$\nabla.\overline{H} = 0 \qquad (8)$$

Let us take the curl of both equations (5) and (6):

$$\nabla\times\nabla\times\overline{E} = -\frac{\mu}{c}\frac{\partial}{\partial t}\left[\nabla\times\overline{H}\right]$$

$$\nabla\times\nabla\times\overline{H} = 4\pi\left[\sigma c\,.\nabla\times\overline{E} + \frac{\varepsilon}{4\pi c}\frac{\partial}{\partial t}(\nabla\times\overline{E})\right]$$

We have the general vectorial relation:

$$\nabla\times\nabla\times\overline{A} = \nabla(\nabla.\overline{A}) - \nabla^2\overline{A}$$

If $\overline{A}$ denotes either $\overline{E}$ or $\overline{H}$: $\nabla.\overline{A} = 0$

thus:
$$\begin{cases} \nabla^2\overline{E} = 4\pi\sigma\mu\dfrac{\partial\overline{E}}{\partial t} + \dfrac{\varepsilon\mu}{c^2}\dfrac{\partial^2\overline{E}}{\partial t^2} \\ \nabla^2\overline{H} = 4\pi\sigma\mu\dfrac{\partial\overline{H}}{\partial t} + \dfrac{\varepsilon\mu}{c^2}\dfrac{\partial^2\overline{H}}{\partial t^2} \end{cases}$$

The terms in $\frac{\partial}{\partial t}$ are <u>diffusion terms</u>.

The terms in $\frac{\partial^2}{\partial t^2}$ are <u>propagation terms</u>.

1. <u>Outside of the wires,</u> $\sigma = 0$ also $\varepsilon = \mu = 1$

$$\begin{cases} \nabla^2\overline{E} = \dfrac{1}{c^2}\dfrac{\partial^2\overline{E}}{\partial t^2} \\ \nabla^2\overline{H} = \dfrac{1}{c^2}\dfrac{\partial^2\overline{H}}{\partial t^2} \end{cases} \qquad (9)$$

We have a pure propagation.

2. Inside of the wires, $\sigma \neq 0$

We may now neglect the displacement current term. This is an exdellent approximation, as can be shown by what follows. Let us assume for $\overline{E}$ and $\overline{H}$ a cissoidal oscillation $\varepsilon^{j\omega t}$, then:

$$\frac{\partial E}{\partial t} = j\omega\, \varepsilon^{j\omega t} \qquad \text{or} \qquad \left|\frac{\partial E}{\partial t}\right| = \omega |E|$$

We have also: $$\left|\frac{\partial^2 E}{\partial t^2}\right| = \omega^2 |E|$$

and $$4\pi\sigma\mu \left|\frac{\partial E}{\partial t}\right| = 4\pi\sigma\mu\,\omega |E|$$

$$\frac{\varepsilon\mu}{c^2}\left|\frac{\partial^2 E}{\partial t^2}\right| = \frac{\varepsilon\mu}{c^2}\,\omega^2 |E|$$

The condition that the propagation term be smaller that the diffusion term is:

$$\frac{\varepsilon\mu}{c^2}\,\omega^2 E < 4\pi\sigma\mu\omega E \qquad \text{or} \qquad \varepsilon\omega < 4\pi\sigma c^2$$

For conductors, ε, though unknown, is small

ω at industrial frequencies, and even voice frequencies, is less that $2\pi \times 1000$

σ conductivity of conductors is always large.

For copper, the resistivity in e.m.u. is 1800.

Thus, σ (in e.m.u.) $= \dfrac{1}{1800}$

and $\sigma c^2 = \dfrac{9.10^{20}}{1800} = 5.14 \times 10^{17}$ (conductivity in e.s.u.)

For iron, the resistivity in e.m.u. is 10 000.

Thus σ (in e.m.u.) $= \frac{1}{10000}$

and $\sigma c^2 = \frac{9.10^{20}}{10000} = .9 \times 10^{17}$ (conductivity in e.s.u.)

For copper, $\varepsilon\omega$ should be much less than 6.45×10^{18} which is the case $\varepsilon\omega << 4\pi\sigma c^2$

For iron, $4\pi\sigma c^2 = 1.13 \times 10^{18}$

$$\varepsilon\omega << 1.13 \times 10^{18}$$

This justifies our approximation. We neglect the displacement current by putting simply $\varepsilon = 0$, wherever it may appear.

Thus:
$$\nabla^2\overline{E} = 4\pi\sigma\mu \frac{\partial\overline{E}}{\partial t}$$
$$\nabla^2\overline{H} = 4\pi\sigma\mu \frac{\partial\overline{H}}{\partial t} \qquad (10)$$

The physical meaning of these equations is that if any modification occurs at any given point, its effects are felt at the very same instant at any other point. Diffusion phenomenon is therefore a particular case of propagation with an infinite velocity.

The general formula for the Laplacian of a vector in curvilinear coordinates, applied to the cylindrical coordinates, is:

$$\nabla^2\overline{A} = \frac{1}{z}\frac{\partial}{\partial z}\left(z\frac{\partial\overline{A}}{\partial z}\right) + \frac{1}{z^2}\frac{\partial^2\overline{A}}{\partial\alpha^2} + \frac{\partial^2\overline{A}}{\partial x^2}$$

Now, we suppose a cylindrical symmetry.

Therefore: $\nabla^2 \bar{A} = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \bar{A}}{\partial r}\right)$

Consequently, outside of the wires:

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \bar{E}}{\partial r}\right) = \frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \bar{H}}{\partial r}\right) = \frac{1}{c^2} \frac{\partial^2 \bar{H}}{\partial t^2}$$

exact equations

and, inside of the wires:

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \bar{E}}{\partial r}\right) = 4\pi\sigma\mu \frac{\partial \bar{E}}{\partial t}$$

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \bar{H}}{\partial r}\right) = 4\pi\sigma\mu \frac{\partial \bar{H}}{\partial t}$$

very approximate, since: $\varepsilon\omega << 4\pi\sigma c^2$

Determination of $\bar{E}$ and $\bar{H}$ inside of the wire.

A. First method.

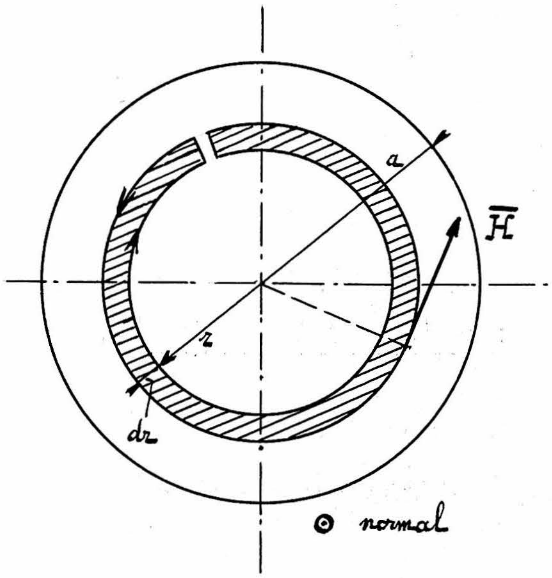
The return wire for the current is supposed to be at an infinite distance.

Our two Maxwell's equations are :

$$\nabla \times \bar{E} = -\frac{\mu}{c} \frac{\partial \bar{H}}{\partial t}$$

$$\nabla \times \bar{H} = 4\pi\sigma c \bar{E} + \frac{\varepsilon}{c} \frac{\partial \bar{E}}{\partial t}$$

Let us consider the following closed path, consisting of the circles of radii r and r dr, plus two elements of length dr. We apply Stokes' theorem to that path.



$$\oint \overline{H}\cdot\overline{ds} = \int_S \nabla\times\overline{H}\cdot\overline{dS}$$

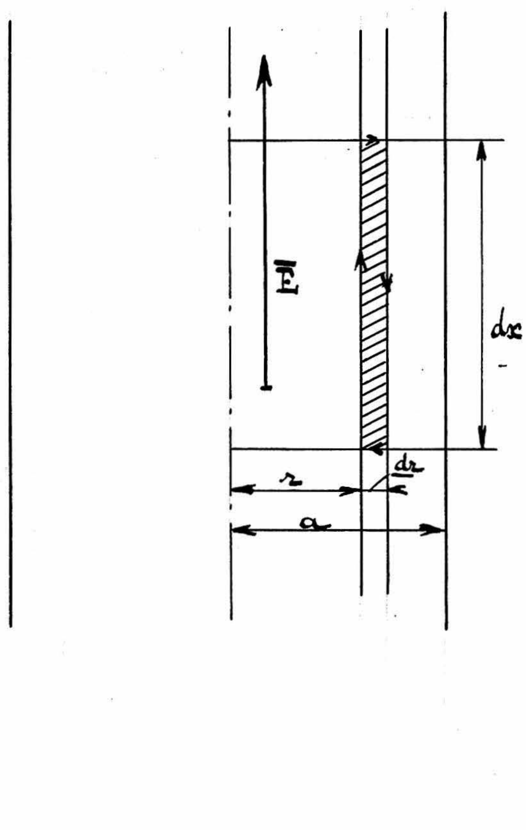
The positive normal to this surface points toward us, in accordance with the direction of circulation along the path. The given direction of $\overline{H}$ corresponds also to a current pointing toward us.

Therefore:

$$-H\cdot 2\pi r + \left[H\cdot 2\pi r + \frac{\partial}{\partial r}(2\pi r H)\,dr\right] = \nabla\times\overline{H}\cdot dS$$

$$= \left(4\pi\sigma c E + \frac{\varepsilon}{c}\frac{\partial E}{\partial t}\right)\cdot 2\pi r\cdot dr$$

or
$$\frac{1}{r}\frac{\partial}{\partial r}(rH) = 4\pi\sigma c E + \frac{\varepsilon}{c}\frac{\partial E}{\partial t} \qquad (11)$$



Another closed path is shown here. The supposed direction for $\overline{E}$ implies a magnetic field into the paper. In order that the positive normal to the circuit be in the same direction as $\overline{H}$, we have to consider the direction of circulation as shown by arrows on the sketch.

$$\oint \overline{E}\,.\,\overline{ds} = \int \nabla \times \overline{E}\,.\,\overline{dS}$$

$$E.dx - \left[E + \frac{\partial E}{\partial r}\,dr\right]dx = \nabla \times \overline{E}\,.\,dS = -\frac{\mu}{c}\,\frac{\partial H}{\partial t}\,dx\,dr$$

or
$$\frac{\partial E}{\partial r} = \frac{\mu}{c}\,\frac{\partial H}{\partial t} \qquad (12)$$

The equations (11) and (12) can then be transformed as follows:

$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial H}{\partial t}\right) = 4\pi\sigma c\,\frac{\partial E}{\partial t} + \frac{\varepsilon}{c}\,\frac{\partial^2 E}{\partial t^2}$$

or
$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial E}{\partial r}\right) = 4\pi\sigma\mu\,\frac{\partial E}{\partial t} + \frac{\varepsilon\mu}{c^2}\,\frac{\partial^2 E}{\partial t^2}$$

also, by (11):
$$\frac{\partial H}{\partial r} + \frac{1}{r}\,H = 4\pi\sigma c\,E + \frac{\varepsilon}{c}\,\frac{\partial E}{\partial t}$$

Thus, by differentiation with respect to r:

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\,\frac{\partial H}{\partial r} - \frac{1}{r^2}\,H = 4\pi\sigma c\,\frac{\partial E}{\partial r} + \frac{\varepsilon}{c}\,\frac{\partial}{\partial t}\left(\frac{\partial E}{\partial r}\right)$$

or
$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial H}{\partial r}\right) - \frac{1}{r^2}\,H = 4\pi\sigma\mu\,\frac{\partial H}{\partial t} + \frac{\varepsilon\mu}{c^2}\,\frac{\partial^2 H}{\partial t^2}$$

Neglecting the displacement current inside of the wires, our equations become:

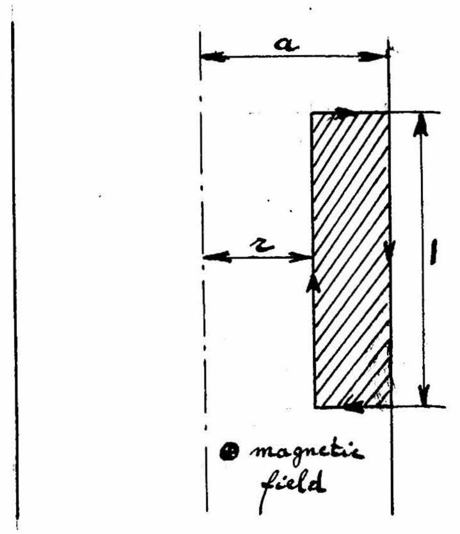
$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial E}{\partial r}\right) = 4\pi\sigma\mu\,\frac{\partial E}{\partial t} \qquad (13)$$

$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial H}{\partial r}\right) - \frac{1}{r^2}\,H = 4\pi\sigma\mu\,\frac{\partial H}{\partial t} \qquad (14)$$

B. Second method.

Let λ be the conduction current density at a distance r from the center, λ_a at the periphery.

