A Kakeya estimate for sticky sets using a planebrush

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ABSTRACT

A Besicovitch set is defined as a compact subset of $\mathbb{R}^n$ which contains a line segment of length 1 in every direction. The Kakeya conjecture says that every Besicovitch set has Minkowski and Hausdorff dimensions equal to n. This thesis gives an improved Hausdorff dimension estimate, $d \geqslant 0.60376707287\, n + O(1)$, for Besicovitch sets displaying a special structural property called "stickiness." The improved estimate comes from using an incidence geometry argument called a "k-planebrush," which is a higher dimensional analogue of Wolff's "hairbrush" argument from 1995.

In addition, an x-ray transform estimate is obtained as a corollary of Zahl's k-linear estimate in 2019. The x-ray estimate, together with the estimate for sticky sets, implies that all Besicovitch sets in $\mathbb{R}^n$ must have Minkowski dimension greater than $(2 - \sqrt{2} + \varepsilon)n$. Though this Minkowski dimension estimate is not as good as one previously known from Katz-Tao (2000), it provides a new proof of the same result.

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Chapter 1

INTRODUCTION

1.1 Outline of thesis and summary of notation

Chapter 1 of this document contains a brief exposition of the Kakeya problem. It introduces the Kakeya set conjecture and the Kakeya maximal conjecture along with the concept of a "sticky set," providing context for the results proved in later chapters. Chapter 2 introduces some variants of the Kakeya problem, including the multilinear Kakeya problem and the k-linear Kakeya problem, and outlines some recent results in the k-linear Kakeya problem (Hickman-Rogers-Zhang [12] and Zahl [27]) which are used in Chapters 3 and 4.

Chapter 3 gives a Kakeya maximal estimate, at a dimension depending on a parameter k, for sets of δ-tubes obeying a special structural property called k-planiness. Chapter 4 combines the results of Chapter 3 with recent results in the k-linear Kakeya conjecture to prove an improved Hausdorff dimension estimate for sticky sets. In the final section of Chapter 4, an x-ray transform estimate is obtained as a corollary of Zahl ([27]). The x-ray transform estimate is used to give a Minkowski dimension estimate, $d \geqslant (2-\sqrt{2}) + 10^{-10})n + O(1)$, for all Besicovitch sets. Though this Minkowski dimension estimate is not as good as one previously known (from Katz and Tao [18]), it provides a different proof of the result.

Table 1.1: Summary of notation

$A \lesssim B$	$\exists C > 0 : A \leqslant CB$
$A \sim B$	$A \lesssim B$ and $B \lesssim A$
$A \lessapprox B$	$\exists C_\varepsilon > 0 : A \leqslant C_\varepsilon \delta^{-\varepsilon} B$
$A \approx B$	$A \lessapprox B$ and $B \lessapprox A$
$\dim_H(E)$	Hausdorff dimension of E
$\underline{\dim_M}(E)$	Minkowski dimension of E
$\overline{\dim_M}(E)$	upper Minkowski dimension of E
$\dim_{\mathcal{P}}(E)$	Packing dimension of E
$\lvert E \rvert$	Lebesgue measure of E
ν	normalized surface measure on $\mathbb{S}^{n-1}$
$\#E$	Cardinality of E
$N_\delta(E)$	δ-neighborhood of E

1.2 From Kakeya needle sets to the Kakeya maximal conjecture

In 1920, the Russian mathematician A.S. Besicovitch studied the following problem in a paper ([3]) published in a Russian monthly periodical:

Given a function of two variables, Riemann integrable on a plane domain, does there always exist a pair of mutually perpendicular directions such that the repeated simple integration along the two directions exists and gives the value of the integral over the domain?

This problem reduced to proving the existence a set of Lebesgue plane measure 0 which contained a line segment of length $\geqslant 1$ in every direction. Besicovitch solved the problem, answering the above question in the negative, by demonstrating the construction of such a set. Later, in 1928, Besicovitch wrote that he had become aware of a "twin problem" posed by the Japanese mathematician S. Kakeya in 1917 ([2]). The problem had never before come to his attention because of the isolation of Russia following the civil war. Kakeya's problem was this:

In the class of figures in which a line segment of length 1 can be rotated by 360^o *without ever leaving the figure, which one has the smallest area?*

Each of these two problems was concerned with sets containing a line segment of length 1 in every direction, with Kakeya's problem imposing the additional constraint that there should be a continuous transition.

Definition 1. *A* Kakeya needle set *is a set in* $\mathbb{R}^n$ *in which a line segment of length 1 can be rotated continuously by* 360^o *without leaving the set.*

Definition 2. *A* Besicovitch set *is a compact set in* $\mathbb{R}^n$ *which contains a line segment of length 1 in every direction.*

In his paper [2], Besicovitch gave a modified version of his construction of a measure zero Besicovitch set in [3] to solve Kakeya's problem, thereby disproving Kakeya's original conjecture that the smallest figure in which one could continuously rotate a unit line segment by 360^o was the hypocycloid of area $\pi/8$ ([13]).

Decades later, in 1971, Besicovitch sets resurfaced in the work of Charles Fefferman, who was studying the following question.

If T is an operator on $L^p(\mathbb{R}^n)$ defined by $\widehat{Tf} = \chi_B \hat{f}$, where χ_B is the characteristic function of the unit ball in $\mathbb{R}^n$, for which values of p is it the case that, for all $f \in L^p(\mathbb{R}^n)$,

$$||Tf||_{L^p(\mathbb{R}^n)} \leqslant C||f||_{L^p(\mathbb{R}^n)}?$$

The "disc conjecture" stated the the above estimate holds for all

$$\frac{2n}{n+1} < p < \frac{2n}{n-1};$$

these necessary conditions were demonstrated by Hertz using the example where f is a Schwartz function whose Fourier transform is identically 1 on the unit ball. Prior to Fefferman's work, the disc conjecture was known to be true in $\mathbb{R}^1$. It thus came as a surprise when Fefferman showed that the disc conjecture is false whenever $p \neq 2$ ($n > 1$) using, as a lemma, Besicovitch's construction of measure zero Besicovitch sets ([9]). The work of Fefferman and Hertz led to the following further conjecture, which has come to be called the Kakeya set conjecture, about Besicovitch sets:

Conjecture 3 (Kakeya set conjecture)**.** *Every Besicovitch set in $\mathbb{R}^n$ has Minkowski and Hausdorff dimensions equal to n.*

The Kakeya set conjecture in $\mathbb{R}^2$ was solved by Davies ([8]) but remains open for $n \geqslant 3$. The best known results are due to Katz-Zahl ([19]) in $\mathbb{R}^3$ and due to Katz-Zahl ([14]) in $\mathbb{R}^4$. The best known bounds in higher dimensions are due to Katz-Tao ([18]). Corollary 47 of this thesis gives an new proof of a Minkowski dimension estimate for Besicovitch sets which was previously known from [18].

The current state-of-the-art for the Kakeya set conjecture (Hausdorff dimension version) is given in Table 1.2. The motivation for the Kakeya set conjecture is discussed in Section 1.3 below.

1.3 Formulation of the Kakeya maximal conjecture

The Kakeya maximal conjecture is a more quantitative formulation of the Kakeya set conjecture in terms of a maximal operator. The formulation of the conjecture given below was given by Bourgain [5], though a similar formulation had been given by Córdoba [7], who also proved the conjecture in $\mathbb{R}^2$.

Table 1.2: State of the art for the Kakeya conjecture

n	$\dim_H(K) \geqslant$	Proved by
3	$2.5+\varepsilon$	Katz-Zahl [15]
4	$3.058\ldots$	Katz-Zahl [14]
5	3.6	Hickman-Rogers-Zhang [12]
6	$7-2\sqrt{2}$	Katz-Tao [18]
7	34/7	Hickman-Rogers-Zhang [12]
$\vdots$	$\vdots$	

For any small number $\delta > 0$, $e \in \mathbb{S}^{n-1}$ and $a \in \mathbb{R}^n$, define $T_e^\delta(a)$ to be the δ-neighborhood of a line segment pointing in direction e and centered at point a, so $T_e^\delta(a)$ is a tube (or cylinder) of radius δ and length 1. If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, its Kakeya maximal function $f_\delta^* : \mathbb{S}^{n-1} \to \mathbb{R}$ is defined by

$$f_\delta^*(e) = \delta^{1-n} \sup_{a\in\mathbb{R}^n} \int_{T_e^\delta(a)} |f|.$$

If f is the indicator function of the set $N_\delta(B)$, where B is any Besicovch set, any bound of the type

$$||f_\delta^*||_q \leqslant C||f||_p$$

with $1 < q < \infty$, $p < \infty$, and C independent of δ, fails to hold. In addition, if f is the indicator function of the set $N_\delta(B)$, where B is any Besicovch set of zero Lebesgue measure, a bound of the form

$$||f_\delta^*||_p \leqslant C_\varepsilon \delta^{-\varepsilon}||f||_p$$

cannot hold if $p > n$. Thus the strongest estimate which may be expected is of the following form.

Conjecture 4 (Kakeya maximal conjecture, version 1)**.** *For every $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that for all $\delta > 0$ and $f \in L^n(\mathbb{R}^n)$,*

$$||f_\delta^*||_{L^n(\mathbb{S}^{n-1})} \leqslant C_\epsilon \delta^{-\varepsilon}||f||_{L^n(\mathbb{R}^n)}.$$

Interpolating this (by the Riesz-Thorin interpolation theorem) with the trivial estimate

$$||f_\delta^*||_{L^\infty(\mathbb{S}^{n-1})} \lesssim \delta^{1-n}||f||_{L^1(\mathbb{R}^n)},$$

one obtains a conjecture

$$||f_\delta^*||_{L^q(\mathbb{S}^{n-1})} \lessapprox \delta^{1-n/p}||f||_{L^p(\mathbb{R}^n)} \qquad \text{where } 1 \leqslant p \leqslant n \text{ and } q = (n-1)p'.$$

Partial progress in the Kakeya conjecture is given by estimates of this form. The full Kakeya maximal conjecture is solved only in $\mathbb{R}^2$ ([7]) and remains open in higher dimensions. The Kakeya maximal conjecture implies the Kakeya set conjecture, as is demonstrated in Section 1.5 below. Table 1.3 gives the best known results for the Kakeya maximal conjecture in higher dimensions.

Table 1.3: State of the art for the Kakeya maximal conjecture

n	$\dim_H(K) \geqslant$	Result proved by
2	2	Córdoba
3	$5/2+\varepsilon$	Katz-Zahl
4	3.057...	Katz-Zahl
5	18/5	Hickman-Rogers-Zhang
6	4	Wolff
7	34/7	Hickman-Rogers-Zhang
8	21/4	Hickman-Rogers-Zhang
9	6	Hickman-Rogers-Zhang
10	13/2	Hickman-Rogers-Zhang
⋮		⋮

1.4 Importance of the Kakeya maximal conjecture

There are many prominent open conjectures in diverse areas of math which have been shown, if they are true, to each imply that the Kakeya set conjecture and the Kakeya maximal conjecture are true. Some of these conjectures are mentioned below.

Restriction conjecture.

If σ is the surface measure on the unit sphere $\mathbb{S}^{n-1}$, is there an estimate

$$||\widehat{f d\sigma}||_{L^p(\mathbb{R}^n)} \lesssim ||f||_{L^p(d\sigma)}$$

for all $p \geqslant \frac{2n}{n-1}$. (There are also versions of the restriction conjecture for other hypersurfaces with boundary, rather than the sphere.)

It was proved by Fefferman in [9] that the restriction conjecture implies the Kakeya maximal conjecture.

Bochner-Riesz conjecture.

If $\delta > 0$, and if S^δ is an operator on $L^p(\mathbb{R}^n)$ defined by $\widehat{S^\delta f}(\xi) = (1-|\xi|^2)^\delta \chi_B(\xi)\hat{f}(\xi)$, where χ_B is the characteristic function of the unit ball in $\mathbb{R}^n$, then

$$||S^\delta f||_{L^p(\mathbb{R}^n)} \leqslant C||f||_{L^p(\mathbb{R}^n)}$$

for all p satisfying $\left|\frac{1}{p} - \frac{1}{2}\right| \leqslant \frac{2\delta+1}{2n}$.

Tao showed that the Bochner Riesz conjecture implies the restriction conjecture ([22]). The greater importance of the Kakeya maximal conjecture is that there are methods, first developed by Bourgain in 1991, whereby progress on the Kakeya maximal conjecture would imply progress on the Bochner-Riesz conjecture and restriction conjecture ([6]). However, even if the Kakeya maximal conjecture were proved in full, the Bochner Riesz and restriction conjectures would not be completely solved.

Local smoothing conjecture.
Let u be the solution of the initial value problem for the wave equation in n space dimensions:

$$\Box u = 0, \quad u(\cdot, 0) = f, \quad \frac{\partial u}{\partial t}(\cdot, 0) = 0.$$

Then

$$\forall \varepsilon > 0 \quad \exists C_\varepsilon > 0 : \quad ||u||_{L^p(\mathbb{R}^n \times [1,2])} \leqslant C_\varepsilon ||f||_{p,\varepsilon},$$

for all $p \in [2, \frac{2n}{n-1}]$. *Here* $||\cdot||_{p,\varepsilon}$ *is the inhomogeneous* L^p *Sobolev norm with* ε *derivatives.*

This conjecture was made by Sogge and is known to imply the Bochner Riesz conjecture ([20]).

Montgomery's conjecture.
Assume $T \leqslant N^2$. *Let*

$$D(s) = \sum_{n=1}^{N} a_n n^{is}$$

where $||\{a_n\}||_{\ell^\infty} \leqslant 1$. *Let* $\mathcal{T}$ *be a 1-separated subset of* $[0, T]$. *Then is there an estimate*

$$\forall \varepsilon > 0 \quad \exists C_\varepsilon > 0 : \quad \sum_{t \in \mathcal{T}} |D(t)|^2 \leqslant C_\varepsilon N^{1+\varepsilon}(N + |\mathcal{T}|)?$$

Bourgain showed in [4] that Montgomery's conjecture implies the Kakeya set conjecture.

1.5 Equivalent versions of the Kakeya maximal conjecture

For the following sections, it may be more convenient to work with some equivalent formulations of the Kakeya maximal conjecture, which are written below.

Conjecture 5 (Kakeya maximal conjecture, version 2)**.** *For every $\varepsilon > 0$, if $\mathbb{T}$ is a family of δ-tubes pointing in δ-separated directions, there is a number $C_\varepsilon > 0$ such that*

$$||\sum_{T\in\mathbb{T}} \chi_T||_{L^n(\mathbb{R}^{n-1})} \leqslant C_\varepsilon \delta^{-\varepsilon}\big(\sum_{T\in\mathbb{T}} |T|\big)^{(n-1)/n}.$$

Conjecture 6 (Kakeya maximal conjecture, version 3)**.** *For every $\varepsilon > 0$, if*

(i) *$\mathbb{T}$ is a set of δ-tubes in $\mathbb{R}^n$ that point in δ-separated directions, and*

(ii) *$Y(T)$ is a subset of T for every $T \in \mathbb{T}$, and $0 \leqslant \lambda \leqslant 1$ is a number such that $|Y(T)| \geqslant \lambda|T|$ for every $T \in \mathbb{T}$,*

then there exists a constant $C_\varepsilon > 0$ such that

$$\left|\bigcup_{T\in\mathbb{T}} Y(T)\right| \geqslant C_\epsilon \delta^\varepsilon \lambda^n \sum_{T\in\mathbb{T}} |T|. \tag{1.1}$$

Theorem 7. *The three versions of the Kakeya conjecture are equivalent.*

It will be shown that version 3 $\Rightarrow$ version 2 $\Rightarrow$ version 1 $\Rightarrow$ version 3.

Proof that version 3 implies version 2. Assume that version 3 of the Kakeya conjecture is true. Let $\mathbb{T}$ be a set of δ-tubes pointing in δ-separated directions. For every nonnegative integer k, define

$$E_k = \{x \in \mathbb{R}^n : 2^k \leqslant \#\mathbb{T}(x) < 2^{k-1}\}.$$

The statement of version 2 of the Kakeya conjecture is equivalent to

$$\sum_k |E_k| 2^{kn/(n-1)} \leqslant C_\epsilon \delta^{-\epsilon/(n-1)} \sum_{T\in\mathbb{T}} |T|$$

By the pigeonhole principle, there exists a value of k for which

$$|E_k| 2^{kn/(n-1)} \gtrsim \frac{1}{\log(1/\delta)} \sum_k |E_k| 2^{kn/(n-1)}.$$

Since $\log(1/\delta) \lesssim \delta^\varepsilon$ for any $\varepsilon > 0$, it is sufficient to show that for this value of k,

$$|E_k|^{n-1} 2^{kn} \leqslant C_\epsilon \delta^{-\varepsilon} \left(\sum_{T\in\mathbb{T}} |T|\right)^{n-1}$$

For each $T \in \mathbb{T}$, define a shading $Y_k(T)$ by

$$Y_k(T) = T \cap E_k.$$

Note that for a certain value of ℓ, we have

$$|E_k| \cdot 2^k \sim \sum_{T \in \mathbb{T}} |Y_k(T)| \sim \frac{1}{\log \delta} \sum_{T \in \mathbb{T}_\ell} 2^{-\ell}|T| \sim \frac{1}{\log \delta} |\bigcup_{T \in \mathbb{T}_\ell} Y_k(T)| \cdot 2^k,$$

where

$$\mathbb{T}_\ell = \{T \in \mathbb{T} : 2^{-\ell-1} < |Y_k(T)| \leqslant 2^{-\ell}\}.$$

This leads to two conclusions, first

$$|E_k| \approx |\bigcup_{T \in \mathbb{T}_\ell} Y_k(T)|$$

and second

$$2^k \approx \frac{2^{-\ell}\delta^{n-1}\#\mathbb{T}_\ell}{|E_k|}.$$

From the second conclusion, the statement to show becomes

$$\frac{2^{-\ell d}(\sum_{T \in \mathbb{T}_\ell} |T|)^n}{|E_k|} \leqslant C_\epsilon \delta^{-\epsilon} \left(\sum_{T \in \mathbb{T}} |T| \right)^{n-1}.$$

Rearranging and using the first conclusion above, this is

$$|\bigcup_{T \in \mathbb{T}_\ell} Y_k(T)| \gtrsim \delta^\epsilon 2^{-\ell n} \sum_{T \in \mathbb{T}_\ell} |T|,$$

which is true from version 3 of the Kakeya conjecture. □

Proof that version 2 implies version 1. Let $\varepsilon > 0$ be given, and let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let $\Omega \subseteq \mathbb{S}^{n-1}$ be a maximal δ-separated set of directions. For every $e \in \Omega$ define T_e as the δ-tube centered at the position that achieves the supremum

$$\sup_{a \in \mathbb{R}^n} \int_{T_e^\delta(a)} |f(x)| \, dx,$$

and then set $\mathbb{T} := \{T_e : e \in \Omega\}$. From version 2 of the conjecture,

$$|| \sum_{T \in \mathbb{T}} \chi_T ||_{L^n(\mathbb{R}^n)} \lesssim \delta^{-\varepsilon}.$$

The crucial observation used is that, provided $|e - e'| < \delta$, there is some $C > 0$ such that $|f^*_\delta(e)| \sim |f^*_{C_\delta}(e')|$. Thus

$$||f^*_\delta||^n_{L^n(\mathbb{S}^{n-1})} \lesssim \sum_{e \in \Omega} \int_{N_\delta(\{e\})} |f^*_\delta(\omega)|^n \, d\omega \lesssim \delta^{n-1} \sum_{e \in \Omega} |f^*_{C\delta}(e)|^n.$$

Using duality between ℓ_n and $\ell_{n'}$, there exist numbers $\{y_e\}_{e\in\Omega}$ such that

$$\delta^{n-1}\sum_{e\in\Omega} y_e f_\delta^*(e) = \left(\delta^{n-1}\sum_{e\in\Omega}|f_\delta^*(e)|^n\right)^{1/n}\left(\delta^{n-1}\sum_{e\in\Omega}|y_e|^{n'}\right)^{1/n'}$$

and furthermore

$$\delta^{n-1}\sum_{e\in\Omega}|y_e|^{n'} = 1.$$

In the following estimates we will abuse notation by using y_e interchangably with y_T, where $T \in \mathbb{T}$ is understood to be T_e as defined above in the proof.

$$\begin{aligned}
||f_\delta^*||_{L^n(\mathbb{S}^{n-1})} &\lesssim \left(\delta^{n-1}\sum_{e\in\Omega}|f_\delta^*(e)|^n\right)^{1/n} \\
&\lesssim \delta^{n-1}\sum_{e\in\Omega} y_e \cdot |f_\delta^*(e)| \\
&\lesssim \int \sum_{T\in\mathbb{T}} \chi_T(x)\cdot y_T \cdot f(x)\, dx \\
&\leqslant ||\sum_{T\in\mathbb{T}} y_T\cdot\chi_T||_{n/(n-1)} \cdot ||f||_{L^n(\mathbb{R}^n)}.
\end{aligned}$$

It now remains to show that

$$||\sum_{T\in\mathbb{T}} y_T\cdot\chi_T||_{n/(n-1)} \lesssim \delta^{-\varepsilon}.$$

