Appendix B

Sensitivity kernels for different model parameterizations

Note

Here we consider three different parameterizations of seismic structure. The formulas are summarized in Table B.1 and shown for a 2D synthetic example in Figure 3.10.

B.1 General formulas

For an *isotropic earth*, the variation of the misfit function F with respect to model parameters κ-μ-ρ or α-β-ρ or c-β-ρ can be expressed in the following forms:

$$\delta F = \int_V \left[K_{\kappa(\mu\rho)} \frac{\delta\kappa}{\kappa} + K_{\mu(\kappa\rho)} \frac{\delta\mu}{\mu} + K_{\rho(\kappa\mu)} \frac{\delta\rho}{\rho} \right] d^3\mathbf{x} \,, \tag{B.1}$$

$$\delta F = \int_V \left[K_{\alpha(\beta\rho)} \frac{\delta\alpha}{\alpha} + K_{\beta(\alpha\rho)} \frac{\delta\beta}{\beta} + K_{\rho(\alpha\beta)} \frac{\delta\rho}{\rho} \right] d^3\mathbf{x} \,, \tag{B.2}$$

$$\delta F = \int_V \left[K_{c(\beta\rho)} \frac{\delta c}{c} + K_{\beta(c\rho)} \frac{\delta\beta}{\beta} + K_{\rho(c\beta)} \frac{\delta\rho}{\rho} \right] d^3\mathbf{x} \,. \tag{B.3}$$

Each kernel, $K(\mathbf{x})$, should be read as follows: $K_{\kappa(\mu\rho)}$ is "the sensitivity kernel for κ with μ and ρ fixed." In Equations (B.1)–(B.3) the kernels are defined according to model parameters defined as fractional perturbations, as used in *Tromp et al.* (2005). Below we derive expressions for the kernels.

B.1.1 Kernels for different model parameterizations

To get from Equation (B.1) to Equation (B.2), we insert Equations (B.12) and (B.13) into the integrand in Equation (B.1):

$$\begin{aligned}
& \frac{\delta\kappa}{\kappa} K_{\kappa(\mu\rho)} + \frac{\delta\mu}{\mu} K_{\mu(\kappa\rho)} + \frac{\delta\rho}{\rho} K_{\rho(\kappa\mu)} \\
= & \left[\frac{\delta\rho}{\rho} \;+\; 2 \left(\alpha^2 - \tfrac{4}{3}\beta^2 \right)^{-1} \left(\alpha^2 \, \frac{\delta\alpha}{\alpha} - \tfrac{4}{3}\beta^2 \, \frac{\delta\beta}{\beta} \right) \right] K_{\kappa(\mu\rho)} \;+\; \left[\frac{\delta\rho}{\rho} \;+\; 2 \, \frac{\delta\beta}{\beta} \right] K_{\mu(\kappa\rho)} \;+\; \frac{\delta\rho}{\rho} K_{\rho(\kappa\mu)} \\
= & \frac{\delta\rho}{\rho} K_{\kappa(\mu\rho)} \;+\; 2 \, \frac{\delta\alpha}{\alpha} \left(\alpha^2 - \tfrac{4}{3}\beta^2 \right)^{-1} \alpha^2 \, K_{\kappa(\mu\rho)} \;-\; 2 \, \frac{\delta\beta}{\beta} \left(\alpha^2 - \tfrac{4}{3}\beta^2 \right)^{-1} \tfrac{4}{3}\beta^2 \, K_{\kappa(\mu\rho)} \\
& + \frac{\delta\rho}{\rho} K_{\mu(\kappa\rho)} \;+\; 2 \, \frac{\delta\beta}{\beta} K_{\mu(\kappa\rho)} \;+\; \frac{\delta\rho}{\rho} K_{\rho(\kappa\mu)} \\
= & \frac{\delta\alpha}{\alpha} \left\{ 2\alpha^2 \left(\alpha^2 - \tfrac{4}{3}\beta^2 \right)^{-1} K_{\kappa(\mu\rho)} \right\} \;+\; \frac{\delta\beta}{\beta} \left\{ 2 \, K_{\mu(\kappa\rho)} - 2 \left(\alpha^2 - \tfrac{4}{3}\beta^2 \right)^{-1} \tfrac{4}{3}\beta^2 \, K_{\kappa(\mu\rho)} \right\} \\
& + \frac{\delta\rho}{\rho} \left\{ K_{\rho(\kappa\mu)} + K_{\kappa(\mu\rho)} + K_{\mu(\kappa\rho)} \right\} \\
= & \frac{\delta\alpha}{\alpha} \left\{ 2 \frac{\kappa + \frac{4}{3}\mu}{\rho} \frac{\rho}{\kappa} K_{\kappa(\mu\rho)} \right\} \;+\; \frac{\delta\beta}{\beta} \left\{ 2 \, K_{\mu(\kappa\rho)} - \frac{8}{3} \beta^2 \frac{\rho}{\kappa} \, K_{\kappa(\mu\rho)} \right\} \;+\; \frac{\delta\rho}{\rho} \left\{ K_{\rho(\kappa\mu)} + K_{\kappa(\mu\rho)} + K_{\mu(\kappa\rho)} \right\} \\
= & \frac{\delta\alpha}{\alpha} \left\{ \left(2 + \frac{8}{3} \frac{\mu}{\kappa} \right) K_{\kappa(\mu\rho)} \right\} \;+\; \frac{\delta\beta}{\beta} \left\{ 2 \, K_{\mu(\kappa\rho)} - \frac{8}{3} \frac{\mu}{\kappa} \, K_{\kappa(\mu\rho)} \right\} \;+\; \frac{\delta\rho}{\rho} \left\{ K_{\rho(\kappa\mu)} + K_{\kappa(\mu\rho)} + K_{\mu(\kappa\rho)} \right\} \\
\equiv & \frac{\delta\alpha}{\alpha} K_{\alpha(\beta\rho)} \;+\; \frac{\delta\beta}{\beta} K_{\beta(\alpha\rho)} \;+\; \frac{\delta\rho}{\rho} K_{\rho(\alpha\beta)} \; ,
\end{aligned}$$

