THE SELF-ENERGY OF THE SCALAR NUCLEON

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ABSTRACT

The analogue of the variational principle was applied on the path integral to obtain the self-energy of the scalar nucleon interacting with the neutral scalar meson (with vanishing mass) field for variable coupling constant g^2. To obtain the lowest energy of the nucleon from the asymptotic form of its kernel it is essential to replace it (t:time) by another variable τ, say. Therefore it was first shown that the equation for a scalar nucleon in a given external field has the form which can be obtained from the Klein-Gordon equation by formally changing it by τ. It is known that the Klein-Gordon equation may be described, though indirectly, by Lagrangian form by introducing a fifth parameter u and a differential equation for a new function of the space-time variables and u. Then the kernel of this equation is represented by a path integral over trajectories in space-time. To apply the variational technique on this kernel and find the best value of the lowest "energy" corresponding to the "Hamiltonian" of this equation, here again we need to replace iu by σ. The kernel corresponding to the transition in which there are no mesons present initially and finally was obtained by integrating out the meson field in the kernel for the case of a given potential. The "energy" contains the logarithmic divergence which was cut off by the analogy to electrodynamics. The kernel of a nucleon for the same transition can be given in terms of the kernel just explained. From the asymptotic form of this kernel the best value of the lowest (or self-) energy M of the nucleon was obtained. M was given in terms of preliminarily renormalized mass m'. It was found that no solution exists for too large value of g^2. For practical purposes the procedure to find the theoretical mass m in terms of M and g^2 was also explained.

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I. INTRODUCTION

The meson theory of nuclear forces[1] was first established on the basis of the analogy to the Coulomb force between the charged particles interacting through the electromagnetic field. However, as the experimental findings are accumulated it was revealed that these two forces are quite different in many respects and that the nuclear forces are far more complicated than the Coulomb force. Furthermore a number of experimental data pertaining especially to a system of nucleons, such as the binding energy, the magnetic moment, and the quadrupole moment of the deuteron, the cross-section for neutron-proton scattering, etc., supplied important information for the development of the meson theory. In an attempt to explain these findings various types of meson theories and their mixtures were introduced. In these theories the nucleon was treated as a point source to make the theory relativistically invariant, and the perturbation technique was invariably employed despite the experimental indication of not-weak coupling between the nucleon and the meson. The predictions of these relativistic theories were not very satisfactory in explaining the experimental data.

Now the methods of approximation used in these theories are either the analogue of the classical electrodynamics which determines the fields produced by given sources neglecting the reaction of the field on the source, or that of the perturbation method of the quantum electrodynamics based on the weak coupling between the field and the particle. However, in the meson theory, the nucleon recoil may not legitimately be neglected on the one hand, while the coupling between the meson and the nucleon, inferred from the experiments, can not be said to be small enough to justify the use of the perturbation technique on the other. Therefore the application of these methods, especially the latter, to the meson theory may be objected to, because of this inconsistency.

In addition to this, the point source model, with the exception of the pseudoscalar meson theory, results in a singularity and infinity of higher order as the order of the approximation increases. To avoid these difficulties inherent to the point source people tried to assume that the source is extended over a finite size. This makes the theory convergent only at the expense of the relativistic invariance of the theory. It was shown for this theory that the perturbation approximation is not valid for the values of the size of the source and the coupling constant that are required from the experiments, but the other approximation of strong coupling must be employed. It is clear that the stronger the meson-nucleon coupling the smaller the energy difference between the successive excited states so that for a very strong coupling case the excitation energy is much less than the rest mass of the meson. These excited states are isobars of the ground state of the nucleon with anomalous charge and spin. On the other hand, for a very weak coupling case the minimum excitation energy will be so large that the excited states of the nucleon can not be found unless the energy involved is very high.

Various consequences of the strong coupling approximation were extensively investigated by many workers(2), especially the isobar separations, the nuclear forces, the anomalous magnetic moment of the nucleon, and the scattering cross-section of the meson by the nucleon. The cross-section for the meson scattering by the nucleon favors the strong coupling theory inasmuch as it gives a smaller value than the inacceptably too large value predicted by the weak coupling theory. However, the strong coupling theory predicts that the magnetic moment of the deuteron is only a few percent of the observed value, and it also predicts the instability of highly charged nuclei contradicting the experimental findings. Because of these fatal deficiencies and the lack of relativistic covariance, it was once considered that the strong coupling theory based on the extended sources should be abandoned, and the weak coupling theory should be examined more carefully.

Recently a useful method has been developed which is applicable to a class of problems in which the field-particle coupling is neither weak nor strong and in which the recoil of the source particle may be neglected[3]. This is widely known as "the intermediate coupling approximation" and has been applied to such problems as the self-energy of the nucleon, the nuclear forces, the photo-meson production, the meson-nucleon scattering, and the slow electrons in a polar crystal, etc. However, this neglect of recoil is difficult to justify so that the validity of these calculations is not established.

What has been reviewed so far tells us the necessity of employing the appropriate approximation according to the strength of the coupling between the field and the particle of the problem under consideration. However, the most desirable is to invent a method, if any, which is valid for the entire range of the coupling strength. Although it may not be possible for such a method to be able to explain all the experimental findings consistently, it is not impossible to find a method which gives answers for any value of the coupling constant when we select some particularly simple problem for investigation. In fact, Professor Richard P. Feynman[4] treated very recently the problem of the slow electrons in a polar crystal by a variational technique extended to path integrals over trajectories for the arbitrary strength of the phonon-electron coupling. Besides this feature of giving answers for arbitrary magnitude of the coupling constants this method has another great advantage in that it avoids the objectionable approximation of "neglecting the recoil" of the source particle. He obtained the lowest energy of the system which is at least as accurate as previously known results. The effective mass of the polaron was also calculated.

In the present work the self-energy of the scalar nucleon interacting with the scalar neutral meson field was calculated as a function of the coupling constant using the same variational technique as mentioned above. For the sake of simplicity the rest mass of the meson was assumed to be zero. Although it is necessary to treat nucleons as Dirac particles and to consider, for example, the pseudoscalar symmetric meson theory, such a realistic problem accompanies a greater complexity indicating the necessity of more detailed study of this technique to be applied to such a problem. For this reason the case of the scalar nucleon interacting with the neutral scalar meson field was investigated here, in which no further complications pertaining to the operators such as Dirac matrices γ_μ, spin operator $\vec{\sigma}$, and isotopic spin operator $\vec{\tau}$ will occur. This work is meant as a preliminary test of the method to see whether the work needed to extend it to the real problem is justified.

In order to make use of the variational technique it is essential to replace it (t is the physical time) by another variable τ , say. The purpose of this is to arrange that the kernel K will have the asymptotic form e^{-ET} (T is a very large interval of τ). In this case it is the easiest to determine the lowest energy E of the system considered. Therefore, in Section II, the scalar nucleon field in an external neutral meson potential was treated by the theory of second quantization starting from the equation $-\partial\Omega/\partial\tau = H\Omega$ instead of the conventional one $-\partial\Omega/i\partial t = H\Omega$. Here Ω is the wave function describing the state of the scalar nucleon field and H is the operator Hamiltonian of the system. The argument here is similar to that of the Dirac electron in an external electromagnetic potential[5]. The differential equation satisfied by the scalar nucleon was obtained. This has the form which one would expect from the conventional (Klein-Gordon) equation of the scalar particle by substitution $it = \tau$. The contribution to the transition matrix element from processes of various order in the perturbation method was also calculated.

The Klein-Gordon equation satisfied by the scalar nucleon with wave function $\psi(x_\mu)$, where μ runs from 1 to 4 and $x_4 = \tau$, is, unfortunately, not one of those which permit the direct application of the Lagrangian formulation. For this reason we introduced a fifth parameter σ and described the trajectories in space-time (time here refers to τ) by giving x_μ as functions of this parameter. Then a differential equation for a new function $\phi(x_\mu, \sigma)$ was introduced in such a way that it allows the application of the Lagrangian formulation and the variational technique. At the same time the function Ψ is expressed in terms of ϕ in a simple manner. The kernel associated with ϕ corresponding to the transition from the vacuum to the vacuum of the meson field in the quantum mechanical case was obtained by first finding the kernel for the case when there is an external potential present and then integrating out the meson field operator remaining in it. This kernel contains all the effects of the self-action of the scalar nucleon, except for the effects of closed loops, interacting with the neutral scalar meson field. After having found the kernel K for the transition mentioned above in the form $\int e^S \mathcal{D} x_\mu(\sigma)$, where S is the action, we next need to find its asymptotic form to determine the lowest "energy" E_0 corresponding to "the Hamiltonian" of the equation satisfied by ϕ. Such path integrals as those involved in the evaluation of the asymptotic form of K are ordinarily very difficult to work out, but we can do this by means of an approximation method whose validity does not require that the coupling constant be particularly small or particularly large. The method used is the analogue of the variational technique mentioned before. We will sketch the variational procedure briefly here. In this technique, instead of the trial wave function in the conventional method, we choose a trial action S_1, say, which is a real quantity, quadratic in coordinates of the particle, and contains one or more parameters in it. It is essential that S_1 is a real quantity in order to make use of an important inequality valid for averages over any real quantities. Also it is very convenient to have S_1 quadratic in particle coordinates because for such a case the path integrals involved can be worked out most easily. In the present calculation we borrowed the trial action S_1 from the three-dimensional case of the polaron problem[4] with the direct extension to the four dimensions because our kernel K associated with ϕ is expressed as a path integral over trajectories in four dimensions. There are two parameters contained in S_1 which are to be varied at the later stage to obtain the best value of the lowest "energy" E_o. The kernel for arrival at x_μ starting from x_μ' has the asymptotic form $e^{-E_o\Sigma}$ (Σ is a very large interval of σ) because of our choice of the dependence of the differential equation for ϕ on one of the five independent variables σ. Let us now assume that we know the lowest "energy" E_1 corresponding to S_1 so that we can write $\int e^{S_1} \mathcal{D}x_\mu(\sigma) \sim e^{-E_1\Sigma}$, where the symbol $\sim$ means "asymptotically equal for large Σ". Then we see that

$$e^{-E_o\Sigma} \sim \int (e^{S-S_1}) e^{S_1} \mathcal{D}x_\mu(\sigma) \sim \left[\int (e^{S-S_1}) e^{S_1} \mathcal{D}x_\mu(\sigma) / \int e^{S_1} \mathcal{D}x_\mu(\sigma) \right] e^{-E_1\Sigma}.$$

The factor in square bracket may be looked upon as the average of e^{S-S_1} with a positive weight e^{S_1} and we shall write this $\langle e^{S-S_1} \rangle$. In the following the symbol $\langle \ldots\ldots \rangle$ will be used to denote the average in the sense just explained. By virtue of an inequality $\langle e^f \rangle \geqslant e^{\langle f \rangle}$ which holds for the averages over any real quantities f the relation obtained above can be written as $e^{-E_o\Sigma} \geqslant e^{\langle S-S_1 \rangle} e^{-E_1\Sigma}$. Here $\langle S-S_1 \rangle$ is defined by $\int (S-S_1) e^{S_1} \mathcal{D}x_\mu(\sigma) / \int e^{S_1} \mathcal{D}x_\mu(\sigma)$. When Σ is very large $\langle S-S_1 \rangle$ is proportional to Σ so we write $\langle S-S_1 \rangle = s\Sigma$. Calling $E = E_1 - s$ we finally have $E_o \leqslant E$. E contains free parameters through S_1 and hence the best value of the lowest "energy" E_o can be obtained by minimizing E with respect to these parameters. This is a brief account of how the variational technique works for the path integrals. The values of E_1 and s depend, in addition to their dependence on the parameters, on the values of the coordinates x_μ at the initial and the final space-time points. The expression for E corresponding to the transition $x_\mu = x_\mu' = 0$ was obtained first. E contains an integral which is divergent logarithmically.

In the case of a problem in four dimensions, however, the kernel arriving at $x_\mu = 0$ from $x_\mu' = 0$ is of no interest since this does not correspond to the actual motion of the particle in space-time. We are concerned with the case in which $x_\mu \neq 0$ holds at least for $\mu = 4$ at the final point when it started from $x_\mu' = 0$. The value of E corresponding to arrival at $x_\mu = X_\mu$ from $x_\mu' = 0$ can be obtained by partly making use of the result for the case $x_\mu = x_\mu' = 0$. This expression also contains a logarithmically divergent integral. The divergence was removed by a relativistic cut-off analogous to the one applied to quantum electrodynamics[6]. By this, E now contains the cut-off, Λ, logarithmically. However, the actual computation of minimizing E will not be performed at this stage and E will be left as it stands.

Finally, by combining the relation existing between ϕ and ψ with the fact that K is the kernel associated with ϕ it is easy to obtain the kernel $\mathcal{K}$ associated with ψ in terms of K. This relation enables us to evaluate the asymptotic form of $\mathcal{K} \sim e^{-\mathcal{E}_0 T}$ ($\mathcal{E}_0$ is the self-energy of the scalar nucleon and T is a very large interval of τ) from the corresponding form of $K \sim e^{-E_0 \Sigma}$. Hence the best estimate M of the self-energy $\mathcal{E}_0$ can be given in terms of the best value E of E_0 obtained by minimizing it with respect to the free parameters. In the course of calculation it turned out to be convenient to renormalize the rest mass m of the scalar nucleon to define the renormalized mass m'. An integral left unperformed in E was carried out for the case in which two parameters available are close to each other. Finally the ratio $(M/m')^2$ was computed numerically as a function of

$\mu = g^2/8\pi m'^2$ for the charged scalar nucleon (or a function of $\mu = g^2/2\pi m'^2$ for the neutral scalar nucleon), where M is the best value of the self-energy of the scalar nucleon. This ratio decreases, starting from 1 at $\mu = 0$, with increasing μ, as is expected, until it becomes about 0.88 near $\mu = 0.30$. And there can be no solution found beyond that value of μ. The possible reason why no solution appears for too large value of g^2 may be the following. That is, for strong enough attraction possibly the binding of a pair of nucleon and antinucleon (analogue of positronium) exceeds their rest masses so that pairs would be found ad infinitum releasing the energy by radiating mesons. Thus no sensible theory may exist for too large g^2 in the scalar theory. For the practical purposes a procedure to find the theoretical mass m of the nucleon from M, which is to be identified as the experimental mass, and the coupling constant g^2 was also given.

II. SCALAR NUCLEON IN AN EXTERNAL MESON POTENTIAL

In this section we will study the scalar nucleon in a given neutral scalar meson field by the theory of second quantization. This can be done in a manner similar to that of the Dirac electron in an external field[5]. However, since we are eventually interested in the calculation of the self-energy of such a particle by the variational technique we will develop the argument in terms of new fourth variable τ instead of t ($\tau = it$). The equation satisfied by the scalar nucleon and the transition matrix element will be obtained. The answer is in most cases simply what would result from the conventional expression (in which t is the variable instead of τ) replacing it by τ, as one expects. Nevertheless it was thought worth while to make sure, at least once, that such a prescription would give the correct result by following necessary steps in detail. Though we are not going to make use of the perturbation method in this work it is interesting to see how the contributions to the transition element coming from various orders of the approximation look; and to compare them with those of a Dirac electron in a given external potential.

The Hamiltonian for a charged scalar nucleon in a given neutral scalar meson field is given by ($\hbar = c = 1$)

$$H(t) = \int \Big\{ \pi^*(\vec{x}\,')\pi(\vec{x}\,') + \nabla\psi^*(\vec{x}\,')\cdot\nabla\psi(\vec{x}\,') + (m^2 + \sqrt{4\pi}\,g\,\chi(\vec{x}\,', t)) \times \psi^*(\vec{x}\,')\,\psi(\vec{x}\,') \Big\}\, d^3\vec{x}\,' , \qquad \text{(II. 1)}$$

with the commutation relations

$$\left[\pi(\vec{x}, t),\ \psi(\vec{x}\,', t)\right] = \left[\pi^*(\vec{x}, t),\ \psi^*(\vec{x}\,', t)\right] = \frac{1}{i}\,\delta(\vec{x} - \vec{x}\,') \quad . \qquad \text{(II. 2)}$$

Here ψ is the nucleon field operator and π is its canonically conjugate momentum, m is the rest mass of the nucleon, χ is the given meson potential, and g is the coupling constant. If we consider the neutral scalar nucleon instead of the charged one we will have

$$H(t) = \int \tfrac{1}{2}\left\{\pi^2 + (\nabla\psi)^2 + (m^2+2\sqrt{4\pi}\, g\chi)\psi^2\right\} d^3\vec{x}\,, \qquad \text{(II. 1')}$$

so that the difference is a factor of 2 in appropriate places. The state vector Ω describing the motion of the system is given by

$$-\frac{1}{i}\frac{\partial\Omega}{\partial t} = H\Omega\,.$$

Anticipating the use of the variational technique to obtain the self-energy of the scalar nucleon, as was briefly mentioned in the introduction, (see also Eq. (IV. 1) et. seq.) we replace it by τ and consider the following equation

$$-\frac{\partial\Omega}{\partial\tau} = H\Omega\,,$$

without asking about the physical meaning of this new variable τ. For the sake of the later reference it will prove convenient to establish some of the relations existing between corresponding quantities in both formalisms in which t or τ is the independent variable (t- or τ-formalism, respectively).

If a function f of $\vec{x}$ and t is of the form $f(\vec{x},t) = F(i\vec{x}, it)$ then it is $F(i\vec{x},\tau)$ in terms of $\vec{x}$ and τ and we will denote this by $\tilde{f}(\vec{x},\tau)$ with the tilde on f indicating explicitly that it is now considered as a quantity in τ-formalism. However, in the following the tilde will often be omitted whenever it is apparent that there will be no confusion by doing so. For the complex conjugate of f we have $f^*(\vec{x},t)=F(-i\vec{x},-it)=F(-i\vec{x},-\tau)$

and we will call the last expression $\tilde{f}^{\dagger}(\vec{x},\tau)$:

$$f^*(\vec{x},t) = \tilde{f}^{\dagger}(\vec{x},\tau). \qquad \text{(II. 3)}$$

This simply means that to take the complex conjugate (*) of a quantity in t-formalism corresponds to taking the complex conjugate of the spatial part and changing the sign of τ variable in τ-formalism ($\dagger$ operation). To the time derivatives in t-formalism, correspond the following:

$$\frac{\partial f(\vec{x},t)}{\partial t} = \frac{d\tau}{dt}\frac{\partial \tilde{f}(\vec{x},\tau)}{\partial \tau} = i\,\frac{\partial \tilde{f}(\vec{x},\tau)}{\partial \tau},$$

$$\frac{\partial f^*(\vec{x},t)}{\partial t} = \frac{d\tau}{dt}\frac{\partial \tilde{f}^{\dagger}(\vec{x},\tau)}{\partial \tau} = i\,\frac{\partial \tilde{f}^{\dagger}(\vec{x},\tau)}{\partial \tau}. \qquad \text{(II. 4a, b)}$$

We note that the commutation relations in τ-formalism are of the same form as those in t-formalism with * replaced by $\dagger$, since τ and t have a one-to-one correspondence. They are

$$\left[\tilde{\pi}(\vec{x},\tau),\ \tilde{\psi}(\vec{x}',\tau)\right] = \left[\tilde{\pi}^{\dagger}(\vec{x},\tau),\ \tilde{\psi}^{\dagger}(\vec{x}',\tau)\right] = \frac{1}{i}\,\delta(\vec{x}-\vec{x}'). \qquad \text{(II. 5)}$$

The theory of scalar particles was developed by Pauli and Weisskopf(7) who showed that the oppositely charged particles in this theory can be interpreted as corresponding to the electron and the positron of Dirac's hole theory. Let us arbitrarily call the positively charged particle "the electron" and the negatively charged one "the positron". We attempt to calculate the transition matrix element of a process in which there is present only one "electron" with wave function $g(\vec{x})$ at $\tau = T$ if there was only one "electron" with wave function $f(\vec{x})$ at $\tau = 0$.

The case when several "electrons" and "positrons" are involved in the initial as well as in the final state, the argument will be given in a similar fashion as was done for the case of Dirac particles[5]. If the wave functionals representing the initial and the final states are called Ω_i and Ω_f, we are to evaluate the transition matrix element

$$\Upsilon = (\Omega_f^{\dagger}\, e^{-\int_0^T H d\tau}\, \Omega_i).$$

Writing $\Omega_i = F^{\dagger}\,\Omega_o$ and $\Omega_f = G^{\dagger}\,\Omega_o$ in terms of the state vector Ω_o of the vacuum we rewrite Υ as follows:

$$\Upsilon = (\Omega_o^{\dagger}\; G\; S\; F^{\dagger}\; \Omega_o), \tag{II.6}$$

where $S = e^{-\int_0^T H d\tau}$. In the case of Dirac particles the field operator $\psi^*(x)$ is responsible for both the creation of an electron and the annihilation of a positron at the position $\vec{x}$ so that a state with an electron with spinor function $f(\vec{x})$ can be obtained from Ω_o by operating with $F^* = \int \psi^*(\vec{x})\, f(\vec{x})\, d^3\vec{x}$ on it. However, for a scalar particle, the expression for F^* in t-formalism (or $F^{\dagger}$ in τ-formalism) is a little bit more complicated, due to the fact that it obeys a differential equation which is second order in the time. Since it turns out, for the scalar particle, that both $\psi^*(\vec{x})$ and $\pi(\vec{x})$ are responsible for the creation of an "electron" F^* must be expressed in terms of ψ^*, π, and f. If the creation operator of an "electron" with momentum $\vec{k}$ is a^*_k and the Fourier transform of $f(\vec{x})$ is $\phi(\vec{k})$, then F^* can as well be written as $\int a^*_k\, \phi(\vec{k})\, d^3\vec{k}$. To find the expression for F^* we need to study the work of Pauli and Weisskopf. In going to the momentum space they consider a cube of side L (volume $V = L^3$) on whose surface the periodic boundary conditions are satisfied. Then they expand the field operators ψ, π, ψ^*, and π^* in the following manner:

$$\psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_k q_k \, e^{i\vec{k}\vec{x}}, \quad \psi^*(\vec{x}) = \frac{1}{\sqrt{V}} \sum_k q_k^* \, e^{-i\vec{k}\vec{x}},$$

$$\pi^*(\vec{x}) = \frac{1}{\sqrt{V}} \sum_k p_k^* \, e^{i\vec{k}\vec{x}}, \quad \pi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_k p_k \, e^{-i\vec{k}\vec{x}},$$

where the summation over k extends over all the components of a vector $\vec{k}$. Solving for q, p, q* and p* gives

$$q_k = \frac{1}{\sqrt{V}} \int \psi(\vec{x}) \, e^{-i\vec{k}\vec{x}} \, d^3\vec{x}, \quad q_k^* = \frac{1}{\sqrt{V}} \int \psi^*(\vec{x}) e^{i\vec{k}\vec{x}} \, d^3\vec{x},$$

$$p_k^* = \frac{1}{\sqrt{V}} \int \pi^*(\vec{x}) \, e^{-i\vec{k}\vec{x}} \, d^3\vec{x}, \quad p_k = \frac{1}{\sqrt{V}} \int \pi(\vec{x}) \, e^{i\vec{k}\vec{x}} \, d^3\vec{x}.$$

Again they write

$$p_k = \sqrt{\frac{\omega_k}{2}} \, (a_k^* + b_k) \; ; \qquad q_k = \frac{-i}{\sqrt{2\omega_k}} \, (-a_k + b^*_k),$$

$$p_k^* = \sqrt{\frac{\omega_k}{2}} \, (a_k + b^*_k), \qquad q^*_k = \frac{-i}{\sqrt{2\,\omega_k}} \, (a_k^* - b_k),$$

where a_k and a^*_k are the annihilation and creation operator of an "electron" with momentum $\vec{k}$ and b_k and b^*_k are the corresponding quantities for a "positron", and $\omega_k = +\sqrt{\vec{k}^{\,2} + m^2}$. From these last equations we have

$$a_k = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{\omega_k}} \, p^*_k - i\sqrt{\omega_k} \, q_k \right), \quad a^*_k = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{\omega_k}} \, p_k + i\sqrt{\omega_k} \, q^*_k \right),$$

$$b_k = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{\omega_k}} \, p_k - i\sqrt{\omega_k} \, q^*_k \right), \qquad b^*_k = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{\omega_k}} \, p^*_k + i\sqrt{\omega_k} \, q_k \right).$$

Now we can write a^*_k in terms of ψ^* and π as

$$a^*_k = \frac{1}{\sqrt{V}} \int \frac{1}{\sqrt{2\omega_k}} \left(\pi(\vec{x}) + i\,\omega_k \, \psi^*(\vec{x}) \right) e^{i\vec{k}\vec{x}} \, d^3\vec{x}.$$

Since the wave function $f(\vec{x})$ of a free "electron" with energy ω_k varies with time as $e^{-i\omega_k t}$ we can regard the coefficient of $\psi^*(\vec{x})$, $i\omega_k$, as coming from $-\frac{\partial f(\vec{x})}{\partial t}$. Making use of the Fourier transform of a^*_k given above we can now write F* as

$$\int \frac{1}{\sqrt{2\,\omega_k}} \left(\pi\,(\vec{x}_1)\, f(\vec{x}_1) - \psi^*(\vec{x}_1)\,\frac{\partial f(\vec{x}_1)}{\partial t_1}\right) d^3\vec{x}_1 \,.$$

However, it proves to be more convenient to use

$$F^* = \int \left(\pi(\vec{x}_1)\, f(\vec{x}_1) - \psi^*(\vec{x}_1)\,\frac{\partial f(\vec{x}_1)}{\partial t_1}\right) d^3\vec{x}_1 \,, \qquad \text{(II. 7)}$$

leaving the factor $\frac{1}{\sqrt{2\omega_k}}$ which can be taken care of as the normalization constant (cf. footnote on p. 20). It can be shown in general that this F^* creates "electrons" and annihilates "positrons" when the wave function $f(\vec{x})$ is the mixture of both kinds of particles with various values of momentum*.

