A NEW METHOD OF STRESS ANALYSIS FOR ARCH DAMS

by

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FOREWORD

In the past most of the arch dams have been designed according to the cylinder formula. When this formula is used the stresses are figured in the same way as the stress in a pipe is figured. $S = PR/T$ where S denotes the stress in pounds per square inch, P denotes the pressure in the pipe in pounds per square inch, R denotes the radius in inches, and T denotes the thickness in inches. It has been shown that this formula is entirely inadequate when applied to the design of arch dams.

F. Noetzli, A. Nadai, and others have devised methods for computing the stresses in arch dams. These methods which have been devised either involve a great deal of labor, or they are based on a trial and error solution. The method of analysis discussed in this thesis attempts to put the design of arch dams on a more scientific basis. The method is both new and unique. Because the method is new, few men in the world understand it at the present time.

The author claims little originality in the preparation of this thesis. Credit is due mainly to Dr. Eugenio Kalman, visiting professor at the California Institute of Technology. Dr. Kalman devised the method of analysis; the author has merely made the computations under his guidance and direction.

In making the computations the author has tried to attain accuracy. The work has been checked. Complicated terms to be integrated have been checked by the proof of integration, viz., differentiating the integrated result, and showing that it is equivalent to the expression under the integral sign.

The author wishes to express his thanks to the civil engineering members of the California Institute faculty for helpful criticism and advice.

Pasadena, California, May 1931. T.J.N.

A NEW METHOD OF STRESS ANALYSIS OF ARCH DAMS

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The purpose of this thesis is to present the mathematical treatment of the Kalman method of stress analysis of arch dams. Before presenting the computations, the salient points of the Kalman method will be discussed. For a more comprehensive discussion of the method the reader is referred to the "Proceedings of the American Society of Civil Engineers for March 1931, pages 440 to 460."

When this method is applied to the analysis of arch dams, the dam is divided into cantilever elements by vertical, radial planes, and it is divided into arch elements by horizontal planes. The intersections of these horizontal and vertical planes form parallelepiped elements. We shall presently consider the forces acting on these parallelepiped elements.

The water pressure against a dam varies, of course, with the depth of the water. The pressure at any given depth is constant. Since all points of an arch element are at the same depth, one might think that the arch elements are under uniform load. However, such is not the case. The arch elements would be under uniform load if it were not for the action of the cantilever elements and the shearing forces due to adjacent arches. The cantilever elements put bending moments in the arches, and complicate the loading considerably.

The question arises "At any given point, how much of the load is carried by the arch, and how much of the load is carried by the cantilever?"

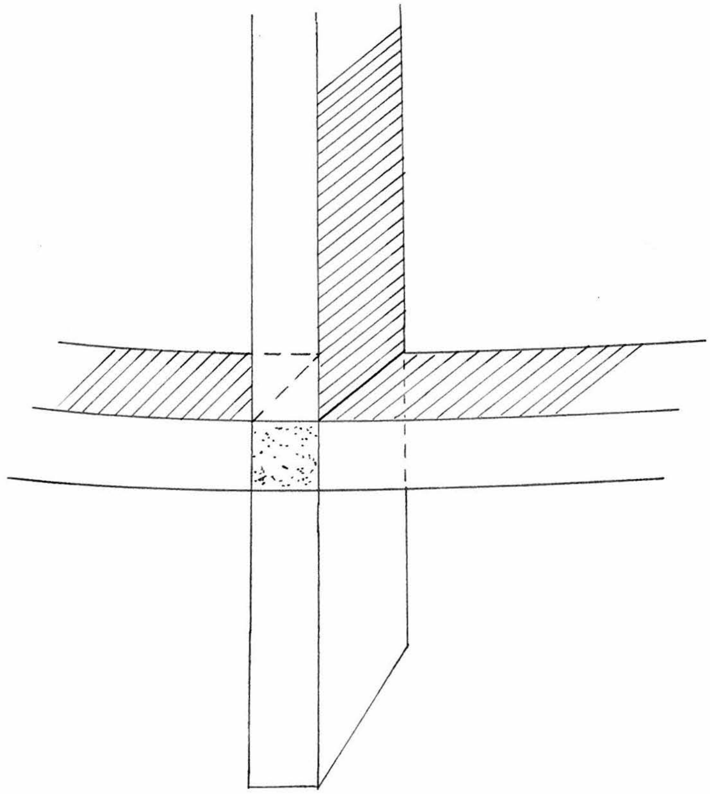


Fig. 1.

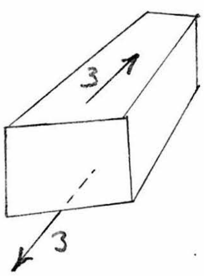


Fig. 2.

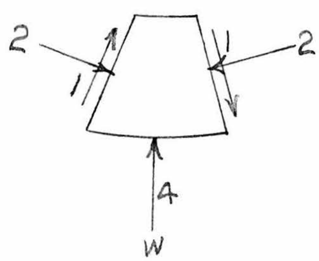


Fig. 3.

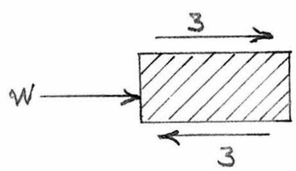


Fig. 4.

All arch dam analyses are concerned with this problem of the division of load between the arches and cantilevers.

Figure 1 shows an element formed by the intersection of an arch and a cantilever. Figure 2 is an end view of the element. Figure 3 shows a plan view of the element, and Figure 4 shows a side view.

There are four sets of forces that act on the element. They are:

1. The forces (1), Figure 3. These are shearing forces between the element and adjacent cantilevers.
2. The compressive forces (2), Figure 3. These forces are caused by the adjacent cantilevers pushing against the element.
3. The forces (3), Figure 2. These are shearing forces between the top and bottom of the element and the adjacent arches. The forces are also shown in Fig. 4.
4. The load (4) due to the water pressure. This load is proportional to the distance below the surface of the water.

Since the element is in equilibrium, the resultant of the sets of forces acting upon it must be zero. That is:

$$(1)+(2)+(3)+(4)=0 \quad \ldots\ldots\ldots\ldots\ldots \quad (a)$$

From the figures on the preceeding page we see that the forces which act on the cantilever are the forces (1), (2), and (4). Let us call f(x) the resultant cantilever force, then:

$$f(x)=(1)+(2)+(4) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (b)$$

or $$(1)+(2)=f(x)-(4) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (c)$$

From the figures on the preceeding page we also see that the forces acting on the arch are the forces (3) and (4). But from equation (a) we obtain:

$$(3) + (4) = -(1) - (2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (d)$$

Substituting the value of (1) + (2) from equation (c) in equation (d), we obtain:

$$(4) - f(x) \quad \text{for the load on the arch.}$$

But (4) is the force due to the water pressure and is equal to kx, where x is the distance below the surface of the water. Therefore the load on the arch is:

$$kx - f(x).$$

We see that the load on the arch consists of two parts.

1. The load due to the water pressure, and,
2. The correction load f(x) due to the cantilever action.

The correction load f(x) may be expressed by a Fourier series of the form:

$$f(x) = (a + bx + cx^2)\cos\frac{\pi}{2\phi_0}\phi + (A + Bx + Cx^2)\cos\frac{3\pi}{2\phi_0}\phi + - - -$$

where a, b, c, and A, B, C, are parameters to be determined.
ϕ = angle measured from center, ϕ_0 = angle from center to abutement

In evaluating the paramaters two things are considered:

1. The deformation in the bottom arch is zero, and
2. The paramaters minimize the work of deformation.

THE CHIEF DIFFERENCE BETWEEN THE KALMAN METHOD OF ANALYSIS AND OTHER METHODS IS THAT THE CORRECTING LOAD IN THE KALMAN METHOD IS EXPRESSED BY A FOURIER SERIES. In other methods of analysis the distribution of load between the arches and cantilevers is determined by approximations involving the solution of simultaneous equations. At any point in the dam, the deflection of the cantilever is equal to the deflection of the arch. There is an equation for each point that is taken. The more points that are taken, the better the approximation.

By the Kalman method a correcting load is chosen containing a few parameters. The stresses in the arches and cantilevers due to the load thus chosen are then computed. The integral of work is expressed in terms of the stresses acting on the arch and cantilever slices. The equations for the parameters are derived from the condition that the work of deformation is a minimum, and finally the linear equations for the parameters are solved.

In presenting the computations for the Kalman analysis, some trivial steps and transformations have been omitted. However, the author has tried to make the computations sufficiently complete so that they may be followed.

In order to codify the mass of computations the author has arranged them in seven steps as follows:

STEP 1: The determination of the moment at any point in an arch. Three conditions of loading are considered:

(a) $p(\phi) = \text{Constant}.$

(b) $p(\phi) = \cos\frac{\pi}{2\phi_0}\phi$

(c) $p(\phi) = \cos\frac{3\pi}{2\phi_0}\phi$

In each of these three cases the Kalman formulas have been applied. (Proceedings A.S.C.E, March, 1931, p. 437)

STEP 2: Choice of an additional load on the arches $f(x, \phi)$.
(The additional load on the arch is the load on the cantilever.)

STEP 3: Elimination of deformations in bottom arch since the bottom arch undergoes no deformation.

STEP 4: Evaluation of Step 3. Application of the moment equations to the bottom arch.

STEP 5: Determination of the stresses in the arches and cantilevers for the assumed additional load.

STEP 6: Integration of the expressions occuring in the integral of work.

STEP 7: Determination of the parameters a,b,c, and A,B,C, which minimize the work of deformation.

After the parameters have been obtained the analysis may be applied to a particular dam by substituting numerical values in the formulas. The author attempted to compare results obtained by using different central angles, and different ratios of thickness to radius of dam. The central angles of 120 degrees, 140 degrees and 180 degrees ($\phi_0 = 60°, 70°, 80°$) were chosen, and t/r ratios of 1/5, 1/10, and 1/20 were chosen.

Time did not permit the making of as many numerical calculations as the author would like to have made. Numerical calculations which have been made will be found in the appendix. A discussion of the Kalman method will be found after the computations, just preceding the appendix.

