CORRELATION OF INDUCTION MOTOR DESIGN FACTORS

AND

DETERMINATION OF THE END TURN REACTANCE OF POLYPHASE MACHINES

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SUMMARY

My research work has consisted of a study of the design of single phase and polyphase induction motors, and the determination of the end turn reactance of polyphase machines.

The work on the single phase condenser motor was reported in a thesis for the degree of Master of Science.

The work on polyphase machines has consisted of the determination of the slot area, wire area, stator conductivity, pullout torque, motor rating, and stator losses as functions of the stator dimensions. The condition for maximum stator conductivity is obtained and the pullout torque is derived for any value of stator conductivity as a function of its value for the condition of maximum stator conductivity. Evidence is submitted why a motor should be designed in the region of maximum stator conductivity. This work is reported in the paper entitled "Correlation of Induction Motor Design Factors".

In the paper entitled "End Turn Reactance of Polyphase Machines", the end turn reactance is considered as a problem in self and mutual inductance. The reactance between any two line segments is derived and a method of calculating end turn reactance in terms of coefficients of this type is developed.

CORRELATION OF INDUCTION MOTOR DESIGN FACTORS

INTRODUCTION

An analysis of stator resistance, wire size, horsepower rating, and air gap density as a function of the flux per pole per unit length of lamination stacking, stator slot shape and slot insulation, rotor diameter and flux densities in the stator teeth and core.

The usual method of design is to begin with the rotor radius, flux per pole, and ampere conductors per inch of rotor circumference. The flux density in the air gap is determined when the stator tooth density and the ratio of slot pitch and tooth width are set. The depth of slot is determined by the ampere conductors per inch and the circular mills per ampere. The depth of stator core and the stator radius are then determined by setting the value of the flux density in the stator core.

The relations of, the rotor and stator radius, the width of slot and tooth, the depth of slot and core, the flux per pole and the flux densities, to each other and to the motor performance are quite vague. To determine their best proportions is difficult unless a large amount of experimental data are available.

In this paper an attempt is made to determine the relation of these factors to each other and their proportions for the best design.

A brief outline of the analysis is as follows. The area of the stator slot is developed as a function of the shape of the slot and the area occupied by the slot insulation. The effect of the slot insulation on the net area of the stator slot is taken into account by using a corrected value for the stator radius. The area of the wire used for the stator coils is developed as a function of the flux per pole. The stator resistance is expressed as a function of flux per pole and slot shape. The rotor radius is determined by setting the ratio of the flux densities in the teeth and core. A sample calculation is made using this method of design for motors from two to twelve poles. The maximum torque as a function of the flux densities and the flux per pole is discussed. The effect of stator resistance on maximum torque is calculated and shows that the flux per pole used should be very nearly that for minimum stator resistance.

In the equations which are developed factors not pertinent to the discussion are omitted in which case the equations are more in the form of proportions. This procedure when followed will be evident from the text.

TABLE OF SYMBOLS

Number of stator slots	N
Slot pitch in radians	K_1
Minimum width of stator tooth	W_t
Width of stator slot	W_{os}
Depth of stator slot	D_{os}
Depth of stator core	D_c
Stator outside radius	R_{os}
Rotor radius	R_{or}
Maximum possible depth of stator slot	H_o
Distance added to depth of slot for slot insulation	D_v
Distance added to width of slot for slot insulation	D_h
Gross length of lamination	L_g
Flux per pole	Φ_p
Number of poles	P
Flux density in the stator core	B_c
Flux density in the stator teeth	B_t
Average flux density in the air gap	B_g
Area of stator conductors	A_w
Number of conductors per slot	n
Slot pitch	λ_o
Factor proportional to stator resistance	R
Factor proportional to stator conductance	G
Factor proportional to maximum torque	T
Factor proportional to stator copper loss	W

Rotor resistance . . . . r_r

Slip . . . . S

Motor reactance per phase. x

Ratio of flux per pole to flux per pole at which available slot area is equal to zero for constant flux densities. . . . . Z

Ratio of flux density in the stator core to flux density in the stator teeth. . . . . B

Watts delivered per phase to the rotor . . . . W_r

Ratio of stator resistance when $(Z = 1/2)$ to motor reactance when $(Z = 1/2)$. . . . . γ

Ratio of maximum torque to maximum torque when $(Z = 1/2)$ . . . . T_o

Greatest value of (T_o) when Z is increased . . . . T_o max.

$$K_2 = K_3 + K_4$$

$$K_3 = 0.556/B_c$$

$$K_4 = 0.278\ P/B_t$$

$$\alpha = K_4/K_3$$

DIMENSIONS CORRECTED FOR SLOT INSULATION

Width of stator slot	W_s
Depth of stator slot	D_s
Stator radius	R_s
Rotor radius	R_r
Maximum possible depth of stator slot.	H
Slot pitch	λ
Minimum possible rotor radius.	R_1
Slot area.	A_s
Ratio of depth of slot to maximum depth possible	Y
Maximum slot area.	A_s max.
Ratio of slot area to maximum slot area for constant flux per pole . .	A
Ratio of slot area to maximum slot area possible	A_1
Ratio of depth of slot to stator radius.	X

DETERMINATION OF PROPORTIONS FOR MAXIMUM AREA OF A RECTANGULAR SLOT

If (N) is the number of stator slots then the slot pitch in radians is ($2\pi/N$). Let K_1 be this angle, or $K_1 = 2\pi/N$. (Fig. 1.)

Fig. 1 shows a portion of a stator with the dimensions indicated by the symbols which will be used in the discussion. Referring to Fig. 1.:

W_t = Minimum width of stator tooth

W_s = Width of stator slot

D_s = Depth of stator slot

R_s = Stator radius

R_r = Rotor radius

R_1 = Minimum rotor radius possible

H = Maximum depth of stator slot possible

K_1 = Slot pitch in radians

From the figure it is evident that the wedge shaped area of height (H) available for the stator slot has the same vertex angle (K_1) as the total area of height (R_s). Then for any depth of core (D_c) and width of tooth (W_t) the problem of finding the shape of slot for maximum slot area reduces to that of finding the largest rectangle that can be constructed in a triangle of height (H) and vertex angle (K_1).
(Fig. 2.)

In Fig. 2 the shaded areas represent slots of various shapes inclosed by the available triangular area. From Fig. 2 the following proportions are derived:

$W_s = K_1 H_1$

$D_s = H - H_1$

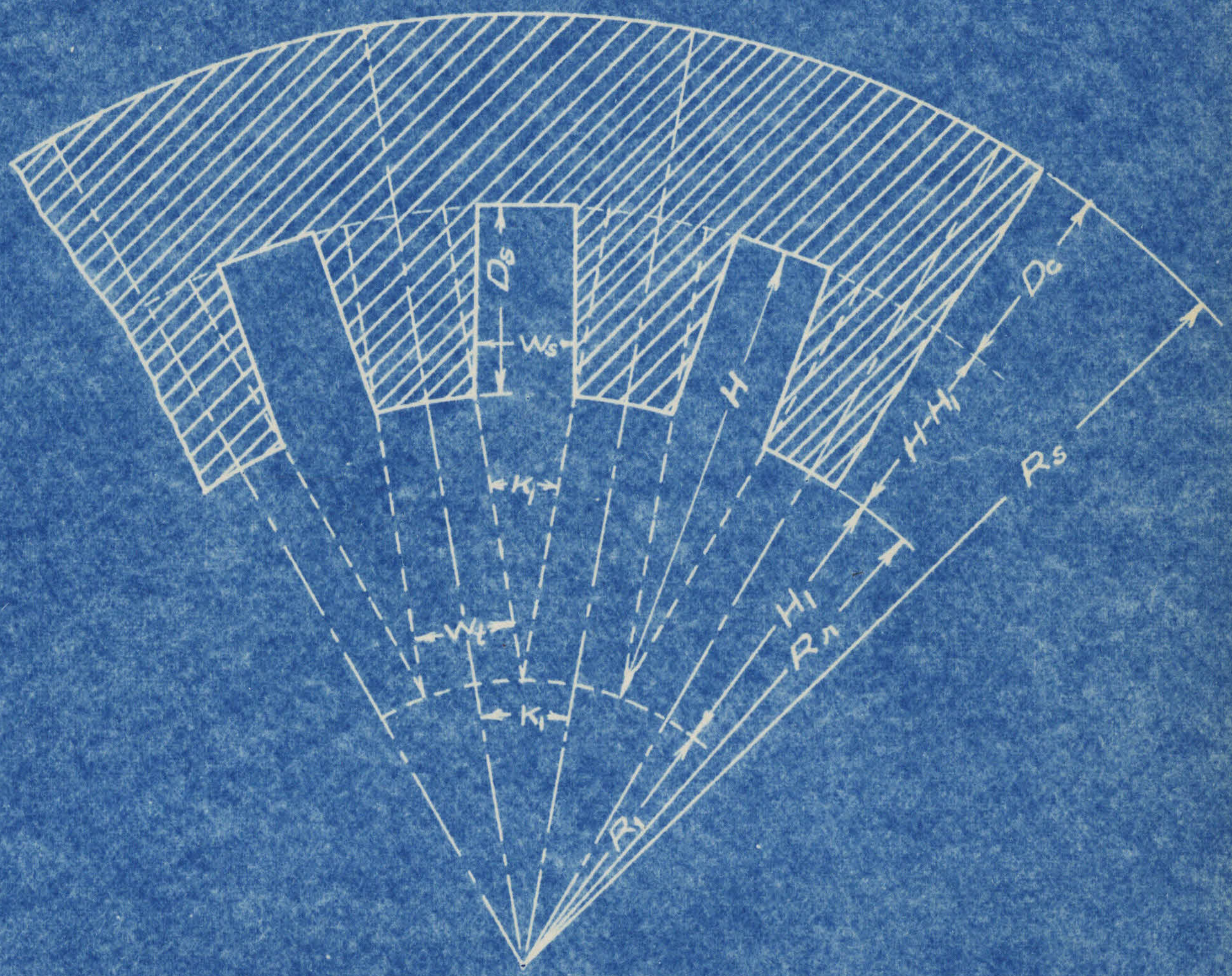


FIGURE 1

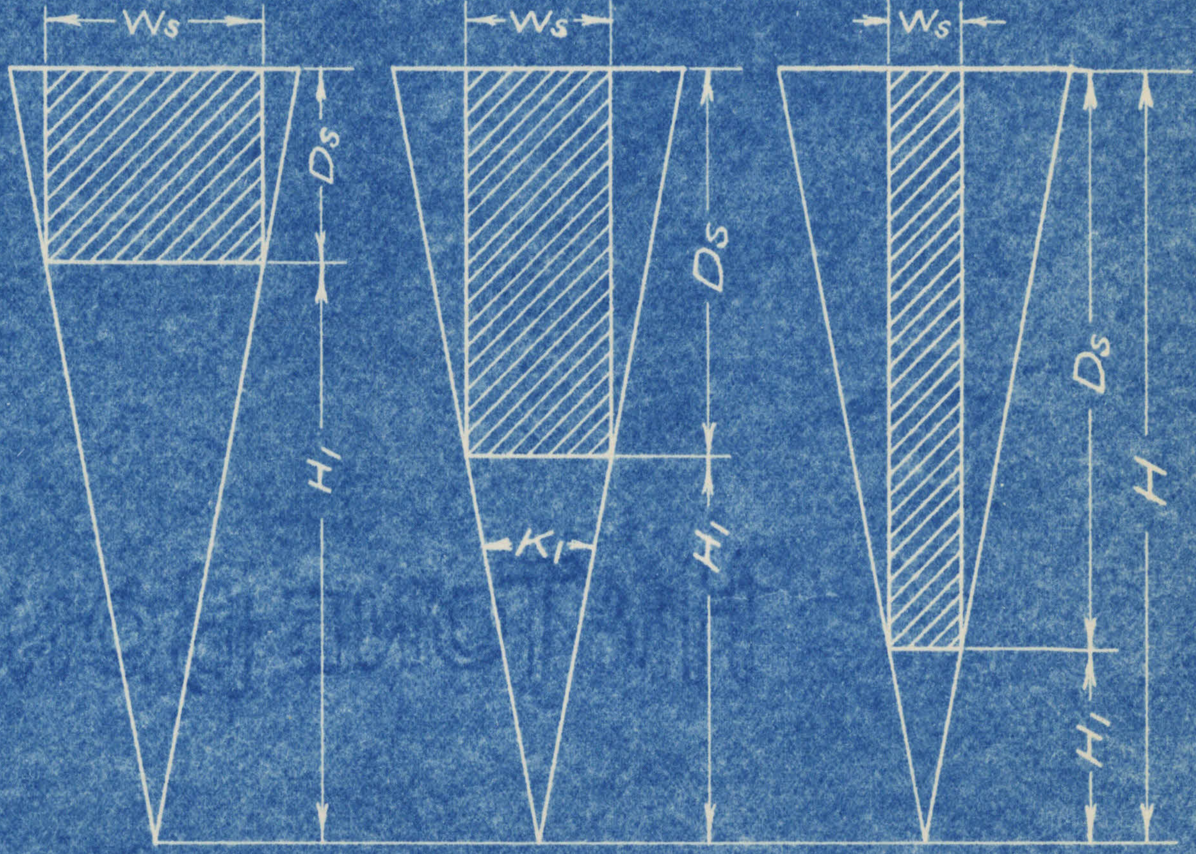


FIGURE 2

$A_s = (W_s)\ (D_s)$ $\quad A_s$ = Slot area.

$A_s = (K_1)\ (H_1)\ (H - H_1)$

$A_s = (K_1)\ (H_1\ H - H_1^{\ 2})$ (1)

To find the value of (H_1) for the maximum area possible differentiate (A_s) with respect to (H_1) and equate to zero.

$$\frac{d(A_s)}{d(H_1)} = K_1\ (H - 2H_1) = 0$$

$H_1 = H/2$ For maximum slot area (2)

Substitute (2) in (1)

$$A_s \text{ max.} = \frac{K_1 H^2}{4} \tag{3}$$

Where A_s max. is the maximum value of the slot area.

Since we are not interested in actual values but rather in their proportions, let (Y) be the ratio of the depth of slot to the maximum depth possible.

$$Y = \frac{D_s}{H} = \frac{(H - H_1)}{H} \tag{4}$$

Let

$$A = \frac{A_s}{A_s \text{ max.}} \tag{5}$$

Then

$$A = \frac{K_1\ (H_1 H - H_1^2)}{\frac{K_1\ H^2}{4}} = \frac{4\ (H_1)\ (H - H_1)}{H^2} \tag{6}$$

Substitute (4) in (6)

$$A = 4(1 - Y)\ Y \tag{7}$$

(Curve 1.)

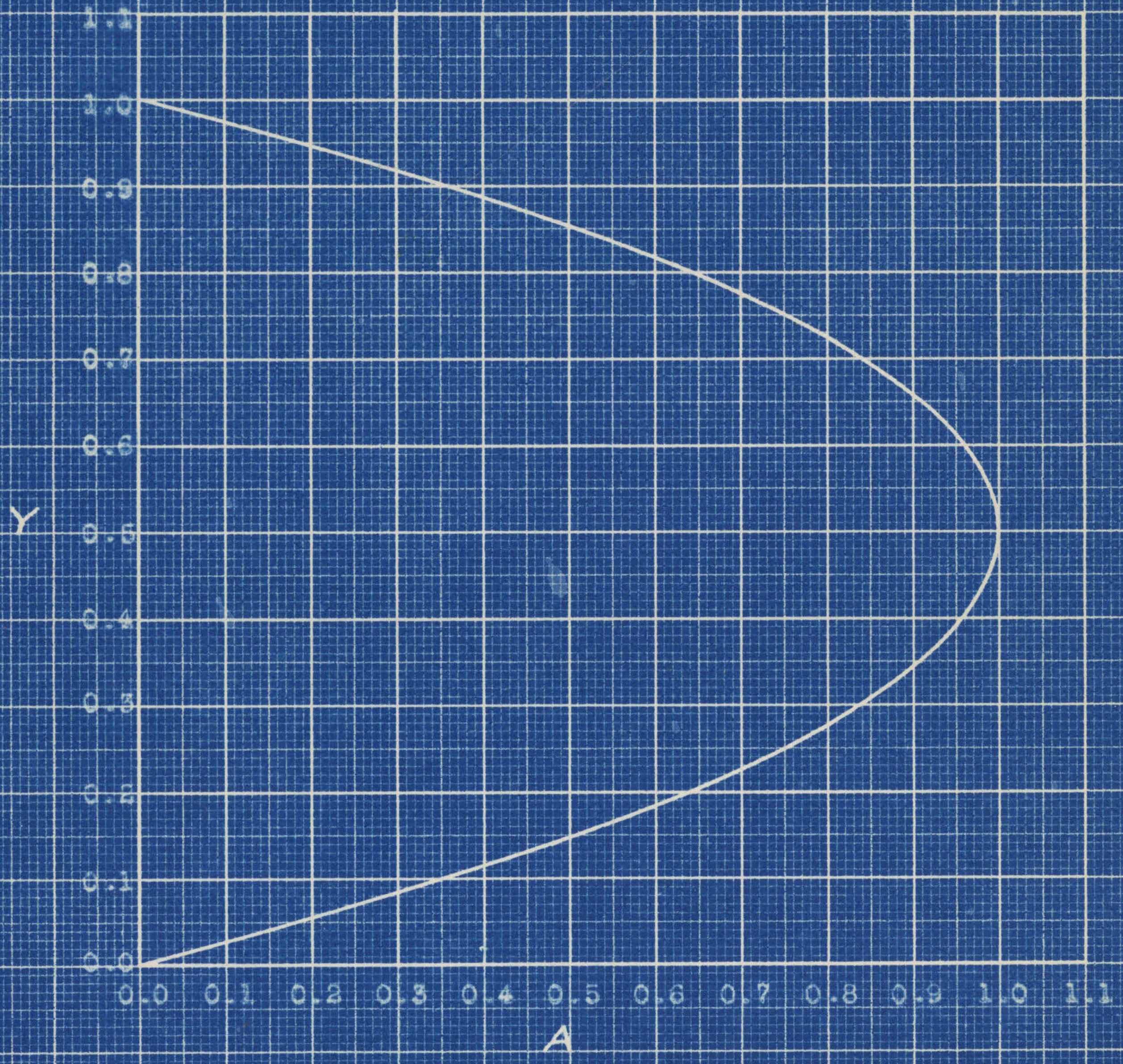
PLATE I

Equation (7)

A = 4 (1 - Y) Y

(A) IS THE RATIO OF THE SLOT AREA TO THE MAXIMUM SLOT AREA

(Y) IS THE RATIO OF THE DEPTH OF SLOT TO THE MAXIMUM DEPTH POSSIBLE



Equation (7) is plotted on Plate 1. The curve is symmetrical with respect to ($Y = 1/2$). The curve shows that (A) is a maximum at ($Y = 1/2$) and that it does not change rapidly for small changes in (Y) near this value. Since for shallow slots the reactance is lower and the rotor diameter is greater, it is evident that the depth of slot should be made as small as possible without sacrificing too much slot area.

TABLE I.

$Y = D_s/H$	$A = \frac{\text{Slot Area}}{\text{Maximum slot area}}$
0.50	1.00
0.45	0.99
0.40	0.96
0.35	0.91
0.30	0.84

Table I. shows that decreasing the slot depth 10% from that for maximum area decreases the slot area only 1%. A reduction in slot depth of 20% of that for maximum area decreases the slot area only 4%. If the slot depth is made 80 to 90% of that for maximum area, a material decrease in slot reactance is obtained without producing any appreciable loss in slot area.

EFFECT OF SLOT INSULATION ON SLOT SHAPE

In the preceding derivation the assumption was made that all of the slot area be occupied by the windings. Taking account of the insulation for the slot walls and the top wedge does not change the analysis. The area occupied by all the slot insulation except that surrounding the individual conductors can be accounted for by considering the depth of the stator core and the width of the stator teeth increased by these amounts; the remaining area can be treated in the manner described in the preceding paragraphs.

(Fig. 3).

In Fig. 3 (D_1) and (D_2) are the net height of the stator coils and (W_s) is the net width of the stator coils. (W_{os}) and (D_{os}) are the actual slot dimensions. The net area of the coils in the slot is ($D_1 + D_2$) (W_s).

Let

$D_s = D_1 + D_2$ Net height of stator coils (8)

$D_v = D_{os} - D_s$ Total height of insulation (9)

$D_h = W_{os} - W_s$ Total width of insulation (10)

If (D_v) is added to the depth of the stator core and (D_h) is added to the width of the stator teeth, the remaining area having height (H) can be used rather than the actual area of height (H_o). The values of (D_s) and (W_s) which are then derived will be the net dimensions of the coils in the stator slot. To obtain the actual slot dimensions (D_v) is added to (D_s) and (D_h)is added to (W_s).

$$H = H_o - (D_v + D_h/K_1) \qquad (11)$$

Fig. 4.)

