COMPARISON OF THE PASCHEN AND THE BALMER SERIES OF HYDROGEN LINES IN STELLAR SPECTRA

Thesis by

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ABSTRACT

Introduction. The structure of the hydrogen spectrum, and previous observations of the Paschen series are briefly described. The behavior of the Paschen lines m = 12 to m = 14, recently photographed at Mount Wilson in various types of stellar spectra resembles in general that of the Balmer lines in the same types. The Paschen series is very conspicuous in the c stars β Orionis and α Cygni.

Photometric data. Measurements of structure and intensity have been made by the usual photometric methods on numerous lines in both hydrogen series in the spectra of α Leonis B8n, α Lyrae A0, α Canis Majoris A0, β Orionis cB8, and α Cygni cA2.

Structure of lines. Most lines not affected by the overlapping of neighboring lines have contours of the simple exponential form, although in α Leonis the centers are flattened. In the broad-line stars α Leonis and α Lyrae considerable departure from the exponential form is exhibited by the wings of lines near the heads of the series. In β Orionis, α Cygni, α Leonis, and the emission-line stars γ Cassiopeiae and P Cygni the shapes of Balmer and Paschen lines are similar, with the dimensions proportional to

wavelength; but this is not true of α Lyrae in whose spectrum the Balmer lines have relatively more intense wings.

Central intensities. Measured values for numerous lines are given in Table VI. In general the central intensities of Paschen lines are less than those of the corresponding Balmer lines. In α Leonis the Paschen series is relatively intense and has a slow decrement. In β Orionis and α Cygni both series exhibit slow decrements and little overlapping.

Problem of overlapping lines. The shapes as well as the central intensities of many lines, particularly in the Balmer series, are probably seriously modified by the overlapping wings of adjacent lines. The problem is to find the shapes and intensities of the "true" or original lines which by their mutual interaction produce the observed curve. A physically reasonable sequence of lines has been found which yields close approximations to the observed shapes and intensities in α Lyrae but gives too great reduction of the general intensity of the whole spectrum toward the head of the series. This difficulty which appears to be of a rather general nature is briefly discussed.

Total line absorption. Measured values for lines of both series in α Cygni, β Orionis, and α Leonis are given in Table VII. From these data Unsold's equation gives the numbers of atoms in the second and third levels. The numbers computed for a given level from various lines differ systematically, indicating that the theory is incomplete. The maximum numbers, used in Boltzmann's equation for thermal equilibrium, give reasonable values of the temperature of the absorbing hydrogen.

Intensities of emission lines. Several bright lines in both series were measured in γ Cassiopeiae and P Cygni. The relative importance of induced and spontaneous emission is discussed. The photospheric temperatures computed from the intensities of pairs of lines having a common upper level depend on the assumed distribution of the atoms with respect to azimuthal quantum number. Assuming distribution corresponding to thermal equilibrium leads to reasonable temperatures for both stars. The possibility of determining the amount of space absorption by comparison with other data is pointed out.

Introduction

The spectrum of the hydrogen atom consists essentially of a number of series of lines whose wave numbers are represented by the formulae $\nu = R_H(\frac{1}{n^2} - \frac{1}{m^2})$. For the lines of any one series the total quantum number of the lower level, n, is constant and m ranges through integral values from $n + 1$ to infinity. The series having n = 1, 2, 3, 4 and 5 have been observed, at least partially, in laboratory sources and are commonly designated by the names Lyman, Balmer, Paschen, Brackett and Pfund respectively. Until recently only the second of these series, lying within the approximate limits of 6600 to 3600 A, had been observed in celestial sources. Since 1932, however, Dr. P. W. Merrill of the Mt. Wilson Observatory has succeeded in obtaining good spectrograms of several stars in the region of 8900 to 8200 A. This range includes a number of lines of the Paschen series and makes possible for the first time a photometric study of them. Further details of this work and a qualitative description of the results may be found in a recent paper by Merrill[1).

The Paschen lines available for study extend from P 11 to about P 25[2). In order to obtain comparable material for the Balmer series, the writer took a number of spectrograms of the same stars with a three-prism ultra-violet spectrograph and 15-inch camera

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1) Ap. J., 79, 183, 1934

2) In this paper the Balmer and Paschen lines having m as the total quantum number of this upper level will be designated as Hm and Pm respectively.

attached to the Mt. Wilson 60 inch telescope. Eastman 33 plates were used for the latter work. The spectrograms of the Balmer series permitted a study of the lines from about H 6 (H δ) on out into the violet as far as higher members could be seen, thus corresponding to the portion of the Paschen series covered by the infra-red plates. The Balmer spectrograms have dispersions at λ 3771, H 11, of 12.5 A/mm; at λ 3671, H 24, 10.4 A/mm. For photometric calibration the U. V. spectrograph is provided with an auxiliary lamp and step-slit which permits ten standard continuous spectra of known intensity ratios to be impressed on the plate during a stellar exposure[3]. The infra-red plates, made with a plane grating spectrograph and 18-inch camera, have a dispersion of 33.4 A/mm. They were calibrated by exposures with a tube sensitometer through a deep red filter. The two methods of standardizing have been compared on ordinary emulsions and are in good agreement. In Table I are given, for convenience, the wavelengths of the Balmer and Paschen lines relevant to this investigation, together with the corresponding values of f, the oscillator strength (to be referred to later).

Before proceeding to the more detailed discussion of the results obtained it is well to state that, speaking qualitatively, the general appearance and behavior of the Paschen series in the spectra of the stars studied is quite similar to that of the Balmer

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3) For a description of this standardizing arrangement see: E. G. Williams, Mt. W. Contr., No. 487; Ap. J., 79, 280, 1934.

Table I

Wave-Lengths and f's of Balmer and Paschen Lines

Upper Level m	Balmer Series H IA	Balmer Series f	Paschen Series IA	Paschen Series f
3	6562.79	0.641	....	...
4	4861.33	.119	18751.05	0.842
5	4340.47	.045	12818.11	.151
6	4101.74	.022	10938.12	.056
7	3970.08	.013	10049.39	.028
8	3889.05	.0080	9545.98	.016
9	3835.39	.0054	9229.03	.0102
10	3797.90	.0038	9014.91	.0069
11	70.63	.0028	8862.79	.0048
12	50.15	.0021	8750.47	.0036
13	34.37	.0017	8665.03	.0027
14	21.94	.0014	8598.40	.0021
15	11.97	.0011	8545.39	.0017
16	3703.86	.00088	8502.50	.0014
17	3697.15	.00072	8467.27	.0012
18	91.56	.00061	8437.96	.00097
19	86.83	.00052	8413.33	.00083
20	82.81	.00044	8392.41	.00071

Table I (continued)

21	79.36	.00038	74.49	.00061
22	76.36	.00033	59.01	.00053
23	73.76	.00029	45.55	.00047
24	71.48	.00026	33.79	.00041
25	69.47	.00023	23.43	.00036
26	67.68	.00020	8314.27	.00032
.	...		...	
.	...		...	
.	...		...	
∞	3645.98		8203.58	

series. This fact is evident from an inspection of the microphotometer tracings in Fig. 1, and is also indicated by the contents of Table II. In the latter are listed all the stars having absorption lines of hydrogen which have been studied, together with the m's for the highest members of the two series which are visible on the original spectrograms. It is apparent that both series terminate at very nearly the same point.