We shall neglect the displacement current density, i.e. we shall consider the only λ density.



We consider the finite circuit between the radii r and a. By Maxwell's first law:

$$\frac{\lambda}{\sigma} - \frac{\lambda_a}{\sigma} = -\frac{\partial \varphi}{\partial t} \quad (15)$$

φ being the magnetic flux per unit length between the radii r and a.

We have also:

$$-\frac{\partial \varphi}{\partial r} = -\mu H \quad (16)$$

where the minus sign is due to the fact that if r increases, φ decreases.

Now let i be the total current in the section of radius r : $di = \lambda . 2\pi r\, dr$ or $\frac{\partial i}{\partial r} = 2\pi r \lambda$ (17)

Furthermore:

$$H = \frac{2i}{r} \quad (18)$$

and $\lambda = \sigma c E$ (19) (E in e.s.u., σ and λ in e.m.u)

The five equations (15) up to (19) contain six parameters : E , H , λ , I , φ , and r , It is therefore possible to eliminate four of them and to find a relation between the two other parameters. We want the relations between E and r, and H and r.

From (18): $\mu \frac{\partial H}{\partial t} = \frac{2\mu}{r} \frac{\partial i}{\partial t}$ (18')

and from (16): $\mu \frac{\partial H}{\partial t} = - \frac{\partial^2 \varphi}{\partial r \partial t}$

But: $\frac{1}{\sigma} \cdot \frac{\partial \lambda}{\partial r} = - \frac{\partial^2 \varphi}{\partial r \partial t}$ from (15)

thus $\frac{2\mu}{r} \cdot \frac{\partial i}{\partial t} = \frac{1}{\sigma} \cdot \frac{\partial \lambda}{\partial r}$ or $\frac{\partial i}{\partial t} = \frac{r}{2\mu\sigma} \cdot \frac{\partial \lambda}{\partial r}$

Differentiating with respect to r :

$$\frac{\partial^2 i}{\partial t \partial r} = \frac{1}{2\mu\sigma} \frac{\partial}{\partial r}\left(r \frac{\partial \lambda}{\partial r}\right)$$

but, by (17): $\frac{\partial^2 i}{\partial r \partial t} = 2\pi r \frac{\partial \lambda}{\partial t}$

Consequently: $2\pi r \frac{\partial \lambda}{\partial t} = \frac{1}{2\mu\sigma} \frac{\partial}{\partial r}\left(r \frac{\partial \lambda}{\partial r}\right)$

or $\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial E}{\partial r}\right) = 4\pi\sigma\mu \frac{\partial E}{\partial t}$ (20)

identical with (13).

To find H, we proceed as follows:

$\frac{\partial i}{\partial t} = \frac{r}{2\mu\sigma} \frac{\partial \lambda}{\partial r}$ but $i = \frac{r}{2} H$

thus $\frac{r}{2} \frac{\partial H}{\partial t} = \frac{r}{2\mu\sigma} \cdot \sigma c \frac{\partial E}{\partial r}$

or $\frac{\partial H}{\partial t} = \frac{c}{\mu} \cdot \frac{\partial E}{\partial r}$

identical with (12).

We have also: $\frac{\partial H}{\partial t} = \frac{1}{\mu\sigma} \cdot \frac{\partial \lambda}{\partial r}$

We must evaluate $\frac{\partial \lambda}{\partial r}$.

$$\lambda = \frac{1}{2\pi r} \cdot \frac{\partial i}{\partial r} \qquad \text{from (17)}$$

or $$\lambda = \frac{1}{4\pi r} \frac{\partial}{\partial r}\left(r H\right) = \frac{1}{4\pi}\left[\frac{H}{r} + \frac{\partial H}{\partial r}\right] \text{ from (14)}$$

thus $$\frac{\partial \lambda}{\partial r} = \frac{1}{4\pi}\left[\frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial H}{\partial r} - \frac{1}{r^2} H\right]$$

or $$\frac{\partial \lambda}{\partial r} = \frac{1}{4\pi}\left[\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial H}{\partial r}\right) - \frac{1}{r^2} H\right]$$

Finally: $$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial H}{\partial r}\right) - \frac{1}{r^2} H = 4\pi\mu\sigma \frac{\partial H}{\partial t}$$

identical with (14).

We arrive thus at the same equations by a completely different method. To determine E and H, it is not necessary to solve both equations (13) and (14). We can solve for E, for instance, and determine H from the equation (12).

<u>Solution of the electric field equation.</u>

We have: $$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r} \frac{\partial E}{\partial r} = 4\pi\sigma\mu \frac{\partial E}{\partial t}$$

This is a <u>Bessel-type equation of zeroth order</u>.

We use now the symbolic representation $p = \frac{d}{dt}$

It is customary, in all skin effect problems, to introduce the quantity: $\alpha = \sqrt{4\pi\sigma\mu}$

then: $$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r} \frac{\partial E}{\partial r} = \alpha^2 p E$$

Let X be a new variable related to r by: $X = j r \alpha \sqrt{p}$

and let : $Y = r\alpha\sqrt{p}$

so that: $X = jY$

Our equation becomes:

$$\frac{\partial^2 E}{\partial X^2} + \frac{1}{X} \cdot \frac{\partial E}{\partial X} + E = 0 \qquad (21)$$

This is a Bessel equation under its usual form.

The most general solution of this equation is:

$$E \doteqdot A J_0(X) + B K_0(X)$$

where A and B are constants of integration and J_0 and K_0 are Bessel functions of the zeroth order and <u>first and second kind</u> respectively.

The electric field is a continuous function of r or X, and since $K_0(X)$ is discontinuous at $X = 0$, it follows that $B = 0$, so that:

$$E \doteqdot A J_0(X) = A J_0(jY) = A I_0(Y) \qquad (22)$$

where $$I_0(Y) = \sum_{K=0}^{\infty} \frac{Y^{2K}}{2^{2K} (K!)^2}$$

<u>Relation between $E_\sim$ and I .</u>

The electric field E is thus expressed by the operational equation (22). The equation (12) can be written:

$$H \doteqdot \frac{c}{\mu p} \cdot \frac{\partial E}{\partial z} = \frac{c}{\mu p} \cdot \frac{\partial E}{\partial X} \cdot j\alpha\sqrt{p}$$

or $$H \doteqdot \frac{c}{\mu p} \cdot j\alpha\sqrt{p} \cdot A J_0'(X)$$

Let $\chi = \chi_a$ and $\Upsilon = \Upsilon_a$ for $r = a$, i.e. at the periphery.

We know that : $\mathcal{H}_a = \frac{2I}{a}$, where I is the total current in the section.

Thus : $$I \doteq \frac{a}{2} \cdot \frac{c}{\mu\rho} \cdot j\alpha\sqrt{\rho} \cdot A\, J_0'(\chi_a)$$

or $$I \doteq -2\pi\sigma a^2 c\, A \cdot \frac{J_0'(\chi_a)}{\chi_a}$$

Let R denote the d.c. resistance per unit length of the double line, the two wires being placed in series:

$$\mathcal{R} = \frac{1}{\sigma} \cdot \frac{2}{\pi a^2} \qquad (23) \qquad \text{in e.m.units.}$$

so that $$I \doteq -\frac{4c}{\mathcal{R}} \cdot A \cdot \frac{J_0'(\chi_a)}{\chi_a}$$

also : $$2E_a \doteq 2A \cdot J_0(\chi_a)$$

thus : $$\frac{2E_a}{I} \doteq -\mathcal{R} \cdot \frac{\chi_a}{2c} \cdot \frac{J_0(\chi_a)}{J_0'(\chi_a)} \qquad (24)$$

<u>Notice</u>: Before going any further, we may check on the dimensions of this last expression (Units-Jeans.p.530)

$$[\mathcal{R}] = [L\,T^{-1} L^{-1}] = [T^{-1}] \qquad \text{in e.m.u.}$$

$$\left[\chi_a \frac{J_0(\chi_a)}{J_0'(\chi_a)}\right] = [\chi_a^2] = [L^2 L^{-2} T\, T^{-1}] = 1 \qquad \text{in e.m.u.}$$

$$[I] = [M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}] \qquad \text{in e.m.u.}$$

$$[c] = [L\,T^{-1}]$$

then: $$\left[\mathcal{R} \cdot \frac{\chi_a}{c} \cdot \frac{J_0(\chi_a)}{J_0'(\chi_a)} \cdot I\right] = \left[\frac{T^{-1} M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}}{L\,T^{-1}}\right] = \left[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}\right]$$

which represents $[E_e]$ in e.s.u. of potential per unit length. This shows the consistency of the formula (24) in which magnetic quantities are expressed in magnetic units, and electric quantities in electric units.

Transformation of the equation (24).

We have the general algebraic relation:

$$\frac{d}{dX}\left[X^{-n} J_n(X)\right] = -X^{-n} J_{n+1}(X)$$

thus $$\frac{d}{dX} J_0(X) = J_0'(X) = -J_1(X)$$

and $$\frac{2E_a}{I} \doteqdot R \cdot \frac{X_a}{2c} \cdot \frac{J_0(X_a)}{J_1(X_a)}$$

In the " Funktionentafeln" of E. Jahnke and F. Emde, we have a general formula :

$$j^{-n} J_n(jx) = \frac{\varepsilon^x}{\sqrt{2\pi x}} S_n(-2x) \qquad (25)$$

where $S_n(x)$ is a series function defined as follows :

$$S_n(x) = 1 + \frac{4n^2-1}{1!\,4x} + \frac{(4n^2-1)(4n^2-9)}{2!\,(4x)^2} + \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!\,(4x)^3} + \cdots$$

or $$S_n(x) = 1 + \sum_{\nu=1}^{\nu=\infty} \frac{(4n^2-1^2)(4n^2-3^2)\cdots\left[4n^2-(2\nu-1)^2\right]}{\nu!\,(4x)^\nu}$$

For $n=0$ $$J_0(jY) = \frac{\varepsilon^Y}{\sqrt{2\pi Y}} S_0(-2Y)$$

$n=1$ $$J_1(jY) = j\frac{\varepsilon^Y}{\sqrt{2\pi Y}} S_1(-2Y)$$ with $Y = 2\alpha\sqrt{p}$

The series $S_0(-2Y)$ and $S_1(-2Y)$ are given by :

$$S_0(-2Y) = 1 + \frac{1}{1!\,8Y} + \frac{1^2.3^2}{2!(8Y)^2} + \frac{1^2.3^2.5^2}{3!(8Y)^3} + \cdots\cdots$$

$$S_1(-2Y) = 1 - \frac{3}{1!\,8Y} + \frac{3.5}{2!(8Y)^2} - \frac{3.5.21}{3!(8Y)^3} - \cdots\cdots$$

By dividing S_0 by S_1, we easily derive the new series development :

$$\frac{S_0(-2Y)}{S_1(-2Y)} = 1 + \frac{1}{2Y} + \frac{3}{8Y^2} + \frac{3}{8Y^3} + \cdots\cdots$$

We notice also that :

$$\lim_{Y\to\infty} J_0(jY) = \frac{\varepsilon^Y}{\sqrt{2\pi Y}} \qquad \text{and} \qquad \lim_{Y\to\infty} J_1(jY) = j\,\frac{\varepsilon^Y}{\sqrt{2\pi Y}}$$

Finally, we have :

$$\frac{2E_a}{I} \doteq \frac{R}{2c}\cdot jY_a\cdot\frac{J_0(jY_a)}{J_1(jY_a)} = \frac{R}{2c}\cdot Y_a\cdot\frac{S_0(-2Y_a)}{S_1(-2Y_a)}$$

or $$\frac{2E_a}{I} \doteq \frac{R}{2c}\left[Y_a + \frac{1}{2} + \frac{3}{8Y_a} + \frac{3}{8Y_a^2} + \cdots\right] \quad (26)$$

Explicitly:

$$\frac{2E_a}{I} \doteq \frac{R}{2c}\left[a\alpha\sqrt{p} + \frac{1}{2} + \frac{3}{8a\alpha\sqrt{p}} + \frac{3}{8a^2\alpha^2 p} + \cdots\right]$$

or $$\frac{2E_a}{I} \doteq \frac{R}{4c} + \frac{R}{2c}\,a\alpha\sqrt{p} + \frac{1}{c}\,P_1\left(\frac{1}{\sqrt{p}}\right) \quad (27)$$

where $P_1\left(\frac{1}{\sqrt{p}}\right)$ is an analytic function of $\frac{1}{\sqrt{p}}$.

<u>Variation of the difference of potential V between the wires.</u>

Using the symbolic operator p, we have now :

$$- \frac{dV}{dx} = 2 E_a + L p I$$

$$- \frac{dI}{dx} = (G + Cp) V$$

$$2E_a = \left[\frac{R}{4c} + \frac{R}{2c} a\alpha\sqrt{p} + \frac{1}{c} P_1\left(\frac{1}{\sqrt{p}}\right)\right] I$$

We have thus three equations and three unknowns, and proceed to solve for V.

