Bost–Connes–Marcolli system for the Siegel Modular Variety

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Ismail Abouamal

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ABSTRACT

Bost–Connes–Marcolli systems are an important type of quantum statistical mechanical systems which provide connections between the ergodic theory of dynamical systems, class field theory and von Neumann algebras. In this thesis, we generalize the $GL_{2,\mathbb{Q}}$-system in [CM04] to the case of the Siegel modular variety of degree two and study its various properties. We show that this dynamical system undergoes a spontaneous symmetry breaking phase transition and classify its equilibrium extremal states at different inverse temperatures β. We next study its symmetry group and derive an intertwining equality between the action by symmetries and a subgroup of the Galois group of the Siegel modular field. Finally, we study the von Neumann algebras associated to the equilibrium states and classify their types.

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Chapter 1

INTRODUCTION

Classical mechanics describes systems of finitely many interacting particles. The algebra of observables is given by the *commutative* algebra $C^\infty(M)$ where (M, ω) is a symplectic manifold of dimension $2n$ called the *phase space*. The time evolution is determined by a smooth real-valued functions H on M called the *Hamiltonian*. For every observable $f \in C^\infty(M)$, the time evolution is given by Hamilton's equation:

$$\frac{df}{dt} = \{H, f\}.$$

In classical statistical mechanics a state is determined by a probability measure μ on a symplectic manifold M of dimension $2n$ that assigns to each observable $f \in C^\infty(M)$ an expectation value given by

$$\phi(f) = \int_M f \, \mathrm{d}\mu.$$

The Gibbs canonical ensemble is a measure defined in terms the Hamiltonian $H : M \to \mathbb{R}$ and the symplectic structure of M. We let $\beta = 1/k_B T$ be the inverse temperature parameter, where k_B is the Boltzmann constant (we will set the Boltzmann constant to one in the rest of this thesis). The Gibbs measure is given by

$$\mathrm{d}\mu_G = \frac{1}{Z} e^{-\beta H} \, \mathrm{d}\Omega,$$

where Ω is the Liouville volume form on M and $Z = \int_M e^{-\beta H} \, \mathrm{d}\Omega$.

In the quantum statistical mechanical framework, the observables form (in general) a noncommutative C^*-algebra $\mathcal{A}$, which we define formally in §2.1. In this quantum setting, Hamilton's equation is replaced with the *Hamiltonian flow*, that is, a continuous group homomorphism

$$\sigma_t : \mathbb{R} \to \mathrm{Aut}(\mathcal{A}),$$

which specifies the time evolution of the system. The observables correspond to self-adjoint elements of a C^*-algebra $\mathcal{A}$ and a state of the quantum system corresponds to a state ϕ on the algebra (cf. §2.1) $\mathcal{A}$:

$$\phi : \mathcal{A} \to \mathbb{C},$$

where $\phi(a)$ corresponds to the expected value of the observable $a \in \mathcal{A}$.

The quantum analogue for the Gibbs measure is given by the KMS_β condition at inverse temperature β (cf. Definition 2.1.4). The table below summarizes the correspondence between the classical and quantum settings.

Classical system	Quantum system
$f \in C^\infty(M)$ (M, ω) is a symplectic manifold	$a \in \mathcal{A}, a = a^*$ $\mathcal{A}$ is a C^* algebra
Poisson Bracket: $\{a_1, a_2\}$	Commutator: $[a_1, a_2]$
Hamilton equations: $\frac{df}{dt} = \{H, f\}$	Hamiltonian flow: $\sigma_t : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$
Probability measure μ on M: $\phi(X) = \int_M f d\mu$	Positive linear functional of norm 1: $\phi : \mathcal{A} \to \mathbb{C}$
Partition function: $Z(\beta) = \int_M e^{-\beta H} d\Omega$	$Z(\beta) = \mathrm{T}r(e^{-\beta H})$
Gibbs measure: $d\mu = \frac{e^{-\beta H} d\Omega}{Z(\beta)}$	KMS_β condition: $\phi(ab) = \phi(b\sigma_{i\beta}(a))$

By the end of the last century, Bost and Connes [BC95], motivated by the work of B. Julia [Jul90], constructed a C^*-dynamical system $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}})$ which provided a quantum statistical reinterpretation of the class field theory of $\mathbb{Q}$. This system admits the Riemann zeta function $\zeta(\beta)$ as its partition function and exhibits a spontaneous symmetry braking phase transition at the critical inverse temperature $\beta_c = 1$. On the other hand, the extremal equilibrium states (called KMS_β states) at zero temperature are parametrized by the points on the zero-dimensional Shimura variety $\mathrm{Sh}(\pm 1, GL_1)$.

The symmetry group of the BC system is $GL_1(\hat{\mathbb{Z}})$, which acts only by automorphisms. Bost and Connes defined a dense rational subalgebra $\mathcal{A}_\mathbb{Q} \subset \mathcal{A}$ such that the evaluation of equilibrium states at low temperatures on $\mathcal{A}_\mathbb{Q}$ generates the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ of $\mathbb{Q}$. It was then shown that the natural action by symmetries of the Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ through the class field theory isomorphism is intertwined with the action on the values of equilibrium states.

Over the past several years, several generalizations of the Bost-Connes system have been studied. The original construction was first generalized by Connes and Marcolli in [CM04] where they introduced the $GL_{2,\mathbb{Q}}$-system. This system exhibits several new properties that were not present in the original BC system.

The partition function of the $GL_{2,\mathbb{Q}}$-system is expressed as the product $\zeta(\beta)\zeta(\beta-1)$ and the system exhibits two phase transitions, the first at inverse temperature $\beta_{c_1} = 1$ and $\beta_{c_2} = 2$. The emergence of the region between $\beta_{c_1} = 1$ and $\beta_{c_2} = 2$ is unique to the $GL_{2,\mathbb{Q}}$-system. The analysis of the set of equilibrium states within this region turns out to be rich mathematically. The symmetry group of the $GL_{2,\mathbb{Q}}$-system consists in this case of automorphisms and endormorphisms and coincides with the group $GL_2(\mathbb{A}_{\mathbb{Q},f})/\mathbb{Q}^\times$. This result provided a connection with the Galois theory of the modular field F through the construction of an arithmetic subalgebra $\mathcal{A}_{2,\mathbb{Q}}$ (which in this case is a subalgebra of unbounded multipliers) such that the evaluation of equilibrium states at low temperatures on $\mathcal{A}_{2,\mathbb{Q}}$ generate a field isomorphic to the modular field, (this copy is realized via a choice of a generic state as an embedded subfield of $\mathbb{C}$). The evaluation of states on elements of the subalgebra $\mathcal{A}_{2,\mathbb{Q}}$ intertwines the action by symmetries and the Galois group of the modular field.
In a subsequent work by Connes, Marcolli and Ramachandran [CMR06], the authors constructed a new system which can viewed as either a generalization of the original $GL_{1,\mathbb{Q}}$-system to an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ or a specialization of the $GL_{2,\mathbb{Q}}$-system to elliptic curves with complex multiplication by K. The partition function of this new system is the Dedekind zeta function $\zeta_{\mathbb{K}}(\beta)$ and its group of symmetries is the Galois group of the maximal abelian extension of K, providing once again a quantum statistical interpretation of the class field theory of K. The following table summarizes some arithmetic properties of these systems.

System	$\mathbb{G}_{m,\mathbb{Q}}$	$\mathrm{GL}_{2,\mathbb{Q}}$	$\mathbb{G}_{m,\mathbb{K}}, \mathbb{K} = \mathbb{Q}(\sqrt{-d})$
Partition function	$\zeta(\beta)$	$\zeta(\beta)\zeta(\beta-1)$	$\zeta_{\mathbb{K}}(\beta)$
Symmetries	$\mathbb{A}^\times_{\mathbb{Q},f}/\mathbb{Q}^\times$	$GL_2(\mathbb{A}_{\mathbb{Q},f})/\mathbb{Q}^\times$	$\mathbb{A}^\times_{\mathbb{K},f}/\mathbb{K}^\times$
Automorphisms	$\hat{\mathbb{Z}}^\times$	$GL_2(\hat{\mathbb{Z}})$	$\hat{O}^\times/O^\times$
Endomorphisms	-	$GL_2^+(\mathbb{Q})$	$\mathrm{Cl}(O)$
Galois group	$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$	Aut(F)	$\mathrm{Gal}(\mathbb{K}^{\mathrm{ab}})/\mathbb{K}$
Extremal KMS_∞	$\mathrm{Sh}(\pm 1, GL_1)$	$\mathrm{Sh}(\mathbb{H}^\pm, GL_2)$	$\mathbb{A}^\times_{\mathbb{K},f}/\mathbb{K}^\times$

The construction of Connes and Marcolli was further generalized to arbitrary reductive groups by Ha and Paugam in [HP05]. The former authors reformulated the $GL_{2,\mathbb{Q}}$ system in the adelic language, making explicit its relation to an important class of moduli spaces in arithmetic geometry, namely Shimura varieties. They generalized the construction of Bost, Connes and Marcolli to an arbitrary Shimura datum (G, X) and over any number field. They introduced a formal definition of the abstract Bost-Connes-Marcolli system associated to the pair (G, X) and were successful in showing, among other features, that these systems admit the Dedekind zeta function as the partition function and the group of connected components of the idèle class group acts as the symmetry group.

In this thesis, we generalize the construction of Connes and Marcolli and study the Bost–Connes-Marcolli (BCM) system associated to the two dimensional Siegel modular variety. After reviewing the background material in Chapter I, we investigate in Chapter II the thermodynamical properties of this system and study its pure (extremal) equilibrium states at various inverse temperatures $\beta > 0$. We compute its partition function and show that it is a rational function of shifted zeta functions. We show that two spontaneous phase transitions happen at $\beta_{c_1} = 3$ and $\beta_{c_2} = 4$. We provide an explicit description of all extremal states within various ranges of β. More precisely, we show that the system does not admit an equilibrium state in the range $\beta < 3$ and provide an explicit formula for the Gibbs states in the range $\beta > 4$ in terms of the partition function. Within the range $3 < \beta \leq 4$, we show that a unique equilibrium state exists and can be constructed out of the Haar measures on the locally compact spaces underlying the system.

In Chapter III we compute the symmetry group and make the connection with the theory of Siegel modular forms. We then construct an arithmetic subalgebra of the algebra of multipliers of the algebra of observables and prove an equality which intertwines the action by symmetries and the Galois action of a subgroup of the Galois Group of the Siegel modular field.

Finally, in Chapter IV we focus on the type of von Neumann algebras generated by the states in the GNS construction. We study the type of these algebras at various inverse temperatures β and show that a phase transition occurs at the level of von Neumann algebras. We prove that the type of these algebras is I_∞ for the low temperature region and transitions to a III_1 factor as β decreases to the range $3 < \beta \leq 4$.

Chapter 2

BACKGROUND

The goal of this chapter is to provide an overview of different results and techniques that will be useful throughout this work. It is divided into three main sections. We first review the theory of C^*-dynamical system including the notion of observables, compactification, multipliers, time evolution and KMS_β states. Readers already familiar with this topic should feel free to skip this section. The second part of this chapter provides a brief review of the theory of von Neumann algebras and their classification. We will focus on crossed products and their classification in terms of the properties of the group action. The last section reviews the adelic construction of BCM systems associated to an arbitrary Shimura datum (G, X).

2.1 C^*-dynamical systems

Given a C^*-algebra $\mathcal{A}$, we know from Gelfand–Naimark theorem ([BR87, Theorem 2.1.10]) that the algebra $\mathcal{A}$ is isomorphic to a norm-closed self-adjoint algebra of bounded operators on a Hilbert space. This result, together with the axioms of quantum mechanics, motivates the following operator algebraic formulation of quantum statistical mechanical systems.

Definition 2.1.1. A quantum statistical mechanical system $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}})$ is a C^*-algebra $\mathcal{A}$ together with a strongly continuous one-parameter group of automorphisms $(\sigma_t)_{t\in\mathbb{R}}$; that is, the map

$$t \to \sigma_t(a)$$

is norm continuous for every $a \in \mathcal{A}$.

In practice, the algebra $\mathcal{A}$ may not be unital. Although any C^*-algebra contains an approximate unit, it is necessary sometimes to realize the algebra $\mathcal{A}$ as a subalgebra of a unital C^*-algebra. There are two ways one can achieve this. The first consists of considering the "one-point compactification" of $\mathcal{A}$. Let

$$\tilde{\mathcal{A}} = \mathbb{C} \oplus \mathcal{A},$$

and define a product, involution and a norm on $\tilde{\mathcal{A}}$ as follows:

$$(\alpha_1, a)(\alpha_2, b) := (\alpha_1\alpha_2, ab + \alpha_1 b + \alpha_2 a)$$
$$(\alpha, a)^* := (\bar{\alpha}, a^*).$$

One then obtains that $\tilde{\mathcal{A}}$ is a unital C^*-algebra such that $\mathcal{A}$ is a maximal ideal in $\tilde{\mathcal{A}}$. There exists another type of compactification of $\mathcal{A}$ which requires the construction the *multiplier algebra* of $\mathcal{A}$.

Definition 2.1.2. Given a non-unital C^*-algebra $\mathcal{A}$, the multiplier algebra $M(\mathcal{A})$ is a unital C^*-algebra with the property that $\mathcal{A} \subset M(\mathcal{A})$ is an essential ideal[1].

The multiplier algebra $M(\mathcal{A})$ can be constructed using double centralizers and satisfies the following universal property ([Bla06]): whenever D is a unital C^*-algebra containing $\mathcal{A}$ as an ideal, there exists a unique *-homomorphism $D \to M(\mathcal{A})$ with kernel

$$\mathcal{A}^\perp = \{d \in D : \forall a \in \mathcal{A}, da = 0\},$$

and restricts to the identity map on $\mathcal{A}$.

This type of compactification corresponds to the Stone–Čech compactification in the commutative case. In fact, if $\mathcal{A}$ is a commutative C^*-algebra then it is necessarily isomorphic to the algebra $C_0(X)$ for a locally compact Hausdorff space X and $M(A) = C(\beta X)$ where βX is the Stone–Čech compactification of X.

A bounded multiplier on a C^*-algebra $\mathcal{A}$ is a bounded linear map

$$T : \mathcal{A} \to \mathcal{A},$$

such that $T(ab) = aT(b)$ for all $a, b \in \mathcal{A}$. Similarly, an unbounded multiplier on $\mathcal{A}$ is a linear operator

$$T : \mathcal{D}(T) \to \mathcal{A},$$

defined on a dense ideal $\mathcal{D}(T)$ of $\mathcal{A}$ with the property that $T(ab) = aT(b)$ for all $a \in \mathcal{D}(T)$ and $b \in \mathcal{A}$.

Certain types of positive linear functionals play a key role in the representation theory of C^*-algebras and their quantum physical interpretation. We next recall the

[1] That is, for $x \in M(\mathcal{A})$, if $xa = a$ for all $a \in \mathcal{A}$, then $x = 0$.

definition and properties of a particular type of continuous linear functionals over $\mathcal{A}$.

Definition 2.1.3. A state ϕ on a unital C^*-algebra $\mathcal{A}$ is a linear functional over $\mathcal{A}$ satisfying the following normalization and positivity conditions:

$$\phi(e) = 1, \quad \phi(a^*a) \geq 0.$$

Note that we did not require from the functional ϕ to be continuous as this a consequence of the positivity condition. The state ϕ is said to be faithful if $\phi(a^*a) = 0$ implies $a = 0$.

To obtain a state on a non-unital C^*-algebra $\mathcal{A}$, we replace the normalization condition $\phi(e) = 1$ by

$$\|\phi\| := \sup_{x \in \mathcal{A}, \|x\| \leq 1} |\phi(x)| = 1. \tag{2.1}$$

The Gelfand–Neumark–Segal construction is an important method used to build representations of any C^*-algebra from states.

Theorem 2.1.1 (The GNS representation)**.** *let ϕ is a state of a C^*-algebra $\mathcal{A}$. Then there exists a unique (up to unitary equivalence) cyclic representation π_ϕ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\phi$ and a unit cyclic vector x_ϕ for π_ϕ (i.e., $\pi_\phi(\mathcal{A})x_\phi$ is dense in $\mathcal{H}_\phi$) such that*

$$\phi(a) = (\pi_\phi(a)x_\phi, x_\phi), \quad a \in \mathcal{A}.$$

Proof. See [BR87, Theorem 2.3.16] □

KMS$_\beta$ states on C^*-algebras

In statistical mechanics, we are interested in the so-called thermal equilibrium states at inverse temperature $\beta = 1/T$. To motivate the definition of a KMS$_\beta$ state, we consider first a finite dimensional C^*-dynamical system $(Mat_n(\mathbb{C}), (\sigma_t)_{t\in\mathbb{R}})$, where the time evolution is given by

$$\sigma_t(a) = e^{itH} a e^{-itH}, \quad a \in Mat_n(\mathbb{C}),$$

and $H = H^*$ is the Hamiltonian. Any state ϕ on $Mat_n(\mathbb{C})$ has the form

$$\phi(a) = Tr(Qa),$$

where Q is a positive semi-definite matrix in $Mat_n(\mathbb{C})$ of trace one (density matrix). The free energy at inverse temperature β associated to the state ϕ is given by $F(\phi) = -Tr(Q\log(Q)) + \beta\phi(H)$. The equilibrium state is the state of minimal free energy and is given by the Gibbs state

$$\phi_\beta(a) = \frac{Tr(e^{-\beta H}a)}{Tr(e^{-\beta H})}.$$

Define the function

$$f(t) := \phi_\beta(a\sigma_t(b)), \quad t \in \mathbb{R}. \tag{2.2}$$

By analytic continuation from $\mathbb{R}$ to $\mathbb{R} + i\beta$ and the cyclicity of the trace, we can see that

$$f(t + i\beta) = \phi_\beta(\sigma_t(b)a), \quad a, b \in Mat_n(\mathbb{C}). \tag{2.3}$$

Equations (2.2) and (2.3) motivate the following definition, proposed by Haag, Winnik and Hugenholtz [HHW67; Rud92] as an equilibrium condition in the C^*-algebraic setting of quantum statistical mechanics.

Definition 2.1.4. Let $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}})$ be a C^*-dynamical system and ϕ a state on $\mathcal{A}$. For $0 < \beta < \infty$, we say that ϕ is a KMS$_\beta$ state if for all $a, b \in \mathcal{A}$, there exists a bounded continuous function f on the strip

$$\Omega_\beta = z \in \mathbb{C} : 0 \leq \mathrm{Im}(z) \leq \beta,$$

such that f is holomorphic in the interior of Ω_β and

$$f(t) = \phi(a\sigma_t(b)), \quad f(t + i\beta) = \phi(\sigma_t(b)a), \quad \forall t \in \mathbb{R}. \tag{2.4}$$

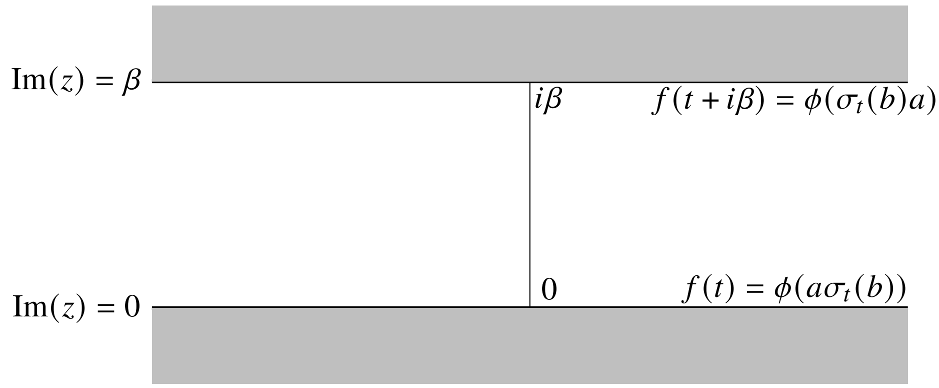


Figure 2.1: The strip Ω_β in the KMS_β condition.

In practice we will often use the following equivalent characterization of the KMS$_\beta$ condition. We recall that given a C^*-dynamical system $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}})$, an element $a \in \mathcal{A}$ is said to be σ-analytic if the function $t \longrightarrow \sigma_t(a)$ extends to an entire function of $z \in \mathbb{C}$. The set $\mathcal{A}_\sigma$ of σ-analytic elements of $\mathcal{A}$ is dense in the norm topology.

Proposition 2.1.1. Let ϕ be state over a C^*-dynamical system $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}})$. Then ϕ is a KMS$_\beta$ state if and only if

$$\phi(ab) = \phi(b\sigma_{i\beta}(a)), \tag{2.5}$$

for all a, b in a norm dense σ-invariant *-subalgebra of $\mathcal{A}_\sigma$.

Proof. See [BR96, Proposition 5.3.7] □

Using this condition, we can show that the function $g : z \mapsto \phi(\sigma_z(a))$ where $a \in \mathcal{A}_\sigma$ is analytic and bounded on $\mathbb{C}$. By Liouville's theorem this implies that the function g is constant and by the density of $\mathcal{A}_\sigma$ this can be extended to all observables $a \in \mathcal{A}$. Hence the KMS$_\beta$ condition guarantees that the state ϕ is stationary:

$$\phi = \phi \circ \sigma_t, \quad t \in \mathbb{R},$$

which is the simplest condition for an equilibrium.

For $\beta < \infty$, the set of KMS$_\beta$ states, which we denote by $\mathcal{E}_\beta$, is a convex, weak*-compact simplex. It is then natural to consider the set of its extremal points as these states describe the pure thermodynamic phases of the system $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}})$.

When working with non-unital C^*-dynamical system $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}})$, we will often need to extend a state on $\mathcal{A}$ to the multiplier algebra $M(\mathcal{A})$. The following result guarantees that it is always possible to perform such an extension. We denote by $M(\mathcal{A})_\sigma \subset M(\mathcal{A})$ the C^*-subalgebra of elements on $M(\mathcal{A})$ such that the map $t \to \sigma_t(x)$ is norm continuous.

Proposition 2.1.2. Let ϕ be a state on $\mathcal{A}$. Then ϕ admits a canonical extension $\tilde{\phi}$ to the multiplier algebra $M(\mathcal{A})$. If ϕ is a KMS$_\beta$ state, then $\tilde{\phi}$ still satisfies the KMS$_\beta$ condition on the subalgebra $M(\mathcal{A})_\sigma$.

Proof. See [CM19, Proposition 3.10] □

When $\beta = \infty$, there are two ways we can define the notion of an equilibrium state ϕ on the algebra $\mathcal{A}$ at zero temperature. The first consists of requiring the existence, for each $a, b \in \mathcal{A}$, of a bounded holomorphic function $F_{a,b}(z)$ on the upper half plane such that

$$F_{a,b}(t) = \phi(a\sigma_t(b)).$$

In the literature, a state ϕ satisfying this property is called a *ground state*. In this thesis, we use the following natural and stronger definition of a KMS$_\infty$ state.

Definition 2.1.5. KMS$_\infty$ states on the system $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}})$ are weak limits of KMS$_\beta$ states as $\beta \to \infty$, that is,

$$\phi_\infty(a) = \lim_{\beta\to+\infty} \phi_\beta(a), \quad \forall a \in \mathcal{A}.$$

Symmetries of C^*-dynamical systems

We review from [CM04] some general facts on symmetry groups of quantum mechanical statistical systems.

We first introduce symmetries by *automorphisms*.

Definition 2.1.6. We say that a locally compact group G is a group of symmetries by automorphisms of a C^*-dynamical system $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}})$ if there is an embedding $\alpha : G \hookrightarrow \mathrm{Aut}(\mathcal{A})$ such that

$$\sigma_t \alpha_g = \alpha_g \sigma_t, \quad \forall g \in G, \forall t \in \mathbb{R}. \tag{2.6}$$

There is a natural action of G on the set of KMS$_\beta$ states. This action is given by pull backs, that is,

$$(g^* \phi)(a) = \phi(\alpha_g(a)),$$

so that

$$(g_1 g_2)^* \phi = g_2^*(g_1^* \phi), \quad g_1, g_2 \in G.$$

Denote by $\mathcal{A}_U^\sigma$ the unitary group of the fixed point algebra of σ_t:

$$\mathcal{A}_U^\sigma := \{u \in \mathcal{A} : u^* u = u u^* = 1, \, \sigma_t(u) = u, \, \forall t \in \mathbb{R}\}.$$

Every element $u \in \mathcal{A}_U^\sigma$ acts on the dynamical system $(\mathcal{A}, (\sigma_t)_{t \in \mathbb{R}})$ by inner automorphisms as follows:

$$\mathrm{ad}(u)x = u x u^*, \quad \forall x \in \mathcal{A}. \tag{2.7}$$

It turns out that the induced action of $\mathrm{ad}(u)^*, u \in \mathcal{A}_U^\sigma$ on the set of KMS$_\beta$ states is trivial.

Proposition 2.1.3. Let ϕ be any KMS$_\beta$ state of the system $(\mathcal{A}, (\sigma_t)_{t \in \mathbb{R}})$ and $u \in \mathcal{A}_U^\sigma$. Then

$$\mathrm{ad}(u)^*(\phi) = \phi.$$

Proof. Recall that the KMS$_\beta$ condition implies that

$$\phi(ab) = \phi(b \sigma_{i\beta}(a)), \tag{2.8}$$

for all a, b in a norm dense and σ-invariant $*$-subalgebra of $\mathcal{A}$. Hence for have

$$\mathrm{ad}(u)^*(\phi)(x) = \phi(u x u^*) = \phi(x u^* \sigma_{i\beta}(u)) = \phi(x), \quad \forall x \in \mathcal{A},$$

where $\sigma_{i\beta}(a)$ denotes the analytic continuation $z \longrightarrow \sigma_z(a)$ of $t \longrightarrow \sigma_t(a)$. □

Next we introduce symmetries by *endomorphisms*, which is a more general type of symmetries.

Definition 2.1.7. An endomorphism α of a C^*-dynamical system $(\mathcal{A}, \sigma_t)$ is a $*$-homomorphism $\alpha : \mathcal{A} \to \mathcal{A}$ such that

$$\sigma_t \alpha = \alpha \sigma_t, \quad \forall t \in \mathbb{R}. \tag{2.9}$$

An endomorphism α of $(\mathcal{A}, \sigma_t)$ acts on a KMS$_\beta$ state ϕ by pullback, provided that the idempotent element $\alpha(1)$ satisfies $\phi \circ \alpha(1) \neq 0$:

$$(\alpha^*\phi)(a) := \frac{\phi \circ \alpha(a)}{\phi \circ \alpha(1)}, \quad \forall a \in \mathcal{A}. \tag{2.10}$$

Denote by $\mathcal{A}_I^\sigma$ the set of isometries that are eigenvectors of the time evolution, namely

$$\mathcal{A}_I^\sigma := \{u \in \mathcal{A} \mid u^*u = 1, \quad \exists \lambda \in \mathbb{R}_+^*, \quad \sigma_t(u) = \lambda^{it} u, \forall t \in \mathbb{R}\}.$$

Any element $u \in \mathcal{A}_I^\sigma$ defines an inner endomorphism $\mathrm{ad}(u)$ of $(\mathcal{A}, \sigma_t)$ as in (2.7). Similarly to the case of symmetries by automorphisms we have the following:

Lemma 2.1.1. Let ϕ be any KMS$_\beta$ state of the system $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}})$ and $u \in \mathcal{A}_I^\sigma$. Then the inner endomorphism $\mathrm{ad}(u)$ acts trivially on the set of KMS$_\beta$ states.

Proof. We obtain from the KMS$_\beta$ condition (2.8) that $\phi(uu^*) = \lambda^{-\beta} \neq 0, \lambda \in \mathbb{R}_+^*$. Hence the pullback $\mathrm{ad}(u)^*(\phi)$ of any KMS$_\beta$ state ϕ is well defined. We then have

$$\mathrm{ad}(u)^*(\phi)(x) = \frac{\phi(uxu^*)}{\phi(uu^*)} = \lambda^\beta \phi(xu^* \sigma_{i\beta}(u)) = \phi(x).$$

□

Groupoid C^*-algebras

In this section, we briefly review the basic construction of groupoid C^*-algebras. We refer the reader to the work by J. Renault in [Ren06] for a more detailed treatment. We first recall the notion of groupoid.

Definition 2.1.8. A groupoid is a set $\mathcal{G}$ endowed with a product map $(x, y) \mapsto xy : \mathcal{G}^2 \to \mathcal{G}$, where $\mathcal{G}^2$ is a subset of $\mathcal{G} \times \mathcal{G}$, called the set of composable pairs, and an inverse map $x \mapsto x^{-1} : \mathcal{G} \to \mathcal{G}$ such that the following relations are satisfied:

(i). $(x^{-1})^{-1} = x$ for all $x \in \mathcal{G}$.

(ii). Given $(x, y), (y, z) \in \mathcal{G}^2$, then $(xy, z), (x, yz) \in \mathcal{G}^2$ and $(xy)z = x(yz)$

(iii). $(x^{-1}, x) \in \mathcal{G}^2$ for all $x \in \mathcal{G}$ and if $(x, y) \in \mathcal{G}^2$, then $x^{-1}(xy) = y$

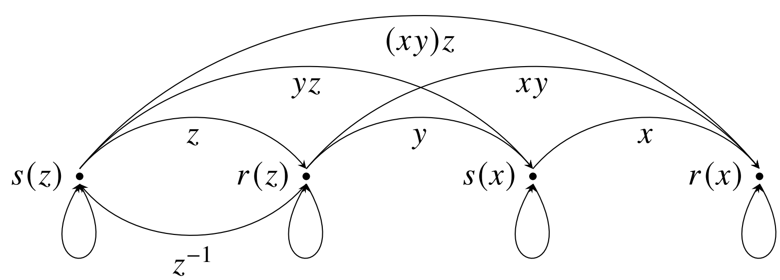
(iv). $(x, x^{-1}) \in \mathcal{G}^2$ for all $x \in \mathcal{G}$ and if $(z, x) \in \mathcal{G}^2$, then $(zx)x^{-1} = z$

For $x \in \mathcal{G}$, the map $s : x \mapsto x^{-1}x$ is called the *source map* and $r : x \mapsto xx^{-1}$ is the *range map*. The pair (x, y) is composable iff $r(y) = s(x)$. The set $\mathcal{G}^0 = s(\mathcal{G}) = r(\mathcal{G})$ is the *unit space* of $\mathcal{G}$ (since $xs(x) = r(x)x = x$). A groupoid $\mathcal{G}$ is said to be *principal* if the map $x \mapsto (r(x), s(x)) : \mathcal{G} \to \mathcal{G}^0 \times \mathcal{G}^0$ is one-to-one. The groupoid $\mathcal{G}$ is said to be *transitive* if this map is onto. For $x, y \in \mathcal{G}^0$ we set $\mathcal{G}^x = r^{-1}(x)$, $\mathcal{G}_y = s^{-1}(y)$ and $\mathcal{G}^x_y = \mathcal{G}^x \cap \mathcal{G}_y$. The set $\mathcal{G}^x_x$, which is a group, is called the *isotropy group* at x.

Given a groupoid $\mathcal{G}$, we consider the small category whose set of objects is $\mathcal{G}^0$ and for any $x, y \in \mathcal{G}^0$, we put

$$\mathrm{Hom}(x, y) = \{z \in \mathcal{G} \mid s(z) = x,\ r(z) = y\}.$$

If we define the composition of morphisms to be the groupoid composition operation, we easily see that a groupoid $\mathcal{G}$ is simply a small category in which all morphisms are invertible.



A topological groupoid is a groupoid $\mathcal{G}$ endowed with a topology under which the inverse map is continuous and the product map is continuous with respect to the subspace topology on $\mathcal{G}^2$. To ensure that the topology is well behaved, we will always assume that a topological groupoid $\mathcal{G}$ is locally compact, second countable and Hausdorff.

Definition 2.1.9. A locally compact, second countable and Hausdorff topological groupoid $\mathcal{G}$ is said to be *étale* if the range map $s : \mathcal{G} \to \mathcal{G}$ is a local homeomorphism.

Remark 1. It follows from the definition that the source map s is also a local homeomorphism since $s(x) = r(x^{-1})$.

We fix a locally compact, second countable and Hausdorff étale groupoid $\mathcal{G}$ and consider the set $C_c(\mathcal{G})$ of continuous compactly supported complex functions on $\mathcal{G}$. To build a C^*-algebra out of $C_c(\mathcal{G})$, we need a $*$-structure on $C_c(\mathcal{G})$. We define the product of two elements $f_1, f_2 \in C_c(\mathcal{G})$ by the convolution

$$(f_1 * f_2)(g) := \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2) = \sum_{h \in \mathcal{G}^{r(g)}} f_1(h) f_2(h^{-1}g), \quad \forall g \in \mathcal{G}, \tag{2.11}$$

and the involution of an element $f \in C_c(\mathcal{G})$ by:

$$f^*(g) = \overline{f(g^{-1})}, \quad \forall g \in \mathcal{G}. \tag{2.12}$$

Note that the sum in (2.11) is finite since f_1 and f_2 are compactly supported and the fibers $\mathcal{G}^{r(g)}$ of an étale groupoid are discreet.

Let u be a unit on an étale groupoid. For each a in $C_c(\mathcal{G})$ and $\xi \in l^2(s^{-1}\{u\}) = l^2(\mathcal{G}_u)$, the formula

$$(\pi_u(a)\xi)(g) = \sum_{h \in \mathcal{G}^{r(g)}} a(h)\xi(h^{-1}g), \quad g \in \mathcal{G}_u,$$

defines a representation $\pi_u : C_c(\mathcal{G}) \to \mathcal{B}(l^2(\mathcal{G}_u))$ of $C_c(\mathcal{G})$ and the norm of $\pi_u(a)$ is bounded by a constant that depends only on a and not u. We then have the following definition.

Definition 2.1.10. Let $\mathcal{G}$ be a locally compact, second countable and Hausdorff étale groupoid. Its reduced C^*-algebra, which we denote by $C_r^*(\mathcal{G})$, is the completion of $C_c(\mathcal{G})$ in the norm

$$\|a\| := sup_{u \in \mathcal{G}^0} \|\pi_u(a)\|.$$

2.2 Von Neumann algebras arising from C^*-dynamical systems

von Neumann algebras

In what follows we shall review some basic facts about von Neumann algebras and how they naturally arise from quantum statistical mechanical systems. A detailed exposition of this material can be found in [BR87] and [BR96].