The effect of slot insulation can be entirely eliminated from the calculations if instead of using the actual stator radius (R_{os}) a stator radius of ($R_{os} - (D_v + D_h/K_1)$ is used.

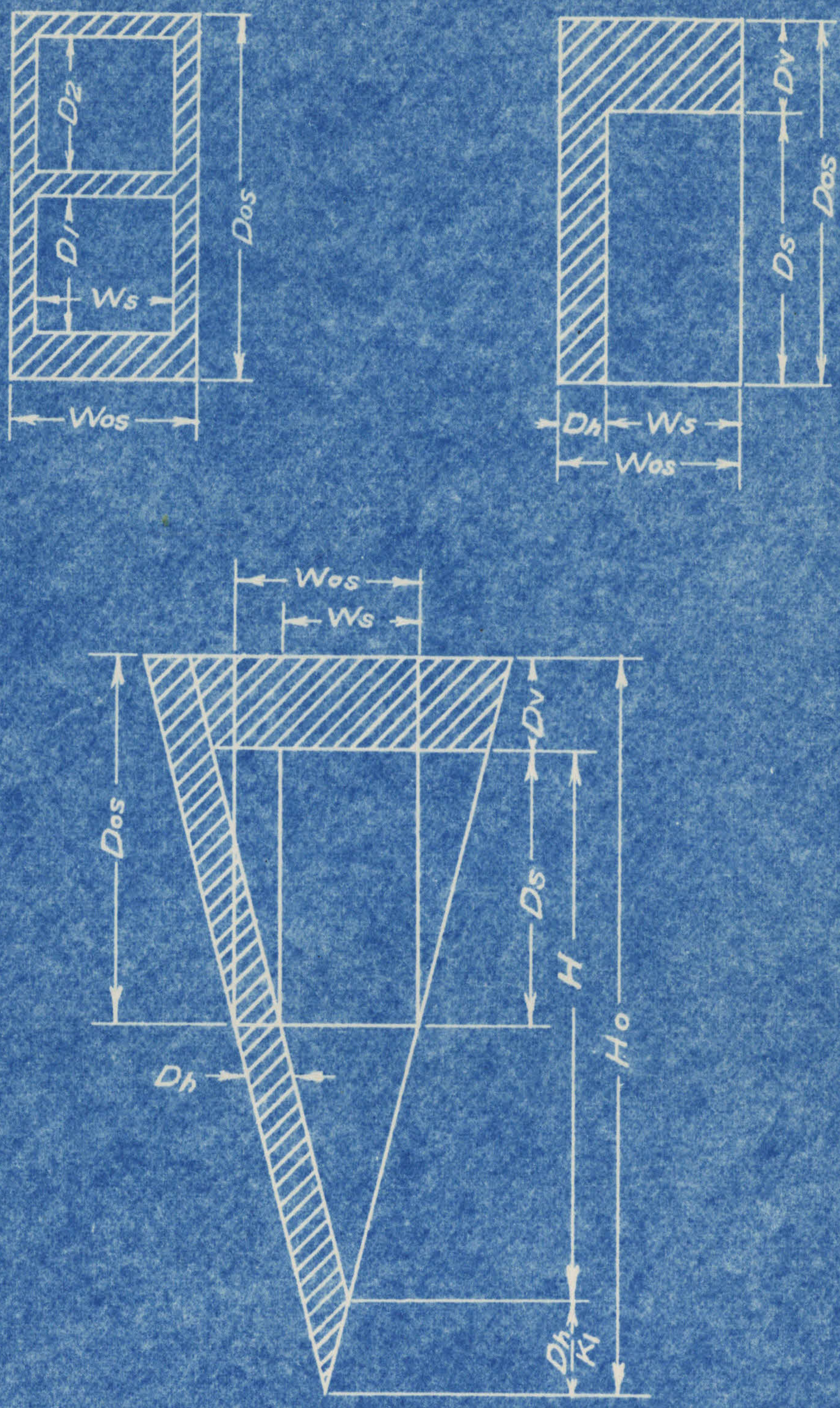


FIGURE 3

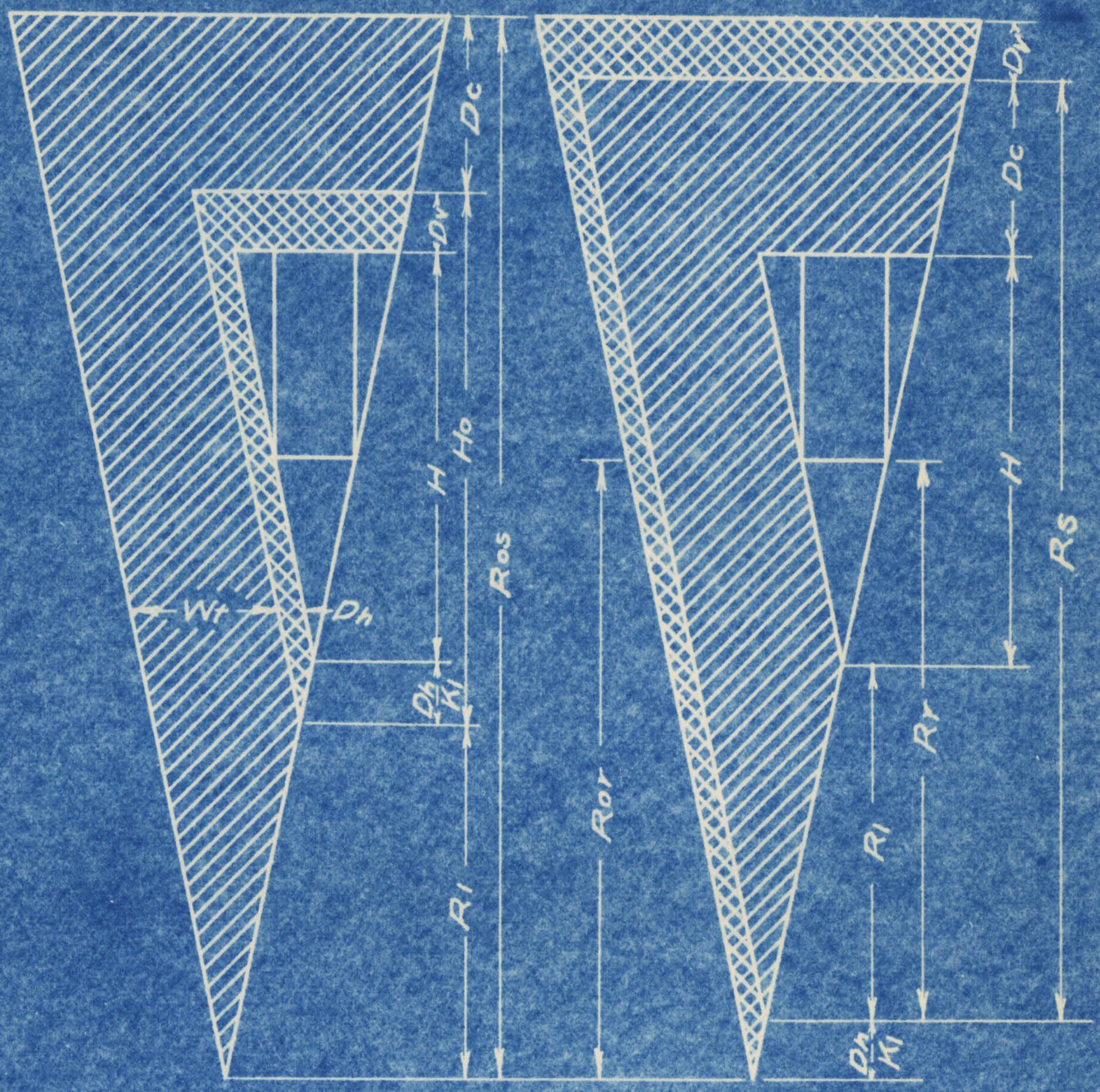


FIGURE 4

From Fig. 4 the following relations are obvious:

$$R_s = R_{os} - (D_v + D_h/K_1) \tag{12}$$

$$R_r = R_{or} - D_h/K_1 \tag{13}$$

In the discussion to follow the net stator dimensions will be used, the actual values can be obtained from the preceding equations.

Let

R_r = Net rotor radius

R_s = Net stator radius

B_c = Stator core flux density

B_t = Maximum stator tooth density

D_c = Depth of stator core

W_t = Minimum width of stator teeth

Φ_p = Flux per pole

N = Number of stator slots

L_g = Gross length of lamination

P = Number of poles

Using the above symbols the common equations for the flux densities in the teeth and core are:

$$B_t = \frac{\Phi_p \pi P}{L_g N 2(0.9) W_t}$$

Introducing K_1 we have

$$\frac{\Phi_p (K_1 P (0.278)}{L_g W_t} = B_t$$

$$B_c = \frac{\Phi_p}{L_g 2(0.9) D_c} = \frac{\Phi_p (0.556)}{L_g D_c}$$

From the above equations (D_c) and (W_t) are given by equations (14) and (15).

$$D_c = \frac{\Phi_p\ (0.556)}{L_g\ B_c} \tag{14}$$

$$W_t = \frac{\Phi_p\ K_1\ P\ (0.278)}{L_g\ \ B_t} \tag{15}$$

From Fig. 1.

$$H = R_s - (D_c + W_t/K_1)$$

Substituting the values of (D_c) and (W_t) given by equations (14) and (15).

$$H = R_s - \frac{\Phi_p}{L_g} \left(\frac{(0.556)}{B_c} + \frac{P\ (0.278)}{B_t} \right) \tag{16}$$

Let

$$K_2 = \frac{0.556}{B_c} + \frac{P\ (0.278)}{B_t} \tag{17}$$

$$K_3 = \frac{(0.556)}{B_c} \tag{18}$$

$$K_4 = \frac{P\ (0.278)}{B_t} \tag{19}$$

Substituting equations (17), (18), and (19) in (14), (15) and (16) the following expressions are derived:

$$D_c = \frac{\Phi_p\ K_3}{L_g} \tag{20}$$

$$W_t = \frac{\Phi_p\ K_1\ K_4}{L_g} \tag{21}$$

$$H = R_s - \frac{\Phi_p\ K_2}{L_g} \tag{22}$$

$$K_2 = K_3 + K_4 \tag{23}$$

Equation (22) shows that (H) decreases linearly as $\left(\frac{\Phi_p}{L_g}\right)$ increases.

When $\frac{(\Phi_p K_2)}{L_g} = R_s$, (H) becomes equal to zero. That is, at this value of flux per pole all of the available space is occupied by the stator core and teeth. This then is the greatest value of flux per pole which is theoretically possible.

From (3)

$$A_s \text{ max.} = \frac{K_1 H^2}{4}$$

From (7)

$$A_s = K_1 H^2 (1 - Y) Y$$

Substitute (22) in (7)

$$A_s = K_1 \left(R_s - \frac{\Phi_p K_2}{L_g}\right)^2 (1 - Y) Y \qquad (24)$$

Equation (24) is the general equation for the slot area. But here again the proportions are of greater interest than the actual values. The slot area reduces to zero when (H) is equal to zero.

For (H = O)

$$\frac{\Phi_p}{L_g} = R_s/K_2$$

Let

$$Z = \frac{\frac{\Phi_p}{L_g} \text{ for any value of H.}}{\frac{\Phi_p}{L_g} \text{ at } (H = O)}$$

$$Z = \frac{\Phi_p K_2}{L_g R_s} \qquad (25)$$

The variable (Z) is then equal to the ratio of the flux per pole to the maximum flux per pole possible with the flux densities determined by (K_2). Substitute (25) in (22) and (24)

$$H = R_s\ (1 - Z) \qquad (26)$$

$$A_s\ \text{max.} = \frac{K_1\ (R_s)^2\ (1 - Z)^2}{4} \qquad (27)$$

$$A_s = K_1\ R_s^{\ 2}\ (1 - Z)^2\ (1 - Y)\ Y \qquad (28)$$

The greatest slot area would occur when the flux per pole is equal to zero. This is equivalent to putting (Z) equal to zero in equation (27). When ($Z = 0$)

$$A_s\ \text{max.} = \frac{K_1\ (R_s)^2}{4}$$

Let

$$A_1 = \frac{A_s}{K_1 R_s^{\ 2}/4}$$

$$A_1 = 4(1 - Z)^2\ (1 - Y)\ Y \qquad (29)$$

The variable (A_1) is then the ratio of the slot area to the maximum slot area possible when the flux per pole and the slot shape both are allowed to vary. It is a maximum for ($Z = 0$) and ($Y = 1/2$).

Plate 2.)

Plate 2 shows values of (Y) and (Z) for which (A_1) is constant.

DETERMINATION OF FLUX PER POLE FOR MAXIMUM WIRE SIZE

Let (A_w) be the area of the conductor and (n) the number of conductors per slot.

Then

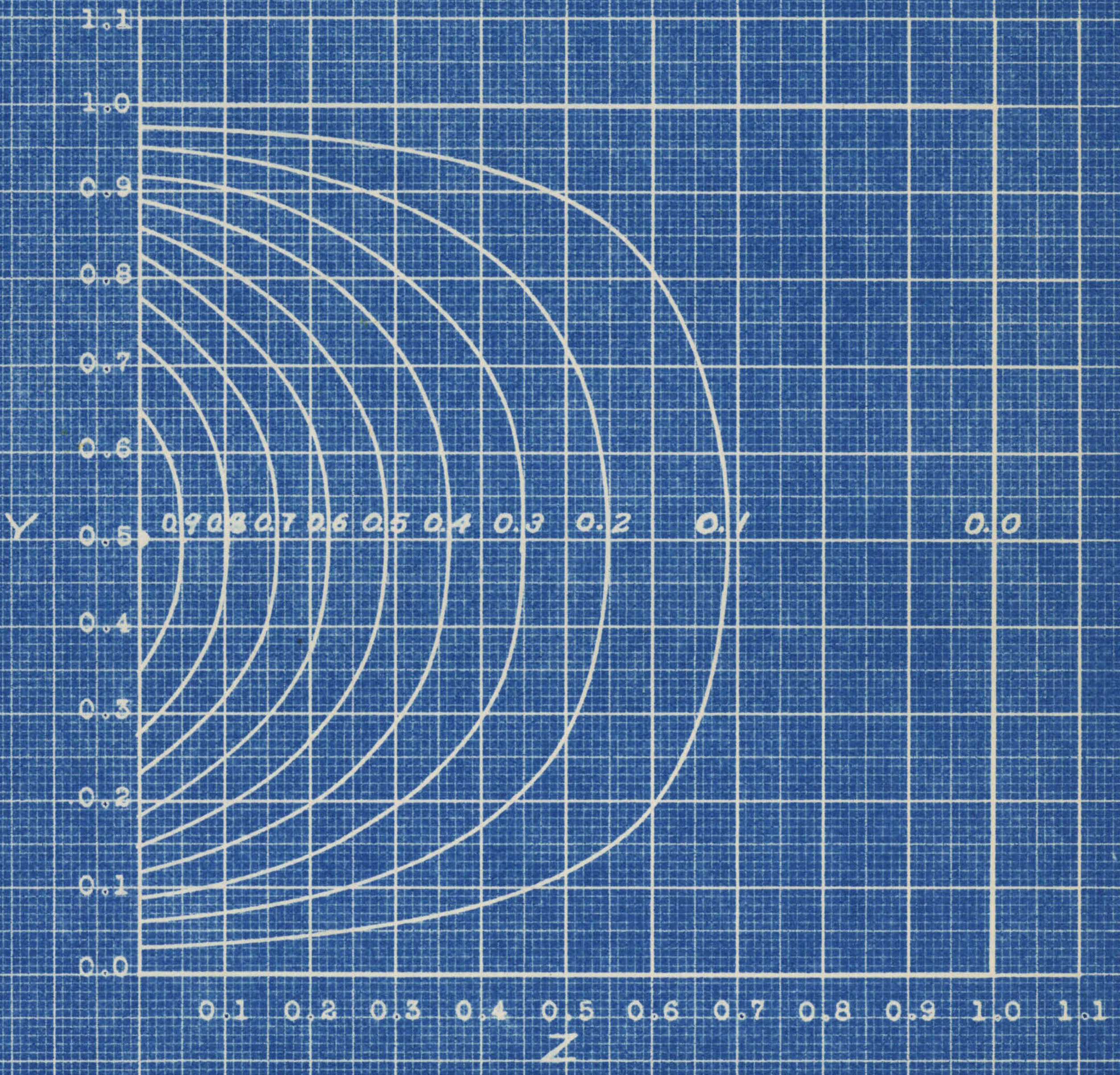
PLATE 2

EQUATION (29)

$$A_1 = 4 (1 - Z)^2 (1 - Y) Y$$

(A_1) IS THE RATIO OF THE SLOT AREA TO THE MAXIMUM SLOT AREA POSSIBLE

(Z) IS THE RATIO OF THE FLUX PER POLE TO THE FLUX PER POLE FOR ZERO SLOT AREA



$$A_w = \frac{A_s}{n}$$

The flux per pole varies inversely as the number of conductors, so (A_w) varies directly as $(A_s)\left(\frac{\Phi_p}{L_g}\right)$. But (Z) is proportional to $\left(\frac{\Phi_p}{L_g}\right)$ so we may write the proportion.

$$A_w = A_s Z$$

The slot area (A_s) is proportional to (A_1). The slot area is a maximum for ($Y = 1/2$). Substituting this value of (Y) in equation (29) for (A_1) results in the proportion.

$$A_w = (1 - Z)^2 Z \qquad \text{For } (Y = 1/2) \tag{30}$$

To find the value of (Z) for maximum wire size differentiate (A_w) with respect to (Z) and equate to zero.

$$\frac{d(A_w)}{d(Z)} = 1 - 3Z = 0$$

and

$$Z = 1/3$$

When ($Z = 1/3$), or when the flux per pole is one-third of that at which all of the available space is occupied by the iron in the stator core and teeth, the area available for each conductor is a maximum.

Z is equal to one when ($\Phi_p/L_g = R_s/K_2$), so the wire size is a maximum for

$$\frac{\Phi_p}{L_g} = \frac{R_s}{3K_2}$$

If ($Z = 1/3$) is substituted in equation (30), the value of (A_w) given by this equation is 4/27. In order to make (A_w) the ratio of the wire size to the maximum wire size, the value of (A_w) given by equation (30) must be increased by 27/4. This results in the equation

$$A_w = \frac{27\,(1 - Z)^2\,Z}{4} \tag{31}$$

Where (A_w) is the ratio of the wire area to the maximum wire area. Equation (31) is plotted on Plate 3.

(Plate 3.)

Plate 3 shows that (A_w) is greater than 0.5 from $(Z = 0.09)$ to $(Z = 0.66)$. In larger machines two or more conductors are often used in parallel, so that in the range $(0.5 < A_w < 1)$, the actual copper area of the conductors may be considered to be proportional to the area available for each conductor.

DETERMINATION OF FLUX PER POLE FOR MINIMUM STATOR RESISTANCE

The resistance of the stator will vary directly as the number of conductors in series per phase, and inversely as the area of the conductor.

$R = n/A_w$ R is proportional to the stator resistance.

From (30)

$A_w = (1 - Z)^2\,Z$ When $(Y = 1/2)$. (A_w) is proportional to the wire area.

The number of conductors varies inversely as the flux per pole.

$n = 1/Z$

The stator resistance is then expressed by the proportion

$$R = \frac{1}{(1 - Z)^2\,Z^2}$$

and the conductance, (G), by the proportion

$$G = (1 - Z)^2\,Z^2$$

To find the maximum value of (G) differentiate with respect to (Z) and equate to zero.