But, if V is expressed in volts, I in amperes, and so on, we must express $2E_a$ in volts per cm., which corresponds to expressing R in ohms.

Then :

$$2E_a = \left[\frac{R}{4} + \frac{R}{2} a\alpha\sqrt{p} + P_1\left(\frac{1}{\sqrt{p}}\right)\right] I \qquad \text{in volts cm}$$

Incidentally, $(a\alpha\sqrt{p})$ is dimensionless and may be evaluated in any system of units.

Thus :

$$- \frac{dV}{dx} = \left[pL + \frac{R}{4} + \frac{R}{2} a\alpha\sqrt{p} + P_1\left(\frac{1}{\sqrt{p}}\right)\right] I$$

$$- \frac{dI}{dx} = \left[G + Cp\right] V$$

and $$\frac{d^2V}{dx^2} = \gamma^2 V \qquad (28)$$

with $$\gamma^2 = (Cp + G)\left(pL + \frac{R}{4} + \frac{R}{2} a\alpha\sqrt{p} + P_1\left(\frac{1}{\sqrt{p}}\right)\right) \qquad (29)$$

Let us introduce $v = \frac{1}{\sqrt{LC}}$. As we shall see, v is the velocity of propagation of the waves along the line.

Consequently:

$$\gamma = \frac{p}{v}\sqrt{\left(1 + \frac{G}{Cp}\right)\left(1 + \frac{R}{4L}\cdot\frac{1}{p} + \frac{R}{2L}\cdot\frac{a\alpha}{\sqrt{p}} + \frac{1}{pL} P_1\left(\frac{1}{\sqrt{p}}\right)\right)} \qquad (30)$$

where : $P_1\left(\frac{1}{\sqrt{p}}\right) = \frac{R}{2}\left[\frac{3}{8a\alpha\sqrt{p}} + \frac{3}{8a^2\alpha^2 p} + \cdots\cdots\right]$

The equation (30) may be transformed into:

$$\gamma = \frac{p}{v} + \beta + \delta\sqrt{p} + P_2\left(\frac{1}{\sqrt{p}}\right) \quad (31)$$ (See note 1)

with $\beta = \frac{R}{8}\sqrt{\frac{C}{L}}\left(1 - \frac{R}{L}\cdot\frac{a^2\alpha^2}{4}\right) + \frac{G}{2}\sqrt{\frac{L}{C}}$ (32)

and $\delta = \frac{R}{4}\sqrt{\frac{C}{L}}\, a\alpha$ (33)

The general solution of the equation (28) is :

$$V \doteq M\varepsilon^{-\gamma x} + N\varepsilon^{+\gamma x}$$

where γ is supposed to be constant.

It is logical to take $N = 0$, since the voltage will not increase indefinitely with large x.

M is an unknown function of time, say E(t),

then : $V \doteq \varepsilon^{-\gamma x} E(t)$ (34)

It is purposely that we write $\varepsilon^{-\gamma x}.E(t)$ and not $E(t).\varepsilon^{-\gamma x}$, since $\varepsilon^{-\gamma x}$ is an ~~operand~~ operator operating on the time function $E(t)$.

According to the value of γ , we write :

$$V \doteq \varepsilon^{-\frac{p}{v}x} . \varepsilon^{-\beta x} . \varepsilon^{-\delta\sqrt{p}\,x} . \varepsilon^{-x P_2\left(\frac{1}{\sqrt{p}}\right)} . E(t)$$

Let : $E_1(t) = \varepsilon^{-x P_2\left(\frac{1}{\sqrt{p}}\right)} . E(t)$

$E_1(t)$ being, of course, an unknown function of time.

Thus : $V \doteq \varepsilon^{-\frac{p}{v}x} . \varepsilon^{-\beta x} . \varepsilon^{-\delta\sqrt{p}\,x} . E_1(t)$

Now : $\varepsilon^{-\frac{p}{v}x} E_1(t) = E_1\left(t - \frac{x}{v}\right)$ good for $t \geqslant \frac{x}{v}$, according to a well known <u>operational transfer formula</u> (See note 2).

Thus : $V = \varepsilon^{-\beta x} . \varepsilon^{-\delta\sqrt{p}\,x} . E_1\left(t - \frac{x}{v}\right)$

In the note 3, we prove the following operational formula :

$$\varepsilon^{-\delta\sqrt{p}.x} f(t) \doteq \frac{2}{\sqrt{\pi}} \int_{\frac{\delta x}{2\sqrt{t}}}^{\infty} f\left(t - \frac{\delta^2 x^2}{4z^2}\right) \varepsilon^{-z^2} dz$$

Thus :

$$V = \frac{2}{\sqrt{\pi}} \varepsilon^{-\beta x} \int_{\frac{\delta x}{2\sqrt{t - \frac{x}{v}}}}^{\infty} E_1\left(t - \frac{x}{v} - \frac{\delta^2 x^2}{4z^2}\right) \varepsilon^{-z^2} dz \qquad (35)$$

valid only for $t \geqslant \frac{x}{v}$.

For $t < \frac{x}{v}$, $V = 0$. This shows clearly that v is the <u>velocity of propagation</u> along the line.

In the expression (35), everything is known but the time function $E_1(t)$. But, let $x = 0$:

$$V = \frac{2}{\sqrt{\pi}} E_1(t) \int_0^{\infty} \varepsilon^{-z^2} dz = E_1(t)$$

From (34), at $x = 0$, $V = E(t)$. Thus : $E_1(t) = E(t)$

Our expression (35) is entirely determined. The constants β, δ and v can be computed and $E_1(t)$ is the applied voltage function at the beginning of the line at $t = 0$. For any type of voltage applied at $x = 0$, the formula (35) gives us the means of evaluating the difference of potential between the wires as a function of time at a given point x.

Discussion of the voltage equation (35).

We shall discuss first the voltage equation for a rectangular applied voltage at the beginning of the line. This is the most unfavorable case; thus, the conclusions at which we shall arrive will put us on the safe side in designing protective apparatus.

For $t = \frac{x}{v}$, $V = 0$. Therefore, the difference of potential at the front of the wave is zero. This property is quite general and does not depend on the special type of applied voltage $E_1(t)$.

We have established the formula (35) on the basis that the skin effect in one wire depends on the wire itself, that there is no action from the other conductors. Therefore, the formula (35) holds for a three phase line where V and $E_1(t)$ are either composed voltages or phase voltages.

Now, however, L and C are the industrial self inductance and capacitance (note 4). In particular, we recall that the self inductance of the three phase line is exactly half of that of the double wire line.

For a uniform distribution of current in the wires, we know that, in the case of a double wire line :

$L = \mu_i + 4\mu_e \ln \frac{D}{2}$ per cm. of length, where μ_i and μ_e are the internal and external permeability. The first term is due to the self induction in both wires, the second term is introduced by the self induction between the wires (note 4).

Consequently, we may doubt the accuracy of this value of L when there is a pronounced skin effect; even more, doubt that L be constant at all. But, if we look back at the way in which we obtained our equation (A), we see that L is the self inductance for AA'BB', i.e. <u>exteriorly to the wires.</u> Therefore, in this case : $L = 4\mu_e \ln \frac{D}{r}$ For iron, where μ_i is very large, this remark has its full importance.

<u>Deformation of an initially rectangular wave.</u>

This is a particular case of our equation (35), in which $E_1(t)$ is a constant, say V_0 . Therefore :

$$V = \varepsilon^{-\beta x} \cdot V_0 \cdot \frac{2}{\sqrt{\pi}} \int_{\frac{\delta x}{2\sqrt{t - \frac{x}{v}}}}^{\infty} \varepsilon^{-z^2} dz$$

or
$$V = V_0 \cdot \varepsilon^{-\beta x} \left[1 - \operatorname{erf} \left(\frac{\delta x}{2\sqrt{t - \frac{x}{v}}} \right) \right] \qquad (36)$$

The larger β and δ are, the smaller V is at a given point. We see that δ is <u>directly proportional to</u> $\sqrt{\frac{C}{L}}$, <u>i.e. the surge admittance.</u> Thus, δ will be larger for a cable than for an aerial line. Consequently, the front of a wave in a cable flattens down much more quickly that along an aerial line.

Let : $vt - x = \eta$

$$\frac{1}{2}\sqrt{v} \cdot \delta \cdot x = \xi$$

then : $V = V_0 \cdot \varepsilon^{-\beta x}\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{z_1} \varepsilon^{-z^2} dz\right]$

where : $z_1 = \frac{\xi}{\sqrt{\eta}}$

A. <u>Variation of V in function of time at a given point x.</u>

x, and thus ξ, is a constant.

Consequently: $d\eta = v\,dt$

and : $\frac{dV}{d\eta} = V_0 \cdot \varepsilon^{-\beta x} \cdot \varepsilon^{-\frac{\xi^2}{\eta}} \cdot \frac{\xi}{\eta^{\frac{3}{2}}} \cdot \frac{1}{\sqrt{\pi}}$

The maximum gradient of potential is obtained for:

$\frac{d^2V}{d\eta^2} = 0$ or $-\frac{3}{2}\cdot\frac{1}{\eta^{\frac{5}{2}}} + \frac{\xi^2}{\eta^{\frac{7}{2}}} = 0$ i.e. $\eta = \frac{2}{3}\xi^2$

This can be written as : $vt - x = \frac{2}{3}\cdot\frac{1}{4}\,v\,\delta^2 x^2$

and $t = \frac{x}{v} + \frac{1}{6}\,\delta^2 x^2$ (37)

<u>At a given point x, the gradient of potential is maximum $(\frac{1}{6}\delta^2 x^2)$ seconds after the impinging wave has struck the point.</u> Explicitly, this time is : $\frac{1}{6\pi}\cdot\frac{\mu}{\sigma}\cdot\frac{x^2}{a^2}\cdot\frac{C}{L}$ (C, L, σ, μ in e.m.u. as we decided).

The farther we are along the line, the longer that time is; this time increases with the square of the distance to the origin.

The corresponding maximum slope is then :

$$\left(\frac{dV}{d\eta}\right)_{max} = V_0 \cdot \varepsilon^{-\beta x} \cdot \varepsilon^{-\frac{3}{2}} \cdot \frac{\xi}{(\frac{2}{3})^{\frac{3}{2}}\xi^3} \cdot \frac{1}{\sqrt{\pi}}$$

$$\left(\frac{dV}{d\eta}\right)_{max} = V_0 \cdot \varepsilon^{-\beta x} \cdot \varepsilon^{-\frac{3}{2}} \cdot \frac{\cancel{\xi^4}}{\left(\frac{2}{3}\right)^{\frac{3}{2}} \cancel{\xi^3}\, v\delta^2 x^2} \cdot \frac{1}{\sqrt{\pi}}$$

or $$\left(\frac{dV}{dt}\right)_{max} = V_0 \cdot \varepsilon^{-\beta x} \cdot \frac{0.924}{\delta^2 x^2} \qquad (38)$$

<u>At a given point x, the steepest slope of the voltage wave is inversely proportional to the square of the distance to the origin.</u>

We see that $\left(\frac{dV}{dt}\right)_{max}$ is proportional to $\frac{1}{\delta^2}$, or to $\frac{L}{C}$; the slope is therefore much smaller for a cable than for an aerial line.