Let $\mathcal{H}$ be a separable Hilbert space and denote by $B(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. Recall that the weak operator topology on $B(\mathcal{H})$ is the topology induced by the semi-norms $T \mapsto |y(Tx)|$ where $y \in H^*$ (continuous dual) and $x \in \mathcal{H}$. We then have the following definition.

Definition 2.2.1. A von Neumann algebra is a weakly closed $*$-subalgebra $\mathcal{M}$ of $B(\mathcal{H})$.

The von Neumann double commutant theorem provides the following algebraic characterization of a von Neumann algebra: $\mathcal{M} \subset B(\mathcal{H})$ is a von Neumann algebra iff $\mathcal{M}$ is closed under the *-operation and equal to its double commutant, i.e.,

$$\mathcal{M} = \mathcal{M}'',$$

where $\mathcal{M}' := \{a \in B(\mathcal{H}) \,|\, am = ma, \forall m \in \mathcal{M}\}$ is the commutant of $\mathcal{M}$.

The von Neumann algebra $C_{\mathcal{M}} = M \cap \mathcal{M}'$ is called the center of $\mathcal{M}$ and a von Neumann algebra with a trivial center is called a factor.

We say that $p \in \mathcal{M}$ is a projection if $p = p^* = p^2$ and we denote by $P(\mathcal{M})$ the set of projections in $\mathcal{M}$. For $p \in P(\mathcal{M})$, we say that $p\mathcal{M}p$ is a compression (or a corner) of M. It is a standard fact that $p\mathcal{M}p$ is itself a von Neumann algebra acting on $p\mathcal{H}$.

For $p, q \in \mathcal{M}$, we write $p \leq q$ if $pq = p$. The set of $P(\mathcal{M})$ can be equipped with the following equivalence relation $\sim$:

$$p_1 \sim p_2 \Leftrightarrow \exists u \in \mathcal{M} \text{ such that } p_1 = u^*u, p_2 = uu^*,$$

where u is a partial isometry (i.e., an isometry on $\ker(u))^\perp$).

A projection $p \in P(\mathcal{M})$ is said to be:

- *minimal* if $p \neq 0$ and $q \leq p$ implies $q = 0$ or $q = p$.

- *abelian* if pMp is abelian.
- *finite* if $q \leq p$ and $q \sim p$ implies $p = q$.
- *semi-finite* if there exists a family $\{p_i\}_{i \in I}$ of pairwise orthogonal, finite projections such that $p = \sum_{i \in I} p_i$.
- *purely infinite* if $p \neq 0$ and there does not exist any nonero finite projections $q \leq p$.
- *properly infinite* if $p \neq 0$ and for all nonzero central projections z (i.e., $zx = xz$ for all $x \in \mathcal{M}$) we have that zp is not finite.

The von Neumann algebra $\mathcal{M}$ is said to be finite, semi-finite, purely infinite or properly infinite if $1 \in \mathcal{M}$ has the corresponding property.

It is known that every von Neumann algebra can be decomposed as a direct integral of factors (See [KR86] for a detailed discussion of the decomposition theory of von Neumann algebras). Hence, factors form the building blocks of von Neumann algebras and a classification of factors is enough to classify von Neumann algebras.

Let $\mathcal{M}$ be a factor and consider its unique (up to a constant) tracial weight ϕ (i.e., $\phi(aa^*) = \phi(a^*a)$). Denote by D the restriction of ϕ to the projections in $\mathcal{M}$. Then the map D is an injection of equivalence classes of projections of $\mathcal{M}$ in $[0, +\infty]$ such that

$$D(p+q) = D(p) + D(q), \quad \text{whenever } \; pq = qp = 0.$$

The following result by Murray and Neumann gives a classification of factors in terms of the range of the dimension function D. We refer the reader to [Tak02] for a comprehensive and detailed treatment of Murray-von Neumann classification theory.

Theorem 2.2.1. *Let $\mathcal{M}$ be a factor, $P(\mathcal{M})$ the set if all its projections and D the dimension function introduced above. Then one of the following cases occurs:*

- *Type I_n: $D(P(\mathcal{M}))$ coincides (up to normalization) with the set $\{1, \dots, n\}$.*
- *Type I_∞: $D(P(\mathcal{M}))$ coincides (up to normalization) with the set $\{1, \dots, \infty\}$.*
- *Type II_1: $D(P(\mathcal{M}))$ coincides (up to normalization) with the set $[0, 1]$.*

- *Type II_∞: $D(P(\mathcal{M}))$ coincides with the set* $[0,\infty]$.
- *Type III: $D(P(\mathcal{M}))$ coincides with the set* $\{0,\infty\}$.

The type I_n factors are the finite dimensional von Neumann algebras, which are all isomorphic to a matrix algebra. Type I_∞ are all isomorphic to the algebra of bounded operators on some Hilbert space. Examples of type II_1 and II_∞ and III can be constructed from crossed products von Neumann algebras.

Connes Classification of Type III Factors

Let $\mathcal{M}$ be a von Neumann algebra. A state ω on $\mathcal{M}$ is called normal if for every monotone net T_α of operators bounded above by S, we have that $\sup(\omega(T_\alpha)) = \omega(\sup T_\alpha)$. Let ω be a faithful normal state on $\mathcal{M}$ and assume that $\mathcal{M}$ is already in its GNS representation on $\mathcal{H}$ and let η the associated cyclic vector. We define the following operator

$$\begin{aligned} S_0 :\ & \mathcal{M}\eta \to \mathcal{H} \\ & a\eta \mapsto a^*\eta. \end{aligned}$$

The operator S_0 is preclosed [KR86] and we denote its closure by the S. We consider the polar decomposition of S:

$$S = J\Delta_\omega^{1/2},$$

where J is an anti-unitary operator and $\Delta_\omega^{1/2} = \sqrt{S^*S}$ is a positive self-adjoint operator. We then have the following theorem by Tomita and Takesaki.

Theorem 2.2.2. *Let $\mathcal{M}$ be a von Neumann algebra together with a faithful normal state ω. Let S, J and Δ be the operators defined above. Then*

$$\begin{aligned} & J\mathcal{M}J = \mathcal{M}' \\ & \Delta_\omega^{it}\mathcal{M}\Delta_\omega^{-it} = \mathcal{M}, \quad t \in \mathbb{R}. \end{aligned}$$

We can then define a canonical automorphism group associated to $(\mathcal{M},\omega)$:

Definition 2.2.2. The one-parameter automorphism group defined by

$$\sigma_t(a) := \Delta_\omega^{it} a \Delta_\omega^{-it}, \quad t \in \mathbb{R},$$

is called the modular automorphism group associated to $(\mathcal{M},\omega)$.

In [Con73], Connes refined the classification of type III factors using the invariant given by:

$$S(\mathcal{M}) := \bigcap_{\phi}\{Spec(\Delta_\phi) : \phi \text{ is a faithful normal state on } \mathcal{M}\} \tag{2.13}$$

We summaries this classification in the following theorem.

Theorem 2.2.3. *Let $\mathcal{M}$ be a factor. Then M is of type III if and only if $0 \in S(\mathcal{M})$. If $\mathcal{M}$ is of type III, then one of the following cases occurs:*

1. *Type III_0:* $S(\mathcal{M}) = \{0, 1\}$
2. *Type III_λ:* $S(\mathcal{M}) = \{\lambda^n; n \in \mathbb{Z}\} \cup \{0\}$, $0 < \lambda < 1$.
3. *Type III_1:* $S(\mathcal{M}) = [0, \infty)$

Crossed Products

Let $\mathcal{M}$ be a von-Neumann algebra acting on the Hilbert space $\mathcal{H}$, α a continuous homomorphism of a locally compact group (resp. countable) G into $Aut(\mathcal{M})$. We will now construct a new von Neumann algebra acting on the space $\tilde{\mathcal{H}} = L^2(G, \mathcal{H})$ (resp. $\tilde{\mathcal{H}} = l^2(G, \mathcal{H})$). Consider the following two representations, π_α and λ_G of $\mathcal{M}$ and G on $\tilde{\mathcal{H}}$:

$$\begin{aligned}
(\pi_\alpha(a)\xi)(s) &:= \alpha_{s^{-1}}(a)\xi(s), \quad \xi \in \tilde{\mathcal{H}}, a \in \mathcal{M}, s \in G,\\
(\lambda_G(t)\xi)(s) &:= \xi(t^{-1}s), \quad t, s \in G.
\end{aligned}$$

Definition 2.2.3. The von Neumann algebra

$$\mathcal{M} \rtimes_\alpha G := (\pi_\alpha(\mathcal{M}) \cup \lambda_G(G))''$$

is called *the crossed product* of $\mathcal{M}$ and G (w.r.t α).

We consider the case where $\mathcal{M} = L^\infty(X, \mathcal{F}, \mu)$ acting on with X is a standard Borel space equipped with a σ-finite measure μ and G acts on the measure space (X, μ) via non-singular transformations (i.e., preserve the class of sets of μ-measure 0). We then have an action of G on $\mathcal{M}$ defined as

$$\alpha_g(f) := f \circ g^{-1}.$$

Definition 2.2.4. The action of G on the measure space $(X, \mathcal{F}, \mu)$ is *ergodic* if the following holds: If A is any G-invariant Borel subset of X, then $\mu(A) = 0$ or $\mu(A^c) = 0$. The action of G is said to be essentially free if for every $g \in G$ such that $g \neq 1$, then $\mu(\{x \in X : g \cdot x = x\}) = 0$.

We then have the following important result (cf. [Sun87]).

Theorem 2.2.4. *The von Neumann algebra $\mathcal{M} = L^\infty(X, \mathcal{F}, \mu) \rtimes_\alpha G$ is a factor if and only if the action of G is essentially free and ergodic. In this case, the type of $\mathcal{M}$ is determined as follows:*

- *$\mathcal{M}$ is of type I or II if and only if there exists a G-invariant measure ν which is mutually absolutely continuous with respect to μ.*
- *$\mathcal{M}$ is of type I_n precisely when the ν above is totally atomic and $\mathcal{F}$ is the disjoint union of n atoms for ν.*
- *$\mathcal{M}$ is of type II when the ν as above is non-atomic.*
- *$\mathcal{M}$ is type III if and only if there exists no ν as above.*

2.3 Bost–Connes–Marcolli systems

The goal of this section is to introduce the general construction given in [HP05] of the abstract Bost-Connes-Marcolli system attached to an arbitrary Shimura variety. Since this material lies at the intersection of Operator Algebras and Algebraic Number Theory, we propose to briefly review some number theoretic facts which also serves the purpose of fixing some notations. For a more detailed exposition we refer the reader to [Pla69].

Let F be a number field (a finite extension of the rational numbers $\mathbb{Q}$). Recall that an absolute value $|\ | : F \to \mathbb{R}^+$ on F is non-Archimedean if and only if the condition $|x + y| \leq \max(|x|, |y|)$ holds for all $x, y \in F$. We use M_F to denote the set of places of F (equivalence classes of absolute values on F). A place υ is called finite if the corresponding absolute value is non-Archimedean. We denote the set of finite places by $M_{F,f}$ and we write $\upsilon < \infty$ if $\upsilon \in M_{F,f}$. Otherwise we say it is infinite and we denote by $M_{F,\infty}$ the set of infinite places and we write $\upsilon|\infty$ if $\upsilon \in M_{F,\infty}$. For $\upsilon \in M_F$ we denote by F_υ the completion of F with respect to υ. For $\upsilon < \infty$ we use O_υ to denote the valuation ring of F_υ:

$$O_\upsilon = \{x \in F_\upsilon : |x|_\upsilon \leq 1\}.$$

Define $\hat{\mathcal{O}} := \prod_{v<\infty} \mathcal{O}_v$ and let $\mathcal{O}^\times = \prod_{v<\infty} \mathcal{O}_v^\times$ be its group of units.

For $F = \mathbb{Q}$, Ostrowski's theorem provides an explicit description of nontrivial absolute values in $M_\mathbb{Q}$. In this case, every infinite place is equivalent to the ordinary Archimedean absolute value $|\ |_\infty$ while any finite place is equivalent to the p-adic absolute value $|\ |_p$ associated to a prime number p. Writing any rational number $x \neq 0$ as $p^r m/n$ where $r, m, n \in \mathbb{Z}$ and m and n are not divisible by p, the p-adic absolute value is given by $|x|_p = p^{-r}$ and $|0|_p = 0$.

The set of adèles $\mathbb{A}_F$ of F is the subset of the direct product $\prod_{v \in M_F} F_v$ consisting of those elements $x = (x_v)$ such that $x_v \in \mathcal{O}_v$ for almost all v in $M_{F,f}$. $\mathbb{A}_F$ is a ring with the respect to the usual operations in the direct product. We endow this ring with the adèle topology. Namely, the base of the open sets consists of sets of the form $\prod_{v \in S} W_v \times \prod_{v \notin S} \mathcal{O}_v$ where $S \subset M_F$ is a finite subset containing $M_{F,\infty}$ and $W_v \subset F_v$ are open subsets for each $v \in S$. This topology turns $\mathbb{A}_F$ into a locally compact topological ring called the *ring of adeles*.

The canonical embedding $F \hookrightarrow F_v$ induces a diagonal embedding

$$\begin{aligned} F &\hookrightarrow \mathbb{A}_F \\ x &\mapsto (x, x, x, \dots), \end{aligned}$$

since if $x \in F$ then $x \in \mathcal{O}_v$ for almost all $v \in M_{F,f}$. The image of F in $\mathbb{A}_F$ forms a ring called the *principal adeles* which can be shown to be discrete in $\mathbb{A}_F$.

The group of invertible elements $\mathbb{I}_F$ of $\mathbb{A}_F$ is called the *group of idèles*. We note that $\mathbb{I}_F$ is not a topological group with respect to the topology induced from $\mathbb{A}_F$. To turn $\mathbb{I}_F$ into a locally compact group one has to introduce the idèle topology which is the multiplicative analogue of the adele topology (we refer the reader to [Pla69, Section 1.2] for the details).

Let $S \subset M_F$ be a finite set of places and denote by $\mathbb{A}_{F,S}$ the ring of S-adèles, that is the image of $\mathbb{A}_F$ under the projection onto the product $\prod_{v \notin S} F_v$. A topology is introduced on $\mathbb{A}_{F,S}$ in the natural way: for a base of opens sets we take the sets of the form $\prod_{v \in T} W_v \times \prod_{v \notin S \cup T} \mathcal{O}_v$ where T is a subset of $M_F \backslash S$ and W_v is an open subset of F_v for every $v \in T$. We then have $\mathbb{A}_F = \mathbb{A}_{F,S} \times \prod_{v \in S} F_v$ as topological rings, where the product $\prod_{v \in S} F_v$ is given the product topology. In particular, for

$S = M_{F,\infty}$ we can write

$$\mathbb{A}_F = A_{F,f} \times \prod_{v|\infty} F_v,$$

and the ring $\mathbb{A}_{F,f}$ is called the *ring of finite adeles*.

Note that we still have an embedding of F in $A_{F,S}$ under the diagonal map. We then have the following result (See [Pla69, Theorem 1.5])

Theorem 2.3.1 (Strong Approximation)**.** *If $S \neq \emptyset$ then the image of F under F under the diagonal embedding is dense in $\mathbb{A}_{F,S}$*

Given an algebraic group G over F, there is a natural way to define $G(\mathbb{A}_F)$ and $G(\mathbb{A}_{F,S})$. We fix a faithful representation $G \longrightarrow GL_n$ and let $G(\mathcal{O}_v)$ be the intersection of $GL_n(\mathcal{O}_v)$ with (the image of) $G(F_v)$ in $GL_n(F_v)$:

$$G(\mathcal{O}_v) = GL_n(\mathcal{O}_v) \cap G, \quad v < \infty.$$

Let S be a finite set of places and consider the ring of S-adèles of G:

$$G(\mathbb{A}_S) := \{g = (g_v) \in \prod_{v \notin S} G(F_v) \mid g_v \in G(\mathcal{O}_v) \text{ for almost all } v \notin S\}.$$

With the natural S-adelic topology, the topological ring $G(\mathbb{A}_S)$ is locally compact and contains F via the canonical embedding $G(F) \hookrightarrow G(\mathbb{A}_S)$. We then have the following definition.

Definition 2.3.1. An algebraic group G over a number field F has the strong approximation with respect to S if $G(F)$ is dense in $G(\mathbb{A}_S)$.

The strong approximation property will be crucial in the study of Bost-Connes-Marcolli systems. The following useful result provides a necessary and sufficient condition for this property to hold for certain class of algebraic groups.

Theorem 2.3.2. *(See [Kne65] and [Pla69]) Let G be an absolutely almost simple simply connected algebraic group over a field F with zero characteristic and S a finite nonempty set of places of F. Then G has the strong approximation with respect to S if and only if the group $G_S = \prod_{v \in S} G(F_v)$ is noncompact.*

We now introduce the general construction of Bost-Connes-Marcolli systems associated to a reductive group G over $\mathbb{Q}$ and a Shimura datum(G, X, h). We refer the reader to [HP05] for a more detailed discussion of Shimura varieties and in particular the definition of a Shimura datum. Here we only recall that X is a left homogeneous space under $G(\mathbb{R})$ and $h : X \to \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ a $G(\mathbb{R})$-equivariant map with additional axioms from which it follows that X has a unique structure of a complex manifold and can be expressed as a disjoint union of Hermitian symmetric domains. We also recall that an enveloping semigroup for G is a multiplicative semigroup M which is irreducible, normal and such that $M^{\times} = G$.

A BCM datum is a tuple $\mathcal{D} = (G, X, V, M)$ with (G, X) a Shimura datum, (V, ψ) a faithful representations of G and M an enveloping semigroup for G contained in $\mathrm{End}(V)$. A level structure on $\mathcal{D}$ is a triple $\mathcal{L} = (L, K, K_M)$ with $L \subseteq V$ a lattice, $K \subseteq G(\mathbb{A}_{\mathbb{Q},f})$ a compact subgroup and $K_M \subseteq M(\mathbb{A}_{\mathbb{Q},f})$ a compact open subsemigroup such that

- K_M stabilizes $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$
- $\psi(K)$ is contained in K_M.

The pair $(\mathcal{D}, \mathcal{L})$ is called a BCM pair. For the purpose of this exposition, we assume that the Shimura datum (G, X) is classical, that is, the Shimura variety associated to (G, X) is described as:

$$\mathrm{Sh}(G, X) \simeq G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathbb{Q},f}).$$

We let

$$Y_{\mathcal{D},\mathcal{L}} := K_M \times \mathrm{Sh}(G, X),$$

and denote the points of $Y_{\mathcal{D},\mathcal{L}}$ by $y = (\rho, [z, l])$. We let $Y^{\times}_{\mathcal{D},\mathcal{L}} = K^{\times}_M \times \mathrm{Sh}(G, X)$ be the invertible part of $Y_{\mathcal{D},\mathcal{L}}$. We then have a partially defined action of $G(\mathbb{A}_{\mathbb{Q},f})$ on $Y_{\mathcal{D},\mathcal{L}}$:

$$g \cdot y = (g\rho, [z, lg^{-1}]) \quad \text{for} \;\; y = (\rho, [z, l]).$$

Consider the subspace

$$\mathcal{U}_{\mathcal{D},\mathcal{L}} \subseteq G(\mathbb{A}_{\mathbb{Q},f}) \times Y_{\mathcal{D},\mathcal{L}},$$

of pairs (g, y) such that $gy \in Y_{\mathcal{D},\mathcal{L}}$ (i.e., $g\rho \in K_M$). This space is a groupoid with source and target maps $s : \mathcal{U}_{\mathcal{D},\mathcal{L}} \to Y_{\mathcal{D},\mathcal{L}}$ and $t : \mathcal{U}_{\mathcal{D},\mathcal{L}} \to Y_{\mathcal{D},\mathcal{L}}$ given by $s(g, y) = y$ and $t(g, y) = gy$. The unit space is $Y_{\mathcal{D},\mathcal{L}}$ and composition is given by

$$(g_1, y_1) \circ (g_2, y_2) = (g_1 g_2, y_2) \quad \text{if} \;\; y_1 = g_2 y_2.$$

There is an action of K^2 on the groupoid $\mathcal{U}_{\mathcal{D},\mathcal{L}}$ given by

$$(\gamma_1, \gamma_2) \cdot (g, y) := (\gamma_1 g \gamma_2^{-1}, \gamma_2 y)$$

and the quotient stack $\mathfrak{Z}_{\mathcal{D},\mathcal{L}} = [K^2 \backslash \mathcal{U}_{\mathcal{D},\mathcal{L}}]$ has the structure of a stack-groupoid (see [HP05, Appendix A]).

Next, we let $\Gamma = G(\mathbb{Q}) \cap K$ and consider

$$\mathcal{U}^{\text{princ}} := \{(g, \rho, z) \in G(\mathbb{Q}) \times K_M \times X \mid g\rho \in K_M)\}.$$

Finally, let X^+ be a connected component of X, $G^+(\mathbb{Q}) = G(\mathbb{Q}) \cap G^+(\mathbb{R})$ (where $G^+(\mathbb{R})$ is the identity component of $G(\mathbb{R})$) and $\Gamma_+ = G^+(\mathbb{Q}) \cap K$ and consider the groupoid

$$\mathcal{U}^+ = \{(g, \rho, z) \in G^+(\mathbb{Q}) \times K_M \times X^+ \mid g\rho \in K_M)\},$$

The composition in $\mathcal{U}^{\text{princ}}$ and $\mathcal{U}^+$ is given by

$$(g_1, \rho_1, z_1) \circ (g_2, \rho_2, z_2) = (g_1 g_2, \rho_2, z_2) \quad \text{if } (\rho_1, z_1) = (g_2 \rho_2, \rho_2 z_2).$$

There is a natural action of Γ^2 (resp. Γ_+^2) on $\mathcal{U}^{\text{princ}}$ (resp. $\mathcal{U}^+$) given by

$$(\gamma_1, \gamma_2) \cdot (g, \rho, z) := (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 z),$$

and the quotient $\mathfrak{Z}^{\text{princ}}_{\mathcal{D},\mathcal{L}}$ (resp. $\mathfrak{Z}^{+}_{\mathcal{D},\mathcal{L}}$) of $\mathcal{U}^{\text{princ}}$ (resp. $\mathcal{U}^+$) by Γ^2 (resp. Γ_+^2) has again the structure of a stack-groupoid.

For an arbitrary BCM pair $(\mathcal{D}, \mathcal{L})$, the relation between the three groupoids $\mathfrak{Z}_{\mathcal{D},\mathcal{L}}$ and $\mathfrak{Z}^{princ}_{\mathcal{D},\mathcal{L}}$, $\mathfrak{Z}^{+}_{\mathcal{D},\mathcal{L}}$ is given by the following important result obtained in [HPB05, Propositions 5.2, Proposition 5.3].

Proposition 2.3.1. We denote by $h(G,K)$ the cardinality of the set $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$. Assume that $h(G,K) = 1$ and the natural map $\Gamma \to G(\mathbb{Q})/G^+(\mathbb{Q})$ is surjective. Then the natural maps

$$\mathfrak{Z}^+_{\mathcal{D},\mathcal{L}} \longrightarrow \mathfrak{Z}^{princ}_{\mathcal{D},\mathcal{L}}, \quad \mathfrak{Z}^{princ}_{\mathcal{D},\mathcal{L}} \longrightarrow \mathfrak{Z}_{\mathcal{D},\mathcal{L}}$$

are isomorphisms.

The conditions of Proposition 2.3.1 are satisfied for both the original Bost-Connes system and the GL_2-case. It is then natural to consider the algebra $\mathcal{H}(\mathcal{D},\mathcal{L}) := C_c(Z_{\mathcal{D},\mathcal{L}})$ of continuous compactly supported functions on the coarse quotient $Z_{\mathcal{D},\mathcal{L}}$ of $\mathcal{U}_{\mathcal{D},\mathcal{L}}$ by the action of K^2. We view its elements as functions on $\mathcal{U}_{\mathcal{D},\mathcal{L}}$ satisfying the following properties:

$$f(\gamma g, y) = f(g,y), \quad f(g\gamma, y) = f(g, \gamma y), \quad \forall \gamma \in K, \;\; g \in G(\mathbb{A}_{\mathbb{Q},f}), \;\; y \in Y_{\mathcal{D},\mathcal{L}}.$$

The convolution product on $\mathcal{H}(\mathcal{D},\mathcal{L})$ is defined by the expression

$$(f_1 * f_2)(g,y) := \sum_{\substack{h \in K\backslash G(\mathbb{A}_f) \\ hy \in Y_{\mathcal{D},\mathcal{L}}}} f_1(gh^{-1}, hy) f_2(h,y), \tag{2.14}$$

and the involution is given by

$$f^*(g,y) := \overline{f(g^{-1}, gy)}.$$

Let $y = (\rho, [z,l]) \in Y_{\mathcal{D},\mathcal{L}}$ and we put $G_y = \{g \in G(\mathbb{A}_f) \mid g\rho \in K_M\}$. We define a *-representation $\pi_y : \mathcal{H}(\mathcal{D},\mathcal{L}) \to \mathcal{B}(l^2(K\backslash G_y))$ by

$$(\pi_y(f)\xi)(g) := \sum_{h \in K\backslash G_y} f(gh^{-1}, hy)\xi(h) \;\text{ for }\; f \in \mathcal{H}(\mathcal{D},\mathcal{L}),$$

where ξ is the standard basis of $l^2(K\backslash G_y)$. The operators $\pi_y(f)$, for $y \in Y_{\mathcal{D},\mathcal{L}}$ and $f \in \mathcal{H}(\mathcal{D},\mathcal{L})$, are uniformly bounded [HP05, Lemma 4.16]. We then obtain a C^*-algebra $\mathcal{A}$ after completing $\mathcal{H}(\mathcal{D},\mathcal{L})$ in the norm

$$\|f\| = \sup_{y \in Y_{\mathcal{D},\mathcal{L}}} \left\|\pi_y(f)\right\|.$$

Given a homomorphism

$$N : \mathrm{GL}(V) \to \mathbb{R}_+^*,$$

we define a time evolution on $\mathcal{H}(\mathcal{D}, \mathcal{L})$ by

$$\sigma_t(f)(g, y) = N(\psi(g))^{it} f(g, y),$$

so that the operator on $l^2(K\backslash G_y)$ given by

$$(H_y\zeta)(g) = \log N(\psi(g))\zeta(g)$$

is the Hamiltonian. The resulting C^*-dynamical system $(\mathcal{A}, \sigma_t)$ is the Bost-Connes-Marcolli system associated to the BCM pair $(\mathcal{D}, \mathcal{L})$. In the rest of this thesis, we shall explore the rich structure of these systems in the special case of the Siegel modular varieties which, together with the Hilbert modular varieties, constitutes an important class of Shimura varieties.

Chapter 3

PHASE TRANSITION IN THE BOST–CONNES–MARCOLLI GSp_4-SYSTEM

In this chapter we introduce and study the Connes-Marcolli system associated to the Siegel modular variety of degree two. We classify its KMS$_\beta$-states for inverse temperatures $\beta > 0$ and show that a spontaneous phase transition occurs at $\beta = 3$. More precisely, we prove that the system does not admit a KMS$_\beta$ state for $\beta < 3$ with $\beta \neq 1$, construct the explicit extremal Gibbs states for $\beta > 4$ and show that a unique KMS$_\beta$ state exists for every $\beta > 0$ with $3 < \beta \leq 4$.

To prove this result, we combine the approach of Laca, Larsen and Neshveyev in [LLN07] and generalize their ideas to the explicit case of the Shimura variety $(GSp_4^+, \mathbb{H}_2^+)$. There are several complications compared to the $GL_{\mathbb{Q},2}$-system. First, the action of the group $Sp_4(\mathbb{Z})$ on the set $\mathbb{H}_2^+ \times GSp_4(\mathbb{A}_{\mathbb{Q},f})$ is not free and one can not easily resolve this issue by excluding the subset $\mathbb{H}_2^+ \times \{0_4\}$. Our approach consists of first replacing the homogeneous space $\mathbb{H}_2^+$ by the quotient $K \backslash PGSp_4^+(\mathbb{R})$ (where K is a compact subgroup of $PGSp_4^+(\mathbb{R})$) and then prove a one-to-one correspondence of the KMS$_\beta$ states between the two systems. We first establish the correspondence between the set of KMS$_\beta$ states and Borel measures, which allows us to study the properties of those measures instead of working directly with the KMS$_\beta$ states. The second difficulty arises from the structure of the Hecke pair $(GSp_{2n}(\mathbb{Q}), Sp_{2n}(\mathbb{Z}))$. As we will show in this chapter, the case $n = 2$ is already computationally demanding and even in this case it is not always possible to directly apply some techniques used in [LLN07] (especially in the critical interval $3 < \beta \leq 4$). As a first result we show in Theorem 3.2.1 that the GSp_4-system does not admit any KMS$_\beta$ state for $0 < \beta < 3$ and $\beta \notin \{1, 2\}$. We next show that the extremal states in the region $\beta > 4$ correspond to Gibbs states and give an explicit construction of these states in Theorem 3.2.2. The final main result (Theorem 3.2.5) is a uniqueness theorem: we show that in the region $3 < \beta \leq 4$, the GSp_4-system admits a unique KMS$_\beta$ state. To show this, we split the proof into two parts. The main ingredient of the first is the convergence of Dirichlet L-functions for nontrivial characters. The second part relies on a variant of the technique used in [LLN07]. As stated above, the structure of the Hecke pair $(GSp_{2n}(\mathbb{Q}), Sp_{2n}(\mathbb{Z}))$ becomes less explicit for $n \geq 2$ and in order to compute the number of right representatives in a double coset one has to work with upper bounds instead of explicit formulas. We achieve this by using the root datum of the group GSp_{2n} and use the equidistribution of Hecke points for the group GSp_{2n} to establish the second ergodicity result.

Throughout this chapter, we fix the following notations.
We use the common notations $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ together with $\mathbb{R}_+^* = (0, +\infty)$ and $\mathbb{Z}^+ = \mathbb{R}_+^* \cap \mathbb{Z}$;
If R is a ring, we denote its group of multiplicative units by $R^\times$;
We use the notation $\mathrm{Mat}_n(R)$ for the ring of square matrices with entries in R. We denote by E_{ij} the usual elementary matrix with 1 in the (i, j) position and 0 elsewhere;
The group of units in the ring $\mathrm{Mat}_n(R)$ is denoted by $GL_n(R)$. If A is a square matrix, then A^t stands for its transpose. If $A_1, \dots, A_n$ are square matrices we denote by $\mathrm{diag}(A_1, \dots, A_n)$ the square matrix with $A_1, \dots, A_n$ as diagonal blocks and 0's otherwise;
Whenever there is an ambiguity, we shall use 1_n and 0_n to denote the $n \times n$ identity matrix and the a rectangular zero matrix;
$|F|$ denotes the cardinality of a finite set F;
the set of prime numbers is denoted by $\mathcal{P}$;
given a nonempty finite set of prime numbers $F \subset \mathcal{P}$, we denote by $\mathbb{N}(F)$ the unital multiplicative subsemigroup of $\mathbb{N}$ generated by $p \in F$;
for two sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n \sim b_n$ if $\lim_n (a_n/b_n) = 1$ and

$$\sum_n a_n \sim \sum_n b_n$$

if the two series are simultaneously divergent or convergent;
if Y is subset of X, we denote $Y^c = X \backslash Y = \{a \in X : a \notin Y\}$;
for a number field K, we denote by $\mathbb{A}_K = \mathbb{A}_{K,f} \times \mathbb{A}_{K,\infty}$ the adèle ring of K, where $\mathbb{A}_{K,f}$ is the ring of finite adèles and $\mathbb{A}_{K,\infty}$ the infinite adèles of K. The ring of integers of K is denoted by $\mathcal{O}_K$.

Recall from Chapter 1 that given a countable group G acting on a locally compact second countable topological space X, one can construct the transformation groupoid as the space $G \times X$ with unit space X with the source and target maps given by $s(g, x) = x$ and $t(g, x) = gx$ and the composition law is defined by

$$(g, x)(h, y) = (gh, y) \quad \text{if } x = hy.$$

If Γ is a subgroup of G and the action of Γ is free and proper, we introduce a new groupoid $\Gamma\backslash G \times_\Gamma X$ by taking the quotient of $G \times X$ by the following action of $\Gamma \times \Gamma$:

$$(\gamma_1, \gamma_2)(g, x) := (\gamma_1 g \gamma_2^{-1}, \gamma_2 x). \tag{3.1}$$

In all our settings, the main motivation for taking the quotient by this action is physical. In fact, to obtain a well behaved partition function of the C^*-dynamical system we will introduce shortly, one should necessarily take the quotient by the group $\Gamma \times \Gamma$ (See also [CM04] for another motivation based on the $\mathbb{Q}$-lattice picture). With these assumptions, the groupoid $\mathcal{G} = \Gamma\backslash G \times_\Gamma X$ is étale. We then introduce the algebra $C_c(\mathcal{G})$ of continuous compactly supported functions on the quotient space $\Gamma\backslash G \times_\Gamma X$ with the convolution and involution as in (2.11) and (2.12).