STEP 1. :

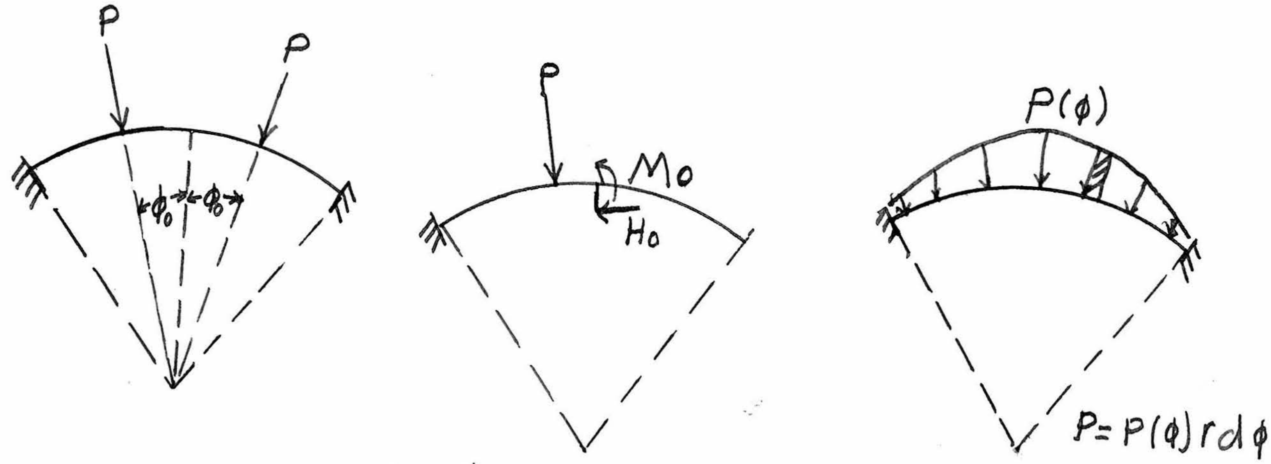
In this step the moment at any point in the arch is determined. Three cases are considered:

(a) $P(\phi) = \text{Constant}$

(b) $P(\phi) = \text{Cos}\ \frac{3\pi}{2\phi_0}$

(c) $P(\phi)\ \ \text{Cos}\ \frac{\pi}{2\phi_0}$

In each of the three cases, the Kalman formulas are applied. (See Proceedings of A.S.C.E. for March 1931 p. 457)



$$M_o = Pr\,\frac{1-\cos(\phi_0-\phi)}{\phi_0}$$

$$H_o = 2\,\frac{P}{D}\left\{\frac{1-\cos(\phi_0-\phi)}{\phi_0} + \left(1+\frac{t^2}{3r^2}\right)\frac{\phi_0-\phi}{2\sin\phi_0}\sin\phi_0 - \frac{1}{2}\left(1-\frac{t^2}{6r^2}\right)\sin(\phi_0-\phi)\right]$$

$$\underset{(\text{TOTAL})}{M_o} = \int_0^{\phi}\frac{r^2}{\phi}\left\{1-\cos(\phi_0-\phi)\right\}d\phi\cdot p(\phi) \qquad \underset{\text{TOTAL}}{H_o} = \frac{2r}{D}\int_0^{\phi}\left\{\qquad\qquad\right\}P(\phi)\,d\phi$$

$$P(\phi) = \cos 2\phi$$

$$\therefore M = \int_0^{\phi}\frac{r^2}{\phi_0}\left(1-\cos(\phi_1-\phi)\right)\cos 2\phi - r\left(\cos\phi - \frac{\sin\phi_0}{\phi_0}\right)\frac{2r}{D}\int_0^{\phi}\{\cdots\}\cos 2\phi\,d\phi$$

$$- r^2\int_{\alpha=0}^{\phi}\cos 2\phi\,\sin(\phi-\alpha)\,d\alpha$$

$$\int_{\alpha=0}^{\phi}\cos 2\phi\,\sin(\phi-\alpha)\,d\alpha = \int\frac{\sin(\phi-\alpha+2\alpha)+\sin(\phi-\alpha-2\alpha)}{2}$$

$$= \int\frac{\sin[(2-1)\alpha+\phi]}{2} - \frac{\sin[(2+1)\alpha-\phi]}{2}$$

$$-\frac{\cos[(2-1)\alpha+\phi]}{2(2-1)} + \frac{\cos[(2+1)\alpha-\phi]}{2(2+1)}\Bigg|_0^{\phi}$$

STEP 1 (CONT'D)

$$= -\frac{1}{2(2-1)}\left\{\underset{\cos 2\phi - \cos\phi}{\cos(2\phi-\phi+\phi)-\cos\phi}\right\} + \frac{1}{2(2+1)}\left\{\cos 2\phi - \cos\phi\right\}$$

$$\frac{\cos 2\phi - \cos\phi}{2}\left\{\frac{1}{2+1}-\frac{1}{2-1}\right\} = \frac{\cos 2\phi - \cos\phi}{1-2^2}$$

$$\therefore -r^2\int_{\alpha=0}^{\phi}\cos 2\alpha \sin(\phi-\alpha)\,d\alpha = \frac{r^2}{2^2-1}\left(\cos 2\phi - \cos\phi\right) \quad (3)$$

$$\frac{r^2}{\phi_0}\int_0^{\phi_0}\left\{1-\cos(\phi_0-\phi)\right\}\cos 2\phi\,d\phi = \frac{r^2}{\phi_0}\left\{\frac{1}{2^2-1}\sin\phi_0 - \frac{1}{2(2^2-1)}\sin 2\phi_0\right\}$$

---

(2) $$\int_0^{\phi}\left\{\frac{1-\cos(\phi_0-\phi)}{\phi_0} + \left(1+\frac{t^2}{3r^2}\right)\frac{\phi_0-\phi}{2\sin\phi_0}\sin\phi - \frac{1}{2}\left(1-\frac{t^2}{6r^2}\right)\sin(\phi_0-\phi)\right\}\cos 2\phi\,d\phi$$

(a) $$\int_0^{\phi_0}\frac{1-\cos(\phi_0-\phi)}{\phi_0}\cos 2\phi\,d\phi = \frac{1}{\phi_0}\left(\frac{1}{2^2-1}\sin\phi_0 - \frac{1}{2(2^2-1)}\sin 2\phi_0\right)$$

(b) $$\int_0^{\phi_0}\left(1+\frac{t^2}{3r^2}\right)\frac{\phi_0-\phi}{2\sin\phi_0}\sin\phi\cos 2\phi\,d\phi = \left(1+\frac{t^2}{3r^2}\right)\frac{1}{2\sin\phi_0}\cdot\left(-\frac{1}{2}\frac{\sin(1+2)\phi_0}{(1+2)^2} - \frac{1}{2}\frac{\sin(1-2)\phi_0}{(1-2)^2} + \frac{\phi_0}{1-2^2}\right)$$

(C) $$\int_\phi^{\phi_0} -\frac{1}{2}\left(1-\frac{t^2}{6r^2}\right)\sin(\phi_0-\phi)\cos 2\phi\,d\phi = \frac{1}{2}\left(1-\frac{t^2}{6r^2}\right)\left(\cos\phi_0 - \cos 2\phi_0\right)\cdot\frac{1}{1-2^2}$$

Hence (2) becomes

$$-r\left(\cos\phi - \frac{\sin\phi_0}{\phi_0}\right)\frac{2r}{D}\int_0^{\phi_0}\{\,---\,\}\cos 2\phi\,d\phi =$$

$$-\frac{2r^2}{D}\left(\cos\phi - \frac{\sin\phi_0}{\phi_0}\right)\Bigg\{\frac{1}{2^2-1}\frac{\sin\phi_0}{\phi_0} - \frac{1}{2(2^2-1)}\frac{\sin 2\phi_0}{\phi_0}$$

$$+\frac{1}{2}\left(1-\frac{t^2}{6r^2}\right)\frac{1}{1-2^2}\left(\cos\phi_0 - \cos 2\phi_0\right) - \left(1+\frac{t^2}{3r^2}\right)\left[\frac{1}{4(1+2)^2}\right.$$

$$\left.\cdot\frac{\sin(1+2)\phi_0}{\sin\phi_0} + \frac{1}{4(1-2)^2}\frac{\sin(1-2)\phi_0}{\sin\phi_0} - \frac{1}{2(1-2^2)}\cdot\frac{\phi_0}{\sin\phi_0}\right.$$

STEP 1 (CONTD)

$$M = \frac{r^2}{\lambda^2-1}\frac{\sin\phi_0}{\phi_0} - \frac{r^2}{\lambda(\lambda^2-1)}\frac{\sin\lambda\phi_0}{\phi_0} + \frac{r^2}{\lambda^2-1}(\cos\lambda\phi - \cos\phi)$$

$$-\frac{2r^2}{D}\left(\cos\phi - \frac{\sin\phi_0}{\phi_0}\right)\Big\{\frac{1}{\lambda^2-1}\frac{\sin\phi_0}{\phi_0} - \frac{1}{\lambda(\lambda^2-1)}\frac{\sin\lambda\phi_0}{\phi_0}$$

$$+\frac{1}{2}\left(1-\frac{t^2}{6r^2}\right)\frac{1}{1-\lambda^2}(\cos\phi_0 - \cos\lambda\phi_0)$$

$$-\left(1+\frac{t^2}{3r^2}\right)\Big[\frac{1}{4(1+\lambda)^2}\frac{\sin(1+\lambda)\phi_0}{\sin\phi_0} + \frac{1}{4(1-\lambda)^2}\frac{\sin(1-\lambda)\phi_0}{\sin\phi_0}$$

$$-\frac{1}{2(1-\lambda^2)}\frac{\phi_0}{\sin\phi_0} \quad \text{-----------} \quad (1)$$

Three values of λ will now be substituted in equation (1).

