THE AILERON INFLUENCE ON WING FLUTTER

Thesis

by

Dalimil Kybal

In Partial Fulfillment of the Requirements for the Degree of Master of Science

California Institute of Technology
Pasadena, California

1940

SUMMARY

That the aileron has a large influence on wing flutter was found as early as 1928 when Frazer and Duncan[5,6] published a "List of Design Recommendations relative to Ailerons". Since these Recommendations were stated in a general form it was felt by the author that it would be of interest to investigate this influence of the aileron for an actual modern airplane.

Therefore as main part of this work the case of flutter for three degrees of freedom (torsion-flexure-aileron) is investigated for six different positions of the aileron center of gravity. These results are then compared with the case of two degrees of freedom of flexure-torsion. Also the subcase of torsion-aileron flutter is investigated.

One of the conclusions reached is that the flutter velocity increases almost linearly with the distance of the aileron center of gravity forward from the position of dynamic balance to a position of about five percent overbalance, but that further increase of overbalance will cause only a very small increase of flutter velocity.

A calculation scheme for the determination of the cubic and quadratic coefficients is presented together with a number of tables for the various numerical coefficients.

ACKNOWLEDGEMENT

The author wishes to express his debt of gratitude to the Vultee Aircraft Corporation, in particular to Mr. R. W. Palmer, Chief Engineer, and Mr. W. C. Rockefeller, Head of the Aerodynamics Department, for letting him use the wing parameters of an early version of the Vultee Vanguard. It has to be stated from the beginning that these wing parameters have been modified in order to comply better with the problem undertaken, and therefore the conclusions reached in this paper are not to be applied directly to the above mentioned airplane.

The author wishes to express his gratitude to Dr. Theodore von Karman, Director of the Daniel Guggenheim Laboratory, for the interest he has shown in this particular problem.

The author also wishes to express his appreciation to Dr. A. E. Lombard, Jr., who suggested the problem and extended helpful ideas.

Finally the author wishes to express his gratitude to Dr. Theodore Theodorsen , as the work is based on equations set up by him.

TABLE OF CONTENTS

TABLE OF FIGURES

NOTATIONS

These notations in general agree with Theodorsen.

α = angle of wing relative to wing direction (fig. 1)

β = aileron angle relative to wing chord (fig. 1)

h = vertical distance of center of wing chord (fig. 1)

$\dot{\alpha} = \frac{d\alpha}{dt}$, $\ddot{\alpha} = \frac{d^2\alpha}{dt^2}$, etc.

b = half chord of wing used as reference unit length (ft.)

a = distance of torsional axis aft of mid-chord as fraction of semi-chord (fig.1). Location of stiffness axis in percent total chord measured from the leading edge is

$$100\,\frac{1+a}{2} \qquad \text{or}$$

$$a = \frac{2(\text{stiffness axis})}{100} - 1$$

c = distance of the aileron hinge axis aft of mid-chord as fraction of semi-chord. Location of aileron hinge axis in percent total chord measured from leading edge is

$$100\,\frac{1+c}{2} \qquad \text{or}$$

$$c = \frac{2(\text{aileron hinge})}{100} - 1$$

$\tau = \frac{1-c}{2}$, ratio of aileron chord to wing chord

σ = distance of center of gravity of aileron mass aft of hinge line, as fraction of wing chord.

ρ = mass of air per unit of volume

M = mass of wing per unit of span

m = mass of aileron per unit of span

m' = additional mass per unit of span added on to the aileron to vary its center of gravity location.

κ = $\frac{\pi \rho b^2}{M}$, ratio of mass of a cylinder of air of a diameter equal to the chord of the wing, both taken at the reference section.

x_α = location of center of gravity of wing-aileron system from a. (fig. 1)

x_β = reduced location of center of gravity of aileron referred to c. (fig. 1)

r_α = radius of gyration of wing-aileron system referred to a. (fig. 1)

r_β = reduced radius of gyration of aileron referred to c. (fig. 1)

S_α = static moment of wing-aileron per unit span length referred to a.

S_β = static moment of aileron per unit span length referred to c.

I_α = moment of inertia of wing-aileron about a, per unit span length.

I_β = moment of inertia of aileron about c, per unit span length.

C_α = torsional stiffness of wing around a, per unit span length.

C_β = torsional stiffness of aileron around c, per unit span length.

C_h = stiffness of wing in deflection, per unit span length.

ω = circular frequency of wing vibrations. (rad/sec.)

$\omega_\alpha = \sqrt{\frac{C_\alpha}{I_\alpha}}$, natural circular frequency of torsional vibration of wing-system around a, as of one degree of freedom in vacuum.

$\omega_\beta = \sqrt{\frac{C_\beta}{I_\beta}}$, natural frequency of torsional vibration of aileron around c.

$\omega_h = \sqrt{\frac{C_h}{M}}$, natural frequency of wing in deflection.

t = time

v = velocity of airplane

V_{cr}, V_{cr_0} = flutter or critical velocities
V_{cr} designates either the flexure-torsion-aileron flutter velocity or the aileron-torsional flutter velocity,
V_{cr_0} represents the flexure-torsion flutter velocity which is independent of the aileron system.

$\mu = \frac{V_{cr}}{V_{cr_0}}$, parameter used to indicate the influence of the aileron on wing flutter.

$k = \frac{b\omega}{v}$, reduced frequency equal number of waves in the wake in a distance equal to the semichord $\times 2\pi$.

$\frac{1}{k}$ = reduced wave length equal to the length of one wave of the wake in terms of a distance equal to the semichord $\times 2\pi$.

X = function of flutter frequency

for Flexure-Torsion $X = \frac{r_\alpha^2}{\kappa} \left(\frac{\omega_\alpha^2}{\omega}\right)^2$

for Flexure-Aileron $X = \frac{1}{\kappa} \left(\frac{\omega_h}{\omega}\right)^2$

for Torsion-Aileron $X = \frac{r_\alpha^2}{\kappa} \left(\frac{\omega}{\omega_\alpha}\right)^2$

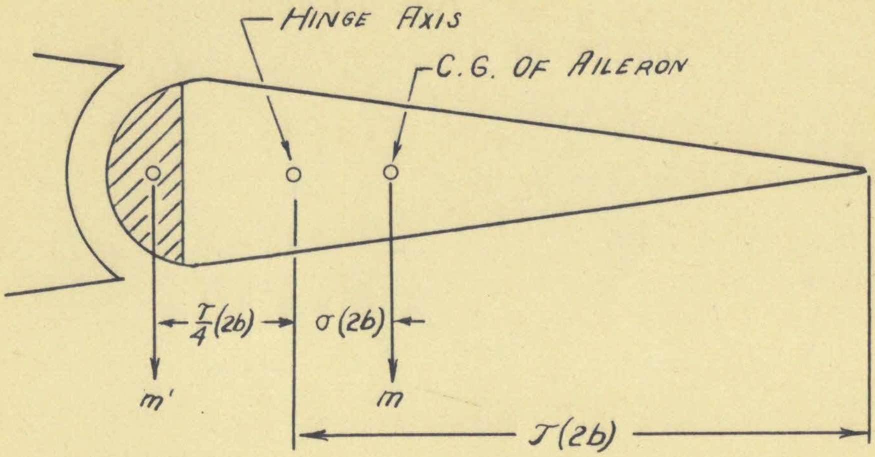
for Flexure-Torsion-Aileron $X = \frac{r_\alpha^2}{\kappa} \left(\frac{\omega}{\omega_\alpha}\right)^2$

$C(k) = F + iG$, complex lift vector, tabulated in References 1, 2, 3, 4.

T_1, T_2, etc. constants given in References 1 and 2.

DISCUSSION OF THE SPECIFIC PROBLEM UNDERTAKEN

The purpose of this paper is to show the influence of the aileron on wing flutter while changing the location of the aileron center of gravity. This was done by incorporating and additional mass $\underline{m}'$ into the nose of the aileron



in order to cause a change in the location of the aileron center of gravity. This practice was adopted since it corresponds exactly to the procedure followed in aircraft industry.

The basic assumption was that the total mass M of the wing per ft. of span, including the aileron, did not change when additional mass was added to the aileron.

Another assumption was that the center of gravity of the additional mass $\underline{m}'$ was located at twenty five percent of the aileron chord forward of the hinge axis. The aileron chord being defined as the distance from its hinge axis to the trailing edge.

If $(\ )_o$ represents the notation used for a particular location of the center of gravity for which all the wing parameters were determined by tests on the actual wing $(m' = 0)$, then for any other position of the center of gravity the moment of inertia equation will become:

$$I_\beta = Mr_\beta^2 b^2 = \left[Mr_{\beta_o}^2 b^2 - m'(\tfrac{1}{2}\tau b)^2 \right] .$$

from which

$$r_\beta^2 = r_{\beta_o}^2 - \left[\frac{m'}{M}\right] \frac{1}{4}\tau^2 \qquad (1)$$

Similarly for the static moment relation:

$$S_\beta = S_{\beta_o} - m' \frac{\tau b}{2}$$

since $x_\beta = \frac{S_\beta}{Mb}$ and $x_{\beta_o} = \frac{S_{\beta_o}}{Mb}$

$$x_\beta = x_{\beta_o} - \frac{m'}{M}\frac{\tau}{2} \qquad (2)$$

The frequency of the aileron will vary according to the relation

$$\frac{\omega_\beta}{\omega_{\beta_o}} = \frac{r_\beta}{r_{\beta_o}} \qquad (3)$$

As an auxilliary expression we have:

$$x_\beta = 2\sigma \frac{m}{M}$$

Since $m = m_o + m'$,

$$x_\beta = 2\sigma \frac{m_o + m'}{M}$$

Eliminating x_β between this last equation and equation (2) we obtain:

$$\frac{m'}{M} = \frac{x_{\beta_o} - 2\sigma \frac{m_o}{M}}{\frac{\tau}{2} + 2\sigma} \qquad (4)$$

Six different stations were investigated for which the distance from the center of gravity of the aileron to the hinge axis $\left(\frac{\xi}{\tau}\right)$ amounted to -10%; -5%; +1.4%; +10%; +20% and +40% of the aileron chord. The last value corresponds to the case of an unbalanced aileron.

The variation of the parameters $\frac{m'}{M}$, x_β, r_β and ω_β with respect to $\left(\frac{\xi}{\tau}\right)$ is shown in Figure 2.

For each of these six cases the flutter velocities of flexure-torsion-aileron and torsion-aileron were determined and compared to the flexure-torsion case of the wing.

For the determination of flutter velocities Theodorsen's equations (A,B,C pg.10, Ref.1) of equilibrium were used. These equations represent the equilibrium of the moments about <u>a</u> of the entire airfoil, of the moments on the aileron about <u>c</u>, and of the vertical forces. In these expressions the total aerodynamic forces and moments were obtained by Theodorsen from the theory of nonstationary potential flow.

The underlying assumption of the above mentioned equations is that the large oscillations are not considered and only infinitely small oscillations about the position of equilibrium need be considered.