Fig. 1a

Microphotometer Tracings

Balmer Series

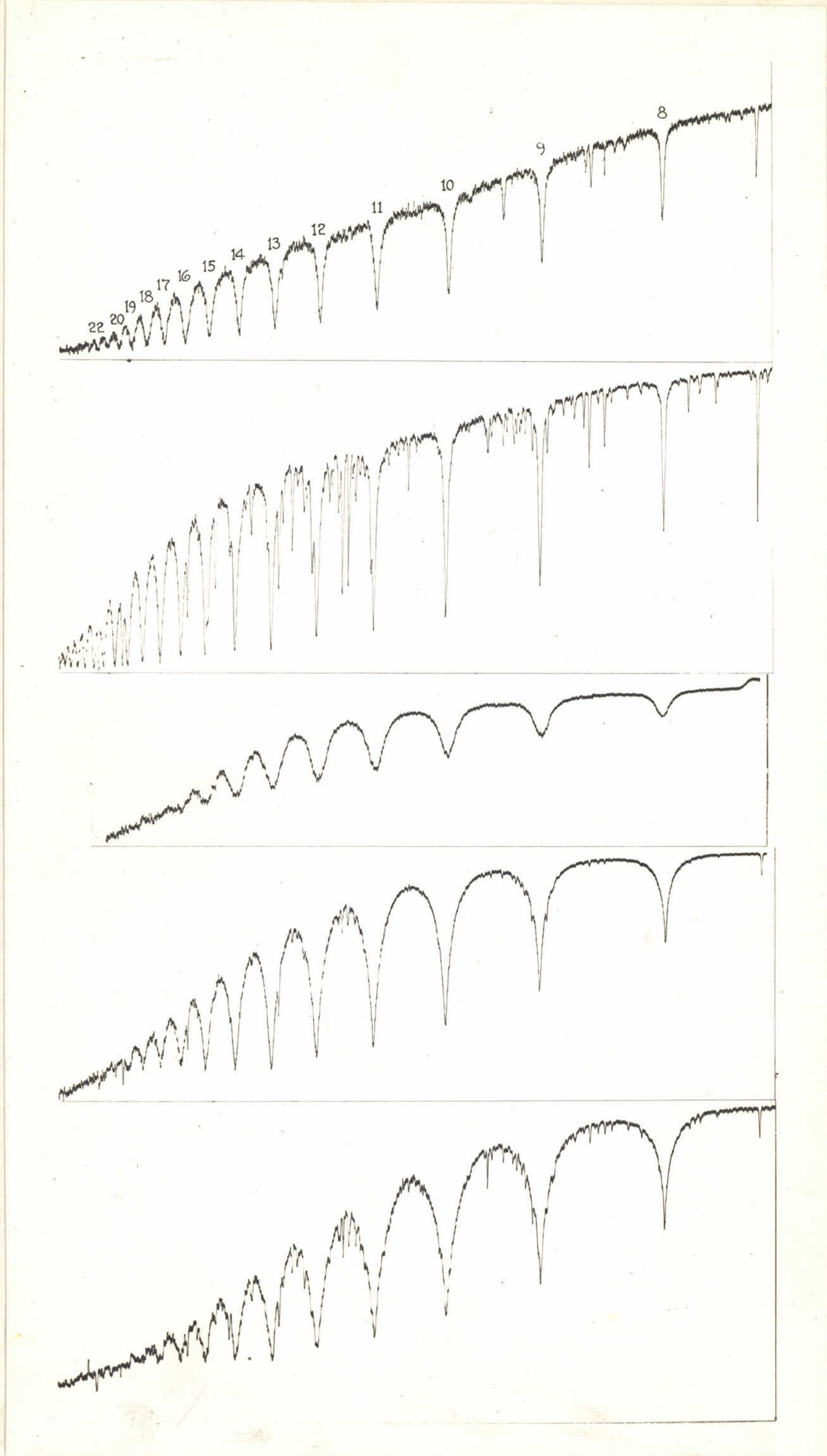


Fig. 1b

Microphotometer Tracings
Paschen Series

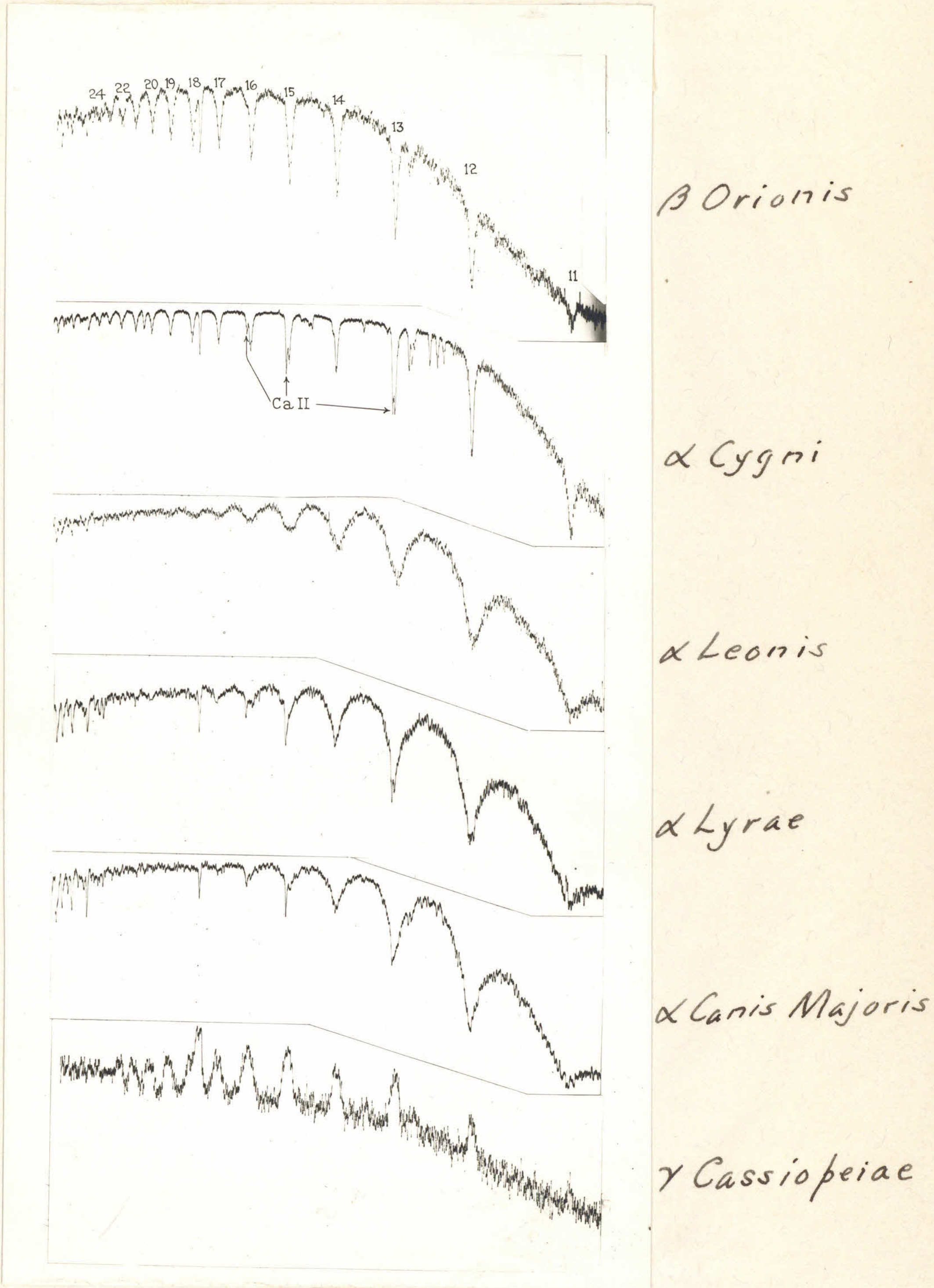


Table II

Lines of Shortest Wave-Length Observed in Balmer and Paschen Series

Star; Type	Balmer Series		Paschen Series	
	m		m	
α Leonis	16	distinct	19	
B8n	17	doubtful		
β Orionis	23		24	fairly well marked*
cB8				
α Lyrae	19	well marked	18	
A0	20	doubtful		
α Canis Majoris	18		17	present
A0			18	probable
α Cygni	28	fairly well	24	strong and sharp
	29	(marked	25	probable, but atmospheric
cA2	30	doubtful	26	lines interfere

* Following lines, if present, are lost among atmospheric lines.

Two other stars, γ Cassiopeiae and P Cygni, whose spectra show the Balmer and Paschen lines in emission, have also been investigated, but will be discussed separately later on.

The results of the investigation are now taken up in more detail under the headings

Structure of lines

Central intensities

Problem of overlapping wings

Total line absorptions

Shapes and intensities of emission lines.

Structure of Lines

The shapes of the lines have been determined as follows. On the microphotometer tracing the apparent continuous background is sketched in across each line so as to form part of the smooth curve joining the maximum between the successive lines. From this apparent background two or more sets of points are laid off in given intensity steps by means of the calibration curve. Smooth curves parallel to the background are then drawn in, joining the corresponding members of these sets of points. Then the distance, measured in the direction of the dispersion, between the two points where one of these curves cuts the tracing of the line, gives the width of the latter at that particular intensity. In this way the two wings of the line are not treated separately, and the result of plotting the width so obtained (divided by two) against the intensity may be termed the mean contour of the line. A number of these contours are shown in Fig. 2. In this drawing the abscissae are in Angstroms and the ordinates are the percent absorptions on a logarithmic scale. The percent absorption at the wavelength λ , denoted by $(PA)_\lambda$, is defined thus:

$$\frac{(PA)_\lambda}{100} = \frac{I^c_\lambda - I_\lambda}{I^c_\lambda}$$

where I_λ is the measured intensity in the line at λ , and I^c_λ is the intensity of the apparent continuous background at the same wave-

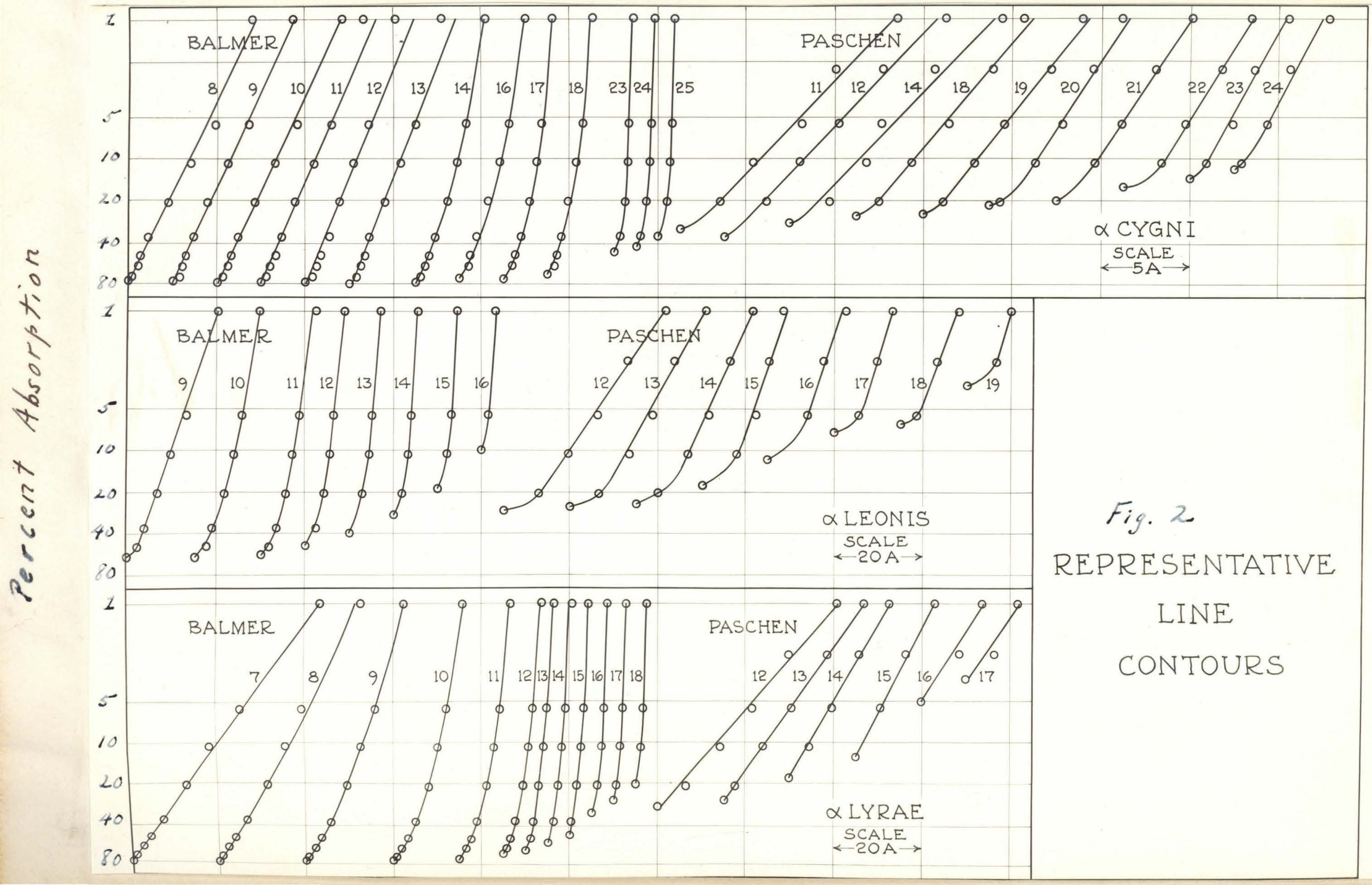


Fig. 2
REPRESENTATIVE
LINE
CONTOURS

length. It will be noted that the continuous background has thus far been referred to as apparent. The reason for this is that in regions of the spectrum where strong lines become crowded close together (relatively speaking) there are no points between them completely free of absorption. In other words the lines overlap. The continuous background which one draws on the tracing in such a case is, therefore, evidently lower than it would be were only one line present, the spectrum on either side of it being free of line absorption. Of course in such a case, i.e. of an isolated line, the apparent and true backgrounds become synonomous. More will be said of this effect later in the paper.