By dyadic pigeonholing, it is possible to choose a subset $\mathbb{T}' \subseteq \mathbb{T}$ for which

$$||\sum_{T\in\mathbb{T}'} y_T\cdot\chi_T||_{n/(n-1)} \gtrapprox ||\sum_{T\in\mathbb{T}} y_T\cdot\chi_T||_{n/(n-1)}$$

and for which there is number $N \in \mathbb{N}$ such that $y_T \sim 2^N$ for all $T \in \mathbb{T}'$. It now follows that

$$\begin{aligned}
||\sum_{T\in\mathbb{T}} y_T\cdot\chi_T||_{n/(n-1)} &\lessapprox 2^N \cdot ||\sum_{T\in\mathbb{T}'}\chi_T||_{n/(n-1)} \\
&\lessapprox 2^N\,\delta^{-\varepsilon}(\delta^{n-1}\#\mathbb{T}')^{1/n'} \\
&\sim \delta^{-\varepsilon}\left(\delta^{n-1}\sum_{T\in\mathbb{T}'}|y_T|^{n'}\right)^{1/n'} \\
&\leqslant \delta^{-\varepsilon},
\end{aligned}$$

and this completes the proof. □

Proof that version 1 implies version 3. Let $\varepsilon > 0$ be given, and let $\mathbb{T}$ be a set of δ-tubes which point in δ-separated directions. Let $Y(T) \subseteq T$ be given for every $T \in \mathbb{T}$ and let $0 < \lambda < 1$ be chosen so that $|Y(T)| \geqslant \lambda |T|$ for every $T \in \mathbb{T}$. By dyadic pigeonholing, it is possible to find a number λ' and a subset $\mathbb{T}' \subset \mathbb{T}$ for which

$$\lambda' |T| \leqslant |Y(T)| < 2\lambda' |T| \qquad \text{for all } T \in \mathbb{T}'$$

and

$$\sum_{T \in \mathbb{T}'} |Y(T)| \gtrsim \frac{1}{\log(1/\delta)} \sum_{T \in \mathbb{T}} |Y(T)|.$$

The above statement implies that

$$\lambda' \gtrsim \lambda / \log(1/\delta). \tag{1.2}$$

Let f be the indicator function of $\bigcup_{T \in \mathbb{T}'} Y(T)$. Let $\Omega = \{\mathrm{dir}(T) : T \in \mathbb{T}'\}$ and let $T_e \in \mathbb{T}'$ be the δ-tube pointing in direction e for every $e \in \Omega$. Now

$$\begin{aligned} \lambda' \delta^{n-1} \# \mathbb{T}' \sim \sum_{T \in \mathbb{T}'} |Y(T)| &\lesssim \sum_{e \in \Omega} \int_{T_e} f(x) \, dx \\ &\lesssim \sum_{e \in \Omega} \delta^{n-1} f_\delta^*(e) \\ &\lesssim \sum_{e \in \Omega} \int_{N_\delta(\{e\})} f_\delta^*(\omega) \, d\sigma(\omega) \\ &\lesssim ||f_\delta^*||_{L^n(\mathbb{S}^{n-1})} \cdot (\delta^{n-1} \# \Omega)^{(n-1)/n}. \end{aligned}$$

Thus, from version 1 of the conjecture,

$$||f||_{L^n(\mathbb{R}^n)} \gtrapprox \lambda' (\delta^{n-1} \# \mathbb{T}')^{1/n}.$$

From (1.2),

$$|\bigcup_{T \in \mathbb{T}} Y(T)| \gtrsim \delta^\varepsilon \lambda^n \left(\sum_{T \in \mathbb{T}} |T| \right),$$

which is the conclusion of version 3 of the Kakeya maximal conjecture. $\square$

1.6 Proof of the Kakeya maximal conjecture in $\mathbb{R}^2$

In this section, it will be shown that version 2 of the Kakeya conjecture is true for $\mathbb{R}^2$. The statement to prove is the following. For every $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that whenever $\mathbb{T}$ is a set of δ-tubes pointing in δ-separated directions in $\mathbb{R}^2$,

$$\int_{\mathbb{R}^n} |\sum_{T \in \mathbb{T}} \chi_T|^2 \leqslant C_\varepsilon \delta^{-\varepsilon} \delta \# \mathbb{T}.$$

The above statement may be re-written as

$$\int_{\mathbb{R}^n} \sum_{T_1,T_2\in\mathbb{T}} \chi_{T_1}\chi_{T_2} = \sum_{T_1,T_2\in\mathbb{T}} |T_1 \cap T_2| \leqslant C_\varepsilon \delta^{-\varepsilon} \left(\delta\#\mathbb{T}\right)^2. \tag{1.3}$$

Let $\varepsilon > 0$ be given. Observe that for any two δ-tubes T_1 and T_2, the area of the intersection $T_1 \cap T_2$ depends on the angle $\theta = \angle(\text{dir}(T_1), \text{dir}(T_2))$:

$$|T_1 \cap T_2| \leqslant \frac{\delta^2}{\sin\theta} \lesssim \frac{\delta^2}{\theta}.$$

For each dyadic number θ, let

$$\mathbb{T}_\theta[T_1] = \{T_2 \in \mathbb{T} : \theta \leqslant \angle(\text{dir}(T_1), \text{dir}(T_2)) < 2\theta\}.$$

Since the tubes in $\mathbb{T}$ point in δ-separated directions, $\#\mathbb{T}_\theta[T_1] \lesssim \theta\delta^{-1}$ for every $T_1 \in \mathbb{T}$. Thus,

$$\begin{aligned} \sum_{T_1,T_2\in\mathbb{T}} |T_1 \cap T_2| &\lesssim \sum_{T_1\in\mathbb{T}} \sum_{\theta} \sum_{T_2\in\mathbb{T}_\theta[T_1]} \frac{\delta^2}{\theta} \\ &\lesssim \sum_{T_1\in\mathbb{T}} \sum_{\theta} \delta. \end{aligned}$$

Since there are $\sim \log(1/\delta)$ possible values of θ, and since there always exists a constant $C_\varepsilon > 0$ such that $\log(1/\delta) \leqslant C_\varepsilon\delta^{-\varepsilon}$, this proves (1.3).

1.7 Proof that the Kakeya maximal conjecture implies the Kakeya set conjecture

Slightly different versions of the following proof have been written down by Wolff ([24]) and by Tao ([21]) in collections of lecture notes. The proof will show that version 3 of the Kakeya maximal conjecture implies that every Besicovitch set in $\mathbb{R}^n$ has Hausdorff dimension equal to n. Since the Minkowski dimension of a set is never less than its Hausdorff dimension, this will prove that that every Besicovitch set in $\mathbb{R}^n$ also has Minkowski dimension n. The reader will observe that the full strength of the Kakeya maximal conjecture (version 3) will not be needed in the proof; rather, the subcase of the Kakeya maximal conjecture where $\lambda \gtrsim (1/\log(1/\delta))^2$ will be sufficient.

Let B be a Besicovitch set in $\mathbb{R}^n$ and let $\varepsilon > 0$ be arbitrary. The goal is to prove that the $(n-\varepsilon)$-dimensional Hausdorff measure $\mathcal{H}^{n-\varepsilon}(B)$ is infinite. For each direction $\omega \in \mathbb{S}^{n-1}$, let ℓ_ω be a unit line segment which is contained in B and which points in the direction ω. For any set $E \subseteq \mathbb{R}^n$, define a measure μ by

$$\mu(E) = \int_{\mathbb{S}^{n-1}} |E \cap \ell_\omega| \, d\omega,$$

where $|E \cap \ell_\omega|$ is the 1-dimensional Lebesgue measure of $E \cap \ell_\omega$. Thus $\mu(B) = 1$. This measure is sometimes called the canonical measure associated to a Besicovitch set.

Let $\delta_0 > 0$ be given and let $\mathcal{U}$ be a covering of B by balls of any radius smaller than δ_0. For any $k \in \mathbb{Z}$, let $\mathcal{U}_k$ be the subcollection of balls given by

$$\mathcal{U}_k = \{U \in \mathcal{U} : 2^{-k-1} < \operatorname{diam}(U) \leqslant 2^{-k}\}$$

and define B_k be the union of all $U \in \mathcal{U}_k$. Since

$$\sum_{k \in \mathbb{Z}\,:\,2^{-k} < \delta_0} \frac{1}{k^2} \lesssim 1,$$

there must exist a value of k for which

$$\mu(B_k) \gtrsim \frac{1}{k^2}.$$

Fix $\delta := 2^{-k}$. Let $\mathbb{T}_0$ be the collection of δ-tubes T_ω, each defined as the δ-neighborhood of the line segment ℓ_ω, for every $\omega \in \mathbb{S}^{n-1}$. Observe that

$$\mu(B_k) = \delta^{1-n} \int_{\mathbb{S}^{n-1}} \int_{B_k} 1_{T_\omega}(x)\; dx\; d\omega.$$

Consider the following tiling of $\mathbb{S}^{n-1}$: let $\mathbb{R}^{n-1}$ be tiled by δ-cubes given by translates $(0,\delta]^{n-1}$ placed end-to-end and let a "tile" on the sphere be given by the image of one of these δ-cubes via the map $\xi \mapsto (\xi, \sqrt{1-|\xi|^2})$ sending $\mathbb{R}^{n-1}$ to the northern hemisphere. Thus the sphere $\mathbb{S}^{n-1}$ may be covered by tiles θ so that

$$\mu(B_k) = \delta^{1-n} \sum_{\theta} \int_{\theta} \int_{B_k} 1_{T_\omega}(x)\; dx\; d\omega.$$

Further, it may arranged that half of all the tiles θ are removed from the sum above without reducing the right hand side above by a factor of more than one half, so that any two tiles are separated by a distance of at least δ. For every remaining tile θ, there must exist $\omega \in \theta$ such that $\int_{B_k} 1_{T_\omega}(x)\; dx \geqslant \int_\theta \int_{B_k} 1_{T_\omega}(x)\; dx\; d\omega$. Let $\mathbb{T}$ be the subcollection of $\mathbb{T}_0$ consisting of this choice of ω for every tile θ; then there are $\sim \delta^{1-n}$ tubes in $\mathbb{T}$ and they point in δ-separated directions. Define a shading for each tube $T \in \mathbb{T}$ by

$$Y(T) := T \cap B_k,$$

where $k = \log(1/\delta)$ is the chosen value of k above, for which $\mu(B_k) \gtrsim k^{-2}$. Thus

$$\sum_{T \in \mathbb{T}} |Y(T) \cap B_k| \gtrsim \mu(B_k) \geqslant \left(\frac{1}{\log(1/\delta)}\right)^2.$$

Using dyadic pigeonholing, it is possible to find a refinement $\mathbb{T}' \subseteq \mathbb{T}$ and a number $0 < \lambda < 1$ such that $|Y(T)| \sim \lambda|T|$ for every $T \in \mathbb{T}'$, and further $\sum_{T\in\mathbb{T}'} |Y(T)| \gtrsim \frac{1}{\log(1/\delta)} \sum_{T\in\mathbb{T}} Y(T)$. From this, and the fact that $\mathbb{T}'$ continues to be direction separated, it follows that $\lambda \gtrsim (1/\log(1/\delta))^3$. With an appropriate implicit constant, it may then be arranged that $\lambda \gtrsim \delta^{\varepsilon/4n}$, and if the Kakeya maximal conjecture is true,

$$|\bigcup_{T\in\mathbb{T}} Y(T)| \gtrsim \delta^{\varepsilon/2},$$

then it follows that $|B_k| \gtrsim \delta^{\varepsilon}$. Since each ball in $\mathcal{U}_k$ has volume $\sim \delta^n$, it must be that $\#\mathcal{U}_k \gtrsim \delta^{-n+\varepsilon/2}$. Thus,

$$\sum_{U\in\mathcal{U}_k} \delta^{n-\varepsilon} \geqslant \delta^{-\varepsilon/2},$$

and $\mathcal{H}^{n-\varepsilon}(B) = \infty$. This completes the proof.

1.8 Introduction to the x-ray transform estimate and sticky sets

In 1998, Wolff observed a certain estimate involving the x-ray transform, defined below, had a close connection to the Kakeya problem ([25]). The question considered by Wolff about the x-ray transform is written below:

Let $\mathcal{G}$ be the space of all lines $\ell = \ell_{x,v} \subset \mathbb{R}^n$, where x, v are points within the unit ball centered at the origin in $\mathbb{R}^{n-1}$, and $\ell_{x,v}$ is parametrized by

$$\ell_{x,v} = \{(x + vt, t) : t \in [0, 1]\}.$$

For any function $f \in L^1(\mathbb{R}^n)$, the x-ray transform $Xf : \mathcal{G} \to \mathbb{R}$ *is defined by*

$$Xf(\ell) = \int_\ell f.$$

For which $1 \leqslant p, q, r \leqslant \infty$ and $\alpha \geqslant 0$ does the estimate

$$||Xf||_{L^q_v L^r_x} \leqslant ||f||_{L^p_\alpha},$$

hold? Here L^p_α is the Sobolev space $(1 + \sqrt{-\Delta})^{-\alpha} L^p$.

Various estimates of this kind have been proved (e.g. [25]) and more are conjectured to hold. The relation of this question to the Kakeya problem becomes more evident in the following equivalent formulation of a certain mixed norm x-ray estimate:

Definition 8. *We say there is an* x-ray estimate at dimension d *if there exist* $0 \leqslant \alpha, \gamma \leqslant 1$ *for which the following statement holds: for any* δ*-separated set* Ω *of directions and any collection* $\mathbb{T}$ *of* δ*-tubes pointing in directions in* Ω*, of which no tube is contained in the tenfold dilate of another tube, the following estimate holds:*

$$||\sum_{T\in\mathbb{T}} \chi_T||_{d'} \lessapprox \delta^{1-\frac{n}{d}} m^{1-\alpha} (\delta^{n-1}\#\Omega)^{\gamma},$$

where m *is the directional multiplicity of* $\mathbb{T}$*.*

It was first observed by Katz, Łaba and Tao that an x-ray transform estimate could be used to deduce the prevalence of special structure in "extremal" Besicovitch sets. More precisely, the following result is proved in [17]:

Proposition 9. *Suppose that an x-ray transform estimate holds at dimension* d *in* $\mathbb{R}^n$*. If there exists a Besicovitch set* B *satisfying* $\dim_M(B) \leqslant d + \varepsilon$*, then there exists a set of* δ*-tubes* $\mathbb{T}$ *with the following properties:*

(i) The tubes in $\mathbb{T}$ *point in* δ*-separated directions,*

(ii) There exists a set of $\delta^{1/2}$*-tubes* $\mathbb{T}_{\delta^{1/2}}$ *and a partition of* $\mathbb{T}$ *into sets* $\mathbb{T}[T_{\delta^{1/2}}]$ *given by*

$$\mathbb{T}[T_{\delta^{1/2}}] = \{T \in \mathbb{T} : T \subseteq T_{\delta^{1/2}}\};$$

satisfying the following properties:

$$\begin{aligned} \#\mathbb{T} &\approx \delta^{1-n}, \\ \#\mathbb{T}_{\delta^{1/2}} &\approx \delta^{(1/2)(1-n)} \quad \textit{and} \\ \#\mathbb{T}[T_{\delta^{1/2}}] &\approx \delta^{(1/2)(1-n)} \end{aligned}$$

(iii) $|\bigcup_{T\in\mathbb{T}} T| \lessapprox \delta^{n-d-\varepsilon}$.

(iv) $|\bigcup_{T_{\delta^{1/2}}\in\mathbb{T}_\delta} T| \lessapprox (\delta^{1/2})^{n-d-\varepsilon}$.

The proof of a slightly different version of this proposition is given in Section 4.5.

Sets of δ-tubes satisfying properties (i) and (ii) in the list above are called *sticky sets*. Katz-Łaba-Tao ([17]) demonstrated that sticky sets have a self-similar structure and exploited the structural properties of sticky sets to give an improved Minkowski dimension estimate for Besicovitch sets in $\mathbb{R}^3$.

Very recently, the Kakeya conjecture was proved by Wang and Zahl ([23]) in the special case of sticky sets in $\mathbb{R}^3$. Chapter 4 in this thesis gives an improved Hausdorff dimension estimate for sticky sets in higher dimensions.

Chapter 2

RECENT PROGRESS IN THE k-LINEAR KAKEYA CONJECTURE

2.1 The multilinear Kakeya theorem

The most recent results in the higher-dimensional Kakeya problem have been proved by means of progress on a variant of the Kakeya problem called the multilinear Kakeya problem. The multilinear variant of the Kakeya problem was first introduced by Benett, Carbery and Tao in [1]. The following theorem was proved up to endpoint in [1] and the endpoint was proved by Guth in [10].

Theorem 10 (Multilinear Kakeya theorem). *Let $2 \leqslant k \leqslant n$. There exists a constant C so that, if $\mathbb{T}_1, \ldots, \mathbb{T}_k$ are collections of δ-tubes in $\mathbb{R}^n$, then*

$$\|\sum_{T_1 \in \mathbb{T}_1} \cdots \sum_{T_k \in \mathbb{T}_k} \big(\chi_{T_1} \cdots \chi_{T_k} \, |\mathrm{dir}(T_1) \wedge \cdots \wedge \mathrm{dir}(T_k)|\big)^{1/k}\|_{k/(k-1)} \leqslant C\delta^{1-\frac{n}{k}} \prod_{i=1}^{k} \big(\sum_{T_i \in \mathbb{T}_i} |T_i|\big)^{1/k}.$$

As was observed by Bourgain and Guth in 2010, the multlinear Kakeya estimates could lead to estimates for the Kakeya maximal function, but the resulting estimates were weaker than the maximal estimates proved previously by Wolff ([26]) using incidence geometry techniques.

Though the multilinear Kakeya theorem is sharp, the multilinear variant of the Kakeya problem does not impose the requirement that the δ-tubes point in δ-separated directions. The direction-separated multilinear Kakeya conjecture is still open. The best known estimates for this problem are due to Zahl ([28]). A very similar though slightly weaker estimate was proved by Hickman-Rogers-Zhang in [12].

Theorem 11 (Zahl, k-linear estimate). *Let $2 \leqslant k \leqslant n$. For every $\epsilon > 0$, if $\mathbb{T}$ is a set of δ-tubes pointing in δ-separated directions, then there is a constant $C_\epsilon > 0$ such* *that, for all* $d \leqslant \frac{n^2+n+k^2-k}{2n}$,

$$||(\sum_{T_1,\dots,T_k\in\mathbb{T}} 1_{T_1}\cdots 1_{T_k}|\mathrm{dir}(T_1)\wedge\cdots\wedge\mathrm{dir}(T_k)|^{k/d})^{1/k}||_{d/(d-1)} \leqslant C_\epsilon \delta^{1-\frac{n}{d}-\epsilon}(\sum_{T\in\mathbb{T}}|T|)^{n(d-1)/(n-1)d}.$$

The next two sections (on polynomial partitioning and the polynomial Wolff axioms) mention some of the prominent insights in the proof of the above theorem. Directly below is a corollary of Zahl's k-linear estimate that will be applied in Chapter 4.

Corollary 12. *Let* $2 \leqslant k \leqslant n$. *For every* $\epsilon > 0$, *if* $\mathbb{T}$ *is a set of* δ*-tubes satisfying the following hypotheses:*

i. *The tubes in* $\mathbb{T}$ *point in* δ*-separated directions,*

ii. Y *is a shading of* $\mathbb{T}$ *satisfying* $|Y(T)| \geqslant \lambda|T|$ *for every* $T \in \mathbb{T}$,

iii. *At every point* $x \in \bigcup_{T\in\mathbb{T}} Y(T)$,

$$\#\{(T_1,\dots,T_k)\in\mathbb{T}^k(x)) : |\mathrm{dir}(T_1)\wedge\cdots\wedge\mathrm{dir}(T_k)| > \beta\} \gtrsim \frac{1}{\log(1/\delta)}\#\mathbb{T}^k(x),$$

where $\beta > 0$ *is a fixed constant that does not depend on* δ,

then there is a constant $C_\epsilon > 0$ *such that for all* $d \leqslant \frac{n^2+n+k^2-k}{2n}$,

$$\left|\bigcup_{T\in\mathbb{T}} Y(T)\right| \geqslant C_\epsilon \beta\lambda^d\delta^{n-d+\epsilon}(\sum_{T\in\mathbb{T}}|T|)^{(n-d)/(n-1)}.$$

Proof of Corollary 12 *from Zahl's* k*-linear theorem.* Let $\epsilon > 0$ be given, and let $\mathbb{T}$ be a set of δ-tubes that obeys hypotheses i, ii and iii. Then, from condition ii,

$$||(\sum_{T_1,\dots,T_k\in\mathbb{T}} 1_{Y(T_1)}\cdots 1_{Y(T_k)}|\mathrm{dir}(T_1)\wedge\cdots\wedge\mathrm{dir}(T_k)|^{k/d})^{1/k}||_{L^1} \geqslant \beta^{1/d}\sum_{T\in\mathbb{T}}\lambda|T|.$$

By Hölder's inequality,

$$||(\sum_{T_1,\dots,T_k\in\mathbb{T}} 1_{Y(T_1)}\cdots 1_{Y(T_k)}|\mathrm{dir}(T_1)\wedge\cdots\wedge\mathrm{dir}(T_k)|^{k/d})^{1/k}||_1 \leqslant ||(\sum_{T_1,\dots,T_k\in\mathbb{T}} 1_{T_1}\cdots 1_{T_k}|\mathrm{dir}(T_1)\wedge\cdots\wedge\mathrm{dir}(T_k)|^{k/d})^{1/k}||_{d/(d-1)}\cdot|\bigcup_{T\in\mathbb{T}}Y(T)|^{1/d}.$$

Applying Theorem 11 to the statement above, one obtains

$$\beta\lambda^d(\sum_{T\in\mathbb{T}}|T|)^{d-n(d-1)/(n-1)} \leqslant C_\epsilon\delta^{d-n-\epsilon}|\bigcup_{T\in\mathbb{T}}Y(T)|.$$

Rearranging this gives the statement

$$|\bigcup_{T\in\mathbb{T}}Y(T)| \geqslant C_\epsilon\beta\delta^{n-d+\epsilon}\lambda^d(\sum_{T\in\mathbb{T}}|T|)^{(n-d)/(n-1)}.$$

□

2.2 Polynomial partitioning method

The techniques used for proving Zahl's k-linear estimate involve polynomial partitioning, which was introduced by Guth and Katz ([11]) in 2014.

