# Scalable Analysis of Nonlinear Systems Using Convex Optimization

Thesis by

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To Nayia

Σα βγείς στον πηγαιμό για την Ιθάκη
να εύχεσαι νάναι μακρύς ο δρόμος
γεμάτος περιπέτειες, γεμάτος γνώσεις.
Ιθάκη, *Κωνσταντίνος Καβάφης, 1911*

When setting out upon your way to Ithaca,
wish always that your journey be long,
full of adventure, full of lore.
Ithaca, *Constantinos Kavafis, 1911*

# Acknowledgements

I was once told that this section takes longer to write than any other part of a thesis – I never believed it. Now I see why this is so: it's hard to find words great enough to express gratitude to everyone that contributed to the completion of this journey.

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However I put it, my time in the US was empty in one particular aspect. Long distance relationships are hard, this one wasn't easy either. I want to express my deepest love to Nayia, who even though was 5,500 miles away, always patiently supported me in my endeavor. I dedicate this thesis to you, Nayia.

# Abstract

In this dissertation, we investigate how convex optimization can be used to analyze different classes of nonlinear systems at various scales algorithmically. The methodology is based on the construction of appropriate Lyapunov-type certificates using sum of squares techniques.

After a brief introduction on the mathematical tools that we will be using, we turn our attention to robust stability and performance analysis of systems described by Ordinary Differential Equations. A general framework for constrained systems analysis is developed, under which stability of systems with polynomial, non-polynomial vector fields and switching systems, as well as estimating the region of attraction and the $L_2$ gain can be treated in a unified manner. Examples from biology and aerospace illustrate our methodology.

We then consider systems described by Functional Differential Equations (FDEs), i.e., time-delay systems. Their main characteristic is that they are infinite dimensional, which complicates their analysis. We first show how the complete Lyapunov-Krasovskii functional can be constructed algorithmically for linear time-delay systems. Then, we concentrate on delay-independent and delay-dependent stability analysis of nonlinear FDEs using sum of squares techniques. An example from ecology is given.

The scalable stability analysis of congestion control algorithms for the Internet is investigated next. The models we use result in an arbitrary interconnection of FDE subsystems, for which we require that stability holds for arbitrary delays, network topologies and link capacities. Through a constructive proof, we develop a Lyapunov functional for FAST – a recently developed network congestion control scheme – so that the Lyapunov stability properties scale with the system size. We also show how

other network congestion control schemes can be analyzed in the same way.

Finally, we concentrate on systems described by Partial Differential Equations. We show that axially constant perturbations of the Navier-Stokes equations for Hagen-Poiseuille flow are globally stable, even though the background noise is amplified as $R^3$ where $R$ is the Reynolds number, giving a ‘robust yet fragile’ interpretation. We also propose a sum of squares methodology for the analysis of systems described by parabolic PDEs.

We conclude this work with an account for future research.

# Contents

# List of Figures

# List of Tables

# Chapter 1

# Introduction

Ἀρχή ἥμισυ παντός
Ἀριστοτέλης
Well begun is half done
Aristotle

One of the primary concerns in control system design is *guaranteed functionality and performance* under varying environmental conditions and uncertain system parameters. These properties should be verified before the design is implemented, so as to avoid possible malfunctions which are usually a result of bad design or a design based on an inadequate model. Nowadays, these objectives and specific system limitations are well understood, something that is reflected into the design of reliably functional chemical plants, nuclear power stations, aircraft, etc. A major future challenge that is suggested by the technological advances of the 20th century [49], is the design and analysis of large-scale networks-of-systems that inevitably adds an important adjective to the design objectives: the desired robust functionality and performance also need to be *scalable*, meaning that the system properties should scale with the system size and be independent of the introduction of new technologies, different network topologies and varying system parameters [61].

In extracting valuable information about the functionality of designed systems, it is basic good practice to construct truthful, robust, differential equation models based on theoretical principles or experimental data. For example, we tend to model simple mechanical systems or electrical circuits by a set of Ordinary Differential Equations (ODEs); simple communication networks or predator-prey models in ecology by

Functional Differential Equations (FDEs); and systems involving heat transfer, fluid motion or wave propagation using Partial Differential Equations (PDEs). In going from finite dimensional models (ODEs) to infinite dimensional ones (FDEs or PDEs), the richness of modeling tools increases, and delay/aftereffect and distributed systems can be modeled adequately. What is indeed remarkable is the wealth of modeling frameworks that are available for describing the world around us; what is distressing is that the more complicated the system description, the more apparent is the absence of efficient algorithmic tools to answer questions of interest about them.

What makes the problem more interesting, but at the same time almost intractable, is the requirement that the functionality and performance of an arbitrary interconnection of such components–modules to form large-scale networks be *scalable*. In this case, the components themselves may be described by finite or infinite dimensional models. A particular example of a large-scale networked system in which the modules themselves are infinite dimensional is the Internet [89]: the source/link dynamics are adequately described by Functional (Delay) Differential Equations, and the interconnection topology of the sources and links is arbitrary. Here, by 'scalable stability' we mean the stability of an infinite-dimensional system on one hand, and a large-scale interconnection on the other, which should also be robust to the sizes of the round trip times and the capacities of the links in the network. Such questions will appear frequently in the future, as the advances and merging of computing, communications and control create new challenges for system analysis and design.

It is indeed true that in most cases, the questions we wish to answer about such systems fall into complexity classes that are computationally difficult to answer. Take as an example the question of robust stability of a linear system under structured uncertainty: it is known that the $\mu$ recognition problem with either purely real or mixed real/complex uncertainties is $\mathcal{NP}$-hard [11], implying that most probably there is no polynomial-time algorithm to solve it exactly, unless $\mathcal{P} = \mathcal{NP}$. However, simple, algorithmically verifiable criteria can yield valuable information about the system's robustness properties - in this case, in the form of upper bounds on the value of $\mu$.

In this thesis, we take this viewpoint. Even though the questions we need to answer about our models may be computationally expensive, we seek algorithmically verifiable tests/answers to analysis questions for nonlinear systems described by Ordinary, Functional and Partial Differential Equations of either small-scale or large-scale. At this point, we should stress that this methodology is not based on simulations. Simulations can only be used to give an idea of the system behavior, but can never guarantee that the system is flawless for all initial conditions and parameters, however fine a gridding of these spaces may be. The situation is even more complicated for infinite dimensional systems; not only is gridding of the initial condition space impossible, but also simulations are hard to set up and take a lot of computational power, depending on the fidelity required.

To present the methodology we wish to follow, consider the following stability-related question: 'Do all trajectories of the following system, starting from an initial condition in the initial set $\mathcal{X}_0$, go to the origin?'

$$\frac{dx(t)}{dt} = f(x(t)), \quad x(0) = x_0 \in \mathcal{X}_0 \tag{1.1}$$

Here, $f$ is known exactly and $f(0) = 0$. $\mathcal{X}_0$ is a domain that contains the origin, and we assume that solutions to this system exist and are unique, properties that are guaranteed locally if $f$ is Lipschitz in $\mathcal{X}_0$. Note that the question we are interested in – whether *for all initial conditions in the set* $\mathcal{X}_0$ the trajectories of the system tend to the origin – cannot be answered by simulation alone. Solving the differential equation symbolically may answer this question, but most ODE systems do not have solutions in closed form, let alone infinite dimensional ones or differential equations describing large-scale network interactions. On the contrary, the method that A. M. Lyapunov suggested in 1892 follows a complementary, ingenious approach [107]: by constructing an energy-like function whose level curves are 'trapping regions' for the system trajectories – i.e., whenever a trajectory enters one such trapping region, it can never escape it – the boundedness of the trajectories is ensured. Under some stronger conditions, the convergence of the trajectory to the equilibrium is guaranteed. If every initial condition in $\mathcal{X}_0$ is contained in a level set of this energy function, the question is answered *exactly*; the distinct feature of this approach is that it *does not require the solution* of the underlying differential equation. Still, a problem remains: how does one construct this energy-like function, a function of state, that *proves* stability of the equilibrium? For years, this was left to the imagination of the researcher and general guidelines suggested considering energy-based candidates first. However, intuition alone was never enough to allow their construction and an efficient algorithmic methodology was needed. The technical conditions that a Lyapunov function $V(x)$ has to satisfy for asymptotic stability are a positivity condition $V(x) > 0$, along with a negative definite time derivative along the system's trajectories, $\dot{V}(x) < 0$, properties that are inherently difficult to test.

For the special class of systems described by linear Ordinary Differential Equations of the form $\dot{x} = Ax$, Lyapunov functions can be constructed by solving a set of Linear Matrix Inequalities [10], i.e., a semi-definite programme [96]. This is because it is necessary and sufficient to choose $V(x) = x^T P x$, with $P > 0$; the derivative condition then becomes $A^T P + PA < 0$. The matrix $A$ being Hurwitz is equivalent to the existence of a feasible solution to the semidefinite programme with constraints $P > 0$ and $A^T P + PA < 0$. Semidefinite programming in general, and Linear Matrix Inequalities in particular, have been an attractive algorithmic tool for robust systems analysis for years [106], due to the fact that they are worst-case polynomial time complex to solve [51].

Consider now the special class of nonlinear systems that are described by ODEs with polynomial vector fields $f$ and consider Lyapunov function candidates $V(x)$ of polynomial form. Even in this restricted case, the two Lyapunov conditions are polynomial non-negativity conditions, and testing them is known to be $\mathcal{NP}$-hard when they are of a degree of at least 4. In part, this explains the lack of efficient algorithms for the construction of Lyapunov functions. Nonetheless, if we relax the non-negativity conditions to the existence of a sum of squares decomposition – a method that was introduced in Pablo A. Parrilo's thesis [66] – the problem reduces to the solution of a semidefinite programme just as in the case of linear systems. This observation has opened the way for algorithmic analysis of nonlinear systems described by ODEs.

Lyapunov theory now forms the basis of nonlinear control and dynamical systems methodologies to investigate equilibrium stability, input-to-state and performance calculations, estimating basins of attractions, synthesizing control laws, etc. It is readily applicable to other system classes, such as stochastic systems, hybrid systems, systems described by Functional Differential Equations and systems described by Partial Differential Equations. For scalable functionality of nonlinear systems, a 'scalable' Lyapunov function argument is usually employed, i.e., we seek a function which satisfies the Lyapunov conditions independent of the size of the network and the interconnection topology.

Inevitably, therefore, the availability of efficient algorithmic tools for the analysis of nonlinear systems described by ODEs has opened the way for the efficient analysis of other classes of systems. This thesis is about the algorithmic analysis of nonlinear systems ranging from a more general class of finite dimensional (ODE) models to infinite-dimensional ones described by FDEs and PDEs. In a later chapter, we will also consider the analysis of a large-scale network interconnection of FDE systems, modeling sources and links in the Internet. The scope is to ensure scalable stability of network congestion control for arbitrary networks, delays and link capacities which we achieve by constructing a Lyapunov-type certificate.

## 1.1 Outline and Contributions

A schematic of the structure of this thesis is shown in Figure 1.1. It covers analysis of systems along two axes related to scale as they were outlined in the previous section: From Ordinary Differential Equations (finite dimensional) to Functional and Partial Differential Equations (infinite dimensional); and from small-scale ODE/FDE systems to large-scale, interconnected ones related to network congestion control for the Internet. Here, we summarize the contents of each chapter, emphasizing the main contributions.

- In Chapter 2, we review the theory behind polynomial non-negativity, the sum of squares decomposition and its algorithmic verifiability. We introduce positivstellensatz, a central theorem in real algebraic geometry, giving examples of how it can be used, and we present key results stemming from positivstellensatz that will be used in the rest of this thesis.

- In Chapter 3, we concentrate on systems described by ordinary differential equations, and show how small-scale dynamical systems can be analyzed effectively using sum of squares. We investigate robust stability of nonlinear and switching systems as well as performance analysis, applying our results to examples ranging from biology to aerospace.

- In Chapter 4, we extend our results to systems of infinite dimension described by Functional Differential Equations. We first present how Lyapunov functionals can be constructed even for the case of linear systems – something that was difficult before as it involves the solution of parameterized Linear Matrix Inequalities. We then consider the stability and robust stability of nonlinear time delay systems, both delay-independent and delay-dependent, based on the construction of Lyapunov functionals. We end the chapter with an illustrative example from ecology.

- In Chapter 5, we investigate the problem of stability analysis of network congestion control schemes for the Internet for arbitrary network topologies. The subsystem dynamics are modeled by Functional Differential Equations, i.e., the effect of heterogeneous delays in the network is accounted for, and so is the fact that the system is an arbitrary interconnection of such subsystems. We present a Lyapunov argument for the analysis of the linearization, as well as the full global stability analysis for arbitrary topologies, delays and link capacities. The proof is constructive and the structure of the system helps greatly in the choice of the Lyapunov certificate.

- In Chapter 6, we consider the stability analysis of systems described by Partial Differential Equations. These equations are usually used to describe spatially

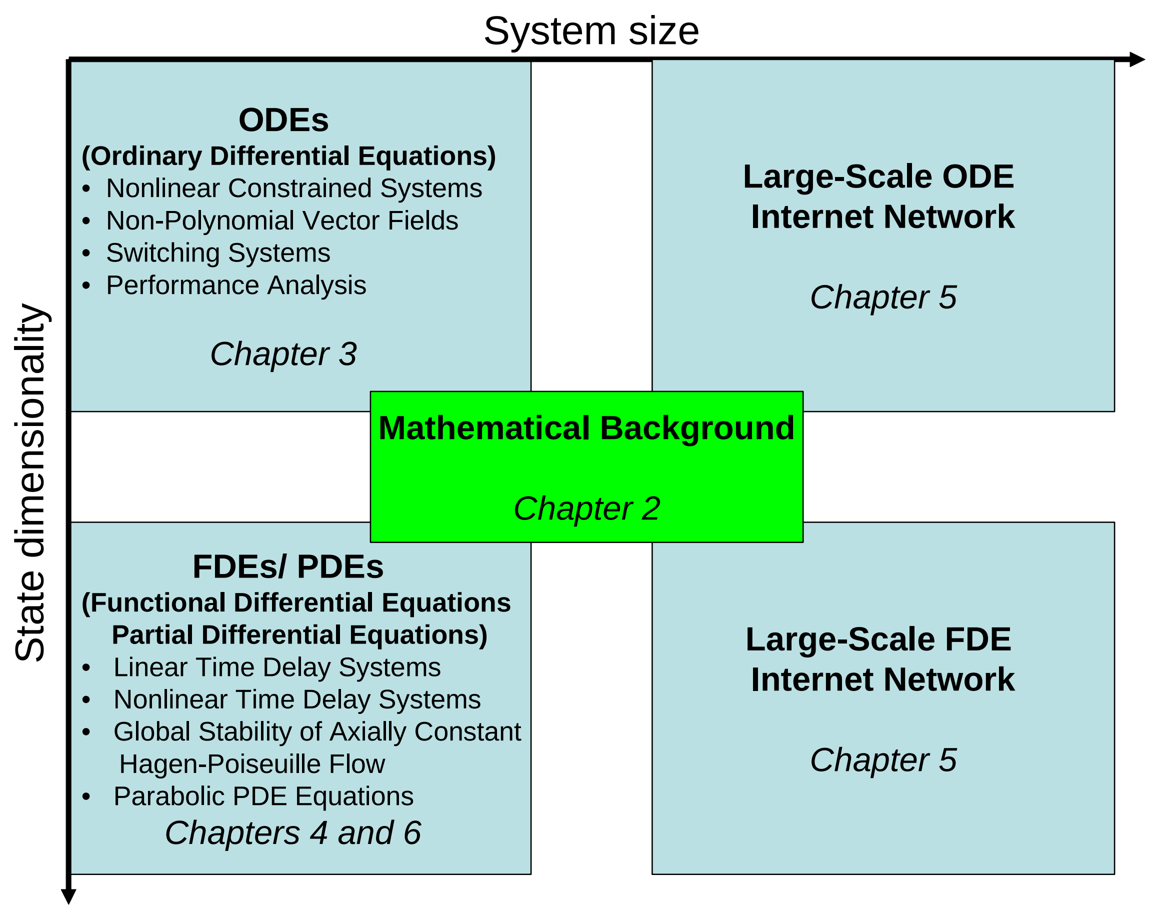



Figure 1.1: The outline of this thesis.

distributed systems, such as systems arising in fluid mechanics and heat transfer. We show how the Navier Stokes equations with axially constant perturbations and initial conditions for Hagen-Poiseuille (pipe) flow are globally stable, while they retain an $R^3$ growth on the background noise where $R$ is the Reynolds number. The 'robust yet fragile' properties of the system are evident, in that streamlining the flow can prohibit bifurcations to instabilities at the expense of increased sensitivity to disturbances and uncertainties. We then develop an algorithmic methodology for constructing Lyapunov functionals for PDE systems using the sum of squares decomposition.

- We conclude the thesis in Chapter 7 with future research directions.

# Chapter 2

# Mathematical Preliminaries

Πάντα κατ᾽ ἀριθμόν γίγνονται
Πυθαγόρας
Everything is made of numbers
Pythagoras

In this chapter, we present some of the mathematical ideas and algorithmic methodologies that will be employed in the rest of this thesis, based on the work of Pablo A. Parrilo [66]. These tools are an assemblage of important notions and machinery from algebraic geometry, optimization and control theory and find application in fields ranging from systems analysis to combinatorial optimization, physics etc. It will be appreciated later through particular remarks, that they do not only unify known results in many fields in these areas, but also extend them in a natural way. A particular example is Yakubovich's S-procedure – an important tool in robust control theory – for which better conditions can now be obtained.

We begin this chapter by introducing tools from algebraic and polynomial geometry such as polynomial non-negativity and the Sum of Squares decomposition and describe some of the properties of polynomials that possess such a decomposition and how it can be computed algorithmically. We then briefly describe SOSTOOLS, a software that facilitates the search for a Sum of Squares decomposition given a polynomial structure (i.e., monomials that are present). Positivstellensatz – a theorem central in Real Algebraic Geometry – is presented next, followed by a discussion on some applications from diverse fields.

## 2.1 Nonnegativity and the Sum of Squares Decomposition

One of the most important differences between real and ordinary algebra is the notion of "positivity". In this section we will be concerned with two subsets of the commutative ring of polynomials $\mathbb{R}[x] \triangleq \mathbb{R}[x_1, \ldots, x_n]$, i.e., of polynomials in $(x_1, \ldots, x_n)$ with real coefficients: non-negative forms and sums of squares. Let us start with a few basic definitions.

**Definition 2.1** *Let $x = (x_1, \ldots, x_n)$, $x \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha \in \mathbb{N}^n$. We call the function $z_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ a* monomial *in $(x_1, \ldots, x_n)$ of degree $|\alpha| = \sum_{i=1}^n \alpha_i$. A* polynomial *$p$ in $x$ with coefficients in $\mathbb{R}$ is a linear combination of a finite set of monomials:*

$$p(x) = \sum_\alpha c_\alpha x^\alpha = \sum_\alpha c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad c_\alpha \in \mathbb{R} \tag{2.1}$$

*The degree of the polynomial, deg $\{p(x)\}$, is the maximum degree of the monomials in it.*

A problem of great interest in Real Algebraic Geometry is whether a given polynomial takes non-negative values.

**Notation 2.2** *Let $\mathcal{P}_{n,m}$ denote the set of nonzero forms (i.e., polynomials of homogeneous degree) in $n$ variables of degree $m$, with coefficients in $\mathbb{R}$ that are non-negative on $\mathbb{R}^n$ ($m$ is necessarily even).*

Non-negativity conditions appear frequently in control theory. For example, the stability of an equilibrium of a nonlinear system can be concluded by verifying non-negativity of certain conditions. However, even if we restrict our attention to the case in which these conditions are polynomial, we are faced with a difficult problem as testing polynomial non-negativity when the degree of the polynomial is greater than or equal to 4 is $\mathcal{NP}$-hard [50]. This has led researchers to seek sufficient conditions for non-negativity that are algorithmically verifiable in polynomial time. A particularly attractive condition is the existence of a sum of squares decomposition, introduced in [66].

By definition, a polynomial $p(x) \in \mathbb{R}[x]$ admits a *sum of squares (SOS)* decomposition, if there exists a set of polynomials $f_i$, $i = 1, \ldots, M$ such that:

$$p(x) = \sum_{i=1}^{M} f_i^2(x). \tag{2.2}$$

It is obvious from the above expression that all polynomials that are sums of squares are indeed non-negative in the whole of $\mathbb{R}^n$. The converse is not true: not all non-negative polynomials can be written as sums of squares, apart from three special cases, which were identified by Hilbert himself [80]:

- Polynomials in 1 variable;
- Polynomials of 2nd order;
- Polynomials of 4th order in 2 variables.

A celebrated example of a non-negative polynomial that is not a SOS is the *Motzkin form*:

$$M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 \tag{2.3}$$

Hilbert was aware of the non-equivalence between non-negativity and sum of squares, and he therefore posed the following question, now known as 'Hilbert's 17th problem': Can a non-negative polynomial over a real closed field be written as a sum of squares of *rational* functions? The answer is affirmative, and the solution was given by Emil Artin in 1922 which marked the birth of real algebraic geometry [9].

**Notation 2.3** *We denote by $\Sigma_{n,m}$ the subset of $P_{n,m}$ of those forms which are sums of squares of polynomials.*

It was mentioned that testing polynomial non-negativity is, in general, $\mathcal{NP}$-hard [50], and that means that there is no known polynomial-time algorithm for deciding polynomial non-negativity. On the other hand, testing the existence of a

SOS decomposition is computationally more tractable [73, 14, 87]; in fact, it has worst-case polynomial time complexity, as it is reducible to the solution of a semidefinite program [51]. The following proposition shows why this is indeed so.

**Proposition 2.4** *A polynomial $p(x)$ of degree $2d$ is a sum of squares if and only if there exists a positive semidefinite matrix $Q$ and a vector of monomials $Z(x)$ containing monomials in $x$ of degree less than or equal to $d$ such that*

$$p = Z(x)^T Q Z(x). \tag{2.4}$$

**Proof.** $\Rightarrow$: Denote by $\mathbf{f}(x) = [f_i(x)] = LZ(x)$. The $f_i$ are given by Equation (2.2), $Z(x)$ is a vector containing all monomials in $\mathbf{f}(x)$ and $L$ is a compatible coefficient matrix. Then

$$p(x) = \mathbf{f}(x)^T \mathbf{f}(x) = Z(x)^T L^T L Z(x),$$

and $L^T L = Q \geq 0$.

$\Leftarrow$: Suppose the decomposition (2.4) is given. Then perform a Cholesky factorization on $Q = R^T R$. Now write

$$p(x) = (RZ(x))^T (RZ(x)) = \mathbf{g}(x)^T \mathbf{g}(x) = \sum_{i=1}^{M} g_i^2(x),$$

where $\mathbf{g}(x) = [g_i(x)] = RZ(x)$. Obviously $p$ is a sum of squares. ■

In general, the monomials in $Z(x)$ are not algebraically independent. Expanding $Z(x)^T Q Z(x)$ and equating the coefficients of the resulting monomials to the ones in $p(x)$, we obtain a set of affine relations in the elements of $Q$. Since $p(x)$ being SOS is equivalent to $Q \geq 0$, the problem of finding a $Q$ which proves that $p(x)$ is an SOS is a Linear Matrix Inequality [10]. An alternative formulation is in [39].

The following is an example of how this is done.

**Example 2.5** *Consider the quartic form in two variables described below, and define*

*$Z(x) = [\ x_1^2 \quad x_2^2 \quad x_1x_2\ ]^T$:*

$$
\begin{aligned}
p(x_1, x_2) &= 5x_1^4 + 2x_2^4 - x_1^2x_2^2 - 2x_1^3x_2 - 2x_1x_2^3 = Z(x)^T \overbrace{\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}}^{Q} Z(x) \\
&= q_{11}x_1^4 + q_{22}x_2^4 + (2q_{12} + q_{33})x_1^2x_2^2 + 2q_{13}x_1^3x_2 + 2q_{23}x_1x_2^3,
\end{aligned}
$$

*from which we get the following relations:*

$$
q_{11} = 5, \qquad q_{22} = 2, \qquad 2q_{12} + q_{33} = -1, \qquad q_{13} = -1, \qquad q_{23} = -1.
$$

*Now, decomposing $p(x)$ as an SOS amounts to searching for $q_{12}$ and $q_{33}$ satisfying $2q_{12} + q_{33} = -1$, such that $Q \geq 0$. For $q_{12} = -1$ and $q_{33} = 1$, the matrix $Q$ will be positive semidefinite and we have*

$$
Q = L^T L, \text{ where } L = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.
$$

*This immediately yields the following SOS decomposition:*

$$
p(x) = (2x_1^2 - x_2^2)^2 + (x_1^2 + x_2^2 - x_1x_2)^2.
$$

The fact that Linear Matrix Inequalities are constraints in semidefinite programs [96] allows us to optimize linear functionals of decision variables that appear in a problem. As a first application of using the sum of squares decomposition as a substitute to polynomial non-negativity, consider the problem of finding the minimum of a polynomial function [69]. Here, a decision variable is introduced which can be maximized to yield a good lower bound on the polynomial function.

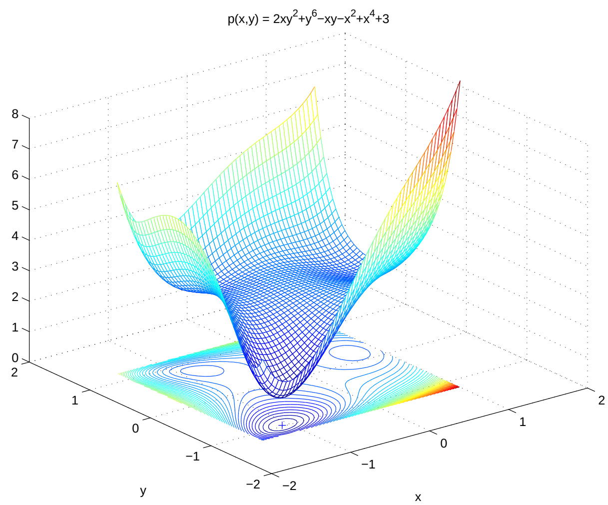



Figure 2.1: Finding minima of polynomial functions.

**Example 2.6** *Consider the following function:*

$$p(x, y) = 2xy^2 + y^6 - xy - x^2 + x^4 + 3$$

*A graph of this function in* $\mathbb{R}^3$ *is shown in Figure 2.1. In order to find a lower bound for the function* $p(x, y)$*, we can try to maximize* $\gamma$ *such that*

$$p(x) - \gamma \text{ is a sum of squares.}$$

$\gamma$ *appears as a new decision variable in the relevant semidefinite programme, which we can optimize over. Indeed, implementation of the sum of squares programme results in the maximum allowable value of* $\gamma = 0.9468$ *achieved at (-1.0799,-0.9752) – information that may be retrieved from the dual solution of the semidefinite programme.*

The above example shows that even if some coefficients of the polynomial are unknown or constrained to lie within certain intervals, checking the sum of squares decomposition can still be done using semidefinite programming. This is helpful, for example when searching for Lyapunov functions for nonlinear systems.

As suggested by Example 2.5, the construction of an equivalent semidefinite program for computing the SOS decomposition can be quite involved when the degree of the polynomials is high. For this reason, conversion of SOS conditions to the corresponding semidefinite program has been automated in SOSTOOLS, a software developed for this purpose. This software calls SeDuMi [90] or SDPT3 [93], semidefinite programming solvers, to solve the resulting semidefinite program, and converts the solutions back to the solutions of the original SOS programs. These software packages are used for solving all of the examples in this thesis - Example 2.5 is solved, for example, by using the command `findsos` and Example 2.6 by using the command `findbound`. A more detailed description of the software can be found in [75]. Moreover, in many examples, the polynomials possess special properties [67] or structure: they are sparse, or bipartite, as we will see in Chapter 4. In this case, we can characterize what monomials are required in $Z(x)$, which reduces significantly the computational burden since the size of the LMIs is reduced, but it also helps improve numerical conditioning. Such structure-exploiting algorithms are available in SOSTOOLS.

## 2.2 The Positivstellensatz

Real algebraic geometry studies real algebraic sets, i.e., subsets of $\mathbb{R}^n$ defined by polynomial equations. There is a fundamental difference between real and complex algebraic geometry, as the field of real numbers is not algebraically closed. Real algebraic geometry deals not only with the zeros of polynomials, but also with domains where the polynomials have a constant sign.

An important result in real algebraic geometry is *positivstellensatz*, a theorem that provides an equivalence relation between the emptiness of a semi-algebraic set (i.e., a finite set of polynomial equalities and inequalities), to an algebraic relationship being valid. Along with the algorithmic verifiability of the sum of squares decomposition, they form the pillars of the theory that will be used in the rest of the thesis, unifying and extending known results not only in optimization and control, but also in physics and euclidean geometry [68].

We begin with a few definitions that are used in the theorem. Here, $x \in \mathbb{R}^n$.

**Definition 2.7** *Given polynomials* $\{h_1, \ldots, h_u\} \in \mathbb{R}[x]$*, the Multiplicative Monoid generated by the* $h_i$ *is the set of all finite products of* $h_i$*, including 1. We will denote this by* $\mathcal{M}(h_1, \ldots, h_u)$*.*

**Definition 2.8** *Given polynomials* $\{f_1, \ldots, f_s\} \in \mathbb{R}[x]$*, the Algebraic Cone generated by the* $f_j$ *is the set:*

$$\mathcal{C}(f_1, \ldots, f_s) = \left\{ \lambda_0 + \sum_i \lambda_i F_i | F_i \in \mathcal{M}(f_1, \ldots, f_s), \lambda_i \in \Sigma_n \right\}. \quad (2.5)$$

**Definition 2.9** *Given polynomials* $\{g_1, \ldots, g_t\} \in \mathbb{R}[x]$*, the Ideal generated by the* $g_i$ *is the set:*

$$\mathcal{I}(g_1, \ldots, g_t) = \left\{ \sum_l \mu_l g_l | \mu_l \in \mathbb{R}[x] \right\}. \quad (2.6)$$

Now can now proceed by quoting Positivstellensatz, a theorem that we will be using frequently in the sequel.

**Theorem 2.10** *Let* $\mathbb{R}$ *be a real closed field. Let* $(f_j)_{\{j=1,\ldots,s\}}$*,* $(g_l)_{\{l=1,\ldots,t\}}$ *and* $(h_k)_{\{k=1,\ldots,u\}}$ *be finite families of polynomials in* $\mathbb{R}[x_1, \ldots, x_n]$*. Denote by* $\mathcal{C}$ *the Algebraic Cone generated by* $(f_j)_{\{j=1,\ldots,s\}}$*,* $\mathcal{M}$ *the Multiplicative Monoid generated by* $(h_k)_{\{k=1,\ldots,u\}}$ *and* $\mathcal{I}$ *the ideal generated by* $(g_l)_{\{l=1,\ldots,t\}}$*. Then the following properties are equivalent:*

- *The set*

$$\{x \in \mathbb{R}^n | f_j(x) \geq 0, j = 1, \ldots, s, \quad g_l = 0, l = 1, \ldots, t, \quad h_k(x) \neq 0, h = 1, \ldots, u\} \quad (2.7)$$

  *is empty.*

- *There exist* $f \in \mathcal{C}$*,* $g \in \mathcal{I}$ *and* $h \in \mathcal{M}$ *such that:*

$$f + g + h^2 = 0. \quad (2.8)$$

A few things should be emphasized about this theorem. First, it is an *equivalence* relation between a *geometric* object, the set defined in (2.7) and an *algebraic* relationship, given by (2.8). It can be seen as a generalization of the separation theorem in convex optimization, where the intersection of two convex sets is empty if and only if there is a hyperplane separating them that certifies the emptiness. Here, the set (2.7) need not be convex, in which case the certificate is not necessarily a hyperplane; the algebraic relationship (2.8) certifies this emptiness.

It should be stressed that there is no guidance as to how, for example, the cone $\mathcal{C}$ should be formed — what the degree or structure of $f$ should be. Putting an upper bound on these degrees and checking whether (2.8) holds, one can create a series of tests for the emptiness of (2.7); each of these tests requires the construction of some sum of squares and polynomial multipliers, resulting in a sum of squares programme that can be solved using SOSTOOLS.

Let us give an example of a problem from combinatorial optimization whose decision version is $\mathcal{NP}$-hard, and for which positivstellensatz results in a series of tests.

**Example 2.11** ***Number Partitioning Problem***. *Consider the optimization version of Partition:*

***Problem 2.12*** *Given a set of $n$ non-negative numbers $\{a_1, \ldots, a_n\}$, separate them into two disjoint sets such that the difference of the subset sums is minimized.*

*The above question can be converted into finding $x_i = \pm 1$, $i = 1, \ldots, n$ such that:*

$$F(x) = \left| \sum_{i=1}^{n} x_i a_i \right|$$

*is minimized. This is the same problem as*

$$\begin{aligned} \min \quad & F^2(x) = \left( \sum_{i=1}^{n} x_i a_i \right)^2 \\ s.t. \quad & x_i^2 = 1. \end{aligned}$$

*We can turn this combinatorial optimization problem into an emptiness of a set as* *follows:*

$$\begin{array}{ll} \max & \gamma \\ s.t. & \left\{ x \in \mathbb{R}^n \;\middle|\; \left( \sum_{i=1}^{n} x_i a_i \right)^2 - \gamma < 0, x_i^2 = 1, i = 1, \ldots, n \right\} = \emptyset. \end{array}$$

*In order to generate the ideal of the polynomials* $h_i(x) \triangleq x_i^2 - 1 = 0$*, we look for polynomials* $\tilde{p}_i(x)$ *such that:*

$$\mathcal{I}(h_1, \ldots, h_n) = \sum_{i=1}^{n} \tilde{p}_i(x) h_i(x). \tag{2.9}$$

*The cone of* $f(x) \triangleq \gamma - F^2(x)$ *is*

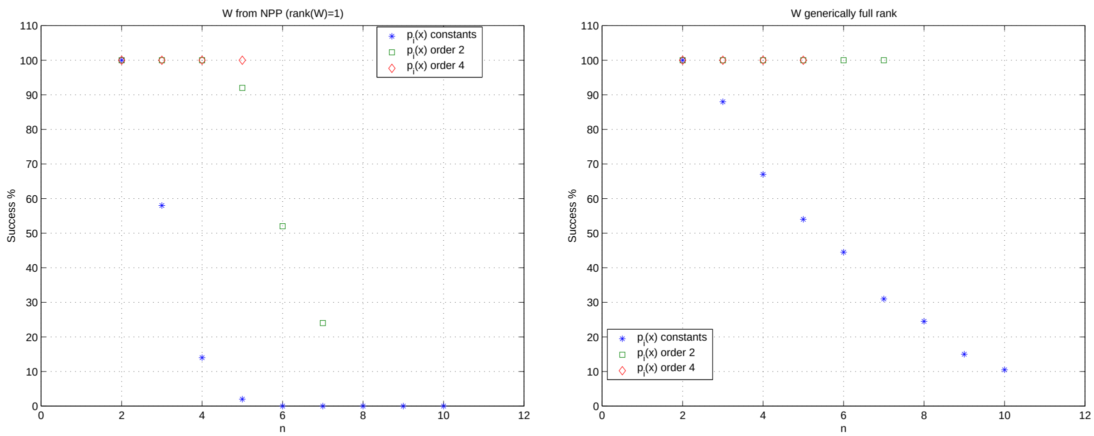
$$\mathcal{C}(f) = \sigma_0(x) + \sigma_1(x)(\gamma - F^2(x)). \tag{2.10}$$

*where* $\sigma_0(x), \sigma_1(x)$ *are SOS. The monoid of* $f(x)$ *is* $f^{2k}$ *where* $k$ *is a non-negative integer. Choosing* $\sigma_0(x) = 0$ *and* $k = 1$*, and structuring the multipliers* $\tilde{p}_i(x) = f(x)p_i(x)$*, the overall condition becomes:*

$$\begin{array}{ll} \max & \gamma \\ s.t. & F^2(x) - \gamma + \sum_{i=1}^{n} p_i(x) h_i(x) \text{ is SOS.} \end{array}$$

*While the SOS condition is satisfied and* $h_i(x) = 0$ *(i.e.,* $x_i = \pm 1$*),* $F^2(x) \geq \gamma$*, i.e.,* $\gamma$ *is a lower bound on the optimal cost.*

*When the* $p_i(x)$ *are constants, the standard SDP relaxation that was investigated by Goemans and Williamson is retrieved [25] (see the remark at the end of this example). For the Number Partitioning Problem,* $F^2(x) = x^T a a^T x = x^T W x$ *where* $W$ *is rank-1.*




(a) (b)

Figure 2.2: Performance of Positivstellensatz tests for the Ising spin glass problem.