Returning to Fig. 2 it will be noted that many of the hydrogen lines have contours represented in such a plot by straight lines. Since the ordinates are on a logarithmic scale, this means that such contours may be expressed by the expontential formula

$$(PA)_{\lambda} = (PA)_{\lambda_0} \, e^{-k|\lambda - \lambda_0|}$$

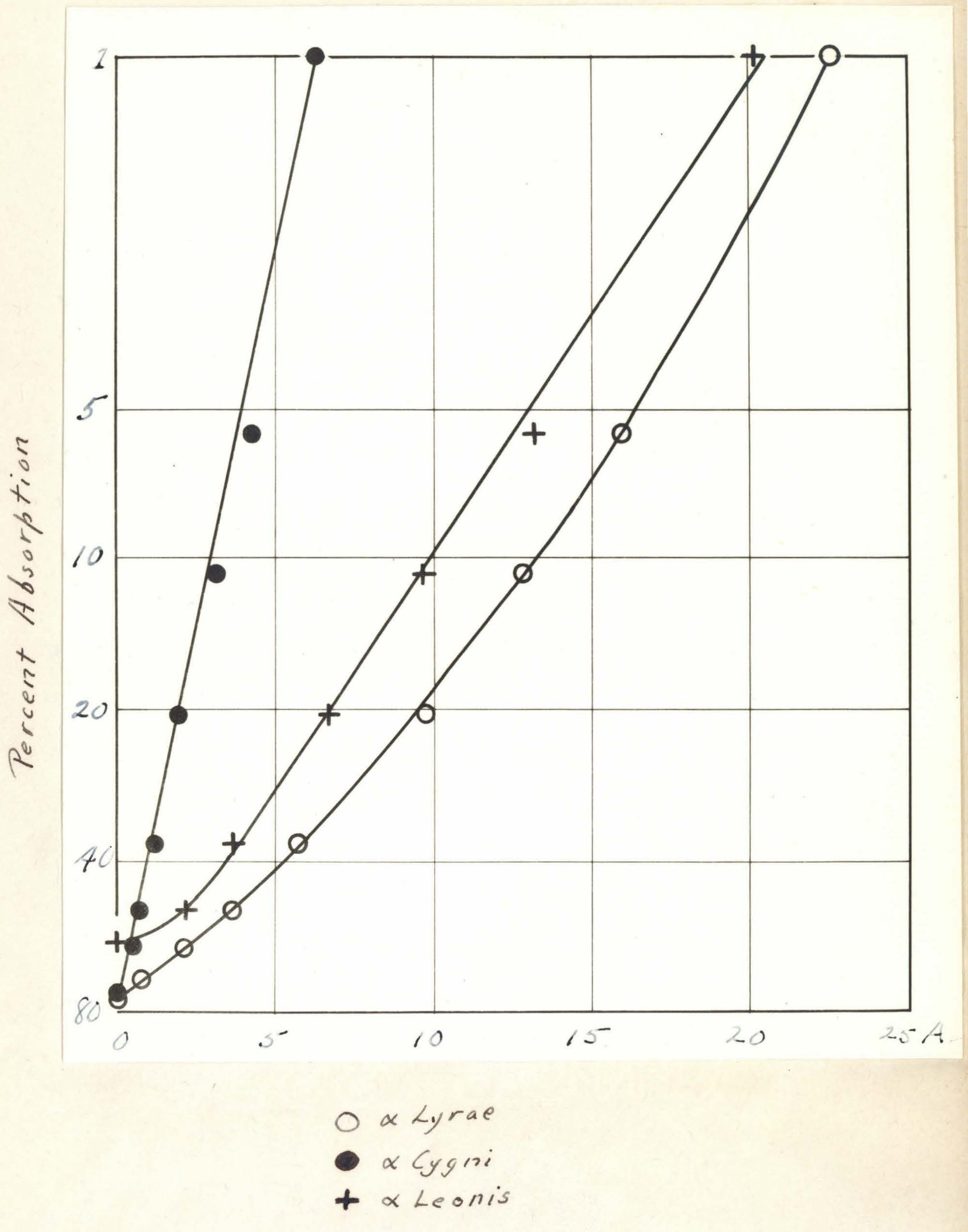
where λ_0 is the wavelength of the line center and k is a constant. This is in keeping with the results of numerous observers on earlier members of the Balmer series in these and similar stars. It is well to mention that in drawing these contours the widths where the line seems to merge into the continuous background have been arbitrarily plotted against a percent absorption of 1 instead of 0 as they should be, strictly speaking. It is necessary to do this in order to get these points in the finite region of the logarithmic scale. This is quite allowable on other grounds, however, since a difference of percent absorptions of one unit is quite outside the possibility of measurement.

The centers of most of the lines exhibit less absorption than that called for by a continuation of the exponential form of the wings. In α Leonis the effect amounts to a decided flattening of the centers of the lines. In α Lyrae, on the other hand, the lines while very wide have well marked cores. In the Balmer lines of α Cygni and β Orionis the exponential form continues practically to the center. A slight departure at the very center must be expected, of course, because the finite purity of the spectrograms would round off the sharp central cusp given by the formula, even if the latter were strictly accurate up to λ_0 .

For the narrow-line stars, β Orionis and α Cygni, the plotted points define nearly straight lines whose slopes for the first few members are essentially constant and then slowly increase in advancing from line to line toward the heads of the series. In the broad-line stars, α Leonis and α Lyrae, on the other hand, considerable curvature appears in many lines, although for one or two lines of smallest quantum numbers the contours are practically straight. In advancing down the series the wings apparently become less and less extensive as evidenced by the increasing curvature exhibited by the plotted points, an effect, doubtless, of overlapping.

Fig 3

H9 in α Lyrae, α Cygni and α Leonis

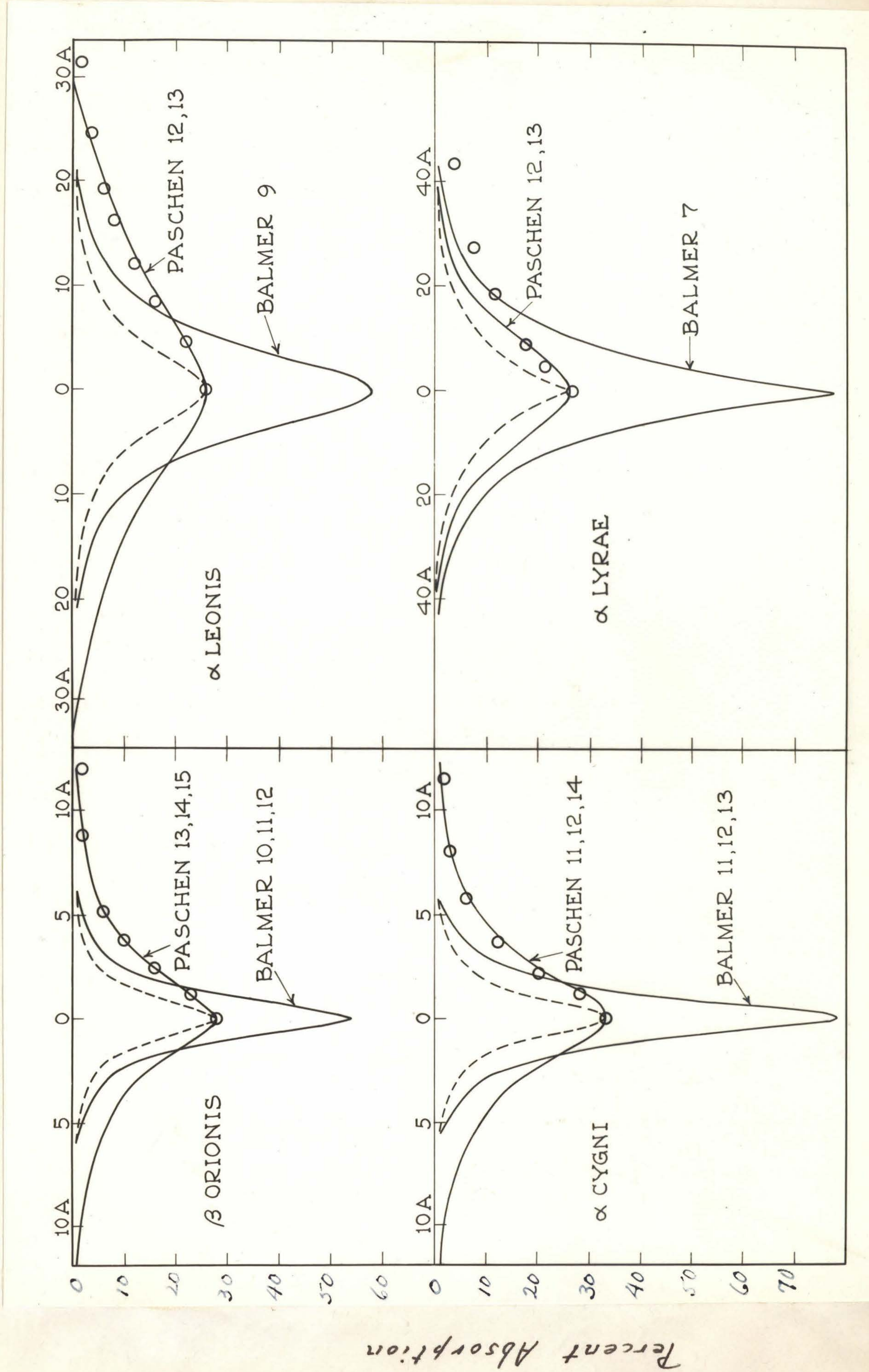


Hence it is necessary not only to distinguish between true and apparent background as previously mentioned, but also between the true and apparent shapes of lines in such regions of the spectrum.

The contours of H 9, λ 3835, in the spectra of α Cygni, α Lyrae and α Leonis are plotted on the same scale in Fig. 3.