For any polynomial $P : \mathbb{R}^n \to \mathbb{R}$, the zero set of P, $Z(P) := \{x \in \mathbb{R}^n : P(x) = 0\}$, partitions $\mathbb{R}^n$ into "cells", which are the disjoint connected components of $\mathbb{R}^n \backslash Z(P)$. Further, a *$\delta$-shrunken cell* is a connected component of $\mathbb{R}^n \backslash N_\delta(Z(P))$. The "idea" of the polynomial partitioning technique is to consider the contributions to the norm of the function to be bounded (in this case, $||(\sum_{T_1,\ldots,T_k\in\mathbb{T}} 1_{T_1}\cdots 1_{T_k}|\mathrm{dir}(T_1)\wedge\cdots\wedge\mathrm{dir}(T_k)|^{k/d})^{1/k}||_{d/(d-1)})$ from the different cells, and from $N_\delta(Z(P))$, which is the δ-neighborhood of an algebraic variety. If the polynomial P is correctly chosen, the contribution to the function norm in every cell is roughly equal. If the contribution from the cells dominates, the "cellular case" is said to hold; alternatively, if the dominant contribution comes from the δ-neighborhood of the zero set of the polynomial, then the "algebraic case" is said to hold.

The following polynomial partitioning theorem is due to Guth.

Theorem 13. *Fix* $0 < \delta < r$, $x_0 \in \mathbb{R}^n$ *and suppose* $F \in L^1(\mathbb{R}^n)$ *is nonnegative and supported on* $B(x_0, r) \cap N_{4\delta}\mathbf{Z}$ *where* $\mathbf{Z}$ *is an* m*-dimensional transverse complete intersection with* $\deg \mathbf{Z} \leqslant d$. *At least one of the following cases holds:*

1. *Cellular case. There exists a polynomial* $P : \mathbb{R}^n \to \mathbb{R}$ *of degree* $O(d)$ *with the following properties:*
   - a) *#cell*$(P) \sim d^m$ *and each* $O \in$ *cell*(P) *has diameter at most* $r/2$.
   - b) *One may pass to a refinement of cell*(P) *such that if* $\varnothing$ *is defined as a family of* δ*-shrunken cells as above, then*

$$\int_O F \sim d^{-m}\int_{\mathbb{R}^n} F \textit{ for all } O \in \varnothing.$$

2. *Algebraic case. There exists an* $(m-1)$*-dimensional algebraic variety* $\boldsymbol{Y}$ *of degree at most* $O(d)$ *such that*

$$\int_{B(x_0,r)\cap N_{4\delta}\boldsymbol{Z}} F \lesssim \log d \int_{B(x_0,r)\cap N_{\delta}Y} F.$$

In the cellular case, Bezout's theorem is used to deduce that the number of δ-tubes entering each cell can be controlled since by Bezout's theorem, a δ-tube T can cross $Z(P)$ at most $\deg P$ times. Thus, each δ-tube T can enter at most $\deg P+1$ shrunken cells. As long as the cellular case holds, each cell can be re-partitioned into smaller cells, until the cells have radius $\sim \delta$ and the contribution from each cell is bounded.

In the algebraic case, one can obtain a chain of nested algebraic varieties, of decreasing dimensions, through which passing δ-tubes makes the dominant contribution to the norm of the function to be estimated. In this case, a "multi-scale" polynomial Wolff axiom, discussed briefly in the next section, is used to estimate the maximum number of δ-tubes that lie at a tangent to several nested algebraic varieties.

2.3 Polynomial Wolff axioms

The following estimate for the cardinality of a family of direction-separated δ-tubes partially contained in a semialgebraic subset of $\mathbb{R}^n$ is a key result used in proving Theorem 11. The following theorem was proved for $n=3$ by Larry Guth, for $n=4$ by Zahl ([27]), and for $n \geqslant 5$ by Katz and Rogers ([16]).

Theorem 14 (Polynomial Wolff axioms)**.** *Let n and E be integers, with* $n \geqslant 2$*, and let* $\epsilon > 0$*.Then there is a constant* $C(n,E,\epsilon)$ *so that for every semialgebraic set* $S \subset \mathbb{R}^n$ *of complexity at most E and for every set* $\mathbb{T}$ *of direction-separated* δ*-tubes, we have*

$$\#\{T \in \mathbb{T} : T \cap S| \geqslant \lambda |T|\} \leqslant C(n,E,\epsilon)|S|\delta^{1-n-\epsilon}\lambda^{-n}$$

The new feature used in Zahl's k-linear estimate, and in Hickman-Rogers-Zhang ([12]), is the "multiscale polynomial Wolff axiom", which provides a sharp bound for the cardinality of a set of δ-separated δ-tubes that intersect several algebraic varieties of different dimensions at once.

Theorem 15 (Multiscale polynomial Wolff axioms)**.** *For all* $n \geqslant m \geqslant k \leqslant 1$ *and* $\epsilon > 0$*, there is a constant* $C_{n,d,\epsilon}$ *such that*

$$\#\bigcap_{j=k}^{m}\{T \in \mathbb{T} : |T \cap B_{\lambda_j} \cap N_\rho \boldsymbol{Z}_j| \geqslant \lambda_j |T|\} \leqslant C_{n,d,\epsilon}\left(\prod_{j=k}^{m-1}\frac{\rho}{\lambda_j}\right)\left(\frac{\rho}{\lambda_m}\right)^{n-m}\delta^{-(n-1)-\epsilon}$$

whenever $0 < \delta \leqslant p \leqslant \lambda_k \leqslant \cdots \leqslant \lambda_m \leqslant 1$*,* $\mathbb{T}$ *is a direction-separated family of* δ*-tubes,* $\mathbf{Z}_j \subset \mathbb{R}^n$ *are* j*-dimensional algebraic varieties of degree* $\leqslant d$ *and the balls* B_{λ_j} *are nested:* $B_{\lambda_k} \subseteq \ldots \subseteq B_{\lambda_m} \subset \mathbb{R}^n$*.*

This theorem is proved using tools from real algebraic geometry. There are some open questions related to the polynomial Wolff axioms which, if solved, could lead to progress in proving x-ray transform estimates at higher dimensions. Namely, is there a cardinality bound for sets of δ-tubes partially contained in the neighborhood of an algebraic variety, which do not necessarily satisfy the hypothesis that the tubes point in δ-separated directions, but instead, satisfy the hypothesis that the directional multiplicity of tubes is at most $\delta^{-\beta n}$ for some number β between 0 and 1? Such a set of δ-tubes would also have to satisfy the condition that no tube in the set contains the 5-fold dilate of another one.

Chapter 3

NEW KAKEYA ESTIMATE FOR k-PLANY SETS

The Kakeya estimate obtained in this chapter is proved using a geometric structure called the "k-planebrush". A planebrush structure has been used before to prove Kakeya maximal estimates in $\mathbb{R}^4$ by Katz and Zahl in [14]. The planebrush is a higher-dimensional analogue of the "hairbrush" appearing in Wolff ([26]). Wolff's hairbrush consisted of all δ-tubes intersecting a single, central δ-tube T at large angles in $\mathbb{R}^3$. Each δ-tube passing through T at a large angle lies in the δ-neighborhood of a plane that contains the axis of T. The δ-thickened planes containing the axis of T are disjoint from each other away from T, and the each tube in the hairbrush lies in exactly one thickened k-plane. The Kakeya maximal estimate for the plane is applied separately to each thickened plane to get a Kakeya estimate for all the tubes in the hairbrush.

In the argument used in this chapter, the k-planebrush consist of all tubes T' which intersect at least one of many tubes T lying tangent to a $(k-1)$-plane V. The tubes in the planebrush each lie in exactly one of up to δ^{-n+k} δ-thickened k-planes that all contain V. Away from a neighborhood of V, the different thickened k-planes are disjoint, and the best known result for the Kakeya maximal conjecture in $\mathbb{R}^k$ is applied to separately to each thickened k-plane.

In order to ensure that each k-planebrush contains enough δ-tubes to give a good estimate, the k-planebrush technique requires the hypothesis that the collection of $\mathbb{T}$ is $(k-1)$-plany. The meaning of this term is explained below:

Definition 16 (m-Planiness)**.** *A family of δ-tubes $\mathbb{T}$ with shading Y is called m-*plany *if for every δ-cube Q, there exists an m-plane $V(Q)$ such that for every $T \in \mathbb{T}$ satisfying $Y(T) \cap Q \neq \emptyset$,*

$$\angle(T, V(Q)) \lesssim \delta.$$

The main result in this chapter is the following:

Proposition 17. *Let $3 \leqslant k \leqslant n$. For any $\varepsilon > 0$, if $\mathbb{T}$ is a set of $(k-1)$-plany δ-tubes pointing in δ-separated directions, and if each $T \in \mathbb{T}$ has a shading $Y(T) \subseteq T$*

satisfying $|Y(T)| \geqslant \lambda|T|$, *then there exists a constant* $C_\varepsilon > 0$ *such that*

$$|\bigcup_{T\in\mathbb{T}} Y(T)| \gtrsim \lambda^d \delta^{n-d+\varepsilon} \sum_{T\in\mathbb{T}} |T|,$$

where $d \leqslant \frac{2n+2-(\sqrt{2}-1)k}{3}$.

The above result is obtained by combining the next proposition with the Kakeya maximal estimate obtained from Hickman-Rogers-Zhang. The proof of Proposition 17 is given at the end of this chapter.

Proposition 18. *Let* $\varepsilon > 0$. *If* $\mathbb{T}$ *is a set of* δ*-tubes satisfying the following hypotheses:*

i. *The tubes in* $\mathbb{T}$ *point in* δ*-separated directions,*

ii. *Each tube* $T \in T$ *has a subset* $Y(T)$ *and* $|Y(T)| \geqslant \lambda|T|$ *for all* $T \in \mathbb{T}$*,*

iii. *The family* $\mathbb{T}$ *with shading* Y *is (*k*-1)-plany,*

then there exists a constant $C_\epsilon > 0$ *such that*

$$\left|\bigcup_{T\in\mathbb{T}} Y(T)\right| \geqslant C_\epsilon \delta^{(n-2+k-d(k))/3+\epsilon} \lambda^{d(k)/3+1} (\sum_{T\in\mathbb{T}} |T|), \tag{3.1}$$

where $d(k)$ *is the largest exponent for which a Kakeya maximal estimate holds in* $\mathbb{R}^k$.

This proposition is proved in section 3.4 of this chapter.

3.1 Two-ends reduction

The "two-ends reduction" was first introduced by Wolff ([25]) and has been reused by many others in proving Kakeya estimates. The proof given below imitates one given by Tao's lecture notes [21].

Definition 19 (Two-ends condition)**.** *Let* $\epsilon > 0$. *A set* $F \subseteq \mathbb{R}^n$ *obeys the* "two-ends" condition with exponent ϵ *if*

$$|F \cap B(x,r)| \lesssim r^\epsilon |F|$$

for all balls $B(x,r)$.

Proposition 20. *Let $\epsilon > 0$. If $\mathbb{T}$ is a set of δ-tubes satisfying the following hypotheses:*

i. The tubes in $\mathbb{T}$ point in δ-separated directions,

ii. Each tube $T \in T$ has a subset $Y(T)$ and $|Y(T)| \geqslant \lambda|T|$ for all $T \in \mathbb{T}$,

iii. The family $\mathbb{T}$ with shading Y is (k-1)-plany,

iv. For every $T \in \mathbb{T}$, $Y(T)$ obeys the two-ends conjecture with exponent ϵ,

then there exists a constant $C_\epsilon > 0$ such that

$$\left|\bigcup_{T\in\mathbb{T}} Y(T)\right| \geqslant C_\epsilon \delta^{(n-2+k-d(k))/3+\epsilon} \lambda^{d(k)/3+1} (\sum_{T\in\mathbb{T}} |T|),$$

where $d(k)$ is the largest exponent for which a Kakeya maximal estimate holds in $\mathbb{R}^k$.

Lemma 21 (Two-ends reduction)**.** *Proposition* 20 *implies Proposition* 18.

Proof. Let $\epsilon > 0$ be given. Suppose that Proposition 20 is true, and let $\mathbb{T}$ be a set of δ-tubes obeying hypotheses i, ii and iii in the statement of Proposition 18. For any tube $T \in \mathbb{T}$, consider the ratio

$$\frac{|Y(T) \cap B(x,r)|}{r^\epsilon} \tag{$\star$}$$

where $B(x,r)$ is a ball of radius r centered at point x. If $\delta \leqslant r \leqslant 1$, then $|Y(T) \cap B(x,r)| \leqslant \delta^{n-1} r \leqslant r^\epsilon$, so ($\star$) is bounded above by 1. If $r < \delta$, then $|Y(T) \cap B(x,r)| \lesssim r^n \leqslant r^\epsilon$. Thus, the set

$$\left\{\frac{|Y(T) \cap B(x,r)|}{r^\epsilon} : B(x,r) \text{ is a ball with } r \leqslant 1\right\}.$$

is bounded above and has a supremum. If $r = 1$, and the point x is chosen so that $B(x,r)$ contains $Y(T)$, one can see that the supremum must exceed $\lambda|T|$. For each $T \in \mathbb{T}$, it is possible to choose a ball $B(x_T, r_T)$ that comes within a factor of 2 of the supremum, so that

$$|Y(T) \cap B(x_T, r_T)| \geqslant \frac{r_T^\epsilon \lambda |T|}{2}. \tag{3.2}$$

Since $|Y(T) \cap B(x_T, r_T)| \lesssim r_T^n$, and since $\lambda \geqslant \delta^{1-\epsilon}$, the above statement implies that $r_T \gtrsim \delta$. For each T, define a new shading $Y_1(T)$ by

$$Y_1(T) = Y(T) \cap B(x_T, r_T).$$

By performing dyadic pigeonholing on r_T, we may choose a subcollection $\mathbb{T}_1 \subseteq \mathbb{T}$ of tubes such that

$$\rho \leqslant r_T < 2\rho \text{ for every } T \in \mathbb{T}_1,$$

where ρ is a number between $\delta^{1-\epsilon}$ and 1, and

$$\#\mathbb{T}_1 \gtrsim \frac{1}{\log 1/\delta}\#\mathbb{T}.$$

Then for every $T \in \mathbb{T}_1$, (3.2) implies that

$$|Y_1(T)| \gtrsim \rho^\epsilon \lambda |T| \geqslant \delta^\epsilon \lambda |T| \gtrsim \delta^\epsilon \lambda \frac{|T \cap B(x_T, r_T)|}{\rho}. \tag{3.3}$$

The intersection $T \cap B(x_T, r_T)$ is contained in a tube of radius δ and length 2ρ. By means of an isotropic linear rescaling σ which takes $x \mapsto x/2\rho$, each set $T \cap B(x_T, r_T)$ may be scaled to a (δ/ρ)-tube $\tilde{T}$. Then $\sigma(Y_1(T)) \subseteq \tilde{T}$ and from (3.3),

$$\sigma(Y_1(T))| \gtrapprox \frac{\lambda}{\rho}|\tilde{T}|.$$

We will now show that each δ/ρ-tube $\tilde{T}$ with shading $\sigma(Y_1(T))$ satisfies the two-ends condition. For any ball $B(x', r')$, we have

$$\frac{|Y_1(T) \cap B(x', r')|}{(r')^\epsilon} \leqslant \frac{|Y(T) \cap B(x', r')|}{(r')^\epsilon} \leqslant \frac{2 \cdot |Y(T) \cap B(x_T, r_T)|}{(r_T)^\epsilon} \leqslant 2\rho^{-\epsilon}|Y_1(T)|,$$

because of the way $B(x_T, r_T)$ were previously chosen. Re-arranging the previous statement gives

$$|Y_1(T) \cap B(x', r')| \lesssim \left(\frac{r'}{\rho}\right)^\epsilon |Y_1(T)|.$$

This implies that

$$\left|\sigma\left(Y_1(T) \cap B(x', r')\right)\right| \lesssim \left(\frac{r'}{\rho}\right)^\epsilon \left|\sigma\left(Y_1(T)\right)\right|,$$

which means that $\sigma(Y_1(T))$ obeys the two-ends condition because $\sigma(B(x', r'))$ is a ball of radius r'/ρ. Since the number of δ/ρ tubes $\tilde{T}$ is $\#\mathbb{T}_1$, which is $\sim 1/\log(1/\delta) \cdot \#\mathbb{T}$, we will need to choose a subcollection $\mathbb{T}_2 \subseteq \mathbb{T}_1$ so that the rescaled tubes $\tilde{T} = \sigma(T)$ for $T \in \mathbb{T}_2$ will point in δ/ρ-separated directions. Observe that

$$\#\mathbb{T}_2 \gtrsim \rho^{n-1}\#\mathbb{T}_1 \gtrapprox \rho^{n-1}\#\mathbb{T}.$$

We may now apply the conclusion of Proposition 20 to the rescaled tubes from $\mathbb{T}_2$ to get

$$\left|\bigcup_{T \in \mathbb{T}_2} \sigma\left(Y_1(T)\right)\right| \geqslant C_\epsilon \left(\frac{\lambda}{\rho}\right)^d \left(\frac{\delta}{\rho}\right)^{n-d+\epsilon} (\sum_{T \in \mathbb{T}_2} |\tilde{T}|).$$

Reversing the rescaling σ, we now find

$$\left|\bigcup_{T\in\mathbb{T}_2} Y(T)\right| \geqslant C_\epsilon \lambda^d \delta^{n-d+\epsilon}(\#\mathbb{T}_2)(\delta/\rho)^{n-1} \gtrapprox C_\epsilon \lambda^d \delta^{n-d+\epsilon}(\sum_{T\in\mathbb{T}} |T|),$$

which is the desired conclusion for Proposition 18. □

3.2 Broad-narrow reduction

If $\mathbb{T}$ is a family of δ-tubes with shading Y, we will say that a point $x \in \bigcup_{T\in\mathbb{T}} Y(T)$ is *s-narrow* if there exists a unit vector $v_x \in \mathbb{S}^{n-1}$ such that

$$\{T \in \mathbb{T}(x) : \angle(\mathrm{dir}(T), v_x) \leqslant s\} \geqslant \frac{1}{2}\#\mathbb{T}(x).$$

If this condition fails at a point $x \in \bigcup_{T\in\mathbb{T}} Y(T)$, then we say that the point x is *s-broad*.

Proposition 22. *Let $\epsilon > 0$. If $\mathbb{T}$ is a set of δ-tubes satisfying the following hypotheses:*

i. The tubes in $\mathbb{T}$ point in δ-separated directions,

ii. Each tube $T \in T$ has a subset $Y(T)$ and $|Y(T)| \geqslant \lambda|T|$ for all $T \in \mathbb{T}$,

iii. The family $\mathbb{T}$ with shading Y is (k-1)-plany,

iv. For every $T \in \mathbb{T}$, $Y(T)$ obeys the two-ends conjecture with exponent ϵ,

v. There is a set $E \subseteq \bigcup_{T\in\mathbb{T}} Y(T)$ satisfying the condition

$$\sum_{T\in\mathbb{T}} |Y(T) \cap E| \gtrsim \frac{1}{\log(1/\delta)} \sum_{T\in\mathbb{T}} |Y(T)|,$$

for which any point $x \in E$ satisfies:

$$\#\{(T_1, T_2) \in \mathbb{T}(x) \times \mathbb{T}(x) : \angle(\mathrm{dir}(T_1), \mathrm{dir}(T_2)) \gtrsim \delta^\epsilon\} \gtrsim \frac{1}{\log(1/\delta)}\#\left(\mathbb{T}(x)\right)^2;$$

then there exists a constant $C_\epsilon > 0$ such that

$$\left|\bigcup_{T\in\mathbb{T}} Y(T)\right| \geqslant C_\epsilon \delta^{(n-2+k-d(k))/3+\epsilon} \lambda^{d(k)/3+1}(\sum_{T\in\mathbb{T}} |T|)$$

where $d(k)$ is the largest exponent for which a Kakeya maximal estimate holds.

Lemma 23 (Broad-narrow reduction)**.** *Proposition* 22 *implies Proposition* 20.

To prove the broad-narrow reduction, we will first make some observations on broad and narrow points in the following simple lemmas.