Since there is no necessity for solving a general case of damped or divergent motion, only the border case of a pure sinusoidal motion need be determined. Therefore by introducing into the equations

$$\alpha = \alpha_0 e^{ik\frac{s}{b}}$$

$$\beta = \beta_0 e^{i(k\frac{s}{b} + \varphi_1)}$$

$$h = h_0 e^{i(k\frac{s}{b} + \varphi_2)}$$

where $s = vt$ and φ_1 and φ_2 are phase angles of β and h with respect to α, the equations can be written into the following complex form:

Moments about torsional axis:

$$\alpha\left[A_{a\alpha} + \Omega_\alpha X\right] + \beta\left[A_{a\beta}\right] + h\left[A_{ah}\right] = 0$$

Moments about hinge axis:

$$\alpha\left[A_{b\alpha}\right] + \beta\left[A_{b\beta} + \Omega_\beta X\right] + h\left[A_{bh}\right] = 0$$

Vertical forces:

$$\alpha\left[A_{c\alpha}\right] + \beta\left[A_{c\beta}\right] + h\left[A_{ch} + \Omega_h X\right] = 0$$

where $A_{a\alpha} = R_{a\alpha} + iI_{a\alpha}$, etc. All A terms are function of $\frac{1}{k}$.

The solution of the instability problem as contained in the system of these equations is given by the vanishing of a third order determinant of complex numbers representing the coefficients of α, β, and h :

$$D = \Delta R + i\Delta I = \begin{vmatrix} A_{a\alpha} + \Omega_\alpha X & A_{a\beta} & A_{ah} \\ A_{b\alpha} & A_{b\beta} + \Omega_\beta X & A_{bh} \\ A_{c\alpha} & A_{c\beta} & A_{ch} + \Omega_h X \end{vmatrix} = 0$$

The solution of this determinantal equation resolves itself into the solution of two simultaneous equations, one real and one imaginary.

For the major case, i.e. three degrees of freedom, the solution results in a real cubic equation and an imaginary quadratic equation in X, while the particular subcases of two degrees of freedom are characterized by a real quadratic equation and an imaginary linear equation in X.

The solution of these equations determines X and $\frac{1}{k}$, from which the unknown velocity and flutter frequency can be obtained.

The problem reduces itself to the finding of values of X satisfying both the real and imaginary parts of the determinant. For this, the procedure consists in plotting graphically the roots X against $\frac{1}{k}$ for both equations, the points of intersection being the solutions of the determinantal equation.

In Figures 3 to 10 are shown the curves of the roots of the real and imaginary equations plotted against $\frac{1}{k}$ for the various cases considered. The results are presented in Fig. 12, which shows the flutter velocity V_{cr}, expressed as a ratio to the flutter speed for flexure and torsion, V_{cr_0}, plotted against the aileron center of gravity location.

RESULTS AND CONCLUSIONS

Three main conclusions can be drawn from the results obtained in this research, as presented in Fig. 12.

For an approximate five percent increase of mass overbalance from the position of dynamic balance ($\frac{\sigma}{\tau} = 0$) the increase in flutter speed is quite considerable. For a further overbalance this increase is very small and probably the flutter velocity will actually decrease. Therefore it is felt that only the first five percent of overbalance contributes appreciably towards the increase of flutter speed. It is seen that this mass overbalance of the aileron is also beneficial since it eliminates the case of Torsion-Aileron flutter.

It is also seen from the results that for the region of $\frac{\sigma}{\tau} = 0$ to $\frac{\sigma}{\tau} = +10\%$, for the case of Torsion-Aileron, the flutter speed decreases very rapidly to a speed of 10 ft/sec. for a center of gravity location of +10%. It is believed that this fact can be explained by the low value of the aileron frequency in the present example. Also one must not forget that throughout this analysis the internal or solid friction has not been taken into account. It is believed that this factor will have a large influence for the regions of low velocity. The neglecting of this internal friction can also explain the fact that no flutter was found for the +10% position of the aileron center of gravity. The effect of the internal friction may result in causing an intersection of the real and imaginary roots at a low value of the reduced wave length, and therefore instead of no flutter velocity a low flutter velocity may actuall exist. This would give to the three degrees of freedom curve a trend similar to that of the lower curve of the Torsion-Aileron case for the region of $\frac{\sigma}{\tau} = 0$ to $\frac{\sigma}{\tau} = +40\%$.

The third conclusion which can be reached is that as the aileron center of gravity is moved aft of the hinge axis, the inertia terms, which vary as the square of the distance $\sigma(2b)$, become prevalent in the equations of equilibrium. This increase in effective mass moment of inertia causes an increase in flutter speed. This corresponds to the results shown in Fig. 12.

PLAN OF FUTURE WORK

Lack of time has prevented the carrying of the present work to a more complete state. It is intended that further calculations be carried out in the near future to include the following investigations:

1.) The region between $\frac{\sigma}{\tau} = 0$ and $+20$ will be investigated for more positions of the aileron center of gravity location.

2.) The influence of friction damping will be investigated, especially in the neighborhood of low flutter velocities.

3.) Since it is felt that the influence of control stiffness will have a large influence in increasing the flutter velocity, this effect will be thoroughly investigated. The only parameter which will be affected by variation of this parameter will be the aileron frequency (ω_β), since the controls are assumed to be in locked position.

REFERENCES

1. Theodorsen, Theodore: General Theory of Aerodynamic Instability and the Mechanism of Flutter. T.R. No. 496, N.A.C.A., 1934.

2. Theodorsen, Theodore and Garrick, I.E.: Mechanism of Flutter. A Theoretical and Experimental Investigation on the Flutter Problem. T.R. No. 685, N.A.C.A., 1940.

3. Lombard, A.E., Jr.: An investigation of the conditions for the occurence of flutter in aircraft and the development of criteria for the prediction and elimination of such flutter. Doctor's Thesis, California Institute of Technology, 1939.

4. Kassner, R., and Fingado, H.: Das ebene Problem der Flugelschwingung. Luftfahrtforschung, Bd. 13, Nr. 11, 20 Nov. 1936, S. 374-387.

5. Frazer, R.A., and Duncan, W.J.: The Flutter of Aeroplane Wings. R. and M. No. 1155, August 1928.

6. Frazer, R.A., and Duncan, W.J.: The Flutter of Monoplanes, Biplanes and Tail Units. (A sequel to R. and M. No. 1155) R. and M. 1255, January 1931.

CALCULATION SCHEME FOR DETERMINING COEFFICIENTS

This calculation scheme is given in such a form that, proceeding progressively from the determined wing parameters, one can determine the coefficients of the Real and Imaginary Equations for the Case of Flexure-Torsion-Aileron Flutter. These equations, as already stated, appear in the form:

Real Equation $\Delta R = \mathcal{A}_1 X^3 + \mathcal{B}_1 X^2 + \mathcal{C}_1 X + \mathcal{D}_1 = 0$

Imaginary Equation $\Delta I = \mathcal{A}_2 X^2 + \mathcal{B}_2 X + \mathcal{C}_2 = 0$

where $\mathcal{A}_1$, $\mathcal{B}_2$, $\mathcal{C}_1$ ... are the coefficients to be determined.

The following eleven wing parameters will have to be determined by preliminary ground tests or calculations:

$a,\ c,\ x_\alpha,\ x_\beta^*,\ K,\ r_\alpha,\ r_\beta^*,\ b,\ \omega_h,\ \omega_\alpha,\ \omega_\beta^*$

Note that three varying parameters have been written with an ( )* in order to denote them as variable, and the same denomination will be used for all the coefficients affected by these variable parameters. This procedure will be followed throughout this calculation scheme.

From the first seven original parameters a first set of constants is determined:

$$A_{\alpha_1} = \frac{r_\alpha^2}{K} - \left(\frac{1}{8} + a^2\right)$$

$$A_{\alpha_2} = \left(\frac{1}{2} - a\right)$$

$$A_{\beta_1}^{*} = \frac{r_\beta^{2\,*}}{K} - \frac{T_7}{\pi} + (c-a)\left(\frac{x_\beta^{*}}{K} - \frac{T_1}{\pi}\right)$$

$$A_{\beta_2} = \frac{1}{\pi}\left[-2p - \left(\frac{1}{2} - a\right) T_4\right]$$

$$A_{\beta_3} = \frac{1}{\pi}\left(T_4 + T_{10}\right)$$

$$A_{h_1} = \frac{x_\alpha}{K} - a$$

$$B_{\alpha_1}^{*} = \frac{r_\beta^{2\,*}}{K} - \frac{T_7}{\pi} + (c - a)\left(\frac{x_\beta^{*}}{K} - \frac{T_1}{\pi}\right) \quad (= A_{\beta_1})$$

$$B_{\alpha_2} = \frac{1}{\pi}\left(p - T_1 - \frac{1}{2}T_4\right)$$

$$B_{\beta_1}^{*} = \frac{r_\beta^{2\,*}}{K} - \frac{1}{\pi^2} T_3$$

$$B_{\beta_2} = -\frac{1}{2\pi^2} T_4 T_{11}$$

$$B_{\beta_3} = \frac{1}{\pi^2}\left(T_5 - T_4 T_{10}\right)$$

$$B_{h_1}^{*} = \frac{x_\beta}{K} - \frac{1}{\pi} T_1$$

$$C_{\alpha_1} = \frac{x_\alpha}{K} - a \quad (= A_{h_1})$$

$$C_{\alpha_2} = 1$$

$$C_{\beta_1}^{*} = \frac{x_\beta}{K} - \frac{1}{\pi} T_1 \quad (= B_{h_1})$$

$$C_{\beta_2} = -\frac{1}{\pi} T_4$$

$$C_{\beta_3} = 0$$

$$C_{h_1} = \frac{1}{K} + 1$$

The values of T are obtained from the T table given on page 16 Ref. 1.

The second set of coefficients is given by:

$$R_{a\alpha} = - A_{\alpha_1} + R''_{a\alpha} = - A_{\alpha_1} + \left[(\tfrac{1}{4} - a^2)\frac{2G}{k} - (\tfrac{1}{2} + a)\frac{2F}{k^2}\right]$$

$$I_{a\alpha} = \frac{1}{k}\left[I''_{a\alpha}\right] = \frac{1}{k}\left[A_{\alpha_2} - (\tfrac{1}{2} + a)\frac{2G}{k} - (\tfrac{1}{4} - a^2)2F\right]$$

$$R^*_{b\beta} = - B^*_{\beta_1} + R_{b\beta''} = - B^*_{\beta_1} + \left[\frac{1}{k} B_{\beta_3} - \frac{T_{12}}{2\pi}\frac{T_{11}}{2\pi}\frac{2G}{k} + \frac{T_{12}}{2\pi}\frac{T_{10}}{\pi}\frac{2F}{k^2}\right]$$

$$I_{b\beta} = \frac{1}{k}\left[I''_{b\beta}\right] = \frac{1}{k}\left[B_{\beta_2} + \frac{T_{12}}{2\pi}\frac{T_{10}}{\pi}\frac{2G}{k} + \frac{T_{12}}{2\pi}\frac{T_{11}}{2\pi}2F\right]$$

$$R_{ch} = - C_{h_1} + R''_{ch} = - C_{h_1} + \left[- \frac{2G}{k}\right]$$

$$I_{ch} = \frac{1}{k} 2F$$

A. Flexure-Torsion-Aileron.

From the first set of coefficients one determines:

