Appendix C

Multitaper measurements for adjoint tomography

C.1 Introduction

A multitaper measurements uses a set of optimal tapers design to extract frequency-dependent measurements of traveltime and amplitude differences. Theory and examples of multitaper measurements can be found in *Zhou et al.* (2004, 2005); *Ekström et al.* (1997); *Laske and Masters* (1996); *Thomson* (1982), and also *Percival and Walden* (1993, p. 333–347). The objective of this Appendix is to state the multitaper misfit functions (both traveltime and amplitude), and then to derive the corresponding adjoint source, ala *Tromp et al.* (2005).

C.2 Misfit functions, measurements, and adjoint sources

The conventions for the measurement, the Fourier transform, and the transfer function used for multitaper measurements are related. First we specify the measurement convention (Figure C.1), then we specify the Fourier convention, and these determine the convention for the transfer function (Section C.3.2).

C.2.1 The misfit function and measurement convention

Our misfit functions consists of a set of discrete time windows covering the seismic dataset of S earthquake sources, with N_s time windows for source index s. The total number of measurements is

$$N = \sum_{s=1}^{S} N_s \; . \tag{C.1}$$

For traveltime and amplitude tomography we seek to minimize the objective functions

$$F_{\mathrm{P}}(\mathbf{m}) = \tfrac{1}{2} \sum_{s=1}^{S} \sum_{p=1}^{N_s} \frac{1}{F_{sp}} \int_{-\infty}^{\infty} W_{sp}(\omega) \left[\frac{\tau_{sp}^{\mathrm{obs}}(\omega) - \tau_{sp}(\omega, \mathbf{m})}{\sigma_{\mathrm{P}sp}(\omega)} \right]^2 \mathrm{d}\omega \; , \tag{C.2}$$

$$F_{\mathrm{Q}}(\mathbf{m}) = \tfrac{1}{2} \sum_{s=1}^{S} \sum_{p=1}^{N_s} \frac{1}{F_{sp}} \int_{-\infty}^{\infty} W_{sp}(\omega) \left[\frac{\ln A_{sp}^{\mathrm{obs}}(\omega) - \ln A_{sp}(\omega, \mathbf{m})}{\sigma_{\mathrm{Q}sp}(\omega)} \right]^2 \mathrm{d}\omega \; , \tag{C.3}$$

where p is the index for measurement window "pick," and P and Q are labels denoting measures of traveltime and amplitude, respectively. For example, $\tau_{sp}^{\mathrm{obs}}(\omega) - \tau_{sp}(\omega, \mathbf{m})$ represents the frequency-dependent traveltime difference between synthetics and data for one time-windowed waveform on a single seismogram for source s. The frequency-dependent uncertainty associated with the traveltime measurement is estimated by $\sigma_{\mathrm{P}sp}(\omega)$. The function $W_{sp}(\omega)$ denotes a windowing filter whose width corresponds to the frequency range over which the measurements are assumed reliable. A normalization factor G_{sp} is defined as[1]

$$G_{sp} = \int_{-\infty}^{\infty} W_{sp}(\omega) \, \mathrm{d}\omega. \tag{C.4}$$

We can incorporate the measurement uncertainty and normalization factor into the frequency filter by defining

$$W_{\mathrm{P}sp}(\omega) \equiv \frac{W_{sp}(\omega)}{G_{sp} \, \sigma_{\mathrm{P}sp}^2(\omega)} \; , \tag{C.5}$$

$$W_{\mathrm{Q}sp}(\omega) \equiv \frac{W_{sp}(\omega)}{G_{sp} \, \sigma_{\mathrm{Q}sp}^2(\omega)} \; , \tag{C.6}$$

[1] In practice we define our measurement window according to positive angular frequencies only. Thus we can write Equation (C.4) as

$$G_{sp} = \int_{-\infty}^{\infty} W_{sp}(\omega) \, \mathrm{d}\omega = 2 \int_{0}^{\infty} W_{sp}(\omega) \mathrm{d}\omega,$$

where $W_{sp}(\omega)$ is defined over positive frequencies only.

and then the misfit functions (Eqs. C.2 and C.3) become

$$F_{\mathrm{P}}(\mathbf{m}) = \tfrac{1}{2}\sum_{s=1}^{S}\sum_{p=1}^{N_s}\int_{-\infty}^{\infty} W_{\mathrm{P}sp}(\omega)\left[\tau_{sp}^{\mathrm{obs}}(\omega) - \tau_{sp}(\omega,\mathbf{m})\right]^2 \mathrm{d}\omega\ , \tag{C.7}$$

$$F_{\mathrm{Q}}(\mathbf{m}) = \tfrac{1}{2}\sum_{s=1}^{S}\sum_{p=1}^{N_s}\int_{-\infty}^{\infty} W_{\mathrm{Q}sp}(\omega)\left[\ln A_{sp}^{\mathrm{obs}}(\omega) - \ln A_{sp}(\omega,\mathbf{m})\right]^2 \mathrm{d}\omega\ . \tag{C.8}$$

The variations of Equations (C.7) and (C.8) are given by

$$\delta F_{\mathrm{P}}(\mathbf{m}) = -\tfrac{1}{2}\sum_{s=1}^{S}\sum_{p=1}^{N_s}\int_{-\infty}^{\infty} W_{\mathrm{P}sp}(\omega)\,\Delta\tau_{sp}(\omega,\mathbf{m})\,\delta\tau_{sp}(\omega,\mathbf{m})\,\mathrm{d}\omega\ , \tag{C.9}$$

$$\delta F_{\mathrm{Q}}(\mathbf{m}) = -\tfrac{1}{2}\sum_{s=1}^{S}\sum_{p=1}^{N_s}\int_{-\infty}^{\infty} W_{\mathrm{Q}sp}(\omega)\,\Delta\ln A_{sp}(\omega,\mathbf{m})\,\delta\ln A_{sp}(\omega,\mathbf{m})\,\mathrm{d}\omega\ , \tag{C.10}$$

where

$$\Delta\tau_{sp}(\omega,\mathbf{m}) \equiv \tau_{sp}^{\mathrm{obs}}(\omega) - \tau_{sp}(\omega,\mathbf{m})\ , \tag{C.11}$$

$$\Delta\ln A_{sp}(\omega,\mathbf{m}) \equiv \ln A_{sp}^{\mathrm{obs}}(\omega) - \ln A_{sp}(\omega,\mathbf{m})\ , \tag{C.12}$$

are the measured traveltime difference and amplitude difference between data and synthetics, and $\delta\tau(\omega,\mathbf{m})$ and $\delta\ln A(\omega,\mathbf{m})$ are the traveltime and amplitude perturbations with respect to changes in the model parameters. The conventions in Equations (C.11) and (C.12) are such that $\Delta\tau(\omega) > 0$ corresponds to a delay in the data, i.e., the data at frequency ω arrive late with respect to the synthetics at frequency ω (Figure C.1). Similarly, $\Delta\ln A(\omega) > 0$ corresponds to an amplification of the data with respect to the synthetics, for frequency ω. These are the same conventions used in defining the cross-correlation measurements of (*Dahlen et al.*, 2000; *Dahlen and Baig*, 2002).