Lemma 24. *If a point $x \in \bigcup_{T\in\mathbb{T}} Y(T)$ is s-broad, then*

$$\#\left\{(T_1, T_2) \in \mathbb{T}(x) \times \mathbb{T}(x) : \angle(\mathrm{dir}(T_1), \mathrm{dir}(T_2) > s\right\} \geqslant \frac{3}{4}\#\left(\mathbb{T}(x)\right)^2. \tag{3.4}$$

Proof. If a point x is s-broad with respect to $(\mathbb{T}, Y)$, then one may find a set of $4s$-separated directions $\{v_i \in \mathbb{S}^{n-1}\}_{i=1}^{Cs^{1-n}}$, where $C \sim 1$ is a fixed constant, such that for every i,

$$\#\{T \in \mathbb{T}(x) : \angle(\mathrm{dir}(T), v_i) \leqslant s\} \leqslant \frac{1}{2}\#\mathbb{T}(x).$$

Since the set of directions $\{v_i \in \mathbb{S}^{n-1}\}_{i=1}^{Cs^{1-n}}$ is $4s$-separated, we observe that if $T_i \in \{T \in \mathbb{T} : \angle(\mathrm{dir}(T), v_i) \leqslant s\}$, $T_j \in \{T \in \mathbb{T} : \angle(\mathrm{dir}(T), v_j) \leqslant s\}$, and $i \neq j$, then

$$\angle(\mathrm{dir}(T_i), \mathrm{dir}(T_j)) \geqslant s.$$

The total number of ordered pairs $(T_1, T_2) \in \mathbb{T}(x) \times \mathbb{T}(x)$ in which both T_1 and T_2 come are in the s-neighborhood of the same direction is less than $1/4\ (\#\mathbb{T}(x))^2$, and the conclusion follows. □

Lemma 25. *If $\mathbb{T}$ is a family of δ-tubes with shading Y, there is a number s and a set $E \subseteq \bigcup_{T\in\mathbb{T}} Y(T)$ such that*

$$\sum_{T\in\mathbb{T}} |Y(T) \cap E| \gtrsim \frac{1}{\log(1/\delta)} \sum_{T\in\mathbb{T}} |Y(T)|$$

and every point $p \in E$ satisfies two conditions:

(i) p is s-narrow, and

(ii) we have

$$\#\left\{(T_1, T_2) \in \mathbb{T}(x) \times \mathbb{T}(x) : \angle(\mathrm{dir}(T_1), \mathrm{dir}(T_2)) \geqslant \frac{s}{4}\right\} \geqslant \frac{3}{4}\left(\#\mathbb{T}(x)\right)^2.$$

Proof. For every point p, define k_p by

$$k_p = \max\left\{k \in \mathbb{Z} : \exists v_p \in \mathbb{S}^{n-1} \text{ such that } \#\{T \in \mathbb{T}(p) : \angle(v_p, \mathrm{dir}(T)) \lesssim 2^{-k}\} \geqslant \frac{1}{2}\#\mathbb{T}(p).\right\} \tag{3.5}$$

Note that this maximum must always exist since k must lie between 0 and $\sim \log(\delta^{-1})$. By dyadic pigeonholing, we may choose a value of s such that

$$\frac{s}{2} \leqslant 2^{-k_p} < s \tag{3.6}$$

for every point p in a set $E \subseteq \bigcup_{T \in \mathbb{T}} Y(T)$, such that the Lebesgue measure of E satisfies

$$\sum_{T \in \mathbb{T}} |Y(T) \cap E| \gtrsim \frac{1}{\log(1/\delta)} \sum_{T \in \mathbb{T}} |Y(T)|.$$

We claim that every point $p \in E$ satisfies the two required conditions. To see this, note from (3.5) and (3.6) that p satisfies the first condition. For the second condition, we observe from the definition of k_p that at every $p \in E$,

$$\forall v \in \mathbb{S}^{n-1} : \#\left\{T \in \mathbb{T}(p) : \angle(\mathrm{dir}(T), v) \geqslant \frac{s}{4}\right\} \geqslant \frac{1}{2}\#\mathbb{T}(p),$$

which means p is $s/4$-broad. By applying the result of Lemma 24, we have

$$\#\left\{(T_1, T_2) \in \mathbb{T}(x) \times \mathbb{T}(x) : \mathrm{dir}(T_1), \mathrm{dir}(T_2) > \frac{s}{4}\right\} \geqslant \frac{3}{4} \left(\#\mathbb{T}(p)\right)^2.$$

□

Proof of Lemma 23. Assume that Proposition 22 is true. Let $\varepsilon > 0$ be given and let $\mathbb{T}$ be a family of δ-tubes that satisfies hypotheses (i)–(iv) of Proposition 20. Apply Lemma 25 and let E and s be as in the statement of that lemma. For every point $p \in E$, let $v_p \in \mathbb{S}^{n-1}$ be as in (3.5). Define a new shading by

$$Y'(T) = \{p \in Y(T) \cap E : \angle(v_p, \mathrm{dir}(T) \leqslant s\},$$

and correspondingly define

$$\mathbb{T}'(p) = \{T \in \mathbb{T} : p \in Y'(T)\}.$$

From Lemma 25 we have, for every $p \in E$,

$$\#\{T \in \mathbb{T} : p \in Y'(T)\} \geqslant \frac{1}{2}\#\mathbb{T}(p),$$

and from this we can see that

$$\sum_{T \in \mathbb{T}} |Y'(T)| \sim \int_E \#\mathbb{T}'(p)\, dp \gtrsim \int_E \#\mathbb{T}(p)\, dp \sim \sum_{T \in \mathbb{T}} |Y(T) \cap E|.$$

Also from Lemma 25,

$$\sum_{T\in\mathbb{T}} |Y(T) \cap E| \gtrsim \frac{1}{\log(1/\delta)} \sum_{T\in\mathbb{T}} |Y(T)|.$$

Combining the above two statements, we obtain

$$\sum_{T\in\mathbb{T}} |Y'(T)| \gtrsim \frac{1}{\log(1/\delta)} \sum_{T\in\mathbb{T}} |Y(T)|. \tag{3.7}$$

Next we may choose a shading Y'' of $\mathbb{T}$ such that $Y''(T) \subseteq Y'(T)$ for every $T \in \mathbb{T}$, there is a number $0 < \lambda' < 1$ such that

$$|Y''(T)| \sim \lambda'|T| \text{ for all } T \in \mathbb{T}$$

and in addition

$$\sum_{T\in\mathbb{T}} |Y''(T)| \gtrsim \frac{1}{\log(1/\delta)} \sum_{T\in\mathbb{T}} |Y'(T)|. \tag{3.8}$$

From (3.7), (3.8) and hypothesis (ii) of Proposition 20, we can conclude that

$$\lambda' \gtrsim \frac{\lambda}{(\log(1/\delta))^2}. \tag{3.9}$$

It is possible to cover the upper hemisphere of $\mathbb{S}^{n-1}$ with $\sim s^{1-n}$ finitely overlapping tiles of side length s, where a tile of side length s means the image via the map $\xi \mapsto (\xi, \sqrt{1-|\xi|^2})$ of translates of tiles $[0,s)^{n-1}$ tiling $\mathbb{R}^{n-1}$. For each tile τ of radius s covering the sphere, define

$$\mathbb{T}(\tau) := \{T \in \mathbb{T} : \text{dir}(T) \in \tau\}.$$

Let τ_1 and τ_2 be two non-adjacent tiles so $\text{dist}(\tau_1, \tau_2) > 2s$. If $T_1 \in \mathbb{T}(\tau_1)$ and $T_2 \in \mathbb{T}(\tau_2)$, the angle between the tubes would be greater than $2s$:

$$\angle(T_1, T_2) > 2s.$$

By the definition of the shading Y'', this means that $Y''(T_1) \cap Y''(T_2) = \emptyset$. Under the shading Y'', the collections $\mathbb{T}(\tau_1)$ and $\mathbb{T}(\tau_2)$ are disjoint. Within each s-tile τ covering $\mathbb{S}^{n-1}$, there are up to $(s/\delta)^{n-1}$ δ-tubes. Without loss of generality, we may assume that the center of the tile τ is the north pole of the sphere, i.e. the point $(0,\ldots,0,1)$. Perform a non-isotropic rescaling ϕ which takes $(x_1,\ldots,x_n) \mapsto (x_1/s,\ldots,x_{n-1}/s,x_n)$. After this rescaling, $\phi(T)$ is contained in the n-fold dilate of a δ/s-tube $\tilde{T}$ that points in the same direction as T. Define a shading $\tilde{Y}$ for each $\tilde{T}$ by

$$\tilde{Y}(\tilde{T}) := \phi(Y''(T)).$$

From (3.9), we have

$$|\tilde{Y}(\tilde{T})| \sim \lambda'|\tilde{T}|.$$

We now claim that $(\tilde{T}, \tilde{Y})$ satisfies hypotheses (i) — (v) in the statement of Proposition 22. Hypothesis (ii) has already been shown, and hypothesis (i) holds because of the non-isotropic way in which the re-scaling ϕ was done. The family $\tilde{T}$ is $(k-1)$-plany because the family $\mathbb{T}$ is assumed to be $(k-1)$-plany. For hypothesis (iv), we observe that since $Y(T)$ obeys the two-ends condition with exponent ε for every $T \in \mathbb{T}$, so does $Y''(T)$, and it follows that $\tilde{Y}(\tilde{T})$ obeys the two-ends condition with exponent ε as well. To show that $\tilde{T}, \tilde{Y}$ satisfy hypothesis (v), consider that at every point $p \in \bigcup_{T \in \mathbb{T}[\tau]} Y'(T)$, we know from Lemma 25 that

$$\#\left\{(T_1, T_2) \in \mathbb{T}'(p) \times \mathbb{T}'(p) : \angle(\mathrm{dir}(T_1), \mathrm{dir}(T_2)) \gtrsim \frac{s}{4}\right\} \geqslant \frac{3}{4}\# \left(\mathbb{T}'(p)\right)^2.$$

Finally, we observe that after the rescaling,

$$\angle(\mathrm{dir}(T_1), \mathrm{dir}(T_2)) \geqslant s/4 \text{ implies that } \angle(\mathrm{dir}(\tilde{T}_1, \tilde{T}_2) \geqslant c_\epsilon \delta^\epsilon$$

for some $c_\epsilon > 0$. Thus the family $(\tilde{\mathbb{T}}, \tilde{Y})$ satisfies hypothesis (v), and by Proposition 22, there exists $C_\varepsilon > 0$ such that

$$\left|\bigcup_{\tilde{T} \in \tilde{\mathbb{T}}} \tilde{Y}(\tilde{T})\right| \geqslant C_\varepsilon (\delta/s)^{n-d+\varepsilon} \lambda^d \sum_{\tilde{T} \in \tilde{\mathbb{T}}} |\tilde{T}|.$$

Undoing the rescaling ϕ (which has the effect of multiplying both sides of the above equation by s^{n-1}), we obtain the estimate

$$\left|\bigcup_{T \in \mathbb{T}(\tau)} Y''(T)\right| \geqslant C_\varepsilon \delta^{n-d+\varepsilon} s^{-n+d-\varepsilon} \lambda^d \sum_{T \in \mathbb{T}[\tau]} |T|.$$

Sum over a collection of pairwise non-adjacent tiles covering the sphere to get

$$\left|\bigcup_{T \in \mathbb{T}} Y''(T)\right| \geqslant C_\varepsilon \delta^{n-d+\varepsilon} s^{-n+d-\epsilon} \lambda^d \sum_{T \in \mathbb{T}} |T|.$$

Since $s < 1$, this estimate is stronger than the desired estimate by a factor of $s^{-n+d-\varepsilon}$. This completes the proof. □

3.3 Volume of a planebrush

In this section, an estimate will be given for the volume of δ-tubes in the k-planebrush. It will first be needed to give an estimate for the number of tubes contained in the k-planebrush.

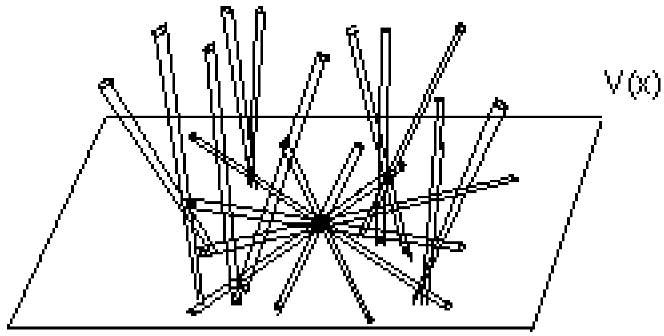

Figure 3.1: A k-planebrush at a point x

Definition 26 (k-planebrush)**.** *If* $p \in \bigcup_{T\in\mathbb{T}} Y(T)$ *and* $0 < \rho < 1$*, a* k-planebrush at point p and angle ρ *is a set of tubes*

$$\mathbb{T}_{\text{planebrush}}(p,\rho) = \{T \in \mathbb{T} : \angle(T, V(p)) \sim \rho \text{ and } \exists T' \in \mathbb{T}(p) : T \cap T' \neq \varnothing\}.$$

Lemma 27 (Constant multiplicity refinement)**.** *Let* $\mathbb{T}$ *be a family of* δ*-tubes pointing in* δ*-separated directions. Assume that every tube* $T \in \mathbb{T}$ *has a subset* $Y(T)$ *and* $0 < \lambda < 1$ *is a number such that* $|Y(T)| \geqslant \lambda|T|$ *for every* $T \in \mathbb{T}$*. Then there exists a refinement* $\mathbb{T}' \subset \mathbb{T}$ *and a new shading* $Y'(T) \subseteq Y(T)$ *for each* $T \in \mathbb{T}'$*, which satisfy the following properties:*

i There is a dyadic number μ *such that for every point* $x \in \bigcup_{T\in\mathbb{T}'} Y'(T)$*,* $\mu \leqslant \mathbb{T}(x) < 2\mu$*.*

ii $\sum_{T\in\mathbb{T}'} |Y'(T)| \gtrsim \left(\sum_{T\in\mathbb{T}} |Y(T)|\right)/\log(1/\delta)^2$.

iii $\mu \gtrsim \lambda\delta^{n-1}\#\mathbb{T}/|\bigcup_{T\in\mathbb{T}} Y(T)|$.

Proof. For every integer k between 1 and $\log\delta^{1-n}$, let $E_k \subseteq \bigcup_{T\in\mathbb{T}} Y(T)$ be the set of points with multiplicity around 2^k, i.e.

$$E_k = \left\{p \in \bigcup_{T\in\mathbb{T}} Y(T) : 2^k \leqslant \#\mathbb{T}(p) < 2^{k+1}\right\}.$$

For each k, define a shading Y_k by $Y_k(T) := Y(T) \cap E_k$. For each integer ℓ between 0 and $\log(\lambda/\delta)$, let $\mathbb{T}_{k,\ell}$ be the collection of tubes T whose for which the shading $Y_k(T)$ has density around $2^{-\ell}\lambda$, i.e.

$$\mathbb{T}_{k,\ell} = \left\{T \in \mathbb{T} : 2^{-\ell}\lambda|T| \leqslant |Y_k(T)| < 2^{-\ell+1}\lambda|T|\right\}.$$

From the definitions of k and ℓ, one can observe that

$$\sum_{T\in\mathbb{T}}|Y(T)| \sim \#\mathbb{T}\cdot\lambda\delta^{n-1} \sim \sum_{k=0}^{\log(1/\delta)}\sum_{\ell=0}^{\log(1/\delta)}\#\mathbb{T}_{k,\ell}2^{-\ell}\lambda\delta^{n-1}.$$

By the pigeonhole principle, there exists at least one pair (k,ℓ) that satisfies

$$\#\mathbb{T}_{k,\ell}2^{-\ell}\lambda\delta^{-1} \gtrsim \frac{1}{(\log(1/\delta))^2}\#\mathbb{T}\cdot\lambda\delta^{-1}. \tag{3.10}$$

Take $\mu = 2^k$, define the shading Y' by Y_k and define $\mathbb{T}'$ as $\mathbb{T}_{k,\ell}$ for this choice of k and ℓ. From here, points (i) and (ii) in the statement of the lemma follow immediately from (3.10). One can also observe that

$$\mu = 2^k \geqslant \frac{\lambda\cdot\#\mathbb{T}_{k,\ell}\cdot 2^{-\ell}\delta^{n-1}}{|\bigcup_{T\in\mathbb{T}}Y(T)|} \gtrapprox \frac{\lambda}{|\bigcup_{T\in\mathbb{T}}Y(T)|},$$

which is statement (iii). □

Lemma 28 (Planebrush multiplicity)**.** *Let $\mathbb{T}$ be a family of direction-separated δ-tubes with shading Y that obeys the hypotheses of Proposition* 22*. In addition, assume that*

$$\sum_{T\in\mathbb{T}}|Y(T)| \gtrsim \lambda.$$

Then there exist an angle $\rho > 0$ and a point p such that

$$\#\mathbb{T}_{\text{planebrush}}(p,\rho) \gtrapprox \frac{\lambda^3\rho\,\delta^{-1}}{|\bigcup_{T\in\mathbb{T}}Y(T)|^2}.$$

Proof. Let $\mathbb{T}', Y'$ be the constant multiplicity refinement of $\mathbb{T}, Y$ given by Lemma 27. Let p be any point for which $\#\mathbb{T}'(p) \gtrsim \mu$. To see that such a point p must exist, suppose to the contrary that at all points, $\#\mathbb{T}'(p) < c\mu$ for a fixed constant c to be chosen later. Then it follows that

$$\sum_{T\in\mathbb{T}'}|Y'(T)| \leqslant c\mu\cdot\left|\bigcup_{T\in\mathbb{T}'}Y'(T)\right| < c\lambda \lesssim \frac{c\sum_{T\in\mathbb{T}}|Y(T)|}{\log(1/\delta)^2}. \tag{3.11}$$

On the other hand, by point (ii) in Lemma 27,

$$\sum_{T\in\mathbb{T}'}|Y'(T)| \gtrsim \frac{\sum_{T\in\mathbb{T}}|Y(T)|}{\log(1/\delta)^2}. \tag{3.12}$$

This contradicts (3.11) if the constant c is chosen to be smaller than the implicit constant in (3.12). For such a point p, one can count the number of triples (T_1, Q_1, T_2)

where T_1 is a tube whose shading intersects p and T_2 is a tube whose shading intersects the shading of T_1 in a δ-cube Q_1. More precisely,

$$\begin{aligned}\#\{(T_1, Q_1, T_2) \in \mathbb{T}' \times \mathcal{Q} \times \mathbb{T} : T_1 \in \mathbb{T}'(p), Q_1 \subseteq Y'(T_1) \cap Y'(T_2)\} &\gtrsim \mu \cdot \lambda' \delta^{-1} \cdot \mu \\ &\gtrapprox \mu^2 \lambda \delta^{-1}.\end{aligned}$$

By the pigeonhole principle, there exists an angle ρ with $\delta \leqslant \rho \leqslant 1$ such that

$$\begin{aligned}\#\{(T_1, Q_1, T_2) \in \mathbb{T}' \times \mathcal{Q} \times \mathbb{T} : T_1 \in \mathbb{T}'(p), Q_1 \subseteq Y'(T_1) \cap Y'(T_2), \angle(T_2, V(p)) \sim \rho\} \\ \gtrapprox \mu^2 \lambda \delta^{-1}.\end{aligned}$$

Since the tube T_1 must pass through both p and Q_1, and since the tubes passing through p are assumed to be $(1, \delta^{\epsilon})$-broad, it follows that the choice of Q_1 determines T_1 entirely. Thus, for each Q_1, there is precisely one T_1 that is a triple counted in the previous step.

$$\#\{(Q_1, T_2) \in \mathcal{Q} \times \mathbb{T} : T_1 \in \mathbb{T}'(p), (T_1, Q_1, T_2) \text{ is a triple in the previous step } \} \gtrsim \mu^2 \lambda \delta^{-1}.$$

Since the tube T_2 makes angle ρ with $V(p)$, there are $\sim \rho^{-1}$ δ-cubes contained in T_2 that are also contained in a δ-neighborhood of $V(p)$. Thus, for each δ-tube T_2 that makes angle ρ with $V(p)$, there are potentially up to ρ^{-1} cubes Q_1 for which (Q_1, T_2) is a pair counted in the previous statement, i.e. a pair for which $Q_1 \subseteq Y'(T_2) \cap N_{\delta}(V(p))$ and $Q_1 \subseteq Y'(T_1)$ for some $T_1 \in \mathbb{T}(p)$. It follows that

$$\begin{aligned}\#\{T_2 \in \mathbb{T} : \exists T_1 \in \mathbb{T}(p) \text{ such that } Y'(T_1) \cap Y'(T_2) \neq \emptyset \text{ and } \angle(T_2, V(p)) \sim \rho\} \\ \gtrapprox \mu^2 \lambda \delta^{-1} \rho.\end{aligned}$$

This precisely counts the tubes in $\mathbb{T}_{\text{planebrush}}(p, \rho)$ and demonstrates the correct multiplicity of tubes in the planebrush $\mathbb{T}_{\text{planebrush}}(p, \rho)$. □

Lemma 29 (Volume of a single planebrush)**.** *Let* $\mathbb{T}_{\text{planebrush}}(p, \rho)$ *be a planebrush at point p and angle* ρ*. Then*

$$\bigcup_{T \in \mathbb{T}_{\text{planebrush}}(p, \rho)} Y(T) \gtrsim C_{\epsilon} \delta^{(n-2+k-d(k))/3+\epsilon} \lambda^{d(k)/3+1} \rho^{1/3}.$$

Proof. Choose a maximal set of δ/ρ-separated points $\{\alpha_j\}_j$ in the unit sphere in the space $V(p)^{\perp} = \{v \in \mathbb{R}^n : \langle v, V(p) \rangle = 0\}$, which is an $(n - k + 1)$-dimensional space. For each α_j, define a k-plane V_j by $V_j = \text{span}(V(p), \alpha_j)$. The resulting collection of k-planes $\{V_j\}_j$ satisfies the following two conditions.