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* For in this case $f(\vec{x})$ and $\partial f(\vec{x})/\partial t$ are given by $f(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{k'} (\phi^e_{k'} + \phi^p_{k'})\, e^{i\vec{k}'\vec{x}}$ and $\partial f(\vec{x})/\partial t = \frac{1}{\sqrt{V}} \sum_{k'} (-i\omega_{k'})(\phi^e_{k'} - \phi^p_{k'}) \cdot e^{i\vec{k}'\vec{x}}$, where $\phi^e_{k'}$ and $\phi^p_{k'}$ are the Fourier coefficients of the wave function of "electrons" and "positrons", respectively. Since

$$\psi^*(\vec{x}) = \frac{1}{\sqrt{V}} \sum_k \sqrt{\frac{1}{2\omega_k}}\,(-i)(a^*_k - b_k)\, e^{-i\vec{k}\vec{x}}$$

and

$$\pi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_k \sqrt{\frac{\omega_k}{2}}\,(a^*_k + b_k)\, e^{-i\vec{k}\vec{x}}$$

Eq. (II. 7) now gives

$$F^* = \frac{1}{V} \int \sum_{k,k'} \left\{ \sqrt{\frac{\omega_k}{2}}\,(a^*_k + b_k)(\phi^e_{k'} + \phi^p_{k'}) - \sqrt{\frac{1}{2\omega_k}}\,(-i)(a^*_k - b_k) \cdot (-i\omega_{k'})(\phi^e_{k'} - \phi^p_{k'}) \right\} e^{-i(\vec{k} - \vec{k}')\vec{x}} = \sum_k \sqrt{2\omega_k}\,(a^*_k \phi^e_k + b_k \phi^p_k),$$

which proves the statement above.

G can be obtained by taking the complex conjugate of G* and is given by

$$G = \int (\pi^*(\vec{x}_2) g^*(\vec{x}_2) - \psi(\vec{x}_2) \frac{\partial g^*(\vec{x}_2)}{\partial t_2})\, d^3\vec{x}_2 \,.$$

In τ-formalism the use of Eqs. (II. 3) and (II. 4) gives

$$F^{\dagger} = \int (\pi(\vec{x}_1)\, f(\vec{x}_1) - \psi^{\dagger}(\vec{x}_1)\, i\, \frac{\partial f(\vec{x}_1)}{\partial \tau_1})\, d^3\vec{x}_1 \qquad \text{(II. 8)}$$

and
$$G = \int (\pi^{\dagger}(\vec{x}_2)\, g^{\dagger}(\vec{x}_2) - \psi(\vec{x}_2)\, i\, \frac{\partial g^{\dagger}(\vec{x}_2)}{\partial \tau_2})\, d^3\vec{x}_2 \,. \qquad \text{(II. 9)}$$

With this preparation we proceed to the evaluation of Υ. Multiplying $F^{\dagger}$ by $e^{-\int_0^{\tau} H(\tau')d\tau'}$ and $e^{+\int_0^{\tau} H(\tau')d\tau'}$ from left and right, respectively, we write it as

$$e^{-\int_0^{\tau} H(\tau')d\tau'}\, F^{\dagger}\, e^{+\int_0^{\tau} H(\tau')d\tau'} = \int \left\{ \pi(\vec{x}_1) f(\vec{x}_1, \tau) - i\, \psi^{\dagger}(\vec{x}_1) \underline{f}(\vec{x}_1, \tau) \right\} d^3\vec{x}_1 . \qquad \text{(II. 10)}$$

This defines $f(\vec{x}_1, \tau)$ and $\underline{f}(\vec{x}_1, \tau)$. Multiply this by $e^{+\int_0^{\tau} H(\tau')d\tau'}$ ..... $e^{-\int_0^{\tau} H(\tau')d\tau'}$ to get back to $F^{\dagger}$ in the form

$$F^{\dagger} = \int \left\{ \pi(\vec{x}_1, \tau)\, f(\vec{x}_1, \tau) - i\, \psi^{\dagger}(\vec{x}_1, \tau)\, \underline{f}(\vec{x}_1, \tau) \right\} d^3\vec{x}_1 \,, \qquad \text{(II. 11)}$$

where

$$\psi^{\dagger}(\vec{x}_1, \tau) = e^{+\int_0^{\tau} H(\tau')d\tau'}\, \psi^{\dagger}(\vec{x}_1)\, e^{-\int_0^{\tau} H(\tau')d\tau'} \qquad \text{(II. 12)}$$

and
$$\pi(\vec{x}_1, \tau) = e^{+\int_0^{\tau} H(\tau')d\tau'}\, \pi(\vec{x}_1)\, e^{-\int_0^{\tau} H(\tau')d\tau'} \,. \qquad \text{(II. 13)}$$

Differentiations of these equations give

$$\frac{\partial \psi^{\dagger}(\vec{x}_1, \tau)}{\partial \tau} = e^{+\int_0^{\tau} H(\tau')d\tau'} \left[H(\tau),\, \psi^{\dagger}(\vec{x}_1) \right] e^{-\int_0^{\tau} H(\tau')d\tau'} ,$$

$$\frac{\partial^2 \psi^\dagger(\vec{x}_1,\tau)}{\partial \tau^2} = e^{+\int_0^\tau H(\tau')d\tau'} \left[H(\tau), \left[H(\tau), \psi^\dagger(\vec{x}_1)\right]\right] e^{-\int_0^\tau H(\tau')d\tau'} ,$$

$$\frac{\partial \pi(\vec{x}_1,\tau)}{\partial \tau} = e^{+\int_0^\tau H(\tau')d\tau'} \left[H(\tau), \pi(\vec{x}_1)\right] e^{-\int_0^\tau H(\tau')d\tau'} ,$$

$$\frac{\partial^2 \pi(\vec{x}_1,\tau)}{\partial \tau^2} = e^{+\int_0^\tau H(\tau')d\tau'} \left[H(\tau), \left[H(\tau), \pi(\vec{x}_1)\right]\right] e^{-\int_0^\tau H(\tau')d\tau'} .$$

From Eq. (II. 5) and the Hamiltonian in τ -formalism

$$H(\tau) = \int \left\{\pi^\dagger(\vec{x}')\pi(\vec{x}') + \nabla\psi^\dagger(\vec{x}')\cdot\nabla\psi(\vec{x}') + (m^2 + \sqrt{4\pi}\, g\, \chi(\vec{x}',\tau))\,\psi^\dagger(\vec{x}')\psi(\vec{x}')\right\} d^3\vec{x}'$$

the commutators are evaluated as follows:

$$\left[H(\tau), \psi^\dagger(\vec{x}_1)\right] = -i\,\pi(\vec{x}_1),$$

$$\left[H(\tau), \left[H(\tau), \psi^\dagger(\vec{x}_1)\right]\right] = \Lambda(\vec{x}_1,\tau)\,\psi^\dagger(\vec{x}_1),$$

where $\Lambda(\vec{x}_1,\tau) = -\nabla_1^2 + m^2 + \sqrt{4\pi}\, g\, \chi(\vec{x}_1,\tau)$. (II. 14)

Also $\left[H(\tau), \pi(\vec{x}_1)\right] = i\,\Lambda(\vec{x}_1,\tau)\,\psi^\dagger(\vec{x}_1)$,

$$\left[H(\tau), \left[H(\tau), \pi(\vec{x}_1)\right]\right] = \Lambda(\vec{x}_1,\tau)\pi(\vec{x}_1).$$

Using these we now have

$$\frac{\partial \psi^\dagger(\vec{x}_1,\tau)}{\partial \tau} = -i\pi(\vec{x}_1,\tau), \qquad \frac{\partial^2 \psi^\dagger(\vec{x}_1,\tau)}{\partial \tau^2} = \Lambda(\vec{x}_1,\tau)\,\psi^\dagger(\vec{x}_1,\tau), \qquad \text{(II.15, a, b)}$$

$$\frac{\partial \pi(\vec{x}_1,\tau)}{\partial \tau} = i\,\Lambda(\vec{x}_1,\tau)\psi^\dagger(\vec{x}_1,\tau), \qquad \frac{\partial^2 \pi(\vec{x}_1,\tau)}{\partial \tau^2} = \Lambda(\vec{x}_1,\tau)\pi(\vec{x}_1,\tau). \qquad \text{(II. 16, a, b)}$$

We notice here that $\psi^\dagger(\vec{x},\tau)$ and $\pi(\vec{x},\tau)$ satisfy an equation of the form

$$(\boxplus^2 - m^2 - \sqrt{4\pi}\, g\, \chi(\vec{x},\tau))\,\Phi(\vec{x},\tau) = 0, \qquad \text{(II. 17)}$$

where the differential operator $\boxplus^2$ is defined by

$$\boxplus^2 = \nabla^2 + \frac{\partial^2}{\partial \tau^2} . \qquad \text{(II. 18)}$$

Eq. (II. 17) can be obtained from the corresponding equation for ψ^* and π in the t-formalism by replacing the D'Alembertian operator $\Box^2 = \nabla^2 - \partial^2/\partial t^2$ by new operator defined by Eq. (II. 18) which can formally be obtained from $\Box^2$ by substitution $it = \tau$.

Since $F^\dagger$ given by Eq. (II. 8) is independent of τ, this must also be the case when it is written in the form of Eq. (II. 11). Hence the differentiation of $F^{\dagger}$ with respect to τ gives

$$0 = \int \Big\{ \frac{\partial \pi(\vec{x}_1,\tau)}{\partial \tau} f(\vec{x}_1,\tau) + \pi(\vec{x}_1,\tau) \frac{\partial f(\vec{x}_1,\tau)}{\partial \tau} - i \frac{\partial \psi^{\dagger}(\vec{x}_1,\tau)}{\partial \tau} \underline{f}(\vec{x}_1,\tau)$$

$$- i \psi^{\dagger}(\vec{x}_1,\tau) \frac{\partial \underline{f}(\vec{x}_1,\tau)}{\partial \tau} \Big\} d^3 \vec{x}_1$$

$$= \int \Big\{ i \psi^{\dagger}(\vec{x}_1,\tau) \Big(\Lambda(\vec{x}_1,\tau) f(\vec{x}_1,\tau) - \frac{\partial \underline{f}(\vec{x}_1,\tau)}{\partial \tau} \Big)$$

$$+ \pi(\vec{x}_1,\tau) \Big(\frac{\partial f(\vec{x}_1,\tau)}{\partial \tau} - \underline{f}(\vec{x}_1,\tau) \Big) \Big\} d^3 \vec{x}_1 ,$$

in which use has been made of Eqs. (II. 15, a) and (II. 16, a) and an integration by parts was performed in order to operate $\Lambda(\vec{x}_1,\tau)$ on $f(\vec{x}_1,\tau)$ instead of $\psi^{\dagger}(\vec{x}_1,\tau)$. From the last expression we have the following relations existing between f and $\underline{f}$:

$$\underline{f}(\vec{x}_1,\tau) = \frac{\partial f(\vec{x}_1,\tau)}{\partial \tau} \; ; \; \frac{\partial \underline{f}(\vec{x}_1,\tau)}{\partial \tau} = \Lambda(\vec{x}_1,\tau) f(\vec{x}_1,\tau). \qquad \text{(II. 19, a, b)}$$

By combining these two equations we see that $f(\vec{x},\tau)$ also satisfies an equation of the form of Eq. (II. 17), that is,

$$(\boxplus^2 - m^2 - \sqrt{4\pi}\, g\, \chi(\vec{x},\tau)\,)\, f(\vec{x},\tau) = 0 \,. \qquad \text{(II. 20)}$$

However, it can easily be seen that this is no longer true for $\underline{f}(\vec{x},\tau)$. Substituting Eq. (II. 19, a) in Eq. (II. 10) and letting $\tau = T$ we have

$$S F^{\dagger} S^{-1} = F'^{\dagger} , \qquad \text{(II. 21)}$$

where

$$S = e^{-\int_0^T H(\tau')d\tau'} ,$$

and

$$F'^{\dagger} = \int \Big\{ \pi(\vec{x}_1)\, f(\vec{x}_1, T) - i\, \psi^{\dagger}(\vec{x}_1) \frac{\partial f(\vec{x}_1, T)}{\partial T} \Big\} d^3 x_1 \,. \qquad \text{(II. 22)}$$

The use of Eq. (II. 21) now permits us to write r of Eq. (II. 6) as

$$r = (\Omega_o^{\dagger}\, G\, F'^{\dagger}\ S\, \Omega_o).$$

To calculate r we want to interchange G and $F'^{\dagger}$. For this reason we need to evaluate the commutator of G and $F'^{\dagger}$.

From Eqs. (II. 9) (with $\tau_2 = T$) and (II. 22) it is given by

$$[G, F'^{\dagger}] = \Big[\int \Big\{ \pi^{\dagger}(\vec{x}_2) g^{\dagger}(\vec{x}_2) - i\psi(\vec{x}_2) \frac{\partial g^{\dagger}(\vec{x}_2)}{\partial T} \Big\} d^3\vec{x}_2,\ \int \Big\{ \pi(\vec{x}_1) f(\vec{x}_1, T) - i\,\psi^{\dagger}(\vec{x}_1) \frac{\partial f(\vec{x}_1, T)}{\partial T} \Big\}\ d^3\vec{x}_1 \Big],$$

which becomes, by virtue of the commutation relations, Eq. (II. 5),

$$= - \int \Big\{ g^{\dagger}(\vec{x}_2) \frac{\partial f(\vec{x}_2, T)}{\partial T} - \frac{\partial g^{\dagger}(\vec{x}_2)}{\partial T} f(\vec{x}_2, T) \Big\} d^3\vec{x}_2 \ .$$

Using this equation the transition matrix element r is now given by

$$r = - \int \Big\{ g^{\dagger}(\vec{x}_2) \frac{\partial f(\vec{x}_2, T)}{\partial T} - \frac{\partial g^{\dagger}(\vec{x}_2)}{\partial T} f(\vec{x}_2, T) \Big\} d^3\vec{x}_2 \cdot C_v + (\Omega_o^{\dagger}\, F'^{\dagger}\, G\, S\, \Omega_o),$$

where $C_v = (\Omega_o^{\dagger}\, S\, \Omega_o)$ is the amplitude for having a vacuum at $\tau = T$, if we had one at $\tau = 0$. In the second term on the right-hand side $\Omega_o^{\dagger} F'^{\dagger}$ is the "complex conjugate" (in τ -formalism) of $F'\Omega_o$ and it will vanish if $f(\vec{x}, T)$ contained only positive energy ("electron") components. For such a case the transition element reduces to*

$$r = -\int \left\{ g^{\dagger}(\vec{x}_2) \frac{\partial f(\vec{x}_2, T)}{\partial T} - \frac{\partial g^{\dagger}(\vec{x}_2)}{\partial T} f(\vec{x}_2, T) \right\} d^3\vec{x}_2 \cdot C_v \ , \qquad \text{(II. 23)}$$

which is a factor times C_v. However, F' contains negative energy components created in the potential χ so that we need to modify[5] what was obtained above. We do this by adding another operator $F''^{\dagger}$ corresponding to a function $f''(\vec{x}, T)$ containing only negative energy components in such a way that the resulting $f(\vec{x}, T)$ contains only positive components, i.e., instead of Eq. (II. 21) we want

$$S(F^{\dagger}_{Pos} + F''^{\dagger}_{Neg}) = F'^{\dagger}_{Pos} \ S \ .$$

Here the subscripts "Pos" and "Neg" refer to the sign of the energy components contained in the operators. From this we have

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*In this case the normalization integral becomes

$$-\int \left\{ f^{\dagger}(\vec{x},\tau) \frac{\partial f(\vec{x},\tau)}{\partial \tau} - \frac{\partial f^{\dagger}(\vec{x},\tau)}{\partial \tau} f(\vec{x},\tau) \right\} d^3\vec{x} \ .$$

For a free "electron", for example, with momentum $\vec{k}$ we have $\partial f/\partial\tau = -\omega_k f$ and $\partial f^{\dagger}/\partial\tau = +\omega_k f$. Therefore if the amplitude of the wave function is normalized to 1 so that $f = 1 \cdot e^{i\vec{k}\vec{x} - \omega_k \tau}$, then the integral above is equal to $2\omega_k$ x Volume. It is convenient to normalize the amplitude of the wave function to 1, but the normalization integration (if it is normalized to the unit volume) becomes $2\omega_k$. This is the reason why we have to divide the probability of transition per second by the product of twice the energy of each free particle involved in the initial and the final states[8].

$$SF^{\dagger}_{Pos} = F'^{\dagger}_{Pos} S - S F''^{\dagger}_{Neg} \quad .$$

Substituting this in Eq. (II. 6) gives

$$r = (\Omega_o^{\dagger} \; GF'^{\dagger}_{Pos} S \Omega_o) - (\Omega_o^{\dagger} \; G \; S \; F''^{\dagger}_{Neg} \; \Omega_o).$$

The first term is identical with the expression given by Eq. (II. 23) while the second vanishes since the creation operator of the negative energy components operating on the vacuum gives zero. And again the result is a factor times C_v. We will not discuss this matter further because the detail of the rest of the argument is essentially given in Reference (5).

The expression of transition element r obtained here is not so simple as the case of the Dirac electron and contains time derivatives of wave functions at the initial and the final states. This is due to the fact that the Klein-Gordon equation is second order in the time and hence in order to specify the state completely we need not only the value of the wave function itself but also that of the time derivative.

In Eq. (II. 23) the dependence of r on the kernel of the Klein-Gordon equation is implicitly given through $f(\vec{x}_2, T)$ and $\partial f(\vec{x}_2, T)/\partial T$. In the following we will find its explicit dependence on the kernel and also will calculate the contribution to the transition matrix element from processes of various order in the perturbation method. To this end let us first consider a free particle whose wave function f is described by

$$(\boxplus^2 - m^2) f = 0.$$

The wave function at point 2 inside a space-time region is given, in terms of the kernel $\mathrm{I}^{(0)}$ associated with the free particle which is defined by

$$(\boxplus_2^2 - m^2)\, \mathrm{I}^{(0)}(2,1) = \delta^4(2,1)\ ,$$

as*

$$f(2) = -\int \sum_{q_\mu(1)=x_1, y_1, z_1, \tau_1} \left\{ f(1)\frac{\partial \mathrm{I}^{(0)}(2,1)}{\partial q_\mu(1)} - \mathrm{I}^{(0)}(2,1)\frac{\partial f(1)}{\partial q_\mu(1)} \right\} N_\mu(1) d^3V_1 .$$

Here $N_\mu(1)$ is the inward drawn unit normal at point 1 and d^3V_1 is the volume element of the closed three-dimensional surface of a space-time region containing point 2. When the region of space-time consists of the three-dimensional surfaces at $\tau = \tau_1 < \tau_2$ and $\tau = \tau_{1'} > \tau_2$ and the cylinder connecting these two surfaces expression above reduces to

$$f(2) = -\int \left\{ f(1)\frac{\partial \mathrm{I}^{(0)}(2,1)}{\partial \tau_1} - \mathrm{I}^{(0)}(2,1)\frac{\partial f(1)}{\partial \tau_1} \right\} d^3\vec{x}_1$$
$$+ \int \left\{ f(1')\frac{\partial \mathrm{I}^{(0)}(2,1')}{\partial \tau_{1'}} - \mathrm{I}^{(0)}(2,1')\frac{\partial f(1')}{\partial \tau_{1'}} \right\} d^3\vec{x}_{1'}\ . \qquad \text{(II. 24)}$$

The first term represents the contribution from the "electron" components of f(1), while the second represents that from the "positron" components of f(1'). Incidentally the function $\mathrm{I}^{(0)}(x_\mu)$ has the following form:

$$\mathrm{I}^{(0)}(x_\mu) = \frac{-1}{(2\pi)^4}\int \frac{e^{i(\vec{k}\vec{x} + k_4\tau)}}{k_4^2 + \vec{k}^2 + m^2}\, d^4k$$
$$= \frac{-\pi}{(2\pi)^4}\int e^{i\vec{k}\vec{x}}\, \frac{e^{-\sqrt{\vec{k}^2+m^2}\,|\tau|}}{\sqrt{\vec{k}^2+m^2}}\, d^3\vec{k}\ . \qquad \text{(II. 25)}$$

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*Cf. Section 9 of Reference (6). The function $\mathrm{I}^{(0)}$ is a modified form of function I_+ defined by Eq. (32) of Reference (5).

From now on we will deal with the scalar nucleon in a given potential χ which obeys Eq. (II. 20) :

$$(\boxplus^2 - m^2 - \sqrt{4\pi}\, g\, \chi(\vec{x},\tau))\, f(\vec{x},\tau) = 0 \; .$$

If we can define a function $\mathcal{I}^{(\chi)}$ in terms of $\mathcal{I}^{(0)}$ and χ such that it satisfies the equation

$$(\boxplus_2^2 - m^2 - \sqrt{4\pi}\, g\chi(2))\, \mathcal{I}^{(\chi)}(2,1) = \delta^4(2,1), \tag{II. 26}$$

then by the similar argument as for the case of free particle we will have

$$f(2) = -\int\left\{ f(1)\frac{\partial \mathcal{I}^{(\chi)}(2,1)}{\partial\tau_1} - \mathcal{I}^{(\chi)}(2,1)\frac{\partial f(1)}{\partial\tau_1}\right\} d^3\vec{x}_1$$

$$+\int\left\{ f(1')\frac{\partial \mathcal{I}^{(\chi)}(2,1')}{\partial\tau_{1'}} - \mathcal{I}^{(\chi)}(2,1')\frac{\partial f(1')}{\partial\tau_{1'}}\right\} d^3\vec{x}_{1'} \; . \tag{II. 27}$$

Now suppose $\mathcal{I}^{(\chi)}$ is expanded in the following form:

$$\mathcal{I}^{(\chi)}(2,1) = \mathcal{I}^{(0)}(2,1) + \int \mathcal{I}^{(0)}(2,3)\,(\sqrt{4\pi}\, g\chi(3))\, \mathcal{I}^{(0)}(3,1)\, d^4\omega_3$$

$$+ \iint \mathcal{I}^{(0)}(2,4)(\sqrt{4\pi}\, g\,\chi(4))\, \mathcal{I}^{(0)}(4,3)(\sqrt{4\pi}\, g\chi(3))\, \mathcal{I}^{(0)}(3,1) d^4\omega_4 d^4\omega_3 + \ldots, \tag{II. 28}$$

where $d^4\omega$ refers to the four-dimensional volume element. When we operate $\boxplus_2^2 - m^2 - \sqrt{4\pi}\, g\chi(2)$ on the right-hand side of Eq. (II. 28) we have, in view of $(\boxplus_2^2 - m^2)\, \mathcal{I}^{(0)}(2,1) = \delta^4(2,1)$,

$$\begin{aligned}
&\delta(2,1) - \sqrt{4\pi}\, g\chi(2)\, \Phi^{(0)}(2,1) \\
&+ \int \delta(2,3)(\sqrt{4\pi}\, g\chi(3))\, \Phi^{(0)}(3,1) d^4\omega_3 - \sqrt{4\pi}\, g\chi(2) \int \Phi^{(0)}(2,3)(\sqrt{4\pi}\, g\chi(3))\, \Phi^{(0)}(3,1) d^4\omega_3 \\
&+ \iint \delta(2,4)(\sqrt{4\pi}\, g\chi(4))\, \Phi^{(0)}(4,3)(\sqrt{4\pi}\, g\chi(3))\, \Phi^{(0)}(3,1) d^4\omega_4\, d^4\omega_3 \\
&\quad - \sqrt{4\pi}\, g\chi(2) \iint \Phi^{(0)}(2,4)(\sqrt{4\pi}\, g\chi(4))\, \Phi^{(0)}(4,3)(\sqrt{4\pi}\, g\chi(3)) \Phi^{(0)}(3,1) d^4\omega_4 d^4\omega_3 + \ldots,
\end{aligned}$$

which reduces to $\delta^4(2,1)$ since the second and the third, the fourth and the fifth, etc., terms cancel each other. It shows, therefore, that the expansion given by Eq. (II. 28) of $\Phi^{(\chi)}$ satisfies Eq. (II. 27).