Reduction for a single measurement ($N = 1$)

The notation is simpler if we consider a single event, a single receiver, a single component, and a single phase. In that case, the misfit functions are

$$F_{\mathrm{P}}(\mathbf{m}) = \tfrac{1}{2}\int_{-\infty}^{\infty} W_{\mathrm{P}}(\omega)\left[\tau^{\mathrm{obs}}(\omega) - \tau(\omega,\mathbf{m})\right]^2 \mathrm{d}\omega\ , \tag{C.13}$$

$$F_{\mathrm{Q}}(\mathbf{m}) = \tfrac{1}{2}\int_{-\infty}^{\infty} W_{\mathrm{Q}}(\omega)\left[\ln A^{\mathrm{obs}}(\omega) - \ln A(\omega,\mathbf{m})\right]^2 \mathrm{d}\omega\ , \tag{C.14}$$

where

$$W_{\rm P}(\omega) \equiv \frac{W(\omega)}{G\,\sigma_{\rm P}^2(\omega)}\,, \tag{C.15}$$

$$W_{\rm Q}(\omega) \equiv \frac{W(\omega)}{G\,\sigma_{\rm Q}^2(\omega)}\,. \tag{C.16}$$

The variations of Equations (C.13) and (C.14) are given by

$$\delta F_{\rm P}(\mathbf{m}) = -\int_{-\infty}^{\infty} W_{\rm P}(\omega)\,\Delta\tau(\omega,\mathbf{m})\,\delta\tau(\omega,\mathbf{m})\,{\rm d}\omega\,, \tag{C.17}$$

$$\delta F_{\rm Q}(\mathbf{m}) = -\int_{-\infty}^{\infty} W_{\rm Q}(\omega)\,\Delta\ln A(\omega,\mathbf{m})\,\delta\ln A(\omega,\mathbf{m})\,{\rm d}\omega\,, \tag{C.18}$$

where

$$\Delta\tau(\omega,\mathbf{m}) \equiv \tau^{\rm obs}(\omega) - \tau(\omega,\mathbf{m})\,, \tag{C.19}$$

$$\Delta\ln A(\omega,\mathbf{m}) \equiv \ln A^{\rm obs}(\omega) - \ln A(\omega,\mathbf{m})\,. \tag{C.20}$$

Reduction for frequency-independent measurements

For frequency-independent measurements (and uncertainties), we have

$$\Delta\tau(\omega) = \tau^{\rm obs} - \tau(\mathbf{m})\,, \tag{C.21}$$

$$\Delta\ln A(\omega) = \ln A^{\rm obs} - \ln A(\mathbf{m})\,, \tag{C.22}$$

$$\sigma_{\rm P}(\omega) = \sigma_{\rm P}\,, \tag{C.23}$$

$$\sigma_{\rm Q}(\omega) = \sigma_{\rm Q}\,. \tag{C.24}$$

The misfit functions (Eqs. C.13 and C.14) become

$$F_{\rm P}(\mathbf{m}) = \tfrac{1}{2}\left[\tau^{\rm obs} - \tau(\mathbf{m})\right]^2 \int_{-\infty}^{\infty} W_{\rm P}(\omega)\,{\rm d}\omega = \tfrac{1}{2}\left[\frac{\tau^{\rm obs} - \tau(\mathbf{m})}{\sigma_{\rm P}}\right]^2 \tag{C.25}$$

$$F_{\rm Q}(\mathbf{m}) = \tfrac{1}{2}\left[\ln A^{\rm obs} - \ln A(\mathbf{m})\right]^2 \int_{-\infty}^{\infty} W_{\rm Q}(\omega)\,{\rm d}\omega = \tfrac{1}{2}\left[\frac{\ln A^{\rm obs} - \ln A(\mathbf{m})}{\sigma_{\rm Q}}\right]^2 \tag{C.26}$$

which are the traveltime and amplitude cross-correlation misfit functions, $F_{\rm T}(\mathbf{m})$ and $F_{\rm A}(\mathbf{m})$, shown in *Tromp et al.* (2005), but also including measurement uncertainties.

The variations in Equations (C.25) and (C.26) are then

$$\delta F_{\mathrm{P}}(\mathbf{m}) = \left[-\frac{\Delta\tau(\mathbf{m})}{\sigma_{\mathrm{P}}^2}\right]\delta\tau(\mathbf{m}) \,, \tag{C.27}$$

$$\delta F_{\mathrm{Q}}(\mathbf{m}) = \left[-\frac{\Delta\ln A(\mathbf{m})}{\sigma_{\mathrm{Q}}^2}\right]\delta\ln A(\mathbf{m}) \,. \tag{C.28}$$

Uncertainty estimate (σ) based on cross-correlation measurements

Suppose we measure traveltime and amplitude anomalies based on the cross-correlation

$$\Gamma(\tau) = \int s(t-\tau)\, d(t)\, \mathrm{d}t, \tag{C.29}$$

where d denotes the observed seismogram and s the synthetic. Let δT denote the cross-correlation traveltime anomaly and $\delta\ln A$ the amplitude anomaly. We seek to determine σ_{T} and σ_A for these quantities. Therefore we write

$$d(t) = (1+\delta\ln A \pm \sigma_A)\, s(t-\delta T \pm \sigma_{\mathrm{T}}) \,. \tag{C.30}$$