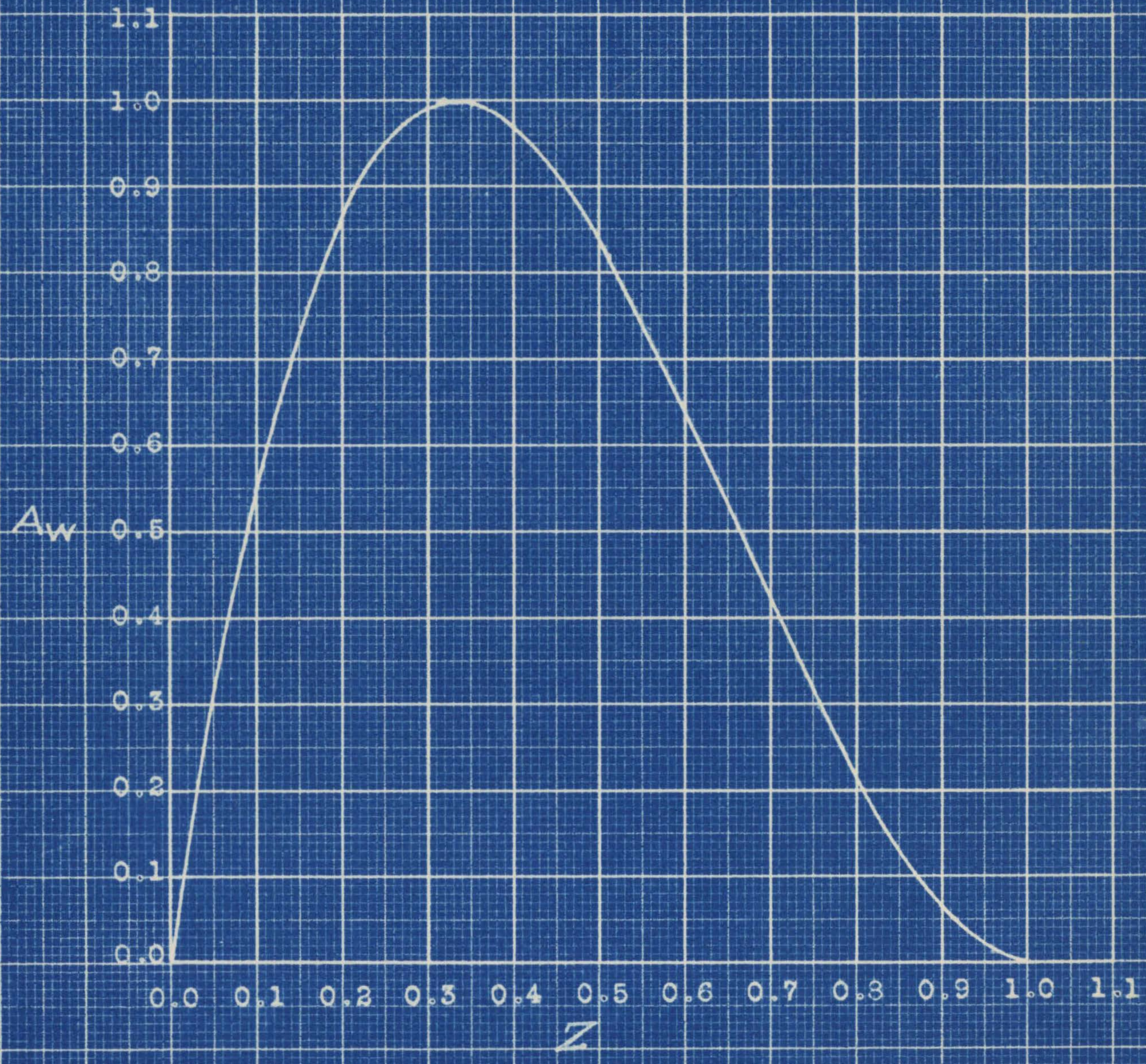
PLATE 3

EQUATION (31)

$$A_w = \frac{27}{4} (1 - z)^2 z$$

(A_w) IS PROPORTIONAL TO THE AREA AVAILABLE FOR EACH CONDUCTOR

(z) IS THE RATIO OF THE FLUX PER POLE TO THAT FOR ZERO SLOT AREA



$$\frac{d\,(G)}{d\,(Z)} = 1 - 2Z = 0.$$

and

$$Z = 1/2 \tag{34}$$

The maximum stator conductivity occurs at $(Z = 1/2)$. When (Z) equals one, $(\Phi_p/L_g = R_s/K_2)$, so for minimum stator resistance

$$\frac{\Phi_p}{L_g} \qquad \frac{R_s}{2\,K_2}$$

If $(Z = 1/2)$ is substituted in the expression for (G), the resultant value of (G) is $1/16$. In order to make (G) the ratio of the stator conductivity to the maximum conductivity, the value of (G) given by this expression must be increased by 16. This results in the equation

$$G = 16\,(1 - Z)^2\, Z^2 \tag{33}$$

The ratio of the stator resistance to the minimum stator resistance is given by the equation

$$R = \frac{1}{16\,(1 - Z)^2\, Z^2} \tag{32}$$

(Plates 4 & 5.)

Equations (32) and (33) are plotted on Plates 4 and 5.

TABLE II.
From Plate 5.

Z	R
0.2	2.40
0.3	1.42
0.4	1.08
0.5	1.00
0.6	1.08
0.7	1.42
0.8	2.40

Table II shows that a small change in (Z) near $(Z = 1/2)$ does not appreciably affect the stator resistance, but for values of (Z) less than 0.4

PLATE 4

EQUATION (33)

$$G = 16 (1 - Z)^2 Z^2$$

(G) IS THE RATIO OF THE STATOR CONDUCTIVITY TO THE MAXIMUM

(Z) IS THE RATIO OF THE FLUX PER POLE TO THAT FOR ZERO SLOT AREA

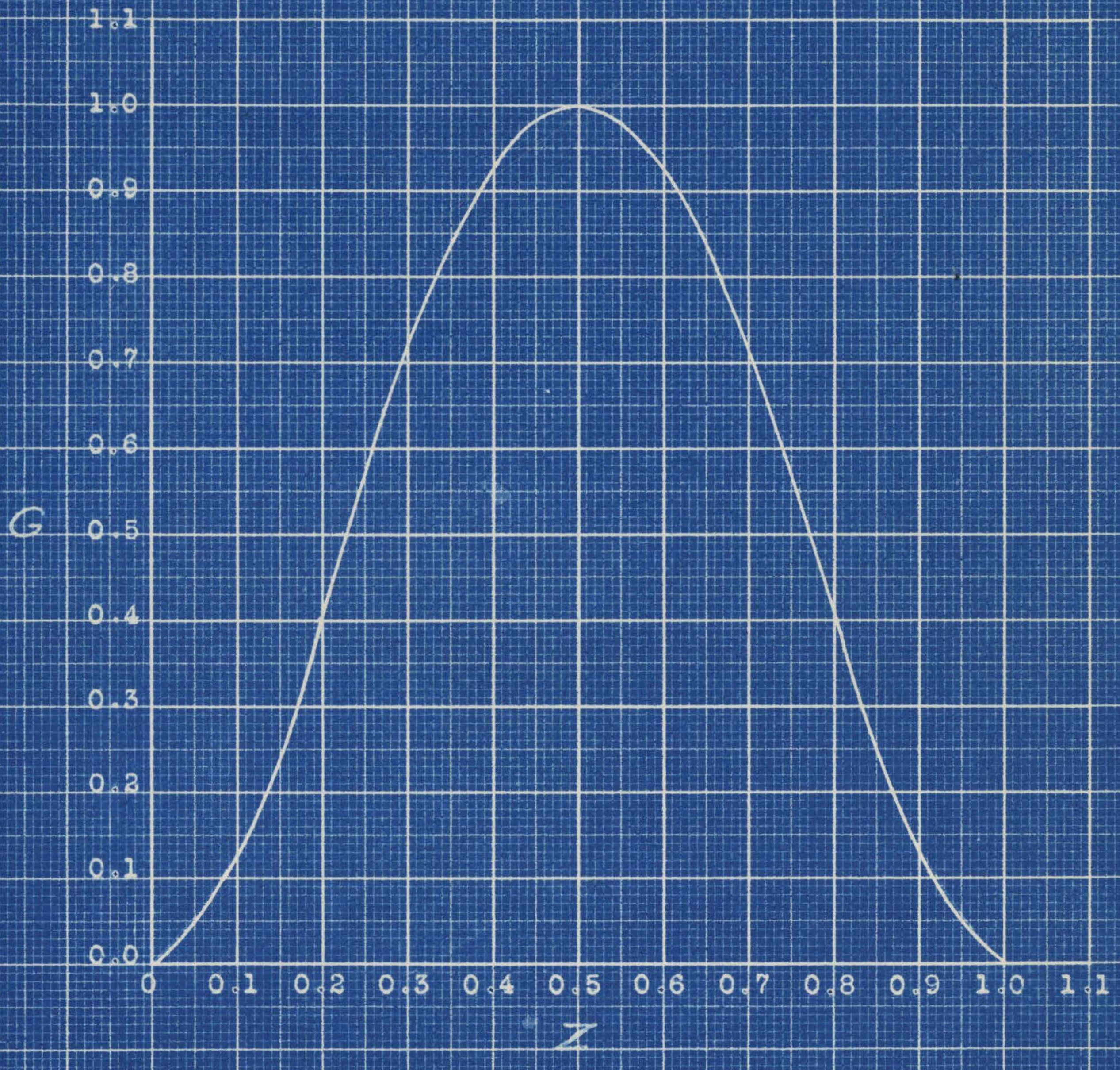


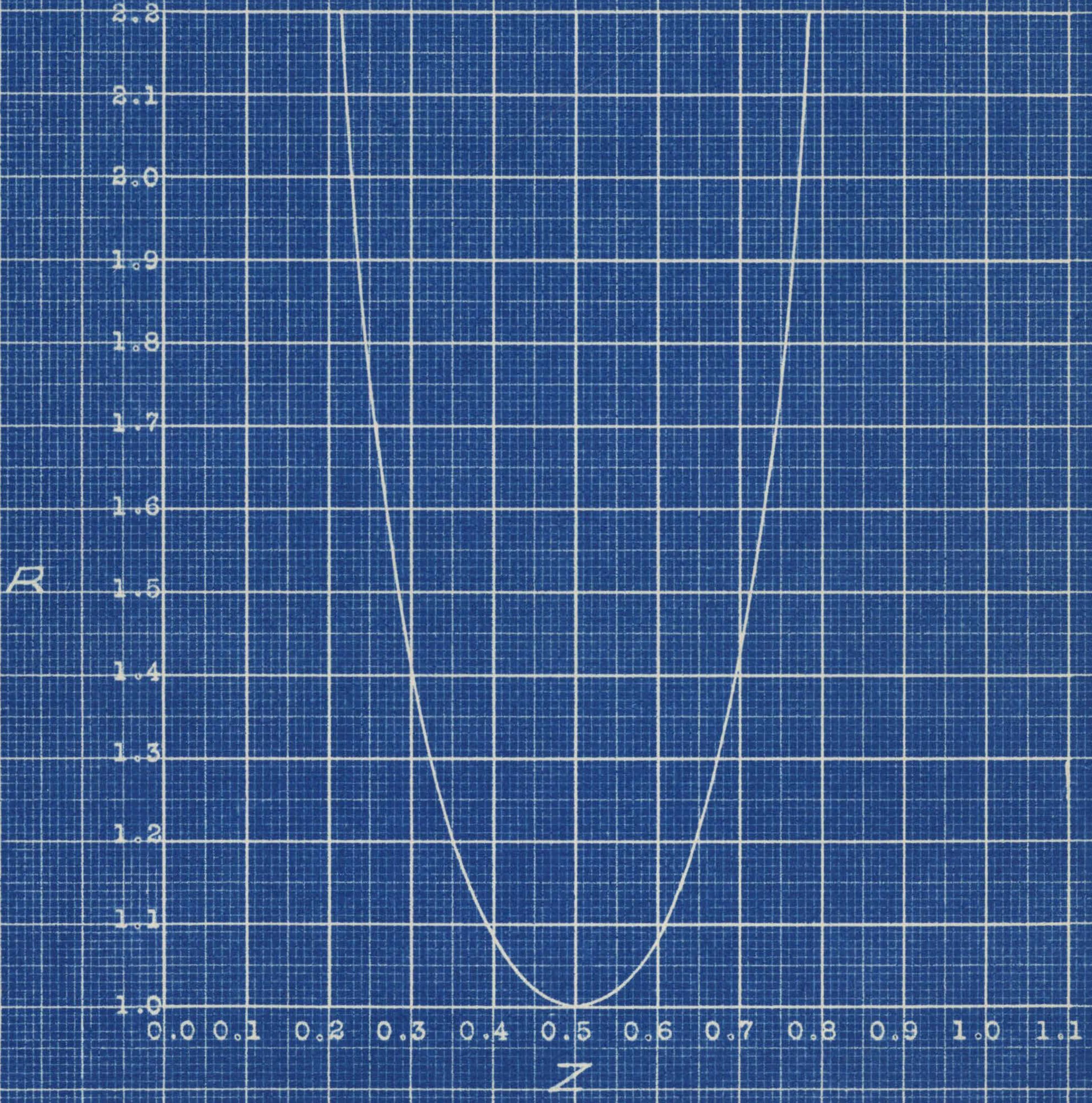
PLATE 5

EQUATION (32)

$$R = \frac{1}{16\,(1 - Z)^2\, Z^2}$$

(R) IS THE RATIO OF THE STATOR RESISTANCE TO THE MINIMUM

(Z) IS THE RATIO OF THE FLUX PER POLE TO THAT FOR ZERO SLOT AREA



or greater than 0.6, the resistance increases rapidly.

In deriving equations (32) and (33) the shape of slot was assumed to be that for maximum slot area, that is (Y) was given the value 1/2. The values of (G) and (R) for any value of (Y) are given by the following equations.

$$R = \frac{1}{64\ (1 - Z)^2\ Z^2\ (1 - Y)\ Y} \qquad (35)$$

$$G = 64\ (1 - Z)^2\ Z^2\ (1 - Y)\ Y \qquad (36)$$

(Plate 6.)

Equation (36) is plotted on Plate 6.

In equation (36)

$$Y = D_s/H$$

From (26)

$$H = R_s\ (1 - Z)$$

Then

$$Y = \frac{D_s}{R_s\ (1 - Z)}$$

Let

$$X = D_s/R_s$$

Then

$$Y = \frac{X}{(1 - Z)}$$

Substituting in equation (36)

$$G = 64\ Z^2\ (1 - Z - X)\ X \qquad (37)$$

PLATE 6

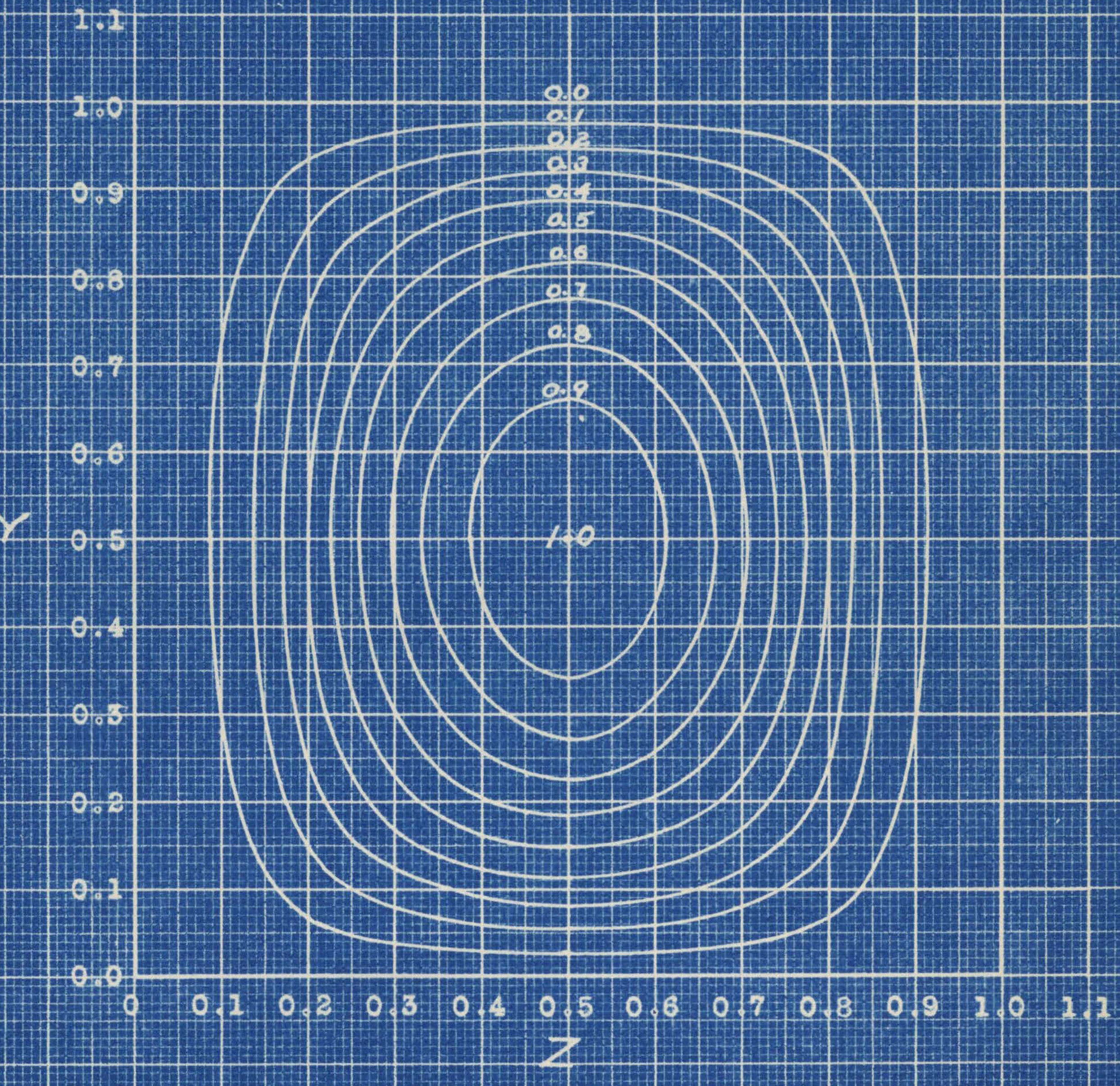
EQUATION (36)

$$G = 64 (1 - Z)^2 Z^2 (1 - Y) Y$$

(G) IS THE RATIO OF THE STATOR CONDUCTIVITY TO THE MAXIMUM. VALUES OF (G) ARE MARKED ON THE CURVES.

(Z) IS THE RATIO OF THE FLUX PER POLE TO THAT FOR ZERO ZLOT AREA

(Y) IS THE RATIO OF THE DEPTH OF SLOT TO THE MAXIMUM DEPTH POSSIBLE FOR CONSTANT (Z)



(Plate 7.)

Equation (37) is plotted on Plate 7. Since (X) is the ratio of the depth of slot to the stator radius, the depth of slot for any value of (Z) and stator radius (R_s) can be taken from this curve. Plate 7 shows that the stator conductivity is a maximum when the depth of the stator slot is equal to one-fourth of the stator radius, and ($Z = 1/2$). Reducing the depth of slot to 0.225 of the stator radius reduces the conductivity only one percent.

In the derivation of these formulas no mention has been made of the actual values of the flux densities in the teeth and core so that the shape of the stator slot for maximum conductivity is independent of the values of the flux densities.