where the misfit kernels $K_{\alpha(\beta\rho)}$, $K_{\beta(\alpha\rho)}$, and $K_{\rho(\alpha\beta)}$ represent Fréchet derivatives with respect to compressional-wave speed, shear-wave speed, and density, respectively (*Tromp et al.*, 2005, Eq. 20):

$$K_{\alpha(\beta\rho)} \;=\; \left(2 + \frac{8}{3} \frac{\mu}{\kappa} \right) K_{\kappa(\mu\rho)} \;=\; 2 \, K_{\kappa(\mu\rho)} + A \, K_{\kappa(\mu\rho)} \; , \tag{B.4}$$

$$K_{\beta(\alpha\rho)} \;=\; 2 \, K_{\mu(\kappa\rho)} - \frac{8}{3} \frac{\mu}{\kappa} \, K_{\kappa(\mu\rho)} \;=\; 2 \, K_{\mu(\kappa\rho)} - A \, K_{\kappa(\mu\rho)} \; , \tag{B.5}$$

$$K_{\rho(\alpha\beta)} \;=\; K_{\rho(\kappa\mu)} + K_{\kappa(\mu\rho)} + K_{\mu(\kappa\rho)} \; , \tag{B.6}$$

where $A \equiv 8\mu/(3\kappa)$. For the (constant) κ and μ values listed in Section 2.3.2, $A = 1.36$. Note that in explicit notation $A(\mathbf{x}) = [8 \, \mu(\mathbf{x})] \, / \, [3 \, \kappa(\mathbf{x})]$. Note that in deriving Equations (B.4)–(B.6), we could have stopped earlier, leaving the kernels in terms of α-β-ρ parameters rather than κ-μ-ρ parameters.

To get from Equation (B.1) to Equation (B.3), we substitute the expressions in Section B.2 into Equation (B.1):

$$
\begin{aligned}
& \frac{\delta\kappa}{\kappa} K_{\kappa(\mu\rho)} + \frac{\delta\mu}{\mu} K_{\mu(\kappa\rho)} + \frac{\delta\rho}{\rho} K_{\rho(\kappa\mu)} \\
= & \left[2\frac{\delta c}{c} + \frac{\delta\rho}{\rho}\right] K_{\kappa(\mu\rho)} \; + \; \left[2\frac{\delta\beta}{\beta} + \frac{\delta\rho}{\rho}\right] K_{\mu(\kappa\rho)} \; + \; \frac{\delta\rho}{\rho} K_{\rho(\kappa\mu)} \\
= & \frac{\delta c}{c}\left\{2K_{\kappa(\mu\rho)}\right\} \; + \; \frac{\delta\beta}{\beta}\left\{2K_{\mu(\kappa\rho)}\right\} \; + \; \frac{\delta\rho}{\rho}\left\{K_{\kappa(\mu\rho)} + K_{\mu(\kappa\rho)} + K_{\rho(\kappa\mu)}\right\} \\
= & \frac{\delta c}{c}\left\{2K_{\kappa(\mu\rho)}\right\} \; + \; \frac{\delta\beta}{\beta}\left\{2K_{\mu(\kappa\rho)}\right\} \; + \; \frac{\delta\rho}{\rho}\left\{K_{\kappa(\mu\rho)} + K_{\mu(\kappa\rho)} + K_{\rho(\kappa\mu)}\right\},
\end{aligned}
$$