If the action of Γ is proper but not free, the quotient space $\Gamma\backslash G \times_\Gamma X$ is no longer a groupoid (cf. Proposition 3.1.5). Under the assumption that X is a homogeneous space of the form $\tilde{X}/H$, where now the action of G on $\tilde{X}$ is free and proper, we can still define a natural convolution algebra from the groupoid algebra $C_c(\Gamma\backslash G \times_\Gamma \tilde{X})$ as was done in the GL_2-case in [CM04]. More specifically, viewing the elements of $C_c(\Gamma\backslash G \times_\Gamma \tilde{X})$ as $\Gamma \times \Gamma$-invariant functions, we rewrite the convolution product as

$$(f_1 * f_2)(g, x) = \sum_{s \in \Gamma\backslash G} f_1(gs^{-1}, sx) f_2(s, x). \tag{3.2}$$

We then define a convolution algebra on the quotient $\Gamma\backslash G \times_\Gamma X$ by restricting the convolution product (3.2) to weight zero functions on $C_c(\Gamma\backslash G \times_\Gamma \tilde{X})$, namely functions satisfying

$$f(g, x\alpha) = f(g, x), \quad \forall \alpha \in H.$$

For each $x \in X$, we have a $*$-representation π_x of $C_c(\Gamma\backslash G \times X)$ on the Hilbert space $l^2(\Gamma\backslash G)$ defined by

$$(\pi_x)(f)\delta_{\Gamma h} = \sum_{g \in \Gamma\backslash G} f(gh^{-1}, hx)\delta_{\Gamma g}, \quad f \in C_c(\Gamma\backslash G \times_\Gamma X).$$

As in the general groupoid construction introduced in Chapter 1, one can explicitly show that the operators $\pi_x(f)$ are uniformly bounded [HP05; LLN07] and we denote by $\mathcal{B} = C_r^*(\Gamma\backslash G \times X)$ the completion of $C_c(\Gamma\backslash G \times X)$ in the reduced norm

$$\|f\| = \sup_{x\in X} \|\pi_x(f)\|. \tag{3.3}$$

Let Y be any clopen Γ-invariant subset of X and denote by $\Gamma\backslash G \boxtimes_\Gamma Y$ the quotient of the space

$$\{(g, y) \mid g \in G, \quad y \in Y, \quad gy \in Y\},$$

by the action of $\Gamma \times \Gamma$ defined in (3.1). We denote by $C_c(\Gamma\backslash G \boxtimes_\Gamma Y)$ the algebra of compactly supported functions on $\Gamma\backslash G \boxtimes_\Gamma Y$ with the convolution product given by

$$(f_1 * f_2)(g, y) = \sum_{\substack{s\in\Gamma\backslash G \\ sy\in Y}} f_1(gs^{-1}, sy) f_2(s, y),$$

and involution

$$f^*(g, y) = \overline{f(g^{-1}, gy)}.$$

We let $\mathcal{A}$ be the corner algebra $e\mathcal{B}e$, where e is the $\Gamma \times \Gamma$-invariant function on $G \times X$ defined by

$$e(g, x) = \begin{cases} 1 & \text{if } (g, x) \in \Gamma \times Y \\ 0 & \text{otherwise.} \end{cases}$$

Given $x \in X$, we put

$$G_x = \{g \in G \mid gx \in Y\}. \tag{3.4}$$

Then we have a representation of $C_c(\Gamma\backslash G \boxtimes_\Gamma Y)$ on the Hilbert space $\mathcal{H}_x = l^2(\Gamma\backslash G_x)$ given by

$$\pi_x(f)\delta_{\Gamma h} = \sum_{g\in\Gamma\backslash G_x} f(gh^{-1}, hx)\delta_{\Gamma g}, \quad f \in C_c(\Gamma\backslash G \boxtimes_\Gamma Y),$$

and the algebra $\mathcal{A}$ coincides ([LLN07]) with the completion of $C_c(\Gamma\backslash G \boxtimes_\Gamma Y)$ in the norm defined by

$$\|f\| = \sup_{y\in Y} \left\|\pi_y(f)\right\|.$$

We denote this completion by $C_r^*(\Gamma\backslash G \boxtimes_\Gamma Y)$. Next, assume that we are given a homomorphism

$$N : G \longrightarrow \mathbb{R}_+^*,$$

such that $\Gamma \subseteq \ker(N)$. We then define a one-parameter group of automorphisms of $\mathcal{B}$ by

$$\sigma_t(f)(g,x) = N(g)^{it} f(g,x), \quad \text{for } f \in C_c(\Gamma\backslash G \times X).$$

The operator on $l^2(\Gamma\backslash G_x)$ given by

$$H_x \delta_{\Gamma g} = \log N(g) \cdot \delta_{\Gamma g}$$

is the Hamiltonian and the dynamics σ_t is then spatially implemented as

$$\pi_x(\sigma_t(a)) = e^{itH_x} \pi_x(a) e^{-itH_x}, \quad \forall x \in X, \forall a \in \mathcal{B}.$$

The following result will be the starting point of our KMS$_\beta$-analysis of the dynamical system $(\mathcal{A}, \sigma_t)$.

Proposition 3.0.1. Let G, X and Y as described earlier and suppose Γ acts freely on X. Then for $\beta > 0$ there exists a one-to-one correspondence between KMS$_\beta$ weights ϕ on $\mathcal{A}$ with domain of definition containing $C_c(\Gamma\backslash Y)$ and Radon measures μ on Y such that

$$\mu(gB) = N(g)^{-\beta} \mu(B)$$

for every $g \in G$ and every Borel compact subset $B \subseteq Y$ such that $gB \subseteq Y$. If ν denotes the induced measure on $\Gamma\backslash Y$, then this correspondence is given by

$$\int_Y f(y) d\mu(y) = \int_{\Gamma\backslash Y} \Big(\sum_{y \in p^{-1}(t)} f(y) \Big) d\nu(t) \ \text{ for } \ f \in C_c(Y),$$

where $p : Y \to \Gamma\backslash Y$ is the quotient map. The induced weight ϕ is given by

$$\phi(f) = \int_{\Gamma\backslash Y} f(e,y) d\nu(y) \ \text{ for } \ f \in C_c(\Gamma\backslash G \boxtimes_\Gamma Y). \tag{3.5}$$

Proof. See [LLN07, Proposition 2.1] □

Recall that if G be a group and Γ a subgroup, the pair (G, Γ) is called is called a *Hecke pair* if for any $g \in G$ we have

$$[\Gamma : \Gamma \cap g^{-1}\Gamma g] < \infty.$$

If (G, Γ) is a Hecke pair then every double coset of Γ contains finitely many right and left cosets of Γ:

$$\Gamma g \Gamma = \bigsqcup_{\gamma \in \Gamma \backslash (\Gamma \cap g\Gamma g^{-1})} \gamma g \Gamma = \bigsqcup_{\gamma \in (\Gamma \cap g^{-1}\Gamma g)\backslash \Gamma} \Gamma g \gamma,$$

so that $|\Gamma \backslash \Gamma g \Gamma| = [\Gamma : \Gamma \cap g^{-1}\Gamma g]$. We denote the cardinality of this set by $\deg_\Gamma(a)$. Let $\beta \in \mathbb{R}$ and S is a semisubgroup of G containing Γ. Then we define

$$\zeta_{S,\Gamma}(\beta) := \sum_{s \in \Gamma \backslash S} N(s)^{-\beta} = \sum_{s \in \Gamma \backslash S / \Gamma} N(s)^{-\beta} \deg_\Gamma(s). \tag{3.6}$$

If G is a group acting on a set X and (G, Γ) is a Hecke pair, the *Hecke operator* associated to $g \in G$ is the operator T_g acting on Γ-invariant functions on X defined by

$$(T_g f)(x) = \frac{1}{\deg_\Gamma(g)} \sum_{h \in \Gamma \backslash \Gamma g \Gamma} f(hx). \tag{3.7}$$

3.1 Bost–Connes–Marcolli system for the Siegel Modular Variety

The Symplectic Group

We recall that a linear connected algebraic group G over $\mathbb{Q}$ is called reductive if any faithful semisimple representation remains semisimple over its the algebraic closure of $\mathbb{Q}$. Let R be a commutative $\mathbb{Q}$-algebra and $n \in \mathbb{N}$. The *symplectic group of similitude of degree n* is the reductive group defined by

$$GSp_{2n}(R) = \{g \in GL_{2n}(R) : \exists\, \lambda(g) \in R^\times \mid g^t\, \Omega\, g \;=\; \lambda(g)\, \Omega\},$$

where

$$\Omega = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \qquad 1_n \text{ is the } n \times n \text{ identity matrix.}$$

The function $\lambda : GSp_{2n}(R) \to R^\times$ is called the multiplier homomorphism. Its kernel is the symplectic group $Sp_{2n}(R)$ and there is an exact sequence

$$1 \longrightarrow Sp_{2n}(R) \longrightarrow GSp_{2n}(R) \longrightarrow R^\times \longrightarrow 1. \tag{3.8}$$

If $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_{2n}(R)$, the following assertions are equivalent:

(i) $\lambda(g) = \lambda(g^t)$

(ii) The inverse of the matrix g is given by:

$$g^{-1} = \lambda(g)^{-1} \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}. \tag{3.9}$$

(iii) The blocks A, B, C, D satisfy the conditions

$$A^t C = C^t A, \quad B^t D = D^t B, \quad A^t D - C^t B = \lambda(g) 1_n. \tag{3.10}$$

(iv) The blocks A, B, C, D satisfy the conditions

$$AB^t = BA^t, \quad CD^t = DC^t, \quad AD^t - BC^t = \lambda(g) 1_n. \tag{3.11}$$

For $r \in R^\times$, we put

$$S_n(r) := \{g \in GSp_{2n}(R) \mid \lambda(g) = r\}.$$

We then obtain an embedding of symplectic groups of different degrees as follows. Given $r \in \mathbb{N}$ and $0 < j < n$, define the map

$$\begin{aligned} S_j(q) \times S_{n-j}(q) &\to S_n(q) \\ (g_1, g_2) &\mapsto g_1 \odot g_2, \end{aligned}$$

where

$$g_1 \odot g_2 := \begin{pmatrix} A_1 & 0_{j\times(n-j)} & B_1 & 0_{j\times(n-j)} \\ 0_{(n-j)\times j} & A_2 & 0_{(n-j)\times j} & B_2 \\ C_1 & 0_{j\times(n-j)} & D_1 & 0_{j\times(n-j)} \\ 0_{(n-j)\times j} & C_2 & 0_{(n-j)\times j} & D_2 \end{pmatrix},$$

and

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$

Under the appropriate conditions on the size, we note that

$$(M_1 \odot M_2) \cdot (N_1 \odot N_2) = (M_1 N_1) \odot (M_2 N_2). \tag{3.12}$$

Consider the following elements of $Sp_{2n}(R)$:

$$\begin{pmatrix} 1_n & \alpha_1 E_{ii} \\ 0_n & 1_n \end{pmatrix}, \quad \begin{pmatrix} 1_n & 0_n \\ \alpha_2 E_{ii} & 1_n \end{pmatrix}, \quad \begin{pmatrix} 1_n & \alpha_3(E_{ij} + E_{ji}) \\ 0_n & 1_n \end{pmatrix}, \tag{3.13}$$
$$\begin{pmatrix} 1_n & 0_n \\ \alpha_4(E_{ij} + E_{ji}) & 1_n \end{pmatrix}, \quad \begin{pmatrix} 1_n + \alpha_5 E_{ij} & 0_n \\ 0_n & 1_n - \alpha_5 E_{ji} \end{pmatrix},$$

where $\alpha_1, \ldots, \alpha_5 \in R$. If F is a field, then the group $Sp_{2n}(F)$ is generated [OMe78] by the matrices given in (3.13) with $\alpha_1, \ldots, \alpha_5 \in F$.

An algebraic group is called a torus if it is isomorphic to a product of copies of $\mathbb{G}_m$ over a finite separable extension of $\mathbb{Q}$. It is called split if it is isomorphic to a product of copies of $\mathbb{G}_m$ over $\mathbb{Q}$. For the group GSp_{2n}, the center Z consists of scalar matrices and the standard maximal torus is

$$T = \{\mathrm{diag}(u_1, \ldots, u_n, v_1, \ldots, v_n) \, : \, u_1 v_1 = \cdots = u_n v_n \neq 0\}.$$

If $t \in T$, we often write

$$t = \mathrm{diag}(u_1, \ldots, u_n, u_1^{-1}\lambda(t), \ldots, u_n^{-1}\lambda(t)), \tag{3.14}$$

We also recall that a root datum consists of a quadruple $(X, \Phi, X^\vee, \Phi^\vee)$ consisting of the lattice X of characters of the maximal torus, the dual lattice $X^\vee$, a set of roots Φ and coroots $\Phi^\vee$. We refer the unfamiliar reader to [Spr98] for a thorough exposition of this material. We shall now describe the root datum of the group GSp_{2n}.

We fix the following characters $e_i \in \mathrm{Hom}(T, G_m)$:

$$e_i(t) = u_i, \quad i = 0, 1, \ldots, n \ \text{ where } \quad u_0 := \lambda(t).$$

and cocharacters $f_i \in \mathrm{Hom}(G_m, T)$:

$$
\begin{aligned}
f_0(u) &= \mathrm{diag}(\underbrace{1,\dots,1}_{n}, \underbrace{u,\dots,u}_{n}),\\
f_1(u) &= (\underbrace{u,1,\dots,1}_{n}, \underbrace{u^{-1},1,\dots,1}_{n}),\\
&\vdots\\
f_n(u) &= (\underbrace{1,\dots,1,u}_{n}, \underbrace{1,\dots,1,u^{-1}}_{n}).
\end{aligned}
$$

Proposition 3.1.1. The root datum of GSp_{2n} is described as follows. We set

$$
\begin{aligned}
X &= \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \dots \mathbb{Z}e_n,\\
X^\vee &= \mathbb{Z}f_0 \oplus \mathbb{Z}f_1 \oplus \dots \mathbb{Z}f_n,
\end{aligned}
$$

and let $\langle \cdot, \cdot \rangle$ the natural pairing on $X \times X^\vee$:

$$\langle e_i, f_j \rangle = \delta_{ij}.$$

Then we have the following set of simple roots:

$$\alpha_1(t) = u_{n-1}^{-1} u_n, \quad \dots \quad \alpha_{n-1}(t) = u_1^{-1} u_2, \quad \alpha_n(t) = u_1^2 u_0^{-1},$$

where t has the form in (3.14). In terms of the basis e_i, $i = 0, 1, \dots, n$, we have

$$\alpha_1 = e_n - e_{n-1}, \quad \dots \quad \alpha_{n-1} = e_2 - e_1, \quad \alpha_n = 2e_1 - e_0.$$

The corresponding coroots are

$$\alpha_1^\vee = f_n - f_{n-1}, \quad \dots \quad \alpha_{n-1}^\vee = f_2 - f_1, \quad \alpha_n^\vee = f_1.$$

Let $R = \{\alpha_1, \dots, \alpha_n\}$ and $R^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$. Then

$$(X, R, X^\vee, R^\vee).$$

is the root datum of GSp_{2n}. The Cartan matrix is given by

$$\langle \alpha_i, \alpha_j^\vee \rangle = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -2 & 2 \end{pmatrix}.$$

Proof. See [Tad94, page 134-136] □

As noted in section 2.3, the definition of the Bost-Connes-Marcolli system associated to a general Shimura datum (G, X) requires the notion of an enveloping semigroup which plays the role of $\mathrm{Mat}_{2,\mathbb{Q}}$ in the case of the $GL_{2,\mathbb{Q}}$-system. For a given a reductive group G, it is always possible to construct an enveloping semigroup M (see [HP05, Appendix B.2]). For the case $G = GSp_{2n}$ we are considering in this work, we have the following explicit description of M. Given a commutative $\mathbb{Q}$-algebra R we have

$$M(R) := MSp_{2n}(R) = \{m \in \mathrm{Mat}_{2n}(R) \mid \exists\, \lambda(m) \in R, m^t\Omega m = \lambda(m)\Omega\}, \quad (3.15)$$

since $m \in MSp_{2n}(R)^\times$ if and only if $\lambda(m) \in R^\times$.

The group $\Gamma_n = Sp_{2n}(\mathbb{Z})$ is called *the Siegel modular group of degree n*. For $m \in \mathbb{N}$, we denote by $U_m = GL_m(\mathbb{Z})$ the unimodular group of degree m and note that $\Gamma_n \subseteq U_{2n}$ with equality if $n = 1$. We let $MSp^+_{2n}(\mathbb{Z}) = \{M \in MSp_{2n}(\mathbb{Z}) \mid \lambda(M) > 0\}$. Then given any $M \in MSp^+_{2n}(\mathbb{Z})$, we denote by $\Gamma_n M \Gamma_n$ the double coset generated by M and put

$$\mathcal{D}(\Gamma_n M \Gamma_n) := \{D \in \mathrm{Mat}_n(\mathbb{Z}) \text{ such that there exists } \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_n M \Gamma_n\}.$$

For each $D \in \mathcal{D}(\Gamma M \Gamma)$, we set

$$\mathcal{B}(D, \Gamma_n M \Gamma_n) := \{B \in \mathrm{Mat}_n(\mathbb{Z}) \text{ such that there exists } \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_n M \Gamma_n\}.$$

We define the following equivalence relation on $\mathcal{B}(D)$:

$$B \sim B' \Leftrightarrow (B - B')D^{-1} \in \mathrm{Sym}_n(\mathbb{Z}), \quad (3.16)$$

and we write $B \equiv B' \mod D$ if $B \sim B'$. For $r \in \mathbb{N}$, we put

$$S_n(r) = \{g \in GSp_{2n}(\mathbb{Z}) \mid \lambda(g) = r\}.$$

Let $M \in \mathrm{Mat}_m(\mathbb{Z})$ with $\det(M) > 0$ and $N \in S_n(r)$. Then by the Elementary Divisor Theorem (See [Kri90, Theorem 2.2, Chapter V]) the double cosets $U_m M U_m$ and $\Gamma_n N \Gamma_n$ contain unique representatives of the form

$$\mathrm{Elm}(M) = \mathrm{diag}(a_1, a_2, \dots, a_m), \quad a_1, a_2, \dots, a_n \in \mathbb{N}, \tag{3.17}$$

with $a_1 \mid a_2 \mid \cdots \mid a_m$ and

$$\mathrm{Elm}(N) = \mathrm{diag}(a_1, \dots, a_n, d_1, \dots, d_n), \quad a_1, \dots, a_n, d_1, \dots, d_n \in \mathbb{N}, \tag{3.18}$$

such that $a_i d_i = r$, $i = 1, \dots, n$ and $a_1 \mid a_2 \mid \dots a_n \mid d_n \mid \cdots \mid d_{n-1} \mid d_1$.

Theorem 3.1.1. *Let $M \in MSp^+_{2n}(\mathbb{Z})$. Then a set of representatives of the right cosets relative to Γ_n in $\Gamma_n M \Gamma_n$ is given by the matrices*

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad A = \lambda(M)(D^t)^{-1},$$

where

1. *D runs through a set of representatives of $GL_n(\mathbb{Z}) \backslash \mathcal{D}(\Gamma_n M \Gamma_n)$;*
2. *B runs through a set of representatives of $\mod D$ incongruent matrices in $\mathcal{B}(D, \Gamma_n M \Gamma_n)$.*

Proof. See [Kri90, Theorem 3.4, Chapter VI] □

Proposition 3.1.2. Let p be a prime number and $l \in \mathbb{N}$. Then the set $S_n(p^l)$ decomposes into finitely many right cosets relative to Γ_n. A set of representatives is given by

$$\begin{pmatrix} p^l (D^t)^{-1} & B \\ 0 & D \end{pmatrix}, \tag{3.19}$$

where D runs through a set of representatives of

$$GL_n(\mathbb{Z}) \backslash \left\{ D \in \mathrm{Mat}_n(\mathbb{Z}) \mid \mathrm{Elm}(D) = \mathrm{diag}(d_1, d_2, \dots, d_n), \ \ d_i \mid p^l, \ \ i = 1, 2, \dots, n \right\},$$

and B runs through a set of representatives of $\mod D$ incongruent matrices in

$$B(D) := \{B \in \mathrm{Mat}_n(\mathbb{Z}) \mid B^t D = D^t B\}. \tag{3.20}$$

Proof. Every right coset $\Gamma_n M$ contains a representatives of the form in (3.19). Suppose we have two representatives N and M of this form:

$$N = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad M = \begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix},$$

with $\Gamma_n N = \Gamma_n N$. Then since

$$\left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_n \right\} = \left\{ \begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix} \begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix} \mid U \in GL_n(\mathbb{Z}) \, , S \in \mathrm{Sym}_n(\mathbb{Z}) \right\}, \tag{3.21}$$

we obtain from the conditions in (3.10)-(3.11) that there exists $U \in GL_n(\mathbb{Z})$ such that $D(A')^t = \lambda(M)U^{-1}$ and hence

$$D(D')^{-1} = U^{-1}.$$

This shows that $D = D'$ and consequently $A = A'$. Moreover from the equality (3.21) we know that there exists $S \in \mathrm{Sym}_n(\mathbb{Z})$ such that

$$\begin{aligned} -A(B')^t + B(A)^t &= -A(B')^t + AB^t, \\ &= \lambda(M)S. \end{aligned}$$

Since $A^t D = \lambda(M) = \lambda(N)$ we obtain that $B - B' = SD$, i.e., $B \equiv B' \mod D$. □

Lemma 3.1.1. Let p be a prime number, $l \in \mathbb{N}$, $D \in \mathrm{Mat}_n(\mathbb{Z})$ such that $\mathrm{Elm}(D) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with $d_i | p^l$ for $i = 1, 2, \ldots, n$ and $B(D)$ is as in equation (3.20). Given $U, V \in GL_n(\mathbb{Z})$, then

1. $|B(D) \mod D| = |B(UDV) \mod UDV|$,
2. $|B(D) \mod D| = d_1^n d_2^{n-1} \ldots d_n$.

Proof. Since $U, V \in \mathcal{U}_n(\mathbb{Z})$, we have the following bijection

$$\begin{aligned} B(D) &\to B(UDV) \\ M &\mapsto (U^t)^{-1} MV. \end{aligned}$$

This proves the first claim. Hence we can suppose that $D = \mathrm{Elm}(D)$ to prove the second assertion. Since $d_1 | \ldots | d_n$, we can write

$$B(D) = \{M = (b_{jk}) \mid b_{jk} \in \mathbb{Z}, b_{jk} = b_{kj} \frac{d_k}{d_l} \text{ for } j \leq k\}.$$

By definition of the relation in (3.16) the entries b_{jk} may be reduced $\mod d_k$. This shows that $B(D)$ consists exactly of $d_1^n d_2^{n-1} \ldots d_n$ equivalence classes $\mod D$. □

Let $r \in \mathbb{N}$ and define

$$R_{\Gamma_n}(r) := \sum_{g \in \Gamma_n \backslash S_n(r)/\Gamma_n} \deg_{\Gamma_n}(g).$$

Proposition 3.1.3. The function $R_{\Gamma_n} : \mathbb{N} \to \mathbb{N}$ is a multiplicative function, i.e., given relatively prime numbers $q, r \in \mathbb{N}$, we have

$$R_{\Gamma_n}(qr) = R_{\Gamma_n}(q) R_{\Gamma_n}(r).$$

Proof. Observe first that if $g \in S_n(qr)$, then $\mathrm{Elm}(g) = \mathrm{Elm}(g_1)\mathrm{Elm}(g_2)$ for some $g_1 \in S_n(q)$ and $g_2 \in S_n(r)$. Hence

$$R_{\Gamma_n}(qr) = \sum_{g \in \Gamma_n \backslash S_n(qr)/\Gamma_n} \deg_{\Gamma_n}(g) = \sum_{g_2 \in \Gamma_n \backslash S_n(q)/\Gamma_n} \sum_{g_1 \in \Gamma_n \backslash S_n(r)/\Gamma_n} \deg_{\Gamma_n}(g_1 g_2),$$

hence it is enough to show that $\deg(g_1 g_2) = \deg(g_1)\deg(g_2)$. We decompose the double cosets $\Gamma_n \backslash g_1 / \Gamma_n$ and $\Gamma_n \backslash g_2 / \Gamma_n$ into finitely many right cosets: $\Gamma_n Q_i, i = 1, \ldots, \deg(g_1)$ and $\Gamma_n R_i, i = 1, \ldots, \deg(g_2)$. Then consider the right cosets given by $\Gamma Q_i R_j$. Suppose hat $\Gamma Q_i R_j = \Gamma Q_k R_l$. Then there exists some $\gamma \in \Gamma_n$ and a matrix $M \in GSp_{2n}(\mathbb{Q})$ such that

$$M = Q_k^{-1} \gamma Q_i = R_l R_j^{-1}, \quad \lambda(Q_k) = \lambda(Q_i) = q, \quad \lambda(R_k) = \lambda(R_i) = r.$$

Recall from (3.9) that $Q_k = \lambda(Q_k)^{-1} \Omega^{-1} Q_k^t \Omega$ and so after writing $M = \{\frac{n_{ij}}{m_{ij}}\}_{ij}$ where $(n_{ij}, m_{ij}) = 1$, we see that the integers m_{ij} divide $\lambda(Q_k) = q$ and similarly m_{ij} divide $\lambda(R_j) = r$. By assumption $(q, r) = 1$ so $M \in \Gamma_n$ since $\lambda(A) = 1$. This shows that $\Gamma_n Q_k = \Gamma_n Q_i$ and $\Gamma_n R_l = \Gamma_n R_k$, in other words $i = k$ and $l = j$. To conclude we simply observe that the cosets $\Gamma Q_i R_j$ form a partition of $\Gamma_n \backslash g_1 g_2 / \Gamma_n$. ◻

Let p be a prime and $l \in \mathbb{N}$. For arbitrary n, it is in general not possible to obtain a closed formula for $\deg(a)$ if $a \in S_{p^l}$ is given in its elementary form. On the other hand, an upper bound of $\deg(a)$ will be enough in most of our calculations. We first suppose that a is given by

$$a = \mathrm{diag}(p^{k_1}, p^{k_2}, \ldots, p^{k_n}, p^{l-k_1}, \ldots, p^{l-k_n}), \quad [l/2] \le k_1 \le k_2 \le \cdots \le k_n$$

Note that this is *not* the elementary symplectic form of a since $k_1 \geq [l/2]$. We shall use the root datum of GSp_{2n} given in Proposition 3.1.1. The set Φ^+ of positive roots is given by (see [Tad94, page 167])

$$\begin{aligned} e_j - e_i, &\quad 1 \leq i < j \leq n \\ e_j + e_i - e_0, &\quad 1 \leq i < j \leq n \\ 2e_i - e_0, &\quad 1 \leq i \leq n. \end{aligned}$$

where we have used our choice of the basis $e_i, i = 1, \ldots, n$ (the choice of the basis used in [Tad94] is different but the computations are essentially the same). Hence

$$2\rho = \sum_{\alpha \in \Phi^+} \alpha = 2 \sum_{i=0}^{i=n-1} (n-i) e_{n-i} - \frac{1}{2} n(n+1) e_0.$$

We set

$$\lambda = \sum_{i=1}^{i=n} k_i f_i + l f_0.$$

Observe that $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+$. Then using the degree formula in Proposition 7.4 in [Gro98] (see also the proof of Corollary 1.9 in [COU01] for a similar result in the case of GL_n) we obtain

$$\deg(a) = p^{(\sum_{i=0}^{i=n-1} 2(n-i) k_{n-i}) - \frac{1}{2} n(n+1) l} (1 + O(p^{-1})). \tag{3.22}$$

where the big O depends only on n. If $a \in S_{p^l}$ is given in its elementary symplectic form (3.17) , we apply left and right (symplectic) permutations matrices to a (which leaves invariant the degree) and use formula (3.22).

Structure theorems of the symplectic group

For a prime p, we denote by $\mathbb{Q}_p$ the field of p-adic numbers and $\mathbb{Z}_p$ its compact subring of p-adic integers. We consider $\mathbb{A}_{\mathbb{Q},f}$ the ring of finite adèles of $\mathbb{Q}$ and we denote by $I_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q},f}^{\times}$ the idèle group. We denote by $\hat{\mathbb{Z}} = \varprojlim_{N>1} \mathbb{Z}/N\mathbb{Z}$ the ring of profinite integers and we write $M_{2n}(\hat{\mathbb{Z}})$ for the ring of $2n \times 2n$-matrices with coefficients in $\hat{\mathbb{Z}}$. Note that by the Chinese remainder theorem we have that $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, which is a maximal compact subring of $\mathbb{A}_{\mathbb{Q},f}$. The profinite compact group $GL_{2n}(\hat{\mathbb{Z}}) = \varprojlim_{N>1} GL_{2n}(\mathbb{Z}/N\mathbb{Z})$ is a subset of $M_{2n}(\hat{\mathbb{Z}})$ and consists of invertible matrices. The subgroup $Sp_{2n}(\hat{\mathbb{Z}}) \triangleleft GL_{2n}(\hat{\mathbb{Z}})$ is defined by exactness of the sequence (3.8).

As seen in the previous section, the class number $h(G, K)$ plays an important role in the definition of the abstract BCM system. The aim of this section is to show that for $G = GSp_{2n}$, we have that $h(G, K) = 1$ where K is any open compact subgroup of $GSp_4(\mathbb{A}_{\mathbb{Q},f})$. The proof relies on the notion of strong approximation in algebraic groups (cf. Section 2.3).

We have from Theorem 2.3.2 that the symplectic group $G = Sp_{2n}$ (over $\mathbb{Q}$) has the strong approximation with respect to $S = \{\infty\}$. We provide an elementary proof of this result using matrix factorization. For this, we first put

$$\Gamma_n(N) = \{\gamma \in \Gamma_n \mid \gamma \equiv 1_{2n} \mod N\}, \tag{3.23}$$

for every positive integer N.

Lemma 3.1.2. Let π_N be the projection $\pi_N : Sp_{2n}(\mathbb{Z}) \to Sp_{2n}(\mathbb{Z}/N\mathbb{Z})$ defined by $\pi_N(\gamma) = \gamma \mod N$. Then the sequence

$$1_{2n} \longrightarrow \Gamma_n(N) \longrightarrow \Gamma_n \xrightarrow{\pi_N} Sp_{2n}(\mathbb{Z}/N\mathbb{Z}) \longrightarrow 1_{2n}$$

is exact.

Proof. The only nontrivial part is the surjectivity of the map π_N. This follows directly from [NS64, Theorem 1] □

Proposition 3.1.4. $Sp_{2n}(\mathbb{Z})$ is dense in $Sp_{2n}(\hat{\mathbb{Z}})$.

Proof. Since $M_{2n}(\hat{\mathbb{Z}}) = \varprojlim_{N>1} M_{2n}(\mathbb{Z}/N\mathbb{Z})$ is a profinite ring, a system of neighborhoods of the zero matrix is given by $\{NM_{2n}(\hat{\mathbb{Z}}) \mid N \in \mathbb{N}\}$ and thus a system of neighborhood of $\mathbb{1}_{2n}$ in $Sp_{2n}(\hat{\mathbb{Z}}))$ is given by $\{U_N \mid N \in \mathbb{N}\}$ where $U_N = \{1 + NM_{2n}(\hat{\mathbb{Z}})\} \cap Sp_{2n}(\hat{\mathbb{Z}})\}$. Given $N \in \mathbb{N}$, consider the projection map $\pi_N : Sp_{2n}(\hat{\mathbb{Z}}) \to Sp_{2n}(\mathbb{Z}/N\mathbb{Z})$ and note that $\ker \pi_N = U_N$. By Lemma 3.1.2 the projection π_N restricted to $Sp_{2n}(\mathbb{Z})$ is surjective. Hence for any given $x \in Sp_{2n}(\hat{\mathbb{Z}})$ and $N \in \mathbb{N}$, we choose $\gamma_N \in Sp_{2n}(\mathbb{Z})$ such that $\pi_N(\gamma_N) = \pi_N(x)$.Then $x^{-1}\gamma_N \in U_N$, that is, $\gamma_N \in xU_N$. Taking N large enough shows that $Sp_{2n}(\mathbb{Z})$ is dense in $Sp_{2n}(\hat{\mathbb{Z}})$. □

Theorem 3.1.2. *The algebraic group $G = Sp_{2n}$ (over $\mathbb{Q}$) has the strong approximation with respect to the infinite place $S = \{\infty\}$.*

Proof. We denote by H the closure of $Sp_{2n}(\mathbb{Q})$ in $Sp_{2n}(\mathbb{A}_{\mathbb{Q},f})$. We have a dense diagonal embedding of $\mathbb{Q}$ inside $\mathbb{A}_{\mathbb{Q},f}$, whence the subgroup H contains the group generated by the matrices of the form (3.13) with $\alpha_1, \ldots, \alpha_5 \in \mathbb{A}_{\mathbb{Q},f}$. In particular, given any prime number p, the subgroup H contains the set of matrices of the form (3.13) with $(\alpha_i)_q = 1$ for $q \neq p$ and $i = 1, \ldots, 5$. Since $Sp_4(\mathbb{Q}_p)$ is generated by these type of matrices, we see that H contains the elements $M = (M_p)_p \in Sp_{2n}(\mathbb{A}_{\mathbb{Q},f})$, with $M_p \in Sp_{2n}(\mathbb{Q}_p)$ and $M_q = 1$ for $q \neq p$. Hence for any finite set of primes F, we have the inclusion

$$\{(x)_q \in Sp_{2n}(\mathbb{A}_{\mathbb{Q},f}) \mid \forall q \in F, x_q \in Sp_{2n}(\mathbb{Q}_p) \text{ and } x_q = 1_{2n} \text{ if } q \notin F\} \subseteq H.$$

The result follows since the union of these subsets over all finite set of primes F is dense in $Sp_{2n}(\mathbb{A}_{\mathbb{Q},f})$. □

We can now establish the following important Corollary.

Corollary 3.1.1. Let K be an open compact subgroup of $GSp_{2n}(\mathbb{A}_f)$. Then $\lambda(K) \subseteq \hat{\mathbb{Z}}^\times$ and if $\lambda(K) = \hat{\mathbb{Z}}^\times$, we have

$$GSp_{2n}(\mathbb{A}_f) = K \cdot GSp_{2n}^+(\mathbb{Q}) = GSp_{2n}^+(\mathbb{Q}) \cdot K \tag{3.24}$$

In particular we get that $h(G, K) = 1$ for the maximal open compact subgroup $K = GSp_{2n}(\hat{\mathbb{Z}})$.