DETERMINATION OF THE DEPTH OF STATOR CORE, WIDTH OF STATOR TEETH AND ROTOR RADIUS

In order to determine the depth of the stator core, the width of the stator teeth and the rotor radius, the flux densities must be considered. Referring again to equations (17) to (25).

$$(17) \quad K_2 = \frac{0.556}{B_c} + \frac{P\ (0.278)}{B_t} = K_3 + K_4$$

$$(18) \quad K_3 = \frac{0.556}{B_c}$$

$$(19) \quad K_4 = \frac{P\ (0.278)}{B_t}$$

$$(20) \quad D_c = \frac{\Phi_p\ K_3}{L_g}$$

$$(25) \quad Z = \frac{\Phi_p\ K_2}{L_g\ R_s}$$

PLATE 7

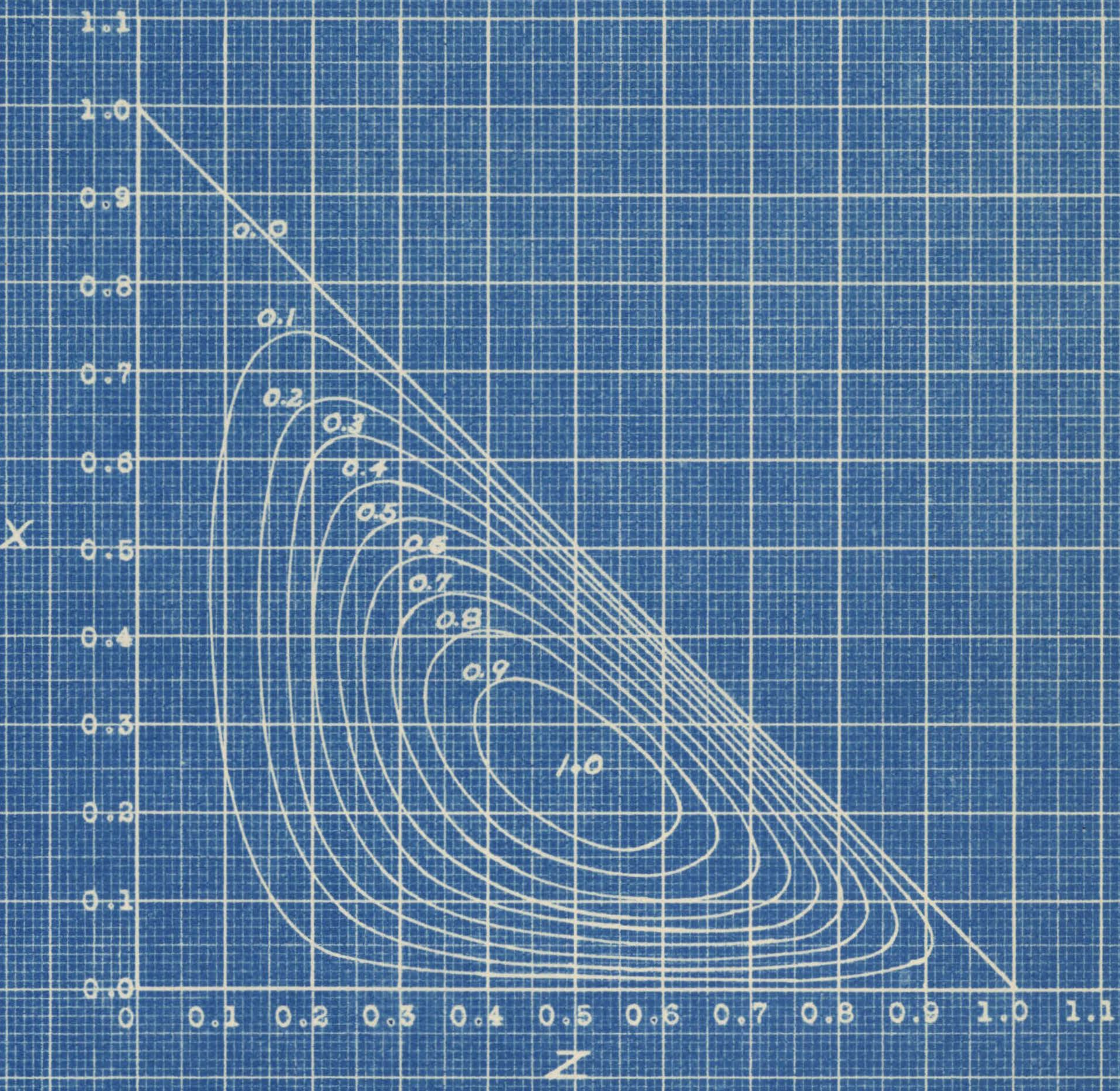
EQUATION (37)

$$G = 64\, Z^2\, (1 - Z - X)\, X$$

(G) IS THE RATIO OF THE STATOR CONDUCTIVITY TO THE MAXIMUM. VALUES OF (G) MARKED ON CURVES

(Z) IS THE RATIO OF THE FLUX PER POLE TO THAT FOR ZERO SLOT AREA

(X) IS THE RATIO OF THE DEPTH OF SLOT TO THE STATOR RADIUS



Substituting (25) in (20)

$$D_c/R_s = Z\ K_3/K_2 \tag{38}$$

(26) $H/R_s = (1 - Z)$

$$\frac{R_r}{R_s} = 1 - \frac{Dc}{R_s} + \frac{Y\ H}{R_s}$$

Substituting equation (26) and (38)

$$\frac{R_r}{R_s} = 1 - Z\ \frac{K_3}{K_2} - Y\ (1 - Z) \tag{39}$$

The equation for R_r/R_s may also be written as

$$R_r/R_s = R_1/R_s + (1 - Y)\ H/R_s$$

But $R_1/R_s = Z\ K_4/K_2$

and $H/R_s = (1 - Z)$

So

$$R_r/R_s = Z\ K_4/K_2 + (1 - Y)(1 - Z) \tag{40}$$

Let

$$\alpha = K_4/K_3 = \frac{P\ (0.278)\ B_c}{B_t\ (0.556)}$$

$$\alpha = \frac{P\ \ B_c}{2\ B_t}$$

Let

$$B = B_c/B_t$$

Then

$$\alpha = P\ B/2$$

$$K_2 = K_3 + K_4 = K_3\ (1 + \alpha)$$

So

$$K_4/K_2 = \frac{\alpha}{(1 + \alpha)} \tag{41}$$

Substitute (41) in (40)

$$R_r/R_s = \frac{Z \alpha}{(1 + \alpha)} + (1 - Y)(1 - Z) \qquad (42)$$

Equation (42) shows that in order to determine the rotor radius for any values of (Y) and (Z) it is only necessary to fix the ratio of the flux densities in the core and teeth and not their actual values. A curve of R_r/R_s against (α) is plotted on Plate 8. for (Z = 1/2) and (Y = 1/2), this being the condition for minimum stator resistance. Any change in (Y) does not alter the shape of the curve it merely displaces all points on the curve an equal distance. If (Y) is made 0.45 instead of 0.50 the slot area is decreased only one percent, but (R_r/R_s) is increased by 0.025 for all values of (α) when (Z = 1/2).

The value of (B_c/B_t) will in general lie between 0.50 and 1.0 so that (α) will assume values from 0.25 P to 0.50 P.

(Plate 9).

Plate 9 shows curves of (R_r/R_s) for values of (B_c/B_t) from 0.5 to 1.0 for (Z = 1/2) and (Y = 1/2). The values of (Y) and (Z) being those for minimum stator resistance. The curves are plotted for motors having from two to twelve poles.

For an open slot motor a core density of 70,000 line per square inch and a tooth density of 90,000 may be taken as suitable values. This would give a ratio of (B_c/B_t) near 0.8. The probable variation from this value would perhaps be from 0.7 to 0.9.

PLATE 8

EQUATION (42)

$$R_r / R_s = \frac{Z\alpha}{(1+\alpha)} + (1 - Y)(1 - Z)$$

(R_r / R_s) IS THE RATIO OF THE ROTOR RADIUS TO THE STATOR RADIUS

(α) IS ($P B_c / 2 B_t$)

CURVE PLOTTED FOR ($Y = 1/2$) AND ($Z = 1/2$)

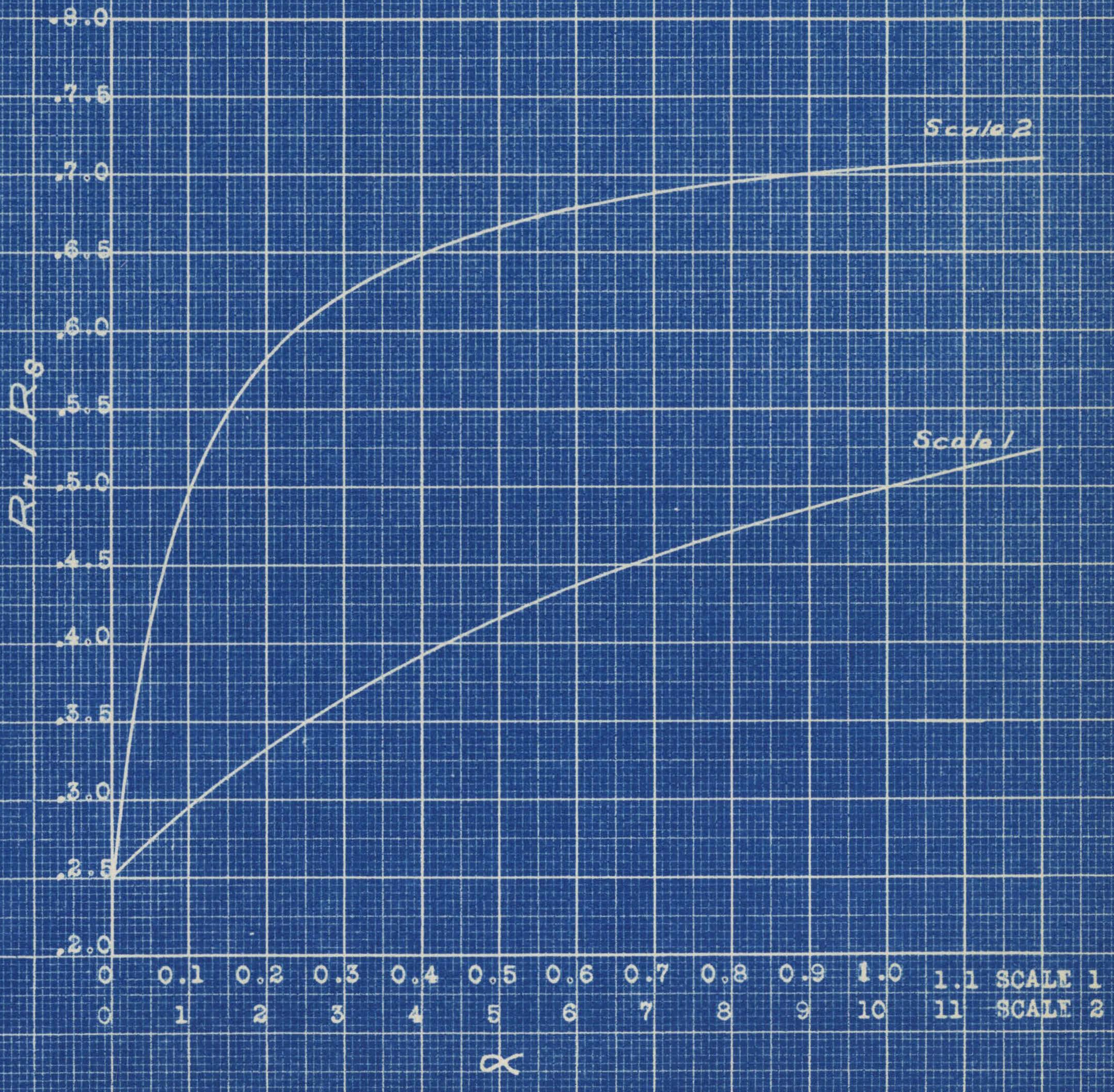


PLATE 9

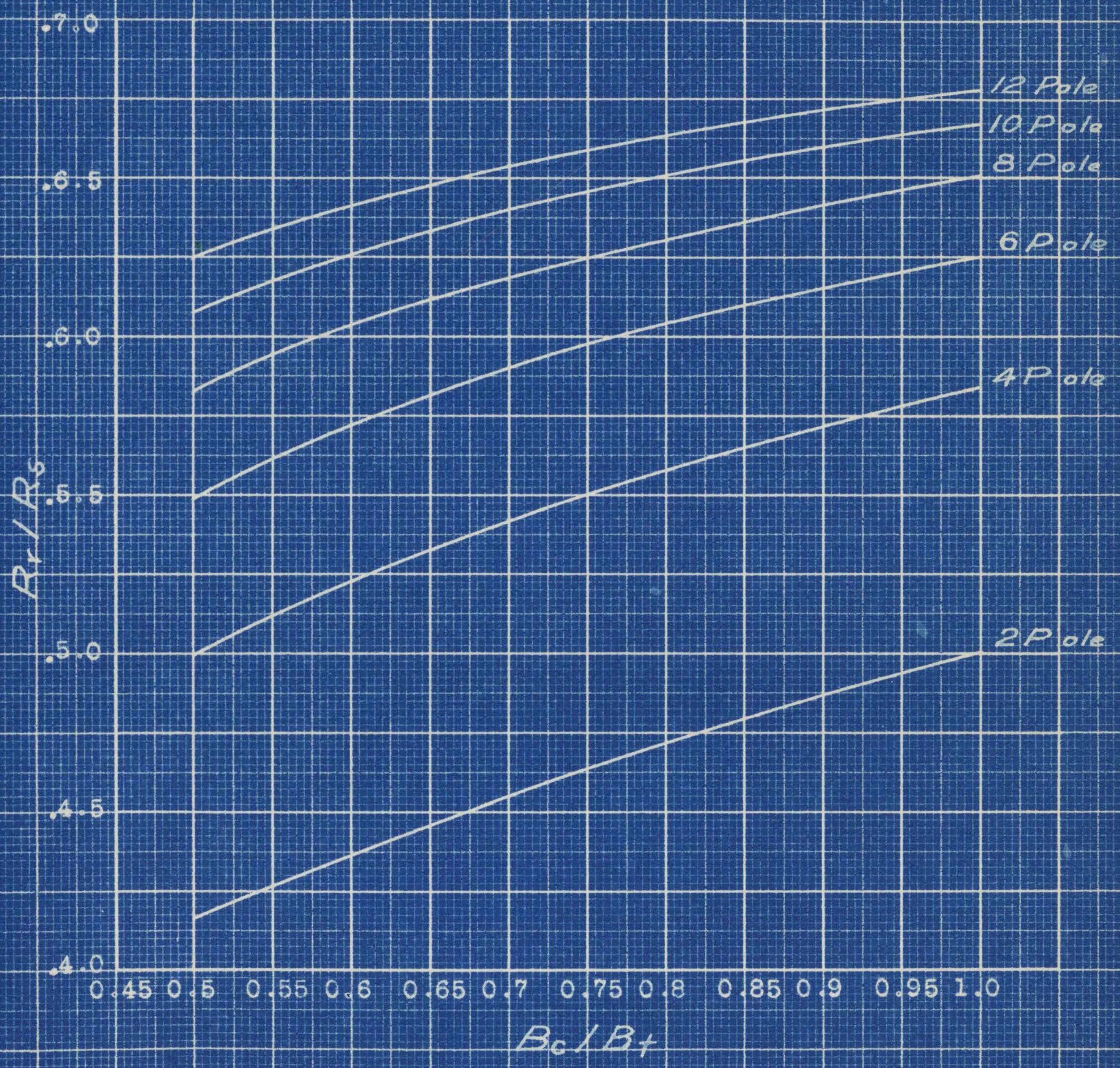
EQUATION (42)

$$R_r / R_s = \frac{Z \alpha}{(1+\alpha)} + (1 - Y)(1 - Z)$$

(R_r / R_s) IS THE RATIO OF THE ROTOR RADIUS TO THE STATOR RADIUS

(α) IS ($P B_c / 2 B_t$)

CURVES PLOTTED FOR ($Y = 1/2$) AND ($Z = 1/2$)



<u>TABLE III.</u>

	Ratio of Rotor Radius to Stator Radius					
	(Y = 1/2), (Z = 1/2)			(Z = 1/2), (Y = 0.45)		
B_c/B_t	0.7	0.8	0.9	0.7	0.8	0.9
Poles						
2	0.455	0.472	0.487	0.480	0.497	0.512
4	0.542	0.557	0.572	0.567	0.582	0.597
6	0.590	0.604	0.615	0.615	0.629	0.640
8	0.617	0.630	0.642	0.642	0.655	0.667
10	0.640	0.650	0.660	0.665	0.675	0.685
12	0.653	0.662	0.670	0.678	0.687	0.695

Table III gives values of R_r/R_s for motors from two to twelve poles for ratios of B_c/B_t of 0.7, 0.8, 0.9. Two conditions are considered, one, that for minimum stator resistance (Z = 1/2) and (Y = 1/2), the other having the same flux per pole but a slightly shallower slot (Z = 1/2) and (Y = 0.45). For a four pole motor with (Y = 0.45) and (B_c/B_t = 0.9), R_r/R_s is equal to 0.597. For a six pole motor with (Y = 1/2) and B_c/B_t = 0.7), R_r/R_s is equal to 0.590. By slightly increasing the core density of the four pole motor and decreasing its tooth density the same diameter rotor can be used for both the four and six pole motors. For an eight pole motor with (Y = 0.45) and (B_c/B_t = 0.8), the value of R_r/R_s is 0.655, while for a twelve pole motor with (Y = 1/2) and (B_c/B_t = 0.7), R_r/R_s takes the value 0.653. Here by decreasing the depth of the slot for the eight pole motor and increasing the tooth density of the twelve pole, the same diameter rotor can be used for motors from eight to twelve poles.

In order to determine all of the mechanical dimensions of the stator as a function of the stator resistance, it is only necessary to set the ratio of the flux densities in the teeth and core. In the discussion to follow an attempt will be made to prove that for economical design the motor should be designed very near the point where it has minimum stator resistance.

SAMPLE DESIGN

For a sample design let the following dimensions be assumed.

Stator radius = R_{os} = 10 inches.

Vertical slot insulation = D_v = 0.3 inches.

Horizontal slot insulation = D_h = 0.1 inches.

Number of stator slots for two pole motor = N = 54.

Number of stator slots for 4 and 6 pole motors = N = 72.

Number of stator slots for 8, 10, and 12 pole motors = N = 96.

Ratio of flux density in core to density in teeth = B_c/B_t = 7/9.

To correct for the area occupied by the slot insulation equations (9) to (13) are used.

(9) $D_s = D_{os} - D_v$.

(10) $W_s = W_{os} - D_h$.

(12) $R_s = R_{os} - (D_v + D_h/K_1)$.

(13) $R_r = R_{or} - D_h/K_1$.

$$K_1 = \frac{2\pi}{N}$$

Table IV. gives the values of R_s determined by equation (12):

TABLE IV.

Poles	2	4 & 6	8, 10, & 12.
N	54	72	96
K_1	0.1161	0.0871	0.0654
D_h/K_1	0.860	1.15	1.53
$D_v + D_h/K_1$	1.16	1.45	1.83
R_s	8.84	8.55	8.17

Because of the space occupied by the slot insulation the actual stator radius is reduced to the effective value (R_s) given by Table IV. Equation (42) is used to find the rotor radius.

$$(42) \quad R_r/R_s = \frac{Z\,\alpha}{1+\alpha} + (1 - Y)\,(1 - Z)$$

Where $\alpha = \dfrac{P\,B_c}{2\,B_t}$

Let two designs be considered; one for minimum stator resistance, (Y = 1/2), (Z = 1/2), and the other having the flux per pole for minimum resistance but a slot a little shallower than that for maximum slot area, (Z = 1/2), (Y = 0.45).

The results of equation (42) and (13) are given by Table V.

TABLE V.

	R_r/R_s		R_r		Actual rotor radius, R_{or}	
Poles	(Y = 1/2)	(Y = 0.45)	(Y = 1/2)	(Y = 0.45)	(Y = 1/2)	(Y = 0.45)
2	0.468	0.493	4.13	4.36	4.99	5.22
4	0.554	0.579	4.73	4.95	5.88	6.10
6	0.601	0.626	5.14	5.35	6.29	6.50
8	0.627	0.652	5.12	5.32	6.65	6.85
10	0.648	0.673	5.29	5.50	6.82	7.03
12	0.661	0.686	5.40	5.61	6.93	7.14

The values of the actual rotor radius given by Table V agree with those found in general practice. For $(Y = 0.45)$ the conductivity is 99% of the maximum, but the rotor radius is greater, so this design would be preferable to that for which $(Y = 1/2)$.

To find the remaining dimensions the following equations are used.

$$D_s = Y\ H$$

By (26)

$$H = R_s\ (1 - Z)$$

So $$D_s = R_s\ Y(1 - Z)$$

$$W_s = K_1\ H_1$$

Combining this with equations (4) and (26)

$$W_s = R_s\ K_1\ (1 - Y)(1 - Z)$$

Let λ and λ_o be the slot pitch

$$\lambda = K_1\ R_r$$

$$\lambda_o = K_1\ R_{or}$$

Then $$W_t = \lambda - W_s$$

Or $$W_t = \lambda_o - W_{os}$$

The results of these equations for the two designs $(Y = 1/2)$, $(Z = 1/2)$, and $(Y = 0.45)$, $(Z = 1/2)$ are given by Table VI.

TABLE VI.

For $(Y = 1/2)$ and $(Z = 1/2)$

Poles	D_s	W_s	λ	D_c	D_{os}	W_{os}	λ_o	W_t	λ_o/W_t
2	2.21	0.257	0.480	2.51	2.51	0.357	0.580	0.223	2.60
4	2.14	0.187	0.412	1.68	2.44	0.287	0.512	0.225	2.27
6	2.14	0.187	0.447	1.27	2.44	0.287	0.547	0.260	2.10
8	2.04	0.133	0.335	1.01	2.34	0.233	0.435	0.202	2.15
10	2.04	0.133	0.346	0.84	2.34	0.233	0.446	0.213	2.09
12	204	0.133	0.355	0.73	2.34	0.233	0.453	0.220	2.06

(Y = 0.45) , (Z = 1/2)

Poles	D_s	W_s	λ	D_c	D_{os}	W_{os}	λ_o	W_t	λ_o/W_t
2	1.99	0.283	0.506	2.49	2.29	0.382	0.606	0.224	2.71
4	1.92	0.205	0.430	1.68	2.22	0.305	0.530	0.225	2.37
6	1.92	0.205	0.465	1.28	2.22	0.305	0.565	0.260	2.17
8	1.84	0.147	0.349	1.01	2.14	0.247	0.449	0.202	2.22
10	1.84	0.147	0.360	0.83	2.14	0.247	0.460	0.213	2.16
12	1.84	0.147	0.367	0.72	2.14	0.247	0.467	0.220	2.12

The values of D_s and W_s are the net dimensions of the stator coils in each slot. D_{os} and W_{os} are the actual slot dimensions. The ratio of slot pitch to tooth width is a function of the slot insulation as well as the flux densities, so it is difficult to make comparisons. However, the results of this calculation fall in the range found in practice.

By setting the ratio of the flux densities in the stator core and teeth it has been possible to determine all of the mechanical dimensions of the stator.

In order to determine the flux per pole and the flux density in the air gap, the actual values of the flux densities must be considered. The value of B_c/B_t was taken as 7/9.

