EXPERIMENTAL VERIFICATION OF DR. BIOT'S METHOD OF PREDICTION OF EARTHQUAKE STRESSES IN A BUILDING

THESIS BY

CLIFFORD C. CAWLEY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA

1933

INDEX

REFERENCES

"Calculation of the stresser occuring in a building during earthquakes," by Dr. M. Biot, Guggenheim Aeronautics Laboratory, California Institute of Technology, Pasadena, Publication No. 24. (Copy included in Thesis.)

"Theory of Elastic Systems Vibrating under Transient Impulse with an Application to Earthquake-Proof Buildings." M. Biot, Department of Aeronautics, California Institute of Technology. Reprinted from the Proceedings of the National Academy of Sciences, Vol. 19, No. 2, pp. 262-268, February, 1933.

"Reoprt on Shaking Table Research," Vito A. Vanoni, Department of Civil Engineering, California Institute of Technology. (1932.)

"Transient Oscillations in Elastic Systems," Thesis, M. Biot, California Institute of Technology, Ph. D., 1932.

"The Effects of Earthquakes on Buildings with a Flexible First Story," R. R. Martel, Associate Professor of Civil Engineering, California Institute of Technology, Pasadena, Reprinted from the Bulletin of the Seismological Society of America, Vol.19, No. 3, September, 1929.

"The Dynamic Behaviour of Some Simple Bents Subjected to Established Simple Harmonic Motion," (Paper No. 490) By R. R. Martel, S. B., Ass. Professor of Civil Engineering, California Institute of Technology.

Annales De La Societe Scientifique De Bruxelles, Extrait, "Propriete generale des systemes elastiques Soumis a impulsion transitoire, Notes de M. Maurice Biot. 1932.

"Design and Construction of an Harmonic Analyzer," Thesis, Jesse W. M. Du Mond, 1916, California Institute of Technology.

"The Overturn of Blocks during EArthquakes," Dr. Maurice M. Biot, Civil Engineering Office, California Institute of Technology, 1933.

"Earthquake Damage and Earthquake Insurance," John R. Freeman. (1932.)

FIG. 1.

RECORDING SHAKING TABLE, IN THE CIVIL ENGINEERING DEPARTMENT OF THE CALIFORNIA INSTITUTE OF TECHNOLOGY.

INTRODUCTION

In June, 1932, Dr. Maurice M. Biot, of the California Institute of Technology. presented a paper at the National Applied Mechanics Meeting entitled "Calculation of the Stresses Occurring in a Building during Earthquakes." This paper follows. On the assumption that a building behaves as an elastic system, he arrives at an expression for the motion due to an arbitrary horizontal motion applied at the base. The Recording Shaking Table, Fig. 1., brain-child of Professor R. R. Martel, offered a means of carrying out Dr. Biot's prediction for model bents. This Thesis is the summary of this work.

A very important correction in Dr. Biot's paper:

"K" is the force necessary to displace two consecutive floors unit horizontal distance relative to each other.

$\mu = Kh_1$, NOT $\dfrac{K}{h_1}$.

CALCULATION OF THE STRESSES OCCURING IN A BUILDING DURING EARTHQUAKES.

By Dr. M. Biot,[1] Pasadena, Calif.

SUMMARY

The motion of a building affected by an earthquake has the character of a transient oscillation. A general method of analysis of such an oscillation is here developed and applied to this special case. The principles involved are the same as those proposed by Heaviside for the analysis of transient electric currents. We consider here only shearing deformations.

The calculation is carried out completely for a building of constant mass and rigidity at every floor except the first. The influence of an "elastic first floor" is investigated and it is shown that the oscillation depends fundamentally on a parameter α. This parameter is a product of the number of upper floors n by the ratio R of the rigidity of the first floor to that of the others. The solution of the problem for a sudden constant acceleration leads to a simple calculation of the effect of resonance.

For the purpose of applying the theory to actual earthquake accelerations a general theorem for elastic systems is established which gives a very simple method of calculating the oscillation amplitudes under a random impulse by considering its spectrum.

A practical method is given to calculate the motion of any type of building.

This study was made at the California Institute of Technology at the suggestion of Prof. Th.von Kármán and the author wishes to express his appreciation of the continual interest Prof. von Kármán has taken in its progress.

CHARACTER OF EARTHQUAKE ACCELERATIONS

When an harmonic force acts upon an elastic system the oscillation of the system tends to a steady state. In case of resonence the maximum amplitude is determined by the internal friction. But when the elastic system is affected by a short wave or a random impulse, the steady harmonic motion has no sufficient time to develop and the damping has very little effect.

This is generally a character of earthquake accelerations. It is therefore interesting to investigate the response of an undamped building to such transitory forces.

The method which shall be used is, applied to mechanical vibrations, that which is well known to Electrical Engineers, and shows to be so powerful in transient electrical currents analysis. We will further establish a general theorem, introducing the notion of earthquake spectrum, which gives a very adequate method of treating the problem.

As the influence of the so called "elastic first floor" is actually very much discussed, we shall start the analysis by considering a building of the following simple type.

Let it be of rectangular shape. The most important deformation is an horizontal shear as shown in fig. 1. This would naturally not be true for very high buildings where the bending would have to be taken into account. Furthermore the shearing rigidity and the mass of each floor are supposed to be constant from the second floor to the top. Only the first floor will be of a different rigidity.

Let h be the height of the building without the first floor and x the coordinate counted downwards from the top as origin. We may consider this part of the building as an elastic continuous beam whose only possible deformation is shear (fig. 2). If M is its total mass, $m=\frac{M}{h}$ will be the mass per unit length. The number of upper floors being n and $h_1=\frac{h}{n}$ their height and calling K the force that is necessary to displace two consecutive floors so that the relative glide would be equal to the unit length, the coefficient of corresponding shearing rigidity of the continuous beam will be $\mu=\frac{K}{h}$. The rigidity of the first floor is characterised by a coefficient G defined in the same way as K. note correction

The ground is supposed to move horizontally with a variable acceleration j(t). The equation of relative motion of the beam is

$$\mu\frac{\partial^2 u}{\partial t^2}=m\frac{\partial^2 u}{\partial t^2}+mj(t)$$

We define the following notations:

$c=\sqrt{\frac{\mu}{m}}$ propagation speed of a shear wave

$\frac{h}{c}=t_0$

$\frac{t}{t_0}=\tau$

$j(t)=j_0\varphi(\tau)$

$\frac{x}{h}=\xi$

$\frac{u}{j_0 t_0^2}=y$

The equation of motion becomes,

$$\frac{\partial^2 y}{\partial \xi^2}=\frac{\partial^2 y}{\partial \tau^2}+\varphi(\tau)$$

All the quantities figuring in this equation are dimensionless.

[1] California Institute of Technology

(Biot,b-1)

Contributed by the Applied Mechanics Division for presentation at the National Applied Mechanics Meeting, New Haven, June 23-25, 1932. All papers are subject to revision. Note: Statements and opinions advanced in papers are to be understood as individual expressions of their authors, and not those of the Society.

§1. FREE OSCILLATION

This motion is given by the homogeneous equation

$$\frac{\partial^2 y}{\partial \xi^2}=\frac{\partial^2 y}{\partial \tau^2}$$

which by putting $y=z(\xi)e^{i\lambda\tau}$, becomes

$$\frac{d^2 z}{d\xi^2}+\lambda^2 z=0.$$

The general solution is,

$$z=A\cos\lambda\xi+B\sin\lambda\xi.$$

Consider the boundary conditions

$$\frac{\partial u}{\partial x}=0 \qquad x=0$$

$$\mu\frac{\partial u}{\partial x}=-Gu \qquad x=h$$

put $R=\frac{Gh_1}{K}$ ratio of the rigidity of the first floor to that of the others, and $\alpha=\frac{Gh}{\mu}=Rn$, n being the number of these other floors, these conditions take the form,

$$\frac{dz}{d\xi}=0 \qquad \xi=0 \qquad (1)$$

$$\frac{dz}{d\xi}+\alpha z=0 \quad \xi=1 \qquad (2)$$

From equation (1) $B=0$ and $z=A\cos\lambda\xi$

From condition (2) $\lambda\,tg\,\lambda=\alpha$

The roots λ_k of this equation correspond to the free oscillation frequencies of the building. We choose certain values of α corresponding to certain simple values of R and n as follows:

n \ R	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$	1
15	1.66	2.50	5	
10	1.11	1.66	3.33	10
5	0.556	0.834	1.66	

The volume of λ_k as a function of α are given in the following table:

α	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
0	0	π	2π	3π	4π	5π
0.556	0.68	3.31	6.31	9.48	12.60	15.73
0.834	0.80	3.38	6.41	9.51	12.62	15.75
1.11	0.89	3.45	6.45	9.54	12.65	15.77
1.66	1.03	3.58	6.53	9.59	12.69	15.80
2.50	1.15	3.73	6.65	9.67	12.76	15.85
3.33	1.23	3.86	6.74	9.75	12.82	15.91
5.0	1.32	4.04	6.91	9.90	12.93	16.0
10.0	1.44	4.30	7.22	10.18	13.20	16.24
∞	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{2}$	$\frac{7\pi}{2}$	$\frac{9\pi}{2}$	$\frac{11\pi}{2}$

In fig.3 are plotted the values of λ'_k where $\lambda_k=k\pi+\lambda'_k$

The period T_k corresponding to λ_k is,

$$T_k=\frac{2\pi}{\lambda_k}t_0$$

It is interesting to compare the fundamental period T_0 to that T'_0 which would occur if the building would be perfectly rigid from the second floor to the top, the only elasticity being due to the first floor. We get,

$$M\frac{d^2u}{dt^2}+Gu=0 \qquad T'_0=2\pi\sqrt{\frac{M}{G}}=2\pi\frac{t_0}{\sqrt{\alpha}}$$

The ratio of frequencies $\frac{f_0}{f'_0}=\frac{T'_0}{T_0}=\frac{\lambda_0}{\sqrt{\alpha}}$ is a function of α

α	f_0/f'_0
0	1
0.556	0.910
0.834	0.875
1.11	0.840
1.66	0.800
2.50	0.725
3.33	0.674
5.0	0.590
10.	0.455
∞	0

§ 2. FORCED HARMONIC OSCILLATION

In this case $\varphi(\tau) = e^{i\lambda\tau}$ and the equation of motion becomes

$$\frac{\partial^2 y}{\partial \xi^2} = \frac{\partial^2 y}{\partial \tau^2} + e^{i\lambda\tau}$$

which by putting $y = z(\xi) e^{i\lambda\tau}$ takes the form

$$\frac{d^2 z}{d\xi^2} + \lambda^2 z = 1$$

The solution of this equation can be expressed as a sum of the orthogonal functions $z_k = \cos\lambda_k\xi$ satisfying the corresponding homogeneous equation and the given boundary conditions,

$$z = \sum_0^\infty A_k z_k$$

Carrying this expression into the differential equation

$$\sum_0^\infty \left[A_i \frac{d^2 z_i}{d\xi^2} + \lambda^2 A_i z_i\right] = 1$$

Taking into account the identity,

$$\frac{d^2 z_i}{d\xi^2} = -\lambda_i^2 z_i$$

If we multiply both sides of that equation by z_k and integrate from 0 to 1 with respect to ξ we get,

$$\sum_0^\infty A_i\left[\lambda^2 - \lambda_i^2\right] \int_0^1 z_i z_k \, d\xi = \int_0^1 z_k \, d\xi$$

The condition of orthogonality $\int_0^1 z_k z_i \, d\xi = 0$ for $k \neq i$ gives,

$$A_k = \frac{\sigma_k}{\lambda^2 - \lambda_k^2} \qquad \sigma_k = \frac{\int_0^1 z_k \, d\xi}{\int_0^1 z_k^2 \, d\xi}$$

The required solution is,

$$z = \sum_0^\infty \frac{\sigma_k}{\lambda^2 - \lambda_k^2} z_k .$$

The value of σ_k may be given more explicitly,

$$\sigma_k = 2\alpha \frac{\beta_k}{\lambda_k^2} , \quad \beta_k = \frac{\cos\lambda_k}{1 + \alpha \frac{\cos^2\lambda_k}{\lambda_k^2}}$$

$$z = 2\alpha \sum_0^\infty \frac{\beta_k}{\lambda_k^2} \cdot \frac{z_k}{\lambda^2 - \lambda_k^2}$$

In this series the values of the coefficients β_k tend to unity when k increases indefinitely and the convergence is absolute and uniform.

