ON THE INTEGRATION OF ABSTRACT FUNCTIONS

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1. Introductory.

The concept of integration has been extended to abstract spaces by three distinct methods. Saks (1) considers numerical-valued functions of an abstract variable and defines a generalization of the Lebesgue integral for such functions. Kerner (2) considers abstract-valued functions of a single real variable and defines a generalization of the Riemann integral for such functions. Lastly, Bochner (3) considers abstract-valued functions of n real variables and defines a generalization of the Lebesgue integral for such functions.

This paper, as shown in section 4, presents a unification of these three theories by generalizing the Lebesgue integral to the case of abstract-valued functions of an abstract variable . The essentially new contributions are contained in Theorems 3.4 and 3.5, where the integral is shown to exist for a large class of functions and bounded continuous functions in certain spaces are shown to be integrable. It was found necessary to postulate the existence of a measure function, as none could be discovered for general vector spaces. If such a function can be shown to exist, then we shall have a true generalization of the theory of integration.

This paper was written under the guidance of Professor A. D. Michal.

(1). Saks, THEORIE DE L' INTEGRALE, Warsaw, 1933.

(2). Kerner, Prace Matematyczne Fizyczne, 40, Part 1, 1933.

(3). Bochner, Fundamenta Mathematicae, 20, 1933.

2. Spaces.

In the course of the paper we shall have occasion to refer to the following types of spaces:

B; a complete, normed, linear space, usually called a Banach space, whose elements will be denoted by small Latin letters from the first part of the alphabet.

R; the real number system, whose elements will be denoted by small Greek letters.

X; a space satisfying the following postulates:

(a). X is a B.

(b). Corresponding to each set, A, of elements of X there exists a non-negative real number, denoted by $|A|$, called the measure of A, with the properties:

(1). If $A=\sum_1^\infty A_n$, then $|A|\leq \sum_1^\infty |A_n|$; equality obtaining if and only if the A_n are non-overlapping.

(2). $|A|=\infty$ if and only if there exists an infinite sequence $\{A_n\}$ such that $A_{n+1}\supseteq A_n$, for all n, and such that $A=\sum_1^\infty A_n$ and $\lim_{n\to\infty} |A_n|=\infty$. In such a case A is said to be the outer limiting set of the sequence $\{A_n\}$.

V; a metric space in which the triangular inequality may not hold, but such that the distance between any two points approaches zero as the distance between each of them and an arbitrary third point approaches zero.

The elements of the space X will be denoted by small Latin letters from the end of the alphabet. Hence, the nature of a function will be completely determined by the letters used in its expression; eg. $f(x)$ denotes a function on X to B, $\alpha(e)$ a function on B to R, etc. The letter A will be used to denote a set of elements of the space X. The letters i, j, m, n, will denote positive integers.

3. Integration. Definition and Properties.

Definition 3.1. A function $g(x)$ defined over A is said to be elementary if and only if: $.A=\sum_n A_n : i\neq j . \supset . A_i \cdot A_j = 0$, where 0 is the null set. $x \epsilon A_n . \supset . g(x)=g_n$. g_n constant for each n. (Throughout, the letter $\sum$ with no range indicated will denote finite sums).

Definition 3.2. A function $f(x)$ defined over A is said to be measurable if and only if: If $A^* \subset A$ is the entire set of points for which some sequence $\{f_n(x)\}$ of elementary functions does not converge to $f(x)$, then $|A^*|=0$. That is, there exists a sequence $\{f_n(x)\}$ which converges to $f(x)$ almost everywhere in A.

Hence, every elementary function is measurable. If a function is measurable over each of a finite number of sets, it is measurable over their sum set. If a function is measurable over a set A, it is measurable over every sub-set of A. If $f(x)$ is measurable, so is $\|f(x)\|$, because of the triangular property of the norm. Finally, the sum of any finite number of measurable functions is a measurable function, and the product of a measurable function by a real-valued measurable function is measurable.