Proof. The first assertion follows from the fact that the map $\lambda : GSp_{2n}(\mathbb{A}_f) \longrightarrow \mathbb{A}_f^\times$ is continuous and $\mathbb{Z}_p^\times$ is the unique maximal subgroup of $\mathbb{Q}_p^\times$. To show (3.24), let $g \in GSp_{2n}(\mathbb{A}_f)$ so that $\lambda(g) \in \mathbb{A}_f^\times$. Since $\mathbb{A}_{\mathbb{Q},f}^\times = \mathbb{Q}^\times \hat{\mathbb{Z}}^\times$, we can write

$$\lambda(g) = \alpha \cdot x,$$

for some $\alpha \in \mathbb{Q}^\times$ (we choose $\alpha > 0$ if necessary) and $x \in \hat{\mathbb{Z}}^\times = \lambda(K)$. Thus we can choose $k \in K$ such that $\lambda(k) = x$. Consider the matrix

$$g' = \mathrm{diag}(\underbrace{\alpha^{-1}, 1, \ldots, \alpha^{-1}, 1}_{n}, \underbrace{1, \alpha^{-1}, \ldots, 1, \alpha^{-1}}_{n})\, gk^{-1}.$$

Observe that $g' \in Sp_{2n}(\mathbb{A}_f)$ and by Theorem 3.1.2 the open set $g' \cdot Sp_{2n}(\hat{\mathbb{Z}}) \subseteq Sp_{2n}(\mathbb{A}_f)$ contains $\eta \in Sp_{2n}(\mathbb{Q})$ such that $\eta = g' \cdot h$ for some $h \in Sp_{2n}(\hat{\mathbb{Z}})$. Moreover by Proposition 3.1.4 the group $Sp_{2n}(\mathbb{Z})$ is dense in $Sp_{2n}(\hat{\mathbb{Z}})$, hence we can find $\gamma \in Sp_{2n}(\mathbb{Z})$ such that $\gamma \in g'^{-1}\eta U$, where $U = K \cap Sp_{2n}(\hat{\mathbb{Z}})$. This shows that $g \in K \cdot GSp_{2n}^+(\mathbb{Q})$ as desired. Considering the automorphism $x \mapsto x^{-1}$ for $x \in GSp_{2n}(\mathbb{A}_f)$, we see that $GSp_{2n}(\mathbb{A}_f) = GSp_{2n}^+(\mathbb{Q}) \cdot K$. □

Siegel upper half plane

Definition 3.1.1. The Siegel upper half plane of degree n consists of all symmetric complex $n \times n$-matrices whose imaginary part is positive definite:

$$\mathbb{H}_n^+ = \{\tau = \tau_1 + i\tau_2 \in \mathrm{Mat}_n(\mathbb{C}) \mid \tau^t = \tau, \quad \tau_2 > 0\}. \tag{3.25}$$

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_{2n}^+(\mathbb{R})$ and $\tau \in \mathbb{H}_n^+$. Then the matrix $C\tau + D$ is invertible and if we define

$$g \cdot \tau := (A\tau + B)(C\tau + D)^{-1},$$

then the map

$$\tau \mapsto g \cdot \tau \tag{3.26}$$

is an action of $GSp_{2n}^+(\mathbb{R})$ on $\mathbb{H}_n^+$ [Pit19] . If we write $\tau = \tau_1 + i\tau_2 \in \mathbb{H}_n^+$ and $d\tau = d\tau_1 d\tau_2$ is the Euclidean measure, then the element of volume on $\mathbb{H}_n^+$ given by

$$d^*\tau := \det(\tau_2)^{-(n+1)} d\tau, \tag{3.27}$$

is invariant under all transformations of the group $GSp_{2n}^+(\mathbb{R})$, i.e.,

$$d^*(g \cdot \tau) = d^*\tau, \quad \text{for all } g \in GSp_{2n}^+(\mathbb{R}).$$

Given an element $\tau = \tau_2 + i\tau_2 \in \mathbb{H}_n^+$, the relation

$$\begin{pmatrix} 1_n & \tau_1 \\ 0_n & 1_n \end{pmatrix} \begin{pmatrix} \tau_2^{1/2} & 0_n \\ 0_n & \tau_2^{-1/2} \end{pmatrix} \cdot i1_n = \tau$$

shows that the action of $GSp_{2n}^+(\mathbb{R})$ is transitive. The stabilizer of $i1_n$ is the subgroup

$$S = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in GL_{2n}(\mathbb{R}) \right\} \cap GSp_{2n}^+(\mathbb{R}), \tag{3.28}$$

Hence the group $Z(\mathbb{R})\backslash GSp^+_{2n}(\mathbb{R})$ (where $Z(\mathbb{R})$ denotes the center of $GSp^+_{2n}(\mathbb{R})$) acts transitively on $\mathbb{H}^+_n$ and we have the following identification

$$\mathbb{H}^+_n = PGSp^+_{2n}(\mathbb{R})/K, \tag{3.29}$$

where K is the compact group $K = Z(\mathbb{R})\backslash S \simeq \mathbb{U}^n/\{\pm 1_n\}$ and $\mathbb{U}^n$ is isomorphic to the unitary group of order n. The restriction of the action defined in (3.26) to the arithmetic subgroup $Sp_{2n}(\mathbb{Z})$ will be of special importance to us. Note that from the identification (3.29) we see that the action of Γ_n on $\mathbb{H}^+_n$ is proper since $\Gamma_n = Sp_{2n}(\mathbb{Z})$ is discrete and the subgroup K in (3.29) is compact. The action of $Sp_{2n}(\mathbb{Z})$ on $\mathbb{H}^+_n$ admits a fundamental domain and thanks to a result by Siegel [Sie64], it has the following concrete description [Nam06]. Let U_n be the subset of matrices $\tau = \tau_1 + i\tau_2 \in \mathbb{H}^+_n$ satisfying the following conditions:

1. $|\det(C\tau + D)| \geq 1$ for every $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{Z})$;

2. $\tau_2 = (\tau_2)_{ij}$ is Minkowski reduced, i.e.,

$$a^t \tau_2 a \geq (\tau_2)_{kk}, \quad 1 \leq k \leq n \quad \text{for all} \quad a = \begin{pmatrix} a_1 \\ \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{Z}^n \ \text{ where } \ (a_1, \ldots, a_n) = 1;$$

3. $\left|(\tau_1)_{ij}\right| \leq 1/2$.

Then U_n is a fundamental domain of $Sp_{2n}(\mathbb{Z})$ on $\mathbb{H}^+_n$ of finite volume with the respect to the element of volume (3.27):

$$\mathrm{vol}(U_n) = 2 \prod_{i=1}^{n} \pi^{-k} \Gamma(i) \zeta(2i),$$

where $\Gamma(s)$ denotes the Gamma function and $\zeta(s)$ is the Riemann zeta function.

Bost-Connes-Marcolli system: the GSp_{2n}-case

We consider the connected Shimura datum $(GSp^+_{2n}, \mathbb{H}^+_n)$ together with the BCM pair

$$G^+ = GSp^+_{2n}, \quad X^+ = \mathbb{H}^+_n, \quad V = \mathbb{Q}^{2n}, \quad M = MSp_{2n}(\hat{\mathbb{Z}}),$$
$$L = \mathbb{Z}^{2n}, \quad K = GSp_{2n}(\hat{\mathbb{Z}}), \quad K_M = MSp_{2n}(\hat{\mathbb{Z}}).$$

Let $M \in GSp_{2n}^+(\mathbb{Q}) \cap GSp_{2n}(\hat{\mathbb{Z}})$. Since $\hat{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$ it is clear that $M \in MSp_{2n}(\mathbb{Z})$ and there exists $M' \in MSp_{2n}^+(\mathbb{Z})$ such that $MM' = 1$ so that $\lambda(M) = 1$, that is $\Gamma_+ = GSp_{2n}^+(\mathbb{Q}) \cap GSp_{2n}(\hat{\mathbb{Z}}) = \Gamma_n$. From Corollary 3.1.1 we have that $h(G, K) = 1$, hence by Proposition 2.3.1 it is enough to consider the space $\mathfrak{Z}_{\mathcal{D},\mathcal{L}}^+$. As in the case of the GL_2 system, the first difficulty comes from the presence of points in $\mathbb{H}_n^+$ with nontrivial stabilizers:

Proposition 3.1.5. The groupoid structure on $\mathfrak{U}_{\mathcal{D},\mathcal{L}}^+$ does not pass to the quotient by the action of $\Gamma_n \times \Gamma_n$.

Proof. Let $g = \begin{pmatrix} 1_n & 0_n \\ 0_n & \frac{1}{2}1_n \end{pmatrix} \in GSp_{2n}^+(\mathbb{Q})$ and assume the groupoid composition is defined when we pass to the quotient. Since $g \cdot \frac{1}{2}i1_{2n} = i1_{2n}$ we obtain that

$$(g, 0, i1_{2n})(g, 0, \frac{1}{2}i1_{2n}) = (g^2, 0, \frac{1}{2}i1_{2n}),$$

where the equality holds in the quotient. On the other hand, let $\gamma = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$ so that $g\gamma^{-1}g = -\frac{1}{2}\gamma$ and $\gamma \cdot i1_{2n} = g \cdot \frac{1}{2}i1_{2n}$. We then have the following equality in the quotient:

$$(g^2, 0, \frac{1}{2}i1_{2n}) = (g\gamma^{-1}, 0, \gamma \cdot i1_{2n})(g, 0, \frac{1}{2}i1_{2n}) = (g\gamma^{-1}g, 0, \frac{1}{2}i1_{2n}).$$

Hence there exist $\gamma_1, \gamma_2 \in \Gamma_n = Sp_{2n}(\mathbb{Z})$ satisfying the following two conditions:

$$\gamma_2 \cdot i1_{2n} = i1_{2n},$$

$$\gamma_1 g^2 \gamma_2^{-1} = \frac{1}{2}\begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}.$$

The first condition implies that γ_2 is of the form $\gamma_2 = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ while the second condition gives

$$\gamma_1 = \begin{pmatrix} \frac{1}{2}B & -2A \\ \frac{1}{2}A & 2B \end{pmatrix}.$$

Note that since $\gamma_1 \in Sp_{2n}(\mathbb{Z})$ we get

$$\begin{pmatrix} \frac{1}{4}(BA - AB) & A^2 + B^2 \\ -B^2 - A^2 & 4(BA - AB) \end{pmatrix} = \Omega,$$

that is, $I = 4(A'^2 + B'^2)$ for some $A', B' \in \mathrm{Mat}_n(\mathbb{Z})$, which is a contradiction. $\square$

The $GSp_{4,\mathbb{Q}}$-system

We restrict our attention to the case $n = 2$ and fix the following notation. We let $Y = \mathbb{H}_2^+ \times MSp_4(\hat{\mathbb{Z}})$, $X = GSp_4^+(\mathbb{Q})Y = \mathbb{H}_2^+ \times MSp_4(\mathbb{A}_{\mathbb{Q},f})$ and $\Gamma_2 = Sp_4(\mathbb{Z})$. As observed above the action of Γ_2 on Y is not free and it turns out that the set of points in Y with nontrivial stabilizers strictly contains $\mathbb{H}_2^+ \times \{0_4\}$. Let

$$F_Y = \{h \in MSp_4(\hat{\mathbb{Z}}) \mid \mathrm{rank}_{\mathbb{Q}_p}(h_p) \leq 2 \quad \text{for all primes } p\}.$$

Then the action of Γ_2 on $\tilde{Y} = Y \backslash (\mathbb{H}_2^+ \times F_Y)$ is free. To see this, we suppose that for some $\gamma \in \Gamma_2$ we have $\gamma \cdot \tau = \tau$ and $\gamma h_p = h_p$ for some $\tau \in \mathbb{H}_2^+$ and $h_p \in MSp_4(\mathbb{Q}_p)$ with $\mathrm{rank}_{\mathbb{Q}_p}(h_p) > 2$ for some prime p. Then we can find $T \in GL_4(\mathbb{Q}_p)$ such that

$$T\gamma T^{-1} = \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & x_4 \end{pmatrix},$$

for some $x_1 \ldots, x_4 \in \mathbb{Q}_p$. Since the entries of γ are in $\mathbb{Z}$ we see that $x_4 \in \mathbb{Q}$ and thus $C_{\mathbb{Q},\gamma}(x) = (x-1)^3(x-x_4)$. On the other hand, since γ fixes a point in $\mathbb{H}_2^+$, then there exists $P \in Sp_4(\mathbb{R})$ such that

$$P\gamma P^{-1} = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \odot \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix}, \quad a_i, b_i \in \mathbb{R} \quad a_1^2 + b_1^2 = a_2^2 + b_2^2 = 1,$$

So $C_{\mathbb{C},\gamma}(x) = (x-\lambda_1)(x-\overline{\lambda}_1)(x-\lambda_2)(x-\overline{\lambda}_2)$ where $\lambda_1 = a_1 + ib_1$ and $\lambda_2 = a_2 + ib_2$. Hence $\lambda_1 = \lambda_2 = 1$ and $\gamma = 1$. It is easy to see that the set of points in Y with nontrivial stabilizers is strictly larger than $\mathbb{H}_2^+ \times \{0\}$.

The fact that F_Y is not invariant under scalar matrices in $GSp_4^+(\mathbb{Q})$ creates a new difficulty that was not present in the GL_2-system. For this reason, and for the purpose of KMS$_\beta$ analysis, instead of working with the quotient $\mathbb{H}_2^+ = GSp_4^+(\mathbb{R})/K$, we consider first the quotient $PGSp_4^+(\mathbb{R}) = GSp_4^+(\mathbb{R})/Z(\mathbb{R})$, where $Z(\mathbb{R})$ is the center of the group $GSp_4^+(\mathbb{R})$. From now on we refer to this system as the GSp_4-system and we call the original dynamical system (corresponding to the Shimura datum $(GSp_4^+, \mathbb{H}_2^+)$) the *Connes-Marcolli GSp_4-system*. We will show later that the two systems have the same thermodynamical properties. Since now $PGSp_4^+(\mathbb{R})$ is a group, we get the following:

Proposition 3.1.6. For $\beta \neq 0$, there exists a correspondence between KMS$_\beta$ states on the GSp_4-system and Γ_2-invariant measures μ on $PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_{\mathbb{Q},f})$

such that

$$\nu(\Gamma_2\backslash PGSp_4^+(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}})) = 1, \quad \mu(gB) = \lambda(g)^{-\beta}\mu(B) \tag{3.30}$$

for any $g \in GSp_4^+(\mathbb{Q})$ and Borel compact subset $B \subset PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_{\mathbb{Q},f})$. Here ν denotes the measure on $\Gamma_2\backslash PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_{\mathbb{Q},f})$ corresponding to μ.

Proof. Since the action of Γ_2 on $PGSp_4^+(\mathbb{R})$ is free, Proposition 3.0.1 applied to the group $G = GSp_4^+(\mathbb{Q})$ and the spaces $X = PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_{\mathbb{Q},f})$ and $Y = PGSp_4^+(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}})$ gives a one-to-one correspondence between the set of KMS$_\beta$ states on $(\mathcal{A}, \sigma)$ and Γ_2-invariant measures μ on Y such that $\mu(gB) = \lambda(g)^{-\beta}\mu(B)$ if gZ and Z are measurable subsets of Y. The equality $MSp_4(\mathbb{A}_f) = GSp_4^+(\mathbb{Q})MSp_4(\hat{\mathbb{Z}})$ allows us to extend ([LLN07, Lemma 2.2]) this measure to a Radon measure on X such that $\mu(gB) = \lambda(g)^{-\beta}\mu(B)$ for every Borel subset $B \subseteq X$. Since the algebra $\mathcal{A}$ is not unital, from the normalization condition (2.1) and equation (3.5) we obtain that ν is a probability measure on $\Gamma_2\backslash Y$. □

From now on, we let $Y = PGSp_4^+(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}})$ and $X = PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_{\mathbb{Q},f})$. For $\beta > 0$, we denote by $\mathcal{E}_\beta$ the set of Radon measures on X satisfying the properties in Proposition 3.1.6. Note that the extremal KMS$_\beta$ states correspond to point mass measures.

3.2 KMS$_\beta$ states analysis

High temperature region

We begin the KMS$_\beta$-analysis of the GSp_4-system by first considering the high temperature region $0 < \beta < 3$. Our first goal is to show that the GSp_4-system constructed above does not admit a KMS$_\beta$ state for $0 < \beta < 3$ with $\beta \notin \{1, 2\}$. We first show some useful lemmas.

Lemma 3.2.1. Let F be a finite set of prime numbers and let $g = (g_p)_{p\in F}$ be an element of $\prod_{p\in F} MSp_4(\mathbb{Z}_p) \subset \prod_{p\in F} MSp_4(\mathbb{Q}_p)$ with $\lambda(g_p) \neq 0$ for all $p \in F$. Then there exist $g_1 \in S_F$ and $g_2 \in \prod_{p\in F} GSp_4(\mathbb{Z}_p)$ such that $g = g_1 g_2$.

Proof. It follows from Corollary 3.1.1 that we can find $g_1 \in GSp_4^+(\mathbb{Q})$ and $g_2 \in GSp_4(\mathbb{Z}_p)$ such that $g = g_1 g_2$ with $g_1 \in GSp_4(\mathbb{Z}_q)$, $q \neq p$ and $g_1 \in MSp_4(\mathbb{Z}_p)$ and $\lambda(g_1) \in \mathbb{N}_F$, that is, $g_1 \in S_F$. □

For $k_0, k_1, k_2 \in \mathbb{Z}$, we set

$$P_{k_0} := \begin{pmatrix} 0 & 0 \\ 0 & p^{k_0} \end{pmatrix} \odot 0_2, \quad P_{k_1,k_2} := \begin{pmatrix} 0 & 0 \\ 0 & p^{k_1} \end{pmatrix} \odot \begin{pmatrix} 0 & 0 \\ 0 & p^{k_2} \end{pmatrix}.$$

$$Z^{(0)}_{k_0} := Sp_4(\mathbb{Z}_p)\, P_{k_0}\, GSp_4(\mathbb{Z}_p), \quad Z^{(1)}_{k_1,k_2} := Sp_4(\mathbb{Z}_p)\, P_{k_1,k_2}\, GSp_4(\mathbb{Z}_p). \tag{3.31}$$

Lemma 3.2.2. The sets $Z^{(0)}_{k_0}$ and $Z^{(1)}_{k_1,k_2}$, $k_0, k_1, k_2 \in \mathbb{Z}$ are pairwise disjoint. Moreover, given any nonzero matrix $a \in MSp_4(\mathbb{Q}_p)$ with $\lambda(a) = 0$, then $a \in Z^{(0)}_{k_0} \cup Z^{(1)}_{k_1,k_2}$ for some $k_0, k_1, k_2 \in \mathbb{Z}$.

Proof. We first fix some notations. For $1 \leq i, j \leq 4$, let E_{ij} be the elementary matrix with coefficient 1 at the position (i, j) and 0 otherwise. For $U \in GL_2(\mathbb{Z}_p)$ and $S \in \mathrm{Sym}_2(\mathbb{Z}_p)$, we put

$$J(U) = \begin{pmatrix} U^t & 0_2 \\ 0_2 & U^{-1} \end{pmatrix}, \quad J(S) = \begin{pmatrix} 1_2 & S \\ 0_2 & 1_2 \end{pmatrix}.$$

Consider $g \in MSp_4(\mathbb{Q}_p)$ with $\lambda(g) = 0$. Let g_0 be any entry of g with maximal p-adic valuation and we write $g_0 = a_0 p^{k_0}$, where $a_0 \in \mathbb{Z}_p^{\times}$. Using the matrices Ω and $J_1(P)$ (where P is a permutation matrix), we may assume that $g_{11} = g_0$. If a is an entry of the matrix g, we set

$$U_a = 1_2 - g_0^{-1} a E_{21}, \quad S_a = -g_0^{-1} a E_{11}, \quad \tilde{S}_a = -g_0^{-1} a (E_{12} + E_{21}).$$

Observe that by maximality, these matrices are in $Sp_4(\mathbb{Z}_p)$. We multiply g from the right by

$$J(U_{g_{12}}) J(S_{g_{13}+g_{11}g_{12}g_{14}}) J(\tilde{S}_{g_{14}}).$$

to obtain a matrix whose first row is $g_0\mathbf{e_1}$. Taking the transpose and repeating this process, we obtain a matrix whose first column is equal to $g_0\mathbf{e_1}^t$. The symplectic relations 3.10 and 3.11 imply that this matrix has the following form:

$$\begin{pmatrix} g_0 & 0 \\ 0 & 0 \end{pmatrix} \odot M, \quad M \in \mathrm{Mat}_2(\mathbb{Q}_p), \quad \det(M) = 0.$$

If $M = 0$, then one has

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \odot 1_2 \cdot \begin{pmatrix} g_0 & 0 \\ 0 & 0 \end{pmatrix} \odot M \cdot \begin{pmatrix} 0 & a_0^{-1} \\ 1 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & a_0^{-1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & p^{k_0} \end{pmatrix} \odot 0_2 \in P_{k_0}.$$

Otherwise, we can use right and left multiplication to find $\gamma_1 \in SL_2(\mathbb{Z}_p)$ and $\gamma_2 \in GL_2(\mathbb{Z}_p)$ such that $\gamma_1 M \gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & p^{k_1} \end{pmatrix}$, for some $k_1 \in \mathbb{Z}$. Then the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \odot \gamma_1 \cdot \begin{pmatrix} g_0 & 0 \\ 0 & 0 \end{pmatrix} \odot M \cdot \begin{pmatrix} 0 & a_0^{-1} \\ -a_o \det(\gamma_2) & 0 \end{pmatrix} \odot \gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & p^{k_0} \end{pmatrix} \odot \begin{pmatrix} 0 & 0 \\ 0 & p^{k_1} \end{pmatrix} \in P_{k_0,k_1}$$

has the desired form. One can easily check that this decomposition is unique. □

Lemma 3.2.3. Let p be a prime and we put

$$g_{1,p} := \operatorname{diag}(1,1,p,p), \quad g_{2,p} := \operatorname{diag}(p,p,p,p), \quad g_{3,p} := \operatorname{diag}(1,p,p^2,p).$$

A set of representatives of the right cosets relative to Γ_2 in $\Gamma_2 g_{1,p} \Gamma_2$ is given by the matrices

$$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1 & 0 & k_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \quad \begin{pmatrix} 1 & -k_2 & k_3 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & k_2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & k_4 & k_5 \\ 0 & 1 & k_5 & k_6 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix},$$

where $0 \le k_1, k_2, \dots, k_6 < p$.

A set of representatives of the right cosets relative to Γ_2 in $\Gamma_2 g_{3,p} \Gamma_2$ is given by the matrices

$$\begin{pmatrix} p^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \quad \begin{pmatrix} p & -pr_1 & 0 & 0 \\ 0 & p^2 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & r_1 & 1 \end{pmatrix}, \quad \begin{pmatrix} p & 0 & 0 & pr_2 \\ 0 & 1 & r_2 & r_3 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -r_4 & r_5r_4 + r_6 & r_5 \\ 0 & p & pr_5 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & pr_7 & p \end{pmatrix}, \quad \begin{pmatrix} p & 0 & r_8 & r_9 \\ 0 & p & r_9 & r_{10} \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix},$$

where $1 \le r_1, r_2, r_4, r_5 < p$, $1 \le r_3, r_6 < p^2$, and $0 \le r_8, r_9, r_{10} < p$ are such that $r_p\begin{pmatrix} r_8 & r_9 \\ r_9 & r_{10} \end{pmatrix} = 1$, where $r_p(B)$ denotes the rank of the matrix $B \in \operatorname{Mat}_2(\mathbb{Z})$ over

$\mathbb{Z}/p\mathbb{Z}$. In particular we have:

$$\begin{aligned}
\deg_{\Gamma_2}(g_{1,p}) &= (1+p)(1+p^2)\\
\deg_{\Gamma_2}(g_{2,p}) &= 1\\
\deg_{\Gamma_2}(g_{3,p}) &= p+p^2+p^3+p^4.
\end{aligned}$$

Proof. Recall that $S_2(p)$ denotes the set of matrices $g \in GSp_4(\mathbb{Z})$ such that $\lambda(g) = p$. This set consists of a single coset:

$$S_2(p) = \Gamma g_{1,p}\Gamma. \tag{3.32}$$

From this we can see that

$$\mathcal{D}(\Gamma_2 g_{1,p}\Gamma_2) = \{\Gamma_1\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Gamma_1, \Gamma_1\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_1, \Gamma_1\begin{pmatrix}p & 0\\ 0 & p\end{pmatrix}\Gamma_1\}.$$

The decomposition of $\Gamma g_{1,p}\Gamma$ into right cosets follows then from applying Theorem 3.21 with $n = 2$ and $n = 1$ (notice that we are using the convention that for n impair, the matrices B are under the diagonal).

The decomposition of $\Gamma g_{2,p}\Gamma$ is trivial. To decompose the double cosets $\Gamma g_{3,p}\Gamma$, we use the following simple criterion:

$$M \in \Gamma_2 g_{3,p}\Gamma_2 \Leftrightarrow r_p(M) = 1 \text{ and } M \in GSp_4(\mathbb{Z}).$$

We then obtain

$$\mathcal{D}(\Gamma_2 g_{3,p}\Gamma_2) = \{\Gamma_1\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_1, \Gamma_1\begin{pmatrix}p & 0\\ 0 & p\end{pmatrix}\Gamma_1, \Gamma_1\begin{pmatrix}p & 0\\ 0 & p^2\end{pmatrix}\Gamma_1\}.$$

For each $D \in U_2\backslash\mathcal{D}(\Gamma_2 g_{3,p}\Gamma_2)$, the set $\mathcal{B}(D, \Gamma_2 g_{3,p}\Gamma_2)$ is then obtained by applying again Theorem 3.21 and using the relations

$$A^t D = p^2 1_2, \quad B^t D = D^t B, \quad AB^t = BA^t.$$

□

Lemma 3.2.4. Let p be a prime and denote by G_p the subgroup of $GSp_4^+(\mathbb{Q})$ generated by Γ_2 and the matrices $g_{1,p}$, $g_{2,p}$ and $g_{3,p}$ defined in Lemma 3.2.3. Suppose that μ_p is a Γ_2-invariant measure on $PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Q}_p)$ such that

1. $\mu_p(PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Z}_p)) < \infty$

2. $\mu_p(gZ) = \lambda(g)^{-\beta}\mu_p(Z)$ for all $g \in G_p$ and Borel $Z \subseteq PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Q}_p)$.

If $\beta \notin \{1,2,3\}$ then $PGSp_4^+(\mathbb{R}) \times GSp_4(\mathbb{Q}_p)$ is a subset of full measure in $PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Q}_p)$

Proof. We denote ν_p the corresponding measure on $\Gamma_2\backslash PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Q}_p)$. Then for a Γ_2-invariant Borel set $Z \subseteq MSp_4(\mathbb{Q}_p)$, we define a measure $\tilde{\mu}_p$ on the σ-algebra of Γ_2-invariant Borel sets of on $MSp_4(\mathbb{Q}_p)$ by

$$\tilde{\mu}_p(Z) = \nu_p(\Gamma_2\backslash PGSp_4^+(\Gamma_2\mathbb{R}) \times Z).$$

Note that by assumption we have $\tilde{\mu}_p(MSp_4(\mathbb{Z}_p)) < \infty$. For any $g \in G_p$ and any positive integrable Γ_2-invariant function on $MSp_4(Q_p)$ we get from the second condition and [LLN07, Lemma 2.6] that

$$\int_{MSp_4(\mathbb{Q}_p)} T_g f \,\mathrm{d}\tilde{\mu}_p = \lambda(g)^{\beta} \int_{MSp_4(\mathbb{Q}_p)} f \,\mathrm{d}\tilde{\mu}_p. \tag{3.33}$$

For $k_0, k_1, k_2 \in \mathbb{Z}$, consider the functions $f_{k_0}^{(0)} = I_{Z_{k_0}^{(0)}}$ and $f_{k_1,k_2}^{(1)} = I_{Z_{k_1,k_2}^{(1)}}$, where the sets $Z_{k_0}^{(0)}$ and $Z_{k_1,k_2}^{(1)}$ are as in equation (3.31). Given $g \in GSp_4^+(\mathbb{Q})$, we have that the function $T_g f_{0,0}^{(1)}$ is continuous and Γ-invariant. By Proposition 3.1.4 the group Γ is dense in $Sp_4(\mathbb{Z}_p)$, whence $T_g f_{0,0}^{(1)}$ is left $Sp_4(\mathbb{Z}_p)$-invariant. For $k_1, k_2 \in \mathbb{Z}$, we can expand the expression $T_{g_{2,p}^{-1}g_{1,p}} f_{0,0}^{(1)}(P_{k_1,k_2})$ using the explicit representatives given in Lemma 3.2.3 as follows:

$$\begin{aligned}
&\deg(g_{2,p}^{-1}g_{1,p})^{-1}\left(f_{0,0}^{(1)}\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p^{k_1-1} & 0 \\ 0 & 0 & 0 & p^{k_2-1} \end{pmatrix}\right) + \sum_{k=0}^{p-1} f_{0,0}^{(1)}\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & kp^{k_2-1} \\ 0 & 0 & p^{k_1-1} & 0 \\ 0 & 0 & 0 & p^{k_2} \end{pmatrix}\right)\right. \\
&+ \sum_{0\le k',a\le p-1} f_{0,0}^{(1)}\left(\begin{pmatrix} 0 & 0 & k'p^{k_1-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p^{k_1} & 0 \\ 0 & 0 & ap^{k_1-1} & p^{k_2-1} \end{pmatrix}\right) + \left.\sum_{0\le a,b,n\le p-1} f_{0,0}^{(1)}\left(\begin{pmatrix} 0 & 0 & bp^{k_1-1} & np^{k_2-1} \\ 0 & 0 & np^{k_1-1} & bp^{k_2-1} \\ 0 & 0 & p^{k_1} & 0 \\ 0 & 0 & 0 & p^{k_2} \end{pmatrix}\right)\right).
\end{aligned}$$

Since $T_{g_{2,p}^{-1}g_{1,p}} f_{0,0}^{(1)}(\gamma_1 P_{k_1,k_2}\gamma_2) = T_{g_{2,p}^{-1}g_{1,p}} f_{0,0}^{(1)}(P_{k_1,k_2}\gamma_2)$ for $\gamma_1 \in Sp_4(\mathbb{Z}_p)$ and $\gamma_2 \in GSp_4(\mathbb{Z}_p)$, it follows that $\deg(g_{2,p}^{-1}g_{1,p})T_{g_{2,p}^{-1}g_{2,p}} f_{0,0}^{(1)}$ is equal to

$$\begin{aligned}
&f_{1,1}^{(1)} + f_{1,0}^{(1)} + (p-1)f_{1,1}^{(1)} + f_{0,1}^{(1)} + (p^2-1)f_{1,1}^{(1)} + (p-1)f_{1,0}^{(1)} + (p-1)^2 f_{1,1}^{(1)} \\
&+ f_{0,0}^{(1)} + (p-1)f_{1,1}^{(1)} + (p-1)f_{0,1}^{(1)} + (p-1)^2 f_{1,1}^{(1)} + (p-1)^2 f_{1,1}^{(1)} + (p-1)^3 f_{1,1}^{(1)} \\
&= f_{0,0}^{(1)} + p f_{1,0}^{(1)} + p f_{0,1}^{(1)} + (p^3+p^2-p) f_{1,1}^{(1)}.
\end{aligned}$$

Similarly, the expansion of $T_{g_{2,p}^{-2}g_{3,p}} f_{0,1}(P_{k_1,k_2})$ is given by

$$\begin{aligned}
&\deg(g_{2,p}^{-2}g_{3,p})^{-1}\Bigg(f_{0,1}\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p^{k_1-2} & 0 \\ 0 & 0 & 0 & p^{k_2-1} \end{pmatrix}\right) + \sum_{0\le a\le p-1} f_{0,1}\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p^{k_1-1} & 0 \\ 0 & 0 & ap^{k_1-2} & p^{k_2-2} \end{pmatrix}\right) \\
&+ \sum_{\substack{0\le b\le p-1 \\ 0\le c\le p^2-1}} f_{0,1}\left(\begin{pmatrix} 0 & 0 & 0 & bp^{k_2-1} \\ 0 & 0 & bp^{k_1-2} & cp^{k_2-2} \\ 0 & 0 & p^{k_1-1} & 0 \\ 0 & 0 & 0 & p^{k_2} \end{pmatrix}\right) + \sum_{r_p\begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}=1} f_{0,1}\left(\begin{pmatrix} 0 & 0 & b_1p^{k_1-2} & b_2p^{k_2-1} \\ 0 & 0 & b_2p^{k_1-2} & b_3p^{k_2-2} \\ 0 & 0 & p^{k_1-1} & 0 \\ 0 & 0 & 0 & p^{k_2-1} \end{pmatrix}\right) \\
&+ \sum_{\substack{0\le l\le p^2-1 \\ 0\le k,d\le p-1}} f_{0,1}\left(\begin{pmatrix} 0 & 0 & (kd+l)p^{k_1-2} & kp^{k_2-2} \\ 0 & 0 & kp^{k_1-1} & 0 \\ 0 & 0 & p^{k_1} & 0 \\ 0 & 0 & dp^{k_1-1} & p^{k_2-1} \end{pmatrix}\right)\Bigg).
\end{aligned}$$

Counting the number of instances where the indices are divisible by p, we find that $\deg(g_{2,p}^{-2}g_{3,p})T_{g_2^{-2}g_3} f_{0,0}^{(1)}$ is equal to

$$\begin{aligned}
&f_{2,1}^{(1)} + f_{1,2}^{(1)} + (p-1)f_{2,2}^{(1)} + f_{1,0}^{(1)} + (p-1)f_{1,1}^{(1)} + (p^2-p)f_{1,2}^{(1)} \\
&+ (p-1)f_{2,1}^{(1)} + (p-1)^2 f_{2,1}^{(1)} + \Big((p-1)(p^2-1) - (p-1)^2\Big) f_{2,2}^{(1)} \\
&+ (p-1)f_{1,2}^{(1)} + (p-1)f_{2,1}^{(1)} + (p-1)^2 f_{2,2}^{(1)} + f_{0,1}^{(1)} + (p-1)f_{1,2}^{(1)} \\
&+ (p-1)f_{1,1}^{(1)} + (p-1)^2 f_{2,2}^{(1)} + (p-1)f_{1,1}^{(1)} + \Big((p^2-1) - (p-1)\Big) f_{2,1}^{(1)} \\
&+ (p-1)^2 f_{1,2}^{(1)} + \Big((p^2-p)(p-1)\Big) f_{2,2}^{(1)} + (p-1)^2 f_{1,1}^{(1)} \\
&+ \Big((p^2-p)(p-1)\Big) f_{2,1}^{(1)} + p(p-1)^2 f_{1,2}^{(1)} + \Big((p-1)^2(p^2-1) - p(p-1)^2\Big) f_{2,2}^{(1)} \\
&= f_{0,1}^{(1)} + f_{1,0}^{(1)} + (p^2+p-2)f_{1,1}^{(1)} + p^3 f_{1,2}^{(1)} + p^3 f_{2,1}^{(1)} + (p^4-p^3) f_{2,2}^{(1)}.
\end{aligned}$$