§ 3. EFFECT OF A SUDDEN CONSTANT ACCELERATION

With the purpose of finding a more general solution we shall use the preceding results for building a solution which corresponds to

$$\varphi(\tau) = 0 \qquad \tau < 0$$

$$\varphi(\tau) = 1 \qquad \tau > 0 \qquad \text{(FIG. 5)}$$

Consider the following integral in the complex plane,

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i\lambda\tau}}{\lambda} d\lambda$$

taken along the real axis (fig.4). For $\tau < 0$ the real part of $i\lambda\tau$ is negative on the half circle of infinite radius A D C. Hence we may add this path to the contour of integration without any effect on the value of the integral. For $\tau < 0$ it can then be written,

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i\lambda\tau}}{\lambda} d\lambda = \frac{1}{2\pi i} \oint_{ACDA} \frac{e^{i\lambda\tau}}{\lambda} d\lambda = 0$$

For the same reason, when $\tau > 0$ we may add the path A B C and

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i\lambda\tau}}{\lambda} d\lambda = \frac{1}{2\pi i} \oint_{ACBA} \frac{e^{i\lambda\tau}}{\lambda} d\lambda = 1$$

The function $\varphi(\tau)$ defined by

$$\varphi(\tau) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i\lambda\tau}}{\lambda} d\lambda$$

is discontinuous at the origin where it jumps from 0 to 1 and is equal to a constant anywhere else. (fig. 5). The solution corresponding to each element of this integral is

$$d\zeta = 2\alpha \sum_0^\infty \frac{\beta_k}{\lambda_k^2(\lambda^2 - \lambda_k^2)} \cdot z_k e^{i\lambda\tau} \frac{d\lambda}{\lambda}$$

Integrating term by term we get the required solution in the form of contour integrals

$$\zeta = 2\alpha \sum_0^\infty \frac{\beta_k z_k}{\lambda_k^2} \frac{1}{2\pi i} \oint_{ACBA} \frac{e^{i\lambda\tau}}{\lambda(\lambda^2 - \lambda_k^2)} d\lambda ,$$

And by taking the residues,

$$\frac{1}{2\pi i} \oint_{ACBA} \frac{e^{i\lambda\tau}}{\lambda(\lambda^2 - \lambda_k^2)} d\lambda = \frac{1}{2\pi i \lambda_k^2} \oint \left[\frac{1}{2}\left(\frac{e^{i\lambda\tau}}{\lambda + \lambda_k} + \frac{e^{i\lambda\tau}}{\lambda - \lambda_k}\right) - \frac{e^{i\lambda\tau}}{\lambda}\right] d\lambda$$

$$= \frac{1}{\lambda_k^2}(\cos\lambda_k\tau - 1).$$

we obtain finally, $\zeta = 2\alpha \sum_0^\infty \frac{\beta_k z_k}{\lambda_k^4} (\cos\lambda_k\tau - 1)$

By putting $B_k = \frac{2\alpha\beta_k}{\lambda_k^4}$,

$$\zeta = \sum_0^\infty B_k \cos\lambda_k\xi (\cos\lambda_k\tau - 1)$$

Let us study the convergence of this series and compute the order of magnitude of the rest. Each of its terms, the first one excepted, satisfies the inequality

$$\left|\frac{\beta_k z_k(\cos\lambda_k\tau - 1)}{\lambda_k^4}\right| < \frac{2}{\lambda_k^4} < \frac{2}{(\pi k)^4}$$

The following inequality gives an upper limit for the rest

$$|R_p| < \sum_p^\infty \left|\frac{\beta_k z_k(\cos\lambda_k\tau - 1)}{\lambda_k^4}\right| < \frac{2}{\pi^4} \sum_p^\infty \frac{1}{k^4}$$

The value of $\sum_1^\infty \frac{1}{k^4}$ is given by the Bernouilli number B_2,

$$\sum_1^\infty \frac{1}{k^4} = \frac{\pi^4}{90} = 1.080$$

On the other hand

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} = 1.074 ,$$

so that $\sum_4^\infty \frac{1}{k^4} = 0.006$, and finally $|R_4| < \frac{2}{\pi^4} \times 0.006$

Hence if we take only the four initial terms of the series

$$\zeta = \sum_0^3 B_k \cos\lambda_k\xi \left[\cos\lambda_k\tau - 1\right]$$

the error due to the terms neglected will be smaller than

$$\frac{4\alpha}{\pi^4} \times 0.006 = \alpha \times 0.00025$$

We also see that this series is absolutely and uniformly convergent.

Displacement

The dimensionless function ζ enables us to calculate the motion due to a sudden acceleration j_0. The real displacement is

$$u = j_0 t_0^2 \zeta$$

The values of B_k are only functions of α , and are given in the following table::

α	B_0	B_1	B_2	B_3
0	$\infty(\frac{1}{\alpha})$	0	0	0
0.556	2.336	-0.00890	0.000692	-0.000138
0.834	1.734	-0.0116	0.000970	-0.000208
1.11	1.425	-0.0137	0.00122	-0.000272
1.66	1.091	-0.0166	0.00177	-0.000386
2,50	0.902	-0.0191	0.00234	-0.000544
3.33	0.802	-0.0203	0.00274	-0.000682
5.0	0.710	-0.0210	0.00334	-0.000912
10.0	0.612	-0.0212	0.00404	-0.00125
∞	0.520	-0.0192	0.00417	-0.00151

The maximum displacement takes place at the top and its value is given very accurately by,

$$u_0 = 2 j_0 t_0 B_0$$

It is practically twice the statical deformation that would occur under the same acceleration.

Stresses

The total shearing force due to the shearing deformation is

$$F = \mu \frac{\partial u}{\partial x} = j_0 M \frac{\partial \zeta}{\partial \xi}$$

It is the total inertia force $j_0 M$ multiplied by a dimensionless function

$$\frac{\partial \zeta}{\partial \xi} = -2\alpha \sum_0^\infty \frac{\beta_k}{\lambda_k^3} \sin\lambda_k\xi (\cos\lambda_k\tau - 1)$$

which, by putting $B_k' = -\frac{2\alpha\beta_k}{\lambda_k^3} = \lambda_k B_k$ takes the form,

$$\frac{\partial \zeta}{\partial \xi} = \sum_0^\infty \sin\lambda_k\xi (\cos\lambda_k\tau - 1)$$

The values of the B_k' as functions of α are given in the following table:

α	B_0'	B_1'	B_2'	B_3'
0	$-\infty(1/\sqrt{\alpha})$	0	0	0
0.556	-1.588	0.0294	-0.00440	0.00131
0.834	-1.388	0.0394	-0.00622	0.00198
1.11	-1.268	0.0473	-0.00788	0.00259
1.66	-1.123	0.0598	-0.0115	0.00370
2.55	-1.036	0.0712	-0.0155	0.00526
3.33	-0.987	0.0774	-0.0184	0.00665
5.0	-0.937	0.0850	-0.0235	0.00912
10.0	-0.880	0.0912	-0.0291	0.0127
∞	-0.816	0.0905	-0.0328	0.0166

As previously we can easily compute the rest of this series. We encounter the expression - - - - $\sum_1^\infty \frac{1}{k^3} = 1.20205$ (Stieltjes Act. Math. vol. 10 p. 299).
But $\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3} = 1.162$ --Hence

$$\sum_4^\infty \frac{1}{k^3} = 0.04 .$$

The error made by taking only the four first terms is smaller than,

$$\frac{4\alpha}{\pi^3} \times 0.04 = 0.0055\alpha$$

The maximum possible value of $\left|\frac{\partial\zeta}{\partial\xi}\right|$ is.

$$\left|\frac{\partial\zeta}{\partial\xi}\right|_{max} = 2\sum_0^3 \left|B'_k \sin\lambda_k\right| = 2S$$

The following table gives the values of these terms:

α	$\lvert B'_0 \sin\lambda_0\rvert$	$\lvert B'_1 \sin\lambda_1\rvert$	$\lvert B'_2 \sin\lambda_2\rvert$	$\lvert B'_3 \sin\lambda_3\rvert$	S
0.	1.	0	0	0	1.00
0.556	0.999	0.0050	0.00048	0.00077	1.004
0.834	0.994	0.0095	0.00081	0.00017	1.005
1.11	0.986	0.0144	0.00132	0.00031	1.00
1.66	0.965	0.0254	0.00245	0.00064	0.99
2.50	0.946	0.0398	0.00562	0.00132	0.98
3.33	0.931	0.0511	0.00825	0.00218	0.98
5.0	0.908	0.0666	0.0138	0.00420	0.97
1.0	0.870	0.0830	0.0234	0.00875	0.97
∞	0.816	0.0905	0.0328	0.0166	0.95

We could have proved by direct considerations that the maximum value of the sheering force must be twice the static force j_0M . We find actually $2Sj_0M$ for this maximum which is a very good approximation. The above calculation shows furthermore that this maximum sheering force is very nearly reached during the first fundamental oscillation.

§ 4. OSCILLATIONS DUE TO ARBITRARY HORIZONTAL ACCELERATION

Let the horizontal acceleration of the ground be

$$j = j_0 \psi(\tau)$$

We will calculate the corresponding motion of the building by considering the curve $\psi(\tau)$ as composed of an infinite number of small jumps. (fig. 6). Each of these increments can be written $d\psi = \frac{d\psi}{d\tau}d\tau$. By using then the function ζ previously mentioned we get the elementary solution corresponding to the increment at the point θ.

$$\zeta(\tau-\theta)\,\frac{d\psi(\theta)}{d\theta}\,d\theta$$

The function giving the total motion is the integral

$$\eta(\xi\tau) = \int_0^\tau \zeta(\tau-\theta)\,\frac{d\psi(\theta)}{d\theta}\,d\theta .$$

Integrating by parts and noting that $\psi(0) = 0$ we get the more convenient form

$$\eta(\xi\tau) = \int_0^\tau \psi(\theta)\,\frac{d}{d\theta}\zeta(\tau-\theta)\,d\theta,$$

Where $\frac{d}{d\theta}\left[\zeta(\tau-\theta)\right] = -\sum B_k\lambda_k \cos\lambda_k\xi \sin\lambda_k(\tau-\theta).$

Hence more explicitly

$$\eta(\xi\tau) = 2\alpha\sum_0^\infty \frac{\beta_k \cos\lambda_k\xi}{\lambda_k^3}\int_0^\tau \sin\lambda_k(\tau-\theta)\,\psi(\theta)\,d\theta. \qquad (3)$$

The real displacement is $u = j_0 t_0 \eta(\xi\tau)$

Application to resonance

This formula may be applied to the case where the acceleration is a sudden harmonic function of the time. If the frequency of this harmonic acceleration is one of the free oscillation frequencies of the building we have resonance; the amplitudes increase indefinitely. The preceding formula gives the possibility to calculate the rate of increase of the amplitudes.

