A COMPARISON AND EXPERIMENTAL TEST OF TWO TYPES OF ANALYSIS

OF

THE MOTIONS OF BUILDINGS EFFECTED BY TRANSIENT OSCILLATIONS

THESIS BY

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I N D E X

INTRODUCTION.

The purpose of this paper is three fold: First, to set forth the practical applications of two methods of analyzing the motions of buildings effected by an earthquake. Second, to compare the results of these two methods in some of their peculiarities. And, third, by actual shaking table experiments, to compare the results of analysis to real recorded motions.

The first method is that formulated by Dr. M. Biot at the California Institute of Technology. It was presented in full to the National Applied Mechanics Meeting in June, 1932. Since that time the practical application of the theory has been developed at the California Institute of Technology. The full derivation is a complex problem in mathematics and will not be presented here, thus limiting this to an explanation of the practical methods to be employed.

The second analysis is that formulated by Professor H. M. Westergaard and presented in full in the Engineering News Record of November, 1933. To our knowledge a full application of this theory has not previously been made.

The conclusions put forth in this paper are based on eight complete applications of the Biot analysis and two applications of the Westergaard theory. Examples of these calculations are shown in this paper.

CHAPTER I.

The Biot Analysis.

Dr. M. Biot in his paper states that "the motion of a building effected by an earthquake has the character of a transient oscillation." Using the same principle as applied by Heaviside in the analysis of transient electric currents, Dr. Biot, considering shearing forces only, applies the general method of analysis to the oscillations in a building.

Briefly the theory is, that when an harmonic force acts upon an elastic system the oscillation tends toward a steady state and in the case of resonance the maximum amplitude is determined by the internal friction. When the system is effected by a transient umpulse, however, the harmonic motion has not sufficient time to develop. Dr. Biot investigates the response of an undamped building to such impulses in four distinct phases. First, free oscillation, second, forced harmonic oscillation, third, sudden constant accelerations, and fourth, oscillations due to arbitrary horizontal accelerations. Having treated these cases separately, Dr. Biot then formulates a general theorem which when fully developed takes the form -

$$U_k(x) = K_k F(\nu_k)$$

where $U_k(x)$ equals the displacement and K_k and $F(\nu_k)$ are functions involving constants and the actual earthquake function. If the analysis is made considering the building vibrating, primarily in the fundamental period, this equation becomes

$$U_0(x) = K_0 \, F(\nu_0)$$

in this case

$$K_0 = 2\pi\nu_0 \, j_0 \, t_0^2 \, B_0 \, \cos(\lambda_0 \xi)$$

$$F(\nu_0) = (2\pi\nu_0 T_0)^2 \sqrt{\left(\int_0^T G(t) \sin 2\pi\nu_0 t \, dt\right)^2 + \left(\int_0^T G(t) \cos 2\pi\nu_0 t \, dt\right)^2}$$

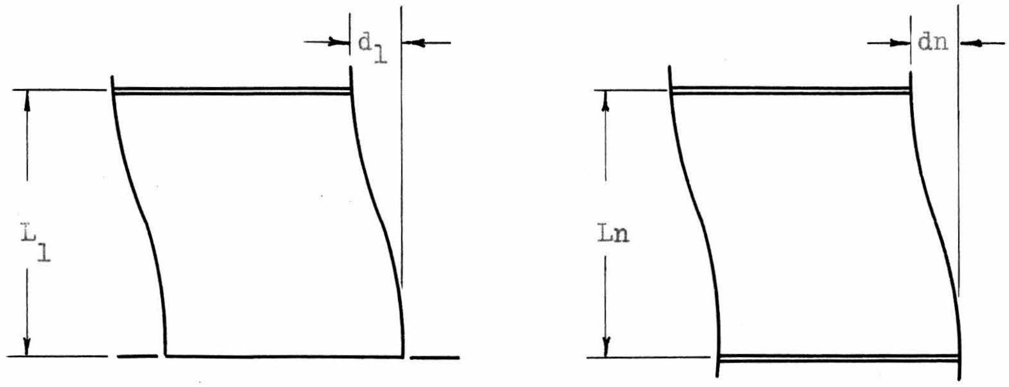
Where

T_0 = the fundamental period of the building in seconds per oscillation as found by experiment.

