LINEAR FUNCTIONAL EQUATIONS ON A COMPOSITE RANGE.

Thesis by

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Introduction and Summary.

In Part I of this paper we develop systematically a general theory of linear functional equations, where the unknowns are a function and n independent variables. The Fredholm theory of integral equations , and the classical theory of linear algebraic equations are included if we specialize the equations. We may regard the set of equations as a linear transformation on the composite range consisting of the function and the n variables. It is proved that the transformations for which the bordered Fredholm determinant, which is defined in §2, do not vanish form a group. A generalization of the transformation is brought in at the end of Part I.

In Part II we apply the generalized transformation to the invariant theory of quadratic forms with continuous coefficients on the composite range.[1] The algebraic theory, and the theories worked out by A. D. Michal and T. S. Peterson (references on page 40), are special cases of this. The theory of quadratic forms with continuity of order one is reduced to the case studied earlier in Part II.

In Part III the theory worked out in Part I is applied to projective transformations in function space. In particular all the results given by L. L. Dines (reference on page 7) are obtained somewhat simpler.

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[1] Part II includes some unpublished work of Professor A. D. Michal.

It is proved for the first time that the one-parameter family of finite transformations generated by an infinitesimal projective transformation in function space forms a one-parameter continuous group.

I wish to put on record my indebtedness to Professor A. D. Michal, who suggested the problem and has kindly supervised and assisted me in carrying it through.

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Part I

Linear Functional Equations in a Function and n Independent Variables.

§1 Introduction: In the first part of this paper the theory of the set of n + 1 linear integral equations:

$$y(x) = \bar{y}(x) + \int_a^b K(x,s)\bar{y}(s)ds + \sum_{j=1}^{n} K_j(x)\bar{y}^j$$
$$y^i = \bar{y}^i + \int_a^b K^i(s)\bar{y}(s)ds + \sum_{j=1}^{n} K_j^i \bar{y}^j \qquad (i = 1, 2, \cdots, n)$$

is developed from first principles, the results being given in explicit form that will be suitable for applications.

We assume once and for all that the functions $K(x,s)$, $K_j(x)$, $K^i(s)$ are bounded and integrable in the Riemann sense on both x and s on (a,b). For most of the applications the assumption of continuity is also made. A consideration of this is deferred to §9.

We shall use the convention of writing the continuous arguments above as indices of position in the alphabet after q, and shall imply Riemann integration on (a,b) with respect to any such index which occurs as both sub- and super-script in a single term. Indices before q shall take on integral values 1 to n unless otherwise stated, and the repetition of such an index as sub- and super-script in any term shall imply summation from 1 to n. With this notation we may write the above as

$$(1.1) \qquad \begin{aligned} y^x &= \bar{y}^x + K_s^x \bar{y}^s + K_j^x \bar{y}^j \\ y^i &= \bar{y}^i + K_s^i \bar{y}^s + K_j^i \bar{y}^j \end{aligned}$$

If α and β pass over the composite range consisting of the continuous interval (a,b) followed by the discrete range $(1,n)$ we may write (1.1) as

$$(1.2) \qquad y^\alpha = \bar{y}^\alpha + K_\beta^\alpha \bar{y}^\beta$$

The summation and integration conventions do not apply to subscripts of indices.

Dines[1] has indicated an explicit solution for the case n=1, and has applied this. He gives the relations (4.1 -.8) below for that case in different notation. Hildebrandt[2] indicates that the existence theorem for inversion, the product theorem and the group property follow from the fact that the bordered Fredholm determinants are actual Fredholm determinants.

We shall prove these by direct methods analogous to those of Fredholm's classic paper.[3]

§ 2 <u>Definitions:</u> We define the function $\Delta^{x_1 \cdots x_m}_{s_1 \cdots s_m}$ as follows:

$$
\Delta^{x_1 \cdots x_m}_{s_1 \cdots s_m} = \begin{vmatrix} K^{x_1}_{s_1} & \cdots & K^{x_1}_{s_m} & K^{x_1}_{1} & \cdots & K^{x_1}_{n} \\ \vdots & & \vdots & \vdots & & \vdots \\ K^{x_m}_{s_1} & \cdots & K^{x_m}_{s_m} & K^{x_m}_{1} & \cdots & K^{x_m}_{n} \\ K^{1}_{s_1} & \cdots & K^{1}_{s_m} & 1+K^{1}_{1} & \cdots & K^{1}_{n} \\ \vdots & & \vdots & \vdots & & \vdots \\ K^{n}_{s_1} & \cdots & K^{n}_{s_m} & K^{n}_{1} & \cdots & 1+K^{n}_{n} \end{vmatrix} \qquad (2.1)
$$

Also the following functions:

$\Delta^{x\, x_1 \cdots x_m}_{i\, s_1 \cdots s_m}$ is the cofactor of K^{i}_{s} in $\Delta^{x\, x_1 \cdots x_m}_{s\, s_1 \cdots s_m}$

$\Delta^{j\, x_1 \cdots x_m}_{s\, s_1 \cdots s_m}$ is the cofactor of K^{x}_{j} in $\Delta^{x\, x_1 \cdots x_m}_{s\, s_1 \cdots s_m}$

$\Delta^{j\, x_1 \cdots x_m}_{i\, s_1 \cdots s_m}$ is the cofactor of $\delta^{i}_{j}+K^{i}_{j}$ in $\Delta^{x_1 \cdots x_m}_{s_1 \cdots s_m}$

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(1) L. L. Dines: "Projective Transformations in Function Space." Trans. Am. Math. Soc. V. 20(1919) p. 45.

(2) T. H. Hildebrandt: "On Bordered Fredholm Determinants." Bull. Am. Math. Soc. V. 26(1920) p. 400.

(3) I. Fredholm: "Sur une Classe d'Equations Functionelles." Acta Math. V. 27(1903) p. 365.

Let us define ~~as~~

~~as the cofactor of~~ ~~in~~

~~as the cofactor of~~ ~~in~~

~~as the cofactor of~~ ~~in~~

Δ as the determinant $|\delta^i_j + K^i_j|$ where δ^i_j is the usual Kronecker delta.

~~We further define~~ ~~as~~ ~~etc.~~

We notice that

$$(2.3)\qquad \Delta^j_s = -K^i_s \Delta^j_i \qquad \Delta^x_i = -K^x_j \Delta^j_i \qquad \Delta^x_s = K^x_s \Delta + K^x_j \Delta^j_s \cancel{=} K^x_s \Delta + K^i_s \Delta^x_i$$

Now define:

$$(2.41)\qquad D = \Delta + \sum_{m=1}^{\infty} \frac{1}{m!} \Delta^{x_1 \cdots x_m}_{x_1 \cdots x_m} = \sum_{m=0}^{\infty} \frac{1}{m!} \Delta^{x_1 \cdots x_m}_{x_1 \cdots x_m}$$

$$(2.42)\qquad D^x_s = \Delta^x_s + \sum_{m=1}^{\infty} \frac{1}{m!} \Delta^{x\, x_1 \cdots x_m}_{s\, x_1 \cdots x_m} = \sum_{m=0}^{\infty} \frac{1}{m!} \Delta^{x\, x_1 \cdots x_m}_{s\, x_1 \cdots x_m}$$

$$(2.43)\qquad D^x_j = -\Delta^x_j - \sum_{m=1}^{\infty} \frac{1}{m!} \Delta^{x\, x_1 \cdots x_m}_{j\, x_1 \cdots x_m} = -\sum_{m=0}^{\infty} \frac{1}{m!} \Delta^{x\, x_1 \cdots x_m}_{j\, x_1 \cdots x_m}$$

$$(2.44)\qquad D^i_s = -\Delta^i_s - \sum_{m=1}^{\infty} \frac{1}{m!} \Delta^{i\, x_1 \cdots x_m}_{s\, x_1 \cdots x_m} = -\sum_{m=0}^{\infty} \frac{1}{m!} \Delta^{i\, x_1 \cdots x_m}_{s\, x_1 \cdots x_m}$$

$$(2.45)\qquad D^i_j = D\delta^i_j - \Delta^i_j - \sum_{m=1}^{\infty} \frac{1}{m!} \Delta^{i\, x_1 \cdots x_m}_{j\, x_1 \cdots x_m} = D\delta^i_j - \sum_{m=0}^{\infty} \Delta^{i\, x_1 \cdots x_m}_{j\, x_1 \cdots x_m}$$

After Dines and Hildebrandt we call the bordered Fredholm determinant of (1.1) and $D^x_s,\ D^x_j,\ D^i_s,\ D^i_j$ its first minors.

§3 <u>Convergence of the Series Defined:</u> The general term of each of the series in (2.4) is the repeated integral of a determinant of order m + n. Now let M be such that $|K^x_s|,\ |K^x_j|,\ |K^i_s|,\ |\delta^i_j + K^i_j|$ are all less than M. This is possible since all the kernels are bounded, and there are only a finite number of them. Then by

Hadamard's lemma the general term is less in absolute value than $\dfrac{(m+n)^{\frac{m+n}{2}} M^{m+n} (b-a)^{m-p}}{m!}$

where p = 0 or 1 according to the series taken.

The dominating series thus obtained converges for all values of M, for the ratio of the (m+1)st term to the mth term is

$$\frac{M(b-a)}{m+1}\left(\frac{m+n+1}{m+n}\right)^{\frac{m+n}{2}}(m+n+1)^{1/2}$$

$$= \frac{M(b-a)}{(m+1)^{1/2}}\left[\left(1+\frac{1}{m+n}\right)^{m+n}\right]^{1/2}\left(1+\frac{n}{m+1}\right)^{1/2}$$

Now $\underset{m\to\infty}{L}\left[\left(1+\frac{1}{m+n}\right)^{m+n}\right]^{1/2} = e^{1/2}$

$\underset{m\to\infty}{L}\left(1+\frac{n}{m+1}\right)^{1/2} = 1$

therefore the limit of the ratio is $e^{1/2}M(b-a)\underset{m\to\infty}{L}\dfrac{1}{(m+1)^{1/2}} = 0.$

Hence these series converge absolutely and uniformly in (a,b), and the definitions are justified.

§4 <u>The Eight Relations:</u> If we take $\Delta^{x\,x_1\cdots x_m}_{s\,x_1\cdots x_m}$ and expand by minors of the first row, we get

$$\Delta^{x\,x_1\cdots x_m}_{s\,x_1\cdots x_m} = K^x_s\Delta^{x_1\cdots x_m}_{x_1\cdots x_m} - K^x_{x_1}\Delta^{x_1\,x_2\cdots x_m}_{s\,x_2\cdots x_m} + \cdots + (-1)^m K^x_{x_m}\Delta^{x_1\cdots x_m}_{s\,x_1\cdots x_{m-1}} + K^x_j\Delta^{j\,x_1\cdots x_m}_{s\,x_1\cdots x_m}$$

Since the x_i are variables of integration we may rearrange them, giving

$$\Delta^{x\,x_1\cdots x_m}_{s\,x_1\cdots x_m} = K^x_s\Delta^{x_1\cdots x_m}_{x_1\cdots x_m} - m K^x_t\Delta^{t\,x_1\cdots x_{m-1}}_{s\,x_1\cdots x_{m-1}} + K^x_j\Delta^{j\,x_1\cdots x_m}_{s\,x_1\cdots x_m}$$

If we multiply by $\frac{1}{m!}$, and sum from m = 1 to m = ∞,

we get $D^x_s - \Delta^x_s = K^x_s(D-\Delta) - K^x_t D^t_s + K^x_j(D^j_s - \Delta^j_s)$

Using (2.3) and transposing, we get

(4.1) $D^x_s - DK^x_s + K^x_t D^t_s + K^x_j D^j_s = 0.$

If we expand similarly by cofactors of the first column we get

(4.2) $D^x_s - DK^x_s + D^x_t K^t_s + D^x_j K^j_s = 0.$

As above let us expand $\Delta^{x\, x_1 \cdots x_m}_{i\, x_1 \cdots x_m}$ by cofactors of the first row, getting

$$\Delta^{x\, x_1 \cdots x_m}_{i\, x_1 \cdots x_m} = -m\, K^x_t \Delta^{t\, x_1 \cdots x_{m-1}}_{i\, x_1 \cdots x_{m-1}} - K^x_j \Delta^{j\, x_1 \cdots x_m}_{i\, x_1 \cdots x_m}$$

Taking $\sum_{m=1}^{\infty} \frac{1}{m!}$ of both sides, we have

$$-D^x_i - \Delta^x_i = K^x_t D^t_i + K^x_j (D^j_i - D\delta^j_i + \Delta^j_i)$$

Recalling (2.3) and transposing:

$$(4.3) \qquad D^x_i - DK^x_i + K^x_t D^t_i + K^x_j D^j_i = 0.$$

Expanding $\Delta^{j\, x_1 \cdots x_m}_{s\, x_1 \cdots x_m}$ by the first column similarly, we have

$$(4.6) \qquad D^j_s - DK^j_s + D^j_t K^t_s + D^j_\ell K^\ell_s = 0.$$

Taking the sum of members of the (m + j)th row by cofactors of the first row of $\Delta^{x\, x_1 \cdots x_m}_{s\, x_1 \cdots x_m}$, we have

$$0 = K^j_s \Delta^{x_1 \cdots x_m}_{x_1 \cdots x_m} - m\, K^j_t \Delta^{t\, x_1 \cdots x_{m-1}}_{s\, x_1 \cdots x_{m-1}} + (\delta^j_\ell + K^j_\ell) \Delta^{\ell\, x_1 \cdots x_m}_{s\, x_1 \cdots x_m}$$

Taking $\sum_{m=0}^{\infty} \frac{1}{m!}$, and changing sign, we have

$$(4.5) \qquad D^j_s - DK^j_s + K^j_t D^t_s + K^j_\ell D^\ell_s = 0$$

Similarly multiplying the (m + i)th column by cofactors of the first, we get

$$(4.4) \qquad D^x_i - DK^x_i + D^x_t K^t_i + D^x_\ell K^\ell_i = 0$$

For our final expansion of this kind sum the elements of the (m + i)th row times the cofactors of the (m + j)th row of $\Delta^{x_1 \cdots x_m}_{x_1 \cdots x_m}$ giving

$$\delta^i_j \Delta^{x_1 \cdots x_m}_{x_1 \cdots x_m} = m\, K^i_t \Delta^{t\, x_1 \cdots x_{m-1}}_{j\, x_1 \cdots x_{m-1}} + (\delta^i_\ell + K^i_\ell) \Delta^{\ell\, x_1 \cdots x_m}_{j\, x_1 \cdots x_m}$$

Taking $\sum_{m=0}^{\infty} \frac{1}{m!}$, we have

$$\delta^i_j D = -K^i_t D^t_j - (\delta^i_\ell + K^i_\ell)(D^\ell_j - D\delta^\ell_j)$$

Cancelling $D\delta^i_j = D\delta^i_\ell \delta^\ell_j$ and changing sign, we get

$$(4.7) \qquad D^i_j - DK^i_j + K^i_t D^t_j + K^i_\ell D^\ell_j = 0.$$

Similarly by columns

$$(4.8) \qquad D^i_j - DK^i_j + D^i_t K^t_j + D^i_\ell K^\ell_j = 0.$$

We collect the results of this paragraph into

Theorem I: Under the conditions in §1 and with the definitions in §2, we have the following relations:

$$(4.1)\quad D_s^x - DK_s^x + K_t^x D_s^t + K_\ell^x D_s^\ell = 0$$

$$(4.2)\quad D_s^x - DK_s^x + D_t^x K_s^t + D_\ell^x K_s^\ell = 0$$

$$(4.3)\quad D_j^x - DK_j^x + K_t^x D_j^t + K_\ell^x D_j^\ell = 0$$

$$(4.4)\quad D_j^x - DK_j^x + D_t^x K_j^t + D_\ell^x K_j^\ell = 0$$

$$(4.5)\quad D_s^i - DK_s^i + K_t^i D_s^t + K_\ell^i D_s^\ell = 0$$

$$(4.6)\quad D_s^i - DK_s^i + D_t^i K_s^t + D_\ell^i K_s^\ell = 0$$

$$(4.7)\quad D_j^i - DK_j^i + K_t^i D_j^t + K_\ell^i D_j^\ell = 0$$

$$(4.8)\quad D_j^i - DK_j^i + D_t^i K_j^t + D_\ell^i K_j^\ell = 0$$

Since the odd and even numbered relations associate naturally, we may refer to them later as (4. odd) and (4. even) respectively,

§ 5 The Inversion of the Transformation ($D \neq 0$): We May write (1.1) as $S_K(\bar{y}) = y$, a transformation from y to $\bar{y}$. If S_G is the product $S_L(S_K) = S_L \cdot S_K$, the kernels of S_G are as follows:

$$(5.1)\quad \begin{aligned} G_s^x &= L_s^x + K_s^x + L_t^x K_s^t + L_\ell^x K_s^\ell \\ G_j^x &= L_j^x + K_j^x + L_t^x K_j^t + L_\ell^x K_j^\ell \\ G_s^i &= L_s^i + K_s^i + L_t^i K_s^t + L_\ell^i K_s^\ell \\ G_j^i &= L_j^i + K_j^i + L_t^i K_j^t + L_\ell^i K_j^\ell \end{aligned}$$

We will now consider the case $D \neq 0$.

Assuming that (1.1) has a solution $(\bar{y})$, apply the transformation S_K^{-1} to (1.1) where the kernels of S_K^{-1} are

$$-\frac{D_s^x}{D},\ -\frac{D_j^x}{D},\ -\frac{D_s^i}{D},\ -\frac{D_j^i}{D}.$$

(5.2) This gives us $S_K^{-1} y = S_K^{-1} \cdot S_K \bar{y}$

Applying (5.1) to the right side of (5.2) the kernels of the product $S_K^{-1} \cdot S_K$ are $-\frac{1}{D}$ times the left hand sides of (4. even). Hence $S_K^{-1} \cdot S_K = S_0$, the identity transformation, and we have

<u>Theorem II: If a bounded and integrable solution $(\bar{y})$ of (1.1) exists when $D \neq 0$, it is unique and is given by</u>

$$(5.3) \quad \begin{aligned} \bar{y}^x &= y^x - \frac{D_s^x}{D} y^s - \frac{D_j^x}{D} y^j \\ \bar{y}^i &= y^i - \frac{D_s^i}{D} y^s - \frac{D_j^i}{D} y^j \end{aligned}$$

Substituting (5.3) in (1.1), we get

$$(5.4) \quad y \overset{?}{=} S_K \cdot S_K^{-1} \bar{y}$$

Applying (5.1), the kernels of $S_K \cdot S_K^{-1}$ are $-\frac{1}{D}$ times the left sides of (4. odd). Hence $S_K \cdot S_K^{-1} = S_0$ and (5.4) is verified. This gives us

<u>Theorem III: The equation (1.1) has a unique solution when $D \neq 0$, given by (5.3).</u>

§6 <u>Continuity and Fréchet Differentiability of Functionals of Several Functions:</u>

A functional $F^{z_1 \cdots z_m}[\varphi_1, ---, \varphi_n]$ is said to be linear and homogeneous in the n functions φ_i, if

$$(6.1) \quad F^{z_1 --- z_m}[\lambda_1 \varphi_1 + \mu_1 \psi_1, ---, \lambda_n \varphi_n + \mu_n \psi_n] = A_1 \lambda_1 + B_1 \mu_1 + --- + A_n \lambda_n + B_n \mu_n,$$

where the A's and B's are functions of the .

Now define

$$(6.2) \quad F_i^{z_1 --- z_m}[\varphi_i] = F^{z_1 --- z_m}[\varphi_1, ---, \varphi_n]\Big)_{\varphi_j = 0 \, (j \neq i)}$$

<u>Theorem IV: A necessary and sufficient condition that $F^{z_1 --- z_m}[\varphi_1, ---, \varphi_n]$ be linear and homogeneous in the n functions φ_i is that</u>

(6.3)
1) <u>$F_i^{z_1 --- z_m}[\varphi_i]$ be linear and homogeneous in φ_i</u>
2) <u>$F^{z_1 --- z_m}[\varphi_1, ---, \varphi_n] = \sum_{i=1}^{n} F_i^{z_1 --- z_m}[\varphi_i]$.</u>

Suppose (6.1) to hold.