The function ψ takes the form,

$$\psi(\theta) = 0 \qquad \theta < 0$$

$$\psi(\theta) = \sin\lambda\theta \quad \theta > 0$$

Put

$$\phi_k(\tau) = \int_0^\tau \sin\lambda_k(\tau-\theta)\sin\lambda\theta\,d\theta.$$

$$= \frac{1}{\lambda_k+\lambda}\left[\sin\left(\frac{\lambda+\lambda_k}{2}\tau\right)\cos\left(\frac{\lambda-\lambda_k}{2}\tau\right)\right] + \frac{1}{\lambda_k-\lambda}\left[\sin\left(\frac{\lambda-\lambda_k}{2}\tau\right)\cos\left(\frac{\lambda+\lambda_k}{2}\tau\right)\right]$$

$$\eta(\xi\tau) = -2\alpha\sum_{k=0}^\infty \frac{\beta_k}{\lambda_k^3}\cos\lambda_k\xi\,\phi_k(\tau)$$

If $\lambda = \lambda_k$ we have resonance and the value of $\phi_k(\tau)$ becomes,

$$\phi_k(\tau) = \frac{\tau}{2}\cos\lambda_k\tau + \frac{1}{2\lambda_k}\sin\lambda_k\tau$$

After a certain time the value of $\eta(\xi\tau)$ is reduced to its principal term. The amplitude increases indefinitely.

$$\eta_k(\xi\tau) = -\alpha\frac{\beta_k\tau}{\lambda_k^3}\cos\lambda_k\xi\cos\lambda_k\tau$$

Put

$$C_k = \frac{\alpha\beta_k}{\lambda_k^3} = -\frac{1}{2}B'_k$$

The maximum amplitude takes place at the top and its actual value is

$$u = j_0 t_0^2 C_k \tau = (j_0 t_0 C_k)t$$

(Biot b-3)

The coefficients C_k are given in the following table:

α	C_0	C_1	C_2	C_3
0.	$\infty(1/\sqrt{2})$	0	0	0
0.556	0.794	-0.0147	0.00220	-0.000656
0.834	0.694	-0.0197	0.00311	-0.000990
1.11	0.633	-0.0237	0.00394	-0.00123
1.66	0.562	-0.0299	0.00578	-0.00185
2.5	0.518	-0.0356	0.00778	-0.00263
3.33	0.493	-0.0387	0.00924	-0.00332
5.0	0.468	-0.0425	0.0117	-0.00456
10.0	0.440	-0.0456	0.0145	-0.00635
∞	0.408	-0.0452	0.0164	-0.00830

We have shown previously that the stress is,

$$j_0 M\frac{\partial\eta_k}{\partial\xi} = j_0 M C'_k \tau \sin\lambda_k\xi\cos\lambda_k\tau,$$

where $C'_k = \frac{\alpha\beta_k}{\lambda_k^2} = -\frac{1}{2}B'_k\lambda_k$.

The maximum shearing force occurs between the first and the second floor for the first harmonic and has the value $j_0 M\tau C'_0 \sin\lambda_0$

The maximum shearing forces due to higher harmonics are $j_0 M\tau C'_k$

These can be easily computed from the following table or fig. 7.

α	C'_0	$C'_0 \sin\lambda_0$	C'_1	C'_2	C'_3
0	-0.500	0	0	0	0
0.556	-0.540	-0.340	0.0486	-0.0139	0.00622
0.834	-0.555	-0.397	0.0665	-0.0199	0.00942
1.11	-0.564	-0.438	0.0818	-0.0254	0.0123
1.66	-0.578	-0.496	0.107	-0.0377	0.0177
2.50	-0.596	-0.544	0.132	-0.0517	0.0254
3.33	-0.606	-0.572	0.150	-0.0623	0.0324
5.0	-0.619	-0.600	0.172	-0.0812	0.0451
10.0	-0.635	-0.629	0.196	-0.105	0.071
∞	-0.642	-0.642	0.212	-0.129	0.091

Upper limit of resonance stresses; their rate of increase

Consider the case of resonance with the fundamental harmonic,

$$\frac{\partial\eta}{\partial\xi} = 2\alpha\frac{\beta_0}{\lambda_0^2}\sin\lambda_0\xi\cdot\frac{\tau}{2}\cos\lambda_0\tau + \frac{1}{2}\cdot\frac{2\alpha\beta_0}{\lambda_0^3}\sin\lambda_0\xi\sin\lambda_0\tau + 2\alpha\sum_1^\infty\frac{\beta_k}{\lambda_k^2}\sin\lambda_k\xi\,\phi_k(\tau)$$

The last term may be neglected, for

$$\left|\phi_k(\tau)\right| < \frac{1}{\lambda_k+\lambda_0} + \frac{1}{\lambda_k-\lambda_0} = \frac{2\lambda_k}{\lambda_k^2-\lambda_0^2} = \frac{2}{\lambda_k\left[1-\left(\frac{\lambda_0}{\lambda_k}\right)^2\right]}$$

On the other hand, $\frac{\lambda_0}{\lambda_k} < \frac{1}{3}$

hence,

$$\left|2\alpha\sum_1^\infty\frac{\beta_k}{\lambda_k^2}\sin\lambda_k\xi\,\phi_k(\tau)\right| < \sum_1^\infty\left|B'_k\right|\frac{2}{1-\left(\frac{\lambda_0}{\lambda_k}\right)^2}$$

Noting the values of B in this expression we see that it is comparatively small. We can write,

$$\left|\frac{d\eta}{d\xi}\right| < \tau\left|C'_0\sin\lambda_0\right| + \frac{1}{2}$$

In this case the sheering force is never higher than

$$j_0 M\left[\tau\left|C'_0\sin\lambda_0\right| + \frac{1}{2}\right]$$

The values of $\tau\left|C'_0\sin\lambda_0\right| + \frac{1}{2}$ are plotted in fig. 8.

In case of resonance with an harmoric of higher order

$$\left|\phi_k(\tau)\right| < \frac{2\lambda_i}{\lambda_i^2-\lambda_k^2} \quad \text{if } k<i,$$

$$\left|\phi_k(\tau)\right| < \frac{2\lambda_k}{\lambda_k^2-\lambda_i^2} \quad \text{if } k>i,$$

hence

$$\left|\frac{d\eta_i}{d\xi}\right| < \left|C'_i\right|\tau + \frac{1}{2}\left|B'_i\right| + 4\sum_0^{i-1}C'_k\frac{\lambda_i}{\lambda_i^2-\lambda_k^2} + 4\sum_{i+1}^\infty C'_k\frac{\lambda_k}{\lambda_k^2-\lambda_i^2}$$

§ 5. ACTION OF EARTHQUAKE ACCELERATIONS DERIVED FROM THE SPECTRUM OF A SISMOGRAM

Most of the sismograms show the random character of earthquake acceleration, with very often series of quasi periodic oscillations. It is therefore difficult to identify the action of an earthquake to that of the simple acceleration curves studied till now. We might of course apply equation (3) and calculate the displacement corresponding to a given sismogram.

But it is much more convenient to divide the problem and to analyse separately the elastic properties of the building and the frequency distribution of the earthquake. We need therefore a general theorem which we shall prove here in the special case of building oscillations. As we are more interested in the stresses we will not give the corresponding equations of the displacement; the method being exactly the same as before we get the stresses from the displacement by a simple derivation.

By differentiation, equation (3) may be written

$$\frac{\partial\eta(\xi\tau)}{\partial\xi} = 2\alpha\sum_0^\infty\frac{\beta_k\sin\lambda_k\xi}{\lambda_k^2}\left[\sin\lambda_k\tau\int_0^\tau\cos\lambda_k\theta\,\psi(\theta)\,d\theta - \cos\lambda_k\tau\int_0^\tau\sin\lambda_k\theta\,\psi(\theta)\,d\theta\right] \quad (4)$$

Putting $j(t) = g\psi(\theta)$ the acceleration

and
$$F_1(\omega) = \frac{g t_0}{\pi}\int_0^{\tau} \cos\lambda\theta\,\psi(\theta)d\theta = \frac{1}{\pi}\int_0^{t}\cos\omega t\; j(t)dt.$$
$$F_2(\omega) = \frac{g t_0}{\pi}\int_0^{\tau} \sin\lambda\theta\,\psi(\theta)d\theta = \frac{1}{\pi}\int_0^{t}\sin\omega t\; j(t)dt.$$

Considering then the Fourier integral
$$j(t) = \frac{1}{\pi}\int_0^{\infty} d\omega \int_0^{t} j(\theta)\cos\omega(\theta - t)d\theta,$$
where $j(t)$ represents an acceleration record located in the time interval (Ot) we get
$$j(t) = \int_0^{\infty} F_1(\omega)\cos\omega t\, d\omega + \int_0^{\infty} F_2(\omega)\sin\omega t\, d\omega$$

The expression $\Delta(\omega) = F_1^2(\omega) + F_2^2(\omega)$ represents what might be called the spectral intensity as a function of ω. Coming back to equation (4), we note that $\frac{\partial\eta}{\partial\xi}$ is the resultant of free oscillations of amplitude.
$$\frac{2\alpha\beta_K\cos\lambda_K\xi}{\lambda_K^2}\,\frac{\pi}{g t_0}\sqrt{F_1^2(\omega) + F_2^2(\omega)}$$

The shearing force at a certain moment is the sum of the forces due to each free oscillation existing at that moment. Each of the shearing forces may be written $Mg\frac{d\eta_K}{d\xi}$.

or
$$M.\frac{2\alpha\beta_K\sin\lambda_K\xi}{\lambda_K^2}\cdot\frac{\pi}{t}\sqrt{\Delta(\omega_K)}\cdot\frac{t}{t_0}$$

M is the total mass of the upper floors.

This result can be expressed in a more complete manner as follows:

When an acceleration $j(t)$ acts upon a building during a time interval (Ot), the maximum total shearing force at the final moment is the sum of the shearing forces due to every free oscillation existing at that moment. Each of those shearing forces is a product of three factors;

an effective mass, $M_K(\xi) = M.\frac{2\alpha\beta_K\sin\lambda_K\xi}{\lambda_K^2} = 2MC'_K\sin\lambda_K\xi$

an effective acceleration $\frac{\pi}{t}\sqrt{\Delta(\omega_K)}$

and a coefficient $\frac{t}{t_0}$

The first factor depends only on the elastic properties of the building. The second factor which has the dimension of an acceleration depends essentially on the shape or spectrum of the siemogram. The third one is the relative length of the earthquake. The problem is thus divided in two parts:

1) Calculation of the effective masses for a given building.

2) Calculation of the function $\Delta(\omega)$ or spectrum for a certain number of siemograms of the region. The use of a Fourier analyser will make this part of the work very easy. In Fig. 7 it is shown very clearly that the effective masses decrease very rapidly for values of α smaller than 1.

An advantage of this method is the fact that the effective acceleration may be derived in a very simple way from a ground displacement record. Let $d(t)$ be the displacement.

Putting
$$F_{1d}(\omega) = \frac{1}{\pi}\int_0^{t}\cos\omega t\; d(t)\,dt,$$
$$F_{2d}(\omega) = \frac{1}{\pi}\int_0^{t}\sin\omega t\; d(t)\,dt,$$
we get
$$d(t) = \int_0^{\infty} F_{1d}(\omega)\cos\omega t\, d\omega + \int_0^{\infty} F_{2d}(\omega)\sin\omega t\, d\omega$$

The acceleration takes the form
$$j(t) = d''(t) = -\int_0^{\infty}\omega^2 F_{1d}(\omega)\cos\omega t\, d\omega \int_0^{\infty}\omega^2 F_{2d}(\omega)\sin\omega t\, d\omega,$$
hence
$$\omega^2 F_{1d}(\omega) = F_1(\omega)$$
$$\omega^2 F_{2d}(\omega) = F_2(\omega)$$
Putting then, $\Delta_d(\omega) = F_{1d}^2(\omega) + F_{2d}^2(\omega)$ we find the required formula
$$\sqrt{\Delta(\omega)} = \omega^2\sqrt{\Delta_d(\omega)}$$

This enables us to determine very simply the spectrum of the acceleration record, when we know the spectrum of the displacement record.