ν_0 = the frequency of the fundamental of the building in oscillations per second equals $\frac{1}{T_0}$.

j_0 = $\frac{1}{T_0^2}$.

λ_0 = is found from the equation $\lambda_0 \tan \lambda_0 = \alpha$ where $\alpha = Rn$ where n is the number of stories above the first and R is the ratio of the rigidity of the first floor to that of the others. This may be calculated if the columns have very constant cross-sections by the formula $R = L_n^3 \;\; L_1^3$ (L_n is the height of one of the upper stories and L_1 the height of the first story), however it was found more satisfactory to determine R. experimentally. A measurable lateral force was applied at the first floor and a deflection measured as shown.



The same force was then applied to one of the upper floors, the floor immediately below being held stationary, and the deflection measured as shown.

then

$$R = \frac{d_n}{d_1}$$

t_o = is found from the equation $t_o = \frac{T_o \lambda_o}{2\pi}$

B_o = is found from the equation $B_o = \frac{2\alpha\beta_o}{\lambda_o^4}$

where $\beta_o = \frac{\cos\lambda_o}{1 + \frac{\alpha\cos\lambda_o}{\lambda_o^2}}$

$\xi = \frac{x}{h}$ where x is the distance down from the top at which the displacement is desired and h is the height of the building above the first story.

t = is the independent variable, time.

$G(t)$ = is a function of t expressing the ground motion in terms of t.

This nomenclature is the same as used in Dr. Biot's paper and the terms are the same; however, from these equations the exact physical significance of the terms does not appear. This is due to the fact that the physical values are not as easily obtained as the ones given above.

In this manner all of the constants can now be obtained. The difficulty lies in evaluating the two integrals under the radical sign. This was done by means of graphical integration. The sine and cosine functions when integrated become cosine and sine functions respectively. These curves are plotted to appropriate scale and then the curve of the earthquake plotted at right angles to both of them. It will be seen that the values of the two integrals are then expressed by the accumulative, algebraic sums of the areas up to the point in question. Care, however, must be taken to distinguish between positive and negative areas.

With this information it is possible to evaluate the entire expression obtaining values for deflections at every time during the earthquake and at every point in the building. These values can be either positive or negative and thus they provide an envelope within which the curve for the actual relative displacement (due to oscillations in the fundamental) should remain.

The above formulated methods as previously mentioned, have been for oscillations in the fundamental period of the building. If it is desired to make an analysis for movement in any of the harmonics, the subscripts in the above equations should be changed to 1, 2, or 3 for the first, second, or third harmonics, respectively, the necessary constants again calculated and values of deflections computed. By the theory of superposition these deflections may be added to those of the fundamental oscillations and thus a second envelope is obtained within which the displacements due to fundamental and harmonic oscillations should remain.

In applying this method, Dr. Biot's assumptions must be remembered. That is, the building must be of regular shape, the shearing rigidity and the mass of each story above the first must be constant, and the most important deformation must be in horizontal shear.

CHAPTER II.

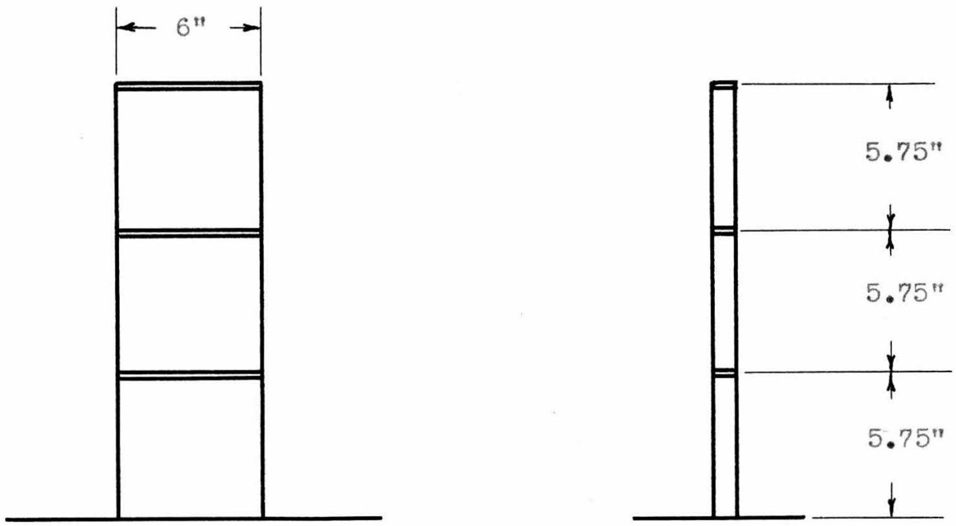
Example of the Biot Analysis.