The value of V, <u>when the slope is maximum</u>, is:

$$V = V_0 \cdot \varepsilon^{-\beta x}\left[1 - erf\sqrt{\frac{3}{2}}\right] \quad \text{or} \quad V = 0.083 \cdot V_0 \cdot \varepsilon^{-\beta x} \qquad (39)$$

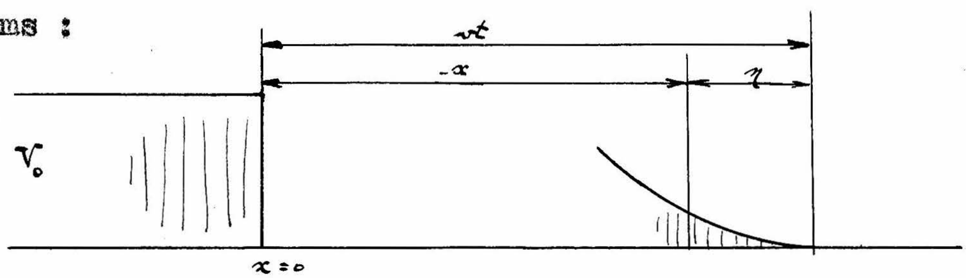
From the expression for $\frac{dV}{d\eta}$, we see that, at time $t = \frac{x}{v}$ or $\eta = 0$:

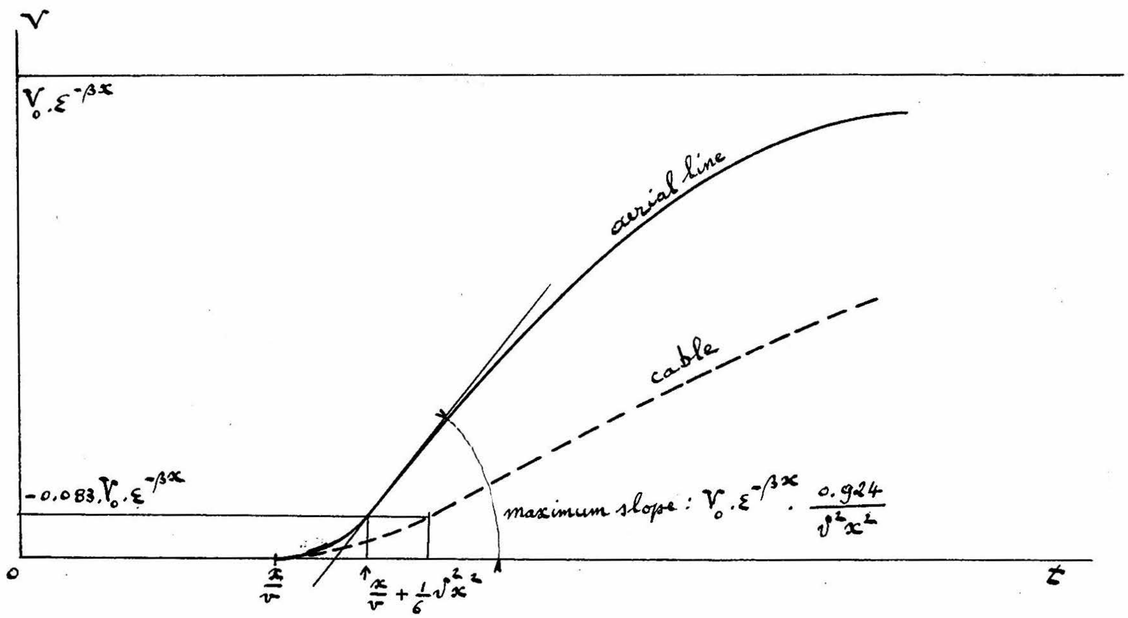
$$\left(\frac{dV}{d\eta}\right)_{\eta=0} = V_0\, \varepsilon^{-\beta x} \frac{\xi}{\sqrt{\pi}} \lim_{\eta = 0}\left[\frac{\varepsilon^{-\frac{\xi^2}{\eta}}}{\eta^{\frac{3}{2}}}\right] \equiv 0 \qquad (\text{ note } 5\ )$$

Thus, <u>when the voltage wave strikes the point x, the initial values of the difference of potential V and of the initial slope $\left(\frac{dV}{dt}\right)_{t=\frac{x}{v}}$ are zero.</u>

This holds for any type of wave.

In the case we actually consider, we have the diagrams :





B. Variation of V along the line at a given time t.

The calculations are somewhat longer.

Since, now, t is constant : $dy = -dx$

$$V = V_o\,\varepsilon^{-\beta x}\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{z_1}\varepsilon^{-z^2}dz\right]$$

$$\frac{dV}{dx} = -V_o\,\beta\,\varepsilon^{-\beta x}\left[1 - \operatorname{erf} z_1\right] + V_o\,\varepsilon^{-\beta x}\frac{2}{\sqrt{\pi}}\,\varepsilon^{-z_1^2}.\,2z_1.\,\frac{dz_1}{dx}$$

but $\quad z_1 = \frac{1}{2}.\sqrt{v}.\delta.\frac{x}{\sqrt{vt - x}}$

thus $\quad \frac{dz_1}{dx} = \frac{1}{2}\,\delta\sqrt{v}\;\frac{2vt - x}{vt - x}$

and $\quad \frac{dV}{dx} = V_o\,\varepsilon^{-\beta x}\left[-\beta\left(1 - \operatorname{erf} z_1\right) + \frac{1}{\sqrt{\pi}}\,\varepsilon^{-z_1^2}\delta^2 x v\,\frac{2vt - x}{(vt - x)^{\frac{3}{2}}}\right]$

From this expression, we check easily that:

$\frac{dV}{dx} = 0 \quad$ at $\quad x = 0 \quad$ and $\quad x = vt$

We must, therefore, have a maximum gradient somewhere between $x = 0$ and $x = vt$.

Unfortunately, the expression for $\frac{d^2V}{dx^2}$ is too intricate to be handled simply. However, this part B of the study of a rectangular applied voltage is less important, since what matters is to know how V varies in function of time at a given point x, as was found out in the part A.

<u>Numerical example</u>.

We have : $\alpha = \sqrt{4\pi\sigma\mu}$

$$\delta = \frac{R}{4}\sqrt{\frac{C}{L}}\,a\alpha$$

$$\beta = \frac{R}{8}\sqrt{\frac{C}{L}}\left(1 - \frac{R}{L}\frac{a^2\alpha^2}{4}\right) + \frac{G}{2}\sqrt{\frac{L}{C}}$$

For a <u>double wire line</u> : $L = 4\ln\frac{D}{a}$ cm/cm

$$C = \frac{1}{4\ln\frac{D}{a}} \quad \text{cm/cm}$$

We remember that : $R = \frac{2}{\sigma.\pi a^2}$

thus $\frac{R}{L}\frac{\alpha^2 a^2}{4} = \frac{2}{\sigma\pi a^2}\frac{1}{4\ln\frac{D}{a}}\frac{4\pi\sigma\mu a^2}{4} = \frac{\mu}{2\ln\frac{D}{a}}$

and $\beta = \frac{R}{8}\sqrt{\frac{C}{L}}\left(1 - \frac{\mu}{2\ln\frac{D}{a}}\right) + \frac{G}{2}\sqrt{\frac{L}{C}}$

For a <u>three phase line</u> : $L = 2\ln\frac{D}{a}$

$$C = \frac{1}{2\ln\frac{D}{a}}$$

but now $R = \frac{1}{\sigma\pi a^2}$ and β has the same value.as for the double wire line. This, of course, could be expected from a physical point of view, as the steady state value of the voltage must be the same. In the same way, there is no change for δ .

Let us consider a three phase aerial line :

let : $D = 10$ meter

$a = 1$ cm.

$\sigma = \dfrac{1}{1800}$ in e.m.u.

Resistance : $R = \dfrac{1}{\sigma \pi a^2} = \dfrac{1800}{\pi \times 10^9} \dfrac{ohm}{cm}$

$R = \dfrac{1}{\pi} \times \dfrac{1800}{10^9} \times 10^5 \dfrac{ohm}{Km}$ or $R = 0.0574$ Ohm / km.

Self inductance : $L = 2 \ln \dfrac{D}{a} \times 10^{-9} \dfrac{Henrys}{cm}$

$L = 2 \times 2.3 \times \log 1000 \times 10^{-9} \times 10^5 \dfrac{Henry}{Km}$ or $L = 1.38 \times 10^{-3}$ Henry / km.

Capacitance : $C = \dfrac{1}{2 \ln 1000} \times \dfrac{1}{9 \times 10^{11}} \cdot \dfrac{Fd}{cm}$

$C = \dfrac{1}{2 \times 2.3 \times 3} \times \dfrac{10^5}{9 \times 10^{11}} \dfrac{Fd}{Km}$ $C = 0.805 \times 10^{-8}$ Farad / km.

Leakage conductance corresponding to corona losses :

an average value is $G = 0.1 \times 10^{-6}$ Mho / km.

Velocity of waves : $v = \dfrac{1}{\sqrt{LC}} = 300.000$ Km / sec.

Surge impedance : $\sqrt{\dfrac{L}{C}} = \sqrt{\dfrac{1.38}{10^3} \times \dfrac{10^8}{.805}} = 415$ Ohms.

Evaluation of β :

$$\beta = \frac{0.0574}{8} \times \frac{1}{415}\left(1 - \frac{1}{13.8}\right) + \frac{0.05}{10^6} \times 415$$

$\beta = (16 \times 10^{-6}) + (20.75 \times 10^{-6})$ or $\beta = 37 \times 10^{-6}$/km

Evaluation of δ :

$$\delta = \frac{0.0574}{4} \times \frac{1}{415} \times \sqrt{4\pi \times \frac{1}{1800}} \qquad \text{or } \delta = 2.89 \times 10^{-6} \sqrt{\text{sec}} / \text{km.}$$

Time for the wave to reach the maximum potential gradient after striking the point x = 100 km. :

$$\frac{1}{6}\delta^2 x^2 = \frac{1}{6} \times 8.35 \times 10^{-12} \frac{\text{sec}}{\text{Km}^2} \times 10^4 \text{ Km}^2 \qquad \text{or } \frac{1}{6}\delta^2 x^2 = 0.014 \,\mu\text{sec.}$$

Maximum voltage gradient :

$$\left(\frac{dV}{dt}\right)_{max} = V_0 \varepsilon^{-\beta x} \frac{0.924}{\delta^2 x^2} = V_0 \cdot \varepsilon^{-37 \times 10^{-4}} \cdot \frac{0.924}{0.084 \,\mu\text{sec}}$$

but $\varepsilon^{-37\times 10^{-4}} = 1 - \frac{37}{10^4} + \frac{37^2}{10^8} \cdots \approx 0.9963$

thus $\left(\frac{dV}{dt}\right)_{max} = V_0 \times 11 / \mu\text{sec}$

During 1 μ sec, the wave travels 300 meter, and :

$$\left(\frac{dV}{v\,dt}\right)_{max} = \frac{11\, V_0}{300 \text{ meter}} \qquad \text{or } \left(\frac{dV}{v\,dt}\right)_{max} = 3.67 \times 10^{-2} \times V_0 / \text{meter}$$

If $V_0 = 1000$ Kv , then : $\left(\frac{dV}{v\,dt}\right)_{max} = 36.7 \frac{\text{Kv}}{\text{meter}}$

This value, 36.7 Kvolt meter, is the critical voltage gradient at a distance of 100 km. for a 1000 Kv. initial impulse.

Time for the wave to reach half of its steady state value after it has struck the point x = 100 km.

$$V = \frac{1}{2} V_0 \varepsilon^{-\beta x} = V_0 \varepsilon^{-\beta x} \left[1 - \text{erf} \frac{\delta x}{2\sqrt{t - \frac{x}{v}}} \right]$$

$$\text{erf} \frac{\delta x}{2\sqrt{t - \frac{x}{v}}} = 0.5 \qquad \text{thus } \frac{\delta x}{2\sqrt{t - \frac{x}{v}}} = 0.477$$

and $t - \frac{x}{v} = 0.0925$ μ sec.

This gives an average voltage gradient:

$$\frac{500 \text{ Kv}}{0.0925 \times 300 \text{ meter}} \qquad \text{or} \quad \left(\frac{dV}{v\,dt}\right)_{av} = 18 \ \frac{\text{Kv}}{\text{meter}}$$

General Case. Deformation of a wave of general form $E_1(t)$.

Our equation (35) proved to be in a convenient form for a rectangular wave. However, in the general case, the integration that has to be carried out from a given value to infinity is, of course, a graphical impossibility.

We shall discuss the general equation (35) and transform it :

$$V = \frac{2}{\sqrt{\pi}} \varepsilon^{-\beta x} \int_{\frac{\delta x}{2\sqrt{t-\frac{x}{v}}}}^{\infty} E_1\left(t - \frac{x}{v} - \frac{\delta^2 x^2}{4z^2}\right) \varepsilon^{-z^2} dz$$

Let ; $\qquad z_0 = \frac{\delta x}{2\sqrt{t - \frac{x}{v}}}$

For $z = z_0 \qquad E_1\left(t - \frac{x}{v} - \frac{\delta^2 x^2}{4z^2}\right) = E_1(0)$

$z = \infty \qquad E_1\left(t - \frac{x}{v} - \frac{\delta^2 x^2}{4z^2}\right) = E_1\left(t - \frac{x}{v}\right)$

For $z_0 < z < \infty$; $\quad 0 < t - \frac{x}{v} - \frac{\delta^2 x^2}{4z^2} < t - \frac{x}{v}$

This corresponds to the physical truth, since, at time, t, only the values of E_1 for which the argument lies between 0 and $\left(t - \frac{x}{v} \right)$ can affect the voltage.

This proves in a new way that v must be the velocity of the traveling waves.

We see that : $t - \frac{x}{v} = \frac{\delta^2 x^2}{4 z_0^2}$

thus $V = \frac{2}{\sqrt{\pi}} \varepsilon^{-\beta x} \int_{z_0}^{\infty} E_1 \left[\frac{\delta^2 x^2}{4} \left(\frac{1}{z_0^2} - \frac{1}{z^2} \right) \right] \varepsilon^{-z^2} dz$

Let $z = \frac{1}{u}$ $\qquad dz = - \frac{1}{u^2} du$

For $z = z_0$ $\qquad u_0 = \frac{1}{z_0} = \frac{2\sqrt{t - \frac{x}{v}}}{\delta x}$

$z = \infty$ $\qquad u = 0$

After an easy transformation, we find :

$$V = \frac{2}{\sqrt{\pi}} \varepsilon^{-\beta x} \int_0^{u_0} E_1 \left[\frac{\delta^2 x^2}{4} (u_0^2 - u^2) \right] \frac{\varepsilon^{-\frac{1}{u^2}}}{u^2} du$$

Under this form, the graphical integration can be performed in a most convenient way. Before the integration, we must effectuate the product of the curves $\frac{1}{u^2} \varepsilon^{-\frac{1}{u^2}}$ and $E_1 \left[\frac{\delta^2 x^2}{4} (u_0^2 - u^2) \right]$

1. $E_1 \left[\frac{\delta^2 x^2}{4} (u_0^2 - u^2) \right]$ will take values from $E_1 \left[\frac{\delta^2 x^2}{4} u_0^2 \right]$ (or $E_1 (t - \frac{x}{v})$) to $E_1 (0)$. These values are finite and can be calculated.