Mean contours of Paschen lines in several stars are compared with those of Balmer lines in Fig. 4. The earliest available members of the Paschen series have been selected because they are the most intense and have the most accurately determined shapes. In the narrow-line stars, Balmer lines having the same or slightly smaller m's have been used; in the broad line stars, to avoid effects of overlapping, those with m's about 4 units smaller than the Paschen lines have been used. The individual lines chosen for each star are noted in the figure. The full lines represent the original mean contours; the dashed lines the Balmer contours with ordinates reduced by a factor which gives the curve the same central intensity as that of the Paschen contour. The circles are determined by multiplying the abscissae of the reduced curve by a factor, about 2.3, which is the ratio of the wavelengths of the Paschen and Balmer lines involved. The circles, therefore, represent the hypothetical contours which the Balmer lines would have at the wavelength of the Paschen lines if their structure were on a pattern proportional to wavelength, and the central intensity were reduced to that of the Paschen lines. It is interesting to find that for

Fig. 4



α LEONIS
PASCHEN 12,13
BALMER 9
α LYRAE
PASCHEN 12,13
BALMER 7
β ORIONIS
PASCHEN 13,14,15
BALMER 10,11,12
α CYGNI
PASCHEN 11,12,14
BALMER 11,12,13
30A
20
10
0
10
20
30A
40A
20
0
20
40A
10A
5
0
5
10A
10A
5
0
5
10A
0
10
20
30
40
50
60
0
10
20
30
40
50
60
70
Percent Absorption

the c stars, β Orionis and α Cygni, the circles lie remarkably near to the actual Paschen contours; and that for α Leonis, in whose spectrum the lines are extremely wide and diffuse, the correspondence is almost as good. The widths of the hydrogen lines in these three stars might, therefore, as far as the above mentioned facts are concerned, be due to radial motion. The hypothesis of stellar rotation does not seem applicable to β Orionis and α Cygni however, since lines of elements other than hydrogen are quite sharp in their spectra. Possibly some sort of turbulence, more or less restricted to the hydrogen atoms in the atmospheres of these stars, is the cause of this. Stark effect is another possibility, although hitherto it has not been supposed to be very considerable in stars of this type whose atmospheres are likely to be extensive and of low density. In the case of α Leonis, since all the lines present are wide and shallow, the rotational explanation is very likely the correct one.

In contrast to the above stars, the relation of the line shapes in the two series is evidently quite different in α Lyrae.[4)] In this star the circles representing the reduced Balmer contours do not fit those of the Paschen lines, but indicate considerably more intense wings. Thus it appears impossible to ascribe the wide

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4) α Lyrae and α Canis Majoris are so nearly identical as regards the shapes and intensities of the hydrogen lines that the latter is omitted from further separate discussion. Any statements made here or elsewhere in the paper about one of these stars may be considered to apply equally to the other.

wings of the hydrogen lines in these stars to any sort of radial motion and one must look for a widening mechanism which does not produce displacements proportional to wavelength. For some time it has been supposed that Stark effect is the cause of the great extent of the wings in stars of this type. Unfortunately, however, no general theory of the shapes of lines widened in this way has yet been given and so for the present it is impossible to state whether or not Stark effect would lead to the relations between Balmer and Paschen lines observed here.

Central Intensities

In Table III are collected the values of the percent absorption at the centers of the lines measured. These are the apparent central absorptions, as noted previously, and in the majority of cases differ more or less from the true absorptions which would be found if the lines could be observed separately.

The general run of the central absorptions is well shown in Fig. 5 in which are plotted the data of Table III together with a few values for earlier members of the Balmer series which are means of measures by C. T. Elvey[5] and E. G. Williams[6]. Attention may be called to certain points of interest exhibited by this figure.

In the first place, the Paschen lines have in general

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5) Ap. J., 71, 191, 1930

6) Annals of the Solar Physics Observatory, Cambridge, 2, 25, 1932

Fig 5
Central Absorptions

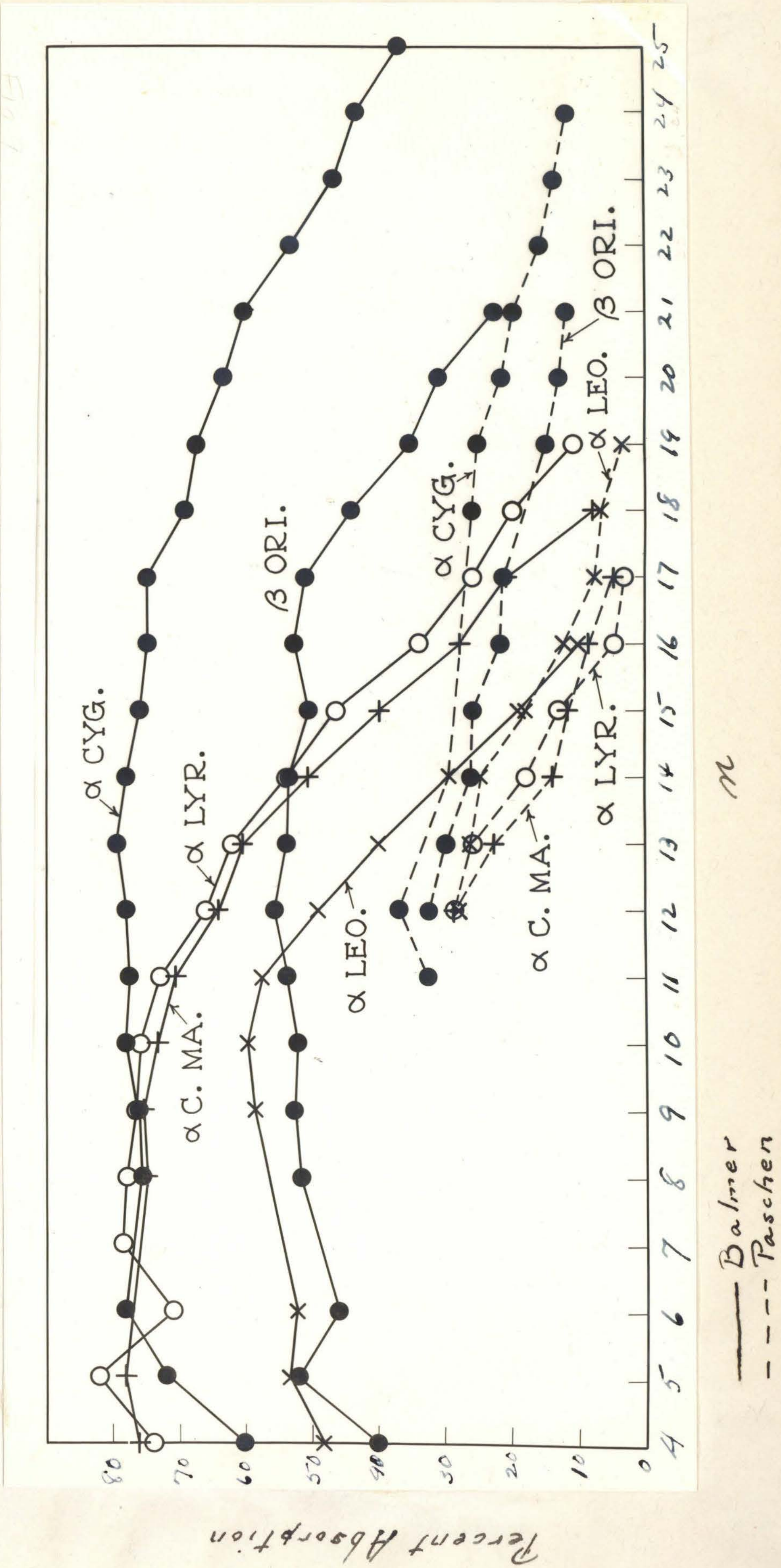


Table III

Apparent Central Percent Absorptions

m	α Cygni cA2		β Orionis cB8		α Leonis B8n		α Can Maj A0		α Lyrae A0	
	Balmer	Paschen	B	P	B	P	B	P	B	P
6							80			
7							80		79	
8	76		52				75		78	
9	76		52		58		75		77	
10	78		52		60		73		76	
11	78	32	54		58		70		73	
12	78	37	56	32	49	28	64	28	66	28
13	80		54	30	40	26	60	22	62	26
14	78	30	53	26	30	24	50	14	54	18
15	76		50	26	19	18	40	12	46	13
16	75		53	22	10	12	28	8	34	4
17	75		51	21		8	21	4	26	3
18	69	26	44			6	8		20	
19	68	25	36	15		3			11	
20	63	22	31	13						
21	60	20	22	12						
22	53	16								
23	47	14								
24	43	12								
25	37									

smaller central absorptions than the corresponding Balmer lines. The one exception to this statement is α Leonis in whose spectrum the Paschen lines are deeper after m = 16. Secondly it will be noted that for the narrow line stars, α Cygni and β Orionis, both series show slower decrements and can be observed to correspondingly shorter wavelengths than for the others. This must be due, partly at least, to the much smaller effect of the overlapping of the lines when the latter are narrow. In the next section is described an attempt to answer this question.

Problem of Overlapping Lines

In comparing the two hydrogen series it is evidently desirable to use lines having, as nearly as possible, the same upper levels, and, with the material available thus far, this leads at once to the problem of overlapping wings. Although P 11 has been observed a few times, P 12 is, in general, the first Paschen line which yields results of value. Thus we wish to make use of Balmer lines with $m \geq 12$. For stars with narrow lines this can be done fairly well, as a reference to Fig. 2 shows that no serious distortion of the exponential shapes of the lines in the Balmer series sets in until m is about 14. But for stars like α Lyrae it is evident that, at least from m = 8 onward, the lines deviate more and more from the simple exponential form. That this deviation must be due, at least in part, to the overlapping of adjacent lines is supported by the following considerations.

Considering α Cygni, for example, we find the mean measured widths at the background of the earlier Balmer lines to be about 12 or 13 A. Then, if the curling inward of the contours as we proceed down the series is due to overlapping, we should expect this effect to appear at that point in the series where the distance between successive lines first becomes equal to the mean width. This occurs at about m = 13 or 14 for α Cygni and it is, in fact, at just this point that the distortion first becomes appreciable. The same argument applies to β Orionis, and, with more uncertainty, to the wide-line stars. In the Paschen series, owing to the smaller depth and wider spacing of the lines, this effect is less noticeable and apparently does not occur until farther along in the series than for the Balmer lines.