The records used in the calculations given below were obtained from a shaking table designed by Professor R. R. Martel. The tapes taken were enlarged to a desirable size in a photographer's enlarging camera. In doing this it was necessary to carefully synchronize the various traces and note the enlargement factor (E). The sine and cosine curves were then plotted having a period of $(2\pi V_0 E)$, and an amplitude of any desirable size, for example, y.

The ground motion was then plotted at right angles to these curves as shown in Plate #1. The convention adopted for distinguishing between positive and negative areas was as follows. The areas are positive when the ordinate of the ground motion and the slope of the sine or cosine curves have like signs. These areas are then plotted to the left of the sine or cosine curves. The areas are negative when the ordinates and slopes have unlike signs and are plotted to the right of the sine or cosine curves.

These areas were obtained by means of a planimeter and the algebraic sums noted at points where values of deflections were desired.

In the example given herein, the model used was of the dimensions shown below:



The calculations for constants for the equations of oscillations in the fundamental are shown below:

<u>Calculations for Constants for Oscillations in the Fundamental</u>

$$T_o = \frac{12.026 \times 4.30}{160} = 3.23 \text{ sec.}$$ By Experiment.

$$\nu_o = \frac{1}{T_o} = \frac{1}{.323} = 3.10$$ oscillations/second.

$$j_o = \frac{1}{T_o^2} = \frac{1}{.323^2} = 9.60$$

$$n = 2$$

$$R = \frac{L_2^3}{L_1^3} = 1$$

$$\alpha = Rn = 2$$

$$\lambda_o \quad \tan \lambda_o = \alpha = 2 \qquad \lambda_o = 1.077^{r}$$

$$t_o = \frac{T_o \ \lambda_o}{2\pi} = \frac{.323 \times 1.077}{6.28} = .0554$$

$$\beta_o = \frac{\cos \lambda_o}{1 + \dfrac{\alpha \cos \lambda_o}{\lambda_o^2}} = \frac{.4741}{1 + \dfrac{2 \times .4741}{1.077^2}} = .342$$

$$B_o = \frac{2\alpha\beta_o}{\lambda_o^4} = \frac{2 \times 2 \times .342}{1.077^4} = 1.019$$

$$K_o = 2\pi \nu_o j_o \ t_o^2 \ B_o \cos(\lambda_o \xi) = 2\pi \times 3.10 \times 9.6 \times .0554^2$$

$$\times 1.019 \ \cos(\lambda_o \xi) = .584 \cos(\lambda_o \xi)$$

$$F(\nu_o) = (2\pi \nu_o T_o^2) \sqrt{\left(\int_o^T G(t) \sin 2\pi\nu_o t \ dt\right)^2 + \left(\int_o^T G(t) \sin 2\pi\nu_o t \ dt\right)^2}$$

$$= 39.4 \sqrt{(\text{Area } \#1)^2 + (\text{Area } \#2)^2}$$

$$U_o = \frac{K_o \ F(\nu_o)}{E_y \ 2\pi\nu_o \ P} = \frac{.584 \cos(\lambda_o \xi) \times 39.4 \sqrt{A_1^2 + A_2^2}}{1 \times 1.5 \times 2\pi \times 3.1 \times .65}$$

$$= 1.213 \cos(\lambda_o \xi) \sqrt{A_1^2 + A_2^2}$$

For Roof	$\cos(\lambda_o 0)$	=	$\cos(1.077 \times 0)$	= 1
For 3rd Floor	$\cos(\lambda_o 1/2)$	=	$\cos(1.077 \times 1/2)$	= .859
For 2nd Floor	$\cos(\lambda_o \times 1)$	=	$\cos(1.077 \times 1)$	= .475

The graphical integration is shown in Plate #1; the results are tabulated in Table #1 and plotted in Plate #2.