Let B_c = 70,000 lines per square inch.

B_t = 90,000 lines per square inch.

By (25)

$$\frac{\Phi_p}{L_g \; R_s} = \frac{Z}{K_2}$$

By (17)

$$K_2 = K_3 + K_4$$

(18) $$K_3 = \frac{0.556}{B_c}$$

(19) $$K_4 = \frac{P\,(0.278)}{B_t}$$

$$B_g = \frac{\Phi_p \; P}{L_g \; 2\pi \; R_{or}}$$

The air gap density is given by the usual equation shown above.

TABLE VII.

Poles	K_4	K_2	$\frac{\Phi_p}{L_g R_s}$	$\frac{\Phi_p}{L_g}$	B_g (Y = 1/2)	B_g (Y = 0.45)
2	6.18 (10^{-6})	1.41 (10^{-6})	3.54 (10^{4})	3.12 (10^{5})	19,900	19,000
4	12.36	2.03	2.46	2.11	22,800	22,000
6	18.54	2.65	1.89	1.62	24,600	23,800
8	24.72	3.27	1.33	1.25	23,900	23,200
10	30.90	3.89	1.29	1.05	24,600	23,900
12	37.08	4.50	1.11	0.908	25,000	24,200

Table VII gives the air gap density in lines per square inch. The values given here agree very closely with those found in practice.

If higher values had been taken for the flux densities, the resulting air gap density would have been proportionately higher.

From the results of this calculation it is evident that the general practice is to design the motor near the point where the stator resistance is a minimum. In the discussion to follow reasons will be given why this should be so.

RELATION OF THE PULLOUT TORQUE TO THE FLUX PER POLE AND THE FLUX DENSITIES

For a constant stator radius the flux per pole is determined by the two variables (Z) and (K_2).

(25)

$$\frac{\Phi_p}{L_g} = \frac{Z \; R_s}{K_2}$$

A change in the variable (K_2) represents a change in the value of the flux densities. The value of (Z) depends on the depth of core and width of teeth. Thus the flux per pole can be increased either by increasing the flux densities or by increasing the area of the stator core and teeth. The motor performance and pullout torque will depend not only on the flux per pole but also on the way in which that flux per pole is obtained. That is, they are functions of the variables (Z) and (K_2).

From equation (25) it is evident that the flux per pole varies as $(1/K_2)$ for constant values of stator radius and (Z).

(17) $K_2 = 0.556 \left(\frac{1}{B_c} + \frac{P}{2\ B_t} \right)$

Let (T) be proportional to the maximum torque.

(T) varies as $(\Phi_p)^2$.

(Φ_p) varies as $(1/K_2)$

So $T = (1/K_2)^2$.

If curves for constant values of (K_2) are plotted against $(1/B_c)$ and $(p/2\ B_t)$ the result is a series of straight lines. The curves for constant values of $(1/K_2)$ and $(1/K_2)^2$ will also be straight lines. Curves for constant values of $(1/K_2)^2$ from $1(10^9)$ to $15(10^9)$ are plotted against $(1/B_c)$ and $(P/2\ B_t)$ on Plate 10. Curves for these same values of $(1/K_2)^2$ are plotted against (B_c) and $(2\ B_t/P)$ on Plate 11. These curves will be curves for constant values of maximum torque. Then for any value of torque the values of the flux densities can be changed along the curve to the point where the total iron loss and magnetizing current are a minimum.

PLATE 10

EQUATION (17)

$$\left\{\frac{1}{K_2}\right\}^2 = \left\{\frac{1}{0.555}\right\}^2 \left\{\frac{(B_c)(2B_t/P)}{B_c + 2B_t/P}\right\}^2$$

THE PULLOUT TORQUE IS PROPORTIONAL TO $(1/K_2)^2$ FOR CONSTANT VALUES OF (Z) AND (Y)

$(1/K_2)^2$ IS EQUAL TO VALUE MARKED ON CURVES TIMES $(10)^9$

(B_c) IS THE FLUX DENSITY IN THE STATOR CORE

(B_t) IS THE FLUX DENSITY IN THE STATOR TEETH

(P) IS THE NUMBER OF POLES

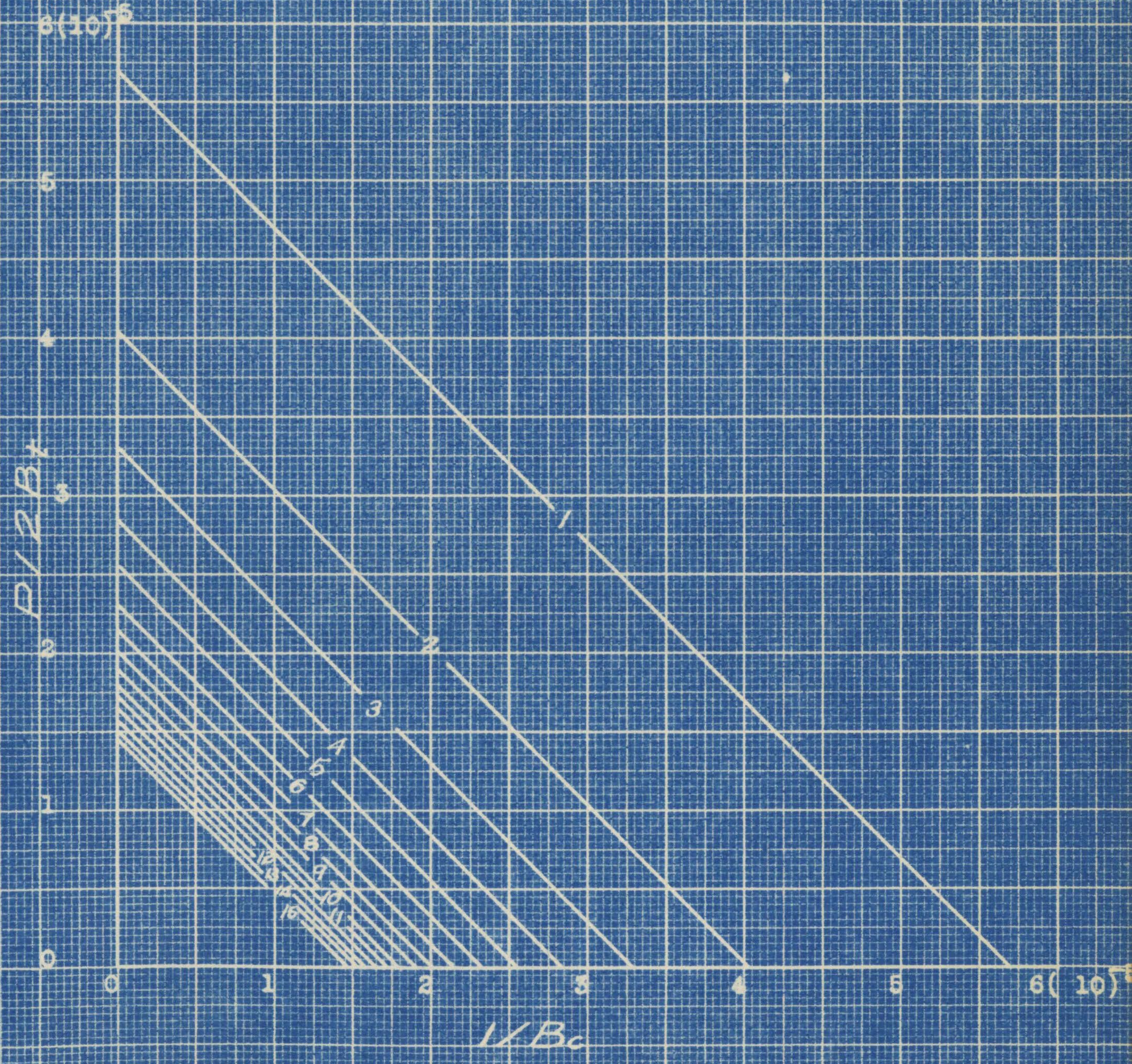


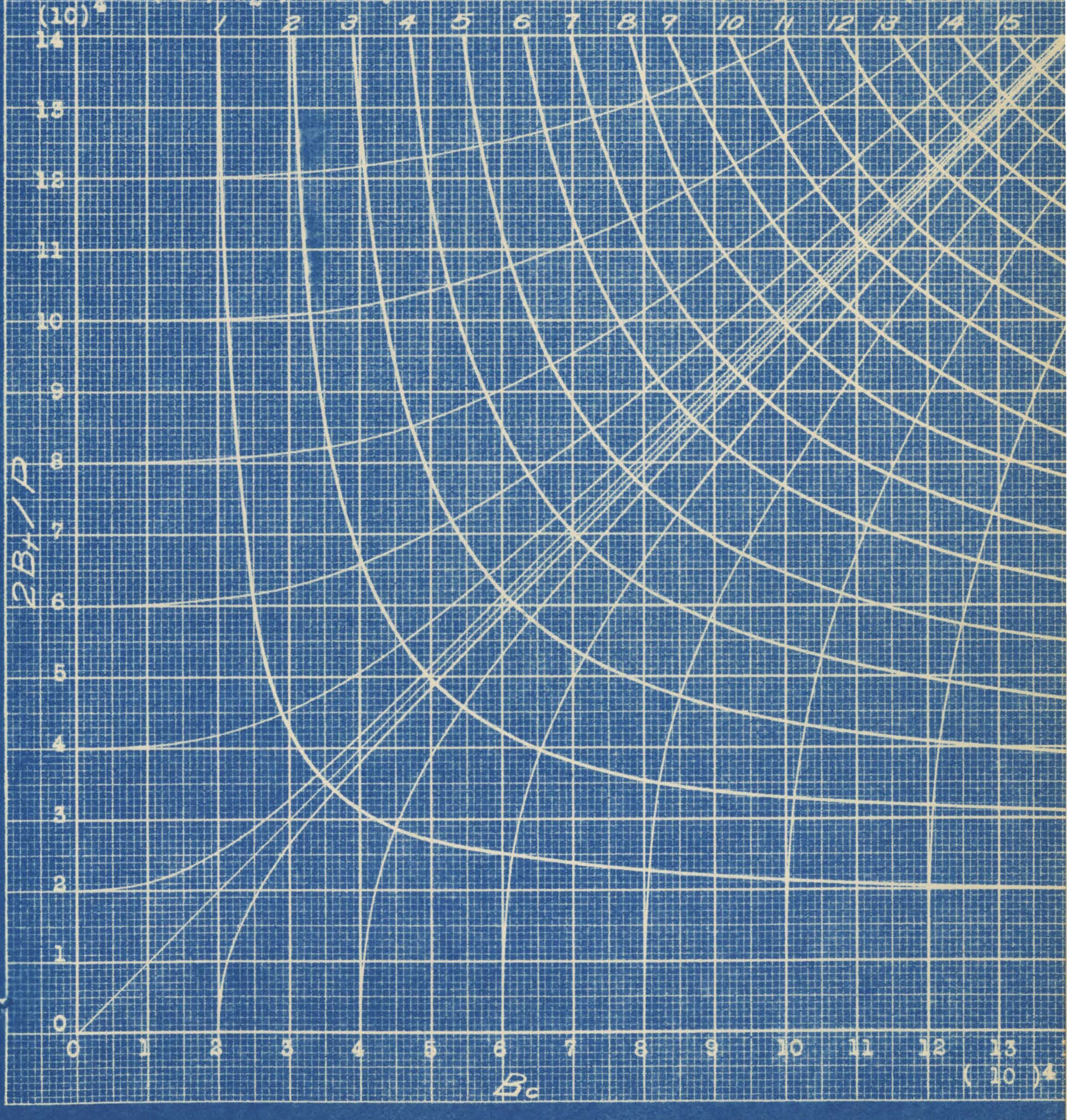
PLATE II

EQUATION (17)

$$\left\{\frac{1}{K_2}\right\}^2 = \left\{\frac{1}{0.556}\right\}^2 \left\{\frac{(B_c)(2B_t/P)}{B_c + 2B_t/P}\right\}^2$$

THE PULLOUT TORQUE IS PROPORTIONAL TO $(1/K_2)^2$ FOR CONSTANT VALUES OF (Y) AND (Z)

$(1/K_2)^2$ IS EQUAL TO VALUE MARKED ON CURVES TIMES $(10)^9$



DETERMINATION OF THE GRADIENT OF THE MAXIMUM TORQUE

The torque varies as $(1/K_2)^2$ so if the gradient of this function is determined, it will be the gradient of the torque.

(17) $K_2 = 0.556\ (1/B_c + P/2\ B_t)$

$$(K_2)^2 = (0.556)^2 \left(\frac{2\ B_t/P + B_c}{(B_c)\ (2\ B_t/P} \right)^2$$

$$T = (1/K_2)^2$$

$$T = (1.8)^2 \left(\frac{(B_c)(2\ B_t/P}{B_c + 2\ B_t/P} \right)^2$$

Taking the partial of (T) with respect to $(2\ B_t/P)$

$$\frac{\partial T}{\partial (2\ B_t/P} = \frac{(1.8)^2\ 2(B_c)(2\ B_t/P)(B_c)^2}{(2\ B_t/P + B_c)^3}$$

Similarly

$$\frac{\partial T}{\partial (B_c)} = \frac{(1.8)^2\ 2(B_c)(2\ B_t/P)(2\ B_t/P)^2}{(2\ B_t/P + B_c)^3}$$

The direction of the gradient will be given by the ratio of the two partial derivatives.

$$\frac{\frac{\partial T}{\partial (2\ B_t/P)}}{\frac{\partial T}{\partial\ (B_c)}} = \frac{(B_c)^2}{(2\ B_t/P)^2}$$

$$\frac{d\ (B_c)}{d\ (2\ B_t/P)} = \frac{(2\ B_t/P)^2}{(B_c)^2}$$

Integrating

$$B_c = \sqrt[3]{(2\ B_t/P)^3 + A} \qquad (43)$$

Equation (43) is the equation of the line which everywhere has the direction of the gradient of $(1/K_2)^2$. In the equation (A) is a constant which may assume any value. A few of these lines are plotted on Plate 11. For any value of (B_c) and $(2\ B_t/P)$ the pullout torque can be increased most rapidly by increasing (B_c) and $(2\ B_t/P)$ along the gradient lines.

EFFECT OF INCREASING THE FLUX PER POLE ON THE HORSE POWER RATING

If the value of (Y) and (Z) are kept constant, but the flux per pole is increased by increasing the flux densities, there will be no change in the shape or size of the stator slots. The stator resistance will then vary as the number of conductors in series per slot squared or as $(1/\Phi_p)^2$. The rotor resistance will also vary in the same manner. If we consider the ratio of current and torque to horse power constant, then all of these will increase as $(\Phi_p)^2$. Thus the horse power rating of the motor could be increased indefinitely, as far as these factors are concerned, by increasing the flux densities. There are, however, two limiting factors, one, the saturation of the iron, the other, heating. For the iron commonly used for induction motors the iron losses vary almost as the square of the flux densities. But the horse power rating would also increase as $(\Phi_p)^2$. The ratio of iron loss to rating would then remain constant. For normal values of flux densities the noload current increases as $(\Phi_p)^2$, so here again the ratio of noload to full-load current remains fixed. For high values of flux densities the magnetizing current increases very rapidly, lowering the power factor and materially increasing the full load current and losses.

The conductivity of the stator and rotor, and the full load current all increase as $(\Phi_p)^2$, so the full load copper loss would also increase as $(\Phi_p)^2$. The total losses will then vary in the same manner. The amount of heat which the motor can dissipate, however, will not increase. The losses and consequently the horse power rating of the machine will be thus limited.

If the rating of the machine is kept constant but the flux per pole is increased the motor resistance is decreased. But the no-load current, full load current, locked rotor current, and iron losses are all increased. There is no object then in increasing the flux per pole above that required by the pullout torque, unless the rating of the machine can be increased or its losses decreased.

The locked rotor current increases as $(\Phi_p)^2$. For line start motors the starting current is limited. If a low reactance rotor is used the flux per pole must be low to limit the starting current. Since the motor resistance varies inversely as $(\Phi_p)^2$, this will cause the motor resistance to be high and the efficiency at full load consequently low. However, if a high reactance rotor is used the flux per pole for the same starting current can be increased. The resistance will be decreased and the efficiency increased. The increase in no-load current and iron losses will be limiting factors.

EFFECT OF INCREASING THE FLUX PER POLE FOR CONSTANT FLUX DENSITIES

To change the flux per pole without changing the flux densities corresponds to a change in the variable (Z). If the effect of stator resistance on the pullout torque is neglected, then the torque, rating and current will increase as $(\Phi_p)^2$. The rotor conductance will be given by equation (36).

(36) $G = 64\ (1 - Z)2\ \ Z^2\ (1 - Y)\ Y$

For $(Y = 1/2)$

$$G = 16\ (1 - Z)^2\ Z^2$$

For constant flux densities and stator radius (Φ_p) varies as (Z).

Let (W) be proportional to the stator copper loss and (I) be the full load current.

Then $W = I^2/G$

(I) is proportional to $(Z)^2$

$$W = \frac{Z^4}{16\ (1 - Z)^2\ Z^2}$$

If the value of (W) is made equal to one when $(Z = 1/2)$, then (W) will be the ratio of the stator copper loss to that for $(Z = 1/2)$.

This relation is expressed by equation (44).

$$W = \frac{Z^2}{(1 - Z)^2} \tag{44}$$

Equation (44) is plotted on Plate 12.

TABLE VIII.

Z	W
0.3	0.183
0.4	0.444
0.5	1.000
0.6	2.250
0.7	5.450

Since the torque varies as (Z^2) an increase in (Z) from 0.5 to 0.6 represents an increase in torque of 44% but at the same time the stator

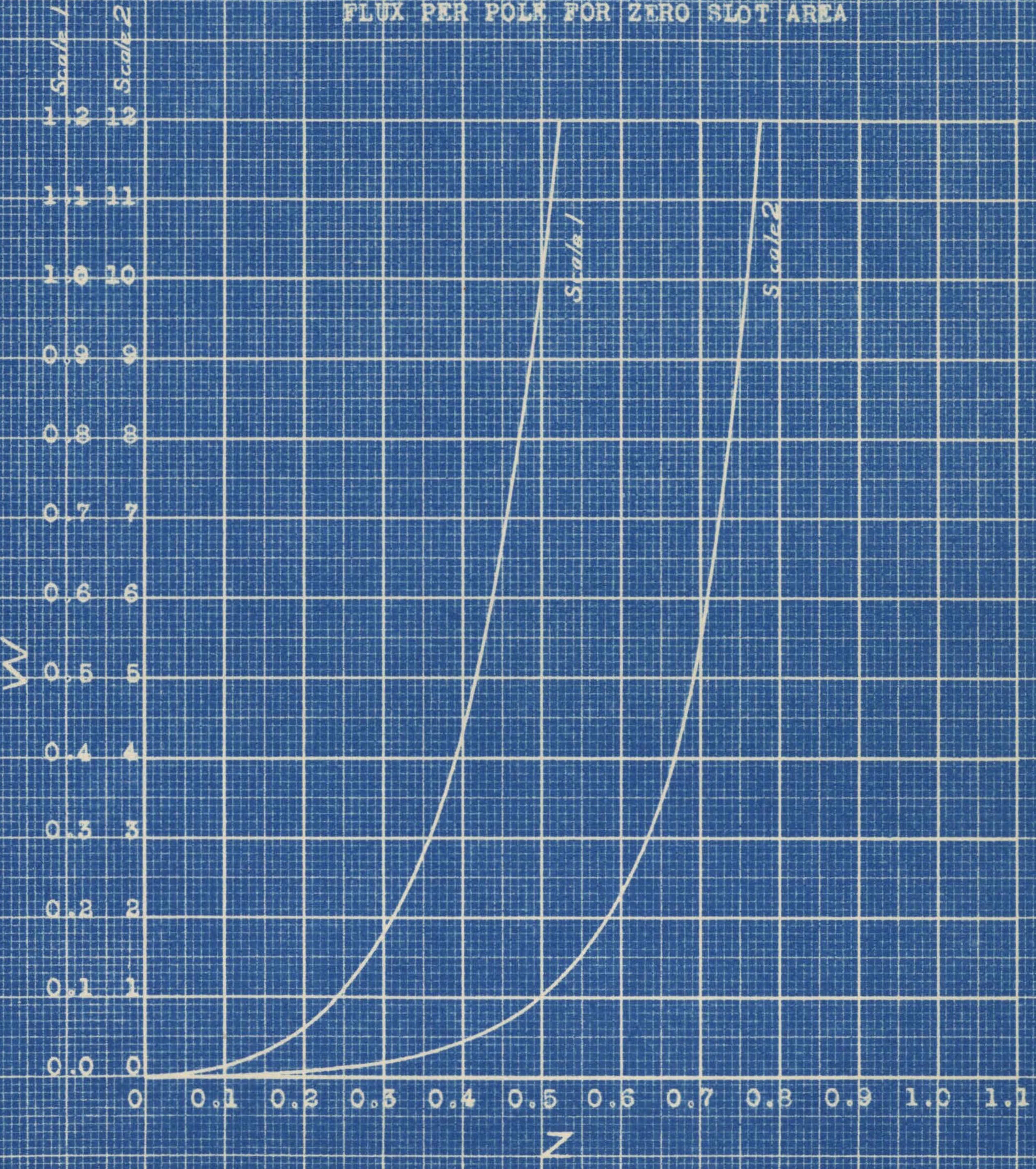
PLATE 12

EQUATION (44)

$$W = Z^2 / (1 - Z)^2$$

(W) IS THE RATIO OF THE STATOR COPPER LOSS TO THAT AT (Z=1/2) WHEN THE RATING IS INCREASED AS (Z^2)

(Z) IS THE RATIO OF THE FLUX PER POLE TO THE FLUX PER POLE FOR ZERO SLOT AREA



copper loss is increased 125%. Increasing (Z) from 0.5 to 0.7 increases the torque 96% and the stator copper loss 445%. On Plate 13 (W) is plotted against (Z^2), this gives a better idea of how the loss varies with the torque. For values of (Z) greater than 0.5 both the current and the resistance are increasing so the stator copper loss increases very rapidly. Since the total heat which the motor can dissipate is limited it is not practical to use values of (Z) much greater than 0.5.