Then let $\lambda_j = \mu_j = 0 \quad (j \neq i)$

$F_i^{z_1 \cdots z_m}[\lambda_i \varphi_i + \mu_i \psi_i] = A_i \lambda_i + B_i \mu_i$ which proves 1)

Now let $\lambda_i = 1$, $\lambda_j = 0 \quad (j \neq i)$, $\mu_j = 0$ (all j)

Then $F_i^{z_1 \cdots z_m}[\varphi_i] = A_i$

Now let $\lambda_i = 1$, $\mu_i = 0$ (all i)

$F^{z_1 \cdots z_m}[\varphi_1, \cdots, \varphi_n] = \sum_{i=1}^{m} A_i = \sum_{i=1}^{m} F_i^{z_1 \cdots z_m}[\varphi_i]$ which proves 2). Hence the condition is necessary.

Now if 1) and 2) hold, we have

$$F^{z_1 \cdots z_m}[\lambda_1 \varphi_1 + \mu_1 \psi_1, \cdots, \lambda_m \varphi_m + \mu_m \psi_m] =$$
$$\sum_{i=1}^{m} F_i^{z_1 \cdots z_m}[\lambda_i \varphi_i + \mu_i \psi_i] = \sum_{i=1}^{m} A_i \lambda_i + \mu_i B_i$$

Hence the condition is also sufficient.

Let $F^{z_1 \cdots z_m}[Y_i^{xt}, Y_j^x, Y_\ell]$ be a functional of the p functions Y_i^{xt} of two variables, of the q functions Y_j^x of one variable, and of the r independent variables Y_ℓ. Give each of the Y's an increment δY, such that η is the largest of the set $\max|\delta Y|$.

Define ΔF as

$$(6.4) \quad F^{z_1 \cdots z_m}[Y_i^{xt} + \delta Y_i^{xt}, Y_j^x + \delta Y_j^x, Y_\ell + \delta Y_\ell] - F^{z_1 \cdots z_m}[Y_i^{xt}, Y_j^x, Y_\ell]$$

We say that F is a continuous functional of the Y's at the point of p+q+r-fold function space, $(Y_i^{xt}, Y_j^x, Y_\ell)$ if for every $\epsilon > 0$ there exists a δ such that if $\eta < \delta$ $|\Delta F| < \epsilon$. We say that F is a continuous functional in a region of the above space if it is continuous at every point of the region.

The Fréchet differential $\delta F^{z_1 \cdots z_m}[Y_i^{xt}, Y_j^x, Y_\ell | \delta Y_i^{xt}, \delta Y_j^x, \delta Y_\ell]$ of F is defined to have the properties:

1) <u>It is linear and homogeneous in the δY's.</u>

2) $\left|\frac{\Delta F - \delta F}{\eta}\right|$ <u>approaches zero with η.</u>

The distinction should be noticed between F as depending on the variables Y_ℓ, and on the variable $z_1, ---, z_m$. The Y_ℓ are to be regarded as independent of the Y_i^{xt} and Y_j^x, while the z's enter as a rule as arguments of the Y_i^{xt} or Y_j^x. ~~and in general depend on the latter.~~ For this reason we regard F as a <u>functional</u> of the Y_ℓ and as a <u>function</u> of the z's. If we take the differential of F defined in the differential calculus, it is linear and homogenous in $dz_1, ---, dz_m$ and the Y_ℓ, likewise the Y_i^{xt} and Y_j^x are regarded as fixed. On the other hand, the Fréchet differential of F will not involve the dz's, but will, as defined above, be linear and homogeneous in $\delta Y_i^{xt}, \delta Y_j^x, \delta Y_\ell$.

An example is furnished by the bordered Fredholm first minor $D_s^x \begin{bmatrix} K_s^x & K_j^x \\ K_s^i & \delta_j^i + K_j^i \end{bmatrix}$ which is a function of x and s, and a functional of the K_j^i.

The above definitions and remarks readily extend to the case of a functional of n functions, each of an arbitrary number (including zero as above) of variables.

<u>Theorem V:</u> <u>Let $F^{z_1 ---z_m, x}[Y_1, ---, Y_n]$ be a functional of the n functions Y_i, each of an arbitrary number (including zero) of variables, and a function of the m+1 variables $z_1, ---, z_m, x$. Let F possess a Fréchet differential, $\delta F^{z_1 --- z_m, x}[Y_1, ---, Y_n / \delta Y_1, ---, \delta Y_n]$.</u>

Then

$$(6.5)\quad \delta \int_a^b F^{z_1 \cdots z_m, x} [Y_1, \cdots, Y_n] \, dx$$
$$= \int_a^b \delta F^{z_1 \cdots z_m, x} [Y_1, \cdots, Y_n] \delta Y_1, \cdots, \delta Y_n] \, dx$$

Let us call

$$\delta F^{z_1 \cdots z_m, x} [Y_1, \cdots, Y_n | 0, \cdots, 0, \delta Y_i, 0, \cdots, 0] \qquad \delta F_i^{z_1 \cdots z_m, x}$$

Then by Theorem IV, we have

$$\delta F^{z_1 \cdots z_m, x} [Y_1, \cdots, Y_n | \delta Y_1, \cdots, \delta Y_n] = \sum_{i=1}^n \delta F_i^{z_1 \cdots z_m, x}$$

Then

$$\int_a^b \delta F^{z_1 \cdots z_m, x} [Y_1, \cdots, Y_n | \delta Y_1, \cdots, \delta Y_n] = \sum_{i=1}^n \int_a^b \delta F_i^{z_1 \cdots z_m, x}$$

and this integral is linear and homogeneous in the δY_i.

$$\Delta \int_a^b F^{z_1 \cdots z_m, x} [Y_1, \cdots, Y_n] \, dx = \int_a^b F^{z_1 \cdots z_m, x} [Y_1 + \delta Y_1, \cdots, Y_n + \delta Y_n] \, dx$$
$$- \int_a^b F^{z_1 \cdots z_m, x} [Y_1, \cdots, Y_n] \, dx = \int_a^b \Delta F^{z_1 \cdots z_m, x} \, dx$$

But

$$\Delta F^{z_1 \cdots z_m, x} = \delta F^{z_1 \cdots z_m, x} [Y_1, \cdots, Y_n | \delta Y_1, \cdots, \delta Y_n] + \theta \max |\delta Y_i|$$

where $\theta \to 0$ with $\max |\delta Y_i|$.

Hence

$$\Delta \int_a^b F^{z_1 \cdots z_m, x} [Y_1, \cdots, Y_n] \, dx = \int_a^b \delta F^{z_1 \cdots z_m, x} [Y_1, \cdots, Y_n | \delta Y_1, \cdots, \delta Y_n] \, dx + \max|\delta Y_i| \int_a^b \theta \, dx$$

Since $\theta \to 0$ with $\max |\delta Y_i|$, so does $\int_a^b \theta \, dx$, and the theorem is proved.

Now let $F_p^{z_1 \cdots z_m} [Y_1, \cdots, Y_n]$ $(p = 1, 2, \cdots)$ be a set of functionals as before.

Define $S_P^{z_1 \cdots z_m} [Y_1, \cdots, Y_n] = \sum_{p=1}^{P} F_p^{z_1 \cdots z_m} [Y_1, \cdots, Y_n]$

If $\underset{P \to \infty}{L} S_P$ exists, we call it $S^{z_1 \cdots z_m} [Y_1, \cdots, Y_n]$. In this case define $R_P^{z_1 \cdots z_m} [Y_1, \cdots, Y_n]$ as $S - S_P$.

We say that the series $\sum_{p=1}^{\infty} F_p^{z_1 \cdots z_m}[Y_1, \cdots, Y_n]$ converges to the value $S^{z_1 \cdots z_m}[Y_1, \cdots, Y_n]$ uniformly for a region T of n-fold function space and m-dimensional space if there is an N such that whenever $p > N$, $|R_p| < \epsilon$ for every point $(Y_1, \cdots, Y_n | z_1, \cdots z_m)$ in T.

Theorem VI: If $F_p^{z_1 \cdots z_m}[Y_1, \cdots, Y_n]$ $(p = 1, 2, \cdots)$ are a set of functionals of the n functions Y_i, each of an arbitrary number (including zero) of variables, also functions of the m variables $z_1, \cdots z_m$, and if the Fréchet differential $\delta F_p^{z_1 \cdots z_m}[Y_1, \cdots, Y_n | \delta Y_1, \cdots, \delta Y_n]$ exists for each p and if the series $\sum_{p=1}^{\infty} F_p$ converges uniformly for a region T, then if the series

(6.6) $$\delta S^{z_1 \cdots z_m}[Y_1, \cdots, Y_n | \delta Y_1, \cdots, \delta Y_n] = \sum_{p=1}^{\infty} \delta F_p^{z_1 \cdots z_m}[Y_1, \cdots, Y_n | \delta Y_1, \cdots, \delta Y_n]$$

converges uniformly for T and for δY_i bounded, this series is the Fréchet differential of

$$S^{z_1 \cdots z_m}[Y_1, \cdots, Y_n] = \sum_{p=1}^{\infty} F_p^{z_1 \cdots z_m}[Y_1, \cdots, Y_n]$$

Proof: Define $\delta F_{p,i}$ to be $\delta F_p^{z_1 \cdots z_m}[Y_1, \cdots, Y_n | 0, \cdots, 0, \delta Y_i, 0, \cdots, 0]$ and δS_i to be $\delta S^{z_1 \cdots z_m}[Y_1, \cdots, Y_n | 0, \cdots, 0, \delta Y_i, 0, \cdots, 0]$.

Then $\delta S^{z_1 \cdots z_m}[Y_1, \cdots, Y_n | \delta Y_1, \cdots, \delta Y_n] = \sum_{p=1}^{\infty} \delta F_p$

$$= \sum_{p=1}^{\infty} \sum_{i=1}^{n} \delta F_{p,i} = \sum_{i=1}^{n} \sum_{p=1}^{\infty} \delta F_{p,i} = \sum_{i=1}^{n} \delta S_i$$

Hence δS is linear and homogeneous in the δY_i.

$$\delta S = \sum_{p=1}^{P} \delta F_p + {}'R_P = \delta S_P + {}'R_P$$ where S_P is defined as above and ${}'R_P$, defined by this equation, has the properties of the R_P defined above.

$$\Delta S - \delta S = \Delta S_P + \Delta R_P - \delta S_P - {}'R_P$$

Now by definition of δS_P, for every $\epsilon > 0$, there is a δ, such that $\left| \frac{\Delta S_P - \delta S_P}{\eta} \right| < \frac{\epsilon}{4}$ for all finite P, whenever max $|\delta Y_i| = \eta < \delta$.

$$\Delta R_p = R_p^{z_1 \cdots z_m}[Y_1 + \delta Y_1, \cdots, Y_n + \delta Y_n] - R_p[Y_1, \cdots, Y_n]$$

Since the convergence is uniform, there is an N_1 such that $|R_p[Y_i + \delta Y_i]|$ and $|R_p[Y_i]|$ are each less than $\frac{\epsilon\eta}{4}$ when $P > N_1$, and similarly there is an N_2 such that $|'R_p| < \frac{\epsilon\eta}{4}$ if $P > N_2$.

Now taking P greater than both N_1 and N_2, we have

$$\left|\frac{\Delta S - \delta S}{\eta}\right| \leq \left|\frac{\Delta S_p - \delta S_p}{\eta}\right| + \left|\frac{R_p[Y_i + \delta Y_i]}{\eta}\right| + \left|\frac{R_p[Y_i]}{\eta}\right| + \left|\frac{'R_p}{\eta}\right| < \epsilon$$

Hence the $\delta S^{z_1 \cdots z_m}[Y_1, \cdots, Y_n | \delta Y_1, \cdots, \delta Y_n]$ satisfies both conditions and is the Fréchet differential of S.

Theorems V and VI are brought in to show that the process of taking the Fréchet differential is commutative with integration and summation to infinity, under certain conditions.

§ 7 <u>The Fréchet Differential of the Bordered Fredholm Determinant:</u>

<u>Theorem VII: The Fréchet differential of D (§2) is given by :</u>

$$\delta D = D\,\delta K_s^s - D_t^s\,\delta K_s^t - D_j^s\,\delta K_s^j - D_j^s\,\delta K_s^j - D_j^i\,\delta K_i^j + D\,\delta K_i^i \tag{7.1}$$

$\Delta_{s_1 \cdots s_m}^{x_1 \cdots x_m}$ is a function of the kernels in the ordinary sense. Hence its Fréchet differential is given by the ordinary rule for determinants, as follows:

$$\delta\Delta_{s_1 \cdots s_m}^{x_1 \cdots x_m} = \sum_{i=1}^{m} \Delta_{s_1 \cdots (s_i) \cdots x_m}^{x_1 \cdots (x_i) \cdots x_m}\,\delta K_{s_i}^{x_i} + \sum_{\substack{i,j=1 \\ (j \neq i)}}^{n} (-1)^{i+j}\,\Delta_{s_1 \cdots (s_j) \cdots s_m}^{x_1 \cdots (x_j) \cdots x_m}\,\delta K_{s_j}^{x_i}$$

$$+ \sum_{i=1}^{m} (-1)^i\,\Delta_{s_1 \cdots \cdots s_m}^{j\,x_1 \cdots (x_i) \cdots x_m}\,\delta K_j^{x_i} + \sum_{i=1}^{m} (-1)^i\,\Delta_{j s_1 \cdots (s_i) \cdots s_m}^{x_1 \cdots x_m}\,\delta K_{s_i}^j + \Delta_{i\,x_1 \cdots x_m}^{j\,x_1 \cdots x_m}\,\delta K_j^i$$

the parentheses here indicate omission of the index contained.

Setting $s_i = x_i$, integrating, and rearranging the variables of integration, we have

$$\delta \Delta^{x_1 \cdots x_m}_{x_1 \cdots x_m} = m \Delta^{x_1 \cdots x_{m-1}}_{x_1 \cdots x_{m-1}} \delta K^s_s - m(m-1) \Delta^{s\, x_1 \cdots x_{m-2}}_{t\, x_1 \cdots x_{m-2}} \delta K^t_s$$

$$+ m \Delta^{j\, x_1 \cdots x_{m-1}}_{s\, x_1 \cdots x_{m-1}} \delta K^s_j + m \Delta^{s\, x_1 \cdots x_{m-1}}_{j\, x_1 \cdots x_{m-1}} \delta K^j_s + \Delta^{j\, x_1 \cdots x_m}_{i\, x_1 \cdots x_m} \delta K^i_j.$$

Now take $\sum\limits_{m=0}^{\infty} \frac{1}{m!}$ of both sides, and it is easily seen that the convergence is uniform if the $K's$ are all bounded. Hence by Theorem VI,

$$\delta D = D \delta K^s_s - D^s_t \delta K^t_s - D^j_s \delta K^s_j - D^s_j \delta K^j_s + (D \delta^j_i - D^j_i) \delta K^i_j$$

which is the same as (7.1).

<u>Corollary: Dividing by D we have if $D \neq 0$,</u>

$$\text{(7.2)} \qquad \frac{\delta D}{D} = \delta \log D = \delta K^s_s - \frac{D^s_t}{D} \delta K^t_s - \frac{D^j_s}{D} \delta K^s_j - \frac{D^s_j}{D} \delta K^j_s - \frac{D^j_i}{D} \delta K^i_j + \delta K^i_i$$

§ 8 The Product Theorem and the Group Property: If $S_{\bar{\bar K}}$ is the product transformation $S_{\bar K} \cdot S_K$ the kernels of $S_{\bar{\bar K}}$ are given by (5.1) as follows:

$$(8.11) \quad \bar{\bar K}^x_s = \bar K^x_s + K^x_s + \bar K^x_t K^t_s + \bar K^x_\ell K^\ell_s$$

$$(8.12) \quad \bar{\bar K}^x_j = \bar K^x_j + K^x_j + \bar K^x_t K^t_j + \bar K^x_\ell K^\ell_j .$$

$$(8.13) \quad \bar{\bar K}^i_s = \bar K^i_s + K^i_s + \bar K^i_t K^t_s + \bar K^i_\ell K^\ell_s$$

$$(8.14) \quad \bar{\bar K}^i_j = \bar K^i_j + K^i_j + \bar K^i_t K^t_j + \bar K^i_\ell K^\ell_j .$$

If D and $\bar D$, the determinants of S_K and $S_{\bar K}$ are both not zero, we can solve these for the kernels of S_K and $S_{\bar K}$.

From (8.11) and (8.13) we have

$$(8.21) \quad K^x_s = (\bar{\bar K}^x_s - \bar K^x_s) - \frac{\bar D^x_t}{\bar D}(\bar{\bar K}^t_s - \bar K^t_s) - \frac{\bar D^x_\ell}{\bar D}(\bar{\bar K}^\ell_s - \bar K^\ell_s)$$

$$(8.23) \quad K^i_s = (\bar{\bar K}^i_s - \bar K^i_s) - \frac{\bar D^i_t}{\bar D}(\bar{\bar K}^t_s - \bar K^t_s) - \frac{\bar D^i_\ell}{\bar D}(\bar{\bar K}^\ell_s - \bar K^\ell_s)$$

From (8.12) and (8.14) we have

$$(8.22) \quad K^x_j = (\bar{\bar K}^x_j - \bar K^x_j) - \frac{\bar D^x_t}{\bar D}(\bar{\bar K}^t_j - \bar K^t_j) - \frac{\bar D^x_\ell}{\bar D}(\bar{\bar K}^\ell_j - \bar K^\ell_j)$$

$$(8.24) \quad K^i_j = (\bar{\bar K}^i_j - \bar K^i_j) - \frac{\bar D^i_t}{\bar D}(\bar{\bar K}^t_j - \bar K^t_j) - \frac{\bar D^i_\ell}{\bar D}(\bar{\bar K}^\ell_j - \bar K^\ell_j)$$

Using Theorem XI below, which is independent of §6-8, we have from (8.11) and (8.12)

$$(8.31) \quad \bar K^x_s = (\bar{\bar K}^x_s - K^x_s) + (\bar{\bar K}^x_t - K^x_t)\left(-\frac{D^t_s}{D}\right) + (\bar{\bar K}^x_\ell - K^x_\ell)\left(-\frac{D^\ell_s}{D}\right)$$

$$(8.32) \quad \bar K^x_j = (\bar{\bar K}^x_j - K^x_j) + (\bar{\bar K}^x_t - K^x_t)\left(-\frac{D^t_j}{D}\right) + (\bar{\bar K}^x_\ell - K^x_\ell)\left(-\frac{D^\ell_j}{D}\right)$$

From (8.13) and (8.14) we have

$$(8.33) \quad \bar K^i_s = (\bar{\bar K}^i_s - K^i_s) + (\bar{\bar K}^i_t - K^i_t)\left(-\frac{D^t_s}{D}\right) + (\bar{\bar K}^i_\ell - K^i_\ell)\left(-\frac{D^\ell_s}{D}\right)$$

$$\bar K^i_j = (\bar{\bar K}^i_j - K^i_j) + (\bar{\bar K}^i_t - K^i_t)\left(-\frac{D^t_j}{D}\right) + (\bar{\bar K}^i_\ell - K^i_\ell)\left(-\frac{D^\ell_j}{D}\right)$$

Theorem VIII: The determinant $\bar{\bar D}$ of the transformation $S_{\bar{\bar K}} = S_{\bar K} \cdot S_K$ equals $\bar D \cdot D$.

We shall prove this for the case when D and $\bar{D}$ are both not zero. If either or both vanish, the theorem will follow from the continuity of the as functionals of the kernels.