Values of Envelope for Oscillations in the Fundamental

Point #	Area #1	Area #2	$\sqrt{A_1^2+A_2^2}$	U_o Roof	U_o 3rd	U_o 2nd
1	.02	.08	.08	.10	.09	.05
2	.01	.21	.21	.26	.22	.13
3	.39	.16	.42	.51	.49	.24
4	.16	.16	.23	.27	.23	.13
5	.36	.19	.42	.51	.44	.24
6	.25	.19	.31	.38	.33	.18
7	.34	.38	.51	.62	.53	.29
8	.38	1.12	1.19	1.44	1.24	.68
9	1.09	.41	1.16	1.41	1.21	.67
10	.37	.34	.50	.61	.52	.29
11	.37	.43	.55	.67	.58	.32
12	.40	1.16	1.23	1.49	1.28	.71
13	1.15	.34	1.20	1.46	1.26	.69
14	1.07	.01	1.07	1.30	1.12	.62
15	.72	.62	.95	1.15	.99	.55
16	1.00	1.09	1.48	1.80	1.55	.86
17	1.40	.52	1.49	1.81	1.56	.86
18	1.47	.38	1.52	1.84	1.58	.87
19	1.56	.23	1.57	1.91	1.64	.91
20	1.52	.12	1.53	1.86	1.60	.88
21	1.81	.05	1.82	2.21	1.90	1.05
22	1.66	.25	1.68	2.04	1.75	.97
23	1.46	.55	1.56	1.89	1.62	.90
24	1.57	.64	1.69	2.05	1.76	.98

TABLE #1.

It was believed necessary to make an analysis for the harmonic vibrations. The calculations for the constants for this case are shown on the next page.

Calculations for Constants for Oscillations in the First Harmonic

$T_1 = .108$ By Experiment.

$\nu_1 = \frac{1}{.108} = 9.30$ Oscillations/second.

$j_1 = \frac{1}{T_1^2} = \frac{1}{.108^2} = 85.7$

$n = 2$

$R = 1$

$\alpha = Rn = 2$

$\lambda_1 \tan \lambda_1 = \alpha = 2 \qquad \lambda_1 = 3.643$

$t_1 = \frac{T_1 \lambda_1}{2\pi} = \frac{.108 \times 3.643}{6.28} = .0627$

$\beta_1 = \frac{\cos \lambda_1}{1 + \frac{\alpha \cos \lambda_1}{\lambda_1^2}} = \frac{.8774}{1 + \frac{2 \times .8774}{3.643}} = .592$

$B_1 = \frac{2\alpha\beta_1}{\lambda_1^4} = \frac{2 \times 2 \times .592}{3.643^4} = .0134$

$K_1 = 2\pi \nu_1 \; j_1 \; t_1^2 \; B_1 \cos(\lambda_1 \xi) = 2\pi \times 9.30 \times 85.7 \times .0627^2$

$\times .0134 \cos(\lambda_1 \xi) = .264 \cos(\lambda_1 \xi)$

$F(\nu_1) = (2\pi \nu_1 T_1)^2 \sqrt{\left(\int_0^T G(t) \sin 2\pi\nu_1 t \, dt\right)^2 + \left(\int_0^T G(t) \sin 2\pi\nu_1 t \, dt\right)^2}$

$= 39.4\sqrt{A_3^2 + A_4^2}$

$$U_1 = \frac{K_1 \; F(\gamma_1)}{E \; y \; 2\pi\gamma_o P} = \frac{.264 \cos(\lambda_1 \xi) \times 39.4}{1 \times 1.5 \times 2\pi \, 3.1 \times .65} \sqrt{A_3^{\,2} + A_4^{\,2}}$$

$$= .552 \cos(\lambda_1 \xi) \sqrt{A_3^{\,2} + A_4^{\,2}}$$

For Roof	$\cos(\lambda_1 \xi)$	=	$\cos(3.643 \times 0)$	=	1
For 2nd Floor	$\cos(\lambda_1 \xi)$	=	$\cos(3.643 \times \frac{1}{2})$	=	.988
For 3rd Floor	$\cos(\lambda_1 \xi)$	=	$\cos(3.643 \times 1)$	=	.876

The graphical integration is also shown in Plate #1. The results are tabulated in Table #2 and the envelope for the combined fundamental and harmonic oscillations is plotted in Plate #2.