If it is desired to limit the stator copper loss to a certain percent, then the conductivity must increase as the rating or as (Z^2).

Let (G') be the required conductivity,

Then

$$G' = k\,Z^2$$

Where (k) is a constant dependent on the percent stator copper loss.

The conductivity which it is possible to have is again given by equation (36).

Letting ($Y = 1/2$) in this equation

$$G = 16\,(1 - Z)^2\,Z^2$$

In order to show how the conductivity varies with the torque, (G) is plotted against (Z^2) on Plate 14. The equation ($G' = k\,Z^2$), for the required conductivity, is also plotted on the same plate for values of (k) from 1 to 10. These curves intersect with the curve for (G) at the points A, B, ---- M. For values of (Z^2) less than 1/4, (Z) less than 1/2, the horse power rating would not be limited to the values of (Z^2) corresponding to the points E,F,H, K,L,M, but rather by the values of (Z^2) determined by the points E',F',H',K' L',M', at which ($G' = 1$), since by increasing (Z^2) to 1/4 the conducti-

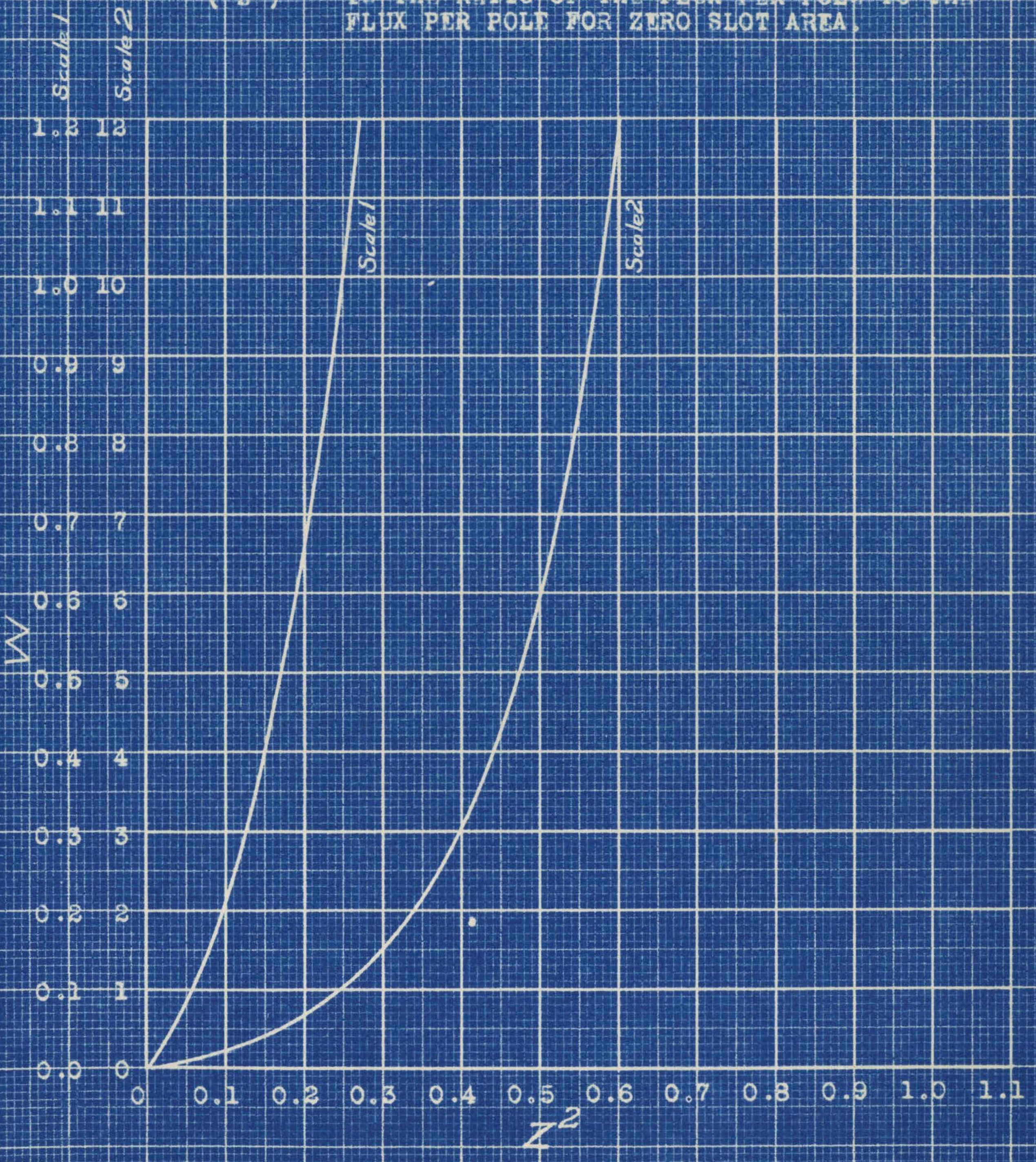
PLATE 13

EQUATION (44)

$$W = Z^2 / (1 - Z)^2$$

(W) IS THE RATIO OF THE STATOR COPPER LOSS TO THAT AT (Z = 1/2) WHEN THE RATING IS INCREASED AS (Z)^2.

(Z) IS THE RATIO OF THE FLUX PER POLE TO THE FLUX PER POLE FOR ZERO SLOT AREA.



vity can be raised to 1. Then for values of (k) greater than four in the equation ($G' = k\ Z^2$), the actual value of (Z) used in the motor would be 1/2, but the horse power rating of the machine would be determined by the value of (Z^2) for which ($G' = 1$).

For the points A,B,C, for which (Z^2) is greater than 1/4, the horse power rating would be definitely limited to the values of (Z^2) corresponding to the points A,B,C, since any further increase in (Z) would lower the conductivity still more.

It then appears that from the standpoint of losses it is not economical to use values of (Z) either much greater or lower than 1/2.

EFFECT OF STATOR RESISTANCE ON THE PULLOUT TORQUE

So far in the treatment of the pullout torque the effect of stator resistance has been neglected. When the flux per pole is increased by increasing the flux densities, the ratio of stator resistance to motor reactance remains constant, so the torque developed is still proportional to $(\Phi_p)^2$. However, when the flux densities are kept constant and the flux per pole is increased by increasing the value of (Z), the stator resistance plays an important part.

If the magnetizing current is neglected, the watts per phase delivered to the rotor will be,

$$W_r = \frac{E^2\ r_r/s}{(\ r_r/s + r_s\ )^2 + x^2}$$

PLATE 14

EQUATION (33)

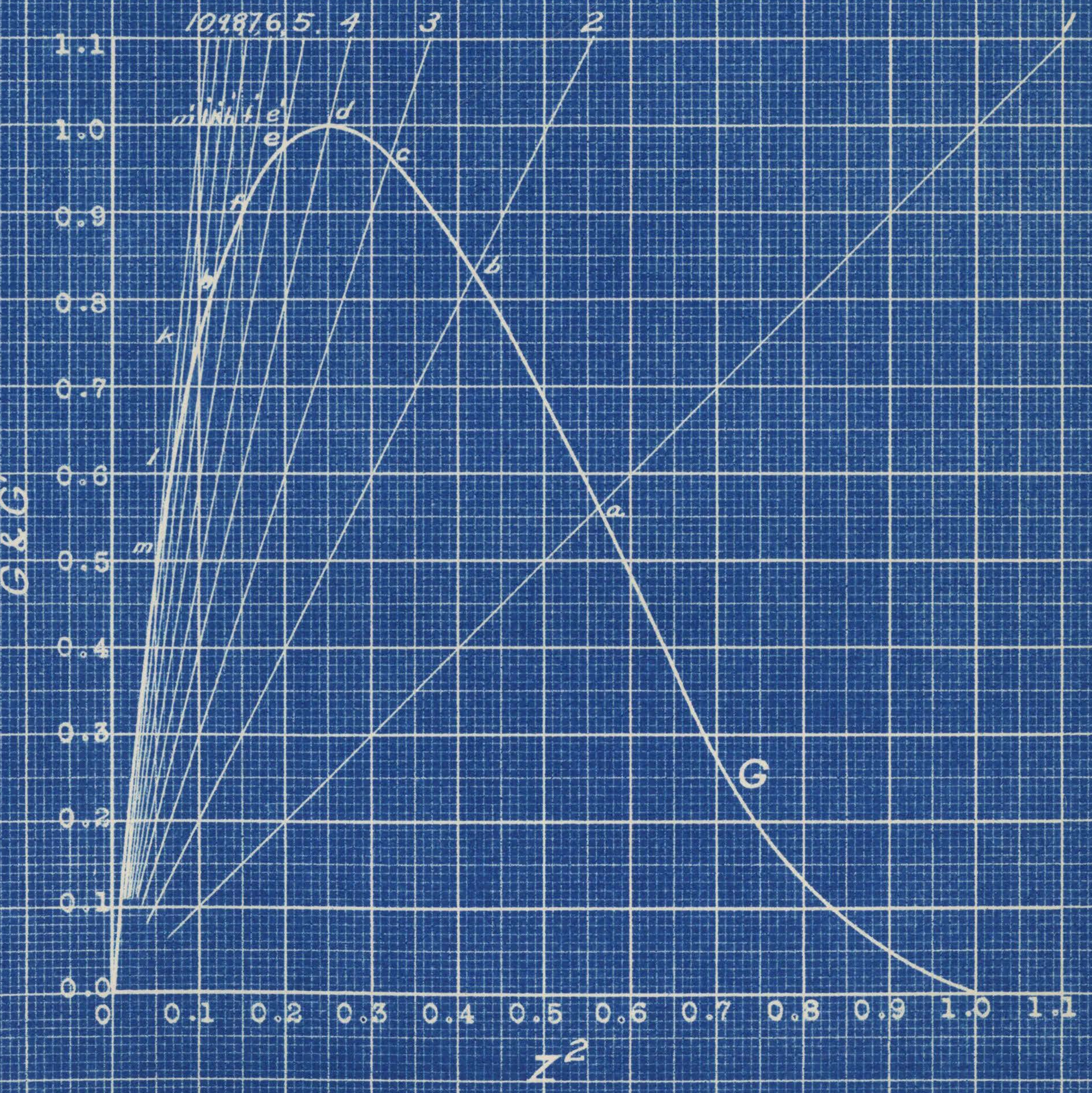
$$G = 16 (1 - Z)^2 Z^2$$

$$G' = k Z^2$$

(G) IS STATOR CONDUCTIVITY AVAILABLE

(G') IS STATOR CONDUCTIVITY REAQUIRED FOR CONSTANT PER CENT STATOR COPPER LOSS WHEN RATING IS INCREASED AS (Z^2)

CURVES PLOTTED FOR VALUES OF (k) FROM ONE TO TEN



E = Volts per phase
s = Slip
r_r = Rotor resistance per phase
r_s = Stator resistance per phase
x = Motor reactance per phase

(W_r) will be a maximum when

$$r_r/s = \sqrt{r_s^2 + x^2}$$

Let (T) be proportional to the maximum torque. Then (T) will vary directly as the maximum value of (W_r).

$$T = \frac{\sqrt{r_s^2 + x^2}}{(r_s + \sqrt{r_s^2 + x^2})^2 + x^2}$$

$$T = \frac{1}{\sqrt{r_s^2 + x^2} + r_s} \qquad (45)$$

The stator resistance will be proportional to (R) in equation (35). Letting (Y = 1/2) in this equation.

$$R = r_s = \frac{1}{(1 - Z)^2 Z^2 16}$$

The motor reactance (x) varies as $(1/Z)^2$.

$$x = c/Z^2$$

Where (c) is a constant.

Equation (45) may be written in the following forms:

$$T = \frac{1}{r_s\left(\sqrt{1 + (x/r_s)^2} + 1\right)} \qquad (46)$$

$$T = \frac{1}{x\left(\sqrt{(r_s/x)^2 + 1} + r_s/x\right)} \qquad (47)$$

$$\frac{x}{r_s} = \frac{16\, c\, (1 - Z)^2 Z^2}{Z^2}$$

Simplifying

$$\frac{x}{r_s} = 16\ c\ (1 - Z)^2 \tag{48}$$

Substituting equation (48) in (46):

$$T = \frac{16\ (1 - Z)^2\ Z^2}{\sqrt{1 + (16\ c\ (1 - Z)^2)^2} + 1} \tag{49}$$

Substituting equation (48) in (47):

$$T = \frac{Z^2}{c\left(\sqrt{\left(\frac{1}{c\ (1 - Z)^2 16}\right)^2 + 1} + \frac{1}{16\ c\ (1 - Z)^2}\right)} \tag{50}$$

Let (ψ) be the ratio of (r_s) to (x) when (Z = 1/2).

Substituting (Z = 1/2) in equation (48):

$$\psi = \frac{1}{4\ c} \tag{51}$$

Substituting (51) in (50):

$$T = \frac{16\ Z^2\ (1 - Z)^2\ \psi}{\sqrt{\psi^2 + 16\ (1 - Z)^4} + \psi}$$

When (Z = 1/2)

$$T = \frac{\psi}{\sqrt{\psi^2 + 1} + \psi}$$

Let (T_o) be the ratio of the pullout torque to that when (Z = 1/2)

$$T_o = \frac{T}{(\ T\ )\ \text{for}\ (Z = 1/2)}$$

$$T_o = \frac{(16\ Z\ \ (1 - Z)\ \psi)(\sqrt{\psi^2 + 1} + \psi)}{(\sqrt{\psi^2 + 16\ (1 - Z)^4} + \psi)(\ \psi\ )}$$

$$T_o = \frac{16\ Z^2\ (1 - Z)^2\ (\sqrt{\psi^2 + 1} + \psi)}{\sqrt{\psi^2 + 16\ (1 - Z)^4} + \psi} \tag{52}$$

PLATE 15

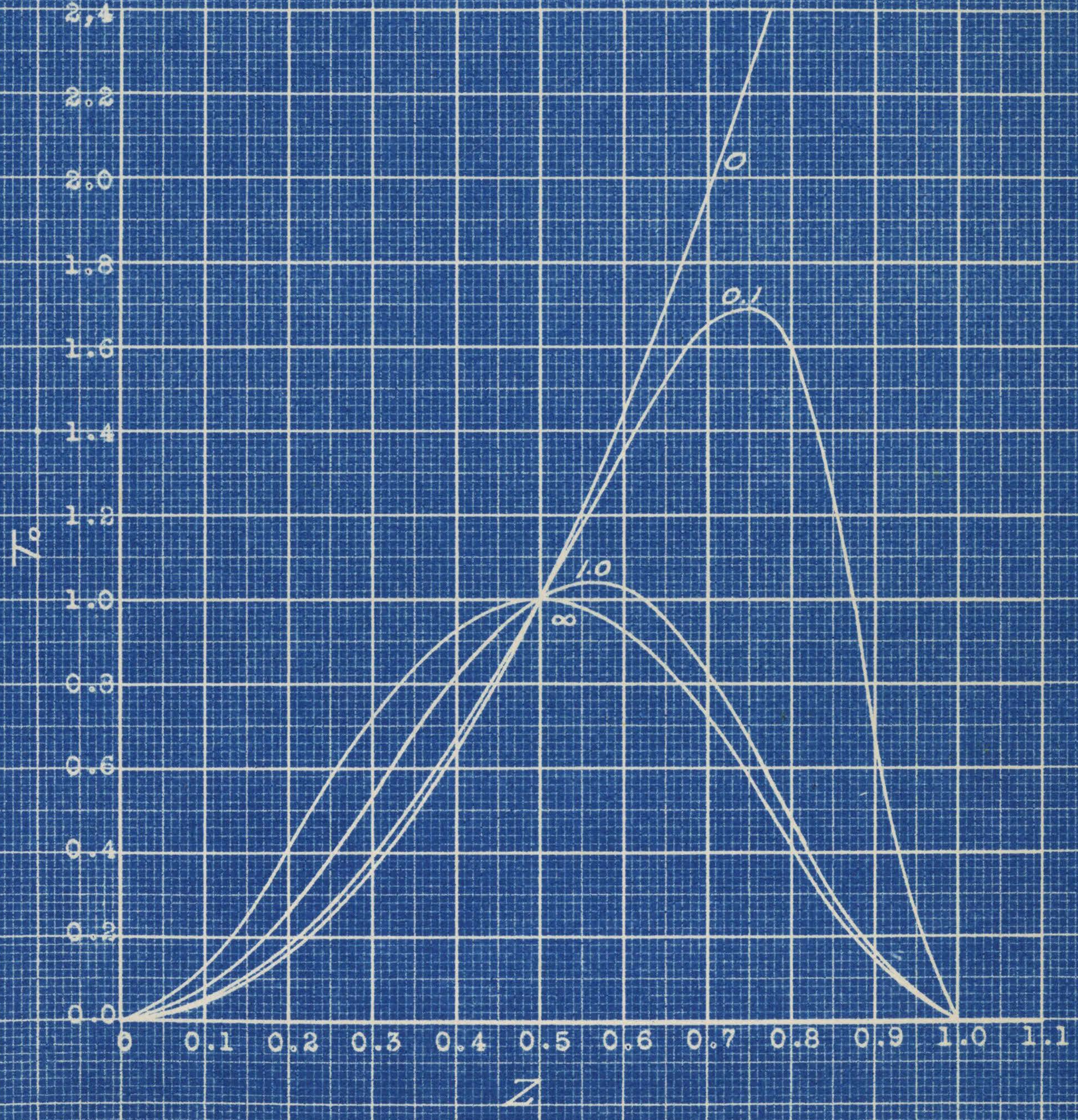
EQUATION (52)

$$T_o = \frac{(16 z^2)(1 - z)^2 \left(\sqrt{\gamma^2 + 1} + \gamma\right)}{\sqrt{\gamma^2 + 16(1 - z)^4} + \gamma}$$

(T_o) IS THE RATIO OF THE PULLOUT TORQUE TO THAT AT ($z = 1/2$)

(γ) IS THE RATIO OF THE STATOR RESISTANCE TO THE MOTOR REACTANCE AT ($z = 1/2$)

VALUES OF (γ) ARE MARKED ON THE CURVES



In equation (52), (ψ) is the ratio of the stator resistance to the motor reactance at ($Z = 1/2$), (T_o) is the ratio of the pullout torque to that for ($Z = 1/2$). Equation (52) is plotted on Plate 15 for values of (ψ) from zero to infinity. For ($\psi = 0$) the maximum value of (T_o) is four, the maximum occurring for ($Z = 1$). In all practical cases (ψ) is greater than zero. From Plate 15 it is evident that for all values of (ψ) greater than zero ($T_o = 0$) at ($Z = 1$) and the maximum value of (T_o) occurs at some point between ($Z = 1/2$) and ($Z = 1$). For ($\psi = 0.1$) the maximum value of (T_o) is 1.68 at ($Z = 0.737$). For high values of stator resistance the maximum value of (T_o) occurs for values of (Z) only slightly greater than 1/2.

The maximum value of (T_o) for any value of (ψ) can be obtained by differentiating (T_o) with respect to (Z) and equating to zero. This gives the value of (Z) at which the maximum occurs. By substituting this value of (Z) in equation (52) the maximum value of (T_o) can be expressed as a function of (ψ).