(A) Every tube $T \in \mathbb{T}_{\text{planebrush}}(p,\rho)$ is contained in $\mathcal{N}_{C_2\delta}V_j$, with $C_2 \sim 1$, for some j.

(B) If a point $x \in \mathbb{R}^n$ satisfies $\text{dist}(x, V(p)) \geqslant C_3\rho)$, then $x \in \mathcal{N}_{C_2\delta}V_j$ for at most $\sim (C_2/C_3)^{n-k+1}$ choices of k-plane V_j.

To show that (A) holds, let $T \in \mathbb{T}_{\text{planebrush}}(p,\rho)$ and write $\text{dir}(T') = v^{||} + v^{\perp}$, where $v^{||} \in V(p)$ and $v^{\perp} \in (V(p))^{\perp}$. Choose α_j such that $\angle(\alpha_j, v^{\perp}) \leqslant \beta/\rho$. This is possible because of the maximality of the set $\{\alpha_j\}$. Write $v^{\perp} = v_1 + v_2$, where v_1 lies in the direction of α_j and $\langle v_2, \alpha_j\rangle = 0$. Since $\angle(T', V(Q_0)) \leqslant \rho$, $|v^{\perp}| = \sin\left(\angle(T', V(p))\right) \lesssim \rho$. Now

$$\sin\left(\angle(\text{dir}(T), V_j)\right) = |v_2| = \sin\left(\angle(v_2, v^{\perp})\right) \cdot |V^{\perp}| \lesssim \delta,$$

so (A) is proved.

To show (B), let a point x satisfy $\text{dist}(x, V(p)) \geqslant C_3\rho)$. If $x = x^{||} + x^{\perp}$, where $x^{||}$ lies in $V(p)$ and $\langle x^{\perp}, x^{||}\rangle = 0$, then $|x_{\perp}| \gtrsim C_3\rho$. Suppose that $x \in \mathcal{N}_{C_2\delta}V_j$ for some j. Then

$$\text{dist}\left(\frac{x^{\perp}}{|x^{\perp}|}, \alpha_j\right) \lesssim \frac{C_2\beta}{C_3 p}. \tag{3.13}$$

Since the directions $\{\alpha_j\}$ are δ/ρ-separated, there are only at most $\sim (C_2/C_3)^{n-k+1}$ choices of α_j that can satisfy (3.13). This proves (B).

For each k-plane V_j, let $\mathbb{T}_j$ be the family of tubes given by

$$\mathbb{T}_j = \{T \in \mathbb{T}_p b(p,\rho) : T \subset \mathcal{N}_{C_2\delta}V_j\}.$$

Define the set E_j by

$$E_j = \bigcup_{T\in\mathbb{T}_j} Y(T) \backslash \mathcal{N}_{C_3\rho}V(Q_0).$$

By the conclusion of Lemma 21, we may assume that each tube $T \in \mathbb{T}_j$ satisfies

$$|Y(T) \cap B(x, C_2\rho)| \lesssim (C_2\rho)^{\epsilon_1}|Y(T)|$$

for every $\epsilon_1 > 0$. By choosing ϵ_1 small enough,

$$|Y(T) \cap E_j| \gtrsim \frac{\lambda}{2}|T|.$$

Summing over all $T \in \mathbb{T}_j$, we obtain

$$\int_{E_j} \sum_{T\in\mathbb{T}_j} \chi_{Y(T)} \geqslant \frac{\lambda}{2}\beta^{n-1}\#\mathbb{T}_j. \tag{3.14}$$

On the other hand, by applying Hölder's inequality we obtain

$$\int_{E_j} \sum_{T\in\mathbb{T}_j} \chi_{Y(T)} \leqslant || \sum_{T\in\mathbb{T}_j} \chi_T ||_{L^{d(k)'}(\mathbb{R}^n)} \cdot |E_j|^{1/d(k)}, \tag{3.15}$$

where $d(k)$ is chosen to be the highest dimension for which a Kakeya maximal estimate is known to hold in $\mathbb{R}^k$, and $d(k)'$ is the Hölder conjugate of $d(k)$. Specifically,

$$d(k) = \max_{2\leqslant \ell \leqslant k} \min\left(k - \ell + 2, \frac{k^2 + k + \ell^2 - \ell}{2k}\right).$$

We will now apply a Kakeya maximal estimate to $|| \sum_{T\in\mathbb{T}_j} \chi_T ||_{L^{d(k)'}}$. Let ρ_{V_j} be the orthogonal projection function from $\mathbb{R}^n$ onto V_j. Since each $T \in \mathbb{T}_j$ satisfies $T \subset \mathcal{N}_{C_2\delta} V_j$, we observe that $\angle(T, V_j) \lesssim C_2\delta$. This implies that $\rho_{V_j}(T)$ is a cylinder of radius β and length $\gtrsim C_2^{-1}$ and is contained in a $\sim C_2^{-1}$-dilation of a δ-tube in $\mathbb{R}^k$. Thus, we may write

$$\begin{aligned} || \sum_{T\in\mathbb{T}_j} \chi_T ||_{L^{d(k)'}(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} | \sum_{T\in\mathbb{T}_j} \chi_T|^{d(k)'}\right)^{1/d(k)'} \\ &= \left(\int_{[-C_2\delta, C_2\delta]^{n-k}} \int_{V_j} | \sum_{T\in\mathbb{T}_j} \chi_T|^{d(k)'}\right)^{1/d(k)'} \\ &\lesssim \delta^{(n-k)/d(k)'} \left(\int_{V_j} | \sum_{T\in\mathbb{T}_j} \chi_{C_2\rho_{V_j}(T)}|^{d(k)'}\right)^{1/d(k)'} . \end{aligned}$$

After applying the Kakeya maximal estimate, this gives

$$|| \sum_{T\in\mathbb{T}_j} \chi_T ||_{L^{d(k)'}(\mathbb{R}^n)} \lessapprox \delta^{(n-k)/d(k)'} \delta^{-k/d(k)+1-\epsilon} (\delta^{k-1} \#\mathbb{T}_j)^{1/d(k)'}.$$

Combining this estimate with (3.14) and (3.15), we obtain

$$\frac{\lambda}{2} \delta^{n-1} \#\mathbb{T}_j \lessapprox \delta^{(n-k)/d(k)'} \delta^{-k/d(k)+1-\epsilon} (\delta^{k-1} \#\mathbb{T}_j)^{1/d(k)'} \cdot |E_j|^{1/d(k)},$$

which simplifies to

$$|E_j| \gtrapprox \delta^{n-1} \lambda^{d(k)} \delta^{k-d(k)} \#\mathbb{T}_j. \tag{3.16}$$

Summing (3.16) over all j and using properties (A) and (B) above, we obtain

$$\left| \bigcup_{T\in\mathbb{T}_{\text{planebrush}}(p,\rho)} Y(T) \right| \gtrapprox \delta^{n-1} \lambda^{d(k)} \delta^{k-d(k)} \#\mathbb{T}_{\text{planebrush}}(p, \rho). \tag{3.17}$$

Combining (3.17) with Lemma 28 gives

$$\left| \bigcup_{T \in \mathbb{T}_{\text{planebrush}}(p,\rho)} Y(T) \right|^3 \gtrapprox \delta^{n-2} \lambda^{d(k)+3} \delta^{k-d(k)} \rho. \tag{3.18}$$

□

3.4 Proof of *k*-planebrush estimate

The goal of this section is to prove Proposition 22. Below is described an algorithm that decomposes the set of δ-tubes $\mathbb{T}$ into a collection of "planebrushes."

Algorithm:

Input. The input to the algorithm consists of:

- (i) A number $\delta > 0$.
- (ii) A family of δ-tubes $\mathbb{T}$ with shading Y satisfying $|Y(T)| \geqslant \lambda |T|$ for every $t \in \mathbb{T}$, for some number $0 \leqslant \lambda < 1$.
- (iii) A shading Y' of $\mathbb{T}$ satisfying $Y'(T) \subseteq Y(T)$ for every $T \in \mathbb{T}$ and

$$\sum_{T \in \mathbb{T}} |Y'(T)| \geqslant \frac{\lambda}{2}.$$

- (iv) An index i, initially equal to 1.

Output. The output of the algorithm will be:

- (i) A sequence $(p_i, \rho_i)_{i=1}^N$ where p_i is a point in $\mathbb{R}^n$ and $0 < \rho_i < 1$ for every $1 \leqslant i \leqslant N$
- (ii) A collection of shadings $\{Y_i\}_{i=0}^{N-1}$ of $\mathbb{T}$

together satisfying the following properties:

- (P1) The sets ("planebrushes") $\bigcup_{T \in \mathbb{T}_{\text{planebrush}}(p_i,\rho_i)} Y_{i-1}(T)$ are pairwise disjoint, and
- (P2) The "planebrushes" together cover more than half the "incidences", i.e.

$$\sum_{i=1}^{N} \sum_{T \in \mathbb{T}_{\text{planebrush}}(p_i,\rho_i)} |Y_{i-1}(T)| \geqslant \frac{\lambda}{2}.$$

The algorithm consists of the following steps:

Step 1. Apply Lemma 28 to the family $(\mathbb{T}_{i-1}, Y_{i-1})$ to obtain p_i and ρ_i such that

$$\#\mathbb{T}_{\text{planebrush}}(p_i, \rho_i) \gtrapprox \frac{\lambda^3 \rho_i \delta^{-1}}{|\bigcup_{T\in\mathbb{T}} Y(T)|^2}.$$

Step 2. Define $\mathbb{T}_i := \mathbb{T}_{i-1} \backslash \mathbb{T}_{\text{planebrush}}(p_i, \rho_i)$. Define the shading Y_i as follows:

$$Y_i(T) = Y_{i-1}(T) \backslash N_{\rho_i}(V(p_i)).$$

For any angle θ, the number of tubes making angle θ or less to the $(k-1)$-plane $V(p_i)$ is $\lesssim \theta_i^{-n+k-1}\delta^{1-n}$. If a tube $T \in \mathbb{T}_i$ makes an angle θ to the $(k-1)$-plane $V(p_i)$, then a ρ_i/θ-fraction of T is contained in a ρ_i-neighborhood of $V(p_i)$. Therefore the maximum number of incidences removed is

$$\sum_{\theta} \sum_{T\in\mathbb{T}, \angle(T,V(p_i))\sim\theta} |Y_i(T) \cap N_\rho(V(p_i))| \leqslant \sum_\theta \theta^{-n+k-1} \frac{\rho_i}{\theta} \delta^{n-1} \leqslant 2\rho_i \delta^{k-1}.$$

On the other hand, by the two ends condition,

$$\sum_{\theta} \sum_{T\in\mathbb{T}, \angle(T,V(p_i))\sim\theta} |Y_i(T) \cap N_\rho(V(p_i))| \leqslant \sum_\theta \theta^{-n+k-1} \left(\frac{\rho_i}{\theta}\right)^{\epsilon} \lambda\delta^{n-1} \leqslant 2\rho_i^{\epsilon} \lambda \delta^{k-1}.$$

Taking the geometric mean of the two estimates above,

$$\sum_{\theta} \sum_{T\in\mathbb{T}, \angle(T,V(p_i))\sim\theta} |Y_i(T) \cap N_\rho(V(p_i))| \leqslant 2\rho_i^{(1+\epsilon)/2} \lambda^{1/2} \delta^{k-1}.$$

Terminate the algorithm if

$$\sum_{j=1}^{i} \rho_j^{(1+\epsilon)/2} \lambda^{1/2} \delta^{k-1} \geqslant \frac{\lambda}{2},$$

otherwise we observe that

$$\sum_{T\in\mathbb{T}_i} |Y_i(T)| \geqslant \sum_{T\in\mathbb{T}_0} |Y_0(T)| - \sum_{j=1}^{i} \rho_j^{(1+\epsilon)/2} \lambda^{1/2} \geqslant \frac{\lambda}{2},$$

so we return to Step 1 with i increased by 1 and repeat.

Proof of Proposition 22. Define $\mathbb{T}_0 := \mathbb{T}$ and $Y_0 := Y$. Then apply the above algorithm with beginning input $(\mathbb{T}_0, Y_0)$ and starting with $i = 1$.

After applying the algorithm, a collection of "planebrushes" $(p_i, \rho_i)_{i=1}^N$ is obtained, and from (P1), it follows that

$$|\bigcup_{T\in\mathbb{T}} Y(T)| \geqslant \sum_{i=1}^{N} |\bigcup_{T\in\mathbb{T}_{\text{planebrush}}(p_i,\rho_i)} Y_{i-1}(T)|.$$

By applying Lemma 29,

$$|\bigcup_{T\in\mathbb{T}} Y(T)| \geqslant C_\epsilon \delta^{(n-2+k-d(k))/3+\epsilon} \lambda^{d(k)/3+1} \left(\sum_{i=1}^{N} \rho_i^{d(k)/3}\right).$$

Since

$$\sum_{j=1}^{i} \rho^k \geqslant \sum_{j=1}^{i} \rho_j^{(1+\epsilon)/2} \lambda^{1/2} \delta^{k-1} \geqslant \frac{1}{2},$$

we have $\sum_{i=1}^{N} \rho_i^{d(k)/3} \geqslant \frac{1}{2}$, and this completes the proof. □

Proof of Proposition 18. This is given by combining Proposition 22 with Lemma 21 and Lemma 23. □

Proof of Proposition 17. Let $\varepsilon > 0$ and let $\mathbb{T}, Y$ be as stated in the hypothesis of the proposition. By applying Proposition 18 with

$$d(k) = (2 - \sqrt{2})k,$$

which is the estimate given by Hickman-Rogers-Zhang, we obtain the estimate

$$|\bigcup_{T\in\mathbb{T}} Y(T)| \gtrsim \delta^{(n-2+(\sqrt{2}-1)k)/3+\varepsilon} \lambda^{(2-\sqrt{2})k/3+1} \sum_{T\in\mathbb{T}} |T|,$$

which is the result needed. □

Chapter 4

NEW KAKEYA ESTIMATE FOR STICKY SETS

Definition 30 (Sticky set)**.** *A set $\mathbb{T}$ of δ-tubes pointing in δ-separated directions is* sticky *if, for any $\varepsilon > 0$, whenever $\delta^{1-\varepsilon} \lesssim \theta < 1$, there is a family $\mathbb{T}_\theta$ of θ-tubes pointing in θ-separated directions so that $\mathbb{T}$ and $\mathbb{T}_\theta$ satisfy the following conditions:*

i. $\#\mathbb{T} \gtrsim \delta^{1-n+\varepsilon}$,

ii. $\#\mathbb{T}_\theta \gtrsim \theta^{1-n+\varepsilon}$, *and*

iii. $\mathbb{T}$ *can be partitioned into sets*

$$\mathbb{T}[T_\theta] := \{T \in \mathbb{T} : T \subseteq T_\theta\},$$

with

$$\#\mathbb{T}[T_\theta] \gtrsim (\delta/\theta)^{1-n+\varepsilon}$$

for every $T_\theta \in \mathbb{T}_\theta$.

*The family $\mathbb{T}_\theta$ in this definition will be called the θ-*parents *of $\mathbb{T}$.*

The major result of the this chapter is the following proposition.

Proposition 31. *For every $\varepsilon > 0$, whenever $\mathbb{T}$ is a sticky family of δ-tubes with shading Y, which satisfies $|Y(T)| \gtrsim \delta^\varepsilon |T|$ for all $T \in \mathbb{T}$,*

$$|\bigcup_{T\in\mathbb{T}} Y(T)| \gtrsim \delta^{n-d+(d+1)\varepsilon},$$

where

$$d \leqslant 0.60376707287\, n + O(1).$$

Sections 4.1, 4.2 and 4.3 are devoted to the proof of this proposition. Section 4.4 in this chapter gives the relation between sticky sets as defined in this thesis and sticky Besicovitch sets as defined recently by [23]. Section 4.5 proves an x-ray transform estimate as a corollary of Zahl's k-linear Kakeya estimate in [28].

4.1 Shading and refinement of sticky sets

Lemma 32 (Shading of sticky sets)**.** *Let $\epsilon > 0$ and let $\mathbb{T}$ be a sticky family of δ-tubes. Suppose that every $T \in \mathbb{T}$ has a shading $Y(T) \subseteq T$ such that*

$$|Y(T)| \geqslant C_\varepsilon \delta^\varepsilon |T|.$$

Then for any $\delta \leqslant \theta \leqslant \delta^{1-\epsilon}$, there is a shading Y of the θ-parent family $\mathbb{T}_\theta$ such that for every $T_\theta \in \mathbb{T}_\theta$,

$$|Y(T_\theta)| \geqslant C_\varepsilon \theta^{\varepsilon/(1-\varepsilon)} |T_\theta|$$

and $Y(T) \subseteq Y(T_\theta)$ for every $T \in \mathbb{T}[T_\theta]$.

Proof. For any $T_\theta \in \mathbb{T}_\theta$, define a shading by

$$Y(T_\theta) = \bigcup \left\{ Q_\theta : Q_\theta \cap T_\theta \cap \left(\bigcup_{T \in \mathbb{T}[T_\theta]} Y(T) \right) \neq \varnothing \right\}.$$

Since $|Y(T)| \geqslant C_\varepsilon \delta^\epsilon |T|$ for every $T \in \mathbb{T}[T_\theta]$, there are at least $C_\varepsilon \delta^{-1+\epsilon}$ δ-cubes in each $Y(T)$. Thus,

$$\#\{(Q, T) : T \in \mathbb{T}[T_\theta], Q \subseteq Y(T)\} \sim C_\varepsilon \theta^{n-1} \delta^{-n+\varepsilon}.$$

Each θ-cube $Q_\theta \subseteq T_\theta$ contains at most (θ/δ) δ-cubes in any one δ-tube T, so

$$\#\{(Q_\theta, T) : T \in \mathbb{T}[T_\theta], Q_\theta \cap T \neq \varnothing\} \leqslant C_\varepsilon \theta^{-1} (\theta^{n-1} \delta^{1-n+\varepsilon}).$$

Since there are $\sim \theta^{n-1-\varepsilon} \delta^{1-n+\varepsilon}$ tubes in $\mathbb{T}[T_\theta]$, we can conclude that there are at least

$$\sim \delta^\varepsilon \theta^{-1}$$

θ-cubes Q_θ in $Y(T_\theta)$. Since $\theta \leqslant \delta^{1-\varepsilon}$, we have

$$\delta^\varepsilon \leqslant \theta^\varepsilon \leqslant (\delta^{1-\varepsilon})^\varepsilon = \delta^{c\varepsilon},$$

where $c = 1 - \varepsilon$. This shows that $|Y(T_\theta)| \gtrsim \delta^{\varepsilon/c}$. □

Remark 33. *Lemma* 32 *will need to be applied inductively, by covering a family $\mathbb{T}$ of δ-tubes by $\delta^{1-\varepsilon}$-tubes, and then covering these by $(\delta^{1-\varepsilon})^{1-\varepsilon}$-tubes, and so on, until we can work with a family of β-tubes, where $\beta \sim 1$. This means that Lemma 32 will be applied* $\log(1/\delta)$ *many times. For small enough δ,*

$$(1-\varepsilon)^{\log(1/\delta)} \leqslant (1-\varepsilon)^{1/\varepsilon} \lesssim \frac{1}{e}.$$

Therefore

$$\frac{1}{(1-\varepsilon)^{\log(1/\delta)}} \gtrsim e,$$

and the family of β-tubes will have a shading of density $\beta^{e\varepsilon}$.