(1) $\underline{\lambda = 0}$

$$M_1 = -\frac{r^2\sin\phi_0}{\phi_0} + r^2 - r^2(1-\cos\phi_0) - \frac{2r^2}{D}\left(\cos\phi - \frac{\sin\phi_0}{\phi_0}\right)$$

$$\left\{-\frac{\sin\phi_0}{\phi_0} + 1 - \frac{1}{2}\left(1-\frac{t^2}{6r^2}\right)(1-\cos\phi_0) - \left(1+\frac{t^2}{3r^2}\right)\left(\frac{1}{4}+\frac{1}{4}-\frac{1}{2}\frac{\phi_0}{\sin\phi_0}\right)\right\}$$

$$= r^2\left(\cos\phi_0 - \frac{\sin\phi_0}{\phi_0}\right) + \frac{2r^2}{D}\left(\cos\phi_0 - \frac{\sin\phi_0}{\phi_0}\right)\Big\{\frac{\sin\phi_0}{\phi_0} - 1$$

$$+\frac{1}{2}\left(1-\frac{t^2}{6r^2}\right)(1-\cos\phi_0) + \frac{1}{2}\left(1+\frac{t^2}{3r^2}\right)\left(1-\frac{\phi_0}{\sin\phi_0}\right)\Big\}$$

$$= \frac{2r^2}{D}\Big\{\overset{①}{\frac{D}{2}} - 1 + \overset{②}{\frac{\sin\phi_0}{\phi_0}} + \overset{③}{\frac{1}{2}\left(1-\frac{t^2}{6r^2}\right)(1-\cos\phi_0)} + \overset{④}{\frac{1}{2}\left(1+\frac{t^2}{3r^2}\right)\left(1-\frac{\phi_0}{\sin\phi_0}\right)}\Big\}$$

$$\left(\cos\phi - \frac{\sin\phi_0}{\phi_0}\right) \quad \text{---------} \quad (I)$$

In the above formula:

$$D = \frac{\phi_0}{\sin\phi_0} + \cos\phi_0 - 2\frac{\sin\phi_0}{\phi_0} + \frac{t^2}{6r^2}\left(2\frac{\phi_0}{\sin\phi_0} - \cos\phi_0\right)$$

STEP 1 (cont'd)

Case II $\lambda = \frac{\pi}{2\phi_0}$ Sub. in (1) we get:

$$M_2 = r^2 \cdot \frac{4\phi_0}{\pi^2 - 4\phi_0^2} \frac{\sin\phi_0}{\phi_0} - r^2 \frac{8\phi_0^3}{\pi(\pi^2 - 4\phi_0^2)} \cdot \frac{1}{\phi_0} - r^2 \cdot \frac{4\phi_0^2}{\pi^2 - 4\phi_0^2}\left(\cos\phi - \cos\frac{\pi}{2\phi_0}\phi\right)$$

$$- \frac{2r^2}{D}\left(\cos\phi - \frac{\sin\phi_0}{\phi_0}\right)\Big\{\frac{4\phi_0^2}{\pi^2 - 4\phi_0^2} \frac{\sin\phi_0}{\phi_0} - \frac{8\phi_0^3}{\pi(\pi^2 - 4\phi_0^2)} \cdot \frac{1}{\phi_0}$$

$$+ \frac{1}{2}\left(1 - \frac{t^2}{6r^2}\right) \cdot \frac{4\phi_0^2}{4\phi_0^2 - \pi^2} \cos\phi_0$$

$$- \left(1 + \frac{t^2}{3r^2}\right)\left[\frac{\cot\phi_0}{4}\left(\frac{1}{(1+\lambda)^2} - \frac{1}{(1-\lambda)^2}\right) + \frac{2\phi_0^2}{\pi^2 - 4\phi_0^2} \cdot \frac{\phi_0}{\sin\phi_0}\right]\Big\}$$

Simplifying we get:

$$M_2 = r^2 \frac{4\phi_0}{\pi^2 - 4\phi_0^2} \sin\phi_0 - r^2 \cdot \frac{8\phi_0^2}{\pi(\pi^2 - 4\phi_0^2)} - r^2 \frac{4\phi_0^2}{\pi^2 - 4\phi_0^2} \cos\phi_0$$

$$- \frac{2r^2}{D}\left(\cos\phi - \frac{\sin\phi_0}{\phi_0}\right)\Big\{\frac{4\phi_0^{\cancel{2}}}{\pi^2 - 4\phi_0^2} \cdot \frac{\sin\phi_0}{\cancel{\phi_0}} - \frac{8\phi_0^2}{\pi(\pi^2 - 4\phi_0^2)}$$

$$+ \frac{1}{2}\left(1 - \frac{t^2}{6r^2}\right) \cdot \frac{4\phi_0^2}{4\phi_0^2 - \pi^2} \cos\phi - \left(1 + \frac{t^2}{3r^2}\right)\left[\frac{32\phi_0^3\pi}{(\pi^2 - 4\phi_0^2)^2} \cot\phi_0\right.$$

$$\left. - \frac{2\phi_0^2}{\pi^2 - 4\phi_0^2} \cdot \frac{\phi_0}{\sin\phi_0}\right]\Big\} \qquad \text{----- (II)}$$

(II) may be written in the form

$$M_2 = r^2\, II - \frac{2r^2}{D}\, III$$

CASE III $\lambda = \frac{3\pi}{2\phi_0}$ sub. in (1) we get:

$$M_3 = r^2 \cdot \frac{4\phi_0^2}{9\pi^2 - 4\phi_0^2} \frac{\sin\phi_0}{\phi_0} + r^2 \cdot \frac{8\phi_0^3}{3\pi(9\pi^2 - 4\phi_0^2)} \cdot \frac{1}{\phi_0}$$

$$+ r^2 \frac{4\phi_0^2}{9\pi^2 - 4\phi_0^2}\left(\cos\frac{3\pi}{2\phi_0}\phi - \cos\phi\right) -$$

(cont'd next pg.)

STEP 1 (contd.)

$$- \frac{2r^2}{D}\left(\cos\phi_0 - \frac{\sin\phi_0}{\phi_0}\right)\left\{\frac{4\phi_0^2}{9\pi^2-4\phi_0^2}\frac{\sin\phi_0}{\phi_0} + \frac{8\phi_0^3}{3\pi(9\pi^2-4\phi_0^2)}\frac{1}{\phi_0}\right.$$

$$- \frac{1}{2}\left(1-\frac{t^2}{6r^2}\right)\frac{4\phi_0^2}{9\pi^2-4\phi_0^2}\cos\phi_0 - \left(1+\frac{t^2}{3r^2}\right)\left[\cot\phi_0 + \frac{4a}{(1-a^2)^2}\right.$$

$$+ \frac{2\phi_0^2}{9\pi^2-4\phi_0^2} \cdot \frac{\phi_0}{\sin\phi_0}$$

Simplifying we get.

$$M_3 = r^2\left[\frac{4\phi_0}{9\pi^2-4\phi_0^2} \cdot \sin\phi_0\right] + r^2\left[\frac{8\phi_0^2}{3\pi(9\pi^2-4\phi_0^2)}\right]$$

$$+ r^2 \cdot \frac{4\phi_0^2}{9\pi^2-4\phi_0^2}\left(\cos\frac{3\pi}{2\phi_0} - \cos\phi\right) -$$

$$- \frac{2r^2}{D}\left(\cos\phi - \frac{\sin\phi_0}{\phi_0}\right)\left\{\frac{4\phi_0}{9\pi^2-4\phi_0^2} \cdot \sin\phi_0 + \frac{8\phi_0}{3\pi(9\pi^2-4\phi_0^2)}\right.$$

$$+ \frac{1}{2}\left(1-\frac{t^2}{6r^2}\right) \cdot \frac{4\phi_0^2}{4\phi_0^2-9\pi^2}\cos\phi_0 - \left(1+\frac{t^2}{3r^2}\right)\left[\cot\phi_0 \cdot\right.$$

$$\left.\left. \cdot \frac{96\phi_0^3\pi}{9\pi^2-4\phi_0^2} + \frac{2\phi_0^2}{9\pi^2-4\phi_0^2} \cdot \frac{\phi_0}{\sin\phi_0}\right]\right\} \quad --- (3)$$

(3) may be written in the form

$$M_3 = r^2 II^* - \frac{2r^2}{D} III^*$$

STEP 2:

Choice of additional load on arches, $f(\chi, \phi)$.

(The additional load on the arch is the load on the cantilever).

We may let a Fourier series represent the additional load on an arch. Thus:

$$f(\chi, \phi) = (a + b + c\chi^2)\cos\frac{\pi}{2\phi_0}\phi + (A + B\chi + C\chi^2)\cos\frac{3\pi}{2\phi_0}\phi.$$

$f(\chi, \phi)$ is load on cantilever.

a, b, c, and A, B, C are parameters that are to be determined later.

STEP 3: Elimination of deformation in the bottom arch (since the bottom arch undergoes no deformation.)

If M denotes the vertical moment in the bottom arch, then there exists a horizontal moment of magnitude $\frac{\mu M}{E}$ where μ is Poisson's ratio, E is modulus of elasticity.

$$\text{and } M + \frac{\mu M}{E} = 0 \quad \text{------- (e)}$$

M is the Moment which acts on the arch and $\frac{\mu M}{E}$ the moment that acts on the cantilever.

STEP 4: In this step we evaluate step 3. The moment equations for M_1, M_2 and M_3 which we derived in step 1 are applied to the bottom arch and the results are substituted in equation (e) of step 3.

The Cantilever moment is;

$$M_{x=H} = \int_{x=0}^{H} f(\chi, \phi)(H - \chi)\,d\chi = \cos\frac{\pi}{2\phi_0}\phi \cdot H\left(aH + \frac{bH^2}{2} + \frac{cH^3}{3}\right)$$

$$- \frac{aH^2}{2} - \frac{bH^3}{3} - \frac{cH^4}{4} + \cos\frac{3\pi}{2\phi_0}\phi\, H\left(AH + \frac{BH^2}{2} + \frac{CH^3}{3}\right)$$

STEP 4 (CONTD)

$$-A\frac{H^2}{2}-\frac{BH^3}{3}-\frac{CH^4}{4}.$$

$$\text{or } M_X=\left(\frac{aH^2}{2}+\frac{bH^3}{6}+\frac{CH^4}{12}\right)\cos\frac{\pi}{2\phi_0}+\left(\frac{AH^2}{2}+\frac{BH^3}{6}+\frac{CH^4}{12}\right)\cos\frac{3\pi}{2\phi_0}\phi$$

The load on the arch is:

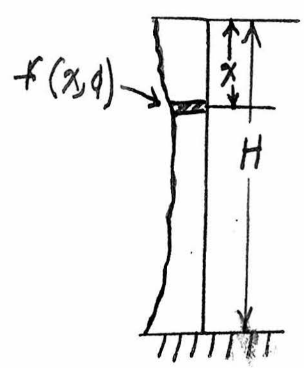
$$P(\phi)=\gamma_w H-(a+bH+CH^2)\cos\frac{\pi}{2\phi_0}\phi-(A+BH+CH^2)\cos\frac{3\pi}{2\phi_0}\phi.$$

STEP 5:

In this step we shall compute the stresses in the arches and cantilevers for the assumed additional arch load.