We have by Corollary to Theorem XVI below

$$\delta \frac{\bar{\bar{D}}}{D\bar{D}} = \frac{D\bar{D}\delta\bar{\bar{D}} - \bar{\bar{D}}D\delta\bar{D} - \bar{\bar{D}}\bar{D}\delta D}{(D\bar{D})^2}$$

From (5.1) we have

$$\begin{aligned}
&\delta\bar{\bar{K}}{}^x_s = \delta\bar{K}^x_s + \delta K^x_s + \bar{K}^x_t\delta K^t_s + K^t_s\delta\bar{K}^x_t + \bar{K}^x_\ell\delta K^\ell_s + K^\ell_s\delta\bar{K}^x_\ell \\
(8.4)\quad &\delta\bar{\bar{K}}{}^x_j = \delta\bar{K}^x_j + \delta K^x_j + \bar{K}^x_t\delta K^t_j + K^t_j\delta\bar{K}^x_t + \bar{K}^x_\ell\delta K^\ell_j + K^\ell_j\delta\bar{K}^x_\ell \\
&\delta\bar{\bar{K}}{}^i_s = \delta\bar{K}^i_s + \delta K^i_s + \bar{K}^i_t\delta K^t_s + K^t_s\delta\bar{K}^i_t + \bar{K}^i_\ell\delta K^\ell_s + K^\ell_s\delta\bar{K}^i_\ell \\
&\delta\bar{\bar{K}}{}^i_j = \delta\bar{K}^i_j + \delta K^i_j + \bar{K}^i_t\delta K^t_j + K^t_j\delta\bar{K}^i_t + \bar{K}^i_\ell\delta K^\ell_j + K^\ell_j\delta\bar{K}^i_\ell
\end{aligned}$$

Substituting from (8.4) and (7.1) we have

$$\begin{aligned}
(8.5)\quad & D\bar{D}\delta\bar{\bar{D}} - \bar{\bar{D}}D\delta\bar{D} - \bar{\bar{D}}\bar{D}\delta D = \\
& D\bar{D}\bar{\bar{D}}[\delta\bar{K}^s_s + \delta K^s_s + \bar{K}^s_t\delta K^t_s + K^t_s\delta\bar{K}^s_t + \bar{K}^s_\ell\delta K^\ell_s + K^\ell_s\delta\bar{K}^s_\ell] \\
& - D\bar{D}\bar{\bar{D}}{}^s_t[\delta\bar{K}^t_s + \delta K^t_s + \bar{K}^t_u\delta K^u_s + K^u_s\delta\bar{K}^t_u + \bar{K}^t_\ell\delta K^\ell_s + K^\ell_s\delta\bar{K}^t_\ell] \\
& - D\bar{D}\bar{\bar{D}}{}^s_j[\delta\bar{K}^j_s + \delta K^j_s + \bar{K}^j_t\delta K^t_s + K^t_s\delta\bar{K}^j_t + \bar{K}^j_\ell\delta K^\ell_s + K^\ell_s\delta\bar{K}^j_\ell] \\
& - D\bar{D}\bar{\bar{D}}{}^j_s[\delta\bar{K}^s_j + \delta K^s_j + \bar{K}^s_t\delta K^t_j + K^t_j\delta\bar{K}^s_t + \bar{K}^s_\ell\delta K^\ell_j + K^\ell_j\delta\bar{K}^s_\ell] \\
& - D\bar{D}\bar{\bar{D}}{}^j_i[\delta\bar{K}^i_j + \delta K^i_j + \bar{K}^i_t\delta K^t_j + K^t_j\delta\bar{K}^i_t + \bar{K}^i_\ell\delta K^\ell_j + K^\ell_j\delta\bar{K}^i_\ell] \\
& + D\bar{D}\bar{\bar{D}}[\delta\bar{K}^i_i + \delta K^i_i + \bar{K}^i_t\delta K^t_i + K^t_i\delta\bar{K}^i_t + \bar{K}^i_\ell\delta K^\ell_i + K^\ell_i\delta\bar{K}^i_\ell] \\
& - \bar{\bar{D}}D[\bar{D}\delta\bar{K}^s_s - \bar{D}^s_t\delta\bar{K}^t_s - \bar{D}^j_s\delta\bar{K}^s_j - \bar{D}^s_j\delta\bar{K}^j_s - \bar{D}^j_i\delta\bar{K}^i_j + \bar{D}\delta\bar{K}^i_i] \\
& - \bar{\bar{D}}\bar{D}[D\delta K^s_s - D^s_t\delta K^t_s - D^j_s\delta K^s_j - D^s_j\delta K^j_s - D^j_i\delta K^i_j + D\delta K^i_i] = \\
& \delta K^s_s[D\bar{D}\bar{\bar{D}} - \bar{\bar{D}}D\bar{D}] + \delta\bar{K}^s_s[D\bar{D}\bar{\bar{D}} - \bar{\bar{D}}\bar{D}D] \\
& + \delta K^t_s[D\bar{D}\bar{\bar{D}}\bar{K}^s_t - D\bar{D}\bar{\bar{D}}{}^s_t - D\bar{D}\bar{\bar{D}}{}^s_u\bar{K}^u_t - D\bar{D}\bar{\bar{D}}{}^s_j\bar{K}^j_t + \bar{\bar{D}}\bar{D}D^s_t] \\
& + \delta\bar{K}^t_s[D\bar{D}\bar{\bar{D}}K^s_t - D\bar{D}\bar{\bar{D}}{}^s_t - D\bar{D}\bar{\bar{D}}{}^u_t K^s_u - D\bar{D}\bar{\bar{D}}{}^j_t K^s_j + \bar{\bar{D}}D\bar{D}^s_t] \\
& + \delta K^s_j[-D\bar{D}\bar{\bar{D}}{}^j_s - D\bar{D}\bar{\bar{D}}{}^j_t\bar{K}^t_s - D\bar{D}\bar{\bar{D}}{}^j_i\bar{K}^i_s + D\bar{D}\bar{\bar{D}}\bar{K}^j_s + \bar{\bar{D}}\bar{D}D^j_s] \\
& + \delta\bar{K}^s_j[-D\bar{D}\bar{\bar{D}}{}^j_s - D\bar{D}\bar{\bar{D}}{}^\ell_s K^j_\ell - D\bar{D}\bar{\bar{D}}{}^u_s K^j_u + \bar{\bar{D}}D\bar{D}^j_s + D\bar{D}\bar{\bar{D}}K^j_s]
\end{aligned}$$

$$+\delta K^j_s[-D\bar{D}\bar{\bar{D}}^s_j + D\bar{D}\bar{\bar{D}}\bar{K}^s_j - D\bar{D}\bar{\bar{D}}^s_t \bar{K}^t_j - D\bar{D}\bar{\bar{D}}^s_l \bar{K}^l_j + \bar{\bar{D}}\bar{D}D^s_j]$$
$$+\delta\bar{K}^j_s[-D\bar{D}\bar{\bar{D}}^s_j + D\bar{D}\bar{\bar{D}}K^s_j - D\bar{D}\bar{\bar{D}}^t_j K^s_t - D\bar{D}\bar{\bar{D}}^l_j K^s_l + \bar{\bar{D}}D\bar{D}^s_j]$$
$$+\delta K^i_i[D\bar{D}\bar{\bar{D}} - \bar{\bar{D}}D\bar{D}] + \delta\bar{K}^i_i[D\bar{D}\bar{\bar{D}} - \bar{\bar{D}}\bar{D}D]$$
$$+\delta K^i_j[-D\bar{D}\bar{\bar{D}}^j_i - D\bar{D}\bar{\bar{D}}^j_s \bar{K}^s_i - D\bar{D}\bar{\bar{D}}^j_l \bar{K}^l_i + D\bar{D}\bar{\bar{D}}\bar{K}^j_i + \bar{\bar{D}}\bar{D}D^j_i]$$
$$+\delta\bar{K}^i_j[-D\bar{D}\bar{\bar{D}}^s_i K^j_s - D\bar{D}\bar{\bar{D}}^j_i - D\bar{D}\bar{\bar{D}}^l_i K^j_l + D\bar{D}\bar{\bar{D}}K^j_i + \bar{\bar{D}}D\bar{D}^j_i]$$

In the coefficients of the variations of the K's eliminate the $\bar{K}$'s by (8.3). This gives for that of δK^t_s, after removing $\bar{D}$

$$\bar{\bar{D}}[D\bar{K}^s_t - DK^s_t - \bar{K}^s_u D^u_t + K^s_u D^u_t - \bar{K}^s_l D^l_t + K^s_l D^l_t]$$
$$-\bar{\bar{D}}^s_u[D\bar{K}^u_t - DK^u_t - \bar{K}^u_x D^x_t + K^u_x D^x_t - \bar{K}^u_l D^l_t + K^u_l D^l_t]$$
$$-\bar{\bar{D}}^s_j[D\bar{K}^j_t - DK^j_t - \bar{K}^j_u D^u_t + K^j_u D^u_t - \bar{K}^j_l D^l_t + K^j_l D^l_t]$$
$$-D\bar{\bar{D}}^s_t + \bar{\bar{D}}D^s_t =$$
$$-D[\bar{\bar{D}}^s_t - \bar{\bar{D}}\bar{K}^s_t + \bar{\bar{D}}^s_u\bar{K}^u_t + \bar{\bar{D}}^s_j\bar{K}^j_t]$$
$$+\bar{\bar{D}}[D^s_t - DK^s_t + K^s_u D^u_t + K^s_l D^l_t]$$
$$+D^u_t[\bar{\bar{D}}^s_u - \bar{\bar{D}}\bar{K}^s_u + \bar{\bar{D}}^s_x\bar{K}^x_u + \bar{\bar{D}}^s_j\bar{K}^j_u]$$
$$-\bar{\bar{D}}^s_u[D^u_t - DK^u_t + K^u_x D^x_t + K^u_l D^l_t]$$
$$+D^l_t[\bar{\bar{D}}^s_l - \bar{\bar{D}}\bar{K}^s_l + \bar{\bar{D}}^s_u\bar{K}^u_l + \bar{\bar{D}}^s_j\bar{K}^j_l]$$
$$-\bar{\bar{D}}^s_l[D^l_t - DK^l_t + K^l_u D^u_t + K^l_j D^j_t] = 0$$

The coefficient of δK^s_j is $\bar{D}$ times

$$+\bar{\bar{D}}[D\bar{K}^j_s - DK^j_s - \bar{K}^j_u D^u_s + K^j_u D^u_s - \bar{K}^j_l D^l_s + K^j_l D^l_s]$$
$$-\bar{\bar{D}}^j_t[D\bar{K}^t_s - DK^t_s - \bar{K}^t_u D^u_s + K^t_u D^u_s - \bar{K}^t_l D^l_s + K^t_l D^l_s]$$
$$-\bar{\bar{D}}^j_i[D\bar{K}^i_s - DK^i_s - \bar{K}^i_u D^u_s + K^i_u D^u_s - \bar{K}^i_l D^l_s + K^i_l D^l_s]$$
$$-D\bar{\bar{D}}^j_s + \bar{\bar{D}}D^j_s =$$
$$\bar{\bar{D}}[D^j_s - DK^j_s + K^j_u D^u_s + K^j_l D^l_s]$$
$$-\bar{\bar{D}}^j_u[D^u_s - DK^u_s + K^u_t D^t_s + K^u_l D^l_s]$$
$$-\bar{\bar{D}}^j_i[D^i_s - DK^i_s + K^i_u D^u_s + K^i_l D^l_s]$$
$$+D^u_s[\bar{\bar{D}}^j_u - \bar{\bar{D}}\bar{K}^j_u + \bar{\bar{D}}^j_t\bar{K}^t_u + \bar{\bar{D}}^j_i\bar{K}^i_u$$
$$+D^l_s[\bar{\bar{D}}^j_l - \bar{\bar{D}}\bar{K}^j_l + \bar{\bar{D}}^j_t\bar{K}^t_l + \bar{\bar{D}}^j_i\bar{K}^i_l]$$
$$-D[\bar{\bar{D}}^j_s - \bar{\bar{D}}\bar{K}^j_s + \bar{\bar{D}}^j_t\bar{K}^t_s + \bar{\bar{D}}^j_i\bar{K}^i_s] = 0$$

The coefficient of δK^j_s is $\bar{D}$ times

$$\bar{\bar{D}}[D\bar{\bar{K}}^s_j - DK^s_j - \bar{\bar{K}}^s_t D^t_j + K^s_t D^t_j - \bar{\bar{K}}^s_l D^l_j + K^s_l D^l_j]$$
$$-\bar{\bar{D}}^s_t[D\bar{\bar{K}}^t_j - DK^t_j - \bar{\bar{K}}^t_u D^u_j + K^t_u D^u_j - \bar{\bar{K}}^t_l D^l_j + K^t_l D^l_j]$$
$$-\bar{\bar{D}}^s_l[D\bar{\bar{K}}^l_j - DK^l_j - \bar{K}^l_u D^u_j + K^l_u D^u_j - \bar{\bar{K}}^l_i D^i_j + K^l_i D^i_j]$$
$$-D\bar{\bar{D}}^s_j + \bar{\bar{D}} D^s_j =$$
$$\bar{D}[D^s_j - DK^s_j + K^s_t D^t_j + K^s_l D^l_j]$$
$$-\bar{\bar{D}}^s_u[D^u_j - DK^t_j + K^t_u D^u_j + K^t_l D^l_j]$$
$$-\bar{\bar{D}}^s_l[D^l_j - DK^l_j + K^l_u D^u_j + K^l_i D^i_j]$$
$$-D[\bar{\bar{D}}^s_j - \bar{\bar{D}}\bar{K}^s_j + \bar{\bar{D}}^s_t \bar{\bar{K}}^t_j + \bar{\bar{D}}^s_l \bar{\bar{K}}^l_j]$$
$$+D^u_j[\bar{\bar{D}}^s_u - \bar{\bar{D}}\bar{\bar{K}}^s_u + \bar{\bar{D}}^s_t \bar{\bar{K}}^t_u + \bar{\bar{D}}^s_l \bar{\bar{K}}^l_u]$$
$$+D^l_j[\bar{\bar{D}}^s_l - \bar{D}\bar{\bar{K}}^s_l + \bar{\bar{D}}^s_t \bar{\bar{K}}^t_l + \bar{\bar{D}}^s_i \bar{\bar{K}}^i_l] = 0.$$

The coefficient of δK^i_j is $\bar{D}$ times

$$\bar{\bar{D}}[D\bar{\bar{K}}^j_i - DK^j_i - \bar{\bar{K}}^j_t D^t_i + K^j_t D^t_i - \bar{\bar{K}}^j_l D^l_i + K^j_l D^l_i]$$
$$-\bar{\bar{D}}^j_s[D\bar{\bar{K}}^s_i - DK^s_i - \bar{\bar{K}}^s_t D^t_i + K^s_t D^t_i - \bar{\bar{K}}^s_l D^l_i + K^s_l D^l_i]$$
$$-\bar{\bar{D}}^j_l[D\bar{\bar{K}}^l_i - DK^l_i - \bar{K}^l_t D^t_i + K^l_t D^t_i - \bar{\bar{K}}^l_k D^k_i + K^l_k D^k_i]$$
$$-D\bar{\bar{D}}^j_i + \bar{\bar{D}} D^j_i =$$
$$\bar{D}[D^j_i - DK^j_i + K^j_t D^t_i + K^j_l D^l_i]$$
$$-\bar{\bar{D}}^j_s[D^s_i - DK^s_i + K^s_t D^t_i + K^s_l D^l_i]$$
$$-\bar{\bar{D}}^j_l[D^l_i - DK^l_i + K^l_t D^t_i + K^l_k D^k_i]$$
$$-D[\bar{\bar{D}}^j_i - \bar{D}\bar{\bar{K}}^j_i + \bar{\bar{D}}^j_t \bar{\bar{K}}^t_i + \bar{\bar{D}}^j_l \bar{\bar{K}}^l_i]$$
$$+D^t_i[\bar{\bar{D}}^j_t - \bar{D}\bar{\bar{K}}^j_t + \bar{\bar{D}}^j_s \bar{\bar{K}}^s_t + \bar{\bar{D}}^j_l \bar{\bar{K}}^l_t]$$
$$+D^l_i[\bar{\bar{D}}^j_l - \bar{D}\bar{\bar{K}}^j_l + \bar{\bar{D}}^j_t \bar{\bar{K}}^t_l + \bar{\bar{D}}^j_k \bar{\bar{K}}^k_l] = 0.$$

In the coefficients of the variations of the $\bar{K}$'s, eliminate the K's by (8.2). The coefficient of $\delta\bar{K}^t_s$ is

D times

$$\bar{\bar D}[\bar D \bar{\bar K}^s_t - \bar D \bar K^s_t - \bar D^s_u \bar{\bar K}^u_t + \bar D^s_u \bar K^u_t - \bar D^s_\ell \bar{\bar K}^\ell_t + \bar D^s_\ell \bar K^\ell_t]$$
$$-\bar{\bar D}^u_t[\bar D \bar{\bar K}^s_u - \bar D \bar K^s_u - \bar D^s_x \bar{\bar K}^x_u + \bar D^s_x \bar K^x_u - \bar D^s_\ell \bar{\bar K}^\ell_u + \bar D^s_\ell \bar K^\ell_u]$$
$$-\bar{\bar D}^j_t[\bar D \bar{\bar K}^s_j - \bar D \bar K^s_j - \bar D^s_u \bar{\bar K}^u_j + \bar D^s_u \bar K^u_j - \bar D^s_\ell \bar{\bar K}^\ell_j + \bar D^s_\ell \bar K^\ell_j]$$
$$-\bar D \bar{\bar D}^s_t + \bar{\bar D} \bar D^s_t =$$
$$\bar{\bar D}[\bar D^s_t - \bar D \bar K^s_t + \bar D^s_u \bar K^u_t + \bar D^s_\ell \bar K^\ell_t]$$
$$-\bar{\bar D}^u_t[\bar D^s_u - \bar D \bar K^s_u + \bar D^s_x \bar K^x_u + \bar D^s_\ell \bar K^\ell_u]$$
$$-\bar{\bar D}^j_t[\bar D^s_j - \bar D \bar K^s_j + \bar D^s_u \bar K^u_j + \bar D^s_\ell \bar K^\ell_j]$$
$$-\bar D[\bar{\bar D}^s_t - \bar{\bar D} \bar{\bar K}^s_t + \bar{\bar K}^s_u \bar{\bar D}^u_t + \bar{\bar K}^s_j \bar{\bar D}^j_t]$$
$$+\bar D^s_u[\bar{\bar D}^u_t - \bar{\bar D} \bar{\bar K}^u_t + \bar{\bar K}^u_x \bar{\bar D}^x_t + \bar{\bar K}^u_j \bar{\bar D}^j_t]$$
$$+\bar D^s_j[\bar{\bar D}^j_t - \bar{\bar D} \bar{\bar K}^j_t + \bar{\bar K}^j_u \bar{\bar D}^u_t + \bar{\bar K}^j_\ell \bar{\bar D}^\ell_t] = 0.$$

The coefficient of $\delta \bar{K}^s_j$ is D times

$$\bar{\bar D}[\bar D \bar{\bar K}^j_s - \bar D \bar K^j_s - \bar D^j_t \bar{\bar K}^t_s + \bar D^j_t \bar K^t_s - \bar D^j_\ell \bar{\bar K}^\ell_s + \bar D^j_\ell \bar K^\ell_s]$$
$$-\bar{\bar D}^u_s[\bar D \bar{\bar K}^j_u - \bar D \bar K^j_u - \bar D^j_t \bar{\bar K}^t_u + \bar D^j_t \bar K^t_u - \bar D^j_\ell \bar{\bar K}^\ell_u + \bar D^j_\ell \bar K^\ell_u]$$
$$-\bar{\bar D}^\ell_s[\bar D \bar{\bar K}^j_\ell - \bar D \bar K^j_\ell - \bar D^j_t \bar{\bar K}^t_\ell + \bar D^j_t \bar K^t_\ell - \bar D^j_i \bar{\bar K}^i_\ell + \bar D^j_i \bar K^i_\ell]$$
$$-\bar D \bar{\bar D}^j_s + \bar{\bar D} \bar D^j_s =$$
$$\bar{\bar D}[\bar D^j_s - \bar D \bar K^j_s + \bar D^j_t \bar K^t_s + \bar D^j_\ell \bar K^\ell_s]$$
$$-\bar{\bar D}^u_s[\bar D^j_u - \bar D \bar K^j_u + \bar D^j_t \bar K^t_u + \bar D^j_\ell \bar K^\ell_u]$$
$$-\bar{\bar D}^\ell_s[\bar D^j_\ell - \bar D \bar K^j_\ell + \bar D^j_t \bar K^t_\ell + \bar D^j_i \bar K^i_\ell]$$
$$-\bar D[\bar{\bar D}^j_s - \bar{\bar D} \bar{\bar K}^j_s + \bar{\bar K}^j_u \bar{\bar D}^u_s + \bar{\bar K}^j_\ell \bar{\bar D}^\ell_s]$$
$$+\bar D^j_u[\bar{\bar D}^u_s - \bar{\bar D} \bar{\bar K}^u_s + \bar{\bar K}^u_t \bar{\bar D}^t_s + \bar{\bar K}^u_\ell \bar{\bar D}^\ell_s]$$
$$+\bar D^j_\ell[\bar{\bar D}^\ell_s - \bar{\bar D} \bar{\bar K}^\ell_s + \bar{\bar K}^\ell_u \bar{\bar D}^u_s + \bar{\bar K}^\ell_i \bar{\bar D}^i_s] = 0$$