From Egoroff's theorem (1), and its converse (2), both of which are valid for functions of the type considered here... since the norm, by means of which convergence is defined, and the measure-function both have the triangular property, and because of postulate b (2) for the spaces X... we have:

Theorem 3.1. A function f(x) defined over A is measurable if and only if there exists a sequence of elementary functions which converges "asymptotically"... or "on the average"... to f(x); that is, in every sub-set M, of finite measure, of A, the sequence converges uniformly to f(x) except in a set of arbitrarily small measure. Symbolically: $M \leq A$. $|M|$ finite. $\delta > 0 :\supset: \exists M_\delta \leq M. |M_\delta| > |M| - \delta.$ $f_n(x) \rightarrow f(x\ )$ uniformly in M_δ.

This theorem is essentially the same as that of S. Bochner. The only properties of R_n that he uses are those of the measure of sets, and these have been preserved by our postulates. (2). This property of measurable functions is a fundamental one, but we have taken the other as our definition because it is more similar to the definition in the case of functions of a real variable. (See Saks).

(1). Titchmarsh. THE THEORY OF FUNCTIONS. Page 339.
(2). Hobson. FUNCTIONS OF A REAL VARIABLE. Vol. 2. Page 239.

Theorem 3.2. If $f(x)$ is measurable in A and bounded almost everywhere in A, then there exists an approximating sequence $\{f_n(x)\}$ which is bounded uniformly in A for all n.

Proof.

Let $\varphi \geq \|f(x)\|$ almost everywhere in A, and let $\varphi_1 > \varphi$.

Let $\{g_n(x)\}$ be a sequence of elementary functions converging to $f(x)$ almost everywhere in A.

Definition $f_n(x)$: $f_n(x)=g_n(x) .\equiv. \|g_n(x)\| < \varphi_1$.
$f_n(x)=0 .\equiv. \|g_n(x)\| \geq \varphi_1$.

Then, clearly $\|f_n(x)\| < \varphi_1$ for all x in A and for all n.

Next, Definition A^*: $x \in A^* \leq A :\equiv: \{f_n(x)\}$ does not converge to $f(x)$.

Definition A_1^*: $x \in A_1^* \leq A^* .:\equiv:. \exists$ an infinite sequence S of n's : $n \in S . \supset . f_n(x)=0$.

Definition A_2^* : $x \in A_2^* \leq A^* .:\equiv:. \exists$ an m: $n > m . \supset . f_n(x)=g_n(x)$.

Then, $A_1^* \cdot A_2^* = 0$. $A^* = A_1^* + A_2^*$. $\therefore |A^*| = |A_1^*| + |A_2^*|$.

Now, $x \in A_1^* .:\equiv:. \exists$ an infinite sequence S of n's: $n \in S . \supset . \|f(x) - g_n(x)\| \geq \big| \|f(x)\| - \|g_n(x)\| \big| \geq \varphi_1 - \varphi$, almost everywhere. $\therefore$ $\{g_n(x)\}$ does not converge to $f(x)$ anywhere in A_1^* except possibly in a sub-set of measure zero. $\therefore$ $|A_1^*| = 0$.

Also, $x \in A_2^* .:\equiv:. \exists$ m: $n > m . \supset . \{g_n(x)\}$ does not converge to $f(x)$. $\therefore$ $|A_2^*| = 0$.

$\therefore$ $|A^*| = 0$.

$\therefore$ $\{f_n(x)\}$ converges to $f(x)$ almost everywhere in A.

Also, since $\|f_n(x)\| < \varphi_1$ for any $\varphi_1 > \varphi$, $\|f_n(x)\| \leq \varphi$.

Definition 3.3 If $g(x)$ is an elementary function defined over A, where $|A|$ is finite, we define the symbol $I[g(x), A]$ by the relation:

$$I[g(x), A] = \sum_n g_n \cdot |A_n|$$

where g_n and A_n satisfy Definition 3.1.

Definition 3.4. If $f(x)$ is measurable over A and bounded almost everywhere in A, and if for every sequence $\{f_n(x)\}$ defined by Theorem 3.2 the sequence $\{I[f_n(x), A]\}$ is convergent, then $f(x)$ is said to be integrable over A, and we write

$$\int_A f(x)dx \equiv \lim_{n\to\infty} I[f_n(x), A].$$

Theorem 3.3. If $\int_A f(x)dx$ exists, it is unique; ie., independent of the particular choice of the approximating sequence $\{f_n(x)\}$.