Expanding the second term to first order, we obtain

$$\begin{aligned} d(t) &\approx (1+\delta\ln A \pm \sigma_A)\,\left[s(t-\delta T) \pm \sigma_{\mathrm{T}}\,\dot{s}(t-\delta T)\right] \\ &= (1+\delta\ln A \pm \sigma_A)\, s(t-\delta T) \pm \sigma_{\mathrm{T}}\,(1+\delta\ln A \pm \sigma_A)\,\dot{s}(t-\delta T) \\ &= (1+\delta\ln A)\, s(t-\delta T) \pm \sigma_A\, s(t-\delta T) \pm \sigma_{\mathrm{T}}\,(1+\delta\ln A)\,\dot{s}(t-\delta T) \pm \sigma_{\mathrm{T}}\,\sigma_A\,\dot{s}(t-\delta T). \end{aligned}$$

Thus, to first order in σ_{T} and σ_A, this may be written as

$$d(t) - (1+\delta\ln A)\, s(t-\delta T) = \pm\sigma_{\mathrm{T}}(1+\delta\ln A)\,\dot{s}(t-\delta T) \pm \sigma_A\, s(t-\delta T). \tag{C.31}$$

If we assume the errors are uncorrelated, we find that

$$\sigma_{\mathrm{T}}^2 = \frac{\int \left[d(t) - (1+\delta\ln A)\, s(t-\delta T)\right]^2 \mathrm{d}t}{\int \left[(1+\delta\ln A)\,\dot{s}(t-\delta T)\right]^2 \mathrm{d}t} \,, \tag{C.32}$$

$$\sigma_A^2 = \frac{\int \left[d(t) - (1+\delta\ln A)\, s(t-\delta T)\right]^2 \mathrm{d}t}{\int \left[s(t-\delta T)\right]^2 \mathrm{d}t} \,. \tag{C.33}$$

Because $\sigma_{\rm T}$ and σ_A appear in the denominator of the adjoint source, one must specify a nonzero water-level value for each. Otherwise, for a perfect cross-correlation measurement, $\sigma_{\rm T} = \sigma_A = 0$, and the adjoint source (and therefore event kernel) will blow up. The water-level is an input parameter in `mt_measure_adj.f90`.

C.2.2 Multitaper measurements

Each windowed pulse on an individual seismogram is characterized by a (complex) transfer function from the modeled synthetics to the observed data:

$$T(\omega)\, s(\omega) = d(\omega). \tag{C.34}$$

Note that the transfer function is the same, whether the data and synthetics are in displacement, velocity, or acceleration, etc:

$$\begin{aligned} T(\omega)\, i\omega s(\omega) &= i\omega d(\omega)\ , \\ -T(\omega)\, \omega^2 s(\omega) &= -\omega^2 d(\omega)\ . \end{aligned}$$

Here, the convention $ds/dt \leftrightarrow i\omega s(\omega)$ is consistent with the Fourier convention in Section C.3.2.

Consider a single record of synthetics and data, $s(t)$ and $d(t)$, windowed in time over a particular phase and *both preprocessed in the same way*, e.g., filtered over a particular frequency window. The tapered versions are given by

$$s_j(t) = s(t)\, h_j(t)\ , \tag{C.35}$$

$$d_j(t) = d(t)\, h_j(t)\ , \tag{C.36}$$

where $h_j(t)$ is the taper.

Following *Laske and Masters* (1996), we use the multitaper method of *Thomson* (1982), which uses prolate spheroidal eigentapers (*Slepian*, 1978) for the $h_j(t)$. The transfer function $T(\omega)$ between the data and synthetics is given by (see Section C.3.1)

$$T(\omega) = \frac{\sum_j d_j(\omega)\, s_j^*(\omega)}{\sum_j s_j(\omega)\, s_j^*(\omega)}\ . \tag{C.37}$$

This function may be computed directly from the data and synthetics, and then represented in terms of the real functions, $\Delta\tau(\omega)$ and $\Delta \ln A(\omega)$:

$$T(\omega) = \frac{\sum_j d_j(\omega)\, s_j^*(\omega)}{\sum_j s_j(\omega)\, s_j^*(\omega)} = \exp[-i\omega\, \Delta\tau(\omega)]\,[1 + \Delta \ln A(\omega)]\ , \tag{C.38}$$

$$\Delta\tau(\omega) = \frac{-1}{\omega} \tan^{-1}\left(\frac{\mathrm{Im}\,[T(\omega)]}{\mathrm{Re}\,[T(\omega)]}\right), \tag{C.39}$$

$$\Delta \ln A(\omega) = |T(\omega)| - 1\ . \tag{C.40}$$

The sign convention in (C.38) is consistent with (C.19)–(C.20) and with the Fourier convention $\partial_t \leftrightarrow i\omega$ (Sections C.3.2 and C.3.3).

C.2.3 Multitaper adjoint sources

The units on various quantities in this section are shown in Table C.1.

Following *Tromp et al.* (2005), we must express the misfit function variations in (C.17)–(C.18) in terms of the perturbed seismograms δs. The tapered data, $d_j(t)$, can be expressed as

$$d_j = d\, h_j = (s + \delta s)\, h_j = s\, h_j + \delta s\, h_j = s_j + \delta s_j, \tag{C.41}$$

where we have defined the tapered, perturbed synthetics as

$$\delta s_j = \delta s\, h_j. \tag{C.42}$$

Substituting (C.41) into (C.37), we obtain

$$T(\omega) = \frac{\sum_j [s_j(\omega) + \delta s_j(\omega)]\ s_j^*(\omega)}{\sum_j s_j(\omega)\, s_j^*(\omega)} = 1 + \frac{\sum_j \delta s_j\, s_j^*}{\sum_j s_j\, s_j^*}\ . \tag{C.43}$$

If we write

$$T(\omega) = \exp[-i\omega\, \delta\tau(\omega)]\,[1 + \delta \ln A(\omega)]\ , \tag{C.44}$$

and make a first-order approximation for $\exp[-i\omega\, \delta\tau(\omega)]$, we have

$$T(\omega) \approx [1 - i\omega\, \delta\tau(\omega)]\,[1 + \delta \ln A(\omega)] \approx 1 - i\omega\, \delta\tau(\omega) + \delta \ln A(\omega)\ , \tag{C.45}$$

and thus, from (C.43) and (C.45),

$$T(\omega) - 1 = \frac{\sum_j \delta s_j\, s_j^*}{\sum_j s_j\, s_j^*} \approx -i\omega\, \delta\tau(\omega) + \delta \ln A(\omega)\,, \tag{C.46}$$

$$\delta\tau(\omega) = \frac{-1}{\omega}\,\mathrm{Im}\left(\frac{\sum_j \delta s_j\, s_j^*}{\sum_j s_j\, s_j^*}\right), \tag{C.47}$$

$$\delta \ln A(\omega) = \mathrm{Re}\left(\frac{\sum_j \delta s_j\, s_j^*}{\sum_j s_j\, s_j^*}\right). \tag{C.48}$$