$$a^*_1 = B^*_{h_1} C_{\alpha_1} - B^*_{\alpha_1} C_{h_1}$$

$$b_1 = C_{h_1} A_{\alpha_1} - C_{\alpha_1} A_{h_1}$$

$$c^*_1 = A_{h_1} B^*_{\alpha_1} - A_{\alpha_1} B^*_{h_1}$$

$$a^*_2 = B^*_{h_1} C_{\alpha_2} - B_{\alpha_2} C_{h_1}$$

$$b_2 = C_{h_1} A_{\alpha_2} - C_{\alpha_2} A_{h_1}$$

$$c^*_2 = A_{h_1} B_{\alpha_2} - A_{\alpha_2} B^*_{h_1}$$

$$r_a = A_{\alpha_1} - (\tfrac{1}{2} - a) A_{h_1}$$

$$r^*_b = B^*_{\alpha_1} - (\tfrac{1}{2} - a) B^*_{h_1}$$

$$r_c = C_{\alpha_1} - (\tfrac{1}{2} - a) C_{h_1}$$

$$s_a = A_{h_1} - A_{\alpha_2}$$

$$s_b^* = B_{h_1}^* - B_{\alpha_2}$$

$$s_c = C_{h_1} - C_{\alpha_2}$$

Constant Term Coefficients:

$$R^* = -1\left\{A^*_{\beta_1}(a_1)^* + B^*_{\beta_1}(b_1) + C^*_{\beta_1}(c_1)^*\right\}$$

$$\bar{R}^* = 1\left\{A_{\beta_2}(a_2)^* + B_{\beta_2}(b_2) + C_{\beta_2}(c_2)^* + A_{\beta_3}(a_1)^* + B_{\beta_3}(b_1)^* + C_{\beta_3}(c_1)^*\right\}$$

$$S^* = (\tfrac{1}{2} + a)\left\{C^*_{\beta_1}(r_b)^* - B^*_{\beta_1}(r_c) + \tfrac{T_{11}}{2\pi}(a_1)^*\right\}$$

$$- \tfrac{T_{12}}{2\pi}\left\{A^*_{\beta_1}(r_c) - C^*_{\beta_1}(r_a) + \tfrac{T_{11}}{2\pi}(b_1)\right\}$$

$$- 1\left\{B^*_{\beta_1}(r_a) - A^*_{\beta_1}(r_b)^* + \tfrac{T_{11}}{2\pi}(c_1)^*\right\}$$

$$\bar{S}^* = -(\tfrac{1}{2} + a)\left\{C_{\beta_3}(r_b)^* - B_{\beta_3}(r_c) + B_{\beta_2}(s_c) - C_{\beta_2}(s_b)^* + \tfrac{T_{10}}{\pi}(a_2)^*\right\}$$

$$+ \tfrac{T_{12}}{2\pi}\left\{A_{\beta_3}(r_c) - C_{\beta_3}(r_a) + C_{\beta_2}(s_a) - A_{\beta_2}(s_c) + \tfrac{T_{10}}{\pi}(b_2)\right\}$$

$$+ 1\left\{B_{\beta_3}(r_a) - A_{\beta_3}(r_b)^* + A_{\beta_2}(s_b)^* - B_{\beta_2}(s_a) + \tfrac{T_{10}}{\pi}(c_2)^*\right\}$$

$$T^* = -(\tfrac{1}{2} + a)\left\{C_{\beta_2}(r_b)^* - B_{\beta_2}(r_c) + B^*_{\beta_1}(s_c) - C^*_{\beta_1}(s_b)^* + \tfrac{T_{10}}{\pi}(a_1)^* + \tfrac{T_{11}}{2\pi}(a_2)^*\right\}$$

$$+ \tfrac{T_{12}}{2\pi}\left\{A_{\beta_2}(r_c) - C_{\beta_2}(r_a) + C^*_{\beta_1}(s_a) - A^*_{\beta_1}(s_c) + \tfrac{T_{10}}{\pi}(b_1) + \tfrac{T_{11}}{2\pi}(b_2)\right\}$$

$$+ 1\left\{B_{\beta_2}(r_a) - A_{\beta_2}(r_b)^* + A^*_{\beta_1}(s_b)^* - B^*_{\beta_1}(s_a) + \tfrac{T_{10}}{\pi}(c_1)^* + \tfrac{T_{11}}{2\pi}(c_2)^*\right\}$$

$$\bar{T}^* = (\tfrac{1}{2} + a)\left\{B_{\beta_3}(s_c) - C_{\beta_3}(s_b)^*\right\},$$

$$- \tfrac{T_{12}}{2\pi}\left\{C_{\beta_3}(s_a) - A_{\beta_3}(s_c)\right\}$$

$$- 1\left\{A_{\beta_3}(s_b)^* - B_{\beta_3}(s_a)\right\}$$

$$U^* = i \left\{ A^*_{\beta_1}(a_2)^* + B^*_{\beta_1}(b_2) + C^*_{\beta_1}(c_2)^* + A_{\beta_2}(a_1) + B_{\beta_2}(b_1) + C_{\beta_2}(c_1)^* \right\}$$

$$\bar{U}^* = -i \left\{ A_{\beta_3}(a_2)^* + B_{\beta_3}(b_2) + C_{\beta_3}(c_2)^* \right\}$$

From these coefficients the constant terms D_1^* and C_2^* of the real and imaginary equations can be determined.

$$D_1^* = R^* + \bar{R}^*\frac{1}{k^2} + \frac{2G}{k}(c)^* + \frac{2F}{k^2}(d)^*$$

$$C_2^* = \frac{1}{k}\left[U^* + \bar{U}^*\frac{1}{k^2} + \frac{2G}{k}(d)^* - 2F(c)^*\right] \qquad (I)$$

where

$$(c)^* = S^* + \bar{S}^*\frac{1}{k^2}$$

$$(d)^* = T^* + \bar{T}^*\frac{1}{k^2}$$

<u>Coefficients of X :</u>

For these coefficients the following terms have to be determined:

$$A_1 = b_1$$

$$B_1 = r_a + (\tfrac{1}{2} + a) r_c$$

$$C_1 = s_a + (\tfrac{1}{2} + a) s_c$$

$$D_1 = -b_2$$

$$A_2^* = B_{\beta_1}^* C_{h_1} - C_{\beta_1}^* B_{h_1}^*$$

$$\bar{A}_2 = -B_{\beta_3} C_{h_1}$$

$$B_2^* = B_{\beta_1}^* - \frac{T_{12}}{2\pi} C_{\beta_1}^* + \frac{T_{11}}{2\pi}(t)^*$$

$$\bar{B}_2 = -B_{\beta_3}$$

$$C_2^* = -\left(B_{\beta_2} - \frac{T_{12}}{2\pi} C_{\beta_2}\right) - \frac{T_{10}}{\pi}(t)^*$$

$$D_2^* = -\left(B_{\beta_2} C_{h_1} - B_{h_1}^* C_{\beta_2}\right)$$

$$A_3^* = A_{\alpha 1} B_{\beta 1}^* - B_{\alpha 1}^* A_{\beta 1}^*$$

$$\bar{A}_3^* = -(A_{\alpha 1} B_{\beta 3} - B_{\alpha 1}^* A_{\beta 3}) - (A_{\alpha 2} B_{\beta 2} - B_{\alpha 2} A_{\beta 2})$$

$$B_3^* = (\tfrac{1}{2} - a)(u)^* + \frac{T_{11}}{2\pi}(v)^*$$

$$\bar{B}_3 = -(\tfrac{1}{2} - a)(q) - \frac{T_{10}}{\pi}(w) - (p)$$

$$C_3^* = -(\tfrac{1}{2} - a)(p) - \frac{T_{11}}{2\pi}(w) - \frac{T_{10}}{\pi}(v)^* - (u)^*$$

$$\bar{C}_3 = (q)$$

$$D_3^* = -\left\{ A_{\alpha 1} B_{\beta 2} - B_{\alpha 1}^* A_{\beta 2} \right\} - \left\{ A_{\alpha 2} B_{\beta 1}^* - B_{\alpha 2} A_{\beta 1}^* \right\}$$

$$\bar{D}_3 = -(A_{\alpha 2} B_{\beta 3} - B_{\alpha 2} A_{\beta 3})$$

where:

$$(t)^* = \frac{T_{12}}{2\pi} C_{h_1} - B_{h_1}^*$$

$$(u)^* = -(\tfrac{1}{2} + a) B_{\beta 1}^* - \frac{T_{12}}{2\pi} A_{\beta 1}^*$$

$$(v)^* = \frac{T_{12}}{2\pi} A_{\alpha 1} + (\tfrac{1}{2} + a) B_{\alpha 1}^*$$

$$(q) = -(\tfrac{1}{2} + a) B_{\beta 3} - \frac{T_{12}}{2\pi} A_{\beta 3}$$

$$(w) = \frac{T_{12}}{2\pi} A_{\alpha 2} + (\tfrac{1}{2} + a) B_{\alpha 2}$$

$$(p) = -(\tfrac{1}{2} + a) B_{\beta 2} - \frac{T_{12}}{2\pi} A_{\beta 2}$$

The next coefficients are:

$$I_1^{R*} = \Omega_\alpha A_2^* + \Omega_\beta A_1 + \Omega_h A_3^*$$

$$J_1^{R*} = \Omega_\alpha \bar{A}_2 + \Omega_h \bar{A}_3^*$$

$$I_2^{*} = \Omega_\alpha B_2^{*} + \Omega_\beta B_1 + \Omega_h B_3^{*}$$

$$J_2 = \Omega_\alpha \bar{B}_2 + \Omega_h \bar{B}_3$$

$$I_3^{*} = \Omega_\alpha C_2^{*} + \Omega_\beta C_1 + \Omega_h C_3$$

$$J_3 = \Omega_h \bar{C}_3$$

$$I_1^{I*} = \Omega_\alpha D_2^{*} + \Omega_\beta D_1 + \Omega_h D_3^{*}$$

$$J_1^{I} = \Omega_h \bar{D}_3$$

where $\Omega_\alpha = 1$

$$\Omega_\beta = \left(\frac{\omega_\beta^{*}}{\omega_\alpha}\right)^2 \left(\frac{r_\beta^{*}}{r_\alpha}\right)^2$$

$$\Omega_h = \left(\frac{\omega_h}{\omega_\alpha}\right)^2 \frac{1}{r_\alpha^2}$$

The coefficients of X for the real and imaginary equations are finally given by:

$$\begin{aligned} \mathfrak{C}_1^{*} &= I_1^{R*} + J_1^{R*} \frac{1}{k^2} + \frac{2G}{k}(a)^{*} + \frac{2F}{k^2}(b)^{*} \\ \mathfrak{B}_2^{*} &= \frac{1}{k}\left[I_1^{I*} + J_1^{I} \frac{1}{k^2} + \frac{2G}{k}(b)^{*} - 2F(a)^{*} \right] \end{aligned} \qquad \text{(II)}$$

where:

$$(a)^{*} = I_2^{*} + J_2 \frac{1}{k^2}$$

$$(b)^{*} = I_3^{*} + J_3 \frac{1}{k^2}$$

<u>Coefficients of X^2:</u>

$$\text{Real Equation} \qquad \mathfrak{B}_1^{*} = C_{X^2}^{R*} + E_1'' \qquad \text{(III)}$$

$$\text{Imaginary Equation} \qquad \mathcal{A}_2 = \Omega_\alpha \Omega_\beta I_{ch} + \Omega_\beta \Omega_h I_{a\alpha} + \Omega_h \Omega_\alpha I_{b\beta}$$

where:

$$C_{x^2}^{R\,*} = -\left[\Omega_n \Omega_\alpha B_{\beta 1}^{*} + \Omega_\beta (\Omega_\alpha C_{h1} + \Omega_n A_{a1})\right]$$

$$B_1'' = \Omega_\alpha \Omega_\beta \left(\frac{2G}{k}\right) + \Omega_\beta \Omega_n R_{a\alpha}'' + \Omega_n \Omega_\alpha R_{b\beta}''$$

<u>Coefficient of Real Equation X^3:</u>

$$\mathcal{A}_1 = \left(\frac{\omega_\beta}{\omega_\alpha^2}\right)^2 \left(\frac{r_\beta}{r_\alpha^2}\right)^2 \qquad \text{(IV)}$$

The following cases are a slightly modified form of those given by Theodorsen(, and are listed here more for convenience than necessity.