A *refinement* of a set of δ-tubes $\mathbb{T}$ with shading Y is a set of δ-tubes $\mathbb{T}'$ with shading Y' such that $\mathbb{T}' \subseteq \mathbb{T}$, $Y(T) \subseteq Y'(T)$ for every $T \in \mathbb{T}'$, and

$$\sum_{T\in\mathbb{T}'} |Y'(T)| \gtrsim \frac{1}{\log(1/\delta)} \sum_{T\in\mathbb{T}} |Y(T)|.$$

Lemma 34. *Let $\mathbb{T}$ be a sticky family of tubes with shading Y. Suppose that, for any $\varepsilon > 0$, Y satisfies the condition $|Y(T)| \gtrsim \delta^{\varepsilon}|T|$ for every $T \in \mathbb{T}$. If $\mathbb{T}', Y'$ is a refinement of $\mathbb{T}, Y$, then $\mathbb{T}'$ is also sticky.*

Proof. Let $\varepsilon > 0$ be given. Suppose $\mathbb{T}$ be a sticky family of tubes with shading Y satisfying

$$|Y(T)| \gtrsim \delta^{\varepsilon}|T|$$

for every $T \in \mathbb{T}$. Suppose that $\mathbb{T}', Y'$ is a refinement of $\mathbb{T}, Y$, i.e. $\mathbb{T}' \subseteq \mathbb{T}$ and $Y'(T) \subseteq Y(T)$ for every $T \in \mathbb{T}'$ and

$$\sum_{T\in\mathbb{T}'} |Y'(T)| \gtrsim \frac{1}{\log(1/\delta)} \sum_{T\in\mathbb{T}} |Y(T)|.$$

First, consider the cardinality of $\mathbb{T}'$. Since

$$\sum_{T\in\mathbb{T}'} |Y'(T)| \gtrsim \delta^{n-1+2\varepsilon}\#\mathbb{T},$$

we know that

$$\#\mathbb{T}' \gtrsim \delta^{2\varepsilon}\#\mathbb{T} \gtrsim \delta^{1-n+3\varepsilon}.$$

Next, let $\delta^{1-\varepsilon} \geqslant \theta < 1$ be any number and let $\mathbb{T}_\theta$ be the θ-parent family of $\mathbb{T}$. Since $\mathbb{T}'$ is a subset of $\mathbb{T}$, there is an inherited partition of $\mathbb{T}'$ into sets

$$\mathbb{T}'[T_\theta] = \{T \in \mathbb{T}' : T \subseteq T_\theta\}.$$

Now define a refinement $\mathbb{T}'_\theta \subseteq T_\theta$ by

$$\mathbb{T}'_\theta = \{T_\theta \in \mathbb{T}_\theta : \mathbb{T}'[T_\theta] \gtrsim 2(\delta/\theta)^{1-n+3\varepsilon}\}.$$

Since $\#\mathbb{T}' \gtrsim \delta^{1-n+3\varepsilon}$, it follows that

$$\#\left(\mathbb{T}_\theta \backslash \mathbb{T}'_\theta\right) \lesssim \frac{1}{2}\,\theta^{1-n+3\varepsilon},$$

so $\#\mathbb{T}'_\theta \gtrsim \frac{1}{2}\,\theta^{1-n+3\varepsilon}$, which makes $\mathbb{T}'_\theta$ a valid set of θ-parents for $\mathbb{T}'$. This proves that $\mathbb{T}_\theta$ is sticky. □

4.2 Extremal sticky sets

The following definitions build up the concept of an "extremal sticky set," which will be used in the following propositions to prove a Hausdorff dimension estimate for sticky sets of δ-tubes. Define $M(\delta)$ by

$$M(\delta) := \inf_{\mathbb{T},Y} \left| \bigcup_{T\in\mathbb{T}} Y(T) \right|$$

where the infimum is taken over all sets $\mathbb{T}, Y$ satisfying the following conditions:

i. $\mathbb{T}$ is a sticky set of δ tubes, and

ii. Y is a shading of $\mathbb{T}$ satisfying $|Y(T)| \gtrsim \delta^{\varepsilon}|T|$ for every $T \in \mathbb{T}$.

Define σ by

$$\sigma := \limsup_{\delta\to 0} \frac{\log M(\delta)}{\log \delta}.$$

Definition 35 (Extremal sticky set)**.** *For any $\varepsilon > 0$, if $\mathbb{T}$ is a sticky set of δ-tubes with shading Y satisfying, firstly, $|Y(T)| \gtrsim \delta^{\varepsilon}|T|$ for every $T \in \mathbb{T}$, and secondly,*

$$|\bigcup_{T\in\mathbb{T}} Y(T)| \lesssim \delta^{n-\sigma+\varepsilon}(\delta^{\varepsilon})^{\sigma}, \tag{$*$}$$

*then $\mathbb{T}$ will be called an ε-*extremal *sticky set, or just an ε-extremal set.*

From the definition of σ, it is immediate that ε-extremal sets of δ-tubes exist, if δ is small enough, for every $\varepsilon > 0$. The following two lemmas will show that ε-extremal sticky sets exhibit a kind of self-similarity, provided ε is small enough. The following notation will be used in the lemmas.

Notation. For any point $x \in \mathbb{R}^n$, we define:

$$\mathbb{T}_\theta(x) := \{T_\theta \in \mathbb{T}_\theta : x \in Y_\theta(T_\theta)\}$$

$$\text{and} \quad \mathbb{T}[T_\theta](x) := \{T \in \mathbb{T}[T_\theta] : x \in Y(T)\}.$$

The following notation will be used to denote multiplicity:

$$\mu_\delta(x) := \#\mathbb{T}(x),$$

$$\mu_\theta(x) := \#\mathbb{T}_\theta(x),$$

$$\text{and } \mu_\delta[T_\theta](x) := \#\mathbb{T}[T_\theta](x).$$

Lemma 36 (Multiplicities in extremal sticky sets)**.** *There exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, if $\mathbb{T}$ is an ε-extremal family of δ-tubes with shading Y, then there exists a refined shading Y' of $\mathbb{T}$ (i.e. $Y'(T) \subseteq Y(T)$ for every $T \in \mathbb{T}$) such that*

$$|Y'(T)| \gtrsim \delta^{2\varepsilon}|T|$$

for every $T \in \mathbb{T}$, and whenever θ is a number satisfying $\delta < \theta \leqslant \delta^{1-\varepsilon}$, there is a family of θ-tubes $\mathbb{T}_\theta$ with shading Y_θ such that

(i) (Fine multiplicity) $\mu_\delta(p) \gtrsim \delta^{-n+\sigma-\varepsilon}$.

(ii) (Coarse multiplicity) $\mu_\theta(p) \gtrsim \theta^{-n+\sigma-\varepsilon}$.

(iii) (Inner multiplicity) $\mu_\delta[T_\theta](p) \gtrsim (\theta/\delta)^{-n+\sigma-\varepsilon}$.

Proof. Let $\varepsilon > 0$ and suppose that $\mathbb{T}$ is an ε-extremal set of δ-tubes with shading Y. Let $\mathbb{T}_1, Y_1$ be a constant multiplicity refinement of $(\mathbb{T}, Y)$ with multiplicity μ_δ. The details of how to obtain this refinement are given below.

For every positive integer k, let E_k be the set of points at which the multiplicity of δ-tubes in $\mathbb{T}$ is $\sim 2^k$, i.e.

$$E_k = \{x \in \bigcup_{T\in\mathbb{T}} Y(T) : 2^{k-1} < \#\mathbb{T}(x) \leqslant 2^k\}.$$

Since $\sum_{k\in\mathbb{N}} |E_k| \cdot 2^k \sim \sum_{T\in\mathbb{T}} |Y(T)| \gtrsim \delta^\varepsilon$, there is an integer k such that

$$|E_k| \cdot 2^k \gtrsim \frac{1}{\log(1/\delta)}\delta^\varepsilon.$$

Set $\mu_\delta := 2^k$ for this value of k. Now define a new shading Y_1 of $\mathbb{T}$ by

$$Y_1(T) = Y(T) \cap E_k, \quad \forall T \in \mathbb{T}.$$

For every $0 < \lambda < 1$, define a new family $\mathbb{T}_{1,\lambda}$ by

$$\mathbb{T}_{1,\lambda} = \{T \in \mathbb{T} : \lambda|T| \leqslant |Y_1(T)| < 2\lambda|T|\}.$$

By the pigeonhole principle, there exists some λ such that

$$\sum_{T\in\mathbb{T}_{1,\lambda}} |Y_1(T)| \geqslant \frac{1}{\log(\theta/\delta)} \sum_{T\in\mathbb{T}} |Y_1(T)| \gtrsim \frac{1}{\log(1/\delta)^2}\delta^\varepsilon. \tag{4.1}$$

Let $\mathbb{T}_1 := \mathbb{T}_{1,\lambda}$ for this chosen value of λ, and further observe from (4.1) that

$$\lambda \gtrsim \frac{1}{(\log(1/\delta)^2}\delta^\varepsilon.$$

This shows how to obtain the constant multiplicity refinement, and we now need to show that this refinement obeys the multiplicity bound in part (i) in the statement of this lemma. Using the hypothesis that $\mathbb{T}$ is an ε-extremal set, we see that

$$|\bigcup_{T\in\mathbb{T}_1} Y_1(T)| \leqslant |\bigcup_{T\in\mathbb{T}} Y(T)| \lessapprox \delta^{n-\sigma+\epsilon}(\delta^{\varepsilon})^{\sigma}.$$

On the other hand, from the definition of σ and because $\mathbb{T}_1$ is also a sticky set by Lemma 34,

$$|\bigcup_{T\in\mathbb{T}_1} Y_1(T)| \gtrapprox \delta^{n-\sigma+\epsilon}(\delta^{\varepsilon})^{\sigma}.$$

Since

$$\mu_\delta \cdot |\bigcup_{T\in\mathbb{T}_1} Y_1(T)| \sim \sum_{T\in\mathbb{T}_1} |Y_1(T)| \gtrapprox \delta^{2\varepsilon},$$

we conclude that

$$\mu_\delta \gtrapprox \delta^{-n+\sigma-\epsilon}(\delta^{\varepsilon})^{1-\sigma}. \tag{4.2}$$

Fix a tube $T_\theta \in \mathbb{T}_\theta$ and consider the set of δ-tubes $\mathbb{T}'[T_\theta]$ with shading Y'. Let $\mathbb{T}_1[T_\theta]$ be a constant multiplicity refinement of $\mathbb{T}'[T_\theta]$ with shading Y_1 and multiplicity $\mu_\delta[T_\theta]$. (The details of how to obtain the constant multiplicity refinement are very similar to those given in detail for $\mathbb{T}$ above, so they are not repeated.) We will prove an upper bound on $\mu_\delta[T_\theta]$ by observing the structure of the set $\mathbb{T}'[T_\theta]$.

Let $e = \mathrm{dir}(T_\theta)$. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be the anisotropic rescaling defined by

$$\phi(x) = x \cdot e + \frac{x \cdot e^{\perp}}{\theta}.$$

Observe that $\phi(T_\theta)$ equals a cylinder of radius 1 and length 1. Every δ-tube T in $\mathbb{T}'[T_\theta]$ no longer has the shape of a tube. Since $\angle(\mathrm{dir}(T), e) \lessapprox \theta$, we know that $|\mathrm{dir}(T) \cdot e^{\perp}| \lessapprox \theta$, and thus $|\phi(\mathrm{dir}(T))| \leqslant 2$. This implies that $\phi(T)$ is contained in the 2-fold dilate of a δ/θ-tube $\tilde{T}$ which has the same center as T but points in the direction as the vector $\phi(\mathrm{dir}(T))$. Similarly $\tilde{T}$ is contained in the 2-fold dilate of $\phi(T)$. Through the identification $\phi(T) \mapsto \tilde{T}$, one may speak of $\phi(T)$ as a δ/θ-tube, which we will now do by an abuse of notation. Let $\tilde{\mathbb{T}}_{\delta/\theta}$ denote the collection of δ/θ-tubes $\phi(T)$ contained in $\phi(T_\theta)$.

Since $\mathbb{T}'[T_\theta]$ consists of tubes pointing in δ-separated directions, $\tilde{T}_{\delta/\theta}$ consists of (δ/θ)-tubes which are (δ/θ)-separated, It will be now shown that $\tilde{\mathbb{T}}_{\delta/\theta}$ is a sticky family of δ/θ-tubes. To see this, consider the set $\mathbb{T}'[T_\theta]$ and let η be an "intermediate" scale, i.e. let $\delta < \eta < \theta$. Since $\mathbb{T}$ is sticky, there is an eta-parent family $\mathbb{T}_\eta$ of

η-tubes covering every $T \in \mathbb{T}$. There are $\sim \eta^{1-n+\epsilon}$ η-tubes in $\mathbb{T}_\eta$, which means that for all except at most $\theta^{-\epsilon}$ θ-tubes T_θ, there are $\theta^{n-1}\eta^{1-n}$ η-tubes covering all $\mathbb{T}(T_\theta)$. Finally, observe that if an η-tube T_η contains a δ-tube $T \in \mathbb{T}'[T_\theta]$, then $\angle(T_\eta, T_\theta) \lesssim \theta$, and thus T_η is contained in a 2-fold dilate of T_θ.

Applying the rescaling ϕ to each T_η, one sees that $\phi(T_\eta)$ is an η/θ-tube containing $\approx \eta^{n-1}(\delta/\theta)^{1-n}$ (δ/θ)-tubes in it. This shows that the family $\tilde{T}_{\delta/\theta}$ is sticky.

For every $\tilde{T} \in \tilde{\mathbb{T}}_{\delta/\theta}$, let $\tilde{Y}$ be the shading given by

$$\tilde{Y}(\tilde{T}) = \phi(Y_1(T)).$$

By the definition of σ, and also because $|Y(T)| \gtrsim \delta^{2\varepsilon}|T|$ for every $T \in \mathbb{T}[T_\theta]$, we know that

$$|\bigcup_{\tilde{T}\in\tilde{\mathbb{T}}_{\delta/\theta}} \tilde{Y}(\tilde{T})| \gtrsim (\delta/\theta)^{n-\sigma+\epsilon}(\delta^{2\varepsilon})^{\sigma}.$$

Since

$$\mu_\delta[T_\theta] \cdot |\bigcup_{\tilde{T}\in\tilde{\mathbb{T}}_{\delta/\theta}} \tilde{Y}(\tilde{T})| \approx \sum_{\tilde{T}\in\tilde{T}_{\delta/\theta}} |\tilde{Y}(\tilde{T})| \sim \delta^{2\varepsilon},$$

we conclude that

$$\mu_\delta[T_\theta] \lesssim (\delta/\theta)^{-n+\sigma-2\varepsilon}(\delta^{2\varepsilon})^{1-\sigma}. \tag{4.3}$$

Now consider the set of tubes $\mathbb{T}_\theta$, which is the θ-parent family to the set $\mathbb{T}'$. Let Y_θ be the shading for $\mathbb{T}_\theta$ that is given by applying Lemma 32 to $(\mathbb{T}', Y')$. Let $\mathbb{T}_{\theta,1}, Y_{\theta,1}$ be a constant multiplicity refinement of $(\mathbb{T}_\theta, Y_\theta)$ with multiplicity μ_θ. Then since $\mathbb{T}_{\theta,1}$ is sticky, we have, using the definition of σ as above,

$$\mu_\theta \lesssim \theta^{-n+\sigma-\epsilon}(\theta^{e\epsilon})^{1-\sigma}. \tag{4.4}$$

Finally, by taking another refinment of $\mathbb{T}_{\theta,2}$ of $\mathbb{T}_{\theta,1}$, it is possible to ensure that for all $T_\theta \in \mathbb{T}_{\theta,2}$,

$$\mu_\delta[T_\theta] \sim \mu_\delta[\mathbb{T}_\theta]$$

for a constant $\mu_\delta[\mathbb{T}_\theta]$, which also satisfies

$$\mu_\delta[\mathbb{T}_\theta] \lesssim (\delta/\theta)^{-n+\sigma-2\varepsilon}(\delta^{2\varepsilon})^{1-\sigma}. \tag{4.5}$$

We will use the observation that at a point x,

$$\mu_\delta(x) \leqslant \mu_\theta(x) \cdot \mu_\delta[T_\theta](x).$$

For this observation to be applicable, it is necessary to take a refinement $\mathbb{T}_2, Y_2$ of $\mathbb{T}_1, Y_1$ such that

$$\bigcup_{T\in\mathbb{T}_2} Y_2(T_2) \subseteq \bigcup_{T_\theta\in\mathbb{T}_{\theta,2}} Y_\theta(T_\theta).$$

Now putting together (4.2), (4.4) and (4.5), and using the observation at any point x, and for any tube $T_\theta \in \mathbb{T}_\theta$, we can conclude the following statements:

$$\mu_\delta \gtrsim \delta^{-n+\sigma-\epsilon}(\delta^\varepsilon)^{1-\sigma}$$

$$\mu_\theta \gtrsim \theta^{-n+\sigma-\epsilon}(\delta^\varepsilon)^{1-\sigma}$$

$$\mu_\delta[\mathbb{T}_\theta](x) \gtrsim (\delta/\theta)^{-n+\sigma-\varepsilon}(\delta^\varepsilon)^{1-\sigma}.$$

□

Lemma 37 (Self-similarity of extremal sticky sets)**.** *There exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$*, if* $\mathbb{T}$ *is an* ε*-extremal family of* δ*-tubes with shading* Y*, then the* θ*-parent family* $\mathbb{T}_\theta$*, with the induced shading from Lemma* 32*, is* $\varepsilon(1-\varepsilon)$*-extremal, provided* $\delta < \theta \leqslant \delta^{1-\varepsilon}$*.*

Proof. This is a direct consequence of Lemma 36. Let $\varepsilon > 0$ and suppose that $\mathbb{T}$ is an ε-extremal family of δ-tubes with shading Y. Let $\mathbb{T}_\theta$ be the θ-parent set of $\mathbb{T}$ and let Y_θ be the shading given by Lemma 32. By Lemma 36, there is a refinement $\mathbb{T}', Y'$ of $\mathbb{T}, Y$ and a refinement $\mathbb{T}'_\theta, Y'_\theta$ of $\mathbb{T}_\theta, Y_\theta$ such that for all $x \in \bigcup_{T\in\mathbb{T}'} Y'(T)$, and any $T_\theta \in \mathbb{T}'_\theta(x)$,

$$\mu(x) \sim \mu \gtrsim \delta^{-n+\sigma-\varepsilon}$$
$$\mu_\theta(x) \sim \mu_\theta \gtrsim \theta^{-n+\sigma-\varepsilon}$$
$$\mu_\delta[T_\theta](x) \sim \mu_\delta[\mathbb{T}'_\theta] \gtrsim (\delta/\theta)^{-n+\sigma-\varepsilon},$$

for some numbers μ, μ_θ and $\mu_\delta[\mathbb{T}'_\theta]$. Since we have

$$\mu_\theta \cdot |\bigcup_{T\in\mathbb{T}'} Y'(T)| \sim \sum_{T\in\mathbb{T}'} |Y'(T)| \gtrsim \delta^\varepsilon \sum_{T\in\mathbb{T}} |Y(T)| \gtrsim \delta^{2\varepsilon},$$

it follows that

$$|\bigcup_{T\in\mathbb{T}'} Y'(T)| \lesssim \theta^{2\varepsilon(1-\varepsilon)}\theta^{n-\sigma+\varepsilon},$$

which is the desired conclusion. □

Lemma 38 (Balanced cover)**.** *For any $\varepsilon > 0$, if $\mathbb{T}$ is an ε-extremal sticky set of δ-tubes with shading Y and if $\mathbb{T}_\theta$ is the θ-parent set of $\mathbb{T}$ for some $\delta^{1-\varepsilon} \leqslant \theta \leqslant \delta^{1-2\varepsilon}$, and if Y_θ is the shading of $\mathbb{T}_\theta$ induced by Lemma 32, then there are refinements $(\mathbb{T}', Y')$ of $(\mathbb{T}, Y)$ and $(\mathbb{T}'_\theta, Y'_\theta)$ of $(\mathbb{T}_\theta, Y_\theta)$ such that every θ-cube $Q_\theta \in \bigcup_{T_\theta \in \mathbb{T}'_\theta} Y'_\theta(T_\theta)$ contains $\sim (\theta/\delta)^{\sigma-\varepsilon}$ δ-cubes in $\bigcup_{T\in\mathbb{T}} Y(T)$.*

Proof. Let $\varepsilon > 0$ be given. Let $\mathbb{T}, Y, \mathbb{T}_\theta$ and Y_θ be as described in the hypothesis in the lemma statement. Since $(\mathbb{T}, Y)$ is ε-extremal, the number of δ-cubes contained in $\bigcup_{T\in\mathbb{T}} Y(T)$ is $\lesssim \delta^{-\sigma+\varepsilon}$. By Lemma 37, the number of θ-cubes in $\bigcup_{T_\theta\in\mathbb{T}_\theta} Y_\theta(T_\theta)$ is $\lesssim \theta^{-\sigma+\varepsilon}$. By dyadic pigeonholing, there is a set $E \subseteq \bigcup_{T_\theta\in\mathbb{T}_\theta} Y_\theta(T_\theta)$ such that every θ-cube Q_θ contained in E contains $\sim N$ δ-cubes and

$$\sum_{T_\theta\in\mathbb{T}_\theta} |Y_\theta(T_\theta) \cap E| \gtrsim \theta^\varepsilon \sum_{T\in\mathbb{T}_\theta} |Y_\theta(T_\theta)|.$$

Define a new shading of $\mathbb{T}_\theta$ by $Y'_\theta(T_\theta) := Y_\theta(T_\theta) \cap E$ for every $T_\theta \in \mathbb{T}_\theta$. Define a subset $\mathbb{T}'_\theta \subseteq \mathbb{T}_\theta$ so that $|Y'_\theta(T_\theta)| \sim \lambda_\theta |T_\theta|$ for all $T_\theta \in \mathbb{T}_\theta$ and

$$\sum_{T_\theta\in\mathbb{T}'_\theta} |Y'_\theta(T_\theta)| \gtrsim \theta^{2\varepsilon} \sum_{T\in\mathbb{T}_\theta} |Y_\theta(T_\theta)|.$$

It follows that $(\#\mathbb{T}'_\theta) \cdot \lambda_\theta \cdot |T_\theta| \gtrsim \theta^{3\varepsilon}$, which allows us to conclude that $\#\mathbb{T}'_\theta \gtrsim \theta^{3\varepsilon} \#\mathbb{T}_\theta$ and $\lambda_\theta \gtrsim \theta^{3\varepsilon}$. Now define a refined shading of $\mathbb{T}, Y$ by

$$Y'(T) := Y(T) \cap E,$$

and take $\mathbb{T}' \subseteq \mathbb{T}$ such that $|Y'(T)| \sim \lambda |T|$ for all $T \in \mathbb{T}'$, all so that

$$\sum_{T\in\mathbb{T}'} |Y'(T)| \gtrsim \delta^\varepsilon \sum_{T\in\mathbb{T}} |Y(T)|.$$

