On arithmetic invariants of special families of K3-type surfaces

Thesis by

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In Partial Fulfillment of the Requirements for the
Degree of
Doctor of Philosophy

Caltech

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2024
Defended May 15, 2024

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ACKNOWLEDGEMENTS

First and foremost, I would like to thank my advisor Elena Mantovan. It has been an absolute honor to be your student and learn so much from your expertise. Thank you for taking me under your wing, and for your unwavering support, guidance, and patience in years.

I am grateful to all the teachers and mentors who have taught and inspired me to do mathematics. I would also like to thank Tom Graber, Matthias Flach, and Roy Zhao for being part of my thesis committee, and Nam Ung for constant reassurance and kindness.

I want to thank my Caltech friends Jeck, Yuxin, Deepesh, Bjarne, Alex, and Abishek for their camaraderie and support on this journey. I will miss our shared food, laughter, and intellectual conversations the most. Thank you Jeck for being you: a goofy roommate, a great cook, and a good-hearted person.

I want to thank my Vietnamese friends Linh, Hai, Quang, Chi, Nghia, and The Anh for always being there and reminding me of who I am over the years. I also want to thank my Caltech mentees and PiMA teammates for allowing me to pass on some of my knowledge and wisdom.

Finally, I thank my family for their unconditional love, endless support, and constant encouragement. I dedicate this thesis to them. This PhD journey is not easy, and I thank Trung Can for never giving up on your hopes and dreams.

ABSTRACT

This thesis studies applications of Shimura varieties in positive characteristic to questions on arithmetic invariants of special families of K3-type surfaces.

In the first part, we consider a K3-type surface X over $\mathbb{Q}$ that arises from cyclic covers of the projective line $\mathbb{P}^1$ and a prime p of good reduction. Studying the geometry of the associated Shimura varieties and Hurwitz spaces, our main theorem determines all possible Newton polygons of X at p, at least when p is sufficiently large. This establishes several new instances of the Manin problem for K3-type surfaces. When X is a generic supersingular K3 surface in the family, we also calculate the corresponding Artin invariant.

In the second part, we discuss some applications as corollaries of our calculation and recent works [3, 17]. For instance, let X be a K3 surface over a number field L for which the endomorphism algebra of its transcendental lattice is an abelian field. Then the set of μ-ordinary primes of L has density 1 under some conditions. We remark that the proof is discovered by the authors of [3] for a certain class of abelian varieties. Our contribution, if any, is simply realizing that with minor adjustments their techniques also work for K3 surfaces. For another result, we deduce the existence of infinitely many basic primes for one family of K3 surfaces over $\mathbb{Q}$ which can be thought of an analog of Elkies's supersingular prime theorem in our setting.

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Chapter 1

INTRODUCTION

1.1 Motivation

Shimura varieties are higher-dimensional analogues of modular curves. They play an important role in the theory of automorphic forms and the Langlands program. In recent years, Shimura varieties have found new applications in arithmetic geometry such as the Schottky problem in positive characteristics [15, 16], generalized Elkies's supersingular theorem [17], or Serre's ordinariness conjecture [3]. These questions can be formulated in terms of a discrete invariant in positive characteristics called Newton polygon. Here Shimura varieties enter the picture as moduli spaces of abelian varieties with additional structure in characteristic p. One of the most beautiful properties of these moduli spaces is that they admit natural stratifications by discrete invariants such as the Newton polygon [4, 26, 34]. Studying the resulting geometry often leads to helpful insights to solve problems. For example, the authors study Newton stratification of certain Shimura varieties of PEL type to calculate all possible Newton polygons that occur for special families of Jacobians of cyclic covers of the projective line [16]. To us, special means the image of the family under the classifying map to the moduli Shimura variety is open and dense in a Shimura subvariety[1] (see [20, Section 1] for equivalent definitions). It is a natural question to ask whether their framework can be applied to other settings.

And the answer is positive. The first part of this thesis carries out a similar calculation for the occurrence of Newton polygons in special families of K3-type surfaces arising from cyclic covers of the projective line. Our methods are similar to the framework in [15, 16]. As input for our theorem we use the construction of special K3 type families from [21]. An analogue of the Manin problem for abelian varieties asks which Newton polygons occur for a K3-type surface with given degree over a fixed characteristic p. Earlier work [27] shows that all possible polygons for a K3 surface occur in any given characteristic when the degree is sufficiently large. Our result establishes new instances of Newton polygons for k3-type surfaces that occur in positive characteristic with fixed degrees.