2. $\frac{1}{u^2} \varepsilon^{-\frac{1}{u^2}}$. Let this function be f(u).

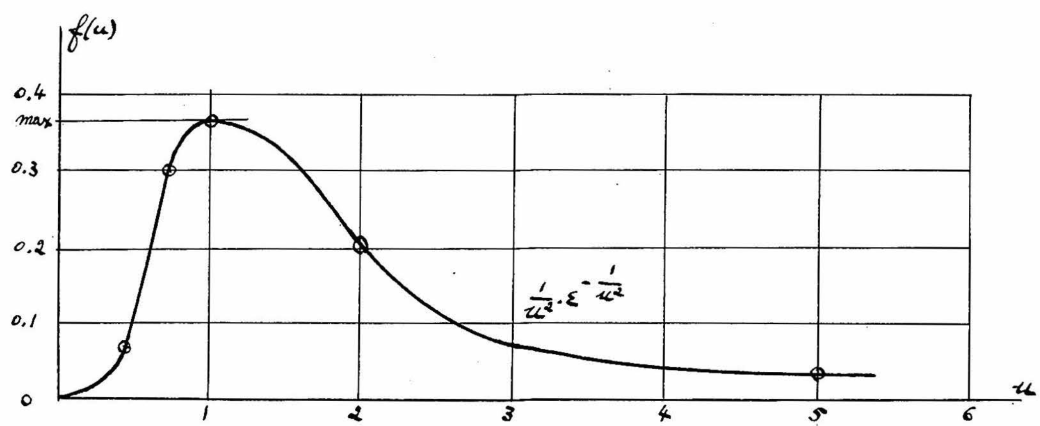
For $u = 0$ $\qquad f(u) = 0$

$u = \infty$ $\qquad f(u) = 0$

For $u = 1$, f(u) has a maximum equal to $\frac{1}{\varepsilon} = 0.368$

It is convenient to build the following table :

u	$\frac{1}{u^2}$	$\varepsilon^{-\frac{1}{u^2}}$	$f(u) = \frac{1}{u^2}\varepsilon^{-\frac{1}{u^2}}$
0.25	16	0.112	1.792
0.50	4	0.0182	0.0728
0.75	1.78	0.168	0.299
1.00	1	0.368	0.368 (max)
2.00	0.25	0.784	0.196
5.00	0.04	0.963	0.0385
10.00	0.01	0.99	0.01
50.00	0.0004	1	0.0004
100.00	0.0001	1	0.0001



In order to see what is the order of magnitude of giving the range of integration, let us take the line we already considered.

We found: $\delta = 2.89 \times 10^{-6} \frac{\sqrt{sec}}{Km}$

At a distance x = 100 km : $\delta x = 2.89 \times 10^{-4} \sqrt{sec}$

also ; $\frac{x}{v} = \frac{100}{300\,000}$ sec = 333 μ sec thus $u_0 = \frac{2\sqrt{t - \frac{x}{v}}}{\delta x} = \frac{2 \times 10^4}{2.89 \times 10^3}\sqrt{t - 333} = 6.93\sqrt{t - 333}$

where t is expressed in microseconds.

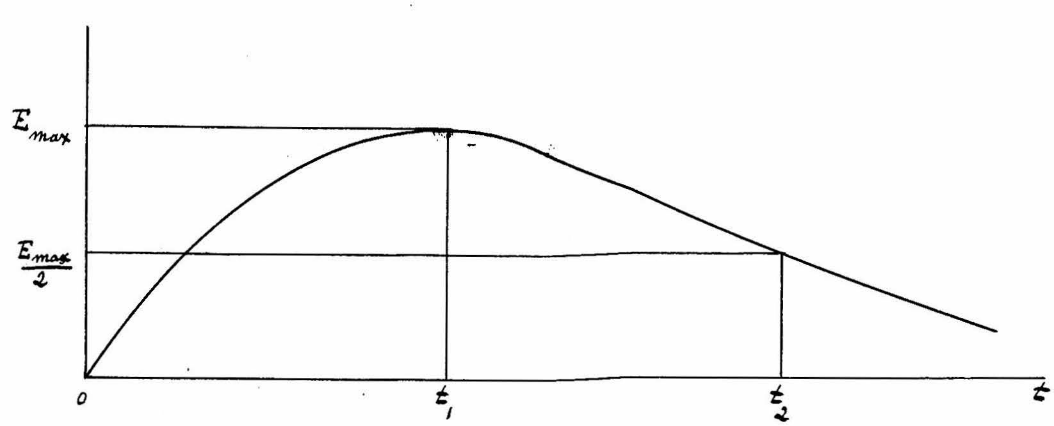
Let t_1 be the time after which the wave has reached the point x = 100 km, i.e. $t_1 = t - 333$ $u_0 = 6.93\sqrt{t_1}$

In all practical cases where lightning surges are concerned, it will be quite sufficient to take values of up to 30 μsec, since after that time, the value of the tail has come down to less than half its peak value.

t_1 in μsec	u_0
1	6.93
4	13.86
10	21.90
20	31.00
30	38.00

Within this range of values of u, the graphical integration can be carried out for any particular case.

Lightning Surges.

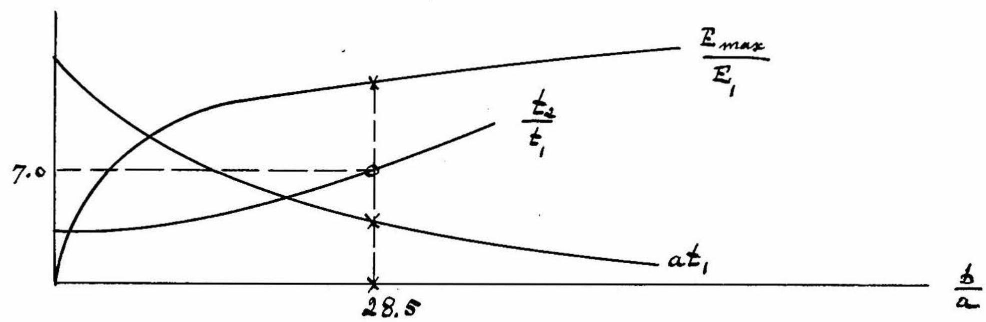


All the lightning surges induced on lines have a very close shape to the double exponential wave :

$$e = E_1\left(\varepsilon^{-at} - \varepsilon^{-bt}\right) \qquad (41)$$

By giving to a and b different combination values, either real or complex, this wave takes all the possible practical shapes. It is thus well worth to investigate the behavior of such a wave. From an oscillogram, the equation can be obtained by reading E_{max} , t_1 and t_2 .

In Bewley's " Traveling Waves ", we see how curves of $\frac{t_2}{t_1}$, $\frac{E_{max}}{E_1}$ and at_1 can be drawn in function of $\frac{b}{a}$ (t_1 and t_2 in μ sec.)



Any wave is defined as $E_{max}/t_1/t_2/\pm$, the sign corresponding to the polarity of the wave. As an example, let us find the equation of the wave (oscillogram):

$1000\,kV/3.0/21/+$. We have thus : $\frac{t_2}{t_1} = \frac{21}{3} = 7.0$

The graph gives then : $\frac{b}{a} = 28.5$

$$at_1 = 0.122$$

$$\frac{E_{max}}{E_1} = 0.852$$

thus $a = \frac{0.122}{3} = 0.041$

$$\delta = 0.041 \times 28.5 = 1.15$$

$$E_1 = \frac{E_{max}}{0.852} = \frac{1000}{0.852} = 1175 \text{ and } e = +1175\left(\varepsilon^{-0.041t} - \varepsilon^{-1.15t}\right)$$

The physical reason of the double exponential in the lightning waves.

A cloud hanging over a line creates an electric field which may reach 50 Kvolts per foot of height. This field induces charges on the different wires; these charges may be calculated from the Maxwell's equations for a system of conductors.

When the cloud is discharged by direct stroke, two waves are sent in opposite directions. However, the cloud does not discharge instantaneously, but according to an exponential discharge function. It can take up to 100 μsec. to discharge the cloud to 95% of its initial value.

By application of the superposition theorem, it can then be shown that the two waves sent in opposite directions on the line have the double exponential shape.
(See Bewley : " Transmission Lines ").

Deformation of the double exponential wave.

We just have determined the shape of a lightning wave. Let us investigate now its attenuation, as it travels along the line.

Our equation (40) becomes : (42)

$$V = \frac{2E_1}{\sqrt{\pi}}\,\varepsilon^{-\beta x}\int_0^{u_0}\left[\varepsilon^{-a\frac{\delta^2 x^2}{4}(u_0^2-u^2)} - \varepsilon^{-b\frac{\delta^2 x^2}{4}(u_0^2-u^2)}\right]\frac{\varepsilon^{-\frac{1}{u^2}}}{u^2}\,du$$

(43)

or $$V = \frac{2E_1}{\sqrt{\pi}}\,\varepsilon^{-\beta x}\left[\int_0^{u_0}\varepsilon^{-a\frac{\delta^2 x^2}{4}(u_0^2-u^2)}\,\frac{\varepsilon^{-\frac{1}{u^2}}}{u^2}\,du - \int_0^{u_0}\varepsilon^{-b\frac{\delta^2 x^2}{4}(u_0^2-u^2)}\,\frac{\varepsilon^{-\frac{1}{u^2}}}{u^2}\,du\right]$$

The two integrals must be evaluated by a graphical method. With the same line as before :

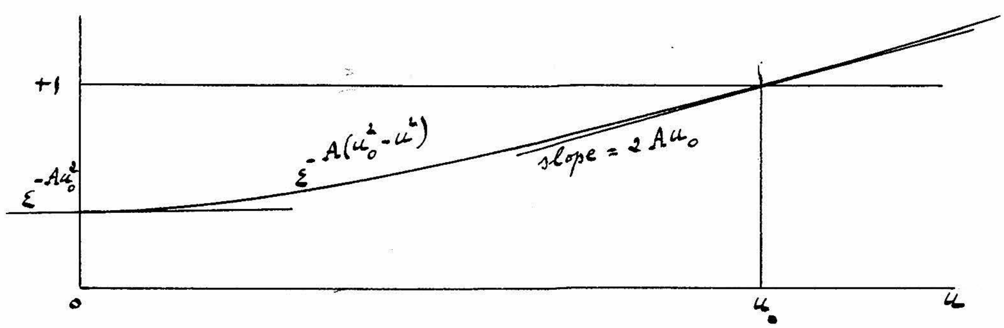
$$a\cdot\frac{\delta^2 x^2}{4} = 0.041 \times \frac{0.084}{4} = 0.861 \times 10^{-3}$$

$$b\cdot\frac{\delta^2 x^2}{4} = 1.15 \times \frac{0.084}{4} = 24.15 \times 10^{-3}$$

For u = 0, the integrand of each term of (43) is

u = u_0, the integrand of each term is $\dfrac{\varepsilon^{-\frac{1}{u_0^2}}}{u_0^2}$

The function $\varepsilon^{-A(u_0^2-u^2)}$ has the following aspect:



Within the range of integration, this function is finite and the graphical integration is easy to carry out. After an infinite time, $u_0 = \infty$, and the curve $\varepsilon^{-A(u_0^2-u^2)}$ reduces to zero. Thus V = 0 for t = ∞ , as could be expected.

<u>Conclusion.</u> Our voltage equation (35), modified into the form (40) can thus solve the problem of the

attenuation and distortion of a wave of any shape, under corona conditions, when G is constant. We shall see very soon now how the variation of G comes in. It is a known fact that above the corona critical voltage, the waves attenuate rapidly until their voltage falls down to the corona starting voltage. Further damping takes place much more slowly.

Assuming a line without distortion and having thus a mere damping, we can find out the rate of change of the electromagnetic energy, and put it equal to the rate of change of the energy absorbed by corona losses. According then to the assumption made for the losses, different laws can be found for the attenuation of voltage waves. These well known laws have been established by Skilling, Peeks, and Foust and Menger. They are generally well sufficient when attenuation only is concerned. However, if the real behavior of the wave, attenuated and distorted, is desired, then the general equation (40) must be used.

Corona distortion only was considered by E.W. Boehne (A.I.E.E. Trans. Vol.50, p.558). His theory of different velocities for the wave in function of corona seems very doubtful. It may serve as an explanation; it can not lead to any computable result.

If G is treated as a constant, it will have the average value : 0.1×10^{-6} mhos per km.