The analysis of a siemogram gives a function spectral intensity or effective acceleration $\frac{\pi}{t}\sqrt{\Delta(\omega)}$ having one or more maxima as shown in fig.9. In a given region there generally exist certain characteristic frequencies of the soil which appear in many siemograms. These frequencies will cause the mentioned maxima to be located in the vicinity of fixed values of ω in different records. If three critical values, ω_1, ω_2, ω_3, for the building are in the region of those maxima we should, if possible, place them between or outside the peaks. The maximum stresses are given by,
$$\pi\frac{t}{t_0}\left[M_1(\xi)\frac{\sqrt{\Delta(\omega_1)}}{t} + M_2(\xi)\frac{\sqrt{\Delta(\omega_2)}}{t} + M_3(\xi)\frac{\sqrt{\Delta(\omega_3)}}{t}\right]$$
$$= \frac{2\pi M}{t_0}\left[C'_1\sin\lambda_1\xi\sqrt{\Delta(\omega_1)} + C'_2\sin\lambda_2\xi\sqrt{\Delta(\omega_2)} + C'_3\sin\lambda_3\sqrt{\Delta(\omega_3)}\right]$$

In order to illustrate this method we will calculate the spectrum of a finite sine-wave of total length T :
$$j(t) = 0 \qquad t < -\frac{T}{2} \qquad t > \frac{T}{2}$$
$$j(t) = j_0\sin\Omega t \qquad -\frac{T}{2} < t < \frac{T}{2}$$

(Biot, b-4)

$$F_1(\omega) = \frac{j_0}{\pi}\int_{-\frac{T}{2}}^{+\frac{T}{2}}\cos\omega t\,\sin\Omega t\, dt = 0$$
$$F_2(\omega) = \frac{j_0}{\pi}\int_{-\frac{T}{2}}^{+\frac{T}{2}}\sin\omega t\,\sin\Omega t\, dt = \frac{j_0}{\pi}\cdot\left[\frac{\sin(\omega-\Omega)\frac{T}{2}}{\omega-\Omega} - \frac{\sin(\omega+\Omega)\frac{T}{2}}{\omega+\Omega}\right]$$

Near the maximum we have approximately,
$$F_2(\omega) = \frac{j_0}{\pi}\,\frac{\sin(\omega-\Omega)\frac{T}{2}}{\omega-\Omega}$$
$$\sqrt{\Delta(\omega)} = \frac{j_0}{\pi}\left|\frac{\sin(\omega-\Omega)\frac{T}{2}}{\omega-\Omega}\right|$$

This function is plotted in fig. 10. The shearing forces corresponding to different free oscillations are,
$$M_K(\xi)\frac{j_0}{t_0}\left|\frac{\sin(\omega_K-\Omega)\frac{T}{2}}{\omega_K-\Omega}\right|.$$

In case of resonance with the harmonic of order k the corresponding shearing force is,
$$M_K(\xi)\frac{j_0}{t_0}\,\frac{T}{2}$$

The effective mass has the value
$$M_K = M.2C'_K\sin\lambda_K\xi$$

The shearing force may be put in the form already obtained by a previous method,
$$Mj_0C'_K\sin\lambda_K\xi\frac{T}{t_0}$$

§ 6. GENERALIZED METHOD FOR BUILDINGS HAVING VARIABLE MASS AND RIGIDITY AT THE DIFFERENT FLOORS

The method is absolutely general and could be used for any type of building considering both bending and shearing deformations. We will restrict ourselves to the shear in the case of variable $\mu(x)$ and $m(x)$. The equation of motion takes the form,
$$\frac{\partial}{\partial x}\left[\mu(x)\frac{\partial u}{\partial x}\right] = m(x)\frac{\partial^2 u}{\partial t^2} + m(x)j(t).$$

If the acceleration is an harmonic function of time $j = j_0\sin\omega t$, the solution is $u = y\sin\omega t$, and
$$\frac{d}{dx}\left[\mu(x)\frac{dy}{dx}\right] + m(x)\omega^2 y = m(x)j_0.$$

Expressing the solution in terms of the orthogonal functions corresponding to the free oscillations and defined by the equations,
$$\frac{d}{dx}\left[\mu(x)\frac{dy_i}{dx}\right] + m(x)\omega_i^2 y_i = 0,$$
we get $y = \sum_0^{\infty} A_i y_i$,
$$\sum_0^{\infty} A_i\left[\omega^2 - \omega_i^2\right]y_i = j_0$$

After multiplying both sides by y_i and integrating along the total height of the building
$$A_i = \frac{j_0 C_i}{\omega^2 - \omega_i^2} \quad \text{with} \quad C_i = \frac{\int_0^h y_i\, dh}{\int_0^h y_i^2\, dh}$$

Finally, $y = j_0\sum_0^{\infty}\frac{C_i}{\omega^2 - \omega_i^2}y_i$

The displacement corresponding to a sudden acceleration is
$$u = j_0\sum_0^{\infty}\frac{C_i}{\omega_i^2}y_i(x)\left[\cos\omega_i t - 1\right]$$

The amplitude of each free oscillation is given as a function of the spectral intensity $\Delta(\omega)$ by,
$$u_i = j_0\frac{C_i}{\omega_i^2}y_i(x)\,\pi\sqrt{\Delta(\omega_i)},$$

and the stress at the coordinate x due to that oscillation by,
$$\mu(x)\frac{\partial u_i}{\partial x} = j_0\frac{C_i}{\omega_i^2}\,\frac{dy_i}{dx}\cdot\pi\sqrt{\Delta(\omega_i)}$$

The total maximum stress is the sum of all these expressions.

The preceding considerations show that the problem is solved whenever we know the set of orthogonal functions y_i multiplied by any factor of proportionality, i.e. when we know the shape of the free oscillations.

From y_i we easily deduce the frequency or the corresponding ω_i by simple energy considerations. Consider the building oscillating with that frequency. The motion is supposed to be free. When the amplitude is maximum the kinetic energy is equal to zero and the potential energy is
$$\frac{1}{2}\int_0^h \mu(x)\left(\frac{dy_i}{dx}\right)^2 dx.$$

On the other hand, the potential energy passes through zero when the strain disappears; at that moment the kinetic energy is maximum and has the value
$$\frac{1}{2}\int_0^h m(x)\omega_i^2 y_i^2\, dx.$$

Equating those two expressions we get the value

$$\omega_i^2 = \frac{\int_0^h \mu(x)\left(\frac{dy_i}{dx}\right)^2 dx}{\int_0^h m(x)\, y_i^2\, dx} \qquad (5)$$

which is independent of any arbitrary constant multiplying y_i

THE CALCULATION OF THE ORTHOGONAL FUNCTIONS

The orthogonal functions y_i may be found by two methods. One is semi empirical and very simple. We note that those functions are defined by equation, which by a change of the independent variable

$$z = \int \frac{dx}{\mu(x)}$$

becomes

$$\frac{1}{\mu[x(z)]}\,\frac{d^2}{dz^2}\left[y_i(z)\right] + m\left[x(z)\right]\omega^2 y_i = 0. \qquad (6)$$

This is the equation of buckling under a load P of a beam of moment of inertia.

$$I(z) = \frac{P}{Em\mu\omega^2}$$

If we consider an elastic strip of uniform thickness h, its moment of inertia will have the value I under the condition that the variable width satisfies the equation,

$$\frac{\ell h^3}{12} = I = \frac{P}{Em\mu\omega^2} = \frac{A}{\mu m}$$

Where A is an arbitrary constant, we may choose for z any scale convenient for we are only interested in the shape of the functions y_i

In order to realize the given boundary conditions, the strip will be repeated symmetrically around a point representing the top of the building. (fig. 11). The deformation of the half strip under different loads will give the different functions y_i. Only the first deformation is stable, so that the others will have to be stabilized by a special but very simple device. We then compute by the energetic method the corresponding values of the frequencies. (equation 5).

Another method is analytical and might be useful in case of only slight deviations from the case of constant rigidity and mass previously investigated. It is known in atomic physics as "perturbation calculus."

Putting

$$m(x) = m_o + \epsilon(x),$$
$$\mu(x) = \mu_o + \gamma(x),$$
$$y_i(x) = y_{oi} + \delta_i(x),$$
$$\omega_i^2 = \omega_{io}^2 + \rho_i,$$

where $m_o \mu_o y_{oi} \omega_{io}^2$ are known and $\epsilon, \gamma, \delta_i, \rho_i^2$ are small variations,

Equation (6) becomes

$$\frac{d}{dx}\left[(\mu_o+\gamma(x))\frac{d}{dx}(y_{io}(x)+\delta(x))\right] + \left[(m_o+\epsilon(x))(\omega_{io}^2+\rho_i)(y_{io}+\delta(x))\right] = 0$$

Taking into account the identity

$$\frac{d}{dx}\left[\mu_o \frac{dy_{oi}}{dx}\right] + m_o\omega_{io}^2 y_i = 0.$$

and neglecting the small terms of higher order, we finally get

$$\frac{d}{dx}\left[\mu_o \frac{d}{dx}\delta(x)\right] + m_o\omega_{io}^2\delta(x) = -\frac{d}{dx}\left[\gamma(x)\frac{d}{dx}(y_{io}(x))\right] - y_{io}\left[\epsilon(x)\omega_{io}^2 + \rho_i m_o\right]$$

This equation has only finite solutions if the function of the left side is orthogonal to the characteristic functions of the equation obtained by equating the left side to zero (1).

This condition may be written,

$$\int_0^h y_{io}\frac{d}{dx}\left[\gamma(x)\frac{d}{dx}y_{io}(x)\right] + \omega_{io}^2\int_0^h y_{io}\,\epsilon(x) + \rho_i m_o\int_0^h y_{io}^2\,dx = 0.$$

(Biot, b-5) and gives the values of ρ_i.

(1) Hilbert Courant Chap. V p. 277

The second side of the equation is then entirely known and δ_i is determined by

$$\frac{d}{dx}\left[\mu_o \frac{d}{dx}\delta(x)\right] + m_o\omega_{oi}^2\,\delta(x) = f_i(x)$$

Expanding the solution in terms of the orthogonal functions

$$\delta(x) = \sum A_K y_{oK},$$

putting this expression back into the equation,

$$\sum A_K m_o\left[\omega_{oi}^2 - \omega_{oK}^2\right] y_{oK} = f_i(x),$$

multiplying then both sides by y_{K_o} and integrating from 0 to h we get,

$$A_K = \frac{C_K}{\omega_{io}^2 - \omega_{Ko}^2} \qquad i \neq k$$

where

$$C_K = \frac{\int_0^h y_{oK} f_i(x)}{\int_0^h m_o y_{oK}^2(x)\,dx}$$

Hence

$$\delta_i(x) = \sum_0^\infty \frac{C_K}{\omega_{io}^2 - \omega_{Ko}^2}\, y_{oK} \qquad k \neq i.$$

and the required orthogonal functions may finally be written,

$$y_i = y_{oi} + \sum_0^\infty \frac{C_K}{\omega_{io}^2 - \omega_{oK}^2}\, y_{ko} \qquad k \neq i.$$