Proof.

Let $\{g_n(x)\}$ and $\{h_n(x)\}$ be bounded approximating sequences, and let $\{I[g_n(x), A]\}$ and $\{I[h_n(x), A]\}$ converge to g and h respectively. Consider the sequence $\{k_n(x)\}$, where $k_{2n}(x) \equiv g_n(x)$ and $k_{2n+1}(x) \equiv h_n(x)$. Then clearly $\{k_n(x)\}$ is a bounded approximating sequence. Hence, by Definition 3.4 $\{I[k_n(x), A]\}$ converges. This can be true if and only if $g=h$.

We are now in a position to prove the existence of the integral for a large class of functions, namely:

Theorem 3.4 If $f(x)$ is measurable over A and bounded almost everywhere in A, then $\int_A f(x)dx$ exists, if $|A|$ is finite. Proof.

Let $\{f_n(x)\}$ be any sequence satisfying the conditions of Theorem 3.2. Let $\varphi \geq \|f(x)\|$ almost everywhere in A. Hence, $\varphi \geq \|f_n(x)\|$ for all x and n. By Theorem 3.1, given any $\varepsilon > 0$, there exists a set $A_\varepsilon \leq A$, such that $|A_\varepsilon| > |A| - \varepsilon$, and such that in A_ε $\{f_n(x)\}$ approaches $f(x)$ uniformly; ie., there exists an $n_0 = n_0(\varepsilon)$ such that if $m, n > n_0$, then $\|f_n(x) - f_m(x)\| < \varepsilon$ for all x in A_ε. Write $F_\varepsilon = A - A_\varepsilon$. Then, $|F_\varepsilon| < \varepsilon$.

Now clearly, for any n, $I[f_n(x), A] = I[f_n, A_\varepsilon] + I[f_n, F_\varepsilon]$ since $A_\varepsilon \cdot F_\varepsilon = 0$ and $A = A_\varepsilon + F_\varepsilon$. Since the $f_n(x)$ are elementary functions, we have that for each n, $A_\varepsilon = \sum_m {}_\varepsilon A_{nm}$, where ${}_\varepsilon A_{ni} \cdot {}_\varepsilon A_{nj} = 0$ if $i \neq j$, and $f_n(x) = f_{nm}$ if x is in ${}_\varepsilon A_{nm}$. Similarly, $F_\varepsilon = \sum_m {}_\varepsilon F_{nm}$, ${}_\varepsilon F_{ni} \cdot {}_\varepsilon F_{nj} = 0$ if $i \neq j$, and $f_n(x) = \overline{f}_{nm}$ if x is in ${}_\varepsilon F_{nm}$.

Definition I_{nm}: $I_{nm} \equiv \| I[f_n(x), A] - I[f_m(x), A] \|$.

Hence $I_{nm} \leq \| I[f_n(x), A_\varepsilon] - I[f_m(x), A_\varepsilon] \|$

$$+ \| I[f_n(x), F_\varepsilon] - I[f_m(x), F_\varepsilon] \|.$$

By Definition 3.3

$$I_{nm} \leq \left\| \sum_i f_{ni} |{}_\varepsilon A_{ni}| - \sum_j f_{mj} |{}_\varepsilon A_{mj}| \right\| + \left\| \sum_i \overline{f}_{ni} |{}_\varepsilon F_{ni}| - \sum_j \overline{f}_{mj} |{}_\varepsilon F_{mj}| \right\|.$$

Let us consider the second term of the right member. By Theorem 3.2 we have that $\|\overline{f}_n(x)\| \leq \varphi$.

$$\left\|\sum_i \bar{f}_{ni}|_\varepsilon F_{ni}| - \sum_j \bar{f}_{mj}|_\varepsilon F_{mj}|\right\| \le \left\|\sum_i \bar{f}_{ni}|_\varepsilon F_{ni}|\right\| + \left\|\sum_j f_{mj}|_\varepsilon F_{mj}|\right\|$$
$$\le \varphi\left(\sum_i |_\varepsilon F_{ni}| + \sum_j |_\varepsilon F_{mj}|\right) = 2\varphi|F_\varepsilon|.$$

(1). $\therefore \left\|\sum_i f_{ni}|_\varepsilon F_{ni}| - \sum_j f_{mj}|_\varepsilon F_{mj}|\right\| < 2\varphi\varepsilon.$

Now consider the first term of the right member.