Similarly, it follows that $\#\mathbb{T} \gtrsim \delta^\varepsilon \#\mathbb{T}$ and $\lambda \gtrsim \delta^\varepsilon$. From the definition of σ, we know that $\bigcup_{T\in\mathbb{T}'} Y'(T)$ contains at least $\delta^{-\sigma+\varepsilon}$ δ-cubes. But since $\mathbb{T}, Y$ was ε-extremal, the number of δ-cubes in $\bigcup_{T\in\mathbb{T}'} Y'(T)$ must be $\sim \delta^{-\sigma+\varepsilon}$. Similarly, the number of θ-cubes in $\bigcup_{T_\theta\in\mathbb{T}'_\theta} Y'_\theta(T_\theta)$ is $\sim \theta^{\sigma_\varepsilon}$. Every θ-cube in $\bigcup_{T_\theta\in\mathbb{T}'_\theta} Y'_\theta(T_\theta)$ contains the same number of delta cubes, so $N \sim (\delta/\theta)^{-d+\epsilon}$. □

4.3 Kakeya estimate for sticky sets

Lemma 39. *If $\mathbb{T}_\beta$ is a set of β-tubes that point in β-separated directions, with $\beta \sim 1$, and Y_β is a shading of $\mathbb{T}_\beta$, then there is a refinement $\mathbb{T}'_\beta, Y'_\beta$ such that **one** of the following cases holds:*

(a) $(\mathbb{T}'_\beta, Y'_\beta)$ *is* $(k-1)$*-plany.*

(b) At every point $x \in \bigcup_{T_\beta \in \mathbb{T}'_\beta} Y'_\beta(T_\beta)$,

$$\#\left\{((T_1, \ldots, T_k) \in \left(\mathbb{T}'_\beta(x))^k\right) : |\mathrm{dir}(T_1) \wedge \cdots \wedge \mathrm{dir}(T_k)| > \beta^k\right\} \gtrsim \#(\mathbb{T}'_\beta(x))^k. \tag{4.6}$$

Proof. Let $\mathbb{T}_\beta$ be a set of β-tubes that point in β-separated directions, and let Y_β be a shading of $\mathbb{T}_\beta$. We will say that a point $x \in \bigcup_{T_\beta \in \mathbb{T}_\beta} Y_\beta(T_\beta)$ *satisfies the k-plane condition* if there exists a $(k-1)$-plane $V(x)$ such that

$$\#\{T_\beta \in \mathbb{T}_\beta(x) : \angle(\mathrm{dir}(T_\beta), V(x)) \lesssim \beta\} \gtrsim \frac{1}{N}\#\mathbb{T}_\beta(x), \tag{4.7}$$

where N is any number greater than k. For example, one could use $N = n$. Observe that at any point $x \in \bigcup_{T_\beta \in \mathbb{T}_\beta} Y_\beta(T_\beta)$ which does not satisfy the k-plany condition, the condition (4.6) is satisfied. To see this, let $\{V_i\}$ be a maximal collection of $\sim \beta$-separated $(k-1)$-planes, all passing through the point x, in $\mathbb{R}^n$. We may partition $\mathbb{T}_\beta(x)$ into sets

$$\mathbb{T}_\beta(x)[V_i] := \{T_\beta \in \mathbb{T}_\beta(x) : \angle(\mathrm{dir}(T_\beta, V_i) \lesssim \beta\}.$$

If T_1 is chosen from $\mathbb{T}_\beta(x)[V_{i_1}]$, …, and T_k is chosen from $\mathbb{T}_\beta(x)[V_{i_k}]$, and if $i_1, \ldots, i_k$ are pairwise distinct, then

$$|\mathrm{dir}(T_1) \wedge \cdots \wedge \mathrm{dir}(T_k)| \gtrsim \beta^k.$$

Since x does not satisfy the k-plany condition, there are at least

$$(1 - \frac{1}{N}) \cdot (1 - \frac{2}{N}) \cdots (1 - \frac{k}{N}) \cdot (\#\mathbb{T}_\beta(x))^k$$

such k-tuples that can be chosen, and this shows (4.6). Denote

$$E = \{x \in \bigcup_{T_\beta \in \mathbb{T}_\beta} Y_\beta(T_\beta) : x \text{ satisfies the } k\text{-plany condition}\}.$$

Now we will consider two cases separately. First, suppose it is the case that

$$\sum_{T_\beta \in \mathbb{T}_\beta} |Y_\beta(T_\beta) \cap E| \geqslant \frac{1}{2} \sum_{T_\beta \in \mathbb{T}_\beta} |Y_\beta(T_\beta)|. \tag{4.8}$$

Then we will define

$$Y'_\beta(T_\beta) = \{x \in Y_\beta(T_\beta) \cap E : \angle(\mathrm{dir}(T_\beta), V(x)) \lesssim \beta\}$$

and define $\mathbb{T}'_\beta = \mathbb{T}_\beta$, and now $(\mathbb{T}'_\beta, Y'_\beta)$ is $(k-1)$-plany, so condition (a) holds. On the other hand, if (4.8) is not true, then it must be that

$$\sum_{T_\beta \in \mathbb{T}_\beta} |Y_\beta(T_\beta) \backslash E| \geqslant \frac{1}{2} \sum_{T_\beta \in \mathbb{T}_\beta} |Y_\beta(T_\beta)|.$$

In this case we will define

$$Y'_\beta(T_\beta) = Y_\beta(T_\beta) \backslash E$$

and $\mathbb{T}'_\beta = \mathbb{T}_\beta$, and now condition (b) is satisfied. □

Proposition 40. *Let $3 \leqslant k \leqslant n$. For every $\varepsilon > 0$, whenever $\mathbb{T}$ is a sticky family of δ-tubes with shading Y, which satisfies $|Y(T)| \gtrsim \delta^\varepsilon |T|$ for all $T \in \mathbb{T}$, there exists $C > 0$ such that*

$$|\bigcup_{T \in \mathbb{T}} Y(T)| \gtrsim \delta^{n-d+C\varepsilon},$$

where

$$d \leqslant \max_{3 \leqslant k \leqslant n} \min \left(\frac{2n+2-(\sqrt{2}-1)k}{3}, \frac{n^2+k^2+n-k}{2n} \right).$$

Proof. Fix $3 \leqslant k \leqslant n$. The proof is by induction on the scale δ. In the base case, the hypothesis is that $\mathbb{T}$ is a family of β-tubes, where $\beta \sim 1$ is the constant appearing in the definition of k-linear tubes. By Lemma 39, $\mathbb{T}$ is either $(k-1)$-plany, or else satisfies (4.6). If $\mathbb{T}$ is $(k-1)$-plany, Proposition 18 gives the result. If $\mathbb{T}$ satisfies (4.6), the result is given by Corollary 12 with $C = d+1$.

The induction hypothesis is the following: for every $\varepsilon > 0$ and every θ in the interval $[\delta/\beta, \beta)$, if

(i) $\mathbb{T}_\theta$ is a sticky set of θ-tubes with shading Y_θ, and

(ii) $|Y_\theta(T_\theta)| \gtrsim \theta^\varepsilon |T_\theta|$ for all $T_\theta \in \mathbb{T}_\theta$,

then there is a constant $C_\varepsilon > 0$ such that

$$\left| \bigcup_{T_\theta \in \mathbb{T}_\theta} Y_\theta(T_\theta) \right| \geqslant C_\varepsilon \theta^{n-d+(d+1)\varepsilon}.$$

Let $\varepsilon > 0$ and assume that $\mathbb{T}$ is a sticky family of δ-tubes, with shading Y, such that $|Y(T)| \geqslant \delta^\varepsilon |T|$ for all $T \in \mathbb{T}$.

From the definition of σ, it is enough to consider the case where $(\mathbb{T}, Y)$ is an ε-extremal family of δ-tubes. Since $\mathbb{T}$ is sticky, it is possible to cover $\mathbb{T}$ by a family

$\mathbb{T}_\theta$ of θ-tubes pointing in θ-separated directions, with $\theta = \delta/\beta$. Let Y_θ be the shading given by Lemma 32. By Lemma 37, the family $(\mathbb{T}_\theta, Y_\theta)$ admits a refinement $(\mathbb{T}'_\theta, Y'_\theta)$ which is $\varepsilon(1-\varepsilon)$-extremal. Since $(\mathbb{T}, Y)$ is ε-extremal and $(\mathbb{T}'_\theta, Y'_\theta)$ is $\epsilon(1-\varepsilon)$-extremal,

$$|\bigcup_{T\in\mathbb{T}} Y(T)| \lesssim \delta^{n-\sigma+(\sigma+1)\varepsilon} \text{ and } |\bigcup_{T_\theta\in\mathbb{T}'_\theta} Y'_\theta(T_\theta)| \lesssim \theta^{n-\sigma+(\sigma+1)\varepsilon(1-\varepsilon)}. \tag{4.9}$$

By Lemma 38, there is a refinement $(\mathbb{T}', Y')$ of $(\mathbb{T}, Y)$ and a refinement $(\mathbb{T}''_\theta, Y''_\theta)$ of $(\mathbb{T}'_\theta, Y'_\theta)$, and a number $N > 0$ such that

$$|\bigcup_{T\in\mathbb{T}'} Y'(T) \cap Q_\theta| \sim N\delta^n \tag{4.10}$$

for all θ-cubes $Q_\theta \subset \bigcup_{T_\theta\in\mathbb{T}''_\theta} Y''_\theta(T_\theta)$. From the definition of σ,

$$|\bigcup_{T_\theta\in\mathbb{T}''_\theta} Y''_\theta(T_\theta)| \gtrsim \theta^{n-\sigma+(\sigma+1)\varepsilon}.$$

This statement, together with (4.10) and (4.9), gives the conclusion that

$$(\delta/\theta)^{-\sigma+(\sigma+1)\varepsilon} \sim \beta^{-\sigma+(\sigma+1)\varepsilon} \lesssim N \lesssim \beta^{-\sigma+(\sigma+1)\varepsilon}\theta^{-(\sigma+1)\varepsilon^2}.$$

By the induction hypothesis,

$$|\bigcup_{T_\theta\in\mathbb{T}''_\theta} Y''_\theta(T_\theta)| \geqslant C_\varepsilon \theta^{n-d+(d+1)\varepsilon}.$$

With (4.9), this implies that $\sigma - (\sigma-1)\varepsilon + (\sigma-1)\varepsilon^2 \geqslant d - (d-1)\varepsilon$ and thus $N \gtrsim \beta^{-d+(d+1)\varepsilon-(\sigma+1)\varepsilon^2}$. Together with (4.10) and the induction hypothesis, this gives

$$|\bigcup_{T\in\mathbb{T}} Y(T)| \gtrsim C_\epsilon \delta^{n-d+(d+1)\epsilon}\beta^{(\sigma+1)\varepsilon^2}.$$

If β is chosen to be small enough that $\beta^{(\sigma+1)\varepsilon^2}$ times the implicit constant in the statement above is smaller than 1, this closes the induction. □

The proof of Proposition 31 follows immediately from this result.

Proof of Proposition 31. Let $\varepsilon > 0$ and let $\mathbb{T}, Y$ be as in the hypothesis of the proposition. The result follows immediately by finding the value of k for which the maximum is obtained in the following expression:

$$\max_{3\leqslant k\leqslant n} \min\left(\frac{2n+2-(\sqrt{2}-1)k}{3}, \frac{n^2+k^2+n-k}{2n}\right).$$

The optimal value of k is given by

$$k = \frac{1 - \sqrt{2} + \sqrt{6 - 2\sqrt{2}}}{3} n + O(1) \approx 0.45555915723\, n + O(1).$$

This gives

$$d \geqslant 0.60376707287\, n + O(1).$$

In the worst case scenario, given any $\varepsilon > 0$, there are infinitely many dimensions for which $d \leqslant 0.60376707287\, n + O(1) - \varepsilon$. □

4.4 Relation to sticky sets in Wang-Zahl

Recently, Wang and Zahl introduced the concept of a *sticky Besicovitch set* using the packing definition of Besicovitch sets ([23]). In this section of the paper, the relation between sticky sets of δ-tubes and sticky Besicovitch sets as defined by Wang and Zahl is outlined. A version of this result is proved in [23].

Definition 41. *A* sticky Besicovitch set *is a compact set $K \subseteq \mathbb{R}^n$ obeying the following hypotheses: there is a set of lines L in $\mathbb{R}^n$ such that*

i. L has a line in every direction,

ii. $K \cap \ell$ contains a unit line segment for every $\ell \in \mathcal{L}$, and

iii. L has packing dimension $n - 1$.

Let $\mathcal{L}$ be the set of all lines in $\mathbb{R}^n$. Every non-vertical line $\ell \in \mathbb{L}$ can be represented as

$$\ell = (\mathbf{a}, 0) + \mathbb{R}e$$

where $e \in \mathbb{S}^{n-1}$ and $\mathbf{a} \in \mathbb{R}^{n-1}$. In this section ℓ will be represented as

$$\ell = (\mathbf{a}_\ell, e_\ell).$$

The distance between two lines ℓ and ℓ' in $\mathcal{L}$ is defined by

$$\operatorname{dist}(\ell, \ell') = |\mathbf{a}_\ell - \mathbf{a}_{\ell'}| + \angle(e_\ell, e_{\ell'}).$$

A measure λ is defined on $\mathcal{L}$ to be the product of the Lebesgue measure on $\mathbb{R}^{n-1}$ and the normalized surface measure ν on $\mathbb{S}^{n-1}$. With these definitions, the packing dimension of a set $L \subseteq \mathcal{L}$ agrees with its upper modified box dimension.

Proposition 42. *For any $\varepsilon > 0$, if K is a sticky Besicovitch set in $\mathbb{R}^n$, then there exists $\delta_0 > 0$ such that*

(1) for any $\delta < \delta_0$, there is a set of δ-tubes $\mathbb{T}$ which is sticky; and

(2) if $\dim_H(K) = d$, then there is a shading $Y(T)$ satisfying $Y(T)| \gtrsim \big(\log(1/\delta)\big)^{-2}|T|$ for every $T \in \mathbb{T}$, such that

$$\bigcup_{T\in\mathbb{T}} Y(T) \lesssim \delta^{n-d+\varepsilon}.$$

Proof. Let $\epsilon > 0$ be given and let K be a sticky Besicovitch set. Since the packing dimension of L coincides with its upper modified box dimension, L can be written as a countable union of closed sets $\{L_i\}$ such that

$$\overline{\dim_M}(L_i) \leqslant n - 1 + \frac{\varepsilon}{4}$$

for all i. There exists an index i for which $\nu\big(\mathrm{dir}(L_i)\big) > 0$. From the upper Minkowski dimension of L_i, there eixsts $\delta_1 > 0$ such that for all $\delta < \delta_1$,

$$\lambda(N_\delta(L_i)) \leqslant \delta^{(2n-2)-(n-1)-\varepsilon/2}.$$

We will say that a direction $e \in \mathbb{S}^{n-1}$ is *over-represented at scale δ* if

$$|\{\mathbf{a} \in \mathbb{R}^{n-1} : (\mathbf{a}, e) \in N_\delta(L_i)\}| > \delta^{n-1-\varepsilon}.$$

By Fubini's theorem, it follows that

$$\nu\big(\{e \in \mathbb{S}^{n-1} : v \text{ is over-represented at scale } \delta\}\big) \leqslant \delta^{\varepsilon/2}.$$

By making δ_1 smaller if necessary, we can then ensure that

$$\sum_{\text{dyadic } \delta:\delta<\delta_1} \nu\big(\{e \in \mathbb{S}^{n-1} : v \text{ is over-represented at scale } \delta\}\big) \leqslant \frac{1}{2}\nu(\mathrm{dir}(L_i)).$$

Now define a subset $L' \subseteq L_i$ by

$$L' = L_i \backslash \bigcup_{\text{dyadic } \delta:\delta<\delta_0} \mathrm{dir}^{-1}\big(\{e \in \mathbb{S}^{n-1} : v \text{ is over-represented at scale } \delta\}\big).$$

By the definition of Hausdorff dimension, there exists a number $\delta_0 > 0$ (which we can ensure is smaller than δ_1 above) and a cover Q of K, where

$$Q = \bigcup_{\text{dyadic}\delta:\delta<\delta_0} Q_\delta,$$

where Q_δ is a collection of δ-cubes and

$$\#Q_k \lesssim \delta^{-d+\varepsilon/2}.$$

For each $\ell \in L'$, since $|K \cap \ell| \geqslant 1$, there exists at least one δ such that

$$\Big|\ell \cap \big(\bigcup_{Q \in Q_\delta} Q\big)\Big| \geqslant \frac{1}{100\big(\log(1/\delta)\big)^2}. \tag{4.11}$$

For each dyadic number δ, let $L_\delta \subseteq L'$ be the set of lines for which (4.11) holds. Then there exists a dyadic scale $\delta < \delta_0$ for which $\nu(L_\delta) > \nu(L')/100\big(\log(1/\delta)\big)^2$. Fix this value of δ and rename L_δ as L''. Now we define $\mathcal{E}$ to be a δ-separated subset of $\mathrm{dir}(L'')$, then

$$\#\mathcal{E} \geqslant \frac{\nu(L')\delta^{1-n}}{100\big(\log(1/\delta)\big)^2}.$$

Define $\mathbb{T}$ be a set of δ-tubes pointing in directions in $\mathcal{E}$, with centers and shading $Y(T)$ chosen so that, using (4.11),

$$\bigcup_{T \in \mathbb{T}} Y(T) = \bigcup_{Q \in Q_\delta} Q$$

and $|Y(T)| \gtrsim |T|/\big(\log(1/\delta)\big)^2$ for every $T \in \mathbb{T}$. Since this implies that

$$|\bigcup_{T \in \mathbb{T}} Y(T)| \lesssim \delta^{n-d+\varepsilon},$$

conclusion (2) in the statement of the proposition is shown. It now remains to show that $\mathbb{T}$ is sticky. Let ρ be a number such that $\delta < \rho < 1$.

If $\rho \geqslant \delta_0$: since δ_0 may be considered an admissible constant, if $\mathbb{T}_\rho$ is any set of maximal ρ-separated tubes, it is trivial that $\#\mathbb{T}_\rho \approx \rho^{1-n}$ and $\#\mathbb{T}[T_\rho] \approx (\delta/\rho)^{1-n}$.

If $\rho < \delta_0$: We will greedily cover L'' with ρ-balls, so that $\rho/3$-balls at the same centers would be disjoint. Define $\mathbb{T}_\rho$ to be the set of tubes pointing in the directions of the lines in the cover and with centers inside the set on the left-hand-side of (4.11). From this definition, it is clear that every $T \in \mathbb{T}$ is contained in some $T_\beta \in \mathbb{T}_\rho$.

Since $\delta_0 < \delta_1$, and since $\mathrm{dir}(L'')$ has no directions over-represented at any scale less than δ_1, there are at $\lesssim \delta^{-\varepsilon}$ tubes $T_\rho \in \mathbb{T}_\rho$ that point in the same direction. To make the directions ρ-separated, choose any one tube out of the possible $\delta^{-\varepsilon}$ tubes in the cover pointing in a given direction, and remove the other tubes pointing in that direction from $\mathbb{T}_\rho$. Then $\#\mathbb{T}_\rho \approx \rho^{1-n}$. To ensure $\mathbb{T}$ is still covered by $\mathbb{T}_\rho$, we will take a refinement $\mathbb{T}' \subset \mathbb{T}$, losing at most a $\rho^{-\varepsilon} \leqslant \delta^{-\varepsilon}$-fraction of $\mathbb{T}$. Then $\mathbb{T}'$ will be a sticky set, and it can be easily seen that conclusion (2) still holds for this refinement. □

Corollary 43. *The Hausdorff dimension of every sticky Besicovitch set in* $\mathbb{R}^n$ *must exceed* $0.60376707287\,n + O(1)$.

4.5 An x-ray estimate derived as a corollary of Zahl's k-linear estimate

In this section, an x-ray estimate is obtained as a corollary of Zahl's k-linear estimate. The proof is similar to that of Proposition 4.2 in [12].

For convenience, the definition of an x-ray estimate is repeated here. An x-ray estimate at dimension d holds in $\mathbb{R}^n$ if there exist $0 \leqslant \alpha, \gamma < 1$ such that the following statement is true. For any $\varepsilon > 0$, if $\mathcal{E}$ is a set of δ-separated directions in $\mathbb{S}^{n-1}$ and $\mathbb{T}$ is a set of δ-tubes obeying the following properties:

i. No tube in $\mathbb{T}$ is contained in the 10-fold dilate of another one.

ii. Every tube in $\mathbb{T}$ points in a direction in $\mathcal{E}$ and no more than $\delta^{-\beta}$ tubes in $\mathbb{T}$ point in any given direction,

then there is a constant $C_\varepsilon > 0$ such that

$$||\sum_{T\in\mathbb{T}} \chi_T||_{d'} \leqslant C_\varepsilon \delta^{1-\frac{n}{d}-\varepsilon}(\delta^{-\beta})^{1-\alpha}(\delta^{n-1}\#\Omega)^{\gamma}$$

Proposition 44. *An x-ray estimate holds in* $\mathbb{R}^n$ *at dimension* $d = \max_{2\leqslant k\leqslant n}\min\{n^2 + n + k^2 - k)/2n, n-k+2\}$.