Using the identities in Appendix C.3.4, we obtain

$$\begin{aligned}
\delta\tau(\omega) &= \frac{-1}{\omega}\,\mathrm{Im}\left(\frac{\sum_j \delta s_j\, s_j^*}{\sum_j s_j\, s_j^*}\right) \\
&= \mathrm{Im}\left(\frac{-1}{\omega}\frac{\sum_j \delta s_j\, s_j^*}{\sum_j s_j\, s_j^*}\right) \\
&= \mathrm{Re}\left(\frac{i}{\omega}\,\frac{\sum_j \delta s_j\, s_j^*}{\sum_j s_j\, s_j^*}\right) \\
&= \mathrm{Re}\left(-\frac{i}{\omega}\,\frac{\sum_j s_j\, \delta s_j^*}{\sum_j s_j\, s_j^*}\right)\,,
\end{aligned} \tag{C.49}$$

$$\begin{aligned}
\delta \ln A(\omega) &= \mathrm{Re}\left(\frac{\sum_j \delta s_j\, s_j^*}{\sum_j s_j\, s_j^*}\right) \\
&= \mathrm{Re}\left(\frac{\sum_j s_j\, \delta s_j^*}{\sum_j s_j\, s_j^*}\right)\,.
\end{aligned} \tag{C.50}$$

Inserting these expressions for the transfer function into (C.17)–(C.18), and omitting the explicit dependence on $\mathbf{m}$, we have

$$\begin{aligned}
\delta F_{\mathrm{P}} &= -\int_{-\infty}^{\infty} W_{\mathrm{P}}(\omega)\,\Delta\tau(\omega)\,\mathrm{Re}\left(\frac{-i}{\omega}\,\frac{\sum_j s_j\, \delta s_j^*}{\sum_j s_j\, s_j^*}\right) \mathrm{d}\omega \\
&= \mathrm{Re}\left[\int_{-\infty}^{\infty} W_{\mathrm{P}}(\omega)\,\Delta\tau(\omega) \sum_j \left(\frac{i}{\omega}\frac{s_j}{\sum_k s_k s_k^*}\right) \delta s_j^*(\omega)\,\mathrm{d}\omega\right] \\
&= \mathrm{Re}\left[\int_{-\infty}^{\infty} W_{\mathrm{P}}(\omega)\,\Delta\tau(\omega) \sum_j p_j(\omega)\,\delta s_j^*(\omega)\,\mathrm{d}\omega\right],
\end{aligned} \tag{C.51}$$

$$
\begin{aligned}
\delta F_{\mathrm{Q}} &= -\int_{-\infty}^{\infty} W_{\mathrm{Q}}(\omega)\,\Delta\ln A(\omega)\;\mathrm{Re}\left(\frac{\sum_j s_j\,\delta s_j^*}{\sum_j s_j\,s_j^*}\right)\mathrm{d}\omega \\
&= \mathrm{Re}\left[\int_{-\infty}^{\infty} W_{\mathrm{Q}}(\omega)\,\Delta\ln A(\omega)\sum_j\left(\frac{-s_j}{\sum_k s_k s_k^*}\right)\delta s_j^*(\omega)\,\mathrm{d}\omega\right] \\
&= \mathrm{Re}\left[\int_{-\infty}^{\infty} W_{\mathrm{Q}}(\omega)\,\Delta\ln A(\omega)\sum_j q_j(\omega)\,\delta s_j^*(\omega)\,\mathrm{d}\omega\right]\;, \qquad \text{(C.52)}
\end{aligned}
$$

where

$$
p_j(\omega) \equiv \frac{i}{\omega}\frac{s_j}{\sum_k s_k s_k^*} = \frac{i\omega\,s_j}{\sum_k(i\omega s_k)(-i\omega s_k^*)} = \frac{i\omega\,s_j}{\sum_k(i\omega s_k)(i\omega s_k)^*}\;, \qquad \text{(C.53)}
$$

$$
q_j(\omega) \equiv \frac{-s_j}{\sum_k s_k s_k^*} = i\omega\,p_j(\omega), \qquad \text{(C.54)}
$$

where in (C.53) we have used the property (Eq. C.87) $-iz^* = (iz)^*$. Note that p_j and q_j are based on the (tapered) synthetics alone, and that, based on our Fourier convention in Section C.3.2,

$$
q_j(t) = \dot{p}_j(t). \qquad \text{(C.55)}
$$

Furthermore, note that the time-domain terms in (C.53) are all $\dot{s}_j(t)$, the derivative of the tapered synthetics.

We now use Plancherel's theorem (Section C.3.4), one version of which is

$$
\int_{-\infty}^{\infty} f(\omega)\,g^*(\omega)\,\mathrm{d}\omega = 2\pi\int_{-\infty}^{\infty} f(t)\,g^*(t)\,\mathrm{d}t, \qquad \text{(C.56)}
$$

to convert (C.51)–(C.52) into the time domain:

$$\begin{aligned}
\delta F_{\mathrm{P}} &= \mathrm{Re}\left[\int_{-\infty}^{\infty} W_{\mathrm{P}}(\omega)\,\Delta\tau(\omega)\sum_j p_j(\omega)\,\delta s_j^*(\omega)\,\mathrm{d}\omega\right] \\
&= \mathrm{Re}\left[\sum_j \int_{-\infty}^{\infty} \left[W_{\mathrm{P}}(\omega)\,\Delta\tau(\omega)\,p_j(\omega)\right]\delta s_j^*(\omega)\,\mathrm{d}\omega\right] \\
&= \sum_j \int_{-\infty}^{\infty} 2\pi\left[W_{\mathrm{P}}(t)*\Delta\tau(t)*p_j(t)\right]\delta s_j(t)\,\mathrm{d}t \\
&= \sum_j \int_{-\infty}^{\infty} 2\pi\left[W_{\mathrm{P}}(t)*\Delta\tau(t)*p_j(t)\right]h_j(t)\,\delta s(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\left\{2\pi\sum_j h_j(t)\left[W_{\mathrm{P}}(t)*\Delta\tau(t)*p_j(t)\right]\right\}\delta s(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f_{\mathrm{P}}^{\dagger}(t)\;\delta s(t)\,\mathrm{d}t\;, \qquad \text{(C.57)}
\end{aligned}$$