where we have defined the c-β-ρ kernels in terms of the κ-μ-ρ kernels:

$$K_{c(\beta\rho)} = 2K_{\kappa(\mu\rho)} \, , \tag{B.7}$$

$$K_{\beta(c\rho)} = 2K_{\mu(\kappa\rho)} \, , \tag{B.8}$$

$$K_{\rho(c\beta)} = K_{\kappa(\mu\rho)} + K_{\mu(\kappa\rho)} + K_{\rho(\kappa\mu)}. \tag{B.9}$$

Note that these expressions do not contained factors of μ or κ, as in Equations (B.4)–(B.6).

B.2 Model parameterizations: α-β-ρ or κ-μ-ρ or c-β-ρ

There are several different variables that can be used to describe an isotropic elastic structure for wave propagation. Here we consider three sets of model variables: α-β-ρ, κ-μ-ρ, and c-β-ρ. Going back and forth among them, or their perturbation formulas, is a matter of algebra.

B.2.1 Perturbations

We can derive the perturbations $\delta\kappa$ and $\delta\mu$ in terms of perturbations in α-β-ρ:

$$
\begin{aligned}
\kappa &= \rho\, c^2 = \rho\left(\alpha^2 - \tfrac{4}{3}\beta^2\right), \\
\delta\kappa &= \delta\rho\left(\alpha^2 - \tfrac{4}{3}\beta^2\right) \; + \; \rho\left(2\alpha\,\delta\alpha - \tfrac{4}{3}\,2\beta\,\delta\beta\right) \\
&= \delta\rho\left(\alpha^2 - \tfrac{4}{3}\beta^2\right) \; + \; 2\rho\left(\alpha\,\delta\alpha - \tfrac{4}{3}\beta\,\delta\beta\right),
\end{aligned}
\tag{B.10}
$$

and

$$\begin{aligned}\mu &= \rho\,\beta^2\,, \\ \delta\mu &= \delta\rho\,\beta^2 \;+\; 2\rho\beta\,\delta\beta\,. \end{aligned} \tag{B.11}$$

B.2.2 Fractional Perturbations

We can derive the fractional perturbations $\delta\ln\kappa = \frac{\delta\kappa}{\kappa}$ and $\delta\ln\mu = \frac{\delta\mu}{\mu}$ in terms of fractional perturbations in α-β-ρ by working with Equations (B.10) and (B.11):

$$\begin{aligned}
\left(\frac{1}{\kappa\,\rho\,\alpha\,\beta}\right)[\delta\kappa &= \delta\rho\left(\alpha^2-\tfrac{4}{3}\beta^2\right) \;+\; 2\rho\left(\alpha\,\delta\alpha-\tfrac{4}{3}\beta\,\delta\beta\right)] \\
\left(\frac{1}{\rho\,\alpha\,\beta}\right)\frac{\delta\kappa}{\kappa} &= \frac{1}{\kappa\,\alpha\,\beta}\left(\alpha^2-\tfrac{4}{3}\beta^2\right)\frac{\delta\rho}{\rho} \;+\; 2\rho\,\alpha\,\frac{1}{\kappa\,\rho\,\beta}\,\frac{\delta\alpha}{\alpha} \;-\; 2\rho\,\tfrac{4}{3}\beta\,\frac{1}{\kappa\,\rho\,\alpha}\,\frac{\delta\beta}{\beta} \\
\frac{\delta\kappa}{\kappa} &= \frac{\rho}{\kappa}\left(\alpha^2-\tfrac{4}{3}\beta^2\right)\frac{\delta\rho}{\rho} \;+\; 2\rho\,\alpha\,\frac{\alpha}{\kappa}\,\frac{\delta\alpha}{\alpha} \;-\; 2\rho\,\tfrac{4}{3}\beta\,\frac{\beta}{\kappa}\,\frac{\delta\beta}{\beta} \\
&= \frac{\rho}{\kappa}\frac{\kappa}{\rho}\frac{\delta\rho}{\rho} \;+\; 2\,\frac{\rho}{\kappa}\,\alpha^2\,\frac{\delta\alpha}{\alpha} \;-\; 2\,\frac{\rho}{\kappa}\,\tfrac{4}{3}\,\beta^2\,\frac{\delta\beta}{\beta} \\
&= \frac{\delta\rho}{\rho} \;+\; 2\,\frac{\rho}{\kappa}\left(\alpha^2\,\frac{\delta\alpha}{\alpha}-\tfrac{4}{3}\beta^2\,\frac{\delta\beta}{\beta}\right) \\
&= \frac{\delta\rho}{\rho} \;+\; 2\left(\alpha^2-\tfrac{4}{3}\beta^2\right)^{-1}\left(\alpha^2\,\frac{\delta\alpha}{\alpha}-\tfrac{4}{3}\beta^2\,\frac{\delta\beta}{\beta}\right)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{1}{\rho\,\beta\,\mu}\right)[\delta\mu &= \delta\rho\,\beta^2 \;+\; 2\rho\beta\,\delta\beta] \\
\frac{1}{\rho\,\beta}\frac{\delta\mu}{\mu} &= \frac{\delta\rho}{\rho}\,\frac{\beta}{\mu} \;+\; \frac{2\beta}{\mu}\,\frac{\delta\beta}{\beta} \\
\frac{\delta\mu}{\mu} &= \frac{\delta\rho}{\rho}\,\frac{\rho\,\beta^2}{\rho\,\beta^2} \;+\; \frac{2\rho\,\beta^2}{\rho\,\beta^2}\,\frac{\delta\beta}{\beta} \\
\frac{\delta\mu}{\mu} &= \frac{\delta\rho}{\rho} \;+\; 2\,\frac{\delta\beta}{\beta}\,.
\end{aligned}$$