Determination of $\sigma_x, \sigma_\phi, \tau(x,r), \tau(\phi,r)$.

(Notation is the same as in the article by Kalman in Proceedings of A.S.C.E. for March 1931, p. 446. The notation is used by a number of authors.)



$$f=K+\delta x+\eta x^2=\left(a\cos\frac{\pi}{2\phi_0}\phi+A\cos\frac{3\pi}{2\phi_0}\phi\right)$$
$$+\left(b\cos\frac{\pi}{2\phi_0}\phi+B\cos\frac{3\pi}{2\phi_0}\phi\right)$$
$$+\left(c\cos\frac{\pi}{2\phi_0}\phi+C\cos\frac{\pi}{2\phi_0}\phi\right)$$

$$\text{or } f=K+\delta x+\eta x^2,$$

$$M=\int_{\xi=0}^{x}f(\xi)(x-\xi)\,d\xi=\int_{\xi=0}^{x}(K+\delta\xi+\eta\xi^2)(x-\xi)\,d\xi.$$

STEP 5 (CONTD).

or $M = \int_{\xi=0}^{x} \left(Kx + (\delta x - K)\xi + (\eta x - \xi)\xi^2 - \eta\xi^2\right) d\xi$

$$= \left| Kx\xi + (\delta x - K)\frac{\xi^2}{2} + (\eta x - \delta)\frac{\xi^3}{3} - \eta\frac{\xi^4}{4} \right|_{\xi=0}^{x}$$

$$= Kx^2 + (\delta x - K)\frac{x^2}{2} + (\eta x - \delta)\frac{x^3}{3} - \frac{\eta x^4}{4}$$

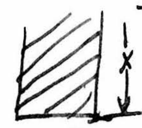
$$M_x = \left(a\cos\frac{\pi}{2\phi_0}\phi + A\cos\frac{3\pi}{2\phi_0}\phi\right)x^2 + ------$$

The shear $V_{xr} = \int_{x=0}^{x} f\,dx = \int_0^x (K + \delta x + \eta x^2)\,dx$

$$= Kx + \frac{\delta x^2}{2} + \frac{\eta x^3}{3} = \left(a\cos\frac{\pi}{2\phi_0}\phi + A\cos\frac{3\pi}{2\phi_0}\phi\right)x + ---$$

$\sigma_x = \frac{12My}{t^3}$ $\quad \tau(x,r) = \frac{V}{t}$ $\quad \sigma_x = \frac{My}{I}$ $\quad I = \frac{t^3}{12}$.

$\tau = \left(y^2 - \frac{t^2}{4}\right)C$



$C = \frac{6V}{t^3}$ $\quad \therefore t = -\frac{6V}{t^3}\left(y^2 - \frac{t^2}{4}\right)$ $\quad M t^2$

$\gamma_w = 62.4$,

$$M_\phi = \gamma_w x M_1 - (a + bx + cx^2)M_2 - (A + Bx + Cx^2)M_3$$

$$= \gamma_w x \overbrace{N\left(\cos\phi - \frac{\sin\phi_0}{\phi_0}\right)}^{\xi} - (a + bx + cx^2)\overbrace{\left(\Xi + \Gamma\cos\frac{\pi}{2\phi_0}\phi + \Delta\cos\phi\right)}^{\delta} - (A + Bx + Cx^2)\left(\Omega + \mathcal{S}\cos\phi + \Sigma\cos\frac{3\pi}{2\phi_0}\phi\right)$$

The above equation for M_ϕ, the moment in the arch, may be written:

$$(-a\delta - A\varepsilon) + x(\gamma_w\xi - b\delta - B\varepsilon) + x^2(-c\delta - C\varepsilon)$$

$$\sigma_\phi^2 = \frac{\int_0^H \int_{-\phi}^{+\phi} M_\phi^2\, r\, d\phi\, dx}{I}$$

STEP ~~E~~ 5 (cont'd)

$$M_\phi = (-a\delta - A\varepsilon) + x(\gamma w \xi - b\delta - B\varepsilon) + x^2(-c\delta - C\varepsilon)$$

$$M^2_\phi = (-a\delta - A\varepsilon)^2 + x^2(\gamma w \xi - b\delta - B\varepsilon)^2 + x^4(-c\delta - C\varepsilon)$$
$$+ 2(-a\delta - A\varepsilon)(\gamma w \xi - b\delta - B\varepsilon)x + 2(-c\delta - C\varepsilon)\cdot$$
$$(\gamma w \xi - b\delta - B\varepsilon)x^3 + 2(-a\delta - A\varepsilon)(-c\delta - C\varepsilon)x^2$$

$$\therefore \int_0^H \int_{-\phi_0}^{\phi_0} M_\phi^2 \, r \, d\phi \, dx =$$

$$r\int_{-\phi}^{\phi} \Big[\overset{\mathrm{I}}{H(-a\delta - A\varepsilon)^2} + \overset{\mathrm{II}}{\frac{H^3}{3}(\gamma w \xi - b\delta - B\varepsilon)^2} + \overset{\mathrm{III}}{\frac{H^5}{5}(-c\delta - C\varepsilon)^2}$$
$$+ \overset{\mathrm{IV}}{\frac{2H^2}{2}(-a\delta - A\varepsilon)(\gamma w \xi - b\delta - B\varepsilon)} + \overset{\mathrm{V}}{\frac{H^4}{2}(-c\delta - C\varepsilon)\cdot}$$
$$(\gamma w \xi - b\delta - B\varepsilon) + \overset{\mathrm{VI}}{\frac{2H^3}{3}(-a\delta - A\varepsilon)(-c\delta - C\varepsilon)}\Big] d\phi$$

I $\quad (-a\delta - A\varepsilon)^2 = a^2\delta^2 + 2aA\delta\varepsilon + A^2\varepsilon^2$

$$① _{\mathrm{I}} \quad \int a^2\delta^2 d\phi = a^2\int\Big[\Xi^2 + \Gamma^2\cos^2\frac{\pi}{2\phi_0}\phi + \Delta^2\cos^2\phi$$
$$+ 2\Xi\Gamma\cos\frac{\pi}{2\phi_0}\phi + 2\Gamma\Delta\cos\phi\cos\frac{\pi}{2\phi_0}\phi$$
$$\cancel{+2\Gamma\Delta\cos\phi\cos} + 2\Xi\Delta\cos\phi\Big] d\phi$$
$$= \Big[2\Xi^2\phi_0 + \Gamma^2\phi_0 + \Delta^2\Big(\phi_0 + \frac{\sin 2\phi_0}{2}\Big) + \frac{8}{\pi}\Xi\Gamma\phi_0$$
$$+ \Gamma\Delta \cdot \frac{8\pi\phi\cos\phi_0}{\pi^2 - 4\phi_0^2} + 4\Xi\Delta\sin\phi_0\Big]$$

$$= a^2\Big[\phi_0\Big(2\Xi^2 + \Gamma^2 + \Delta^2 + \frac{8}{\pi}\Xi\Gamma\Big) + \frac{\Delta^2}{2}\sin 2\phi_0$$
$$+ \frac{8\pi\phi_0\cos\phi_0\,\Gamma\Delta}{\pi^2 - 4\phi_0^2} + 4\Xi\Delta\sin\phi_0$$

STEP 5 (CONT'D.)

② $\int 2aA\delta\varepsilon\, d\phi = 2aA\int\Big[\Xi\Omega + \Omega\Gamma\cos\frac{\pi}{2\phi_0}\phi$

$+\Omega\Delta\cos\phi + \Xi\mathcal{f}\cos\frac{\pi}{2\phi_0}\cos\phi + \Delta\mathcal{f}\cos^2\phi$

$+\Xi\Sigma\cos\frac{3\pi}{2\phi_0}\phi + \Gamma\Sigma\cos\frac{\pi}{2\phi_0}\phi\cos\frac{3\pi}{2\phi_0}\phi$

$+\Sigma\Delta\cos\frac{3\pi}{2\phi_0}\phi\cos\phi\Big]d\phi$

$= 2aA\Big[2\Xi\Omega\phi_0 + \frac{4\Omega\Gamma\phi_0}{\phi_0} + 2\sin\phi_0(\Omega\Delta + \Xi\mathcal{f}$

$+\Gamma\mathcal{f}\cdot\frac{4\pi\phi_0\cos\phi_0}{\pi^2-4\phi_0^2} + \Delta\mathcal{f}(\phi_0 + \frac{1}{2}\sin 2\phi_0)$

$-\frac{4}{3\pi}\Xi\Sigma\phi_0 - \frac{12\Delta\Sigma\pi\phi_0\cos\phi_0}{9\pi^2-4\phi_0^2}$

③ $A^2\int\varepsilon^2 d\phi = A^2\int_{-\phi_0}^{\phi_0}\Big[\Omega^2 + \mathcal{f}^2\cos^2\phi + \Sigma^2\cos^2\frac{3\pi}{2\phi_0}$

$+2\Omega\mathcal{f}\cos\phi + 2\mathcal{f}\Sigma\cos\frac{3\pi}{2\phi_0}\phi\cos\phi + 2\Omega\Sigma\cos\frac{3\pi}{2\phi_0}\phi\Big]d\phi$

$= A^2\Big[2\Omega^2\phi_0 + \mathcal{f}^2(\phi_0 + \frac{1}{2}\sin 2\phi_0 + \Sigma^2\phi_0 + 4\Omega\mathcal{f}\cos\phi_0$

$+\frac{8\mathcal{f}\Sigma\pi\phi_0\cos\phi_0}{\pi^2-4\phi_0^2} - \frac{8}{3\pi}\Omega\Sigma\phi_0\Big]$

$= A^2\Big[\phi_0(2\Omega^2 + \mathcal{f}^2 + \Sigma^2 - \frac{8}{3\pi}\Omega\Sigma) + \mathcal{f}^2\cdot\frac{1}{2}\sin 2\phi_0$

$+4\Omega\mathcal{f}\cos\phi_0 + \frac{8\mathcal{f}\Sigma\pi\phi_0\cos\phi_0}{\pi^2-4\phi_0^2}\Big].$