These facts led to an attempt to eliminate the effects of overlapping. In order to make even a beginning, however, certain assumptions are necessary. In the first place, we suppose that at any point, λ , in the spectrum where the optical thickness due to any one constituent of the star's atmosphere is $\tau_\lambda^{(1)}$, the intensity which gets through to the outside is given by $I_\lambda^{(1)} = I_0^{(1)} e^{-\tau_\lambda^{(1)}}$ where $I_0^{(1)}$ would be the intensity of the escaping radiation if this particular absorbing constituent of the atmosphere were absent. Then if any other constituent of the atmosphere has the optical thickness $\tau_\lambda^{(2)}$ at λ we will have

$$I_\lambda^{(2)} = I_\lambda^{(1)} e^{-\tau_\lambda^{(2)}} = I_0^{(1,2)} e^{-\tau_\lambda^{(1)} - \tau_\lambda^{(2)}}$$

Thus, generalizing, I_λ , the actual intensity of the radiation escaping through the star's atmosphere will be given by

$$I_\lambda = I_0 e^{-\sum_i \tau_\lambda^{(i)}}$$

where I_0 is now the photospheric intensity, i.e. the intensity which would be emitted if the star's atmosphere were completely transparent at wavelength λ . In this equation the summation extends over all the different constituents of the atmosphere having non-zero absorption coefficients at λ . Thus, if we are considering the Balmer series and suppose the energy levels to be smeared out by any cause, then the various constituents of the atmosphere, in the above sense, are hydrogen atoms in the second energy level which are about to make a transition to any of the upper levels such that the quantum of energy absorbed has wavelength λ . If we knew the value of $\sum\tau$ at every wavelength we could calculate the resultant energy curve of the star and compare it with observation. However, we do not have this information and must proceed differently.

Attention has been called to the fact that as far down the Balmer series as the point where overlapping begins, the slopes of the exponential contours remain sensibly constant. Thus it does not seem unreasonable to assume that the slopes of the true contours maintain this constancy beyond the point where overlapping begins.

The true central absorptions of the lines will presumably decrease down the series at a different rate from that observed, the difference being due to the effect of overlapping.

The first calculation was made to find out what assumed run of true central intensities, together with the above assumption of constant slope for all the lines, would reproduce the observed run of central absorptions for the Balmer series in the case of α Lyrae. No simple direct way of working this out was evident, so recourse was had to a graphical method, as follows.

On a long strip of paper the positions of the Balmer lines and of the points half way between them were plotted on a wavelength scale. Then the slope observed for H 7 was taken to be the true slope for all the lines and the latter were plotted, one above the other, on a sheet of semi-logarithmic paper, with an assumed decrement of the central absorptions down the series. These assumed contours then appeared as a series of straight lines all having the same slope, but with progressively shorter intercepts on the percent absorption axis as the line number, m, increased. Suppose then that the effect of the nth line was to be calculated. The sheet of semi-log paper was laid upon the wavelength plot so that λ_m on the latter coincided with the percent absorption axis of the former. Then the percent absorption due to λ_m could be read off directly at the points λ_m , $\lambda_{m\pm1}$, $\lambda_{m\pm2}$, etc. and at the mid-points $\frac{\lambda_m+\lambda_{m\pm1}}{2}$, $\frac{\lambda_{m\pm1}+\lambda_{m\pm2}}{2}$, etc., in both directions from λ_m as far as the absorption due to the latter extended appreciably. The effect of each line was computed each way as far as it exceeded about 1%.

In this way the percent absorptions due to all relevant lines were obtained at all the line centers λ_n and at all the mid-points $\frac{\lambda_n + \lambda_{n\pm 1}}{2}$. These percent absorptions were then converted into Δ log I values by means of a curve, and summed for each point. This is in accord with the equation

$$I_\lambda = I_0 e^{-\sum_i \tau_\lambda^{(i)}} \tag{1}$$

which is readily seen to be equivalent to

$$(\Delta \log I)_{total} = \sum_i \Delta_i \log I \tag{2}$$

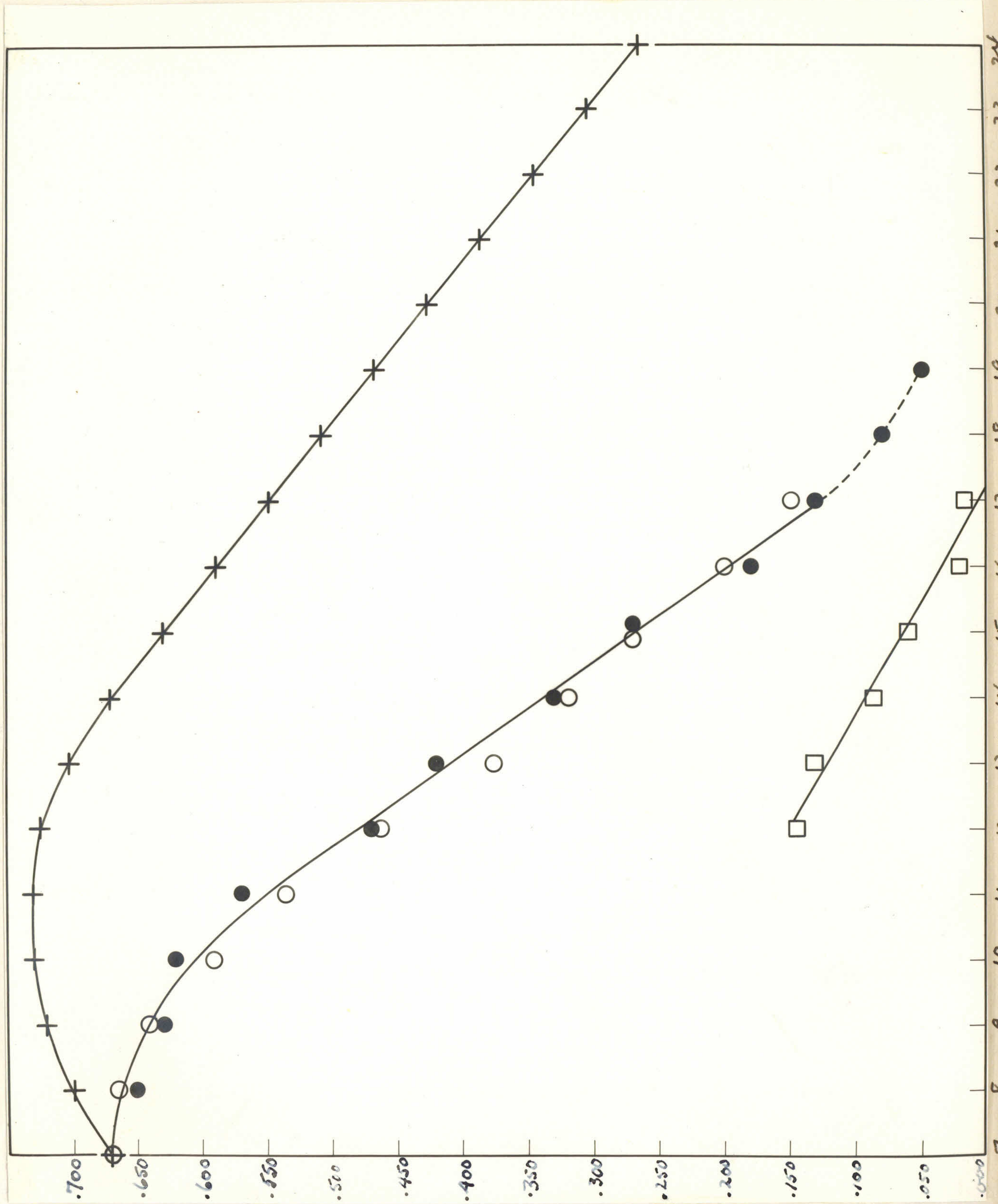
The total Δ log I for the background at the nth line, λ_n , was taken as the mean of the Δ log I's for the mid-points on either side of it, $\frac{\lambda_n + \lambda_{n+1}}{2}$ and $\frac{\lambda_n + \lambda_{n-1}}{2}$; for the later members of the series. For the earlier lines a slight correction was made to allow for the fact that λ_n does not lie exactly half-way between the mid-points as defined above. The quantity to be compared with the observations is then

$$(\Delta \log I)_{apparent} = (\Delta \log I)_{line} - (\Delta \log I)_{background}$$

Although equation (2) is in natural logarithms, the computations were made directly in common logs, since the quantities with which comparison was to be made were also in this form and hence no conversion was necessary.

Fig 6

Observed and computed central intensities in Balmer series of α Lyrae



+ Assumed Balmer
● Observed "
○ Computed "
□ Observed Paschen

Δ log I

n →

Fig. 6 shows the final results of these calculations obtained after a number of trials. The ordinates are the central Δ log I's plotted against the line numbers as abscissae. It is to be noted that this is a considerably more sensitive means of representation than would be had if the percent absorptions were plotted, especially for the larger values. In this diagram the filled circles are the observed apparent Δ log I's for the Balmer lines in α Lyrae. The crosses are the assumed values and the open circles have been calculated from them by the above method. The observations of the Paschen lines are represented by the squares. In the latter, the effect of overlapping is probably much smaller than for the Balmer series and the observed decrement may be a fair approximation to the actual one. It is evident that, of the observed apparent and the assumed actual decrements of the Balmer lines, the latter is in better agreement with the observed Paschen decrement than the former, a not unreasonable result.

Obviously the next step in testing the utility of this method is to see whether the assumed true contours which gave the intensity decrement of Fig. 6 are capable of reproducing the measured apparent contours. To make these calculations it is convenient to modify the previously described method somewhat. If we define, as is customary,