The transition matrix element from the initial state characterized by f(1) and $\partial f(1)/\partial\tau_1$ of an "electron" to the final state with g(2) and $\partial g(2)/\partial\tau_2$ of another "electron" is given by Eq. (II. 23). From Eq. (II. 27), where the second term drops because it corresponds to the "positron" components, we have

$$f(2) = -\int \left\{ \frac{\partial \Phi^{(\chi)}(2,1)}{\partial\tau_1} f(1) - \Phi^{(\chi)}(2,1) \frac{\partial f(1)}{\partial \tau_1} \right\} d^3 \vec{x}_1 , \qquad \text{(II. 29)}$$

and $$\frac{\partial f(2)}{\partial\tau_2} = -\int \left\{ \frac{\partial^2 \Phi^{(\chi)}(2,1)}{\partial\tau_2 \partial\tau_1} f(1) - \frac{\partial \Phi^{(\chi)}(2,1)}{\partial\tau_2} \frac{\partial f(1)}{\partial\tau_1} \right\} d^3\vec{x}_1 .$$

Substitution of these equations in Eq. (II. 23) gives, aside from C_v,

$$\begin{aligned}
r = + \iint \Big\{ & g^\dagger(2) \frac{\partial^2 \Phi^{(\chi)}(2,1)}{\partial\tau_2\partial\tau_1} f(1) - g^\dagger(2) \frac{\partial \Phi^{(\chi)}(2,1)}{\partial\tau_2} \frac{\partial f(1)}{\partial\tau_1} \\
& - \frac{\partial g^\dagger(2)}{\partial\tau_2} \frac{\partial \Phi^{(\chi)}(2,1)}{\partial\tau_1} f(1) + \frac{\partial g^\dagger(2)}{\partial\tau_2} \Phi^{(\chi)}(2,1) \frac{\partial f(1)}{\partial\tau_1} \Big\} d^3\vec{x}_1 d^3\vec{x}_2 .
\end{aligned} \qquad \text{(II. 30)}$$

Corresponding to Eq. (II. 28) let us expand r in powers of the potential χ as $\Upsilon = \Upsilon^{(0)} + \Upsilon^{(1)} + \Upsilon^{(2)} + \ldots\ldots\ldots,$

where $\Upsilon^{(i)}$ is the contribution from the process i-th order in the potential. $\Upsilon^{(0)}$ is self-evident. For $\Upsilon^{(1)}$ we have

$$r^{(1)} = \iiint \Big\{ g^{\dagger}(2) \frac{\partial \Phi^{(0)}(2,3)}{\partial \tau_2} (\sqrt{4\pi}\, g\chi(3)) \frac{\partial \Phi^{(0)}(3,1)}{\partial \tau_1} f(1)$$

$$- g^{\dagger}(2) \frac{\partial \Phi^{(0)}(2,3)}{\partial \tau_2} (\sqrt{4\pi}\, g\chi(3)) \Phi^{(0)}(3,1) \frac{\partial f(1)}{\partial \tau_1}$$

$$- \frac{\partial g^{\dagger}(2)}{\partial \tau_2} \Phi^{(0)}(2,3) (\sqrt{4\pi}\, g\chi(3)) \frac{\partial \Phi^{(0)}(3,1)}{\partial \tau_1} f(1)$$

$$+ \frac{\partial g^{\dagger}(2)}{\partial \tau_2} \Phi^{(0)}(2,3) (\sqrt{4\pi}\, g\chi(3)) \Phi^{(0)}(3,1) \frac{\partial f(1)}{\partial \tau_1} \Big\} d^3\vec{x}_1\, d^3\vec{x}_2 d^4\omega_3$$

$$= \iiint d^3\vec{x}_2 \Big\{ g^{\dagger}(2) \frac{\partial \Phi^{(0)}(2,3)}{\partial \tau_2} - \Phi^{(0)}(2,3) \frac{\partial g^{\dagger}(2)}{\partial \tau_2} \Big\} (\sqrt{4\pi}\, g\chi(3))\, d^4\omega_3$$

$$\Big\{ f(1) \frac{\partial \Phi^{(0)}(3,1)}{\partial \tau_1} - \Phi^{(0)}(3,1) \frac{\partial f(1)}{\partial \tau_1} \Big\}\, d^3\vec{x}_1 \ .$$

The content of Eq. (II. 24) gives, for our case where $\tau_1 < \tau_3 < \tau_2$,

$$f(3) = - \int \Big\{ f(1) \frac{\partial \Phi^{(0)}(3,1)}{\partial \tau_1} - \Phi^{(0)}(3,1) \frac{\partial f(1)}{\partial \tau_1} \Big\}\, d^3\vec{x}_1$$

and

$$g(3) = + \int \Big\{ g(2) \frac{\partial \Phi^{(0)}(3,2)}{\partial \tau_2} - \Phi^{(0)}(3,2) \frac{\partial g(2)}{\partial \tau_2} \Big\}\, d^3\vec{x}_2.$$

Since $\Psi^{(0)\dagger}(3,2) = \Psi^{(0)}(2,3)$ from Eq. (II. 25) the "complex conjugate" of the latter equation is

$$g^{\dagger}(3) = + \int \left\{ g^{\dagger}(2) \frac{\partial \Psi^{(0)}(2,3)}{\partial \tau_2} - \Psi^{(0)}(2,3) \frac{\partial g^{\dagger}(2)}{\partial \tau_2} \right\} d^3 \vec{x}_2 \,.$$

Hence $r^{(1)}$ now becomes

$$r^{(1)} = - \int g^{\dagger}(3) \, (\sqrt{4\pi}\, g\, \chi(3)) \, f(3) \, d^4\omega_3 \,. \qquad \text{(II. 31)}$$

$r^{(2)}$ is obtainable in the similar fashion by simply replacing the rôle of $g^{\dagger}(3)$ in $\gamma^{(1)}$ by $\int g^{\dagger}(4)\, d^4\omega_4 \, (\sqrt{4\pi}\, g\, \chi(4))\, \Psi^{(0)}(4,3)$ and is given by

$$r^{(2)} = - \iint g^{\dagger}(4)(\sqrt{4\pi}\, g\chi(4))\Psi^{(0)}(4,3)(\sqrt{4\pi} g \chi(3)) f(3) d^4\omega_4 d^4\omega_3 \,. \qquad \text{(II. 32)}$$

The higher order terms can now be written down automatically.

Though the expression for Υ given by Eq. (II. 23) is simple, it becomes considerably more complicated when expressed in terms of the kernel $\Psi^{(\chi)}$ as shown in Eq. (II. 30). Eq. (II. 30) corresponds to $\Upsilon = \iint g^*(2) K_+^{(A)}(2,1) \beta f(1) d^3 \vec{x}_1 \, d^3 \vec{x}_2$ for the case of Dirac electron[5]. The cause of the complication is, as can easily be seen, the fact that the Klein-Gordon equation is a differential equation which is of second order in the time variable. Each term $\Upsilon^{(i)}$ permits a simple interpretation exactly in the same way as for Dirac electron . Thus, for example, the transition amplitude to second order, $\Upsilon^{(2)}$, can be integrated in the following way. The particle arrives at 3 with amplitude f(3), gets scattered there by the potential (scattering amplitude $\sqrt{4\pi}\, g\, \chi(3)$), then proceeds to 4 (with amplitude $\Psi^{(0)}(4,3)$), gets scattered again ($\sqrt{4\pi}\, g\, \chi(4)$), then we ask for the amplitude that it is in state g(4). We finally sum over all possible values of space-time coordinates of points 3 and 4. The appearance of the function $I^{(0)}$ in $\gamma^{(i)}$ corresponding to the free propagation of the particle between the scattering potentials shows that the propagator in the momentum space is, in the t-formalism, given by $(k^2 - m^2)^{-1}$.

To summarize: The consequences of the change $it = \tau$ for the case of the scalar nucleon in a given potential were studied in this section using the theory of second quantization. This change, $it = \tau$, is desirable in connection with the later use of the variational principle to obtain the lowest energy of the scalar nucleon. The equation satisfied by the scalar nucleon in τ-formalism, i.e., Eq. (II. 20), has the form which can be obtained from the corresponding equation in t-formalism by formal substitution $t = -i\tau$. This equation is, so to speak, the key equation from which whole developments in the remainder of the present work originate. The expression for the transition matrix element and the contribution to it coming from different order of the perturbation approximation was obtained also though we are not going to utilize it in the following.

III. ELIMINATION OF FIELD OPERATOR IN THE QUANTUM MECHANICAL CASE

The Lagrangian form of the scalar particle in a given external electromagnetic field A_μ was discussed by Professor R. P. Feynman[9]. In the first part of this section the same problem of the scalar nucleon in an external neutral meson potential χ will be examined starting from Eq. (II. 20) obtained in the previous section from the theory of second quantization. After finding the kernel for this case as the path integral over trajectories of the particle in space-time (time hereafter refers to $x_4 = \tau$) the kernel corresponding to the transition from the vacuum to the vacuum of the meson field in the quantum mechanical case will be obtained by integrating out the meson field operator in the same way as was done in quantum electrodynamics (Cf. Sect. 6 of Reference (8)).

In the previous section it was shown that the scalar charged nucleon in a given neutral scalar meson potential χ obeys Eq. (II. 20)

$$(\boxplus^2 - \sqrt{4\pi}\, g\chi(\vec{x},\tau))\, \psi(\vec{x},\tau) = m^2 \psi(\vec{x},\tau), \qquad \text{(III. 1)}$$

with $\boxplus^2 = \nabla^2 + \dfrac{\partial^2}{\partial\tau^2}$.

For the neutral scalar nucleon the coupling term is to be multiplied by a factor of 2 (as can be seen from Eq. (II. 1')). Now Eq. (III. 1), as it stands, cannot directly be cast into the Lagrangian form. However, we can do this in an indirect manner, as is discussed in Reference (9), by introducing a fifth parameter and considering the space and time coordinates $x_\mu = (x_4 = \tau, \vec{x}) = x$ as functions of this parameter. Then a new function of five variables, i.e., the space-time coordinates and the parameter, and a differential equation satisfied by this function will be introduced. This must be done in such a way that it permits the application of the Lagrangian formulation and the variational technique, and it must also allow us to write down the wave function ψ of the scalar nucleon in a simple way in terms of this new function. This can be achieved by choosing the parameter σ, the new function $\phi(x,\sigma)$, and the differential equation as

$$-\frac{\partial\phi(x,\sigma)}{\partial\sigma} = -\frac{1}{2}(\boxplus^2 - \sqrt{4\pi}\, g\chi)\phi(x,\sigma). \qquad \text{(III. 2)}$$

This equation differs from the equation one would obtain from the argument in Reference (9) in that iu there corresponds to σ here. This choice of the left-hand side of Eq. (III. 2) enables us to apply the variational method on the path integral and eventually to calculate the lowest "energy" corresponding to the "Hamiltonian" (the coefficient of ϕ on the right-hand side of Eq. (III. 2)) of the "particle". Eq. (III. 2) has the form of Schrödinger equation in which it is replaced by σ and the spatial variables extended to the four-dimensional variables x_μ. Since the potential χ is independent of σ the solution of Eq. (III. 2) can be obtained by separating variables as $\phi(x,\sigma) = e^{\frac{1}{2}m^2\sigma}\psi(x)$, where ψ satisfies Eq. (III. 1). The eigenvalue $\frac{1}{2}m^2$ of the σ-dependent part $\Sigma(\sigma)$ of the equation $\frac{d\Sigma}{d\sigma} = \frac{1}{2}m^2\,\Sigma$ is related to the particle mass m. Hence Eq. (III. 2) together with this equation is equivalent to Eq. (III. 1). Therefore we may first put Eq. (III. 2) in Lagrangian form without any restriction, and the eigenvalue condition will be called for only in the final solution. That is, if $\phi(x,\sigma)$ is any solution of Eq. (III. 2), the solution of Eq. (III. 1) can be obtained from it by selecting the part corresponding to the eigenvalue $\frac{1}{2}m^2$ as

$$\psi(x) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}m^2\sigma} \phi(x,\sigma)\, d\sigma . \qquad \text{(III. 3)}$$

The kernel $K^{(x)}(x'', \sigma''; x', \sigma')$ of Eq. (III. 2) for arrival at x'', σ'' from x', σ' is defined by $\phi(x'', \sigma'') = \int K^{(x)}(x'', \sigma''; x', \sigma')\phi(x', \sigma') d^4\omega_{x'}$, where $d^4\omega_{x'}$ stands for the four-dimensional volume element at x'. Eq. (III. 3), as it stands, is divergent since the lower limit of the integration is $-\infty$. However, this difficulty can easily be avoided. For we can always let σ' coincide with the origin of σ-coordinate and also choose $K^{(x)}(\sigma, 0) = 0$ for $\sigma < 0$ by definition so that

$$\phi(x,\sigma) = \int K^{(x)}(x, \sigma; x'\, 0)\phi(x', 0) d^4\omega_{x'} = 0 \text{ for } \sigma < 0.$$

For this choice we will have, instead of Eq. (III. 3),

$$\psi(x) = \int_{0}^{\infty} e^{-\frac{1}{2}m^2\sigma} \phi(x,\sigma)\, d\sigma ,$$

which will no longer be divergent. However, as far as we confine ourselves to the study of the kernel $K^{(x)}$ of ϕ-function and not concern ourselves with the relation between ψ and ϕ (like Eq. (III. 3)) we need not worry about the finiteness of ψ and hence we can take any value for σ' and σ''. Therefore, in Sect. IV, where we are exclusively concerned with the kernel K associated with ϕ, we will choose $\sigma' = -\sum/2$ and $\sigma'' = \sum/2$ for convenience. Only in Sect. VI where the relation between ψ and ϕ comes into play we need be careful about the finiteness of ψ. $K^{(\chi)}$ is given by

$$K^{(\chi)}(2,1) = \int e^{S^{(\chi)}} \mathcal{D}\, x_\mu(\sigma), \quad * \tag{III. 4}$$

where $$S^{(\chi)} = \int_{\sigma'}^{\sigma''} \left\{ -\tfrac{1}{2}\left(\frac{dx_\mu}{d\sigma}\right)^2 - \tfrac{1}{2}\sqrt{4\pi}\, g\chi \right\} d\sigma \tag{III. 5}$$

and $$\left(\frac{dx_\mu}{d\sigma}\right)^2 = \sum_{q_\mu = x, y, z, \tau} \left(\frac{dq_\mu}{d\sigma}\right)^2 .$$

This is because if we assume Eq. (III. 4) together with Eq. (III. 5) it gives Eq. (III. 2) correctly. To see this let us suppose that the whole range of σ from σ' to σ'' is divided into a large number of very small interval ϵ ($\sigma_0 = \sigma'$, $\sigma_1 = \sigma' + \epsilon$, $\sigma_2 = \sigma' + 2\epsilon$, ...), and let us suppose that the particle is at $\sigma_k = \sigma$. Then the fact that $K^{(\chi)}$ is the kernel implies that

$$\phi(x_{k+1,\mu}, \sigma+\epsilon) = \iint e^{\epsilon\left[-\frac{1}{2}\left(\frac{x_{k+1,\mu} - x_{k,\mu}}{\epsilon}\right)^2 - \frac{1}{2}\sqrt{4\pi}\, g\chi(x_{k+1,\mu})\right]} \phi(x_{k,\mu}, \sigma)\, \frac{d^3\vec{x}_k}{a^3}\, \frac{d\tau_k}{a}$$

$$= \int e^{-\frac{(x_{k+1,\mu} - x_{k,\mu})^2}{2\epsilon} - \frac{\epsilon}{2}\sqrt{4\pi}\, g\chi(x_{k+1,\mu})} \phi(x_{k,\mu}, \sigma)\, \frac{d^4 x_{k,\mu}}{a^4} ,$$

where a is the normalization constant to be determined later. Here we need not be very careful in writing the exponent because there is not involved linear terms in velocity whose presence in the integrand of $S^{(\chi)}$ makes the situation complicated[(10)]. Let us call

$$(x_{k+1} - x_k)_\mu = \xi_\mu \quad \text{and } x_{k+1,\mu} = x_\mu .$$

Then

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* As to the meaning of the symbol $\mathcal{D}\, x_\mu(\sigma)$ see References (9) and (10).

$$\phi(x_\mu, \sigma+\epsilon) = \int e^{-\frac{\epsilon}{2}\sqrt{4\pi}\, g\, \chi(x_\mu)} \prod_{\mu=1}^{4} e^{-\frac{\xi_\mu^2}{2\epsilon}} \phi(x_\mu-\xi_\mu, \sigma) \frac{d^4\xi_\mu}{a^4} .$$

Since only small values of ξ_μ give important contribution to the integral because of the presence of the factor $e^{-\xi_\mu^2/2\epsilon}$ we can expand $\phi(x_\mu-\xi_\mu, \sigma)$ in powers of ξ_μ and write

$$\phi(x_\mu, \sigma+\epsilon) = e^{-\frac{\epsilon}{2}\sqrt{4\pi}\, g\, \chi(x_\mu)} \int \frac{d\xi_1}{a} e^{-\frac{\xi_1^2}{2\epsilon}} \int \cdots \int \cdots \int \frac{d\xi_4}{a} e^{-\frac{\xi_4^2}{2\epsilon}} \times$$

$$\left[\phi(x_\mu, \sigma) - \left(\sum_{\mu=1}^{4} \xi_\mu \frac{\partial}{\partial x_\mu}\right)\phi(x_\mu, \sigma) + \frac{1}{2!}\left(\sum_{\mu=1}^{4} \xi_\mu \frac{\partial}{\partial x_\mu}\right)^2 \phi(x_\mu, \sigma) + \ldots \right] . \tag{III. 6}$$

Now we need the following integrals:

$$\int_{-\infty}^{\infty} d\xi\, e^{-\frac{\xi^2}{2\epsilon}} = \sqrt{2\pi\epsilon} ,$$

$$\int_{-\infty}^{\infty} d\xi\, \xi e^{-\frac{\xi^2}{2\epsilon}} = 0 ,$$

$$\int_{-\infty}^{\infty} d\xi\, \xi^2 e^{-\frac{\xi^2}{2\epsilon}} = \epsilon\sqrt{2\pi\epsilon} .$$

The third term on the right-hand side of Eq. (III. 6) becomes

$$\frac{1}{2!}\left(\sum_{\mu=1}^{4} \frac{\partial^2\phi(x,\sigma)}{\partial x_\mu^2} \int \xi_\mu^2 e^{-\frac{\xi_\mu^2}{2\epsilon}} \frac{d\xi_\mu}{a} \prod_{\nu\neq\mu} e^{-\frac{\xi_\nu^2}{2\epsilon}} \frac{d\xi_\nu}{a} \right.$$

$$\left. + 2\sum_{\mu<\nu} \frac{\partial^2\phi(x,\sigma)}{\partial x_\mu \partial x_\nu} \int \xi_\mu e^{-\frac{\xi_\mu^2}{2\epsilon}} \frac{d\xi_\mu}{a} \int \xi_\nu e^{-\frac{\xi_\nu^2}{2\epsilon}} \frac{d\xi_\nu}{a} \prod_{\sigma\neq\mu,\nu} \int e^{-\frac{\xi_\sigma^2}{2\epsilon}} \frac{d\xi_\sigma}{a} \right)$$

$$= \frac{1}{2} \sum_{\mu=1}^{4} \frac{\partial^2 \phi(x,\sigma)}{\partial x_\mu^2} \frac{\epsilon \sqrt{2\pi\epsilon}}{a} \left(\frac{\sqrt{2\pi\epsilon}}{a}\right)^3 .$$

Eq. (III. 5) becomes, when its left-hand side and the factor $e^{-\frac{\epsilon}{2}\sqrt{4\pi}\, g\chi}$ are expanded in powers of ϵ up to the first,

$$\phi(x,\sigma) + \epsilon \frac{\partial\phi(x,\sigma)}{\partial\sigma} + \ldots$$

$$= \left[1 - \frac{\epsilon}{2}\sqrt{4\pi}\, g\, \chi(x) + \ldots\right]\left[\left(\frac{\sqrt{2\pi\epsilon}}{a}\right)^4 \phi(x,\sigma) + \frac{1}{2}\epsilon\left(\frac{\sqrt{2\pi\epsilon}}{a}\right)^4 \sum_{\mu=1}^{4} \frac{\partial^2\phi(x,\sigma)}{\partial x_\mu^2} + \ldots\ldots\right].$$

Comparison of the $\phi(x,\sigma)$-terms on both sides gives the normalization constant as

$$a = \sqrt{2\pi\epsilon} .$$

Then the equation above now becomes

$$\phi(x,\sigma) + \epsilon\frac{\phi(x,\sigma)}{\partial\sigma} + \ldots = \left[1 - \frac{\epsilon}{2}\sqrt{4\pi}\, g\chi(x) + \ldots\right]\left[\phi(x,\sigma) + \frac{\epsilon}{2}\sum_{\mu=1}^{4}\frac{\partial^2\phi(x,\sigma)}{\partial x_\mu^2} + \ldots\right].$$

The terms proportional to ϵ give

$$-\frac{\partial\phi(x,\sigma)}{\partial\sigma} = -\frac{1}{2}\left(\sum_{\mu=1}^{4}\frac{\partial^2}{\partial x_\mu^2}\right)\phi(x,\sigma) + \frac{1}{2}\sqrt{4\pi}\, g\chi(x)\,\phi(x,\sigma)$$

$$= -\frac{1}{2}\left(\boxplus^2 - \sqrt{4\pi}\, g\chi(x)\right)\phi(x,\sigma) .$$

This establishes that the kernel $K^{(\chi)}$ is given by Eq. (III. 4) with $S^{(\chi)}$ defined by Eq. (III. 5).

We now proceed to find the kernel corresponding to the transition of the scalar nucleon from the vacuum to the vacuum of the neutral scalar meson field when it is in the quantum mechanical interaction with this field. The corresponding problem in quantum electrodynamics was discussed by making use of the operator calculus[11] or the Lagrangian method[12]. Because of its clarity we will use the operator calculus here. We first expand χ in the momentum eigenfunctions

$$\chi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_k q_k e^{i\vec{k}\vec{x}} .$$

We then write q_k in terms of the creation and annihilation operator a^*_k and a_k as

$$q_k = \frac{1}{\sqrt{2\omega_k}} (a_k + a^*_{-k}),$$

where $\omega_k = +\sqrt{\vec{k}^2 + m'^2}$ when the mass of the meson is m'. Substitution of this gives

$$\chi(\vec{x}) = \frac{1}{\sqrt{V}} \left\{ \sum_k \frac{1}{\sqrt{2\omega_k}} a_k e^{i\vec{k}\vec{x}} + \sum_k \frac{1}{\sqrt{2\omega_k}} a^*_{-k} e^{i\vec{k}\vec{x}} \right\}$$

$$= \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} (a_k e^{i\vec{k}\vec{x}} + a^*_k e^{-i\vec{k}\vec{x}}) ,$$

since $\omega_{-k} = \omega_k$. If we go over to the interaction representation a_k is replaced by $a_k e^{-\omega_k \tau}$ while a^*_k is replaced by $a^*_k e^{+\omega_k \tau}$.*

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*Cf. Eq. (30) of Reference (8).

Therefore in the interaction representation χ is decomposed into two parts

$$\chi(\vec{x},\tau) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{i\vec{k}\vec{x}-\omega_k\tau} + a_k^* e^{-i\vec{k}\vec{x}+\omega_k\tau}\right)$$

$$= \chi^+(\vec{x},\tau) + \chi^-(\vec{x},\tau), \qquad \text{(III. 7)}$$

where $\chi^+(\vec{x},\tau) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} a_k e^{i\vec{k}\vec{x}-\omega_k\tau}$

annihilates mesons, while

$$\chi^-(\vec{x},\tau) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} a_k^* e^{-i\vec{k}\vec{x}+\omega_k\tau}$$

creates them.

The commutator of χ^- *and* χ^+ *is given by*

$$\left[\chi^-(\vec{x}_1,\tau_1), \chi^+(\vec{x}_2,\tau_2)\right]$$
$$= \frac{1}{V} \sum_k \sum_{k'} \frac{1}{\sqrt{2\omega_k}} \cdot \frac{1}{\sqrt{2\omega_{k'}}} e^{-i\vec{k}\vec{x}_1+\omega_k\tau_1} e^{i\vec{k}'\vec{x}_2-\omega_{k'}\tau_2} \left[a_k^*, a_{k'}\right],$$

which, by virtue of $\left[a_{k'}, a_k^*\right] = \delta_{k'k}$, becomes

$$= -\frac{1}{V} \sum_k \frac{1}{2\omega_k} e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)-\omega_k(\tau_2-\tau_1)} .$$

By going to the continuum case from the discrete one we replace

$$\sum_k \ldots \text{ by } \frac{V}{(2\pi)^3} \int d^3\vec{k} \ldots\ldots$$

When we take the meson mass m' to be zero for the reason of simplicity the commutator becomes

$$\left[\chi^-(\vec{x}_1,\tau_1), \chi^+(\vec{x}_2,\tau_2)\right] = -\frac{1}{(2\pi)^3} \int e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)-k(\tau_2-\tau_1)} \frac{d^3\vec{k}}{2k}$$

$$= -\frac{1}{2(2\pi)^3} \int \frac{d^3\vec{k}}{k} e^{-k(\tau_2-\tau_1)} \cos \vec{k}(\vec{x}_1-\vec{x}_2) .$$

Use of an identity ($r = |\vec{x}|$)

$$\int e^{-k|\tau|} \cos \vec{k}\vec{x} \frac{d^3\vec{k}}{(2\pi)^2 k} = \frac{1}{\pi r}\int_0^\infty dk\, e^{-k|\tau|} \sin kr$$

$$= \frac{1}{\pi r}\frac{1}{2i}\int_0^\infty dk \left\{ e^{ikr} - e^{-ikr} \right\} e^{-k|\tau|}$$

$$= \frac{1}{2\pi r i}\left\{ \frac{1}{|\tau| - i r} - \frac{1}{|\tau| + i r} \right\} = \frac{1}{\pi(\tau^2 + r^2)}$$

(III. 8)

reduces the commutator to

$$\left[\chi^-(1), \chi^+(2) \right] = \frac{-1}{4\pi^2} \cdot \frac{1}{(\tau_2 - \tau_1)^2 + (\vec{x}_2 - \vec{x}_1)^2} \qquad \text{(III. 9)}$$

for $\tau_2 > \tau_1$.*

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*This result can of course be obtained from Eq. (46) of Reference (8)

$$A^-_\mu(1)\, A^+_\nu(2) - A^+_\nu(2)\, A^-_\mu(1) = -i\, e^2 \delta_{\mu\nu}\, \delta_+(S^2_{12}).$$

For firstly $\delta_+(S^2_{12}) = (t^2_{12} - r^2_{12}) - \frac{i}{\pi} \cdot \frac{1}{t^2_{12} - r^2_{12}}$ reduces to $\frac{i}{\pi} \cdot \frac{1}{\tau^2_{12} + r^2_{12}}$ by it $= \tau$ and secondly our case is to have the opposite sign to that of the quantum electrodynamics for $\mu = \nu = 4$ because the nuclear force is attractive in contrast to the repulsive Coulomb force. Furthermore the coupling constant e^2 is to be replaced by $(\sqrt{4\pi}\, g)^2$ in our case.