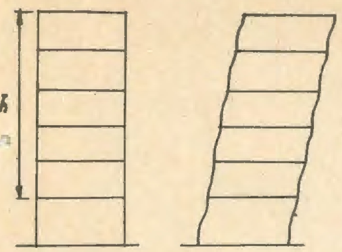


FIG. 1

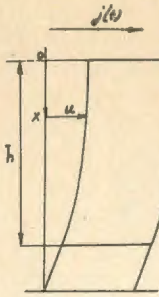


FIG. 2

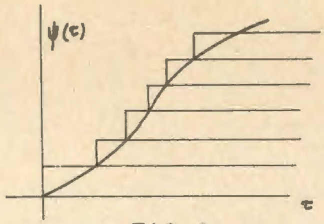


FIG. 6

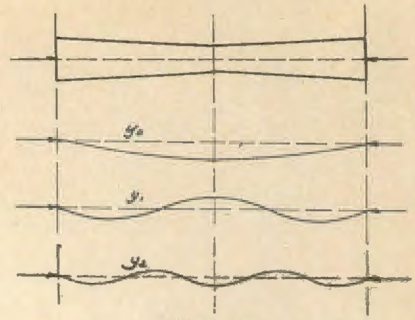


FIG. 11

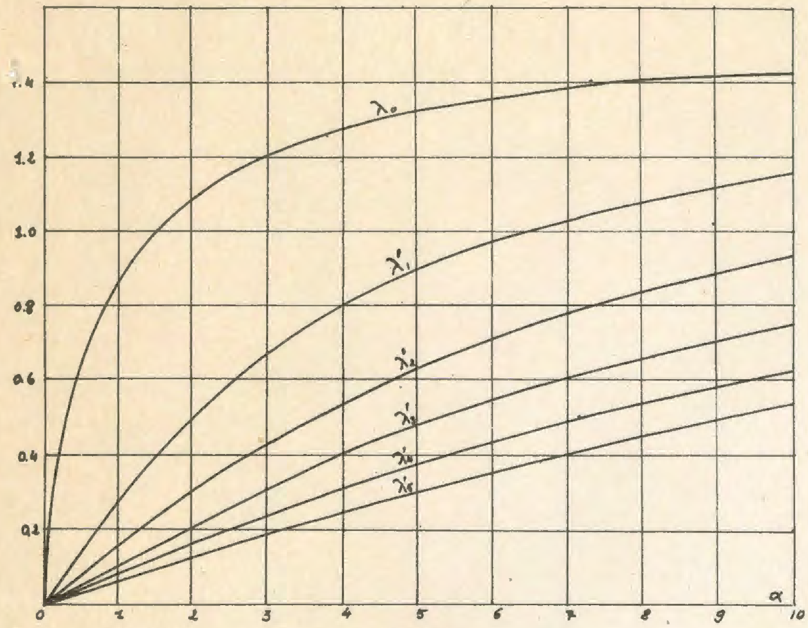


FIG. 3

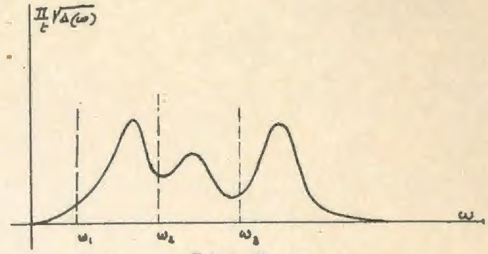


FIG. 9

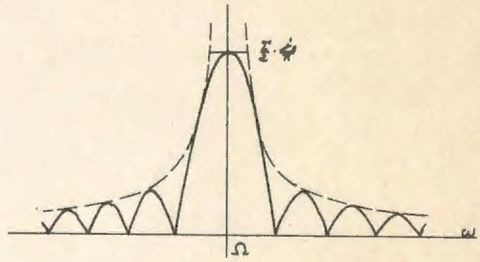


FIG. 10

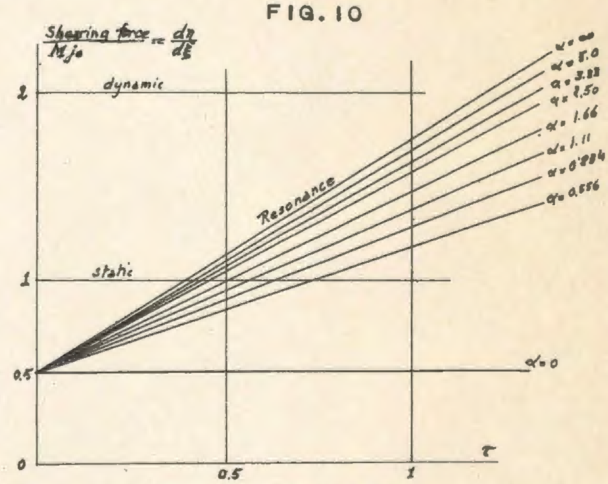


FIG. 8

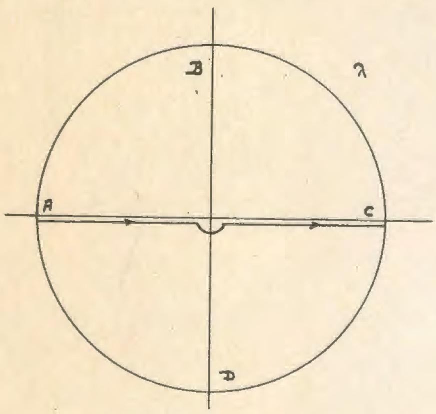


FIG. 4

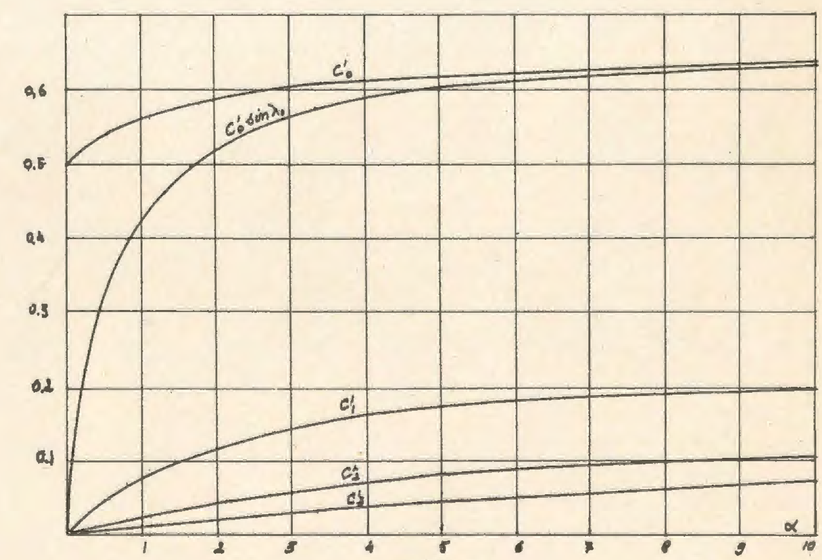


FIG. 7

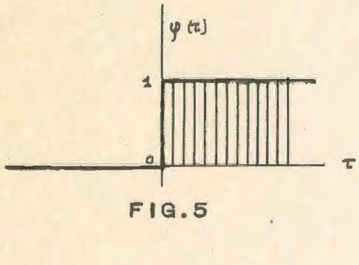


FIG. 5

Biot, b-6

THE RECORDING SHAKING TABLE

Fig. 1., is a photograph of Professor R. R. Martel's Recording Shaking Table, at the California Institute of Technology. The "earthquake" is applied at the base of the bent by means of a "quake stick." The Quake stick is drawn thru a rigid holder, on a level with the base of the bent. One edge of the quake-stick is cut to an irregular earthquake curve, and bears against a roller attached to the base of the bent. The bent is forced against the stick by stiff spring supports from underneath.

The absolute motion of the "ground" and of the upper floors is recorded on tapes perforated by electric sparks. A constant-speed motor and rollers draws the tapes thru the bent between the points of spark gaps attached to the floors. The record is a closely spaced series of perforations, which may be traced out in pencil. The different floor records are synchronized by applying the spark before the "quake" is started.

METHOD OF CALCULATION

The problem is to take a record of ground motion, and from this predict, by Dr. Biot's method, the displacement of the upper floors relative to the base. The check on this prediction is the actual upper floor records obtained.

Figure 2., shows the calculation of the constants for the six-story bent.

For a given bent of say, six stories, the relative deflection of any floor at any instant T during the earthquake will be given by the sum of six terms of the form

$$j_0 t_0^2 B_K \cos \lambda_K \xi \cdot 2\pi \nu_K F(\nu_K) .$$

Here j_0, t_0, B_K, λ_K, ξ, AND ν_K are constants of the bent and earthquake, and $F(\nu_K)$ is

$$(2\pi\nu_K T)^2 \sqrt{\left[\int_0^T G(t)\cos 2\pi\nu_K t \, dt\right]^2 + \left[\int_0^T G(t)\sin 2\pi\nu_K t \, dt\right]^2} .$$

The five harmonic terms are all negligible, usually, so that the first term is all that need be calculated. By calculating this term at intervals during the earthquake, a curve of maximum possible deflections is obtained, and from this the highest possible stresses may be calculated.

Expression (1.) , Fig. 2., gives the displacement at any instant during the quake, at any point along the height of the building (defined by ξ) for the fundamental mode. The quantity under the radical has a definite value for any instant during the earthquake, and is obtained from the areas of the curves shown at the bottom of Fig. 3.

From the table of B 's, page 7 , it can be seen that the calculation for the fundamental frequency is usually the only one that need be made. In the case of this bent, for example, $B_0 = 0.710$, while $B_1 = -0.0210$, $B_2 = 0.00334$, AND $B_3 = -0.000912$.

In obtaining the values of $\int_0^T G(t) \cos 2\pi\nu_0 t \, dt$ and $\int_0^T G(t) \sin 2\pi \nu_0 t \, dt$ up to any instant, the principle of the mechanical harmonic analyzer is used. Cosine and sine curves of frequency ν_0 are drawn at 1. and 2. , Fig. 3. Ground motion ordinates are plotted horizontally from the corresponding points on the cosine and sine curves, taking signs into account. The area on the cosine curve up to a given point, divided by $2\pi\nu_0$, is

$\int_0^T G(t) \sin 2\pi\nu_0 t \, dt$ for that point, and similarly for 2.

Calculations for point A-A, Fig. 3., for example, were as follows:

Planimeter value for 1. up to A-A: = −1.88

for 2. = +0.87

$$\sqrt{(1.88)^2 + (.87)^2} = 2.47 \quad .$$

Multiplying this by 1.11 , the planimeter constant, to reduce the reading to square inches; dividing by 1.5 because the cosine and sine curves were laid out with half amplitudes of 1.5 inches instead of unity, for convenience; and dividing by $2\pi V_0$ from the principle of the harmonic analyzer,

$$u_0 \text{ , SIXTH FL.} = \cos(1.32 \cdot 0) \cdot 40.2 \, \frac{1.11 \cdot .194}{1.5 \cdot 2\pi} \, 2.07$$

$$= \cos(0) \cdot .920 \cdot 2.07$$

$$= 1.91''$$

$$u_0 \text{ , FOURTH FL.} = \cos\left(1.32 \cdot \tfrac{1}{3}\right) \cdot .920 \cdot 2.07$$

$$= .936 \cdot 1.91$$

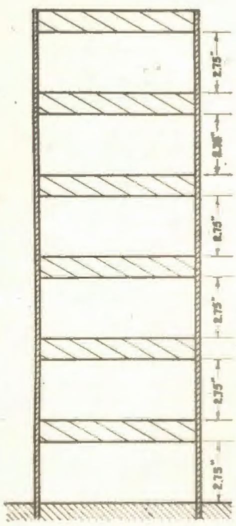
$$= 1.79''$$

$$u_0 \text{ , 2ND. FL.} = \cos\left(1.32 \cdot \tfrac{2}{3}\right) \cdot .920 \cdot 2.07$$

$$= .752 \cdot 1.91$$

$$= 1.43'' \quad .$$

Calculation of Constants for the Six-Story Bent



Six-Story Bent
Uniform Rigidity Throughout

$R = 1$

$n = 5$

$\alpha = Rn = 5$

$T_0 = 0.194$ sec., by experiment.

$\nu_0 = \frac{1}{0.194}$

$K = 33.4^{\#}$, by experiment.

Solving $\lambda \tan \lambda = \alpha$, $\lambda_0 = 1.32$

From $T_k = \frac{2\pi t_m}{\lambda_k}$: $t_0 = \frac{T_0 \lambda_0}{2\pi} = \frac{0.194 \cdot 1.32}{2\pi}$ 0.0408

$B_0 = 0.710$, from table on page 7.

$j_0 = \frac{1}{T^2}$.

From page 7.,

$$u_k(x) = j_0 t_m^2 B_k \cos \lambda_k \xi \cdot 2\pi \nu_k F(\nu_k)$$

From page 9.,

$$F(\nu_k) = (2\pi \nu_k T)^2 \sqrt{\left[\int_0^T G(t) \sin 2\pi \nu_k t\,dt\right]^2 + \left[\int_0^T G(t) \cos 2\pi \nu_k t\,dt\right]^2}$$

Then,

$$u_0 = j_0 t_0^2 B_0 \cos \lambda_0 \xi \cdot 2\pi \nu_0 (2\pi \nu_0 T)^2 \sqrt{\left[\int_0^T G(t) \sin 2\pi \nu_0 t\,dt\right]^2 + \left[\int_0^T G(t) \cos 2\pi \nu_0 t\,dt\right]^2} \qquad (1.)$$

Substituting values from above,

$$u_0 = \frac{1}{T^2} (0.0408)^2 (0.710) \cos 1.32 \xi \cdot 2\pi \frac{1}{0.194} \left(2\pi \frac{1}{0.194} T\right)^2 \sqrt{\left[\int_0^T G(t) \sin 2\pi \nu_0 t\,dt\right]^2 + \left[\int_0^T G(t) \cos 2\pi \nu_0 t\,dt\right]^2}$$

$$u_0 = 40.2 \cos 1.32 \xi \sqrt{\left[\int_0^T G(t) \sin 2\pi \nu_0 t\,dt\right]^2 + \left[\int_0^T G(t) \cos 2\pi \nu_0 t\,dt\right]^2}$$

FIG. 2.