$$\Delta \log I = \log \frac{I^{(c)}}{I} = \log \frac{1}{r}$$

then our equation (1) is equivalent to

$$(r_\lambda)_{total} = \prod_i (r_\lambda)_i$$

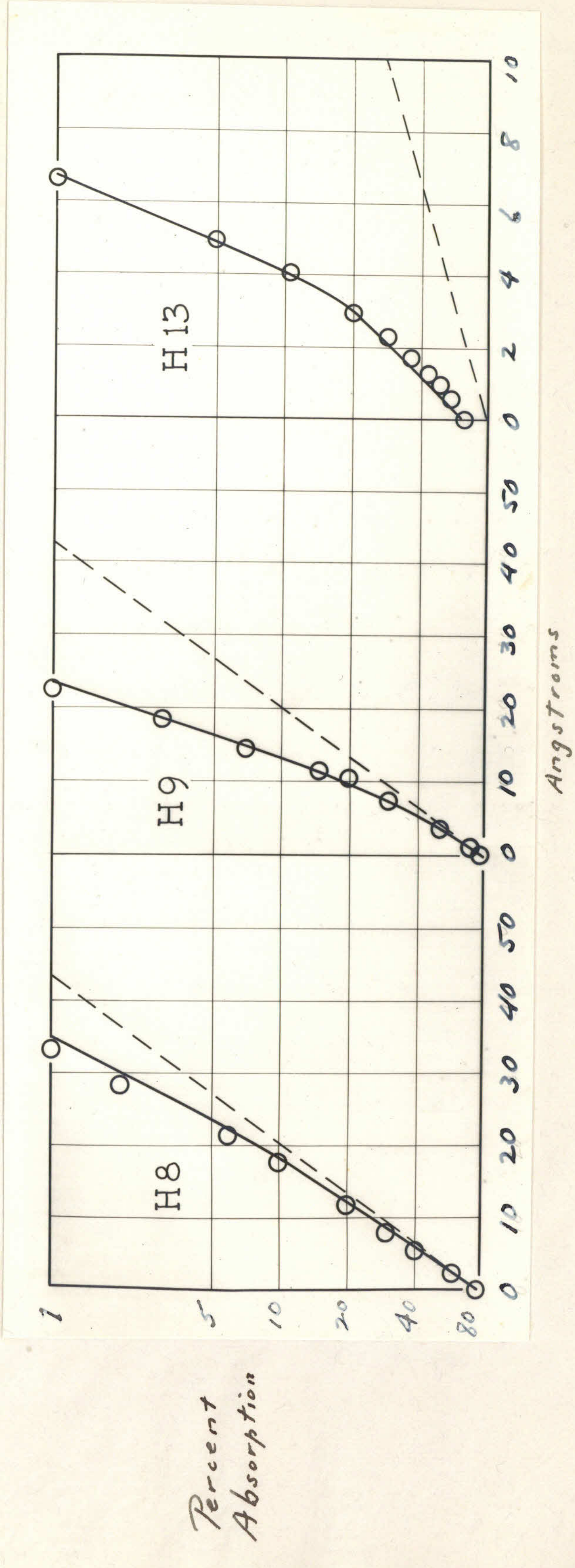
The application of this equation is as follows. The percent absorption scale on the semi-log paper will serve equally well as an r scale if we simply run the numbers in the opposite direction. That is, 20% absorption is denoted by r = .80, 10% by r = .90 etc. So, if one plots a number of assumed overlapping lines on this paper, the total r at any λ is quickly calculated by successive multiplications on a slide rule of the r's corresponding to each line at that point. In this way the relative r values can be plotted on the same sheet of paper. These r values have a minimum at the wavelength of any absorption line and a maximum on either side nearly coinciding with the mid-points between the line in question and its two nearest neighbors. The straight line drawn between the two maxima will thus correspond to the background sketched in across the line on the original microphotometer tracing. Then the final step is, for each λ , to set the relative r for the background equal to .99 on the slide rule and to read off the absolute r corresponding to the relative r in the line at that wavelength. These latter quantities are then plotted in the usual way and are then directly comparable with the observed contours.

These calculations were carried out for H 8, H 9 and H 13 for α Lyrae by making the same assumption as to the true slopes of the lines as previously, and using the assumed central absorptions of Fig. 6. The results are shown in Fig. 7 where half-widths are plotted against percent absorptions. The solid curve is the calcu-

Fig 7

Observed and computed contours

α Lyrae



Solid lines : computed
Circles : observed
Dotted lines : assumed true contour

lated contour and the circles are the observations. The broken line represents the assumed original contours. The agreement is seen to be excellent throughout. It must be pointed out, however, that the coincidence of the calculated with the observed points at the background (i.e. r = .99) can be given no weight whatsoever. A little consideration will show that this must necessarily follow regardless of the assumed central intensities and slopes, provided only that the latter do not change too rapidly from line to line.

Thus far the calculations presented above have been found to be in good agreement with the observations, but now some unfavorable considerations must be mentioned. In the first place, the contours calculated in this way are, unfortunately, not very sensitive to the assumptions made as to the original lines. To test this point, the calculations were repeated for H 13 in α Lyrae by arranging all the relevant hypothetical lines to have only half as great an intercept on the λ axis as in the first attempt, and adjusting their assumed central absorptions so as to give the correct computed value. The contour computed from these assumptions fitted the observed points nearly as well as did the first one. It was definitely a little too low, as might have been expected, but not a great deal. Thus the first criticism to be raised is that the validity of the assumed true contours can not be tested by a comparison of the calculated with the observed shapes.

The second difficulty is the following. The calculations which resulted in Fig. 6 give us the Δ log I at all the points in the spectrum mid-way between the lines as explained previously. These values are such that, with our original assumptions, we find practically no absorption at the point $\frac{H_6 + H_7}{2}$ whereas at $\frac{H_{17} + H_{18}}{2}$ only 1/20 of the photospheric intensity gets through. Yü's[7] measures give for the intensity ratio between these points the value $\frac{H_6 + H_7}{2} \div \frac{H_{17} + H_{18}}{2} \sim 2$. Hence if the present calculations were correct it would mean that the photosphere of α Lyrae radiates about 10 times as much energy at λ3700 as at λ4100. This result is absurd.

Several ways out of this dilemma may be suggested. In the first place, the assumption as to constant slope for the true contours of the lines may be wrong. The lines may actually become narrower in passing down the series and thus produce less absorption in the regions between them. There is, however, no evidence for this and no reason to anticipate such behavior; in fact if the wings were due to Stark broadening one would naively expect just the opposite to occur.

Secondly, it is generally supposed that the light abstracted from the outward beam by an absorption line is returned to the photosphere, raises the temperature of the latter, and reappears as general photospheric radiation. However, if there were some mechanism by which the energy absorbed by the crowded lines near the head of the series was returned spread out through the same general range of

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7) C. S. Yü, L.O.B., 12, 155, 1926

the spectrum, then the whole region might be boosted sufficiently to overcome this difficulty. I am not aware of any mechanism for doing this however.

The third suggestion appears most likely to me. The calculations were based on the assumption that $I = I_o e^{-\Sigma\tau}$ as explained previously. Physically this corresponds to the case of pure absorption, i.e., the radiation completely disappears as such. The predominating process may very well be scattering, i.e., the radiation is merely changing its direction constantly with no essential changes in wavelength once it leaves the photosphere. As has been known since Schuster's paper of 1905, the solution of this problem is, essentially, $I = \frac{I_o}{1 + \Sigma\tau}$ (to the first order). For $\Sigma\tau$ small it is evident that the two expressions give the same result, but for $\Sigma\tau$ large, the exponential will cut down the intensity much faster. This may well be the reason for the large discrepancy mentioned previously. At any rate, owing to these uncertainties, no attempt has been made to correct any of the observations for overlapping in the remainder of the paper.

Total Absorptions and Populations of Second and Third Levels

Unsöld[8] has given a method of calculating the number of hydrogen atoms in the second energy level from the measured total absorptions of the Balmer lines, and has applied it to the solar spectrum. He supposes that in going down the series radiation damp-

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8) A. Unsöld, Zeitschrift für Physik, 59, 353, 1929

ing plays a gradually decreasing part in producing the line contours, until finally it can be neglected altogether. Then, making the further assumption that the light re-emitted in the lines is negligible, i.e., that $I = I_0 e^{-\tau}$, it follows that

$$\int \tau \, d\lambda = \frac{\pi e^2 \lambda^2}{m c^2} f N$$

In this equation f is the so-called oscillator strength and N the number of atoms in the second level. The solar hydrogen lines give low values of N for the earlier members of the series, but the values for successive lines increase at first and then seem to attain a constant value, a behavior in accord with Unsold's assumptions. It is evident that the N so obtained is a lower limit. Unsöld fixes an upper limit by fitting to the observations of $H\alpha$, the line most affected by radiation damping, a theoretical contour involving N. The difference between the two limits is such as to make it probable that either will give a fair approximation to the truth.

In the present work we must proceed by the first method since the earlier Balmer lines have not been observed. It must also be borne in mind that in the solar spectrum the Balmer lines have a rapid decrement and fade out entirely before there is any possibility of overlapping, while just the opposite is true of the spectra studied here. In fact no attempt is made to evaluate N_2 or N_3 for the wide-line stars since it appears that only meaningless results would be obtained. This was one of the difficulties which it was hoped would be overcome by the study of overlapping, but the uncertainties mentioned previously indicate the need of further study of this method before applying it. Therefore, for want of a better means of evaluating N, Unsöld's equation, with its inherent assumptions, is used directly. We have

$$\frac{\pi e^2 \lambda^2}{mc^2} Nf = \int \tau \, d\lambda = \int \Delta \log_e I d\lambda = 2.30 \int \Delta \log_{10} I d\lambda$$

The quantities under the last integral are those directly observed and hence the right hand side of the equation is known for all the lines. The f's are included in Table I. The first nine are taken from Sugiura's[9] paper and the rest were calculated by means of the well known asymptotic formulae

$$f_{n,m} = \frac{\text{const}}{n^3}, \text{ for n large.}$$

In this way N_2 and N_3 were calculated from each of the lines measured in α Cygni, β Orionis and α Leonis. In Table IV are given the absorptions, $\int \log_e \frac{I}{I_0} d\lambda$, and the logarithms of N_2 and N_3 for these three stars. The log N's are also plotted in Fig. 8.