STEP 5 (CONT'D)

$$\text{II} \quad (\gamma_W \xi - b\delta - B\varepsilon)^2 = \overset{①_{II}}{\gamma_W^2 \xi^2} + \overset{②_{II}}{b^2\delta^2} + \overset{③_{II}}{B^2\varepsilon^2} - \overset{④_{II}}{2\gamma_W b \xi \delta} + \overset{⑤_{II}}{2bB\delta\varepsilon} - \overset{⑥_{II}}{2\gamma_W B \xi \varepsilon}.$$

$①_{II}$ $$\int \gamma_W^2 \xi^2 d\phi = \gamma_W^2 ⓝ^2 \int_{-\phi_0}^{\phi_0} \left[\cos^2\phi - \frac{2\sin\phi_0}{\phi_0}\cos\phi_0 + \frac{\sin^2\phi_0}{\phi_0^2}\right] d\phi = \gamma_W^2 ⓝ^2 \left[\phi_0 + \frac{1}{2}\sin 2\phi_0 - \frac{4\sin\phi_0}{\phi_0}\sin\phi_0 + \frac{2\sin^2\phi_0}{\phi_0^2}\cdot\phi_0\right.$$

$$= \gamma_W^2 ⓝ^2\left[\phi_0 + \frac{1}{2}\sin 2\phi_0 - \frac{2\sin^2\phi_0}{\phi_0}\right]$$

$②_{II}$ $$\int b^2\delta^2 d\phi = b^2\left[\phi_0\left(2\Xi^2 + \Gamma^2 + \Delta^2 + \frac{8}{\pi}\Xi\Gamma\right) + \frac{\Delta^2}{2}\sin 2\phi_0 + \frac{8\pi\phi_0\cos\phi_0\,\Gamma\Delta}{\pi^2 - 4\phi_0^2} + 4\Xi\Delta\sin\phi_0\right]$$

$③_{II}$ $$\int B^2\varepsilon^2 d\phi = B^2\left[\phi_0\left(2\Omega^2 + \mathcal{S}^2 + \Sigma^2 - \frac{8}{3\pi}\Omega\Sigma\right) + \mathcal{S}^2\cdot\frac{1}{2}\sin 2\phi_0 + 4\Omega\mathcal{S}\cos\phi_0 + \frac{8\mathcal{S}\Sigma\pi\phi_0\cos\phi_0}{\pi^2 - 4\phi_0^2}\right]$$

$④_{II}$ $$-2b\gamma_W\int\xi\delta = -2b\gamma_W ⓝ \int\left[\Xi\cos\phi_0 + \Gamma\cos\frac{\pi}{2\phi_0}\cos\phi + \Delta\cos^2\phi - \Xi\frac{\sin\phi_0}{\phi_0} - \Gamma\frac{\sin\phi_0}{\phi_0}\cos\frac{3\pi}{2\phi_0} - \Delta\frac{\sin\phi_0}{\phi_0}\cos\phi\right]d\phi$$

$$= -2b\gamma_W ⓝ\left[\cancel{2\Xi\sin\phi_0} + \frac{4\Gamma\pi\phi_0\cos\phi_0}{\pi^2 - 4\phi_0^2} + \Delta\left(\phi_0 + \frac{1}{2}\sin 2\phi_0\right) - \cancel{2\Xi\sin\phi_0} + \frac{4\Gamma}{3\pi}\sin\phi_0 - \frac{2\Delta\sin^2\phi_0}{\phi_0}\right].$$

STEP 5 (cont'd.)

$$⑤_{II}\quad 2bB\int\delta\varepsilon\,d\phi = 2bB\Big[\phi_0\Big(2\Xi\Omega + \frac{4\Omega\Gamma}{\pi} - \frac{4}{3\pi}\Xi\Sigma\Big)$$
$$+2\sin\phi_0(\Omega\Delta+\Xi\delta) + \Gamma\delta\,\frac{4\pi\phi_0\cos\phi_0}{\pi^2-4\phi_0^2}$$
$$+\Delta\delta\Big(\phi_0+\tfrac{1}{2}\sin 2\phi_0\Big) - \frac{12\Delta\Sigma\pi\,\phi_0\cos\phi_0}{9\pi^2-4\phi_0^2}$$

$$⑥_{II}\quad -2\gamma_w B\int\xi\varepsilon\,d\phi = -2\gamma_w B\,N\int\Big[\Omega\cos\phi + \delta\cos^2\phi$$
$$+\Sigma\cos\frac{3\pi}{2\phi_0}\phi\cos\phi - \Omega\frac{\sin\phi_0}{\phi_0} - \delta\frac{\sin\phi_0}{\phi_0}\cos\phi$$
$$-\Sigma\frac{\sin\phi_0}{\phi_0}\cos\frac{3\pi}{2\phi_0}\phi\Big]d\phi.$$
$$= -2\gamma_w B\,N\Big[2\Omega\sin\phi_0 + \delta\Big(\phi_0+\tfrac{1}{2}\sin 2\phi_0\Big)$$
$$-\frac{12\Sigma\pi\,\phi_0\cos\phi_0}{9\pi^2-4\phi_0^2} - 2\Omega\sin\phi_0 - 2\frac{\delta\sin^2\phi_0}{\phi_0}$$
$$+\frac{4\Sigma}{3\pi}\sin\phi_0\Big].$$

$$\text{III}\quad (-c\delta - C\varepsilon)^2 = \overset{①}{c^2\delta^2} + \overset{②}{2cC\delta\varepsilon} + \overset{③}{C^2\varepsilon^2}$$

$$①_{III}\quad c^2\int\delta^2\,d\phi = c^2\Big[\phi_0\Big(2\Xi^2+\Gamma^2+\Delta^2 + \frac{8}{\pi}\Xi\Gamma$$
$$+\frac{\Delta^2}{2}\sin 2\phi_0 + \frac{8\pi\phi_0\cos\phi_0\,\Gamma\Delta}{\pi^2-4\phi_0^2}$$
$$+4\Xi\Delta\sin\phi_0.\Big]$$

$$②_{III}\quad 2cC\int\delta\varepsilon\,d\phi = 2cC\Big[2\Xi\Omega\phi_0 + \frac{4\Omega\Gamma}{\pi}\phi_0$$
$$+2\sin\phi_0(\Omega\Delta+\Xi\delta) + \Gamma\delta\cdot\frac{4\pi\phi_0\cos\phi_0}{\pi^2-4\phi_0^2}$$

STEP 5 (CONT'D)

$$+ \Delta\,\delta\left(\phi_0 + \tfrac{1}{2}\sin 2\phi_0\right) - \frac{4}{3\pi}\,\Xi\,\Sigma\,\phi_0 - \frac{12\,\Delta\,\Sigma\,\pi\,\phi_0\cos\phi_0}{9\pi^2 - 4\phi_0^2}$$

$$③_{III}\quad c^2\int \varepsilon^2\,d\phi = c^2\Big[2\Omega^2\phi_0 + \delta^2\left(\phi_0 + \tfrac{1}{2}\sin 2\phi_0\right) + \Sigma^2\phi_0 + 4\Omega\,\delta\cos\phi + \frac{8\,\delta\,\Sigma\,\pi\,\phi_0\cos\phi_0}{\pi^2 - 4\phi_0^2} - \frac{8}{3\pi}\Sigma\,\phi_0\Big].$$

$$\underline{IV}\quad (-A\delta - A\varepsilon)(\gamma_w - b\delta - B\varepsilon) = \overset{①}{-A\gamma_w\delta\xi} + \overset{②}{A b\delta^2} + \overset{③}{AB\delta\varepsilon} \overset{④}{- A\gamma_w\varepsilon\xi} + Ab\delta\varepsilon + AB\varepsilon^2$$

$$①_{IV}\quad -A\gamma_w\int\delta\xi\,d\phi = A\gamma_w \,Ⓝ\int\Big[\Xi\cos\phi + \Gamma\cos\tfrac{\pi}{2\phi_0}\phi\cos\phi + \Delta\cos^2\phi - \Xi\frac{\sin\phi_0}{\phi_0} - \Gamma\frac{\sin\phi_0}{\phi_0}\cos\frac{\pi}{2\phi_0}\phi - \Delta\frac{\sin\phi_0}{\phi_0}\cos\phi\Big]d\phi.$$

$$= -A\gamma_w\,Ⓝ\Big[2\,\Xi\sin\phi_0 + \frac{4\,\Gamma\,\pi\,\phi_0\cos\phi_0}{\pi^2 - 4\phi_0^2} + \Delta\left(\phi_0 + \tfrac{1}{2}\sin 2\phi_0\right) - 2\,\Xi\sin\phi_0 - \frac{4\,\Gamma\sin\phi_0}{\pi} - \frac{2\Delta\sin^2\phi_0}{\phi_0}\Big]$$

$$②_{IV}\quad Ab\int\delta^2\,d\phi = Ab\Big[\phi_0\left(2\,\Xi^2 + \Gamma^2 + \Delta^2 + \frac{8}{\pi}\,\Xi\,\Gamma\right) + \frac{\Delta^2}{2}\sin 2\phi_0 + \frac{8\pi\,\phi_0\cos\phi_0\,\Gamma\,\Delta}{\pi^2 - 4\phi_0^2} + 4\,\Xi\,\Delta\sin\phi_0.$$

$$③_{IV}\quad A(b+B)\int\delta\varepsilon\,d\phi = A(b+B)\Big[\phi_0\left(2\,\Xi\,\Omega + \frac{4\Omega}{\pi} - \frac{4}{3\pi}\,\Xi\,\Sigma\right) + 2\sin\phi_0\left(\Omega\Delta + \Xi\,\delta\right).$$

STEP 5 (cont'd.)

$$+ \frac{\Gamma \mathcal{J} \cdot 4\pi \phi_0 \cos\phi_0}{\pi^2 - 4\phi_0^2} + \Delta \mathcal{J}\left(\phi_0 + \tfrac{1}{2} \sin 2\phi_0\right)$$

$$- \frac{12 \Delta \Sigma \phi_0 \cos\phi_0}{9\pi^2 - 4\phi_0^2}\Big]$$

④$_{IV}$ $$-A\gamma_w \int \varepsilon \xi \, d\phi = -A\gamma_w \Big[\mathcal{J}\left(\phi_0 + \tfrac{1}{2} \sin 2\phi_0\right) - \frac{12\Sigma\pi\phi_0\cos\phi_0}{9\pi^2-4\phi_0^2} - \frac{2\mathcal{J}\sin^2\phi_0}{\phi_0} + \frac{4}{3\pi}\Sigma\sin\phi_0\Big]$$