<u>B. Torsion-Aileron.</u>

For the two degrees of freedom the two simultaneous equations are reduced to the following form:

Real Equation $\Delta R = \mathcal{A}_1 X^2 + \mathcal{B}_1 X + \mathcal{C}_1 = 0$

Imaginary Equation $\Delta I = \mathcal{A}_2 X + \mathcal{B}_2 = 0$

<u>Real Equation:</u>

Coefficient of X^2: $\mathcal{A}_1 = \Omega_\alpha \Omega_\beta$ where: $\Omega_\alpha = 1$

$$\Omega_\beta = \left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 \left(\frac{r_\beta}{r_\alpha}\right)^2$$

Coefficient of X: $\mathcal{B}_1 = \Omega_\alpha R_{b\beta}^{*} + \Omega_\beta R_{a\alpha}$

Constant: $\mathcal{C}_1 = A_3^{*} + \bar{A}_3^{*} \frac{1}{k^2} + \frac{2G}{k}(a)^{*} + \frac{2F}{k^2}(b)^{*}$

where: $a^* = B_3^* - \bar{B}_3 \frac{1}{k^2}$

$b^* = C_3^* + \bar{C}_3 \frac{1}{k^2}$

Imaginary Equation:

Coefficient of X : $\mathcal{A}_2 = \Omega_\alpha I_{b\beta} + \Omega_\beta I_{a\alpha}$

Constant: $\mathcal{B}_2 = \frac{1}{k}\left[D_3^* + \bar{D}_3 \frac{1}{k^2} + \frac{2G}{k}(b)^* - 2F(a)^* \right]$

C. Flexure-Aileron

Real Equation:

Coefficient of X^2 : $\mathcal{A}_1 = \Omega_\beta \Omega_h$ where: $\Omega_\alpha = 1$

$\Omega_\beta = \left(\frac{\omega_\beta}{\omega_h}\right)^2 r_\beta^2$

Coefficient of X : $\mathcal{B}_1^* = \Omega_\beta R_{ch} + \Omega_h R_{b\beta}^*$

Constant : $\mathcal{C}_1 = A_2^* + \bar{A}_2 \frac{1}{k} + (a)^* \frac{2G}{k} + C_2^* \frac{2F}{k^2}$

where: $(a)^* = B_2^* + \bar{B}_2 \frac{1}{k^2}$

Imaginary Equation:

Coefficient of X : $\mathcal{A}_2 = \Omega_\beta I_{ch} + \Omega_h I_{b\beta}$

Constant : $\mathcal{B}_2 = \frac{1}{k}\left[D_2^* + C_2^* \frac{2G}{k} - (a)^* 2F \right]$

D. Flexure-Torsion.

Real Equation:

Coefficient of X^2: $\mathcal{A}_1 = \Omega_h \Omega_\alpha$ where: $\Omega_\alpha = 1$

$$\Omega_h = \left(\frac{\omega_h}{\omega_\alpha}\right)^2 \frac{1}{r_\alpha^2}$$

Coefficient of X : $\mathcal{B}_1 = \Omega_h R_{a\alpha} + \Omega_\alpha R_{ch}$

Constant : $\mathcal{C}_1 = A_1 + B_1 \frac{2G}{k} + C_1 \frac{2F}{k^2}$

Imaginary Equation:

Coefficient of X : $\mathcal{A}_2 = \Omega_h I_{a\alpha} + \Omega_\alpha I_{ch}$

Constant: $\mathcal{B}_2 = \frac{1}{k}\left[D_1 + C_1 \frac{2G}{k} - B_1 2F \right]$

NUMERICAL EXAMPLE

As already stated in the paper, the following Cases were investigated:

Case I,	which corresponds to	$\frac{\sigma}{\tau} = -\ 10\%$
Case II,	" " "	$\frac{\sigma}{\tau} = -\ 5\%$
Case III,	" " "	$\frac{\sigma}{\tau} = +\ 1.4\%$
Case IV,	" " "	$\frac{\sigma}{\tau} = +\ 10\%$
Case V,	" " "	$\frac{\sigma}{\tau} = +\ 20\%$
Case VI,	" " "	$\frac{\sigma}{\tau} = +\ 40\%$

The following wing parameters, corresponding to Case III, were used:

$a = -\ 0.34$ $\qquad \kappa = \frac{1}{16.5}$

$c = 0.60$ $\qquad M = 0.868$

$\tau = 0.20$ $\qquad m_o = 0.1112$

$b = 2.65$

$x_\alpha = 0.12$ $\qquad r_\alpha^2 = 0.17439$

$x_\beta^* = 0.0007173$ $\qquad (r_\beta^2)^* = 0.0015524$

$$\omega = 100 \text{ rad/sec.}$$

$$\omega_\alpha = 240 \text{ rad/sec.}$$

$$\omega_\beta^* = 105 \text{ rad/sec.}$$

The variation of the three aileron parameters is given in Table I, also Fig. 2, while in Tables II, III, IV and V the different coefficients for the Flexure - Torsion - Aileron Case are tabulated.

TABLE I.

Variation of $(\frac{m'}{M})^*$, x_β^*, $(r_\beta^2)^*$, ω_β^* parameters.

	I	II	III	IV	V	VI
$\frac{\sigma}{\tau}$	-10%	-5%	+1.4%	+10%	+20%	+40%
$(\frac{m'}{M})^*$	.057105	.040898	0	$\overline{.031369}$	$\overline{.052782}$	$\overline{.075641}$
x_β^*	$\overline{.008993}$	$\overline{.003373}$	$\overline{.000717}$	.003854	.005996	.008301
$(r_\beta^2)^*$	.002524	.001961	.001552	.001239	.001025	.000794
ω_β^*	82.35	93.4	105	117.5	129.2	146.8

TABLE II.

	I	II	III	IV	V	VI
A_{α_1}	3.118035	3.118035	3.118035	3.118035	3.118035	3.118035
A_{α_2}	.84	.84	.84	.84	.84	.84
$A^{*}_{\beta_1}$	$\overline{.}08030$	$\overline{.}0025976$	$\overline{.}0542868$	.097764	.127443	.159402
A_{β_2}	.228259	.228259	.228259	.228259	.228259	.228259
A_{β_3}	.40744	.40744	.40744	.40744	.40744	.40744
A_{h_1}	2.32	2.32	2.32	2.32	2.32	2.32
$B^{*}_{\alpha_1} = A_{\beta_1}$	$\overline{.}08030$	$\overline{.}0025976$	$\overline{.}0542868$	.097764	.127443	.159402
B_{α_2}	.04008	.04008	.04008	.04008	.04008	.04008
$B^{*}_{\beta_1}$	.0438678	.0345931	.0278446	.0226686	.0191359	.0153308
B_{β_2}	.02118	.02118	.02118	.02118	.02118	.02118
B_{β_3}	.01651	.01651	.01651	.01651	.01651	.01651
$B^{*}_{h_1}$	$\overline{.}125168$	$\overline{.}0324263$	$\overline{.}0350555$	.0868143	.122146	.160193
$C_{\alpha_1} = A_{h_1}$	2.32	2.32	2.32	2.32	2.32	2.32
C_{α_2}	1	1	1	1	1	1
$C^{*}_{\beta_1} = B_{h_1}$	$\overline{.}125168$	$\overline{.}0324263$	$\overline{.}0350555$	.0868143	.122146	.160193
C_{β_2}	.14238	.14238	.14238	.14238	.14238	.14238
C_{β_3}	0	0	0	0	0	0
C_{h_1}	17.5	17.5	17.5	17.5	17.5	17.5

TABLE III.

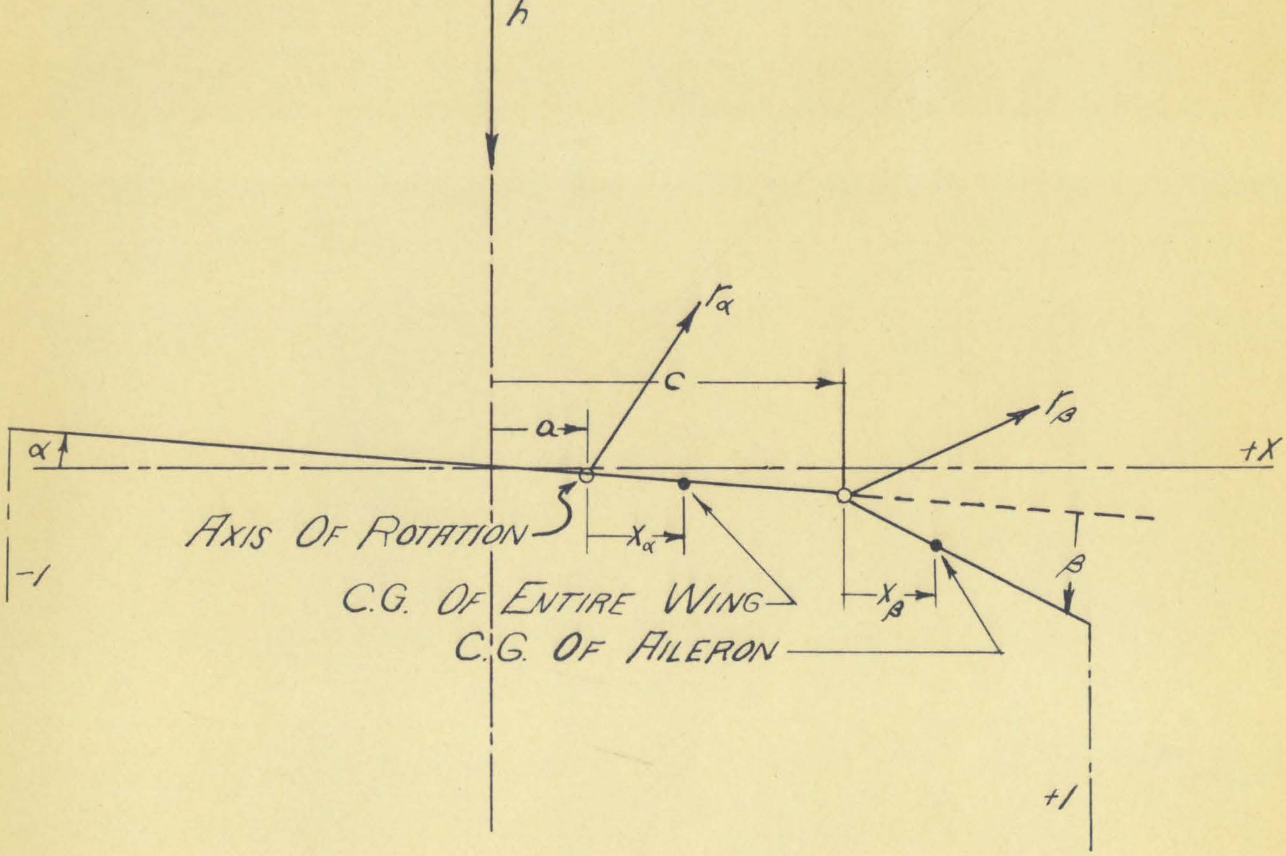
	I	II	III	IV	V	VI
R^*	$\bar{2}.042508$	$\bar{1}.698385$	$\bar{1}.322915$	$\bar{.}863537$	$\bar{.}682647$	$\bar{.}347830$
$\bar{R}^*$	1.3680	0.910284	.577238	.321780	.147401	$\bar{.}040375$
S^*	$\bar{.}024479$	$\bar{.}034075$	$\bar{.}0410546$	$\bar{.}046411$	$\bar{.}050064$	$\bar{.}054000$
$\bar{S}^*$	.0179802	$\bar{.}0$[illegible]9728	$\bar{.}0298908$	$\bar{.}045356$	$\bar{.}055913$	$\bar{.}067282$
T^*	.047553	.090021	.131732	.169897	.199027	.233185
$\bar{T}^*$	.178107	.140320	.112822	.091737	.077341	.061839
U^*	1.909880	1.473836	1.156563	.913202	.747086	.568201
$\bar{U}^*$	.132383	.094596	.067102	.046013	.031617	.016115

TABLE IV.