$$\begin{aligned}
\delta F_{\mathrm{Q}} &= \mathrm{Re}\left[\int_{-\infty}^{\infty} W_{\mathrm{Q}}(\omega)\,\Delta\ln A(\omega)\sum_j q_j(\omega)\,\delta s_j^*(\omega)\,\mathrm{d}\omega\right] \\
&= \mathrm{Re}\left[\sum_j \int_{-\infty}^{\infty} \left[W_{\mathrm{Q}}(\omega)\,\Delta\ln A(\omega)\,q_j(\omega)\right]\delta s_j^*(\omega)\,\mathrm{d}\omega\right] \\
&= \sum_j \int_{-\infty}^{\infty} 2\pi\left[W_{\mathrm{Q}}(t)*\Delta\ln A(t)*q_j(t)\right]\delta s_j(t)\,\mathrm{d}t \\
&= \sum_j \int_{-\infty}^{\infty} 2\pi\left[W_{\mathrm{Q}}(t)*\Delta\ln A(t)*q_j(t)\right]h_j(t)\,\delta s(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\left\{2\pi\sum_j h_j(t)\left[W_{\mathrm{Q}}(t)*\Delta\ln A(t)*q_j(t)\right]\right\}\delta s(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f_{\mathrm{Q}}^{\dagger}(t)\;\delta s(t)\,\mathrm{d}t\;, \qquad \text{(C.58)}
\end{aligned}$$

where $\delta s_j(t) = h_j(t)\,\delta s(t)$ is the tapered, perturbed time series, $\Delta\tau(t)$ and $\Delta\ln A(t)$ are the time domain versions of (C.39)–(C.40), and we have defined our adjoint sources for multitaper traveltime measurements (P) and multitaper amplitude measurements (Q) as[2]

$$f_{\mathrm{P}}^{\dagger}(t) \equiv \sum_j h_j(t)\,[2\pi\, W_{\mathrm{P}}(t) * \Delta\tau(t) * p_j(t)] \ , \tag{C.59}$$

$$f_{\mathrm{Q}}^{\dagger}(t) \equiv \sum_j h_j(t)\,[2\pi\, W_{\mathrm{Q}}(t) * \Delta\ln A(t) * q_j(t)] \ . \tag{C.60}$$

The frequency domain versions of (C.59)–(C.60) are

$$f_{\mathrm{P}}^{\dagger}(\omega) = \sum_j h_j(\omega) * [2\pi\, W_{\mathrm{P}}(\omega)\,\Delta\tau(\omega)\,p_j(\omega)] \ , \tag{C.61}$$

$$f_{\mathrm{Q}}^{\dagger}(\omega) = \sum_j h_j(\omega) * [2\pi\, W_{\mathrm{Q}}(\omega)\,\Delta\ln A(\omega)\,q_j(\omega)] \ . \tag{C.62}$$

We also define the following functions:

$$P_j(t) \equiv 2\pi\, W_{\mathrm{P}}(t) * \Delta\tau(t) * p_j(t) \ , \tag{C.63}$$

$$Q_j(t) \equiv 2\pi\, W_{\mathrm{Q}}(t) * \Delta\ln A(t) * q_j(t) \ , \tag{C.64}$$

$$P_j(\omega) \equiv 2\pi\, W_{\mathrm{P}}(\omega)\,\Delta\tau(\omega)\,p_j(\omega) \ , \tag{C.65}$$

$$Q_j(\omega) \equiv 2\pi\, W_{\mathrm{Q}}(\omega)\,\Delta\ln A(\omega)\,q_j(\omega) \ . \tag{C.66}$$

These lead to succint expressions for the adjoint sources:

$$f_{\mathrm{P}}^{\dagger}(t) \equiv \sum_j h_j(t) P_j(t) \ , \tag{C.67}$$

$$f_{\mathrm{Q}}^{\dagger}(t) \equiv \sum_j h_j(t) Q_j(t) \ , \tag{C.68}$$

$$f_{\mathrm{P}}^{\dagger}(\omega) = \sum_j h_j(\omega) * P_j(\omega) \ , \tag{C.69}$$

$$f_{\mathrm{Q}}^{\dagger}(\omega) = \sum_j h_j(\omega) * Q_j(\omega) \ . \tag{C.70}$$

[2]Note that we have not written the time-dependence using the time-rerversal convention, i.e., $T - t$.

Table C.1: Units for adjoint quantities for multitaper measurements. The misfit functions $F_{\rm P}(\mathbf{m})$ and $F_{\rm Q}(\mathbf{m})$ (and $F_{\rm T}(\mathbf{m})$ and $F_{\rm A}(\mathbf{m})$) are unitless if we take into account the units for the σ terms. In practice, the adjoint sources have a $\rm m^{-3}$ or $\rm m^{-2}$ quantity as well, due to the 3D or 2D volume for the delta function, $\delta(\mathbf{x})$, that is applied at the source location. Bottom two rows are for adjoint sources based on cross-correlation measurements.

frequency domain $[\widetilde{h}(\omega)] = [h(t)]$ s		time domain $[h(t)] = [\widetilde{h}(\omega)]$ $\rm s^{-1}$	
$s(\omega)$, $d(\omega)$, $\delta s(\omega)$	m s	$s(t)$, $d(t)$, $\delta s(t)$	m
$\dot{s}(\omega)$	m	$\dot{s}(t)$	m $\rm s^{-1}$
$W_{\rm P}(\omega)$	$\rm s^{-1}$	$W_{\rm P}(t)$	$\rm s^{-2}$
$W_{\rm Q}(\omega)$	s	$W_{\rm Q}(t)$	none
$\Delta\tau(\omega)$	s	$\Delta\tau(t)$	none
$\Delta \ln A(\omega)$	none	$\Delta \ln A(t)$	$\rm s^{-1}$
$p_j(\omega)$	$\rm m^{-1}$	$p_j(t)$	$\rm m^{-1}$ $\rm s^{-1}$
$q_j(\omega)$	$\rm m^{-1}$ $\rm s^{-1}$	$q_j(t)$	$\rm m^{-1}$ $\rm s^{-2}$
$f_{\rm P}^{\dagger}(\omega)$	$\rm m^{-1}$	$f_{\rm P}^{\dagger}(t)$	$\rm m^{-1}$ $\rm s^{-1}$
$f_{\rm Q}^{\dagger}(\omega)$	$\rm m^{-1}$	$f_{\rm Q}^{\dagger}(t)$	$\rm m^{-1}$ $\rm s^{-1}$
$f_{\rm T}^{\dagger}(\omega)$	$\rm m^{-1}$	$f_{\rm T}^{\dagger}(t)$	$\rm m^{-1}$ $\rm s^{-1}$
$f_{\rm A}^{\dagger}(\omega)$	$\rm m^{-1}$	$f_{\rm A}^{\dagger}(t)$	$\rm m^{-1}$ $\rm s^{-1}$