Ground wires.

Their action is to reduce the peak value of the lightning surges on the power lines. The protective ratio that they introduce can be calculated (See Bewley). The reduced voltage waves are then attenuated and distorted as we have established.

Variation of G.

When corona takes place, the leakage conductance is not a constant any more.

In the general equation (35) or (40), G comes only in the damping factor β . The constant δ is independent of G, therefore the space-time integral :

$$\frac{2}{\sqrt{\pi}}\int_{\frac{\delta x}{2\sqrt{t-x/v}}}^{\infty} E_1\left(t-\frac{x}{v}-\frac{\delta^2 x^2}{4z^2}\right)\varepsilon^{-z^2}dz$$ can be expressed as a function f(t,x) and can be evaluated independently of G.

Thus : $V = \varepsilon^{-\beta x}\cdot f(t,x)$ or $V.\varepsilon^{\beta x} = f(t,x)$

where $\beta = \frac{R}{8}\sqrt{\frac{C}{L}}\left(1-\frac{R}{L}\cdot\frac{a^2\alpha^2}{4}\right) + \frac{G}{2}\sqrt{\frac{L}{C}}$

In all cases the corona losses are given by GV^2.

Experimentally, the following laws have been found:

$GV^2 = A(V-V_c)$ (Skilling) $G = A\cdot\frac{V-V_c}{V^2}$ $V > V_c$

$GV^2 = B(V-V_c)^2$ (Peeks) $G = B\left(1-\frac{V_c}{V}\right)^2$ $V > V_c$

$GV^2 = KV^3$ (Foust and Menger) $G = K.V$

The first term of β is a constant, say β_1 .

Thus : $\beta = \beta_1 + \varphi(V)$

where $$\varphi(V) = \frac{1}{2}\sqrt{\frac{L}{C}}\, A\, \frac{V - V_c}{V^2} \qquad V > V_c$$
$$= \frac{1}{2}\sqrt{\frac{L}{C}}\, B\left(1 - \frac{V_c}{V}\right)^2 \qquad V > V_c$$
$$= \frac{1}{2}\sqrt{\frac{L}{C}}\,.\,KV$$

(V in Kv, t in μsec)

$0.02 \times 10^{-6} \leq K \leq 0.14 \times 10^{-6}$

Thus : $\varepsilon^{\beta_1 x}.V.\varepsilon^{x\varphi(V)} = f(t,x)$

For a certain given point x, the function f(t,x) will look as follows (fig.1). On the other hand, the left hand member of the upper equation will have the aspect of the figure 2 or 2', according to the adopted law.

Then, at a given time t_1 , we find, on fig.1, the value of $f(t_1,x)$. The figure 2, or 2', gives then at once the value of the voltage V, at the considered time.

We see clearly how distortion comes in above the corona critical voltage.

Variation of $\varphi(V)$.

The variation of $\varphi(V)$ is clearly shown in the figures 3,4 and 5.

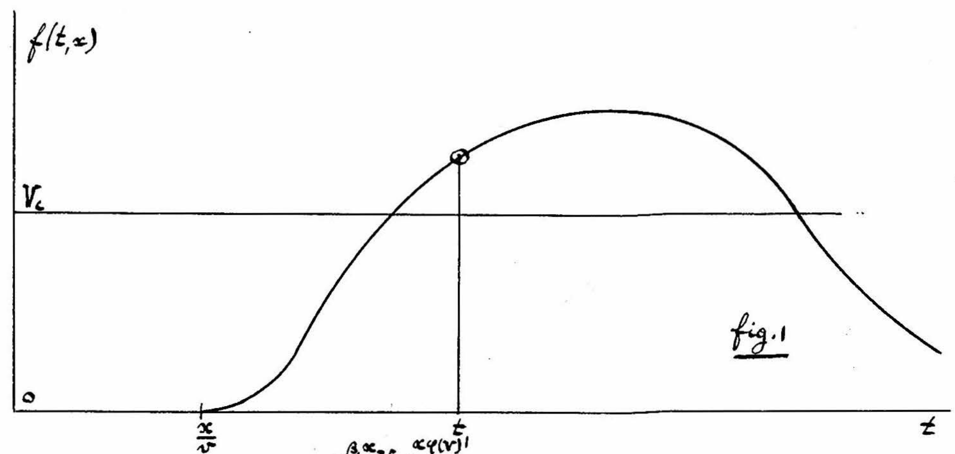
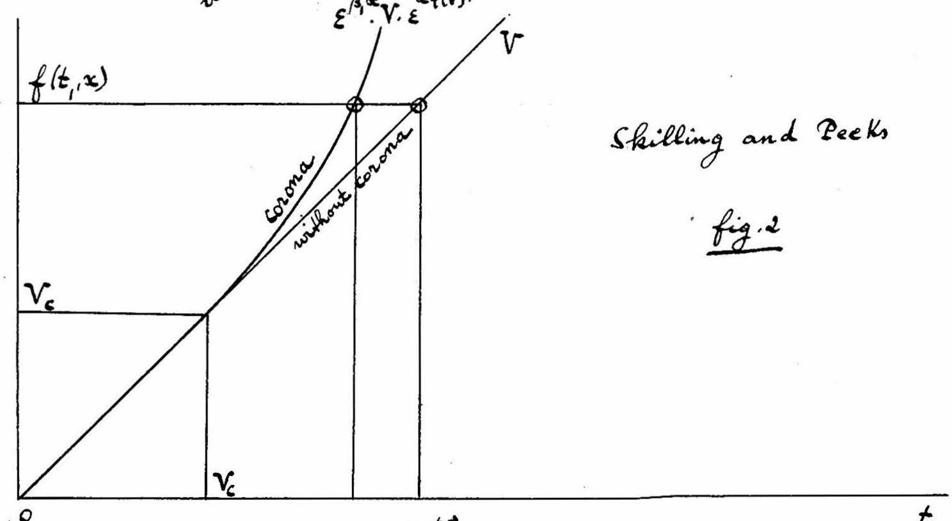
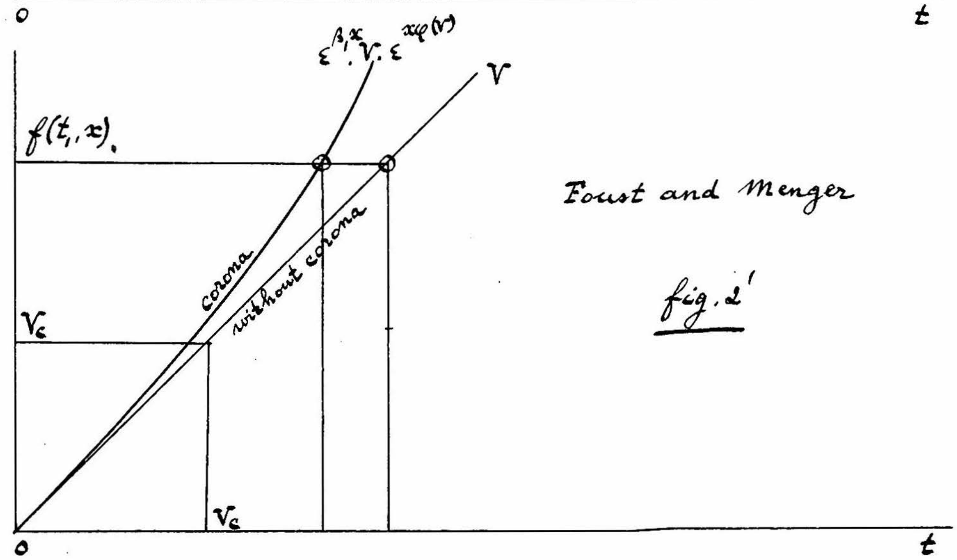


fig. 1



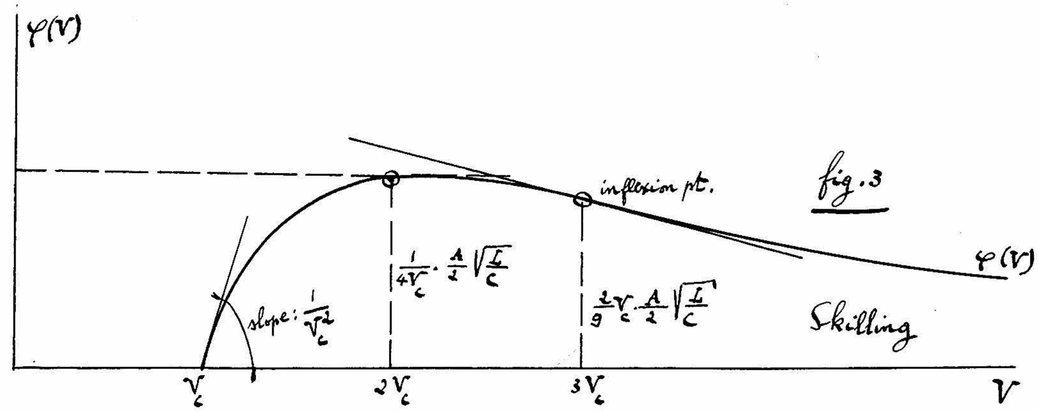
Skilling and Peeks

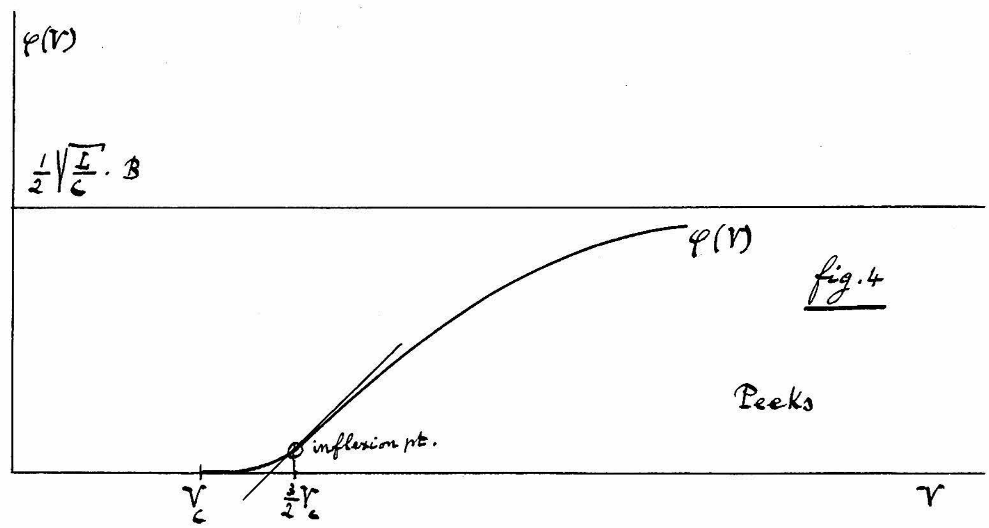
fig. 2

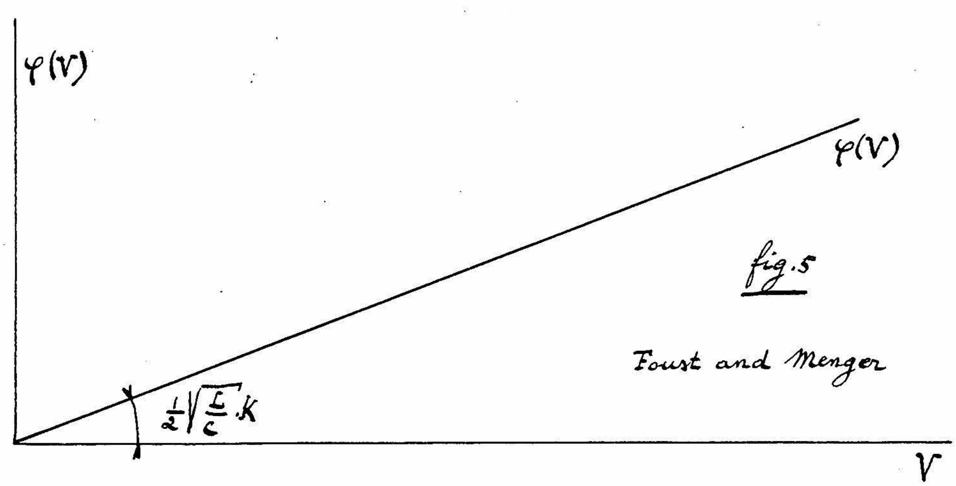


Foust and Menger

fig. 2'


φ(V)
inflexion pt.
fig.3
φ(V)
slope: 1/V_c^2
1/(4V_c) · A/2 √(L/C)
2/9 V_c · A/2 √(L/C)
Skilling
V_c
2V_c
3V_c
V


φ(V)
1/2 √(L/C) · B
φ(V)
fig.4
Peeks
inflexion pt.
V_c
3/2 V_c
V


φ(V)
φ(V)
fig.5
Foust and Menger
1/2 √(L/C) · K
V

General solution of the telegraphists' equations

We just have seen how easy it is to reach a solution when G varies, from the solution for G constant. We could treat our solution (35) as we did, because G influences only β , and not δ .