Values of Corrections for Envelope due to the Harmonic.

Pt.	Area #3	Area #4	$\sqrt{A_3^2 + A_4^2}$	U_1 Roof	U_1 3rd	U_1 2nd
1	.09	.09	.13	.07	.07	.06
2	.28	.45	.53	.29	.29	.25
3	.70	1.10	1.30	.72	.71	.62
4	1.25	1.45	1.92	1.06	1.05	.93
5	.16	1.36	1.38	.76	.75	.67
6	.16	.77	.79	.44	.43	.39
7	.46	1.45	1.52	.84	.83	.74
8	1.08	.83	1.36	.75	.74	.66
9	.46	.21	.51	.28	.28	.25
10	.16	.83	.85	.47	.46	.41
11	.46	1.45	1.52	.84	.83	.74
12	1.08	.83	1.36	.75	.74	.66
13	.46	.21	.51	.28	.28	.25
14	.68	.99	1.21	.67	.66	.59
15	.87	2.06	2.24	1.24	1.23	1.08
16	.25	.82	.86	.47	.46	.41
17	.02	.20	.20	.11	.11	.09
18	.56	.32	.65	.36	.36	.31
19	.60	.17	.62	.34	.34	.30
20	.89	.49	1.02	.56	.55	.48
21	.73	.08	.77	.42	.41	.36
22	.57	.27	.63	.35	.35	.31
23	.73	.46	.86	.47	.46	.40
24	.63	.29	.69	.38	.38	.33

TABLE #2.

CHAPTER III.

The Westergaard Analysis.

The theory formulated by Professor Westergaard is based upon the fact that a sudden horizontal earthquake shock under a tall building will cause a wave of deformations to travel through the building to the top. The wave will be reflected at the top and will return to the foundation and be reflected again. The energy will gradually be dissipated by damping and incomplete reflections until the building again returns to rest.

When placed in mathematical formula the theory is developed as follows. If,

t = time.

x = vertical distance.

y = horizontal deflection.

s = horizontal shear at pt x.

K = total stiffness of the columns.

w = weight of the building per unit of height.

g = acceleration due to gravity.

v = velocity of the shear wave.

h = height of the building.

T = period of the fundamental of the building.

then,

$$S = K \frac{\delta y}{\delta x}$$

By dynamic considerations of forces acting upon a differential of height we obtain the equation

$$\frac{\delta S}{\delta x} = \frac{w}{g} \frac{\delta^2 y}{\delta t^2}$$

combining we get

$$K \frac{\delta^2 y}{\delta x^2} = \frac{w}{g} \frac{\delta^2 y}{\delta t^2}$$

solving this differential equation we get

$$y = f\left(t \pm \frac{x}{V}\right)$$

where

$$V = \sqrt{Kg/w}$$

Physically this means that the deflection ordinate remains the same value but it has been moved along the time axis by the value $\frac{V}{x}$ which is the time it takes a shear wave to travel from the bottom to the position x.

When the wave reaches the top a new wave is created which starts downward. If the shears at the top are now set equal to zero, which is the correct boundary condition, it is found that the deflection y_1 at the top is doubled and the new descending wave is of the form -

$$y_2 = f\left(t + \frac{x - 2h}{V}\right)$$

When this wave reaches the foundation another wave is reflected. This case has the boundary condition that $y_2 = - y_3$

From this we find -

$$y_3 = -f\left(t - \frac{X + 2h}{V}\right)$$

If the ground is in motion at this time the value of the reflected wave must be added to the actual ground motion before analysis can be continued.