The coefficient of $\delta \bar{K}^j_s$ is D times

$$\bar{\bar D}[\bar D \bar{\bar K}^s_j - \bar D \bar K^s_j - \bar D^s_t \bar{\bar K}^t_j + \bar D^s_t \bar K^t_j - \bar D^s_\ell \bar{\bar K}^\ell_j + \bar D^s_\ell \bar K^\ell_j] +$$

$$-\bar{\bar{D}}^t_j[\bar D \bar{\bar K}^s_t - \bar D \bar K^s_t - \bar D^s_u \bar{\bar K}^u_t + \bar D^s_u \bar K^u_t - \bar D^s_\ell \bar{\bar K}^\ell_t + \bar D^s_\ell \bar K^\ell_t]$$

$$-\bar{\bar{D}}^\ell_j[\bar D \bar{\bar K}^s_\ell - \bar D \bar K^s_\ell - \bar D^s_u \bar{\bar K}^u_\ell + \bar D^s_u \bar K^u_\ell - \bar D^s_i \bar{\bar K}^i_\ell + \bar D^s_i \bar K^i_\ell]$$

$$-\bar D \bar{\bar D}^s_j + \bar{\bar D}\, \bar D^s_j =$$

$$\bar{\bar D}[\bar D^s_j - \bar D \bar K^s_j + \bar D^s_t \bar K^t_j + \bar D^s_\ell \bar K^\ell_j]$$

$$-\bar{\bar D}^t_j[\bar D^s_t - \bar D \bar K^s_t + \bar D^s_u \bar K^u_t + \bar D^s_\ell \bar K^\ell_t]$$

$$-\bar{\bar D}^\ell_j[\bar D^s_\ell - \bar D \bar K^s_\ell + \bar D^s_u \bar K^u_\ell + \bar D^s_i \bar K^i_\ell]$$

$$-\bar D[\bar{\bar D}^s_j - \bar{\bar D} \bar{\bar K}^s_j + \bar{\bar K}^s_u \bar{\bar D}^u_j + \bar{\bar K}^s_\ell \bar{\bar D}^\ell_j]$$

$$+\bar D^s_t[\bar{\bar D}^t_j - \bar{\bar D} \bar{\bar K}^t_j + \bar{\bar K}^t_u \bar{\bar D}^u_j + \bar{\bar K}^t_\ell \bar{\bar D}^\ell_j]$$

$$+\bar D^s_\ell[\bar{\bar D}^\ell_j - \bar{\bar D} \bar{\bar K}^\ell_j + \bar{\bar K}^\ell_u \bar{\bar D}^u_j + \bar{\bar K}^\ell_i \bar{\bar D}^i_j] = 0$$

The coefficient of $\delta \bar K^i_j$ is D times

$$\bar{\bar D}[\bar D \bar{\bar K}^j_i - \bar D \bar K^j_i - \bar D^j_t \bar{\bar K}^t_i + \bar D^j_t \bar K^t_i - \bar D^j_k \bar{\bar K}^k_i + \bar D^j_k \bar K^k_i]$$

$$-\bar{\bar D}^s_i[\bar D \bar{\bar K}^j_s - \bar D \bar K^j_s - \bar D^j_t \bar{\bar K}^t_s + \bar D^j_t \bar K^t_s - \bar D^j_k \bar{\bar K}^k_s + \bar D^j_k \bar K^k_s]$$

$$-\bar{\bar D}^\ell_i[\bar D \bar{\bar K}^j_\ell - \bar D \bar K^j_\ell - \bar D^j_t \bar{\bar K}^t_\ell + \bar D^j_t \bar K^t_\ell - \bar D^j_k \bar{\bar K}^k_\ell + \bar D^j_k \bar K^k_\ell]$$

$$-\bar D \bar{\bar D}^j_i + \bar{\bar D}\, \bar D^j_i =$$

$$\bar{\bar D}[\bar D^j_i - \bar D \bar K^j_i + \bar D^j_t \bar K^t_i + \bar D^j_k \bar K^k_i]$$

$$-\bar{\bar D}^s_i[\bar D^j_s - \bar D \bar K^j_s + \bar D^j_t \bar K^t_s + \bar D^j_k \bar K^k_s]$$

$$-\bar{\bar D}^\ell_i[\bar D^j_\ell - \bar D \bar K^j_\ell + \bar D^j_t \bar K^t_\ell + \bar D^j_k \bar K^k_\ell]$$

$$-\bar D[\bar{\bar D}^j_i - \bar{\bar D} \bar{\bar K}^j_i + \bar{\bar K}^j_t \bar{\bar D}^t_i + \bar{\bar K}^j_\ell \bar{\bar D}^\ell_i]$$

$$+\bar D^j_s[\bar{\bar D}^s_i - \bar{\bar D} \bar{\bar K}^s_i + \bar{\bar K}^s_t \bar{\bar D}^t_i + \bar{\bar K}^s_\ell \bar{\bar D}^\ell_i]$$

$$+\bar D^j_\ell[\bar{\bar D}^\ell_i - \bar{\bar D} \bar{\bar K}^\ell_i + \bar{\bar K}^\ell_t \bar{\bar D}^t_i + \bar{\bar K}^\ell_k \bar{\bar D}^k_i] = 0.$$

Hence the right hand side of (8.5) vanishes, and we have

$$\delta \frac{\bar{\bar D}}{D \bar D} = 0$$

Referring to Theorem XVIII below, we have $\bar{\bar D} = C \bar D D$ where C is a constant.

If we make all the kernels of S_K and $S_{\bar{K}}$ zero, those of $S_{\bar{\bar{K}}}$ also are zero, and $D = \bar{D} = \bar{\bar{D}} = 1$.

Hence $C = 1$ and

$$\bar{\bar{D}} = D \cdot \bar{D} \tag{8.6}$$

<u>Theorem IX: The set of all transformations of type (1.1), whose bordered Fredholm determinants are not zero form a group.</u>

The product of two such transformations (D's unrestricted) is a third, with kernels given by (5.1). By Theorem VIII the product of two members of the set with determinants not zero is also a member. This shows the transitive property.

It is readily verified that such transformations are associative.

The identity transformation S_0 exists, whose kernels are all zero. Its determinant is unity.

The unique inverse for $D \neq 0$ exists, and is given explicitly by Theorem III.

§ 9 <u>The Case of Continuous Functions:</u> We see readily from (1.1) that if K_s^x and K_j^x are continuous functions of x, then the transformation S_K upon $(\bar{y}^x, \bar{y}^i)$ where $\bar{y}^x$ is continuous, gives (y^x, y^i) where y^x is continuous. If in addition K_s^i is continuous, the adjoint transformation,

$$(9.1) \qquad \begin{aligned} z_s &= \bar{z}_s + K_s^x \bar{z}_x + K_s^i \bar{z}_i \\ z_j &= \bar{z}_j + K_j^x \bar{z}_x + K_j^i \bar{z}_i \end{aligned}$$

applied to $(\bar{z}_s, \bar{z}_j)$ where $\bar{z}_s$ is continuous, will give (z_s, z_j) where z_s is continuous.

<u>Theorem X: If the kernels K_s^x, K_j^x, K_s^i are continuous in both x and s $(a \le x, s \le b)$, and if $D \ne 0$ then:</u>

1) <u>If $(y^x, y^i) = S_K(\bar{y}^x, \bar{y}^i)$ where y^x is continuous in (a, b), then $\bar{y}^x$ in the solution $(\bar{y}^x, \bar{y}^i) = S_K^{-1}(y^x, y^i)$ is continuous in (a, b).</u>

2) <u>If $(z_s, z_j) = \tilde{S}_K(\bar{z}_s, \bar{z}_j)$ where z_s is continuous in (a, b), then $\bar{z}_s$ in the solution $(\bar{z}_s, \bar{z}_j) = \tilde{S}_K^{-1}(z_s, z_j)$ is continuous in (a, b).</u>

The integrated algebraic determinants used in defining the kernels of S_K^{-1} will be continuous, as they are the integrals of polynomials in the kernels of S_K. By § 3, the infinite series of these integrated determinants converge uniformly in (a, b), so that their sums, which, after dividing by the constant $-D$, are the kernels of S_K^{-1} are continuous in (a, b). Hence $\bar{y}^x = y^x - \frac{D_s^x}{D} y^s - \frac{D_j^x}{D} y^j$ is continuous.

For 2) we see that by interchanging sub-and superscripts, which is a change in notation only, we have a system of type (1.1) with same restrictions as in 1), so that 2) follows from 1), if $\tilde{D} \neq 0$ which is seen to follow from $D \neq 0$ by Theorem XI.

<u>Theorem XI: If $D \neq 0$ for the system (1.1), then the system (9.1) has the same bordered Fredholm determinant and the unique solution is given by :</u>

$$(9.2) \qquad \begin{aligned} \bar{z}_s &= z_s - \frac{D_s^x}{D} z_x - \frac{D_s^i}{D} z_i \\ \bar{z}_j &= z_j - \frac{D_j^x}{D} z_x - \frac{D_j^i}{D} z_i \end{aligned}$$

This may be stated more concisely: The reciprocal of the adjoint is the adjoint of the reciprocal.

By interchanging rows and columns in the determinants which are used to define D, it readily follows that $\tilde{D} = D$. Hence if $D \neq 0$, a unique solution exists. Applying $\tilde{S}_K$ to $(\bar{z}_s, \bar{z}_j)$ in (9.2) we have $(z_s, z_j) = \tilde{S}_{K'} \tilde{S}_K^{-1}(z_s, z_j)$. The kernels of the product transformation $\tilde{S}_{K'} \tilde{S}_K^{-1}$ are given by $-\frac{1}{D}$ times the left hand sides of (4. even). Hence (9.2) is the solution.

§10 <u>Transformations of the Third Kind:</u> Consider a transformation of the third kind on our composite range, as follows:

$$(10.1) \qquad \begin{aligned} y^x &= K^x \bar{y}^x + K_s^x \bar{y}^s + K_j^x \bar{y}^j \\ y^i &= \qquad\quad\; K_s^i \bar{y}^s + K_j^i \bar{y}^j \end{aligned}$$

If $K^x = 1$ and if K_j^i is the old K_j^i (1.1) plus the Kronecker delta δ_j^i, this reduces to (1.1), which we shall refer to as a transformation of second kind. If $K^x \equiv 0$, this is a transformation of first kind, with which we are not concerned. In fact we assume for what follows that K^x does not vanish in (a, b).

This is a generalization to the composite range of the Fredholm transformation of third kind used in an unpublished paper by A. D. Michal and T. S. Peterson.[1]

The product of two such transformations, $S_L^3 \cdot S_K^3$ is also a transformation of third kind S_G^3, whose kernels are given by

$$(10.21) \quad G^x = L^x K^x$$

$$(10.22) \quad G^x_s = L^x K^x_s + L^x_s K^{(s)} + L^x_t K^t_s + L^x_\ell K^\ell_s$$

$$(10.23) \quad G^x_j = L^x K^x_j + L^x_t K^t_j + L^x_\ell K^\ell_j$$

$$(10.24) \quad G^i_s = L^i_s K^{(s)} + L^i_t K^t_s + L^i_\ell K^\ell_s$$

$$(10.25) \quad G^i_j = L^i_t K^t_j + L^i_\ell K^\ell_j$$

Letting $K^x \bar{y}^x = \bar{\varphi}^x$ in (10.1) we have

$$(10.3) \quad \begin{aligned} y^x &= \bar{\varphi}^x + K^x_s/K^s \,\bar{\varphi}^s + K^x_j \bar{y}^j \\ y^i &= K^i_s/K^s \,\bar{\varphi}^s + K^i_j \bar{y}^j \end{aligned}$$

which is a transformation of second kind from $(\bar{\varphi}^x, \bar{y}^i)$ to (y^x, y^i). If the D's are the bordered Fredholm determinant and first minors of $\begin{pmatrix} K^x_s/K^{(s)} & K^x_j \\ K^i_s/K^{(s)} & K^i_j \end{pmatrix}$ and if $D \neq 0$, we have the unique inverse of (10.3),

$$\begin{aligned} \bar{\varphi}^x &= y^x - \frac{D^x_s}{D} y^s - \frac{D^x_j}{D} y^j \\ \bar{y}^i &= y^i - \frac{D^t_s}{D} y^s - \frac{D^i_j}{D} y^j = -\frac{D^i_s}{D} y^s + \frac{D\delta^i_j - D^i_j}{D} y^j \end{aligned}$$

Dividing the upper equation by K^x, we have the unique inverse of (10.1)

$$(10.4) \quad \begin{aligned} \bar{y}^x &= y^x/K^x - \frac{D^x_s/K^x}{D} y^s - \frac{D^x_j/K^x}{D} y^j \\ \bar{y}^i &= -\frac{D^i_s}{D} y^s + \frac{D\delta^i_j - D^i_j}{D} y^j \end{aligned}$$

which is likewise a transformation of third kind.

The relations derived from §4 for this case are:

$$(10.51) \quad K^{(s)} D^x_s - D K^x_s + K^{(s)}/K^t \, K^x_t D^t_s + K^{(s)} K^x_\ell D^\ell_s = 0$$

$$(10.52) \quad D^x_s K^{(s)} - D K^x_s + D^x_t K^t_s + D^x_\ell K^\ell_s = 0$$

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(1) Reference on page 40.

$$(10.53)\quad D^{x}_{j} + K^{x}_{\ell}(D^{\ell}_{j} - D\delta^{\ell}_{j}) + \frac{K^{x}_{t}}{K^{t}} D^{t}_{j} = 0$$

$$(10.54)\quad -DK^{x}_{j} + D^{x}_{t} K^{t}_{j} + D^{x}_{\ell} K^{\ell}_{j} = 0$$

$$(10.55)\quad D^{i}_{s} K^{(s)} + \frac{K^{(s)}}{K^{t}} K^{i}_{t} D^{t}_{s} + K^{(s)} K^{i}_{\ell} D^{\ell}_{s} = 0$$

$$(10.56)\quad D^{i}_{s} K^{(s)} + (D^{i}_{\ell} - D\delta^{i}_{\ell}) K^{\ell}_{s} + D^{i}_{t} K^{t}_{s} = 0$$

$$(10.57)\quad K^{i}_{\ell}(D^{\ell}_{j} - D\delta^{\ell}_{j}) + K^{i}_{t} D^{t}_{j} = -D\delta^{i}_{j}$$

$$(10.58)\quad (D^{i}_{\ell} - D\delta^{i}_{\ell}) K^{\ell}_{j} + D^{i}_{t} K^{t}_{j} = -D\delta^{i}_{j}$$

§11 <u>The Product Theorem and Group Property for Transformations of Third Kind:</u>

<u>Theorem XII: The determinant of the transformation S_G determined by (10.2) is the product of the determinants of S_K and S_L.</u>

That is:

$$(11.1)\quad D\begin{bmatrix} G^{x}_{s}/G^{(s)} & G^{x}_{j} \\ G^{i}_{s}/G^{(s)} & G^{i}_{j} \end{bmatrix} = D\begin{bmatrix} L^{x}_{s}/L^{(s)} & L^{x}_{j} \\ L^{i}_{s}/L^{(s)} & L^{i}_{j} \end{bmatrix} \cdot D\begin{bmatrix} K^{x}_{s}/K^{(s)} & K^{x}_{j} \\ K^{i}_{s}/K^{(s)} & K^{i}_{j} \end{bmatrix}$$

From (10.2) we have

$$(11.21)\quad G^{x}_{s}/G^{(s)} = K^{x}_{s}/K^{(s)}\, L^{x}/L^{s} + L^{x}_{s}/L^{(s)} + L^{x}_{t}/L^{t}\, K^{t}_{s}/K^{(s)}\, L^{t}/L^{s} + L^{x}_{\ell}\, K^{\ell}_{s}/K^{(s)}\, 1/L^{s}$$

$$(11.22)\quad G^{x}_{j} = K^{x}_{j} L^{x} + L^{x}_{t}/L^{t}\, K^{t}_{j} L^{t} + L^{x}_{\ell} K^{\ell}_{j}$$

$$(11.23)\quad G^{i}_{s}/G^{(s)} = L^{i}_{s}/L^{(s)} + L^{i}_{t}/L^{t}\, K^{t}_{s}/K^{(s)}\, L^{t}/L^{s} + L^{i}_{\ell}\, K^{\ell}_{s}/K^{(s)}\, 1/L^{s}$$

$$(11.24)\quad G^{i}_{j} = L^{i}_{t}/L^{t}\, K^{t}_{j} L^{t} + L^{i}_{\ell} K^{\ell}_{j}$$

which may be written

$$\begin{pmatrix} G^{x}_{s}/G^{(s)} & G^{x}_{j} \\ G^{i}_{s}/G^{(s)} & G^{i}_{j} \end{pmatrix} = \begin{pmatrix} L^{x}_{s}/L^{(s)} & L^{x}_{j} \\ L^{i}_{s}/L^{(s)} & L^{i}_{j} \end{pmatrix} \cdot \begin{pmatrix} K^{x}_{s}/K^{(s)}\, L^{x}/L^{s} & K^{x}_{j} L^{x} \\ K^{i}_{s}/K^{(s)}\, 1/L^{s} & K^{i}_{j} \end{pmatrix}$$

Hence by Theorem VIII, we have

$$(11.3)\quad D\begin{bmatrix} G^{x}_{s}/G^{(s)} & G^{x}_{j} \\ G^{i}_{s}/G^{(s)} & G^{i}_{j} \end{bmatrix} = D\begin{bmatrix} L^{x}_{s}/L^{(s)} & L^{x}_{j} \\ L^{i}_{s}/L^{(s)} & L^{i}_{j} \end{bmatrix} \cdot D\begin{bmatrix} K^{x}_{s}/K^{(s)}\, L^{x}/L^{s} & K^{x}_{j} L^{x} \\ K^{i}_{s}/K^{(s)}\, 1/L^{s} & K^{i}_{j} \end{bmatrix}$$

Referring to §2, we have, with an obvious extension of notation

$$\Delta^{x_1 \cdots x_m}_{x_1 \cdots x_m}\begin{bmatrix} K^x_s/K^{(s)} & L^x/L^s & K^x_j L^x \\ K^i_s/K^{(s)} & 1/L^s & K^i_j \end{bmatrix} = \int_a^b dx_i \begin{vmatrix} \frac{K^{x_1}_{x_1}}{K^{x_1}}\frac{L^{x_1}}{L^{x_1}} & \cdots & \frac{K^{x_1}_{x_m}}{K^{x_m}}\frac{L^{x_1}}{L^{x_m}} & K^{x_1}_j L^{x_1} \\ \vdots & & \vdots & \vdots \\ \frac{K^{x_m}_{x_1}}{K^{x_1}}\frac{L^{x_m}}{L^{x_1}} & \cdots & \frac{K^{x_m}_{x_m}}{K^{x_m}}\frac{L^{x_m}}{L^{x_m}} & K^{x_m}_j L^{x_m} \\ \frac{K^i_{x_1}}{K^{x_1}}\frac{1}{L^{x_1}} & \cdots & \frac{K^i_{x_m}}{K^{x_m}}\frac{1}{L^{x_m}} & K^i_j \end{vmatrix}$$

$$= \frac{L^{x_1}\cdots L^{x_m}}{L^{x_1}\cdots L^{x_m}}\Delta^{x_1 \cdots x_m}_{x_1 \cdots x_m}\begin{bmatrix} K^x_s/K^{(s)} & K^x_j \\ K^i_s/K^{(s)} & K^i_j \end{bmatrix}$$

Cancelling the L^{x_i}, multiplying by $\frac{1}{m!}$ and summing on m, we have

$$D\begin{bmatrix} K^x_s/K^{(s)} & L^x/L^s & K^x_j L^x \\ K^i_s/K^{(s)} & 1/L^s & K^i_j \end{bmatrix} = D\begin{bmatrix} K^x_s/K^{(s)} & K^x_j \\ K^i_s/K^{(s)} & K^i_j \end{bmatrix} \quad \text{and hence (11.1)}$$

<u>Theorem XIII: The transformations of third kind (10.1) where K^x does not vanish in (a,b), and where $D\begin{bmatrix} K^x_s/K^{(s)} & K^x_j \\ K^i_s/K^{(s)} & K^i_j \end{bmatrix}$ does not vanish, form a group, of which the group of transformations of second kind (1.1) with $D\neq 0$, is a sub-group.</u>

Firstly, all the transformations of second kind $D\neq 0$ form a group and they belong to this set, so if this set is a group the group of second kind is a sub-group.