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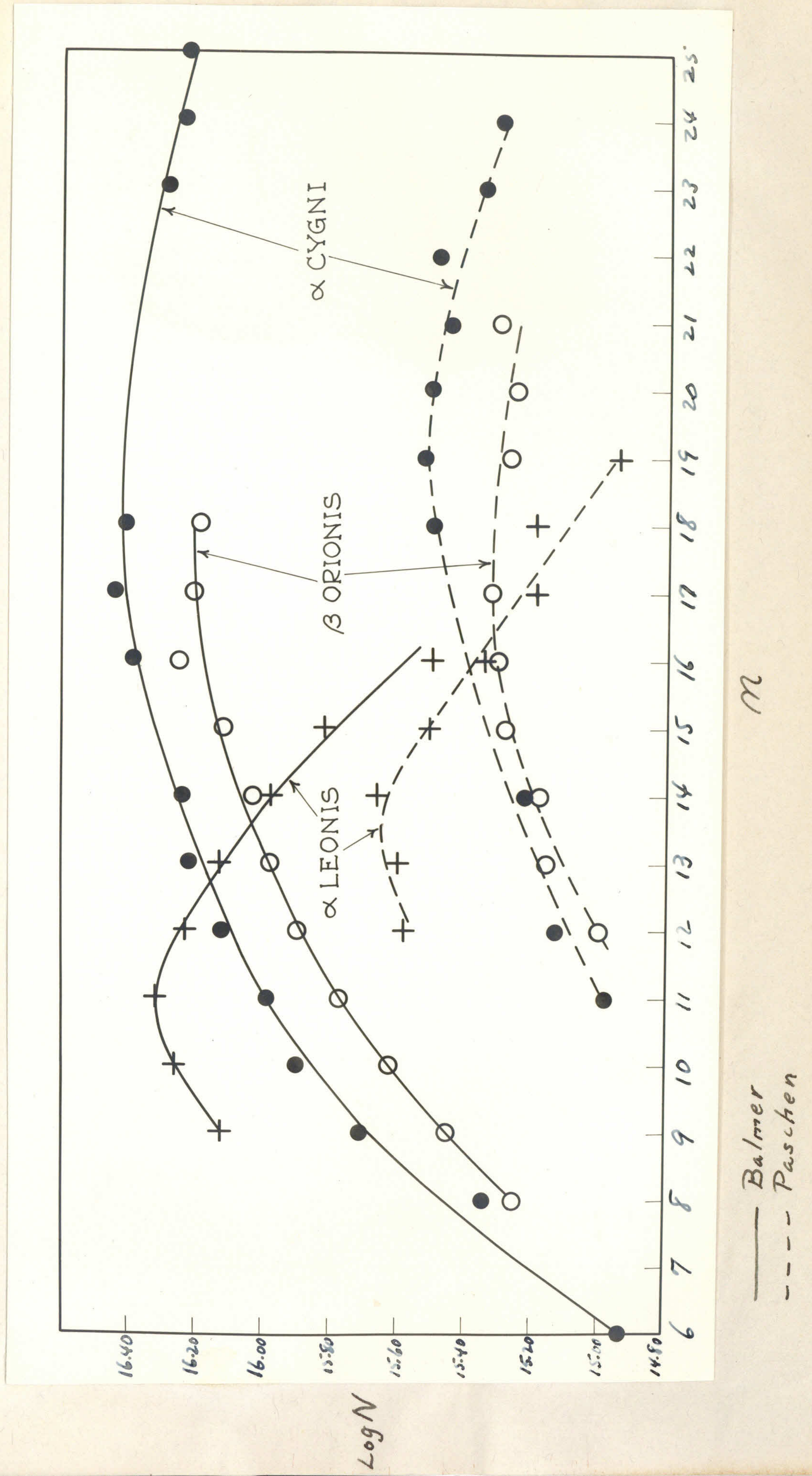
9) Scientific Papers of the Institute of Physical and Chemical Research, Tokyo, 11, 1, 1929

Table IV

n	α Cygni				β Orionis				α Leonis			
	Balmer		Paschen		Balmer		Paschen		Balmer		Paschen	
	Abs	Log N_2	Abs	Log N_3	Abs	Log N_2	Abs	Log N_3	Abs	Log N_2	Abs	Log N_3
6	2.74											
7												
8	2.96	15.34			1.90	15.25						
9	3.59	15.71			1.99	15.45			9.26	16.12		
10	3.84	15.90			2.10	15.62			8.77	16.26		
11	3.41	15.99	2.76	14.98	2.05	15.77			7.19	16.32		
12	3.47	16.12	3.00	15.13	2.11	15.90	2.25	15.00	4.46	16.23	8.60	15.58
13	3.48	16.22			1.98	15.98	2.54	15.16	2.84	16.13	6.95	15.60
14	2.95	16.24	2.26	15.22	1.82	16.03	2.02	15.18	1.62	15.98	6.16	15.66
15					1.69	16.12	2.09	15.28	0.84	15.82	3.52	15.51
16	2.65	16.39			1.89	16.25	1.78	15.30	0.34	15.50	1.96	15.34
17	2.41	16.44			1.40	16.21	1.59	15.32			1.17	15.19
18	1.92	16.41	1.91	15.50	1.14	16.19					0.94	15.19
19			1.75	15.53			0.96	15.27			0.45	14.94
20			1.40	15.51			0.77	15.25				
21			1.08	15.45			0.76	15.30				
22			1.00	15.49								
23	0.68	16.29	0.64	15.35								
24	0.54	16.24	0.50	15.30								
25	0.46	16.23										

Fig 8
Populations of Levels 2 and 3



Roughly speaking, the general form of these curves is similar to that found for the solar lines by Unsold, but instead of asymptotically approaching a limiting value they rise to a maximum and then decrease. The most probable explanation is that after the lines begin to overlap their apparent areas decrease so rapidly as to overcome the natural tendency of the N's to increase. This undoubtedly accounts for the very early turning point reached by the α Leonis curves in comparison with the others.

The best values of N_2 and N_3 will be those read from the maxima of the various curves. As noted previously these N's will necessarily be lower limits. But while both N_2 and N_3 may be considerably in error, the general parallelism of the two curves for each star make it appear fairly likely that the relative values of the numbers of atoms in the two states may be moderately accurate. In any event, assuming this to be the case, it remains to examine the data as to its consistency with thermodynamic equilibrium. If the stars' atmospheres are in thermodynamic equilibrium we have

$$N_3 = N_2 \frac{g_3}{g_2} e^{-\frac{E_3-E_2}{kT_0}}$$

which may be written

$$\frac{9400}{T_0} = \log \frac{N_2}{N_3} + 0.35$$

Table V gives the values of N_2 and N_3 read from the curves and the corresponding values of T_0 calculated from the above equation.

Table V

Numbers of Hydrogen Atoms in the Second and Third Levels

	$\log N_2$	$\log N_3$	$\log N_2/N_3$	T_o	T_e
α Cygni	16.42	15.50	0.92	7400°	8800°
β Orionis	16.21	15.32	0.89	7600	9000
α Leonis	16.30	15.62	0.68	9100	11000

The T_o calculated in this way is analogous to what is known as the boundary temperature in the theory of radiative equilibrium. The relation of the latter to the effective temperature, T_e, which is that judged by color, etc., is $T_e = \sqrt[4]{2}\, T_o$. The corresponding effective temperatures appear in the last column of Table V. The result for α Leonis cannot be given as much weight as the other two. These temperatures are somewhat lower than those usually attributed to these stars, but are not so far off as to invalidate the assumption of thermodynamic equilibrium. In particular, there is no evidence for the effect between the second and third levels as was noted by Unsöld between the first and second. He found, on the assumption of equilibrium, that the number of atoms calculated to be in the ground state from the observed number in the second level was far larger than could be admitted on other grounds. Of course this difficulty remains in the present work.

Intensities of Emission Lines

P Cygni and γ Cassiopeiae have long been known as having bright Balmer lines in their spectra. Infra-red plates of these stars show that all of the Paschen lines in the observable region are also bright. The violet plates for each of these stars were taken about a year later than the red ones and hence the two may not be strictly comparable. This is especially true for γ Cassiopeiae whose Balmer lines are known to consist of two components whose relative intensities show variations from time to time. For P Cygni, however, no variations of the bright lines are known.

In discussing the present observations of the emission lines the usual picture of a Be star is adopted, namely that the bright lines originate in a more or less extended gaseous envelope surrounding an otherwise normal B type star. The density in this envelope is assumed to be sufficiently low so that collisions may be neglected and the material may be supposed transparent to its own radiations. The best procedure appears to be that in which Balmer and Paschen lines having a common upper level are discussed. In this way the difficulty of knowing how the populations of the upper levels vary with the total quantum number can be avoided.

Let π_{mn} be the probability that in unit time an atom in state m will perform the transition $m \rightarrow n$. Then if we have a mass of gas in which the number of atoms in the m^{th} level, N_m, is kept constant by any means, there will be a steady emission of radiation of frequency ν_{mn} from it of amount

$$E_{mn} = h \, \nu_{mn} \, N_m \, \pi_{mn} \quad \text{per unit time}$$

Generalizing this expression a little we find for the ratio of energies emitted in a Balmer and Paschen line of common upper level m the value

$$\frac{E_{m,2}}{E_{m,3}} = \frac{\nu_{m2}}{\nu_{m3}} \left[\frac{N_{mS} \Pi_{mS \to 2P} + N_{mP} \Pi_{mP \to 2S} + N_{mD} \Pi_{mD \to 2P}}{N_{mS} \Pi_{mS \to 3P} + N_{mP}\left(\Pi_{mP \to 3S} + \Pi_{mP \to 3D}\right) + N_{mD} \Pi_{mD \to 3P} + N_{mF} \Pi_{mF \to 3D}} \right]$$
$$= \phi(N, \Pi)$$

The notation is obvious when it is remembered that the ratio on the left will depend on the distribution of atoms with respect to azimuthal quantum number in the upper level and hence that the transitions between the various sub-levels concerned must be taken into account. Of course the total number of atoms in the upper level, N_m, is not necessarily equal to $N_{mS} + N_{mP} + N_{mD} + N_{mF}$ since atoms with $\ell \geq 4$ cannot jump to either the second or third levels and thus can contribute nothing to Balmer or Paschen line.

In order to be able to write the above ratio for a gaseous stellar envelope such as we have assumed, two points must be looked into. First, the relative values of the N_ℓ's might conceivably depend on the distance from the star, and second, the π's certainly will. As to the first, if the mechanism of excitation is the same throughout the gas and varies only as to its intensity, then the ratios of the N_ℓ's will probably remain constant. This seems reasonable and we assume it to be so. The π's must now be examined in more detail. As usual we have

$$\pi_{mn} = A_{mn} + U_\nu B_{mn}$$

where U_ν is the density of radiation of frequency, $\nu = \frac{E_m - E_n}{h}$ and A_{mn}, B_{mn} are, respectively, the Einstein coefficients of spontaneous and induced emission. Moreover

$$U_\nu = \frac{1}{c}\int I_\nu \, d\omega$$

and $$B_{mn} = \frac{c^3}{8\pi h \nu^3} A_{mn}$$

Thus $$\pi_{mn} = A_{mn}\left(1 + \frac{c^3}{8\pi h\nu^3}\,\frac{1}{c}\int I_\nu \, d\omega\right)$$

Inside a hohlraum at temperature T

$$I = \frac{2h\nu^3}{c^2}\left(e^{\frac{h\nu}{kT}} - 1\right)^{-1}$$

and is independent of direction so that

$$\pi_{mn} = A_{mn}\left(1 + \frac{1}{e^{h\nu/kT} - 1}\right).$$

For the frequency of $H\alpha$, for instance, $\frac{h\nu}{k} = 21{,}900$ and so for practically all laboratory experiments involving optical frequencies one can say that

$$\pi_{mn} = A_{mn}$$

However, this will not apply in the present case since we are dealing with temperatures of the order of 20000° or so.

Making use of the above relations it is easily seen that one may write

$$\frac{E_{m2}}{E_{m3}} = \phi(N,\pi) = \Gamma\phi(N,A).$$

The factor Γ depends on the temperature of the star, on the lines considered, and on the structure of the gaseous envelope. Thus, if r be the radius of a thin spherical shell of gas surrounding a star, in terms of the radius of the photosphere as unit, then we compute the following table of Γ, for H_{14} and P_{14}, as a function of T and r.