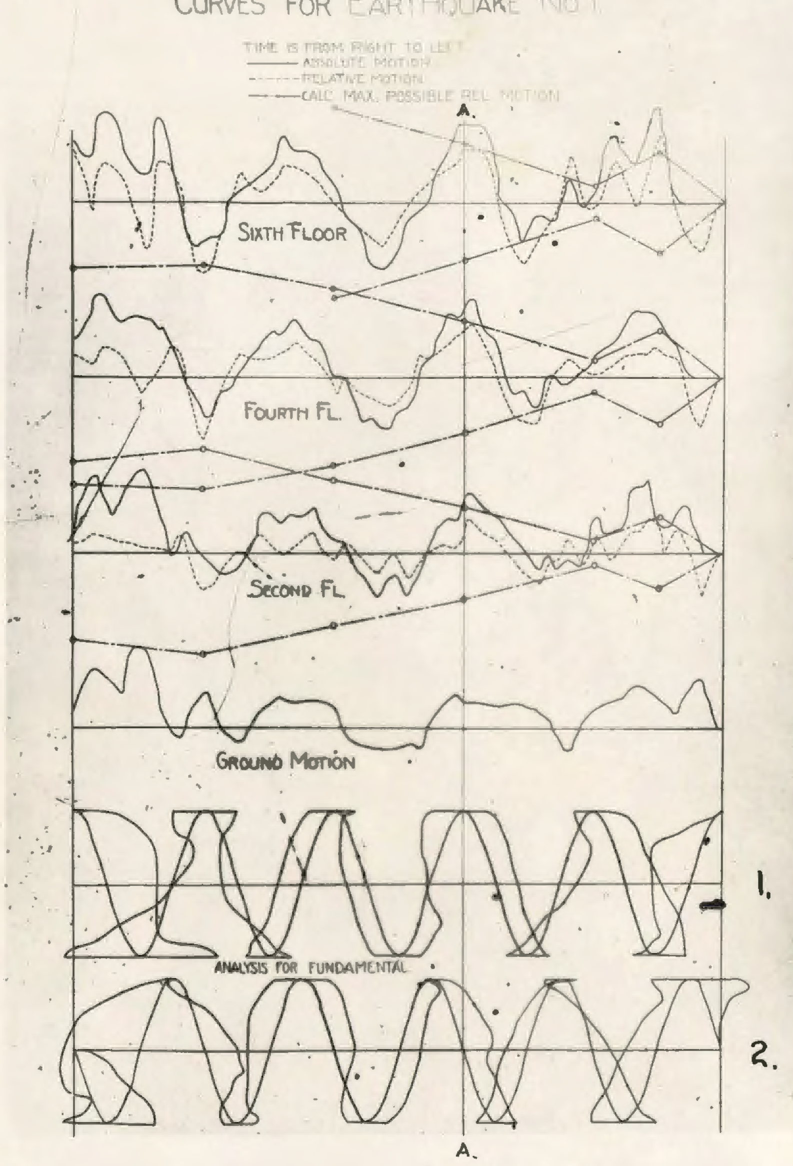


FIG. 3.

CURVES FOR EARTHQUAKE NO. I.

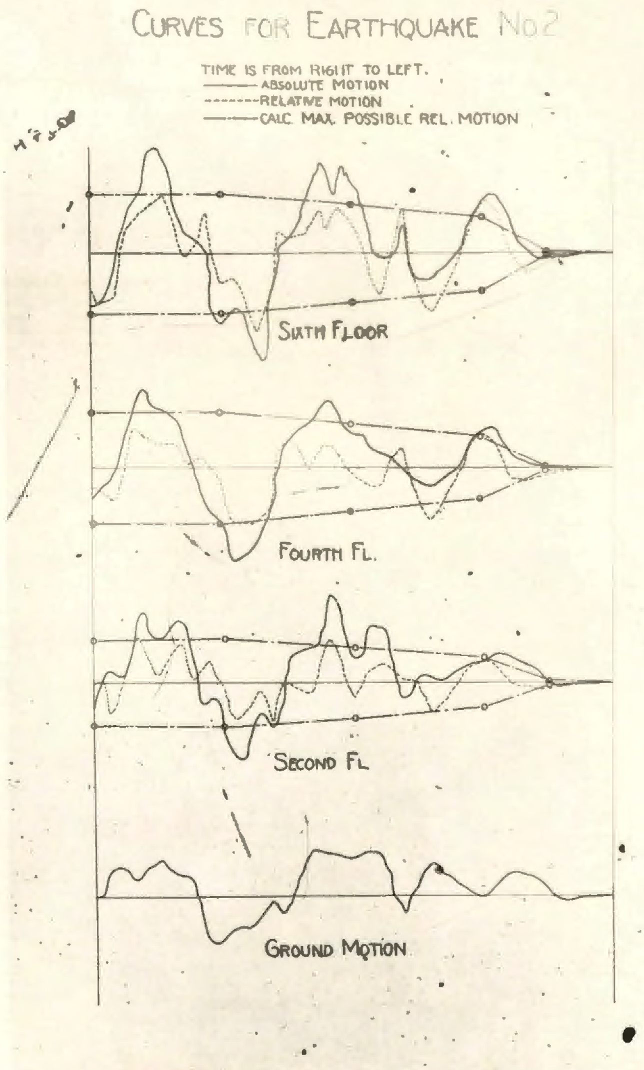


FIG. 4.

CURVES FOR EARTHQUAKE NO. 2.

APPENDIX 1.

GETTING THE $\sqrt{\qquad}$ VALUE

The principle of the Mechanical Harmonic Analyzer is very satisfactorily set forth in the following paper: "Design and Construction of an Harmonic Analyzer," Thesis, Jessie W. M. Du Mond, 1916, California Institute of Technology.

To get the value of

$$\int_0^T G(t)\, \underline{\underline{\sin}}\, 2\pi \nu_0 t\, dt$$

at any time T, the ground motion $G(t)$ ordinates are plotted horizontally out from corresponding $\underline{\underline{\cos}}\ 2\pi\nu_0$ curve points. The area inclosed by these two curves up to the point T, <u>divided by $2\pi\nu_0$</u>, according th the principle of the harmonic analyzer, is the required value. If the half amplitudes of the harmonic curves have been made other than unity, the areas are affected in direct proportion.

If the ground displacement ordinate at any point is <u>positive</u>, and the harmonic curve is <u>ascending</u>, the increment in area is <u>positive</u>.

If the ground displacement is positive and the harmonic curve is <u>descending</u>, the increment in area is <u>negative</u>.

If the ground displacement is <u>negative</u>, and the harmonic curve is <u>ascending</u>, the increment in area is <u>negative</u>.

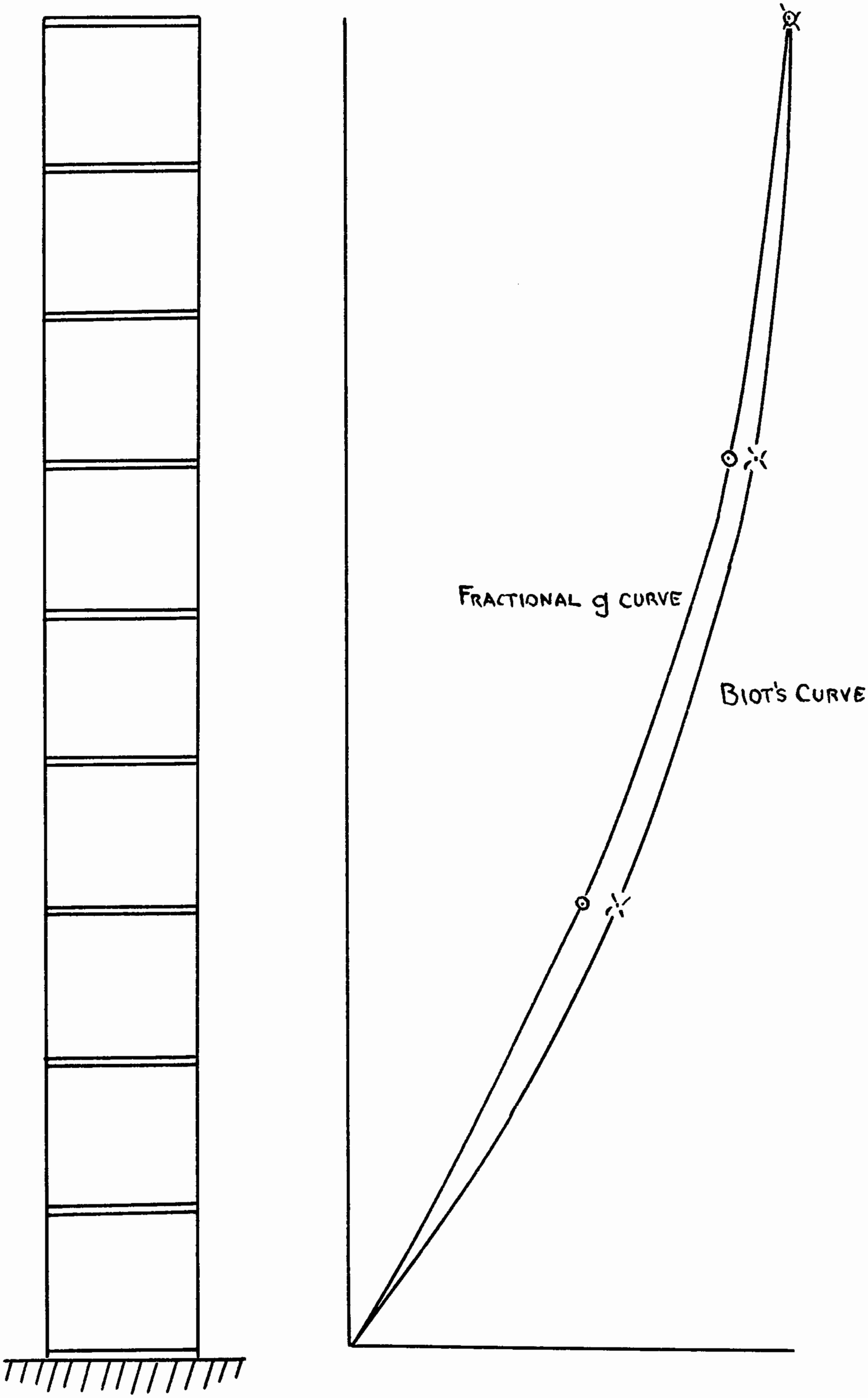
If the ground displacement is negative and the harmonic curve is <u>descending</u>, the increment in area is <u>positive</u>.

APPENDIX 2.