The transitive property is given above; by Theorem XII the determinant of the product is not zero, and by (10.21) $G^x\neq 0$ in (a,b). The unique inverse is given above. The identity transformation $K^x = 1$, $K^x_s = K^x_j = K^i_s = 0$, $K^i_j = \delta^i_j$, is a member of the set and the associative property is readily verified.

§12 Symmetric Transformations of Third Kind: In applications, we shall have occasion to use transformations of type (10.1) where the kernels are continuous and the following equations hold:

$$K^s_x = K^x_s, \quad K^x_i = K^i_x, \quad K^j_i = K^i_j. \tag{12.1}$$

If (12.1) hold, the transformation will be called symmetric. The inverse (10.4) is not in symmetric notation, and it may be an advantage to have it so.

In (10.1) let $\psi^x = \sqrt{K^x}\, y^x$, $\bar\psi^x = \sqrt{K^x}\, \bar y^x$ and divide the upper equation by $\sqrt{K^x}$, giving

$$\begin{aligned} \psi^x/K^x &= \bar\psi^x + K^x_s/\sqrt{K^xK^s}\, \bar\psi^s + K^x_j/\sqrt{K^x}\, \bar y^j \\ y^i &= K^i_s/\sqrt{K^s}\, \bar\psi^s + K^i_j\, \bar y^j \end{aligned} \tag{12.2}$$

If K^x is negative, imaginaries enter.

The unique solution of (12.2) is

$$\begin{aligned} \bar\psi^x &= \psi^x/K^x - \frac{D^x_s/K^s}{D}\, \psi^s - \frac{D^x_j}{D}\, y^j \\ \bar y^i &= -\frac{D^x_s/K^s}{D}\, \psi^s + \frac{D\delta^i_j - D^i_j}{D}\, y^j \end{aligned} \tag{12.3}$$

where D and its minors refer to $\begin{pmatrix} K^x_s/\sqrt{K^xK^{(s)}} & K^x_j/\sqrt{K^x} \\ K^i_s/\sqrt{K^{(s)}} & K^i_j \end{pmatrix}$

This is a symmetric matrix if (10.1) is symmetric.

Expressing (12.3) in terms of y^x and $\bar y^x$, and dividing the upper equation by $\sqrt{K^x}$, we have

$$\begin{aligned} \bar y^x &= y^x/K^x - \frac{D^x_s/\sqrt{K^xK^s}}{D}\, y^s - \frac{D^x_j/\sqrt{K^x}}{D}\, y^j \\ \bar y^i &= -\frac{D^i_s/\sqrt{K^s}}{D}\, y^s + \frac{D\delta^i_j - D^i_j}{D}\, y^j \end{aligned} \tag{12.4}$$

By interchanging rows and columns in the determinants used in defining the bordered Fredholm minors it is easily seen that (12.4) is symmetric if (10.1) is.

Since (10.4) and (12.4) are both unique inverses of (10.1) they must be the same. Hence (12.4) is a real solution, even if K^x is negative, in spite of the fact that it is apparently imaginary in this case. We shall show that the determinant of each is the same; this will follow from the fact that $\Delta^{x_1 \cdots x_m}_{x_1 \cdots x_m}$ for each is the same.

For the matrix $\begin{pmatrix} K^x_s/K^{(s)} & K^x_j \\ K^i_s/K^{(s)} & K^i_j \end{pmatrix}$ we have

$$\Delta^{x_1 \cdots x_m}_{x_1 \cdots x_m} = \int_a^b dx_i \begin{vmatrix} \dfrac{K^{x_1}_{x_1}}{K^{x_1}} & \cdots & \dfrac{K^{x_1}_{x_m}}{K^{x_m}} & K^{x_1}_j \\ \vdots & & \vdots & \vdots \\ \dfrac{K^{x_m}_{x_1}}{K^{x_1}} & \cdots & \dfrac{K^{x_m}_{x_m}}{K^{x_m}} & K^{x_m}_j \\ \dfrac{K^i_{x_1}}{K^{x_1}} & \cdots & \dfrac{K^i_{x_m}}{K^{x_m}} & K^i_j \end{vmatrix}$$

$$= \int_a^b \frac{dx_i}{K^{x_1} \cdots K^{x_m}} \begin{vmatrix} K^{x_1}_{x_1} & \cdots & K^{x_1}_{x_m} & K^{x_1}_j \\ \vdots & & \vdots & \vdots \\ K^{x_m}_{x_1} & \cdots & K^{x_m}_{x_m} & K^{x_m}_j \\ K^i_{x_1} & \cdots & K^i_{x_m} & K^i_j \end{vmatrix}$$

For the matrix $\begin{pmatrix} K^x_s/\sqrt{K^x K^{(s)}} & K^x_j/\sqrt{K^x} \\ K^i_s/\sqrt{K^{(s)}} & K^i_j \end{pmatrix}$

$$\Delta^{x_1 \cdots x_m}_{x_1 \cdots x_m} = \int_a^b dx_i \begin{vmatrix} \dfrac{K^{x_1}_{x_1}}{\sqrt{K^{x_1} K^{x_1}}} & \cdots & \dfrac{K^{x_1}_{x_m}}{\sqrt{K^{x_1} K^{x_m}}} & \dfrac{K^{x_1}_j}{\sqrt{K^{x_1}}} \\ \vdots & & \vdots & \vdots \\ \dfrac{K^{x_m}_{x_1}}{\sqrt{K^{x_m} K^{x_1}}} & \cdots & \dfrac{K^{x_m}_{x_m}}{\sqrt{K^{x_m} K^{x_m}}} & \dfrac{K^{x_m}_j}{\sqrt{K^{x_m}}} \\ \dfrac{K^i_{x_1}}{\sqrt{K^{x_1}}} & \cdots & \dfrac{K^i_{x_m}}{\sqrt{K^{x_m}}} & K^i_j \end{vmatrix}$$

$$= \int_a^b \frac{dx_i}{\sqrt{K^{x_1}\cdots K^{x_m}}\sqrt{K^{x_1}\cdots K^{x_m}}} \begin{vmatrix} K^{x_1}_{x_1} & \cdots & K^{x_1}_{x_m} & K^{x_1}_{j} \\ \vdots & & \vdots & \vdots \\ K^{x_m}_{x_1} & \cdots & K^{x_m}_{x_m} & K^{x_m}_{j} \\ K^{i}_{x_1} & \cdots & K^{i}_{x_m} & K^{i}_{j} \end{vmatrix}$$

Hence $\Delta^{x_1 \cdots x_m}_{x_1 \cdots x_m}$ and likewise D are the same for both.

The notation in this paragraph, while retaining symmetry, has the disadvantage of bringing in an apparent irrationality.

§13 Appendix to Part I: In order to make the work of §6, 7, 8 rigorous we must prove the following theorems. All the functionals referred to are functionals of n functions each of an arbitrary number of variables. The notation of §6 is retained throughout.

Theorem XIV: The Fréchet differential of the sum of two functionals having Fréchet differentials is the sum of the Fréchet differentials of the functionals.

More briefly:

$$\delta(F+G) = \delta F + \delta G \tag{13.1}$$

Requirement 1) for the Fréchet differential is seen to be satisfied immediately.

For 2), we have, where $\eta = \max |\delta Y_i|$

$$\left| \frac{\Delta(F+G) - (\delta F + \delta G)}{\eta} \right| \leq \left| \frac{\Delta F - \delta F}{\eta} \right| + \left| \frac{\Delta G - \delta G}{\eta} \right|$$

which approaches zero with η.

Corollary: The Fréchet differential of the sum of a finite number of functionals having Fréchet differentials is the sum of the Fréchet differentials of the functionals.

Theorem XV: The Fréchet differential of the product of two functionals having Fréchet differentials is the sum of each times the Fréchet differential of the other.

More briefly:

$$\delta(FG) = F\delta G + G\delta F \tag{13.2}$$

Requirement 1) is easily seen to be satisfied.

For 2), we have

$$\Delta(FG) = F\Delta G + G\Delta F + \Delta F\cdot\Delta G$$

Hence

$$\left|\frac{\Delta(FG) - (F\delta G + G\delta F)}{\eta}\right| \leq |F|\left|\frac{\Delta G - \delta G}{\eta}\right| + |G|\left|\frac{\Delta F - \delta F}{\eta}\right| + \left|\frac{\Delta F}{\eta}\right|\cdot|\Delta G|$$

Now $\left|\frac{\Delta G - \delta G}{\eta}\right|$ and $\left|\frac{\Delta F - \delta F}{\eta}\right|$ each approach zero.

For $\left|\frac{\Delta F}{\eta}\right|\cdot|\Delta G|$ we may write $\Delta F = \delta F + \epsilon\eta$ where $\epsilon \to 0$ with η. $\therefore \left|\frac{\Delta F}{\eta}\right| \leq \left|\frac{\delta F}{\eta}\right| + |\epsilon|$.

If we can show that $\left|\frac{\delta F}{\eta}\right|$ is bounded, then $\left|\frac{\Delta F}{\eta}\right|$ is bounded and since G is continuous, $\left|\frac{\Delta F}{\eta}\right|\cdot|\Delta G|$ approaches zero and 2) is satisfied.

<u>Lemma: With F and δF defined as in § 6, $\left|\frac{\delta F}{\eta}\right|$ is bounded.</u>

From Theorem IV this will follow if we can show that $\left|\frac{\delta F_i}{\eta}\right|$ is bounded, where δF_i is $(\delta F)_{\delta Y_j = 0,\ j \neq i}$

In δF_i let δY_i be ϵL_i where L_i is a fixed function of the variables that Y_i depends on, such that max $|L| = 1$, and ϵ is an independent variable greater than zero.

Then $\delta F_i[Y_j/\delta Y_i] = \epsilon\,\delta F_i[Y_j/L]$, since the differential is linear and homogeneous.

$$\eta = \max|\delta Y_j| \geq \epsilon \max|L| = \epsilon$$

Hence $\left|\frac{\delta F_i}{\eta}\right| \leq |\delta F_i[Y_j/L]|$ which is independent of η.

<u>Corollary: The Fréchet differential of the product of a finite number of functionals having Fréchet differentials is the sum of the Fréchet differential of each times the product of all the others.</u>

Fréchet differential of a rational function of a finite number of functionals is simply isomorphic to the process of calculating the ordinary differential of a rational function of a finite number of ~~indepen-dent~~ variables.

Theorem XVII: The Fréchet differential of the logarithm of a functional having a Fréchet differential, taken at a point where the functional does not vanish, is the Fréchet differential of the functional divided by the functional.

More briefly

$$\delta \log F = \frac{\delta F}{F} \tag{13.4}$$

Requirement 1) is seen to be satisfied.

For 2) we have

$$\Delta \log F = \log(F+\Delta F) - \log F = \log\left(1+\frac{\Delta F}{F}\right)$$

Now assume $|\Delta F| < |F|$. Since $F \neq 0$ this is legitimate as above.

$$\log\left(1+\frac{\Delta F}{F}\right) = \frac{\Delta F}{F} - \frac{(\Delta F)^2}{2F^2} + \text{ terms of higher order.}$$

Also $\Delta F = \delta F + \theta\eta$ where $\theta \to 0$ with η.

Hence

$$\Delta \log F = \frac{\delta F}{F} + \frac{\theta\eta}{F} - \frac{(\Delta F)^2}{F^2}\left[\frac{1}{2} - \frac{\Delta F}{3F} + \cdots\right]$$

The series in brackets converges for $\left|\frac{\Delta F}{F}\right| < 1$, hence is bounded, say less than M.

$$\left|\frac{\Delta \log F - \frac{\delta F}{F}}{\eta}\right| \leq \left|\frac{\theta}{F}\right| + \left|\frac{\Delta F}{\eta}\right| \cdot \left|\frac{\Delta F}{F^2}\right| \cdot M$$

As above $\left|\frac{\Delta F}{\eta}\right| \leq \left|\frac{\delta F}{\eta}\right| + |\theta|$ and this is bounded. Then since $\left|\frac{\theta}{F}\right|$ and $\left|\frac{\Delta F}{F^2}\right|$ approach zero with η the

Theorem is proved.

Theorem XVIII: If the Fréchet differential of a functional exists and is zero identically in the arguments and the variations of the arguments, then the functional is a constant functional.

Let

$$F[Y_i + \lambda \delta Y_i] = g[Y_i, \lambda]$$

Gateaux's[1] formula gives us

$$\frac{d}{d\lambda} g[Y_i, \lambda]\Big)_{\lambda=0} = \delta F[Y_i | \delta Y_i]$$

Hence g has a derivative with respect to λ at $\lambda = 0$.

Now $\frac{d}{d\lambda} F[Y_i + \lambda\delta Y_i] = \frac{d}{d\mu} F[Y_i + \lambda \delta Y_i + \mu \delta Y_i]\Big)_{\mu=0} = \delta F[Y_i + \lambda \delta Y_i | \delta Y_i]$

Hence $\frac{d}{d\lambda} g[Y_i, \lambda]$ exists for all λ and is given by

$$\delta F[Y_i + \lambda \delta Y_i | \delta Y_i]$$

Since the derivative exists, we may apply the Theorem of the Mean of the differential calculus giving

$$g[Y_i | 1] - g[Y_i | 0] = \frac{d}{d\lambda} g[Y_i | \lambda] \qquad 0 < \lambda < 1$$

But

$$g[Y_i | 1] - g[Y_i | 0] = F[Y_i + \delta Y_i] - F[Y_i] = \Delta F$$

Hence under the hypothesis, $\Delta F = 0$, and F is a constant functional.

Paul Levy: "Analyse Fonctionnelle" (1922) p. 100.

Part II

Quadratic Functional Forms on a Composite Range.(1)

(1) Part II includes some unpublished work of Professor A. D. Michal.

§1 <u>Introduction:</u> Quadratic functional forms of the type $\int_a^b\int_a^b g(x,s)y(x)y(s)\,dx\,ds + \int_a^b g(x)[y(x)]^2dx$ have been considered.(1) Using the convention of letting a repeated sub- and super-script in the same term stand for Riemann integration with respect to that index, we may write this as:

$$g_{xs}y^xy^s + g_x(y^x)^2$$

In this work we shall consider forms quadratic on the range (y^x, y^i) where y^x is a function continuous in (a,b) and y^i is a set of n independent variables. We may write such a form as

(1.1) $$Q = g_{xs}y^xy^s + g_x(y^x)^2 + 2g_{xi}y^xy^i + g_{ij}y^iy^j$$

The indices x, s, t etc. stand for continuous variables in (a,b) while i, j, l etc. are discrete, running from 1 to n as in §1, Part I, and are to be summed from 1 to n when repeated once as a subscript and once as a superscript in the same term.

A parenthesis about any index suspends the summation or integration convention with respect to that index.

We assume that the g's are continuous, and that $g_x \neq 0$ in (a,b). Hence g_x is of one sign in (a,b).

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(1) A. D. Michal: "Affinely Connected Function Space Manifolds." Am. Jour. Math. V. 50(1928) p. 473.
T. S. Peterson: "A Class of Invariant Functionals of Quadratic Functional Forms." Am. Jour Math. V. 51(1929) p. 417.
A. D. Michal and T. S. Peterson: "The Invariant Theory of Functional Forms under the Group of Linear Transformations of Third Kind." To be published soon.

§ 2 Relation between Forms of this Type and Symmetric Transformations of Third Kind. Consider a transformation of third kind as follows:

$$(2.1)\qquad \begin{aligned} z_x &= g_{(x)}y^x + g_{xs}y^s + g_{xj}y^j \\ z_i &= \qquad\quad\ \ g_{si}y^s + g_{ij}y^j \end{aligned}$$

We may assume without loss of generality in form (1.1) that g_{xs} and g_{ij} are symmetric. Then transformation (2.1) is a symmetric transformation. In terms of the y's and z's we may write (1.1) as

$$(2.2)\qquad Q = y^x z_x + y^i z_i$$

Hence we see a one-to-one reciprocal correspondence between such transformations of the third kind and quadratic forms of this type.

Now consider what happens to Q when the y's undergo a bordered Fredholm transformation of third kind with continuous kernels.

$$(2.3)\qquad \begin{aligned} y^x &= K^x\bar{y}^x + K^x_s\bar{y}^s + K^x_j\bar{y}^j \\ y^i &= \qquad\quad\ K^i_s\bar{y}^s + K^i_j\bar{y}^j \end{aligned}$$

We have

$$Q = y^x z_x + y^i z_i$$

$$= K^x\bar{y}^x z_x + z_x K^x_s\bar{y}^s + z_x K^x_j\bar{y}^j + z_i K^i_s\bar{y}^s + z_i K^i_j\bar{y}^j$$

where the z's are given in terms of the y's by the product transformation of (2.1) and (2.3).

Rearranging, we have

$$(2.4)\qquad Q = \bar{y}^s(K^s z_s + K^x_s z_x + K^i_s z_i) + \bar{y}^j(K^x_j z_x + K^i_j z_i)$$

If we now call $(\bar{z}_s, \bar{z}_j)$ the transform of the z's by the adjoint of (2.3) i. e.

$$(2.5)\quad \begin{aligned} \bar{z}_s &= K^{(s)} z_s + K_s^x z_x + K_s^i z_i \\ \bar{z}_j &= \qquad\qquad K_j^x z_x + K_j^i z_i, \end{aligned}$$

We have

$$(2.6)\qquad Q = \bar{y}^s \bar{z}_s + \bar{y}^j \bar{z}_j$$

But this is a form of type (2.2) where the transformation of third kind representing it, is the product of three transformations of third kind, and may be written symbolically

$$(2.7)\qquad \overset{*}{g} = \tilde{K} \cdot g \cdot K$$

the tilda denoting the adjoint transformation.

§3 <u>The Transformation Induced on the Coefficients:</u>

In order that $\bar{Q}$, which is defined to be

$$\bar{g}_{xs} \bar{y}^x \bar{y}^s + \bar{g}_x (\bar{y}^x)^2 + 2 \bar{g}_{xi} \bar{y}^x \bar{y}^i + \bar{g}_{ij} \bar{y}^i \bar{y}^j$$

shall equal Q, that is, in order that Q be an absolute quadratic form under (2.3) we must have

$$(3.1)\quad 0 \equiv \bar{y}^x \bar{y}^s (\bar{g}_{xs} - \overset{*}{g}_{xs}) + (\bar{y}^x)^2 (\bar{g}_x - \overset{*}{g}_s) + 2 \bar{y}^x \bar{y}^i (\bar{g}_{xi} - \overset{*}{g}_{xi}) + \bar{y}^i \bar{y}^j (\bar{g}_{ij} - \overset{*}{g}_{ij})$$

where the $\overset{*}{g}$ are the coefficients we get by expanding the $\bar{z}$'s in (2.6) and collecting terms.