Table VI

Distance from center	T = 15000°	25000°	35000°
r = 1	.836	.745	.692
2	.972	.950	.935
5	1.000	1.000	.990
10	1.000	1.000	1.000
100	1.000	1.000	1.000

Thus, aside from the layer very near the star, the effect of the density of radiation on the ratio is very small, but without more precise knowledge as to the size and structure of the envelope we cannot calculate it exactly. As a reasonable value we take $\Gamma = 0.9$.

To calculate the Amn's the well known relation

$A_{mn} = \frac{g_n}{g_m}\frac{3}{\tau} f_{mn}$ was used, where $\tau = 4.51\lambda^2$. The sub f's can readily be gotton from the total f's of Table I by a simple algebraic process. Table VII gives the values of $\frac{E_{14,2}}{E_{14,3}}$, for $\Gamma = 0.9$, corresponding to certain simple distributions of the atoms among the sub-levels of the upper state of total quantum number 14.

Table VII

	N_S	N_P	N_D	N_F	$E_{14,2}/E_{14,3}$
a)	1	0	0	0	3.3
b)	0	1	0	0	6.5
c)	0	0	1	0	6.0
d	0	0	0	1	0
e	1	3	5	7	3.1

The last line is the distribution in thermodynamic equilibrium. The observations are now to be discussed in conjunction with this table.

First the matter of line structure must be investigated. Dr. Merrill has measured the widths of the Paschen lines in γ Cassiopeiae and the distance between the emission and absorption components, E - A, in P Cygni. His results on γ Cassiopeiae are summarized in Table VIII.

Table VIII

Widths of Paschen Emission Lines in γ Cassiopeiae

m	Mean	Measured	Calculated (Hγ = 4.4A)
12 - 17	8590 A	9.5 ± 0.5 A	8.7 A
19 - 22	8380	8.5 ± 0.2	8.5 A

The measured widths of the Paschen lines are seen to agree well with those calculated from the measured width of H γ on the assumption that $\frac{\Delta\lambda}{\lambda}$ = const. In the same way for P Cygni he finds E - A = 3.3 A for the Paschen lines and E - A = 1.5 A for five Balmer lines of mean wavelength 3760 A. Multiplying the latter value by the ratio of the mean wavelengths yields E - A(calc.) = 3.3 A for the Paschen lines. Thus it seems quite safe to say that the structure of the emission lines is such that $\frac{\Delta\lambda}{\lambda}$ = const., and we make use of this fact in what follows.

Let I_B and I_P be the central intensities of a Balmer and Paschen line respectively, and I_{BC}, I_{PC} the corresponding intensities in the star's continuous spectrum. Then if E_B and E_P are the total energies emitted in the two lines we have

$$E_B = k\, I_B\, \Delta\lambda_B \quad \text{and} \quad E_P = k\, I_P\, \Delta\lambda_P$$

where the $\Delta\lambda$'s are mean widths of some sort, defined in the same way for both lines.

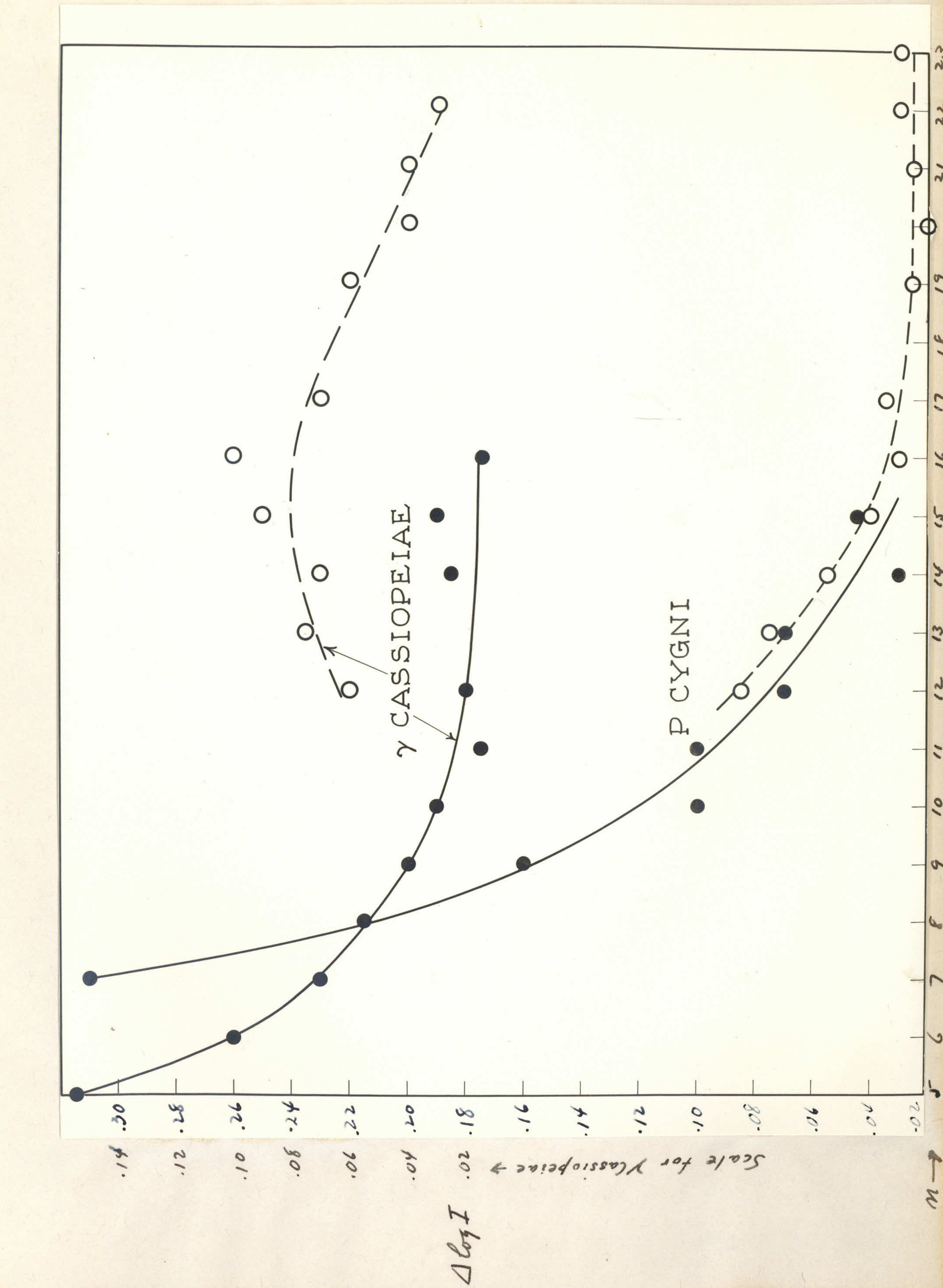
Then $$I_B = \frac{E_B}{k\,\Delta\lambda_B} = \frac{E_B}{\text{const} \times \lambda_B}$$

and $$I_P = \frac{E_P}{\text{const} \times \lambda_P} \quad \text{since } \frac{\Delta\lambda}{\lambda} = \text{const.}$$

Now let $$r_B = \frac{I_B}{I_{BC}}, \qquad r_P = \frac{I_P}{I_{PC}}$$

and $$\rho = \frac{I_{BC}}{I_{PC}}$$

Fig 9
Central Intensities of Emission Lines



Then $\rho = \frac{I_{BC}}{I_{PC}} = \frac{I_B}{r_B} \cdot \frac{r_P}{I_P} = \frac{E_B}{E_P} \cdot \frac{\lambda_P}{\lambda_B} \cdot \frac{r_P}{r_B}$

$\frac{E_B}{E_P}$ we have calculated theoretically, r_P and r_B have been measured (and are plotted in Fig. 9) and so ρ, the ratio of the intensity of the continuous spectrum in the ultra violet to that in the infra-red may be calculated.

Now, both from the hypothetical picture we have adopted and also from the appearance of the tracings, the bright lines are to be considered merely as little additions or humps placed on top of the star's continuous spectrum. Thus, what has been measured as Δ log I is really

$$\Delta \log I = \log \left(\frac{I_c + I}{I_c}\right) = \log \left(1 + \frac{I}{I_c}\right) = \log (1 + r)$$

Since $I \ll I_c$ this gives $\Delta \log I = \log (1 + r) = r$, or in common logarithms, $\Delta \log I = 0.43\, r$.

The data read from the curves of Fig. 9 for H 14 and P 14 are as follows:

	Balmer 14		Paschen 14	
	Δ log I	r	Δ log I	r
γ Cass.	0.04	0.10	0.09	0.23
P Cygni	0.05	0.12	0.06	0.14

From these are calculated the corresponding values of ρ and T, shown in Table IX. These are given for each of the hypothetical distributions of Table VII

Table IX[10)]

Photospheric Temperatures Derived from Intensities of Emission Lines

Hypothesis	γ Cassiopeiae		P Cygni	
	ρ	T	ρ	T
a)	18	27000°	8.6	12000°
b)	36	-	16.9	24000
c)	33	-	15.6	22000
d)	-	-	-	-
e)	17	25000	8.1	12000

Since there seems no reason to suppose that a distribution of atoms similar to case a) is at all likely, the results for γ Cassiopeiae indicate a distribution similar to that in thermal equilibrium. The same conclusion cannot be stated so definitely for P Cygni. However, if the mode of generation of the emission lines is the same for both stars, then P Cygni is intrinsically, as well as apparently, the cooler of the two. Moreover, all of these possible temperatures are higher than those found by direct color measures. The latter average about 15000° for γ Cassiopeiae and about 7000°

10) The blanks in the T column indicate that for the corresponding value of ρ no real positive temperature exists.

for P Cygni. This is, of course, to be expected if there is much space reddening.

In conclusion, it must be stated that too much reliance can not be placed on the results of this section, owing to the scanty material available. Nevertheless, the method presented here appears sound, and, in particular, it should prove of value in determining the cause of the abnormally low color temperatures of many B type stars. Thus, if good observations were available for a number of Be stars, yellowish and otherwise, the simple and plausible assumption that the mechanism responsible for the bright lines was the same for all should settle the question as to whether the yellowish ones appear so due to space reddening, or whether they are intrinsically cooler than the others.

In conclusion, it is a great pleasure to express my heartiest thanks to Dr. P. W. Merrill who invited my collaboration in this research, which is being published jointly elsewhere, and who supplied many stimulating suggestions during the course of the work as well as the infra-red spectrograms on which it is based. I am greatly indebted also to Dr. W. S. Adams, Director of the Mt. Wilson Observatory, for his kindness in permitting me to use this work as a thesis, and to Mr. F. Ellerman for the prints used to illustrate it.