	I	II	III	IV	V	VI
A_1	49.1833	49.1833	49.1833	49.1833	49.1833	49.1833
B_1	$\overline{.81156}$	$\overline{.81156}$	$\overline{.81156}$	$\overline{.81156}$	$\overline{.81156}$	$\overline{.81156}$
C_1	4.12	4.12	4.12	4.12	4.12	4.12
D_1	$\overline{12.38}$	$\overline{12.38}$	$\overline{12.38}$	$\overline{12.38}$	$\overline{12.38}$	$\overline{12.38}$
A_2^*	.752020	.604328	.486052	.382164	.318958	.242627
$\bar{A}_2$	$\overline{.288925}$	$\overline{.288925}$	$\overline{.288925}$	$\overline{.288925}$	$\overline{.288925}$	$\overline{.288925}$
B_2^*	.079836	.0561771	.0389622	.0257585	.0167459	.0070397
$\bar{B}_2$	$\overline{.01651}$	$\overline{.01651}$	$\overline{.01651}$	$\overline{.01651}$	$\overline{.01651}$	$\overline{.01651}$
C_2^*	$\overline{.150289}$	$\overline{.098297}$	$\overline{.0621948}$	$\overline{.0337367}$	$\overline{.0143107}$	$\overline{.006083}$
D_2^*	$\overline{.388471}$	$\overline{.375267}$	$\overline{.365659}$	$\overline{.358290}$	$\overline{.352259}$	$\overline{.347842}$
A_3^*	.130334	.107837	.0838735	.0611238	.0434248	.022393
$\bar{A}_3^*$	$\overline{.092838}$	$\overline{.0610879}$	$\overline{.0380024}$	$\overline{.020288}$	$\overline{.0061956}$	$\overline{.0048258}$
B_3^*	$\overline{.0044282}$	$\overline{.0017439}$	$\overline{.0002093}$	$\overline{.0017073}$	$\overline{.0027299}$	$\overline{.0038311}$
$\bar{B}_3$	.002773	.002773	.002773	.002773	.002773	.002773
C_3^*	$\overline{.0049865}$	$\overline{.0028550}$	$\overline{.0085606}$	$\overline{.0123368}$	$\overline{.0159241}$	$\overline{.019141}$
$\bar{C}_3$	$\overline{.0052329}$	$\overline{.0052329}$	$\overline{.0052329}$	$\overline{.0052329}$	$\overline{.0052329}$	$\overline{.0052329}$
D_3^*	$\overline{.124437}$	$\overline{.0957416}$	$\overline{.0748623}$	$\overline{.0588477}$	$\overline{.0479163}$	$\overline{.0361441}$
$\bar{D}_3$	.0024618	.0024618	.0024618	.0024618	.0024618	.0024618

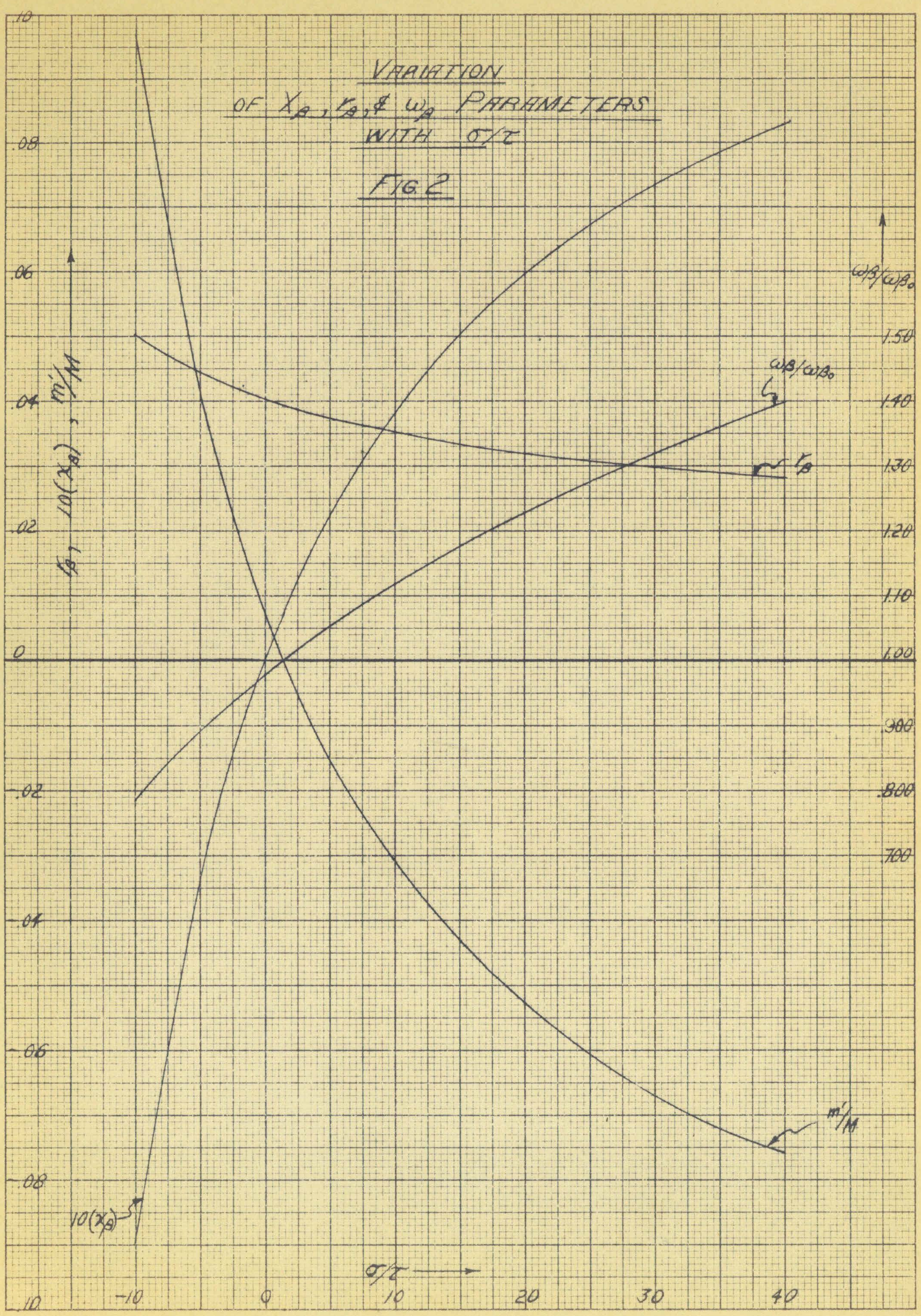
TABLE V.

Cases	I	II	III	IV	V	VI
$C^R_{x^2}$ *	-.0787697	-.0695363	-.0628178	-.0576648	-.0541478	-.0503597
I^R_1 *	.965553	.795484	.653331	.533795	.446969	.348699
J^R_1 *	-.381350	-.349751	-.326758	-.309123	-.297034	-.284121
I_2 *	.074045	.053059	.037788	.026076	.018081	.009471
J_2	-.0137493	-.0137493	-.0137493	-.0137493	-.0137493	-.0137493
I_3 *	-.138307	-.095121	-.0636991	-.039598	-.023146	-.005955
J_3	-.0052096	-.0052096	-.0052096	-.0052096	-.0052096	-.0052096
I^I_1 *	-.533442	-.491671	-.461276	-.437964	-.422050	-.404913
J^I_1	.0024508	.0024508	.0024508	.0024508	.0024508	.0024508