Similar computations lead to the following identities:

$$\begin{aligned}
\deg(g_{2,p}^{-1})T_{g_{2,p}^{-1}}f_{0,0}^{(1)} &= f_{1,1}^{(1)},\\
\deg(g_{2,p}^{-2}g_{1,p})T_{g_{2,p}^{-2}g_{1,p}}f_{0,0}^{(1)} &= f_{1,1}^{(1)} + pf_{2,1}^{(1)} + pf_{1,2}^{(1)} + (p^3+p^2-p)f_{2,2}^{(1)},\\
\deg(g_{2,p}^{-2})T_{g_{2,p}^{-2}}f_{0,0}^{(1)} &= f_{2,2}^{(1)}.
\end{aligned}$$

We then have

$$\begin{aligned}
f_{0,0}^{(1)} &= \deg(g_{2,p}^{-1}g_{1,p})T_{g_{2,p}^{-1}g_{1,p}}f_{0,0}^{(1)} - p\deg(g_{2,p}^{-1}g_{3,p})T_{g_{2,p}^{-1}g_{3,p}}f_{0,0}^{(1)}\\
&\quad - (p+p^3)\deg(g_{2,p}^{-1})T_{g_{2,p}^{-1}}f_{0,0}^{(1)} + p^3\deg(g_{2,p}^{-2}g_{1,p})T_{g_{2,p}^{-2}g_{1,p}}f_{0,0}^{(1)}\\
&\quad - p^6\deg(g_{2,p}^{-2})T_{g_{2,p}^{-2}}f_{0,0}^{(1)}.
\end{aligned}$$

Since $\deg(g_{2,p}^{-i}g_{k,p}) = \deg(g_k)$, $i \in \mathbb{N}$, $k = 1,2,3$, it follows from equation (3.33) that

$$\tilde{\mu}_p(Z_{0,0}^{(1)}) = R(p,\beta)\tilde{\mu}_p(Z_{0,0}^{(1)}), \tag{3.34}$$

where

$$\begin{aligned}
R(p,\beta) &= p^{1-\beta}+p^{2-\beta}+p^{3-\beta}+p^{-\beta}-p^{1-2\beta}-p^{2-2\beta}-2p^{3-2\beta}-p^{4-2\beta}-p^{5-2\beta}\\
&\quad + p^{3-3\beta}+p^{4-3\beta}+p^{5-3\beta}+p^{6-3\beta}-p^{6-4\beta}\\
&= 1-p^6\left(p^{-\beta}-1\right)\left(p^{-\beta}-p^{-1}\right)\left(p^{-\beta}-p^{-2}\right)\left(p^{-\beta}-p^{-3}\right).
\end{aligned}$$

We can repeat the same computations with the functions $T_g f_0^{(0)}$, $g \in GSp_4^+(\mathbb{Q})$ instead. As a summary we get

$$\begin{aligned}
\deg(g_{2,p}^{-1}g_{1,p})T_{g_{2,p}^{-1}g_{1,p}}f_0^{(0)} &= (1+p)f_0^{(0)} + (p^3+p^2)f_1^{(0)}\\
\deg(g_{2,p}^{-1}g_{3,p})T_{g_{2,p}^{-1}g_{3,p}}f_0^{(0)} &= f_0^{(0)} + (p^2-1+p^3+p)f_1^{(0)} + p^4f_2^{(0)}\\
\deg(g_{2,p}^{-1})T_{g_{2,p}^{-1}}f_0^{(0)} &= f_1^{(0)}\\
\deg(g_{2,p}^{-2}g_{1,p})T_{g_{2,p}^{-2}g_{1,p}}f_0^{(0)} &= (1+p)f_1^{(0)} + (p^3+p^2)f_2^{(0)}\\
\deg(g_{2,p}^{-2})T_{g_{2,p}^{-2}}f_0^{(0)} &= f_2^{(0)},
\end{aligned}$$

and

$$\tilde{\mu}_p(Z_0^{(0)}) = R(p,\beta)\tilde{\mu}_p(Z_0^{(0)}). \tag{3.35}$$

Suppose that $\tilde{\mu}_p(Z_{0,0}^{(1)}) \neq 0$. Since $\beta \notin \{1,2,3\}$, by equation (3.34) we get that $\beta = 0$. Hence

$$\tilde{\mu}_p(Z_{k_1,k_2}^{(1)}) = \tilde{\mu}_p(Z_{k_1+2,k_2+2}^{(1)}), \quad k_1, k_2 \in \mathbb{Z}. \tag{3.36}$$

This is a contradiction since $\tilde{\mu}_p(MSp_4(\mathbb{Z}_p)) < \infty$. This shows that $\tilde{\mu}_p(Z_{0,0}^{(1)}) = 0$ and by induction we see that $\tilde{\mu}_p(Z_{k_1,k_2}^{(1)}) = 0$ for $k_1, k_2 \in \mathbb{N}$ (by considering suitable elements of $g \in G_p$). The same argument shows that $\tilde{\mu}_p(Z_k^{(0)}) = 0$ for $k \in \mathbb{Z}$. It follows from Lemma 3.2.2 that the $\tilde{\mu}_p$-measure of the set of nonzero matrices $g \in MSp_4(\mathbb{Q}_p)$ with $\lambda(g) = 0$ is zero. □

Corollary 3.2.1. We denote by $MSp_4(\mathbb{A}_f)^*$ the set of elements $h \in MSp_4(\mathbb{A}_f)$ such that $\lambda(m_p) \neq 0$ for all primes p. Let μ_β be a measure in $\mathcal{E}_\beta$ and $\beta \notin \{0,1,2,3\}$. Then $PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_f)^*$ is a subset of full measure in $PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_f)$.

Proof. Given a prime p, consider the restriction of μ_β to the set

$$PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Q}_p) \times \prod_{p \neq q} MSp_4(\mathbb{Z}_q),$$

and the measure $\mu_{\beta,p}$ on $PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Q}_p)$ obtained from the projection on the first two coordinates. Since $\mu_\beta \in \mathcal{K}_\beta$ we have that $\mu_{\beta,p}(PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Z}_p)) < \infty$. Given any $g \in G_p$ and Borel $Z \in PGSp_4(\mathbb{R})^+ \times MSp_4(\mathbb{Q}_p)$ we get

$$\mu_{\beta,p}(gZ) = \mu_\beta(g(Z \times \prod_{q \neq p} MSp_4(\mathbb{Z}_q))) = \lambda(g)^{-\beta}\mu_{\beta,p}(Z).$$

Thus the measure $\mu_{\beta,p}$ satisfies the conditions of Lemma 3.2.4, whence $PGSp_4^+(\mathbb{R}) \times GSp_4(\mathbb{Q}_p)$ is a subset of full $\mu_{\beta,p}$-measure. This shows that the μ_β-measure of the set

$$\{PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Q}_p) \times \prod_{p \neq q} MSp_4(\mathbb{Z}_q) \mid \lambda(h_p) = 0\}$$

is zero. Finally observe that the complement of $\mathbb{H}_2^+ \times MSp_4^+(\mathbb{A}_f)$ is equal to

$$\bigcup_{p\in\mathcal{P}} GSp_4^+(\mathbb{Q})\{PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Q}_p) \times \prod_{p\neq q} MSp_4(\mathbb{Z}_q) \mid \lambda(h_p) = 0\},$$

which completes the proof. $\square$

Given a prime number $p \in \mathcal{P}$ and $\beta \in \mathbb{R}_+^*$, we consider the subsemigroup of $GSp_4^+(\mathbb{Q})$ given by

$$S_{2,p} = \bigcup_{l\geq 0} S_2(p^l).$$

Note that $\Gamma_2 \subseteq S_{2,p}$ for any prime number p and the corresponding Dirichlet series (cf. Definition 3) is

$$\zeta_{S_{2,p},\Gamma_2}(\beta) = \sum_{g\in\Gamma_2\backslash S_{2,p}/\Gamma_2} \lambda(g)^{-\beta} \deg_{\Gamma_2}(g) = \sum_{l=0}^{\infty} p^{-\beta l} R_{\Gamma_2}(p^l). \tag{3.37}$$

Proposition 3.2.1. Suppose $\beta \in \mathbb{R}_+^*$. Then $\zeta_{S_2,\Gamma_2}(\beta) < \infty$ if and only if $\beta > 3$. In this case we have that

$$\zeta_{S_{2,p},\Gamma_2}(\beta) = \frac{1 - p^{2-2\beta}}{(1 - p^{3-\beta})(1 - p^{2-\beta})(1 - p^{1-\beta})(1 - p^{-\beta})}. \tag{3.38}$$

Proof. We combine the results from Proposition 3.1.2 and Lemma 3.1.1 to first compute $R_{\Gamma_2}(p^l)$:

$$\begin{aligned}
R_{\Gamma_2}(p^l) &= \sum_{d_1|d_2|p^l} \deg_{\Gamma_1}(\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}) d_1^2 d_2 \\
&= \sum_{l_1\leq l_2\leq l} \deg_{\Gamma_1}(\begin{pmatrix} p^{l_1} & 0 \\ 0 & p^{l_2} \end{pmatrix}) p^{2l_1+l_2} \\
&= \sum_{i=0}^{l} \sum_{k=0}^{l-i} \deg_{\Gamma_1}(\begin{pmatrix} p^{i} & 0 \\ 0 & p^{k+i} \end{pmatrix}) p^{2i} p^{i+k} \\
&= \sum_{i=0}^{l} p^{3i} + \sum_{i=0}^{l} \sum_{k=1}^{l-i} p^{k-1}(1+p) p^{3i+k} \\
&= \frac{1}{(1-p)^2(1+p+p^2)} \Big(1 - p^{2l}(p+p^2+p^3) + p^{3l}(p^2+p^4)\Big),
\end{aligned}$$

since $\deg_{\Gamma_1}\begin{pmatrix} p^{l_1} & 0 \\ 0 & p^{l_2} \end{pmatrix} = p^{l_2-l_1-1}$ for $l_2 \neq l_1$ [Kri90, Theorem 4.1, Chapter IV]. It is then clear that the series $\zeta_{S_2,\Gamma_2}(\beta)$ converges if and only if $\beta > 3$ and

$$\begin{aligned}\zeta_{S_{2,p},\Gamma_2}(\beta) &= \frac{1}{(1-p)^2(1+p+p^2)}\Big(\frac{(1-p)^2(1+p+p^2)p^{-\beta}(p+p^{\beta})}{(1-p^{3-\beta})(1-p^{-\beta})(1-p^{2-\beta})}\Big) \\ &= \frac{1-p^{2-2\beta}}{(1-p^{3-\beta})(1-p^{2-\beta})(1-p^{1-\beta})(1-p^{-\beta})},\end{aligned}$$

for $\beta > 3$ as desired. □

We are now ready to prove the main theorem of this section.

Theorem 3.2.1. *The GSp_4-system $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}^+})$ does not admit a KMS_β state for $0 < \beta < 3$ with $\beta \notin \{1, 2\}$.*

Proof. We fix a prime $p \in \mathcal{P}$ and put

$$Y_p := PGSp_4^+(\mathbb{R}) \times GSp_4(\mathbb{Z}_p) \times \prod_{q\neq p} MSp_4(\mathbb{Z}_q), \tag{3.39}$$

and note that

$$PGSp_4^+(\mathbb{R}) \times (MSp_4(\mathbb{Z}_p) \cap GSp_4(\mathbb{Q}_p)) \times \prod_{q\neq p} MSp_4(\mathbb{Z}_q) = \bigcup_{s\in\Gamma_2\backslash S_{2,p}/\Gamma_2} \Gamma_2 s Y_p. \tag{3.40}$$

It is easy to see that the sets $\Gamma_2 s Y_p$ are disjoint for $s \in \Gamma_2\backslash S_{2,p}/\Gamma_2$ and the complement of their union in $\mathbb{H}_2^+ \times MSp_4(\hat{\mathbb{Z}})$ is a subset of $(\mathbb{H}_2^+ \times MSp_4^+(\mathbb{A}_f))^c$, whence by Lemma 3.2.4 it has full measure for $\beta \notin \{1, 2, 3\}$. Let $\mu_\beta \in \mathcal{E}(K_\beta)$ and ν_β its corresponding measure on the quotient space. Note that if $g \in G_p \cap GSp_4(\mathbb{Z}_p)$ then necessarily $\lambda(g) = 1$ and $g \in MSp_4(\mathbb{Z})$, hence $G_p \cap GSp_4(\mathbb{Z}_p) = \Gamma$. We can then apply [LLN07, Lemma 2.7] to the group G_p (a simple calculation shows that any elementary matrix in $S_{2,p}$ is generated by $g_{1,p}, g_{2,p}$ and $g_{3,p}$) and the spaces $\tilde{X}$ and $Y_0 = Y_p$. We obtain that for any $g \in \Gamma_2\backslash S_{2,p}/\Gamma_2$, we have

$$\nu_\beta(\Gamma_2\backslash\Gamma_2 g Y_p) = \lambda(g)^{-\beta} \deg_{\Gamma_2}(g)\nu_\beta(\Gamma_2\backslash Y_p).$$

Observe that

$$\begin{aligned}\nu_\beta(\Gamma_2\backslash PGSp_4^+(\mathbb{R})\times MSp_4(\hat{\mathbb{Z}})) &= \sum_{g\in\Gamma_2\backslash S_{2,p}/\Gamma_2} \nu_\beta(\Gamma_2\backslash\Gamma_2 gY_p)\\ &= \sum_{g\in\Gamma_2\backslash S_{2,p}/\Gamma_2} \lambda(g)^{-\beta}\deg_{\Gamma_2}(g)\nu_\beta(\Gamma_2\backslash Y_p).\end{aligned}$$

Hence

$$1 = \zeta_{S_{2,p},\Gamma_2}(\beta)\nu_\beta(\Gamma_2\backslash Y_p), \tag{3.41}$$

which is not possible for $\beta < 3$ by Proposition 3.2.1. This shows that there are no KMS$_\beta$-states for $\beta < 3$ and $\beta \notin \{0, 1, 2\}$. □

Low temperature region and Gibbs states

Theorem 3.2.2. *For $\beta > 4$, the extremal KMS states of the GSp_4-system are given by the Gibbs states*

$$\phi_\beta(f) = \frac{\zeta(2\beta-2)Tr(\pi_y(f)e^{-\beta H_y})}{\zeta(\beta)\zeta(\beta-1)\zeta(\beta-2)\zeta(\beta-3)}, \tag{3.42}$$

where $y \in PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})$.

Proof. Let F be an arbitrary finite set of primes and denote by G_F the group generated by G_p for $p \in F$. We denote by $MSp_4^+(\mathbb{Z}) = GSp_4^+(\mathbb{Q}) \cap MSp_4(\mathbb{Z})$ and we put

$$S_F := \{m \in MSp_4^+(\mathbb{Z}) \mid \lambda(m) \in \mathbb{N}(F)\},$$

and

$$Y_F = PGSp_4^+(\mathbb{R}) \times \prod_{p\in F} GSp_4(\mathbb{Z}_p) \times \prod_{q\notin F} MSp_4(\mathbb{Z}_q).$$

Similarly to the proof of Theorem 3.2.1 (we replace Y_p by Y_F and $S_{2,p}$ by S_F), we get

$$1 = \nu_\beta(\Gamma_2\backslash Y_F)\zeta_{S_F,\Gamma_2}(\beta) = \nu_\beta(\Gamma_2\backslash Y_F)\prod_{p\in F}\zeta_{S_{2,p},\Gamma_2}.$$

Note that $Y_F \subseteq Y_{F'}$ for $F' \subseteq F$ and the intersection of Y_F over all finite primes is the set $PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})$. Hence for $\beta > 4$, we get

$$\nu_\beta(\Gamma_2 \backslash (PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})) = \zeta_{MSp_4^+(\mathbb{Z}),\Gamma_2}(\beta)^{-1},$$

where

$$\zeta_{MSp_4^+(\mathbb{Z}),\Gamma_2}(\beta) = \prod_{p \in \mathcal{P}} \zeta_{S_{2,p},\Gamma_2} = \frac{\zeta(\beta)\zeta(\beta-1)\zeta(\beta-2)\zeta(\beta-3)}{\zeta(2\beta-2)}. \tag{3.43}$$

On the other hand, the sets $\Gamma_2 s(PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}))$ are disjoints for s in $\Gamma_2 \backslash MSp_4^+(\mathbb{Z})/\Gamma_2$. We thus obtain that the ν_β-measure of $\Gamma_2 \backslash MSp_4^+(\mathbb{Z})(PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}))$ is given by

$$\sum_{s \in \Gamma_2 \backslash MSp_4^+(\mathbb{Z})/\Gamma_2} \nu_\beta(\Gamma_2 \backslash \Gamma_2 s(PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}))),$$

which is equal to

$$\zeta_{MSp_4^+(\mathbb{Z}),\Gamma_2}(\beta)\nu_\beta(\Gamma_2 \backslash PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})) = 1.$$

Hence $MSp_4^+(\mathbb{Z})(PGS_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}))$ has full measure in $PGSp_4^+(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}})$ and by Corollary 3.1.2 the subset $PGSp_4^+(\mathbb{R}) \times GSp_4(\mathbb{A}_f)$ has full measure in $PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_f)$. Conversely, any probability Γ_2-invariant measure on $PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})$ extends [LLN07, Lemma 2.4] uniquely to a measure on $PGSp_4^+(\mathbb{R}) \times GSp_4(\mathbb{A}_f)$ satisfying condition 3.30.

Suppose now that $\mu_\beta \in \mathcal{E}_\beta$ is a Dirac measure centered on $y \in PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})$. Then

$$\begin{aligned}
\phi(f) &= \sum_{s \in \Gamma_2 \backslash MSp_4^+(\mathbb{Z})/\Gamma_2} \int_{\Gamma_2 \backslash \Gamma_2 s(PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}))} f(1,\omega) d\nu_\beta(\omega) \\
&= \zeta_{MSp_4^+(\mathbb{Z}),\Gamma_2}(\beta)^{-1} \sum_{s \in \Gamma_2 \backslash MSp_4^+(\mathbb{Z})/\Gamma_2} \lambda(s)^{-\beta} \sum_{h \in \Gamma_2 \backslash \Gamma_2 s \Gamma_2} f(1, hy) \\
&= \zeta_{MSp_4^+(\mathbb{Z}),\Gamma_2}(\beta)^{-1} \sum_{h \in \Gamma_2 \backslash MSp_4^+(\mathbb{Z})} \lambda(h)^{-\beta} f(1, hy) \\
&= \frac{\mathrm{Tr}(\pi_y(f) e^{-\beta H_y})}{\mathrm{Tr}(e^{-\beta H_y})},
\end{aligned}$$

since the operator H_y is positive and $G_y = MSp_4^+(\mathbb{Z})$ for $y \in GSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})$.

□

The critical region

We denote by $\hat{\mathcal{E}}_\beta$ the subset of right $GSp_4(\hat{\mathbb{Z}})$-invariant measures in $\mathcal{E}_\beta$. The next proposition shows that this set is not empty for $3 \leq \beta < 4$.

Proposition 3.2.2. For each $\beta \in (3, 4]$, the GSp_4-system $\{\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}}\}$ admits at least one KMS_β-state.

Proof. We generalize the construction in [LLN07]. By the correspondence in Proposition 3.1.6, it is enough to construct a measure μ_β on $PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_{\mathbb{Q},f})$ such that $\mu_\beta \in \mathcal{E}_\beta$. For each prime p and $3 < \beta \leq 4$, we consider the normalized Haar measure on $GSp_4(\mathbb{Z}_p)$ so that the total volume is $\zeta_{S_{2,p},\Gamma_2}(\beta)^{-1}$ (we denote this measure by $\mathrm{meas}_{\beta,p}$). Observe that $GSp_4(\mathbb{Q}_p) = G_p GSp_4(\mathbb{Z}_p)$ and hence by [LLN07, Lemma 2.4] we can uniquely extend this measure to a measure $\mu_{\beta,p}$ on $GSp_4(\mathbb{Q}_p)$ such that if Z is a compact measurable subset in $GSp_4(\mathbb{Q}_p)$, then

$$\mu_{\beta,p}(Z) = \sum_{g\in G_p} |\lambda(g)|_p^{-\beta} \mathrm{meas}_{\beta,p}(gZ \cap GSp_4(\mathbb{Z}_p)),$$

where $|a|_p$ denotes the p-adic valuation of a. Since $\mu_{\beta,p}(hZ) = |\lambda(h)|_p^\beta \mu_{\beta,p}(Z)$ for $g \in GSp_4(\mathbb{Q}_p)$, it is clear that $\mu_{\beta,p}$ is left $GSp_4(\mathbb{Z}_p)$-invariant. It is also right $GSp_4(\mathbb{Z}_p)$-invariant since the Haar measure $\mathrm{meas}_{\beta,p}$ is right translation invariant. We extend the measure $\mu_{\beta,p}(Z)$ to a measure on $MSp_4(\mathbb{Q}_p)$ by setting $\mu_{\beta,p}(Z) := \mu_{\beta,p}(Z \cap GSp_4(\mathbb{Q}_p))$ for Borel $Z \subseteq MSp_4(\mathbb{Q}_p)$. To extend this measure to $MSp_4(\mathbb{A}_{\mathbb{Q},f})$ we first check that $\mu_{\beta,p}(MSp_4(\mathbb{Z}_p)) = 1$. The proof of Lemma 3.2.4 applied to the space $MSp_4(\mathbb{Q}_p)$ shows that the set $MSp_4(\mathbb{Z}_p) \cap GSp_4(\mathbb{Q}_p)$ has full measure. Since this set is precisely $S_{2,p}GSp_4(\mathbb{Z}_p)$, then similarly to the calculation in the proof of Theorem 3.2.1 we get

$$\mu_{\beta,p}(MSp_4(\mathbb{Z}_p)) = \sum_{g\in\Gamma_2\backslash S_{2,p}/\Gamma_2} \lambda(g)^{-\beta} \deg_{\Gamma_2}(g)\mu_{\beta,p}(GSp_4(\mathbb{Z}_p)) = 1.$$

We thus define a measure on $MSp_4(\mathbb{A}_{\mathbb{Q},f})$ by $\mu_{\beta,f} = \prod_{p\in\mathcal{P}} \mu_{\beta,p}$. Then we have that for any $g \in GSp_4(\mathbb{Q})$ and measurable subset $Z \subseteq MSp_4(\mathbb{A}_f)$, we get

$$\mu_{\beta,f}(gZ) = \Big(\prod_{p\in\mathcal{P}} |\lambda(g)|_p^\beta\Big)\mu_{\beta,f}(Z) = \lambda(g)^{-\beta}\mu_{\beta,f}(Z).$$

If we denote by $\mu_{\beta,PGSp_4^+(\mathbb{R})}$ the normalized Haar measure on $PGSp_4^+(\mathbb{R})$ such that $\nu_{\beta,PGSp_4^+(\mathbb{R})}$ is a probability measure on $\Gamma_2\backslash PGSp_4^+(\mathbb{R})$, then it is clear that the measure defined by $\mu_\beta := \mu_{\beta, PGSp_4^+(\mathbb{R})} \times \mu_{\beta,f}$ is an element of $\mathcal{K}_\beta$. By construction, μ_β is also right $GSp_4(\hat{\mathbb{Z}})$-invariant; that is, $\mu_\beta \in \hat{\mathcal{E}}_\beta$. □

Our next goal is to show that for $3 < \beta \leq 4$, the KMS$_\beta$ constructed in Proposition 3.2.2 is the unique equilibrium state. We first recall the definition of an ergodic action.

Definition 3.2.1. If $\mu \in \mathcal{E}_\beta$, the action of G on the measure space (X, μ) is ergodic if the following holds: If A is any G-invariant Borel subset of X, then $\mu(A) = 0$ or $\mu(A^c) = 0$.

Recall that if W is a locally compact group, then a character of W is a continuous homomorphisms $\chi : W \to \mathbb{T}$.

Lemma 3.2.5. For $n \in \mathbb{N}$, we let $G_n = 1 + p^n\mathbb{Z}_p^\times \subseteq \mathbb{Z}_p$ and χ be any character of $\mathbb{Z}_p^\times$. Then $G_k \subseteq \ker(\chi)$ for some $k \in \mathbb{N}$.

Proof. Consider the open subset of $\mathbb{T}$ given by

$$V = \{z \in \mathbb{T} \mid \mathrm{Re}(z) > 0\}. \tag{3.44}$$

Observe first that the only subgroup of V is $\{1\}$ and a fundamental system of neighborhood of the neutral element of $\mathbb{Z}_p^\times$ is given by the subgroups

$$G_1 \supset G_2 \supset \cdots \supset G_n \ldots$$

Consider now the open subset of $\mathbb{T}$ given in (3.44). Since the character χ is continuous, there exists an integer $k \geq 1$ such that $\chi(G_k) \subseteq V$. Now χ is homomorphism and therefore the subset $\chi(G_k)$ is a subgroup of V, that is $\chi(G_k) = 1$. □

Lemma 3.2.6. Let m be an integer and B a finite set of prime numbers. Then the set

$$\{(n, \ldots, n) \in \prod_{p \in B} \mathbb{Z}_p^\times \mid n \in \mathbb{Z} \text{ and } (n, p) = 1 \quad \forall p \in B\} \tag{3.45}$$

is dense in $\prod_{p \in B} \mathbb{Z}_p^\times$.

Proof. Any $a \in \mathbb{Z}_p^\times$ admits a p-adic expansion of the form

$$a = \sum_{i \geq 0} c_i p^i, \quad 0 < c_i < p.$$

Hence, given $\epsilon > 0$ and $x = (x_1, \ldots, x_{|B|}) \in \prod_{p \in B} \mathbb{Z}_p^\times$, we choose $a_k \in \mathbb{Z}$ and $e_k \in \mathbb{N}$ such that

$$|x_k - a_k|_{p_k} < \frac{\epsilon}{2}, \quad (a_k, p_k) = 1, \quad p_k^{-e_k} < \frac{\epsilon}{2}$$

for $k = 1, \ldots, |B|$. By the Chinese remainder theorem, the congruence system

$$n \equiv a_k \mod p_k^{e_k}, \quad 1 \leq k \leq |B|$$

has a solution $n \in \mathbb{Z}$. The condition (a_k, p_k) implies that $(n, p_k) = 1$ for all $1 \leq k \leq |B|$. Hence

$$|n - x_k|_{p_k} \leq |n - a_k|_{p_k} + |a_k - x_k|_{p_k} \leq p^{e_k} + \frac{\epsilon}{2} \leq \epsilon,$$

as desired.

□

Given $m \in \mathbb{N}$, a Dirichlet character modulo m is a function $\chi_m : \mathbb{Z} \to \mathbb{C}$ obtained by extending a character of $(\mathbb{Z}/m\mathbb{Z})^\times$ to 0 on $\mathbb{Z}/m\mathbb{Z}$ and lifted to $\mathbb{Z}$ by composition. The corresponding Dirichlet L-function is defined by

$$L(s, \chi_m) = \sum_{n=1}^{\infty} \chi(n)/n^s, \quad s \in \mathbb{C}.$$

Theorem 3.2.3. *Let $\hat{\mu}_\beta \in \hat{\mathcal{E}}_\beta$ and $A = L^\infty(PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_{\mathbb{Q},f}), \hat{\mu}_\beta)$. We have that*

$$A^{PGSp_4^+(\mathbb{R}) \times GSp_4^+(\mathbb{Q})} = \mathbb{C}. \tag{3.46}$$

Proof. It is enough to show that the action of $GSp_4^+(\mathbb{Q})$ on $(MSp_4(\mathbb{A}_{\mathbb{Q},f}), \hat{\mu}_{\beta,f})$ (where $\hat{\mu}_{\beta,f}$ is the measure on $MSp_4(\mathbb{A}_{\mathbb{Q},f})$ obtained by projecting onto the second factor) is ergodic. The strategy extends the ideas in [Nes02] and [LLN07]. Since every $GSp_4^+(\mathbb{Q})$-invariant subset is completely determined by its intersection with $MSp_4(\hat{\mathbb{Z}})$, it is enough to show that the closed subspace

$$H = \{f \in L^2(MSp_4(\hat{\mathbb{Z}}), \hat{\mu}_{\beta,f}) \mid V_m f = f, \forall m \in MSp_4^+(\mathbb{Z})\}, \;\; (V_m f)(x) := f(mx),$$

consists of constant functions. Denote by P the orthogonal projection onto H. Since every function in H is Γ_2-invariant, it is enough to show that P maps Γ_2-invariants functions to constants.

Consider the subspace

$$H_F = \{f \in L^2(MSp_4(\hat{\mathbb{Z}})),\, \mathrm{d}\hat{\mu}_{\beta,f}) \mid V_s f = f, \forall s \in S_F\},$$

and denote by P_F the orthogonal projection onto H_F. Consider the subset

$$W_F = \prod_{p\in F} GSp_4(\mathbb{Z}_p) \times \prod_{q\notin F} MSp_4(\mathbb{Z}_q).$$

Note that by Lemma 3.2.1 and Corollary 3.2.6 the disjoint union $\cup_{s\in\Gamma_2\backslash S_F/\Gamma_2} sW_F$ has full measure. Hence given any Γ_2-invariant function $f \in L^2(MSp_4(\hat{\mathbb{Z}})),\, \mathrm{d}\hat{\mu}_{\beta,f})$, we deduce from [LLN07, Lemma 2.9] that

$$P_F f = \zeta_{S_F,\Gamma_2}(\beta)^{-1} \sum_{s\in\Gamma_2\backslash S_F/\Gamma_2} \lambda(s)^{-\beta} R_{\Gamma_2}(s) T_s f. \tag{3.47}$$

We fix a finite set of primes B such that $B \cap F = \emptyset$ and consider the functions in $L^2(MSp_4(\hat{\mathbb{Z}}),\, \mathrm{d}\mu_{\beta,f})$ of the form

$$\chi_B(x) = \begin{cases} \chi(\lambda((x_p)_{p\in B})) & \text{if } \; x \in W_B \\ 0 & \text{otherwise} \end{cases},$$

where χ is a character of the compact abelian group $\prod_{p\in B} \mathbb{Z}_p^\times$.

We will first show that $P\chi_B(x)$ is constant a.e. This is easy to prove if the character χ is trivial. Indeed, in this cases $\chi_B = \mathbb{1}_B$ and the projection formula (3.47) we get that $P_B\chi_B = \zeta_{S_B,\Gamma_2}(\beta)^{-1}$. Since $P = PP_B$ the result follows. We now consider the case where χ is non-trivial. We first write $\chi = \prod_{p\in B} \chi_p$, where χ_p is a character of $\mathbb{Z}_p^\times$ given by

$$\chi_p(a) = \chi(1, \ldots, a, \ldots, 1).$$

Then by Lemma 3.2.5 for each prime $p \in B$ there exists an integer $k_p \in \mathbb{N}$ such that $G_{k_p} \subseteq \ker(\chi_p)$. Let $m = \prod_{p \in B} p^{k_p}$ and define $\chi_m : \mathbb{Z} \to \mathbb{C}$ by

$$\chi_m(n) = \begin{cases} \chi((n, \dots, n)) & \text{if } (n, m) = 1 \\ 0 & \text{otherwise} \end{cases}$$

It is clear that χ_m is multiplicative and $\chi_m(n+m) = \chi_m(n)$ if $(n,m) \neq 1$. Suppose that $(n,m) = 1$ so that $n \in \mathbb{Z}_p^\times$. Hence

$$\chi_m(n+m) = \prod_{p \in B} \chi_p(n+m) = \prod_{p \in B} \chi_p(n)\chi_p(1 + mn^{-1}) = \prod_{p \in B} \chi_p(n) = \chi_m(n),$$

since p^{k_p} divides m. Hence χ_m is a Dirichlet character modulo m. We claim that χ_m is nontrivial. Indeed, let $a \in \prod_{p \in B} \mathbb{Z}_p^\times$ such that $\chi(a) \neq 1$. By Lemma 3.2.1 the set

$$\{(n, \dots, n) \in \prod_{p \in B} \mathbb{Z}_p^\times \mid n \in \mathbb{Z} \text{ and } (n, m) = 1\}$$

is dense in $\prod_{p \in B} \mathbb{Z}_p^\times$. Since χ is continuous there exists $n_0 \in \mathbb{Z}$ with $(n_0, m) = 1$ and $1 \neq \chi(n_0, \dots, n_0) = \chi_m(n_0)$ as desired.