(52)

$$T_o = \frac{16 Z^2 (1 - Z)^2 \left(\sqrt{\psi^2 + 1} + \psi\right)}{\sqrt{\psi^2 + 16 (1 - Z)^4} + \psi} \qquad (52)$$

$$\frac{d(T_o)}{d(Z)} = \frac{16 \left(\sqrt{\psi^2 + 1} + \psi\right)(2Z (1-Z)(1-2Z)\left(\psi^2 + 16 (1-Z)^4 + \psi\sqrt{\psi^2 + 16 (1-Z)^4}\right)}{\left(\sqrt{\psi^2 + 16 (1 - Z)^4} + \psi\right)^2 \left(\sqrt{\psi^2 + 16 (1 - Z)^4}\right)}$$

$$+ \frac{16 \left(\sqrt{\psi^2 + 1} + \psi\right)\left(32 Z^2 (1 - Z)^5\right)}{\left(\sqrt{\psi^2 + 16 (1 - Z)^4} + \psi\right)^2 \left(\sqrt{\psi^2 + 16 (1 - Z)^4}\right)}$$

$$\frac{d(T_o)}{d(Z)} = 0$$

So

$$(1 - 2Z)\left(\psi^2 + 16 (1 - Z)^4 + \psi\sqrt{\psi^2 + 16 (1 - Z)^4}\right) + 16 Z(1 - Z)^4 = 0$$

$$\psi = \frac{4 (1 - Z)^3}{\sqrt{2Z - 1}} \qquad (53)$$

PLATE 16

EQUATION (53)

$$\chi = \frac{4\,(\,1 - z\,)^3}{\sqrt{2z - 1}}$$

THE VALUE OF (Z) DETERMINED BY THIS EQUATION IS THAT FOR WHICH THE PULLOUT TORQUE IS A MAXIMUM

(χ) IS THE RATIO OF STATOR RESISTANCE TO MOTOR REACTANCE AT (Z = 1/2)

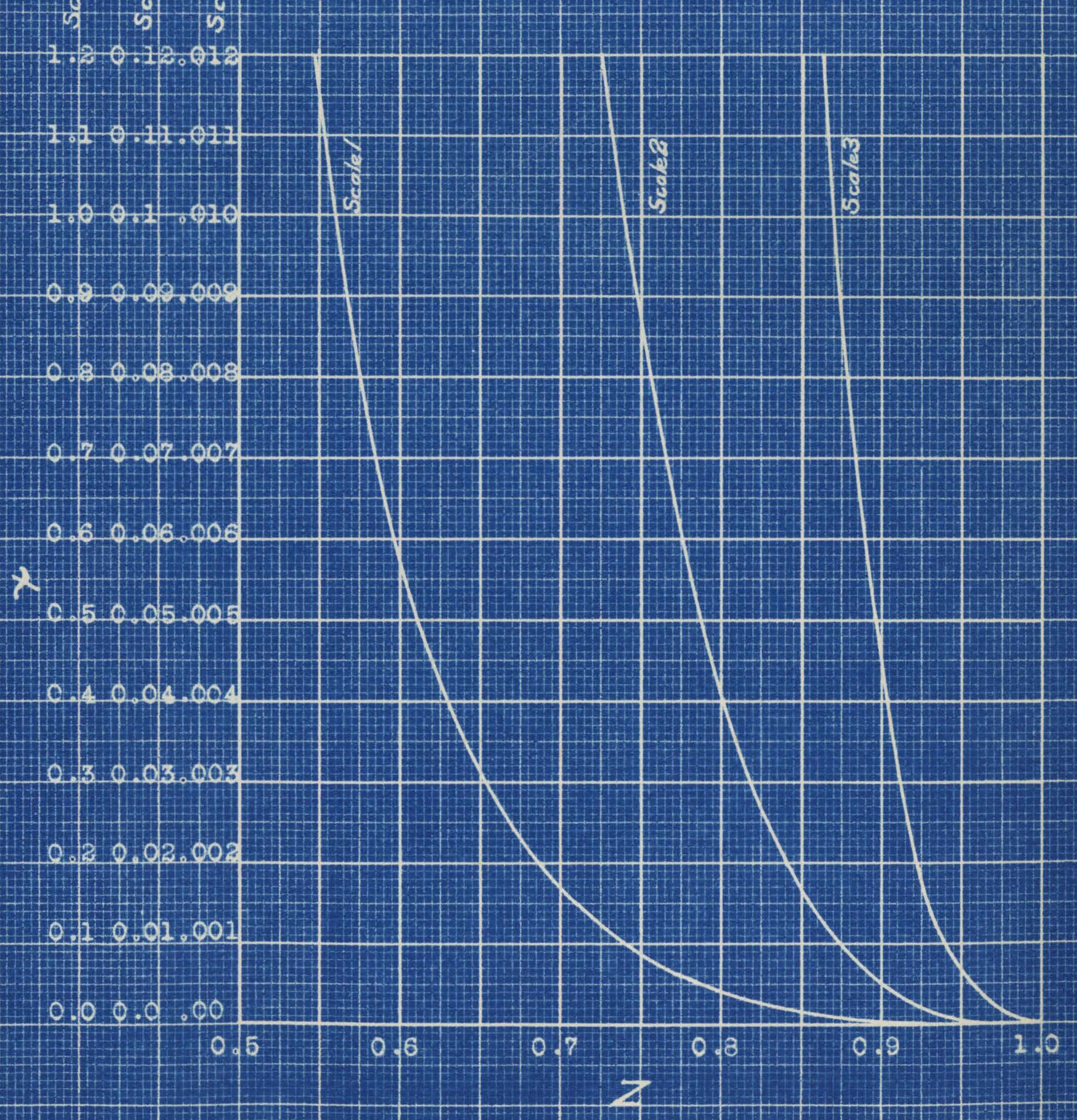
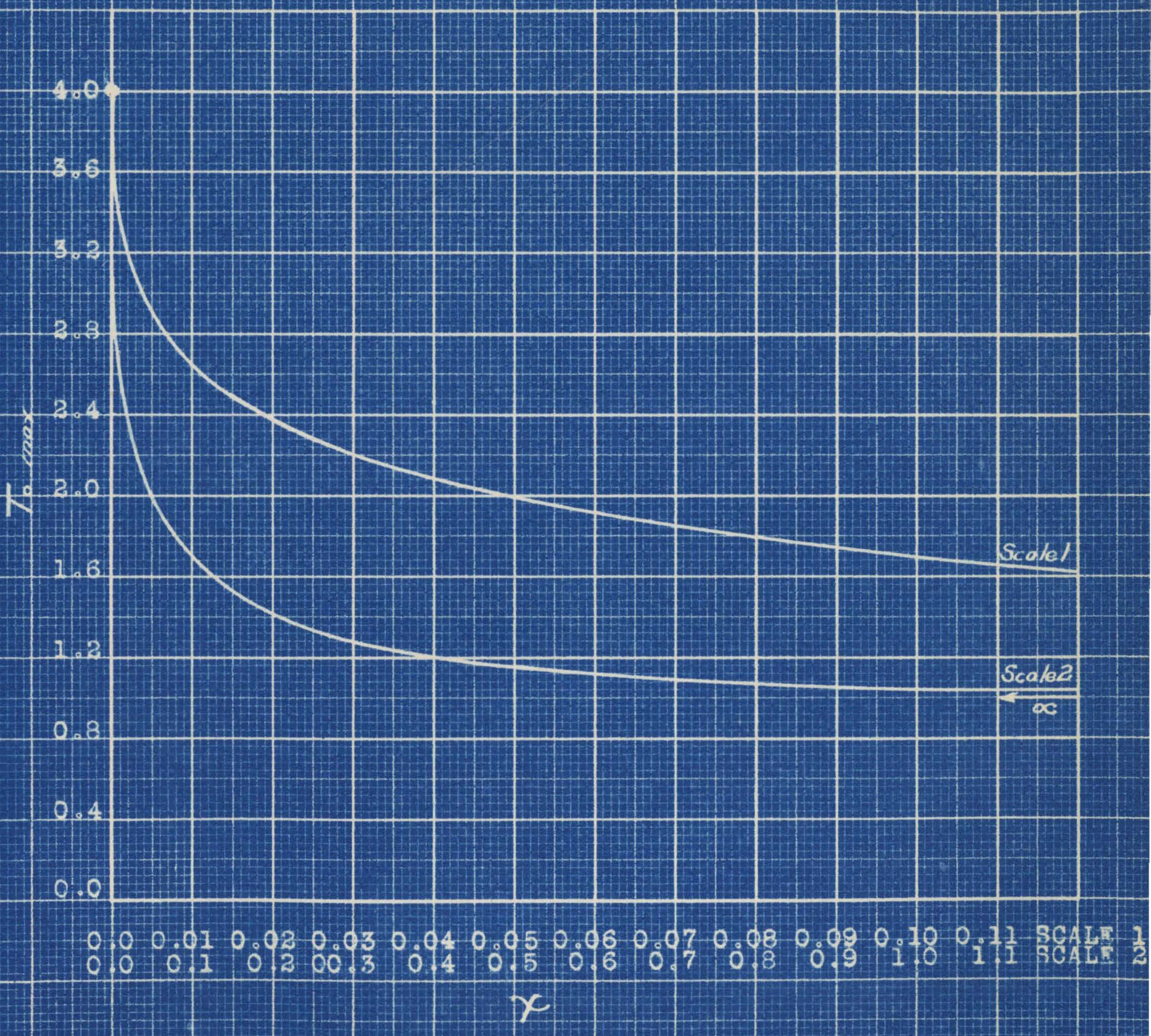


PLATE 17

SOLUTION OF EQUATIONS (52) AND (53)

(T_0 max) IS THE RATIO OF THE MAXIMUM PULLOUT TORQUE TO THE PULLOUT TORQUE AT (Z = 1/2)

(γ) IS THE RATIO OF THE STATOR RESISTANCE TO THE MOTOR REACTANCE AT (Z = 1/2)



The value of (Z) determined by equation (53) is that at which (T_o) is a maximum. Let the maximum value of (T_o) be written ($T_{o\ max.}$). Then ($T_{o\ max.}$) is determined by substituting equation (53) in (52). Perhaps the easiest method of solving equation (53) is to plot it. This is done on Plate 16. These values of (Z) are substituted in (52) and the resultant values of ($T_{o\ max.}$) are plotted against (γ) on Plate 17.

From Plate 17 it is evident that ($T_{o\ max.}$) drops very rapidly as (γ) increases. In an actual motor the value of (γ) would perhaps be about 1/10. For ($\gamma = 1/10$), ($Z = 0.737$) and ($T_{o\ max.} = 1.68$). If the rating is raised in proportion to the torque, the current will be 1.68 times that at ($Z = 1/2$). For ($Z = 0.737$) the conductivity is 0.6. The stator copper loss would then be ($1.68^2/0.6 = 4.70$) times greater than that at ($Z = 1/2$). For an increase in rating of 68% the stator copper loss has been increased 370%. If the effect of stator resistance on the torque is neglected, the value of (Z) for a 68% increase in torque is 0.647 and the increase in stator copper loss is 240%.

The effect of the no-load current on the torque and losses has been neglected. The no-load current increases as (Z^2). Since the torque and rating do not increase this rapidly, the percent no-load current will increase. The increased no-load current will reduce the maximum torque, power factor, and rating, and increase the full load current. Thus the stator losses will increase much more rapidly than the rating.

Then from the standpoint of efficiency, power factor, torque, and heating, the flux per pole used should be such that (Z) is very nearly equal to 1/2.

That is the motor should be designed near the point where the stator resistance is a minimum. If the ratio of stator resistance to motor reactance is low then values of (Z) slightly greater than 1/2 may be used, while for high stator resistance it may be economical to use (Z) slightly less than 1/2.

The sample design checks closely with general practice, showing that the calculated results are experimentally sound.

Briefly the procedure in this method of design is to calculate the slot dimensions from the stator radius. The rest of the mechanical dimensions can be determined when the ratio of the flux densities in the teeth and core are set. After choosing the flux densities the flux per pole can be calculated. With the torque, efficiency, and heating as limiting factors the rating is determined.

DETERMINATION OF THE END TURN REACTANCE

OF POLYPHASE MACHINES

This paper is
not complete and is
now being ~~copped~~
copied and completed
The thesis has however
been accepted on the
basis of the paper
Correlation of induction
Motor design factors

END TURN REACTANCE OF POLYPHASE MACHINES

A stator coil may be considered to consist of ~~straight~~ six straigth line segments. The reactance of the part of the coil which is in the stator slots can be calculated from the slot and air gap deminsions. This ~~reactance~~ reactance is a funtion of the self inductance of the conductors in a single stator slot. It is not materially affected by the conductors in the neighbooring slots. The reactance of that portion of the stator coil which does not lie in the slot is not only a funtion of its self inductance but also of its mutual inductance with every other stator coil. This part of the coil, which is called the end connection , projects from a surface of iron of high permeability in comparison to the sorounding air. The iron may then be considered an equipotential surface for the flux. Each conductor will ~~then~~ have its image in the iron. This image will be similar to the actual end connections at the other end of the stator. The problem is reduced to finding the reactance of a coil in air which has the shape of the two end connections moved together until they touch.

Inorder to set up the problem, consider a three phase machine with (p) poles. If (L) is the self inductance of the end connections of the group of coils per pole per phase , and (L_{21}) is written to mean the self inductance of phase(2) group for pole (1). Then the self inductance of the phase (2) group for pole(3) will be writtem (L_{23}). Let (M_{2132}) be the mutual inductance of the phase (2) group pole(1) with the phase (3) group pole (2).

The total self inductance of the end connections of phase (2) can then be written

$$
\begin{array}{llllllll}
L_2 = & L_{21} & + M_{2122} & + M_{2123} & + M_{2124} & + & + & + M_{212p} \\
 & M_{2221} & + L_{22} & + M_{2223} & + M_{2224} & + & + & + M_{222p} \\
 & M_{2321} & + M_{2322} & + L_{23} & + M_{2324} & + & + & + M_{232p} \\
 & \cdots & + \cdots & + \cdots & + \cdots & + & + & + \cdots \\
 & M_{2p21} & + M_{2p22} & + M_{2p23} & + M_{2p24} & + & + & + L_{2p}
\end{array} \qquad (1)
$$

The mutual inductance between the end connections of phase (2) and phase (1) will be

$$
\begin{array}{llllllll}
M_{21} = & M_{2111} & + M_{2112} & + M_{2113} & + M_{2114} & + & + & + M_{211p} \\
 & M_{2211} & + M_{2212} & + M_{2213} & + M_{2214} & + & + & + M_{221p} \\
 & M_{2311} & + M_{2312} & + M_{2313} & + M_{2314} & + & + & + M_{231p} \\
 & & + & + & + & + & + & + \\
 & M_{2p11} & + M_{2p12} & + M_{2p13} & + M_{2p14} & + & + & + M_{2p1p}
\end{array} \qquad (2)
$$

The mutual inductance between the end connections of phase (2) and phase (3) will be

$$
\begin{array}{llllllll}
M_{23} = & M_{2131} & + M_{2132} & + M_{2133} & + M_{2134} & + & + & + M_{213p} \\
 & M_{2231} & + M_{2232} & + M_{2233} & + M_{2234} & + & + & + M_{223p} \\
 & M_{2331} & + M_{2332} & + M_{2333} & + M_{2334} & + & + & + M_{233p} \\
 & & + & + & + & + & + & + M \\
 & M_{2p31} & + M_{2p32} & + M_{2p33} & + M_{2p34} & + & + & + M_{2p3p}
\end{array} \qquad (3)
$$

Phase (2) has the same mutual inductance with phase (1) as with phase (3). The current in phase(1) and phase (3) is 60° out of phase with the current in phase (2). The sum of the reactance voltages generated by phases (1) and (3) in phase (2) is in phase with the voltage due to the self inductance of phase (2).

The magnitude of the reactance voltage in phase (2) due to currents in phases (1) and (3) is 1/2 of the value it would have if current of phase(2) were flowing in all of the conductors. The total reactance of the end connections of phase (2) may then be written

$$X_2 = 2\pi f \left\{ L_2 + \frac{1}{2} (M_{21} + M_{23}) \right\} \qquad (4)$$

To find the end turn reactance it is then necessary to solve for the values of these coefficients of mutual and self inductance . A single coil consists of six straight line segments. A three phase six pole machine may have 108 stator slots, giving six slots per pole per phase. To find the self inductance for the group of coils for one phase for a single pole will involve the self and mutual inductance of thirty six line segments with each other. For the whole motor each line segment will have (6x 108 = 648) inductance coefficients . From the above example it is evident that it is better to work with current sheets rather than with individual coils.

To find the mutual inductance of two coils it is sufficient to find the flux produced in one coil by current in the other. For calculating the flux density at any point the surface of a coil carrying current may be considered a magnetic sheet . The magnetic potential due to a magnetic sheet is proportional to the solid angle subtended by the sheet. The flux density may then be found by taking the gradient of the magnetic potential. The flux density produced by current in one coil can then be integrated over the surface of the other coil and the coefficient of mutual inductance calculated. However the expressions for the solid angle and its derivatives, and the integration of the flux over the area of the second coil offer difficulties.

A more direct method of procedure is to find the mutual inductance between line elements of the two coils and then integrate this differential mutual inductance over the contours of the two coils.

MUTUAL INDUCTANCE OF TWO LINE ELEMENTS

Let (E) be the angle and (r) the distance between the two line elements (dl_1) and (dl_2). Let (dl_1) have the coordinates (X = X), (Y = 0) and the direction of the (X) axis. Let the coordinates of (dl_2) be (X = 0), (Y = Y).

Then $r = \sqrt{X^2 + Y^2}$

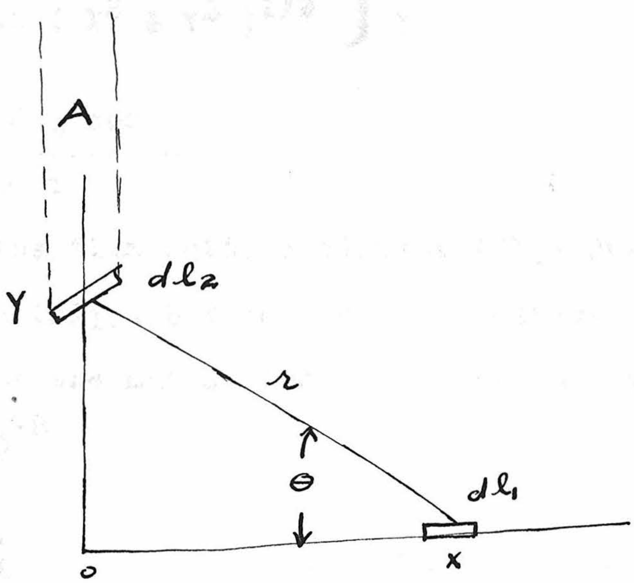


fig 1

The flux density (B) due to a current ~~on~~ of one ampere in (dl_1) is given by,

$$B = \frac{dl_1 \sin \theta}{10\, r^2} \qquad (5)$$

The flux which will sweep across (dl_2) when the current is changed in (dl_1) is contained in the area (A)

$$A = \int_Y^{\infty} dl_2 \cos E \, dY$$

The problem of finding the mutual inductance of the two line elements is that of finding the integral over this area of the flux produced by unit current in (dl_1). Let ($d\Phi$) be the value of this integral.

$$d\Phi = \int_Y^{\infty} \frac{dl_1 \, dl_2 \cos E \sin \theta}{10 \, r^2} \, dY$$

$$\sin \theta = \frac{Y}{\sqrt{X^2 + Y^2}}$$

and $\qquad r = \sqrt{X^2 + Y^2}$

$$d\Phi = \int_Y^{\infty} \frac{dl_1 \, dl_2 \cos E \;\; Y}{10 \, (X^2 + Y^2)^{3/2}} \, dY$$

$$d\Phi = - \left[\frac{dl_1 \, dl_2 \cos E}{10 \, (X^2 + Y^2)^{1/2}} \right]_Y^{\infty}$$

$$d\Phi = \frac{dl_1 \, dl_2 \cos E}{10 \, r} \qquad (6)$$

($d\Phi$) is the flux cutting element (dl_2) due to a unit change in current in (dl_1). Let (de) be the voltage induced in (dl_2) and let (dM) be the mutual inductance of the two elements.

$$de = \frac{d\Phi}{dt} 10^{-8}$$

$$de = dM \frac{dI}{dt}$$

$$dM = \frac{d\Phi}{dI} 10^{-8}$$

Substituting the value of ($d\Phi$) given by equation (6)

$$dM = \frac{dl_1 \, dl_2 \cos E \; 10^{-9}}{r} \qquad (7)$$

Equation (7) gives the mutual inductance between two line elements of lengths (dl_1) and (dl_2) , where (r) is the distance and(E) the angle between them.

To find the mutual inductance of any two conductors it is only necessary to integrate the increment of mutual inductance (dM), over the lengths of the two conductors.

$$M_{12} = 10^{-9} \int_1 \int_2 \frac{dl_1\, dl_2 \cos E}{r} \qquad (8)$$

DETERMINATION OF THE MUTUAL INDUCTANCE OF ANY TWO STRAIGHT LINE SEGMENTS

The end connections may be considered to consist of a combination of straight line segments. To find the inductance of the end connections it is necessary to find the mutual inductance of any two straight line segments.