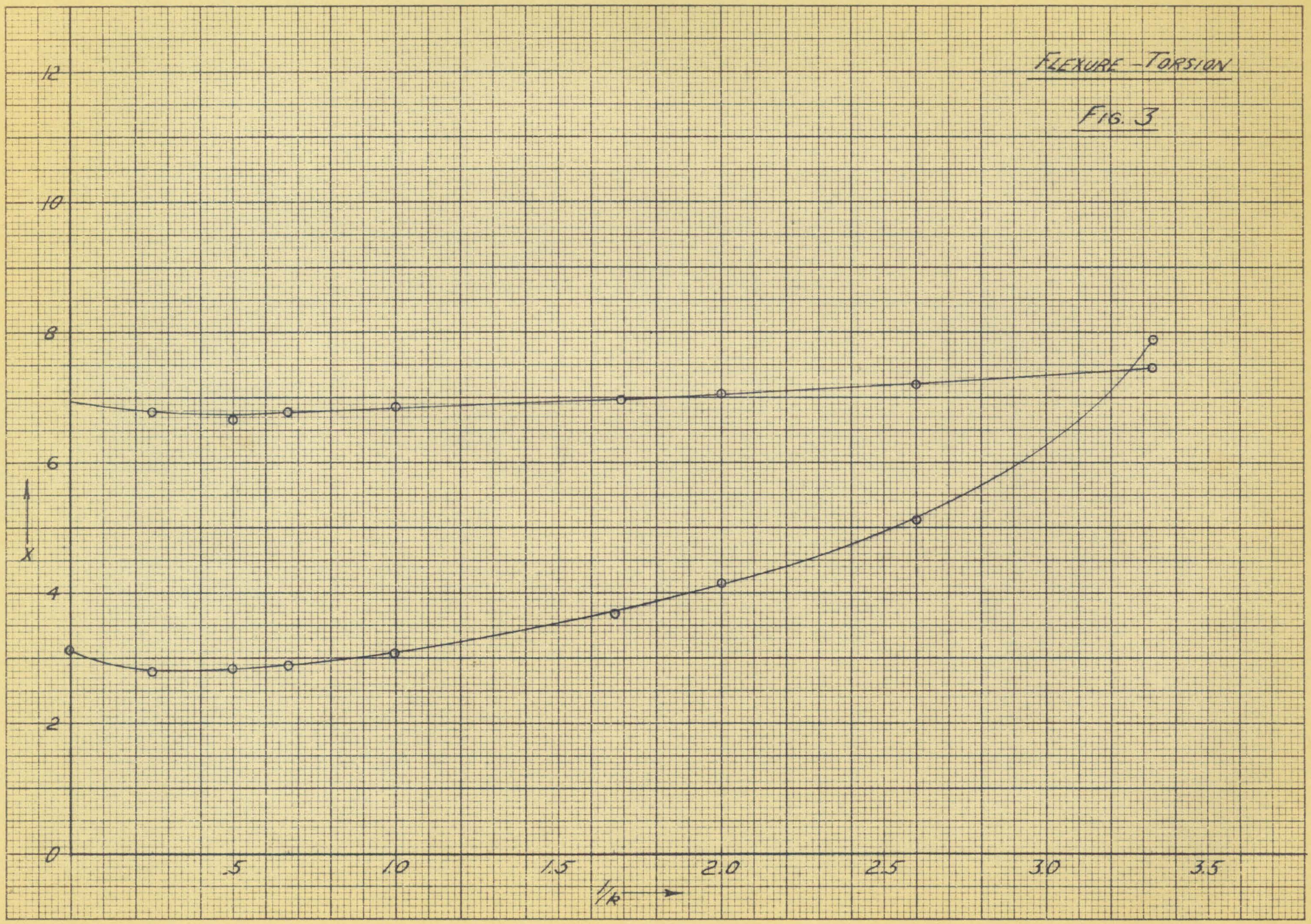


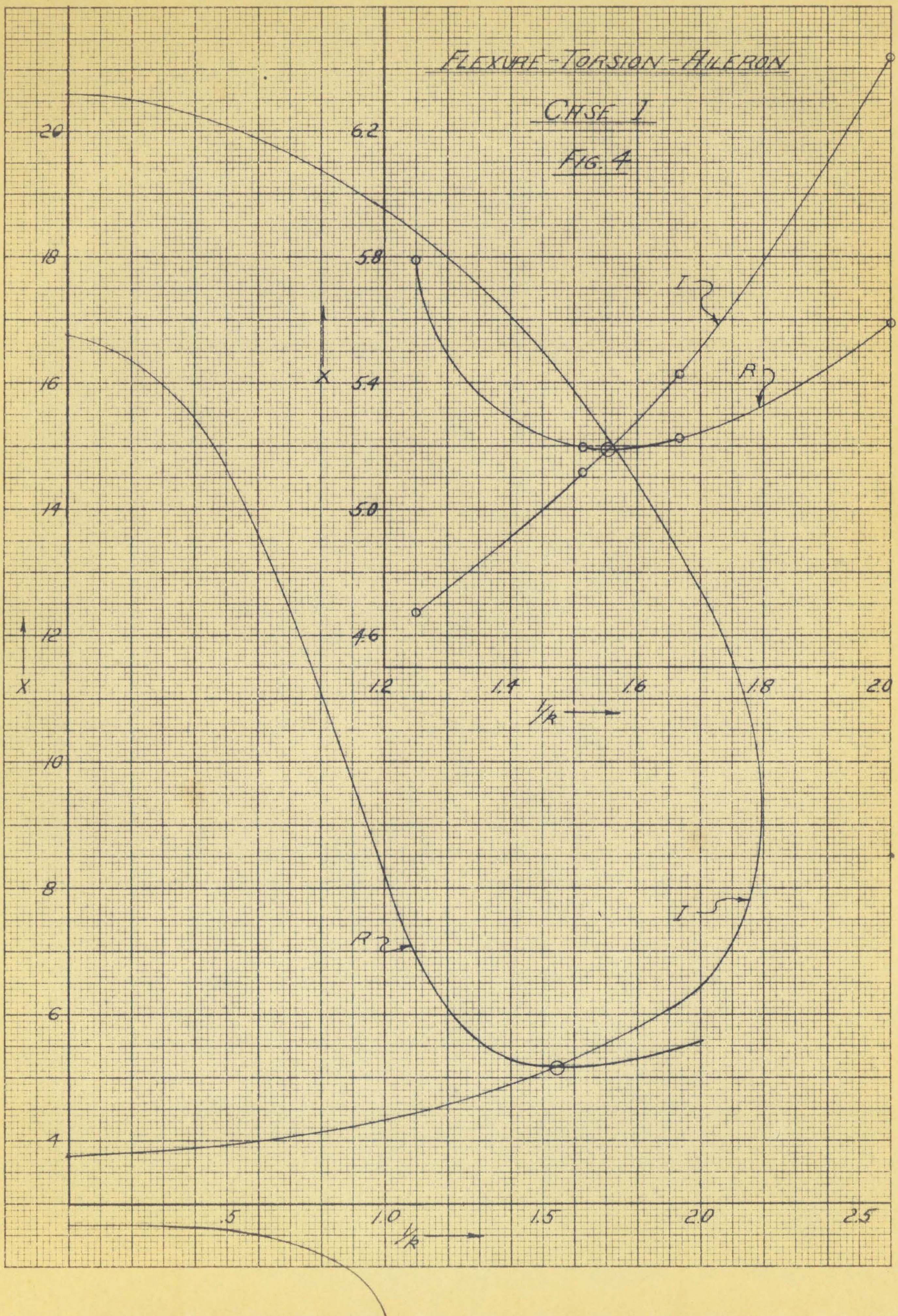
h
r_α
c
r_β
a
α
+X
Axis Of Rotation
x_α
β
-1
C.G. Of Entire Wing
x_β
C.G. Of Aileron
+1

Fig 1

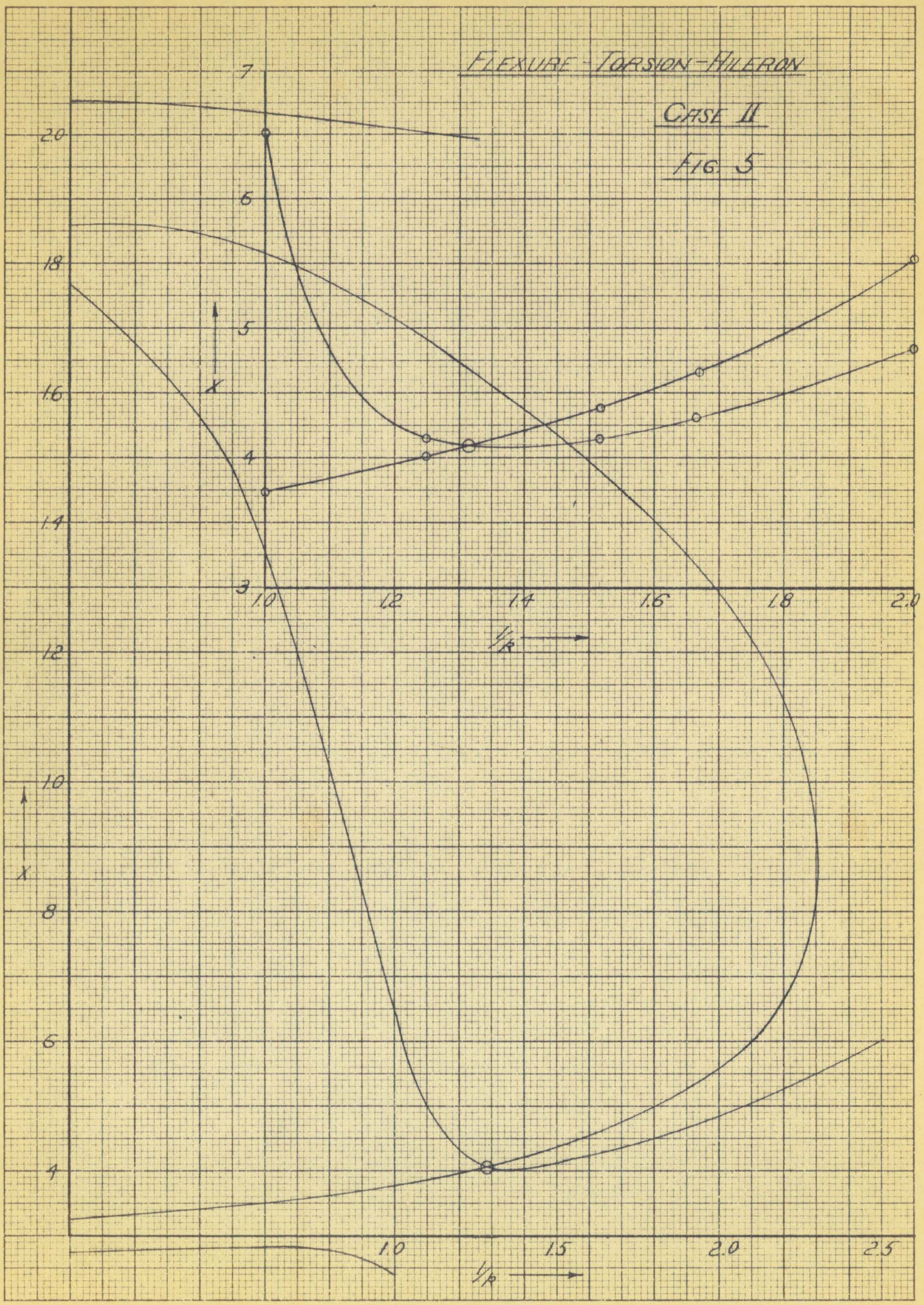


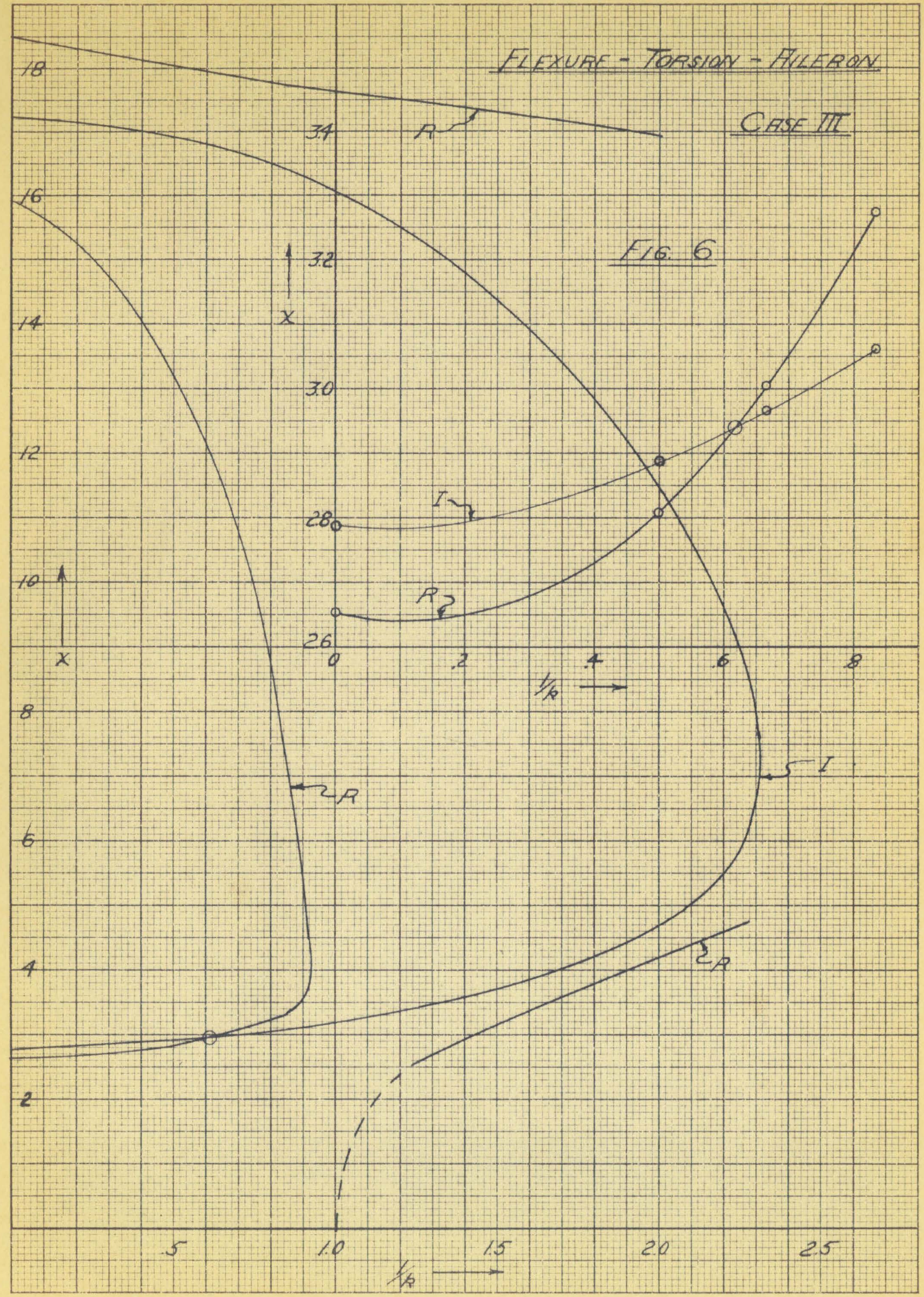
VARIATION
OF X_β, r_β, & ω_β PARAMETERS
WITH σ/τ
FIG 2
r_β, $10(X_\beta)$, m'/M
$\omega_\beta/\omega_{\beta_0}$
$10(X_\beta)$
m'/M
σ/τ
.10
.08
.06
.04
.02
0
-.02
-.04
-.06
-.08
-.10
1.50
1.40
1.30
1.20
1.10
1.00
.900
.800
.700
-10
0
10
20
30
40