In this general manner the entire motion of the building during a period of any length can be plotted. By multiplying by a constant at times of reflections the effect of a damping factor can be easily considered. Thus, a complete analysis is accomplished.

CHAPTER IV.

Example of the Westergaard Analysis.

The records used in the example given below were obtained from the shaking table. The same bent was used as in the previous case and the same ground motion applied.

The necessary calculations are given below -

$$K = \frac{Wl}{d} = \frac{1.007 \times 17.25}{.79} = 21.85 \text{ \#/inch.}$$

$$w = \frac{1.89}{17.25}$$

$$V = \sqrt{\frac{Kg}{w}} = \sqrt{\frac{21.85 \times 32.2 \times 12}{1.89/17.25}} = 273''/\text{sec.}$$

$$t = \frac{17.25}{273} = .0632 \text{ sec.}$$

t is the time it takes a shear wave to travel from the base to the top of the building.

The trace of the ground motion was then plotted as shown in Plate #3, and the time abcissa divided into intervals of .0632 sec. This was done so that shear waves of this length would be handled. If this is not done the paths of the shear waves become a maze of lines in which it is very nearly impossible to distinguish anything.

The shear wave was then traced at the second, third, and fourth floor as a dotted line, as shown in Plate #3. This wave was reflected from the top floor as a dot-dash line (Plate #3). When the wave again reached the ground it was reflected (dotted line) summed with the actual ground motion and taken up along the next wave path.

When these curves had been plotted, it was found that they were similar to the ones recorded upon the shaking table, but that the period of the fundamental as taken from the calculated curve was less than that of the actual bent. For this reason it was thought best for practical purposes, to calculate t from the known period of the building. Therefore -

$$t = \frac{.323}{4} = .081 \text{ sec.}$$

Using this value of t and plotting in the same manner, Plate #4 was prepared. As can be seen the period of the calculated curve coincides with that of the actual curve.

An investigation of the cause of this discrepancy was made and it was traced to the original equation -

$$V = \sqrt{\frac{Kg}{w}}$$

In this expression, w is the weight of the building per unit of height. The equation is good for a case in which the weight is more or less uniformly distributed. In the example chosen, however, the weight is concentrated at the three floor levels and the equation is not good.

Thus in a building of twenty stories or so, the equation is valid but cannot be used in the present example. Since the period was already known, the alternate method could be used.

In cases where the period is not known a more refined method of calculations must be employed.

It is readily seen that the entire process depends upon the accuracy of the plotting. This must be done as carefully as possible if good results are to be obtained.

It is suggested that a stencil, just admitting to the plotter's eye the width of one interval,be prepared. Using this, and templates of the ground motion and reflected waves, a quick and simple routine can be developed.

CHAPTER V.

Conclusions.

The conclusions of this report are drawn chiefly from the curves.

In example #1, Plate #2 shows that the Biot analysis for oscillations in the fundamental fails to consider the quick short oscillations at the beginning of the quake. However, as these iron themselves out, and the building begins to oscillate more periodically, this first simple analysis is valid.

If a refinement is made and an analysis for the harmonic oscillations is computed, we find that this takes care of the short oscillations at the first and then dies out as the building begins to move in the fundamental.

Thus a complete study of the movement can be made and an envelope obtained. It must be remembered, however, that this is a curve which goes through the points of maximum relative displacement and does not give the actual movement of the building.

In example #2, Plates #3 and #4 show two Westergaard analyses for the actual motions of the building. The magnitudes of the deflections seem to be high. These discrepancies are due, first to inaccuracies in plotting and second to the fact that no damping factor was employed in the analysis. It is to the advantage of this method that the results obtained are in the form of the actual movement of the building.

The use of the Biot equations is not difficult, nor is the graphical integration exacting. The method, however, is long and requires a good deal of time to use, especially if a number of different buildings are to be analyzed, since a new set of constants is required for each case.

The use of the Westergaard method is more rapid, but is not as scholarly. The chief difficulty is in obtaining the correct value for t, a problem which will not be taken up in this paper.

In conclusion it may be said that both authors have formulated practical methods of analysis which have been experimentally verified.