C.3 Miscellaneous

C.3.1 Deriving the transfer function

The transfer function, $T(\omega) = T_r(\omega) + i\,T_i(\omega)$ between data, d, and synthetics, s, is found by minimizing the objective function

$$F\left[T(\omega)\right] \;=\; \tfrac{1}{2}\sum_j \left|d_j(\omega) - T(\omega)\,s_j(\omega)\right|^2 \;\;,$$

where the sum over j represents the multitapers of Section C.2.2. Expanding this into real and imaginary parts, we have

$$\begin{aligned}
F(T) &= \tfrac{1}{2}\sum_j |d_j - T s_j|^2 \\
&= \tfrac{1}{2}\sum_j (d_j - T s_j)(d_j^* - T^* s_j^*) \\
&= \tfrac{1}{2}\sum_j \left[d_j - (T_r + i\,T_i)\,s_j\right]\left[d_j^* - (T_r - i\,T_i)\,s_j^*\right] \\
&= \tfrac{1}{2}\sum_j \left[d_j\,d_j^* - (T_r - i\,T_i)\,d_j\,s_j^* - (T_r + i\,T_i)\,d_j^*\,s_j + (T_r^2 + T_i^2)\,s_j\,s_j^*\right] \\
&= \tfrac{1}{2}\sum_j \left[d_j\,d_j^* - T_r\,d_j\,s_j^* + i\,T_i\,d_j\,s_j^* - T_r\,d_j^*\,s_j - i\,T_i\,d_j^*\,s_j + T_r^2\,s_j\,s_j^* + T_i^2\,s_j\,s_j^*\right] \\
&= \tfrac{1}{2}\sum_j \left[d_j\,d_j^* + T_r\left(-d_j\,s_j^* - d_j^*\,s_j\right) + T_i\left(i\,d_j\,s_j^* - i\,d_j^*\,s_j\right) + T_r^2\,s_j\,s_j^* + T_i^2\,s_j\,s_j^*\right] \ .
\end{aligned}$$

The derivatives with respect to the real and imaginary parts of the transfer function are given by

$$\begin{aligned}
\frac{\partial F}{\partial T_r} &= \tfrac{1}{2}\sum_j \left[-d_j\,s_j^* - d_j^*\,s_j + 2\,T_r\,s_j\,s_j^*\right] \;=\; -\tfrac{1}{2}\sum_j \left(d_j^*\,s_j + d_j\,s_j^* +\right) + T_r \sum_j s_j\,s_j^* \ , \\
\frac{\partial F}{\partial T_i} &= \tfrac{1}{2}\sum_j \left[i\,d_j\,s_j^* - i\,d_j^*\,s_j + 2\,T_i\,s_j\,s_j^*\right] \;=\; -\frac{i}{2}\sum_j \left(d_j^*\,s_j - d_j\,s_j^*\right) + T_i \sum_j s_j\,s_j^* \ .
\end{aligned}$$

Setting each equation equal to zero and solving for T_r and T_i gives:

$$\begin{aligned}
T_r &= \frac{\tfrac{1}{2}\sum_j \left(d_j\,s_j^* + d_j^*\,s_j\right)}{\sum_j s_j\,s_j^*} \ , \\
T_i &= \frac{\tfrac{i}{2}\sum_j \left(d_j^*\,s_j - d_j\,s_j^*\right)}{\sum_j s_j\,s_j^*} \ .
\end{aligned}$$

Thus, we obtain (C.37):

$$\begin{aligned}
T &= T_r + i\,T_i \;=\; \frac{\tfrac{1}{2}\sum_j \left(d_j\,s_j^* + d_j^*\,s_j\right)}{\sum_j s_j\,s_j^*} - \frac{\tfrac{1}{2}\sum_j \left(d_j^*\,s_j - d_j\,s_j^*\right)}{\sum_j s_j\,s_j^*} \\
&= \frac{\tfrac{1}{2}\sum_j \left(d_j\,s_j^* + d_j^*\,s_j - d_j^*\,s_j + d_j\,s_j^*\right)}{\sum_j s_j\,s_j^*} \\
&= \frac{\sum_j d_j\,s_j^*}{\sum_j s_j\,s_j^*} \ .
\end{aligned}$$

C.3.2 Conventions for measurements, Fourier transform, and transfer function

We define our measurements according to the conventions in (C.19)–(C.20), such that a positive traveltime measurement, $\Delta\tau > 0$, corresponds to a delay in the data with respect to the synthetics.

We define forward and inverse Fourier transforms

$$\mathcal{F}\left[h(t)\right] = \widetilde{h}(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, \mathrm{d}t \,, \tag{C.71}$$

$$\mathcal{F}^{-1}\left[\widetilde{h}(\omega)\right] = h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{h}(\omega)\, e^{i\omega t}\, \mathrm{d}\omega \,. \tag{C.72}$$

Note that this convention follows that of *Dahlen and Tromp* (1998, p. 109), and that the units conversion is

$$\left[\widetilde{h}(\omega)\right] = \left[h(t)\right]\ \mathrm{s},$$

which is reflected in Table C.1.