FLEXURE - TORSION
FIG. 3
12
10
8
6
4
2
0
X
.5
1.0
1.5
2.0
2.5
3.0
3.5
1/k



FLEXURE-TORSION-AILERON
CASE I
FIG. 4
20
18
16
14
12
10
8
6
4
X
6.2
5.8
5.4
5.0
4.6
1.2
1.4
1.6
1.8
2.0
1/k
I
R
.5
1.0
1.5
2.0
2.5

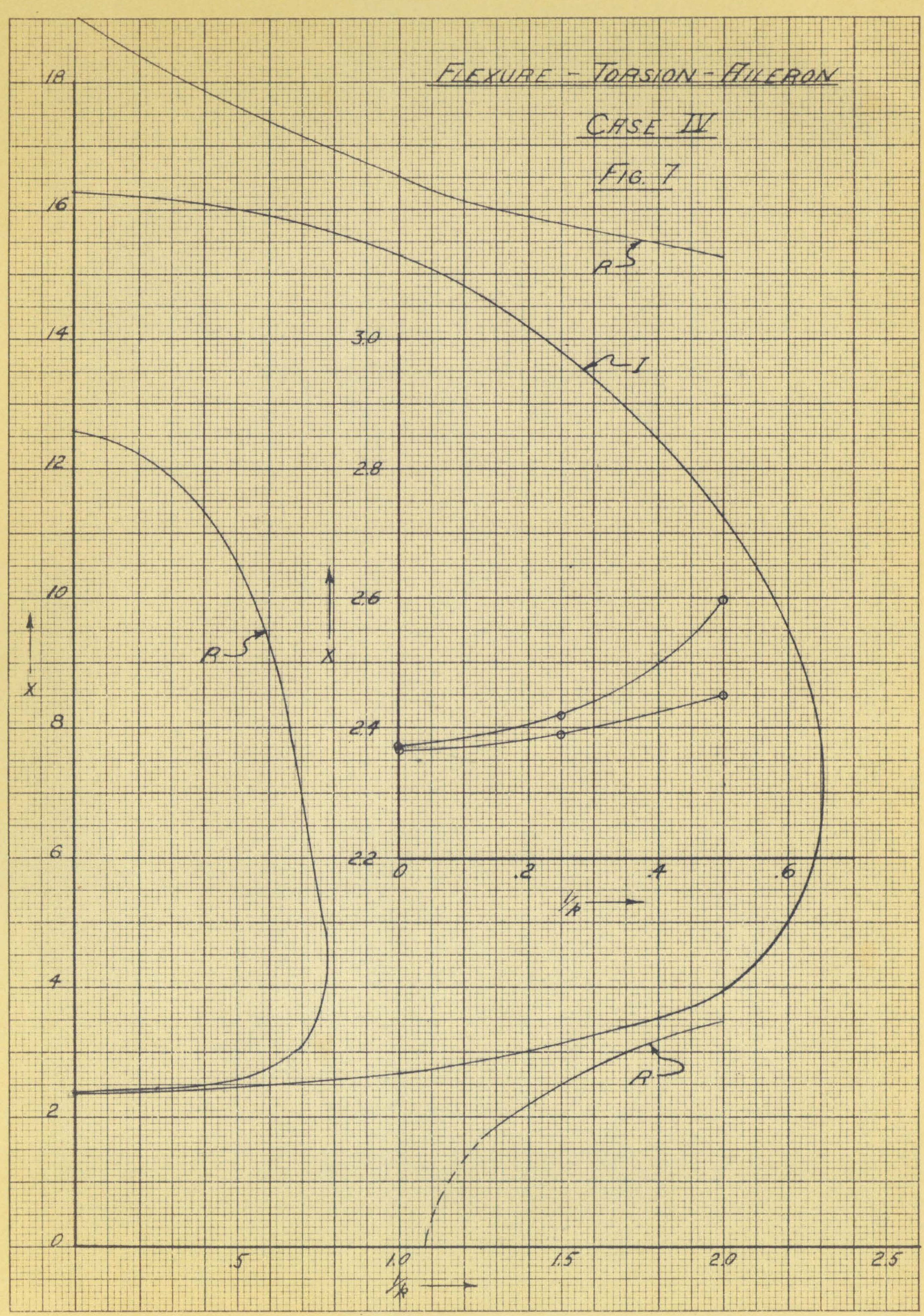

FLEXURE-TORSION-AILERON
CASE II
FIG. 5
7
6
5
4
3
X
20
18
16
14
12
10
8
6
4
X
1.0
1.2
1.4
1.6
1.8
2.0
1/R
1.0
1.5
2.0
2.5
1/R

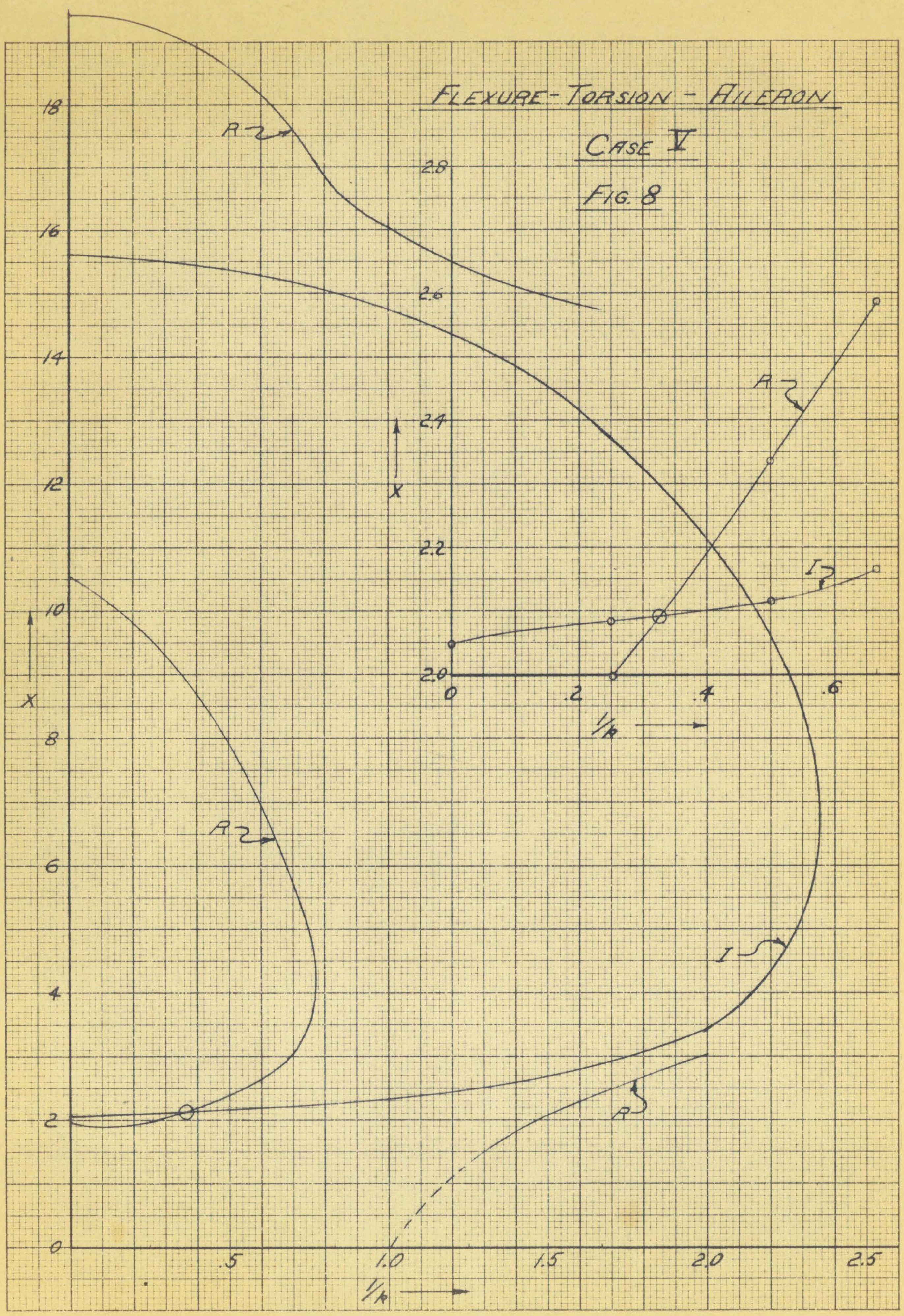

FLEXURE - TORSION - AILERON
CASE III
FIG. 6
R
I
R
I
R
R
X
X
1/R
1/R
18
16
14
12
10
8
6
4
2
3.4
3.2
3.0
2.8
2.6
0
.2
.4
.6
.8
.5
1.0
1.5
2.0
2.5