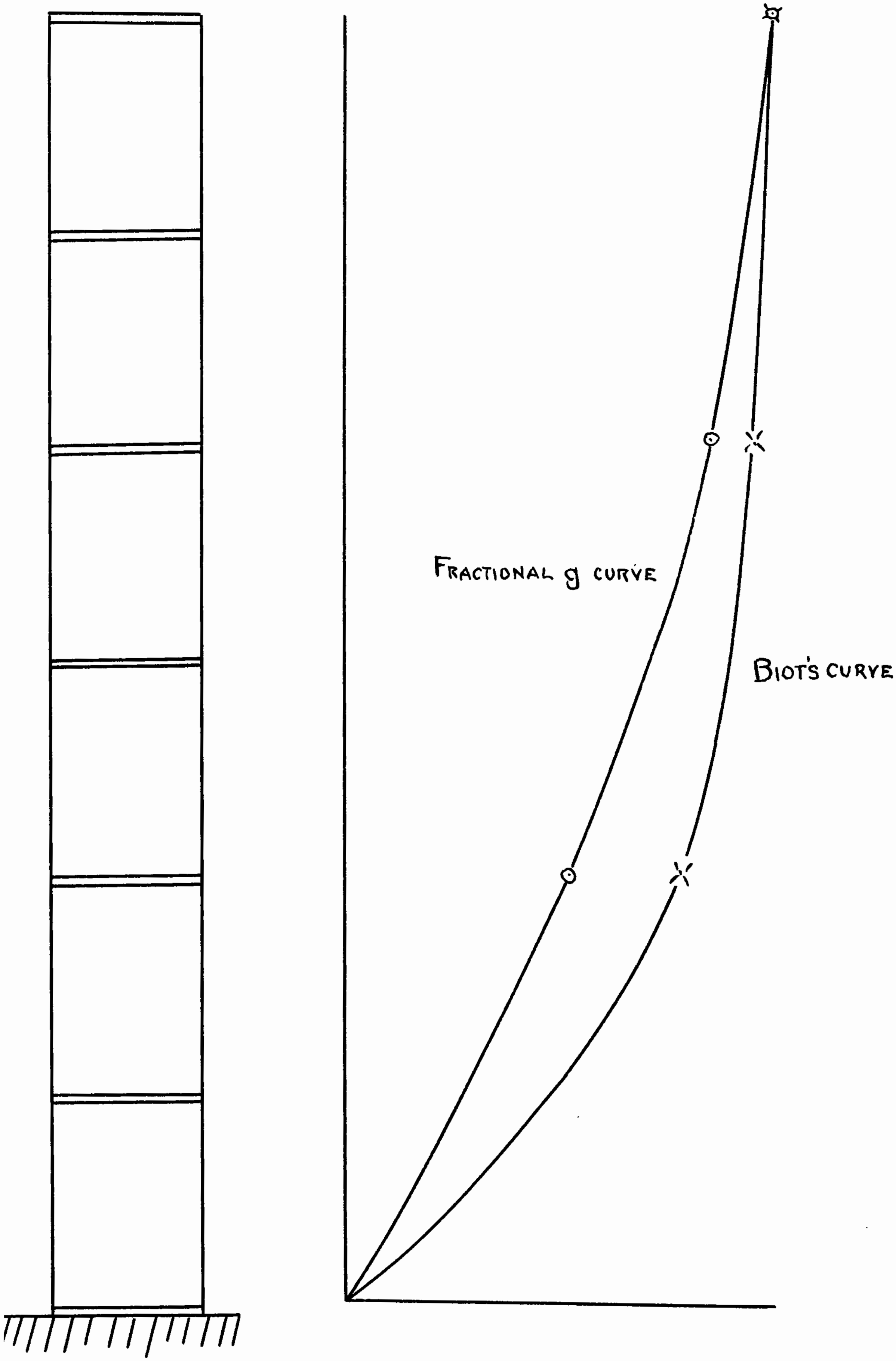
BIOT VS. ONE-TENTH G

If the profile curve of a bent during an earthquake, as predicted by Dr. Biot's method of analysis, has the same SHAPE as that obtained by assuming a constant horizontal acceleration of some fraction of g, then we need not concern ourselves furthur with that difficult method, except in so far as it might lead us to altering the value of the fraction of g we are using.

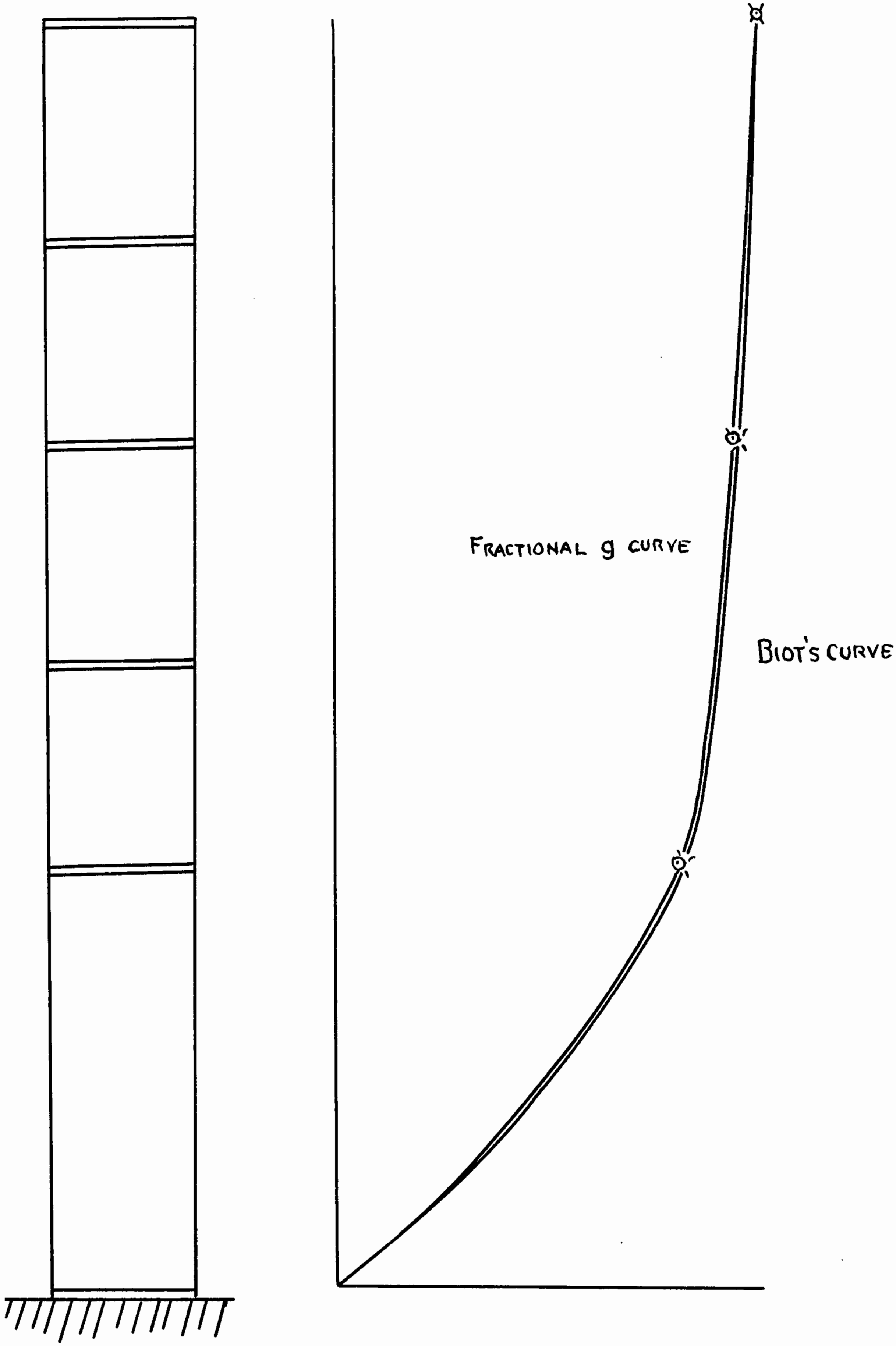
The following curves compare the shape of the profile of the bents obtained by the two methods, for points taken at random during an earthquake. The order of Magnitude of the accelerations that the quake stick produces is much greater than one-tenth g, as was found after calculating the displacements of the bent for this acceleration, and comparing with the records actually obtained with the machine. Hence, in the following graphs, the curves by Dr. Biot's method and the curves by the constant acceleration method have been multiplied by a constant, so that they coincide at the top. This is legitimate for SHAPE considerations.



9-STORY UNIFORM RIGIDITY THROUGHOUT



6-Story   Uniform Rigidity Throughout



5-STORY FLEXIBLE FIRST FLOOR

APPENDIX 3

METHOD OF GETTING WORKING CURVES FROM ORIGINAL RECORDS

It was found not satisfactory to attempt plotting directly from the ground motion tape to the harmonic analyzing curves.

Enlarging the tape records with a pantograph was found not satisfactory.

The method finally used was to trace the tape perforations out in pencil, the tape being laid on a glass with a light shining up from underneath, and then to project these tapes with a magic lantern (grateful acknowledgement to William Beard, Instructor in Technology and Government, and genius in things mechanical) onto a working sheet, on which it was possible to trace them out fairly quickly at the proper enlargement.

An enlargement of five diameters was used, and to guard against either vertical or horizontal foreshortening, each tape was blocked in, this block enlarged five times with the dividers, and drawn at the desired place on the work-sheet. Then adjustment of the lantern was made so that the block image just hit this block already drawn on the work-sheet.

APPENDIX 4

THE "WEST FORMULA"

A recognized method of designing earthquake-resistant buildings is to provide resistance to a horizontal acceleration of some fraction of g, the acceleration due to gravity. This value is arrived at by observing standing and overturned block-like structures in a region just visited by a destructive earthquake. The dimensions of the structures substituted in the "West Formula" give a probable upper value for the magnitude of the horizontal acceleration.

The West formula may be derived as follows: In Fig. 5., suppose the monument with center of gravity C. G. as shown is given a

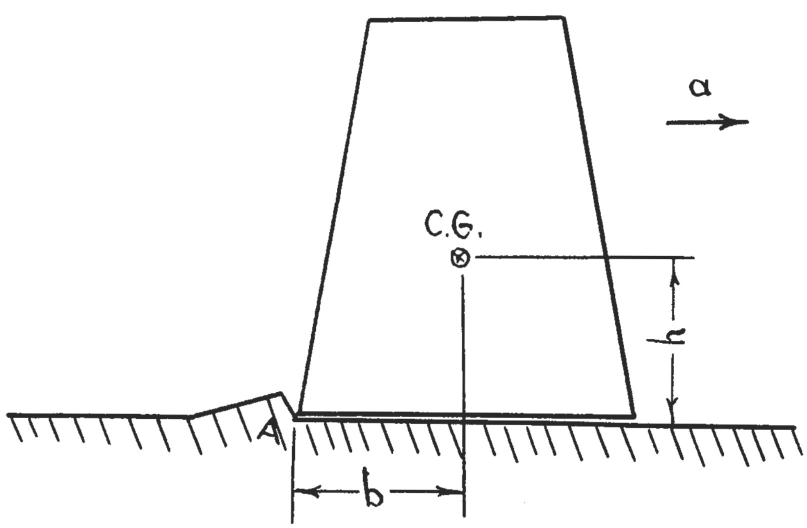


FIG. 5.

horizontal acceleration "a" to the right, by the ground. If this acceleration is of a certain critical amount, the block will be on the verge of rotating in a counter-clockwise direction about A. The free body diagram for the block will be as shown in Fig. 6.

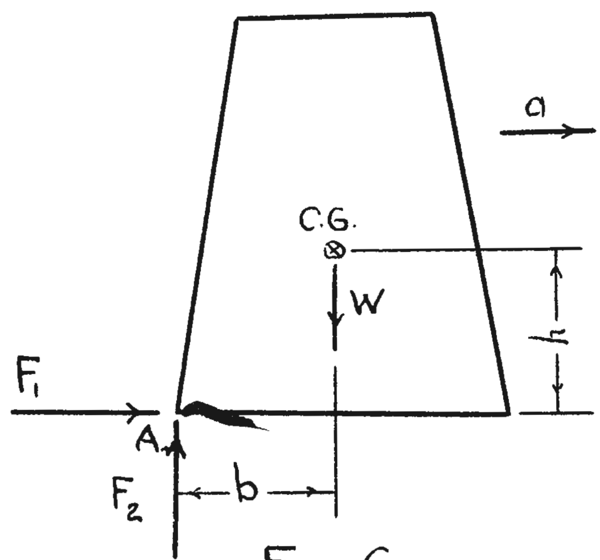


FIG. 6.