<u>Theorem I: The necessary and sufficient condition that</u>

$$(3.2)\quad l_{xs} y^x y^s + l_x (y^x)^2 + l_{xi} y^x y^i + l_{ij} y^i y^j; \quad l_{xs} = l_{sx}, \; l_{ij} = l_{ji}$$

<u>vanish identically for all continuous y^x and all y^i, when the l's are continuous, is that the l's all vanish identically.</u>

The sufficiency of the condition is obvious. To show the necessity, that is, that if the form vanishes for all (y^x, y^i) as stated, the l's are all zero, we will show that the l's are all zero if the form vanish for all (y^x, y^i) as stated such that $y^a = y^t = 0$.

In this case first set all the $y^i = 0$. Then by Lemma 2[1] of Michal and Peterson's paper (l.c.), l_{xs} and l_x are identically zero. Hence the form becomes

$$(l_{xi} y^x + l_{ij} y^j) y^i = 0$$

Since the y^i are independent, we have

$$l_{xi} y^x + l_{ij} y^j = 0$$

Now set $y^x \equiv 0$, and by the independence of the y^i the l_{ij} are zero, leaving only $l_{xi} y^x$ to which we apply the fundamental lemma of the Calculus of Variations.

Applying Theorem I to (3.1) and writing the $\overset{*}{g}$ out in full, we have:

<u>Theorem II: The necessary and sufficient condition that the form (1.1) be an absolute form under (2.3) is that the coefficients transform as follows:</u>

---

[1] A complete statement of this is as follows:

"Lemma 2. A necessary and sufficient condition that
$$\psi_{\alpha\beta} y^\alpha y^\beta + \psi_\alpha (y^\alpha)^2 = 0 \; ; \; \psi_{\alpha\beta} = \psi_{\beta\alpha},$$
be true for all continuous functions y^i, for which $y^a = y^t = 0$ is that
$$\psi_{\alpha\beta} \equiv 0, \; \psi_\alpha \equiv 0 \text{"}$$

$$(3.31)\quad \bar{g}_{xs} = g_{xs}K^{(x)}K^{(s)} + g_{xu}K^{(x)}K^u_s + g_{ts}K^{(s)}K^t_x + g_{tu}K^t_xK^u_s$$
$$+ g_x K^{(x)}K^x_s + g_s K^{(s)}K^s_x + g_t K^t_x K^t_s + g_{x\ell}K^{(x)}K^\ell_s + g_{s\ell}K^{(s)}K^\ell_x$$
$$+ g_{t\ell}K^t_x K^\ell_s + g_{t\ell}K^t_s K^\ell_x + g_{\ell m}K^\ell_x K^m_s,$$

$$(3.32)\quad \bar{g}_x = g_x(K^{(x)})^2$$

$$(3.33)\quad \bar{g}_{xi} = g_{xt}K^{(x)}K^t_i + g_{tu}K^t_xK^u_i + g_xK^{(x)}K^x_i + g_tK^t_xK^t_i$$
$$+ g_{t\ell}K^t_xK^\ell_i + g_{t\ell}K^t_iK^\ell_x + g_{x\ell}K^{(x)}K^\ell_i + g_{\ell m}K^\ell_xK^m_i$$

$$(3.34)\quad \bar{g}_{ij} = g_{tu}K^t_iK^u_j + g_tK^t_iK^t_j + g_{t\ell}K^t_iK^\ell_j + g_{t\ell}K^t_jK^\ell_i + g_{\ell m}K^\ell_iK^m_j$$

It is readily verified that the transformation of third kind represented by the matrix

$$\begin{pmatrix} \bar{g}_x & \bar{g}_{xs} & \bar{g}_{xj} \\ & \bar{g}_{si} & \bar{g}_{ij} \end{pmatrix}$$

is symmetric, and that its kernels are continuous if the g's and K's are.

§4 An Invariant of the Coefficients: The bordered Fredholm determinant of the transformation of third kind (2.3) is the bordered Fredholm determinant of the transformation of of second kind with matrix:

$$\begin{pmatrix} K^x_s/\sqrt{K^xK^{(s)}} & K^x_j/\sqrt{K^x} \\ K^i_s/\sqrt{K^{(s)}} & K^i_j \end{pmatrix}$$

Hence applying Theorem XII, Part I twice, we have:

$$(4.1)\quad D\begin{bmatrix} \bar{g}_{xs}/\sqrt{\bar{g}_x\bar{g}_s} & \bar{g}_{xj}/\sqrt{\bar{g}_x} \\ \bar{g}_{si}/\sqrt{\bar{g}_s} & \bar{g}_{ij} \end{bmatrix} = D\begin{bmatrix} K^s_x/\sqrt{K^xK^s} & K^i_s/\sqrt{K^{(s)}} \\ K^x_j/\sqrt{K^x} & K^i_j \end{bmatrix}\cdot D\begin{bmatrix} g_{xs}/\sqrt{g_xg_s} & g_{xj}/\sqrt{g_x} \\ g_{is}/\sqrt{g_s} & g_{ij} \end{bmatrix}\cdot D\begin{bmatrix} K^x_s/\sqrt{K^xK^{(s)}} & K^x_j/\sqrt{K^x} \\ K^i_s/\sqrt{K^{(s)}} & K^i_j \end{bmatrix}$$

Theorem III: The bordered Fredholm determinant

$$(4.2)\quad D\begin{bmatrix} g_{xs}/g_s & g_{xj} \\ g_{si}/g_s & g_{ij} \end{bmatrix}$$

is an invariant of weight two under the transformation (3.3).

Referring to §12, Part I we see that

$$D\begin{bmatrix} g_{xs}/\sqrt{g_xg_s} & g_{xj}/\sqrt{g_x} \\ g_{si}/\sqrt{g_s} & g_{ij} \end{bmatrix} = D\begin{bmatrix} g_{xs}/g_s & g_{xj} \\ g_{si}/g_s & g_{ij} \end{bmatrix}$$

and the same holds true for the $\bar{g}$'s .

By Theorem XI, Part I

$$D\begin{bmatrix} K^{s}_{x}/\sqrt{K^{(x)}K^{s}} & K^{i}_{s}/\sqrt{K^{(s)}} \\ K^{x}_{j}/\sqrt{K^{x}} & K^{i}_{j} \end{bmatrix} = D\begin{bmatrix} K^{x}_{s}/\sqrt{K^{x}K^{(s)}} & K^{x}_{j}/\sqrt{K^{x}} \\ K^{i}_{s}/\sqrt{K^{(s)}} & K^{i}_{j} \end{bmatrix}$$

and by §12, Part I these are equal to

$$D\begin{bmatrix} K^{x}_{s}/K^{(s)} & K^{x}_{j} \\ K^{i}_{s}/K^{(s)} & K^{i}_{j} \end{bmatrix}$$

Hence (4.1) becomes

$$(4.3)\quad D\begin{bmatrix} \bar{g}_{xs}/\bar{g}_{s} & \bar{g}_{xj} \\ \bar{g}_{si}/\bar{g}_{s} & \bar{g}_{ij} \end{bmatrix} = \left\{ D\begin{bmatrix} K^{x}_{s}/K^{(s)} & K^{x}_{j} \\ K^{i}_{s}/K^{(s)} & K^{i}_{j} \end{bmatrix} \right\}^{2} D\begin{bmatrix} g_{xs}/g_{s} & g_{xj} \\ g_{si}/g_{s} & g_{ij} \end{bmatrix}$$

which is what we mean by saying that the determinant is an invariant of weight two.

If we let $g_{xs} = g_{xi} = 0$, then the determinant (4.2) is independent of g_x . Letting this be zero also, Q becomes the ordinary algebraic quadratic form. If we let $K^x = 1, K^x_s = K^x_{j,} = K^i_s = 0$, then (2.3) becomes a linear algebraic transformation on the In this case Theorem III reduces to the corresponding theorem in the theory of algebraic forms.

If we let $g_{xi} = g_{ij} = 0$, $g_x = 1$, Q becomes a form of type previously studied[1]. Letting $K^i_j = \delta^i_j$, $K^x_j = K^i_s = 0$, $K^x = 1$, the transformation reduces to the ordinary Fredholm type, and Theorems 2.I and 2.II of Michal's paper are special cases of Theorems II and III of this part respectively. If we do not restrict g_x and K^x to be unity, but merely to not vanishing in (a,b), we have another case considered[2]. Here it is the symmetric

(1) A.D.Michal: (l.c.) p. 478.
(2) A.D.Michal and T.S. Peterson (l.c.).

form of the determinant involving square **roots** that is obtained as the invariant rather than the rational form (4.2).

§ 5 <u>Special Case of the Previous:</u> If n=1, equations (3.3) become

$$(5.11)\quad \bar g_{xs} = g_{xs}K^{(x)}K^{(s)} + g_{xu}K^{(x)}K^u_s + g_{ts}K^{(s)}K^t_x + g_{tu}K^t_xK^u_s + g_xK^{(x)}K^x_s + g_sK^{(s)}K^s_x + g_tK^t_xK^t_s + g_{x1}K^{(x)}K^1_s + g_{s1}K^{(s)}K^1_x + g_{t1}K^t_xK^1_s + g_{t1}K^t_sK^1_x + g_{11}K^1_xK^1_s$$

$$(5.12)\quad \bar g_x = g_x(K^{(x)})^2$$

$$(5.13)\quad \bar g_{x1} = g_{xt}K^{(x)}K^t_1 + g_{tu}K^t_xK^u_1 + g_xK^{(x)}K^x_1 + g_tK^t_xK^t_1 + g_{x1}K^{(x)}K^1_1 + g_{t1}K^t_xK^1_1 + g_{t1}K^t_1K^1_x + g_{11}K^1_xK^1_1$$

$$(5.14)\quad \bar g_{11} = g_{tu}K^t_1K^u_1 + g_t(K^t_1)^2 + 2g_{t1}K^t_1K^1_1 + g_{11}(K^1_1)^2$$

§ 6 <u>Forms Quadratic in a Function and its Derivative:</u> Let w^x be a function with a continuous derivative and let y^x be $\frac{d}{dx}w^x$

The quadratic form with continuous coefficients

$$(6.1)\quad Q = A_{xs}w^xw^s + 2B_{xs}w^xy^s + C_{xs}y^xy^s + A_x(w^x)^2 + 2B_xw^xy^x + C_x(y^x)^2\ ;\ A_{xs} = A_{sx},\ C_{xs} = C_{sx}$$

is not an invariantive form. That is Q may be zero for all w^x without having all the coefficients vanish. For example, if $C_{xs} = A_x = B_x = C_x = 0$, $A_{xs} = f_xh'_s$, $2B_{xs} = f_xh_s$, where h_s is a function such that $h_a = h_b = 0$, we have

$$Q = f_xh'_sw^xw^s + f_xh_sw^xy^s$$
$$= f_xw^x(h'_sw^s + h_sw'^s) = f_xw^x(h_bw^b - h_aw^a) = 0$$

We may make A_{xs} symmetric here by letting $f_x = h'_x$.

Now $w^x = w^a + \int_a^x y^s ds$, and if we write Y for w^a, and define the step-functions

$$G^x \equiv 1 \qquad E^x_s = \begin{cases} 0 \text{ for } b \geq s > x \\ 1 \text{ for } x \geq s \geq a \end{cases}$$

we may write

(6.2) $$w^x = G^x Y + E^x_s y^s$$

Substituting (6.2) in (6.1) we have

$$Q = A_{xs}(G^x Y + E^x_t y^t)(G^s Y + E^s_u y^u) + 2B_{xs}(G^x Y + E^x_t y^t) y^s + C_{xs} y^x y^s$$

$$+ A_x (G^x Y + E^x_t y^t)(G^x Y + E^x_u y^u) + 2B_x (G^x Y + E^x_t y^t) y^x + C_x (y^x)^2$$

Collecting terms, we have

(6.3) $$Q = (A_{tu} E^t_x E^u_s + 2B_{ts} E^t_x + C_{xs} + A_t E^t_x E^t_s + 2B_x E^x_s) y^x y^s$$
$$+ C_x (y^x)^2 + (A_{tu} G^t E^u_x + A_{tu} G^u E^t_x + 2B_{tx} G^t + 2A_t G^t E^t_x$$
$$+ 2B_x G^x) y^x Y + (A_{tu} G^t G^u + A_t (G^t)^2) Y^2$$

Without loss of generality we may take A_{xs} and C_{xs} to be symmetric. If we notice that

$$2(B_{ts} E^t_x + B_x E^x_s) y^x y^s$$
$$= (B_{ts} E^t_x + B_{tx} E^t_s + B_x E^x_s + B_s E^s_x) y^x y^s$$

and that

$$y^x (A_{tu} G^t E^u_x + A_{tu} G^u E^t_x) = 2A_{tu} G^u E^t_x y^x;$$

we may rewrite (6.3) as a form of type (1.1) with n=1 and coefficients as follows:

(6.41) $$g_{xs} = A_{tu} E^t_x E^u_s + B_{ts} E^t_x + B_{tx} E^t_s + C_{xs} + A_t E^t_x E^t_s + B_x E^{(x)}_s + B_s E^{(s)}_x$$

(6.42) $$g_x = C_x$$

(6.43) $$g_{x1} = A_{tu} G^u E^t_x + B_{tx} G^t + A_t G^t E^t_x + B_x G^{(x)}$$

(6.44) $$g_{11} = A_{tu} G^t G^u + A_t (G^t)^2$$

These coefficients are continuous if those of (6.1) are; for example:

$$B_{ts} E^t_x = \int_{t=a}^{x} 0 \cdot B_{ts}\, dt + \int_x^b B_{ts}\, dt$$

which is continuous in x and in s. Similarly all others which have E^u_v integrated are continuous, with the possible exception of the term $A_t E^t_x E^t_s$. This term is equal to $\int_z^b A_t\, dt$ where z is the greater of the pair

(x, s) , and is continuous, since z is a continuous function of the two variables x and s .

This accounts for all the terms except $B_x E^{(x)}_s$ and $B_s E^{(s)}_x$, which are both discontinuous. However their sum is continuous and is equal to B_z where z is defined above.

By using z , and by using the convention of a bracket about an index to denote integration over with respect to that index, we may get rid of the and rewrite (6.4) as follows:

$$g_{xs} = A_{tu} E^t_x E^u_s + B_{tx} E^t_s + B_{ts} E^t_x + C_{xs} + A_t E^t_z + B_z \tag{6.51}$$

$$g_x = C_x \tag{6.52}$$

$$g_{x1} = A_{[t]u} E^u_x + B_{[t]x} + A_t E^t_x + B_x \tag{6.53}$$

$$g_{11} = A_{[t][u]} + A_{[t]} \tag{6.54}$$

§ 7 <u>Application of Earlier Work to §6:</u> By Theorem I, we have:

<u>Theorem IV: The necessary and sufficient condition that the form (6.1) vanish for all w^x with continuous derivatives is that the following relations hold among its coefficients:</u>

$$\begin{aligned} &A_{tu} E^t_x E^u_s + B_{tx} E^t_s + B_{ts} E^t_x + C_{xs} + A_t E^t_z + B_z = 0 \\ &C_x = 0 \\ &A_{[t]u} E^u_x + B_{[t]x} + A_t E^t_x + B_x = 0 \\ &A_{[t][u]} + A_{[t]} = 0 \end{aligned} \tag{7.1}$$

It might be interesting to show that the coefficients of the example given above satisfy (7.1). Substituting $\lambda'_x \lambda'_s$ for A_{xs} , $\frac{1}{2}\lambda'_x \lambda_s$ for B_{xs}, and zero for the other coefficients, we have:

$$h'_t E^t_x h'_u E^u_s + \frac{1}{2} h'_t E^t_x h_s + \frac{1}{2} h'_t E^t_s h_x = 0$$

$$0 = 0$$

$$h'_{[t]} h'_u E^u_x + \frac{1}{2} h'_{[t]} h_x = 0$$

$$h'_{[t]} h'_{[u]} = 0$$

Now

$$h'_t E^t_x = \int_x^b h'_t dt = h(b) - h(x) = -h_x$$

$$h'_{[t]} = \int_a^t h'_t dt = h_t - h_a = 0$$

The first of these equations becomes

$$[-h_x][-h_s] + \frac{1}{2}[-h_x]h_s + \frac{1}{2}[-h_s]h_x = 0$$

which is true, and the others reduce to zero, so that (7.1) is satisfied.

Now if we apply a transformation

$$(7.2) \qquad \begin{aligned} y^x &= K^x \bar{y}^x + K^x_s \bar{y}^s + K^x_1 \bar{Y} \\ Y &= \qquad\qquad K^1_s \bar{y}^s + K^1_1 \bar{Y} \end{aligned}$$

we see that the g's defined by (6.5) transform according to (5.1). Since the set (6.5) is not in general uniquely solvable for the original coefficients, this does not give us a definition for $\bar{A}_{xs}$, etc., so that we cannot consider a law of transformation for the original coefficient, without further restrictions.

If $C_x \neq 0$ in (a,b) we may write the invariant (4.2) in terms of the original coefficients, In terms of these it is

$$(7.3) \quad \begin{bmatrix} \dfrac{A_{tu} E^t_x E^u_s + B_{ts} E^t_x + B_{tx} E^t_s + C_{xs} + A_t E^t_z + B_z}{C_s} & A_{[t]u} E^u_x + B_{[t]x} + A_t E^t_x + B_x \\ \dfrac{A_{[t]u} E^u_s + B_{[t]s} + A_t E^t_s + B_x}{C_s} & A_{[t][u]} + A_{[t]} \end{bmatrix}$$

§ 8 Specialization of Transformation (7.2): We assumed that the transformation (7.2) was general. In various particular cases the K's may be related.