The Fourier transform of $\dot{h}(t)$ can be determined using integration by parts,

$$\int u\, dv = [u\, v] - \int v\, du,$$

with $dv = \dot{h}(t)\, dt$, $u = e^{-i\omega t}$, $v = h(t)$, and $du = -i\omega\, e^{-i\omega t} dt$. Thus, we can write

$$\begin{aligned}
\mathcal{F}\left[\dot{h}(t)\right] &= \int_{-\infty}^{\infty} \dot{h}(t)\, e^{-i\omega t}\, \mathrm{d}t \\
&= \left[h(t)\, e^{-i\omega t}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} h(t)\,(-i\omega)\, e^{-i\omega t}\, \mathrm{d}t \\
&= i\omega \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, \mathrm{d}t \\
&= i\omega\, \mathcal{F}\left[h(t)\right] \,.
\end{aligned} \tag{C.73}$$

This process can be iterated for the nth derivative to yield

$$\mathcal{F}\left[h^{(n)}(t)\right] = (i\omega)^n\, \mathcal{F}\left[h(t)\right] \,. \tag{C.74}$$

Note that (C.73)–(C.74) will depend on the Fourier convention.

The transfer function is defined according to $T(\omega)\, s(\omega) = d(\omega)$ (Eq. C.34). The convention for the measurements and the Fourier convention imply that the transfer function is to be written as

$$T(\omega) = e^{-i\omega\Delta\tau},$$

where we have ignored the amplitude measurement and have assumed that $\Delta\tau$ is constant over all ω. Then, the data in the frequency and time domain are given by

$$d(\omega) = s(\omega)\, e^{-i\omega\Delta\tau} \,, \tag{C.75}$$

$$d(t) = \mathcal{F}^{-1}\left[d(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega)\, e^{-i\omega\Delta\tau}\, e^{i\omega t}\, \mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega)\, e^{i\omega(t-\Delta\tau)}\, \mathrm{d}\omega = s(t-\Delta\tau). \tag{C.76}$$

For example, for data arriving early with $\Delta\tau = -3$, we have $d(t) = s(t+3)$, indicating that the data are advanced by 3 seconds with respect to the synthetics.

C.3.3 Implementation of conventions for measurement, Fourier, and transfer function

In practice, the original data are shifted by the cross-correlation measurement, ΔT, prior to making the multitaper measurement. In other words, $d(\omega) = d_0(\omega)\, e^{i\omega\,\Delta T}$, where d_0 are the original, unshifted data. This convention is checked as follows:

$$d(t) = \mathcal{F}^{-1}\left[d(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} d_0(\omega)\, e^{i\omega\Delta T}\, e^{i\omega t}\, \mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} d_0(\omega)\, e^{i\omega\,(t+\Delta T)}\, \mathrm{d}\omega = d_0(t{+}\Delta T) \,, \tag{C.77}$$

that is, $d_0(t) = d(t - \Delta T)$. For example, for (original) data arriving early with respect to the synthetics, with $\Delta T = -3$, then we have $d_0(t) = d(t+3)$, indicating that the original data are advanced by 3 seconds with respect to the shifted data.

In the measurement code, the transfer function we compute is $T'(\omega) = e^{-i\omega\Delta\tau'}$, with $T'(\omega)\, s(\omega) = d(\omega)$. (Again, we ignore amplitudes for clarity.) Thus, the shifted data, $d(t)$, are aligned in phase with the synthetics by a uniform shift with magnitude $|\Delta T|$. Then we can express the *unshifted* data as

$$\begin{aligned} d_0(\omega) &= e^{-i\omega\,\Delta T}\, d(\omega) \\ &= e^{-i\omega\,\Delta T}\, T'(\omega)\, s(\omega) \;=\; e^{-i\omega\,\Delta T}\, e^{-i\omega\Delta\tau'}\, s(\omega) \;=\; e^{-i\omega\,(\Delta T+\Delta\tau')}\, s(\omega) \;=\; T(\omega)\, s(\omega)\;, \end{aligned} \tag{C.78}$$

where $T(\omega)$ represents the transfer function from the synthetics to the unshifted data, and $\Delta\tau'$ represents the frequency-dependent perturbations from the frequency-independent cross-correlation measurement ΔT, such that

$$\Delta\tau(\omega) = \Delta\tau'(\omega) + \Delta T. \tag{C.79}$$

Thus, we can write the phases as

$$-\omega\,\Delta\tau'(\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\,[T'(\omega)]}{\mathrm{Re}\,[T'(\omega)]}\right), \tag{C.80}$$

$$-\omega\,\Delta\tau(\omega) = -\omega\left(\Delta\tau'(\omega) + \Delta T\right) \;=\; \tan^{-1}\left(\frac{\mathrm{Im}\,[T'(\omega)]}{\mathrm{Re}\,[T'(\omega)]}\right) - \omega\,\Delta T\;, \tag{C.81}$$

and the corresponding traveltimes as

$$\Delta\tau'(\omega) = \frac{-1}{\omega}\tan^{-1}\left(\frac{\mathrm{Im}\,[T'(\omega)]}{\mathrm{Re}\,[T'(\omega)]}\right), \tag{C.82}$$

$$\Delta\tau(\omega) = \frac{-1}{\omega}\tan^{-1}\left(\frac{\mathrm{Im}\,[T'(\omega)]}{\mathrm{Re}\,[T'(\omega)]}\right) + \Delta T\;. \tag{C.83}$$

Compare (C.83) with (C.39). The use of $T'(\omega)$ instead of $T(\omega)$ gives rise to the inclusion of ΔT.

C.3.4 Plancherel's theorem

Parseval's theorem applied to Fourier series is known as *Rayleigh's theorem* or *Plancherel's theorem*. The following derivation of Plancherel's theorem is adapted from the Mathworld website.

The exact representation of the theorem depends on the Fourier convention. Using the

Fourier conventions in Equations (C.71) and (C.72), we have

$$\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(\omega)\, e^{i\omega t}\, \mathrm{d}\omega \;, \\
f^*(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}^*(\omega)\, e^{-i\omega t}\, \mathrm{d}\omega \;, \\
\widetilde{f}(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, \mathrm{d}t \;, \\
\widetilde{f}^*(\omega) &= \int_{-\infty}^{\infty} f^*(t)\, e^{i\omega t}\, \mathrm{d}t \;.
\end{aligned}$$

Consider the following derivation:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)\, g^*(t)\, \mathrm{d}t &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \widetilde{f}(\omega)\, e^{i\omega t}\, \mathrm{d}\omega\right] \left[\int_{-\infty}^{\infty} \widetilde{g}^*(\omega')\, e^{-i\omega' t}\, \mathrm{d}\omega'\right] \mathrm{d}t \\
&= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{f}(\omega)\, \widetilde{g}^*(\omega')\, e^{it(\omega-\omega')} \mathrm{d}\omega\, \mathrm{d}\omega'\, \mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{f}(\omega)\, \widetilde{g}^*(\omega') \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(\omega-\omega')} \mathrm{d}t\right] \mathrm{d}\omega\, \mathrm{d}\omega' \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\omega-\omega')\, \widetilde{f}(\omega)\, \widetilde{g}^*(\omega')\, \mathrm{d}\omega\, \mathrm{d}\omega' \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(\omega)\, \widetilde{g}^*(\omega)\, \mathrm{d}\omega.
\end{aligned}$$

If $g(t) = f(t)$ (and thus $g^*(t) = f^*(t)$), then we obtain Plancherel's Theorem,

$$\int_{-\infty}^{\infty} |f(t)|^2\, \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\widetilde{f}(\omega)\right|^2 \mathrm{d}\omega, \tag{C.84}$$

which states that the integral of the squared modulus of a function is equal to the integral of the squared modulus of its spectrum.