Now the kernel in the quantum mechanical case where there are no mesons present in the initial and the final states can be obtained by integrating out the meson field operator χ in the coupling term $e^{-\int \frac{1}{2}\sqrt{4\pi}g\chi(x_\mu(\sigma))d\sigma}$ of Eq. (III. 4). For this reason let us consider an operator

$$R^{(\chi)} = e^{-\int \frac{1}{2}\sqrt{4\pi}g\, j(1)\, \chi(1) d^4\omega_1}, \tag{III. 10}$$

where j(1) can be any numerical function. The study of the operator $R^{(\chi)}$ is exactly analogous to the case of quantum electrodynamics[8] where the electromagnetic potential A_μ in the operator

$$e^{-i\int j_\mu(1) A_\mu(1)\, d^4\omega_1}$$

is integrated out to give

$$e^{-\frac{1}{2}ie^2\iint j_\mu(1) j_\mu(2)\, \delta_+(S_{12}^2)\, d^4\omega_1 d^4\omega_2}$$

by the use of the commutation relation ($\delta_{\mu\nu} = +1$ for $\mu = \nu = 4$, and -1 for $\mu = \nu = 1$, 2, and 3)

$$\left[A_\mu^-(1), A_\nu^+(2) \right] = -i\, e^2\, \delta_{\mu\nu}\, \delta_+(S_{12}^2)$$

for $t_2 > t_1$. The element of the operator $R^{(\chi)}$ corresponding to the meson-vacuum to meson-vacuum transition is then given, using Eq. (III. 9), as

$$R_{00} = e^{\frac{1}{2}\frac{g^2}{4\pi}\iint j(1) j(2) \frac{1}{(\tau_2-\tau_1)^2+(\vec{x}_2-\vec{x}_1)^2}\, d^4\omega_1 d^4\omega_2}. \tag{III. 11}$$

The way Eq. (III. 11) is obtained from Eq. (III. 10) by integration of the field operator χ can be extended to any superposition of the operators of the form of Eq. (III.10). That is, if an operator $R^{(\chi)}$ is given by the linear combination

$$R^{(\chi)} = \sum_i f_i \; e^{-\int \frac{1}{2}\sqrt{4\pi}\, g\, j\,(1;i)\,\chi(1) d^4\omega_1} \qquad \text{(III. 12)}$$

then we have for the vacuum-to-vacuum transition of this operator

$$R_{00} = \sum_i f_i \; e^{\frac{1}{2}\frac{g^2}{4\pi}\iint j(1;i) j(2;i) \frac{1}{\tau_{12}^2 + r_{12}^2} d^4\omega_1\, d^4\omega_2} \;, \qquad \text{(III. 13)}$$

where $\tau_{12} = \tau_1 - \tau_2$ and $r_{12} = |\vec{x}_1 - \vec{x}_2|$.

To perform the integration of the meson field operator χ in Eq. (III. 4) we first notice that its coupling term can be written in the form of Eq. (III. 10) in the following way:

$$e^{-\int \frac{1}{2}\sqrt{4\pi}\, g\, \chi(x_\mu(\sigma)) d\sigma} = e^{-\iint \frac{1}{2}\sqrt{4\pi}\, g\, \chi(x_\mu(1))\; \delta^4(x_\mu(1) - x_\mu(\sigma)) d\sigma\, d^4\omega_1}$$

$$= e^{-\int \frac{1}{2}\sqrt{4\pi}\, g\, j(1)\, \chi(1) d^4\omega_1} \qquad \text{(III. 14)}$$

with

$$j(1) = \int \delta^4\,(x_\mu(1) - x_\mu(\sigma))\, d\sigma \;.$$

Next let us consider $K^{(\chi)}$ given by Eq. (III. 4). Using Eq. (III. 14) and writing $\dot{x}_\mu^2(\sigma)$ for $(d\,x_\mu/d\sigma)^2$ it becomes

$$K^{(\chi)}(2,1) = \int e^{-\frac{1}{2}\int \dot{x}_\mu^2(\sigma) d\sigma}\; e^{-\int \frac{1}{2}\sqrt{4\pi}\, g\, j(1)\, \chi(1)\, d^4\omega_1}\; \mathcal{D}x_\mu(\sigma) \;.$$

Since the integration over $x_\mu(\sigma)$ is the limit of the summation this has the general form given by Eq. (III. 12), where each term inside the summation sign is determined by fixing the value of $x_\mu(\sigma)$. Hence by Eq. (III. 13) we have as the vacuum-to-vacuum element of the operator $K^{(\chi)}$

$$K_{00}(2,1) = \int e^{-\frac{1}{2}\int \dot{x}_\mu^2(\sigma)d\sigma} \, e^{\frac{ig^2}{2\cdot 4\pi}\iint j(1)j(2)\frac{1}{\tau_{12}^2+r_{12}^2} d^4\omega_1 d^4\omega_2} \, \mathcal{D}x_\mu(\sigma) .$$

Since

$$\iint j(1)j(2)\frac{1}{\tau_{12}^2+r_{12}^2} d^4\omega_1 d^4\omega_2$$

$$= \iiiint \frac{\delta^4(x_\mu(1)-x_\mu(\sigma)\,\delta^4(x_\nu(2)-x_\nu(\sigma'))}{(x_\lambda(1)-x_\lambda(2))^2} d\sigma d\sigma' d^4\omega_1 d^4\omega_2$$

$$= \iint \frac{d\sigma d\sigma'}{(x_\mu(\sigma)-x_\mu(\sigma'))^2}$$

we finally have

$$K_{00}(2,1) = \int e^{-\frac{1}{2}\int \dot{x}_\mu^2(\sigma)\, d\sigma + \frac{ig^2}{2\cdot 4\pi}\iint \frac{d\sigma d\sigma'}{(x_\mu(\sigma)-x_\mu(\sigma'))^2}} \mathcal{D}x_\mu(\sigma) . \qquad \text{(III. 15)}$$

This is the kernel associated with function ϕ satisfying Eq. (III. 2) corresponding to the vacuum-to-vacuum transition of the meson field in the quantum mechanical case. Notice that this is not the kernel associated with the scalar particle described by the wave function ψ. However, since ψ can be given, as shown by Eq. (III. 3), in terms of ϕ it becomes possible, as will be discussed in Section VI, to obtain the lowest energy of the scalar nucleon after applying the variational technique on the four-dimensional path integral given by Eq. (III. 15). We also notice that for the case of the neutral scalar nucleon, because of the presence of a factor of 2 in the coupling term of Eq. (III. 1),which has its origin in Eq. (II. 1'), the second term in the exponent of Eq. (III. 15) must also be multiplied by 2.

The evaluation of the kernel (or at least its asymptotic form) given by Eq. (III. 15) represents the central part of our problem. For Eq. (III. 15) contains all the self-action effects, with the exception of the closed loop effects, of the scalar nucleon interacting with the neutral scalar meson field. Such path integrals are ordinarily very difficult to evaluate. However, in the next section we will develop a method of approximate evaluation whose validity does not require that g^2 be particularly small or particularly large.

IV. EVALUATION OF PATH INTEGRALS

In this section we will apply the variational technique on the path integral over the trajectories that has been obtained in the previous section. This part of the work keeps a very close parallelism with the three-dimensional case of polaron problem(4). But before launching the actual calculation it will perhaps be better to expose some of the main features of this technique since this method has never been used before except in the polaron problem which has just been mentioned.

In the conventional variational method a trial wave function, ϕ, involving one or more parameters is chosen and $\int \phi^* H \phi d^3\vec{x} / \int \phi^* \phi d^3\vec{x}$ is minimized with respect to the parameters and this value is considered as the "best" value of the energy of the system. But here in the Lagrangian formulation of the quantum mechanics we do not deal with the wave function but with the kernel which is the path integral over the trajectories of the particle. In the conventional form (that is before we apply a transformation of the form $\tau = it$) the kernel is given by

$$K = \int e^{iS} \mathcal{D}x(t) ,$$

and in turn S, the action, is given in terms of the Lagrangian, L, of the system by

$$S = \int L \, dt.$$

Therefore K is the sum over all the possible trajectories of e^{-iS}. In the τ-formalism (explained in Sect. II) it has the form

$$\underline{K} = \int e^{\underline{S}} \mathcal{D}x(\tau) ,$$

where $\underline{S} = \int \underline{L}\, d\tau.$

We have put underlines to the quantities in the τ -formalism to discriminate them from the corresponding quantities in the t-formalism. Nevertheless, it is most convenient to use the same terminology in both cases so that we will call $\underline{S}$ action, and $\underline{L}$ Lagrangian, etc. Moreover, even the underlines will be dropped hereafter, because we are concerned only with the τ -formalism in the following so that there will be no confusions introduced.

Consider the wave equation in τ -formalism

$$-\frac{\partial\phi}{\partial\tau} = H\phi \; . \tag{IV. 1}$$

If we know the eigenfunctions ϕ_n and the corresponding eigenvalues E_n of the Hamiltonian H, that is,

$$H\phi_n = E_n\phi_n \; ,$$

then any arbitrary solution ϕ of Eq. (IV. 1) can be expanded in terms of these ϕ_n as

$$\phi = \sum_n a_n\phi_n\, e^{-E_n\tau} \; . \tag{IV. 2}$$

If Eq. (IV. 1) permits the Lagrangian formulation of the quantum mechanics the kernel corresponding to it can be given as path integral by

$$K = \int e^S \mathcal{D}x(\tau) \; . \tag{IV. 3}$$

It will turn out later to be essential that the action S is a real quantity in order to apply the variational technique. Since K is a solution of Eq. (IV. 1) it has, in view of Eq. (IV. 2), for a very large interval T of τ the following asymptotic form:

$$K \sim e^{-E_o T} \quad , \tag{IV.4}$$

where E_o is the lowest energy of the system. We are interested in finding the best value for this energy. Let us suppose that we choose an appropriate trial action S_1 and also suppose that we know the lowest energy E_1 corresponding to S_1. Then we can write

$$\int e^{S_1} \mathcal{D}x(\tau) \sim e^{-E_1 T} \quad . \tag{IV.5}$$

From Eqs. (IV.3), (IV.4), and (IV.5) we have

$$e^{-E_o T} \sim \int (e^{S-S_1})\, e^{S_1} \mathcal{D}x(\tau) \sim \left[\frac{\int (e^{S-S_1}) e^{S_1} \mathcal{D}x(\tau)}{\int e^{S_1} \mathcal{D}x(\tau)} \right] e^{-E_1 T} \quad . \tag{IV.6}$$

The factor in the square bracket can be thought of as the average of e^{S-S_1} with a positive weight e^{S_1} and we shall call it $\langle e^{S-S_1} \rangle$. By virtue of an inequality

$$\langle e^{f} \rangle \geqslant e^{\langle f \rangle} , \tag{IV.7}$$

which holds for averages over any real quantities f Eq. (IV.6) can now be written as

$$e^{-E_o T} \geqslant e^{\langle S-S_1 \rangle} \; e^{-E_1 T} , \tag{IV.8}$$

where

$$\langle S-S_1 \rangle = \frac{\int (S-S_1) e^{S_1} \mathcal{D}x(\tau)}{\int e^{S_1} \mathcal{D}x(\tau)} \quad . \tag{IV.9}$$

Since it can be seen from this expression that $\langle S-S_1 \rangle$ is proportional to T we write

$$\langle S-S_1 \rangle = s\, T. \tag{IV.10}$$

When this is substituted in Eq. (IV. 8), by calling

$$E = E_1 - s \; , \tag{IV. 11}$$

it gives

$$E_o \leqslant E \; .$$

Since E_1 and S contain the parameters through S_1 the best value for E_o can be obtained by minimizing E with respect to the parameters contained in it.

For the case of a particle in a potential V, as will be shown below, the new variational method is equivalent to the conventional one. For more complicated situations, however, the relation between these two methods is not known. Thus, it is not clear for a given trial action S_1 what trial function, if any, in the conventional method leads to the relation given by Eq. (IV. 11). The equivalence of both methods for the case of a particle in a potential V can be shown as follows. For this case the Lagrangian L and the action S are given by

$$L = -T - V = -\frac{m}{2} \dot{x}^2 - V \, ,$$

$$S = \int L d\tau = -\int \frac{m}{2} \dot{x}^2(\tau) d\tau - \int V(x_\tau) d\tau \; .$$

Let us suppose that the trial action is given by

$$S_1 = \int L_1 d\tau = -\int \frac{m}{2} \dot{x}^2(\tau)\, d\tau - \int V_1(x_\tau)\, d\tau \; .$$

Then from

$$sT = \langle S - S_1 \rangle = -\left\langle \int (V - V_1) d\tau \right\rangle = -T \langle V - V_1 \rangle \; .$$

s is given by

$$s = -\langle V - V_1 \rangle \; .$$

In estimating s let us take $T > \tau > 0$ and consider first the numerator of

$$\langle V-V_1 \rangle = \frac{\int (V-V_1) e^{S_1} \mathcal{D}x(\tau)}{\int e^{S_1} \mathcal{D}x(\tau)} .$$

Corresponding to $T > \tau > 0$ we split e^{S_1} as

$$e^{S_1} = e^{\int_0^T L_1 d\tau} = e^{\int_\tau^T L_1 d\tau''} \, e^{\int_0^\tau L_1 d\tau'} .$$

Hence

$$\int (V-V_1) e^{S_1} \mathcal{D}x(\tau) = \iiint e^{\int_\tau^T L_1 d\tau''} \mathcal{D}x(\tau'')(V(x_\tau)-V_1(x_\tau)) dx_\tau \, e^{\int_0^\tau L_1 d\tau'} \mathcal{D}x(\tau')$$

$$= \int K(x_T, T; x_\tau, \tau)(V(x_\tau)-V_1(x_\tau)) \, dx_\tau \, K(x_\tau, \tau; x_o, 0).$$

Since both τ and $T-\tau$ are very large we have[5]

$$K(x_\tau, \tau; x_o, 0) = \sum_n \phi_n(x_\tau)\phi_n^*(x_o) e^{-E_n \tau} \sim \phi_o(x_\tau)\phi_o^*(x_o) e^{-E_o \tau}$$

and similarly

$$K(x_T, T; x_\tau, \tau) \sim \phi_o(x_T)\phi_o^*(x_\tau) \, e^{-E_o(T-\tau)} ,$$

where ϕ_o is the eigenfunction of the lowest energy state with energy E_o (this E_o corresponds to E_1 in Eq. (IV.5)). Therefore we have

$$\int (V-V_1) e^{S_1} \mathcal{D}x(\tau) \sim \phi_o(x_T) \phi_o^*(x_o) e^{-E_o T} \int \phi_o^*(x_\tau)(V(x_\tau)-V_1(x_\tau))\phi_o(x_\tau) dx_\tau$$

$$= \phi_o(x_T)\phi_o^* (x_o) e^{-E_o T} \; (V-V_1)_{00} .$$

Similarly we have for the denominator of $\langle V-V_1 \rangle$

$$\int e^{S_1} \mathcal{D}x(\tau) \sim \phi_o(x_T)\phi_o^*(x_o) e^{-E_o T} .$$

Hence $\langle V-V_1 \rangle = -s = (V-V_1)_{00}$.

From $E_1 = (\frac{p^2}{2m})_{00} + (V_1)_{00}$

and Eq. (IV. 11) we finally have

$$E_o = (\frac{p^2}{2m})_{00} + (V)_{00}$$

which is expressing the content of the conventional variational method.

From now on we will deal with the evaluations of the path integrals involved in E_1 and S. To save writing we omit the subscripts 00 attached to the kernel (Eq. (III. 15)) for 0-0 transition in the quantum mechanical case. Thus

$$K(2,1) = \int e^S \mathcal{D} x_\mu(\sigma) \, , \tag{IV. 12}$$

where

$$S = -\frac{1}{2}\int \dot{x}_\mu^2 (\sigma) \, d\sigma + \frac{g^2}{4\pi}\frac{1}{2}\int\int \frac{d\sigma' d\sigma''}{(x_\mu(\sigma')-x_\mu(\sigma''))^2} \, . \tag{IV. 13}$$

As to the trial action we borrow it from the three-dimensional case of polaron problem[4] with direct extension to the four dimensions:

$$S_1 = -\frac{1}{2}\int \dot{x}_\mu^2 (\sigma) d\sigma - C\int\int (x_\mu(\sigma')-x_\mu(\sigma''))^2 \, e^{-a|\sigma'-\sigma''|} d\sigma' d\sigma'' \, , \tag{IV. 14}$$

where a and C are the parameters. The apparent lack of resemblance of the second terms of S and S_1 may not be objected to seriously. In fact for the polaron problem the approximation of $|x(\sigma')-x(\sigma'')|^{-1} \times e^{-|\sigma'-\sigma''|}$ by $(x(\sigma')-x(\sigma''))^2 e^{-a|\sigma'-\sigma''|}$ gave a very satisfactory result. That S_1 is quadratic in x_μ is a very convenient choice because for such a functional S_1 the path integrals can be most easily worked out[13]. The range of σ will conveniently be taken to be from $-\Sigma/2$ to $+\Sigma/2$ but since eventually we go to the limit when Σ tends to infinity all the limits of the σ-integrals can be approximated by $-\infty$ and $+\infty$. In the following we will use Eqs. from (IV. 4) to (IV. 11) replacing T by Σ. Eq. (IV. 10) together with Eqs. (IV. 13) and (IV. 14) gives

$$s = \frac{1}{\Sigma}\langle S - S_1\rangle = \frac{g^2}{4\pi}\frac{1}{2}\int d\sigma'' \left\langle \frac{1}{(x_\mu(\sigma') - x_\mu(\sigma''))^2} \right\rangle \tag{IV. 15}$$
$$+ C\int d\sigma'' \left\langle (x_\mu(\sigma') - x_\mu(\sigma''))^2 \right\rangle e^{-a|\sigma' - \sigma''|} \,.$$

Since it turns out that the second term of this expression can be obtained using the knowledge required for the first term we will deal with the latter first.

To study the first term of Eq. (IV. 15) we write $(x_\mu(\sigma') - x_\mu(\sigma''))^2$ as a Fourier integral. This can be done by combining Eq. (II. 25) with $m = 0$ and Eq. (III. 8), the result being

$$\frac{1}{(x_\mu(\sigma') - x_\mu(\sigma''))^2} = \frac{1}{(2\pi)^2}\int \frac{e^{ik_\mu(x_\mu(\sigma') - x_\mu(\sigma''))}}{k_\mu^2}\, d^4k \,. \tag{IV. 16}$$

Substitution of Eq. (IV. 16) into the first term of Eq. (IV. 15) gives

$$\frac{1}{2}\cdot\frac{g^2}{4\pi}\cdot\frac{1}{(2\pi)^2}\int d\sigma'' \int \frac{d^4k}{k_\mu^2} \left\langle e^{ik_\mu(x_\mu(\sigma') - x_\mu(\sigma''))} \right\rangle. \tag{IV. 17}$$

Hence the problem is now reduced to evaluate

$$\left\langle e^{ik_\mu(x_\mu(\sigma') - x_\mu(\sigma''))} \right\rangle = \frac{\int e^{ik_\mu(x_\mu(\sigma') - x_\mu(\sigma''))}\, e^{S_1}\, \mathcal{D}x_\mu(\sigma)}{\int e^{S_1}\, \mathcal{D}x_\mu(\sigma)} \,. \tag{IV. 18}$$

Writing

$$f_\mu(\sigma) = ik_\mu \left[\delta(\sigma - \sigma') - \delta(\sigma - \sigma'') \right] \tag{IV. 19}$$

the numerator of Eq. (IV. 18) can be written as

$$\int e^{\left[-\frac{1}{2}\int \dot{x}_\mu(\sigma)^2 d\sigma - C\iint (x_\mu(\sigma')-x_\mu(\sigma''))^2 e^{-a|\sigma'-\sigma''|} d\sigma' d\sigma'' + \int f_\mu(\sigma)\, x_\mu(\sigma)\, d\sigma\right]} \mathcal{D}x_\mu(\sigma) . \tag{IV. 20}$$

Since this is the product of components corresponding to four values of μ the integral separates and we only need to consider a one-dimensional problem. The path integral of this scalar problem can be performed by letting

$$x(\sigma) = \bar{x}(\sigma) + y(\sigma) . \tag{IV. 21}$$

Here $\bar{x}(\sigma)$ is the function which makes the exponent $E[x(\sigma)]$ of this problem extremum, and $y(\sigma)$ is the displacement from the path $\bar{x}(\sigma)$. From Eq. (IV. 21) we have

$$\mathcal{D}x(\sigma) = \mathcal{D}y(\sigma).$$

If we choose the boundary conditions such that

$$x(-\tfrac{\Sigma}{2}) = \bar{x}(-\tfrac{\Sigma}{2}) = 0; \quad x(\tfrac{\Sigma}{2}) = \bar{x}(\tfrac{\Sigma}{2}) = 0, \tag{IV. 22}$$

then we have

$$y(-\tfrac{\Sigma}{2}) = y(\tfrac{\Sigma}{2}) = 0 .$$

For these boundary conditions the substitution of Eq. (IV. 21) into the exponent $E[x(\sigma)]$ gives

$$E[x(\sigma)] = E[\bar{x}(\sigma)] - \tfrac{1}{2}\int \dot{y}^2(\sigma) d\sigma - C\iint (y(\sigma')-y(\sigma''))^2 e^{-a|\sigma'-\sigma''|} d\sigma' d\sigma'' .$$

Here the terms linear in y dropped out by integrations by parts assuming the boundary conditions $y(-\Sigma/2) = y(\Sigma/2) = 0$. This is expected because the path $x(\sigma) = \bar{x}(\sigma)$ gives an extremum of $E[x(\sigma)]$

which can only depend on y to second order. Also it is clear that we will not have terms higher than y^2 because $E[x(\sigma)]$ is a quadratic function to begin with. Hence we have

$$\left\langle e^{ik(x(\sigma')-x(\sigma''))}\right\rangle =$$

$$\frac{e^{E[\bar{x}(\sigma)]}\int e^{\left[-\frac{1}{2}\int \dot{y}^2(\sigma)d\sigma - C\iint (y(\sigma')-y(\sigma''))^2 e^{-a|\sigma'-\sigma''|}d\sigma' d\sigma''\right]}\mathcal{D}y(\sigma)}{\int e^{\left[-\frac{1}{2}\int \dot{x}^2(\sigma)d\sigma - C\iint (x(\sigma')-x(\sigma''))^2 e^{-a|\sigma'-\sigma''|}d\sigma' d\sigma''\right]}\mathcal{D}x(\sigma)} \quad \text{(IV. 23)}$$

Therefore for boundary conditions given by Eq. (IV. 22) this equation reduces to

$$\left\langle e^{ik(x(\sigma')-x(\sigma''))}\right\rangle = e^{E[\bar{x}(\sigma)]} . \quad \text{(IV. 24)}$$

Hence we need only to find $\bar{x}(\sigma)$ and substitute it to the right-hand side of this equation. Since $E[x(\sigma)]$ is given by

$$E[x(\sigma)] = -\frac{1}{2}\int \dot{x}^2(\sigma)d\sigma - C\iint (x(\sigma')-x(\sigma''))^2 e^{-a|\sigma'-\sigma''|}d\sigma' d\sigma'' + \int f(\sigma)x(\sigma)d\sigma \quad \text{(IV. 25)}$$

the condition for the extremum of this functional yields

$$\ddot{\bar{x}}(\sigma) - 4C\int (\bar{x}(\sigma) - \bar{x}(\sigma'))\, e^{-a|\sigma-\sigma'|}d\sigma' + f(\sigma) = 0 . \quad \text{(IV. 26)}$$