⑤$_{IV}$ $$AB\int \varepsilon^2 \, d\phi = AB\Big[\phi_0\left(2\Omega^2 + \mathcal{J}^2 + \Sigma^2 - \frac{8}{3\pi}\Omega\Sigma\right) + \mathcal{J}^2 \cdot \tfrac{1}{2} \sin 2\phi_0 + 4\Omega\mathcal{J}\cos\phi_0 + \frac{8\mathcal{J}\Sigma\pi\phi_0\cos\phi_0}{\pi^2-4\phi_0^2}$$

V $$(-c\delta - C\varepsilon)(\gamma_w\xi - b\delta - B\varepsilon) = \overset{①}{-c\gamma_w\delta\xi} + \overset{②}{bc\delta^2} + \overset{③}{Bc\varepsilon\delta} \overset{④}{- C\gamma_w\varepsilon\xi} + bC\varepsilon\delta + \overset{⑤}{BC\varepsilon^2}$$

①$_V$ $$-c\gamma_w\int\delta\xi \, d\phi = -c\gamma_w\Big[\frac{4\Gamma\pi\phi_0\cos\phi_0}{\pi^2-4\phi_0^2} + \Delta\left(\phi_0+\tfrac{1}{2}\sin 2\phi_0\right) - \frac{4}{\pi}\Gamma\sin\phi_0 - \frac{2\Delta\sin^2\phi_0}{\phi_0}\Big]$$

②$_V$ $$bc\int\delta^2 \, d\phi = bc\Big[\phi_0\left(2\Xi^2+\Gamma^2+\Delta^2+\frac{8}{\pi}\Xi\Gamma\right) + \frac{\Delta^2}{2}\sin 2\phi_0 + \frac{8\pi\phi_0\cos\phi_0}{\pi^2-4\phi_0^2}\Gamma\Delta + 4\Xi\Delta\sin\phi_0\Big]$$

③$_V$ $$(Bc+bC)\int\varepsilon\delta = (Bc+bC)\Big[\phi_0\left(2\Xi\Omega + \frac{4\Omega\Gamma}{\pi} - \frac{4}{3\pi}\Xi\Sigma\right) + 2\sin\phi_0(\Omega\Delta+\Xi\mathcal{J}) + \frac{\Gamma\mathcal{J}\cdot 4\pi\phi_0\cos\phi_0}{\pi^2-4\phi_0^2}$$

$$+ \Delta\mathcal{J}\left(\phi_0+\tfrac{1}{2}\sin 2\phi_0\right) - \frac{12\Delta\Sigma\pi\phi_0\cos\phi_0}{9\pi^2-4\phi_0^2}\Big]$$

STEP 5 (CONT'D.)

$$-\frac{2\delta\sin^2\phi_0}{\phi_0}+\frac{4}{3\pi}\Sigma\sin\phi_0\Big].$$

$$⑤_{V}\quad BC\int\varepsilon^2 d\phi = BC\Big[\phi_0\Big(2\Omega^2+\delta^2+\Sigma^2-\frac{8}{3\pi}\Omega\Sigma\Big)$$
$$+\delta^2\cdot\tfrac{1}{2}\sin 2\phi_0+4\Omega\delta\cos\phi_0+\frac{8\delta\Sigma\pi\phi_0\cos\phi_0}{\pi^2-4\phi_0^2}\Big]$$

$$\text{VI}\quad(-a\delta-A\varepsilon)(-c\delta-C\varepsilon)=\overset{①}{ac\delta^2}+\overset{②}{(aC+Ac)\delta\varepsilon}$$
$$+\overset{③}{AC\varepsilon^2}$$

$$①_{VI}\quad ac\int\delta^2 d\phi = ac\Big[\Big(2\Xi^2+\Gamma^2+\Delta^2+\frac{8}{\pi}\Xi\Gamma\Big)\phi_0$$
$$+\frac{\Delta^2}{2}\sin 2\phi_0+\frac{8\pi\phi_0\cos\phi_0\Gamma\Delta}{\pi^2-4\phi_0^2}+4\Xi\Delta\sin\phi_0\Big]$$

$$②_{VI}\quad (aC+Ac)\int\delta\varepsilon\,d\phi=(aC+Ac)\Big[2\Xi\Omega\phi_0+\frac{4\Omega\Gamma\phi_0}{\pi}$$
$$+2\sin\phi_0(\Omega\Delta+\Xi\delta)+\frac{\Gamma\delta\cdot4\pi\phi_0\cos\phi_0}{\pi^2-4\phi_0^2}$$
$$+\Delta\delta\Big(\phi_0+\tfrac{1}{2}\sin 2\phi_0\Big)-\frac{4}{3\pi}\Xi\Sigma\phi_0-\frac{2\Delta\Sigma\pi\phi_0\cos\phi_0}{9\pi^2-4\phi_0^2}\Big]$$

$$③_{VI}\quad AC\int\varepsilon^2 d\phi = AC\Big[2\Omega^2\phi_0+\delta^2\Big(\phi_0+\tfrac{1}{2}\sin 2\phi_0\Big)$$
$$+\Sigma^2\phi_0+4\Omega\delta\cos\phi_0+\frac{8\delta\Sigma\pi\phi_0\cos\phi_0}{\pi^2-4\phi_0^2}-\frac{8}{3\pi}\Omega\Sigma\phi_0\Big]$$

STEP 6: In this step we integrate the quantities occuring in the equation for the work of deformation.

$$W = \iiint \left[\sigma_x^2 + \sigma_\phi^2 - \frac{1}{m+1}(\sigma_x + \sigma_\phi)^2 + 2\left(t_{xr}^2 + t_{\phi r}^2\right)\right] \quad \text{--(1)}$$

$$\iiint \sigma_x^2 = \frac{1}{I}\iiint \frac{M_x^2 y^2}{I^2} = \frac{1}{I}\int_0^H \int_{-\phi_0}^{+\phi_0} M_x^2 \, dx \, r \, d\phi.$$

$$\iiint \sigma_\phi^2 = \frac{1}{I}\int_0^H \int_{-\phi_0}^{\phi_0} M_\phi^2 \, dx \, r \, d\phi.$$

$$\iiint \sigma_x \sigma_\phi = \frac{M_x \cdot y}{I} \cdot \frac{M_\phi y}{I} = \frac{1}{I}\iint M_x M_\phi.$$

$$\iiint t_{xr}^2 = \frac{6}{5t}\iint V_{xr}^2 \, r \, d\phi \, dx.$$

SHEAR CALCULATIONS:

$$V_{xr} = \kappa x + \frac{\gamma x^2}{2} + \frac{\eta x^3}{3}$$

$$(V_{xr})^2 = \kappa^2 x^2 + \frac{\gamma^2 x^4}{4} + \frac{\eta^2 x^6}{9} + \frac{2}{3}\kappa\eta x^4 + \kappa\gamma x^3 + \frac{\gamma\eta x^5}{3}$$
$$+ \frac{\eta^2 x^6}{9},$$

$$\iint V_{xr}^2 = \int_{-\phi_0}^{\phi_0}\int_0^H \left(\kappa^2 x^2 + \kappa\gamma x^3 + \left(\frac{\gamma^2}{4} + \frac{2}{3}\kappa\eta\right)x^4 + \frac{\gamma\eta x^5}{3} + \frac{\eta^2 x^6}{9}\right) r \, d\phi \, dx.$$

$$= \int_{-\phi_0}^{\phi_0}\left[\overset{①}{\frac{\kappa^2 H^3}{3}} + \overset{②}{\frac{\kappa\gamma H^4}{4}} + \frac{\left(\overset{③}{\frac{\gamma^2}{4}} + \overset{④}{\frac{2}{3}\kappa\eta}\right)}{5} + \overset{⑤}{\frac{\gamma\eta H^6}{18}} + \overset{⑥}{\frac{\eta^2 H^7}{63}}\right] d\phi$$

① $$\int_{-\phi_0}^{\phi_0} \frac{\kappa^2 H^3}{3} d\phi = \frac{H^3}{3}\int_{-\phi_0}^{\phi_0}\left(a^2 \cos^2\frac{\pi}{2\phi_0}\phi + 2aA \cdot \cos\frac{\pi}{2\phi_0}\phi \cos\frac{3\pi}{2\phi_0}\phi + A^2 \cos^2\frac{3\pi}{2\phi_0}\phi\right) d\phi.$$

STEP 6 (contd)

$$= \frac{H^3}{3}\left[a^2\left(\frac{\phi}{2} + \frac{\phi_0 \sin\frac{\pi}{\phi_0}\phi}{2\pi}\right) + 2aA\left(\frac{\phi_0 \sin\frac{\pi}{\phi_0}\phi}{2\pi} + \frac{\phi_0 \sin\frac{2\pi}{\phi_0}\phi}{4\pi}\right.\right.$$

$$\left.\left. + A^2\left(\frac{\phi}{2} + \frac{\phi_0 \sin\frac{3\pi}{\phi_0}\phi}{6\pi}\right)\right]_{-\phi_0}^{\phi_0}$$

$$= \frac{H^3}{3}\phi_0\left(a^2 + A^2\right)$$

(2) $$\int \frac{K\gamma H^4}{4} = \frac{H^4}{4}\int_{-\phi_0}^{\phi_0}\left[ab\cos^2\frac{\pi}{2\phi_0}\phi + (bA + aB)\cos\frac{3\pi}{2\phi_0}\phi\cos\frac{\pi}{2\phi_0}\phi\right.$$

$$\left. + AB\cos^2\frac{3\pi}{2\phi_0}\phi\right]d\phi.$$

$$= \frac{H^4}{4}\phi_0(ab + BA)$$

(3) $$\int \frac{\gamma^2}{20}H^5 = \frac{H^5}{20}\int_{-\phi_0}^{\phi_0} b^2\cos^2\frac{\pi}{2\phi_0}\phi + 2bB\cos\frac{3\pi}{2\phi_0}\phi\cos\frac{\pi}{2\phi_0}\phi$$

$$+ B^2\cos^2\frac{3\pi}{2\phi_0}\phi\,]\,d\phi. = \frac{H^5}{20}\left[\phi_0(b^2 + B^2)\right)$$