Since $F \cap B = \emptyset$, for $s \in S_F$ we can write

$$(T_s \chi_B)(x) = \begin{cases} \chi(\lambda((x_p)_{p \in B}))\chi_m(\lambda(s)) & \text{if } x \in W_B \\ 0 & \text{otherwise.} \end{cases}$$

This allows us to obtain an explicit upper bound of the L^2-norm of $P\chi_B$ as follows. Since $P = PP_F$, from the projection formula (3.47) we get

$$\begin{aligned} \|P\chi_B\| = \|PP_F\chi_B\| \le \|P_F\chi_B\| &\le \zeta_{S_F,\Gamma_2}(\beta)^{-1} \left| \sum_{s \in \Gamma \ S_F/\Gamma} \lambda(s)^{-\beta} R_{\Gamma_2}(s)\chi_m(\lambda(s)) \right| \\ &= \zeta_{S_F,\Gamma_2}(\beta)^{-1} \left| \sum_{n \in \mathbb{N}(F)} n^{-\beta} R_{\Gamma_2}(n)\chi_m(n) \right| \\ &= \zeta_{S_F,\Gamma_2}(\beta)^{-1} \left| \prod_{p \in F} \sum_{l=0}^{\infty} p^{-l\beta} R_{\Gamma_2}(p^l)\chi_m(p^l) \right|, \end{aligned}$$

since the function $R_{\Gamma_2}(n)$ is multiplicative by Lemma 3.1.3. As in the proof of Proposition 3.2.1 we get

$$
\begin{aligned}
\|P\chi_B\| &\leq \zeta_{S_F,\Gamma_2}(\beta)^{-1}\prod_{p\in F}\left|\left(\frac{1}{1-\chi_m(p)p^{-\beta}}-\frac{p+p^2+p^3}{1-\chi_m(p)p^{2-\beta}}+\frac{p^2+p^4}{1-\chi_m(p)p^{3-\beta}}\right)\right| \\
&= \prod_{p\in F}\left|\frac{(1+p^{1-\beta}(\chi_m(p)-1)-p^{2-2\beta}\chi_m(p))(1-p^{3-\beta})(1-p^{2-\beta})(1-p^{-\beta})}{(1-\chi_m(p)p^{3-\beta})(1-\chi_m(p)p^{2-\beta})(1-\chi_m(p)p^{1-\beta})(1-\chi_m(p)p^{-\beta})(1-p^{2-2\beta})}\right| \\
&= \left|\frac{(\sum_{n\in\mathbb{N}(F)}\frac{\chi_m(n)}{n^{\beta-3}})(\sum_{n\in\mathbb{N}(F)}\frac{\chi_m(n)}{n^{\beta-2}})(\sum_{n\in\mathbb{N}(F)}\frac{\chi_m(n)}{n^{\beta-1}})}{\zeta_{\mathbb{N}(F)}(\beta-3)\zeta_{\mathbb{N}(F)}(\beta-2)\zeta_{\mathbb{N}(F)}(\beta-1)}\right|\prod_{p\in F}\left|\frac{1+\chi_m(p)p^{1-\beta}}{1+p^{1-\beta}}\right| \\
&\leq \left|\frac{(\sum_{n\in\mathbb{N}(F)}\frac{\chi_m(n)}{n^{\beta-3}})(\sum_{n\in\mathbb{N}(F)}\frac{\chi_m(n)}{n^{\beta-2}})(\sum_{n\in\mathbb{N}(F)}\frac{\chi_m(n)}{n^{\beta-1}})}{\zeta_{\mathbb{N}(F)}(\beta-3)\zeta_{\mathbb{N}(F)}(\beta-2)\zeta_{\mathbb{N}(F)}(\beta-1)}\right| \\
&\leq \frac{L(\beta-3,\chi_m)L(\beta-2,\chi_m)L(\beta-1,\chi_m)}{\zeta_{\mathbb{N}(F)}(\beta-3)\zeta_{\mathbb{N}(F)}(\beta-2)\zeta_{\mathbb{N}(F)}(\beta-1)}.
\end{aligned}
$$

Since the character χ_m is nontrivial and $3<\beta\leq 4$, it follows from [Ser70, Proposition 12, Chapter VI] that the right-hand side above can be made arbitrary small as $F\nearrow\mathcal{P}$ (with $F\cap B=\emptyset$). This shows that $P\chi_B=0$, in particular $P\chi_B$ is again a constant function when χ is a nontrivial character.

Consider now the functions $F_B\in L^2(MSp_4(\hat{\mathbb{Z}}),\hat{\mu}_{\beta,f})$ of the form

$$
F_B(x)=\begin{cases} f((x_p)_{p\in B}) & \text{if } x\in W_B \\ 0 & \text{otherwise.}\end{cases}
$$

where $f\in L^2(\prod_{p\in B}GSp_4(\mathbb{Z}_p),\,\mathrm{d}\hat{\mu}_B)$ and $\hat{\mu}_B=(\pi_B)_*(\hat{\mu}_{\beta,f})$, where π_B is the projection $\pi_B: MSp_4(\hat{\mathbb{Z}})\to\prod_{p\in B}MSp_4(\mathbb{Z}_p)$. We first show that it is enough to assume that f is Γ_2-invariant. Indeed, when this is not the case, we denote by Q the projection onto the space of $Sp_4(\hat{\mathbb{Z}})$-invariant functions. We have that

$$
QF_B(x)=\int_{Sp_4(\hat{\mathbb{Z}})}F_B(gx)\,\mathrm{d}g=\begin{cases}\int_{\prod_{p\in B}Sp_4(\mathbb{Z}_p)}f(gx)\,\mathrm{d}g_B & \text{if } x\in W_B\\ 0 & \text{otherwise,}\end{cases}
$$

Observe that $PQ=P$ since Γ_2 is dense in $Sp_4(\hat{\mathbb{Z}})$), hence we can assume that f is Γ_2-invariant. Then again by the density of Γ_2 in $\prod_{p\in B}Sp_4(\mathbb{Z}_p)$, we see that the function f depends only on $\lambda((x_p)_{p\in B})$, i.e.,

$$F_B(x) = \begin{cases} f'(\lambda(x_p)_{p\in B}) & \text{if } x \in W_B \\ 0 & \text{otherwise,} \end{cases}$$

where f' is a square integrable function in $\prod_{p\in B} \mathbb{Z}_p^\times$ (with the natural pushforward measure). The linear span of χ (where χ is a character of $\prod_{p\in B} \mathbb{Z}_p^\times$) form a dense subspace of such square integrable functions and we have shown that $P\chi_B$ is constant. It follows that PF_B is constant.

We can easily verify that the adjoint of the operator V_s (where $s \in S_B$) is given by

$$(V_s^* h)(x) = \begin{cases} \lambda(s)^\beta h(s^{-1}x) & \text{if } x \in sMSp_4(\hat{\mathbb{Z}}) \\ 0 & \text{otherwise.} \end{cases}$$

Hence the map $\lambda(s)^{-\beta/2} V_s^*$ maps isometrically the functions of the form F_B to functions in $L^2(MSp_4(\hat{\mathbb{Z}}), \mathrm{d}\hat{\mu}_{\beta,f})$ of the form

$$G_B(x) = \begin{cases} g((x_p)_{p\in B}) & \text{if } x \in sW_B \\ 0 & \text{otherwise,} \end{cases} \tag{3.48}$$

where $g \in L^2(s\prod_{p\in B} GSp_4(\mathbb{Z}_p), \mathrm{d}\mu_B)$. It follows then from $P = PV_s^*$ that the projection P maps every function of the form (3.48) to a constant function. The set $\cup_{s\in\Gamma_2\backslash S_B/\Gamma_2} sW_B$ has full measure in $MSp_4(\hat{\mathbb{Z}})$ and thus P maps functions depending only on $(x_p)_{p\in B}$ to constants. Finally observe that as $B \nearrow \mathcal{P}$, the union of $L^2(\prod_{p\in B} MSp_4(\mathbb{Z}_p), \mathrm{d}\mu_B)$ over all finite sets of primes is a dense subspace of $L^2(MSp_4(\hat{\mathbb{Z}}), \mathrm{d}\hat{\mu}_{\beta,f})$ (this follows from weak convergence of measures). Since P maps this dense subspace to constant functions, this finishes the proof.

□

Lemma 3.2.7. Let f be a smooth function on $\Gamma\backslash PGSp_4^+(\mathbb{R})$ with a compact support and let Ω be any compact subset of $\Gamma\backslash PGSp_4^+(\mathbb{R})$. If we denote by μ the normalized Haar measure on $\Gamma\backslash PGSp_4^+(\mathbb{R})$ and $\bar{f} = \int_{\Gamma_2\backslash PGSp_4^+(\mathbb{R})} f d\mu$, then for all $\epsilon > 0$ there exist $\kappa_1(\epsilon) > 0$, $\kappa_2 > 0$ and $M_2 > 0$ (depending on f) such that the inequality

$$\frac{1}{R_{\Gamma_2}(m)} \sum_{a\in\Gamma_2\backslash S_m/\Gamma_2} \left|(T_a f)(\tau) - \bar{f}\right| \deg(a) \le \kappa_1 m^{(2\epsilon-1)} \prod_{i=1}^{i=l} (1+\kappa_2 p_i^{-1}) \frac{1}{(1-p_i^{-2\epsilon-1})^2},$$

holds for all $\tau \in \Omega$ and every integer m with prime factorization of the form $\prod_{i=1}^{i=l} p_i^{l_i}$ where $\min\{p_1, \dots, p_l\} > M_2$.

Proof. Let p be a given prime and $l \in \mathbb{N}$. The formula of $R_{\Gamma_2}(p^l)$ in the proof of Proposition 3.2.1 gives

$$\begin{aligned}(R_{\Gamma_2}(p^l) - p^{3l})(1-p)^2(1+p+p^2) &= 1 - p^{3l} - p^{1+2l} - p^{2+2l} - p^{2+2l} + p^{1+3l} \\ &\quad + p^{3+3l} \\ &\geq 1 + 3p^{3l} - p^{2+2l} \\ &\geq p^{3l}(1-p^{2-l}) \geq 0, \quad \forall l \geq 2.\end{aligned}$$

A simple verification for the case $l = 1$ shows that

$$(R_{\Gamma_2}(p^l) - p^{3l})(1-p)^2(1+p+p^2) \geq 0, \quad \forall l \in \mathbb{N},$$

that is

$$R_{\Gamma_2}(p^l)p^{-3l} \geq 1. \tag{3.49}$$

We suppose now that $m \in \mathbb{N}$ has the form $m = \prod_{i=1}^{i=r} p_i^{l_i}$. Then by [COU01, Theorem 1.7 and section 4.7] together with the calculations for $G = GSp_{2n}, n \geq 2$ in [COU01, pages 22–23] and the degree formula in (3.22) applied to $n = 2$ (recall that the big O depends only on n), we can find $\kappa_1 > 0$, $\kappa_2 > 0$ and $M_2 > 0$ (depending on f) such that

$$\begin{aligned}&\frac{1}{R_{\Gamma_2}(m)} \sum_{a \in \Gamma_2 \backslash S_m / \Gamma_2} \left| (T_a f)(\tau) - \int_{\Gamma_2 \backslash PGSp_4^+(\mathbb{R})} f \, d\mu \right| \deg(a) \\ &\leq \frac{\kappa_1}{R_{\Gamma_2}(m)} \sum_{k_{ij} \leq m_{ij} \leq [l_i/2]} \prod_{i=1}^{i=r} (p^{2l_i - 2k_{ij} - 2m_{ij}})^{\epsilon - \frac{1}{2}} p_i^{3l_i - 4k_{ij} - 2m_{ij}} (1 + O(p_i^{-1})) \\ &\leq \frac{\kappa_1}{R_{\Gamma_2}(m) p^{-3l_i}} \prod_{i=1}^{i=l} \sum_{k_{ij} \leq m_{ij} \leq [l_i/2]} p^{l_i(2\epsilon - 1)} p_i^{(-3-2\epsilon)k_{ij} + (-1-2\epsilon)m_{ij}} (1 + \kappa_2 p_i^{-1}) \\ &\leq \kappa_1 m^{2\epsilon - 1} \prod_{i=1}^{i=l} \sum_{k_{ij} \leq m_{ij} \leq [l_i/2]} p^{-(3+2\epsilon)k_{ij} - (1+2\epsilon)m_{ij}} (1 + \kappa_2 p_i^{-1}) \\ &\leq \kappa_1 m^{2\epsilon - 1} \prod_{i=1}^{i=l} (1 + \kappa_2 p_i^{-1}) \sum_{m_{ij}=0}^{\infty} \sum_{k_{ij}=0}^{\infty} p^{-(3+2\epsilon)k_{ij} - (1+2\epsilon)m_{ij}} \\ &\leq \kappa_1 m^{2\epsilon - 1} \prod_{i=1}^{i=l} (1 + \kappa_2 p_i^{-1}) \frac{1}{(1 - p_i^{-1-2\epsilon})^2},\end{aligned}$$

holds for $\min\{p_1,\ldots,p_l\} > M_2$ as desired. $\square$

For the next proposition, we define the operator T_F acting on the set of bounded functions on $\Gamma_2\backslash PGSp_4^+(\mathbb{R})$ as follows:

$$(T_F f)(\tau) = \zeta_{S_F,\Gamma_2}^{-1} \sum_{s\in\Gamma_2\backslash S_F/\Gamma_2} \lambda(s)^{-\beta} \deg_{\Gamma_2}(s)(T_s f)(\tau).$$

Proposition 3.2.3. Let J be any nonempty finite set of prime numbers, $3 < \beta \leq 4$, f a continuous function on $\Gamma\backslash PGSp_4^+(\mathbb{R})$ with a compact support and Ω any compact subset of $\Gamma\backslash PGSp_4^+(\mathbb{R})$. Then for any $\delta > 0$, there exists a set F consisting of finite prime numbers that are disjoint from J and such that for all $\tau \in \Omega$ we have

$$\left|(T_F f)(\tau) - \int_{\Gamma_2\backslash PGSp_4^+(\mathbb{R})} f d\mu\right| < \delta.$$

Proof. Given $3 < \beta \leq 4$, we fix $0 < \epsilon < \frac{\beta-3}{2} \leq \frac{1}{2}$ and choose κ_1, κ_2 and M_2 as in Lemma 3.2.7. Let $M_1 > 0$ be such that

$$x^{\beta}(1 - x^{2\epsilon-1} - \kappa_2 x^{2\epsilon-2}) > \kappa_2 \quad \forall x > M_1, \tag{3.50}$$

and we set $M := \max\{M_1, M_2\}$. Let F be a finite set of prime numbers with $F \cap J = \emptyset$ and $\min\{p \in F\} > M$. Then by Lemma 3.2.7, we have

$$\begin{aligned}
&\left|(T_F f)(\tau) - \int_{\Gamma_2\backslash PGSp_4^+(\mathbb{R})} f d\mu\right| \\
&\leq \kappa_1 \xi_{S_F,\Gamma}(\beta)^{-1}\Big(\sum_{m\in\mathbb{N}(F)} m^{(2\epsilon-1)} \prod_{i=1}^{i=l}(1+\kappa_2 p_i^{-1})\frac{1}{(1-p_i^{-2\epsilon-1})^2} R_\Gamma(m) m^{-\beta}\Big) \\
&\leq \kappa_1 \xi_{S_F,\Gamma}(\beta)^{-1}\Big(\prod_{p\in F}\sum_{l=0}^{\infty} p^{l(2\epsilon-1)} R_\Gamma(p^l) p^{-l\beta}(1+\kappa_2 p^{-1})\frac{1}{(1-p^{-2\epsilon-1})^2}\Big) \\
&\leq \kappa_1 \xi_{S_F,\Gamma}(\beta)^{-1}\Big(\prod_{p\in F}(1+\kappa_2 p^{-1})\prod_{p\in F}\frac{1}{(1-p^{-2\epsilon-1})^2} \\
&\prod_{p\in F}\frac{1+p^{2\epsilon-1}p^{\beta-1}}{1+p^{\beta-1}}\frac{\zeta_{\mathbb{N}(F)}(\beta-2-2\epsilon)\zeta_{\mathbb{N}(F)}(\beta-1-2\epsilon)\zeta_{\mathbb{N}(F)}(\beta-2\epsilon)}{\zeta_{\mathbb{N}(F)}(\beta-3)\zeta_{\mathbb{N}(F)}(\beta-2)\zeta_{\mathbb{N}(F)}(\beta-1)}.
\end{aligned}$$

Notice that by our choice of M and F, Equation (3.50) gives

$$(1+\kappa_2 p^{-1})\frac{1+p^{2\epsilon-2+\beta}}{1+p^{\beta-1}} < 1, \quad \forall p \in F.$$

Hence

$$\left|(T_F f)(\tau) - \bar{f}\right| \leq \kappa_1 \prod_{p\in F} \frac{\zeta_{\mathbb{N}(F)}(\beta-2-2\epsilon)\zeta_{\mathbb{N}(F)}(\beta-1-2\epsilon)\zeta_{\mathbb{N}(F)}(\beta-2\epsilon)}{(1-p^{(-2\epsilon-1)})\zeta_{\mathbb{N}(F)}(\beta-3)\zeta_{\mathbb{N}(F)}(\beta-2)\zeta_{\mathbb{N}(F)}(\beta-1)}.$$

As $F \nearrow \mathcal{P}$ with $F \cap J = \emptyset$, the right-hand side can be made arbitrary small since $3 < \beta \leq 4$, $\beta - 2 - 2\epsilon > 1$ and $\epsilon > 0$, $\square$

Theorem 3.2.4. *Let $\hat{\mu}_\beta \in \hat{\mathcal{E}}_\beta$ and $A = L^\infty(PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_f), \hat{\mu}_\beta)$. We have that*

$$A^{GSp_4^+(\mathbb{Q})\times GSp_4(\hat{\mathbb{Z}})} = \mathbb{C}. \tag{3.51}$$

Proof. Consider the space $\mathcal{H} = L^2(PGSp_4^+(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}}), \hat{\mu}_\beta)$. Observe that any $GSp_4^+(\mathbb{Q}) \times GSp_4(\hat{\mathbb{Z}})$-invariant subset of $GSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_f)$ is completely determined by its intersection with $PGSp_4^+(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}})$, hence it is enough to show that any $MSp_4^+(\mathbb{Z}) \times GSp_4(\hat{\mathbb{Z}})$-invariant function in $\mathcal{H}$ is constant. We denote by H the closed subspace of $MSp_4^+(\mathbb{Z}) \times GSp_4(\hat{\mathbb{Z}})$-invariant functions in $\mathcal{H}$ and denote by P the orthogonal projection onto H. We will show that the image under P of a dense subspace consists of constant functions. Given any non-empty finite sets of primes F and J, we denote by H_F the closed subspace of S_F-invariant functions in $\mathcal{H}$ and by P_F the orthogonal projection onto H_F. Let $\mathcal{H}_J$ be the subspace of $\Gamma_2 \times \prod_{p\in J} GSp_4(\mathbb{Z}_p)$-invariant functions depending only on $GSp_4^+(\mathbb{R}) \times \prod_{p\in J} MSp_4(\mathbb{Z}_p)$.

Recall that $S_J Y_J$ is a subset of full measure, whence by [LLN07, Lemma 2.9] given any function f in $\mathcal{H}_J$ we get

$$P_J f = \zeta_{S_F,\Gamma_2}(\beta)^{-1} \sum_{\Gamma_2\backslash S_J/\Gamma_2} \lambda(s)^{-\beta} \deg_{\Gamma_2}(s) T_s f. \tag{3.52}$$

It follows that the value $P_J f(\tau, m)$ depends only on $\tau \in GSP_4^+(\mathbb{R})$. We can then write

$$P_J f(x) = \begin{cases} \tilde{f}(\tau) & \text{if } x = (\tau, m) \in Y_J \\ 0 & \text{otherwise} \end{cases}$$

We put $\tilde{f}_J := P_J f$. Observe that since $\tilde{f}_J$ is S_J-invariant we get that $\tilde{f}$ is Γ_2-invariant and therefore can view it as a square integrable function on $\Gamma_2 \backslash PGSp_4^+(\mathbb{R})$. We first suppose that $\tilde{f}$ is smooth with compact support Ω. For $F \cap J = \emptyset$, the projection formula (3.52) gives

$$P_F \tilde{f}_J(x) = \begin{cases} T_F \tilde{f}(\tau) & \text{if } x = (\tau, m) \in Y_J \\ 0 & \text{otherwise.} \end{cases}$$

We put $(T_F \tilde{f})_J := P_F \tilde{f}_J(x)$. Given $\epsilon > 0$, by Proposition 3.2.3 there exists some finite set of primes F disjoint from J such that

$$\left| T_F f(\tau) - \int_{\Gamma_2 \backslash PGSp_4^+(\mathbb{R})} \tilde{f} \, \mathrm{d}\mu \right| < \epsilon, \quad \forall \tau \in \Omega.$$

Since $PP_J = PP_F = P$, we get

$$\begin{aligned} \left\| Pf - P\mathbb{1}_{Y_J} \int_{\Gamma_2 \backslash PGSp_4^+(\mathbb{R})} \tilde{f} \, \mathrm{d}\mu \right\|_2 &= \left\| PP_J f - P\mathbb{1}_{Y_J} \int_{\Gamma_2 \backslash PGSp_4^+(\mathbb{R})} \tilde{f} \, \mathrm{d}\mu \right\|_2 \\ &= \left\| PP_F P_J f - P\mathbb{1}_{Y_J} \int_{\Gamma_2 \backslash PGSp_4^+(\mathbb{R})} \tilde{f} \, \mathrm{d}\mu \right\|_2 \\ &\leq \left\| P_F P_J f - \mathbb{1}_{Y_J} \int_{\Gamma_2 \backslash PGSp_4^+(\mathbb{R})} \tilde{f} \, \mathrm{d}\mu \right\|_2 \\ &\leq \left\| (T_F \tilde{f})_J - \mathbb{1}_{Y_J} \int_{\Gamma_2 \backslash PGSp_4^+(\mathbb{R})} \tilde{f} \, \mathrm{d}\mu \right\|_2 < \epsilon. \end{aligned}$$

Hence using the projection formula 3.52 with $f = \mathbb{1}_{Y_J}$ we get

$$\begin{aligned} Pf = P\mathbb{1}_{Y_J} \int_{\Gamma_2 \backslash PGSp_4^+(\mathbb{R})} \tilde{f} \, \mathrm{d}\mu = PP_J 1_{Y_J} \int_{\Gamma_2 \backslash PGSp_4^+(\mathbb{R})} \tilde{f} \, \mathrm{d}\mu \\ = \zeta_{S_J, \Gamma_2}(\beta)^{-1} \int_{\Gamma_2 \backslash PGSp_4^+(\mathbb{R})} \tilde{f} \, \mathrm{d}\mu, \end{aligned}$$

which is constant. Any integrable functions on $\Gamma_2\backslash PGSp_4^+(\mathbb{R})$ can be approximated by a compactly supported smooth function and hence Pf is constant for all Γ_2-invariant functions in $\mathcal{H}_J$. The results follows since the union of $\mathcal{H}_J$ over all finite set of primes is dense in the space of square integrable $\Gamma_2 \times GSp_4(\hat{\mathbb{Z}})$-invariant functions.

□

Theorem 3.2.5. *For $3 < \beta \leq 4$, The GSp_4-system admits a unique KMS$_\beta$ state.*

Proof. We will show that the set $\mathcal{E}_\beta$ consists of a single point. We first use [LLN07, Proposition 4.6] together with Theorem 3.2.3 and Theorem 3.2.4 to conclude that $A^{GSp_4^+(\mathbb{Q})} = \mathbb{C}$ where $A = L^\infty(PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_{\mathbb{Q},f}), \hat{\mu}_\beta)$ for any $\hat{\mu}_\beta \in \hat{\mathcal{E}}_\beta$ and $3 < \beta \leq 4$, in other words there exists a unique right $GSp_4(\hat{\mathbb{Z}})$-invariant measure $\hat{\mu}_\beta$ in $\mathcal{E}_\beta$. Suppose now that υ_β is any other point of $\mathcal{E}_\beta$. Then the measure defined by

$$\omega = \int_{GSp_4(\hat{\mathbb{Z}})} g \cdot \upsilon_\beta \, \mathrm{d}g$$

is an element of $\mathcal{E}_\beta$ and by unicity we get that $\omega = \hat{\mu}_{\beta,f}$. Since the point $\hat{\mu}_{\beta,f}$ is extremal we conclude that $\hat{\mu}_{\beta,f} = \upsilon_\beta$. This completes the proof. □

Remark 2. We have studied the GSp_4-system in the region $\beta > 0$ with $\beta \notin \{1, 2, 3\}$. Let us now consider the cases where the inverse temperature is a pole of the Dirichlet series (3.38). If $\beta = 2$, it is possible to construct explicit measures $\mu_2 \in \mathcal{E}_2$. We consider the normalized Haar measure on $\mathbb{A}_{\mathbb{Q},f}$ such that meas$(\hat{\mathbb{Z}}) = 1$ and

$$\mathrm{meas}(aE) = \prod_p |a_p|_p \, \mathrm{meas}(E). \tag{3.53}$$

for any $a \in \mathbb{A}_{\mathbb{Q},f}^\times$ and measurable subset $E \subseteq \mathbb{A}_{\mathbb{Q},f}$. Let μ_f be the product measure on $\mathbb{A}_{\mathbb{Q},f}^4$. Since

$$\begin{pmatrix} 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & x_4 \end{pmatrix} \in MSp_4(\mathbb{A}_{\mathbb{Q},f}), \quad x_1, \ldots, x_4 \in A_{\mathbb{Q},f}$$

we may consider μ_f as a measure on $MSp_4(A_{\mathbb{Q},f})$ such that $\mu_f(MSp_4(\hat{\mathbb{Z}})) = 1$. We claim that $\mu_2 = \mu_\infty \times \mu_f \in \mathcal{E}_2$. By construction it is enough to show that μ_2

satisfies the scaling condition (3.30). Given $g \in GSp_4^+(\mathbb{Q})$, we can find $\gamma_1, \gamma_2 \in \Gamma_2$ and a diagonal matrix $D \in GSp_4^+(\mathbb{Q})$ such that $g = \gamma_1 D \gamma_2$. Since $\gamma\hat{\mathbb{Z}}^4 = \hat{\mathbb{Z}}^4$ for any $\gamma \in \Gamma_2$ and the Haar measure is translation invariant we conclude that $\mu_2(gB) = \lambda^{-2}(g)\mu_2(B)$ for any Borel subset of $MSp_4(\mathbb{A}_{\mathbb{Q},f})$.

One would expect to use a similar construction for $\beta = 1$ and $\beta = 3$. However, since the only subspace of $\mathbb{A}^4_{\mathbb{Q},f}$ stable under the action of $GSp_4^+(\mathbb{Q})$ is $\mathbb{A}^4_{\mathbb{Q},f}$ itself, this argument fails in the case $\beta = 1$ or $\beta = 3$. We conjecture that the GSp_4-system does not admit any KMS$_\beta$ state in these two cases.

Remark 3. The results we prove in this paper completely classify the KMS$_\beta$ states on the Bost-Connes-Marcolli GSp_4-system. In fact, we will show that given $\beta > 0$ with $\beta \notin \{1, 2, 3\}$, there exists a one-to-one correspondence between KMS$_\beta$ on the Connes-Marcolli GSp_4-system and the GSp_4 system $(\mathcal{A}, \sigma_t)$. Recall the set

$$F_Y = \{h \in MSp_4(\hat{\mathbb{Z}}) \mid \text{rank}_{\mathbb{Q}_p}(h_p) \leq 2 \ \text{ for all } \ p \in \mathcal{P}\},$$

and consider the dynamical system $I = C_r^*(\Gamma_2\backslash GSp_4^+(\mathbb{Q}) \boxtimes_{\Gamma_2} (\mathbb{H}_2^+ \times F_Y))$. We claim that I can not have any KMS$_\beta$ states. To see this, observe that any KMS_β state on I gives rise to a regular Γ_2-invariant measure μ_β on the space $\mathbb{H}_2^+ \times F_Y$ (note that unlike the case where the underlying space is an r-discreet principal groupoid, the support of this measure is not necessarily contained in $\mathbb{H}_2^+ \times F_Y$.) By the KMS$_\beta$ condition, this measure still satisfies the scaling property 3.1.6. Now since $\mathbb{H}_2^+ = (\mathbb{U}^2/\{\pm 1_4\})\backslash PGSp_4^+(\mathbb{R})$, we can define a measure on $PGSp_4^+(\mathbb{R}) \times F_Y$ by the formula

$$\int_{PGSp_4^+(\mathbb{R})\times F_Y} f(x) d\tilde{\mu}_\beta(x) = \int_{\mathbb{H}_2^+\times F_Y} \left(\int_{\mathbb{U}^2/\{\pm 1_4\}} f(xg)\, \mathrm{d}g \right) \mathrm{d}\mu_\beta(x).$$

The measure $\tilde{\mu}_\beta$ satisfies the condition 3.1.6. There is a canonical extension of this measure to a Γ_2-invariant measure $\mu_\beta \in \mathcal{E}_\beta$ on the space $PGp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_f)$. This leads to a contradiction since the set $PGSp_4^+(\mathbb{R}) \times F_Y$ has measure zero by Corollary 3.2.1. This shows that the set $\mathbb{H}_2^+ \times F_Y$ can be ignored in the analysis of KMS$_\beta$ states for $\beta > 0$ and $\beta \notin \{1, 2, 3\}$ and if we let $\tilde{Y} = Y\backslash F_Y$, it is clear that different KMS$_\beta$ state on $C_r^*(\Gamma_2\backslash GSp_4^+(\mathbb{Q}) \boxtimes_{\Gamma_2} \tilde{Y})$ give rise to different KMS$_\beta$ state on the GSp_4-system $(\mathcal{A}, \sigma_t)$.

Chapter 4

SYMMETRIES AND GALOIS ACTION OF THE SIEGEL MODULAR FIELD

In addition to their interesting thermodynamical behavior, BCM systems are closely related to the class field theory of the number field underlying their structure. This connection is made possible thanks to one important object: the symmetry group. In this chapter, we propose to study the symmetry group of the Connes-Marcolli GSp_4-system. The full symmetry group will take into account both actions by automorphisms and endomorphisms. This will allow us to construct an action of a subgroup of the Siegel modular field and derive an equality which intertwines the action by symmetries at zero temperature and the Galois action of a subgroup of the Siegel modular field on the values of the equilibrium states. Throughout this chapter, it will be necessary to work with the algebra $\mathcal{A} = C_r^*(\Gamma_2\backslash G \boxtimes_{\Gamma_2} Y)$, where $Y = \mathbb{H}_2^+ \times MSp_4(\hat{\mathbb{Z}})$.

4.1 Symmetry group of the Bost–Connes–Marcolli GSp₄-system

Recall that the space Y admits a right action by the profinite group $GSp_4(\hat{\mathbb{Z}})$ as follows:

$$h \cdot (\tau, \rho) := (\tau, \rho h), \quad h \in GSp_{2n}(\hat{\mathbb{Z}}). \tag{4.1}$$

Lemma 4.1.1. The action (4.1) induces an action of $GSp_4(\hat{\mathbb{Z}})$ by automorphisms on the Connes-Marcolli GSp_4-system $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}_+})$ as follows:

$$\alpha_h(f)(g, \tau, \rho) = f(g, \tau, \rho h), \quad \forall f \in \mathcal{A}, \quad \forall h \in GSp_4(\hat{\mathbb{Z}}).$$

Proof. Since the action commutes with the left action of $GSp_4^+(\mathbb{Q})$, we get

$$\alpha_h(f_1 * f_2) = \alpha_h(f_1) * \alpha_h(f_2) \quad f_1, f_2 \in \mathcal{A}.$$

Hence the action of $GSp_4(\hat{\mathbb{Z}})$ defines automorphisms of the algebra $\mathcal{A}$. This action is compatible with the time evolution since

$$\sigma_t\alpha_h(f)(g, \tau, \rho) = \lambda(g)^{it} f(g, \tau, \rho h) = \alpha_h\sigma_t(f)(g, \tau, \rho).$$

□

We denote by $G = GSp_4^+(\mathbb{Q})$, $X = \mathbb{H}_2^+ \times MSp_4(\mathbb{A}_{\mathbb{Q},f})$ and $\Gamma_2 = Sp_4(\mathbb{Z})$. Recall that the algebra $\mathcal{A}$ is the reduction $e\mathcal{B}e$ of the algebra $\mathcal{B} = C_r^*(\Gamma_2\backslash G \times_{\Gamma_2} X)$ by the idempotent multiplier e given by the $\Gamma_2 \times \Gamma_2$ invariant function

$$e(g,\tau,h) = \begin{cases} 1 & \text{if } (g,h) \in \Gamma_2 \times MSp_4(\hat{\mathbb{Z}}) \\ 0 & \text{otherwise.} \end{cases}$$

Let $h \in MSp_4(\hat{\mathbb{Z}}) \cap GSp_4(\mathbb{A}_{\mathbb{Q},f})$. Then $\lambda(h) \in \mathbb{A}_{\mathbb{Q},f}^\times$ can be uniquely written [KL06] as

$$\lambda(h) = \lambda_{\mathbb{Q}}(h)\lambda_{\hat{\mathbb{Z}}}(h), \quad \lambda_{\mathbb{Q}}(h) \in \mathbb{Q}_+^\times,\ \lambda_{\hat{\mathbb{Z}}} \in \hat{\mathbb{Z}}^\times. \tag{4.2}$$

In particular the maps $h \mapsto \lambda_{\mathbb{Q}}(h)$ and $h \mapsto \lambda_{\hat{\mathbb{Z}}}(h)$ are well defined. Hence consider the map $\eta : MSp_4(\hat{\mathbb{Z}}) \cap GSp_4(\mathbb{A}_{\mathbb{Q},f}) \to \{\gamma \in GSp_4(\mathbb{A}_{\mathbb{Q},f}), \gamma^{-1} \in MSp_4(\hat{\mathbb{Z}})\}$ defined by

$$\eta(h) := h\lambda_{\mathbb{Q}}(h)^{-1}. \tag{4.3}$$

Proposition 4.1.1. The map η in (4.3) is a semigroup homomorphism.

Proof. We first check that the map (4.3) is well defined. Let $h \in MSp_4(\hat{\mathbb{Z}}) \cap GSp_4(\mathbb{A}_{\mathbb{Q},f})$. Clearly $\eta(h) \in GSp_4(\mathbb{A}_{\mathbb{Q},f})$ so we need to show that $\eta(h)^{-1} \in MSp_4(\hat{\mathbb{Z}})$. From (3.9) we obtain that

$$\begin{aligned} \eta(h)^{-1} &= h^{-1}\lambda_{\mathbb{Q}}(h) \\ &= h^{-1}\lambda(h)\lambda_{\hat{\mathbb{Z}}}(h)^{-1} \\ &= \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}\lambda_{\hat{\mathbb{Z}}}(h)^{-1} \in MSp_4(\hat{\mathbb{Z}}), \quad h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \end{aligned}$$

Finally, the map η is a semigroup homomorphism by the unicity of the decomposition in (4.2). ◻

Lemma 4.1.2. The following action of $GSp_4(\mathbb{A}_{\mathbb{Q},f})$ on the algebra $\mathcal{B}$ defines an action by automorphisms:

$$\alpha_h(f)(g,\tau,\rho) = f(g,\tau,\rho h), \quad \forall f \in \mathcal{B}, \forall h \in GSp_4(\mathbb{A}_{\mathbb{Q},f}). \tag{4.4}$$

If $h \in \{\gamma \in GSp_4(\mathbb{A}_{\mathbb{Q},f}), \gamma^{-1} \in MSp_4(\hat{\mathbb{Z}})\}$, the restriction $\alpha_h|_{\mathcal{A}}$ defines an endomorphism of the algebra $\mathcal{A}$.