Noticing that for a very large value of Σ

$$\int_{-\frac{\Sigma}{2}}^{+\frac{\Sigma}{2}} e^{-a|\sigma-\sigma'|}d\sigma' \approx \int_{-\infty}^{+\infty} e^{-a|\sigma-\sigma'|}d\sigma' = \int_{-\infty}^{\sigma} e^{-a(\sigma-\sigma')}d\sigma' + \int_{\sigma}^{\infty} e^{-a(\sigma'-\sigma)}d\sigma'$$

$$= \frac{2}{a} , \quad \text{(IV. 27)}$$

Eq. (IV. 26) now can be written as

$$\ddot{\overline{x}}(\sigma) - \frac{8C}{a}\,\overline{x}(\sigma) + 4C\int \overline{x}(\sigma')e^{-a|\sigma-\sigma'|}\,d\sigma' + f(\sigma) = 0. \qquad \text{(IV. 28)}$$

The easiest way to solve this is to introduce another function defined by

$$z(\sigma) = \frac{a}{2}\int \overline{x}(\sigma')\,e^{-a|\sigma-\sigma'|}\,d\sigma' \qquad \text{(IV. 29)}$$

and consider Eqs. (IV. 28) and (IV. 29) as a system of differential equations. The differentiations of Eq. (IV. 29) gives

$$\ddot{z}(\sigma) + a^2(\overline{x}(\sigma) - z(\sigma)) = 0, \qquad \text{(IV. 30)}$$

while Eq. (IV. 28) now reads, in terms of $z(\sigma)$,

$$\ddot{\overline{x}}(\sigma) - \frac{8C}{a}(\overline{x}(\sigma) - z(\sigma)) + f(\sigma) = 0. \qquad \text{(IV. 31)}$$

Subtraction of Eq. (IV. 30) from Eq. (IV. 31) yields

$$\ddot{F}(\sigma) - \left(\frac{8C}{a} + a^2\right)F(\sigma) + f(\sigma) = 0, \qquad \text{(IV. 32)}$$

where

$$F(\sigma) = \overline{x}(\sigma) - z(\sigma).$$

Eq. (IV. 32) has exactly the same form as Eq. (IV. 30), so that by calling

$$\beta^2 = \frac{8C}{a} + a^2 \qquad \text{(IV. 33)}$$

its solution is given immediately as

$$\overline{x}(\sigma) - z(\sigma) = F(\sigma) = \frac{1}{2\beta}\int f(\sigma_1)\,e^{-\beta|\sigma-\sigma_1'|}\,d\sigma_1$$

$$= \frac{\pm ik}{2\beta}\left(e^{-\beta|\sigma-\sigma'|} - e^{-\beta|\sigma-\sigma''|}\right). \qquad \text{(IV. 34)}$$

By adding $\frac{8C}{a^3}$ times Eq. (IV.30) to Eq. (IV.31) we have

$$\frac{d^2}{d\sigma^2}\left(\bar{x}(\sigma) + \frac{8C}{a^3}\, z(\sigma)\right) + f(\sigma) = 0.$$

Integrating once gives

$$\frac{d}{d\sigma}\left(\bar{x}(\sigma) + \frac{8C}{a^3}\, z(\sigma)\right) = -\int^{\sigma} f(\sigma_1)d\sigma_1 + K_1$$

$$= -ik \int^{\sigma} \left\{ \delta(\sigma_1 - \sigma') - \delta(\sigma_1 - \sigma'') \right\} d\sigma_1 + K_1$$

$$= -ik \left\{ H(\sigma - \sigma') - H(\sigma - \sigma'') \right\} + K_1 , \qquad \text{(IV.35)}$$

where K_1 is the integration constant and $H(\sigma)$ is defined by

$$H(\sigma) = \begin{cases} 0 & \text{for} \quad \sigma < 0 \\ 1 & \text{for} \quad \sigma > 0 . \end{cases}$$

One more integration results in

$$\bar{x}(\sigma) + \frac{8C}{a^3}\, z(\sigma) = -ik \int^{\sigma} \left\{ H(\sigma_2 - \sigma') - H(\sigma_2 - \sigma'') \right\} d\sigma_2 + K_1\sigma + K_2$$

$$= -ik \left[(\sigma - \sigma')H(\sigma - \sigma') - (\sigma - \sigma'')H(\sigma - \sigma'') \right] + K_1\sigma + K_2, \qquad \text{(IV.36)}$$

where K_2 is another integration constant. To determine K_1 and K_2 we notice that for a very large value of $\Sigma/2$ the defining equation, Eq. (IV.29), of $z(\sigma)$ gives

$$z\left(-\frac{\Sigma}{2}\right) = z\left(\frac{\Sigma}{2}\right) = 0.$$

These values together with Eq. (IV.22) show that

$$\bar{x}(\sigma) + \frac{8C}{a^3}\, z(\sigma) = 0$$

for $\sigma = \pm \frac{\Sigma}{2}$. Substitution of these in Eq. (IV.36) determines

$K_1 = K_2 = 0.$

Hence the solution satisfying the boundary conditions Eq. (IV. 23) is given by

$$\bar{x}(\sigma) + \frac{8C}{a^3} z(\sigma) = -ik\left[(\sigma-\sigma')H(\sigma-\sigma')-(\sigma-\sigma'')H(\sigma-\sigma'')\right]. \tag{IV. 37}$$

From Eqs. (IV. 34) and (IV. 37) we obtain

$$\bar{x}(\sigma) = \frac{+ik}{1+\frac{8C}{a^3}}\left[-(\sigma-\sigma')H(\sigma-\sigma')+(\sigma-\sigma'')H(\sigma-\sigma'') + \frac{1}{2\beta}\frac{8C}{a^3}\left(e^{-\beta|\sigma-\sigma'|} - e^{-\beta|\sigma-\sigma''|}\right)\right]. \tag{IV. 38}$$

Before substituting this directly into Eq. (IV. 24) let us show that

$$E[\bar{x}(\sigma)] = \frac{1}{2}\int f(\sigma)\,\bar{x}(\sigma)\,d\sigma . \tag{IV. 39}$$

To prove this multiply Eq. (IV. 26) by $\bar{x}(\sigma)$ and integrate over σ to give

$$\int \bar{x}(\sigma)\ddot{\bar{x}}(\sigma)d\sigma - 4C\iint(\bar{x}(\sigma')-\bar{x}(\sigma''))\bar{x}(\sigma')e^{-a|\sigma'-\sigma''|}\,d\sigma' d\sigma'' + \int f(\sigma)\bar{x}(\sigma)d\sigma = 0. \tag{IV. 40}$$

The first term gives (when integrated by parts)

$$\bar{x}(\sigma)\dot{\bar{x}}(\sigma)\Big|_{-\Sigma/2}^{\Sigma/2} - \int \dot{\bar{x}}^2(\sigma)\,d\sigma$$

and the first term of this expression drops because of Eq. (IV. 22). If we interchange σ' and σ'' in the second term of Eq. (IV. 40) and take the average of this and the original expression it reduces to

$$2C\iint(\bar{x}(\sigma') - \bar{x}(\sigma''))^2\, e^{-a|\sigma'-\sigma''|}\, d\sigma' d\sigma'' .$$

Hence Eq. (IV.40) now reads

$$-\int \dot{\bar{x}}^2(\sigma)d\sigma \;-2C\iint(\bar{x}(\sigma')-\bar{x}(\sigma''))^2\, e^{-a|\sigma'-\sigma''|}\,d\sigma' d\sigma'' = -\int f(\sigma)\bar{x}(\sigma)d\sigma \quad . \tag{IV.41}$$

The substitution of Eq. (IV.41) into Eq. (IV.25) results in Eq. (IV.39). Therefore Eq. (IV.24) now becomes

$$\left\langle e^{ik(x(\sigma')-x(\sigma''))}\right\rangle = e^{\frac{1}{2}\int f(\sigma)\bar{x}(\sigma)d\sigma} \quad . \tag{IV.42}$$

For the integral in the exponent of Eq. (IV.42) we obtain from Eqs. (IV.19 and (IV.38)

$$\int f(\sigma)\bar{x}(\sigma)d\sigma = \frac{+ik}{1+\frac{8C}{a^3}}\int\Big[-(\sigma-\sigma')H(\sigma-\sigma')+(\sigma-\sigma'')H(\sigma-\sigma'')+\frac{1}{2\beta}\frac{8C}{a^3}(e^{-\beta|\sigma-\sigma'|}-e^{-\beta|\sigma-\sigma''|})\Big]$$

$$\times\,(+ik)\Big[\delta(\sigma-\sigma')-\delta(\sigma-\sigma'')\Big]\quad d\sigma$$

$$=\frac{-k^2}{1+\frac{8C}{a^3}}\Big[(\sigma'-\sigma'')H(\sigma'-\sigma'')+(\sigma''-\sigma')H(\sigma''-\sigma')$$

$$+\frac{1}{\beta}\frac{8C}{a^3}(1-e^{-\beta|\sigma'-\sigma''|})\Big]$$

$$=\frac{-k^2}{1+\frac{8C}{a^3}}\Big[|\sigma'-\sigma''| \quad +\frac{1}{\beta}\frac{8C}{a^3}(1-e^{-\beta|\sigma'-\sigma''|})\Big], \tag{IV.43}$$

since

$$x(H(x)-H(-x)) = \begin{cases} x\,H(x) = x = |x| & \text{for } x > 0 \\ -x\,H(-x) = -x = |x| & \text{for } x < 0 \end{cases} \quad . \tag{IV.44}$$

From Eq. (IV.33) $1 + 8C/\alpha^3 = \beta^2/\alpha^2$ and the substitution of this in Eq. (IV.43) gives

$$\int f(\sigma)\bar{x}(\sigma)d\sigma = \frac{-k^2}{\beta^2}\left[\alpha^2|\sigma' - \sigma''| + \frac{\beta^2 - \alpha^2}{\beta}(1 - e^{-\beta|\sigma' - \sigma''|})\right]$$
$$= -2F(|\sigma' - \sigma''|)k^2,$$

where

$$F(\sigma) = \frac{1}{2\beta^2}\left[\alpha^2\sigma + \frac{\beta^2 - \alpha^2}{\beta}(1 - e^{-\beta\sigma})\right]. \tag{IV.45}$$

Then Eq. (IV.42) becomes

$$\left\langle e^{ik(x(\sigma') - x(\sigma''))}\right\rangle = e^{-F(|\sigma' - \sigma''|)k^2}. \tag{IV.46}$$

When we take the four rectangular components of the form of Eq. (IV.46) we have for Eq. (IV.17)

$$\frac{1}{2}\cdot\frac{g^2}{4\pi}\cdot\frac{1}{(2\pi)^2}\int d\sigma''\int\frac{d^4k}{k_\mu{}^2}\, e^{-F(|\sigma' - \sigma''|)k_\mu^2}. \tag{IV.47}$$

The integration over σ'' can be rewritten in the following manner:

$$\int_{-\frac{\Sigma}{2}}^{\frac{\Sigma}{2}} d\sigma''\, e^{-F(|\sigma' - \sigma''|)k_\mu^2} \tag{IV.48}$$

$$= \int_{-\frac{\Sigma}{2}}^{\sigma'} d\sigma''\, e^{-F(\sigma' - \sigma'')k_\mu^2} + \int_{\sigma'}^{\frac{\Sigma}{2}} d\sigma''\, e^{-F(\sigma'' - \sigma')k_\mu^2}.$$

By letting $\sigma' - \sigma'' = \sigma$ for the first term and $\sigma'' - \sigma' = \sigma$ for the second we have

$$= \int_0^{\sigma' + \frac{\Sigma}{2}} d\sigma \, e^{-F(\sigma) k_\mu^2} + \int_0^{-\sigma' + \frac{\Sigma}{2}} d\sigma \, e^{-F(\sigma) k_\mu^2} .$$

However, because of our assumption of very large $\Sigma/2$ we can practically extend both upper limits to $+\infty$ and hence it gives

$$= 2 \int_0^{\infty} d\sigma \, e^{-F(\sigma) k_\mu^2} . \qquad \text{(IV. 49)}$$

This result can of course be obtained without detailed argument by simply observing that when Σ tends to infinity Eq. (IV. 48) is symmetric about $\sigma'' = \sigma'$.

The substitution of Eq. (IV. 49) into Eq. (IV. 47) gives

$$\frac{1}{2} \cdot \frac{g^2}{4\pi} \cdot \frac{1}{(2\pi)^2} \cdot 2 \int_0^{\infty} d\sigma \int \frac{d^4k}{k_\mu^2} \, e^{-F(\sigma) \, k_\mu^2} . \qquad \text{(IV. 50)}$$

The integration over k does not separate into four rectangular components but we can do this by the use of four-dimensional polar coordinates[14], k, θ_1, θ_2, and ϕ. We write

$$k_1 = k \cos \theta_1 \qquad (0 \leq \theta_1 \leq \pi) ,$$

$$k_2 = k \sin \theta_1 \cos \theta_2 \qquad (0 \leq \theta_2 \leq \pi) ,$$

$$\left. \begin{aligned} k_3 &= k \sin \theta_1 \sin \theta_2 \cos \phi \\ k_4 &= k \sin \theta_1 \sin \theta_2 \sin \phi \end{aligned} \right\} \qquad (0 \leq \phi \leq 2\pi) .$$

The volume element is given by

$$\begin{aligned} d^4k &= k^3 dk \, d\omega_k \\ &= k^3 \, dk \sin^2 \theta_1 \sin \theta_2 \, d\theta_1 d\theta_2 d\phi . \end{aligned}$$

where $d\omega_k$ is the solid angle element. Also

$$k_\mu^2 = k^2\,(\cos^2\theta_1 + \sin^2\theta_1\cos^2\theta_2 + \sin^2\theta_1\,\sin^2\theta_2) = k^2.$$

Hence we have

$$\int \frac{d^4k}{k_\mu^2}\; e^{-F(\sigma)k_\mu^2} = \int_0^\infty \frac{k^3dk}{k^2}\; e^{-F(\sigma)k^2}\int_0^\pi \sin^2\theta_1 d\theta_1\int_0^\pi \sin\theta_2 d\theta_2\int_0^{2\pi} d\phi$$

$$= \frac{1}{2F(\sigma)}\cdot\frac{\pi}{2}\cdot 2\cdot 2\pi = \frac{\pi^2}{F(\sigma)}\quad . \tag{IV. 51}$$

Thus finally the first term of Eq. (IV. 15) is given by

$$\frac{1}{2}\cdot\frac{g^2}{4\pi}\int d\sigma''\left\langle\frac{1}{(x(\sigma')-x(\sigma''))^2}\right\rangle = \frac{1}{2}\frac{g^2}{4\pi}\cdot\frac{2}{(2\pi)^2}\cdot\pi^2\int_0^\infty\frac{d\sigma}{F(\sigma)}\quad . \tag{IV. 52}$$

Now we proceed to the second term of Eq. (IV. 15)

$$C\int d\sigma''\left\langle (x_\mu(\sigma')-x_\mu(\sigma''))^2\right\rangle e^{-a|\sigma'-\sigma''|}\quad . \tag{IV. 53}$$

A factor in the integrand, $\langle(x(\sigma')-x(\sigma''))\rangle$, can be obtained by

$$\left\langle (x_\mu(\sigma')-x_\mu(\sigma''))^2\right\rangle = -\left[\boxplus_k^2\left\langle e^{ik_\mu(x_\mu(\sigma')-x_\mu(\sigma''))}\right\rangle\right]_{k_\mu=0}\quad , \tag{IV. 54}$$

where $\boxplus^2 = \nabla^2 + \frac{\partial^2}{\partial\tau^2}$.

Again considering a typical component of Eq. (IV. 54) we have from Eq. (IV. 46)

$$\left\langle (x(\sigma')-x(\sigma''))^2\right\rangle = -\left[\frac{d^2}{dk^2}\; e^{-k^2F(|\sigma'-\sigma''|)}\right]_{k=0} = 2F(|\sigma'-\sigma''|)\quad .$$

Hence for Eq. (IV. 54) we have

$$\left\langle (x_\mu(\sigma')-x_\mu(\sigma''))^2\right\rangle = 4\times 2F(|\sigma'-\sigma''|) = 8F(|\sigma'-\sigma''|)\ .$$

and Eq. (IV.53) becomes now

$$8C\int_{-\frac{\Sigma}{2}}^{\frac{\Sigma}{2}} d\sigma'' \; F(|\sigma'-\sigma''|)\, e^{-\alpha|\sigma'-\sigma''|} \quad .$$

By the same kind of argument which led Eq. (IV.48) to Eq. (IV.49) this reduces to

$$16C\int_0^\infty d\sigma\, F(\sigma) e^{-\alpha\sigma} = \frac{8C}{\beta^2}\int_0^\infty d\sigma \;\; e^{-\alpha\sigma}\left[\alpha^2\sigma + \frac{\beta^2-\alpha^2}{\beta}\,(1 - e^{-\beta\sigma})\right].$$

After the integrations are performed it gives $8C/\alpha\beta = (\beta^2-\alpha^2)/\beta$. Hence for the second term of Eq. (IV.15) we obtain

$$C\int d\sigma''\left\langle (x_\mu(\sigma') - x_\mu(\sigma''))^2\right\rangle e^{-\alpha|\sigma'-\sigma''|} = \frac{8C}{\alpha\beta} = \frac{\beta^2-\alpha^2}{\beta} \quad . \qquad \text{(IV.55)}$$

Combining Eqs. (IV.52) and (IV.55) we have

$$S = \frac{g^2}{4\pi}\frac{1}{2}\int d\sigma''\left\langle\frac{1}{(x_\mu(\sigma')-x_\mu(\sigma''))^2}\right\rangle + C\int d\sigma''\left\langle (x_\mu(\sigma')-x_\mu(\sigma''))^2\right\rangle e^{-\alpha|\sigma'-\sigma''|}$$

$$= \frac{g^2\beta^2}{4\pi}\frac{1}{2}\int_0^\infty \frac{d\sigma}{\alpha^2\sigma + \frac{\beta^2-\alpha^2}{\beta}(1-e^{-\beta\sigma})} + \frac{\beta^2-\alpha^2}{\beta} \quad . \qquad \text{(IV.56)}$$

Next, we need to find E_1 of Eq. (IV.5). To this end consider E_1 and S_1 as functions of C. From Eq. (IV.5) we have (replacing T by Σ)

$$\ln\int e^{S_1(C)}\,\mathcal{D}x_\mu(\sigma) \sim -E_1(C)\,\Sigma \;.$$

A differentiation gives

$$-\sum \frac{dE_1(C)}{dC} \sim \frac{\int \frac{\partial S_1(C)}{\partial C} e^{S_1(C)} \mathcal{D} x_\mu(\sigma)}{\int e^{S_1(C)} \mathcal{D} x_\mu(\sigma)}$$

$$= \frac{-\iint d\sigma' d\sigma'' \int (x_\mu(\sigma')-x_\mu(\sigma''))^2 e^{-a|\sigma'-\sigma''|} e^{S_1(C)} \mathcal{D} x_\mu(\sigma)}{\int e^{S_1(C)} \mathcal{D} x_\mu(\sigma)} .$$

Hence we have

$$\frac{dE_1(C)}{dC} \sim \frac{1}{\Sigma} \iint d\sigma' d\sigma'' \ e^{-a|\sigma'-\sigma''|} \left\langle (x_\mu(\sigma')-x_\mu(\sigma''))^2 \right\rangle$$

$$= \int d\sigma'' \ e^{-a|\sigma'-\sigma''|} \left\langle (x_\mu(\sigma')-x_\mu(\sigma''))^2 \right\rangle$$

$$= \frac{8}{a\beta}$$

by Eq. (IV. 55). By Eq. (IV. 33), in which β is considered as functions of a and C, this becomes

$$\frac{dE_1(C)}{dC} = \frac{8}{a} \frac{1}{\sqrt{a^2 + \frac{8C}{a}}} .$$

Integration gives

$$E_1(C) = 2\sqrt{a^2 + \frac{8C}{a}} + K,$$

where K is the integration constant. Since $E_1 = 0$ for $C = 0$* we find

---

*For a free particle the kernel going from $x_\mu(0) = 0$ to $x_\mu(\Sigma) = X_\mu$ is given by $K \sim e^{-X_\mu^2/2\Sigma}$. (Cf. p. 73). This can be written as $\sim e^{-E_1\Sigma}$ by letting $X_\mu = v_\mu \Sigma$, where $E_1 = v_\mu^2/2$. For the present case of $X_\mu = 0$ we have $v_\mu = 0$ and hence $E_1 = 0$.

$K = -2\alpha$ and hence

$$E_1 = 2\sqrt{\alpha^2 + \frac{8C}{\alpha}} - 2\alpha = 2(\beta-\alpha). \qquad \text{(IV. 57)}$$

We finally obtain from Eqs. (IV. 56) and (IV. 57) for E of Eq. (IV. 11)

$$E = E_1 - S$$

$$= \frac{(\beta-\alpha)^2}{\beta} - \frac{1}{2}\frac{g^2\beta^2}{4\pi}\int_0^\infty \frac{d\sigma}{\alpha^2\sigma + \frac{\beta^2-\alpha^2}{\beta}(1-e^{-\beta\sigma})} \quad . \qquad \text{(IV. 58)}$$

This is the expression to be minimized to obtain the lowest value of E considering now α and β as free parameters.

The integral in Eq. (IV. 58) which came from the second term of S is logarithmically divergent at $\sigma = 0$ and $\sigma = \infty$. Although E is not the energy $\mathcal{E}$ of the scalar particle (ψ-field) but that of the ϕ-field (Cf. the end of Sect. III) it will be seen (Cf. the beginning of Sect. VI) that $\mathcal{E}$ is closely related to E so that $\mathcal{E}$ is directly influenced by E. The conventional perturbation-theoretic treatment of the self-energy of a scalar particle with 4-momenta p will give in the g^2-approximation (aside constant)

$$\int \frac{1}{(\not{p}-\not{k})^2 - m^2} \cdot \frac{d^4k}{\not{k}^2} = \int \frac{1}{\not{k}^2 - 2p\cdot k}\frac{d^4k}{\not{k}^2}$$

which also shows the logarithmic divergence at large values of k.

To remove the divergence in S an analogous treatment for the case of quantum electrodynamics will be applied later.

Notice that this value of E is the energy corresponding to the boundary conditions described by Eq. (IV. 22). However, these boundary conditions are of no interest to us because the initial and the final space-time points coincide so that no information as to the energy, say, can be obtained. For that reason we have to change the boundary conditions Eq. (IV. 22) such that the particle goes from

$$x_\mu(-\frac{\Sigma}{2}) = 0 \qquad \text{to} \qquad x_\mu(\frac{\Sigma}{2}) = X_\mu \ . \tag{IV. 59}$$

Let us introduce v_μ defined by

$$X_\mu = v_\mu \Sigma \ . \tag{IV. 60}$$

We are now faced with a problem of solving Eq. (IV. 26) under the conditions given by Eq. (IV. 59). This can be done by taking

$$\bar{x}_\mu(\sigma) = \bar{\bar{x}}_\mu(\sigma) + v_\mu(\sigma + \frac{\Sigma}{2}) \ , \tag{IV. 61}$$

where $\bar{\bar{x}}_\mu(\sigma)$ is the previous solution satisfying the conditions of Eq. (IV. 22), because the substitution of Eq. (IV. 61) into Eq. (IV. 26) satisfies the new condition given by Eq. (IV. 59). This change leads to the addition of a term $\int \sigma v_\mu f_\mu \, d\sigma$ in Eq. (IV. 25). Then by Eq. (IV. 19) we have

$$\int \sigma v_\mu f_\mu \, d\sigma = i k_\mu v_\mu \ (\sigma' - \sigma''), \tag{IV. 62}$$

and Eq. (IV. 46) must now be supplemented by a corresponding term in the expnent as

$$\left\langle e^{ik(x(\sigma')-x(\sigma''))} \right\rangle = e^{-F(|\sigma'-\sigma''|)k^2 + ikv(\sigma'-\sigma'')} \ , \tag{IV. 63}$$

and correspondingly for Eq. (IV. 47). Hence instead of Eq. (IV. 50) we obtain

$$\frac{1}{2}\frac{g^2}{4\pi}\int d\sigma'' \left\langle \frac{1}{(x_\mu(\sigma')-x_\mu(\sigma''))^2} \right\rangle = \frac{g^2}{4\pi} \cdot \frac{1}{(2\pi)^2}\int_0^\infty d\sigma \int \frac{d^4k}{k_\mu^2} \, e^{-F(\sigma)k_\mu^2 + ik_\mu v_\mu \sigma} \ . \tag{IV. 64}$$

Anticipating the relativistic cut-off mentioned before we will leave the right-hand side of this equation as it stands for the moment.