(4) $$\int_{-\phi_0}^{\phi_0}\frac{2Kn}{15}H^5 = \frac{2H^5}{15}\int_{-\phi_0}^{\phi_0}\left[ac\cos^2\frac{\pi}{2\phi_0}\phi + (cA + aC)\cdot\right.$$

$$\left.\cdot\cos\frac{3\pi}{2\phi_0}\phi\cos\frac{\pi}{2\phi_0}\phi + AC\cos^2\frac{3\pi}{2\phi_0}\phi\right]d\phi.$$

$$= \frac{2H^5}{15}\left[\phi_0(ac + AC)\right].$$

(5) $$\int \frac{\gamma n H^5}{18} = \frac{H^6}{18}\int_{-\phi_0}^{\phi_0}\left[bc\cos^2\frac{\pi}{2\phi_0}\phi + (cB + bC)\cos\frac{3\pi}{2\phi_0}\phi\cdot\right.$$

$$\left.\cdot\cos\frac{\pi}{2\phi_0}\phi + BC\cos^2\frac{3\pi}{2\phi_0}\phi\right]d\phi.$$

$$= \frac{H^6}{18}\left[\phi_0(bc + BC)\right].$$

STEP 6 (Cont'd).

$$⑥ \int_{-\phi_0}^{\phi_0} \frac{n^2 H^7}{63} d\phi = \frac{H^7}{63} \int_{-\phi_0} \left(c^2 \cos^2 \frac{\pi}{2\phi_0}\phi + 2cC \cos\frac{\pi}{2\phi_0}\phi \cos\frac{3\pi}{2\phi_0}\phi \right.$$

$$\left. + C^2 \cos^2 \frac{3\pi}{2\phi_0}\phi \right) d\phi = \frac{H^7}{63}\left[\phi_0 (c^2 + C^2)\right].$$

Combining eqs. ①, ② - - - ⑥ we get:

$$\iint V_x{}^2 r = \phi_0 \left[\frac{H^3}{3}(a^2 + A^2) + \frac{H^4}{4}(ab + AB) + \frac{H^5}{20}(b^2 + B^2) \right.$$

$$\left. + \frac{2H^5}{15}(ac + AC) + \frac{H^6}{18}(bc + BC) + \frac{H^7}{63}(c^2 + C^2) \right].$$

$$= \frac{\phi_0}{60}\left[20H^3(a^2 + A^2) + 15H^4(ab + AB) + 3H^5(b^2 + B^2) \right.$$

$$\left. + 8H^5(ac + A C) + 3.33H^6(bc + BC) + .9523(c^2 + C^2) \right].$$

$$\frac{6}{5t}\iint V^2 r\, d\phi\, dx = \frac{r}{50t}\left[20H^3(a^2 + A^2) + 15H^4(ab + AB) \right.$$

$$+ 3H^5(b^2 + B^2) + 8H^5(ac + AC) + 3.33H^6(bc + BC)$$

$$+ .92538(c^2 + C^2).$$

Moment Calculations;

$$M_x = Kx^2 + (\gamma x - K)\frac{x^2}{2} + (nx - \gamma)\frac{x^3}{3} - \frac{n x^4}{4}$$

$$M_x{}^2 = \frac{K^2 x^4}{4} + \frac{\gamma K x^5}{6} + \left(\frac{Kn}{12} + \frac{\gamma^2}{36}\right) x^6 + \frac{\gamma n}{36} x^7 + \frac{n^2}{144} x^8,$$

$$\int_{-\phi_0}^{\phi_0}\int_0^H M_x{}^2\, d\phi\, dx = \int_{-\phi_0}^{\phi_0} \left[\frac{K^2 H^5}{20} + \frac{\gamma K H^6}{36} + \left(\frac{Kn}{84} + \frac{\gamma^2}{252}\right) H^7 \right.$$

$$\left. + \frac{\gamma n}{8 \times 36} H^8 + \frac{n^2}{9 \times 144} H^9 \right] d\phi.$$

From previous work;

$$① \int_{-\phi_0}^{\phi_0} K^2 d\phi = \phi_0 (a^2 + A^2)$$

STEP 6 (cont'd.)

② $\int_{-\phi_0}^{\phi_0} \gamma K \, d\phi = \phi_0(ab + AB)$

③ $\int_{-\phi_0}^{\phi_0} K\eta \, d\phi = \phi_0(ac + AC)$

④ $\int_{-\phi_0}^{\phi_0} \gamma^2 \, d\phi = \phi_0(b^2 + B^2)$

⑤ $\int_{-\phi_0}^{\phi_0} \gamma\eta \, d\phi = \phi_0(bc + BC)$

⑥ $\int_{-\phi_0}^{\phi_0} \eta^2 \, d\phi = \phi_0(c^2 + C^2)$

$$\therefore \iint M_x^2 \, dx \, r \, d\phi = \frac{r\phi_0}{1296}\Big[64.8000H^5(a^2+A^2) + 36.000H^6(ab+AB) + 15.4285H^7(ac+AC) + 5.1429H^7(b^2+B^2) + 4.5000H^8(bc+BC) + H^9(c^2+C^2)\Big].$$

$$\frac{1}{I}\iint M_x^2 \, dx \, r \, d\phi = \frac{rt^3\phi_0}{15,552}\Big[64.8000H^5(a^2+A^2) + 36.0000 \cdot H^6(ab+AB) + 15.4285H^7(ac+AC) + 5.1429H^7(b^2+B^2) + 4.5000H^8(bc+BC) + H^9(c^2+C^2)\Big].$$

STEP 7: In this step we determine the paramaters in the Fourier series which minimize the work of deformation.

We may find the moments in the bottom arch by substituting $x = H$ in the formulas for M_1, M_2, and M_3 in Step 1.

M_1, the moment in the bottom arch due to constant load, may be written:

$$M_1 = r^2(R_1 + S_1 \cos\phi)$$

M_2, the moment in the bottom arch due to the $\cos\frac{\pi}{2\phi_0}\phi$ load, may be written:

$$M_2 = r^2\left(R_2 + S_2\cos\phi + T_2\cos\frac{\pi}{2\phi_0}\phi\right)$$

M_3, the moment in the bottom arch due to the $\cos\frac{3\pi}{2\phi_0}\phi$ load, may be written:

$$M_3 = r^2\left(R_3 + S_3\cos\phi + W_3\cos\frac{3\pi}{2\phi_0}\phi\right.$$

The load really applied to the bottom arch is:

$$P(\phi) = \gamma_W H - (a + bH + cH^2)\cos\frac{\pi}{2\phi_0} - (A + BH + CH^2)\cos\frac{3\pi}{2\phi_0}\phi.$$

The total vertical moment in the bottom arch is:

$$M_{total} = KH\cdot r^2(R_1 + S_1\cos\phi) - (a + bH + cH^2)\left(R_2 + S_2\cos\phi + T_2\cos\frac{\pi}{2\phi_0}\phi\right)r^2 - (A + BH + CH^2)\left(R^3 + S_3\cos\phi + W_3\cos\frac{3\pi}{2\phi_0}\phi\right)r^2$$

The Horizontal moment is; $\frac{\mu}{E}M$ where μ is poisson's ratio.

The combined horizontal and vertical moments is;

$$M + \frac{\mu}{E}M = KHR_1 - (a + bH + cH^2)R_2 - (A + BH + CH^2)R_3$$
$$+ KHS_1 - (a + bH + cH^2)S_2 - (A + BH + CH^2)S_3\cos\phi$$
$$- (a + bH + cH^2)T_2\cos\frac{\pi}{2\phi_0}\phi - (A + BH + CH^2)W_3\cos\frac{3\pi}{2\phi_0}\phi$$
$$+ \frac{\mu}{r^2E}\left(\frac{aH^2}{2} + \frac{bH^3}{6} + \frac{cH^4}{12}\right)\cos\frac{\pi}{2\phi_0}\phi + \frac{\mu}{r^2E}\left(\frac{AH^2}{2} + \frac{BH^3}{6} + \frac{CH^4}{12}\right)\cos\frac{3\pi}{2\phi_0}\phi = 0$$

STEP 7 (Cont'd.)

If: $\lambda + K \cos\phi + \gamma \cos\frac{\pi}{2\phi_0}\phi + \varepsilon \cos\frac{3\pi}{2\phi_0}\phi \equiv 0$

then $\lambda = K = \gamma = \varepsilon = 0$.

From this relation we obtain:

$$KHR_1 - (a + bH + cH^2)R_2 - (A + BH + CH^2)R_3 = 0 \quad \text{---} (1)$$

$$KHS_1 - (a + bH + cH^2)(S_2) - (A + BH + CH^2)S_3 = 0 \quad \text{---} (2)$$

$$-(a + bH + cH^2)T_2 + \frac{w}{r^2E}\left(\frac{aH^2}{2} + \frac{bH^3}{6} + \frac{cH^4}{12}\right) = 0 \quad \text{---} (3)$$

$$-(A + BH + CH^2)W_3 + \frac{w}{r^2E}\left(\frac{AH^2}{2} + \frac{BH^3}{6} + \frac{CH^4}{12}\right) = 0 \quad \text{---} (4)$$

If $P = \frac{w}{r^2ET_2}$ and $Q = \frac{w}{r^2EW_3}$ then:

$$a\left(\frac{PH^2}{2} - 1\right) + b\left(\frac{PH^3}{6} - H\right) + c\left(\frac{PH^4}{12} - H^2\right) = 0 \quad \text{---} (3^*)$$

$$\text{and } A\left(\frac{QH^2}{2} - 1\right) + B\left(\frac{QH^3}{6} - H\right) + C\left(\frac{QH^4}{12} - H^2\right) = 0 \quad \text{---} (4^*)$$

From (1), (2), (3*) and (4*) we obtain the important equations for the paramaters.