It may turn out naturally that the y^x and Y transform independently, that is

$$y^x = K^x \bar{y}^x + K^x_s \bar{y}^s, \quad Y = K^1_1 \bar{Y} \tag{8.1}$$

In this case the determinant of our transformation becomes $D\begin{bmatrix} K^x_s/K^{(s)} & 0 \\ 0 & K^1_1 \end{bmatrix} = K^1_1 \, D\left[K^x_s/K^{(s)}\right]$

If in particular we consider functions w^x that all have the same value at a, $K^1_1 = 1$, and our determinant is

$$D\left[K^x_s/K^{(s)}\right]$$

Perhaps the most interesting specialization of (7.2) is when it is derived from an ordinary Fredholm transformation of second kind on w^x:

$$w^x = \bar{w}^x + K^x_s \bar{w}^s \tag{8.2}$$

If $D[K^x_s] \neq 0$ then to each $\bar{w}^x$, corresponds a unique w^x, so that to each $(\bar{y}^x, \bar{Y})$ corresponds a unique (y^x, Y), and we expect the determinant of (7.2) derived from (8.2) to be different from zero, when $D[K^x_s]$ is. We shall prove the following theorem:

<u>Theorem V:</u> <u>The bordered Fredholm determinant of the transformation (7.2) determined by (8.2), where the kernel has a continuous derivative in (a,b) with respect to x, equals the ordinary Fredholm determinant of (8.2).</u>

We must first show that (8.2) does determine a transformation of type (7.2). To do this, set $x = a$, giving:

$$Y = \bar{Y} + K^{a}_{s}\bar{w}^{s} = \bar{Y} + K^{a}_{s}(G^{s}\bar{Y} + E^{s}_{t}\bar{y}^{t})$$

Differentiate (8.2) with respect to x. If we denote by a prime before a function of a superscript and a subscript, partial differentiation with respect to the superscript, we have:

$$\begin{aligned} y^{x} &= \bar{y}^{x} + {}'K^{x}_{s}\bar{w}^{s} \\ &= \bar{y}^{x} + {}'K^{x}_{s}G^{s}\bar{Y} + {}'K^{x}_{t}E^{t}_{s}\bar{y}^{s}. \end{aligned}$$

Collecting these, we have the transformation of type (7.2) determined by (8.2) as follows:

$$(8.3)\qquad \begin{aligned} y^{x} &= \bar{y}^{x} + [{}'K^{x}_{t}E^{t}_{s}]\bar{y}^{s} + [{}'K^{x}_{[t]}]\bar{Y} \\ Y &= \qquad [K^{a}_{t}E^{t}_{s}]\bar{y}^{s} + [1 + K^{a}_{[t]}]\bar{Y} \end{aligned}$$

The bordered Fredholm determinant of this is

$$D\begin{bmatrix} {}'K^{x}_{t}E^{t}_{s} & {}'K^{x}_{[t]} \\ K^{a}_{t}E^{t}_{s} & 1 + K^{a}_{[t]} \end{bmatrix}$$

Referring to the definition of D, we have with notation of Part I § 2:

$$\Delta^{x_1 \cdots x_m}_{s_1 \cdots s_m} =$$

$$\begin{vmatrix} {}'K^{x_1}_{t_1}E^{t_1}_{s_1} & \cdots & {}'K^{x_1}_{t_m}E^{t_m}_{s_m} & {}'K^{x_1}_{[t]} \\ \vdots & & \vdots & \vdots \\ {}'K^{x_m}_{t_1}E^{t_1}_{s_1} & \cdots & {}'K^{x_m}_{t_m}E^{t_m}_{s_m} & {}'K^{x_m}_{[t]} \\ K^{a}_{t_1}E^{t_1}_{s_1} & \cdots & K^{a}_{t_m}E^{t_m}_{s_m} & 1 + K^{a}_{[t]} \end{vmatrix}$$

$$= \begin{vmatrix} \int_{t_1=s_1}^{b} {}'K^{x_1}_{t_1}\,dt_1 & \cdots & \int_{t_m=s_m}^{b} {}'K^{x_1}_{t_m}\,dt_m & \int_{a}^{b} {}'K^{x_1}_{t}\,dt \\ \vdots & & \vdots & \vdots \\ \int_{t_1=s_1}^{b} {}'K^{x_m}_{t_1}\,dt_1 & \cdots & \int_{t_m=s_m}^{b} {}'K^{x_m}_{t_m}\,dt_m & \int_{a}^{b} {}'K^{x_m}_{t}\,dt \\ \int_{t_1=s_1}^{b} K^{a}_{t_1}\,dt_1 & \cdots & \int_{t_m=s_m}^{b} {}'K^{a}_{t_m}\,dt_m & 1 + \int_{a}^{b} K^{a}_{t}\,dt \end{vmatrix}$$

We may split this into two determinants as follows:

$$\Delta^{x_1 \cdots x_m}_{s_1 \cdots s_m} = \int_{t_i=s_i}^{b} dt_i \begin{vmatrix} {}'K^{x_1}_{t_1} & \cdots & {}'K^{x_1}_{t_m} \\ \vdots & & \vdots \\ {}'K^{x_m}_{t_1} & \cdots & {}'K^{x_m}_{t_m} \end{vmatrix} + \int_{t_i=s_i}^{b} dt_i \int_a^b dt \begin{vmatrix} {}'K^{x_1}_{t_1} & \cdots & {}'K^{x_1}_{t_m} & {}'K^{x_1}_{t} \\ \vdots & & \vdots & \vdots \\ {}'K^{x_m}_{t_1} & \cdots & {}'K^{x_m}_{t_1} & {}'K^{x_m}_{t} \\ K^{a}_{t_1} & \cdots & K^{a}_{t_m} & K^{a}_{t} \end{vmatrix}$$

Setting and integrating, we have

$$\Delta^{s_1 \cdots s_m}_{s_1 \cdots s_m} = \int_a^b ds_i \int_{t_i=s_i}^{b} dt_i \begin{vmatrix} {}'K^{s_1}_{t_1} & \cdots & {}'K^{s_1}_{t_m} \\ \vdots & & \vdots \\ {}'K^{s_m}_{t_1} & \cdots & {}'K^{s_m}_{t_m} \end{vmatrix} + \int_a^b ds_i \int_{t_i=s_i}^{b} dt_i \int_a^b dt \begin{vmatrix} {}'K^{s_1}_{t_1} & \cdots & {}'K^{s_1}_{t_m} & {}'K^{s_1}_{t} \\ \vdots & & \vdots & \vdots \\ {}'K^{s_m}_{t_1} & \cdots & {}'K^{s_m}_{t_m} & {}'K^{s_m}_{t} \\ K^{a}_{t_1} & \cdots & K^{a}_{t_m} & K^{a}_{t} \end{vmatrix}$$

Applying Dirichlet's formula

$$\int_{s_i=a}^{b} ds_i \int_{t_i=s_i}^{b} dt_i (\quad) = \int_{t_i=a}^{b} dt_i \int_{s_i=a}^{t_i} ds_i (\quad)$$

we have

$$\Delta^{s_1 \cdots s_m}_{s_1 \cdots s_m} = \int_a^b dt_i \int_a^{t_i} ds_i \begin{vmatrix} {}'K^{s_1}_{t_1} & \cdots & {}'K^{s_1}_{t_m} \\ \vdots & & \vdots \\ {}'K^{s_m}_{t_1} & \cdots & {}'K^{s_m}_{t_m} \end{vmatrix}$$

$$+ \int_a^b dt_i \int_a^b dt \int_a^{t_i} ds_i \begin{vmatrix} {}'K^{s_1}_{t_1} & \cdots & {}'K^{s_1}_{t_m} & {}'K^{s_1}_{t} \\ \vdots & & \vdots & \vdots \\ {}'K^{s_m}_{t_1} & \cdots & {}'K^{s_m}_{t_m} & {}'K^{s_m}_{t} \\ K^{a}_{t_1} & \cdots & K^{a}_{t_m} & K^{a}_{t} \end{vmatrix}$$

$$= \int_a^b dt_i \begin{vmatrix} K^{t_1}_{t_1} - K^{a}_{t_1} & \cdots & K^{t_1}_{t_m} - K^{a}_{t_m} \\ \vdots & & \\ K^{t_m}_{t_1} - K^{a}_{t_1} & \cdots & K^{t_m}_{t_m} - K^{a}_{t_m} \end{vmatrix} +$$

$$+ \int_a^b dt_i \int_a^b dt \begin{vmatrix} K_{t_1}^{t_1} & \cdots & K_{t_m}^{t_1} & K_t^{t_1} \\ \vdots & & \vdots & \vdots \\ K_{t_1}^{t_m} & \cdots & K_{t_m}^{t_m} & K_t^{t_m} \\ K_{t_1}^{a} & \cdots & K_{t_m}^{a} & K_t^{a} \end{vmatrix}$$

The first determinant may be split up into 2^m determinants, of which all but m + 1 have two proportional columns. Omitting these, which vanish, and rearranging the variables of integration, we have

$$\Delta_{s_1 \cdots s_m}^{s_1 \cdots s_m} = \int_a^b dt_i \begin{vmatrix} K_{t_1}^{t_1} & \cdots & K_{t_m}^{t_1} \\ \vdots & & \vdots \\ K_{t_1}^{t_m} & \cdots & K_{t_m}^{t_m} \end{vmatrix} - m \int_a^b dt_i \begin{vmatrix} K_{t_1}^{a} & K_{t_2}^{t_1} & \cdots & K_{t_m}^{t_1} \\ \vdots & \vdots & & \vdots \\ K_{t_1}^{a} & K_{t_2}^{t_m} & \cdots & K_{t_m}^{t_m} \end{vmatrix}$$

$$+ \int_a^b dt_i \int_a^b dt \begin{vmatrix} K_t^{a} & K_{t_1}^{a} & \cdots & K_{t_m}^{a} \\ K_t^{t_1} & K_{t_1}^{t_1} & \cdots & K_{t_m}^{t_1} \\ \vdots & \vdots & & \vdots \\ K_t^{t_m} & K_{t_1}^{t_m} & \cdots & K_{t_m}^{t_m} \end{vmatrix}$$

$$= \overset{*}{\Delta}{}_{t_1 \cdots t_m}^{t_1 \cdots t_m} + \overset{*}{\Delta}{}_{[t]\, t_1 \cdots t_m}^{a \;\; t_1 \cdots t_m} - m \int_a K_{t_1}^{a} dt_i \begin{vmatrix} K_{t_2}^{t_2} & \cdots & K_{t_m}^{t_2} \\ \vdots & & \\ K_{t_2}^{t_m} & \cdots & K_{t_m}^{t_m} \end{vmatrix}$$

$$+ m(m-1) \int_a^b dt_i \, du \, K_u^{a} \begin{vmatrix} K_{t_1}^{u} & K_{t_2}^{u} & \cdots & K_{t_{m-1}}^{u} \\ K_{t_1}^{t_2} & K_{t_2}^{t_2} & \cdots & K_{t_{m-1}}^{t_2} \\ \vdots & \vdots & & \vdots \\ K_{t_1}^{t_{m-1}} & K_{t_2}^{t_{m-1}} & \cdots & K_{t_{m-1}}^{t_{m-1}} \end{vmatrix}$$

$$= \overset{*}{\Delta}{}_{t_1 \cdots t_m}^{t_1 \cdots t_m} + \overset{*}{\Delta}{}_{[t]\, t_1 \cdots t_m}^{a \;\; t_1 \cdots t_m} - m K_{[t]}^{a} \overset{*}{\Delta}{}_{t_1 \cdots t_{m-1}}^{t_1 \cdots t_{m-1}} + m(m-1) K_u^{a} \overset{*}{\Delta}{}_{[t]\, t_1 \cdots t_{m-2}}^{u \;\; t_1 \cdots t_{m-2}}$$

where $\overset{*}{\Delta}{}^{x_1 \cdots x_m}_{s_1 \cdots s_m}$ refer to the kernel K^x_s.

Multiplying by $\frac{1}{m!}$ and summing from m=1 to ∞, we have

$$D - [1 + K^a_{[t]}] = \overset{*}{D} - 1 + \overset{*}{D}{}^a_{[t]} - K^a_{[t]} - K^a_{[t]} \overset{*}{D} + K^a_u \overset{*}{D}{}^u_{[t]}$$

(8.4) i.e.

$$D = \overset{*}{D} + [\overset{*}{D}{}^a_{[t]} - \overset{*}{D} K^a_{[t]} + K^a_u \overset{*}{D}{}^u_{[t]}]$$

Taking the first of Fredholm's relations between D and its first minor, setting $x = a$, and integrating with respect to s on (a, b) we have

(8.5) $$\overset{*}{D}{}^a_{[t]} - \overset{*}{D} K^a_{[t]} + K^a_u \overset{*}{D}{}^u_{[t]} = 0$$

Hence $D = \overset{*}{D}$ and our theorem is proved.

It may be useful to note that the kernel $-\frac{D^x_s}{D}$ of the reciprocal transformation to (8.2) has a continuous partial derivative with respect to x under the assumption that $\frac{\partial K^x_s}{\partial x} = {}'K^x_s$ is continuous in (a, b).

For

$$\frac{\partial}{\partial x} \Delta^{x\, x_1 \cdots x_m}_{s\, x_1 \cdots x_m} = \int_a^b dx_i \begin{vmatrix} {}'K^x_s & {}'K^x_{x_1} & \cdots & {}'K^x_{x_m} \\ K^{x_1}_s & K^{x_1}_{x_1} & \cdots & K^{x_1}_{x_m} \\ \vdots & \vdots & & \vdots \\ K^{x_m}_s & K^{x_m}_{x_1} & \cdots & K^{x_m}_{x_m} \end{vmatrix}$$

which is continuous under the assumptions.

Since ${}'K^x_s$ is continuous in (a, b) it is bounded. Now if M be the greater of the bounds of $|K^x_s|$ and $|{}'K^x_s|$, we have by Hadamard's lemma the dominating series

$$\sum_{m=0}^{\infty} \frac{(m+1)^{\frac{m+1}{2}} M^{m+1} (b-a)^m}{m!}$$

for $$\frac{\partial D^x_s}{\partial x} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial}{\partial x} \Delta^{x\, x_1 \cdots x_m}_{s\, x_1 \cdots x_m}$$

As in Part I §3, this converges uniformly in (a, b) for all M, so that $\frac{\partial}{\partial x}\left(-\frac{D^x_s}{D}\right)$ exists and is continuous in (a, b).

Part III

Projective Transformations in Function Space.

§1 Introduction: In the paper of Dines (l.c.), he considers the function-space analogue of the general projective transformation in n-space.

$$x_i = \frac{a_i + b_i\bar{x}_i + \sum_{j=1}^{m} c_{ij}\bar{x}_j}{d + \sum_{j=1}^{m} e_j\bar{x}_j}$$

If we consider the Fredholm transformation as the function-space analogue of the transformation

$$x_i = \bar{x}_i + \sum_{j=1}^{m} c_i^j\bar{x}_j$$

we have that the above analogue is

$$(1.1)\quad \varphi'(x) = \frac{\alpha(x) + \beta(x)\varphi(x) + \int_0^1 \gamma(x,s)\varphi(s)ds}{\delta + \int_0^1 \epsilon(s)\varphi(s)ds}$$

In this paper we shall reduce the consideration of this transformation to that of a bordered Fredholm transformation of the type considered above with m=1. This reduction is suggested by Hildebrandt (l.c.). We shall get Dines' results somewhat simpler, and shall show that the one-parameter family of transformations generated by the infinitesimal transformation is a group in the Lie sense.

§2 Reduction of Dines' Form to Type Studied Earlier.

Let us write (1.1) in a form more convenient for our use, and introduce the convention, used earlier, of letting a repeated sub- and super-script indicate Riemann integration on (a,b). Then (1.1) becomes

$$(2.1)\quad \varphi^x = \frac{K^x\bar{\varphi}^x + K_s^x\bar{\varphi}^s + K_1^x}{K_s^1\bar{\varphi}^s + K_1^1}$$

The retention of the 1's as indices will later emphasize the connection with bordered Fredholm transformations of third kind.

Let $\varphi^x = \frac{y^x}{y}$, $\bar{\varphi}^x = \frac{\bar{y}^x}{\bar{y}}$, where the pairs $(y^x, y)(\bar{y}^x, \bar{y})$

are homogeneous. Substituting in (2.1), simplifying and equating numerators and denominators, we have:

$$(2.2)\qquad \begin{aligned} y^x &= K^x\bar{y}^x + K^x_s\bar{y}^s + K^x_1\bar{y} \\ y &= \phantom{K^x\bar{y}^x +{}} K^1_s\bar{y}^s + K^1_1\bar{y} \end{aligned}$$

This is a bordered Fredholm transformation of third kind, which has been already studied and is entirely equivalent to (2.1). Referring to Part I §10 we have the definition of the determinant of this transformation as

$$(2.3)\qquad D\begin{bmatrix} K^x_s/K^{(s)} & K^x_{\bar{y}} \\ K^1_s/K^{(s)} & K^1_{\bar{y}} \end{bmatrix}$$

As before, a parenthesis about any index suspends the integration convention with respect to that index.

Expression (2.3) is the same as the B defined by Dines to be the determinant of the transformation. The following theorems given by Dines follow from Part I §11.

<u>Theorem I:</u> <u>The determinant of the product of two projective transformations is equal to the product of their determinants.</u>

<u>Theorem II:</u> <u>If the determinant of the transformation (1.1) is different from zero, then (1.1) has a unique solution for φ in terms of φ', namely</u>

$$\varphi^x = \frac{\alpha'(x) + \beta'(x)\varphi'(x) + \int_0^1 \gamma'(x,s)\varphi'(s)ds}{\delta' + \int_0^1 \epsilon'(s)\varphi'(s)ds}$$

where $\alpha', \beta', \gamma', \delta', \epsilon'$ are given in terms of $\alpha, \beta, \gamma, \delta, \epsilon$.

We shall restate the latter as follows:

<u>Theorem II'</u> <u>If the determinant D (2.3) is not zero then (1.1) has a unique solution for $\bar{\varphi}^x$ in terms</u>

of φ^x, namely

$$(2.4) \qquad \bar{\varphi}^x = \frac{D/K^x\varphi^x - D^x_s/K^x\varphi^s - D^x_1/K^x}{D' - D'_s\varphi^s}$$

where $D' = D - D'_1$ and D^x_s, D^x_1, D'_s, D'_1 are the bordered Fredholm first minors of (2.3).

Dines gives two corollaries to Theorem II, the first, that the product of the determinants of a transformation and of its inverse is unity, following from the fact that the determinant of the identity transformation is unity. In our notation, the second becomes:

Corollary 2′:

$$(2.5) \qquad [D' - D'_s\varphi^s]\cdot[K'_1 + K'_s\bar{\varphi}^s] = D$$

For

$$D' - D'_s\varphi^s = D' - \frac{D'_s K^s\bar{\varphi}^s + D'_u K^u_s\bar{\varphi}^s + D'_u K^u_1}{K'_1 + K'_s\bar{\varphi}^s}$$

so that $[K'_1 + K'_s\bar{\varphi}^s][D' - D'_s\varphi^s] =$

$$D'K'_1 + D'K'_s\bar{\varphi}^s - D'_sK^s\bar{\varphi}^s - D'_uK^u_s\bar{\varphi}^s - D'_uK^u_1 =$$

$$D'K'_1 - D'_uK^u_1 - \bar{\varphi}^s[D'_sK^s - D'K'_s + D'_uK^u_s]$$

Referring to (2.68) and (2.66) below, we get (2.5).

The eight relations that Dines gives between B and its first minors, are in our notation:

$$(2.61) \qquad K^{(s)}D^x_s - DK^x_s + K^{(s)}/K^t\, K^x_t D^t_s + K^{(s)}K^x_1 D'_s = 0$$

$$(2.62) \qquad D^x_s K^{(s)} - DK^x_s + D^x_t K^t_s + D^x_1 K'_s = 0$$

$$(2.63) \qquad D^x_1 + K^x_1 D' + K^x_t/K^t\, D^t_1 = 0$$

$$(2.64) \qquad -DK^x_1 + D^x_t K^t_1 + D^x_1 K'_1 = 0$$

$$(2.65) \qquad -DK'_s + K^{(s)}/K^t\, K'_t D^t_s + K^{(s)}K'_1 D'_s = 0$$

$$(2.66) \qquad D'_s K^{(s)} - D'K'_s + D'_t K^t_s = 0$$

$$(2.67) \qquad -K'_1 D' + K'_t D^t_1 = -D$$

$$(2.68) \qquad -D'K'_1 + D'_t K^t_1 = -D$$

These are derived from (10.5) Part I by letting n=1.

Dines proves, without stating it as a theorem that the set of transformations (1.1) whose determinants are not zero form a group. This follows readily from Part I.

§ 3 The Infinitesimal Projective Transformation:

For the identity transformation,

$$K^x = K_1^1 = 1, \quad K_s^x = K_1^x = K_s^1 = 0$$

Hence for an infinitesimal transformation we can give each of these a small increment. Following Dines we take $K_1^1 = 1$ for the infinitesimal transformation, which on account of the fractional character is not less general.

Hence our infinitesimal transformation will have kernels as follows:

$$(3.1) \quad K^x = 1 + L^x \delta t, \; K_s^x = L_s^x \delta t, \; K_1^x = L_1^x \delta t, \; K_s^1 = L_s^1 \delta t, \; K_1^1 = 1,$$

and our infinitesimal transformation is

$$(3.2) \qquad \varphi^x = \frac{\bar{\varphi}^x + L^x \bar{\varphi}^x \delta t + L_s^x \bar{\varphi}^s \delta t + L_1^x \delta t}{1 + L_s^1 \bar{\varphi}^s \delta t}$$

expanding the denominator and dropping powers of δt higher than the first, we have:

$$(3.3) \qquad \varphi^x = \bar{\varphi}^x + \delta t [L^x \bar{\varphi}^x + L_s^x \bar{\varphi}^s + L_1^x - \bar{\varphi}^x L_s^1 \bar{\varphi}^s]$$

A family of transformations generated by continuous application of (3.3),

$$(3.4) \qquad \bar{\varphi}^x(t) = \frac{K^x(t)\varphi^x + K_s^x(t)\varphi^s + K_1^x(t)}{K_s^1(t)\varphi^s + K_1^1(t)}$$

will then transform φ^x into $\bar{\varphi}^x(t)$ where $\bar{\varphi}^x(t)$ satisfies the integro-differential equation,

$$(3.5) \qquad \frac{\partial \bar{\varphi}^x(t)}{\partial t} = L^x \bar{\varphi}^x(t) + L_s^x \bar{\varphi}^s(t) + L_1^x(t) - \bar{\varphi}^x(t) L_s^1 \bar{\varphi}^s(t)$$

with the initial condition $\bar{\varphi}^x(0) = \varphi^x$

The term "regular infinitesimal projective transformation" has been given to (3.3) by Kowalewski.