Lemma 45. *Suppose that* $\delta_0 > 0$ *is a small number (e.g.* $\delta_0 < n/10^9$*). Let* $2 \leqslant k \leqslant n$ *and let* $d \leqslant n-k+2+\delta_0$*. Suppose that for every* $\varepsilon > 0$*, whenever* $\mathbb{T}$ *is a set of* δ*-tubes pointing in* δ*-separated directions, then there is a constant* $C_\epsilon > 0$ *such that*

$$\begin{aligned} ||(\sum_{T_1,\ldots,T_k\in\mathbb{T}} 1_{T_1}\cdots 1_{T_k}|\mathrm{dir}(T_1)\wedge\cdots\wedge\mathrm{dir}(T_k)|^{k/d})^{1/k}||_{d/(d-1)} \\ \leqslant C_\epsilon\delta^{1-\frac{n}{d}-\epsilon}(\sum_{T\in\mathbb{T}}|T|)^{n(d-1)/(n-1)d}. \end{aligned} \tag{4.12}$$

Then an x-ray estimate holds at any dimension $d_0 := d - \delta_0$.

Proof of Lemma 45. Fix $2 \leqslant k \leqslant n$ and let $d \leqslant n-k+2$. Assume that the k-linear estimate in the lemma hypothesis holds. Let $\varepsilon > 0$ be given. The argument uses induction on scale δ.

The induction hypothesis is that there exist $0 < \alpha, \gamma < 1$ and $C'_\varepsilon > 0$, whose values will be specified later, such that the following statement holds:

If $\tilde{\mathbb{T}}$ is a set of (δ/s)-tubes obeying properties i and ii in the definition of an x-ray estimate, then there exists $C_\varepsilon > 0$ such that

$$\|\sum_{\tilde{T}\in\tilde{T}} \chi_{\tilde{T}}\|_{d'} \leqslant C_\varepsilon(\delta/s)^{1-\frac{n}{d}-\varepsilon}(\delta/s)^{-\beta})^{1-\alpha}((\delta/s)^{n-1}\#\mathcal{E})^{\gamma}.$$

Let $\mathcal{E}$ be a set of δ-separated directions in $\mathbb{S}^{n-1}$ and let $\mathbb{T}$ be a set of δ-tubes obeying properties i and ii in the definition of the x-ray estimate. Let $s \sim 1$ be an admissible constant whose value will be specified later.

Define a set $\text{Narrow}_k \subseteq \bigcup_{T\in\mathbb{T}} T$ as a set of all points x for which there exists a $(k-1)$-plane $V(x)$ such that

$$\#\{T \in \mathbb{T}(x) : \angle(\text{dir}(T), V(x)) \lesssim s\} \gtrsim \#\frac{1}{10n}\mathbb{T}(x).$$

Define another set $\text{Broad}_k \subseteq \bigcup_{T\in\mathbb{T}} T$ as the set of all points $x \in \bigcup_{T\in\mathbb{T}}$ at which this condition is not satisfied. Observe that for every point $x \in \text{Broad}_k$, this leads to the conclusion

$$\#\left\{(T_1,\ldots,T_k) \in (\mathbb{T}(x))^k : |\text{dir}(T_1)\wedge\cdots\wedge\text{dir}(T_k)| > s^k\right\} \gtrsim (\#\mathbb{T}'(x))^k. \quad (4.13)$$

(For a more detailed justification of the last claim, see the proof of Lemma 39, where the same claim is explained in detail.) Now

$$\|\sum_{T\in\mathbb{T}}\chi_T\|_{d/(d-1)}^{d/(d-1)} \leqslant \int_{\text{Broad}_k} |\sum_{T\in\mathbb{T}}\chi_T|^{d/(d-1)} + \int_{\text{Narrow}_k} |\sum_{T\in\mathbb{T}}\chi_T|^{d/(d-1)}.$$

In the right-hand-side above, we will call the first term the "k-broad contribution" and the second term the "k-narrow contribution". If the k-broad contribution dominates, then it can be estimated by using the the k-linear estimate on $\delta^{-\beta}$ sets of direction-separated tubes partitioning $\mathbb{T}$, as follows.

$$\int_{\text{Broad}_k} |\sum_{T\in\mathbb{T}}\chi_T|^{d/(d-1)} = \int_{\text{Broad}_k} \Big|\Big(\sum_{T_1,\ldots,T_k\in\mathbb{T}} \chi_{T_1}\cdots\chi_{T_k}\Big)^{1/k}\Big|^{d/(d-1)}.$$

From property 4.13 above, we can ignore the contribution of k-tuples $(T_1,\ldots,T_k)$ unless $|\text{dir}(T_1)\wedge\cdots\wedge\text{dir}(T_k)| > s^k$. Then, by using the k-linear estimate on the norm applied to these k-tuples alone,

$$\|\Big(\sum_{T_1,\ldots,T_k\in\mathbb{T}} \chi_{T_1}\cdots\chi_{T_k}\Big)^{1/k}\|_{L^{d/(d-1)}(\text{Broad}_k)} \leqslant C_\epsilon\delta^{(1-\frac{n}{d}-\epsilon)}(\delta^{-\beta})(\delta^{n-1}\#\mathcal{E})^{\frac{n(d-1)}{(n-1)d}}.$$

Using Riesz-Thorin interpolation between this estimate and a known x-ray estimate at dimension $(n+2)/2$, from Theorem 1.2 of [**{\L }aba1999xray**], we obtain

$$||(\sum_{T_1,\ldots,T_k\in\mathbb{T}} \chi_{T_1}\cdots\chi_{T_k})^{1/k}||_{L^{d_0/(d_0-1)}(\mathrm{Broad}_k)} \leqslant C_\epsilon \delta^{(1-\frac{n}{d_0}-\epsilon)}(\delta^{-\beta})^\alpha(\delta^{n-1}\#\mathcal{E})^{\frac{n(d-1)}{(n-1)d}}$$

provided that α and γ have the following values:

$$\alpha = \left(\frac{\frac{1}{d_0}-\frac{1}{d}}{\frac{2}{n+2}-\frac{1}{d}}\right)\frac{(n+1)}{(n-1)(n+2)} \text{ and } \gamma = \frac{d_0-1}{d_0}.$$

This shows the conclusion if the k-broad contribution dominates. On the other hand, if the k-narrow contribution dominates, we will use the following tiling of $\mathbb{S}^{n-1}$: let $\mathbb{R}^{n-1}$ be tiled by s-cubes given by translates $(0,s]^{n-1}$ placed end-to-end and let a "tile" θ on the sphere be given by the image of one of these s-cubes via the map $\xi \mapsto (\xi, \sqrt{1-|\xi|^2})$ sending $\mathbb{R}^{n-1}$ to the northern hemisphere. Let $\mathbb{T}$ be partitioned into sets

$$\mathbb{T}[\theta] = \{T \in \mathbb{T}' : \mathrm{dir}(T) \in \theta\}.$$

Then we may write

$$\int_{\mathrm{Narrow}_k} \sum_{T\in\mathbb{T}'} Y'(T) \sim \int_{\mathrm{Narrow}_k} |\sum_\theta \sum_{T\in\mathbb{T}_\theta} \chi_{Y'(T)}(x)|\, dx.$$

Consider that at any point $x \in \mathrm{Narrow}_k$, the following statements hold, first by the definition of Narrow_k, and then by Hölder's inequality,

$$\begin{aligned}\sum_\theta \sum_{T\in\mathbb{T}_\theta} \chi_T(x) &= \sum_{\theta:\angle(\theta,V(x))\lesssim s} \sum_{T\in\mathbb{T}[\theta]} \chi_T(x) \\ &\leqslant \left(\sum_{\theta:\angle(\theta,V(x))\lesssim s} 1\right)^{1/d_0} \cdot \left(\sum_\theta |\sum_{T\in\mathbb{T}[\theta]} \chi_T(x)|^{d_0/(d_0-1)}\right)^{(d_0-1)/d_0}.\end{aligned}$$

For any $(k-1)$-plane, the number of caps θ making angle $\lesssim s$ with the $(k-1)$-plane is $\sim s^{-k+2}$. Thus,

$$\int_{\mathrm{Narrow}_k} |\sum_\theta \sum_{T\in\mathbb{T}_\theta} \chi_{Y'(T)}(x)|^{d_0/(d_0-1)} \leqslant s^{(-k+2)/(d_0-1)} \sum_\theta ||\sum_{T\in\mathbb{T}[\theta]} \chi_T||_{d_0/(d_0-1)}^{d_0/(d_0-1)}$$

and

$$||\sum_{T\in\mathbb{T}'} \chi_T||_{d_0/(d_0-1)} \leqslant s^{(-k+2)/d_0}\left(\sum_\theta ||\sum_{T\in\mathbb{T}[\theta]} \chi_T||_{d_0/(d_0-1)}^{d_0/(d_0-1)}\right)^{(d_0-1)/d_0}. \tag{4.14}$$

To estimate the quantity on the right hand side above, we will use the inductive hypothesis. For any tile θ, let $e_\theta \in \mathbb{S}^{n-1}$ be the center of θ. Perform an anisotropic re-scaling of all tubes $\mathbb{T}[T_\theta]$ via the map

$$\xi \mapsto (\xi \cdot e_\theta) + s^{-1} \cdot s^{-1}(\xi - \xi \cdot e_\theta).$$

The image of every δ-tube T is contained in a $2\delta/s$-tube which will be denoted $\tilde{T}$. To ensure that the set of directions in which the tubes $\tilde{T}$ are (δ/s)-separated, we will take a subset $\mathcal{E}' \subset \mathcal{E}$ such that $\mathcal{E}'$ is δ/s-separated. Then

$$\#\mathcal{E}' \sim s^{n-1}\#\mathcal{E}.$$

Let $\tilde{\mathbb{T}}$ be the set of all tubes $\tilde{T}$, where $T \in \mathbb{T}$ and $\mathrm{dir}(T) \in \mathcal{E}'$. Now it is immediate that $\tilde{T}$ satisfies properties i and ii in the hypothesis of the x-ray transform, and that $\tilde{T}$ has the following relation to $\mathbb{T}[\theta]$:

$$|| \sum_{T \in \mathbb{T}[\theta]} \chi_T ||_{d_0/(d_0-1)} = s^{(n-1)(d_0-1)/d_0} || \sum_{\tilde{T} \in \tilde{\mathbb{T}}} \chi_{\tilde{T}} ||_{d_0/(d_0-1)}.$$

By the induction hypothesis, the right-hand-side above is majorized by

$$\begin{aligned} &s^{(n-1)(d_0-1)/d_0} (\delta/s)^{1-\frac{n}{d_0}-\varepsilon} (\delta^{-\beta})^\alpha \left((\delta/s)^{n-1} (\#\mathcal{E}' \cap \theta) \right)^\gamma \\ &\sim s^{(n-1)(d_0-1)/d_0} (\delta/s)^{1-\frac{n}{d_0}-\varepsilon} (\delta^{-\beta})^\alpha \left(\delta^{n-1} (\#\mathcal{E} \cap \theta) \right)^\gamma. \end{aligned}$$

After putting this together with (4.14), and our choice of $\gamma = (d_0 - 1)/d_0$, we find that $|| \sum_{T \in \mathbb{T}} \chi_T ||_{d_0/(d_0-1)}$ is majorized by

$$\begin{aligned} &s^{\varepsilon + (n-k+2-d_0)/d_0} \delta^{-1+\frac{n}{d_0}-\varepsilon} (\delta^{-\beta})^\alpha \left(s^{n-1} \sum_\theta \left(\delta^{n-1} \#(\#\mathcal{E} \cap \theta) \right)^{\gamma d_0/(d_0-1)} \right)^{(d_0-1)/d_0} \\ &= s^{\varepsilon + (n-k+2-d_0)/d_0} \delta^{-1+\frac{n}{d_0}-\varepsilon} (\delta^{-\beta})^\alpha \left(\delta^{n-1} \#\mathcal{E} \right)^\gamma. \end{aligned}$$

By hypothesis, $d_0 \leqslant n - k + 2$, and the exponent of s is positive. Thus s may be chosen to satisfy

$$s^{\varepsilon + (n-k+2-d_0)/d_0} \leqslant \frac{1}{2} C'_\varepsilon.$$

If C'_ε is chosen such that

$$C_\varepsilon > 2 \max\{C_\varepsilon, C''_\varepsilon\},$$

then the induction hypothesis closes after adding the estimates obtain above for the k-broad and the k-narrow contributions. □

Proof of Proposition 44. Let $\delta_0 > 0$ be a small number (e.g. smaller than $n/10^9$). Let k_0 be the value of k for which the maximum

$$\max_{2\leqslant k\leqslant n} \min\left\{\frac{n^2+n+k^2-k}{2n}, n-k+2\right\} =: d_0$$

is achieved. Let k_1 be the value of k for which the maximum

$$\max_{2\leqslant k\leqslant n} \min\left\{\frac{n^2+n+k^2-k}{2n}, n-k+2+\delta_0\right\} =: d_1.$$

is achieved. Note that

$$d_0 < d_1 < d_0 + \delta_0.$$

By Theorem 11, the k_1-linear estimate (4.12) holds for

$$d = \frac{n^2+n+k_1^2-k_1}{2n} \leqslant d_1.$$

Now, by Lemma 45, an x-ray estimate holds at dimension d_0, since $d_0 \geqslant d_1 - \delta_0$. □

The following proposition is essentially the same as Proposition 3.3 from [17]. It has been provided here for completeness since the definition of stickiness is slightly different from the definition appearing in that paper.

Proposition 46. *Suppose that there is an x-ray estimate at dimension d and that there exists a Besicovitch set B with $\overline{\dim}_M(E) \leqslant d+\varepsilon$, for some $\varepsilon \geqslant 10^{-10}$. Then, for any sufficiently small $\delta > 0$, there is a sticky collection $\mathbb{T}$ of δ-tubes satisfying*

$$|\bigcup_{T\in\mathbb{T}} T| \lessapprox \delta^{n-d+\varepsilon}.$$

In the following proof, the same notation as in previous section will be used to parametrize lines in $\mathbb{R}^n$.

Proof. Let B be a Besicovitch set with $\overline{\dim}_M(E) \leqslant d+\varepsilon$. From the definition of upper Minkowski dimension, this means

$$|N_\theta(B)| \lessapprox \theta^{n-d+\varepsilon} \text{ for all } \delta \leqslant \theta < 1. \tag{4.15}$$

Let L be a set of lines such that

i. L contains a line in every direction (except the vertical direction), and

ii. $B \cap \ell$ contains a unit line segment for every $\ell \in L$.

Every line in ℓ can be parametrized as

$$\ell = (\mathbf{a}, 0) + \mathbb{R}e$$

where $\mathbf{a} \in \mathbb{R}^{n-1}$ is a point and $e \in \mathbb{S}^{n-1}$ is the direction in which ℓ points; and thus ℓ will be denoted (a, e).

Call a direction $e \in \mathbb{S}^{n-1}$ *sticky* at scale θ if

$$|\{\mathbf{a} \in \mathbb{R}^{n-1} : (\mathbf{a}, e) \in N_\theta(L)\}| \lesssim \theta^{n-1-C_1\varepsilon}. \tag{4.16}$$

Let $\mathcal{E} \subseteq \mathbb{S}^{n-1}$ be the set of directions given by

$$\mathcal{E} = \bigcap_{k \in \mathbb{Z}: 1 \lesssim k \lesssim \log(1/\delta)} \{e \in \mathbb{S}^{n-1} : e \text{ is sticky at scale } 2^{-k}\}.$$

For each $e \in \mathbb{S}^{n-1} \backslash \mathcal{E}$, there exists a scale $\theta_e > 0$ such that (4.16) fails. By dyadic pigeonholing, there is a set $\mathcal{E}_0 \subseteq \mathbb{S}^{n-1} \backslash \mathcal{E}$ and a number $\theta > 0$ such that $\theta_e \sim \theta$ for all $e \in \mathcal{E}_0$ and

$$\nu(\mathcal{E}_0) \gtrsim \frac{1}{\log(1/\delta)} \nu(\mathbb{S}^{n-1} \backslash \mathcal{E}).$$

Let $\mathcal{E}_1$ be a maximal θ-separated subset of $\mathcal{E}_0$. Since every $e \in \mathcal{E}_1$ is fails to be sticky at scale θ, there are $\gtrsim \theta^{-C_1\varepsilon}$ θ-separated points $\mathbf{a}$ for which the line $(\mathbf{a}, e) \in N_\theta(L)$. Let $\mathbb{T}_0$ be the set of θ-tubes given by taking the θ-neighborhoods of the unit line segments contained in $B \cap \ell$ for each of these lines $\ell = (\mathbf{a}, e)$. By the x-ray transform estimate, there exist numbers $0 < \alpha, \gamma < 1$ such that

$$|| \sum_{T \in \mathbb{T}_0} \chi_T ||_{d'} \lesssim \theta^{1-\frac{n}{d}-\varepsilon} (\theta^{-C_1\varepsilon})^{1-\alpha} (\theta^{n-1} \# \mathcal{E}_1)^{1-\gamma}.$$

By an application of Hölder's inequality, this implies that

$$| \bigcup_{T \in \mathbb{T}_0} T| \gtrsim \theta^{n-d+\varepsilon} \theta^{-dC_1\varepsilon\alpha} (\theta^{n-1} \# \mathcal{E}_1)^{d\gamma}.$$

Since $| \bigcup_{T \in \mathbb{T}_0} T| \subseteq N_{2\theta}(B)$, it follows from (4.15) that

$$\# \mathcal{E}_1 \lesssim \theta^{1-n} \theta^{C_1\varepsilon\alpha/\gamma}.$$

Since $\mathcal{E}_1$ is a maximal θ-separated subset of $\mathcal{E}_0$, this implies that $\nu(\mathcal{E}_0) \lesssim \theta^{C_1\varepsilon\alpha/\gamma}$ and thus

$$\nu(\mathcal{E}) \gtrsim 1 - \log(1/\delta) \theta^{C_1\varepsilon\alpha/\gamma} \gtrsim 1,$$

provided C_1 is chosen to be sufficiently large.

Now let $\mathcal{E}_\delta$ be a maximal δ-separated subset of $\mathcal{E}$. Let $\mathbb{T}$ be the set of δ-tubes given by the δ-neighborhoods of the unit line segments contained in $B \cap \ell$ for every $\ell \in L$ pointing in a direction $e \in \mathcal{E}_\delta$, then $\mathbb{T} \sim \delta^{1-n}$. It will be shown that a subset $\mathbb{T}'$ of $\mathbb{T}$ is sticky.

Choose a set of θ-tubes T_θ so every tube $T \in \mathbb{T}$ is contained in at least one $T_\theta \in \mathbb{T}_\theta$. It can be ensured that $\mathbb{T}_\theta$ is minimal, i.e. every $T_\theta \in \mathbb{T}_\theta$ contains at least one $T \in \mathbb{T}$ which is not contained in any other θ-tube in $\mathbb{T}_\theta$. Consider the set of coaxial lines of tubes in T_θ. Suppose that there are several of these lines $\ell_1, \cdots, \ell_N$ that are "almost parallel,", i.e. all the lines $\ell_1, \cdots, \ell_N$ point in directions making at most angle θ with a single direction $e \in \mathbb{S}^{n-1}$. For each $1 \leqslant i \leqslant N$, the line $\ell_i = (\mathbf{a}_i, e_i)$ is contained in $N_\theta(\mathrm{dir}^{-1}(\mathcal{E}))$. Thus, the θ-tubes with coaxial lines $\ell_1, \ldots, \ell_N$ all lie in

$$\mathrm{dir}^{-1}(N_{2\theta}(\{e\})) \cap N_{2\theta}(\mathrm{dir}^{-1}(\mathcal{E})).$$

It follows that

$$\lambda\left(\mathrm{dir}^{-1}(N_{2\theta}(\{e\}) \cap N_{2\theta}(\mathrm{dir}^{-1}(\mathcal{E}))\right) \gtrsim N\theta^{2(n-1)}$$

and then, by Fubini's theorem, there must exist $e' \in N_{2\theta}(\{e\}) \cap \mathcal{E}$ such that

$$|\{\mathbf{a} \in \mathbb{R}^{n-1} : (\mathbf{a}, e') \in N_\theta(L)\}| \gtrsim N\theta^{n-1}.$$

From (4.16), $N \lessapprox \theta^{-C_1\varepsilon}$. Thus, it is possible to choose a θ-direction-separated subset $\mathbb{T}'_\theta \subseteq \mathbb{T}_\theta$ such that $\#\mathbb{T}'_\theta \gtrapprox \theta^{C_1\varepsilon}\#\mathbb{T}_\theta$. After taking another refinement $\mathbb{T}''_\theta$, it is possible to ensure that every $T_\theta \in \mathbb{T}''_\theta$ contains $\gtrapprox (\theta/\delta)^{1-n+C_2\varepsilon}$ δ-tubes for a suitable constant $C_2 > 0$. If $\mathbb{T}'$ is chosen to be

$$\mathbb{T}' := \{T \in \mathbb{T} : \exists T_\theta \in \mathbb{T}''_\theta \text{ such that } T \subseteq T_\theta\},$$

it follows that $\mathbb{T}'$ is sticky and that $\#\mathbb{T}' \gtrapprox \delta^{1-n+C_3\varepsilon}$. □

Together with Proposition 46 and Proposition 31, Proposition 44 gives the following corollary.

Corollary 47. *The Minkowski dimension of every Besicovitch set in* $\mathbb{R}^n$ *must exceed* $d \geqslant (2\sqrt{2} + \varepsilon)n$*, for a certain value of* ε *greater than* 10^{-10}.

Though this estimate is not as strong as one previously proved in [18], it provides a new proof of the result.

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