To get the modified form of Eq. (IV. 55) due to conditions given by Eq. (IV. 59) we differentiate Eq. (IV. 63) and find

$$\left\langle (x_\mu(\sigma')-x_\mu(\sigma''))^2 \right\rangle = -\left[\boxplus_k^2 \left\langle e^{ik_\mu(x_\mu(\sigma')-x_\mu(\sigma''))} \right\rangle \right]_{k_\mu=0}$$

$$= 8\, F(|\sigma'-\sigma''|) + v_\mu^2(\sigma'-\sigma'')^2, \tag{IV. 65}$$

where $F(\sigma)$ is defined by Eq. (IV. 45). Using Eq. (IV. 65) we now have

$$C\int d\sigma'' \left\langle (x_\mu(\sigma')-x_\mu(\sigma''))^2 \right\rangle e^{-a|\sigma'-\sigma''|}$$

$$= 8C\int_{-\frac{\Sigma}{2}}^{\frac{\Sigma}{2}} d\sigma'' F(|\sigma'-\sigma''|)\, e^{-a|\sigma'-\sigma''|} + C\int_{-\frac{\Sigma}{2}}^{\frac{\Sigma}{2}} d\sigma''\, e^{-a|\sigma'-\sigma''|}\, v_\mu^2(\sigma'-\sigma'')^2 .$$

The first term is simply equal to the answer of the previous case and gives the right-hand side of Eq. (IV. 55). The second term is also simple. Because of the assumption of very large value of $\Sigma/2$ it is practically equal to

$$C\int d\sigma''\, e^{-a|\sigma'-\sigma''|}\, v_\mu^2(\sigma'-\sigma'')^2 = 2Cv_\mu^2 \int_0^\infty d\sigma\, e^{-a\sigma}\sigma^2 = \frac{4Cv_\mu^2}{a^3} = \frac{\beta^2-a^2}{2a^2}\, v_\mu^2 .$$

Hence

$$C\int d\sigma'' \left\langle (x_\mu(\sigma')-x_\mu(\sigma''))^2 \right\rangle e^{-a|\sigma'-\sigma''|} = C\left[\frac{8}{a\beta} + \frac{4v_\mu^2}{a^3}\right]$$

$$= \frac{\beta^2-a^2}{\beta} + \frac{\beta^2-a^2}{2a^2}\, v_\mu^2 . \tag{IV. 66}$$

The determination of E_1 follows the previous pattern. We have

$$\frac{dE_1(C)}{dC} = \int d\sigma'' \; e^{-a|\sigma'-\sigma''|} \langle (x_\mu(\sigma')-x_\mu(\sigma''))^2 \rangle$$

$$= \frac{8}{a\beta} + \frac{4v_\mu^2}{a^3} \quad .$$

Hence

$$E_1 = 2\sqrt{a^2 + \frac{8C}{a}} + \frac{4}{a^3} v_\mu^{\;2} C + K \, ,$$

where K is the integration constant. This time we have $E_1 = v_\mu^2/2$ for $C = 0$ (Cf. footnote on p. 58) so that

$$K = \frac{v_\mu^{\;2}}{2} - 2a \, ,$$

and

$$E_1 = 2(\beta-a) + \frac{v_\mu^{\;2}}{2}\left(1 + \frac{8C}{a^3}\right) \quad . \qquad \text{(IV. 67)}$$

From Eqs. (IV. 64), (IV. 66), and (IV. 67)

$$E = E_1 - s$$

$$= \frac{(\beta-a)^2}{\beta} + \frac{v_\mu^{\;2}}{2} - \frac{1}{2}\frac{g^2}{8\pi^3} \int_0^\infty d\sigma \int \frac{d^4k}{k_\mu^{\;2}} \, e^{-F(\sigma)k_\mu^2 + ik_\mu v_\mu \sigma} \quad . \qquad \text{(IV. 68)}$$

V. REMOVAL OF DIVERGENCE BY CUT-OFF

The integral involved in Eq. (IV. 68) is logarithmically divergent. We will remove the divergence by the analogue of a procedure applied to quantum electrodynamics[15]. That is, we replace $1/k_\mu^2$ in Eq. (IV. 68) by $1/k_\mu^2 - 1/(k_\mu^2 + \Lambda^2) = C(k_\mu^2)/k_\mu^2$, where Λ, the cut-off, is taken to be very large compared to unity and $C(k_\mu^2) = \Lambda^2/(k_\mu^2 + \Lambda^2)$ may be called the convergence factor.

Using the representation

$$\frac{1}{k_\mu^2} = \int_0^\infty e^{-\gamma k_\mu^2} \, d\gamma \qquad \text{(V. 1)}$$

the integral in Eq. (IV. 68) can be written as

$$\int_0^\infty d\sigma \int_0^\infty d\gamma \int d^4k \; e^{-(\gamma + F(\sigma))k_\mu^2 + i\, k_\mu v_\mu \sigma} . \qquad \text{(V. 2)}$$

The integration over k can be performed yielding

$$\int d^4k \; e^{-(\gamma + F(\sigma)) \left[k_\mu + \frac{i v_\mu \sigma}{2(\gamma + F(\sigma))} \right]^2} e^{-\frac{v_\mu^2 \sigma^2}{4(\gamma + F(\sigma))}}$$

$$= \left(\frac{\sqrt{\pi}}{\sqrt{\gamma + F(\sigma)}} \right)^4 e^{-\frac{v_\mu^2 \sigma^2}{4(\gamma + F(\sigma))}} = \pi^2 \frac{1}{(\gamma + F(\sigma))^2} \, e^{-\frac{v_\mu^2 \sigma^2}{4(\gamma + F(\sigma))}} .$$

Then the integration over γ of Eq. (V. 2) can also be carried out:

$$\pi^2 \int_0^\infty d\sigma \int_0^\infty \frac{d\gamma}{(\gamma + F(\sigma))^2} \, e^{-\frac{v_\mu^2 \sigma^2}{4(\gamma + F(\sigma))}} = 4\pi^2 \int_0^\infty \frac{d\sigma}{v_\mu^2 \sigma^2} \left(1 - e^{-\frac{v_\mu^2 \sigma^2}{4F(\sigma)}} \right) . \qquad \text{(V. 3)}$$

Incidentally when $v_\mu^{\ 2} \to 0$ Eq. (V.3) agrees with the integral in Eq. (IV.58) as it should. When $v_\mu^2 \neq 0$ we can see that the divergence comes from $\sigma = 0$. Since

$$\frac{C(k_\mu^{\ 2})}{k_\mu^{\ 2}} = \frac{1}{k_\mu^{\ 2}} - \frac{1}{k_\mu^{\ 2} + \Lambda^2} = \int_0^\infty d\gamma\, e^{-\gamma k_\mu^{\ 2}} \left(1 - e^{-\gamma\Lambda^2}\right)$$

the replacement of $1/k_\mu^2$ by $C(k_\mu^2)/k_\mu^2$ in the integral of Eq. (IV.68) gives, in view of the first expression in Eq. (V.3),

$$\int_0^\infty d\sigma \int \frac{C(k_\mu^{\ 2}) d^4 k}{k_\mu^{\ 2}}\, e^{-F(\sigma) k_\mu^{\ 2} + i k_\mu v_\mu \sigma}$$

$$= \pi^2 \int_0^\infty d\sigma \int_0^\infty \frac{d\gamma}{(\gamma + F(\sigma))^2}\, e^{-\frac{v_\mu^{\ 2}\sigma^2}{4(\gamma + F(\sigma))}} \left(1 - e^{-\gamma\Lambda^2}\right)$$

$$= \pi^2 J.$$

Thus corresponding to Eq. (IV.68) we have ($v^2 = v_\mu^2$, $F = F(\sigma)$)

$$E = \frac{(\beta - \alpha)^2}{\beta} + \frac{v^2}{2} - \frac{g^2}{16\pi^3} \cdot \pi^2 \int_0^\infty d\sigma \int_0^\infty \frac{d\gamma}{(\gamma + F)^2}\, e^{-\frac{v^2\sigma^2}{4(\gamma + F)}} \left(1 - e^{-\gamma\Lambda^2}\right)$$

$$= \frac{(\beta - \alpha)^2}{\beta} + \frac{v^2}{2} - \frac{g^2}{16\pi} J \ . \qquad \text{(V.4)}$$

In order to evaluate E the integral J involved in Eq. (V.4) must now be performed. We will do this in the following for the limiting case where $\Lambda^2 \gg 1$. Noticing that when σ tends to zero (where the divergence in J comes)

$$F(\sigma) = \frac{1}{2\beta^2}\left[\alpha^2\sigma + \frac{\beta^2 - \alpha^2}{\beta}\left(1 - e^{-\beta\sigma}\right)\right] \to \frac{\sigma}{2} \qquad \text{(V.5)}$$

we write J in the following form:

$$J = \iint d\sigma\, d\gamma \left[\frac{e^{-\frac{v^2\sigma^2}{4(\gamma+F)}}}{(\gamma+F)^2} - \frac{e^{-\frac{v^2\sigma^2}{4(\gamma+\frac{\sigma}{2})}}}{(\gamma+\frac{\sigma}{2})^2} \right] (1-e^{-\gamma\Lambda^2})$$

$$+ \iint d\sigma\, d\gamma\, \frac{e^{-\frac{v^2\sigma^2}{4(\gamma+\frac{\sigma}{2})}}}{(\gamma+\frac{\sigma}{2})^2} (1-e^{-\gamma\Lambda^2})$$

$$= J_1 + J_2 \quad . \tag{V.6}$$

Eq. (V.6) is still exact. The first term J_1 is now convergent even without the factor $(1 - e^{-\gamma\Lambda^2})$ in view of Eq. (V.5) so that we can drop this factor, whereas in J_2 we have to still maintain it.

After integrating over γ J_1 gives

$$J_1 = \int_0^\infty \frac{4d\sigma}{v^2\sigma^2} \left[e^{-\frac{v^2\sigma}{2}} - e^{\frac{v^2\sigma^2}{4F}} \right] \quad .$$

By changing variable by

$$v^2\sigma = \tau \tag{V.7}$$

and calling

$$\alpha = av^2 \quad , \quad \beta = bv^2 \tag{V.8}$$

J_1 results in, remembering the definition of F given by Eq. (IV.45),

$$J_1 = \int_0^\infty \frac{4d\tau}{\tau^2} \left[e^{-\frac{\tau}{2}} - e^{-\frac{\tau^2 b^2}{2} \frac{1}{\left[a^2\tau + \frac{b^2-a^2}{b}(1-e^{-b\tau})\right]}} \right] \quad . \tag{V.9}$$

The limiting value of J_2 for very large Λ^2 will be shown in the next paragraphs to be just

$$J_2 = 2 \ \ln\left(\frac{\Lambda^2 e}{v^2}\right)$$

so that we will have

$$J = J_1 + J_2 = J_1 + 2 \ \ln\left(\frac{\Lambda^2 e}{v^2}\right) \quad . \tag{V.10}$$

Hence the use of Eqs. (V.8) and (V.10) will give for Eq. (V.4)

$$E(a,b,v^2) = \frac{(b-a)^2}{b} v^2 + \frac{v^2}{2} - \frac{g^2}{16\pi}\left(J_1(a,b) \quad +2\ln\left(\frac{\Lambda^2 e}{v^2}\right)\right) . \qquad (V.11)$$

In the next section we will consider the self-energy of the scalar nucleon resulting from this expression.

The limiting value of J_2 given above for large value of Λ^2 can be worked out as follows. We first note that for a very rough estimate of J_2 we may approximate the factor $1 - e^{-\gamma\Lambda^2}$ in J_2 by 0 for $\gamma < \gamma_o$ and by 1 for $\gamma > \gamma_o$ where γ_o is of order Λ^{-2}. A still better estimate of J_2 can be obtained by choosing a number c such that $c \gg 1$ yet the condition

$$\gamma_m = \frac{c}{\Lambda^2} \ll 1 \qquad (V.12)$$

is still satisfied. Let us now split the integration over γ into two pieces.

$$J_2 = \int_0^\infty d\sigma \int_0^{\gamma_m} \frac{d\gamma}{(\gamma + \frac{\sigma}{2})^2}\, e^{-\frac{v^2\sigma^2}{4(\gamma+\frac{\sigma}{2})}} \left(1 - e^{-\gamma\Lambda^2}\right)$$

$$+ \int_0^\infty d\sigma \int_{\gamma_m}^{\infty} \frac{d\gamma}{(\gamma + \frac{\sigma}{2})^2}\, e^{-\frac{v^2\sigma^2}{4(\gamma+\frac{\sigma}{2})}} \left(1 - e^{-\gamma\Lambda^2}\right)$$

$$= A + B . \qquad (V.13)$$

Because of our choice of Eq. (V.12) the factor $1-e^{-\gamma\Lambda^2}$ in B can be approximated by 1. Let us first consider A. Here again we choose $b \gg 1$ such that the following is still valid.

$$\sigma_m = b\gamma_m \ll 1 . \qquad (V.14)$$

Accordingly we split the integration over σ into two parts by

$$A = \int_0^{b\gamma_m} d\sigma \int_0^{\gamma_m} d\gamma \cdots + \int_{b\gamma_m}^{\infty} d\sigma \int_0^{\gamma_m} d\gamma \cdots\cdots \qquad (V.15)$$

$$= A' + A'' .$$

In A'' we have $\sigma \gg \gamma$ so that we can neglect γ relative to σ. Hence

$$A'' \approx \int_{b\gamma_m}^{\infty} d\sigma \int_0^{\gamma_m} \frac{4d\gamma}{\sigma^2} e^{-\frac{v^2\sigma}{2}} (1 - e^{-\gamma\Lambda^2})$$

$$= \int_{b\gamma_m}^{\infty} d\sigma \frac{4}{\sigma^2} e^{-\frac{v^2\sigma}{2}} \left(\gamma_m - \frac{1 - e^{-\gamma_m\Lambda^2}}{\Lambda^2} \right).$$

Noticing that $e^{-\gamma_m\Lambda^2} = e^{-c} \approx 0$ an integration by parts gives

$$\approx \left(\frac{4}{b\gamma_m} e^{-\frac{v^2}{2} b\gamma_m} - 2v^2 \int_{b\gamma_m}^{\infty} \frac{d\sigma}{\sigma} e^{-\frac{v^2\sigma}{2}} \right) \left(\gamma_m - \frac{1}{\Lambda^2} \right)$$

$$\approx \left(\frac{4}{b\gamma_m} + 2v^2 \ln(1.781 \cdot \frac{v^2}{2} \cdot b\gamma_m) \right) \gamma_m$$

$$\approx 0 . \qquad (V.16)$$

in which use has been made of an approximation

$$\int_a^{\infty} \frac{dx}{x} e^{-\lambda x} = - \ln(1.781\lambda a) \; ^* \qquad (V.17)$$

valid for $a \ll 1$.

---

*To prove this do, for example, the following. From $\int_0^{\infty} \left[e^{-\lambda x} - \frac{1}{1+\lambda x} \right] \frac{dx}{x} = -C$, where $\lambda > 0$ and $C = 0.57721...$ is Euler's constant, we have $\int_0^{\infty} e^{-\lambda x} \frac{dx}{x} = -C + \int_0^{\infty} \frac{1}{1+\lambda x} \frac{dx}{x}$. Splitting the left-hand side into two parts we write $\int_0^a e^{-\lambda x} \frac{dx}{x} + \int_a^{\infty} e^{-\lambda x} \frac{dx}{x} = -C + \ln \frac{x}{1+\lambda x} \Big|_0^{\infty}$. Now we have $\int_0^a e^{-\lambda x} \frac{dx}{x} \approx \int_0^a \frac{dx}{x} = \ln x \Big|_0^a$ for $a \ll 1$, while

$$\ln \frac{x}{1+\lambda x} \Big|_0^{\infty} = \ln \frac{1}{\lambda} - \ln x \Big|_{x=0} .$$

Hence $\int_a^{\infty} e^{-\lambda x} \frac{dx}{x} = - C - \ln\lambda a$. Since $C = \ln 1.781$ the proof is complete.

In A' $\sigma \ll 1$ and $\gamma \ll 1$, so that $e^{-\frac{v^2\sigma^2}{4(\gamma+\frac{\sigma}{2})}} \approx 1$. Hence

$$A' \approx \int_0^{b\gamma_m} d\sigma \int_0^{\gamma_m} \frac{d\gamma}{(\gamma+\frac{\sigma}{2})^2}\,(1-e^{-\gamma\Lambda^2})$$

$$= \int_0^{\gamma_m} d\gamma\; 2\left(\frac{1}{\gamma} - \frac{1}{\gamma+\frac{b\gamma_m}{2}}\right)(1-e^{-\gamma\Lambda^2}) \;.$$

Neglecting γ in $(\gamma + b\gamma m/2)^{-1}$ this becomes

$$= 2\left[\int_0^{\gamma_m} \frac{d\gamma}{\gamma}(1-e^{-\gamma\Lambda^2}) - \frac{2}{b\gamma_m}\left(\gamma + \frac{e^{-\gamma\Lambda^2}}{\Lambda^2}\right)\Bigg|_0^{\gamma_m}\right]. \qquad (V.18)$$

In the first term the integral tends to zero when γ tends to zero, so we replace the lower limit by a very small number ϵ. Then

$$\int_0^{\gamma_m} \frac{d\gamma}{\gamma}(1-e^{-\gamma\Lambda^2}) = \int_\epsilon^{\gamma_m}\frac{d\gamma}{\gamma} - \int_\epsilon^{\gamma_m}\frac{d\gamma}{\gamma}e^{-\gamma\Lambda^2}.$$

Noticing that in the second integral $e^{-\gamma\Lambda^2} = e^{-\gamma_m\Lambda^2} = e^{-c} \approx 0$ when $\gamma = \gamma_m$ we can extend the upper limit to $+\infty$ and by doing so we can make use of Eq. (V.17). Hence

$$\int_\epsilon^{\gamma_m} \frac{d\gamma}{\gamma}(1-e^{-\gamma\Lambda^2}) \approx \ln\frac{\gamma_m}{\epsilon} + \ln(1.781\cdot\epsilon\Lambda^2) = \ln(1.781\cdot\gamma_m\Lambda^2).$$

The second term of (V.18) becomes

$$\frac{2}{b\gamma_m}\left(\gamma_m + \frac{e^{-\gamma_m\Lambda^2}-1}{\Lambda^2}\right) \approx \frac{2}{b} - \frac{2}{b\gamma_m\Lambda^2} = \frac{2}{b} - \frac{2}{bc} \approx 0.$$

Therefore

$$A' \approx 2\ln(1.781\,\gamma_m\Lambda^2). \qquad (V.19)$$

Now let us proceed to the study of B. As was mentioned before the factor $1 - e^{-\gamma\Lambda^2}$ may be replaced by 1. So we have

$$B \approx \int_0^\infty d\sigma \int_{\gamma_m}^\infty \frac{d\gamma}{(\gamma + \frac{\sigma}{2})^2} \; e^{-\frac{v^2 \sigma^2}{4(\gamma + \frac{\sigma}{2})}}$$

$$= \int_0^\infty d\sigma \frac{4}{v^2 \sigma^2} \; (1 - e^{-\frac{v^2\sigma^2}{4(\gamma_m + \frac{\sigma}{2})}}) \; .$$

Let $\sigma_m = b\,\gamma_m$ with $\sigma_m \ll 1$ and $b \gg 1$ as before, and split the integration over σ into two parts.

$$B = \int_0^{b\gamma_m} d\sigma \; \ldots\ldots + \int_{b\gamma_m}^\infty d\sigma \; \ldots \;\; = B' + B'' \; . \qquad \text{(V.20)}$$

Consider B' first. Since $\sigma \ll 1$ in B' let us expand the exponential. Then

$$B' = \int_0^{b\gamma_m} d\sigma \frac{4}{v^2\sigma^2} \; (1 - 1 + \frac{v^2\sigma^2}{4(\gamma_m + \frac{\sigma}{2})} - \ldots\ldots)$$

$$= 2 \; \ln(\gamma_m + \frac{\sigma}{2})\Big|_0^{b\gamma_m} = 2\,\ln\Big(\frac{{}_m + \frac{b\gamma_m}{2}}{\gamma_m}\Big) \approx 2 \; \ln\frac{b}{2} \; . \qquad \text{(V.21)}$$

In the denominator of the exponent of B'' we neglect γ_m as compared to $\sigma/2$. Then we have

$$B'' \approx \int_{b\gamma_m}^\infty d\sigma \frac{4}{v^2\sigma^2} \; (1 - e^{-\frac{v^2\sigma}{2}})$$

$$= \frac{4}{v^2}\Big[-\frac{1}{\sigma}(1 - e^{-\frac{v^2\sigma}{2}})\Big|_{b\gamma_m}^\infty + \int_{b\gamma_m}^\infty \frac{d\sigma}{\sigma} \cdot \frac{v^2}{2} e^{-\frac{v^2\sigma}{2}} \Big]$$

$$= \frac{4}{v^2} \cdot \frac{1 - e^{-\frac{v^2}{2} b\gamma_m}}{b\gamma_m} - 2 \; \ln(1.781 \cdot \frac{v^2}{2} \cdot b\gamma_m) \; .$$

Expanding the exponential we obtain

$$B'' = 2 - 2 \; \ln(1.781 \cdot \frac{v^2}{2} \cdot b\,\gamma_m). \qquad \text{(V.22)}$$

Collecting Eqs. (V.16), (V.19), (V.21), and (V.22) gives

$$J_2 = A' + A'' + B' + B''$$

$$= 2 + 2 \ln\left(\frac{\Lambda^2}{v^2}\right) = 2 \ln\left(\frac{\Lambda^2 e}{v^2}\right) . \qquad (V.23)$$

This is the limiting value of J_2 for large Λ^2 which was used in Eq. (V.11).

VI. SELF-ENERGY OF THE SCALAR NUCLEON

The self-energy of the scalar nucleon described by ψ can be obtained from the asymptotic form of its kernel $\mathcal{K}$ corresponding to the transition in which there are no mesons present in the initial and the final states. The expression for E given by Eq. (V.11) is to give the best value of E_o which appears in the asymptotic form $e^{-E_o \Sigma}$ of the kernel K associated with ϕ for the same transition. This does not give self-energy of the scalar nucleon directly, because for that we need the asymptotic form of $\mathcal{K}$, not K. In the following, however, we will show that the kernel associated with ψ is so closely related to K that the best value of the self-energy of the scalar nucleon can be given in terms of E_o.

To see how $\mathcal{K}$ is related to K let us first consider the case of a scalar nucleon in a given external potential χ (Sect. III). Throughout this section we shall conveniently take the range of σ to be from 0 to $+\infty$ (Cf. above Eq. (III.4)). Then the kernel $K^{(\chi)}$ of Eq. (III.2) satisfies the following equation:

$$\phi(x,\sigma) = \int K^{(\chi)}(x,\sigma;x',0)\phi(x',0)d^4\omega_{x'} \quad . \qquad \text{(VI.1)}$$

Now consider a function $\psi(x)$ defined by

$$\psi(x) = \int_0^\infty e^{-\frac{1}{2}m^2\sigma} \phi(x,\sigma)d\sigma \ . \qquad \text{(VI.2)}$$

By operating $\boxplus^2 - \sqrt{4\pi}\, g\chi$ on ψ we find

$$(\boxplus^2 - \sqrt{4\pi}\, g\, \chi\,)\psi(x) = \int_0^\infty e^{-\frac{1}{2}m^2\sigma} (\boxplus^2 - \sqrt{4\pi}\, g\chi)\, \phi(x,\sigma)\, d\sigma$$

$$= \int_0^\infty e^{-\frac{1}{2}m^2\sigma}\, 2\, \frac{\partial\phi(x,\sigma)}{\partial\sigma}\, d\sigma$$

$$= 2\, e^{-\frac{1}{2}m^2\sigma} \phi(x,\sigma)\Big|_0^\infty + m^2 \int_0^\infty e^{-\frac{1}{2}m^2\sigma} \phi(x,\sigma) d\sigma$$

$$= -2\, \phi(x,0) + m^2\, \psi(x),$$

in which use has been made of Eq. (III. 2). Hence ψ obeys the following differential equation:

$$(\boxplus^2 - m^2 - \sqrt{4\pi}\, g\, \chi\,)\, \psi(x) = -2\, \phi(x,0) \quad . \qquad \text{(VI. 3)}$$

On the other hand, the kernel $I^{(\chi)}$ of Eq. (II. 20) satisfies

$$(\boxplus^2 - m^2 - \sqrt{4\pi}\, g\, \chi(x)\,)\, I^{(\chi)}(x,x') = \delta^4(x-x') \quad . \qquad \text{(II. 26)}$$

From Eqs. (VI. 3) and (II. 26) it can be seen that $I^{(\chi)}(x,x')$ is given by

$$I^{(\chi)}(x,x') = \int_0^\infty e^{-\frac{1}{2}m^2\sigma} \phi(x,\sigma) d\sigma \qquad \text{(VI. 4)}$$

in which

$$-2\phi(x,0) = \delta^4(x-x') \quad . \qquad \text{(VI. 5)}$$

Substituting Eq. (VI. 5) into Eq. (VI. 1) we have

$$-2\phi(x,\sigma) = K^{(\chi)}(x,\sigma;x',0) \quad . \qquad \text{(VI. 6)}$$

Hence Eq. (VI.4) becomes*

$$-2\,\mathcal{I}^{(\chi)}(x, x') = \int_0^\infty e^{-\frac{1}{2}m^2\sigma}\, K^{(\chi)}(x, \sigma; x', 0)\,d\sigma \ . \tag{VI.7}$$

Calling

$$\mathcal{K}^{(\chi)}(x;x') = -2\,\mathcal{I}^{(\chi)}(x, x') \tag{VI.8}$$

for convenience we now have

$$\mathcal{K}^{(\chi)}(x;x') = \int_0^\infty e^{-\frac{1}{2}m^2\sigma}\, K^{(\chi)}(x, \sigma; x', 0)\, d\sigma \ . \tag{VI.9}$$

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*The corresponding relation for the case of a free particle

$$-2\,\mathcal{I}^{(0)}(x, x') = \int_0^\infty e^{-\frac{1}{2}m^2\sigma}\, K^{(0)}(x, \sigma; x', 0)\, d\sigma$$

can also be verified in the following way. The kernel $K^{(0)}$ is defined by (Cf. Eq. (III.2)) $\left(-\frac{\partial}{\partial\sigma_2} + \frac{1}{2}\,\boxplus_2^2\right) K^{(0)}(2,1) = -\,\delta^5(2,1)$, where $\delta^5(2,1) = \delta(\sigma_2 - \sigma_1)\ \delta^4(x_{\mu 2} - x_{\mu 1})$. Expressing $\delta^5(2,1)$ as the five-dimensional Fourier integral we have

$$K^{(0)}(x) = (2\pi)^{-5}\, i \int e^{ik_\mu x_\mu - ik_5\sigma}\, d^5k/(k_5 - ik_\mu^2/2).$$

Integration over k_5 gives $K^{(0)}(x) = (2\pi)^{-4}\int e^{-k_\mu^2\sigma/2 + ik_\mu x_\mu}\, d^4k$, and this can be integrated out to give $e^{-x_\mu^2/2\sigma}/(2\pi\sigma)^2$. Hence we have

$$\int_0^\infty e^{-\frac{1}{2}m^2\sigma} K^{(0)}(x, \sigma; x', 0)\, d\sigma = (2\pi)^{-4}\int_0^\infty e^{-\frac{1}{2}m^2\sigma}\, d\sigma \int e^{-k_\mu^2\sigma/2 + ik_\mu(x-x')_\mu}\, d^4k$$

$$= (2\pi)^{-4}\int e^{ik_\mu(x-x')_\mu}\, d^4k \int_0^\infty e^{-(k_\mu^2 + m^2)\sigma/2}\, d\sigma = (2\pi)^{-4}\, 2\int (k_\mu^2 + m^2)^{-1}\, e^{ik_\mu(x-x')_\mu}\, d^4k$$

$= -2\,\mathcal{I}^{(0)}(x;x')$ by Eq. (II.25).