Write ${}_\varepsilon A_{ni}\cdot {}_\varepsilon A_{mj} = {}_\varepsilon A^{nm}_{ij}$. Then ${}_\varepsilon A^{nm}_{ij}\cdot {}_\varepsilon A^{nm}_{kl} = {}_\varepsilon A_{ni}\cdot {}_\varepsilon A_{mj}\cdot {}_\varepsilon A_{nk}\cdot {}_\varepsilon A_{ml} = {}_\varepsilon A_{ni}\cdot {}_\varepsilon A_{nk}\cdot {}_\varepsilon A_{mj}\cdot {}_\varepsilon A_{ml}$

$= 0$ if $i \neq k$ or if $j \neq l$.

Furthermore, ${}_\varepsilon A_{ni} = \sum_j {}_\varepsilon A^{nm}_{ij}$. $\therefore$ $|{}_\varepsilon A_{ni}| = \sum_{i,j} |{}_\varepsilon A^{nm}_{ij}|$ for any n,m and $|A_\varepsilon| = \sum_{i,j} |{}_\varepsilon A^{nm}_{ij}|$ for any n,m. Now, $f_n(x) = f_{ni}$ if $x \in {}_\varepsilon A_{ni}$.

$\therefore$ $f_n(x) = f_{ni}$ if $x \in {}_\varepsilon A^{nm}_{ij}$ for any m, j.

$\therefore \left\|\sum_i f_{ni}|_\varepsilon A_{ni}| - \sum_j f_{mj}|_\varepsilon A_{mj}|\right\| = \left\|\sum_{i,j} f_{ni}|_\varepsilon A^{nm}_{ij}| - \sum_{i,j} f_{mj}|_\varepsilon A^{mn}_{ji}|\right\|.$

Now since ${}_\varepsilon A_{ni}\cdot {}_\varepsilon A_{mj} = {}_\varepsilon A_{mj}\cdot {}_\varepsilon A_{ni}$, we have ${}_\varepsilon A^{nm}_{ij} = {}_\varepsilon A^{mn}_{ji}$.

$\therefore$ the right member becomes: if $n, m > n_0(\varepsilon)$,

(2) $\left\|\sum_{i,j}(f_{ni} - f_{mj})|_\varepsilon A^{nm}_{ij}|\right\| \le \sum_{i,j}\|f_{ni} - f_{mj}\||_\varepsilon A^{nm}_{ij}| < \varepsilon\sum_{i,j}|_\varepsilon A^{nm}_{ij}| = \varepsilon|A_\varepsilon| \le \varepsilon|A|$.

Thus from (1) and (2) we have that

$$I_{nm} < \varepsilon(2\varphi + |A|).$$

Since φ and $|A|$ are finite, we have that the sequence $\{I[f_n(x), A]\}$ converges. But the choice of $\{f_n(x)\}$ was arbitrary. Hence every such sequence $\{I[f_n(x), A]\}$ converges, and by Theorem 3.3 to the same limit.

Hence, the expression $I[g(x), A]$, where $g(x)$ is an elementary function can be written $\int_A g(x)dx$.

Before proceeding with the properties of the integral, we wish to prove a theorem on the measurability of continuous functions. For this purpose we need a theorem proved by Fréchét in an article in the Bull. de la Soc. Math. de France (45).

The statement of the theorem which we need is the following: If A is compact and closed, and if $\mathcal{I}$ is a family of sets I such that every element of A is an interior element of some I, then there exists a finite sub-family $\mathcal{I}_1$ of $\mathcal{I}$ having the same property. A may be in any space V.

From this we have

Theorem 3.5. If A is compact and closed and if $f(x)$ is continuous over A, then $f(x)$ is measurable over A.

Proof.