IMPORTANT EQUATIONS

$$a+b+c = \gamma_w H \times \frac{\begin{vmatrix} R_1 & R_3 \\ S_1 & S_3 \end{vmatrix}}{\begin{vmatrix} R_2 & R_3 \\ S_2 & S_3 \end{vmatrix}} = \gamma_w H \, \frac{R_1 S_3 - R_3 S_1}{R_2 S_3 - R_3 S_2} = K$$

$$A+B+C = \gamma_w H \, \frac{\begin{vmatrix} R_2 & R_1 \\ S_2 & S_1 \end{vmatrix}}{\begin{vmatrix} R_2 & R_3 \\ S_2 & S_3 \end{vmatrix}} = \gamma_w H \, \frac{R_2 S_1 - R_1 S_2}{R_2 S_3 - R_3 S_2} = K_{\text{(capital)}}$$

$$b = \frac{KH^2(PH^2-12) - a\{PH^4 - 12H^2 - 6PH^2 + 12\}}{PH^4 - 12H^2 - 2PH^3 + 12H.}$$

$$c = -\frac{6}{H^2} \cdot \frac{PH^2-2}{PH^2-12}\,a - \frac{2KH(PH^2-6)}{PH^2-12H^2-2PH^3+12H} + \frac{2}{H} \cdot \frac{PH^2-6}{PH^2-12} \cdot \frac{PH^4-12H^2-6PH^2+12}{PH^4-12H^2-2PH^3+12H}\,a$$

$$B = \frac{KH^2(QH^2-12) - A\{QH^4 - 12H^2 - 6QH^2 + 12\}}{QH^4 - 12H^2 - 2QH^3 + 12H.}$$

$$C = \frac{-6}{H^2} \cdot \frac{QH^2-2}{QH^2-12}\,A - \frac{2KH(QH^2-6)}{QH^4+12H^2-2QH^3+12H} + \frac{2}{H}\,\frac{QH^2-6}{QH^2-12} \cdot \frac{QH^4-12H^2-6QH^2+12}{QH^4-12H^2-2QH^3+12H}\,A.$$

$$P = \frac{w}{r^2 E T_2} \qquad w = 7. \qquad \gamma_w = 62.4$$

$$Q = \frac{w}{r^2 E W_3}$$

DISCUSSION

A first glance at the computations of the Kalman method of analysis may give one the impression that the method is considerably involved. A great deal of time and patience is required for one to follow through the computations and thoroughly understand each step.

However, the method is not nearly as complicated as it appears to be at first sight. Most of the computations, while long, are quite simple. Other methods of analysis also involve a great deal of labor. Some methods require for good approximations the solving of sixty to a hundred simultaneous equations. Solving this many simultaneous equations, even if trial and error methods are adapted, is no small task.

When the Kalman method is used, and the general equations for the parameters are derived, the parameters for any particular dam may be obtained by merely substituting numerical values in the formulas on page 32. The general equations do not have to be derived for each dam.

The author believes that the Kalman method has real merit. The Kalman method expresses in <u>one</u> <u>equation</u> the cantilever loading at <u>any</u> point in the dam. The equation for the cantilever load is:

$$f(x, \phi) = (a + bx + cx^2)\cos\frac{\pi}{2\phi_0}\phi + (A + Bx + Cx^2)\cos\frac{3\pi}{2\phi_0}\phi + \cdots$$

Any point in the dam is located by its coordinates x and ϕ. "x" is the distance of the point below the surface of the water, and ϕ is the angle to the point from the center. When these two quantities are known the cantilever loading may be obtained from the above equation. The load on the arch element is simply the load due to the water pressure minus this correction load $f(x, \phi)$.

Just how accurate the method is, the author does not know, but he believes that the method will give good results. It would be interesting to analyze a particular dam by several methods, and compare the results. Such a research would involve a great deal of labor, but the results would be valuable.

The problem of determining the stress distribution in arch dams is an interesting one, and one which offers considerable opportunity for future research.

APPENDIX

Results of numerical calculations.

TABLE OF CONSTANTS

$\log \sin 60^\circ = 9.9375306$

$\log \sin 70^\circ = 9.9729858 - 10.$

$\log \sin 80^\circ = 9.9933515 - 10.$

$\log \pi = 0.4971499$ $\qquad$ $\operatorname{colog} \pi = 9.5028501.$

$\log \frac{8\phi_0^2}{\pi(\pi^2 4\phi_0)^2} = .97481$ $\qquad$ $\phi_0 = 70^\circ$

$\log \pi^2 - 4\phi_0^2 = .5909644$ $\qquad$ $\phi_0 = 70^\circ$

$\pi^2 = 9.8696044$ $\qquad$ $\cos 60^\circ = .5000.$

$\log \frac{\pi}{3} = .0200286$ $\qquad$ $\cos 70^\circ = .34202$

$\log \frac{70\pi}{180} = 0.0869754$ $\qquad$ $\cos 80^\circ = .17365.$

$\log \frac{80\pi}{180} = .1449674$ $\qquad$ $\log \left(\frac{\pi}{3}\right)^3 = .0600858$

$\log \left(\frac{\pi}{3}\right)^2 = .0400572$ $\qquad$ $\log \left(\frac{7}{18}\pi\right)^3 = .2609262$

$\log \left(\frac{7}{18}\pi\right)^2 = .1739508$ $\qquad$ $\log \left(\frac{8}{18}\pi\right)^3 = .4349022.$

$\log \left(\frac{8}{18}\pi\right)^2 = .2899348$

$\log 4\left(\frac{\pi}{3}\right)^2 = .6421172.$

$\log 4\left(\frac{7\pi}{18}\right)^2 = .7760108.$

$\log 4\left(\frac{8\pi}{18}\right)^2 = .8919948.$

$9\pi^2 = 88.8264$

$9\pi^2 - \left(\frac{4\pi}{3}\right)^2 = 84.4399$

$9\pi^2 - 4\left(\frac{7\pi}{18}\right)^2 = 82.8559$

$9\pi^2 - 4\left(\frac{8\pi}{18}\right)^2 = 81.0282$

NUMERICAL CALCULATIONS

$\frac{t}{r}$	$\frac{t^2}{r^2}$	$\left(1+\frac{t^2}{3r^2}\right)$	$\frac{1}{2}\left(1+\frac{t^2}{3r^2}\right)$	$\frac{1}{2}\left(1-\frac{t^2}{6r^2}\right)$	$\frac{t^2}{6r^2}$
$\frac{1}{5}$	.04	1.01333	.50666	.49667	.00667
$\frac{1}{10}$	.01	1.00333	.50166	.49916	.001667
$\frac{1}{20}$	.0025	1.00083	.50041	.49979	.000416.

$\frac{t}{r}$	log $\left(1+\frac{t^2}{3r^2}\right)$	log $\frac{1}{2}\left(1+\frac{t^2}{3r^2}\right)$	log $\frac{1}{2}\left(1-\frac{t^2}{6r^2}\right)$	log. $\frac{t^2}{6r^2}$
$\frac{1}{5}$	0.0057509	9.7047166	9.6960679	7.8229087
$\frac{1}{10}$	0.0014438	9.7004095	9.6982398	7.220847
$\frac{1}{20}$	0.0003603.	9.6993260	9.6987876	6.6187887

ϕ_0	$\frac{t}{r}$	Values of D.	$\frac{D}{2}$	$\frac{2}{D}$.
	$\frac{1}{5}$	.0552+.0004 = .0556.	.0278	35.9712
60°	$\frac{1}{10}$	.0552+.0001 = .0553.	.0276	36.2319.
	$\frac{1}{20}$	.0552+.00002 = .0552.	.0276	36.2319.
	$\frac{1}{5}$	.1068+.0007 = .1075.	.0537	18.6220.
70°	$\frac{1}{10}$	.1068+.0002 = .1070.	.0535	18.6916.
	$\frac{1}{20}$	.1068+.00004 = .1068.	.0534.	18.7226.
	$\frac{1}{5}$	.1809+.0012 = .1821	.0910.	10.9890.
80°	$\frac{1}{10}$	.1809+.0003 = .1812	.0906	11.0375
	$\frac{1}{20}$	.1809+.0001 = .1810.	.0905	11.0497.

RESULTS OF NUMERICAL SUBSTITUTIONS--STEP 1.

TERM	1	2	3	4	5	6	7
$= 60^\circ$	$\frac{D}{2}$	.82699	.50000	-.2092	.66160	.50930	$.8000 \cos\phi_0$
$\phi_0 = 70^\circ$	$\frac{D}{2}$	.76916	.65798	-.2974	1.1778	.97483	$1.5313 \cos\phi_0$
$\phi_0 = 80^\circ$	$\frac{D}{2}$	.70532	.82635	-.4178	2.65533	2.3967	$3.7647 \cos\phi_0$

	8	9	10	11	12	13	14
$\phi_0 = 60^\circ$	2.2170	.48368	.04296	.01102	$.05195 \cos\phi_0$	.02805	.03141
$= 70^\circ$	4.3890	.99542	.055424	.01529	$.07205 \cos\phi_0$	.02915	.04684
$\phi_0 = 80^\circ$	11.2459	2.6688	.06788	.02042	$.09624 \cos\phi_0$	.022048	.06822.

RESULTS OF NUMERICAL SUBSTITUTIONS -- STEP 7.

ϕ_0	t/r	R_1	S_1	R_2	S_2	T_2
60°	$\frac{1}{5}$	−1.5763	1.1906	57.0216	−62.3104	+.8000
	$\frac{1}{10}$	+.02397	−.02898	50.8422	−62.0936	"
	$\frac{1}{20}$	+.005992	−.007246	50.7040	−61.9270	"
70°	$\frac{1}{5}$	+.014325	−.01862	48.6429	−64.5072	+1.5313
	$\frac{1}{10}$	−.025776	+.03351	48.3186	−64.0841	"
	$\frac{1}{20}$	−.036014	+.04682	48.2805	−64.0345	"
80°	$\frac{1}{5}$	.034878	−.049451	67.1093	−98.5481	+37.647
	$\frac{1}{10}$	.0093417	−.013245	66.7278	−98.0072	
	$\frac{1}{20}$	.002338	−.003315	66.6294	−97.8676	

ϕ_0	t/R	R_3	S_3	W_3
60°	$\frac{1}{5}$	.24972	−.28864	+.05195
	$\frac{1}{10}$	.27211	−.31571	"
	$\frac{1}{20}$	.27811	−.32297	"
70°	$\frac{1}{5}$	.15537	−.18211	+.07205
	$\frac{1}{10}$	.16747	−.19785	"
	$\frac{1}{20}$	.17068	−.20202	"
80°	$\frac{1}{5}$	.12760	−.07776	+.02205
	$\frac{1}{10}$	.13719	−.09137	"
	$\frac{1}{20}$	.13732	−.09155	"