*In the case of constant* $p_i$*'s, the problem reduces to*

$$\begin{array}{ll} \max & \gamma \\ s.t. & \begin{bmatrix} W - diag p & 0 \\ 0 & -\gamma + \sum_i p_i \end{bmatrix} \geq 0 \end{array}$$

*and using the result by [40] the solution to the above is* $\gamma = \left[a_1 - \sum_{j=2}^{n} a_j\right]^+$*, where* $[\cdot]^+$ *denotes positive projection. When* $n$ *is large, then this first positivstellensatz test is most likely going to give a trivial lower bound* $\gamma$*, as the probability that* $a_1 \geq \sum_{j=2}^{n} a_j$ *is vanishingly small; this is suggested by Figure 2.2(a). Positivstellensatz allows us to write other conditions for nonnegativity, by increasing the order of the polynomials* $p_i(x)$*. Qualitative results are also shown in Figure 2.2(a) when the* $p_i(x)$ *are allowed to be of higher degree. In this figure, comparison is made to the true ground states. Alternatively, if* $W$ *is allowed to be a generically full rank matrix, then better results can be obtained, as shown in Figure 2.2(b). This case has another interpretation: it is related to the problem of finding the ground state of an infinite-range Ising spin glass with couplings* $W_{ij}$ *drawn from some probability distribution. Such a model is the Sherrington-Kirkpatrick (SK) [94] spin glass model where the* $W_{ij}$ *are independent random Gaussian variables with zero mean and variance* $1/n$*,* $n$ *being the total number of spins.*

**Remark 2.13** *The special case in which the multipliers $p_i$'s are constants corresponds exactly to the convex relaxation obtained by standard Lagrangian duality. To make our argument more concrete, consider the general program:*

$$\begin{aligned} \min \quad & x^T W x \\ s.t. \quad & x_i^2 - 1 = 0 \end{aligned}$$

*Denote $P = diag(p_1, \ldots, p_n)$ and $Tr(P)$ the trace of $P$. The Lagrangian of this problem is:*

$$L(x,p) = x^T W x - \sum_{i=1}^{n} p_i(x_i^2 - 1) = x^T (W - P)x + Tr(P) \tag{2.11}$$

*The dual SDP is therefore:*

$$\begin{aligned} \max \quad & Tr(P) \\ s.t. \quad & W - P \geq 0 \end{aligned}$$

*which gives a lower bound on the optimal value of the primal problem. The 'dual of the dual' is easily found to be:*

$$\begin{aligned} \min \quad & Tr(WX) \\ s.t. \quad & X \geq 0 \\ & X_{ii} = 1 \end{aligned}$$

*Now from the original problem, denoting $X = xx^T$ we see that this can be* exactly *rewritten as:*

$$x^T W x = Tr(W x x^T) = Tr(WX) \tag{2.12}$$

*where $X_{ii} = 1$, $X \geq 0$ and $X$* is rank-1. *This last, rank-1 condition is what is missing from the 'dual of the dual' formulation; the SDP relaxation is obtained by leaving this non-convex condition out. If it turns out that the rank of $X$ is 1, then the original problem is solved exactly.*

A nested family of conditions obtained to test emptiness of sets, under which the more computationally expensive ones are at least as good as the previous ones, can be applied to a tool commonly used in robust control theory, the S-procedure.

**Example 2.14** ***S-procedure***. *Many times in control theory we must ensure that a certain condition holds whenever some other condition holds. A common test for conditional satisfiability is the S-procedure. We can easily turn the same problem into an emptiness of a set and seek a positivstellensatz test.*

*In particular, given* $m$ *symmetric matrices* $P_1, \ldots, P_m$*, consider whether*

$$\mathcal{D} := \left\{x \in \mathbb{R}^n : p_i(x) = x^T P_i x \geq 0, \|x\| = 1, i = 1, \ldots, m\right\} = \emptyset.$$

*Using Positivstellensatz, with* $g(x) = \|x\|^2 - 1$ *and* $f(x) = \sum_i \sigma_i p_i(x) + \sigma_0(x)$*, with* $\sigma_0(x)$ *an SOS and* $\sigma_i \geq 0$ *constants, a sufficient condition for the set* $\mathcal{D}$ *to be empty is given by the existence of* $\sigma_i$ *that satisfy the condition:*

$$-\sum_{i=1}^{m} \sigma_i x^T P_i x - \sum_{i=1}^{n} x_i^2 \text{ is SOS}, \quad \sigma_i \geq 0, \tag{2.13}$$

*which is the standard S-procedure [104]. It is well known that this condition may be conservative, depending on the structure of the set* $\mathcal{D}$*. With Positivstellensatz, a hierarchy of polynomial-time computable stronger conditions can be obtained, depending on how the cone* $\mathcal{C}$ *and ideal* $\mathcal{I}$ *are constructed. For example, a second test is the following: Assume there exist solutions* $\sigma_i(x)$ *quadratic polynomials that are SOS and* $\rho_{ij} \geq 0$ *constants to:*

$$-\sum_{i=1}^{m} \sigma_i(x) p_i(x) - \sum_{i=1}^{m} \sum_{j=i+1}^{m} \rho_{ij} p_i(x) p_j(x) - \sum_{i=1}^{m} x_i^4 \text{ is SOS}. \tag{2.14}$$

*Then, the set* $\mathcal{D}$ *is empty. The search for the* $\sigma_i(x)$ *and* $\rho_{ij}$ *and the verification of the sum of squares condition can be done efficiently and in a simultaneous fashion, using semidefinite programming methods and SOSTOOLS. In this way, we obtain a* nested hierarchy *of polynomial-time computable tests for the emptiness of the set* $\mathcal{D}$*; each* *test is* always *at least as powerful as the standard one, and often strictly stronger.*

In the sequel, we will come across various S-procedure type conditions which we test using SOSTOOLS in the framework given above. Instead of using constant multipliers such as the unknowns $\sigma_i$ in (2.13), we will be using higher order multipliers such as the $\sigma_i(x)$ in (2.14) – condition that is at least as strong as (2.13) to test the emptiness of the set $\mathcal{D}$.

## 2.3 Conclusion

In this chapter, we have developed the algorithmic and mathematical tools that will be used in the rest of the thesis for the analysis of nonlinear systems. These center on the sum of squares decomposition and positivstellensatz – tools that may be used to produce new tests that generalize the S-procedure for testing conditional satisfiability.

In the next chapters we will use the SOS decomposition to test nonnegativity and we will formulate Lyapunov and S-procedure type SOS conditions for analysis of nonlinear systems. We will then search for Lyapunov functions or multipliers that satisfy those conditions using semidefinite programming.

# Chapter 3

# Systems Described by Ordinary Differential Equations

Τὰ ψυχρὰ θέρεται, θερμὸν ψύχεται,
ὑγρὸν αὐαίνεται, καρφαλέον νοτίζεται.
Ἡράκλειτος
Cold things become warm, and what is warm cools;
what is wet dries, and the parched is moistened.
Heracletus

In this chapter, we will investigate how various analysis questions for nonlinear systems described by Ordinary Differential Equations (ODEs) can be answered using sums of squares (SOS). ODEs have been an important tool for modeling the physical world, ranging from simple mechanical and electrical systems to chemical processes and simplified aircraft dynamics. They have also been the primary tool for modeling components in biological networks or multi-agent systems. Usually, the far-from-equilibrium behavior of such systems is of greater interest than the local 'linearized' properties, and most analysis tools for such systems center in what are now known as 'Lyapunov methods', named after A. M. Lyapunov. The main feature of these techniques is that system properties are assessed without solving the underlying model equations, but rather through the construction of a function of state (a Lyapunov function) that satisfies certain conditions.

In this chapter, we develop an efficient algorithmic procedure to analyze nonlinear systems described by ODEs that evolve under constraints such as equality, inequality and integral type. This allows robust stability analysis, input-output analysis, as well as analysis of non-polynomial systems to be performed in a unified manner. The techniques we will be using are based on the sum of squares decomposition and Positivstellensatz, as they were introduced in Chapter 2.

## 3.1 Introduction

Systems which appear in electrical or mechanical engineering and even biology are usually modeled by a finite number of coupled first-order Ordinary Differential Equations (ODEs) of the form:

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, \ldots, x_n), \\ \dot{x}_2 &= f_2(t, x_1, \ldots, x_n), \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, \ldots, x_n), \end{aligned}$$

where $\dot{x}_i$ denotes the derivative of $x_i$ with respect to time, and $x_i$ are the state variables. We use vector notation to describe this system. Let $x \in \mathbb{R}^n$ and let $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$. Then, the above can be written as:

$$\dot{x} = f(t, x). \tag{3.1}$$

If $f(t, x)$ is piecewise continuous in $t$ and satisfies a local Lipschitz condition in $x$, then the existence and uniqueness of the solutions is guaranteed locally [35].

In this chapter, we will concentrate on systems that are autonomous, i.e., take the form

$$\dot{x} = f(x), \tag{3.2}$$

where $f : D \to \mathbb{R}^n$ is locally Lipschitz in a domain $D \subset \mathbb{R}^n$. Suppose $x^*$ is an equilibrium point of (3.2), i.e. $f(x^*) = 0$. Without loss of generality, we assume that 0 is an equilibrium – a simple change of coordinates can achieve this – and we are interested in the stability properties of this equilibrium. There are different notions of stability of equilibria which are usually characterized using Lyapunov arguments. Here, we concentrate on *stability* and *asymptotic stability*; $\|\cdot\|$ denotes a norm in $\mathbb{R}^n$.

**Definition 3.1** *The equilibrium $x = 0$ of (3.2) is:*

- *Stable, if for each $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that*

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\ t \geq 0.$$

- *Asymptotically stable if it is stable and $\delta$ can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0.$$

We can see that these definitions of stability involve $\epsilon - \delta$ formulations, which at first give the impression that a complete description of the flow of the vector field is required to answer stability questions. It is fortunate that in many cases stability can be proved directly by exhibiting an *energy-like* function, now called a *Lyapunov function* [35, 107]. This is Lyapunov's direct method. Under some technical conditions, the existence of this function was also proved necessary for asymptotic stability [27]. More precisely, the conditions are stated in the following theorem:

**Theorem 3.2 ([35])** *Consider the system (3.2), and let $D \subseteq \mathbb{R}^n$ be a neighborhood of the origin. If there is a continuously differentiable function $V : D \to \mathbb{R}$ such that the following two conditions are satisfied:*

1. *$V(x) > 0$ for all $x \in D \setminus \{0\}$ and $V(0) = 0$, i.e., $V(x)$ is positive definite in $D$;*

2. *$-\dot{V}(x) = -\frac{\partial V}{\partial x} f(x) \geq 0$ for all $x \in D$, i.e., $\dot{V}(x)$ is negative semidefinite in $D$;*

*then the origin is a* stable *equilibrium. If in condition (2) above, $\dot{V}(x)$ is* negative definite *in $D$, then the origin is* asymptotically stable. *If $D = \mathbb{R}^n$ and $V(x)$ is radially unbounded, i.e., $V(x) \to \infty$ as $\|x\| \to \infty$, then the result holds globally.*

In the case of linear time-invariant systems

$$\dot{x} = Ax, \tag{3.3}$$

the stability properties can be characterized by the locations of the eigenvalues $\lambda_i$ of the matrix $A$, or equivalently, through a Lyapunov argument as follows:

**Theorem 3.3** *[35] The matrix $A$ is a stability matrix; that is $Re\lambda_i < 0$ for all eigenvalues $\lambda_i$ of $A$ if and only if for any given positive definite matrix $Q$ there exists a positive definite matrix $P$ that satisfies:*

$$PA + A^T P = -Q. \tag{3.4}$$

*$P$ is unique, and $V = x^T Px$ is a Lyapunov function for (3.3).*

We see that the construction of the Lyapunov function in the case of linear systems is reduced to solving an appropriate Algebraic Lyapunov Equation (3.4). Alternatively, $P$ can be obtained by solving two Linear Matrix Inequality (LMI) [10] conditions:

$$\begin{aligned} P &> 0, \\ A^T P + PA &< 0. \end{aligned}$$

A feasible $P$ exists if and only if $A$ is Hurwitz. LMIs are constraints in semidefinite programs [96], which can be solved using algorithms with a worst-case polynomial time complexity. This makes them particularly attractive for computation.

The absence of a direct methodology for constructing Lyapunov functions for nonlinear systems led to the development of other methods for assessing nonlinear system properties. For example, in Lyapunov's indirect method, one proceeds by linearizing the vector field about the equilibrium and the (local) stability properties of the original nonlinear system are inferred from the stability properties of the linearized system. However, this procedure is inconclusive when the linearized system has imaginary axis eigenvalues and the result is valid anyway only locally. Other methodologies involve absolute stability theory [19], Linear Parameter Varying (LPV) embeddings [23, 86, 53], Integral Quadratic Constraint (IQC) formulations [47] and others.

In this chapter, we will build on the methodology introduced in [66] and we will show how to use the sum of squares decomposition to analyze different classes of systems described by ODEs using Lyapunov methodologies. There are mainly two reasons why there has been no algorithmic methodology for constructing the Lyapunov functions $V(x)$ for so long. On one hand, the 'terms' that should appear in $V(x)$ are not known *a priori*, and on the other hand, testing the nonnegativity conditions in Theorem 3.2 is a difficult task even in the case in which they are polynomial. We can get to the bottom of the first problem by resorting to intuition and prior knowledge of energy-like terms that are likely to appear in $V(x)$. As far as the second problem is concerned, this is closely related to the fact that testing polynomial nonnegativity when the degree is greater than or equal to 4 is an $\mathcal{NP}$-hard problem, as was mentioned in the previous chapter [50].

For concreteness, let us assume that $f(x)$ is a polynomial vector field. Suppose that we also wish to construct a $V(x)$ that is also polynomial in $x$. In this case, the two conditions in Theorem 3.2 become polynomial nonnegativity conditions. To circumvent the difficult task of testing them, we can restrict our attention to cases in which the two conditions admit SOS decompositions. Note that even if the coefficients of a polynomial Lyapunov candidate $V$ are unknown, we can still search for them so that the two Lyapunov conditions are satisfied, as was explained in the previous chapter.

To impose that $V(x)$ should be positive *definite* rather than positive semi-definite, we construct an auxiliary positive definite 'shaping' function $\varphi(x)$ as follows:

$$\varphi(x) = \sum_{i=1}^{n} \sum_{j=1}^{d} \epsilon_{ij} x_i^{2j}, \quad \sum_{j=1}^{m} \epsilon_{ij} \geq \gamma \; \forall \; i = 1, \ldots, n, \quad \gamma > 0, \quad \epsilon_{ij} \geq 0 \; \forall \; i, j \qquad (3.5)$$

This makes $\varphi(x) > 0$, i.e., positive definite. If we impose $V(x) - \varphi(x)$ to be a SOS, obviously

$$V(x) - \varphi(x) \geq 0 \Rightarrow V(x) \geq \varphi(x) > 0. \tag{3.6}$$

Therefore, we have

**Proposition 3.4** *Given a polynomial $V(x)$ of degree $2d$, let $\varphi(x)$ be given by Equation (3.5). Then, the condition*

$$V(x) - \varphi(x) \text{ is a sum of squares} \tag{3.7}$$

*guarantees the positive definiteness of $V(x)$.*

In the case of global stability, i.e., for $D = \mathbb{R}^n$, the conditions in Theorem 3.2 can then be formulated directly as SOS conditions. Therefore, we have the following sum of squares program:

**Program 3.5** *To construct a Lyapunov function for system (3.2),*

$$\begin{aligned}
&\textit{Find a polynomial } V(x), \quad V(0) = 0\\
&\textit{and a positive definite function } \varphi(x) \textit{ of the form } (3.5)\\
&\textit{such that}\\
&\quad V(x) - \varphi(x) \textit{ is SOS} && (3.8)\\
&\quad -\frac{\partial V}{\partial x} f(x) \textit{ is SOS} && (3.9)
\end{aligned}$$

*Then $V(x)$ is a Lyapunov function for system (3.2) and the zero equilibrium of (3.2) is globally stable.*

The above program guarantees that $V(x)$ is positive definite and also that $\dot{V}(x)$ is negative semidefinite; therefore $V(x)$ is a Lyapunov function that proves stability of the origin of system (3.2). Note also that by construction $\varphi(x)$ is radially unbounded; therefore, $V(x)$ will also be radially unbounded, and the stability property holds globally [35]. Also, if condition (3.9) is replaced by

$$-\frac{\partial V}{\partial x} f(x) - \psi(x) \text{ is SOS,} \tag{3.10}$$

where $\psi(x)$ is a positive definite polynomial constructed as per (3.5), then $\dot{V}(x)$ is negative definite and the origin is globally asymptotically stable.

Below is an example of how the construction of a Lyapunov function is performed using SOSTOOLS.

**Example 3.6** *Consider the system*

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_2^3 - 3x_3x_4 \\
\dot{x}_2 &= -x_1 - x_2^3 \\
\dot{x}_3 &= x_1x_4 - x_3 \\
\dot{x}_4 &= x_1x_3 - x_4^3,
\end{aligned}$$

*which has the only equilibrium at the origin. As a first attempt, we will try to construct a quadratic Lyapunov function of the form $V = \sum_{i=1}^{4}\sum_{j=i}^{4} a_{ij}x_ix_j$ where the $a_{ij}$'s are the unknowns. We search for $V$ that satisfy the conditions in Program 3.5.*

*It turns out that a Lyapunov function of the above form does not exist (the corresponding semidefinite program is infeasible), so we will next search for a quartic Lyapunov function. One then finds a Lyapunov function that satisfies conditions (3.8) and (3.10), and thus proves global asymptotic stability of the origin. To three significant figures, this reads:*

$$\begin{aligned}
V = {} & 1.12x_1x_2x_3^2 - 0.785x_1x_2 + 0.713x_2^3x_1 + 0.500x_1x_2x_4^2 + 0.768x_4^4 \\
& +1.64x_1^2 + 1.76x_3^2 + 0.392x_2^2 + 1.63x_4^2 + 1.69x_1^2x_2^2 + 0.557x_3^4 \\
& +0.724x_1^3x_2 + 0.181x_1^4 + 1.07x_2^4 + 0.561x_1^2x_3^2 + 1.61x_2^2x_3^2 \\
& +0.525x_1^2x_4^2 + 0.969x_2^2x_4^2 + 0.569x_3^2x_4^2 - 0.251x_4x_3x_1 + 0.432x_4x_3x_2.
\end{aligned}$$

*The above can be obtained by using the* `findlyap` *command in SOSTOOLS.*

## 3.2 Stability of Constrained Systems

In this section, we extend Lyapunov's theorem to systems that evolve under equality, inequality, and integral constraints. This is a very general class of systems, special cases of which are differential algebraic equations, robust stability analysis and performance formulations. It will also allow us to treat non-polynomial vector fields exactly.

Inequality constraints arise naturally when considering *positive systems*: systems with inherently positive states, e.g., a chemical reaction in which the concentrations of the reactants are positive. The same type of constraints can be used to describe *uncertain parameter sets* for the study of robust stability of systems in the presence of parametric uncertainty.

On the other hand, systems evolving over a manifold described by a set of equality constraints arise in a plethora of cases, and are also called differential algebraic equations or descriptor systems [16]. Examples of equality constraints are holonomic (configuration) constraints in mechanical systems and conservation laws – in electrical networks in the form of current balance and in chemical engineering in the form of mass balance. Sometimes it is possible to back-substitute and reduce the system to an ordinary differential equation, but this usually results in more complicated vector fields of higher order. In some other cases, this is not possible and a differential index theory was developed as a measure of this singularity [91]. Equality constraints also prove useful in robust stability analysis where they appear as constraints guaranteeing that the equilibrium of the system is at the origin.

The last type of constraints that are going to be incorporated is of integral type, in particular Integral Quadratic Constraints (IQCs) [47]. They provide a framework rich enough to encapsulate many types of uncertainty and unmodelled dynamics: dynamic, time-varying and $\mathcal{L}_2$ bounded uncertainty, just to name a few. Moreover, one can formulate performance calculations using IQCs such as $\mathcal{L}_2$ input-output gain estimation.

Consider the nonlinear system

$$\dot{x} = f(x,u), \tag{3.11}$$

with the following inequality, equality, and integral constraints that are satisfied by $x$ and $u$:

$$a_{i_1}(x,u) \leq 0, \quad \text{for } i_1 = 1, ..., N_1, \tag{3.12}$$

$$b_{i_2}(x,u) = 0, \quad \text{for } i_2 = 1, ..., N_2, \tag{3.13}$$

$$\int_0^T c_{i_3}(x,u)dt \leq 0, \text{for } i_3 = 1, ..., N_3, \text{ and } \forall\, T \geq 0. \tag{3.14}$$

Here $x \in \mathbb{R}^n$ is the state of the system, and $u \in \mathbb{R}^m$ is a collection of auxiliary variables (such as inputs, non-polynomial functions of states, uncertain parameters, etc). We assume that $f(x,u)$, apart from the required Lipschitz conditions for existence of solutions, has no singularity in $\mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^{n+m}$ is defined as

$$\mathcal{D} = \{(x,u) \in \mathbb{R}^{n+m} \mid a_{i_1}(x,u) \leq 0, b_{i_2}(x,u) = 0, \text{ for all } i_1 \text{ and } i_2\}.$$

Without loss of generality, it is also assumed that $f(x,u) = 0$ for $x = 0$ and $u \in \mathcal{D}_u^0$, where

$$\mathcal{D}_u^0 = \{u \in \mathbb{R}^m | (0,u) \in \mathcal{D}\}.$$

The following theorem is an extension of Lyapunov's stability theorem, and can be used to prove that the origin is a stable equilibrium of the above system. It uses a technique reminiscent of the well-known S-procedure [104] in nonlinear and robust control theory, that was discussed in Example 2.14.

**Theorem 3.7** *[62] Suppose that for system (3.11), there exist functions* $V(x)$, $p_{1_{i_1}}(x,u) \geq$

$0$, $p_{2_{i_1}}(x,u) \geq 0$, $q_{1_{i_2}}(x,u)$, $q_{2_{i_2}}(x,u)$ *and constants* $r_{i_3} \geq 0$ *such that*

$$V(x) + \sum p_{1_{i_1}}(x,u)a_{i_1}(x,u) + \sum q_{1_{i_2}}(x,u)b_{i_2}(x,u) > 0, \tag{3.15}$$

$$-\frac{\partial V}{\partial x}f(x,u) + \sum p_{2_{i_1}}(x,u)a_{i_1}(x,u) + \sum q_{2_{i_2}}(x,u)b_{i_2}(x,u) + \sum r_{i_3}c_{i_3}(x,u) \geq 0 \tag{3.16}$$

*Then the origin of the state space is a stable equilibrium of the system.*

**Proof.** If condition (3.15) is fulfilled, then we have that in $\mathcal{D}$

$$V(x) > -\sum p_{1_{i_1}}(x,u)a_{i_1}(x,u) - \sum q_{1_{i_2}}(x,u)b_{i_2}(x,u) \geq 0,$$

and so $V(x) > 0$ in $\mathcal{D}$, where $a_{i_1}(x,u)$ and $b_{i_2}(x,u)$ satisfy (3.12).

Condition (3.16) can be integrated from time $t = 0$ to $t = T$ to obtain

$$V(0) - V(T) \geq -\sum \int_0^T \{p_{2_{i_1}}(x,u)a_{i_1}(x,u) - r_{i_3}c_{i_3}(x,u)\}dt \geq 0,$$

where we have used the fact that $a_{i_1}(x,u)$, $b_{i_2}(x,u)$ and $c_{i_3}(x,u)$ satisfy (3.12)–(3.14).

This shows that the Lyapunov function is non-increasing along the trajectories of the system, and is positive definite in $\mathcal{D}$. Therefore, the conditions for Lyapunov stability (see Theorem 3.2) are satisfied. The rest of the proof is similar to the proof of Lyapunov's theorem, which can be found in many standard textbooks, e.g., [35].

■

We note that even though the integral constraints used above are required to hold for all $T \geq 0$ (i.e., *hard* integral constraints), most of the ones that one can develop during an analysis method are *soft*, i.e. they need not hold for finite-time intervals. In the case of soft Integral Quadratic Constraints, non-causal multipliers were used for stability analysis. See [47] for more details.

When the vector field $f(x,u)$ is rational, i.e., $f(x,u) = \frac{n(x,u)}{d(x,u)}$ with $d(x,u) \neq 0$ in $\mathcal{D}$, condition (3.16) can be multiplied by the non-vanishing denominator. We will be using Theorem 3.7, along with the SOS decomposition to analyze various cases of systems with constraints. The procedure is similar to the one for the case of unconstrained systems described in the previous section, and is based on relaxing the conditions in Theorem 3.7 to SOS conditions. For this, we need to make some assumptions, some of which will be removed in the sequel.

- The vector field $f_x(x,u)$ is assumed to be polynomial or rational, and the constraint functions $a_{i_1}(x,u)$, $b_{i_2}(x,u)$, $c_{i_3}(x,u)$ are assumed to be polynomial. This assumption will be removed in a later section through a recasting process.

- We search for bounded degree *polynomial* Lyapunov function $V$ and multipliers $p_{i_1}$, $q_{i_1}$, $i_1 = 1,\ldots,N_1$ and $p_{i_2}$, $q_{i_2}$, $i_2 = 1,\ldots,N_2$.

To make the above concrete, in order to use Theorem 3.7 and the sum of squares decomposition, we have the proposition below.

**Proposition 3.8** *Suppose that for system (3.1) with* $f(x,u) = \frac{n(x,u)}{d(x,u)}$ *where* $n(x,u)$ *and* $d(x,u)$ *are polynomials and* $d(x,u) > 0$ *in* $\mathcal{D}$*, there exist polynomial functions* $V(x)$*,* $p_{1_{i_1}}(x,u)$*,* $p_{2_{i_1}}(x,u)$*,* $q_{1_{i_2}}(x,u)$*,* $q_{2_{i_2}}(x,u)$*, a positive definite function* $\varphi(x)$ *of the form given in Equation 3.5 and constants* $r_{i_3} \geq 0$ *such that*

$$V(x) + \sum p_{1_{i_1}}(x,u)a_{i_1}(x,u) + \sum q_{1_{i_2}}(x,u)b_{i_2}(x,u) - \varphi(x) \text{ is SOS,} \tag{3.17}$$

$$p_{1_{i_1}}(x,u), p_{2_{i_1}}(x,u) \text{ are SOS for } i_1 = 1,\ldots,N_1, \tag{3.18}$$

$$d(x,u)\begin{pmatrix} -\frac{\partial V}{\partial x}f(x,u) + \sum p_{2_{i_1}}(x,u)a_{i_1}(x,u) \\ +\sum q_{2_{i_2}}(x,u)b_{i_2}(x,u) + \sum r_{i_3}c_{i_3}(x,u) \end{pmatrix} \text{ is SOS.} \tag{3.19}$$

*Then the origin of the state space is a stable equilibrium of the system.*

The polynomials $V(x)$, $p_{1_{i_1}}(x,u)$, $p_{2_{i_1}}(x,u)$, $q_{1_{i_2}}(x,u)$, $q_{2_{i_2}}(x,u)$, the constants $r_{i_3}$ and the positive definite function $\varphi(x)$ can be constructed using SOSTOOLS [75], and a program similar to Program 3.5 can be constructed.

It was mentioned that the particular class of systems with constraints that we consider is rich enough to include as special cases various important analysis problems in control theory. The rest of the chapter concentrates on some of these problems.

## 3.3 Estimating the Region of Attraction

In many instances, non-global stability analysis may be the objective, i.e., when dealing with physical models with positivity constraints on the states (often referred to as positive systems [42]), or when several equilibria or limit cycles are present. In such cases, one may define regions of interest using inequality constraints.

For example, let us consider local stability analysis of the zero equilibrium of $\dot{x} = f(x)$. Define the following inequality constraint on $x$:

$$a(x) \triangleq x^T x - \xi \leq 0, \tag{3.20}$$

where $\xi$ is a positive constant. Then local stability of the zero equilibrium can be tested using the next corollary.

**Corollary 3.9** *Suppose for the system $\dot{x} = f(x)$ and the inequality constraint $a(x) \leq 0$ given in (3.20) there exist a polynomial function $V(x)$ and SOS polynomials $p_1(x), p_2(x)$ such that*

$$\begin{aligned} V(x) + p_1(x)a(x) - \varphi(x) &\ \text{is SOS}, \\ -\frac{\partial V}{\partial x} f(x) + p_2(x)a(x) &\ \text{is SOS}, \end{aligned}$$

*where $\varphi(x)$ is as defined in Equation (3.5). Then the zero equilibrium of the system is stable.*

A problem of particular interest is estimation of the region of attraction of an equilibrium. An estimate of the region of attraction is the largest level set of $V(x)$ obtained from the previous corollary which can 'fit' in the region described by $a(x) \leq 0$. More specifically, given a Lyapunov function $V(x)$ and a domain $D$ in which a Lyapunov function satisfying the conditions of asymptotic stability was constructed, we seek a $\gamma > 0$ such that $\Omega_\gamma = \{x \in \mathbb{R}^n | V(x) \leq \gamma\}$ is bounded and strictly contained in $D$; then $\Omega_\gamma$ is an estimate of the region of attraction. In order to get the maximum value of $\gamma$, we can formulate the problem as one of testing emptiness of a set as follows:

$$\{x \in \mathbb{R}^n : V(x) < \gamma, a(x) = 0\} = \emptyset \tag{3.21}$$

Positivstellensatz conditions take the form:

$$\left(\sum_{i=1}^{N} x_i^2\right)^r (V(x) - \gamma) + p(x)a(x) \text{ is a SOS} \tag{3.22}$$

where $r$ is a non-negative integer and $p(x)$ is a polynomial. Therefore, the task of finding the maximum $\gamma$ can be formulated as the following sum of squares program:

**Program 3.10** *Program to find the maximum $\gamma$ such that $\{x \in \mathbb{R}^n | V(x) < \gamma\} \subset \{x \in \mathbb{R}^n | a(x) \leq 0\}$:*

*Given $V(x), a(x)$, maximize $\gamma$ and find a non-negative integer $r$*

*and a polynomial $p(x)$ such that*

$$\left(\sum_{i=1}^{N} x_i^2\right)^r (V(x) - \gamma) + p(x)a(x) \text{ is a SOS.}$$

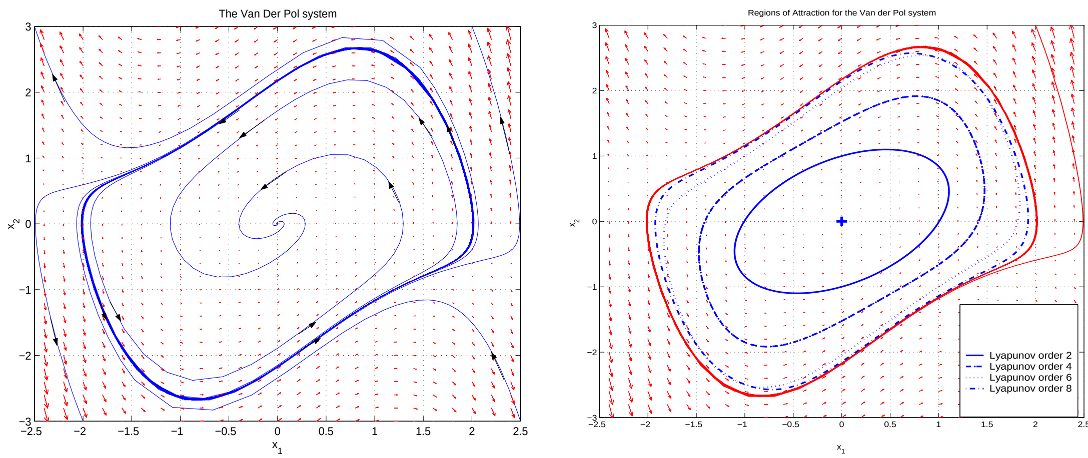
In order to find good estimates of the region of attraction, one has to iterate between $V(x)$ and $a(x)$.

**Example 3.11** *Consider the Van der Pol equation in reverse time:*

$$\dot{x}_1 = -x_2 \tag{3.23}$$

$$\dot{x}_2 = x_1 - (1 - x_1^2)x_2 \tag{3.24}$$

*The phase plane is shown in Figure 3.1(a); the presence of an unstable limit cycle makes the stability of the only equilibrium local. In this case, we are interested in determining how far from the origin we can choose an initial condition and the trajectory will still converge to the origin. To obtain an estimate of the region of attraction of the zero equilibrium of (3.23–3.24), we first initialize $a(x) = x^T x - \xi$, and search for $V(x)$ of order 2 such that $V(x) > 0$ everywhere and $\dot{V}(x) < 0$ in $\mathcal{D}$, performing a search on $\xi$. Obviously, at some point, $\dot{V}(x) = 0$; therefore, we need to*




(a) (b)

Figure 3.1: The Van der Pol system in reverse time, with estimates of the region of attraction of the stable equilibrium.

*find the maximum* $\gamma$ *such that* $V(x) - \gamma$ *is completely contained in the set* $\dot{V}(x) \leq 0$. *We then set* $a(x) = V(x) - \gamma$, *and increase the order of* $V(x)$ *by 2. We can then find a new Lyapunov function of this order adjusting* $\gamma$, *iterating between* $V(x)$ *and* $a(x)$. *Estimates of the region of attraction in the form* $V(x) \leq \gamma$ *were constructed for different degree* $V(x)$, *shown in Figure 3.1(b).*

## 3.4 Analysis of Systems with Non-Polynomial Vector Fields

Thus far, we have concentrated on systems that are described by polynomial vector fields. It is true that physical systems, the functionality of which is in the focus of many research areas, seldom are modeled by polynomial vector fields.

In this section, we will build on a recasting process [82] that produces a polynomial system description from a non-polynomial one, with state dimension at least the same as the original system or higher. The stability properties of the original system can be concluded from the analysis of the recasted system [63]. To describe the original system faithfully, constraints of the form $x_{n+1} = F(x_1, ..., x_n)$ that are created when new variables are introduced should be taken into account. These constraints define an $n$-dimensional manifold on which the solutions to the original differential equations lie. In general such constraints cannot be converted into polynomial forms, even though sometimes there exist polynomial constraints that are induced by the recasting process. For example:

- Two variables introduced for trigonometric functions such as $x_2 = \sin x_1$, $x_3 = \cos x_1$ are constrained via $x_2^2 + x_3^2 = 1$.
- Introducing a variable to replace a power function such as $x_2 = \sqrt{x_1}$ induces the constraints $x_2^2 - x_1 = 0$, $x_2 \geq 0$.
- Introducing a variable to replace an exponential function such as $x_2 = \exp(x_1)$ induces the constraint $x_2 \geq 0$.

We will identify two different classes of systems. We consider systems with non-polynomial vector fields that under a change of variables are transformed into polynomial with:

1. Only polynomial equality constraints;
2. Non-polynomial equality constraints.

A particular example of case (1) above is the simple pendulum, which is described by:

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{l}\sin\theta \end{bmatrix}$$

where $g$ is the gravitational constant, $l$ is the length of the pendulum, $\omega$ its angular velocity and $\theta$ the angular deviation of the bead from the vertical. Setting $x_1 = \sin\theta$ and $x_2 = \cos\theta$, one can easily rewrite the above system as

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ \omega \\ x_2 \end{bmatrix} &= \begin{bmatrix} x_2\omega \\ -\frac{g}{l}x_1 \\ -x_1\omega \end{bmatrix} \\ x_1^2 + x_2^2 &= 1 \end{aligned}$$

where the constraint $x_1^2 + x_2^2 = 1$ is a polynomial equality in $(x_1, x_2)$ that restricts the 3-D recasted system to the original 2-D system.

However, in some cases (case (2) above), this technique results in a series of equality constraints that are not polynomial equalities, for example, relating $\sin(\theta)$ and $\theta$. These appear many times because of modeling descriptions. For example, in order to model enzymatic reactions in biological systems [48], it is common practice to use vector fields with non-rational powers, in the Michaelis-Menten sense. Also, the model of an aircraft in longitudinal flight contains trigonometric nonlinearities of the angle of attack and pitch angle, but in the same equations, one usually captures the coefficients of lift and drag as *polynomial* descriptions of these variables. The stability analysis of the closed loop system using the above methodology becomes difficult, as the same variable appears both in polynomial and non-polynomial terms. The same is true in the case of analysis of chemical processes, where the temperature appears in the energy equation both as a state and also exponentiated in Arrhenius law for the reaction rate.