From the principles of Kinetics, with translation occuring, and rotation not occuring:

$$\sum F_x = \frac{W}{g} a = F_1$$

$$\sum F_y = 0 = F_2 - W \qquad F_2 = W.$$

$$\sum M_{c.G.} = 0 = F_1 h - F_2 b = \frac{W}{g} ah - Wb$$

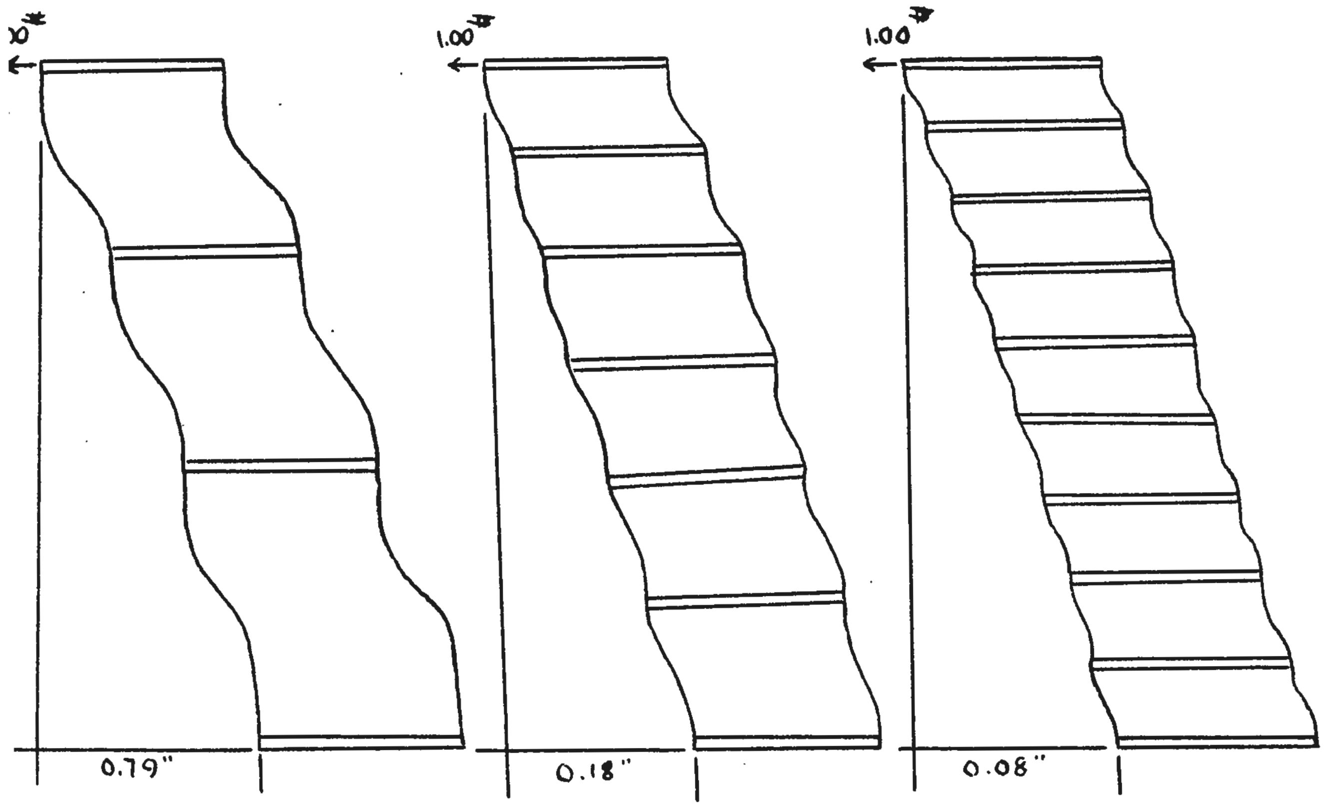
from which we have the West formula:

$$a = \frac{b}{h} g \qquad (1.)$$

APPENDIX 5

EXPERIMENTAL DETERMINATION OF THE "K's"

The stiffnesses of the different bent arrangements were determined by applying a horizontal force at the top with a pulley and weights, and measuring the resulting deflection at the top.



$K = \frac{3 \cdot 1.00}{0.79} = 3.80^{\#}$ $\qquad$ $K = \frac{6 \cdot 1.00}{0.18} = 33.4^{\#}$ $\qquad$ $K = \frac{9 \cdot 1.00}{0.08} = 113.^{\#}$

The values of K obtained using the expression $\frac{12EI}{h^3}$, estimating E, and calculating I from the dimensions of the steel, were found to be quite far from the experimental.

APPENDIX 6

DETERMINATION OF SPEED OF TAPE

To get the time scale of the curves, a tape was run thru the machine for seven minutes, and the length between beginning and ending sparks measured. The resulting calibration was

$$1'' = 0.356 \text{ sec.}$$

The error in observing the time might have been two seconds at both ends, and the tape measurement might have been off a tenth of a foot in the hundred feet, because of the spark record being over an inch long, so that the possible error is about

$$\frac{4}{7 \cdot 60} \cdot 100 + \frac{0.1}{100} \cdot 100 = 1.\%$$

APPENDIX 7

DETERMINATION OF NATURAL FREE PERIODS OF BENTS

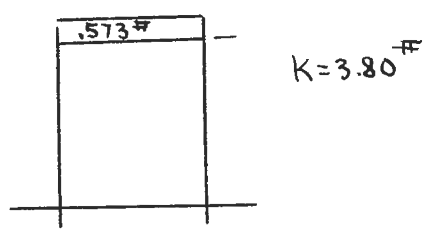
By wedging the bent at different points a variety of bents can be obtained. A bent has as many natural free periods as it has floors. There are three methods of getting these periods:

(1) Tape record

(2) Equations of Kinetics, knowing the constants of the bent

(3) Approximately, from Dr. Biot's equations

Bent No. 1.



1st. method: Period = 0.140 sec.

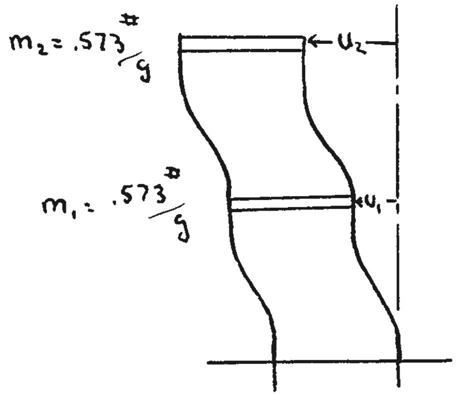
2nd. method:

Simple harmonic motion,

$$\text{Period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{K}{m}}} = 2\pi\sqrt{\frac{.573}{32.2 \cdot 12 \cdot 3.80}} = 0.124 \text{ sec.}$$

3rd. method: cannot be used.

Bent No. 2.



$K = 3.80^{\#}$

1st. method: Fundamental period = 0.240 sec.

First harmonic = 0.089 sec.

2nd. method:

Restoring force $\propto$ displacement, so S. H. M., and acceleration = $\omega^2 u \sin \omega t$.

For maximum displacement

$$\begin{cases} m_2 \omega^2 u_2 = K(u_2 - u_1) \\ m_1 \omega^2 u_1 = K u_1 - K(u_2 - u_1) \end{cases}$$

m's are equal, so let $\dfrac{m\omega^2}{K} = \beta$

Then

$$\begin{cases} \beta u_2 = u_2 - u_1 \\ \beta u_1 = 2u_1 - u_2 \end{cases}$$

$$\begin{cases} \dfrac{u_1}{u_2} = 1 - \beta \\ \dfrac{u_1}{u_2} = \dfrac{-1}{\beta - 2} \end{cases}$$

$$1-\beta = \frac{-1}{\beta-2}$$

$$\beta-1 = \frac{1}{\beta-2}$$

$$\beta^2 - 3\beta + 1 = 0$$

$$\beta = \frac{3 \pm \sqrt{9-4}}{2} = 1.5 \pm \tfrac{1}{2}\sqrt{5} = +2.618 \text{ AND } +0.382$$

SO

$$\omega^2 = \frac{\beta K}{m} = \frac{(2.618 \text{ AND } .382)}{.582} 380 \cdot 32.2 \cdot 12$$

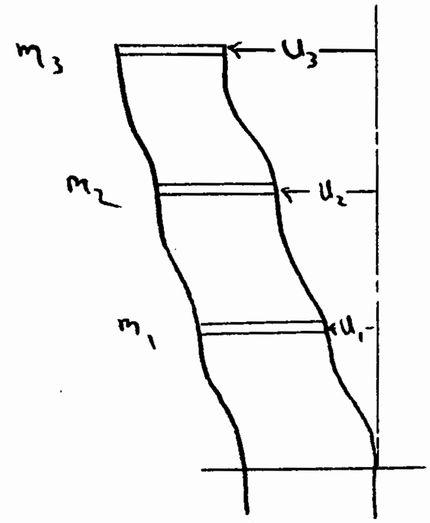
$$= 6600 \text{ AND } 963.$$

$$\text{FUNDAMENTAL PERIOD} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{963}} = 0.202 \text{ SEC.}$$

$$\text{FIRST HARMONIC} = \frac{2\pi}{\sqrt{6600}} = 0.077 \text{ SEC.}$$

3RD. METHOD CAN HARDLY BE USED.

BENT No. 3.



$K = 3.80^{\#}$

METHOD No. 1.

FUNDAMENTAL PERIOD = 0.340 SEC.

FIRST HARMONIC = 0.116 SEC.

SECOND HARMONIC = 0.0757 SEC.

METHOD No. 2.

$$\begin{cases} m\omega^2 u_3 = K(u_3 - u_2) \\ m\omega^2 u_2 = K(u_2 - u_1) - K(u_3 - u_2) \\ m\omega^2 u_1 = K u_1 - K(u_2 - u_1) \end{cases}$$

LET $\dfrac{m\omega^2}{K} = \beta$.

THEN

$$\begin{cases} \beta u_3 = u_3 - u_2 \\ \beta u_2 = u_2 - u_1 - u_3 + u_2 \\ \beta u_1 = u_1 - u_2 + u_1 \end{cases}$$

$$\begin{cases} \dfrac{u_2}{u_3} = 1 - \beta \\ \beta u_2 - 2u_2 = -u_1 - u_3 \\ u_2 = 2u_1 - \beta u_1 \end{cases}$$

$$\beta^3 - 5\beta^2 + 6\beta - 1 = 0$$

$\beta = 0.200$, 1.55 , AND 3.24 .

$$\omega^2 = \frac{\beta K}{m} = \frac{(.200\ ,\ 1.55,\ 3.24) \cdot 32.2 \cdot 12 \cdot 3.80}{.582}$$

$$= 505.\ ,\ 3900.\ ,\ 8150\ .$$

FUNDAMENTAL PERIOD $= \dfrac{2\pi}{\sqrt{505}} = 0.280$ SEC.

FIRST HARMONIC $= \dfrac{2\pi}{\sqrt{3900}} = 0.100$ SEC.

SECOND HARMONIC $= \dfrac{2\pi}{8150} = 0.0695$ SEC.

THIRD METHOD.

$$t_0 = \frac{h}{c} = h\sqrt{\frac{m}{\mu}} = h\sqrt{\frac{m}{Kh_1}} = 2 \cdot 5.75 \sqrt{\frac{.573}{32.2 \cdot 12 \cdot 3.80 \cdot 5.75}} = 0.0948$$

$R = 1.$

$\alpha = Rn = 2.$

$$\begin{cases} \lambda_0 = 1.08 \\ \lambda_1 = 3.64 \\ \lambda_2 = 6.54 \end{cases}$$

$$\text{FUNDAMENTAL PERIOD} = \frac{2\pi t_0}{\lambda_0} = \frac{2\pi\, 0.0948}{1.08} = 0.550 \text{ SEC.}$$

$$\text{FIRST HARMONIC} = \frac{2\pi\, 0.0948}{3.64} = 0.164 \text{ SEC.}$$

$$\text{SECOND HARMONIC} = \frac{2\pi\, 0.0948}{6.54} = 0.091 \text{ SEC.}$$

APPENDIX 8

PERIODS OF ACTUAL BUILDINGS

In Japan, the fundamental periods of actual buildings have been determined experimentally by attaching a cable to the top of the building, pulling it over a little with a powerful wench, and suddenly releasing the cable. Professor Martel is thinking of putting large, variable speed motors, with eccentric weights on the shafts, in actual buildings.

APPENDIX 9

CASH REGISTER TAPE

The redording tape used on the machine is National Cash Register, Size C paper. It comes in rolls, is 1.2" wide, costs four cents per roll, is sold in wrapped cylinders of ten rolls for forty cents, and may be bought at any National Cash Register Store.