$\exists \delta = \delta(x,\varepsilon) :. x, \bar{x} \in A . \|x-\bar{x}\| < \delta(x,\varepsilon) :\supset: \|f(x)-f(\bar{x})\| < \varepsilon$. $\therefore$ the function $\delta(x,\varepsilon)$ determines a family $\mathcal{I}$, as in the previous theorem. $\therefore \exists \mathcal{I}_1$, a finite sub-family with the same property. That is $\exists \{x_n\} \in A$, range of n finite, such that $x \in A :\supset: \exists x_n . \|x-x_n\| < \delta . \|f(x)-f(x_n)\| < \varepsilon$. Definition $g_\varepsilon(x)$: $\|x-x_n\| < \delta \;.\!\equiv\!.\; g_\varepsilon(x) = f(x_n)$. Hence $g_\varepsilon(x)$ is an elementary function, which approaches $f(x)$ everywhere in A as ε approaches zero. Hence, any denumerable sequence from the set $\{g_\varepsilon(x)\}$ is an approximating sequence for $f(x)$.

Henceforth we shall consider only bounded, measurable functions in a set A of finite measure, unless the contrary is explicitly stated.

Theorem 3.6. If $f(x)$ and $g(x)$ are elementary functions, then

$$\int_A (f(x) \pm g(x))\,dx = \int_A f(x)\,dx \pm \int_A g(x)\,dx.$$

Proof:

$A = \sum_n (A'_n)$, where $A'_n \cdot A'_m = 0$ if $m \neq n$, and $f(x) = f_n .\equiv. x \in A_n$

Similarly, $A = \sum_n A^2_n$, where $A^2_n \cdot A^2_m = 0$ if $m \neq n$, and

$g(x) = g_n .\equiv. x \in A^2_n$. Definition A_{nm}: $A_{nm} = A'_n A^2_m$.

As in the proof of Theorem 3.4, $A_{ni} \cdot A_{mj} = 0$ if $m \neq n$, or if $i \neq j$. $A'_m = \sum_n A_{mn}$, $A^2_n = \sum_m A_{mn}$, $A = \sum_{mn} A_{m,n}$.

Hence, $\int_A f(x)dx = \sum_n f_n |A'_n| = \sum_{n,m} f_n |A_{nm}|$.

$\int_A g(x)dx = \sum_m g_m |A^2_m| = \sum_{m,n} g_m |A_{nm}|$.

Since $x \in A_{nm} .\equiv. f(x) = f_n$ and $g(x) = g_m$, then

$$x \in A_{nm} .\equiv. f(x) \pm g(x) = f_n \pm g_m.$$

$$\therefore \int_A f(x)dx \pm \int_A g(x)dx = \sum_{n,m} (f_n \pm g_m)|A_{nm}| = \int_A (f(x) \pm g(x))dx.$$

Theorem 3.7. If $f(x)$ and $g(x)$ are measurable and bounded over A, so is their sum, and

$$\int_A (f(x) \pm g(x))dx = \int_A f(x)dx \pm \int_A g(x)dx.$$

Proof:

$f(x) = \lim_{n\to\infty} f_n(x)$ and $g(x) = \lim_{n\to\infty} g_n(x)$ almost everywhere in A, where $f_n(x)$ and $g_n(x)$ are elementary functions as in Theorem 3.2. $\therefore f(x) \pm g(x) = \lim_{n\to\infty} [f_n(x) \pm g_n(x)]$ almost everywhere in A.

$$\therefore \int_A f(x)dx \pm \int_A g(x)dx = \lim_{n\to\infty} \int_A f_n(x)dx \pm \lim_{n\to\infty} \int_A g_n(x)dx$$

$$= \lim_{n\to\infty} \left(\int_A f_n(x)dx \pm \int_A g_n(x)dx \right).$$

By the preceeding theorem,

$$\int_A f(x)dx \pm \int_A g(x)dx = \lim_{n\to\infty} \int_A (f_n(x) \pm g_n(x))dx = \int_A [f(x) \pm g(x)]dx.$$

By mathematical induction the preceding theorem can be extended to the case of the sum of any finite number of functions.

Theorem 3.8. If $f(x)$ is measurable and bounded, so is $\|f(x)\|$, and $\|\int_A f(x)dx\| \leq \int_A \|f(x)\| dx$.

Proof:

Let $\{f_n(x)\}$ be an approximating sequence.

$A = \sum_m A_{nm}$, $A_{nm} \cdot A_{nl} = 0$ if $m \neq l$ for any n.