Miscellaneous formulas

Considering two complex numbers,

$$
\begin{aligned}
z &= a + bi \ , \\
w &= c + di \ ,
\end{aligned}
$$

we can derive the following expressions:

$$\mathrm{Re}\,[iz] = \mathrm{Re}\,[i(a+bi)] = \mathrm{Re}\,[ai-b] = -b = \mathrm{Im}[-ai-b] = \mathrm{Im}[-z] \ , \tag{C.85}$$

$$\mathrm{Re}\left[i^{-1}z\right] = \mathrm{Re}\,[-iz] = \mathrm{Im}[z] \ , \tag{C.86}$$

$$-iz^* = -i(a-bi) = -b+ai = (-b+ai)^* = [i(a+bi)]^* = (iz)^* \ , \tag{C.87}$$

$$
\begin{aligned}
\mathrm{Re}\,[zw^*] &= \mathrm{Re}\,[(a+bi)(c-di)] \\
&= \mathrm{Re}\,[ac - adi + bci + bd] \\
&= ac + bd \\
&= \mathrm{Re}\,[ac + adi - bci + bd] \\
&= \mathrm{Re}\,[(a-bi)(c+di)] \\
&= \mathrm{Re}\,[z^*w] \ .
\end{aligned} \tag{C.88}
$$

Using Equations (C.87) and (C.88), we obtain

$$\mathrm{Re}\,[izw^*] = \mathrm{Re}\,[(iz)^*w] = \mathrm{Re}\,[-iz^*w] \ . \tag{C.89}$$

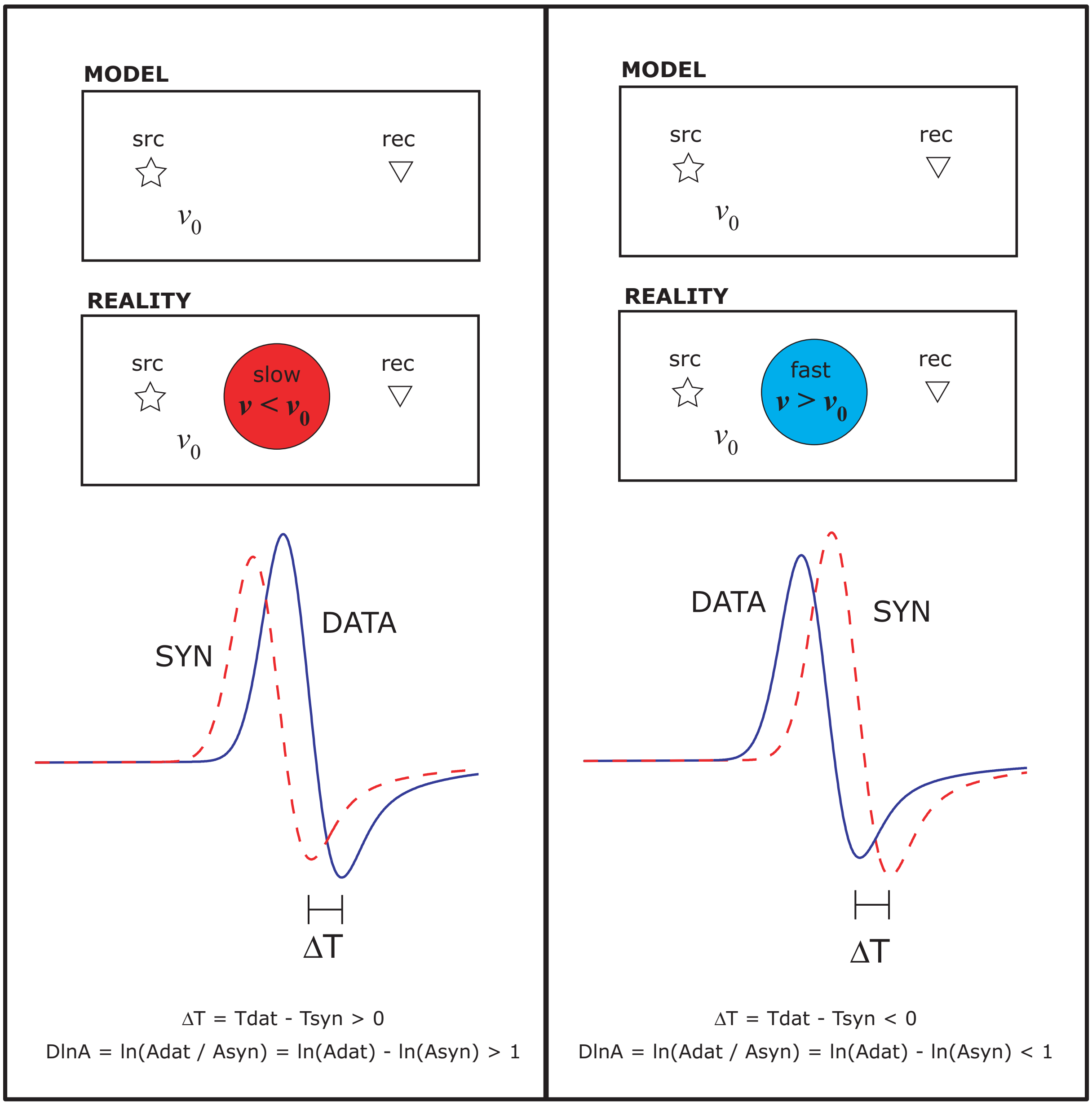

Figure C.1: The measurement convention for traveltime differences, ΔT, and amplitude differences $\Delta \ln A$. See Section C.2.1.