Using the cylinderical system of coordinates, let the common normal to the two lines be taken as the (Z) axis. Then each line will lie in a plane for which (Z) is constant.

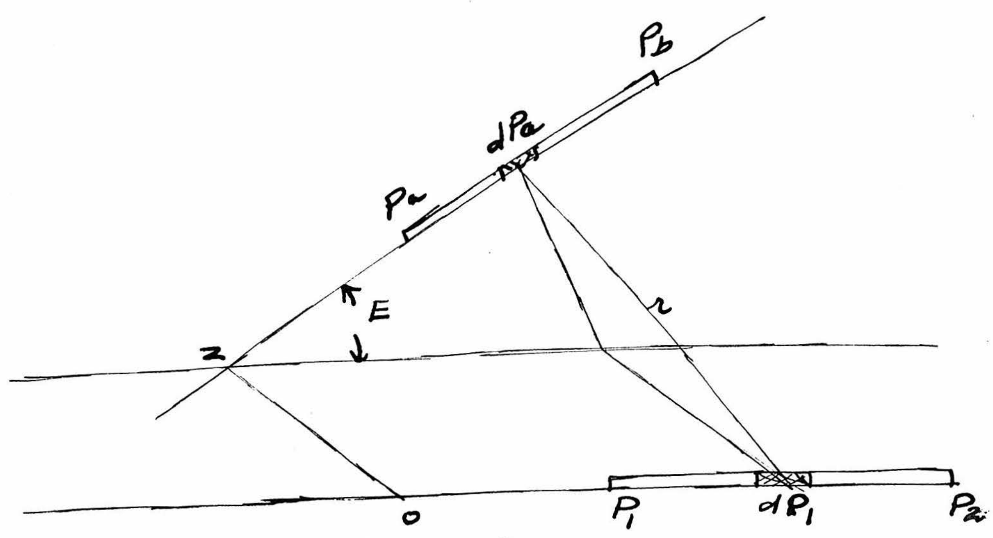


figure 2.

Let the coordinates of line (1) be (Z = 0), (Θ = 0), and let the segment extend from (P_1) to (P_2) . Let the coordinates of line (a) be (Z = Z), (Θ = E), and let the segent of this lie extend from (P_a) to (P_b).

~~The distance (r) between any two line~~ Let (P_1) be the variable along line (1) and let (P_a) be the variable along line (a). The distance (r) between any two line elements will be

$$r = \sqrt{P_1^2 + P_a^2 - 2\,P_1\,P_a \cos E + Z^2} \qquad (9)$$

The mutual inductance of line (a) with line (1) is then expressed by the integral

$$M_{1a} = 10^{-9} \cos E \int_{P_a}^{P_b}\int_{P_1}^{P_2} \frac{dP_a\,dP_1}{\sqrt{P_1^2 + P_a^2 - 2P_1P_a\cos E + Z^2}} \qquad (10)$$

To integrate equation (10) with respect to (P_1)

$$P_1^2 + P_a^2 - 2\,P_1\,P_a\cos E + Z^2 = (P_1 - P_a\cos E)^2 + (Z^2 + P_a^2\sin^2 E)$$

Let

$$P_1 - P_a\cos E = \sqrt{Z^2 + P_a^2\sin^2 E}\;\tan B \qquad (11)$$

Then

$$dP_1 = \sqrt{Z^2 + P_a^2\sin^2 E}\;\sec^2 B\;dB \qquad (12)$$

Substituting equations (11) and (12) in (10)

$$\int \frac{dP_1}{\sqrt{P_1^2 + P_a^2 - 2P_1P_a\cos E + Z^2}} = \int \frac{dB}{\cos B}$$

$$= \frac{1}{2}\operatorname{Log}\left(\frac{1 + \sin B}{1 - \sin B}\right) \qquad (13)$$

But

$$\sin B = \frac{P_1 - P_a\cos E}{\sqrt{P_1^2 + P_a^2 - 2\,P_1\,P_a\cos E + Z^2}}$$

Substituting this value for sin B in equation (13)

$$\int \frac{dP_1}{\sqrt{P_1^2 + P_a^2 - 2P_1P_a\cos E + Z^2}} =$$

$$\frac{1}{2}\operatorname{Log}\left\{\frac{\sqrt{P_1^2 + P_a^2 - 2P_1P_a\cos E + Z^2} + (P_1 - P_a\cos E)}{\sqrt{P_1^2 + P_a^2 - 2P_1P_a\cos E + Z^2} - (P_1 - P_a\cos E)}\right\}$$

$$= \operatorname{Log}\left(\sqrt{P_1^2 + P_a^2 - 2P_1P_a\cos E + Z^2} + P_1 - P_a\cos E\right)$$

$$- \frac{1}{2}\operatorname{Log}\left(Z^2 + P_a^2\sin^2 E\right) \qquad (14)$$

But $\text{Log}(Z^2 + P_a^2 \sin^2 E)$ is not a function of (P_1) so it will vanish when the limits of the integral are put in. The results of the integration over line (1) then is

$$\int_{P_1}^{P_2} \frac{dP_1}{\sqrt{P_1^2 + P_a^2 - 2 P_1 P_a \cos E + Z^2}} =$$

$$\text{Log} \left\{ \frac{\sqrt{P_2^2 + P_a^2 - 2 P_2 P_a \cos E + Z^2} + P_2 - P_a \cos E}{\sqrt{P_1^2 + P_a^2 - 2 P_1 P_a \cos E + Z^2} + P_1 - P_a \cos E} \right\} \quad (15)$$

The mutual inductance of line (1) and line (a) is now expressed by the integral

$$M_{1a} = 10^{-9} \cos E \int_{P_a}^{P_b} \text{Log} \left(\frac{\sqrt{P_2^2 + P_a^2 - 2P_2P_a \cos E + Z^2} + P_2 - P_a \cos E}{\sqrt{P_1^2 + P_a^2 - 2P_1P_a \cos E + Z^2} + P_1 - P_a \cos E} \right) dP_a \quad (16)$$

This integral can be integrated by parts

$$\int \text{Log} \left\{ \sqrt{P_2^2 + P_a^2 - 2 P_2 P_a \cos E + Z^2} + P_2 - P_a \cos E \right\} dP_a$$

$$= P_a \text{Log} \left\{ \sqrt{P_2^2 + P_a^2 - 2 P_2 P_a \cos E + Z^2} + P_2 - P_a \cos E \right) \quad (17)$$

$$- \int P_a \frac{d}{dP_a} \left\{ \text{Log} \left\{ \sqrt{P_2^2 + P_a^2 - 2P_2P_a \cos E + Z^2} + P_2 - P_a \cos E \right\} \right\} dP_a \quad (18)$$

To integrate expression (18) let

$$U = \sqrt{P_2^2 + P_a^2 - 2 P_2 P_a \cos E + Z^2} + P_2 - P_a \cos E \quad (19)$$

Then

$$\left\{ U - (P_2 - P_a \cos E) \right\}^2 = P_2^2 + P_a^2 - 2 P_2 P_a \cos E + Z^2$$

$$P_a^2 \sin^2 E - 2 U P_a \cos E + 2 U P_2 + Z^2 - U^2 = 0$$

Solving for (P_a)

$$P_a = \frac{U \cos E + \sqrt{U^2 - \sin^2 E \, (2 U P_2 + Z^2)}}{\sin^2 E} \quad (20)$$

and

$$dP_a = \frac{\cos E \sqrt{U^2 - (\sin^2 E)(2UP_2 \pm Z^2)} \pm U - P_2 \sin^2 E}{\sin^2 E \sqrt{U^2 - (\sin^2 E)(2U P_2 \pm Z^2)}}\, dU \qquad (21)$$

Substituting (20) and (21) in integral (18)

$$-\int P_a \frac{d}{dP_a}\left\{ \operatorname{Log}\left(\sqrt{P_2^2 \pm P_a^2 - 2 P_2 P_a \cos E \pm Z^2} \pm P_2 - P_a \cos E\right)\right\} dP_a$$

$$= -\int \frac{U \cos E \pm \sqrt{U^2 - (2 U P_2 \pm Z^2) \sin^2 E}}{U \sin^2 E}\, dU \qquad (22)$$

$$-\int \frac{U \cos E \pm \sqrt{U^2 - (2 U P_2 \pm Z^2) \sin^2 E}}{U \sin^2 E}\, dU =$$

$$-\int \frac{\cos E}{\sin^2 E}\, dU \qquad (23)$$

$$-\int \frac{\sqrt{U^2 - 2 U P_2 \sin^2 E - Z^2 \sin^2 E}}{U \sin^2 E}\, dU \qquad (24)$$

Expression (23) can be integrated directly

$$-\int \frac{\cos E}{\sin^2 E}\, dU = -\frac{U \cos E}{\sin^2 E} \qquad (25)$$

To integrate (24) multiply top and bottom by
$\sqrt{U^2 - 2 U P_2 \sin^2 E - Z^2 \sin^2 E}$

$$-\int \frac{\sqrt{U^2 - 2 U P_2 \sin^2 E - Z^2 \sin^2 E}}{U \sin^2 E}\, dU =$$

$$-\int \frac{U^2 - 2 U P_2 \sin^2 E - Z^2 \sin^2 E}{\sqrt{U^2 - 2 U P_2 \sin^2 E - Z^2 \sin^2 E}\;\; U \sin^2 E}\, dU =$$

$$-\int \frac{U - P_2 \sin^2 E}{\sin^2 E \sqrt{U^2 - 2 U P_2 \sin^2 E - Z^2 \sin^2 E}}\, dU \qquad (26)$$

$$\mp \int \frac{P_2 \quad dU}{\sqrt{U^2 - 2\,U\,P_2 \sin^2 E - Z^2 \sin^2 E}} \tag{27}$$

$$\mp \int \frac{Z^2 \quad dU}{U\sqrt{U^2 - 2\,U\,P_2 \sin^2 E - Z^2 \sin^2 E}} \tag{28}$$

Expression (26) can be integrated directly

$$-\int \frac{U - P_2 \sin^2 E}{\sin^2 E \sqrt{U^2 - 2\,U\,P_2 \sin^2 E - Z^2 \sin^2 E}}\, dU =$$

$$-\frac{\sqrt{U^2 - 2\,U\,P_2 \sin^2 E - Z^2 \sin^2 E}}{\sin^2 E} \tag{29}$$

Integral (27) can be integrated by the same method that was used on integral (14)

$$\int \frac{P_2 \quad dU}{\sqrt{U^2 - 2\,U\,P_2 \sin^2 E - Z^2 \sin^2 E}} =$$

$$P_2 \operatorname{Log} \left(\sqrt{U^2 - 2\,U\,P_2 \sin^2 E - Z^2 \sin^2 E} \mp U - P_2 \sin^2 E \right) \tag{30}$$

Integral (28) results in the inverse trigonometric function

$$\int \frac{Z^2 \, dU}{U\sqrt{U^2 - 2\,U\,P_2 \sin^2 E - Z^2 \sin^2 E}} =$$

$$\frac{Z^2}{\sqrt{Z^2 \sin^2 E}} \sin^{-1} \left\{ \frac{(-2\,U\,P_2 \sin^2 E - 2\,Z^2 \sin^2 E)}{U\sqrt{4\,P_2 \sin^4 E \mp 4\,Z^2 \sin^2 E}} \right\} =$$

$$\frac{Z}{\sin E} \sin^{-1} \left\{ \frac{-\sin E\,(U\,P_2 \mp Z^2)}{\sqrt{P_2^2 \sin^2 E \mp Z^2} \quad U} \right\} \tag{31}$$

Integral (22) gives rise to the following terms

$$-\int \frac{U\cos E + \sqrt{U^2 - 2\,U P_2 \sin^2 E - Z^2 \sin^2 E}}{U \sin^2 E}\, dU =$$

$$-\frac{U\cos E + \sqrt{U^2 - 2\,U P_2 \sin^2 E - Z^2 \sin^2 E}}{\sin^2 E} \qquad (25)\ \&\ (29)$$

$$+ P_2 \operatorname{Log}\left(\sqrt{U^2 - 2\,U P_2 \sin^2 E - Z^2 \sin^2 E} + U - P_2 \sin^2 E\right) \qquad (30)$$

$$+ \frac{Z}{\sin E}\sin^{-1}\left(\frac{(\sin E)\,(U P_2 + Z^2)}{U\sqrt{P_2^2 \sin^2 E + Z^2}}\right) \qquad (31)$$

Substituting the value of (U) given by equation (19) and collecting terms expressions (25),(29), (30) and (31) become

$$-P_a$$

$$+ P_2 \operatorname{Log}(1 - \cos E)\left(\sqrt{P_2^2 + P_a^2 - 2\,P_2 P_a \cos E + Z^2} + P_a - P_2 \cos E\right)$$

$$-\frac{Z}{\sin E}\sin^{-1}\left(\frac{\sin E}{\sqrt{P_2^2 \sin^2 E + Z^2}}\right)\left(P_2 + \frac{Z^2}{\sqrt{P_2^2 + P_a^2 - 2P_2P_a\cos E + Z^2} + P_2 - P_a\cos E}\right)$$

Collecting terms and substituting the limits

$$M_{1a} = 10^{-9}\cos E \int_{P_a}^{P_b} \operatorname{Log}\left(\frac{\sqrt{P_2^2 + P_a^2 - 2\,P_2 P_a \cos E + Z^2} + P_2 - P_a \cos E}{\sqrt{P_1^2 + P_a^2 - 2\,P_1 P_a \cos E + Z^2} + P_1 - P_a \cos E}\right) dP_a$$

$$= 10^{-9}\cos E \Bigg($$

$$P_b \operatorname{Log}\left(\frac{\sqrt{P_2^2 + P_b^2 - 2\,P_2 P_b \cos E + Z^2} + P_2 - P_b \cos E}{\sqrt{P_1^2 + P_b^2 - 2\,P_1 P_b \cos E + Z^2} + P_1 - P_b \cos E}\right)$$

$$- P_a \operatorname{Log}\left(\frac{\sqrt{P_2^2 + P_a^2 - 2\,P_2 P_a \cos E + Z^2} + P_2 - P_a \cos E}{\sqrt{P_1^2 + P_a^2 - 2\,P_1 P_a \cos E + Z^2} + P_1 - P_a \cos E}\right)$$

$$+ P_2 \,\text{Log} \left(\frac{\sqrt{P_2^2 + P_b^2 - 2 P_2 P_b \cos E + Z^2} + P_b - P_2 \cos E}{\sqrt{P_2^2 + P_a^2 - 2 P_2 P_a \cos E + Z^2} + P_a - P_2 \cos E} \right)$$

$$- P_1 \,\text{Log} \left(\frac{\sqrt{P_1^2 + P_b^2 - 2 P_1 P_b \cos E + Z^2} + P_b - P_1 \cos E}{\sqrt{P_1^2 + P_a^2 - 2 P_1 P_a \cos E + Z^2} + P_a - P_1 \cos E} \right)$$

$$- \frac{Z}{\sin E} \sin^{-1} \left(\left(\frac{\sin E}{\sqrt{P_2^2 \sin^2 E + Z^2}} \right) \left(\frac{P_2 \left(\sqrt{P_2^2 + P_b^2 - 2 P_2 P_b \cos E + Z^2} + P_2 - P_b \cos E \right) + Z^2}{\sqrt{P_2^2 + P_b^2 - 2 P_2 P_b \cos E + Z^2} + P_2 - P_b \cos E} \right) \right)$$

$$+ \frac{Z}{\sin E} \sin^{-1} \left(\frac{\sin E}{\sqrt{P_2^2 \sin^2 E + Z^2}} \right) \left(\frac{P_2 \left(\sqrt{P_2^2 + P_a^2 - 2 P_2 P_a \cos E + Z^2} + P_2 - P_a \cos E \right) + Z^2}{\sqrt{P_2^2 + P_a^2 - 2 P_2 P_a \cos E + Z^2} + P_2 - P_a \cos E} \right)$$

$$+ \frac{Z}{\sin E} \sin^{-1} \left(\frac{\sin E}{\sqrt{P_1^2 \sin^2 E + Z^2}} \right) \left(\frac{P_1 \left(\sqrt{P_1^2 + P_b^2 - 2 P_1 P_b \cos E + Z^2} + P_1 - P_b \cos E \right) + Z^2}{\sqrt{P_1^2 + P_b^2 - 2 P_1 P_b \cos E + Z^2} + P_1 - P_b \cos E} \right)$$

$$+ \frac{Z}{\sin E} \sin^{-1} \left(\frac{\sin E}{\sqrt{P_1^2 \sin^2 E + Z^2}} \right) \left(\frac{P_1 \left(\sqrt{P_1^2 + P_a^2 - 2 P_1 P_a \cos E + Z^2} + P_1 - P_a \cos E \right) + Z^2}{\sqrt{P_1^2 + P_a^2 - 2 P_1 P_a \cos E + Z^2} + P_1 - P_a \cos E} \right) \Bigg) \qquad (32)$$

Equation (32) expresses the mutual inductance between any two line segments, in cylinderical coordinates. Line (1) has the coordinates, (Z = 0), (θ = 0), and the segment extends from (P_1) to (P_2). Line (A) has the coordinates (Z = Z), (ϕ = E) and the segment extends from (P_a) to (P_b) In equation (32) an expression of the form

$$\sqrt{P_2^2 \pm P_b^2 - 2 P_2 P_b \cos E \pm Z^2}$$

is the distance between the points (2) and (b). ($P_2 - P_b \cos E$) is the projection on line ~~two~~ (2) of the distance from point (2) to point (b). $\sqrt{P_2^2 \sin^2 E \pm Z^2}$ is the perpendicular distance of point (2) from line (a). (E) is the angle between the two lines. (Z) is the length of the common normal of the two lines . (P_a), (P_b), (P_1), and (P_2) are the distance of the points (a), (b), (1), and (2) from the common normal.

The expression for the mutual inductance can be given a ~~bett~~ better geometric interpretation if the distance from point (1) to ~~point~~ point (b) is written (r_{1b}) , and the length of line (1) which is ($P_2 - P_1$) is written r_{21}.

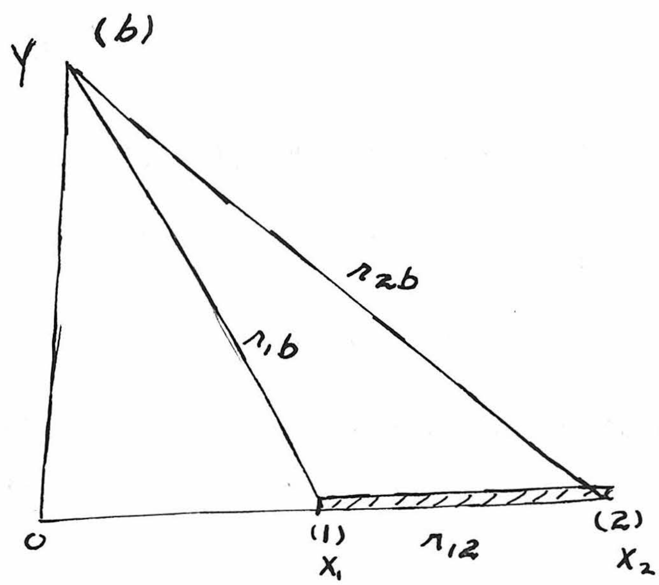


figure 3

Consider the point (b) at ($X = 0$), ($Y = Y$), the point (1) at ($X = X_1$), ($Y = 0$) and point (2) at ($X = X_2$), ($Y = 0$).

Then

$$\ell_{21} = X_2 - X_1$$

An expression in equation (32) of the form

$$\mathrm{Log}\left(\frac{\sqrt{P_2^2 + P_b^2 - 2 P_2 P_b \cos E + Z^2} + P_2 - P_b \cos E}{\sqrt{P_1^2 + P_b^2 - 2 P_1 P_b \cos E + Z^2} + P_1 - P_b \cos E} \right)$$

can then be written

$$\mathrm{Log}\left(\frac{r_{2b} + X_2}{r_{1b} + X_1} \right) \qquad (33)$$

Reffering to figure (3)

$$X_2 = \ell_{21} + X_1$$

$$X_1 = \sqrt{r_{1b}^2 - Y^2}$$

$$X_1 + \ell_{21} = \sqrt{r_{2b}^2 - Y^2}$$

$$X_1 = \frac{r_{2b}^2 - r_{1b}^2 - \ell_{21}^2}{2\ell_{21}} \qquad (34)$$

Substitute (34) in (33)

$$\mathrm{Log}\left(\frac{r_{2b} + X_2}{r_{1b} + X_1} \right) = \mathrm{Log}\left(\frac{r_{2b} + r_{1b} + r_{12}}{r_{2b} + r_{1b} - r_{12}} \right) \qquad (35)$$

The mutual inductance can thus be expressed as a function of the lengths of the conductors and the distance between the ends of the two conductors.