$f_n(x) = f_{nm} .\equiv. x \in A_{nm}$.

$$\therefore \|\int_A f(x)dx\| = \|\lim_{n\to\infty} \int_A f_n(x)dx\| = \lim_{n\to\infty} \|\int_A f_n(x)dx\| .$$

$$= \lim_{n\to\infty} \|\sum_m f_{nm} \cdot |A_{nm}|\| \leq \lim_{n\to\infty} \sum_m \|f_{nm}\| \cdot |A_{nm}|$$

$$\leq \lim_{n\to\infty} \int_A \|f_n(x)\| dx = \int_A \|f(x)\| dx.$$

Theorem 3.9. $\int_A \alpha f(x)dx = \alpha \int_A f(x)dx$.

Proof:

$$\alpha \int_A f(x)dx = \alpha \lim_{n\to\infty} \int_A f_n(x)dx = \lim_{n\to\infty} \alpha \int_A f_n(x)dx$$

$$= \lim_{n\to\infty} \alpha \sum_m f_{nm} \cdot |A_{nm}| = \lim_{n\to\infty} \sum_m \alpha f_{nm} \cdot |A_{nm}|$$

$$= \lim_{n\to\infty} \int_A \alpha f_n(x)dx = \int_A \alpha f(x)dx.$$

Theorem 3.10. If f is a particular element of B, then

$$\int_A f dx = f \cdot |A|.$$

Corollary. If α is any real number, then

$$\int_A \alpha dx = \alpha \cdot |A|.$$

Proof obvious.

Theorem 3.11. If $\|f(x)\| \leq \alpha$, then $\|\int_A f(x)dx\| \leq \alpha|A|$.

Proof:

By Theorem 3.8., $\|\int_A f(x)dx\| \leq \int_A \|f(x)\| dx$

$$\therefore \|\int_A f(x)dx\| \leq \lim_{n\to\infty} \int_A \|f_n(x)\| dx = \lim_{n\to\infty} \sum_m \|f_{nm}\| \cdot |A_{nm}| .$$

Now, by Theorem 3.2, $\|f_{nm}\| \leq \alpha$.

$$\therefore \|\int_A f(x)dx\| \leq \lim_{n\to\infty} \sum_m \alpha |A_{nm}| = \alpha|A| . .$$

Theorem 3.12. If $\varphi(x)$ is an integrable function on A to R, ~~then~~ and if $\varphi(x) \geq \alpha$, then $\int_A \varphi(x)dx \geq \alpha|A|$.

Proof:

$$\int_A \varphi(x)dx = \lim_{n\to\infty} \int_A \varphi_n(x)dx = \lim_{n\to\infty} \sum_m \varphi_{nm} |A_{nm}|$$

$$\geq \lim_{n\to\infty} \sum_m \alpha |A_{nm}| = \alpha|A| .$$

4. Important Special Instances.(1).

a. Saks.

Saks defines a completely additive family of point-sets in a vector-space , and the measure of such sets by means of a group of postulates identical with our set (b). With him an elementary function is one whose set of values is denumerable at most, so that a function which is elementary according to our definition is also elementary according to his. He defines measurability as does Titchmarsh, but proves the theorem that every non-negative measurable function is the limit of a monotonic, non-decreasing sequence

(1). For references, see section 1.

of measurable, non-negative finite functions each of which has only a finite number of values. This is obviously a special instance of our class of measurable functions. He then proceeds to define an integral exactly as we have done. Thus

Theorem 4.1. If B is R, then the integral of Definition 3.4 is an integral as defined by Saks, and conversely.

b. Bochner.

Bochner considers abstract valued functions of n real variables. Measure has already been defined for such spaces (cf. Hobson), and satisfies our postulates. Bochner defines an elementary function, a measurable function and an integral precisely as we do.

Theorem 4.2. If X is R_n, an n-dimensional real space, the integral of Definition 3.4 is an integral as defined by Bochner, and conversely.

c. Kerner.

If X is R, we have immediately from Theorem 3.5

Theorem 4.3. If f(x) is continuous throughout a closed interval, its "Riemann" integral is equal to its "Lebesgue" integral.