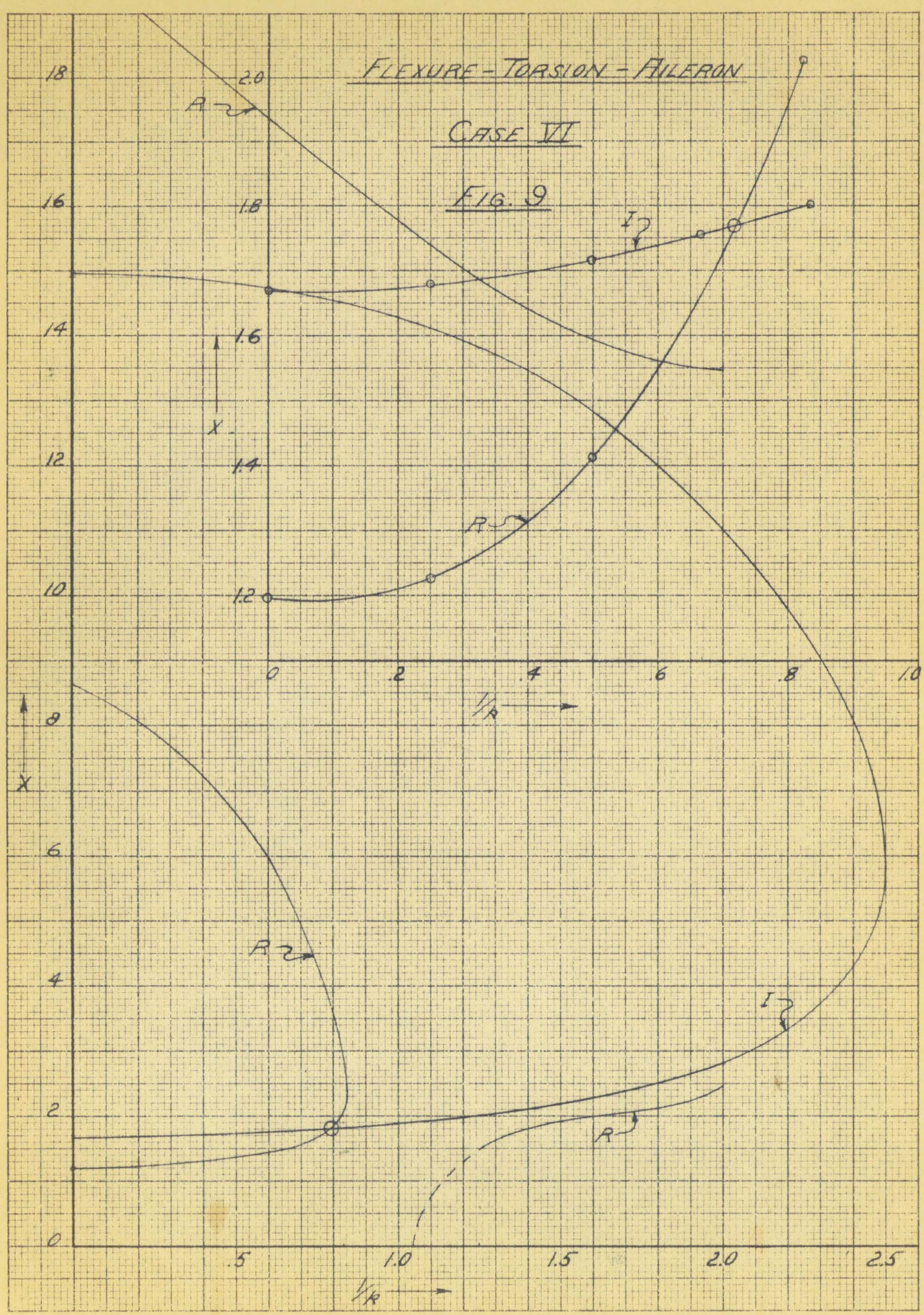
FLEXURE - TORSION - AILERON

CASE IV

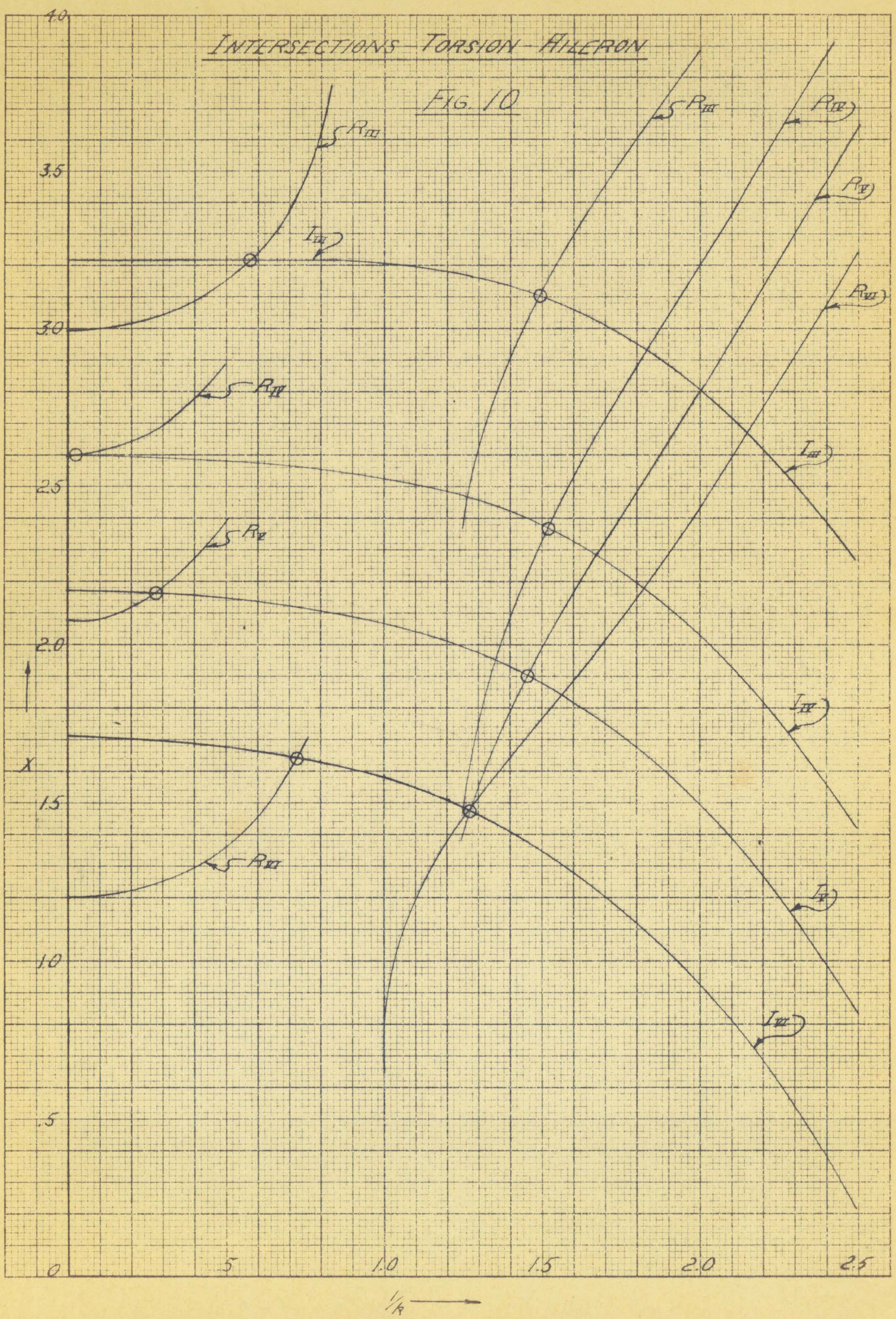
FIG. 7

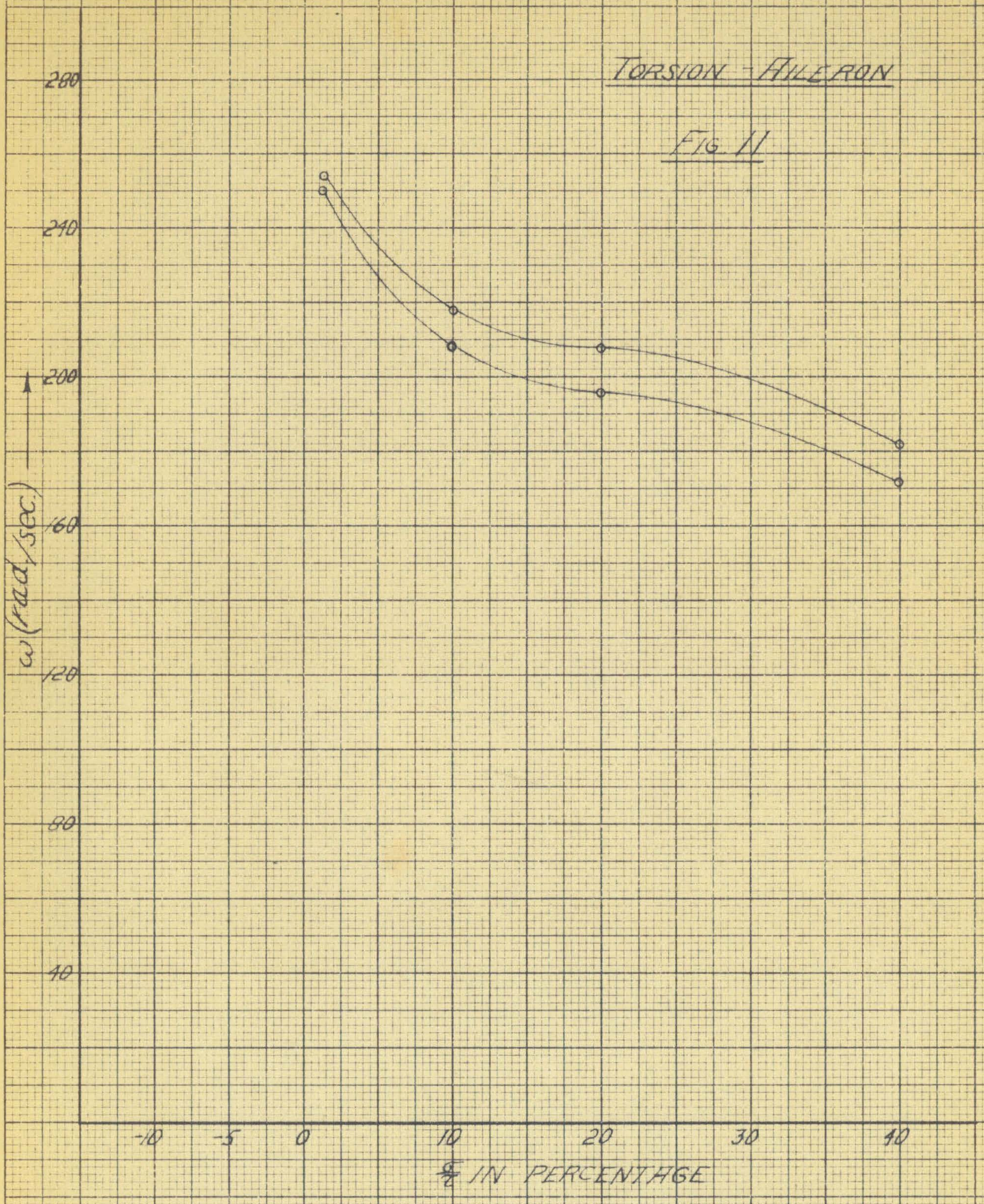



FLEXURE - TORSION - AILERON
CASE V
FIG. 8
R
I
X
1/k
18
16
14
12
10
8
6
4
2
0
.5
1.0
1.5
2.0
2.5
2.8
2.6
2.4
2.2
2.0
.2
.4
.6



FLEXURE - TORSION - AILERON
CASE VI
FIG. 9
R
I
R
X
1/R
1.2
1.4
1.6
1.8
2.0
0
.2
.4
.6
.8
1.0
18
16
14
12
10
8
6
4
2
0
.5
1.0
1.5
2.0
2.5
R
I
R
X
1/R


INTERSECTIONS - TORSION - AILERON
FIG. 10
4.0
3.5
3.0
2.5
2.0
1.5
1.0
.5
0
.5
1.0
1.5
2.0
2.5
X
1/R
R_{III}
R_{IV}
R_V
R_{VI}
I_{III}
I_{IV}
I_V
I_{VI}



FLUTTER FREQUENCY VS. C.G. LOCATION

PLOT OF μ VS. σ/c

FIG. 12