§4 The Finite Transformations Generated by an Infinitesimal Transformation (3.3). The question now to be solved is; given the L's (3.3), can we find a set of K's depending on t, such that $\bar{\varphi}^x(t)$ defined by (3.4) will satisfy (3.5) and the initial condition $\bar{\varphi}^x(0)=\varphi^x$.

If we follow Dines and differentiate (3.4) with respect to t, express φ^x in terms of $\bar{\varphi}^x(t)$ by (2.4), eliminate the denominator by (2.5) and equate corresponding terms of the result to those of (3.5) we have four equations. Adding a fifth to show that $\left.\frac{\partial K_1^1(t)}{\partial t}\right)_{t=0}=0$, we have five integro-differential equations in the five K's Solving these for the derivatives of the K's, we get:

$$(4.11)\quad 'K_s^x = L^x K_s^x + L_s^x K^{(s)} + L_u^x K_s^u + L_1^x K_s^1$$
$$(4.12)\quad 'K_1^x = L^x K_1^x + L_u^x K_1^u + L_1^x K_1^1$$
$$(4.13)\quad 'K_s^1 = L_s^1 K^{(s)} + L_u^1 K_1^u$$
$$(4.14)\quad 'K_1^1 = L_u^1 K_1^u$$
$$(4.15)\quad 'K^x = L^x K^x$$

where the primes, now and henceforth, indicate differentiation with respect to t.

Barnett[1] derives these equations much easier by using homogeneous coördinates as in (2.2). In our notation this is as follows.

Writing $\varphi^x = y^x/y$, $\bar{\varphi}^x(t) = \bar{y}^x(t)/\bar{y}(t)$, (3.4) becomes

$$(4.2)\quad \begin{aligned} \bar{y}^x(t) &= K^x(t)\,y^x + K_s^x\,y^s + K_1^x\,y \\ \bar{y}(t) &= K_s^1(t)\,y^s + K_1^1(t)\,y \end{aligned}$$

[1] I. A. Barnett: "The Transformations Generated by an Infinitesimal Projective Transformation in Function Space." Bull. Am. Math. Soc. XXXVI (1930) p 273.

and (3.5) may be written

$$\frac{\bar{y}(t)\,{}'\bar{y}^x(t) - \bar{y}^x(t)\,{}'\bar{y}(t)}{[\bar{y}(t)]^2} = \frac{L^x\bar{y}^x(t)\bar{y}(t) + L^x_s\bar{y}^s(t)\bar{y}(t) + L^x_1[\bar{y}(t)]^2 - \bar{y}^x(t)L^1_s\bar{y}^s(t)}{[\bar{y}(t)]^2}$$

Cancelling both denominators, letting $\bar{y}(t) = 0$ we have, cancelling $\bar{y}^x(t)$,

$$(4.32) \qquad {}'\bar{y}(t) = L^1_s\bar{y}^s(t)$$

adding $\bar{y}^x(t)$ times (4.32) to the previous we have, cancelling $\bar{y}(t)$

$$(4.31) \qquad {}'\bar{y}^x(t) = L^x\bar{y}^x(t) + L^x_s\bar{y}^s(t) + L^x_1\bar{y}(t)$$

From (4.2) and (4.3) we can get ${}'\bar{y}^x(t)$, ${}'\bar{y}(t)$ in terms of $(y^x_; y)$. Differentiating (4.2) and replacing the left hand sides by this expression, we have

$$(4.41) \qquad L^xK^x(t)y^x + L^xK^x_s(t)y^s + L^xK^x_1(t)y + L^x_sK^s(t)y^s + L^x_uK^u_s(t)y^s$$
$$+ L^x_uK^u_1(t)y + L^x_1K^1_s(t)y^s + L^x_1K^1_1(t)y =$$
$${}'K^x(t)y^x + {}'K^x_s(t)y^s + {}'K^x_1(t)y$$

$$(4.42) \qquad L^1_sK^s(t)y^s + L^1_uK^u_s(t)y^s + L^1_uK^u_1(t)y =$$
$${}'K^1_s(t)y^s + {}'K^1_1(t)y$$

Equating coefficients[1] of y^x, y^s and y in (4.4) gives us (4.1). The initial conditions are

$$K^x(0) = 1, \quad K^x_s(0) = 0, \quad K^x_1(0) = 0, \quad K^1_s(0) = 0, \quad K^1_1(0) = 1.$$

Now let us assume that the K's are expressible as power series in t as follows:

$$(4.5) \qquad K^\alpha_\beta(t) = \sum_{m=0}^{\infty}\frac{t^m}{m!}K^\alpha_\beta[m]$$

where α, β may be any set of indices for which the K's are defined.

Substituting in (4.1) and equating powers of t, we

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(1) That this is legitimate is proved in the unpublished paper by Michal and Peterson already referred to.

have the following recurrence formulas for the $K^{\alpha}_{\beta}[m]$

(4.61) $K^x_s[m+1] = L^x K^x_s[m] + L^x_s K^{(s)}[m] + L^x_u K^u_s[m] + L^x_1 K^1_s[m]$

(4.62) $K^x_1[m+1] = L^x K^x_1[m] + L^x_u K^u_1[m] + L^x_1 K^1_1[m]$

(4.63) $K^1_s[m+1] = L^1_s K^{(s)}[m] + L^1_u K^u_s[m]$

(4.64) $K^1_1[m+1] = L^1_u K^u_1[m]$

(4.65) $K^x[m+1] = L^x K^x[m]$

with the initial conditions

(4.7) $K^x[0] = K^1_1[0] = 1, \quad K^x_s[0] = K^x_1[0] = K^1_s[0] = 0.$

It readily follows that the K's are defined uniquely as functions of t, and that these K's formally satisfy the original condition, namely that $\bar{\varphi}^x(t)$ defined by (3.4) satisfy (3.5) and the initial conditions $\bar{\varphi}^x(0) = \varphi^x$. It remains to show that the infinite series (4.5) converge.

Let L^x, L^x_s, L^x_1, L^1_s all be less in absolute value than M; then we will show by mathematical induction that $|K^{\alpha}_{\beta}[m]| < (4M)^m (b-a)^m$. By (4.6) this is true for $m=0$. Then if it be true for $m=m$ by (4.6) $|K^{\alpha}_{\beta}[m+1]| < 4M(b-a)|K^{\gamma}_{\beta}[m]| < (4M)^{m+1}(b-a)^{m+1}$. Hence series (4.5) are each dominated by $\sum_{m=0}^{\infty} \frac{(4Mt)^m}{m!}(b-a)^m$ which converges for all M and t. This means that the formal solution (4.5) is an actual solution.

It follows from either (4.14) or (4.65) and the initial conditions that $K^x(t) = e^{L^x t} \neq 0$ in (a,b). Furthermore the determinant of the infinitesimal transformation (3.2) is

$$D(\delta t) = 1 + \int_a^b \begin{vmatrix} L^x_x \delta t & L^x_1 \delta t \\ L^1_s \delta t & \end{vmatrix} dx + \text{terms of higher power in } \delta t$$

$$= 1 + L^x_x \delta t + \text{terms of higher power in } \delta t.$$

Now $D(t+\delta t) = D(t) \cdot D(\delta t)$ by the product theorem.

Hence $D(t+\delta t) - D(t) = D(t)[1 + L^x_x \delta t + ---] - D(t)$

$$= D(t)\, L^x_x \delta t$$

Dividing by δt and passing to the limit as $\delta t \to 0$ we have

$$\frac{dD(t)}{dt} = D(t) L^x_x$$

Hence, since $D(0) = 1$

$$D(t) = e^{L^x_x t} \neq 0 \tag{4.8}$$

Hence we have, with Dines,

<u>Theorem III:</u> <u>The finite transformations generated by a regular infinitesimal projective transformation</u>

$$\delta\varphi^x = [L^x\varphi^x + L^x_s\varphi^s + L^x_1 - \varphi^x L^1_s \varphi^s]\delta t$$

<u>constitute a one-parameter family of non-singular projective transformations. For any value of the parameter t, the kernels are given by (4.5), (4.6) and (4.7), and the determinant by (4.8).</u>

§5 <u>The Family Generated Forms a Lie Group:</u> We shall next need a set of formulas concerning the K's which we proceed to prove by mathematical induction. They are:

$$K^x_s[p] = K^x[m]K^x_s[p-m] + K^x_s[m]K^{(s)}[p-m] + K^x_u[m]K^u_s[p-m] + K^x_1[m]K^1_s[p-m] \tag{5.11}$$

$$K^x_1[p] = K^x[m]K^x_1[p-m] + K^x_u[m]K^u_1[p-m] + K^x_1[m]K^1_1[p-m] \tag{5.12}$$

$$K^1_s[p] = K^1_s[m]K^{(s)}[p-m] + K^1_u[m]K^u_s[p-m] + K^1_1[m]K^1_s[p-m] \tag{5.13}$$

$$K^1_1[p] = K^1_u[m]K^u_1[p-m] + K^1_1[m]K^1_1[p-m] \tag{5.14}$$

$$K^x[p] = K^x[m]K^x[p-m] \tag{5.15}$$

For $p=0,\ m=0$, these become

$0=0,\ 0=0,\ 0=0,\ 1=1,\ 1=1,$ respectively,

For $p=1,\ m=0$	$p=1,\ m=1$
$K^x_s[1] = K^x_s[1]$	$K^x_s[1] = K^x_s[1]$
$K^x_1[1] = K^x_1[1]$	$K^x_1[1] = K^x_1[1]$
$K^1_s[1] = K^1_s[1]$	$K^1_s[1] = K^1_s[1]$
$K^1_1[1] = K^1_1[1]$	$K^1_1[1] = K^1_1[1]$
$K^x[1] = K^x[1]$	$K^x[1] = K^x[1]$

provided we remember the initial conditions (4.7).

Now assume that the recurrence formulas are true for all values of p, less than p. Then

$$K^x[m]\,K_s^x[p-m] + K_s^x[m]\,K^{(s)}[p-m] + K_u^x[m]\,K_s^u[p-m] + K_1^x[m]\,K_s^1[p-m]$$

$$= L^x K^x[m-1]\,K_s^x[p-m] + L^x K_s^x[m-1]\,K^{(s)}[p-m] + L_s^x K^{(s)}[m-1]\,K^{(s)}[p-m]$$

$$+ L_u^x K_s^u[m-1]\,K^{(s)}[p-m] + L_1^x K_s^1[m-1]\,K^{(s)}[p-m] + L^x K_u^x[m-1]\,K_s^u[p-m]$$

$$+ L_u^x K^u[m-1]\,K_s^u[p-m] + L_u^x K_v^u[m-1]\,K_s^v[p-m] + L_1^x K_v^1[m-1]\,K_s^v[p-m]$$

$$+ L^x K_1^x[m-1]\,K_s^1[p-m] + L_u^x K_1^u[m-1]\,K_s^1[p-m] + L_1^x K_1^1[m-1]\,K_s^1[p-m]$$

$$= L^x\{K^x[m-1]\,K_s^x[p-m] + K_s^x[m-1]\,K^{(s)}[p-m] + K_u^x[m-1]\,K_s^u[p-m] + K_1^x[m-1]\,K_s^1[p-m]\}$$

$$+ L_s^x\{K^{(s)}[m-1]\,K^{(s)}[p-m]\} + L_u^x\{K_s^u[m-1]\,K^{(s)}[p-m] + K^u[m-1]\,K_s^u[p-m] + K_v^u[m-1]\,K_s^v[p-m]$$

$$+ K_1^u[m-1]\,K_s^1[p-m]\} + L_1^x\{K_s^1[m-1]\,K^{(s)}[p-m] + K_v^1[m-1]\,K_s^v[p-m] + K_1^1[m-1]\,K_s^1[p-m]\}$$

$$= L^x K_s^x[p-1] + L_s^x K^{(s)}[p-1] + L_u^x K_s^u[p-1] + L_1^x K_s^1[p-1] = K_s^x[p] \text{ by (4.61)}$$

Similarly

$$K^x[m]\,K_1^x[p-m] + K_u^x[m]\,K_1^u[p-m] + K_1^x[m]\,K_1^1[p-m] =$$

$$L^x K^x[m-1]\,K_1^x[p-m] + L^x K_u^x[m-1]\,K_1^u[p-m] + L_u^x K^u[m-1]\,K_1^u[p-m]$$

$$+ L_u^x K_v^u[m-1]\,K_1^v[p-m] + L_1^x K_v^1[m-1]\,K_1^v[p-m] + L^x K_1^x[m-1]\,K_1^1[p-m]$$

$$+ L_u^x K_1^u[m-1]\,K_1^1[p-m] + L_1^x K_1^1[m-1]\,K_1^1[p-m] =$$

$$L^x\{K^x[m-1]\,K_1^x[p-m] + K_u^x[m-1]\,K_1^u[p-m] + K_1^x[m-1]\,K_1^1[p-m]\}$$

$$+ L_u^x\{K^u[m-1]\,K_1^u[p-m] + K_v^u[m-1]\,K_1^v[p-m] + K_1^u[m-1]\,K_1^1[p-m]\}$$

$$+ L_1^x\{K_v^1[m-1]\,K_1^v[p-m] + K_1^1[m-1]\,K_1^1[p-m]\}$$

$$= L^x K_1^x[p-1] + L_u^x K_1^u[p-1] + L_1^x K_1^1[p-1] = K_1^x[p] \text{ by (4.62)}$$

Also

$$K_s^1[m]\,K^{(s)}[p-m] + K_u^1[m]\,K_s^u[p-m] + K_1^1[m]\,K_s^1[p-m] =$$

$$L_s^1 K^{(s)}[m-1]\,K^{(s)}[p-m] + L_u^1 K^u[m-1]\,K_s^u[p-m] + L_u^1 K_s^u[m-1]\,K^{(s)}[p-m]$$

$$+ L_u^1 K_v^u[m-1]\,K_s^v[p-m] + L_u^1 K_1^u[m-1]\,K_s^1[p-m] =$$

$$L_s^1\{K^{(s)}[m-1]\,K^{(s)}[p-m]\} + L_u^1\{K^u[m-1]\,K_s^u[p-m] + K_s^u[m-1]\,K^{(s)}[p-m]$$

$$+ K_v^u[m-1]\,K_s^v[p-m] + K_1^u[m-1]\,K_s^1[p-m] = L_s^1 K^{(s)}[p-1] + L_u^1 K_s^u[p-1]$$

$$= K_s^1[p] \text{ by (4.63)}$$

Also $K_1^1[m]K_1^1[p-m]+K_u^1[m]K_1^u[p-m]=$

$$L_u^1K_1^u[m-1]K_1^1[p-m]+L_u^1K^u[m-1]K_1^u[p-m]+L_u^1K_v^u[m-1]K_1^v[p-m]$$

$$=L_u^1\{K_1^u[m-1]K_1^1[p-m]+K^u[m-1]K_1^u[p-m]+K_v^u[m-1]K_1^v[p-m]\}$$

$$=L_u^1K_1^u[p-1]=K_1^1[p] \text{ by } (4.64)$$

$$K^x[m]K^x[p-m]=L^xK^x[m-1]K^x[p-m]=L^xK^x[p-1]=K^x[p] \text{ by } (4.65)$$

Hence the formulas (5.1) are true. Let us write these as

$$(5.2)\qquad K_\beta^\alpha[p]=K_\gamma^\alpha[m]K_\beta^\gamma[p-m]$$

where the Greek letters indicate such and only such indices as appear explicitly in (5.1).

The product of two transformations of type (3.4) with parameters t_1 and t_2 is a projective transformation with kernels as follows:

$$(5.31)\qquad P_s^x(t_1,t_2)=K^x(t_1)K_s^x(t_2)+K_s^x(t_1)K^{(s)}(t_2)+K_u^x(t_1)K_s^u(t_2)+K_1^x(t_1)K_s^1(t_2)$$

$$(5.32)\qquad P_1^x(t_1,t_2)=K^x(t_1)K_1^x(t_2)+K_u^x(t_1)K_1^u(t_2)+K_1^x(t_1)K_1^1(t_2)$$

$$(5.33)\qquad P_s^1(t_1,t_2)=K_s^1(t_1)K^{(s)}(t_2)+K_u^1(t_1)K_s^u(t_2)+K_1^1(t_1)K_s^1(t_2)$$

$$(5.34)\qquad P_1^1(t_1,t_2)=K_u^1(t_1)K_1^u(t_2)+K_1^1(t_1)K_1^1(t_2)$$

$$(5.35)\qquad P^x(t_1,t_2)=K^x(t_1)K^x(t_2)=e^{(t_1+t_2)L^x}$$

These formulas (5.3) may be written

$$(5.4)\qquad P_\beta^\alpha(t_1,t_2)=K_\gamma^\alpha(t_1)K_\beta^\gamma(t_2)$$

where it is important to notice that the Greek indices have exactly the same range as in (5.2).

By (4.5) we have

$$P_\beta^\alpha(t_1,t_2)=\sum_{m=0}^{\infty}\frac{t_1^m}{m!}K_\gamma^\alpha[m]\sum_{n=0}^{\infty}\frac{t_2^n}{n!}K_\beta^\gamma[n]$$

Setting $n=p-m$, we have

$$P_\beta^\alpha(t_1,t_2)=\sum_{m=0}^{\infty}\frac{t_1^m}{m!}K_\gamma^\alpha[m]\sum_{p=m}^{\infty}\frac{t_2^{p-m}}{(p-m)!}K_\beta^\gamma[p-m]$$

Inverting the order of summation, we have

$$P^{\alpha}_{\beta}(t_1,t_2) = \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{m=0}^{p} c_{pm} t_1^{\,m} t_2^{\,p-m} \left(K^{\alpha}_{\gamma}[m]\, K^{\gamma}_{\beta}[p-m]\right)$$

Making use of the fact mentioned above that the range of the Greek indices in (5.4) is that in (5.2), we may apply the latter, giving

(5.5) $$P^{\alpha}_{\beta}(t_1,t_2) = \sum_{p=0}^{\infty} \frac{K^{\alpha}_{\beta}[p]}{p!} \sum_{m=0}^{p} c_{pm} t_1^{\,m} t_2^{\,p-m} = \sum_{p=0}^{\infty} \frac{(t_1+t_2)^p}{p!} K^{\alpha}_{\beta}[p]$$

Hence the product transformation is a member of the family, with parameter t_1+t_2, and we may add to Theorem III of Dines, the following:

<u>Theorem III$'$: The one-parameter family of Theorem III is a one-parameter continuous group in the sense of Lie, and is in canonical form.</u>

The last phrase means that the parameter for the identity transformation is zero, and that that of the product is the sum of those of the factor of the product.

Conversely, if we have a one-parameter continuous group of projective transformations, it may be reduced to canonical form. If it be so reduced to

(5.6) $$\bar{\varphi}^{x}(t) = \frac{K^{x}(t)\varphi^{x} + K^{x}_{1}(t) + K^{x}_{s}(t)\varphi^{s}}{K^{1}_{1}(t) + K^{1}_{s}(t)\varphi^{s}}$$

where t is the canonical parameter, then replace t by δt and expand the K's, neglecting powers of δt higher than the first. Since $\bar{\varphi}^{x}(0) = \varphi^{x}$, we have

$$\bar{\varphi}^{x}(\delta t) = \frac{\varphi^{x} + K^{x}[1]\varphi^{x}\delta t + K^{x}_{1}[1]\delta t + K^{x}_{s}[1]\varphi^{s}\delta t}{1 + K^{1}_{1}[1]\delta t + K^{1}_{s}[1]\varphi^{s}\delta t}$$

where $K^{\alpha}_{\beta}[1]$ is the coefficient of δt in the above expansion.

Expanding the denominator and neglecting higher powers of δt:

$$\bar{\varphi}^x \delta t = \varphi^x + \delta t\left[\left(K^x[1] - K_1^1[1]\right)\varphi^x + K_1^x[1] + K_s^x[1]\varphi^s - \varphi^x K_s^1[1]\varphi^s\right]$$

which is an infinitesimal transformation of type (3.3). This transformation generates a one-parameter continuous group which may be (5.6). Hence, it seems likely that any one-parameter continuous group of projective transformations in canonical form is one generated by continuous application of an infinitesimal projective transformation of type (3.3).