Suppose that for a nonpolynomial system

$$\dot{z} = f(z) \tag{3.25}$$

which has an equilibrium at the origin, the recasted system obtained using a recasting procedure is written as

$$\dot{\tilde{x}}_1 = f_1(\tilde{x}_1, \tilde{x}_2), \tag{3.26}$$

$$\dot{\tilde{x}}_2 = f_2(\tilde{x}_1, \tilde{x}_2), \tag{3.27}$$

where $\tilde{x}_1 = (x_1, ..., x_n) = z$ are the state variables of the original system, $\tilde{x}_2 = (x_{n+1}, ..., x_{n+m})$ are the new variables introduced in the recasting process, and $f_1(\tilde{x}_1, \tilde{x}_2)$, $f_2(\tilde{x}_1, \tilde{x}_2)$ are polynomial in their arguments.

We denote the constraints that arise directly from the recasting process (i.e., the polynomial ones) by

$$\tilde{x}_2 = F(\tilde{x}_1), \tag{3.28}$$

and those that arise indirectly (the non-polynomial ones) by

$$G_1(\tilde{x}_1, \tilde{x}_2) = 0, \tag{3.29}$$

$$G_2(\tilde{x}_1, \tilde{x}_2) \leq 0, \tag{3.30}$$

where $F$, $G_1$, and $G_2$ are column vectors of functions with appropriate dimensions, and the equalities or inequalities hold entry-wise. We should keep in mind that constraints (3.29)–(3.30) are satisfied only when $\tilde{x}_2 = F(\tilde{x}_1)$ are substituted to (3.29)–(3.30). We assume that all functions involved are polynomials in their arguments.

Proving stability of the zero equilibrium of the original system (3.25) amounts to proving that all trajectories starting close enough to $z = 0$ will remain close to this equilibrium point. This can be accomplished by finding a Lyapunov function $V(z)$ that satisfies the conditions of Lyapunov's stability theorem. Here we use the recasted system to construct a Lyapunov function that proves stability of the equilibrium of the original system. Sufficient conditions that guarantee the existence of a Lyapunov function are stated in the following proposition.

**Proposition 3.12** *Let $\mathcal{D}_1 \subset \mathbb{R}^n$ and $\mathcal{D}_2 \subset \mathbb{R}^m$ be open sets such that $0 \in \mathcal{D}_1$ and $F(\mathcal{D}_1) \subseteq \mathcal{D}_2$. Furthermore, define $\tilde{x}_{2,0} = F(0)$. If there exists a function $\tilde{V} : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathbb{R}$ and column vectors of functions $\lambda_1(\tilde{x}_1, \tilde{x}_2)$, $\lambda_2(\tilde{x}_1, \tilde{x}_2)$, $\sigma_1(\tilde{x}_1, \tilde{x}_2)$,* *and* $\sigma_2(\tilde{x}_1, \tilde{x}_2)$ *with appropriate dimensions such that*

$$\tilde{V}(0, \tilde{x}_{2,0}) = 0, \tag{3.31}$$

$$\tilde{V} + \lambda_1^T G_1 + \sigma_1^T G_2 - \phi(\tilde{x}_1, \tilde{x}_2) \geq 0 \quad \forall (\tilde{x}_1, \tilde{x}_2) \in \mathcal{D}_1 \times \mathcal{D}_2, \tag{3.32}$$

$$-\frac{\partial \tilde{V}}{\partial \tilde{x}_1} f_1 - \frac{\partial \tilde{V}}{\partial \tilde{x}_2} f_2 + \lambda_2^T G_1 + \sigma_2^T G_2 \geq 0 \quad \forall (\tilde{x}_1, \tilde{x}_2) \in \mathcal{D}_1 \times \mathcal{D}_2, \tag{3.33}$$

$$\sigma_1(\tilde{x}_1, \tilde{x}_2) \geq 0 \quad \forall (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^{n+m}, \tag{3.34}$$

$$\sigma_2(\tilde{x}_1, \tilde{x}_2) \geq 0 \quad \forall (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^{n+m}, \tag{3.35}$$

*for some scalar function* $\phi(\tilde{x}_1, \tilde{x}_2)$ *with* $\phi(\tilde{x}_1, F(\tilde{x}_1)) > 0 \quad \forall \tilde{x}_1 \in \mathcal{D}_1 \setminus \{0\}$, *then* $z = 0$ *is a stable equilibrium of (3.25).*

**Proof.** Define $V(z) = \tilde{V}(z, F(z))$. From (3.31) and (3.32), it is straightforward to verify that the first Lyapunov condition is satisfied by $V(z)$. In fact, from (3.29)–(3.30), (3.32) and (3.34), we have that

$$V(\tilde{x}_1, \tilde{x}_2) \geq \phi(\tilde{x}_1, \tilde{x}_2) - \lambda_1^T G_1 - \sigma_1^T G_2 \geq \phi(\tilde{x}_1, \tilde{x}_2) \quad \forall (\tilde{x}_1, \tilde{x}_2) \in \mathcal{D}_1 \times \mathcal{D}_2.$$

Since $\phi(z, F(z)) > 0 \quad \forall z \in \mathcal{D}_1 \setminus \{0\}$ and $F(\mathcal{D}_1) \subseteq \mathcal{D}_2$, it follows that $V(z) > 0 \quad \forall z \in \mathcal{D}_1 \setminus \{0\}$.

Finally, by the chain rule of differentiation we have

$$\frac{\partial V}{\partial z}(z) f(z) = \frac{\partial \tilde{V}}{\partial \tilde{x}_1}(z, F(z)) f_1(z, F(z)) + \frac{\partial \tilde{V}}{\partial \tilde{x}_2}(z, F(z)) f_2(z, F(z)),$$

and using the same argument as above in conjunction with (3.29)–(3.30), (3.33) and (3.35), we see that the second Lyapunov condition is satisfied. Therefore, $V(z)$ is a Lyapunov function for (3.25) and $z = 0$ is a stable equilibrium of the system. ■

The above non-negativity conditions can be relaxed to appropriate sum of squares conditions so that they can be algorithmically verified using semidefinite programming, as discussed earlier in this chapter. This leads the way to an algorithmic construction of the Lyapunov function $V$. For this purpose, the set $\mathcal{D}_1 \times \mathcal{D}_2$ is captured as a semialgebraic set:

$$\mathcal{D}_1 \times \mathcal{D}_2 = \{(\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^n \times \mathbb{R}^m : G_{\mathcal{D}}(\tilde{x}_1, \tilde{x}_2) \geq 0\},$$

where $G_{\mathcal{D}}(\tilde{x}_1, \tilde{x}_2)$ is a column vector of polynomials and the inequality is satisfied entry-wise.

Let us now give an example of how this methodology can be used.

**Example 3.13** *(**Continuously Stirred Tank Reactor**) Chemical reactors are the most important unit in a chemical process. Here, we consider the analysis of the dynamics of a perfectly mixed, diabatic, continuously stirred tank reactor (CSTR) [3]. We also assume a constant volume – constant parameter system for simplicity.*

*The reaction taking place in the CSTR is a first-order exothermic irreversible reaction $A \to B$. After balancing mass and energy, the reactor temperature $T$ and the concentration of species $A$ in the reactor $C_A$ evolve as follows:*

$$\dot{C}_A = \frac{F}{V}(C_{A_f} - C_A) - k_0 e^{-\frac{\Delta E}{RT}} C_A \tag{3.36}$$

$$\dot{T} = \frac{F}{V}(T_f - T) - \frac{\Delta H}{\rho c_p} k_0 e^{-\frac{\Delta E}{RT}} C_A - \frac{UA}{V\rho c_p}(T - T_j) \tag{3.37}$$

*where $F$ is the volumetric flow rate, $V$ is the reactor volume, $C_{A_f}$ is the concentration of $A$ in the freestream, $k_0$ is the pre-exponential factor of Arrhenius law, $\Delta E$ is the reaction activation energy, $R$ is the ideal gas constant, $T_f$ is the feed temperature, $-\Delta H$ is the heat of reaction (exothermic), $\rho$ is the density, $c_p$ is the heat capacity, $U$ is the overall heat transfer coefficient, $A$ is the area for heat exchange, and $T_j$ is the jacket temperature. For the analysis, we use the values shown in Table 3.1.*

*The equilibrium of the above system is given by $(C_{A_0}, T_0) = (8.5636, 311.171)$. We employ the following transformation: $x_1 = C_A/C_{A_0} - 1$, $x_2 = T/T_0 - 1$; this serves two purposes: first it moves the equilibrium to the origin, and second, it rescales the*

| Parameter | Units | Nominal Value |
|---|---|---|
| $F/V$ | $\text{hr}^{-1}$ | 1 |
| $k_0$ | $\text{hr}^{-1}$ | $9703\times3600$ |
| $-\Delta H$ | kcal/kgmol | 5960 |
| $\Delta E$ | kcal/kgmol | 11843 |
| $\rho c_p$ | kcal/($\text{m}^3\ ^\circ$C) | 500 |
| $T_f$ | $^\circ$C | 25 |
| $C_{A_f}$ | kgmol/$\text{m}^3$ | 10 |
| $UA/V$ | kcal/($\text{m}^3\ ^\circ$C hr) | 150 |
| $T_j$ | $^\circ$C | 25 |

Table 3.1: Parameter values for the Continuously Stirred Tank Reactor (CSTR) system.

*state to reduce numerical ill-conditioning. The transformed system then becomes:*

$$\begin{aligned}
\dot{x}_1 &= \frac{F}{V}\left(\frac{C_{A_f}}{C_{A_0}} - (x_1+1)\right) - k_0 e^{-\frac{\Delta E}{RT_0(x_2+1)}}(x_1+1) \\
\dot{x}_2 &= \frac{F}{V}\left(\frac{T_f}{T_0} - (x_2+1)\right) - \frac{\Delta H C_{A_0}}{\rho c_p T_0} k_0 e^{-\frac{\Delta E}{RT_0(x_2+1)}}(x_1+1) - \frac{UA}{V\rho c_p}\left((x_2+1) - \frac{T_j}{T_0}\right)
\end{aligned}$$

*Note that the system has an exponential term; the recasting will yield an indirect constraint, as discussed earlier in this section. Define the state* $x_3 = e^{\frac{\Delta E x_2}{RT_0(x_2+1)}} - 1$. *Then an extra equation in the analysis would be*

$$\dot{x}_3 = \frac{\Delta E}{RT_0(x_2+1)^2}(x_3+1)\dot{x}_2$$

*under the constraint that* $x_3 > -1$.

*Then the full system, after we use the equilibrium relationship simplifies to:*

$$\dot{x}_1 = -\frac{F}{V}x_1 - k_0 e^{-\frac{\Delta E}{RT_0}}(x_1x_3 + x_1 + x_3) \tag{3.38}$$

$$\dot{x}_2 = -\frac{F}{V}x_2 - \frac{\Delta H C_{A_0}}{\rho c_p T_0} k_0 e^{-\frac{\Delta E}{RT_0}}(x_1x_3 + x_1 + x_3) - \frac{UA}{V\rho c_p}x_2 \tag{3.39}$$

$$\dot{x}_3 = \frac{\Delta E}{RT_0(x_2+1)^2}(x_3+1)\dot{x}_2 \tag{3.40}$$

*This system is now of the form (3.26)–(3.27), with* $\tilde{x}_1 = (x_1, x_2)$ *and* $\tilde{x}_2 = x_3$. *To*

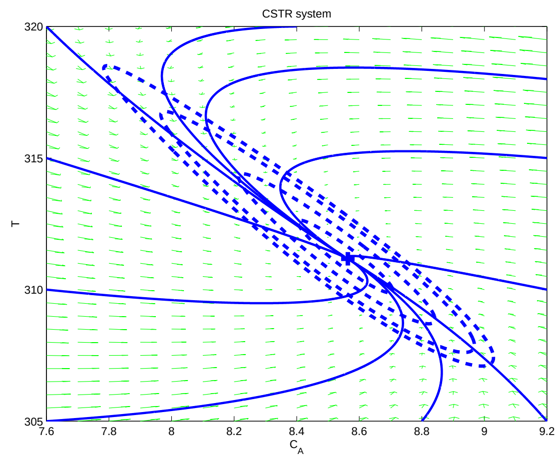



Figure 3.2: Lyapunov function level curves for the Continuously Stirred Tank Reactor system.

*proceed, we define the set* $\mathcal{D}_1$ *as:*

$$\mathcal{D}_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq \gamma_1, |x_2| \leq \gamma_2\}$$

*and then define the set* $\mathcal{D}_2$ *as*

$$\mathcal{D}_2 = \{x_3 \in \mathbb{R} : (x_3 - e^{\frac{-\Delta E \gamma_2}{RT_0(-\gamma_2+1)}} - 1)(x_3 - e^{\frac{\Delta E \gamma_2}{RT_0(\gamma_2+1)}} - 1) \leq 0\}$$

*Then the system is ready for analysis as per Proposition 3.12. For* $\gamma_1 = 0.12$ *and* $\gamma_2 = 0.05$*, a quartic Lyapunov function can be constructed for the system described by Equations (3.38)–(3.40) using Proposition 3.12. Here the following* $\phi(x)$ *is used:*

$$\phi(x) = \sum_{i=1}^{2} \sum_{j=2,4} \epsilon_{i,j} x_i^j + \sum_{j=1}^{4} \epsilon_{3,j} x_3^j,$$

*with*

$$\epsilon_{1,2} + \epsilon_{1,4} - 0.1 \geq 0$$

$$\epsilon_{2,2} + \epsilon_{2,4} + \epsilon_{3,1} + \epsilon_{3,2} + \epsilon_{3,3} + \epsilon_{3,4} - 0.1 \geq 0.$$

*The level curves of the constructed Lyapunov function are shown in Figure 3.2.*

As mentioned earlier, it is common practice in modeling biological reaction pathways to 'lump' enzymatic reactions consisting of three species into a single equation in the Michaelis-Menten sense. The simplified system dynamics are obtained by a methodology similar to singular perturbation methods in nonlinear control. Many times, the 'fitting' is done based on experimental data, which results in non-integer hill function coefficients. We will illustrate the technique with an example of biological significance: regulation and autocatalysis in yeast glycolysis.

**Example 3.14** *Consider a simple biochemical reaction pathway consisting of two metabolites, represented by the state variables* $x_1$ *and* $x_2$*. The product of the first enzymatic reaction is the substrate for the second reaction, and the final output affects the production of* $x_1$*. The equations become:*

$$\dot{x}_1 = x_2^m - \frac{1+k}{x_1+k}x_1 \tag{3.41}$$

$$\dot{x}_2 = \frac{1+k}{x_1+k}x_1 - cx_2^m - (1-c)\frac{1+k}{x_2+k}x_2 \tag{3.42}$$

*The free parameters in the model are* $k$*,* $m$*, and* $c$*. The sign of* $m$ *determines the effect of* $x_2$ *on the production of* $x_1$*. The system has an equilibrium point at* $x_1 = x_2 = x_3 = 1$ *that is independent of the parameters. Linearization about this equilibrium gives the system:*

$$\frac{d}{dt}\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = \begin{bmatrix} -\frac{k}{k+1} & m \\ \frac{k}{k+1} & -cm-(1-c)\frac{k}{k+1} \end{bmatrix}\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

*Linear stability analysis asserts that stability is retained for* $0 \le c \le 1$*,* $k - km - m \ge 0$ *and* $k(2-c) + cm(k+1) \ge 0$*, if we assume that* $k > 0$*. The stability region then becomes:*

$$\left\{ \begin{array}{ll} k \ge \frac{m}{1-m} & \text{for } 0 \le m < 1 \\ k \ge \frac{-cm}{2+cm-c} & \text{for } \frac{c-2}{c} \le m < 0 \end{array} \right\} \tag{3.43}$$

*We can translate the equilibrium of the system given by Equations (3.41–3.42) to* *the origin by employing the transformation* $y_1 = x_1 - 1$, $y_2 = x_2 - 1$ *to get:*

$$\begin{aligned}
\dot{y}_1 &= (y_2+1)^m - \frac{1+k}{1+y_1+k}(y_1+1) \\
\dot{y}_2 &= \frac{1+k}{1+y_1+k}(y_1+1) - c(y_2+1)^m - (1-c)\frac{1+k}{1+y_2+k}(y_2+1)
\end{aligned}$$

*If* $m$ *is integer, then this system is polynomial. If* $m$ *is not integer, then one can proceed as in the CSTR example and try to analyze it directly. To illustrate a special case when having non-integer (but rational) powers, consider* $m = 1/3$. *Denote*

$$z = (y_2+1)^{(1/3)} - 1$$

*Then we have the system*

$$\begin{aligned}
\dot{y}_1 &= z + 1 - \frac{1+k}{1+y_1+k}(y_1+1) && (3.44) \\
\dot{y}_2 &= \frac{1+k}{1+y_1+k}(y_1+1) - c(z+1) - (1-c)\frac{1+k}{1+y_2+k}(y_2+1) && (3.45) \\
\dot{z} &= \frac{z+1}{3(y_2+1)}\dot{y}_2 && (3.46) \\
(z+1)^3 &= (y_2+1) && (3.47)
\end{aligned}$$

*i.e., a 3-D system restricted by one equality constraint. The system is now in a DAE formulation. We set* $c = 0.5$ *and* $k = 1$ *and use SOSTOOLS to construct a Lyapunov function for this system, whose level curves are shown in Figure 3.3. In this figure, '+' denotes the equilibrium point, solid lines are trajectories with initial conditions indicated by 'o', arrows denote vector field and dashed lines are level curves of the Lyapunov function.*

*In this example,* $m$ *is an important parameter, and we wish to analyze robust stability of the above system when* $m$ *and* $k$ *are allowed to vary. If we consider stability analysis for* $-0.2 \le y_1 \le 0.2$ *and* $-0.2 \le y_2 \le 0.2$, *then we can approximate*

$$(y_2+1)^m = 1 + my_2 + \frac{m(m-1)}{2}y_2^2 + u(m, y_2)$$

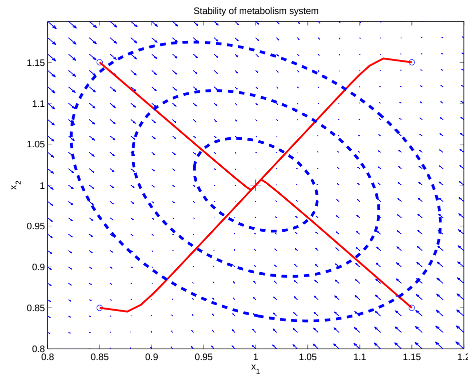



Figure 3.3: Stability of the yeast glycolysis system.

*and we can 'cover' the residual* $u$ *by:*

$$u(m, y_2) = pmy_2^2, \quad p \in [-0.08, 0.06]$$

*The system then becomes:*

$$\dot{y}_1 = 1 + my_2 + \frac{m(m-1)}{2}y_2^2 + pmy_2^2 - \frac{1+k}{1+y_1+k}(y_1+1) \tag{3.48}$$

$$\dot{y}_2 = \frac{1+k}{1+y_1+k}(y_1+1) - c(1 + my_2 + \frac{m(m-1)}{2}y_2^2 + pmy_2^2) - (1-c)\frac{1+k}{1+y_2+k}(y_2+1) \tag{3.49}$$

*There are now three parameters in the system description, namely* $p$*,* $m$ *and* $k$*. We wish to prove stability of the equilibrium for the stable parameter set. To achieve this, we construct two Lyapunov functions as shown in Figure 3.4: one for* $m \geq 0$ *and one for* $-m \geq 0$*. In each region, there are five inequality constraints that we can write: two on the state variables:*

$$a_1 \triangleq (y_1 - 0.2)(y_1 + 0.2) \leq 0$$

$$a_2 \triangleq (y_2 - 0.2)(y_2 + 0.2) \leq 0,$$

*one describing* $p \in [-0.08, 0.06]$*:*

$$a_3 \triangleq (p + 0.08)(p - 0.06) \leq 0,$$

*and two describing the* $m - -k$ *space; for* $m \geq 0$ *we have:*

$$\begin{aligned} a_4 &\triangleq -k + km + m + 1 \leq 0 \\ a_5 &\triangleq m(m - 0.9) \leq 0, \end{aligned}$$

*and for* $m \leq 0$ *we have:*

$$\begin{aligned} a_4 &\triangleq -1.5k - 0.5m(k + 1) + 1 \leq 0 \\ a_5 &\triangleq m(m + 2.5) \leq 0. \end{aligned}$$

*Using these constraints, the sum of squares program can be constructed and solved with the aid of SOSTOOLS. The Lyapunov function* $V(y_1, y_2; m, p, k)$ *for two ranges of parameters, as well as the parameter region are shown in Figure 3.4. In this figure, the shaded region in the center figure shows the parameter set for which a parameterized Lyapunov function was constructed, and the region of stability given by Equation (3.43). The figure on the left corresponds to a parameter set in which* $m$ *is negative, which results in system oscillations — the figure on the right corresponds to another parameter set in which* $m$ *is positive. In both cases, the level curves of a Lyapunov function are shown dashed.*

## 3.5 Robust Stability Analysis

The framework that we presented in this chapter allows the analysis of robustness of systems with parametric and/or dynamic uncertainty.

In the case of parametric uncertainty, some of the auxiliary variables $u$ in Theorem 3.7 may now be taken to be parameters. Additionally, some other auxiliary variables can be used to account for the location of the equilibrium of interest, as the

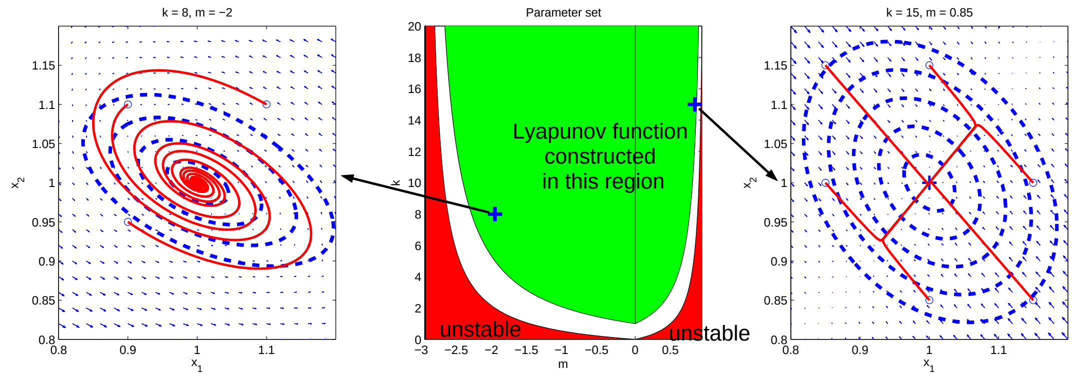



Figure 3.4: Robust stability of the yeast glycolysis system.

location of the equilibrium of a nonlinear system usually changes when the parameters are varied. The use of equality and inequality constraints in this case is natural. The region of the parameter space that is of interest can be described by inequality constraints, and if the equilibrium moves as the parameters change, one can impose an equality constraint on the corresponding auxiliary variables.

Dynamic uncertainty, on the other hand, can be characterized using integral constraints [47]. For example, an uncertain but $\mathcal{L}_2$-norm bounded feedback operator relating $x$ and $u$ can be represented by the IQC

$$\int_0^T (\gamma x^T x - u^T u) dt \geq 0.$$

Stability of the whole system can then be verified using Theorem 3.7.

The Lyapunov functions that we will be constructing may be parameterized by the parameters. The introduction of new, unknown coordinates makes the size of the resulting semidefinite programs grow fast. However, in some special cases, one can structure these multipliers and functions so as to reduce the size of the semidefinite programs and increase the numerical conditioning.

Here we consider a simple model that describes the phenomenon of heat shock in *E-coli*, developed in [20].

| Parameter | Value |
|---|---|
| $K_d$ | 3 |
| $\alpha_d$ | 0.015 |
| $\eta(T)$ | 10 @ $T_1$ & 60 @ $T_2$ |
| $\alpha_0$ | 0.03 |
| $\alpha_s$ | 3 |
| $K_s$ | 0.05 |
| $K_u$ | 0.0254 |
| $K(T)$ | 40 @ $T_1$ & 80 @ $T_2$ |
| $K_{fold}$ | 6000 |
| $P_t$ | $2 \times 10^6$ |

Table 3.2: Parameter values for the model of Heat Shock in *E-coli*.

**Example 3.15** *A simple description for the reduced order heat shock model is:*

$$\begin{aligned}
\frac{dD_t}{dt} &= K_d \frac{S_t}{1 + \frac{K_s D_t}{1+K_u U_f}} - \alpha_d D_t \\
\frac{dS_t}{dt} &= \eta(T) - \alpha_0 . S_t - \alpha_s \frac{\frac{K_s D_t}{1+K_u U_f}}{1 + \frac{K_s D_t}{1+K_u U_f}} S_t \\
\frac{dU_f}{dt} &= K(T)[P_t - U_f] - [K(T) + K_{fold}] D_t
\end{aligned} \tag{3.50}$$

*where $D_t$ is the total number of chaperones, $S_t$ is the total number of $\sigma$ factors and $U_f$ is the amount of unfolded proteins. The rest are parameters that are tabulated in Table 3.2.*

*In this example, we consider the problem of robust stability for the system under parametric uncertainty. We first non-dimensionalize the states of (3.50) by their equilibrium values $(D_{t_0}, S_{t_0}, U_{f0})$, followed by a shifting of the equilibrium of the system to the origin, as it was done for the CSTR example. This yields a system with states $(x_1, x_2, x_3)$ that is better conditioned, in the sense that the states are of the same order of magnitude:*

$$\begin{aligned}
\frac{dx_1}{dt} &= \tilde{f}_1(x_1, x_2, x_3) - \alpha_d x_1 \\
\frac{dx_2}{dt} &= \tilde{\eta}(T) - \alpha_0 x_2 - \tilde{f}_2(x_1, x_2, x_3) \\
\frac{dx_3}{dt} &= K(T)[\tilde{P}_t - x_3] - K_{Tot} x_1
\end{aligned} \tag{3.51}$$

*with* $\tilde{f}_1(x_1, x_2, x_3) = \tilde{K}_d \frac{x_2}{1+\frac{\tilde{K}_s x_1}{1+\tilde{K}_u x_3}}$ *and* $\tilde{f}_2(x_1, x_2, x_3) = \alpha_s x_2 \frac{\frac{\tilde{K}_s x_1}{1+\tilde{K}_u x_3}}{1+\frac{\tilde{K}_s x_1}{1+\tilde{K}_u x_3}}$*, and where* $\tilde{K}_d = K_d S_{t_0}/D_{t_0}$*,* $\tilde{K}_s = K_s D_{t_0}$*,* $\tilde{K}_u = K_u U_{f_0}$*,* $\tilde{\eta} = \eta/S_{t_0}$*,* $\tilde{P}_t = P_t/U_{f_0}$ *and* $K_{Tot} = D_{t_0}(K(T) + K_{fold})/U_{f_0}$*.*

*A simple simulation reveals that the dynamics of the third state* $x_3$ *are much faster than those of* $x_2$ *and* $x_1$*. Such stiffness (large differences in time scales) creates ill-conditioning in the resulting semidefinite programme. In our analysis, we investigate a quasi steady-state approximation by setting* $\frac{dx_3}{dt} = 0$*, which results in a differential-algebraic system.*

*To proceed, we first define the region* $D$ *in the state-space where a Lyapunov function is to be constructed:*

$$D = \{x_i \in \mathbb{R} : (x_i - x_{i_0})^2 - \gamma_i^2 \leq 0, i = 1, 2\} \tag{3.52}$$

*with* $\gamma_i = 0.2$*,* $x_{i_0}$ *denoting the equilibrium of the* $i$*-th state. For robust stability analysis purposes, we pick two important parameters,* $\tilde{\eta}$ *and* $\alpha_s$*.* $\tilde{\eta}$ *depicts the feedforward gain, while* $\alpha_s$ *forms part of the feedback gain. We ask whether the equilibrium of the system described by (3.50) is stable for all values of* $\tilde{\eta}$ *and* $\alpha_s$ *in a certain range for* $T = T_1$*:*

$$P = \{\tilde{\eta}, \alpha_s \in \mathbb{R}^2 : (\tilde{\eta} - \tilde{\eta}_0)^2 - (\gamma_3 \tilde{\eta}_0)^2 \leq 0, (\alpha_s - \alpha_{s_0})^2 - (\gamma_4 \alpha_{s_0})^2 \leq 0\}, \tag{3.53}$$

*with* $\gamma_3$ *and* $\gamma_4$ *measuring the percentage variation. As these parameters change, the equilibrium of the system also changes and the other equilibria may cross through the region* $D$*. In order to fix the equilibrium at the origin, we impose two equality constraints of the form:*

$$\begin{aligned} \alpha_d \alpha_s \tilde{K}_s x_{1_0}^2 - \tilde{K}_d(\tilde{\eta}(T_1) - \alpha_0 x_{2_0}) \left(1 + \tilde{K}_u \left(\tilde{P}_t - \frac{K_{Tot}}{K(T_1)} x_{1_0}\right)\right) &= 0 \\ \tilde{K}_s \tilde{K}_d(\tilde{\eta}(T_1) - \alpha_0 x_{2_0}) - \alpha_s \tilde{K}_s(\tilde{K}_d x_{2_0} - \alpha_d x_{1_0}) &= 0 \end{aligned}$$

*Note that the vector field is rational, but this case can be treated by using Theorem*

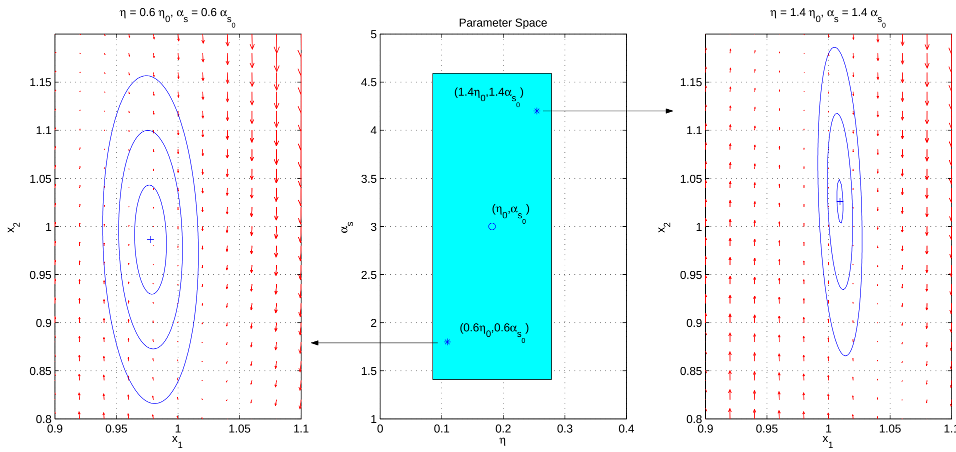



Figure 3.5: Robust stability of the model of Heat Shock in *E-coli*.

*3.7, by multiplying out by the (non-vanishing) common denominator of the vector field. Robust stability analysis is achieved by constructing a parameter-dependent Lyapunov function, using the results in proposition 3.8 and SOSTOOLS. We start with a quadratic Lyapunov function V that is not parameterized by any parameters; in this case, we can prove stability for* $\gamma_3 = \gamma_4 = 0.45$. *When the Lyapunov function is parameterized by* $\alpha_s$ *and* $\tilde{\eta}$, *we can construct a Lyapunov function for* $\gamma_3 = \gamma_4 = 0.53$. *We see that by increasing the complexity of the certificate, we can construct a Lyapunov function for a larger parameter range. The equilibrium is stable for even larger parameter sets, but the other equilibrium in the system (which is unstable) approaches the equilibrium of interest. To get a larger parameter set, we need to reduce the size of the region D and increase the order of the certificate we seek to construct. Figure 3.5 shows the level curves of the Lyapunov function for two sets of parameters in a parameter set with* $\gamma_3 = \gamma_4 = 0.53$.

## 3.6 Stability of Switching Systems

In this section, we show how the sum of squares decomposition can be used to construct Lyapunov-type certificates for switching systems. These are systems that have dynamics that are described by a set of continuous time differential equations in conjunction with a discrete event process. Previous results on stability analysis of switched systems can be found in [12, 18]; here we do not consider hybrid systems, which are treated in a similar manner in [74].

One way of ensuring stability of a switching system is by constructing a Lyapunov function that is 'common' for all subsystems, a notion similar to quadratic stability in Linear Parameter Varying system formulations. Another way is by constructing piecewise quadratic Lyapunov functions [72, 32, 29], a concatenation of several Lyapunov-like functions, each valid for each subsystem with some continuity argument on the switching surfaces. In the special case of switching systems with linear subsystems and with hyperplane switching surfaces, the search for such Lyapunov functions can be performed by solving Linear Matrix Inequalities (LMIs).

Here we will show how the sum of squares decomposition can be used to construct polynomial Lyapunov certificates for switching systems with nonlinear subsystems and polynomial switching surfaces. Uncertain switching systems can also be analyzed in a unified manner.

The systems considered here are of the following form:

$$\dot{x} = f_i(x), \quad i \in I = \{1, ..., N\}, \tag{3.54}$$

where $x \in \mathbb{R}^n$ is the continuous state, $i$ is the discrete state, $f_i(x)$ is the vector field describing the dynamics of the $i$-th mode/subsystem, and $I$ is the index set. We assume that the origin is an equilibrium of the system. Further, we assume that for each $x \in \mathbb{R}^n$, only one $i \in I$ is possible. More specifically, the system is in the $i$-th mode at time $t$ if $x(t) \in X_i$, where $X_i \subset \mathbb{R}^n$ is a region of the state space described by

$$X_i = \{x \in \mathbb{R}^n : G_i(x) \geq 0\}, \tag{3.55}$$

where $G_i : \mathbb{R}^n \rightarrow \mathbb{R}^{k_i}$ is a vector of $k_i$ polynomials and $G_i(x) \geq 0$ means that the inequality holds entry-wisely. Additionally, the state space partition $\{X_i\}$ must satisfy $\bigcup_{i \in I} X_i = \mathbb{R}^n$ and $\text{int}(X_i) \cap \text{int}(X_j) = \emptyset$ for $i \neq j$, i.e., the partitions cover the whole of $\mathbb{R}^n$ and are non-overlapping. A switching surface between the $i$-th and $j$-th modes, i.e., a boundary between $X_i$ and $X_j$, is given by

$$S_{ij} = \{x \in \mathbb{R}^n : h_{ij}(x) = 0\}, \tag{3.56}$$

for some $h_{ij} : \mathbb{R}^n \to \mathbb{R}$. Note that even though the direction of transition for a particular $x \in S_{ij}$ can be determined from the vector fields $f_i(x)$ and $f_j(x)$, it is assumed in our analysis that such a characterization is not performed *a priori*. Also, systems with infinitely fast switching such as those that have sliding modes are excluded from our discussion. We also assume that the functions $f_i$, $G_i$, and $h_{ij}$ are polynomials, for otherwise, we can change coordinates to render the system polynomial as explained earlier.

### 3.6.1 Common Lyapunov Functions

We will first consider stability of the system (3.54) under arbitrary switching. A sufficient condition for such stability is the existence of a common global Lyapunov function for all $f_i$'s, as summarized in the following theorem.

**Theorem 3.16** *Suppose that for the set of vector fields $\{f_i\}$, there exists a polynomial $V(x)$ such that $V(0) = 0$ and*

$$V(x) > 0 \quad \forall x \neq 0, \tag{3.57}$$

$$\frac{\partial V}{\partial x} f_i(x) < 0 \quad \forall x \neq 0,\ i \in I, \tag{3.58}$$

*then the origin of the state space of the system (3.54) is globally asymptotically stable under arbitrary switching.*

The proof of this theorem is obvious and is omitted. If the vector fields are linear and $V(x) = x^T P x$, then the above conditions correspond to the LMIs for quadratic stability [10]. For higher degree polynomial vector fields and Lyapunov functions, the search for $V(x)$ can also be performed using semidefinite programming by formulating

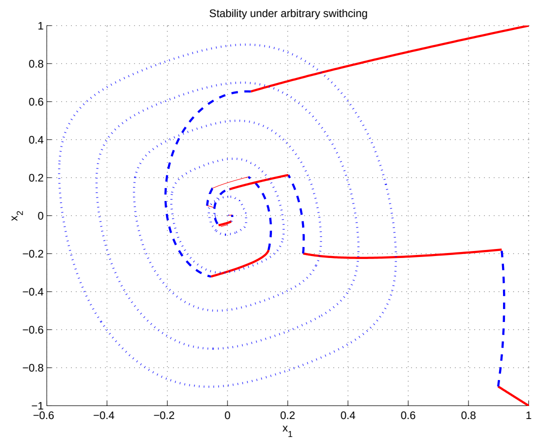



Figure 3.6: Stability of a switching system under arbitrary switching by constructing a common Lyapunov function.

the conditions as sum of squares conditions. The higher degree test is generally less conservative than the quadratic test, as in many cases global higher degree Lyapunov functions exist for systems that do not possess a global quadratic Lyapunov function.