Proof. The proof that (4.4) defines an action by automorphisms is similar to Lemma 4.1.1. Note that the action in (4.4) extends naturally to an action on the algebra of multipliers. Hence for $h \in \{\gamma \in GSp_4(\mathbb{A}_{\mathbb{Q},f}), \gamma^{-1} \in MSp_4(\hat{\mathbb{Z}})\}$, we obtain

$$\begin{aligned}(\alpha_h(e)e)(g,\tau,\rho) &= \sum_{s \in \Gamma_2 \backslash GSp_4^+(\mathbb{Q})} \alpha_h(e)(gs^{-1}, s\cdot\tau, s\rho)e(s,\tau,\rho)\\ &= \alpha_h(e)(g,\tau,\rho)e(1,\tau,\rho)\\ &= \alpha_h(e)(g,\tau,\rho),\end{aligned}$$

where the last equality holds since $h^{-1} \in MSp_4(\hat{\mathbb{Z}})$. This shows that

$$\alpha_h(exe) = e\alpha_h(exe)e \in \mathcal{A}, \quad x \in \mathcal{B},$$

which completes the proof. □

We then obtain an action of $MSp_4(\hat{\mathbb{Z}}) \cap GSp_4(\mathbb{A}_{\mathbb{Q},f})$ on the algebra $\mathcal{A}$:

$$\alpha_h := \alpha_{\eta(h)}|_{\mathcal{A}}, \quad h \in MSp_4(\hat{\mathbb{Z}}) \cap GSp_4(\mathbb{A}_{\mathbb{Q},f})$$

and

$$\alpha_h(f)(g,\tau,\rho) = \begin{cases} f(g,\tau,\rho\eta(h)) & \text{if } (\rho\eta(h), g\rho\eta(h)) \in MSp_4(\hat{\mathbb{Z}}) \times MSp_4(\hat{\mathbb{Z}}) \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

Let G be the group of transformations of KMS$_\beta$ states on the system $(\mathcal{A}, (\sigma_t)_{t\in\mathbb{R}})$ and consider the semigroup homomorphism $\theta : MSp_4(\hat{\mathbb{Z}}) \cap GSp_4(\mathbb{A}_{\mathbb{Q},f}) \longrightarrow G$ given by $\theta(h) := \alpha_h$.

Proposition 4.1.2. For $h = \mathrm{diag}(n,n,n,n)$, $n \in \mathbb{N}$ we have that $\theta(h) = 1$.

Proof. Consider the multiplier element given by

$$\mu_h(g,\tau,\rho) = \begin{cases} 1 & \text{if } g \in \Gamma_2 h \\ 0 & \text{otherwise.} \end{cases}$$

and observe that

$$\mu_n^*(g,\tau,\rho) = \begin{cases} 1 & \text{if } g \in \Gamma_2 h^{-1} \text{ and } h^{-1}\rho \in MSp_4(\hat{\mathbb{Z}}) \\ 0 & \text{otherwise.} \end{cases}$$

For any $f \in \mathcal{A}$ we compute

$$\begin{aligned} \mathrm{ad}(\mu_h)(f)(g,\tau,\rho) &= \sum_{s \in \Gamma_2 \backslash GSp_4^+(\mathbb{Q})} \mu_h(gs^{-1}, s\cdot\tau, s\rho)(f * \mu_h^*)(s,\tau,\rho) \\ &= \sum_{s \in \Gamma_2 \backslash GSp_4^+(\mathbb{Q})} f(h^{-1}gs^{-1}, s\cdot\tau, s\rho)\mu_h^*(s,\tau,\rho) \\ &= f(g,\tau,\rho h^{-1}) \\ &= \alpha_h(f)(g,\tau,\rho), \end{aligned}$$

since we can assume $(\rho h^{-1}, g\rho h^{-1}) \in MSp_4(\hat{\mathbb{Z}}) \times MSp_4(\hat{\mathbb{Z}})$. A direct computation shows that

$$f * (\mu_h^* * \mu) = f, \quad \forall f \in \mathcal{A},$$

and that μ_h is an eigenvector of the time evolution. Hence the action of α_h on the set of KMS states is trivial. □

We now combine the previous results to obtain the following important theorem.

Theorem 4.1.1. *There exists a unique homomorphism $\tilde{\theta} : \mathbb{Q}^\times \backslash GSp_4(\mathbb{A}_{\mathbb{Q},f}) \longrightarrow G$ such that the diagram*

$$\begin{tikzcd} MSp_4(\hat{\mathbb{Z}}) \cap GSp_4(\mathbb{A}_{\mathbb{Q},f}) \arrow[r, "\theta"] \arrow[d, "\pi"] & G \\ \mathbb{Q}^\times \backslash GSp_4(\mathbb{A}_{\mathbb{Q},f}) \arrow[ur, "\tilde{\theta}"'] & \end{tikzcd}$$

commutes. In particular, we obtain an action of the group

$$S = \mathbb{Q}^\times \backslash GSp_4(\mathbb{A}_{\mathbb{Q},f})$$

by symmetries on the set of KMS$_\beta$ states of the GSp_4-Connes-Marcolli system.

Proof. By the decomposition in Corollary 3.1.1 and clearing the denominators if necessary we see that the projection π is a surjective homomorphism. Hence any map $\tilde{\theta}$ such that $\tilde{\theta}(\pi(x)) := \theta(x)$ must be unique.

For $x, y \in MSp_4(\hat{\mathbb{Z}}) \cap GSp_4(\mathbb{A}_{\mathbb{Q},f})$, one has

$$\begin{aligned}\pi(x) = \pi(y) &\Leftrightarrow \exists q_1, q_2 \in \mathbb{Q}^\times \text{ such that } \operatorname{diag}(q_1, \ldots, q_1)x = \operatorname{diag}(q_2, \ldots, q_2)y \\ &\Leftrightarrow \exists n_1, n_2 \in \mathbb{N} \text{ such that } \operatorname{diag}(n_1, \ldots, n_1)x = \operatorname{diag}(n_2, \ldots, n_2)y.\end{aligned}$$

By Proposition 4.1.2 we conclude that $\theta(x) = \theta(y)$ and thus the homomorphism $\tilde{\theta}(y) := \theta(x)$ where $y = \pi(x)$ is well defined and satisfies the desired property. $\square$

Proposition 4.1.3. Let $4 < \beta < \infty$ and consider the extremal KMS_β state ϕ_{β,y_1} associated to $y_1 = (\tau_1, \rho_1) \in \mathbb{H}_2^+ \times GSp_4(\hat{\mathbb{Z}})$. Given $h_1 \in MSp_4^+(\mathbb{Z})$, we write

$$\rho_1 h_1 = h_2 \rho_2 \in GSp_4^+(\mathbb{Q}) GSp_4(\hat{\mathbb{Z}}),$$

and put

$$\tau_2 := h_2^{-1} \cdot \tau_1.$$

Then $h_2 \in MSp_4^+(\mathbb{Z})$ and

$$\phi_{\beta,y_1} \circ \alpha_{h_1} = \lambda(h_1)^{-\beta} \phi_{\beta,y_2}, \tag{4.6}$$

where $y_2 = (\tau_2, \rho_2) \in \mathbb{H}_2^+ \times GSp_4(\hat{\mathbb{Z}})$.

Proof. First observe that $h_2 \in MSp_4^+(\mathbb{Z})$ since $\rho_1 h_1 \rho_2^{-1} \in GSp_4^+(\mathbb{Q}) \cap MSp_4(\hat{\mathbb{Z}}) = MSp_4^+(\mathbb{Z})$. By (4.1) we get

$$\phi_{\beta,y}(\alpha_{h_1}(f)) = \zeta_{MSp_4^+(\mathbb{Z}),\Gamma_2}(\beta)^{-1} \sum_{\substack{h \in \Gamma_2 \backslash MSp_4^+(\mathbb{Z}) \\ h\rho_1\eta(h_1) \in MSp_4(\hat{\mathbb{Z}})}} \lambda(h)^{-\beta} f(1, h \cdot \tau_1, h\rho_1\eta(h_1)). \tag{4.7}$$

Note that $\lambda(h_1) = \lambda(h_2)$ so that

$$h\rho_1\eta(h_1) \in MSp_4(\hat{\mathbb{Z}}) \Leftrightarrow h\eta(h_2)\rho_2 \in MSp_4(\hat{\mathbb{Z}}) \Leftrightarrow h \in MSp_4(\hat{\mathbb{Z}})\eta(h_2)^{-1}.$$

If we write $h = x\eta(h_2)^{-1} \in MSp_4(\mathbb{Z})^+$ with $x \in MSp_4(\hat{\mathbb{Z}})$ then

$$x \in MSp_4(\hat{\mathbb{Z}}) \cap GSp_4^+(\mathbb{Q}) = MSp_4^+(\mathbb{Z}),$$

so $h \in MSp_4^+(\mathbb{Z})\eta(h_2)^{-1}$. Conversely if $h \in MSp_4^+(\mathbb{Z})\eta(h_2)^{-1}$ then $h \in MSp_4^+(\mathbb{Z})$ since $\eta(h_2)^{-1} \in MSp_4(\hat{\mathbb{Z}}) \cap GSp_4^+(\mathbb{Q})$ .We can thus replace the right hand side of (4.7) by the sum

$$\begin{aligned}\phi_{\beta,y}(\alpha_{h_1}(f)) =& \zeta_{MSp_4^+(\mathbb{Z}),\Gamma_2}(\beta)^{-1} \\ & \sum_{h' \in \Gamma_2 \backslash MSp_4^+(\mathbb{Z})} \lambda(h'\eta(h_2)^{-1})^{-\beta} f(1, h'h_2^{-1}\lambda(h_2)\cdot\tau_1, h'h_2^{-1}\rho_1 h_1) \\ =& \lambda(h_1)^{-\beta}\phi_{\beta,y_2},\end{aligned}$$

since $h_2^{-1}\lambda(h_2)\cdot\tau_1 = \tau_2$ (scalar matrices act trivially on $\mathbb{H}_2^+$) and $h_2^{-1}\rho_1 h_1 = \rho_2$. $\square$

4.2 Siegel modular forms

Let $\Gamma_n(N)$ be the arithmetic subgroup of $Sp_{2n}(\mathbb{Z})$ defined in (3.23). A function

$$f : \mathbb{H}_n^+ \to \mathbb{C}$$

is called a *Siegel modular form* of degree (genus) n, weight $k \in \mathbb{N}$ and level N if it is holomorphic[1] and satisfies the following condition:

$$f(g\cdot\tau) = \det(C\tau + D)^k f(\tau), \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n(N), \quad \tau \in \mathbb{H}_n^+.$$

Every Siegel modular form f admits a Fourier expansion (or q-expansion) of the form

$$f(\tau) = \sum_g a_g q^g, \quad a_g \in \mathbb{C}, \quad q^g := \exp(2\pi i \mathrm{Tr}(g\cdot\tau)/N), \tag{4.8}$$

where g runs over all $n \times n$ positive semi-definite symmetric matrices over half integers with integral diagonal entries. The Fourier series is absolutely and uniformly convergent on any compact subset of $\mathbb{H}_n^+$.

[1] One has to require that the function f is holomorphic at the cusps. This condition is automatically satisfied for $n \geq 2$ by Koecher principle [Kli90, Chapter 4, Theorem 1].

We denote by $M_k(\Gamma_n(N))$ the set of Siegel modular forms of weight k and level N. Let $\mathcal{F}_N$ be the field

$$\mathcal{F}_N := \Big\{ \frac{f_1}{f_2} \mid f_1, f_2 \in M_k(\Gamma_n(N)) \text{ for some } k \in \mathbb{N}, \\ \text{with } q\text{-expansion coefficients in } \mathbb{Q}(\zeta_N) \Big\},$$

and

$$\mathcal{F} := \bigcup_{N=1}^{\infty} \mathcal{F}_N.$$

Proposition 4.2.1. There exists a homomorphism

$$\vartheta : GSp_{2n}(\mathbb{A}_{\mathbb{Q},f}) \longrightarrow \mathrm{Aut}(\mathcal{F}),$$

satisfying the following properties:

1. If $f \in \mathcal{F}$ and $g \in GSp^+_{2n}(\mathbb{Q})$, then
$$f^{\vartheta(g)}(\tau) = f(g \cdot \tau), \quad \forall \tau \in \mathbb{H}^+_n. \tag{4.9}$$

2. The sequence
$$1 \longrightarrow \mathbb{Q}^\times \xrightarrow{\iota} GSp_{2n}(\mathbb{A}_{\mathbb{Q},f}) \xrightarrow{\vartheta} \mathrm{Aut}(\mathcal{F}),$$
is exact.

Proof. See [Shi00, Theorem 8.10] □

We define the following left action of $GSp_{2n}(\mathbb{A}_{\mathbb{Q}})$ on $\mathcal{F}$:

$$\mathrm{Aut}(\alpha) \cdot f := f^{\vartheta(\alpha^{-1})}. \tag{4.10}$$

It is known [KSY19] that $\mathcal{F}_N$ is a finite Galois extension over $\mathcal{F}_1$ with

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq GSp_{2n}(\mathbb{Z}/N\mathbb{Z})/(\pm 1_{2n}).$$

Given an even integer k, let

$$E^n_k(\tau) = \sum_{(C,D)} \det(C\tau + D)^k, \quad \tau \in \mathbb{H}^+_n \tag{4.11}$$

be the Siegel Eisenstein series of degree n and weight k. Here the sum is over elements of the form $\begin{pmatrix} * & * \\ C & D \end{pmatrix}$ over a complete set of representatives of $\left\{\begin{pmatrix} * & * \\ 0_n & * \end{pmatrix}\right\}\backslash Sp_{2n}(\mathbb{Z})$. The right-hand side of (4.11) converges absolutely and locally uniformly for $k > n + 1$.

We say that a point $\tau \in \mathbb{H}_n^+$ is a generic point if the evaluation map

$$I_\tau : \mathcal{F} \to \mathbb{C}$$

is an embedding. In this case we have

$$\mathrm{Aut}_\tau(g) \cdot I_\tau(h) = I_\tau(\mathrm{Aut}(g) \cdot h), \quad \forall g \in GSp_{2n}(\mathbb{A}_\mathbb{Q}), \forall h \in \mathcal{F}. \tag{4.12}$$

4.3 The arithmetic subalgebra

A function $f \in C(\Gamma_2\backslash G \boxtimes_{\Gamma_2} Y)$ is called *arithmetic* if the following conditions are satisfied:

1. f has finite support in the variable $g \in GSp_{2n}^+(\mathbb{Q})$
2. The functions $f_{g,\rho}(\tau) := f(g, \tau, \rho)$ fulfill
$$f_{g,\rho} \in \mathcal{F}, \quad \forall (g, \rho) \in GSp_4^+(\mathbb{Q}) \times MSp_4(\hat{\mathbb{Z}}).$$
3. If $g\rho_1 = \rho_2 g_2$, with $g_1, g_2 \in GSp_{2n}^+(\mathbb{Q})$ and $\rho_1, \rho_2 \in GSp_{2n}(\hat{\mathbb{Z}})$, then
$$f_{g,\rho_1\rho} = \mathrm{Aut}(\rho_1) \cdot f_{g_2,\rho}, \quad \rho \in MSp_4(\hat{\mathbb{Z}}). \tag{4.13}$$

We denote by $\mathcal{A}_\mathbb{Q}^{\mathrm{arith}}$ the linear space of arithmetic functions in $C(\Gamma_2\backslash G \boxtimes_{\Gamma_2} Y)$.

Proposition 4.3.1. We have that $\mathcal{A}_\mathbb{Q}^{\mathrm{arith}}$ is a subalgebra of the algebra of unbounded multipliers of the C^*-algebra $\mathcal{A}$.

Proof. It is enough to show that the condition (4.13) is stable under convolution. Consider $f_1, f_2 \in \mathcal{A}_\mathbb{Q}^{\mathrm{arith}}$. By Proposition 4.2.1 we obtain that

$$(f_1 * f_2)_{g,\rho} = \sum_{\substack{s\in\Gamma_2\backslash GSp_4^+(\mathbb{Q}) \\ s\rho\in MSp_4(\hat{\mathbb{Z}})}} \mathrm{Aut}(s^{-1}) \cdot (f_1)_{gs^{-1},s\rho} \ (f_2)_{s,\rho}.$$

Let $\alpha \in GSp_4(\hat{\mathbb{Z}})$ and write $g\alpha = \tilde{\alpha}\tilde{g}$ and $s\alpha = \alpha' s'$ where $\tilde{\alpha}, \alpha' \in GSp_4(\hat{\mathbb{Z}})$ and $\tilde{g}, s' \in GSp_4^+(\mathbb{Q})$. Using the condition (4.13), we need to show that

$$\mathrm{Aut}(\alpha'^{-1}) \cdot (f_1)_{gs^{-1},s\alpha\rho} = (f_1)_{\tilde{g}s'^{-1},s'\rho}.$$

This follows again from condition (4.13) since $\tilde{g}s'^{-1}\alpha'^{-1} = \tilde{g}\alpha^{-1}s^{-1} = \tilde{\alpha}^{-1}gs^{-1}$.

∎

4.4 KMS states at zero temperature

We are now ready to prove the main theorem of this chapter.

Theorem 4.4.1. *Let $y = (\tau,\rho) \in Y$ such that $\rho \in GSp_4(\hat{\mathbb{Z}})$ and τ is a generic point in $\mathbb{H}_2^+$. Then the homomorphism in (2) induces an isomorphism $\vartheta^{-1} : S' \to S$ between a subgroup S' of $Aut(\mathcal{F})$ and the symmetry group S which intertwines the Galois action on the values of the states with the action by symmetries:*

$$\alpha^*_{\vartheta^{-1}(g)}(\phi_{\infty,y})(f) = Aut_\tau(\rho\vartheta^{-1}(g)\rho^{-1}) \cdot \phi_{\infty,y}(f), \quad \forall f \in \mathcal{A}^{arith}_{\mathbb{Q}}, \ \forall g \in S' \quad (4.14)$$

Proof. The first assertion follows from the exactness of the sequence in (2) and

$$S' := \mathrm{Img}(\theta) \simeq S.$$

Suppose first that $\vartheta^{-1}(g)$ is an action by automorphisms, i.e., $\vartheta^{-1}(g) \in GSp_4(\hat{\mathbb{Z}})$. Then

$$\alpha^*_{\vartheta^{-1}(g)}(\phi_{\infty,y})(f) = I_\tau(f_{1,\rho\vartheta^{-1}(g)}) \quad \forall f \in \mathcal{A}^{arith}.$$

Since $f \in \mathcal{A}^{arith}$ and τ is regular, we obtain by (4.12) and property (4.13) that

$$\begin{aligned}\alpha^*_{\vartheta^{-1}(g)}(\phi_{\infty,y})(f) &= I_\tau(\mathrm{Aut}(\rho\vartheta^{-1}(g)\rho^{-1}) \cdot f_{1,\rho}) \\ &= \mathrm{Aut}_\tau(\rho\vartheta^{-1}(g)\rho^{-1}) \cdot I_\tau(f_{1,\rho}) \\ &= \mathrm{Aut}_\tau(\rho\vartheta^{-1}(g)\rho^{-1}) \cdot \phi_{\infty,y}(f).\end{aligned}$$

Consider now the case where $\vartheta^{-1}(g)$ acts by endomorphisms. Since $\rho \in GSp_4(\hat{\mathbb{Z}})$ and $h = \mathrm{diag}(n,n,n,n) \in \ker(\theta)$, we can always find $n \in \mathbb{N}$ large enough so that the action of $\alpha_{\vartheta^{-1}(g)} = \alpha_{h\vartheta^{-1}(g)}$ by pullback on zero-temperature states vanishes identically. To obtain a nontrivial action, we use the warming-up/cooling down process described in [CM04]. Hence the action of $\alpha_{\vartheta^{-1}(g)}$ is obtained by taking the weak limit

$$\alpha^*_{\vartheta^{-1}(g)}(\phi_{\infty,y}) = \lim_{\beta\to+\infty} \alpha^*_{\vartheta^{-1}(g)}(\phi_{\beta,y}).$$

Up to multiplication by $h = \mathrm{diag}(n,n,n,n) \in \ker(\theta)$ for $n \in \mathbb{N}$ large enough we can assume that $\vartheta^{-1}(g) \in MSp_4^+(\mathbb{Z})$. Given any $\beta > 4$ we can now compute the normalized pullback (cf. Equation (2.10)) as follows

$$\alpha^*_{\vartheta^{-1}(g)}(\phi_{\beta,y})(f) = \frac{(\phi_{\beta,y} \circ \alpha_{\vartheta^{-1}(g)})(f)}{\phi_{\beta,y}(e_{\vartheta^{-1}(g)})}, \quad f \in \mathcal{A}^{arith},$$

where $e_{\vartheta^{-1}(g)}$ is the characteristic function of the set $MSp_4(\hat{\mathbb{Z}})\eta^{-1}(\vartheta^{-1}(g))$. We write

$$\rho\vartheta^{-1}(g) = h_2\rho_2 \in GSp_4^+(\mathbb{Q})GSp_4(\hat{\mathbb{Z}}).$$

Since $\lambda(h_2) = \lambda(\vartheta^{-1}(g))$ we obtain

$$\begin{aligned}
\phi_{\beta,y}(e_{\vartheta^{-1}(g)}) &= \zeta_{MSp_4^+(\mathbb{Z}),\Gamma_2}(\beta)^{-1} \sum_{\substack{h\in\Gamma_2\backslash MSp_4^+(\mathbb{Z}) \\ h\rho\eta(\vartheta^{-1}(g))\in MSp_4(\hat{\mathbb{Z}})}} \lambda(h)^{-\beta} \\
&= \zeta_{MSp_4^+(\mathbb{Z}),\Gamma_2}(\beta)^{-1} \sum_{h'\in\Gamma_2\backslash MSp_4^+(\mathbb{Z})} \lambda((h'\eta(h_2)))^{-\beta} \\
&= \lambda(\vartheta^{-1}(g))^{-\beta}.
\end{aligned}$$

Hence applying Proposition 4.1.3 we get

$$\alpha^*_{\vartheta^{-1}(g)}(\phi_{\infty,y})(f) = \lim_{\beta\to+\infty} \phi_{\beta,y_2}(f) = \phi_{\infty,y_2}(f), \quad y_2 = (\tau_2, \rho_2),$$

where $\tau_2 = h_2^{-1} \cdot \tau$.

Letting $g_1 = g_2 = 1 \in GSp_4^+(\mathbb{Q})$, $\rho = 1 \in MSp_4(\hat{\mathbb{Z}})$ and $\rho_1 = \rho_2$ in (4.13), we get

$$\phi_{\infty,y_2}(f) = I_{\tau_2}(f_{1,\rho_2}) = I_{\tau_2}(\mathrm{Aut}(\rho_2) \cdot f_{1,1}) = I_\tau(\mathrm{Aut}(h_2\rho_2) \cdot f_{1,1}),$$

where the last equality follows from Proposition (4.9). Using again property (4.13) and the regularity of the point $\tau \in \mathbb{H}_2^+$, we have

$$\begin{aligned}
\alpha^*_{\vartheta^{-1}(g)}(\phi_{\infty,y})(f) &= I_\tau(\mathrm{Aut}(\rho\vartheta^{-1}(g)\rho^{-1})\mathrm{Aut}(\rho)\cdot f_{1,1}) \\
&= \mathrm{Aut}_\tau(\rho\vartheta^{-1}(g)\rho^{-1})\cdot I_\tau(\mathrm{Aut}(\rho)\cdot f_{1,1}) \\
&= \mathrm{Aut}_\tau(\rho\vartheta^{-1}(g)\rho^{-1})\cdot I_\tau(f_{1,\rho}),
\end{aligned}$$

which concludes the proof since $I_\tau(f_{1,\rho}) = \phi_{\infty,y}(f)$. □

Chapter 5

VON NEUMANN ALGEBRAS ARISING FROM THE BOST–CONNES–MARCOLLI GSP$_4$-SYSTEM

We have seen that the Connes-Marcolli GSp_4-system undergoes a phase transition at the critical inverse temperatures $\beta_{c_1} = 3$ and $\beta_{c_2} = 4$. In this chapter, we will study the structure of the von Neumann algebras generated by the KMS$_\beta$ states and show that a phase transition happens at the level of these algebras as well.

We start by fixing some terminology and notations. The type of a state ϕ on the algebra $\mathcal{A}$ corresponds to the type of the von Neumann algebra $\pi_\phi(\mathcal{A})''$ it generates in the GNS representation. For the rest of this chapter, we set

$$\begin{aligned}&G = GSp_4^+(\mathbb{Q}), \quad \Gamma_2 = Sp_4(\mathbb{Z}),\\ &X = PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_{f,\mathbb{Q}}), \quad Y = PGSp_4^+(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}}).\end{aligned}$$

We denote as usual by $\mathcal{P}$ the set of prime numbers and for any finite set of primes $F \subset \mathcal{P}$, we put

$$\mathbb{Q}_F = \prod_{p\in F} \mathbb{Q}_p, \quad \mathbb{Z}_F = \prod_{p\in F} \mathbb{Z}_p,$$

and

$$X_F = PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Q}_F), \quad Y_F = PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Z}_F).$$

We denote by π_F the factor map $X \longrightarrow X_F$. Given a function f on X_F, we define the function f_F on X by

$$f_F(x) = \begin{cases} f(\pi_F(x)) & \text{if } x_p \in MSp_4(\mathbb{Z}_p) \text{ for all } p \in F^c,\\ 0 & \text{otherwise.}\end{cases}$$

Given a prime $p \in \mathcal{P}$, recall from the proof of Lemma 3.2.3 that

$$\{g \in MSp_4(\mathbb{Z}) : |\lambda(g)| = p\} = \Gamma_2 g_{1,p} \Gamma_2,$$

and

$$\deg_{\Gamma_2}(g_{1,p}) = (1+p)(1+p^2).$$

We put

$$A_p := \{(\tau, x) \in PGSp_4^+(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}}) \text{ such that } x_p \in GSp_4(\mathbb{Z}_p)\}, \tag{5.1}$$
$$B_p := \{(\tau, x) \in PGSp_4^+(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}}) \text{ such that } |\lambda(x)|_p = p^{-1}\}. \tag{5.2}$$

5.1 Ratio set of group actions

Consider the action of the countable group G on the measure space $(X, \mathcal{F}, \mu)$. We recall the following definition from [Kri06].

Definition 5.1.1. The ratio set $r(G)$ of the action of G on $(X, \mathcal{F}, \mu)$ consists of all real numbers $\lambda \geq 0$ such that for every $\epsilon > 0$ and any $A \in \mathcal{F}$ of positive measure, there exists $g \in G$ such that

$$\mu\left(\left\{x \in gA \cap A : \left|\frac{dg_*\mu}{\mathrm{d}\mu}(x) - \lambda\right| < \epsilon\right\}\right) > 0,$$

where the measure $g_*\mu$ is defined by $g_*\mu(B) = \mu(g^{-1}(B))$.

The ratio set depends only on the equivalence relation $\mathcal{R} = \{(x, gx) \mid x \in X, g \in G\} \subset X \times X$ and the measure class of μ (hence will denote the ratio by $r(\mathcal{R}, \mu)$). Moreover one can show that the set $r(\mathcal{R}, \mu)^* := r(\mathcal{R}, \mu) \cap (0, \infty)$ is a closed subgroup of R_+^*. We then have the following result (cf. [Sun87, Proposition 4.3.18]).

Theorem 5.1.1. *Let G be a countable group G acting by automorphisms on a measure space $(X, \mathcal{F}, \mu)$. Assume that the action of G on $(X, \mathcal{F}, \mu)$ is free and ergodic. Then $L^\infty(X, \mathcal{F}, \mu) \rtimes G$ is a factor of type III_1 if and only $r(\mathcal{R}, \mu)^* = R_+^*$.*

This result motivates the following definition.

Definition 5.1.2. The action of G on the measure space $(X, \mathcal{F}, \mu)$ is said to be of type III_1 if $r(\mathcal{R}, \mu) \cap (0, \infty) = \mathbb{R}_+^*$.

5.2 Type I_∞ factors and Gibbs states

Theorem 5.2.1. *Let $y \in PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})$ and $\beta > 4$. Then the KMS$_\beta$ state given by*

$$\phi_{\beta,y}(f) = \frac{\zeta(2\beta-2)Tr(\pi_y(f)e^{-\beta H_y})}{\zeta(\beta)\zeta(\beta-1)\zeta(\beta-2)\zeta(\beta-3)} \ \ for\ \ f \in \mathcal{A}$$

is extremal of type of type I_∞.

Proof. We have seen that the zeta function $\zeta_{\mathcal{D},\mathcal{L}}$ associated to the Connes-Marcolli GSp4-system is given in equation (3.43). Since it converges for $\beta > 4$, the result follows from [HP05, Proposition 6.5]. We reproduce the proof for the reader's convenience. We need to show that the algebra $\mathcal{A}$ associated to the Connes-Marcolli GSp4-system generates a factor in the GNS representation of the state $\phi_{\beta,y}$. Consider the following representation of $\mathcal{A}$ defined by

$$\begin{aligned} \tilde{\pi}_y : \mathcal{A} &\longrightarrow \mathcal{B}(\mathcal{H}_y \otimes \mathcal{H}_y) \\ a &\mapsto \pi_y(a) \otimes id_{\mathcal{H}_y} \end{aligned}$$

and denote by $\Omega_{\beta,y}$ the unitary vector given by

$$\Omega_{\beta,y} = \zeta_{MSp_4(\mathbb{Z}),\Gamma_2}(\beta)^{-1/2} \sum_{h\in\Gamma_2\backslash G_y} \lambda(h)^{-\beta/2}\delta_{\Gamma_2 h} \otimes \delta_{\Gamma_2 h}.$$

A direct computation shows that

$$\phi_{\beta,y} = \langle \tilde{\pi}_y(f)\Omega_{\beta,y}, \Omega_{\beta,y}\rangle, \quad \forall f \in \mathcal{A}.$$

and

$$\tilde{\pi}_y(f)\Omega_{\beta,y} = \zeta_{MSp_4(\mathbb{Z}),\Gamma_2}(\beta)^{-1/2} \sum_{g,h\in\Gamma_2\backslash G_y} \lambda(h)^{-\beta/2} f(gh^{-1}, hy)\delta_{\Gamma_2 g} \otimes \delta_{\Gamma_2 h}.$$

By choosing f with a sufficiently small support, we see that $\tilde{\pi}_y(\mathcal{A})\Omega_{\beta,y}$ is dense in $\mathcal{H}_y \otimes \mathcal{H}_y$. This shows that the GNS representation is equivalent to the triple $(\mathcal{H}_y \otimes \mathcal{H}_y, \tilde{\pi}_y, \Omega_{\beta,y})$. By [Con79, Proposition VII.5 b)] the commutant of $\pi_y(\mathcal{A})$ is generated by the right regular representation of the isotropy group $\mathcal{G}_y^y$ of the groupoid $\mathcal{G} = \Gamma_2\backslash(G \boxtimes Y)$. Since $y \in PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})$, the isotropy group $\mathcal{G}_y^y$ is trivial which implies that $\pi_y(\mathcal{A})' = \mathbb{C}$. Hence

$$\begin{aligned}\tilde{\pi}_y(\mathcal{A})'' &= (\pi_y(\mathcal{A})' \otimes B(\mathcal{H}_y))' \\ &= B(\mathcal{H}_y) \otimes \mathbb{C} \\ &\simeq B(\mathcal{H}_y)\end{aligned}$$

This shows that $\phi_{\beta,y}$ is an extremal state of type I_∞. □

5.3 Type III$_1$ factors

Our next goal is to study the type of the von Neumann algebra generated by the unique KMS$_\beta$ state ϕ_β on the GSp_4-system in the range $3 < \beta \leq 4$. For $\beta > 4$, it was possible to proceed with a constructive approach and compute the type of any Gibbs state by finding an explicit formula for the GNS representation (which is unique up to unitary equivalence). For $3 < \beta \leq 4$, we use a different strategy by extending the approach in [BC95] and [Nes11].

For $3 < \beta \leq 4$, consider the unique KMS state ϕ_β on the Connes-Marcolli GSp_4-system and denote by μ_β the corresponding Γ_2-invariant measure on X given in Proposition 3.1.6. We choose a μ_β-measurable fundamental domain F for the action of Γ_2 on Y. Then (See [FM77] and [LLN07, Remark 2.3]) the algebra $\pi_{\phi_\beta}(\mathcal{A})''$ (recall that $\mathcal{A} = C_r^*(\Gamma_2\backslash G \boxtimes_{\Gamma_2} Y)$) induced by the state ϕ_β is isomorphic to the reduction of the von Neumann algebra of the G-orbit equivalence relation on (X, μ_β) by the projection $\mathbb{1}_F$, that is

$$\pi_{\phi_\beta}(\mathcal{A})'' \simeq \mathbb{1}_F(L^\infty(X, \mu_\beta) \rtimes G)\mathbb{1}_F. \tag{5.3}$$

Denote by λ_∞ the Lebesgue measure on $\mathbb{R}$. We have three commuting actions of G, $\mathbb{R}$ and $GSp_4(\hat{\mathbb{Z}})$ on the space $(\mathbb{R}_+ \times X, \lambda_\infty \times \mu_\beta)$ as follows:

$$\begin{aligned} &g(t,x) = \left(\frac{dg_*\mu}{d\mu}(gx)\, t, gx\right), \quad s(t,x) = (e^{-s}t, x) && \forall g \in G, \forall s \in \mathbb{R}, \\ &h(t,x) = (t, xh) && \forall h \in GSp_4(\hat{\mathbb{Z}}).\end{aligned}$$

The following results will be useful.