However, let us try to obtain a most general solution by the operational treatment of the telegraphists' equations.

$$-\frac{\partial V}{\partial x} = 2E_a + Lp\,I$$

$$-\frac{\partial I}{\partial x} = Cp\,V + GV$$

with : $2E_a = f_1(p).I$ and : $G = f_2(V)$

Thus : $$\frac{d^2V}{dx^2} = \left[f_1(p) + Lp\right]\left[Cp + f_2(V)\right]V$$

or : $$\frac{d^2V}{dx^2} = \gamma^2 V + m^2 f_3(V)$$

If a solution of such an equation can be found, its operational treatment will give the answer.

For example, let : $G = KV$, then :

$$\frac{d^2V}{dx^2} = \gamma^2 V + m^2 V^2$$ with $\gamma^2 = \left[f_1(p) + Lp\right]Cp$

and $m^2 = \left[f_1(p) + Lp\right]K$

To solve this equation, let : $y = \frac{dV}{dx}$

thus : $\frac{d^2V}{dx^2} = \frac{dy}{dx} = y\frac{dy}{dV}$ and : $y\,dy = (\gamma^2 V + m^2 V^2)\,dV$

A first integration gives :

$$\frac{y^2}{2} = \gamma^2 \frac{V^2}{2} + m^2 \frac{V^3}{3} + c^{st}$$

but at $t = \infty$ $\quad V = \dfrac{dV}{dt} = 0$, thus the constant is **zero** .

and $y^2 = \gamma^2 V^2 + \frac{2}{3} m^2 V^3$

or $$\frac{dV}{V\sqrt{\gamma^2 + \frac{2}{3} m^2 V}} = dx$$

by integration : $-\dfrac{2}{\gamma} \tanh^{-1}\left[\dfrac{\sqrt{\gamma^2 + \frac{2}{3} m^2 V}}{\gamma}\right] = -\dfrac{\gamma x}{2}$

After a few transformations:

$$V = -\frac{3}{2}\frac{\gamma^2}{m^2}\frac{1}{\cosh^2 \frac{\gamma x}{2}} \qquad \text{where : } \frac{\gamma^2}{m^2} = \frac{c}{K} \cdot p$$

The solution of this operational formula can be expressed in terms of a series, but is very complicate. The direct solution from the telegraphists' equations is to be avoided. Besides, the integration is impossible if any other law of variation of G is considered.

The conclusion is that the method of variation of G, in the solution for G constant, is the only one leading to the result without any complication whatsoever.

Additional notes. Note 1

We have the equation (30) :

$$\gamma = \frac{p}{v}\sqrt{\left(1+\frac{G}{Cp}\right)\left[1+\frac{R}{4L}\cdot\frac{1}{p}+\frac{R}{2L}\cdot\frac{a\alpha}{\sqrt{p}}+\frac{1}{pL}P_1\left(\frac{1}{\sqrt{p}}\right)\right]}$$

$$\text{or: } \gamma = \frac{p}{v}\sqrt{\begin{aligned}&1+\frac{R}{2L}\cdot\frac{a\alpha}{\sqrt{p}}+\left(\frac{R}{4L}+\frac{G}{C}\right)\frac{1}{p}+\frac{GR}{2LC}\,\frac{a\alpha}{p^{\frac{3}{2}}}+\\&+\frac{GR}{4LC}\cdot\frac{1}{p^2}+\frac{1}{pL}P_1\left(\frac{1}{\sqrt{p}}\right)+\frac{G}{p^2LC}P_1\left(\frac{1}{\sqrt{p}}\right)\end{aligned}}$$

The radical can be expanded in a series by the formula : $\sqrt{1+x} = 1+\frac{1}{2}x-\frac{1}{8}x^2$ -------

Thus, the radical becomes :

$$\begin{aligned}\sqrt{1+\cdots} = {} & 1+\frac{R}{4L}\cdot\frac{a\alpha}{\sqrt{p}}+\frac{1}{2}\left(\frac{R}{4L}+\frac{G}{C}\right)\frac{1}{p}+\frac{GR}{4LC}\cdot\frac{a\alpha}{p^{\frac{3}{2}}}\\ &+\frac{GR}{8LC}\,\frac{1}{p^2}+\frac{1}{2pL}P_1\left(\frac{1}{\sqrt{p}}\right)+\frac{G}{2p^2LC}P_1\left(\frac{1}{\sqrt{p}}\right)+\\ &-\frac{1}{8}\left[\frac{R}{2L}\cdot\frac{a\alpha}{\sqrt{p}}+\left(\frac{R}{4L}+\frac{G}{C}\right)\frac{1}{p}+\frac{GR}{2LC}\,\frac{a\alpha}{p^{\frac{3}{2}}}+\frac{GR}{4LC}\,\frac{1}{p^2}+\right.\\ &\left.+\frac{1}{pL}P_1\left(\frac{1}{\sqrt{p}}\right)+\frac{G}{p^2LC}P_1\left(\frac{1}{\sqrt{p}}\right)\right]^2+ \text{-------}\end{aligned}$$

Independent term : 1

Term in $\frac{1}{\sqrt{p}}$: $\frac{Ra\alpha}{4L}$

Terms in $\frac{1}{p}$: $\frac{1}{2}\left(\frac{R}{4L}+\frac{G}{C}\right)-\frac{R^2}{32L^2}\cdot a^2\alpha^2$

Thus : $\gamma = \frac{p}{v}+\beta+\delta\sqrt{p}+P_2\left(\frac{1}{\sqrt{p}}\right)$

where $\beta = \frac{1}{2v}\left[\frac{R}{4L}+\frac{G}{C}\right]-\frac{R^2}{32L^2}\cdot\frac{a^2\alpha^2}{v}$

$\delta = \frac{Ra\alpha}{4vL}$ and $v = \frac{1}{\sqrt{LC}}$

and where $\mathcal{P}_2\left(\frac{1}{\sqrt{p}}\right)$ is an analytic function of $\frac{1}{\sqrt{p}}$.

We can slightly change the form of β and δ , so that :

$$\delta = \frac{R}{4}\sqrt{\frac{C}{L}} \cdot a\alpha$$

$$\beta = \frac{R}{8}\sqrt{\frac{C}{L}}\left(1 - \frac{R}{L}\cdot\frac{\alpha^2 a^2}{4}\right) + \frac{G}{2}\sqrt{\frac{L}{C}}$$

Note 2.

We shall, following Dr. Bush's textbook, evaluate the operational formula : $\varepsilon^{-hp} f(t)$

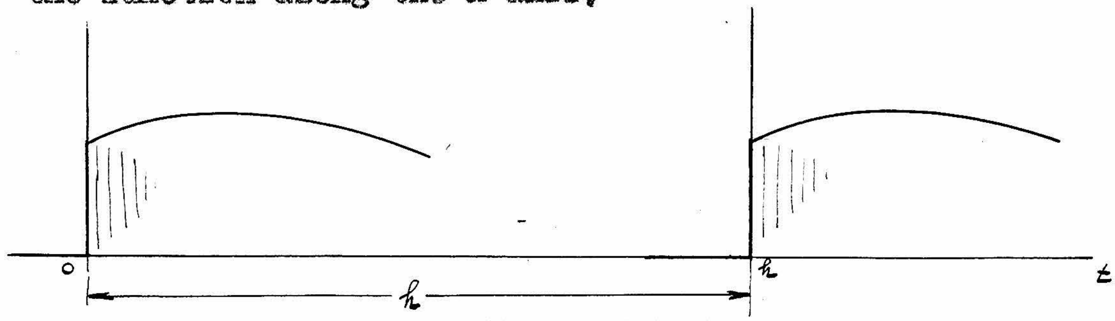
The operand may be expanded in a series :

$$\varepsilon^{-hp} f(t) = \left(1 - hp + \frac{h^2p^2}{2!} - - - -\right) f(t)$$

or : $\varepsilon^{-hp} f(t) = f(t) - hf'(t) + \frac{h^2}{2!} f''(t) - - - -$

i.e. $\varepsilon^{-hp} f(t) = f(t-h)$ valid for $t \geqslant h$

ε^{-hp} is thus a transfer operator. It merely shifts the function along the x-axis.



Note 3.

The object is to find the original time function of the given image : $\varepsilon^{-\sqrt{x}\sqrt{p}} \cdot f(t)$

We shall use one of the several forms of the <u>super-position theorem</u>, for instance :

$$i(t) = f(t) \cdot A(o) + \int_o^t A'(\lambda) f(t-\lambda)\, d\lambda$$

where A(t) is the value of i(t) when f(t) = 1 .

i(t) can be expressed by an operational equation in which the image is f(t) operated on by a certain function F(p), i.e. : $i(t) \doteqdot F(p) \cdot f(t)$

$$A(t) \doteqdot F(p)$$

Consequently :

$$F(p).A(t) \doteqdot f(t) \cdot A(o) + \int_o^t A'(\lambda) f(t-\lambda)\, d\lambda$$

where : $A(t) \doteqdot F(p)$

Let now : $F(p) = \varepsilon^{-\sqrt{x}\sqrt{p}}$

In all operational textbooks, the solution for F(p) is given :

$$F(p) = \varepsilon^{-\sqrt{x}\sqrt{p}} \doteqdot 1 - erf\left[\frac{\sqrt{x}}{2\sqrt{t}}\right] = 1 - \frac{2}{\sqrt{\pi}} \int_o^{\frac{\sqrt{x}}{2\sqrt{t}}} \varepsilon^{-z^2} dz = A(t)$$

For $t = o$, $A(o) = o$ thus :

$$\varepsilon^{-\sqrt{x}\sqrt{p}} f(t) \doteqdot \int_o^t A'(\lambda) f(t-\lambda)\, d\lambda$$

The next step is to evaluate $A'(\lambda)$.

$$A'(t) = \frac{2}{\sqrt{\pi}} \varepsilon^{-\frac{\delta^2 x^2}{4t}} \cdot \frac{\delta x}{2} \cdot \frac{1}{2t\sqrt{t}}$$

then : $$\varepsilon^{-\delta x\sqrt{p}} f(t) \doteqdot \int_0^t f(t-\lambda) \frac{2}{\sqrt{\pi}} \frac{\delta x}{4} \frac{1}{\lambda\sqrt{\lambda}} \varepsilon^{-\frac{\delta^2 x^2}{4\lambda}} d\lambda$$

Let us make a change of variables for the sake of simplification : $\frac{\delta^2 x^2}{4\lambda} = z^2$

thus $\lambda = \frac{\delta^2 x^2}{4z^2}$ and $d\lambda = -\frac{\delta^2 x^2}{4} \cdot \frac{2dz}{z^3}$

As far as the limits are concerned :

$\lambda = 0$ $\qquad z = \infty$

$\lambda = t$ $\qquad z = \frac{\delta x}{2\sqrt{t}}$

Now : $$\frac{2}{\sqrt{\pi}} \frac{\delta x}{4} \frac{d\lambda}{\lambda\sqrt{\lambda}} = -\frac{2}{\sqrt{\pi}} \frac{\delta x}{4} \frac{4z^2}{\delta^2 x^2} \frac{2z}{\delta x} \frac{\delta^2 x^2}{4} \frac{2dz}{z^3} = -\frac{2dz}{\sqrt{\pi}}$$

so that, finally :

$$\underline{\varepsilon^{-\delta x\sqrt{p}} f(t) \doteqdot \int_{\frac{\delta x}{2\sqrt{t}}}^{\infty} f\left(t - \frac{\delta^2 x^2}{4z^2}\right) \varepsilon^{-z^2} dz}$$

<u>Note 4.</u>

The principle of superposition allows us to write the so called <u>Maxwell equations</u> for a system of conductors.

$$V_1 = a_{11} q_1 + a_{12} q_2 + \cdots + a_{1n} q_n$$
$$V_2 = a_{21} q_1 + a_{22} q_2 + \cdots + a_{2n} q_n$$

$$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \mathrm{I}$$

$$V_n = a_{n1} q_1 + a_{n2} q_2 + \cdots\cdots + a_{nn} q_n$$

V : potential

q : charge

a_{jk} : <u>coefficient of potential</u> (having only a geometrical dimension). All a's are positive.