**Example 3.17** *Consider the system* $\dot{x} = f_i(x)$, $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, *with*

$$f_1(x) = \begin{bmatrix} -5x_1 - 4x_2 \\ -x_1 - 2x_2 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} -2x_1 - 4x_2 \\ 20x_1 - 2x_2 \end{bmatrix}.$$

*It can be proven using a dual semidefinite program that no global quadratic Lyapunov function exists for this system [32]. Nevertheless, a global sextic Lyapunov function*

$$V(x) = 19.861x_1^6 + 11.709x_1^5x_2 + 14.17x_1^4x_2^2 + 4.2277x_1^3x_2^3 + 8.3495x_1^2x_2^4 - 1.2117x_1x_2^5 + 1.0421x_2^6$$

*exists, and therefore the system is asymptotically stable under arbitrary switching (see Figure 3.6). In this figure, solid trajectories correspond to system* $f_1$*, dashed ones correspond to system* $f_2$*. The dotted curves are level curves of the common Lyapunov function.*

### 3.6.2 Piecewise Polynomial Lyapunov Functions

Proving stability of an equilibrium of a switching system by constructing a common Lyapunov function may be conservative, in particular when the switching rule is known and included in the system description. In this case, stability can be proven by constructing piecewise polynomial Lyapunov functions, i.e., a concatenation of several polynomial functions $V_i(x)$ (also termed Lyapunov-like functions), typically corresponding to the state space partition $\{X_i\}$. The Lyapunov-like function $V_i(x)$ and its time derivative along the trajectory of the $i$-th mode are required to be positive and negative definite respectively, only within $X_i$. Such conditions can be accommodated using the S-procedure by using the descriptions for the sets $X_i$ given in (3.55) and the switching surfaces (3.56). Note again that the multipliers used to adjoin such conditions can be polynomials instead of just constants as it was mentioned in Example 2.14. Moreover, to ensure continuity of the Lyapunov function on the switching surface between modes $i$ and $j$, we can impose the condition

$$V_i(x) + c_{ij}(x)h_{ij}(x) - V_j(x) = 0,$$

where $c_{ij}(x)$ is a polynomial.

Summing up, we have the following Theorem:

**Theorem 3.18** *Consider the switched system (3.54)–(3.56). Assume that there exist polynomials $V_i(x)$, $c_{ij}(x)$, with $V_i(0) = 0$ if $0 \in X_i$, and vectors of polynomials $a_i(x) \geq 0$, $b_i(x) \geq 0$, such that*

$$V_i(x) - a_i^T(x)G_i(x) > 0 \quad \forall x \neq 0, i \in I, \tag{3.59}$$

$$\frac{\partial V_i}{\partial x} f_i(x) + b_i^T(x)G_i(x) < 0 \quad \forall x \neq 0, i \in I, \tag{3.60}$$

$$V_i(x) + c_{ij}(x)h_{ij}(x) - V_j(x) = 0 \quad \forall i, j. \tag{3.61}$$

*Then the origin of the state space is globally asymptotically stable. A Lyapunov func-*

*tion that proves this is the piecewise polynomial function $V(x)$ defined by*

$$V(x) = V_i(x), \qquad \textit{if } x \in X_i. \tag{3.62}$$

The proof of this theorem is based on the fact that the functions $V_i$ are Lyapunov-like functions for the corresponding subsystems in the regions in which they are defined. Concatenation of these Lyapunov functions to form $V(x)$, and the continuity of the value of the $V_i$'s along the switching surface guarantees that along the trajectories of the system, the value of $V(x)$ does not increase.

Here we present an example that involves a switching system with nonlinear subsystems and a polynomial switching surface.

**Example 3.19** *Consider the system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1^3 - x_2 \\ x_1 - 2x_2x_1^2 \end{bmatrix} \qquad \textit{for } x_1^3 - x_2 > 0$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1x_2^2 - x_2 \\ x_1 + px_1x_2 - x_2 \end{bmatrix} \qquad \textit{for } x_1^3 - x_2 \leq 0$$

*Setting $p = 4$, we can construct a sextic Lyapunov function that proves global stability; the level curves are shown in Figure 3.7. In that figure the trajectories are shown in solid thin lines, and the switching surface as a solid thick line for $p = 4$. Alternatively, consider the case in which $p$ is unknown. A quadratic Lyapunov function proves stability of the zero equilibrium for $p \in [-2.82, 2.8]$. A sextic one gives stability for $p \in [-2.82, 7.6]$ and one of order 8 gives stability for $p \in [-2.82, 15.9]$. Note that for $p < 2.82$, two other equilibria appear in the second system which make the stability of the origin only local.*

## 3.7 Performance Analysis

In this section, we are concerned with performance analysis of uncertain nonlinear models, and how it can be performed using sum of squares techniques. Previous

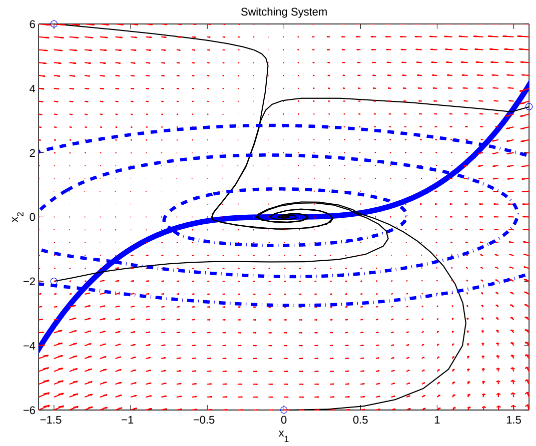



Figure 3.7: Stability of a switching system with predefined switching, using a Lyapunov-like function.

attempts to analyze performance of such systems concentrated on LPV system representations and the discretization of the resulting infinite-dimensional Parameterized LMIs (PLMIs). Again, this approach does not guarantee that the original PLMI problem is feasible, as no discretization is fine enough.

In this section, we wish to obtain upper bounds on the $L_2$ induced norm of the system from an external input to a performance output using SOS. The $L_2$ induced norm is a measure of the worst energy amplification between input and output as the system trajectory and the uncertain parameters vary within their specified bounded regions. For uncertain systems with state-space constraints, it is also important to characterize the reachable set from a bounded energy input. This way, we can associate the set of external inputs to a region in the state-space which is guaranteed to contain the state trajectory for the given amount of uncertainty. Then, for that set of external inputs, the $L_2$ induced norm from input to output can be calculated.

In this section, we concentrate on the following class of systems:

$$\dot{x} = f(x, u; \theta) \tag{3.63}$$

$$y = h(x, u; \theta) \tag{3.64}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^q$, $u \in \mathbb{R}^m$, $\theta \in \Theta \subset \mathbb{R}^p$ where $\Theta$ is given by:

$$\Theta = \{\theta \in \mathbb{R}^p \,|\, g_i(\theta) \leq 0, i = 1, \ldots, k_1\} \tag{3.65}$$

The function $f$ is locally Lipschitz in $(x, u)$ and $h$ is continuous in $(x, u)$ for $x \in D$, where $D \subset \mathbb{R}^n$ be a domain that contains the equilibrium state $x = 0$ that we define by

$$D = \{x \in \mathbb{R}^n \,|\, g_i(x) \leq 0, i = k_1 + 1, \ldots, S\} \tag{3.66}$$

and for $u \in D_0$ where $D_0$ is a domain that contains $u = 0$.

First, consider the problem of calculating reachable sets for the system's state trajectory when it is driven with an input of energy $\epsilon$. We follow the notation in [10], and denote the set $R_{u\epsilon}$ as the set of reachable states with inputs of energy $\epsilon$ for the system (3.63–3.64):

$$R_{u\epsilon} \triangleq \left\{ x(T) \,\middle|\, (x, u) \text{ satisfy (3.63)}, \quad x(0) = 0, \quad \int_0^T u^T u \leq \epsilon, \quad T \geq 0 \right\}. \tag{3.67}$$

An estimate of the reachable set is the $\epsilon$ level set of an appropriate Lyapunov function $V(x)$, i.e., $R_{u\epsilon} \subset \{x \in \mathbb{R}^n | V(x) \leq \epsilon\}$, which is contained in $D$:

**Proposition 3.20** *For the system given by (3.63), let there exist a polynomial function $V(x)$ and a positive definite function $\phi(x)$ so that*

1. $V(x) - \phi(x) \geq 0$
2. $\frac{\partial V}{\partial x} f(x, u; \theta) \leq u^T u$ *for all* $x \in D$ *and* $\theta \in \Theta$
3. $\{V(x) \leq \epsilon\} \subset D$

*Then the set $V(x) \leq \epsilon$ contains the reachable set when the input satisfies $\|u\|^2 < \epsilon$.*

**Proof.** The first two conditions guarantee that $V(x)$ is a Lyapunov function for the unforced system. When $u \neq 0$, condition (2) becomes $\frac{dV}{dt} \leq u^T u$. Integrating this from 0 to $T$, we get:

$$V(x(T)) - V(x(0)) \leq \int_0^T u^T u dt$$

Since $V(x(0)) = V(0) = 0$, we have

$$V(x(T)) \leq \int_0^T u^T u dt \leq \epsilon$$

for every $T \geq 0$ and every $u$ such that $\int_0^T u^T u dt \leq \epsilon$. Condition (3) guarantees that $\{x \in \mathbb{R}^n | V(x) \leq \epsilon\}$ is a subset of the set $D$, and therefore the set $V(x) \leq \epsilon$ is an estimate of the reachable set. ■

The conditions in Proposition 3.20 can be tested using the sum of squares decomposition, by relaxing the non-negativity conditions to the existence of a sum of squares decomposition. The third condition can be easily converted into a set emptiness argument which can be tested using positivstellensatz and the sum of squares decomposition, as it was done in the case of obtaining estimates for the region of attraction. This leads to the following Program:

**Program 3.21** *An algorithm to* bound the reachable set with $\epsilon$-energy inputs*:*

*Maximize* $\epsilon$

*and find a non-negative integer* $r$, *a polynomial* $V(x)$,

*a positive definite function* $\phi(x)$, *sums of squares* $p_i(x,\theta)$, $\quad i = 1, \ldots, S$,

*and sums of squares* $q_i(x)$, $\quad i = k_1, \ldots, S$

*such that*

$$V(x) - \phi(x) \text{ is SOS}$$

$$-\frac{\partial V}{\partial x} f(x,u) + u^T u + \sum_{i=1}^{S} p_i(x,\theta) g_i \text{ is SOS}$$

$$\|x\|^{2r}(V(x) - \epsilon) + \sum_{i=k_1}^{S} q_i(x) g_i \text{ is SOS}$$

*Then the reachable set is contained in the set* $V(x) \leq \epsilon$*.*

Now that we can associate the energy of the external input to a region in the state-space that is guaranteed to contain the state trajectory, we are ready to calculate the input to output gain (the $L_2$ gain). This is the maximum ratio of output to input energy, i.e.,

$$\sup_{\|u\|_2 \neq 0} \frac{\|y\|_2}{\|u\|_2}$$

starting at $x(0) = 0$, where $\|u\|_2^2 = \int_0^\infty u^T u dt$. It can be estimated as follows:

**Proposition 3.22** *Consider the system given by (3.63–3.64). Let there exist a polynomial $V(x)$, a positive number $\gamma$ and a positive definite function $\phi(x)$ such that*

1. $V(x) - \phi(x) \geq 0$,

2. $-\frac{\partial V}{\partial x} f(x, u; \theta) - y^T y + \gamma u^T u \geq 0$

*for $x \in D$ and $\theta \in \Theta$. Then the $L_2$ gain of the system given by (3.63–3.64) is less than $\sqrt{\gamma}$.*

**Proof.** Condition (2) above implies that

$$-\frac{dV}{dt} - y^T y + \gamma u^T u \geq 0.$$

Integrating this from 0 to $T$, we get:

$$V(x(0)) - V(x(T)) \geq \int_0^T (y^T y - \gamma u^T u) dt$$

From condition (1) and from the fact that $x(0) = 0$, we have that $V(x(0)) = V(0) = 0$ and $V(x(T)) \geq 0$. Therefore,

$$\int_0^T (y^T y - \gamma u^T u) dt \leq 0$$

and so

$$\frac{\int_0^T y^T y dt}{\int_0^T u^T u dt} \leq \gamma. \tag{3.68}$$

We conclude that $\sqrt{\gamma}$ is an upper bound on the $L_2$ gain. ■

A similar sum of squares program can be formulated in this case, too. We will demonstrate our results on a model of the short-period dynamics of an F/A-18 aircraft, equipped with a nonlinear dynamic inversion control law.

**Example 3.23** *An uncertain model approximating the F/A-18 short-period dynamics is derived in [64]:*

$$\begin{bmatrix} \dot{q} \\ \Delta\dot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{I_{yy}} M_\alpha(\alpha)_{unc} \\ 1 + \frac{\cos(\alpha)}{mV_o} Z_q(\alpha)_{unc} & \frac{\cos(\alpha)}{mV_o} Z_\alpha(\alpha)_{unc} \end{bmatrix} \begin{bmatrix} q \\ \Delta\alpha \end{bmatrix} + \begin{bmatrix} \frac{1}{I_{yy}} M_\epsilon(\alpha)_{unc} \\ \frac{\cos(\alpha)}{mV_o} Z_\epsilon(\alpha)_{unc} \end{bmatrix} \Delta e. \tag{3.69}$$

*The symbol $q$ denotes the pitch rate, $\alpha$ the angle of attack, $e$ the elevator deflection, $I_{yy}$ the moment of inertia and $m$ the mass of the aircraft. The model is valid for an equilibrium point characterized by the speed $V_0$ and altitude $h_0$ for a range of $\alpha$. The signals $\Delta\alpha = \alpha - \alpha_o$ and $\Delta e = e - e_o$ represent deviations from the equilibrium values $\alpha_o$ and $e_o$.*

*The terms $M_\alpha(\alpha), \ldots, Z_\epsilon(\alpha)$ are aerodynamic derivatives – polynomial functions of $\alpha$ – obtained by data fitting of experimental data (usually obtained from wind tunnel tests). Because of this, their value is, in general, uncertain. It is also dependent on the flight conditions, etc. Therefore, we introduce weighted, additive uncertainty on the aerodynamic coefficients for these moments and forces as shown below.*

$$\begin{aligned}
M_\alpha(\alpha)_{unc} &= M_\alpha(\alpha) + \hat{M}_\alpha w_{M_\alpha} \delta_{M_\alpha}, \\
M_\epsilon(\alpha)_{unc} &= M_\epsilon(\alpha) + \hat{M}_\epsilon w_{M_\epsilon} \delta_{M_\epsilon}, \\
Z_q(\alpha)_{unc} &= Z_q(\alpha) + \hat{Z}_q w_{Z_q} \delta_{Z_q}, \\
Z_\alpha(\alpha)_{unc} &= Z_\alpha(\alpha) + \hat{Z}_\alpha w_{Z_\alpha} \delta_{Z_\alpha}, \\
Z_\epsilon(\alpha)_{unc} &= Z_\epsilon(\alpha) + \hat{Z}_\epsilon w_{Z_\epsilon} \delta_{Z_\epsilon}.
\end{aligned}$$

*The $w$'s are positive numbers and the constant terms $\hat{M}_\alpha, \ldots, \hat{Z}_\alpha, \hat{Z}_\epsilon$ are the average moment or force over the specific range of $\alpha$. The $\delta$'s satisfy $|\delta_i| \leq 1$. Also, consider a simple nonlinear dynamic inversion control law [21, 13] for the uncertain, short-period* *model of (3.69) in order to achieve pitch rate control:*

$$\Delta e = \left(\frac{M_\epsilon(\alpha)}{I_{yy}}\right)^{-1}\left(v - \frac{M_\alpha(\alpha)}{I_{yy}}\Delta\alpha\right). \tag{3.70}$$

*The external signal* $v$ *is the commanded pitch acceleration which is provided according to the proportional control law,* $v = 5(q_{com} - q)$*. The resulting nonlinear controller is valid for the whole of the flight envelope, thus avoiding the use of gain scheduling techniques. Closing the loop, we have the following equation for the uncertain, closed-loop system:*

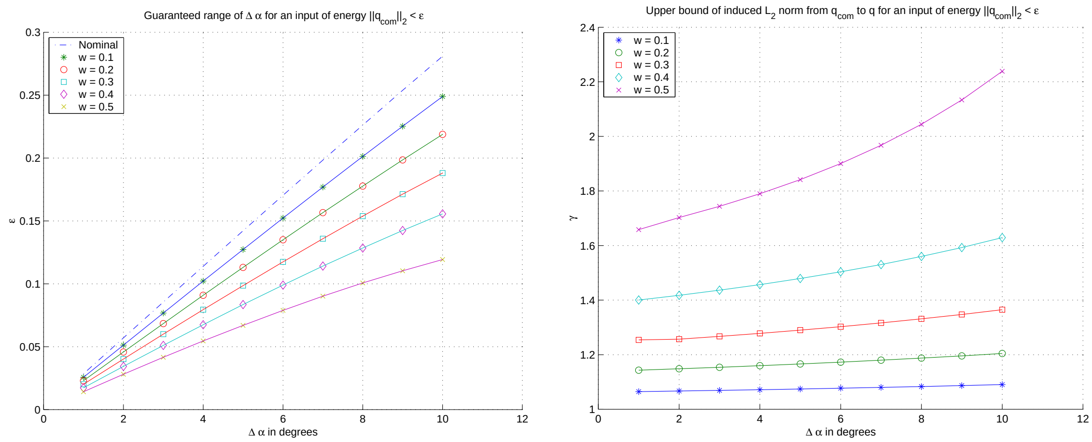
$$\begin{bmatrix} \dot{q} \\ \Delta\dot{\alpha} \end{bmatrix} = A_{cl}(\alpha,\delta)\begin{bmatrix} q \\ \Delta\alpha \end{bmatrix} + B_{cl}(\alpha,\delta)q_{com}, \tag{3.71}$$

*where* $A_{cl}(\alpha,\delta) = \left(A_{inv}(\alpha,\delta) - w_q\begin{bmatrix} B_{inv} & 0_{2\times1}\end{bmatrix}\right)$*,* $B_{cl}(\alpha,\delta) = w_q B_{inv}(\alpha,\delta)$*, the vector* $\delta$ *is a collection of uncertain parameters,* $\delta = \begin{bmatrix} \delta_{m_\alpha}, \delta_{m_\epsilon}, \delta_{z_q}, \delta_{z_\alpha}, \delta_{z_\epsilon}\end{bmatrix}^T$*, and:*

$$\begin{aligned}
A_{inv}(\alpha,\delta) &= \begin{bmatrix} 0 & A_{inv}^{1,2} \\ A_{inv}^{2,1} & A_{inv}^{2,2}\end{bmatrix} \\
A_{inv}^{1,2} &= \frac{\hat{M}_\alpha}{I_{yy}}w_{M_\alpha}\delta_{M_\alpha} - \frac{M_\alpha(\alpha)}{M_\epsilon(\alpha)}\frac{\hat{M}_\epsilon}{I_{yy}}w_{M_\epsilon}\delta_{M_\epsilon} \\
A_{inv}^{2,1} &= 1 + cs(\alpha)\left(\frac{Z_q(\alpha)}{mV_0} + \frac{\hat{Z}_q}{mV_0}w_{Z_q}\delta_{Z_q}\right) \\
A_{inv}^{2,2} &= cs(\alpha)\left(\frac{Z_\alpha(\alpha)}{mV_0} + \frac{\hat{Z}_\alpha}{mV_0}w_{Z_\alpha}\delta_{Z_\alpha}\right) - cs(\alpha)\frac{M_\alpha(\alpha)}{M_\epsilon(\alpha)}\left(\frac{Z_\epsilon(\alpha)}{mV_0} + \frac{\hat{Z}_\epsilon}{mV_0}w_{Z_\epsilon}\delta_{Z_\epsilon}\right) \\
B_{inv}(\alpha,\delta) &= \begin{bmatrix} 1 + \frac{\hat{M}_\epsilon}{M_\epsilon(\alpha)}w_{M_\epsilon}\delta_{M_\epsilon} \\ cs(\alpha)\frac{I_{yy}}{M_\epsilon(\alpha)}\left(\frac{Z_\epsilon(\alpha)}{mV_0} + \frac{\hat{Z}_\epsilon}{mV_0}w_{Z_\epsilon}\delta_{Z_\epsilon}\right)\end{bmatrix}
\end{aligned}$$

*where* $cs(\alpha) = (1 - \frac{1}{2}(\Delta\alpha + \alpha_0)^2)$ *is an approximation to* $\cos(\alpha)$*.*

*We wish to evaluate the input to state and input to output properties of this system. The first task is to associate the* $L_2$ *norm of the input* $q_{com}$ *to a level set that contains the state trajectory and it is contained in the region* $D$*, which in this case is given by*




(a) (b)

Figure 3.8: Performance analysis of an F/A-18 aircraft model: input-to-state and input-to-output gain estimates.

$D = \{\Delta\alpha : -0.166\ rad \leq \Delta\alpha \leq 0.166\ rad\}$. *This can be done using Program 3.21, multiplying out the Lyapunov derivative criterion by the non-vanishing denominator of the vector field to render the expression polynomial. The result, with different sizes of uncertainty, is shown in Figure 3.8(a). For example, when* $w = 0.1$ *for any pitch rate reference satisfying* $||q_{com}||_2 < 0.15$, *it is guaranteed that the angle of attack will not deviate by more than* 6 *degrees from its value at the equilibrium.*

*After the reachable set is obtained, a bound on the input to output* $L_2$ *gain can be found using Proposition 3.22. The results, with different sizes of uncertainty are shown in Figure 3.8(b). The results on the bound of the induced* $L_2$ *norm must be interpreted in conjunction with the previous results in Figure 3.8(a). The combination of the results in Figures 3.8(a) and 3.8(b) (by ignoring the* $\Delta\alpha$ *information) allows us to plot the upper bound on the induced* $L_2$ *norm as a function of the* $L_2$ *norm of the reference input and thus, verify the nonlinear behavior of the controller. This is shown in Figure 3.9, where the degradation of performance with increasing uncertainty is evident.*

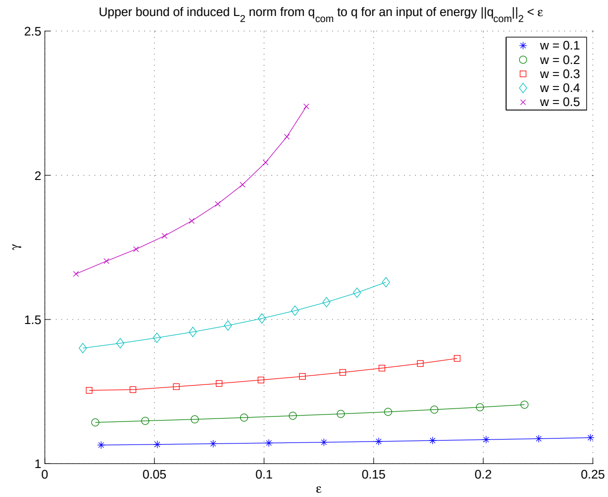



Figure 3.9: Performance analysis of an F/A-18 aircraft model: input-to-output gain estimates.

## 3.8 Conclusion

In this chapter, we have shown how the sum of squares decomposition and positivstellensatz can be used to analyze systems described by Ordinary Differential Equations. Generally speaking, this methodology is an extension of the well-studied Linear Matrix Inequality techniques for the analysis of linear systems to nonlinear systems and most of the results that we presented follow logically from this. We have seen how robust stability analysis, obtaining estimates of the region of attraction, ensuring the stability of switched systems and performance evaluation can be performed in a unified manner.

In the next chapter we will investigate how the sum of squares decomposition can be used to analyze a particular class of infinite dimensional systems: time-delay systems.

# Chapter 4

# Systems Described by Functional Differential Equations

Ἀργία μήτηρ πάσης κακίας
Σόλων
Delay is the mother of all evil
Solon

In this chapter, we concentrate on Functional Differential Equations, another class of differential equations also known as time-delay systems. They are an important modeling tool for systems that involve transport and propagation of data, such as communication systems, but also for modeling maturation and growth in population dynamics. The interest in time delay systems has been intensified recently [81], as they provide the simplest adequate modeling framework for network congestion control for the Internet, which is the topic of the next chapter. In general, the presence of delays may induce undesirable effects such as instabilities and oscillations, so ignoring them may lead to the wrong conclusions about the system's properties.

In this chapter, we will develop algorithmic, Lyapunov-based methodologies for analysis of time-delay systems, based on the sum of squares technique. We start our investigation by considering the construction of Lyapunov-Krasovksii functionals for linear systems, where the structure of the functional necessary and sufficient for stability is known, but its construction is difficult, and we will show how the sum of squares decomposition can be used to construct it. We will then turn to nonlinear time delay systems and consider delay-independent and delay-dependent stability analysis. Robust stability is also addressed, through the construction of parameterized

Lyapunov functionals.

## 4.1 Introduction

Functional Differential Equations (FDEs) differ from Ordinary Differential Equations (ODEs) in that the state belongs to an infinite dimensional space. Inevitably, initial conditions for FDEs are functions on an interval; this is distinct from the initial value problem for ODEs, where initial conditions are points in Euclidean space. The stability question can still be answered through the construction of Lyapunov-type certificates, i.e., functions of state that satisfy certain positivity conditions, as it is done in the case of ODE systems. However, while for the case of ODEs these certificates are functions, the natural certificate in the case of FDEs is a *functional* owing to the fact that the state is a function itself. Let us begin our investigation with some preliminaries.

For $\eta \in [0, +\infty)$, we denote $C([-\eta, 0], \mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[-\eta, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For $\phi \in C([-\eta, 0], \mathbb{R}^n)$, the norm of $\phi$ is defined as $\|\phi\| = \max_{\theta \in [-\eta,0]} |\phi(\theta)|$, where $|\cdot|$ is a norm in $\mathbb{R}^n$. We denote $C^\gamma$ the set defined by $C^\gamma = \{\phi \in C | \|\phi\| < \gamma\}$ with $\gamma > 0$. For $\sigma \in \mathbb{R}, \rho \geq 0$, $x \in C([\sigma - \tau, \sigma + \rho], \mathbb{R}^n)$, and $t \in [\sigma, \sigma + \rho]$, we define $x_t \in C$ as $x_t(\theta) = x(t + \theta), \theta \in [-\tau, 0]$.

Assume $\Omega$ is a subset of $\mathbb{R} \times C$, $f : \Omega \mapsto \mathbb{R}^n$ is a given function, and '$\dot{\ }$' represents the right-hand derivative. Then we call

$$\dot{x}(t) = f(t, x_t) \tag{4.1}$$

a retarded functional differential equation (RFDE) on $\Omega$. For a given $\sigma \in \mathbb{R}$, $\phi \in C$, we say $x(\sigma, \phi)$ is a solution to (4.1) with initial condition $\phi$ at $\sigma$, if there is a $\rho > 0$ such that $x(\sigma, \phi)$ is a solution of (4.1) on $[\sigma - \tau, \sigma + \rho)$ and $x_\sigma(\sigma, \phi) = \phi$. Consider the class of autonomous delayed differential equations, i.e.,

$$\dot{x}(t) = f(x_t). \tag{4.2}$$

We say (4.2) is linear if $f(x_t)$ is linear in $x_t$. In this chapter, we concentrate on autonomous time-delay systems with discrete delays, so $f(x_t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_r))$ where $\tau_i$ are the discrete delays in the system. Without loss of generality, we assume that 0 is a steady-state for the system.

Just as in the case of ODEs, the existence and uniqueness of solutions of (4.2) is guaranteed if $f(\phi)$ is continuous in $\phi$ in an open subset $\Omega$ of $C$. Continuous dependence of the solution on initial conditions and parameters can also be proven under certain assumptions. Here we concentrate on a few stability definitions of the steady-state of (4.2), which we assume to be at 0.

**Definition 4.1** *The trivial solution of (4.2) is called*

1. stable *if for any* $\epsilon > 0$ *there is a* $\delta = \delta(\epsilon)$ *such that* $|x(\phi)| \leq \epsilon$ *for any initial condition* $\phi \in \Omega$*; otherwise it is termed* unstable*;*

2. asymptotically stable *if it is stable and there is a* $\gamma > 0$ *such that* $\lim_{t\to\infty} x(t;\phi) = 0$ *for any initial condition* $\phi \in \Omega^\gamma$*. The set of initial functions* $\phi$ *for which* $\lim_{t\to\infty} x(t;\phi) = 0$ *is called the* domain of attraction *of the trivial solution.*

Establishing the stability of the steady-state is a rather difficult task if $f$ is a nonlinear function of $(x(t), x(t-\tau_1), \ldots, x(t-\tau_r))$. Even in the linear case, establishing the stability boundary, i.e., the exact values of $\tau_i$ for which stability is retained is $\mathcal{NP}$-hard [26]. Just as in the case of nonlinear systems described by Ordinary Differential Equations (ODEs), a Lyapunov argument can be formulated that proves useful when many, incommensurate delays appear in the system.

Consider a continuous functional $V : C \to \mathbb{R}$ and define:

$$\dot{V}(\phi) = \lim_{h\to 0^+} \frac{1}{h}[V(x_h(\phi)) - V(\phi)]$$

the derivative of $V$ along a solution of (4.2). Assume $f : \Omega \to \mathbb{R}^n$ is completely continuous and $f(0) = 0$. We then have the following Lyapunov-Krasovskii theorem [28]:

**Theorem 4.2** *For (4.2), assume $f$ is completely continuous and $f(0) = 0$. Further, assume there exist $a(s)$ and $b(s)$ nonnegative continuous, $a(0) = b(0) = 0$, $\lim_{s\to\infty} a(s) = +\infty$ and $V : \Omega \to \mathbb{R}$ continuous satisfying:*

$$V(\phi) \geq a(|\phi(0)|) \text{ in } \Omega$$
$$\dot{V}(\phi) \leq b(|\phi(0)|) \text{ in } \Omega.$$

*Then the solution $x = 0$ of (4.2) is stable, and every solution is bounded. If in addition, $b(s) > 0$ for $s > 0$, then $x = 0$ is asymptotically stable.*

We note that apart from the Lyapunov-Krasovskii theorem for stability analysis, there is the Lyapunov-Razumikhin theorem which uses functions as the certificates for stability instead of functionals. When using functionals, the Lyapunov-Krasovskii theorem requires that the derivative along solutions decrease monotonically; such a requirement makes finding a Lyapunov functional a rather difficult task, since the space $C$ is much more complicated than $\mathbb{R}^n$, and one has no control of the relationship of $|x(t)|$ and $|x(t+\theta)|$, $\theta \in [-\tau, 0]$. Razumikhin's theorem uses functions and studies their rate of change along solutions.

Let $V : \mathbb{R}^n \to \mathbb{R}$ is a positive definite continuously differentiable function. Its derivative along the solution of (4.2) takes the form:

$$\dot{V}(x(t)) = \frac{\partial V(x)}{\partial x} f(x_t) \tag{4.3}$$

Note that the right hand side is a functional, although $V$ is a function. From the definition of stability, we know that if $x_t$ is initially inside a ball of radius $\delta$ in $C$, then for it to escape that ball, it has to reach the boundary at some time $t^*$. Therefore $|x(t^*)| = \delta$, and $|x(t+\theta)| < \delta$ for $\theta \in [-\tau, 0)$, and $\frac{d|x(t^*)|}{dt} \geq 0$. Razumikhin's theorem rules out this case.

**Theorem 4.3** *Consider the functional differential equation given by (4.2), where $f$ maps bounded sets of $C$ into bounded sets of $\mathbb{R}^n$. Suppose $u$, $v$, $w : \mathbb{R}^+ \mapsto \mathbb{R}^+$ are continuous, nondecreasing functions, $u(s), v(s)$ positive for $s > 0$, $u(0) = 0$, $v(0) = 0$ and $v$ strictly increasing. If there is a continuous function $V : \mathbb{R}^n \mapsto \mathbb{R}$ such that:*

1. $u(\|x\|) \leq V(x) \leq v(\|x\|), \quad x \in \mathbb{R}^n$

2. $\dot{V}(x) \leq -w(\|x\|)$ *if* $V(x(t+\theta)) < p(V(x(t))), \quad \forall\ \theta \in [-\tau, 0]$

*Then the trivial solution of (4.2) is stable.*

In this chapter, as in the previous one, we are interested in *algorithmic* methodologies for constructing Lyapunov-type certificates. The convexity in constructing the Lyapunov functions in the previous chapter was important in this endeavor. Here we have presented two Lyapunov theorems; it is easy to see that the conditions in Theorem 4.2 are convex, whereas those in Theorem 4.3 are not because of the derivative condition and its connection to the Lyapunov certificate itself. The Lyapunov-Krasovskii theorem can be used to construct simple Lyapunov functionals for linear systems by solving a set of Linear Matrix Inequalities [37]. Under convexification assumptions, the Lyapunov-Razuminkin criteria can also be tested by solving LMIs, but are in general more conservative than the Lyapunov-Krasovskii ones [26].

LMIs were used for both *delay-dependent* and *delay-independent* stability analysis of linear time-delay systems, a classification based on the persistence of stability as the delay is increased. We say that a system is *delay-independent* stable, if, generally speaking, the stability property is retained for all positive (finite) values of the delays in the system, i.e., the system is robust with respect to the delay size. On the other hand, we say that a system is *delay-dependent* stable if the stability is preserved for some values of delays and is lost for some others. The existence of complete quadratic Lyapunov-Krasovskii functionals necessary and sufficient for strong delay-independent and delay-dependent stability of linear time-delay systems is well known and so is their structure [78], but there is an inherent difficulty in constructing them. They result in infinite-dimensional parameterized LMIs which are difficult to solve.

In this chapter, we will use the sum of squares technique for the construction of Lyapunov-Krasovskii functionals for linear and nonlinear time delay systems [58]. The SOS decomposition will be used not only for solving the infinite dimensional LMIs in the linear case, but also for constructing Lyapunov functionals with polynomial kernels for nonlinear time delay systems.

## 4.2 Analysis of Linear Time-Delay Systems

A question that the reader may pose at this point, is why do we investigate the stability analysis of linear time-delay systems when it is well known that the sum of squares technique can answer similar questions about nonlinear systems in a unified manner. The answer to this question is twofold. First, the conditions for stability of linear time-delay systems are in the form of parameterized LMIs (PLMIs) which are difficult to solve in general. We will take this opportunity to show how the sum of squares technique can be used to solve PLMIs, and comment on how the sparsity structure that appears naturally can be taken advantage of when solving the sum of squares program. On the other hand, it is important to appreciate that we now seek to construct *functionals* rather than *functions*, and the structure that one should consider is not a polynomial parametrization, but a 'function' of polynomials; we therefore need strong intuition in what structures we should choose. A lot is known about 'complete' (i.e., necessary and sufficient for stability) Lyapunov structures for linear time-delay systems, and therefore we investigate the stability of linear systems first, which can help us choose structures for nonlinear systems later on.

Consider a linear instantiation of (4.2) with one discrete delay of the form:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) = f(x(t), x(t-\tau)) \tag{4.4}$$

with an initial condition $x_0(\theta) = \phi(\theta)$, $\theta \in [-\tau, 0]$, $\phi \in C$. The delay $\tau$ is assumed to be constant, $x(t) \in \mathbb{R}^n$ and $A_0$, $A_1$ are known real constant matrices of appropriate dimensions. The case in which the $A_i$ are parameter-dependent can also be treated in a unified manner.

We have the following version of Theorem 4.2 for the asymptotic stability of the above system.