Proposition 5.3.1. If the action of G on $(X/GSp_4(\hat{\mathbb{Z}}), \mu_\beta)$ is of type III$_1$ then the action of G on $(\mathbb{R}_+ \times X, \lambda_\infty \times \mu)$ is ergodic.

Proof. Theorem 3.2.4 together with the assumption imply that

$$L^\infty(X,\mu_\beta)^G = \mathbb{C}, \quad L^\infty(\mathbb{R}_+ \times X/GSp_4(\hat{\mathbb{Z}}),\mu_\beta)^G = \mathbb{C}.$$

The result follows from [LLN07, Proposition 4.6] since the actions of G, $\mathbb{R}$ and $GSp_4(\hat{\mathbb{Z}})$ on the space $(\mathbb{R}_+ \times X, \lambda_\infty \times \mu_\beta)$ commute, $GSp_4(\hat{\mathbb{Z}})$ is profinite and $\mathbb{R}$ is connected. □

Lemma 5.3.1. Given $3 < \beta \le 4$ and $\omega > 1$, there exist two sequences of distinct primes $\{p_n\}_{n\ge1}$ and $\{q\}_{n\ge1}$ such that

$$\lim_n \frac{q_n^\beta}{p_n^\beta} = \omega, \quad \text{and} \quad \sum_n \frac{1}{p_n^{\beta-3}} = \sum_n \frac{1}{q_n^{\beta-3}} = \infty.$$

Proof. This result follows from the proof of [BZ00, Theorem 2.9]. □

Lemma 5.3.2. Let $3 < \beta \le 4$, $p \in \mathcal{P}$ a prime number and $g_{1,p}$ as in Lemma 3.2.3. Then for the operator $m(A_p)T_{g_{1,p}}m(B_p)$ acting on the space $L^2(\Gamma\backslash X,\nu_\beta)$ we have that

$$\left\| m(A_p)T_{g_{1,p}}m(B_p) \right\| \le \nu_\beta(\Gamma_2\backslash B_p)^{-1/2}.$$

Proof. Combining the factorization in Corollary 3.1.1 and equation (3.32), we have that for any $g \in B_p$ we can find $\gamma_1,\gamma_2 \in Sp_4(\mathbb{Z})$ and $g' \in A_p$ such that $g = \gamma_1 g_{1,p} g' \gamma_2$, in other words we have that $B_p = \Gamma_2 g_{1,p} A_p$. Next recall from Lemma 3.2.3 we have that $\deg_{\Gamma_2}(g_{1,p}) = (1+p)(1+p^2)$. We then fix representatives $\{h_i\}_{1\le i\le(1+p)(1+p^2)}$ of $\Gamma_2\backslash\Gamma_2 g_{1,p}\Gamma_2$ and choose a fundamental domain U for the action of the discreet group Γ_2 on A_p. We claim that the sets $\Gamma_2 h_i U \cap \Gamma_2 h_j U = \emptyset$ for $i \ne j$ and the projection map $\pi : X \longrightarrow \Gamma_2\backslash X$ is injective on the sets $h_i U$. Indeed, if $h_j^{-1}\gamma h_i x_1 = x_2$, for some $\gamma \in \Gamma_2$ and $x_1, x_2 \in A_p$, then necessarily $h_j^{-1}\gamma h_i \in GSp_4(\mathbb{Z}_p) \cap G_p = \Gamma_2$ as shown in the proof of Theorem 3.2.1. Since π is injective on U, we obtain that $x_1 = x_2$ and since the action of Γ_2 on A_p is free, it follows that $i = j$. Given any $f \in L^2(\Gamma_2\backslash X,\nu_\beta)$, we have that $\left|T_g(f)\right|^2 \le T_g(|f|^2)$ pointwise since the function $t \mapsto t^2$ is convex. Since

$$\lambda(h_i) = p \quad \forall i = 1,\dots,(1+p)(1+p^2)$$

and the $\pi(h_i U)$, $i = 1,\dots,(1+p)(1+p^2)$ are disjoint, we obtain

$$
\begin{aligned}
\left\| m(A_p) T_{g_{1,p}} m(B_p)(f) \right\|_2^2 &= \left\| m(A_p) T_{g_{1,p}}(f) \right\|_2^2 \\
&\leq \int_{\Gamma_2 \backslash A_p} \left| T_{g_{1,p}}(f) \right|^2 \mathrm{d}\nu_\beta \\
&\leq \int_{\Gamma_2 \backslash A_p} T_{g_{1,p}}(|f|)^2 \mathrm{d}\nu_\beta \\
&= \frac{1}{\deg_{\Gamma_2}(g_{1,p})} \sum_{i=1}^{(1+p)(1+p^2)} \int_U |f(p(h_i \cdot)|^2 \mathrm{d}\mu_\beta \\
&= \frac{p^\beta}{\deg_{\Gamma_2}(g_{1,p})} \sum_{i=1}^{(1+p)(1+p^2)} \int_{h_i U} (f \circ p)^2 \mathrm{d}\mu_\beta \\
&\leq \frac{p^\beta}{\deg_{\Gamma_2}(g_{1,p})} \|f\|_2^2.
\end{aligned}
$$

Thus

$$
\left\| m(A_p) T_{g_{1,p}} m(B_p) \right\| \leq p^{\beta/2} \deg_{\Gamma_2}(g_{1,p})^{-1/2}.
$$

On the other hand, recall that for $3 < \beta \leq 4$, the measure μ_β om X is given by the proof of Proposition 3.2.2. Using the computation leading to (3.41), we have that

$$
\mu_{\beta,p}(GSp_4(\mathbb{Z}_p)) = \zeta_{S_{2,p},\Gamma_2}(\beta)^{-1}.
$$

Since $B_p = \Gamma_2 g_{1,p} A_p$, we can now compute $\nu_\beta(\Gamma_2 \backslash B_p)$ using the scaling property of $\mu_{\beta,p}$. Hence

$$
\nu_\beta(\Gamma_2 \backslash B_p) = p^{-\beta} \deg_{\Gamma_2}(g_{1,p}) \nu_\beta(\Gamma_2 \backslash A_p) \leq p^{-\beta} \deg_{\Gamma_2}(g_{1,p}),
$$

which concludes the proof since $\deg_{\Gamma_2}(g_{1,p}) = (1+p)(1+p^2)$. □

Lemma 5.3.3. Given $r \in GSp_4(\mathbb{Q}_F)$ and a finite set of primes F, we set

$$
Z := \Gamma_2 \backslash PGSp_4^+(\mathbb{R}) \times (GSp_4(\mathbb{Z}_F) r GSp_4(\mathbb{Z}_F)).
$$

Assume f is a continuous right $GSp_4(\mathbb{Z}_F)$-invariant function on Z with compact support. Then for any $\epsilon > 0$, there exits a constant $C(\epsilon)$ such that for any compact subset Ω of Z and any finite subset S of F^c, we have that

$$
\left| T_g f(x) - \nu_{\beta,F}(Z)^{-1} \int_Z f \, \mathrm{d}\nu_{\beta,F} \right| < C(\epsilon) \prod_{p \in S} p^{2\epsilon - 1} \quad \text{for all } x \in \Omega,
$$

where $g = \prod_{p \in S} g_{1,p}$ and $g_{1,p}$ is as in Lemma 3.2.3.

Proof. We let

$$H = GSp_4(\mathbb{Z}_F) \cap rGSp_4(\mathbb{Z}_F)r^{-1},$$

and

$$K = H \times \prod_{p \in F^c} GSp_4(\mathbb{Z}_p).$$

By viewing $GSp_4(\mathbb{Z}_F)$ and $\prod_{p \in F^c} GSp_4(\mathbb{Z}_p)$ as subgroups of $GSp_4(\hat{\mathbb{Z}})$ (e.g., by considering $GSp_4(\mathbb{Z}_F)$ as the subgroup of $GSp_4(\hat{\mathbb{Z}})$ consisting of elements with coordinates 1 for $p \in F^c$), we obtain the following homeomorphism

$$\Gamma_2 \backslash PGSp_4^+(\mathbb{R}) \times GSp_4(\mathbb{Z}_F)/H \simeq \Gamma_2 \backslash PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})/K.$$

The quotient $GSp_4(\mathbb{Z}_F)/H$ can be unidentified with the $GSp_4(\mathbb{Z}_F)$-space $GSp_4(\mathbb{Z}_F)rGSp_4(\mathbb{Z}_F)$. Hence we can consider f as function on

$$\Gamma_2 \backslash PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})/K.$$

Next, we have that $GSp_4(\hat{\mathbb{Z}}) = \Gamma_2 K$. In fact, since K is an open compact subgroup of $GSp_4(\hat{\mathbb{Z}})$, this follows from the proof of Corollary 3.1.1 if the surjectivity of the map $\lambda : H \to \hat{\mathbb{Z}}^\times$ is assumed.

Let $x \in \mathbb{Z}_F^\times$ and consider a diagonal element $\alpha \in GSp_4(\mathbb{Z}_F)$ such that $\lambda(\alpha) = x$. We choose $\gamma_1, \gamma_2 \in GSp_4(\mathbb{Z}_F)$ and $\tilde{r}$ a diagonal element of $GSp_4(\mathbb{Q}_F)$ such that $r = \gamma_1 \tilde{r} \gamma_2$ (this follows from the proof of the Elementary Divisor Theorem since the p-adic ring of integers is a PID). Then it is clear that $\gamma_1 \alpha \gamma_1^{-1} \in H$ since $\alpha = \tilde{r} \alpha \tilde{r}^{-1}$. Since $\lambda(\gamma_1 \alpha \gamma_1^{-1}) = \lambda(\alpha)$, we conclude that $\lambda(H) = \mathbb{Z}_F^\times$.

We can now proceed as in the proof of Proposition 3.2.3 and use [COU01, Theorem 1.7 and section 4.7] to obtain the upper bound. □

Lemma 5.3.4. Let F be a finite set of primes and f be any positive continuous right $GSp_4(\mathbb{Z}_F)$-invariant function on $\Gamma_2 \backslash (PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Z}_F)) \subset \Gamma_2 \backslash X_F$ with $\int_{\Gamma_2 \backslash X_F} f \, \mathrm{d}\nu_{\beta,F} = 1$. Then given any $0 < \delta < 1$, there exists $M > 0$ such that

$$\int_{\Gamma_2 \backslash X_F} (T_{g_{1,p}} f)(T_{g_{1,q}} f) \, \mathrm{d}\nu_{\beta,F} \geq (1 - \delta)^5 \ \text{ for } \ p, q > M, \quad p, q \in F^c.$$

Proof. Fix $0 < \delta < 1$ and we consider the following decomposition

$$\Gamma_2\backslash PGS_4(\mathbb{R}) \times (GSp_4(\mathbb{Q}_F) \cap MSp_4(\mathbb{Z}_F)) = \bigcup_{k\geq 1} Z_k,$$

where $Z_k = \Gamma_2\backslash(PGS_4^+(\mathbb{R}) \times (GSp_4(\mathbb{Z}_F)g_k GSp_4(\mathbb{Z}_F))$ and $(g_k)_{k\geq 1}$ are representatives of the double coset

$$GSp_4(\mathbb{Z}_F)\backslash(GSp_4(\mathbb{Q}_F) \cap MSp_4(\mathbb{Z}_F))/GSp_4(\mathbb{Z}_F).$$

Given any $N \in \mathbb{N}$ and any compact subsets C_k of Z_k, $k = 1, \dots, N$, we can use Lemma 5.3.3 to find $M > 0$ such that if $p \in F^c$ with $p > M$, then

$$\left| T_{g_{1,p}} f(x) - \nu_{\beta,F}(Z_k)^{-1} \int_{Z_k} f \,\mathrm{d}\nu_{\beta,F} \right| < \delta \nu_{\beta,F}(Z_k)^{-1} \int_{Z_k} f \,\mathrm{d}\nu_{\beta,F}, \quad \forall x \in C_k,$$

where $1 \leq k \leq N$. Hence for two distinct primes p and q such that $p, q > M$, we get

$$\begin{aligned}\int_{\Gamma_2\backslash X_F} (T_{g_{1,p}} f)(T_{g_{1,q}} f) \,\mathrm{d}\nu_{\beta,F} &\geq \sum_{k=1}^{N} \int_{C_k} (T_{g_{1,p}} f)(T_{g_{1,q}} f) \,\mathrm{d}\nu_{\beta,F} \\ &\geq (1-\delta)^2 \sum_{k=1}^{N} \left(\int_{Z_k} f \,\mathrm{d}\nu_{\beta,F} \right)^2 \nu_{\beta,F}(Z_k)^{-2} \nu_{\beta,F}(C_k).\end{aligned}$$

By regularity of the measure $\nu_{\beta,F}$, we can choose the compact subsets C_k such that

$$\nu_{\beta,F}(Z_k) - \nu_{\beta,F}(C_k) < \delta \nu_{\beta,F}(Z_k), \quad 1 \leq k \leq N. \tag{5.4}$$

Moreover, recall that the subset $\cup_k Z_k \subset \Gamma_2\backslash X_F$ has full measure, hence we choose N large such that

$$\int_{\Gamma_2\backslash X_F} f \,\mathrm{d}\nu_{\beta,F} - \sum_{k=1}^{N} \int_{Z_k} f \nu_{\beta,F} < \delta. \tag{5.5}$$

Combining equations (5.4) and (5.5), we obtain by Jensen's inequality that for any $p, q > M$, we have

$$
\begin{aligned}
&\int_{\Gamma_2\backslash X_F} (T_{g_{1,p}} f)(T_{g_{1,q}} f)\, \mathrm{d}\nu_{\beta,F} \\
&\geq (1-\delta)^3 \Big(\sum_{k=1}^{N} \nu_{\beta,F}(Z_k) \Big) \sum_{k=1}^{N} \frac{\nu_{\beta,F}(Z_k)}{\sum_{k=1}^{N} \nu_{\beta,F}(Z_k)} \Big(\frac{1}{\nu_{\beta,F}(Z_k)} \int_{Z_k} f \, \mathrm{d}\nu_{\beta,F} \Big)^2 \\
&\geq \frac{(1-\delta)^3}{\sum_{k=1}^{N} \nu_{\beta,F}(Z_k)} \Big(\sum_{k=1}^{N} \int_{Z_k} f \, \mathrm{d}\nu_{\beta,F} \Big)^2 \\
&\geq (1-\delta)^5,
\end{aligned}
$$

since $\int_{\Gamma_2\backslash X_F} f \, \mathrm{d}\nu_{\beta,F} = 1$ and $\bigcup_{k=1}^{N} Z_k \subset \Gamma_2\backslash PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Z}_F)$.

□

Lemma 5.3.5. Let B be a measurable Γ_2-invariant subset of Y and define $\phi \in L^2(\Gamma_2\backslash X, d\nu_\beta)$ as follows:

$$\phi = \nu_\beta^{-1}(\Gamma_2\backslash B)\, \mathbb{1}_{\Gamma_2\backslash B}.$$

Then there exists a finite set of primes F and a function $f \in L^2(\Gamma\backslash X_F, d\nu_{\beta,F})$ such that

$$\int_{\Gamma_2\backslash X_F} f d\nu_{\beta,F} = 1,$$

and

$$\|f_F - \phi\|_2 \to 0 \quad \text{as} \quad F \nearrow \mathcal{P}.$$

Proof. Let

$$f := \nu_{\beta,F}^{-1}(\Gamma_2\backslash \pi_F(B))\, \mathbb{1}_{\Gamma_2\backslash \pi_F(B)}.$$

Hence

$$
\begin{aligned}
\int_{\Gamma_2\backslash X} |f_F|^2 d\nu_\beta &= \int_{\Gamma_2\backslash Y} |f \circ \pi_F|^2 d\nu_\beta \\
&= \nu_{\beta,F}^{-1}(\Gamma_2\backslash \pi_F(B)).
\end{aligned}
$$

On the other hand we have

$$\int_{\Gamma_2\backslash X} |\phi|^2 d\nu_\beta = \nu_\beta^{-1}(\Gamma_2\backslash B).$$

Hence $\|f_F\|_2 \to \|\phi\|_2$ as $F \nearrow \mathcal{P}$, which concludes the proof since $(f_F, \phi) = \|\phi\|_2$.

$\square$

Lemma 5.3.6. Let $\beta, \omega \in \mathbb{R}_+^*$ such that $3 < \beta \leq 4$ and $\omega > 1$ and set

$$\kappa := \frac{\omega^{(3-\beta)/2\beta}}{1 + \omega^{(3-\beta)/\beta}}.$$

Then given any finite set of primes F and any positive continuous right $GSp_4(\mathbb{Z}_F)$-invariant function on $\Gamma_2 \backslash (PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Z}_F))$ with $\int_{\Gamma_2 \backslash X_F} f \, \mathrm{d}\nu_{\beta,F} = 1$, there exist two sequences of distinct primes $\{p_n\}_{n \geq 1}$ and $\{q_n\}_{n \geq 1}$ in F^c and Γ_2-invariant measurable subsets X_{1n}, X_{2n}, Y_{1n} and Y_{2n}, $n \geq 1$ of X such that:

1. $\lim_n \left| q_n^\beta / p_n^\beta - \omega \right| = 0$
2. The sets Y_{1n} and $Y_{2n}, n \geq 1$ are mutually disjoint;
3. $\sum_{n=1}^\infty \left(\frac{m(X_{1n}) T_{g_n} m(Y_{1n})}{\|m(X_{1n}) T_{g_n} m(Y_{1n})\|} f_F, \frac{m(X_{2n}) T_{h_n} m(Y_{2n})}{\|m(X_{2n}) T_{h_n} m(Y_{2n})\|} f_F \right) \geq \kappa$ where $g_n := g_{1,p_n}$ and $h_n := g_{1,q_n}$.

Proof. Let $F \subset \mathcal{P}$ be any nonempty finite set of primes and f any positive continuous right $GSp_4(\mathbb{Z}_F)$-invariant function on $\Gamma_2 \backslash (PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Z}_F))$ with $\int_{\Gamma_2 \backslash X_F} f \, \mathrm{d}\nu_{\beta,F} = 1$ and fix $\epsilon > 0$. By Lemma 5.3.1 we can find two disjoint sequences of prime numbers $\{p_n\}_{n \geq 1}$ and $\{q_n\}_{n \geq 1}$ in F^c such that

$$\lim_n q_n^\beta / p_n^\beta = \omega,$$

and

$$\sum_{n=1}^\infty \frac{1}{p_n^{\beta-3}} = \infty. \tag{5.6}$$

We let $B_n^{(1)} = \cup_{k=1}^{k=n-1} B_{p_k}$ and $B_n^{(2)} = \cup_{k=1}^{k=n-1} B_{q_k}$ (where B_{p_k} and B_{q_k} are as in (5.2)) and set

$$X_{1n} := A_{p_n} \backslash B_n^{(1)}, \quad Y_{1n} := B_{p_n} \backslash B_n^{(1)},$$
$$X_{2n} := A_{q_n} \backslash B_n^{(2)}, \quad Y_{2n} := B_{q_n} \backslash B_n^{(2)},$$

where A_{p_n}, A_{q_n} are as in (5.1). By construction the sets Y_{1n} and $Y_{2n}, n \geq 1$ are mutually disjoint so it remains to show the last assertion. By Lemma 5.3.4, we choose $M > 0$ and the sequences $\{p_n\}_{n\geq 1}, \{q_n\}_{\geq 1}$ such that

$$\int_{\Gamma_2\backslash X_F} (T_{g_n} f)(T_{h_n} f)\, \mathrm{d}\nu_{\beta,F} \geq (1-\epsilon)^{1/2}, \quad \forall n \geq 1. \tag{5.7}$$

Observe that if $g \in \Gamma_2 g_{1,p_n} \Gamma_2$ then $gX_{1n} \subset Y_{1n}$ since $gA_{p_n} \subset B_{p_n}$ and $|\lambda(g)|_{p_k} = p_k^{-1}$ for all $1 \leq k < n$. By definition of the Hecke operator $T_{g_{1,p_n}}$ we get that

$$m(X_{1n}) T_{g_{1,p_n}} m(Y_{1n}) f_F = m(X_{1n})(T_{g_{1,p_n}} f)_F.$$

Similarly, we have

$$m(X_{2n}) T_{g_{1,q_n}} m(Y_{2n}) f_F = m(X_{2n})(T_{g_{1,q_n}} f)_F.$$

By Lemma 5.3.2 and Equation (5.7) we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \Big(\frac{m(X_{1n}) T_{g_n} m(Y_{1n})}{\|m(X_{1n}) T_{g_n} m(Y_{1n})\|} f_F, \frac{m(X_{2n}) T_{h_n} m(Y_{2n})}{\|m(X_{2n}) T_{h_n} m(Y_{2n})\|} f_F \Big) \\
&\geq \sum_{n=1}^{\infty} (\nu_\beta(\Gamma_2\backslash B_{p_n}) \nu_\beta(\Gamma_2\backslash B_{q_n}))^{1/2} \nu_\beta(\Gamma_2\backslash X_{1n} \cap X_{2n}) \int_{\Gamma_2\backslash X_F} (T_{g_n} f)(T_{h_n} f)\, \mathrm{d}\nu_{\beta,F} \\
&\geq \sum_{n=1}^{\infty} (\nu_\beta(\Gamma_2\backslash B_{p_n}) \nu_\beta(\Gamma_2\backslash B_{q_n}))^{1/2} \\
&\qquad \Big(\prod_{k=1}^{n-1} (1 - \nu_\beta(\Gamma_2\backslash B_{p_k} \cup B_{q_k})) \Big) \nu_\beta(\Gamma_2\backslash A_{p_n}) \nu_\beta(\Gamma_2\backslash A_{q_n}) (1-\epsilon)^{1/2}.
\end{aligned}$$

Since

$$\nu_\beta(\Gamma_2\backslash A_{p_n}) \nu_\beta(\Gamma_2\backslash A_{q_n}) = \zeta_{S_{2,p_n},\Gamma_2}(\beta)^{-1} \zeta_{S_{2,q_n},\Gamma_2}(\beta)^{-1},$$

and

$$\begin{aligned}
&\nu_\beta(\Gamma_2\backslash B_{p_n}) \nu_\beta(\Gamma_2\backslash B_{q_n}) \\
&= (p_n q_n)^{-\beta} \deg_{\Gamma_2}(g_{1,p_n}) \deg_{\Gamma_2}(g_{1,q_n}) \zeta_{S_{2,p_n},\Gamma_2}(\beta)^{-1} \zeta_{S_{2,q_n},\Gamma_2}(\beta)^{-1},
\end{aligned}$$

we obtain that

$$\frac{(\nu_\beta(\Gamma_2\backslash B_{p_n})\nu_\beta(\Gamma_2\backslash B_{q_n}))^{1/2}\nu_\beta(\Gamma_2\backslash A_{p_n})\nu_\beta(\Gamma_2\backslash A_{q_n})}{\nu_\beta(\Gamma_2\backslash B_{p_n}\cup B_{q_n})} \sim \frac{(p_n^\beta q_n^\beta)^{3-\beta/2\beta}}{(p_n^{3-\beta}+q_n^{3-\beta}-(p_nq_n)^{3-\beta})},$$

since $3<\beta\leq 4$. Hence we can choose the sequences $\{p_n\}_{n\geq 1}$ and $\{q_n\}_{\geq 1}$ such that for all $n\geq 1$, we have

$$\frac{(\nu_\beta(\Gamma_2\backslash B_{p_n})\nu_\beta(\Gamma_2\backslash B_{q_n}))^{1/2}\nu_\beta(\Gamma_2\backslash A_{p_n})\nu_\beta(\Gamma_2\backslash A_{q_n})}{\nu_\beta(\Gamma_2\backslash B_{p_n}\cup B_{q_n})} > \frac{\omega^{(3-\beta)/2\beta}}{1+\omega^{(3-\beta)/\beta}}(1-\epsilon)^{1/2}.$$

Since

$$\sum_{n=1}^{\infty}\nu_\beta(\Gamma_2\backslash B_{p_n}\cup B_{q_n}) \geq \sum_{n=1}^{\infty}\nu_\beta(\Gamma_2\backslash B_{p_n}) \sim \sum_{n=1}^{\infty}\frac{1}{p_n^{\beta-3}} = \infty$$

by equation (5.6), we finally obtain that

$$\sum_{n=1}^{\infty}\Big(\frac{m(X_{1n})T_{g_n}m(Y_{1n})}{\|m(X_{1n})T_{g_n}m(Y_{1n})\|}f_F, \frac{m(X_{2n})T_{h_n}m(Y_{2n})}{\|m(X_{2n})T_{h_n}m(Y_{2n})\|}f_F\Big) \geq \frac{\omega^{(3-\beta)/2\beta}}{1+\omega^{(3-\beta)/\beta}}(1-\epsilon),$$

where the last inequality follows from the fact that

$$\sum_{n=1}^{\infty}\nu_\beta(\Gamma_2\backslash B_{p_n}\cup B_{q_n})\Big(\prod_{k=1}^{n-1}(1-\nu_\beta(\Gamma_2\backslash B_{p_k}\cup B_{q_k}))\Big) = 1.$$

Since ϵ was arbitrary, this completes the proof. □

We are now ready to state and prove the main Theorem of this chapter.

Theorem 5.3.1. *Let $3<\beta\leq 4$. Then the unique KMS$_\beta$ state on the Connes-Marcolli GSp_4-system is of type III$_1$.*

Proof. In view of the isomorphism in (5.3) and Theorem 5.1.1, we need to show that the action of G on (X,μ_β) is of type III$_1$. This is the case ([Nes11]) if and only if the action of G on $(\mathbb{R}_+\times X, \lambda_\infty\times\mu)$ is ergodic. Hence by Proposition 5.3.1 it is enough to show that the action of $GSp_4^+(\mathbb{Q})$ on the space $(PGSp_4^+(\mathbb{R})\times MSp_4(\mathbb{A}_f)/GSp_4(\hat{\mathbb{Z}}),\nu_\beta)$ is of type III$_1$. Since $r(\mathcal{R},\mu)^*$ is a closed subgroup of

$\mathbb{R}_+^*$, it is enough to show that any real number $\omega > 1$ belongs to the ratio set $r(\mathcal{R}, \mu_\beta)$ corresponding to this action. Fix $\epsilon > 0$ and let B be any measurable right $GSp_4(\hat{\mathbb{Z}})$-invariant subset of X with positive measure.

Let F be any finite set of primes, f any positive continuous right $GSp_4(\mathbb{Z}_F)$-invariant function with compact support in $\Gamma_2 \backslash (PGSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{Z}_F))$, $X_{1n}, X_{2n}, Y_{1n}, Y_{2n}$ any mutually disjoint Γ_2-invariant measurable subsets of X and $\{p_n\}_{n\geq 1}, \{q_n\}_{n\geq 1}$ any two sequences of distinct primes in F^c. To ease the notation we set

$$T_n^{(1)} = \frac{m(X_{1n})T_{g_n}m(Y_{1n})}{\left\|m(X_{1n})T_{g_n}m(Y_{1n})\right\|}, \quad T_n^{(2)} = \frac{m(X_{2n})T_{h_n}m(Y_{2n})}{\left\|m(X_{2n})T_{h_n}m(Y_{2n})\right\|},$$

$$e_n^{(1)} := m(Y_{1n}), \quad e_n^{(2)} := m(Y_{2n}).$$

Let $\phi \in L^2(\Gamma_2 \backslash X, d\nu_\beta)$. Since $\left\|T_n^{(1)}\right\| = \left\|T_n^{(2)}\right\| = 1$ and e'_n, e''_n are projections, we obtain by Cauchy-Schwartz that

$$\begin{aligned}
\sum_n (T_n^{(1)}\phi, T_n^{(2)}\phi) \geq & \sum_n (T_n^{(1)} f_F, T_n^{(2)} f_F) - \left\|e_n^{(1)}(f_F - \phi)\right\|_2 \left\|e_n^{(2)} f_F\right\|_2 \\
& - \left\|e_n^{(2)}(f_F - \phi)\right\|_2 \left\|e_n^{(1)}\phi\right\|_2 \\
\geq & \sum_n (T_n^{(1)} f_F, T_n^{(2)} f_F) - \Big(\sum_n \left\|e_n^{(1)}(f_F - \phi)\right\|_2^2\Big)^{1/2} \Big(\sum_n \left\|e_n^{(2)} f_F\right\|_2^2\Big)^{1/2} \\
& - \Big(\sum_n \left\|e_n^{(2)}(f_F - \phi)\right\|_2^2\Big)^{1/2} \Big(\sum_n \left\|e_n^{(1)}(\phi)\right\|_2^2\Big)^{1/2} \\
\geq & \sum_n (T_n^{(1)} f_F, T_n^{(2)} f_F) - \|f_F - \phi\|_2(\|f_F\|_2 + \|\phi\|_2).
\end{aligned}$$

Since the subset $GSp_4^+(\mathbb{Q})B$ is completely determined by its intersection with $PGSp_4^+(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}})$, there exists g_0 such that the intersection $B_0 := g_0 B \cap (PGSp_4^+(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}}))$ has positive measure. We set

$$\phi := \nu_\beta(\Gamma_2 \backslash \Gamma_2 B_0) \mathbb{1}_{\Gamma_2 \backslash \Gamma_2 B_0}.$$

Let $\kappa = \frac{\omega^{(3-\beta)/2\beta}}{1+\omega^{(3-\beta)/\beta}}$. By Lemma 5.3.5 there exists f and $F \subset \mathcal{P}$ large enough such that

$$\|f_F - \phi\|_2(\|f_F\|_2 + \|\phi\|_2) < \kappa, \quad \int_{\Gamma_2 \backslash X_F} f d\nu_{F,\beta} = 1.$$

Hence by Lemma 5.3.6 there exists $m \in \mathbb{N}$ such that $(T_m^{(1)}\phi, T_m^{(2)}\phi) > 0$. This implies that $(T_{g_m}\phi, T_{h_m}\phi) > 0$, in particular this shows that the subset $\Gamma_2 g_m^{-1}\Gamma_2 B_0 \cap \Gamma_2 h_m^{-1}\Gamma_2 B_0 \subset X$ has positive measure. Thus there exist $g \in \Gamma_2 g_m \Gamma_2$ and $h \in \Gamma_2 h_m \Gamma_2$ such that $g^{-1}B_0 \cap h^{-1}B_0$ has positive measure, which implies that the set $g_0^{-1}hg^{-1}g_0 B \cap B$ has positive measure. If we set $\tilde{g} := g_0^{-1}hg^{-1}g_0$, we get by the scaling condition 3.30 that

$$\left|\frac{\mathrm{d}\tilde{g}_*\mu_\beta}{\mathrm{d}\mu_\beta}(x) - \omega\right| = \left|\lambda(g_0^{-1}hg^{-1}g_0)^\beta - \omega\right| = \left|\frac{q_m^\beta}{p_m^\beta} - \omega\right| < \epsilon, \quad \forall x \in \tilde{g}B \cap B.$$

This shows that $\omega \in r(\mathcal{R}, \mu_\beta)$, which completes the proof. □

Chapter 6

CONCLUSION

The aim of this thesis was to study three different aspects of the Connes-Marcolli system associated to the Siegel modular variety of degree 2. At inverse temperature $\beta = 3$, we proved that this quantum dynamical system undergoes a spontaneous symmetry breaking phase transition reminiscent of a Higgs mechanism. We then proved an inequality which intertwines the action by symmetries at zero temperature and the Galois action of a subgroup of the Siegel modular field on the values of the equilibrium states. Finally, we showed that a phase transition occurs at the level of the von Neumann algebras generated by the equilibrium states. The following theorem summarizes these results.

Theorem 6.0.1. *For the Connes-Marcolli GSp_4-system, the following assertions hold:*

1. *There is no KMS$_\beta$ state in the range* $0 < \beta < 3$ *and* $\beta \notin \{1, 2\}$.
2. *There exists a unique KMS$_\beta$ state of type III$_1$ in the range* $3 < \beta \leq 4$.
3. *For* $4 < \beta < \infty$, *every extremal KMS$_\beta$ state is of type I_∞ and is given by the explicit formula*

$$\phi_{\beta,y}(f) = \frac{\zeta(2\beta-2)Tr(\pi_y(f)e^{-\beta H_y})}{\zeta(\beta)\zeta(\beta-1)\zeta(\beta-2)\zeta(\beta-3)}, \quad y \in \mathbb{H}_2^+ \times GSp_4(\hat{\mathbb{Z}}), \quad \forall f \in \mathcal{A}.$$

4. *In the range* $4 < \beta \leq \infty$, *the set of extremal states is identified with the Shimura variety* $Sh(GSp_4, \mathbb{H}_2^\pm)$,

$$\mathcal{E}_\beta \simeq GSp_4(\mathbb{Q}) \backslash \mathbb{H}_2^\pm \times GSp_4(\mathbb{A}_{\mathbb{Q},f}).$$

5. *For* $\beta = \infty$ *and a generic point* $\tau \in \mathbb{H}_n^+$, *there exists a subgroup S' of $Aut(F)$ and an isomorphism* $\vartheta^{-1} : S' \to S = \mathbb{Q}^\times \backslash GSp_4(\mathbb{A}_{\mathbb{Q},f})$ *which intertwines the Galois action on the values of the states with the action by symmetries as follows:*

$$\alpha^*_{\vartheta^{-1}(g)}(\phi_{\infty,y})(f) = Aut_\tau(\rho\vartheta^{-1}(g)\rho^{-1}) \cdot \phi_{\infty,y}(f), \quad \forall f \in \mathcal{A}_{\mathbb{Q}}^{arith}, \ \forall g \in S'.$$

Future work

One possible approach to generalize the results of this thesis is to consider the arbitrary case where $n > 2$. For these higher dimensional systems, one needs an alternative approach to obtain a closed formula of 3.37 and prove an analogue of Lemma 3.2.4. Another promising direction would be to utilize the ideas and tools used in this research to study the Connes-Marcolli system associated to the Hilbert Modular Surface. Finally, the author also believes that the tools presented in this thesis can provide a starting point for a full analysis of new systems which can be constructed out of other linear algebraic groups.

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