Eq. (VI. 9) gives the kernel $\mathcal{K}^{(\chi)}$ associated with ψ in terms of K $^{(\chi)}$ of ϕ for the case when there is an external field χ .

When we calculate the kernel $\mathcal{K}_{00}$, say, corresponding to the transition in which there are no mesons present initially and finally we encounter again the problem of eliminating the meson field operator as was done in obtaining Eq. (III. 15) from Eq. (III. 4). In view of Eqs. (III. 4) and (III. 5), where $\sigma' = 0$ and $\sigma'' = \sigma$, Eq. (VI. 9) can be written as

$$\mathcal{K}^{(\chi)}(x;x') = \int_0^\infty e^{-\frac{1}{2}m^2\sigma} d\sigma \int e^{-\int \frac{1}{2}\dot{x}_\mu^2(\sigma)d\sigma - \int \frac{1}{2}\sqrt{4\pi}\, g\chi(x_\mu(\sigma))d\sigma} \mathcal{D}x_\mu(\sigma) \,. \quad \text{(VI. 10)}$$

Then the use of Eqs. from (III. 12) to (III. 15) gives

$$\mathcal{K}_{00}(x;x') = \int_0^\infty e^{-\frac{1}{2}m^2\sigma} d\sigma \int e^{-\int \frac{1}{2}\dot{x}_\mu^2(\sigma)d\sigma + \frac{1}{2}\frac{g^2}{4\pi}\iint \frac{d\sigma' d\sigma''}{(x_\mu(\sigma')-x_\mu(\sigma''))^2}} \mathcal{D}x_\mu(\sigma)$$

$$= \int_0^\infty e^{-\frac{1}{2}m^2\sigma} K_{00}(x,\sigma;\, x',0)\, d\sigma. \quad \text{(VI. 11)}$$

Omitting the subscripts 00 following the convention in the foregoing sections, and considering the case where $x'(0) = 0$ and $x(\Sigma) = X$ we have

$$\mathcal{K}(X;0) = \int_0^\infty e^{-\frac{1}{2}m^2\Sigma} K(X,\Sigma;0,0)\, d\Sigma \,. \quad \text{(VI. 12)}$$

The self-energy of the scalar nucleon can be calculated from the asymptotic form of this kernel $\mathcal{K}$ (X;0). Since $v^2 = v_\mu{}^2$ is anticipated to be of reasonable magnitude our equation $X_\mu = v_\mu \Sigma$

(Eq. (IV.60)) tells us that very large values of X_μ correspond to very large values of Σ. This implies that in evaluating the asymptotic form of $\mathcal{K}(X;0)$, where $X_4 = T$ is to be made to approach to infinity, only the asymptotic, and not the exact, form of $K(X,\Sigma;0,0)$ is required.

In this paragraph we will show in general how the best value of the self-energy can be obtained from Eq. (VI.12). First we recall that what has been done in the previous sections is to calculate $e^{-E\Sigma}$ which is to be minimized to give the best estimate of $K \sim e^{-E_o\Sigma}$. Consequently the best estimate M of the self-energy $\mathcal{E}_o$ of $\mathcal{K} \sim e^{-\mathcal{E}_o T}$ can be found by minimizing E with respect to the parameters available and then having $\int_0^\infty e^{-m^2\Sigma/2}\, e^{-E\Sigma} d\Sigma$ in the form e^{-MT}. In general E is a function of the parameters and v^2. Therefore it has a dependence on Σ through $X_\mu = v_\mu \Sigma$. For the simplicity of the argument let us assume that only one parameter a is contained in E and consider

$$\int_0^\infty e^{-\left[\frac{m^2}{2} + E(a,v^2)\right]\Sigma} d\Sigma = \int_0^\infty e^{-\left[m^2+F(a,v^2)\right]\frac{\Sigma}{2}} d\Sigma \,, \qquad \text{(VI. 13)}$$

where F = 2E. We are interested in the exponent of Eq. (VI.13):

$$\left[m^2 + F(a,v^2) \right] \frac{\Sigma}{2} \,. \qquad \text{(VI. 14)}$$

By varying a first to bring the minimum of E we have ($X^2 = X_\mu^2$)

$$F_a(a_o, \frac{X^2}{\Sigma^2}) = 0 \,. \qquad \text{(VI. 15)}$$

This determines the optimum value a_o in terms of X^2/Σ^2 :

$$a_o = a_o\left(\frac{X^2}{\Sigma^2}\right) \quad . \qquad \text{(VI. 16)}$$

Substituting this value in Eq. (VI. 14) we have

$$\left[m^2+F\left(a_o\left(\frac{X^2}{\Sigma^2}\right), \frac{X^2}{\Sigma^2}\right) \right] \frac{\Sigma}{2} = \left[m^2 + G\left(\frac{X^2}{\Sigma^2}\right) \right] \frac{\Sigma}{2} \quad , \qquad \text{(VI. 17)}$$

where G is defined by

$$G(v^2) = F(a_o(v^2), \ v^2) \quad . \qquad \text{(VI. 18)}$$

We then evaluate Eq. (VI. 13), with its exponent now considered given by Eq. (VI. 17), by the method of stationary phase. That is, we seek for the value Σ_o which contributes most to the integral by minimizing the (negative of) exponent of the integrand. Thus by varying Σ of Eq. (VI. 17) we have

$$m^2 + G(v_o{}^2) = 2\, G'(v_o{}^2)\, v_o{}^2 \quad , \qquad \text{(VI. 19)}$$

where Σ_o and v_o is related by $\Sigma_o = X/v_o$. When this value Σ_o is substituted into Eq. (VI. 17) we have

$$\left[m^2 + G(v_o{}^2) \right] \frac{X}{2v_o} \quad . \qquad \text{(VI. 20)}$$

If we choose X_μ such that $\vec{X} = 0$ and $X_4 = T$ then this becomes

$$\left[m^2 + G(v_o{}^2) \right] \frac{T}{2v_o}$$

Then, since we evaluated Eq. (VI. 13) in the form

$$\int_0^\infty e^{-\left[m^2+F(a, v^2)\right] \frac{\Sigma}{2}} d\Sigma \sim e^{-\left[m^2+G(v_o{}^2)\right] \frac{T}{2v_o}} \quad ,$$

we can identify M as

$$M = \frac{1}{2v_o} \left[m^2 + G(v_o{}^2) \right] \quad . \qquad \text{(VI. 21)}$$

This procedure of finding M may be stated a little more compactly. To this end let us notice that Eq. (VI. 18) gives

$$G'(v^2) = a_o'(v^2) F_a(a_o(v^2), v^2) + F_{v^2}(a_o(v^2), v^2) \quad ,$$

in which the first term on the right-hand side vanishes by Eq. (VI. 15). Then Eq. (VI. 19) reduces to

$$\frac{\partial}{\partial v} \left[\frac{1}{2v} \left[m^2 + F(a, v^2) \right] \right] \Big|_a = 0 \quad ,$$

while Eq. (VI. 15) may be written as

$$\frac{\partial}{\partial a} \left[\frac{1}{2v} \left[m^2 + F(a, v^2) \right] \right] \Big|_v = 0 \quad .$$

Therefore Eq. (VI. 21) can be stated as

$$M = \text{Minimum of } \mathcal{E}(a, v^2)$$

with respect to v and the parameter a, where

$$\mathcal{E}(a, v^2) = \frac{1}{2v} \left[m^2 + F(a, v^2) \right] \quad . \qquad \text{(VI. 22)}$$

The procedure for the case of several parameters is obvious.

Applying this general argument to the present problem we see that the best value of the self-energy of the scalar nucleon, M, is given as the minimum of (Cf. Eqs. (V. 10) and (V . 11))

$$\mathcal{E}(a, b, v^2) = \frac{1}{2v} \left[m^2 + v^2 \left\{ \frac{2(b-a)^2}{b} + 1 \right\} - \frac{g^2}{8\pi} J \right] \quad , \qquad \text{(VI. 23)}$$

where $J = J_1 + J_2 = J_1 + 2 \ln(\Lambda^2 e/v^2)$.

Now it can be shown that for $b \gg a$ J_1 behaves as $2\sqrt{2\pi b}$

and hence the absolute minimum of $\mathcal{E}$ can be made, regardless of the values of m and g^2, as large a negative number as one wishes provided that b is taken to be sufficiently large. Then the integral

$$\int_0^\infty e^{-[m^2+F(a,b,v^2)]\frac{\Sigma}{2}}\, d\Sigma$$

$$\approx \int_0^\infty e^{-\frac{1}{2}[m^2\Sigma+(2b+1)\frac{T^2}{\Sigma}-\frac{g^2}{8\pi}\{J_1+2\ln(\Lambda^2 e\frac{\Sigma^2}{T^2})\}\Sigma]}\, d\Sigma \qquad \text{(VI. 24)}$$

evidently diverges. However this divergence does not make real difficulty. For when we go back to the variables t and u by $t = -i\tau$ and $u = -i\sigma$ (Cf. below Eq. (III. 2)), Eq. (VI. 24) becomes

$$\int_0^\infty e^{-i\frac{1}{2}[m^2U-(2b+1)\frac{T'^2}{U}-\frac{g^2}{8\pi}\{J_1+2\ln(\Lambda^2 e\frac{U^2}{T'^2})\}U]}\, dU, \qquad \text{(VI. 25)}$$

where the large intervals Σ and T of σ and τ are replaced by the corresponding intervals U and T' of u and t, respectively. Written in this form the very large value of J_1 simply makes the integrand oscillate very rapidly and does not introduce any difficulty. All that is significant is the stationary value, if any, of the phase in Eq. (VI. 25), or what amounts to the same thing is the "local" minimum of the (negative of) exponent of Eq. (VI. 24).

Since Λ^2 is assumed to be very large and v^2 is anticipated to be of reasonable size $2\ \ln(\Lambda^2 e/v^2)$ will not be very sensitive to v, whereas J_1 is independent of v. Consequently J may be regarded as a constant in the first approximation.

By varying $\mathcal{E}$ given by Eq. (VI. 23) with respect to v the optimum is found to be

$$v_o = (m^2 - \frac{g^2}{8\pi} J)^{\frac{1}{2}} / (\frac{2(b-a)^2}{b} + 1)^{\frac{1}{2}} . \qquad \text{(VI. 26)}$$

Substituting v_o into Eq. (VI. 23) we have

$$\mathcal{E}(a, b, v_o^2) = (m^2 - \frac{g^2}{8\pi} J)^{\frac{1}{2}} (\frac{2(b-a)^2}{b} + 1)^{\frac{1}{2}} . ^* \qquad \text{(VI. 27)}$$

It turns out to be convenient to use a partly renormalized mass m' and a quantity μ defined by

$$m'^2 = m^2 - \frac{g^2}{8\pi} \cdot 2 \, \ln(\frac{\Lambda^2 e}{v^2}) \qquad \text{(VI. 28)}$$

and

$$\mu = \frac{g^2}{8\pi m'^2} . \qquad \text{(VI. 29)}$$

For the neutral scalar nucleon the coefficient $g^2/8\pi$ in these equations is to be replaced by $g^2/2\pi$. This is because the comparison of Eqs. (II. 1) and (II. 1') shows that the coupling constant for the case of the neutral nucleon corresponds to twice that of the charged one. Since it is easier to deal with $\mathcal{E}^2$ rather than $\mathcal{E}$ itself we will give here the expression for $\mathcal{E}^2$ in terms of m'^2 and μ :

$$\frac{\mathcal{E}^2}{m'^2} = (1 - \mu J_1)(\frac{2(b-a)^2}{b} + 1) . \qquad \text{(VI. 30)}$$

In minimizing $\mathcal{E}^2$ with respect to the parameters a and b, because of the fact that J_1 diverges as $2\sqrt{2\pi b}$ for $b \gg a$, we will only consider the case

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*If we take b = a we have, from Eqs.(IV. 33) and (V. 8), C = 0. If we assume further that $g^2 = 0$, then the problem reduces to the case of a free particle. For this case Eq. (VI. 27) gives the lowest energy, m, correctly, as one expects.

$$b = a\,(1 + \epsilon) \tag{VI. 31}$$

to make $\mathcal{E}^2$ positive, where ϵ is assumed to be very small as compared to 1. Then J_1 will be expanded in powers of ϵ :

$$J_1 = \epsilon\,\mathcal{J}(a) - \epsilon^2\,\mathcal{K}(a) + \ldots. \quad , \tag{VI. 32}$$

where the constant term vanishes as can be seen from Eq. (V. 9) by taking b = a. This approximation will be valid as far as we remain in the range of very small ϵ . We will postpone the actual calculation of $\mathcal{J}(a)$, $\mathcal{K}(a)$, and their derivatives $\mathcal{J}'(a)$ and $\mathcal{K}'(a)$ until the end of this section. The substitution of Eqs. (VI. 31) and (VI. 32) into Eq. (VI. 30) gives the approximate value of $\mathcal{E}^2/m'^2$ as

$$\frac{\mathcal{E}^2}{m'^2} = (1 - \mu(\epsilon\,\mathcal{J} - \epsilon^2\mathcal{K}))\,(2a\,\epsilon^2 + 1)\,. \tag{VI. 33}$$

Introducing

$$\sqrt{2a}\;\epsilon = \eta\,, \quad A = \frac{\mathcal{J}}{\sqrt{2a}}\quad, \text{ and } B = \frac{\mathcal{K}}{2a} \tag{VI. 34}$$

we write

$$\frac{\mathcal{E}^2}{m'^2} = (1 - \mu(\eta A - \eta^2 B)\,(\eta^2 + 1). \tag{VI. 35}$$

By varying a we have

$$\eta = \frac{A'}{B'}\,. \tag{VI. 36}$$

And the variation of η gives

$$\mu^{-1} = \frac{3\eta^2+1}{2\eta}\,A - (2\eta^2 + 1)B\,. \tag{VI. 37}$$

From Eqs. (VI. 34) and (VI. 36) we have

$$\epsilon = \frac{2a\mathcal{J}' - \mathcal{J}}{2(a\mathcal{K}' - \mathcal{K})} \quad . \tag{VI. 38}$$

Correspondingly Eq. (VI. 37) can be expressed as

$$\mu^{-1} = \frac{6a\epsilon^2 + 1}{4a\epsilon}\mathcal{J} - \frac{4a\epsilon^2 + 1}{2a}\mathcal{K} \quad . \tag{VI. 39}$$

Using the optimum values of ϵ and μ given by Eqs. (VI. 38) and (VI. 39) the best value of the self-energy, M, is given by

$$\frac{M^2}{m'^2} = (1 - \mu(\epsilon\mathcal{J} - \epsilon^2\mathcal{K}))\,(2a\epsilon^2 + 1) \quad . \tag{VI. 40}$$

Therefore what we need to do is the following. We first choose a arbitrarily and compute $\mathcal{J}(a)$, $\mathcal{K}(a)$, $\mathcal{J}'(a)$, and $\mathcal{K}'(a)$ numerically. Using these data ϵ and μ are computed from Eqs. (VI. 38) and (VI. 39). Since μ is necessarily positive it is found that there is a smallest allowable value of a. It turns out that μ will become negative for $a < 3.79$. Substituting the values of ϵ and μ thus found into Eq. (VI. 40) M^2/m'^2 will be given as a function of a. The result is given in Table 1 and Figure 1. In addition to these $(v_o/m')^2$ and $\epsilon^2\mathcal{K}/\epsilon\mathcal{J}$ are also given. It can be seen that μ has a maximum value of about 0. 3 near a = 5. 6. Correspondingly $(M/m')^2$ has a minimum of about 0.88 at the same value of a. In Figure 2 $(M/m')^2$ is given as a function of μ . The curve turns back near $\mu = 0.3$ and does not go down toward zero. This turning back is due to the fact that there is no local minimum of $(\mathcal{E}/m')^2$ given by Eq. (VI. 38) beyond that value of μ . The existence of two values for a given value of μ less than 0. 30 can be explained in the following way. When we study the behavior of $(\mathcal{E}/m')^2$ given by Eq. (VI.30) for fixed value of μ it is described by Eq. (VI.33) when ϵ is very small. On the other hand, when ϵ is very large compared to 1, i.e., $b \gg a$, Eq. (VI.33) is no longer valid but J_1 diverges as $2\sqrt{2\pi b}$ as was mentioned before. Therefore, when ϵ becomes very large $(\mathcal{E}/m')^2$ tends to $-\infty$, as can be seen from Eq. (VI.30). Consequently whenever there is a local minimum (in fact there is one for each value of μ less than 0.30) $(\mathcal{E}/m')^2$ must also pass through a maximum, which is necessarily larger in value than the local minimum, before going down to $-\infty$. Since perhaps it is easier to illustrate this behavior of $(\mathcal{E}/m')^2$ by a picture we gave Figure 3. We do not claim that it is exactly to the scale, nevertheless we believe it worth while inserting. The two values of $(\mathcal{E}/m')^2$ for given μ, therefore, represent the smaller value of the local minimum and the larger value of the maximum. Of these two the smaller is the one we are interested in. It is also to be noted that in the entire range where a local minimum is found the size of ϵ remains small. At least $\epsilon^2 \mathcal{K}/\epsilon \mathcal{J}$ is less than 5°/o so that we can say that the expansion we have made of J_1 (Eq. (VI.32)) is a very good approximation.

Although what has been given above is all one needs it is more convenient for the practical purposes to have a direct relation between M, the theoretical mass m, and the coupling constant g^2. In order to do this we define $\bar{\mu}$ by

$$\bar{\mu} = \frac{g^2}{8\pi M^2} \quad . \tag{VI.41}$$

Considering the right-hand side of Eq. (VI. 40) as a function $G(\mu)$ of μ we write

$$\frac{M^2}{m'^2} = G(\mu) \quad . \tag{VI. 42}$$

Then from Eqs. (VI. 42) and (VI. 29) $\bar{\mu}$ can also be written as

$$\bar{\mu} = \frac{g^2}{8\pi M^2} = \frac{g^2}{8\pi m'^2 G(\mu)} = \frac{\mu}{G(\mu)} \quad . \tag{VI. 43}$$

On the other hand, Eq. (VI. 28) gives

$$\begin{aligned} \frac{8\pi m^2}{g^2} &= \frac{1}{\mu} + 2 \ln \frac{M^2 e}{v_o^2} + 2 \ln \frac{\Lambda^2}{M^2} \\ &= D(\mu) + 2 \ln \frac{\Lambda^2}{M^2} \quad , \end{aligned} \tag{VI. 44}$$

where

$$D(\mu) = \frac{1}{\mu} + 2 \ln \frac{M^2 e}{v_o^2} \quad .$$

Eq. (VI. 43) may be considered giving μ as a function of $\bar{\mu}$ and correspondingly we can write $D(\mu)$ as a function of $\bar{\mu}$:

$$D(\mu) = F(\bar{\mu}) \quad .$$

Then Eq. (VI. 44) becomes

$$\frac{8\pi m^2}{g^2} = F(\bar{\mu}) + 2 \ln \frac{\Lambda^2}{M^2} \quad . \tag{VI. 45}$$

Table 2 gives $\bar{\mu}$ and $D(\mu)$ as functions of μ . The graph giving $F(\bar{\mu})$ as a function of $\bar{\mu}$ is given in Figure 4. Using this graph we can easily find the interrelations between m^2, M^2, and g^2. That is, suppose the experimental mass, which is to be identified as M, and the coupling constant g^2 are given. We first compute $\bar{\mu}$ from Eq. (VI. 41). Then by reading Figure 4 we find $F(\bar{\mu})$. Finally the theoretical mass m can be obtained from Eq. (VI. 45) provided that the cut-off Λ is assumed to have a definite value. As an example let us take $\Lambda^2 = 4m^2$. Then Eq. (VI. 45) gives

$$\frac{8\pi m^2}{g^2} = F(\bar{\mu}) + 2 \ln 4 + 2 \ln \frac{m^2}{M^2} \quad . \tag{VI. 46}$$

Using Eq. (VI. 41) we have

$$\frac{m^2}{M^2} = \bar{\mu}(F(\bar{\mu}) + 2 \ln 4 + 2 \ln \frac{m^2}{M^2}) \quad .$$

If we call

$$\frac{m^2}{M^2} = x$$

this becomes

$$x - 2\bar{\mu} \ln x = \bar{\mu}(F(\bar{\mu}) + 2 \ln 4) \quad . \tag{VI. 47}$$

We solve this equation for given $\bar{\mu}$. The solutions for the special values of $\bar{\mu}$ listed in Table 2 are given in Table 3.