**Theorem 4.4** *[26] The system described by Equation (4.4) is asymptotically stable if there exists a bounded quadratic Lyapunov functional $V(x_t)$ such that for some $\epsilon_1 > 0$, $\epsilon_2 > 0$ it satisfies:*

$$V(x_t) \geq \epsilon_1 \|x(t)\|^2 \tag{4.5}$$

*and its derivative along the system trajectory satisfies*

$$\dot{V}(x_t) \leq -\epsilon_2 \|x(t)\|^2. \tag{4.6}$$

For the case of ODEs, it is well known that a Lyapunov function for the generic linear ODE system of the form $\dot{x} = Ax$ with $A$ Hurwitz would be $V = x^T P x$ where $P$ is a positive definite matrix satisfying $A^T P + PA < 0$. $P$ can be found numerically by solving this set of Linear Matrix Inequalities [10].

In the same spirit, the search for structures that are 'complete', i.e., produce necessary and sufficient conditions for stability of linear time-delay systems has been under research in the past. For example, for the case of what is called strong delay-independent stability [7], the class of such Lyapunov functions has been completely characterized:

**Example 4.5** *For system (4.4) a Lyapunov-Krasovskii candidate that would yield a delay-independent condition is*

$$V(x_t) = x(t)^T P x(t) + \int_{-\tau}^{0} x(t+\theta)^T S x(t+\theta) d\theta$$

*Sufficient conditions on $V(x_t)$ to be positive definite are $P > 0$, $S \geq 0$. For $\dot{V}(x_t) < 0$ we require*

$$\begin{bmatrix} A^T P + PA + S & PA_1 \\ A_1^T P & -S \end{bmatrix} < 0,$$

*i.e., the conditions for stability (see Theorem 4.4) can be written as a Linear Matrix Inequality (LMI) with $P$ and $S$ as the unknowns.*

*The Lyapunov functional used above may not suffice to prove stability for a general delay-independent stable system, as the proposed structure is not adequate for the stability proof. Instead, the Lyapunov structure in [6] has been proven to be 'complete' for* strong *delay-independent stability in the case of linear time delay systems. Denoting*

$$z_k(t) = [x(t), x(t-\tau), \ldots, x(t-(k-1)\tau)] \tag{4.7}$$

*the complete structure in the single delay case is*

$$V_k(x_t) = a_1(z_k(t)) + \int_{-\tau}^{0} b_1(z_k(t+\theta))d\theta \tag{4.8}$$

*where $a_1$ and $b_1$ are* quadratic *polynomials in their arguments.*

As far as delay-dependent stability is concerned, the structures of Lyapunov-Krasovskii functionals that are necessary and sufficient for delay-dependent stability are known, but difficult to construct. The complete Lyapunov functional [78, 26] has the following form:

$$\begin{aligned} V(x_t) &= x^T(t)Px(t) + x^T(t)\int_{-\tau}^{0} P_1(\theta)x(t+\theta)d\theta + \int_{-\tau}^{0} x^T(t+\theta)P_1^T(\theta)d\theta x(t) \\ &\quad + \int_{-\tau}^{0}\int_{-\tau}^{0} x^T(t+\theta)P_2(\theta,\xi)x(t+\xi)d\xi d\theta + \int_{-\tau}^{0} x^T(t+\theta)Qx(t+\theta)d\theta \end{aligned} \tag{4.9}$$

with appropriate conditions on $P, P_1, P_2$, etc, to guarantee positivity of $V$. The condition $\dot{V} < 0$ implies that the matrices $P, P_1, P_2$ have to satisfy a complicated system of algebraic, ordinary and partial differential equations with appropriate boundary conditions [78]. If one wanted to write the above conditions as LMI conditions, he would get a set of infinite-dimensional LMIs for which there are not many known algorithms for efficient computation. A rather complicated discretization scheme was proposed in [26] to solve these PLMIs.

In this chapter, we consider structures similar to the complete quadratic Lyapunov-Krasovskii functional (4.9) for which we construct certificates in which the kernels (matrices $P_1, P_2$, etc) are polynomials in the variables $(\theta, \xi)$. As we increase the order of the polynomial kernels, the delay-dependent stability conditions obtained analytically (i.e., using frequency domain methods) can be approached. To proceed, we use Theorem 4.4 and consider the following Lyapunov functional which resembles the complete Lyapunov functional (4.9):

$$\begin{aligned} V(x_t) &= a_0(x(t)) + \int_{-\tau}^{0}\int_{-\tau}^{0} a_1(\theta,\xi,x(t),x(t+\theta),x(t+\xi))d\theta d\xi \\ &\quad + \int_{-\tau}^{0}\int_{t+\theta}^{t} a_2(x(\zeta))d\zeta d\theta + \int_{-\tau}^{0}\int_{t+\xi}^{t} a_3(x(\zeta))d\zeta d\xi \end{aligned} \tag{4.10}$$

where by $a_1(\theta,\xi,x(t),x(t+\theta),x(t+\xi))$, we mean a polynomial in $\theta,\xi,x(t),x(t+\theta)$ and $x(t+\xi)$ of bounded degree, which is quadratic with respect to $x(t),x(t+\theta)$ and $x(t+\xi)$ and allowed to be any order in $\theta$ and $\xi$. Such polynomials are called bipartite and have a special structure [24]. The polynomials $a_0$, $a_2$ and $a_3$ are quadratic in their arguments. In other words,

$$a_1(\theta,\xi,x(t),x(t+\theta),x(t+\xi)) = \begin{bmatrix} x(t) \\ x(t+\theta) \\ x(t+\xi) \end{bmatrix}^T \begin{bmatrix} & & \\ & \tilde{a}_1(\theta,\xi) & \\ & & \end{bmatrix} \begin{bmatrix} x(t) \\ x(t+\theta) \\ x(t+\xi) \end{bmatrix}$$

where $\tilde{a}_1(\theta,\xi)$ is a polynomial matrix in $(\theta,\xi)$. The time derivative of $V(x_t)$ along $f$ given by (4.4) is:

$$\begin{aligned} \dot{V}(x_t) &= \frac{da_0}{dx(t)}f + \int_{-\tau}^{0}\left(a_1(0,\xi,x(t),x(t),x(t+\xi)) - a_1(-\tau,\xi,x(t),x(t-\tau),x(t+\xi))\right)d\xi \\ &\quad + \int_{-\tau}^{0}\left(a_1(\theta,0,x(t),x(t+\theta),x(t)) - a_1(\theta,-\tau,x(t),x(t+\theta),x(t-\tau))\right)d\theta \\ &\quad + \int_{-\tau}^{0}\int_{-\tau}^{0}\left(\frac{\partial a_1}{\partial x(t)}f - \frac{\partial a_1}{\partial \theta} - \frac{\partial a_1}{\partial \xi}\right)d\theta d\xi \\ &\quad + \int_{-\tau}^{0}\left(a_2(x(t)) - a_2(x(t+\theta))\right)d\theta + \int_{-\tau}^{0}\left(a_3(x(t)) - a_3(x(t+\xi))\right)d\xi \end{aligned}$$

$$= \frac{1}{\tau^2}\int_{-\tau}^{0}\left(\begin{array}{c} \frac{da_0}{dx(t)}f + \tau^2\frac{\partial a_1}{\partial x(t)}f - \tau^2\frac{\partial a_1}{\partial \theta} - \tau^2\frac{\partial a_1}{\partial \xi} + \tau a_2(x(t)) + \tau a_3(x(t)) \\ -\tau a_3(x(t+\xi)) - \tau a_1(-\tau,\xi,x(t),x(t-\tau),x(t+\xi)) \\ +\tau a_1(0,\xi,x(t),x(t),x(t+\xi)) + \tau a_1(\theta,0,x(t),x(t+\theta),x(t)) \\ -\tau a_2(x(t+\theta)) - \tau a_1(\theta,-\tau,x(t),x(t+\theta),x(t-\tau)) \end{array}\right) d\theta d\xi$$

In order to ensure positive definiteness of $V$ and negative definiteness of $\dot{V}$, we can impose these conditions on the *kernels* of the integral expressions. Positivity of a kernel of an integral is a conservative test for the positivity of the integral itself and we will see later on how to relax this conservativeness.

By structuring the polynomial $a_1$ and testing positivity of $V$ as explained above, it is easy to see that the resulting SOS conditions will be parameterized LMIs in $(\theta, \xi)$. However, for notational simplicity, we will be working at the polynomial level - This essentially means that PLMIs can be solved by turning them into polynomial expressions and asking for the latter to be SOS.

Sufficient conditions for stability of the system can be found in the following proposition:

**Proposition 4.6** *Consider the system given by Equation (4.4). Denote $x_\tau = x(t-\tau)$, $x_\theta = x(t+\theta)$, $x_\xi = x(t+\xi)$ and $x = x(t)$ for brevity. Suppose we can find polynomials $a_0(x)$ $a_1(\theta,\xi,x,x_\theta,x_\xi)$, $a_2(x(\zeta))$ and $a_3(x(\zeta))$ and a positive constant $\epsilon$ such that the following conditions hold:*

1. $a_0(x) - \epsilon\|x\|^2 \geq 0$,

2. $a_1(\theta,\xi,x,x_\theta,x_\xi) \geq 0$, $\forall \theta,\xi \in [-\tau, 0]$,

3. $a_2(x(\zeta)) \geq 0$, $a_3(x(\zeta)) \geq 0$,

4. $\frac{da_0}{dx}f + \tau^2\frac{\partial a_1}{\partial x}f - \tau^2\frac{\partial a_1}{\partial \theta} - \tau^2\frac{\partial a_1}{\partial \xi} + \tau a_2(x) - \tau a_2(x_\theta) + \tau a_3(x) - \tau a_3(x_\xi) + \tau a_1(0,\xi,x,x,x_\xi) - \tau a_1(-\tau,\xi,x,x_\tau,x_\xi) + \tau a_1(\theta,0,x,x_\theta,x) - \tau a_1(\theta,-\tau,x,x_\theta,x_\tau) \leq -\epsilon\|x\|^2$, $\forall \theta,\xi \in [-\tau, 0]$.

*Then the system described by Equation (4.4) is asymptotically stable.*

**Proof.** The first three conditions impose that:

$$V(x_t) \geq \epsilon \|x(t)\|^2.$$

Similarly, the fourth condition, and the discussion before the statement of the proposition imply that

$$\dot{V}(x_t) \leq -\tilde{\epsilon}\|x(t)\|^2$$

for some $\tilde{\epsilon} > 0$. Therefore, from the statement of Theorem 4.4, the system (4.4) is asymptotically stable. ■

Condition (2) in the above proposition asks for $a_1$ to be non-negative only for a certain $\theta$ and $\xi$ interval. To impose the conditions $\theta \in [-\tau, 0]$, $\xi \in [-\tau, 0]$, we use a process similar to the S-procedure, as it was discussed in Chapter 2. The polynomial $a_1$ is required to be non-negative only when $g_1 \triangleq \theta(\theta + \tau) \leq 0$ and $g_2 \triangleq \xi(\xi + \tau) \leq 0$ are satisfied, which can be tested as follows:

$$a_1 + p_1 g_1 + p_2 g_2 \geq 0 \tag{4.11}$$

where $p_1$ and $p_2$ are sums of squares of degree 2 in $x, x_\theta$ and $x_\xi$ and of bounded degree in $\theta$ and $\xi$. This will retain the bipartite structure of the whole expression, which will be taken advantage of in the computation. The same can be done with Constraint (4) in the above proposition.

In order to reduce conservativeness of the test – in that the positivity of the kernel is too strong a condition for the positivity of the integral – we can introduce terms with polynomial kernels that integrate to zero and add them to these expressions.

Such terms would be, in condition (1):

$$\int_{-\tau}^{0}\int_{-\tau}^{0} b(\theta,\xi,x(t),x(t+\theta),x(t+\xi))d\theta d\xi$$
$$\triangleq \int_{-\tau}^{0}\int_{-\tau}^{0} \begin{bmatrix} x(t) \\ x(t+\theta) \\ x(t+\xi) \end{bmatrix}^T \begin{bmatrix} b_{11}(\theta,\xi) & b_{12}(\xi) & b_{13}(\theta) \\ b_{12}(\xi) & b_{22}(\xi) & 0 \\ b_{13}(\theta) & 0 & b_{33}(\theta) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t+\theta) \\ x(t+\xi) \end{bmatrix} d\theta d\xi = 0 \quad (4.12)$$

and similarly for the derivative condition, we have:

$$\int_{-\tau}^{0}\int_{-\tau}^{0} c(\theta,\xi,x(t),x(t-\tau),x(t+\theta),x(t+\xi))d\theta d\xi$$
$$\triangleq \int_{-\tau}^{0}\int_{-\tau}^{0} \begin{bmatrix} x(t) \\ x(t-\tau) \\ x(t+\theta) \\ x(t+\xi) \end{bmatrix}^T \begin{bmatrix} c_{11}(\theta,\xi) & c_{12}(\theta,\xi) & c_{13}(\xi) & c_{14}(\theta) \\ c_{12}(\theta,\xi) & c_{22}(\theta,\xi) & c_{23}(\xi) & c_{24}(\theta) \\ c_{13}(\xi) & c_{23}(\xi) & c_{33}(\xi) & 0 \\ c_{14}(\theta) & c_{24}(\theta) & 0 & c_{44}(\theta) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \\ x(t+\theta) \\ x(t+\xi) \end{bmatrix} d\theta d\xi = 0 \quad (4.13)$$

The computational complexity of this method increases as the order of the polynomials $a_1$ with respect to $\xi$ and $\theta$ are increased. However, for reducing the size and better conditioning of the SDP, one should take advantage of the bipartite structure of $a_1$ (i.e., the fact that even though $a_1$ might be of degree $n > 2$, it is always of degree 2 in all variables, but possibly $\xi$ and $\theta$). For more details on how sparsity can be exploited, see [66] and the references therein. SOSTOOLS has routines that exploit the sparse bipartite structure to reduce the size of the resulting LMIs.

The system with delay $\tau$ can therefore be tested for stability by solving the following SOS program:

Find polynomials $a_0(x), a_1(\theta, \xi, x, x_\theta, x_\xi), a_2(x(\zeta)), a_3(x(\zeta)), \epsilon > 0$

and Sums of Squares $q_{i,j}(\theta, \xi, x, x_\theta, x_\xi)$ for $i, j = 1, 2$

and polynomials $b(\theta, \xi, x, x_\theta, x_\xi)$ and $c(\theta, \xi, x, x_\tau, x_\theta, x_\xi)$ of the form (4.12) and (4.13)

such that

$$\begin{aligned}
&a_0(x) - \epsilon\|x\|^2 \text{ is SOS,}\\
&a_1(\theta, \xi, x, x_\theta, x_\xi) + \sum_{j=1}^{2} q_{1,j} g_i(\theta, \xi) + b(\theta, \xi, x, x_\theta, x_\xi) \text{ is SOS,}\\
&a_2(x(\zeta)) \text{ is SOS,} \qquad (4.14)\\
&a_3(x(\zeta)) \text{ is SOS,}
\end{aligned}$$

$$\left\{\begin{array}{c}
-\frac{da_0}{dx} f(x_t) - \tau^2 \frac{\partial a_1}{\partial x} f(x_t) + \tau^2 \frac{\partial a_1}{\partial \theta} + \tau^2 \frac{\partial a_1}{\partial \xi} - \tau a_1(0, \xi, x, x, x_\xi)\\
+\tau a_1(-\tau, \xi, x, x_\tau, x_\xi) - \tau a_1(\theta, 0, x, x_\theta, x(t))\\
+\tau a_1(\theta, -\tau, x(t), x_\theta, x_\tau) - \tau a_2(x) - \tau a_3(x) + \tau a_2(x_\theta) + \tau a_3(x_\xi)\\
-\epsilon\|x\|^2 + \sum_{j=1}^{2} q_{2,j} g_i(\theta, \xi) + c(\theta, \xi, x, x_\tau, x_\theta, x_\xi)
\end{array}\right\} \text{ is SOS.}$$

$$\int_{-\tau}^{0}\int_{-\tau}^{0} b(\theta, \xi, x(t), x(t+\theta), x(t+\xi))d\theta d\xi = 0$$

$$\int_{-\tau}^{0}\int_{-\tau}^{0} c(\theta, \xi, x(t), x(t-\tau), x(t+\theta), x(t+\xi))d\theta d\xi = 0$$

Note that the above procedure can be extended to handle systems with more than one delay. A different Lyapunov structure may then be required.

We now present an example to investigate stability of a linear time delay system.

**Example 4.7** *The following two dimensional system that has been analyzed extensively in the past, and the various LMI tests that were developed by researchers were*

| Order of $a_1$ in $\theta$ and $\xi$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| max $\tau$ | 4.472 | 5.17 | 5.75 | 6.02 | 6.09 | 6.15 | 6.16 |

Table 4.1: Constructing a Lyapunov-Krasovksii functional of a linear time-delay system.

*tested against this example.*

$$\begin{aligned}\dot{x}_1(t) &= -2x_1(t) - x_1(t-\tau) \triangleq f_1 \\ \dot{x}_2(t) &= -0.9x_2(t) - x_1(t-\tau) - x_2(t-\tau) \triangleq f_2\end{aligned}$$

*The system is asymptotically stable for $\tau \in [0, 6.17]$. The best bound on $\tau$ that can be obtained without using the discretization method in [26] was $\tau_{\max} = 4.3588$ in [65]. Using $V(x_t)$ given by (4.1), we can test the maximum delays given in Table 4.1. In this table, the values of $\tau_{max}$ for different degree polynomials $a_1$ in $\theta$ and $\xi$ are tabulated. We see that as the order of $a_1$ with respect to $\theta$ and $\xi$ is increased, better delay sizes can be tested that approach the analytical one. We see that as the order of the polynomials is increased, the LMI conditions we can obtain are better, but this is at the expense of computational complexity.*

Robust stability of linear time delay systems with respect to parametric uncertainty can also be treated by constructing parameterized Lyapunov-Krasovskii functionals. We will show how to do this for the case of nonlinear time-delay systems.

## 4.3 Nonlinear Time Delay Systems

In this section, we concentrate on delay-independent and delay-dependent stability analysis of nonlinear time delay systems with or without parametric uncertainty. These quite complex system descriptions can be treated using the sum of squares technique in a unified way.

Previous attempts to analyze stability of nonlinear time delay systems centered on manual constructions of Lyapunov-Razumikhin functions [38] and sometimes Lyapunov-Krasovskii functionals. The main examples in this area were coming from population dynamics, but as we will see in the next chapter, a renewed interest was shown in nonlinear functional differential equations because of research in network congestion control.

### 4.3.1 Delay-Independent Stability

Delay-independent stability for nonlinear systems has been investigated deeply under Lyapunov-Razumikhin conditions [45]. In [92], connections between appropriate Lyapunov-Razumikhin conditions and Input-to-State Stability small-gain are made, and relaxed Razumikhin-type conditions guaranteeing global asymptotic stability are derived. In [7], a relationship between a criterion obtained using a Lyapunov-Krasovskii functional and the delay-independent small gain theorem is established for a special class of nonlinear time-delay systems. The delay-independent stability property is conservative, in the sense that the system may still be stable in a delay-dependent fashion if the condition of stability is violated, but it is used a lot in controller synthesis for delay systems where the size of the delay is uncertain.

In general, finding the proper structure for a Lyapunov functional involves some guessing, especially in the case of nonlinear systems. Building on our work from the linear case, we consider the construction of *polynomial* (i.e., of any order instead of just *quadratic*) kernel Lyapunov-Krasovskii functionals, by formulating relevant sum of squares conditions. For delay-independent stability we will be using the following two structures of Lyapunov functionals,

$$V_{1,k}(x_t) = a_1(z_k(t)) + \int_{-\tau}^{0} b_1(z_k(t+\theta))d\theta \tag{4.15}$$

$$V_2(x_t) = a_2(x(t)) + \int_{-\tau}^{0} b_2(x(t+\theta))d\theta \tag{4.16}$$

where $z_k$ is defined by (4.7) for a system with one discrete delay given by

$$\dot{x}(t) = f(x(t), x(t-\tau)), \tag{4.17}$$

We assume that $f$ is a nonlinear, polynomial function of its arguments, although this is not a significant restriction as argued in the previous chapter. It is assumed without any loss of generality that 0 is a steady-state of the system.

The functional $V_{1,k}$ is the functional given by Equation (4.8), but we will be using it to analyze stability of nonlinear systems. In functional $V_2$, we allow $a_2$ and $b_2$ to be polynomials of bounded degree instead of just quadratics. Related to $V_{1,k}$ we introduce the following notation for *delayed* versions of $f$:

$$f_{j\tau} = f(x_{t-j\tau})$$

where $f_0 = f$. Stability tests for the two cases can be derived, and are summarized in the following proposition:

**Proposition 4.8** *Consider a time-delay system described by Equation (4.17). The steady-state $x = 0$ is globally delay-independent stable if*

1. *Case 1: There exist polynomial functions $a_1(z_k(t))$ and $b_1(z_k(t))$ and a positive definite function $\varphi_1(z_k(t))$ such that:*
   - *(a)* $a_1(z_k(t)) - \varphi_1(z_k(t)) \geq 0,$
   - *(b)* $b_1(z_k(t+\theta)) \geq 0,$
   - *(c)* $\frac{dV_{1,k}}{dt} = \sum_{j=0}^{k-1} \frac{\partial a_1}{\partial x(t-j\tau)} f_{j\tau} + b_1(z_k(t)) - b_1(z_k(t-\tau)) \leq 0.$
2. *Case 2: There exist polynomial functions $a_2(x(t))$ and $b_2(x(t+\theta))$ and a positive definite function $\varphi_2(x(t))$ such that*
   - *(a)* $a_2(x(t)) - \varphi_2(x(t)) \geq 0,$
   - *(b)* $b_2(x(t+\theta)) \geq 0,$
   - *(c)* $\frac{dV_2}{dt} = \frac{\partial a_2}{\partial x(t)} f + b_2(x(t)) - b_2(x(t-\tau)) \leq 0.$

**Proof.** Consider first $V_{1,k}$, and Case 1 above. The fist two constraints impose that $V_{1,k} \geq \varphi_1(z_k(t)) > 0$, so the first Lyapunov-Krasovskii condition is satisfied. The derivative of $V_{1,k}$ along the trajectories of system (4.17) is:

$$\dot{V}_{1,k} = \sum_{j=0}^{k-1} \frac{\partial a_1}{\partial x(t-j\tau)} f_{j\tau} + b_1(z_k(t)) - b_1(z_k(t-\tau))$$

Under the third condition in Case 1, the above derivative is non-positive. Therefore, if all three conditions are satisfied, the system given by (4.17) is stable; since the delay size does not appear explicitly in the above conditions, then the system is stable, independent of delay.

The proof for Case 2 is similar. ■

The conditions in the above proposition can be tested using the sum of squares decomposition, by relaxing the '$\geq$' conditions to the existence of a sum of squares decomposition, as it was done in the previous chapter.

**Example 4.9** *Consider the system:*

$$\dot{x}_1 = -x_1(t) + x_2^2(t-\tau), \quad \dot{x}_2 = -x_2(t)$$

*This system is delay-independent stable, and we prove this by constructing a Lyapunov functional of the form $V_2$ (given by Equation (4.16)) with $a_2$ and $b_2$ polynomials of bounded degree. Note that now $x(t) = [x_1(t), x_2(t)]^T$. When $a_2$ and $b_2$ are second order polynomials, no certificate is found. However, when their order is increased, a certificate of stability is obtained. In fact, the two conditions become*

$$\begin{aligned} V(x_t) &= x_2^2(t) + \frac{3}{4}x_1^2(t) + (0.5x_1(t) + x_2^2(t))^2 + \int_{-\tau}^{0} x_2^4(t+\theta)d\theta. \\ -\dot{V}(x_t) &= (x_1(t) + x_2^2(t) - x_2^2(t-\tau))^2 + 2x_2^2(t) + \\ &\quad + x_2^2(t)x_2^2(t-\tau) + 2(x_2^2(t) + \frac{1}{4}x_1(t))^2 + \frac{14}{16}x_1(t)^2. \end{aligned}$$

*To get this certificate, only a few SOSTOOLS commands are required.*

Nonlinear systems may have more than one equilibria. In this case, we have to use the region $\Omega$ in Theorem 4.2. For this, we define the set:

$$\Omega = \{x_t \in C : \|x_t\| = \sup_{-\tau \le \theta \le 0} |x(t+\theta)| \le \gamma\}.$$

In particular, this means that $|x(t+\theta)| \le \gamma$, $\forall\, \theta \in [-\tau, 0]$, where $|\cdot|$ is the $\infty$-norm. This is a set of inequalities which can be adjoined using the extension to the S-procedure that was introduced in Chapter 2. Suppose for concreteness that one wants to use the Lyapunov functional $V_2(x_t)$ given by (4.16) to prove stability for a system described by (4.17) *locally*, i.e., under the additional constraint that

$$|x(t+\theta)| \le \gamma, \quad \forall\, \theta. \tag{4.18}$$

In particular this gives rise to the following conditions:

$$\begin{aligned} h_{1i} &:= (x_i(t) - \gamma)(x_i(t) + \gamma) \le 0, \\ h_{2i} &:= (x_i(t-\tau) - \gamma)(x_i(t-\tau) + \gamma) \le 0. \end{aligned}$$

Then we have the following result, when we consider the Lyapunov functional (4.16):

**Proposition 4.10** *Let 0 be a steady state of system (4.17), and let there exist polynomial functions* $a_2(x(t))$ *and* $b_2(x(t))$*, a positive definite function* $\phi(x(t))$ *and non-negative multipliers* $p_i(x(t))$, $q_{1i}(x(t), x(t-\tau))$ *and* $q_{2i}(x(t), x(t-\tau))$ *such that:*

1. $a_2(x(t)) - \varphi(x(t)) + \sum_i p_i h_{1i} \ge 0,$

2. $b_2(x(t+\theta)) \ge 0,$

3. $-\frac{dV}{dt} + \sum_i (q_{1i} h_{1i} + q_{2i} h_{2i}) \ge 0.$

*Then the steady state is* delay-independent *stable.*

**Proof.** While $x(t)$ satisfies $h_{1i} \le 0$ and $p_i(x(t))$ is a SOS, we have:

$$V(x_t) = V_0(x(t)) + \int_{-\tau}^{0} V_1(x(t+\theta))d\theta \ge \varphi(x(t)) - \sum_i p_i h_{1i} > 0,$$

and so the first Lyapunov condition is satisfied. The same is true for the derivative condition, so the steady state (4.17) is delay-independent stable. ■

A similar proposition can be written using the structure (4.15). A sum of squares program can be constructed, based on the above proposition. We will now look at delay-dependent stability for these systems.

### 4.3.2 Delay-Dependent Stability

In this case the stability of the system changes as the delay, seen as a parameter, varies. Therefore a different type of Lyapunov functionals has to be used to allow for the delay size to appear explicitly in the stability conditions. For nonlinear time delay systems analysis we can use functionals with structures resembling the complete Lyapunov structure for delay-dependent stability (4.9) for linear systems. Consider the following functional:

$$V(x_t) = a_0(x(t)) + \int_{-\tau}^{0} a_1(\theta, x(t), x(t+\theta))d\theta + \int_{-\tau}^{0}\int_{t+\theta}^{t} a_2(x(\zeta))d\zeta d\theta \quad (4.19)$$

for the system of the form (4.17). The first term is added to impose positive definiteness of $V$ and the last term is added for convenience, as it will be used in the derivative condition to 'complete the squares'. Sufficient conditions for the (global) stability of the zero steady state can then be formulated as follows:

**Proposition 4.11** *Let 0 be a steady state for the system given by (4.17). Denote* $x = x(t)$*,* $x_\tau = x(t-\tau)$*,* $x_\theta = x(t+\theta)$*. Let there exist polynomials* $a_0(x)$*,* $a_1(\theta, x, x_\theta)$ *and* $a_2(x(\zeta))$ *and a positive definite polynomial* $\varphi(x)$ *such that:*

1. $a_0(x) - \varphi(x) \geq 0$,
2. $a_1(\theta, x, x_\theta) \geq 0$ *for* $\theta \in [-\tau, 0]$,
3. $a_2(x(\zeta)) \geq 0$,
4. $\tau\frac{\partial a_1}{\partial x}f + \frac{da_0}{dx}f - \tau\frac{\partial a_1}{\partial \theta} + \tau a_2(x) - \tau a_2(x_\theta) + a_1(0, x, x) - a_1(-\tau, x, x_\tau) \leq 0$ *for* $\theta \in [-\tau, 0]$.

*Then the steady state* 0 *of the system given by (4.17) is* globally stable.

**Proof.** Integrating the second and third conditions and adding the first condition, we get that $V(x_t) \geq \varphi(x(t))$, where $V(x_t)$ is given by (4.19): the first Lyapunov condition is satisfied. The time derivative of $V(x_t)$ is:

$$
\begin{aligned}
\dot{V}(x_t) &= \frac{da_0}{dx(t)} f + a_1(0, x(t), x(t)) - a_1(-\tau, x(t), x(t-\tau)) \\
&+ \int_{-\tau}^{0} \left( \frac{\partial a_1}{\partial x(t)} f - \frac{\partial a_1}{\partial \theta} + a_2(x(t)) - a_2(x(t+\theta)) \right) d\theta \\
&= \frac{1}{\tau} \int_{-\tau}^{0} \left\{ \begin{array}{c} \frac{da_0}{dx(t)} f + \tau \frac{\partial a_1}{\partial x(t)} f - \tau \frac{\partial a_1}{\partial \theta} + \\ + a_1(0, x(t), x(t)) - a_1(-\tau, x(t), x(t-\tau)) + \\ + \tau a_2(x(t)) - \tau a_2(x(t+\theta)) \end{array} \right\} d\theta.
\end{aligned}
$$

Condition (4) above states that the kernel of the above integral is nonpositive for $\theta \in [-\tau, 0]$. Therefore (4.19) is a Lyapunov functional for the system (4.17) and the zero steady state is *stable*. Since there is no constraint on the state-space, the result holds globally. ■

This proposition can be used in a similar way as in the delay-independent and linear delay-dependent cases. The condition $\theta \in [-\tau, 0]$ is adjoined using sum of squares multipliers of bounded degree in all variables. Similarly, we construct terms (4.12) and (4.13) that can be used to reduce the conservativeness of relaxing the positivity of the functional to the positivity of its kernel.

These steps result in four SOS conditions in a relevant sum of squares programme which can be solved using SOSTOOLS [75]. Different Lyapunov-Krasovskii structures can also be used which may have better properties.

**Remark 4.12** *As remarked earlier, when dealing with nonlinear systems with multiple equilibria or with natural constraints on their state-space, it is useful to use a restricted region for which stability is to be proven, in the same way that it was done in the delay-independent case. We will still need to specify* $\Omega = \{x_t \in C : \|x_t\| \leq \gamma\}$, *and adjoin the relevant conditions on* $x(t)$, $x(t-\tau)$ *and* $x(t+\theta)$ $\forall\, \theta \in [-\tau, 0]$ *to the relevant kernels of the Lyapunov functionals using the extended S-procedure, in much* *the same way that the conditions $\theta \in [-\tau, 0]$ can be adjoined in Conditions (2) and (4) of Proposition 4.11.*

### 4.3.3 Robust Stability Analysis Under Parametric Uncertainty

Robust stability under parametric uncertainty can be treated in a unified way. Consider a time-delay system of the form (4.2) with an uncertain parameter $p$:

$$\dot{z}(t) = f(z_t, p), \tag{4.20}$$

where $p \in P$, where $P$ is given by

$$P = \{p \in \mathbb{R}^m \,|\, q_i(p) \geq 0,\ i = 1, \ldots, N\}, \tag{4.21}$$

i.e., the uncertainty set is captured by certain inequalities. Let $x(t) = z(t) - z_0$. Then we have:

$$\dot{x}(t) = f(x_t + z_0, p) \tag{4.22}$$

$$0 = f(z_0, p) \tag{4.23}$$

which has the steady state $x^*$ at the origin. The stability of this system (which has a DAE form) can be handled by constructing a *Parameter Dependent* Lyapunov functional. Consider the functional (modified from (4.19)):

$$V(x_t, p) = a_0(x(t), p) + \int_{-\tau}^{0} a_1(\theta, x(t), x(t+\theta), p) d\theta + \int_{-\tau}^{0}\int_{t+\theta}^{t} a_2(x(\zeta), p) d\zeta d\theta. \tag{4.24}$$

Then we have the following proposition:

**Proposition 4.13** *Consider the system given by (4.22), where $p \in P$ as defined by (4.21). Denote $x = x(t)$, $x_\tau = x(t-\tau)$, $x_\theta = x(t+\theta)$. Suppose that there exist* *polynomials* $a_0(x,p), a_1(\theta,x,x_\theta,p)$ *and* $a_2(x(\zeta),p)$ *and a positive definite function* $\varphi(x)$ *such that the following conditions hold for* $p \in P$*:*

1. $a_0(x,p) - \varphi(x) \geq 0,$

2. $a_1(\theta,x,x_\theta,p) \geq 0 \ \forall \ \theta \in [-\tau, 0],$

3. $a_2(x(\zeta),p) \geq 0,$

4. $a_1(0,x,x,p) - a_1(-\tau,x,x_\tau,p) + \frac{da_0}{dx(t)} f + \tau a_2(x,p) - \tau a_2(x_\theta,p) + \tau \frac{\partial a_1}{\partial x} f - \tau \frac{\partial a_1}{\partial \theta} \leq 0,$ $\forall \ \theta \in [-\tau, 0]$ *and when (4.23) is satisfied.*

*Then the steady state* 0 *of the system given by (4.22–4.23) is* robustly globally uniformly stable *for all* $p \in P$*.*

The proof is similar to the one given earlier. It is based on functional (4.24) and is omitted. The equality constraints given by (4.23) that may arise during the transformation process can be adjoined using appropriate polynomial multipliers, as explained in Chapter 3. State-space constraints for local stability analysis can also be adjoined in a unified manner.

## 4.4 Stability Analysis of a Predator-Prey Model

A simple model of predator-prey interactions is

$$\dot{x} = bx - k_1xy, \quad \dot{y} = k_2xy - \sigma y,$$

where $x$ and $y$ are the prey and predator populations, $b$ is the rate of increase of prey, $k_1$ and $k_2$ are the coefficients of the effect of predation on $x$ and $y$ and $\sigma$ is the death rate of $y$. The cause of death of the prey is due to predation alone, and the growth of the predator population has as the only limitation the number of prey. These equations give rise to Lotka-Volterra predator-prey cycles, but the model is not biologically meaningful because it is *conservative*, giving rise to a family of closed trajectories rather than a single limit cycle [48].

The above equations describe ideal populations that can react instantaneously to any change in the environment. In real populations, this change comes with a delay that represents *maturation* of the predator population. A more realistic set of equations is [101]:

$$\dot{x}(t) = x(t)[b - ax(t) - k_1 y(t)],$$
$$\dot{y}(t) = -\sigma y(t) + k_2 x(t-\tau) y(t-\tau),$$

where $-ax(t)^2$ limits the growth of the prey, and $\tau \geq 0$ is a constant capturing the average period between death of prey and birth of a subsequent number of predators.

**Assumption 4.14** *$a, b, k_1, k_2$ and $\sigma$ are positive.*

The equilibria $(x^*, y^*)$ of the above system are:

$$(x^*, y^*) = (0,0), \quad (x^*, y^*) = (b/a, 0),$$
$$(x^*, y^*) = \left(\frac{\sigma}{k_2}, \frac{bk_2 - a\sigma}{k_1 k_2}\right). \tag{4.25}$$

We are only interested in the steady state given by (4.25).