Thus, we have

$$\frac{\delta\kappa}{\kappa} = \frac{\delta\rho}{\rho} \;+\; 2\left(\alpha^2-\tfrac{4}{3}\beta^2\right)^{-1}\left(\alpha^2\,\frac{\delta\alpha}{\alpha}-\tfrac{4}{3}\beta^2\,\frac{\delta\beta}{\beta}\right), \tag{B.12}$$

$$\frac{\delta\mu}{\mu} = \frac{\delta\rho}{\rho} \;+\; 2\,\frac{\delta\beta}{\beta}\,. \tag{B.13}$$

Table B.1: Sensitivity kernel expressions for three different parameterizations of elastic structure. $\mathbf{D}$ is the deviatoric strain (e.g., *Liu and Tromp*, 2006, Eq. 28).

Model Parameter		Notation	Kernel Expression
Bulk modulus	κ	$K_{\kappa(\mu\rho)}(\mathbf{x})$	$-\kappa \int_0^T [\boldsymbol{\nabla} \cdot \mathbf{s}^\dagger(\mathbf{x}, T-t)][\boldsymbol{\nabla} \cdot \mathbf{s}(\mathbf{x}, t)]\, dt$
Shear modulus	μ	$K_{\mu(\kappa\rho)}(\mathbf{x})$	$-2\mu \int_0^T \mathbf{D}^\dagger(\mathbf{x}, T-t) : \mathbf{D}(\mathbf{x}, t)\, dt$
Density	ρ	$K_{\rho(\kappa\mu)}(\mathbf{x})$	$-\rho \int_0^T \mathbf{s}^\dagger(\mathbf{x}, T-t) \cdot \partial_t^2 \mathbf{s}(\mathbf{x}, t)\, dt$
Bulk sound speed	c	$K_{c(\beta\rho)}(\mathbf{x})$	$2\, K_{\kappa(\mu\rho)}$
S wavespeed	β	$K_{\beta(c\rho)}(\mathbf{x})$	$2\, K_{\mu(\kappa\rho)}$
Density	ρ	$K_{\rho(c\beta)}(\mathbf{x})$	$K_{\rho(\kappa\mu)} + K_{\kappa(\mu\rho)} + K_{\mu(\kappa\rho)}$
P wavespeed	α	$K_{\alpha(\beta\rho)}(\mathbf{x})$	$\left(2 + \frac{8\mu}{3\kappa}\right) K_{\kappa(\mu\rho)}$
S wavespeed	β	$K_{\beta(\alpha\rho)}(\mathbf{x})$	$2\, K_{\mu(\kappa\rho)} - \frac{8\mu}{3\kappa}\, K_{\kappa(\mu\rho)}$
Density	ρ	$K_{\rho(\alpha\beta)}(\mathbf{x})$	$K_{\rho(\kappa\mu)} + K_{\kappa(\mu\rho)} + K_{\mu(\kappa\rho)}$