Finally we return to the computation of the analytical expressions for $\mathcal{J}(a)$ and $\mathcal{K}(a)$. Substituting $b = a(1 + \epsilon)$ in

$$J_1 = \int_0^\infty \frac{4d\tau}{\tau^2} \left[e^{-\frac{\tau}{2}} - e^{-\frac{\tau^2 b^2}{2}\left[a^2\tau + (b^2-a^2)(1-e^{-b\tau})/b\right]^{-1}} \right] \tag{V. 9}$$

consider

$$\frac{b^2-a^2}{b}(1 - e^{-b\tau}) = \frac{a\epsilon(2+\epsilon)}{1+\epsilon}(1-e^{-a\tau}e^{-a\epsilon\tau}) \quad .$$

When expanded in powers of ϵ up to the second it gives

$$= 2a\epsilon(1 - e^{-a\tau}) + a\epsilon^2\left[2a\tau e^{-a\tau} - (1-e^{-a\tau})\right] .$$

The exponent of the second term of Eq. (V. 9) becomes then

$$\frac{\tau^2 b^2}{2} \Big/ \left[a^2\tau + \frac{b^2-a^2}{b}(1 - e^{-b\tau})\right]$$

$$= \frac{\tau^2 a^2(1+2\epsilon+\epsilon^2)}{2} \Big/ a^2\tau\left[1 + \frac{2}{a\tau}(1-e^{-a\tau}) + \epsilon^2\left\{2e^{-a\tau} - \frac{1-e^{-a\tau}}{a\tau}\right\}\right]$$

$$= \frac{\tau}{2} + \epsilon\frac{\tau(2-A_1)}{2} + \epsilon^2\frac{\tau[(A_1-1)^2-A_2]}{2} ,$$

(VI. 48)

where

$$A_1 = \frac{2}{a\tau}(1 - e^{-a\tau}) \qquad \text{(VI. 49)}$$

and

$$A_2 = 2e^{-a\tau} - \frac{1}{a\tau}(1 - e^{-a\tau}) . \qquad \text{(VI. 50)}$$

When Eq. (VI. 48) is substituted and the exponential is again expanded, the second term of Eq. (V. 9) gives

$$e^{-\frac{\tau}{2}}\left[1 - \epsilon\frac{\tau(2-A_1)}{2} + \epsilon^2\left\{\tfrac{1}{2}\left(\frac{\tau(2-A_1)}{2}\right)^2 - \frac{\tau}{2}\left((A_1-1)^2-A_2\right)\right\} + \ldots\right] . \qquad \text{(VI. 51)}$$

The substitution of Eq. (VI. 51) together with Eqs. (VI. 49) and (VI. 50) into Eq. (V. 9) gives

$$J_1 = \epsilon \int_0^\infty \frac{4d\tau}{\tau^2} e^{-\frac{\tau}{2}} \left[\tau - \frac{1}{a}(1 - e^{-a\tau})\right]$$

$$- \epsilon^2 \tfrac{1}{2} \int_0^\infty \frac{4d\tau}{\tau^2} e^{-\frac{\tau}{2}} \left[\left\{ \tau^2 - \frac{2\tau}{a}(1 - e^{-a\tau}) + \frac{1}{a^2}(1 - e^{-a\tau})^2 \right\} \right.$$

$$\left. - \left\{ \frac{4}{a^2\tau}(1-e^{-a\tau})^2 - \frac{3}{a}(1-e^{-a\tau}) + \tau(1-2e^{-a\tau}) \right\} \right] + \ldots$$

$$= \epsilon\, \mathcal{J}(a) - \epsilon^2\, \mathcal{K}(a) + \ldots . \qquad \text{(VI. 52)}$$

To evaluate $\mathcal{J}$ we integrate once by parts to get

$$\mathcal{J} = -2 \int_0^\infty e^{-\frac{\tau}{2}} d\tau + 4\left(1 + \frac{1}{2a}\right) \int_0^\infty \frac{d\tau}{\tau} \left[e^{-\frac{\tau}{2}} - e^{-(\frac{1}{2} + a)\tau} \right] .$$

Using

$$\int_0^\infty \frac{d\tau}{\tau} \left[e^{-a\tau} - e^{-b\tau} \right] = \ln \frac{b}{a} \qquad \text{(VI. 53)}$$

we have

$$\mathcal{J} = 4 \left[-1 + \left(1 + \frac{1}{2a}\right) \ln(1 + 2a) \right] . \qquad \text{(VI. 54)}$$

The evaluation of $\mathcal{K}$ is not hard but very tedious. In calculating the first curly-bracketed term of $\mathcal{K}$ we need

$$\int_0^\infty d\tau\, e^{-\frac{\tau}{2}} = 2,$$

$$\int_0^\infty \frac{d\tau}{\tau} e^{-\frac{\tau}{2}} (1 - e^{-a\tau}) = \ln(1 + 2a) ,$$

and

$$\int_0^\infty \frac{d\tau}{\tau^2}\, e^{-\frac{\tau}{2}} (1 - e^{-a\tau})^2 = -\tfrac{1}{2}\, \ln(1+2a) + (\tfrac{1}{2}+2a)\, \ln(\frac{1+4a}{1+2a}) \;.$$

The result is

$$\tfrac{1}{2} \int_0^\infty \frac{4d\tau}{\tau^2}\, e^{-\frac{\tau}{2}} \left\{ \text{first term of } \mathcal{K} \right\} =$$

$$= 2 \left[2 - \frac{1+4a}{a^2}\, \ln(1 + 2a) + \frac{1+4a}{2a^2}\, \ln(1+4a) \right] . \qquad \text{(VI. 55)}$$

For the evaluation of the second curly-bracketed term we need

$$\int_0^\infty \frac{d\tau}{\tau^2}\, e^{-\frac{\tau}{2}} (1 - e^{-a\tau}) = a - \tfrac{1}{2}\, \ln(1 + 2a) + a \int_0^\infty \frac{d\tau}{\tau}\, e^{-(\frac{1}{2} + a)\tau} \;,$$

$$\int_0^\infty \frac{d\tau}{\tau}\, e^{-\frac{\tau}{2}} (1 - 2\, e^{-a\tau}) = \int_0^\infty \frac{d\tau}{\tau}\, e^{-\frac{\tau}{2}} - 2 \int_0^\infty \frac{d\tau}{\tau}\, e^{-(\frac{1}{2} + a)\tau} \;,$$

and

$$\int_0^\infty \frac{d\tau}{\tau^3}\, e^{-\frac{\tau}{2}} (1 - e^{-a\tau})^2 = \frac{3}{2}\, a^2 + \frac{1}{8}\, \ln(1 + 2a)$$

$$- (\frac{1}{8} + a + a^2)\, \ln(\frac{1+4a}{1+2a}) + a^2 \int_0^\infty \frac{d\tau}{\tau}\, e^{-(\frac{1}{2}+2a)\tau} \;.$$

Substituting these in the second term of $\mathcal{K}$ and rearranging the integrals to apply Eq. (VI. 53) we have

$$-\tfrac{1}{2} \int_0^\infty \frac{4d\tau}{\tau^2}\, e^{-\frac{\tau}{2}} \left\{ \text{second term of } \mathcal{K} \right\}$$

$$= -2 \left[3 + (9 + \frac{11}{2a} + \frac{1}{a^2})\, \ln(1 + 2a) - \frac{(1+4a)^2}{2a^2}\, \ln(1+4a) \right] . \qquad \text{(VI. 56)}$$

The sum of Eqs. (VI. 55) and (VI. 56) gives

$$\mathcal{K}(a) = 2\left[\frac{(1+2a)(1+4a)}{a^2} \ln(1+4a)-1-(9+\frac{19}{2a}+\frac{2}{a^2}) \ln(1+2a)\right]. \quad (VI.57)$$

$\mathcal{J}'$ and $\mathcal{K}'$ are calculated from Eqs. (VI. 54) and (VI. 57).

As the discussion of the results we note first that the dependence of the self-energy on the cut-off Λ is completely determined for our case of the scalar nucleon interacting with the scalar meson. Therefore this theory should be easy to renormalize. For other types of meson theories the Λ-dependence may not be so simple as the present case.

Next we observe that there appears no solution of the problem for too large values of g^2. The possible reason for this is that with strong enough attraction the binding of nucleon and antinucleon will possibly, analogously to the case of the positronium, exceed their rest masses. Thus pairs would be found ad infinitum with release of energy radiated as mesons. Hence no sensible theory may exist for too large values of g^2 in this scalar case.

As was mentioned at the end of Sect. III, the effects of the closed loops are omitted from the self-action effects of the nucleon. This can be seen in the following manner. For the case of a given external field the loops are completely separated from the nucleon line. The contribution of diagrams with no loop, one loop , two loops, and so on, to the amplitude of the nucleon's transition from the initial meson-vacuum state to the final meson-vacuum state can be completely separated from the amplitude of the nucleon line itself to give $C_v = 1 + L + L^2/2 + \dots = e^{+L}$, where L stands for the amplitude for a single closed loop. That C_v can be factored out for the case of an external field is manifested in Eq. (II. 23). However, for the case of the quantum mechanical meson field the closed loops may be connected with the nucleon line and cannot be factored out as above. Since the kernel given by Eq. (III. 15) corresponds to the one excluding the effects of the closed loops the self-energy we obtained represent the self-action of the nucleon omitting the closed loop effects.

In the present calculation the meson mass was assumed to be zero for the sake of simplicity. The effect of non-vanishing meson mass $\varkappa$ will first show up in Eq. (IV. 16) replacing k_μ^2 by $k_\mu^2 + \varkappa^2$. This replacement applies to all the subsequent equations. Thus, for example, Eq. (V. 1) will be replaced by $(k_\mu^2 + \varkappa^2)^{-1} = \int_0^\infty e^{-\gamma(k_\mu^2 + \varkappa^2)} d\gamma$ and correspondingly the integrand of J (Eq. (V. 6)) will be multiplied by $e^{-\gamma\varkappa^2}$. In this case the evaluation of J becomes much more complicated and even the limiting value of J_2 for large Λ^2 may not be obtained in a closed form. Therefore we will probably be forced to resort to the numerical calculation of J.

As to the accuracy it cannot easily be judged because there is no way of comparing our result with the experiment.

An attempt to apply the present technique to the realistic problems encounters a great difficulty. For if the nucleon is assumed to be a Dirac particle and if we consider various types of the meson theory the Lagrangian of the system contain operators such as the Dirac matrices γ_μ, the spin operator $\vec{\sigma}$, and the isotopic spin operator $\vec{\tau}$. Then the amplitude corresponding to Eq. (III. 15) will now contain these as ordered operators in certain manner. In addition to this the action may not have the assurance of being a real quantity which was essential in applying the variational technique. Therefore, the simple path integral treatment will fail for such a problem. However, when the method for this kind of problem is found it will be of some importance to note that the Fock's method of parametrizing the Dirac equation[16] permits the Lagrangian formulation which is, at least in its form, very similar to the present treatment of the Klein-Gordon equation.

Finally we notice that although we have calculated only the self-energy of the nucleon the theory may possibly be extended to another problem such as the scattering of the meson by the nucleon using the best estimate of the real action. However, the accuracy of the result may not be very high.

TABLE 1

a	$\mathcal{J}(a)$	$\mathcal{K}(a)$	$\mathcal{J}'(a)$	$\mathcal{K}'(a)$	μ	ϵ	$\frac{M^2}{m'^2}$	$\frac{v_0^2}{m'^2}$	$\frac{\epsilon^2 \mathcal{K}(a)}{\epsilon \mathcal{J}(a)}$
3.7	5.6628	1.6582	0.7704	-0.0170	-0.0288				
3.8	5.7400	1.6530	0.7536	-0.0210	+0.0098	0.0037	0.9998	0.9998	0.0000
4.0	5.8876	1.6494	0.7252	-0.0292	+0.0660	0.0243	0.9954	0.9859	0.0070
4.5	6.2336	1.6292	0.6612	-0.0450	+0.1995	0.0772	0.9549	0.8596	0.0204
5.0	6.5508	1.6046	0.6080	-0.0558	+0.2750	0.1250	0.9036	0.6762	0.0305
5.5	6.8432	1.5754	0.5628	-0.0636	+0.2992	0.1694	0.8772	0.5067	0.0390
6.0	7.1148	1.5414	0.5244	-0.0686	+0.2949	0.2104	0.8863	0.3778	0.0456
6.5	7.3680	1.5058	0.4904	-0.0716	+0.2777	0.2518	0.9327	0.2803	0.0515

$$\epsilon = \frac{2a\mathcal{J}' - \mathcal{J}}{2(a\mathcal{K}' - \mathcal{K})} \; ; \qquad \mu^{-1} = \frac{6a\epsilon^2+1}{4a\epsilon}\mathcal{J} - \frac{4a\epsilon^2+1}{2a}\mathcal{K} \, .$$

$$\frac{M^2}{m'^2} = (1-\mu(\epsilon\mathcal{J} - \epsilon^2\mathcal{K})(2a\epsilon^2+1) \; ; \; \frac{v_o^2}{m'^2} = \frac{1-(\epsilon\mathcal{J} - \epsilon^2\mathcal{K})}{2a\epsilon^2+1}$$

TABLE 2

μ	$\bar{\mu} = \mu/(1-\mu J_1)(2a\epsilon^2+1)$	$\frac{1}{\mu}$	$M^2/v_o^2 = (2a\epsilon^2+1)^2$	$\ln \frac{M^2}{v_o^2}$	$2\ln \frac{M^2 e}{v_o^2}$	$D(\mu)=F(\bar{\mu})$
0.0098	0.0000	101.9177	1.0000	0.0000	2.0000	103.9177
0.0660	0.0663	15.1529	1.0096	0.0096	2.0192	17.1721
0.1995	0.2089	5.0120	1.1109	0.1052	2.2104	7.2224
0.2750	0.3043	3.6361	1.3363	0.2899	2.5798	6.2159
0.2992	0.3411	3.3421	1.7313	0.5489	3.0978	6.4399

$$D(\mu) = \frac{1}{\mu} + 2 \ln \frac{M^2 e}{v_o^2} = F(\bar{\mu}) \quad .$$

TABLE 3

$\bar{\mu} = \frac{g^2}{8\pi M^2}$	$\bar{\mu}\,(F(\bar{\mu}) + 2\ln 4)$	$\frac{m^2}{M^2}$
0	1	1
0.066	1.32	1.36
0.21	2.10	2.48
0.30	2.70	3.44
0.34	3.15	4.11

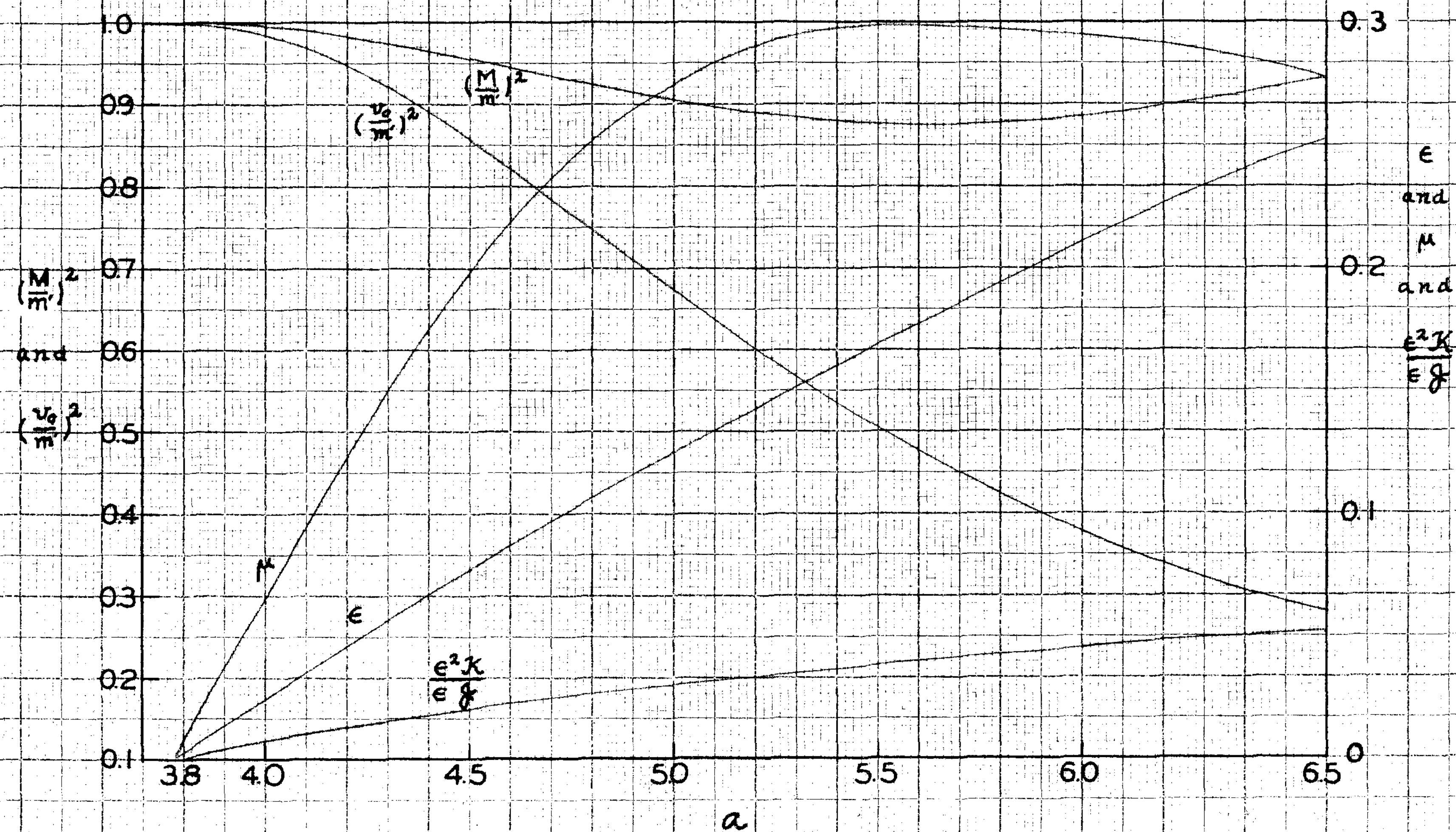
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
$(\frac{M}{m'})^2$ and $(\frac{v_0}{m'})^2$
0.3
0.2
0.1
0
ϵ and μ and $\frac{\epsilon^2 K}{\epsilon J}$
3.8
4.0
4.5
5.0
5.5
6.0
6.5
a
$(\frac{M}{m'})^2$
$(\frac{v_0}{m'})^2$
μ
ϵ
$\frac{\epsilon^2 K}{\epsilon J}$

Figure 1

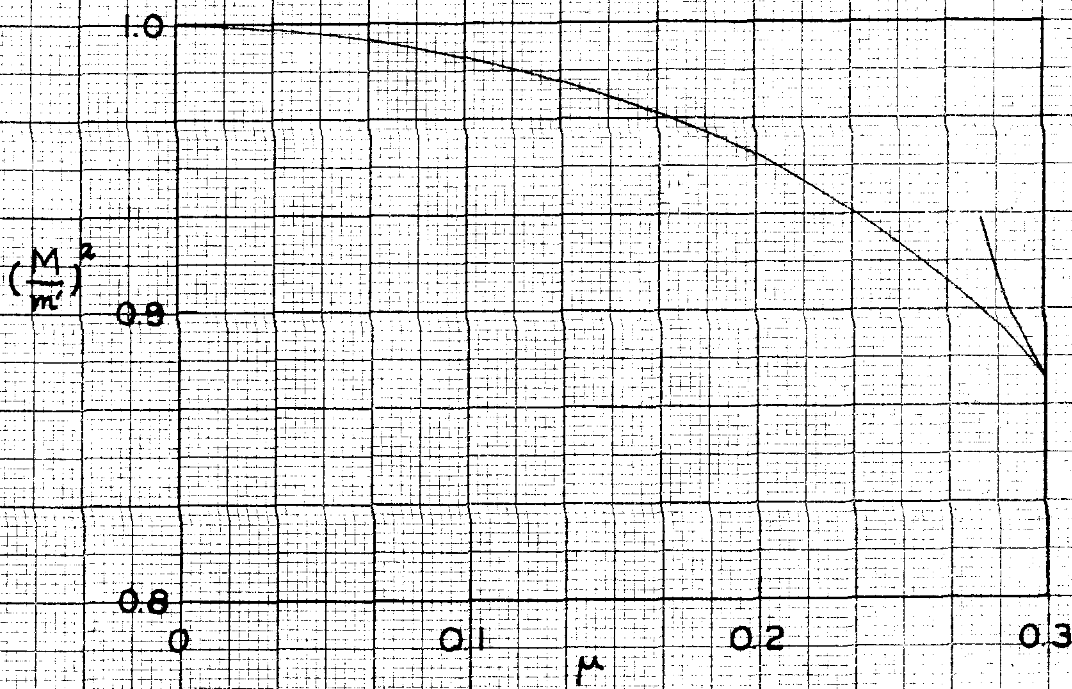

Figure 2

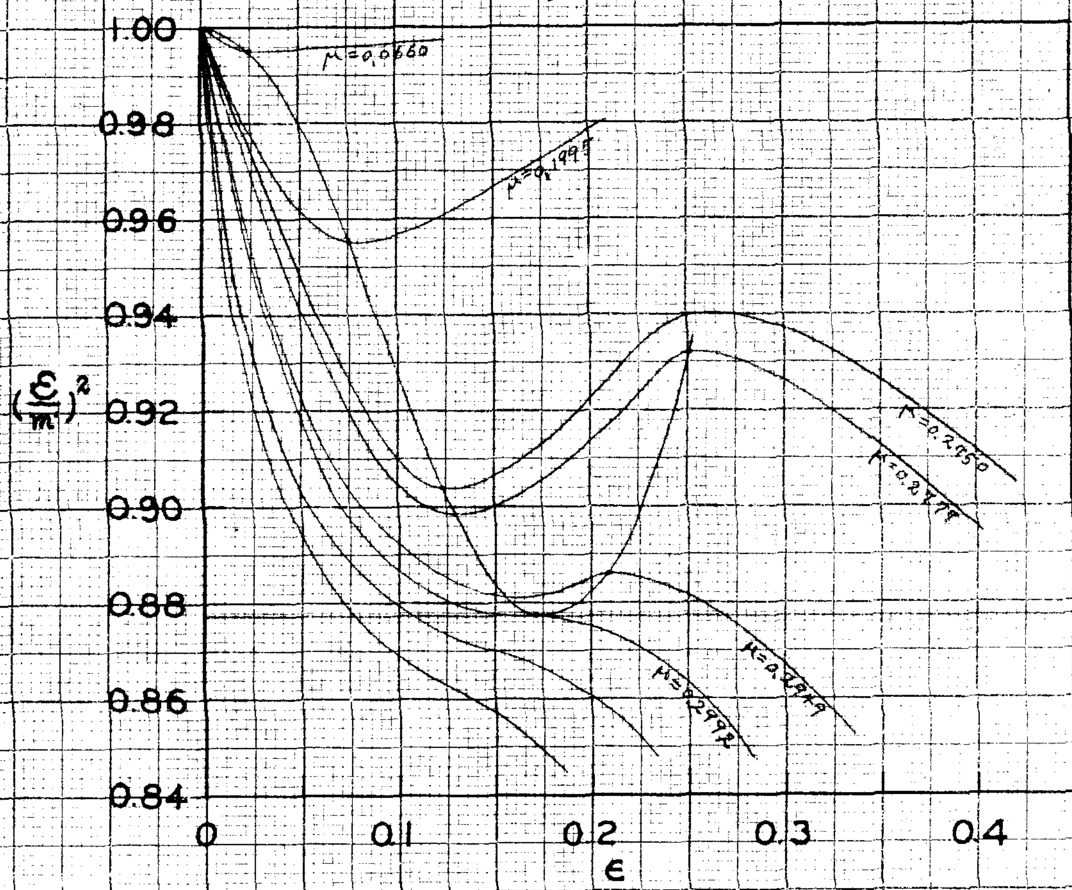

Figure 3

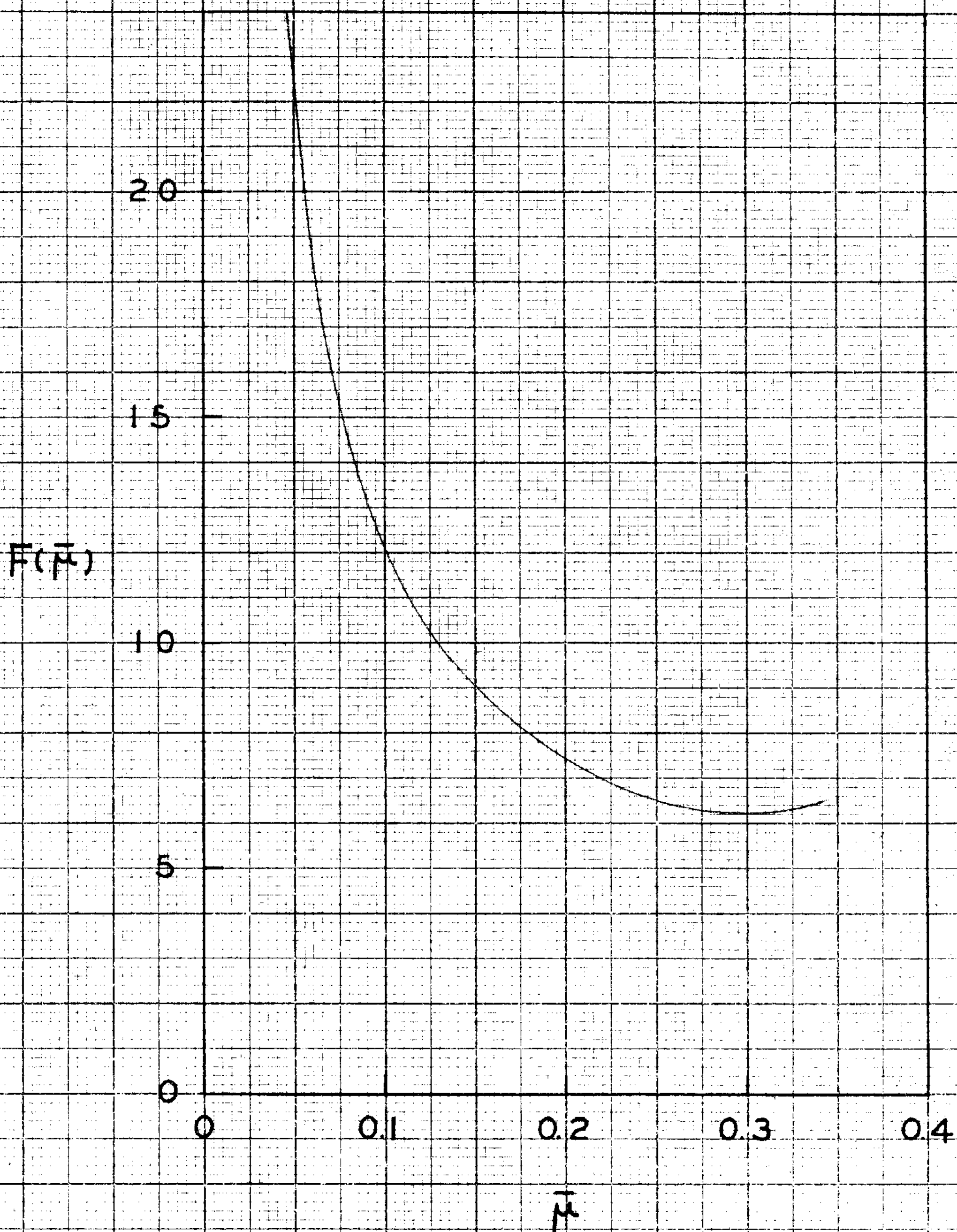

Figure 4

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