**Assumption 4.15** $(bk_2 - a\sigma) > 0$.

Assumption 4.15 ensures that the steady state (4.25) is in the first quadrant. We now shift the coordinates to $(x_1, x_2) = (x - \frac{\sigma}{k_2}, y - \frac{bk_2 - a\sigma}{k_1 k_2})$ to get:

$$\dot{x}_1(t) = \left[x_1(t) + \frac{\sigma}{k_2}\right][-ax_1(t) - k_1 x_2(t)] \tag{4.26}$$

$$\dot{x}_2(t) = -\sigma x_2(t) + \sigma x_2(t-\tau) + \frac{bk_2 - a\sigma}{k_1} x_1(t-\tau) + k_2 x_1(t-\tau) x_2(t-\tau) \tag{4.27}$$

We can linearize the above system about $(0,0)$ to get:

$$\dot{x}_1(t) = \frac{\sigma}{k_2}[-ax_1(t) - k_1 x_2(t)] \tag{4.28}$$

$$\dot{x}_2(t) = -\sigma x_2(t) + \sigma x_2(t-\tau) + \frac{bk_2 - a\sigma}{k_1} x_1(t-\tau) \tag{4.29}$$

For the linearised system, we have the following result:

**Proposition 4.16** *Consider the system (4.28–4.29) under the assumptions (4.14,4.15). Then if* $(bk_2 - 3a\sigma) < 0$*, the zero steady state is stable independent of the delay. If* $(bk_2 - 3a\sigma) > 0$*, the zero steady state is stable if the delay satisfies* $\tau < \tau^*$ *and is unstable otherwise, where* $\tau^*$ *is given by:*

$$\tau^* = \frac{1}{\omega}\ atan\ \left[\omega\frac{(a\sigma^2 - \omega k_2)k_2 - \sigma(2a\sigma + bk_2)(k_2 + a)}{k_2\sigma\omega^2(k_2 + a) + (2a\sigma - bk_2)(a\sigma^2 - \omega k_2)}\right]$$

*and* $\omega$ *solves*

$$\omega^4 + \frac{a^2\sigma^2}{k_2^2}\omega^2 + \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2) = 0.$$

**Proof.** In the absence of delay, and under the two assumptions, the system is *asymptotically stable.* Substituting $s = j\omega$ in the characteristic equation and separating real and imaginary parts we get:

$$\begin{aligned} -\omega^2 + \frac{a\sigma^2}{k_2} &= \sigma\omega\sin(\omega\tau) + \sigma\left(\frac{2a\sigma}{k_2} - b\right)\cos(\omega\tau) \\ \sigma\left[1 + \frac{a}{k_2}\right]\omega &= \sigma\omega\cos(\omega\tau) - \sigma\left(\frac{2a\sigma}{k_2} - b\right)\sin(\omega\tau) \end{aligned}$$

Squaring the two equations and adding we get:

$$\omega^4 + \frac{a^2\sigma^2}{k_2^2}\omega^2 + \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2) = 0. \tag{4.30}$$

Denoting $p_1 = \frac{a^2\sigma^2}{k_2^2}$ and $p_2 = \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2)$, the roots of this equation are:

$$\omega^2 = -\frac{p_1}{2} \pm \frac{\sqrt{p_1^2 - 4p_2}}{2}. \tag{4.31}$$

Under assumption 4.15, if $(bk_2 - 3a\sigma) < 0$ (i.e $p_2 > 0$), then there are no real solutions to (4.30). Since the steady state is stable when the delay is zero, and there is no $\omega$ for which poles cross to the RHP, we conclude that (4.28–4.29) is delay-independent stable.

Under assumption 4.15 and $(bk_2 - 3a\sigma) > 0$, then $p_2 < 0$ and one of the two roots of (4.31) is positive and the other one is negative. Therefore, the poles cross the imaginary axis at only one $\omega$. There is no possibility for *stability reversal*. If $\omega$ is the solution to the above equation, then at $\tau = \tau^*$ given in the statement of the Proposition a Hopf bifurcation occurs; the system is stable for $\tau < \tau^*$ and unstable for $\tau > \tau^*$. ■

We now analyze the nonlinear description of the system (4.26–4.27) using the methodology that was developed in the previous sections. We choose as nominal values for the parameters $\sigma = 10$, $a = 1$, $k_1 = 1$, and $k_2 = 3$.

### 4.4.1 Delay-Independent Stability Analysis

The system (4.26–4.27) has many equilibria, and so we need to define a region around the zero steady state to obtain a stability condition (this is the region $\Omega$ in Theorem 4.2). We let

$$|x_{1_t}| \leq \gamma_1 x^*, \quad |x_{2_t}| \leq \gamma_2 y^*, \tag{4.32}$$

where the steady state $(x^*, y^*)$ is given by (4.25). We consider $b$ to be a parameter in the problem. From Proposition 4.16, the linear version of this system is delay-independent stable when $\frac{a\sigma}{k_2} < b < \frac{3a\sigma}{k_2}$. For the given values of $a, \sigma$ and $k_2$, the system is delay-independent stable for $10/3 < b < 10$. For the purpose of calculating $(x^*, y^*)$, we use a value of $b = 20/3$. The steady state $(0, 0)$ of system (4.26–4.27) does not move as $b$ changes; however, the other two equilibria cross through the region defined by (4.32). If we choose $\gamma_1 = \gamma_2 = 0.1$, then no other steady state enters this region for $11/3 < b < 10$.

We consider the following Lyapunov structure:

$$V(x_t) = a_0(x_1(t), x_2(t), b) + \int_{-\tau}^{0} a_1(x_1(t+\theta), x_2(t+\theta), b)d\theta.$$

We use a variant of Proposition 4.13 to obtain parameter regions for which robust delay-independent stability of the origin can be proven. When the order of $a_0$ is second order and $a_1$ is 4th order, we can construct $V(x_t)$ for $4.56 \leq b \leq 7.11$. When they are 4th order and 6th order respectively, then this region becomes $3.67 \leq b \leq 9.95$, which is essentially the full interval.

### 4.4.2 Delay-Dependent Stability Analysis

Now we will analyze the system in the delay-dependent parameter regime for $b = 15$. Given these parameters, $\tau^* = 0.0541$. The system has several equilibria and so we use the same constraints on $x_1$ and $x_2$ on the state-space given by (4.32) with $\gamma_1 = \gamma_2 = 0.1$.

We can construct the Lyapunov functional $V(x_t)$ given by (4.19) with $a_1$ 0th order with respect to $\theta$, and 2nd order with respect to the rest of the variables for $\tau = 0.04$. When $a_1$ is quartic with respect to all variables but $\theta$ (which is kept at 0 order), then we can construct this $V(x_t)$ for $\tau = 0.053$. The corresponding SDP is bigger as the functional is more complicated, but we see that stability for a delay size close to the stability boundary can be tested.

## 4.5 Conclusion

In this chapter, we presented a methodology to construct Lyapunov-Krasovskii functionals for time delay systems based on the sum of squares decomposition. As far as linear systems are concerned, this is tantamount to solving PLMIs using SOS. For nonlinear systems, the construction is entirely algorithmic and is done using the same tools.

The above methods can be easily extended to systems with many delays, either commensurate or not. Still a judicious choice for the structure of the Lyapunov functional would be required. Functional differential equations of neutral type can also be treated in a unified way.

In the next chapter, we will consider the global stability of an arbitrary interconnection of systems described by FDEs in the framework of network congestion control for the Internet.

# Chapter 5

# Large Scale Systems: Network Congestion Control

$$\overbrace{\text{Ἀεί}}^{3}\ \overbrace{\text{ὁ}}^{1}\ \overbrace{\text{Θεὸς}}^{4}\ \overbrace{\text{ὁ}}^{1}\ \overbrace{\text{Μέγας}}^{5}\ \overbrace{\text{Γεωμετρεῖ}}^{9}$$

Πλάτωνας

$\pi \simeq 3.14159$

God Ever Geometrizes

Plato

This chapter comes to complete the right hand side of the picture shown in Figure 1.1. We illustrate how scalable stability of ODE and FDE system interconnections can be guaranteed for the case of network congestion control for the Internet.

The need for congestion control emerged in the mid-1980s, when a phenomenon known as 'congestion collapse' resulted in unreliable file transfer. In 1988, Jacobson [31] proposed a congestion control scheme for the Internet, an admittedly ingenious design. The shortcomings of this scheme and its successors such as TCP Reno and Vegas have only recently become apparent: they are not scalable to arbitrary networks with very large capacities and time-delays. The technological advances – reflected in the introduction of high bandwidth-delay product links – have revealed that the thus far functional scheme unfortunately results in under-utilization of the network's resources, and theoretical analyses have shown the appearance of instabilities in the transmission rates.

The need for designing new network congestion control schemes for the Internet has steered the development of a mathematically rigorous methodological framework in which the rather complicated design requirements can be accounted for to facilitate design. In this chapter, we review briefly the methodological framework that was developed, as well as how decentralized network congestion control schemes could be designed to be functional, irrespective of the size of the network and the links' capacities. The design is performed at two levels: first, the problem of sharing the available bandwidth in a fair and efficient way is formulated as a resource allocation program, the optimal solution of which is then the desired equilibrium point of the overall system; thereafter, the task is to choose the dynamics of the sources and links in the network so as to drive the system's trajectories to that equilibrium (hence to the optimum) in a scalably stable way.

This design recipe yielded several congestion control schemes, one of which is Fast AQM Scalable TCP (FAST). The desired scalability properties, including scalability with respect to the size of the delays in the network were proven for the delayed but linearized system. This chapter is devoted to the solution of one of the most intriguing problems: that of proving *scalable nonlinear stability in the presence of heterogeneous delays*. The way we provide a proof of this property is through the construction of a Lyapunov-Krasovskii functional, a tool that was presented in the previous chapter. The proof is constructive and can be extended to other network congestion control schemes such as primal schemes, as we will see in the sequel. This construction would not be possible if the structure of the system was not taken into account.

## 5.1 Introduction

Internet congestion control [89] is a distributed algorithm to allocate available resources to competing sources so as to avoid congestion collapse by ensuring that link capacities are not exceeded. This problem can be formulated as a fully centralized resource allocation program which however can be decomposed into a primal and a dual problem by introducing duality-based price signals [34]. In this manner, congestion signals seen as *dual variables* are generated at the Active Queue Management (AQM) part of the algorithm implemented at the links; the congestion measure is usually based on either delay or packet loss. On the other hand, the source rates can be thought of as the *primal variables* which are adapted at the Transmission Control Protocol (TCP) part of the algorithm, according to the size of the price signals. The AQM and TCP algorithms have dynamics that aim to drive the congestion signals and the source rates exactly at, or approximately close to, the optimum point of the distributed resource allocation optimization problem [41].

The structure of the dynamics that are chosen for TCP and AQM are usually based on a gradient algorithm to guarantee convergence [34]; when the delays that are ubiquitously present in the system are ignored, a Lyapunov function argument can be used to verify the global attractivity of the optimal point for arbitrary network sizes. It is nonetheless appreciated that the simplest adequate model for network congestion control is in the form of nonlinear deterministic delay-differential equations [85, 44, 54]: delays cannot be ignored as in general, their presence results in degradation of performance or even instabilities. Indeed, stability is an important measure of the functionality of the system. Without stability, transmission rates oscillate, which could result in a reduction in the link utilization; 'mice' packets, i.e., short-lived small packets on which congestion control is difficult get dropped; and predictability of the behavior of the system is lost. On the other hand, the introduction of increasing bandwidth-delay product links in the network calls for protocols whose performance properties scale with the network size, the Round Trip Time and the link capacities [56].

It is unfortunate that the tools for analysis of the system behavior at the nonlinear model level with heterogeneous delays do not scale well with the problem size. In fact, most scalable analysis procedures in network congestion control center on the investigation of the stability properties of the linearizations of the nonlinear equations, using for example, the generalized Nyquist criterion developed in [97] the result in [56], and [33].

There are significant results on the stability analysis of nonlinear Functional Differential Equations (FDEs), also known as Time Delay Systems (TDSs) [28, 36]. The motivating examples have predominantly come from *population dynamics* – an area in which the effect of delay cannot be neglected [38]. This and other research areas have motivated the development of hand-crafted methodologies for specific classes of time delay systems at the nonlinear level. Lyapunov-based constructions have been used in Internet Congestion Control [99, 100, 17, 46], but also, new tools have been introduced such as passivity theory formulations [102]. The analysis procedure that is usually followed is centered on simple network topologies. The difficulty lies in the construction of the Lyapunov certificates, and the conservativeness of the results is usually due to their type: Lyapunov-Razumikhin functions or Lyapunov-Krasovskii functionals.

In order to obtain a scalable proof methodology for network congestion control schemes, one has to take advantage of the structure of the system and construct Lyapunov type certificates manually. In this chapter, we will describe a methodology based on Lyapunov-Krasovskii functionals to prove stability of so-called 'primal' and 'dual' nonlinear congestion control schemes with delays, making particular reference to Fast AQM Scalable TCP (FAST), a network congestion control scheme that was developed in [56]. We will build up the methodology starting from the linear case with delays and generalizing to the full nonlinear system with/without delays. The stability result we obtain holds for arbitrary network topologies [59]. It is, however, restricted to the structure of the Lyapunov functional constructed and is therefore conservative compared to the result from the linearization.

We then turn to a general framework of primal network congestion control schemes, and show that our result extends to that case in a unified way. However, when specific, simple topologies and non-structured congestion control schemes are investigated, the stability result can be obtained by resorting to the methodology presented in the previous chapter, using the sum of squares decomposition and SOSTOOLS. We present an example at the end of this chapter that illustrates how this can be done.

## 5.2 Problem Formulation

Consider a network of $L$ communication links shared by $S$ sources. For this network, we define a routing matrix $R$ by:

$$R_{li} = \begin{cases} 1 & \text{if source } i \text{ uses link } l \\ 0 & \text{otherwise} \end{cases} . \tag{5.1}$$

Each source $i$ has an associated transmission rate $x_i$. All sources whose flow passes through resource $l$ contribute to the *aggregate rate* $y_l$ for resource $l$, the rates being added with some forward time delay $\tau_{i,l}^f$. Hence, we have:

$$y_l(t) = \sum_{i=1}^{S} R_{li} x_i(t - \tau_{i,l}^f) \triangleq r_f(x_i, \tau_{i,l}^f) \tag{5.2}$$

The resources $l$ react to the aggregate rate $y_l$ by setting congestion information $p_l$, the price at resource $l$. This is the Active Queue Management part of the picture that is to be designed. The prices of all the links that source $i$ uses are aggregated to form $q_i$, the *aggregate price* for source $i$, again through a delay $\tau_{i,l}^b$:

$$q_i(t) = \sum_{l=1}^{L} R_{li} p_l(t - \tau_{i,l}^b) \triangleq r_b(p_l, \tau_{i,l}^b) \tag{5.3}$$

The prices $q_i$ can then be used to set the rate of source $i$, $x_i$, which completes the picture shown in Figure 5.1. The forward and backward delays can be combined to yield the Round Trip Time (RTT):

$$\tau_i = \tau_{i,l}^f + \tau_{i,l}^b, \quad \forall\ l \tag{5.4}$$

The capacity of link $l$ is given by $c_l$. The functions $f$ and $g$ shown in Figure 5.1 are the source law and the link law, respectively. This setting is *universal*, and the only information missing is the structure of the two control laws that describe how the $i$th

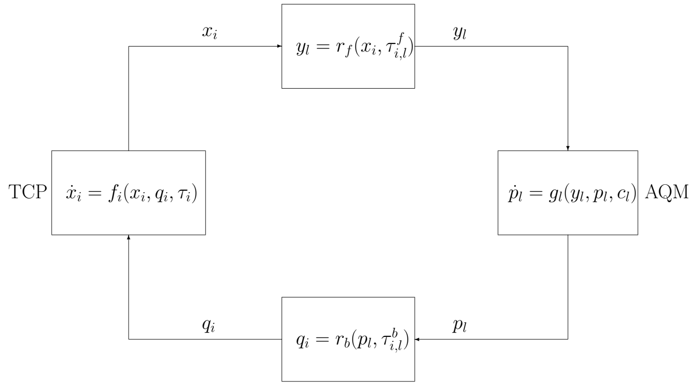



Figure 5.1: The Internet as an interconnection of sources and links through delays.

source reacts to the price signal $q_i$ that it sees

$$\dot{x}_i = f_i(x_i, q_i, \tau_i), \tag{5.5}$$

and how the $l$th router reacts to the aggregate rate $y_l$ it observes

$$\dot{p}_l = g_l(y_l, p_l, c_l). \tag{5.6}$$

Here $f_i$ models TCP algorithms (e.g., Reno or Vegas) and $g_l$ models AQM algorithms (e.g., RED, REM).

In order to understand the meaning of the variables $p_l$, let us consider the properties of the equilibrium of this system, which we assume is given by the vector quantities $x^*, y^*, p^*, q^*$. We have the following relationships from (5.2) and (5.3):

$$y^* = Rx^*, \quad q^* = R^T p^*.$$

We now assume that as the aggregate price signal at equilibrium $q_i^*$ increases, then the demanded transmission rate at the source should decrease, i.e., the two are related by

$$x_i^* = F_i(q_i^*),$$

where $F_i$ is a positive, strictly monotone decreasing function, which is the solution of $f_i(x_i^*, q_i^*) = 0$ where $f_i$ is given by Equation (5.5). Alternatively, one can think that the sources have a certain *utility* if allowed a certain transmission rate, the utility satisfying:

$$U_i'(x_i) = F_i^{-1}(x_i).$$

This relationship implies that $U_i(x_i)$ is a monotonically increasing strictly concave function. Under these assumptions, the equilibrium rate $x_i^*$ for each source solves

$$\max_{x_i \geq 0} U_i(x_i) - x_i q_i^*, \tag{5.7}$$

which means that the sources are trying to maximize their profit: maximize their utility, and at the same time, minimize the cost of having high rates; $q_i^*$ can be thought of as the *price* per unit flow that the sources have to pay. Different protocols correspond to different Utility functions $U_i$, and to different dynamic laws (5.5–5.6) that attempt in a decentralized way to reach the appropriate equilibrium. For this reason this framework allows comparability of different designs.

The role of prices is to coordinate the actions of individual sources so as to align individual optimality with social optimality, i.e., to ensure that the solution of (5.7) also solves the network resource allocation problem

$$\begin{aligned} \max_{x_i \geq 0} \quad & \sum_{i=1}^{S} U_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^{S} R_{li} x_i \leq c_l, \ \forall\, l = 1, \ldots, L, \end{aligned} \tag{5.8}$$

where the inequality constraint is the natural limitation that the sum of all transmission rates through link $l$ has to be less than or equal to its capacity. The uniqueness of the optimal solution to the above problem is guaranteed since the $U_i$ are strictly concave functions and the program is convex. The above optimization problem cannot be solved in a decentralized way, as the source rates are coupled in the shared links through the inequality constraints and solving for $x^*$ would require cooperation among possibly all sources. To solve it in a distributed manner over a large network, we can decompose it into a primal problem that the sources are trying to solve and a dual that the links are trying to solve, regarding the source rates $x_i$ as primal variables and the prices set by the links $p_l$ as the dual variables. Specifically, consider the Lagrangian for program (5.8):

$$\begin{aligned} L(x,p) &= \sum_i U_i(x_i) - \sum_l p_l(y_l - c_l) \\ &= \sum_i \left( U_i(x_i) - x_i \sum_l R_{li} p_l \right) + \sum_l p_l c_l, \end{aligned}$$

where $p_l \geq 0$. The dual problem can then be written as

$$\min_{p_l \geq 0} \max_{x_i \geq 0} \sum_i \left( U_i(x_i) - x_i \sum_l R_{li} p_l \right) + \sum_l p_l c_l.$$

By the properties of the dual solution and the fact that the programs are convex, at the optimum $p^*$ the $x_i$ that maximizes the individual profit (5.7) is the same as the unique solution to the network problem (5.8). If the equilibrium prices $p^*$ are made to align with the Lagrange multipliers, the individual optima – computed in a decentralized fashion by the sources – will align with the global optima of (5.8).

The dynamical system defined by (5.5–5.6) with delays ignored aims to drive the system close to or exactly at the optimal point $(x^*, p^*)$, using well-known gradient algorithms. For example, from the Karush-Kuhn-Tucker conditions, the solution of the dual problem satisfies:

$$\begin{aligned} x_i - U_i'^{-1}(q_i) &= 0 \\ p_l(y_l - c_l) &= 0 \end{aligned}$$

and $p_l \geq 0$, as the dual variables are non-negative. To obtain the $p_l$ and $x_i$, one can use the following algorithm at each link, which is a gradient algorithm:

$$\dot{p}_l(t) = \left[ \frac{y_l - c_l}{c_l} \right]^+_{p_l} \triangleq [g_l]^+_{p_l}, \tag{5.9}$$

$$x_i = U_i'^{-1}(q_i), \tag{5.10}$$

where $[f(x)]^+_x$ means

$$[f(x)]^+_x = \begin{cases} f(x) & x > 0 \\ \max\{(f(x), 0)\}, & x = 0 \end{cases}.$$

After [34], it is customary to call 'dual' the congestion control scheme with dynamics at the links and a static source law - such as the one shown above; and 'primal' the congestion control scheme with dynamics at the sources and a static link law. If both source and link laws have dynamics, the scheme is termed 'primal-dual'.

For both the dual and the primal algorithms, the stability properties of the undelayed systems can be obtained by constructing Lyapunov functions that scale with the system size based on the barrier functions of the optimization strategies. Put another way, the gradient algorithm results in a weighted *gradient system*, i.e., for the primal case, there is a potential function $V$ so that $\dot{x}_i = -\kappa_i \frac{\partial V}{\partial x_i}$ [89]. Taking advantage of this structure, a scalable stability proof can be obtained.

When delays are introduced, the analysis becomes more complicated, and a scalable proof methodology for analysis of such congestion control schemes is difficult in the nonlinear case. The generalized Nyquist criterion can be used for the linearization of these systems, as it is scalable in the case of congestion control as explained in [97]. Here we will show how the analysis of congestion control schemes captured by nonlinear delay-differential equations can be performed in a scalable manner, treating both the presence of heterogeneous delays in the system and the nonlinearities explicitly. The tools we will be using are Lyapunov-Krasovski functionals [36], which are the relevant Lyapunov-based tools for functional differential equations as Lyapunov functions are for ordinary differential equations. We will set them equal to the Lyapunov functions used in the undelayed systems, plus some simple integral terms. We will illustrate how the special structure of the system helps in the scalability of the Lyapunov certificate for the linear case, and then concentrate in the nonlinear case for which sufficient conditions for arbitrary topologies will be derived. The analysis is performed for a dual algorithm and a primal algorithm, and scalable stability conditions are obtained.

The routing matrix $R$ is assumed fixed and full row rank. This means that there are no algebraic constraints between link flows, i.e., they can vary independently by choice of source flows $x_i$. As a consequence, equilibrium prices are uniquely determined.

## 5.3 Dual Congestion Control Schemes

Combining (5.1–5.4) and (5.9–5.10), the system has the following closed loop dynamics:

$$\dot{p}_l(t) = \left[ \sum_{i=1}^{S} \frac{R_{li}}{c_l} U_i'^{-1} \left( \sum_{m=1}^{L} R_{mi} p_m(t - \tau_{i,l}^f - \tau_{i,m}^b) \right) - 1 \right]_{p_l}^{+} \tag{5.11}$$

We also denote $h = \max_{\{i,l,m:R_{mi}=R_{li}=1\}}\{\tau_{i,l}^f + \tau_{i,m}^b\}$, and $C^n = C([-h,0],\mathbb{R}^n)$ the Banach space of continuous functions mapping $[-h,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. We assume that the initial condition is non-negative, i.e.,

$$p(\theta) = \phi(\theta) \geq 0, \quad \theta \in [-h,0], \quad \phi \in C^L.$$

This guarantees that the solutions to (5.11) satisfy $p_l(t) \geq 0$ for all time as a result of the projection nonlinearity.

A particular congestion control scheme that we will be investigating is FAST, in which the sources are assumed to have the following Utility function:

$$U_i(x_i) = \frac{\tau_i M_i}{\alpha_i} x_i \left( 1 - \log \frac{x_i}{\overline{x}_i} \right), \tag{5.12}$$

where $M_i = \sum_{l=1}^{L} R_{li}$, $\alpha_i$ are (positive) source gains and $\overline{x}_i$ are source constants. Since $x_i = U_i'^{-1}(q_i)$, we have:

$$x_i = \overline{x}_i e^{-\frac{\alpha_i q_i}{M_i \tau_i}}$$

which implies that $x_i \leq \overline{x}_i$, since $q_i \geq 0$.

To investigate the linearization of (5.11), we assume that $R$ refers to bottleneck links only, and for non-bottleneck links $p_l = 0$. This gives the system

$$\dot{p}_l(t) = \sum_{i=1}^{S} \sum_{m=1}^{L} \frac{R_{li} R_{mi}}{U_i''(x_i^*) \tilde{c}_l} p_m(t - \tau_{i,l}^f - \tau_{i,m}^b), \tag{5.13}$$

where $\tilde{c}_l$ is a target capacity set just below the true capacity $c_l$ of link $l$ and $^*$ denotes equilibrium quantities. For this system, the following result is known:

**Theorem 5.1** *[56] Let the matrix $R$ that denotes the routing matrix in relation to the bottleneck links be full row rank. Then the system given by (5.13) is asymptotically stable if*

$$\frac{1}{\tilde{c}_l} \sum_{i=1}^{S} \frac{R_{li} M_i \tau_i}{|U_i''(x_i^*)|} < \frac{\pi}{2}. \tag{5.14}$$

In particular, for the Utility function given by (5.12), the above condition reduces to $\alpha_i \leq \pi/2$, which is a decentralized condition on the sources' gains.

In the case of FAST, there are a number of undetermined parameters, $\overline{x}_i$. We can choose

$$\overline{x}_i = x_i^* e^{\frac{\alpha_i q_i^*}{M_i \tau_i}} = x_i^* e^{\frac{\alpha_i \ {}_{m=1}^{L} R_{mi} p_m^*}{M_i \tau_i}} \tag{5.15}$$

so that $y_l^* = \sum_{i=1}^{S} R_{li} x_i^*$. Also at the equilibrium $y_l^* = c_l$ (or $p_l^* = 0$ and $y_l^* < c_l$), we therefore have

$$\sum_{i=1}^{S} R_{li} x_i^* \leq c_l. \tag{5.16}$$

If we wanted to impose fairness so that the sources would have the same rates around some nominal price value $q_i^*$, then the $x_i^*$ would be equal to each other.

### 5.3.1 Stability of the Linearization

In this section, we aim to obtain a Lyapunov-based construction to prove a result similar to Theorem 5.1. It is true that frequency domain methodologies many times result in more accurate descriptions of the stability boundaries and are scalable for the special case of Internet Congestion Control [56]. Lyapunov-based arguments are more difficult to make, and many times are more conservative, but prove useful when considering the stability of nonlinear time-delay systems. In order to construct a scalable Lyapunov-Krasovskii functional, we take advantage of the structure of the system, as will be evident in the proof of the theorem which is constructive. Important in this quest is the following undelayed version of (5.13):

$$\dot{p}_l(t) = \sum_{i=1}^{S}\sum_{m=1}^{L} \frac{R_{li}R_{mi}}{U_i''(x_i^*)\tilde{c}_l} p_m(t) \tag{5.17}$$

**Theorem 5.2** *Let the matrix $R$ that denotes the routing matrix in relation to the bottleneck links be full row rank. Then the system given by (5.13) is asymptotically stable if*

$$\frac{1}{\tilde{c}_l}\sum_{i=1}^{S} \frac{R_{li}M_i\tau_i}{|U_i''(x_i^*)|} < 1. \tag{5.18}$$

**Proof.** Consider the function

$$V_1(p) = \frac{1}{2}\sum_{i=1}^{S}\frac{A_i}{\tau_i}\left(\sum_{l=1}^{L} R_{li}p_l\right)^2 \tag{5.19}$$

First, note that $V_1 > 0$ since $R$ is full rank. The derivative of (5.19) along the trajectories of system (5.13) can be computed to be:

$$\dot{V}_1(p) = -\sum_{l=1}^{L}\tilde{c}_l\dot{p}_l\dot{p}_{l,u} = -\sum_{l=1}^{L}\tilde{c}_l\dot{p}_l^2 - \sum_{l=1}^{L}\tilde{c}_l\dot{p}_l(\dot{p}_{l,u} - \dot{p}_l),$$

where $\dot{p}_{l,u}$ is the undelayed version of (5.13) given by (5.17). To proceed, we use the Leibniz rule to distribute the delay over an interval. For example, we have:

$$p_l(t-\tau) = p_l(t) - \frac{d}{dt}\int_{-\tau}^{0} p_l(t+\theta)d\theta.$$

This gives

$$\dot{p}_l = \sum_{i=1}^{S}\sum_{m=1}^{L}\frac{R_{li}R_{mi}}{U_i''(x_i^*)\tilde{c}_l}p_m(t-\tau_{i,l}^f-\tau_{i,m}^b) = \dot{p}_{l,u} - \sum_{i=1}^{S}\sum_{m=1}^{L}\frac{R_{li}R_{mi}}{U_i''(x_i^*)\tilde{c}_l}\int_{-\tau_{i,l}^f-\tau_{i,m}^b}^{0}\dot{p}_m(t+\theta)d\theta.$$

Putting everything together, we have:

$$\dot{V}_1(p) = -\sum_{l=1}^{L} \tilde{c}_l \dot{p}_l^2 - \sum_{l=1}^{L}\sum_{i=1}^{S}\sum_{m=1}^{L} \frac{R_{li}R_{mi}}{U_i''(x_i^*)} \int_{-\tau_{i,l}^f-\tau_{i,m}^b}^{0} \dot{p}_l \dot{p}_m(t+\theta)d\theta.$$

Now let us concentrate on the second term:

$$\begin{aligned} &-\sum_{l=1}^{L}\sum_{i=1}^{S}\sum_{m=1}^{L} \frac{R_{li}R_{mi}}{U_i''(x_i^*)} \int_{-\tau_{i,l}^f-\tau_{i,m}^b}^{0} \dot{p}_l \dot{p}_m(t+\theta)d\theta \\ \leq\ &\frac{1}{2}\sum_{l=1}^{L}\sum_{i=1}^{S}\sum_{m=1}^{L} \frac{R_{mi}R_{li}(\tau_{i,l}^f+\tau_{i,m}^b)}{|U_i''(x_i^*)|}\dot{p}_l^2 + \frac{1}{2}\sum_{l=1}^{L}\sum_{i=1}^{S}\sum_{m=1}^{L} \frac{R_{mi}R_{li}}{|U_i''(x_i^*)|} \int_{-\tau_{i,l}^f-\tau_{i,m}^b}^{0} \dot{p}_m^2(t+\theta)d\theta, \end{aligned}$$

where the inequality $2ab \leq a^2 + b^2$ was used. Now consider the following functional term:

$$V_2 = \frac{1}{2}\sum_{l=1}^{L}\sum_{i=1}^{S}\sum_{m=1}^{L} \frac{R_{mi}R_{li}}{|U_i''(x_i^*)|} \int_{-\tau_{i,l}^f-\tau_{i,m}^b}^{0} \int_{t+\theta}^{t} \dot{p}_m^2(\zeta)d\zeta d\theta. \tag{5.20}$$

Then

$$\dot{V}_2 = \frac{1}{2}\sum_{l=1}^{L}\sum_{i=1}^{S}\sum_{m=1}^{L} \frac{R_{mi}R_{li}(\tau_{i,l}^f+\tau_{i,m}^b)}{|U_i''(x_i^*)|}\dot{p}_l^2 - \frac{1}{2}\sum_{l=1}^{L}\sum_{i=1}^{S}\sum_{m=1}^{L} \frac{R_{mi}R_{li}}{|U_i''(x_i^*)|} \int_{-\tau_{i,l}^f-\tau_{i,m}^b}^{0} \dot{p}_m^2(t+\theta)d\theta$$

Let $V = V_1 + V_2$. Then we get

$$\dot{V} \leq -\sum_{l=1}^{L} \tilde{c}_l \dot{p}_l^2 + \sum_{l=1}^{L}\sum_{i=1}^{S}\sum_{m=1}^{L} \frac{R_{mi}R_{li}(\tau_{i,l}^f+\tau_{i,m}^b)}{|U_i''(x_i^*)|}\dot{p}_l^2.$$

The coefficient of each term $\dot{p}_l^2$ is

$$-\tilde{c}_l + \sum_{i=1}^{S} \frac{R_{li}M_i\tau_i}{|U_i''(x_i^*)|}$$

Therefore if

$$\frac{1}{\tilde{c}_l}\sum_{i=1}^{S} \frac{R_{li}M_i\tau_i}{|U_i''(x_i^*)|} < 1 \tag{5.21}$$

for all $l$ then $\dot{V} \leq 0$, $V$ is a Lyapunov-Krasovksii functional and the conditions of stability [28] are satisfied. The set $S = \{\phi \in C^L : \dot{V}(\phi) = 0\}$ is the set

$$S = \left\{ \phi : \sum_{i=1}^{S} \sum_{m=1}^{L} \frac{R_{li} R_{mi}}{U_i''(x_i^*)\tilde{c}_l} \phi_m(-\tau_{i,l}^f - \tau_{i,m}^b) = 0, \quad l = 1, \ldots, L \right\}.$$

The largest set in $S$ that is invariant with respect to the system satisfies $\dot{p}_l = 0$ $\forall l = 1, \ldots, L$, i.e. $p_l = K$, a constant. But the only constant $\phi$ that is in $S$ is the zero equilibrium. Therefore, the equilibrium of the system is asymptotically stable by an extension of LaSalle's theorem (Theorem 5.3.1 in [28]) for arbitrary networks and delays, provided condition (5.18) is satisfied and $R$ is full rank. ■

The Lyapunov-Krasovskii functional that we have constructed introduced some conservativeness with respect to the result in Theorem 5.1, but is scalable to the size of the system. In particular, the right hand side of the inequality in Theorem 5.1 is $\pi/2$ and was obtained using frequency domain arguments. The bound that we obtain is 1, and is a result of the structure of the Lyapunov functional that is used. More complicated, 'richer' structures may yield stronger bounds.

In the case of FAST, for which the Utility function is given by (5.12), we have the following corollary:

**Corollary 5.3** *Let the matrix $R$ that denotes the routing matrix in relation to the bottleneck links be full row rank. Then FAST linearized about the equilibrium is asymptotically stable if*

$$\alpha_i < 1.$$

**Proof.** The Utility function for FAST is given by (5.12). This gives:

$$|U_i''(x_i^*)| = \frac{M_i \tau_i}{\alpha_i x_i^*}$$

Condition (5.18) then becomes:

$$\frac{1}{\tilde{c}_l}\sum_{i=1}^{S}\frac{R_{li}M_i\tau_i}{|U_i''(x_i^*)|} < 1$$
$$\Rightarrow \frac{1}{\tilde{c}_l}\sum_{i=1}^{S} R_{li}\alpha_i x_i^* < 1$$

which is guaranteed if $\alpha_i < 1$ since $\sum_{i=1}^{S} R_{li}x_i^* \leq \tilde{c}_l$. ■

## 5.3.2 Nonlinear Stability Analysis

The nonlinear system is given by Equation (5.11). Now $R$ denotes the full routing matrix with non-bottleneck links (rows) included but is still full row rank so that equilibrium prices are uniquely determined. The existence and uniqueness of solutions of (5.11) is assumed.

### 5.3.2.1 Nonlinear Undelayed Model

The closed loop system with delays equal to zero is:

$$\dot{p}_l(t) = \left[\sum_{i=1}^{S}\frac{R_{li}}{c_l}U_i'^{-1}\left(\sum_{m=1}^{L}R_{mi}p_m(t)\right) - 1\right]_{p_l}^{+} = [g_l(p)]_{p_l}^{+} \tag{5.22}$$

We have the following theorem:

**Theorem 5.4** *For fixed full rank $R$, the (unique) equilibrium of (5.22) is asymptotically stable for all non-negative initial conditions.*

**Proof.** Consider the following function:

$$V(p) = \sum_{l=1}^{L}(c_l - y_l^*)p_l + \sum_{i=1}^{S}\int_{q_i^*}^{q_i}(x_i^* - U_i'^{-1}(Q))dQ, \tag{5.23}$$

This function is positive definite for $p \geq 0$, with a minimum at the equilibrium [89]. Differentiating $V(p)$ with respect to time, we get

$$\dot{V}(p) = -\sum_{m=1}^{L} c_m g_m(p)[g_m(p)]^+_{p_m} \leq 0.$$

Now $\dot{V}(p) = 0$ only when $g_m(p) = 0$ or $g_m(p) < 0$ and $p_m = 0$. This can only happen at the equilibrium of interest. Therefore, we obtain asymptotic stability. Moreover, $V$ is radially unbounded; the equilibrium is globally (i.e., for $p_l \geq 0$) asymptotically stable. ■

The above result is the basic building block that ensures the scalability of the results. In the next subsection, we will build on the construction of scalable Lyapunov-Krasovskii functionals for the linear case, to construct similar certificates for the nonlinear case.