The equations I may be solved for the charges :

$$\begin{aligned} q_1 &= c_{11} V_1 + c_{12} V_2 + \cdots\cdots + c_{1n} V_n \\ q_2 &= c_{21} V_1 + c_{22} V_2 + \cdots\cdots + c_{2n} V_n \qquad \mathrm{II} \\ &\vdots \qquad\qquad\qquad \vdots \\ q_n &= c_{n1} V_1 + c_{n2} V_2 + \cdots\cdots + c_{nn} V_n \end{aligned}$$

c_{jk} : <u>coefficient of induction</u> (negative)

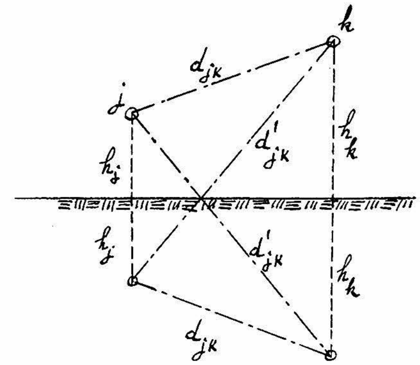
c_{jj} : <u>coefficient of capacity</u> (positive)

also : $c_{11} + c_{12} + \cdots\cdots + c_{1n} > 0$

By successive groundings of the different conductors, the c's are determined quite easily <u>experimentally</u>. The a's, on the other hand, are readily <u>calculated</u>. By the <u>method of images</u>, taking care of the presence of the ground, they are found to be :

$$a_{jj} = \frac{2}{\varepsilon} \ln \frac{2h_j}{r_j}$$

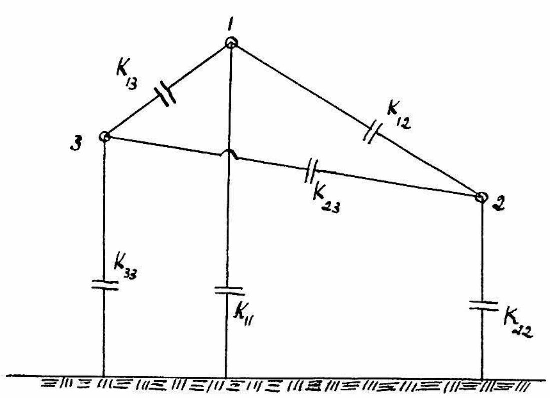
$$a_{jk} = \frac{2}{\varepsilon} \ln \frac{d'_{jk}}{d_{jk}}$$



The Maxwell's equations can also be expressed in function of the partial capacities in the system :

$$\left.\begin{aligned} q_1 &= K_{11}V_1 + K_{12}(V_1 - V_2) + \cdots + K_{1n}(V_1 - V_n) \\ q_2 &= K_{21}(V_2 - V_1) + K_{22}V_2 + \cdots + K_{2n}(V_2 - V_n) \\ &\vdots \\ q_n &= K_{n1}(V_n - V_1) + K_{n2}(V_n - V_2) \cdots + K_{nn}V_n \end{aligned}\right\} \quad \text{III}$$

For a three phase line, these equations correspond to the picture below.



We have :

$$K_{11} + K_{12} + \cdots + K_{1n} = c_{11}$$

$$K_{12} = K_{21} = -c_{12}$$

$$K_{13} = K_{31} = -c_{13}$$

The k's are **positive**.

Case of a three phase line.

The a's are known. Let us express the c's in function of the a's.

$$a_{11} q_1 + a_{12} q_2 + a_{13} q_3 = V_1$$

$$a_{21} q_1 + a_{22} q_2 + a_{23} q_3 = V_2$$

$$a_{31} q_1 + a_{32} q_2 + a_{33} q_3 = V_3$$

$$q_1 = \frac{\begin{vmatrix} V_1 & a_{12} & a_{13} \\ V_2 & a_{22} & a_{23} \\ V_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} = c_{11} V_1 + c_{12} V_2 + c_{13} V_3$$

and : $c_{11} = \frac{1}{D}\left(a_{22}a_{33} - \overline{a}_{23}^{2}\right)$; $c_{12} = \frac{1}{D}\left(a_{31}a_{32} - a_{12}a_{33}\right)$; $c_{13} = \frac{1}{D}\left(a_{12}a_{23} - a_{22}a_{13}\right)$

with : $D = a_{11}a_{22}a_{33} + 2\,a_{12}a_{23}a_{31} - a_{11}\overline{a}_{23}^{2} - a_{22}\overline{a}_{31}^{2} - a_{33}\overline{a}_{12}^{2}$

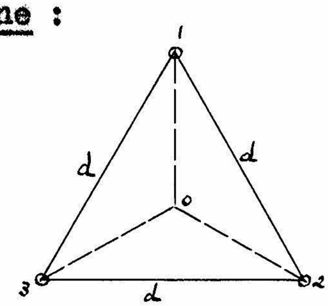
The other c's can be obtained at once by permutation of the subscripts. For all practical purposes, it is well sufficient to take the average value of the c_{11} c_{22} c_{33} : say c, and of the c_{12} c_{23} c_{31} : say c' .

Then : $q_1 = c\,V_1 + c'(V_2 + V_3)$

$$q_2 = c'(V_1 + V_3) + c\,V_2$$

$$q_3 = c V_3 + c'(V_1 + V_2)$$

For a balanced three phase line :



$$V_1 + V_2 + V_3 = 0$$

and $$q_1 = (c - c') V_1$$

or $$q_1 = c_0 . V_1$$

The constant ratio c_0 between the instantaneous charge and voltage of any wire is known as the industrial capacity of the wire.

$$c = \frac{1}{D}(a^2 - a'^2) \qquad c' = \frac{1}{D}(a'^2 - aa')$$

$$D = a^3 + 2a'^3 - 3aa'^2$$

or $$c_0 = c - c' = \frac{(a-a')(a+2a')}{(a-a')^2(a+2a')} = \frac{1}{a-a'}$$

or $$c_0 = \frac{\varepsilon}{2\left[\ln \frac{2h}{r} - \ln \frac{d'}{d}\right]}$$ where $d' = \sqrt{d^2 + 4h^2}$

and $$c_0 = \frac{\varepsilon}{2 \ln \frac{2h}{r} \cdot \frac{d}{d'}} = \frac{\varepsilon}{2 \ln \frac{d}{r} - \ln \frac{d'}{2h}}$$

or $$c_0 = \frac{\varepsilon}{2\left[\ln \frac{d}{r} - \ln \sqrt{1 + \frac{d^2}{4h^2}}\right]}$$

The expression for the capacity per wire shows the influence of the presence of the earth.

However, in general :

$$\frac{d^2}{4h^2} << 1 \qquad \text{and} \qquad c_0 = \frac{\varepsilon}{2 \ln \frac{d}{r}} \qquad (1)$$

For a double wire line :

$$q_1 = c_0 V_1$$

$$-q_1 = c_0 V_2 \qquad \text{and} \qquad 2q_1 = c_0 (V_1 - V_2)$$

The capacity of the line is defined as : $c_1 = \dfrac{q_1}{V_1 - V_2}$

thus : $c_1 = \dfrac{c_0}{2} = \dfrac{\varepsilon}{4 \ln \frac{d}{2}}$ (2)

B. Self inductances of a system of wires.

Inductance of two wires of radii and and of distance D between their centers.

By calculating the self inductance outside of the wires by the definition : $\phi = L i$, and inside of the wires by the energetic relation : $\frac{1}{2} L i^2 = \frac{\mu_i}{8\pi} \int \mathcal{H}^2 d\tau$

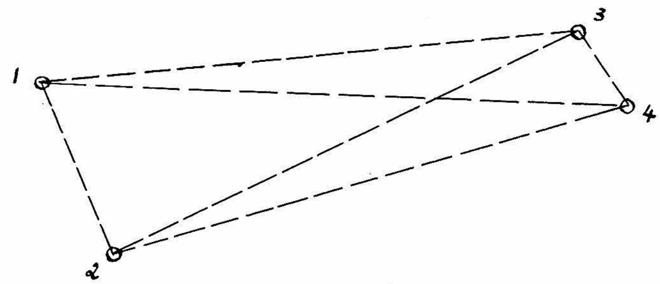
$d\tau$: elementary volume,

we obtain readily : $L = \mu_i + 4 \mu_e \ln \dfrac{D}{\sqrt{r_1 r_2}}$ (3)

μ_i : internal permeability
μ_e : external permeability

The formula (3) corresponds to the equation (2).

Mutual inductance of two double lines.



Let the two loops be (1,2) and (3,4) , having unit length. The mutual inductance is the flux in loop (3,4) due to the current of unit value in 1 and 2.

$$\mathcal{M}_{34,1} = \phi_{14,1} - \phi_{13,1} = \left(\frac{1}{2}\mu_i + 2\mu_e \ln \frac{D_{14}}{r_1}\right) - \left(\frac{1}{2}\mu_i + 2\mu_e \ln \frac{D_{13}}{r_1}\right)$$

or $\mathcal{M}_{34,1} = 2\mu_e \ln \dfrac{D_{14}}{D_{13}}$

and $\mathcal{M}_{34,2} = 2\mu_e \ln \dfrac{D_{24}}{D_{23}}$

Thus : $$\mathcal{M}_{34/12} = \mathcal{M}_{34/1} - \mathcal{M}_{34/2} = 2\mu_e \ln \frac{D_{14}\, D_{23}}{D_{13}\, D_{24}} \quad (4)$$

Suppose now that the wires 4 and 2 are confounded,

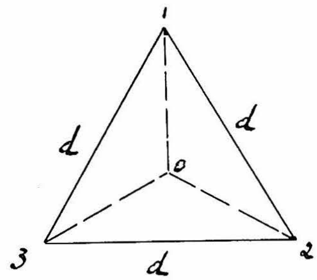
then : $\mathcal{M}_{23/12} = \phi_{32/2} - \phi_{32/1}$

but : $\phi_{32/1} = \phi_{13/1} - \phi_{12/1} = 2\mu_e \ln \dfrac{D_{13}}{D_{12}}$

$$\phi_{32/2} = \frac{1}{2}\mu_i + 2\mu_e \ln \frac{D_{23}}{r_2}$$

Thus : $$\mathcal{M}_{23/12} = \frac{1}{2}\mu_i + 2\mu_e \ln \frac{D_{23}\, D_{12}}{D_{13}\, r_2} \quad (5)$$

<u>Inductance of a three phase balanced line.</u>



It is impossible to speak of the self inductance of a three phase line in general, since there are three wires carrying three different currents.

Each of these wires is the origin of e.m.f's of induction which may be decomposed into an e.m.f. of self-induction and two e.m.f's of mutual induction. We shall see that it is possible to express the e.m.f. in one wire as a function of the current in that same wire, under the condition that we define the industrial self induction of the line.

We may suppose that the neuter wire exists, that it has the same radius than the other wires, and that it occupies the geometrical center of the system.

Since $i_1 + i_2 + i_3 = 0$, we may suppose that the current $(-i_1) + (-i_2) + (-i_3) = 0$ flows in the neuter.

The system is thus equivalent to a sum of three circuits 1-0, 2-0, 3-0 ; in which the currents i_1 i_2 i_3 flow.

$$e_1 = -\left(L_1 \frac{di_1}{dt} + \mathfrak{M}_{20/10} \frac{di_2}{dt} + \mathfrak{M}_{30/10} \frac{di_3}{dt} \right)$$

$$= -\left(L_1 - \mathfrak{M}_{20/10} \right) \frac{di_1}{dt} = -L \frac{di_1}{dt}$$

where L is the industrial self inductance of the wire.

$$\mathcal{L} = \mathcal{L}_1 - \mathcal{M}_{20/10}$$

but $\mathcal{L}_1 = \mu_i + 4\mu_e \ln \frac{d_{10}}{r}$ (formula 3)

and $\mathcal{M}_{20/10} = 2\mu_e \ln \frac{d_{10}d_{20}}{d_{12}\, r} + \frac{\mu_i}{2}$ (formula 5)

$$\mathcal{L} = \frac{\mu_i}{2} + 4\mu_e \ln \frac{d}{2\sqrt{3}} - 2\mu_e \ln \frac{d^2}{3d\,r}$$

thus : $\mathcal{L} = \frac{1}{2}\mu_i + 2\mu_e \ln \frac{d}{r}$ (6)

which is exactly half of the value we found in (3).

The formula (6) corresponds to the relation (1).

<u>Note 5.</u>

We notice that : $\lim\limits_{\eta=0} \left[\frac{\varepsilon^{-\frac{\zeta^2}{\eta}}}{\eta^{\frac{3}{2}}} \right] = \frac{0}{0}$

Let : $\eta = \frac{1}{u}$

then : $\lim \left[\frac{\varepsilon^{-\frac{\zeta^2}{\eta}}}{\eta^{\frac{3}{2}}} \right]_{\eta=0} = \lim \left[\frac{u^{\frac{3}{2}}}{\varepsilon^{\zeta^2 u}} \right]_{u=\infty} = \frac{\infty}{\infty}$

Using now L'Hospital's rule :

$$\lim \left[\frac{u^{\frac{3}{2}}}{\varepsilon^{\zeta^2 u}} \right]_{u=\infty} = \lim \left[\frac{\frac{3}{2}\, u^{\frac{1}{2}}}{\zeta^2\, \varepsilon^{u\zeta^2}} \right]_{u=\infty} = \lim \left[\frac{\frac{3}{2}\cdot\frac{1}{2}}{u^{\frac{1}{2}}\, \zeta^4\, \varepsilon^{\zeta^2 u}} \right]_{u=\infty} \equiv 0$$

In general, $\lim \left[\frac{u^m}{\varepsilon^{u^n}} \right]_{u=\infty} = 0$ if <u>m</u> and <u>n</u> > 0

This means that the exponential function goes to infinity faster than any algebraic function. This is a well known mathematical property, which a series expansion would also bring out.

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