#### 5.3.2.2 Nonlinear Delayed Model

In order to ensure stability for arbitrary topologies with delays, we will use a Lyapunov argument using our result on the linearization. Most handcrafted attempts center on Lyapunov-Razumikhin functions for analysis of simple network topologies [17, 99], or the extraction of delay-independent stability conditions for arbitrary networks [105]. Lyapunov-Razumikhin functions, although an attractive tool for handcrafted stability analysis for nonlinear time delay systems because of their simplicity to construct manually, are known to yield conservative results compared to the ones that Lyapunov-Krasovskii functionals produce [28]. The intuition behind this is that Lyapunov-Krasovskii functionals respect the fact that the system is evolving on a function space, and so the Lyapunov certificate should be a functional; Lyapunov-Razumikhin functions attempt to assess the stability of an infinite dimensional system using finite dimensional arguments – an attempt that is inherently conservative. Here we construct a Lyapunov-Krasovskii functional that scales with the system size.

Recall that the Utility function is a continuously differentiable, non-decreasing, strictly concave function. Therefore, $U_i''(x_i) < 0$ everywhere. Let $\gamma_i$ be the lower bound for $|U_i''(x_i)|$, so that

$$|U_i''(x_i)| \geq \gamma_i > 0, \quad \forall\, i \tag{5.24}$$

$\gamma_i$ here serves as a global (i.e., for $x_i \geq 0$) Lipschitz constant for $U_i^{'-1}$ as:

$$\left|\left(U_i^{'-1}(q)\right)'\right| = \left|\frac{1}{U_i''(x)}\right| \leq \frac{1}{\gamma_i} \tag{5.25}$$

This means that:

$$\left|U_i^{'-1}(q_2) - U_i^{'-1}(q_1)\right| \leq \frac{1}{\gamma_i}\,|q_2 - q_1| \tag{5.26}$$

We then have the following result:

**Theorem 5.5** *The equilibrium of the system described by (5.11) is globally (for $p_l \geq 0$) asymptotically stable for arbitrary delays, provided that*

$$\frac{1}{c_l}\sum_{i=1}^{S}\frac{R_{li}M_i\tau_i}{\gamma_i} < 1 \tag{5.27}$$

*and the matrix $R$ is an arbitrary full rank, fixed routing matrix.*

**Proof.** Consider $V_1 = V$ given by (5.23). From the argument in the proof of Theorem 5.4, $V_1 > 0$ apart at the equilibrium, and is radially unbounded. Now

$$\dot{V}_1(p) = -\sum_{l=1}^{L} c_l g_{l,u}[g_l]^+_{p_l} = -\sum_{l=1}^{L} c_l g_l [g_l]^+_{p_l} - \sum_{l=1}^{L} c_l [g_l]^+_{p_l}(g_{l,u} - g_l),$$

where $g_{l,u}(p)$ corresponds to the undelayed version of unprojected (5.11), the unprojected Equation (5.22). The second term is equal to:

$$
\begin{aligned}
&-\sum_{l=1}^{L} c_l [g_l]^+_{p_l} (g_{l,u} - g_l) \\
&\leq \sum_{l=1}^{L}\sum_{i=1}^{S} R_{li} |[g_l]^+_{p_l}| \left| U_i'^{-1}\left(\sum_{m=1}^{L} R_{mi} p_m(t)\right) - U_i'^{-1}\left(\sum_{m=1}^{L} R_{mi} p_m(t-\tau^f_{i,l}-\tau^b_{i,m})\right)\right| \\
&\leq \sum_{l=1}^{L}\sum_{i=1}^{S}\sum_{m=1}^{L} \frac{R_{mi}R_{li}}{\gamma_i} |[g_l]^+_{p_l}| \left| p_m(t) - p_m(t-\tau^f_{i,l}-\tau^b_{i,m})\right| \\
&\leq \sum_{l=1}^{L}\sum_{i=1}^{S}\sum_{m=1}^{L} \frac{R_{mi}R_{li}}{\gamma_i} \int_{-\tau^f_{i,l}-\tau^b_{i,m}}^{0} |[g_l]^+_{p_l}| |\dot{p}_m(t+\theta)| d\theta,
\end{aligned}
$$

where global Lipschitz continuity and the Leibniz rule were used (note that $p \geq 0$). The rest of the proof is similar to the proof of Theorem 5.2. The candidate Lyapunov-Krasovskii functional is given by $V = V_1 + V_2$ where

$$
V_2(p) = \sum_{l=1}^{L}\sum_{i=1}^{S}\sum_{m=1}^{L} \frac{R_{li}R_{mi}}{2\gamma_i} \int_{-\tau^f_{i,l}-\tau^b_{i,m}}^{0} \int_{t+\theta}^{t} \dot{p}_m^2(\zeta) d\theta d\zeta.
$$

Then we have:

$$
\dot{V} = \sum_{l=1}^{L} \left[ -c_l g_l [g_l]^+_{p_l} + \sum_{i=1}^{S} \frac{R_{li} M_i \tau_i}{\gamma_i} \left\{ [g_l]^+_{p_l} \right\}^2 \right]
$$

Now $\dot{V} \leq 0$ under the condition:

$$
-c_l + \sum_{i=1}^{S} \frac{R_{li} M_i \tau_i}{\gamma_i} < 0
$$

This implies that $\dot{V} \leq 0$, and stability of the equilibrium follows for $p_l \geq 0 \ \forall \ l = 1, \ldots, L$. The set $S = \{\phi \in C^L : \dot{V} = 0\}$ is the set for which $[g_l]^+_{p_l} = 0$, i.e., for $l = 1, \ldots, L$ we have either

$$
\sum_{i=1}^{S} \frac{R_{li}}{c_l} U_i'^{-1} \left( \sum_{m=1}^{L} R_{mi} \phi_m(-\tau^f_{i,l} - \tau^b_{i,m}) \right) = 1
$$

or

$$\begin{cases} \sum_{i=1}^{S} \frac{R_{li}}{c_l} U_i'^{-1} \left( \sum_{m=1}^{L} R_{mi} \phi_m(-\tau_{i,l}^f - \tau_{i,m}^b) \right) < 1, \\ \phi_l(-\tau_{i,l}^f - \tau_{i,m}^b) = 0 \end{cases}$$

for $l = 1, \ldots, L$. The largest invariant set in $S$ satisfies $\dot{p}_l = 0$, i.e. $p_l = K$, a constant. The only constant solution in $S$ is the (unique) equilibrium (either $p_l^* = 0$ and $y_l^* < c_l$ or $p_l = p_l^*$). Therefore, the asymptotic stability of the equilibrium can be proven using an extension to LaSalle's theorem (Theorem 5.3.1 in [28]). Since $V$ is radially unbounded, the equilibrium of the system is globally (i.e., for $p_l \geq 0$) asymptotically stable for arbitrary networks and delays, provided that condition (5.27) is satisfied and $R$ is full rank. ■

For the special case of FAST, we have the following corollary:

**Corollary 5.6** *The equilibrium of FAST is globally (for $p_l \geq 0$) asymptotically stable for arbitrary delays and network topologies, provided that*

$$\alpha_i < \frac{x_i}{\overline{x}_i} \tag{5.28}$$

*and the matrix $R$ is an arbitrary full rank, fixed routing matrix.*

**Proof.** For FAST, the Utility function is given by (5.12); this gives the following value for $\gamma_i$:

$$\gamma_i = \frac{\tau_i M_i}{\alpha_i \overline{x}_i}$$

since $x_i \leq \overline{x}_i$. Therefore, condition (5.27) of Theorem 5.5 becomes:

$$\frac{1}{c_l} \sum_{i=1}^{S} R_{li} \alpha_i \overline{x}_i < 1$$

Therefore, a sufficient condition for stability is

$$\alpha_i < \frac{x_i^*}{\overline{x}_i}.$$

for $R$ full rank. ■

**Remark 5.7** *Another approach would be to get a local result, i.e., valid for all initial conditions in some level set of the Lyapunov functional $V$. In the case of FAST, one can bound the term $(g_{l,u} - g_l)$ using a* local *Lipschitz constant:*

$$|g_{l,u} - g_l| = \left| \sum_{i=1}^{S} \frac{R_{li} x_i^*}{c_l} \left( e^{\frac{-\alpha_i \sum_{m=1}^{L} R_{mi}[p_m(t) - p^*]}{M_i \tau_i}} - e^{\frac{-\alpha_i \sum_{m=1}^{L} R_{mi}[p_m(t-\tau_i) - p^*]}{M_i \tau_i}} \right) \right|$$

$$\leq \sum_{i=1}^{S} \frac{R_{li} \alpha_i x_i^* k_i}{M_i \tau_i c_l} \left| \sum_{m=1}^{L} R_{mi} [p_m(t) - p_m(t - \tau_i)] \right|,$$

*where $k_i > 1$ is a constant resulting from the Lipschitz approximation. If $f(x) = e^{-\frac{\alpha_i}{M_i \tau_i} x}$, then*

$$|f'(x)| = \frac{\alpha_i}{M_i \tau_i} \left| e^{-\frac{\alpha_i}{M_i \tau_i} x} \right| \leq \frac{\alpha_i k_i}{M_i \tau_i}.$$

*The bigger $k_i$ is, the larger the region $x$ around $x = 0$ it can 'cover', which can reflect on the level set of $V$ in which initial conditions will tend to the equilibrium. If the proof methodology that was presented here is used, the bound $\alpha_i < \frac{1}{k_i}$ is obtained; this means that the bigger $k_i$ is, the smaller the value of $\alpha_i$ for which stability can be proven. The maximal level set of the Lyapunov functional (giving a region of attraction) will be a function of the $k_i$.*

**Remark 5.8** *For single-link single-source we get*

$$\dot{p} = \left[ \frac{\overline{x}}{c} e^{-\frac{\alpha}{\tau} p(t-\tau)} - 1 \right]_p^+ .$$

*A simple change of variables $z(t) = \frac{\overline{x}}{c} e^{-\frac{\alpha}{\tau} p(t)} - 1$, ignoring the projection and rescaling of time results in the system gives:*

$$\dot{z} = -\alpha [z(t) + 1] z(t - 1) \tag{5.29}$$

*Equation (5.29) is* Hutchinson's Equation, *a well-known population dynamics model that models a single species striving for a common food. The delay represents the* *maturation of the population. The linearization of this system about the zero equilibrium is stable for* $\alpha < \pi/2 \simeq 1.57$*. In [103], E. M. Wright managed to prove global stability of the equilibrium of the nonlinear system for* $\alpha < 37/24 = 1.54$ *if the initial condition satisfies* $z(t+\theta) \geq -1,\ \theta \in [-1, 0)$ *with* $z(0) > -1$ *(which corresponds to a non-negative price p) by looking at the properties of the solution of (5.29). However, this result is difficult to scale for arbitrary population interactions – a similar problem to the arbitrary network topology case. Existence and uniqueness of solutions of (5.11) can be established in a similar way [38, 55].*

The above treatment may seem to have come from intuition, but in fact, it can be generalized to other congestion control algorithms. In the next section we consider the stability analysis of primal congestion control schemes in a delay-dependent fashion, complimentary to the delay-independent result obtained in [105].

## 5.4 Primal Congestion Control Schemes

In the case of primal congestion control schemes, the sources have dynamics, whereas the pricing function at the links is static. The optimization framework is used in this case too, to provide a means for designing the control laws. The condition for optimality from the resource allocation problem formulation is given by:

$$U_i'(x_i^*) - q_i^* = 0.$$

For the primal case, we consider a gradient control algorithm to drive the source dynamics to achieve this optimality condition. We therefore have the following link and source laws [34, 89]:

$$\begin{aligned} p_l &= f_l(y_l) \\ \dot{x}_i &= \kappa_i(x_i)(U_i'(x_i) - q_i) \end{aligned}$$

where $f_l$ is a strictly increasing function, such that $f_l > 0$, $f_l' > 0$. In particular, the following law has been proposed in [98]:

$$\begin{aligned} p_l(t) &= f_l(y_l(t)) \\ \dot{x}_i &= \kappa_i x_i(t-\tau_i)\left[1 - \frac{q_i}{U_i'(x_i)}\right] \end{aligned}$$

The closed loop dynamics for the whole network then become:

$$\dot{x}_i = \kappa_i x_i(t-\tau_i)\left[1 - \frac{\sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj} x_j(t-\tau_{i,l}^b-\tau_{j,l}^f)\right)}{U_i'(x_i)}\right] \tag{5.30}$$

The initial conditions for (5.30) are non-negative functions defined on $C([-h,0],\mathbb{R}^S)$, where $h = \max h_i$ and

$$h_i = \max_{\{j,l:R_{li}=R_{lj}=1\}} \{\tau_{i,l}^b + \tau_{j,l}^f\}. \tag{5.31}$$

### 5.4.1 Stability of the Linearization

The linearization of (5.30) about the equilibrium $x_i^*$ is:

$$\dot{x}_i = \frac{\kappa_i x_i^*}{q_i^*} U_i''(x_i^*) x_i - \frac{\kappa_i x_i^*}{q_i^*} \sum_{l=1}^{L} R_{li} p_l^{'*} \sum_{j=1}^{S} R_{lj} x_j(t-\tau_{i,l}^b-\tau_{j,l}^f) \tag{5.32}$$

where $q_i^* = U_i^{'-1}(x_i^*)$, $y_l^* = \sum_{l=1}^{L} R_{li} x_i^*$ and $p_l^{'*} = f_l'(y_l^*)$.

We have the following result, similar to the one in Theorem 5.2:

**Theorem 5.9** *System (5.32) is asymptotically stable if $R$ is full rank and*

$$\frac{\kappa_i \tau_i}{q_i^*} \sum_{l=1}^{L} R_{li} y_l^* p_l^{'*} < 1, \quad \forall\ i.$$

We note that the result in [98] has $\pi/2$ in the RHS, and is obtained using frequency domain methods.

**Proof.** Consider the following function:

$$V_1 = -\frac{1}{2}\sum_{i=1}^{S} U_i^{''}(x_i^*)x_i^2 + \frac{1}{2}\sum_{l=1}^{L} p_l^{'*}\left(\sum_{j=1}^{S} R_{lj}x_j(t)\right)^2$$

This is positive definite, as $R$ is full rank. The undelayed version of (5.32) is

$$\dot{x}_i = \kappa_i \frac{x_i^*}{q_i^*}\left[U_i^{''}(x_i^*)x_i - \sum_{l=1}^{L} R_{li}p_l^{'*}\sum_{j=1}^{S} R_{lj}x_j\right].$$

We have:

$$\dot{V}_1 = -\sum_{i=1}^{S}\frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i\dot{x}_{i,u} = -\sum_{i=1}^{S}\frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i^2 - \sum_{i=1}^{S}\frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i(\dot{x}_{i,u} - \dot{x}_i)$$

where $\dot{x}_{i,u}$ denotes the undelayed version. We now manipulate the second term as in the proof of Theorem 5.2 to get:

$$-\sum_{i=1}^{S}\frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i(\dot{x}_{i,u} - \dot{x}_i) = \sum_{i=1}^{S}\sum_{l=1}^{L} R_{li}p_l^{'*}\sum_{j=1}^{S} R_{lj}\int_{-\tau_{i,l}^b - \tau_{j,l}^f}^{0}\dot{x}_i\dot{x}_j(t+\theta)d\theta$$

where the Leibniz rule was used to distribute the delay over an interval. Using the inequality $ab \leq \frac{k}{2}a^2 + \frac{1}{2k}b^2$ for $k > 0$, we have:

$$\begin{aligned}
&-\sum_{i=1}^{S}\frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i(\dot{x}_{i,u} - \dot{x}_i) \\
&\leq \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S}\frac{R_{lj}R_{li}}{2}p_l^{'*}(\tau_{i,l}^b + \tau_{j,l}^f)k_{ij}\dot{x}_i^2 + \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S}\frac{R_{lj}R_{li}}{2k_{ij}}p_l^{'*}\int_{-\tau_{i,l}^b - \tau_{j,l}^f}^{0}\dot{x}_j^2(t+\theta)d\theta
\end{aligned}$$

where $k_{ij}$ are constants. Introduce now the following functional:

$$V_2 = \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S}\frac{R_{lj}R_{li}}{2k_{ij}}p_l^{'*}\int_{-\tau_{i,l}^b - \tau_{j,l}^f}^{0}\int_{t+\theta}^{t}\dot{x}_j^2(\zeta)d\zeta d\theta$$

This then satisfies:

$$\dot{V}_2 = \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}(\tau^b_{i,l}+\tau^f_{j,l})}{2k_{ij}} p_l^{'*}\dot{x}_j^2 - \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}}{2k_{ij}} p_l^{'*} \int_{-\tau^b_{i,l}-\tau^f_{j,l}}^{0} \dot{x}_j^2(t+\theta)d\theta$$

Now let $V = V_1 + V_2$, and $k_{ij} = \frac{x_j^*}{x_i^*}\sqrt{\frac{\tau_i}{\tau_j}}$. Then:

$$\dot{V} \leq -\sum_{i=1}^{S} \frac{q_i^*}{\kappa_i x_i^*}\dot{x}_i^2 + \sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}}{2} p_l^{'*}(\tau_i+\tau_j)\sqrt{\frac{\tau_i}{\tau_j}}\frac{x_j^*}{x_i^*}\dot{x}_i^2$$

Stability is guaranteed if $\dot{V} \leq 0$, i.e., if each $i$ satisfies:

$$\frac{\kappa_i}{q_i^*}\sum_{l=1}^{L}\sum_{j=1}^{S} R_{lj}R_{li}x_j^* p_l^{'*}\frac{(\tau_i+\tau_j)}{2}\sqrt{\frac{\tau_i}{\tau_j}} < 1$$

Since $(\tau_i + \tau_j) \geq 2\sqrt{\tau_i\tau_j}$, stability is also guaranteed by

$$\frac{\kappa_i\tau_i}{q_i^*}\sum_{l=1}^{L}\sum_{j=1}^{S} R_{lj}R_{li}x_j^* p_l^{'*} < 1. \tag{5.33}$$

In order to prove asymptotic stability, we use a LaSalle argument in the same way as it was done in the proof of Theorem 5.2. The set $S = \{\phi \in C^S : \dot{V}(\phi) = 0\}$ is the set

$$S = \left\{ \phi : \sum_{l=1}^{L} R_{li}p_l^{'*}\sum_{j=1}^{S} R_{lj}\phi_j(-\tau^b_{i,l}-\tau^f_{j,l}) = U_i^{''}(x_i^*)\phi_i, \quad i = 1,\ldots,S \right\}.$$

The largest set in $S$ that is invariant with respect to the system satisfies $\dot{x}_i = 0$ $\forall i = 1,\ldots,S$, i.e. $x_i = K$, a constant. But the only constant $\phi$ that is in $S$ is the zero equilibrium. Therefore, the equilibrium of the system is asymptotically stable by an extension of LaSalle's theorem (Theorem 5.3.1 in [28]) provided (5.33) holds and $R$ is full rank. ■

We remark that the proof is very similar to the one for Theorem (5.2), apart from two steps: the way the quadratic inequality $a^2 + b^2 \geq 2ab$ is used, and the last step where, in order to treat the delays $\tau_i$ and $\tau_j$, one has to use another inequality argument.

### 5.4.2 Nonlinear Stability Analysis

The nonlinear system is given by Equation (5.30), and the existence and uniqueness of solutions is assumed. To ensure stability for arbitrary topologies in the nonlinear case with delays, we have to use a Lyapunov based argument similar to the one we used in the linearization. The system to be analyzed is described by Equation (5.30). Further, consider the undelayed version of (5.30):

$$\dot{x}_i = \kappa_i x_i(t-\tau_i)\left[1 - \frac{\sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj} x_j(t)\right)}{U_i'(x_i)}\right] \tag{5.34}$$

Before we proceed, recall that $U_i(x_i)$ are strictly concave, non-decreasing functions and $f_l(y_l)$ are strictly increasing functions, such that $f_l > 0$, $f_l' > 0$. This inevitably means that $x_i$ are upper bounded, as shown in the following proposition:

**Proposition 5.10** *Let $x_i(t)$ be a solution of (5.30). Then there is a $T > 0$ such that, for $t \geq T$, $x_i(t) < U_i'^{-1}\left(\sum_{l=1}^{L} R_{li} f_l(0)\right) = \overline{x}_i$.*

**Proof.** First, note that

$$U_i'^{-1}\left(\sum_{l=1}^{L} R_{li} f_l(0)\right) > U_i'^{-1}\left(\sum_{l=1}^{L} R_{li} f_l(y_l^*)\right) = U_i'^{-1}(q_i^*) = x_i^*.$$

If $x_i(t) \geq x_i^*$ $\forall i$ for all large $t$, say $t \geq T$, then $\dot{x}_i \leq 0$ for $t \geq T$ and hence $\lim_{t\to\infty} x_i(t) = x_i^*$. Thus, we may assume $x_i(t)$ are oscillatory about $x_i^*$. Let $\bar{t} > 2h_i$, where $h_i$ is defined by (5.31) be such that $x_i(\bar{t}) > x_i^*$ and $\dot{x}_i(\bar{t}) = 0$, i.e., a maximum of the trajectory. Then we have:

$$U_i'(x_i(\bar{t})) = \sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj} x_j(\bar{t} - \tau_{i,l}^b - \tau_{j,l}^f)\right)$$

Therefore,

$$x_i(\bar{t}) = U_i'^{-1}\left(\sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj} x_j(\bar{t} - \tau_{i,l}^b - \tau_{j,l}^f)\right)\right) < U_i'^{-1}\left(\sum_{l=1}^{L} R_{li} f_l(0)\right).$$

This completes the proof. ■

Let $x_i(t) \leq \overline{x}_i$ where $\overline{x}_i$ is defined in the above proposition. Similarly, define $\overline{y}_l = \sum_{i=1}^{S} R_{li}\overline{x}_i$ so that $y_l(t) \leq \overline{y}_l$. Then we have:

**Theorem 5.11** *The non-zero equilibrium of (5.30) is globally (i.e., for $x_i > 0$) asymptotically stable, provided $R$ is full rank and*

$$\frac{\kappa_i \tau_i}{U_i'(\overline{x}_i)} \sum_{j=1}^{S} R_{li} \overline{y}_l f_l'(\overline{y}_l) < 1. \tag{5.35}$$

**Proof.** Consider a function of the form:

$$V_1 = -\sum_{i=1}^{S} \left[U_i(x_i) - U_i(x_i^*)\right] + \sum_{l=1}^{L} \int_{y_l^*}^{\sum_{j=1}^{S} R_{lj} x_j} f_l(Y)dY$$

This function is positive definite and radially unbounded. Then we have:

$$\begin{aligned}
\dot{V}_1 &= -\sum_{i=1}^{S}\left[U_i'(x_i) - \sum_{l=1}^{L} R_{li} f_l\left(\sum_{i=1}^{S} R_{lj} x_j\right)\right]\dot{x}_i \\
&= -\sum_{i=1}^{S} \frac{U_i'(x_i)}{\kappa_i x_i(t-\tau)}\dot{x}_{i,u}\dot{x}_i = -\sum_{i=1}^{S} \frac{U_i'(x_i)}{\kappa_i x_i(t-\tau)}\left(\dot{x}_i^2 + \dot{x}_i(\dot{x}_{i,u} - \dot{x}_i)\right)
\end{aligned}$$

where (5.34) is the undelayed version of Equation (5.30). Now

$$\begin{aligned}
-\sum_{i=1}^{S} \frac{U_i'(x_i)}{\kappa_i x_i(t-\tau)}\dot{x}_i(\dot{x}_{i,u} - \dot{x}_i) &= \sum_{i=1}^{S}\sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj} x_j(t)\right)\dot{x}_i \\
&\quad - \sum_{i=1}^{S}\sum_{l=1}^{L} R_{li} f_l\left(\sum_{j=1}^{S} R_{lj} x_j(t - \tau_{i,l}^b - \tau_{j,l}^f)\right)\dot{x}_i \\
&\leq \sum_{i=1}^{S}\sum_{j=1}^{S}\sum_{l=1}^{L} R_{li} R_{lj} f_l'(\overline{y}_l) \int_{-\tau_{i,l}^b - \tau_{j,l}^f}^{0} |\dot{x}_i||\dot{x}_j(t+\theta)|d\theta
\end{aligned}$$

as $f_l$ is globally Lipschitz continuous and strictly increasing. The rest of the proof is the same as in the linear case. Since $x_i < \overline{x}_i$, we have:

$$U_i'(x_i) > U_i'(\overline{x}_i)$$

from the strict concavity of $U_i$ and therefore $\dot{V}$ can be written as follows:

$$\begin{aligned}\dot{V} &\leq -\sum_{i=1}^{S} \frac{U_i'(x_i)}{\kappa_i x_i(t-\tau)} \dot{x}_i^2 + \sum_{i=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}}{2} f_l'(\overline{y}_l)(\tau_i+\tau_j)\sqrt{\frac{\tau_i}{\tau_j}\frac{\overline{x}_j}{\overline{x}_i}}\dot{x}_i^2 \\ &\leq -\sum_{i=1}^{S} \frac{U_i'(\overline{x}_i)}{\kappa_i \overline{x}_i} \dot{x}_i^2 + \sum_{l=1}^{L}\sum_{j=1}^{S} \frac{R_{lj}R_{li}}{2} f_l'(\overline{y}_l)(\tau_i+\tau_j)\sqrt{\frac{\tau_i}{\tau_j}\frac{\overline{x}_j}{\overline{x}_i}}\dot{x}_i^2\end{aligned}$$

Therefore, stability is retained if

$$\frac{\kappa_i \tau_i}{U_i'(\overline{x}_i)} \sum_{j=1}^{S} R_{li}\overline{y}_l f_l'(\overline{y}_l) < 1.$$

Asymptotic stability follows from LaSalle's argument in a similar way as in the proof of Theorem 5.5. Hence, the system is globally asymptotically stable for arbitrary networks and delays provided (5.35) is satisfied and $R$ is full rank. ■

## 5.5 A Primal-Dual Congestion Control Scheme

The above 'manual' constructions of scalable proofs for arbitrary networks have taken advantage of the structure of the dynamics to guide the choice of the Lyapunov certificate. In some cases, this is not possible because the dynamics are designed in an ad-hoc way.

The two congestion control schemes that we analyzed in this chapter have disadvantages. The main drawback of the dual control law is that it puts a restriction on the sources' demand curves, as the source law is *static*. Moreover, a pure primal congestion control scheme has no control over the pricing functions that can be used.

A congestion control scheme that has dynamics, both at the sources and links could alleviate this problem. Such a scheme is called primal-dual, and was developed in [57]. Apart from $q_i$, $y_l$ and $p_l$ given by Equations (5.3), (5.2) and (5.9) respectively, we have:

$$x_i(t) = x_{m,i} e^{\xi_i} e^{\frac{-\alpha_i q_i(t)}{M_i \tau_i}} \tag{5.36}$$

$$\dot{\xi}_i(t) = \frac{\beta_i}{\tau_i}[U_i'(x_i(t)) - q_i(t)] \tag{5.37}$$

where $U_i(x_i)$ is the utility function of source $i$ and $\beta_i$ is a parameter. The following result on the stability of the linearized system is known:

**Theorem 5.12** *[57] Assume that for every source $i$, $\tau_i \leq \tau$. Then the system described by Equations (5.1–5.4), (5.9) and (5.36–5.37), with $\alpha_i < \pi/2$ and $z = \frac{\beta_i M_i}{\alpha_i} = \frac{\eta}{\tau}$ for $\eta \in (0,1)$ small enough depending on $\alpha \geq \alpha_i$, the closed loop system is linearly stable.*

A Lyapunov argument that will scale with the system size may be difficult in this case, as the parameters $\beta_i$ are tuned so as to provide for slow adaptation of the source to its own utility function characteristic. It is fortunate that the tools that we have developed in the previous chapter that use the sum of squares decomposition can be used instead, but only for small network topologies [60]. We note that the system equations are not polynomial in the variables, but if we write the system dynamics in other system variables, we obtain the following description:

$$\begin{aligned}
\dot{x}_i(t) &= x_i \left( \frac{\beta_i}{\tau_i}[U_i'(x_i(t)) - q_i(t)] - \frac{\alpha_i}{M_i \tau_i} \dot{q}_i(t) \right) \\
\dot{q}_i(t) &= \sum_l \frac{R_{li}}{c_l} \left( \sum_j R_{lj} x_j(t - \tau_{j,l}^f - \tau_{i,l}^b) - c_l \right)
\end{aligned}$$

where we have ignored the projection nonlinearity.

This formulation is suitable if the one-way delay is available, although in many special cases, the RTT appears naturally in the delay combinations. We also put a bound on the delay size so that the RTT delay is equal to $\tau$, and we assume the overbound on the one-way delay to be half of the overbound on the RTT. We choose

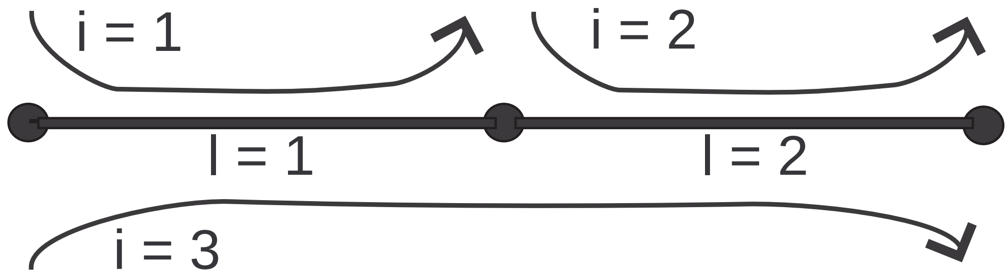



Figure 5.2: A simple network.

$U_i(x_i(t)) = K\log(x_i(t))$, and we analyze a specific topology network that is shown in Figure 5.2. In this case, $R$ is given by

$$R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We assume $\tau_{ij} \leq \tau/2$, where $\tau$ is the overbound on the RTT. We let $\beta_1 = \beta_2 = \beta_3/2 = \beta$, all the $c_l = c$ and $\alpha_i = \alpha$. Denote $\tilde{K}_1 = \frac{K_1+K_2}{K_1+K_2+K_3}$ and $\tilde{K}_2 = \frac{K_3}{K_1+K_2+K_3}$. The equilibrium for this system is

$$(x_{1,0}, x_{2,0}, x_{3,0}, q_{1,0}, q_{2,0}) = \left(c\tilde{K}_1, c\tilde{K}_1, c\tilde{K}_2, \frac{K_1}{c\tilde{K}_1}, \frac{K_2}{c\tilde{K}_1}\right) \tag{5.38}$$

In order to avoid numerical ill-conditioning, we rename variables as follows:

$$z_1 = \frac{1}{c\tilde{K}_1}x_1 - 1, \qquad z_2 = \frac{1}{c\tilde{K}_1}x_2 - 1$$
$$z_3 = \frac{1}{c\tilde{K}_2}x_3 - 1, \qquad z_4 = \frac{c\tilde{K}_1}{K_1}q_1 - 1$$
$$z_5 = \frac{c\tilde{K}_1}{K_2}q_2 - 1$$

This change of variables gives the equations:

$$\begin{aligned}
\dot{z}_1(t) &= -\frac{K_1\beta_1}{\tilde{K}_1 c\tau}[z_1(t)+z_4(t)+z_1(t)z_4(t)] - \frac{\alpha}{\tau}[z_1(t)+1][\tilde{K}_2 z_3(t-\tau)+\tilde{K}_1 z_1(t-\tau)] \\
\dot{z}_2(t) &= -\frac{K_2\beta_2}{\tilde{K}_1 c\tau}[z_2(t)+z_5(t)+z_2(t)z_5(t)] - \frac{\alpha}{\tau}[z_2(t)+1][\tilde{K}_2 z_3(t-\tau)+\tilde{K}_1 z_2(t-\tau)] \\
\dot{z}_3(t) &= -\frac{\beta_3}{\tilde{K}_1 c\tau}[(z_3 z_4 + z_3 + z_4)K_1 + (z_3 z_5 + z_3 + z_5)K_2] \\
&\quad - \frac{\alpha}{2\tau}[z_3(t)+1][z_1(t-\tau)\tilde{K}_1 + z_2(t-\tau)\tilde{K}_1 + 2z_3(t-\tau)\tilde{K}_2] \\
\dot{z}_4(t) &= \frac{\tilde{K}_1 c}{K_1}(\tilde{K}_1 z_1(t-\tau)+\tilde{K}_2 z_3(t-\tau)) \\
\dot{z}_5(t) &= \frac{\tilde{K}_1 c}{K_2}((K_1+K_2)z_2(t-\tau)+K_3 z_3(t-\tau))
\end{aligned}$$

We use $c = 40$, $\alpha = 1$, $\tau = 0.2$, and we calculate $\beta = \frac{0.64\alpha}{\tau M_i}$ and we let $K_1 = 15$, $K_2 = 20$, $K_3 = 25$. Using the technique developed in Chapter 4 based on the sum of squares decomposition, we can construct a Lyapunov functional of the form

$$\begin{aligned}
V_1(z_t) &= a_0(z_1(t), z_2(t), z_3(t), z_4(t)) \\
&\quad + \int_{-\tau}^{0} a_1(z_1(t), z_1(t+\theta), z_2(t), z_2(t+\theta), z_3(t), z_3(t+\theta), z_4(t), z_5(t))d\theta \\
&\quad + \int_{-\tau}^{0}\int_{t+\theta}^{t} a_2(z_1(\zeta), z_2(\zeta), z_3(\zeta))d\zeta d\theta,
\end{aligned}$$

with polynomials $a_0$, $a_1$ of second order and $a_2$ of order four for

$$\begin{aligned}
&0 < x_{1_t} \leq 2.3x_{1,0}, \quad 0 < x_{2_t} \leq 2.3x_{2,0}, \\
&0 < x_{3_t} \leq 2.3x_{3,0}, \quad q_1 > 0, \quad q_2 > 0.
\end{aligned}$$

This proves that the network shown in Figure 5.2 with the aforementioned congestion control algorithm is stable for $\tau = 0.2$.

## 5.6 Conclusion

In this chapter, we have presented a methodology to construct Lyapunov certificates for different congestion control algorithms to verify their stability properties for arbitrary network topologies. The nonlinear conditions are delay dependent and of the same form as the ones for the linearized system.

The structure of the system has helped greatly in obtaining simple scalable stability conditions. More complicated functional structures can be less conservative, at the expense of being more complicated; this has been observed when constructing these functionals algorithmically using LMIs in the linear case, but also in the nonlinear case using SOSTOOLS for simple network topologies.

# Chapter 6

# Systems Described by Partial Differential Equations

Τά πάντα ρεῖ καὶ ουδέν μένει

Ἡράκλειτος

Everything flows, nothing abides

Heracletus

In the previous two chapters, we were primarily concerned with the stability analysis of small-scale and large-scale infinite dimensional systems described by Functional Differential Equations. A natural question to ask is whether the proposed algorithmic methodology, which is based on the sum of squares decomposition, can be used for the algorithmic analysis of systems described by Partial Differential Equations (PDEs). Such system descriptions come about naturally in e.g. fluid dynamics and heat transfer, areas in which the spatial dimension of the problem cannot be ignored.

We begin this chapter with a brief presentation of some background material on the stability analysis of systems described by PDEs. We then consider a problem of particular interest from fluid mechanics, that of stability of shear flows. In particular, we prove that the Navier-Stokes equations in cylindrical coordinates subject to axially constant initial conditions and perturbations are globally stable for all Reynolds numbers $R$. We note, however, that the transient energy growth of the system scales as $R^3$. Therefore even though streamlined flows can be made stable for arbitrary Reynolds numbers, the system becomes susceptible to the effect of disturbances and uncertainties.

We then describe briefly how the analysis of parabolic PDE systems can be performed algorithmically using the sum of squares decomposition of multivariate polynomials.

## 6.1 Introduction

The analysis of systems described by Partial Differential Equations has been considered to be a very difficult task, not only because of their infinite dimensional nature [71, 30, 2], but also because of the complicated and sometimes mystifying underlying theory. A comprehensive discussion on the methods for analysis of linear PDEs can be found in [15]. There are also some results, mostly on a case-by-case basis, on nonlinear PDE systems.

In this section we will present background material on the stability properties of distributed parameter systems, and in the later sections we will investigate the stability of systems arising in fluid mechanics and heat transfer.

First, we define the notion of a dynamical system (nonlinear semigroup) on a complete metric space $C$ with metric 'dist', as the family of maps $\{S(t) : C \to C, t \geq 0\}$ such that:

- For each $t \geq 0$, $S(t)$ is continuous from $C$ to $C$,
- For each $x \in C$, $t \to S(t)x$ is continuous,
- $S(0) = I$ on $C$,
- $S(t)(S(\tau)x) = S(t+\tau)x$ for all $x \in C$ and $\tau, t \geq 0$.

Such a semigroup defines a dynamical system on $C$. For any $x \in C$, let $\gamma(x) = \{S(t)x, t \geq 0\}$ denote the *orbit* (or positive semi-orbit) through $x$. We say $x$ is an *equilibrium point* if $\gamma(x) = \{x\}$. $\gamma(x)$ is a *periodic orbit* if there exists a $p > 0$ such that $\gamma(x) = \{S(t)x, 0 \leq t \leq p\} \neq \{x\}$. An orbit $\gamma(x)$ (or sometimes, the point $x$) is *stable* if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all $t \geq 0$,

$\text{dist}(S(t)x, S(t)y) < \epsilon$ whenever $\text{dist}(x, y) \leq \delta(\epsilon)$, $y \in C$. An orbit $\gamma(x)$ is *unstable* if it is not stable. An orbit $\gamma(x)$ is *uniformly asymptotically stable* if it is stable and also there is a neighbourhood $B = \{y \in C : \text{dist}(x, y) < r\}$ such that

$$\text{dist}(S(t)y, S(t)x) \to 0 \text{ as } t \to \infty$$

uniformly for $y \in B$.

Similarly to the case of ODEs and FDEs, a Lyapunov function can be used to verify stability:

**Definition 6.1** *Let $\{S(t), t \geq 0\}$ be a dynamical system on $C$. A* Lyapunov function *is a continuous real-valued function $V$ on $C$ such that*

$$\dot{V}(x) \triangleq \overline{\lim}_{t \to 0^+} \frac{1}{t} \{V(S(t)x) - V(x)\} \leq 0$$

*for all $x \in C$, not excluding the possibility $\dot{V}(x) = -\infty$. Henceforth we use the notation $\|x - y\| = dist\{x, y\}$, as we will be use the induced topology of some Banach space.*

Having defined what a Lyapunov function is, we now state Lyapunov's theorem for this type of systems:

**Theorem 6.2** *Let $\{S(t), t \geq 0\}$ be a dynamical system on $C$, and let $0$ be an equilibrium point in $C$. Suppose $V$ is a Lyapunov function which satisfies $V(0) = 0$, $V(x) \geq c(\|x\|)$ for $x \in C$, $\|x\| = dist\{x, 0\}$, where $c(\cdot)$ is a continuous strictly increasing function, $c(0) = 0$ and $c(r) > 0$ for $r > 0$. Then $0$ is stable. If in addition $\dot{V}(x) \leq -c_1(\|x\|)$ where $c_1(\cdot)$ is also continuous, increasing and positive with $c_1(0) = 0$, then $0$ is uniformly asymptotically stable.*

Also, in the same way as for ODEs and FDEs, we can define invariant sets, $\omega$-limit sets and state a LaSalle-type theorem. Although there is a converse theorem, finding a Lyapunov function in any particular case is usually a difficult task.

With these definitions in place, we now turn our attention to examples from fluid mechanics and heat transfer. In the next section, we consider a Hagen-Poiseuille pipe flow with axially constant perturbations and initial conditions, for which we show that the laminar flow is globally stable independent of the Reynolds number. We demonstrate nonetheless, that the transient energy amplification of the background disturbances scales as $R^3$. This means that for this model there are no bifurcations to instability, however the large amplification of the background disturbances which was also observed in the linearization is still present. Therefore streamlining the flow has as a consequence an increase in fragility to disturbances and uncertainties.

## 6.2 Global Stability of Axially Constant Perturbations in Hagen-Poiseuille Flow

Hydrodynamic stability is an old research area and came about when the famous pipe experiments performed by Reynolds [79] and his observations on the transition of viscous fluid flow from laminar to turbulent were later on investigated theoretically by Orr [52] and Sommerfeld [88]. More than a century later, these observations still pose one of the most intriguing problems in fluid mechanics: what is the mechanism for transition? In his original paper, Reynolds made an important observation: there is no single 'critical' value below which flow is laminar and above which it becomes turbulent. In his own words:

> ...the critical velocity was very sensitive to disturbance in the water before entering the tubes. This at once suggested the idea that the condition might be one of instability for disturbances of a certain magnitude and stability for smaller disturbances.

In fact, by making sure that the disturbances were minimized, Reynolds was able to keep the flow laminar up to values of Reynolds number $R$ approaching 13000. Subsequently, even more refined experiments have pushed this figure to $R > 10^5$, and all theoretical evidence [83] suggests that fully developed flow down the pipe is stable to disturbances at any finite value of $R$, no matter how large. If no great care is taken to minimize disturbances, transition typically occurs at $R \sim 2000$.

The experimental observation comes to contradict the theoretical explanation for Hagen-Poiseuille flow in a pipe, something that is not experienced for the Rayleigh-Benard convection and Taylor-Couette flow. Classical stability and bifurcation theory could not explain the same phenomenon in plane Couette flow either, i.e. flow between two parallel plates: the solution of the Orr-Sommerfeld equation predicts stability for all Reynolds numbers, which disagrees strongly with the experimental observation. The disagreement between the experiment and linearized theory is usually attributed to nonlinear effects, but recently a new explanation has been asserted.

In a series of papers [95, 1, 22] and in [84] it has been shown that for shear flows, the operator of the linearized Navier-Stokes is non-normal. This means that even for a stable flow, transient disturbances can achieve very large values before eventually decaying, suggesting a mechanism for transition at Reynolds numbers significantly smaller than those predicted by the stability of the Orr-Sommerfeld equation. The large background energy amplification is due to the non-normality of the Orr-Sommerfeld operator, and the corresponding non-orthogonality of its eigenfunctions.

In the case of pipe flow, the mathematical analysis has revealed that the Hagen-Poiseuille laminar flow profile is linearly stable with respect to disturbances for all Reynolds numbers [83]. In particular, it has been demonstrated that maximum background energy amplification is obtained for axially-constant perturbations with an azimuthal wave number of $m = 1$. The initial form of the perturbations is that of streamwise rolls. After a short time, the streamwise velocity component becomes dominant. The perturbations take the form of spanwise modulation of the basic flow leading to the formation of streaks which were observed in other shear flows.

In this section, we concentrate on Hagen-Poiseuille flow in a pipe with axially constant 3-D perturbations. The flow is driven by a constant pressure gradient. The fact that the perturbations are axially constant restricts the model to 2-D with 3 velocity components, which we will call a 2D/3C model. This is in line with the observations that the most energy amplifying disturbances are axially constant and produce axial structures. We show that the laminar solution of the 2D/3C model is globally stable for all Reynolds numbers $R$ by exhibiting a Lyapunov function.

The transient energy growth scales as $R^3$, which agrees with the result from the linearization. A similar result has been obtained for Couette flow [8].

### 6.2.1 The Equations of Motion

We begin our investigation by reformulating the Navier-Stokes equations in cylindrical coordinates into the 2D/3C system of interest. Denote the radial, azimuthal and axial coordinates by $(r, \theta, z)$ and the corresponding velocity components of $v$ by $(v_r, v_\theta, v_z)$. In these coordinates the convective time derivative is:

$$v \cdot \nabla = v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z},$$

and the Laplacian operator is

$$\Delta = \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

The Navier-Stokes equations of motion in these coordinates become:

$$\begin{aligned}
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} &= -\frac{\partial p}{\partial r} + \frac{1}{R} \left( \Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right) \\
\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_\theta v_r}{r} &= -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{R} \left( \Delta v_\theta + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2} \right) \\
\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} &= -\frac{\partial p}{\partial z} + \frac{1}{R} \Delta v_z \\
\frac{1}{r} \frac{\partial (r v_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} &= 0,
\end{aligned}$$

along with boundary conditions for a pipe of radius $r = 1$ and infinite length:

$$\begin{aligned}
&u_z(1, \theta, z, t) = 0, \quad u_z(0, \theta, z, t) = \frac{-dp/dz}{R}, \\
&u_r(1, \theta, z, t) = 0, \quad u_r(0, \theta, z, t) = 0, \\
&u_\theta(1, \theta, z, t) = 0, \quad u_\theta(0, \theta, z, t) = 0.
\end{aligned}$$

In this section we are interested in axially-constant solutions to the above equations; this is ensured if the initial conditions are constant in $z$, $\frac{\partial u}{\partial z} = 0$, and $\frac{\partial p}{\partial z} = \frac{dp}{dz} = c$, a constant:

$$\begin{aligned}
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} &= -\frac{\partial p}{\partial r} + \frac{1}{R}\left(\Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) && (6.1)\\
\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{R}\left(\Delta v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right) && (6.2)\\
\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} &= -\frac{dp}{dz} + \frac{1}{R}\Delta v_z && (6.3)\\
\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} &= 0. && (6.4)
\end{aligned}$$

The above system represents the system evolution in 2D (i.e. $r$ and $\theta$) with three flow components $v_r, v_\theta, v_z$. In order to restrict the evolution of the system on the divergence-free field, we use a streamfunction formulation. In cylindrical coordinates, the streamfunction $\psi$ satisfies:

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{\partial \psi}{\partial r}$$

After some calculations we can rewrite the system equations in $\psi$ and $v_z$:

$$\begin{aligned}
\frac{\partial \Delta\psi}{\partial t} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial \Delta\psi}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \Delta\psi}{\partial \theta} &= \frac{1}{R}\Delta^2\psi && (6.5)\\
\frac{\partial v_z}{\partial t} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial v_z}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial v_z}{\partial \theta} &= -\frac{dp}{dz} + \frac{1}{R}\Delta v_z && (6.6)
\end{aligned}$$

Note that the system is in triangular form, and Equations (6.5–6.6) completely characterize the system evolution. The boundary conditions are:

$$\begin{gathered}
u_z(1,\theta,t) = 0, \quad u_z(0,\theta,t) = \frac{-dp/dz}{R},\\
\frac{\partial \psi}{\partial r}(1,\theta,t) = 0, \quad \frac{\partial \psi}{\partial \theta}(1,\theta,t) = 0,\\
\frac{\partial \psi}{\partial r}(0,\theta,t) = 0, \quad \frac{\partial \psi}{\partial \theta}(0,\theta,t) = 0
\end{gathered}$$

where $dp/dz$ is the pressure drop driving the flow which is constant along the axial direction, as otherwise $v_z$ would be a function of $z$. We further denote $\tau = t/R$, $\Psi(\tau) = R\psi(\tau R)$ and

$$V(\tau) = \frac{Rv_z(\tau R)}{-dp/dz}.$$

Multiplying (6.5) by $R^2$ and (6.6) by $R$, and scaling time by $\frac{1}{R}$ we get:

$$\Delta\Psi_\tau = -\frac{1}{r}\Psi_\theta\Delta\Psi_r + \frac{1}{r}\Psi_r(\Delta\Psi)_\theta + \Delta^2\Psi \quad (6.7)$$

$$V_\tau = -\frac{1}{r}\Psi_\theta V_r + \frac{1}{r}\Psi_r V_\theta + \Delta V + 1 \quad (6.8)$$

where from now on we denote $\Psi_\tau \triangleq \frac{\partial\Psi}{\partial\tau}$, etc. The new boundary conditions are

$$V(1,\theta,\tau R) = 0, \quad V(0,\theta,\tau R) = 1,$$
$$\Psi_r(1,\theta,\tau R) = 0, \quad \Psi_\theta(1,\theta,\tau R) = 0,$$
$$\Psi_r(0,\theta,\tau R) = 0, \quad \Psi_\theta(0,\theta,\tau R) = 0$$

Note that these equations are independent of $R$, and therefore the stability properties of the system, which will be of interest later on, are independent of the Reynolds number.

Before moving on, we present a solution to the above equations that corresponds to $v_r = v_\theta = 0$. We have, from the definition of the $\Psi$ equations, that $\Psi(r,\theta) = \tilde{c}$, a constant. Therefore the stationary solution for $V$ becomes:

$$\Delta\overline{V} = -1. \quad (6.9)$$

Of course $\overline{V} = \overline{V}(r,\theta)$ and is not a function of $z$, a result following from the continuity equation. For the *fully developed* flow, we have, from the momentum equations:

$$0 = -\frac{dp}{dr} = -\frac{dp}{d\theta}$$

$$0 = -\frac{dp}{dz} + \frac{1}{R}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\overline{V}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\overline{V}}{\partial\theta^2}\right)$$

Since $\overline{V}$ does not vary with $z$, it follows that $\frac{dp}{dz}$ must be constant, as mentioned earlier. Taking symmetry into consideration, we have also that $\frac{\partial \overline{V}}{\partial \theta} = 0$. Solving the resulting elliptic equation and applying the boundary condition we get as the steady-state solution:

$$\overline{V} = \frac{1}{4}(1 - r^2)$$

Therefore the steady state of the fully developed laminar flow is a paraboloid.

### 6.2.2 Global Stability Analysis

The energy of the 2-D system describing the $\Psi$ evolution is:

$$\begin{aligned} E_\Psi &= \frac{1}{2}\pi \int_0^1 \int_{-\pi}^{\pi} \left(v_r^2 + v_\theta^2\right) r dr d\theta = \frac{1}{2}\pi \int_0^1 \int_{-\pi}^{\pi} \left(\frac{1}{r^2}\Psi_\theta^2 + \Psi_r^2\right) r dr d\theta \\ &= -\frac{1}{2}\langle \Psi, \Delta\Psi \rangle, \end{aligned}$$

where $\Delta$ is the Laplacian operator in cylindrical coordinates and the inner product is given by:

$$\langle f, g \rangle \triangleq \pi \int_0^1 \int_{-\pi}^{\pi} f(r,\theta) g(r,\theta) r dr d\theta \tag{6.10}$$

Now taking the derivative with respect to $\tau$ we have:

$$\begin{aligned} \frac{dE_\Psi}{d\tau} &= -\frac{1}{2}\frac{d}{d\tau}\langle \Psi, \Delta\Psi \rangle = -\frac{1}{2}\langle \Psi_\tau, \Delta\Psi \rangle - \frac{1}{2}\langle \Psi, \Delta\Psi_\tau \rangle = -\langle \Psi, \Delta\Psi_\tau \rangle \\ &= -\left\langle \Psi, -\frac{1}{r}\Psi_\theta(\Delta\Psi)_r + \frac{1}{r}\Psi_r(\Delta\Psi)_\theta \right\rangle - \left\langle \Psi, \Delta^2\Psi \right\rangle \\ &= -\left\langle \Psi, \Delta^2\Psi \right\rangle = -\langle \Delta\Psi, \Delta\Psi \rangle < 0 \end{aligned}$$

where the last equality follows from a careful integration by parts. The term $< \Psi, -\frac{1}{r}\Psi_\theta(\Delta\Psi)_r + \frac{1}{r}\Psi_r(\Delta\Psi)_\theta >$ is identically zero:

$$\begin{aligned} \left\langle \Psi, \frac{1}{r}\Psi_\theta(\Delta\Psi)_r - \frac{1}{r}\Psi_r(\Delta\Psi)_\theta \right\rangle &= \pi \int_0^1 \int_{-\pi}^{\pi} \Psi \left(\Psi_\theta(\Delta\Psi)_r - \Psi_r(\Delta\Psi)_\theta\right) dr d\theta \\ &= \pi \int_0^1 \int_{-\pi}^{\pi} \left(\frac{d}{dr}\left(\Psi\Psi_\theta\Delta\Psi\right) - \frac{d}{d\theta}\left(\Psi\Psi_r\Delta\Psi\right)\right) dr d\theta \\ &= 0 \end{aligned}$$

which follows by integration by parts and considering the boundary conditions for $\Psi$. This asserts that $V$ is a Lyapunov function, and so global $L_2$ stability of the $\Psi = 0$ solution is proven.

To investigate the stability of the overall system, consider (6.8) with the steady-state $\overline{V}$ removed. Abusing notation, we denote this by $V$:

$$V_\tau = -\frac{1}{r}\Psi_\theta V_r + \frac{1}{r}\Psi_r V_\theta + \Delta V - \frac{1}{r}\overline{V}_r \Psi_\theta$$

with zero boundary conditions. Consider now the kinetic energy:

$$E_V = \frac{1}{2}\pi \int_0^1 \int_{-\pi}^{\pi} V^2 r dr d\theta = \frac{1}{2}\langle V, V\rangle = \frac{1}{2}\|V\|^2$$

with the same inner product as before. The rate of change of $E_V$ with time is:

$$\begin{aligned}
\frac{dE_V}{d\tau} &= \langle V_\tau, V\rangle = \left\langle -\frac{1}{r}\Psi_\theta V_r + \frac{1}{r}\Psi_r V_\theta, V\right\rangle + \left\langle \Delta V - \frac{1}{r}\overline{V}_r\Psi_\theta, V\right\rangle \\
&= \left\langle \Delta V - \frac{1}{r}\overline{V}_r\Psi_\theta, V\right\rangle \\
&\leq \overline{\lambda}(\Delta)\|V\|^2 + \left\|\frac{1}{r}\overline{V}_r\Psi_\theta\right\| \|V\| = \left(\overline{\lambda}(\Delta)\|V\| + \left\|\frac{1}{r}\overline{V}_r\Psi_\theta\right\|\right)\|V\| \\
&\leq \left(\overline{\lambda}(\Delta)\|V\| + \left\|\frac{1}{r}\overline{V}_r\right\|_\infty \|\Psi_\theta\|\right)\|V\|
\end{aligned}$$

where $\overline{\lambda}(\Delta)$ is the maximum eigenvalue of $\Delta$, which is a negative definite operator, i.e. $\overline{\lambda}(\Delta) < 0$. Here we have used that $\left\langle -\frac{1}{r}\Psi_\theta V_r + \frac{1}{r}\Psi_r V_\theta\right\rangle$ is zero identically:

$$\begin{aligned}
\left\langle -\frac{1}{r}\Psi_\theta V_r + \frac{1}{r}\Psi_r V_\theta\right\rangle &= \pi\int_0^1\int_{-\pi}^{\pi} V\left(-\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\frac{\partial V}{\partial r} + \frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{\partial V}{\partial\theta}\right) r dr d\theta \\
&= \pi\int_0^1\int_{-\pi}^{\pi}\left(-V\frac{\partial\Psi}{\partial\theta}\frac{\partial V}{\partial r} + V\frac{\partial\Psi}{\partial r}\frac{\partial V}{\partial\theta}\right) dr d\theta \\
&= \pi\int_0^1\int_{-\pi}^{\pi}\left(-\frac{d}{d\theta}(VV_r\Psi) + \frac{d}{dr}(VV_\theta\Psi)\right) dr d\theta \\
&= 0
\end{aligned}$$

This follows by integration by parts and considering the boundary conditions for $\Psi$

and $V$. Therefore whenever $\|V\|$ is such that:

$$\|V\| \geq -\frac{\left\|\frac{1}{r}\overline{V}_r\right\|_\infty \|\Psi_\theta\|}{\overline{\lambda}(\Delta)} = k\|\Psi_\theta\|$$

where $k$ is a constant, then $E_V$ is non-increasing. We see that $\frac{dE_V}{d\tau}$ does not decrease monotonically to zero, but it may increase before eventually decreasing. However $\|\Psi_\theta\| \to 0$ as time evolves as shown earlier, and therefore $\|V\| \to 0$ eventually. In order to make this argument more concrete, we now show using a Lyapunov argument that a modified energy-based function can be used as a Lyapunov function for the whole system. For this purpose, consider the following functional:

$$W_\alpha(\Psi, V) = \alpha^2 E_\Psi + E_V$$

Then from our earlier calculations we have:

$$\begin{aligned} \frac{d}{d\tau}W_\alpha &= -\alpha^2 \left\langle \Delta^2\Psi, \Psi \right\rangle + \langle \Delta V, V\rangle - \left\langle \frac{1}{r}\overline{V}_r\Psi_\theta, V \right\rangle \\ &\leq \alpha^2\overline{\lambda}(\Delta)(\|\Psi_\theta\|^2 + \|\Psi_r\|^2) + \overline{\lambda}(\Delta)\|V\|^2 + \left\|\frac{1}{r}\overline{V}_r\right\|_\infty \|\Psi_\theta\|\|V\| \end{aligned}$$

Now $-\Delta$ is a strictly coercive operator, and therefore $\overline{\lambda}(\Delta) = -\beta^2 < 0$. We now write

$$\frac{d}{d\tau}W_\alpha \leq -\alpha^2\beta^2\|\Psi_r\|^2 - \beta^2\left(\alpha\|\Psi_\theta\| - \|V\|\right)^2 - 2\beta^2\alpha\|\Psi_\theta\|\|V\| + \left\|\frac{1}{r}\overline{V}_r\right\|_\infty \|\Psi_\theta\|\|V\|$$

Which is guaranteed to be negative definite if $2\beta^2\alpha - \left\|\frac{1}{r}\overline{V}_r\right\|_\infty > 0$, i.e.

$$\alpha > -\frac{\left\|\frac{1}{r}\overline{V}_r\right\|_\infty}{2\overline{\lambda}(\Delta)}.$$

This guarantees that $W_\alpha$ is a Lyapunov function and the equilibrium of the system, i.e. the laminar flow profile $\overline{V}$ is globally asymptotically stable.

### 6.2.3 Energy Scaling

Consider now the total transient energy growth $\mathcal{E}$:

$$\mathcal{E} \triangleq \mathcal{E}_\psi + \mathcal{E}_v = \int_0^\infty E_\psi + E_v dt$$

where $v = v_z - \overline{v}$, and $\overline{v}$ denotes the laminar flow profile. From the nondimensionalization we see that $E_\psi(0) = \frac{1}{R^2} E_\Psi(0)$, and $\int_0^\infty E_\psi(t)dt = \frac{1}{R}\int_0^\infty E_\Psi(\tau)d\tau$. Similarly, $E_V(0) = E_v(0)$, and $\int_0^\infty E_v(t)dt = R\int_0^\infty E_V(\tau)d\tau$.

Consider the case in which $V(0) = 0$. The ratio of output to input energy is:

$$g(\Psi) \triangleq \frac{\int_0^\infty E_V(\tau)d\tau}{\int_0^\infty E_\Psi(\tau)d\tau}$$

This has a finite upper bound as the numerator is bounded, for $\Psi \neq 0$. Because of the triangular structure of the equations of motion, $g(\Psi)$ is only a function of the initial condition in $\Psi$, i.e. $\Psi(0)$. It also satisfies $g(\Psi(0)) = g(\psi(0))$. Therefore we have:

$$\mathcal{E}_v = R\int_0^\infty E_V(\tau)d\tau = Rg(\Psi(0))\int_0^\infty E_\Psi(\tau)d\tau = Rg(\psi(0))\int_0^\infty E_\Psi(\tau)d\tau$$

Moreover we have:

$$\frac{\mathcal{E}_\psi}{E_\psi(0)} = \frac{\frac{1}{R}\int_0^\infty E_\Psi(\tau)d\tau}{\frac{1}{R^2}E_\Psi(0)} = R\frac{\int_0^\infty E_\Psi(\tau)d\tau}{E_\Psi(0)}$$

Similarly:

$$\frac{\mathcal{E}_v}{E_\psi(0)} = \frac{Rg(\Psi(0))\int_0^\infty E_\psi(\tau)d\tau}{\frac{1}{R^2}E_\Psi(0)} = R^3 g(\psi(0))\frac{\int_0^\infty E_\Psi(\tau)d\tau}{E_\Psi(0)}$$

Therefore in total:

$$\frac{\mathcal{E}}{E_\psi(0)} = \left(R + R^3 g(\psi(0))\right)\frac{\int_0^\infty E_\Psi(\tau)d\tau}{E_\Psi(0)}$$

This shows that the total energy growth scales with $R^3$, as the last fraction is independent of $R$.

We summarize our findings in the following theorem:

**Theorem 6.3** *Consider the 2D/3C model (6.5–6.6). Then the following are true:*

1. *The Hagen-Poiseuille flow* $\overline{V} = \frac{1}{4}(1 - r^2)$ *is globally asymptotically stable for all Reynolds numbers* $R$.

2. *For initial conditions* $(v(0) = 0, \psi(0) \neq 0)$ *the total transient energy growth is given by:*

$$\frac{\mathcal{E}}{E_\psi(0)} = k\left(R + R^3 g(\psi(0))\right) \tag{6.11}$$

*for some* $k$ *independent of* $R$.

The above analysis means that streamlining can evade bifurcations to instability, and produce a flow that is stable irrespective of the Reynolds number. At the same time, nonetheless, the amplification of disturbances and background noise scales like $R^3$, i.e. the robustness properties of the system deteriorate significantly. This 'robust yet fragile' principle is observed in many other systems in engineering and biology.

## 6.3 An Algorithmic Analysis Methodology for PDE systems

In this section we will consider the algorithmic construction of Lyapunov functions for systems described by PDEs. The methodology we use is based on the sum of squares decomposition, just as in the case of ODEs and FDEs.

The procedure we consider is as follows. First, we propose a Lyapunov functional structure $V$, and we impose positive definiteness by ensuring non-negativity of its kernel, just as in the case of FDEs. The system dynamics come into play when $\dot{V}$ is considered. The boundary conditions appear naturally when the resulting expression is integrated by parts, which, as seen in the previous section, helps greatly in 'completing' the squares and simplifying terms.

The above methodology is illustrated through an example. Consider the system

$$\begin{aligned} u_t &= u_{xx} + \lambda u^3 + \alpha u \\ u(\pm 1, t) &= 0 \end{aligned} \tag{6.12}$$

with $\lambda \leq 0$. This is the heat equation with forcing. From the linearization of this system about the zero steady state, stability is guaranteed for $\alpha < \pi^2/4$.

Here we are interested in constructing a Lyapunov function of the form

$$V(u) = \int_{-1}^{1} a(u(\eta, t), \eta) d\eta \tag{6.13}$$

where $a$ is a polynomial in its arguments, at least quadratic in $u$. We would expect that as the order of $a$ with respect to $\eta$ and $u$ is increased, better bounds on $\alpha$ can be obtained, and this is the case, as we will see in the sequel.

Consider now $\frac{dV(u)}{dt}$:

$$\frac{dV(u)}{dt} = \int_{-1}^{1} \frac{\partial a}{\partial u} u_t(\eta, t) d\eta = \int_{-1}^{1} \frac{\partial a}{\partial u} \left( u_{\eta\eta} + \lambda u^3 + \alpha u \right) d\eta \tag{6.14}$$

We mentioned earlier that integration by parts of the derivative condition is important in ensuring its non-negativity, as in this way the boundary conditions are taken into consideration. For example, in our case, the following expression is true:

$$\int_{-1}^{1} \left\{ \frac{\partial a}{\partial u} u_{\eta\eta} + u_\eta^2 \frac{\partial^2 a}{\partial u^2} - u \left[ \frac{\partial^3 a}{\partial u^2 \partial \eta} u_\eta + \frac{\partial^3 a}{\partial u_\eta \partial \eta^2} \right] \right\} d\eta = 0$$

This is a result of integration by parts of the term $\int_{-1}^{1} \frac{\partial a}{\partial u} u_{\eta\eta} d\eta$, where we have used the fact that $u \frac{\partial^2 a}{\partial u \partial \eta} \Big|_{-1}^{1} = 0$ and $u_\eta \frac{\partial a}{\partial u} \Big|_{-1}^{1} = 0$ since $a$ is a polynomial at least quadratic in $u$. Therefore the first term in (6.14) will be replaced by the rest of the terms in the above expression, as in itself, it cannot contribute to the negative definiteness of (6.14).

Notice that the following is also true:

$$\int_{-1}^{1} \eta^n u^2 d\eta = -\int_{-1}^{1} n\eta^n u^2 d\eta - 2\int_{-1}^{1} \eta^{n+1} u u_\eta d\eta$$

i.e.,

$$\int_{-1}^{1} \left[(1+n)\eta^n u^2 + 2\eta^{n+1} u u_\eta\right] d\eta = 0$$

We then have the following proposition:

**Proposition 6.4** *Let there be polynomials $a(u,\eta)$ and a positive definite function $\varphi$ such that the following hold:*

1. *$a(u,\eta) - \varphi(u) \geq 0$ when $\eta \in [-1,1]$,*
2. *$\lambda \frac{\partial a}{\partial u} u^3 + \alpha \frac{\partial a}{\partial u} u - u_\eta^2 \frac{\partial^2 a}{\partial u^2} + u\left[\frac{\partial^3 a}{\partial u^2 \partial \eta} u_\eta + \frac{\partial^3 a}{\partial u_\eta \partial \eta^2}\right] \leq 0$ when $\eta \in [-1,1]$ and $(1+n)\eta^n u^2 + 2\eta^{n+1} u u_\eta = 0$ is satisfied.*

*Then the steady-state of system (6.12) is stable.*

The two conditions guarantee the existence of a Lyapunov functional, given by (6.13), that is positive definite and whose derivative is negative definite, guaranteeing stability in $L_2$. The criteria (1) and (2) in the proposition can be tested by adjoining the conditions $\eta \in [-1,1]$ and the equality constraint in (2) using a technique similar to the S-procedure, as it was done earlier in this thesis.

Recall that the stability condition from the linearization is $\alpha < \pi^2/4$. We introduce a parameter measuring conservativeness, $\rho$ as follows:

$$\rho(\text{degree of } a \text{ wrt } \eta) = 1 - \left(\frac{4\alpha}{\pi^2} \text{ for which a } V \text{ can be constructed}\right) \tag{6.15}$$

Then for the functional shown in Equation (6.13), $\rho$ varies as shown in Table 6.1. We see that as the order of the polynomial $a(u,\eta)$ is increased with respect to $\eta$, the conservativeness is reduced, and the construction is entirely algorithmic.

| Order of $a$ wrt $\eta$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 |
|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | 1 | 0.5 | 0.32 | 0.20 | 0.15 | 0.11 | 0.08 | 0.07 | 0.06 |

Table 6.1: Stability analysis of a parabolic PDE system.

The above example shows how stability can be tested for systems described by parabolic PDEs. The boundary conditions come into play through integration by parts, a procedure that can be automated algorithmically. A problem that we face is that many times, certain structures are better than others, and we need to choose these judiciously, which is also the case for nonlinear FDEs.

## 6.4 Conclusion

In this chapter, we considered the problem of investigating the stability properties of systems described by Partial Differential Equations. We have introduced some stability notions, as well as a Lyapunov theorem for testing the stability properties of a system.

We then showed how the axially constant solutions of the Navier Stokes equations in cylindrical shear flow are globally stable, while they still retain an $R^3$ growth on the background noise, giving a 'robust, yet fragile' interpretation to the transition from laminar to turbulent pipe flow.

We also demonstrated how the construction of Lyapunov functionals for PDEs can be done algorithmically, using the sum of squares of multivariate polynomials and semidefinite programming.

# Chapter 7

# Conclusions

Ἐδόκει δὴ μεῖζόν τι εἶναι τῷ Ξενοφῶντι...Καὶ τάχα δὴ ἀκούουσι βοώντων τῶν
στρατιωτῶν **Θάλαττα θάλαττα** καὶ παρεγγυὼντων...
Ἐπεὶ δὲ ἀφίκοντο πάντες ἐπὶ τὸ ἄκρον,
ἐνταῦθα δὴ περιέβαλλον ἀλλήλους δακρύνοντες.
Ξενοφῶντος, Κάθοδος των μυρίων

It seemed to Xenophon that there was something very important... Soon they heard the soldiers shouting: **The Sea! The Sea!** and passing it along... And when they came to the peak, then they embraced one another, weeping.
Xenophon, Exodus

In this thesis, we have shown how the sum of squares technique can be used to analyze nonlinear systems algorithmically, for system descriptions ranging from simple Ordinary Differential Equations to simple Functional and Partial Differential Equations. Then, we investigated the scalable analysis of large-scale interconnections of subsystems described by Ordinary and Functional Differential Equations in relation to congestion control for the Internet.

Before closing, we will give a summary of the main results presented and discuss briefly future research directions.

## 7.1 Summary

In this thesis, we have seen the following results:

- In Chapter 3, we investigated how the algorithmic analysis of nonlinear systems described by Ordinary Differential Equations can be performed using sum of squares techniques. We considered a general class of systems evolving over inequality, equality and integral constraints. As particular cases, we have shown how the region of attraction of an equilibrium can be estimated, how robust stability analysis can be performed and how non-polynomial systems can be treated. Moreover, we have shown how to analyze stability of switching systems and have presented how performance analysis of nonlinear systems can be performed. Illustrative examples from biology and aerospace were used to show the effectiveness of the methodology.

- In Chapter 4, we moved to another class of differential equations, that of Functional Differential Equations. In this case, even the algorithmic analysis of linear system descriptions poses a challenge, and we have shown how the sum of squares technique can be used to construct algorithmically the relevant Lyapunov functionals. We then considered the stability and robust stability of nonlinear time delay systems, which can be performed in a unified way. Finally, an illustrative example from ecology was considered.

- In Chapter 5, we considered network congestion control for the Internet, which is a large-scale interconnection of subsystems described by Functional Differential Equations to form a network. We produced a Lyapunov argument for the analysis of the linearization, as well as the full global stability for arbitrary topologies, delays and link capacities for FAST, a network congestion control scheme. A more general class of TCP/AQM schemes was also considered. The structure of the system and the judicious choice of dynamics have guided the choice of the Lyapunov certificate.

- In Chapter 6, we considered the stability analysis of systems described by Partial Differential Equations. We have introduced some stability notions, as well as a Lyapunov theorem for testing the stability properties of such systems. We then showed how the axially constant solutions of the Navier-Stokes equations in cylindrical shear flow are globally stable, while they still retain an $R^3$

growth on the background noise. This shows how streamlined flows can be made nonlinearly stable for arbitrary Reynolds numbers, at the expense of fragility to disturbances. The chapter concluded with an algorithmic methodology for constructing Lyapunov functionals for PDEs using the sum of squares decomposition.

## 7.2 Future Research Directions

We would now like to mention a few pointers for future research. The work done so far has swept horizontally various system descriptions and has presented algorithmic solutions to analysis type questions. It would be interesting to investigate techniques for algorithmic synthesis of control laws with or without guaranteed cost. The problem of synthesis is, in general, more complex than the problem of systems analysis. In this case, both Lyapunov [76] and density function formulations [77] have been considered, but alternative methodologies should be sought.

In the area of systems described by Ordinary Differential Equations, important remains the concept of stiffness in ODEs and how this can be alleviated when seeking algorithmic constructions using sum of squares. Singular perturbation methods can provide some insight on how the Lyapunov function can be structured so that the resulting semidefinite programming conditions are numerically well-conditioned [35].

In the area of infinite dimensional systems, and in particular systems described by Partial Differential Equations, certain problems from fluid mechanics should be investigated. In particular, for high shear flows [84], still unexplained is the effect of drag reduction due to additives [43]. It is remarkable that a small amount of polymer additives can have a significant effect on the drag reduction [4, 5]. This effect is not entirely due to the change in the viscocity of the polymer/solvent solution – the viscoelastic properties of the continuum are believed to play an important role.

We have seen in the previous chapter how the sum of squares technique opens new possibilities for the analysis of systems described by PDEs, and we expect that these methodologies will have a broader application to systems of infinite dimension in the future. The tools that are currently used for analysis of PDE systems mostly center on discretization, which can never answer questions about the system exactly, and sometimes is computationally expensive. At least conceptually, the PDE description of a system is simpler and richer at the same time. The sum of squares technique is a methodology that opens new, unexplored possibilities and a unique perspective for future research in Partial Differential Equations.

Beyond stability analysis of steady-states for parabolic PDEs, other analysis questions can be answered, such as obtaining functional output estimates for PDEs. This information is important for determining acceptable mesh sizes in computational methods [70]. These and other areas are particularly important, such as obtaining bounds on functional outputs of the Fokker-Planck equation for stochastic systems analysis.

It is appreciated that large scale networks and distributed parameter systems provide the same modeling framework for systems with arbitrarily sized topologies. A way to see this intuitively is as follows. Most analysis procedures for parabolic PDE systems are related to discretization. This results in large-scale systems, each node in the mesh having certain dynamics depending on its neighbors and the discretization scheme. We are definitely more comfortable working with the large-scale system resulting from discretization — at least computers are, since they can simulate finite dimensional descriptions. But most of our questions cannot be answered by discretization alone, and in any case the PDE system description is simpler, at least conceptually. It is perhaps time that we put more effort in understanding the properties of PDE systems directly, producing reliable analysis algorithms that do not use discretization, so that invaluable information about infinitely large systems can be obtained at a low computational cost.

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