[1] In virtue of the recently proven André-Oort conjecture, being special is equivalent to having a Zariski dense set of special points (or CM points).

Our main application is a refined version of a conjecture of Serre for K3 surfaces. For an abelian variety A over a number field L, Serre's conjecture asserts that there exists a finite field extension over which the set of good ordinary primes for the base change of A has density 1. The analogue statement for K3 surface over number field is proven in [2]. Recent works of Sawin [28] and [3] show refined versions of Serre's conjecture for abelian surfaces and certain class of simple abelian varieties of type VI. In particular, they calculate this density explicitly without the passage to a sufficiently large field extension. We follow [3] to get refined results for K3 surfaces over a number field L equipped with an action of an abelian field satisfying some mild assumptions. The families from the first part give examples of K3 surfaces defined over $\mathbb{Q}$ that satisfy all hypotheses of the theorem. Another application is a generalization of Elkies's supersingular prime theorem [10] for one special family of K3 surfaces over $\mathbb{Q}$.

1.2 Overview of the results

First we briefly recall the definition of Newton polygon and Artin invariant.

Let k be a perfect field of characteristic p and $W(k)$ denote the ring of Witt vectors over k. By a K3-type surface X over k we mean a smooth projective surface over k whose Hodge numbers $h^{i,j} := \dim_k H^j(X, \Omega^i_{X/k})$ are given by the following Hodge diamond:

$$\begin{matrix} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,1} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 2n-2 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} .$$

To the second crystalline cohomology $H^2_{cris}(X/W)$, one considers a discrete invariant called the Newton polygon of the F-crystal $H^2_{cris}(X/W)$. It is a lower convex polygon starting at $(0,0)$ and ending at $(2n, 2n)$ with integer breakpoints. From 2.5, there exists an integer $h = h(X) \in \{1, 2, \ldots, n\}$ such that $H^2_{cris}(X/W)$ either has slopes

$$(1 - 1/h, 1, 1 + 1/h)$$

with multiplicities

$$(h, 2n - 2h, h),$$

or the Newton polygon has slopes 1 only in which case we set $h(X) = \infty$. We call this invariant the height of X[2]. If X is a K3 surface then $n = 10$.

X is called ordinary if $h(X) = 1$ and supersingular if $h(X) = \infty$.

Assume that X is a supersingular K3 surface. It follows from the Tate conjecture for K3 surfaces over finite fields[3] that the Tate module T_X has rank 22. The bilinear pairing from Poincaré duality restricted to T_X is a non-degenerate symmetric form

$$\langle , \rangle : T_X \times T_X \longrightarrow \mathbb{Z}_p.$$

One can show that the discriminant of this form has p-valuation $2\sigma_0$ for some integer $\sigma_0 \in \{1, 2, \ldots, 10\}$. The invariant σ_0 is called the Artin invariant of X (see 2.5).

Next we describe the families we consider for our main results.

In [21], motivated by an analogue of Coleman's conjecture for K3-type surfaces, Moonen constructs special families of k3-type surfaces from cyclic covers of the projective line $\mathbb{P}^1$. Each family comes from a tuple $(G, \underline{a}, \underline{b}, N)$ called a K3 type datum (see 3.0.1) where $G \simeq \mathbb{Z}/m\mathbb{Z}$ is the cyclic group of order $m \geq 3$, $\underline{a} = (a_1, \ldots, a_N)$ and $\underline{b} = (b_1, b_2, b_3)$ are monodromy data, and N is the number of branch points. To be more precise, let C and D be smooth curves given by the equations

$$C : y^m = (x - \lambda_1)^{a_1}(x - \lambda_2)^{a_2} \ldots (x - \lambda_N)^{a_N},$$
$$D : z^m = (t - \gamma_1)^{b_1}(t - \gamma_2)^{b_2}(t - \gamma_3)^{b_3},$$

and μ_m be the group scheme of m^{th} roots of unity acting on $C \times D$ via

$$(x, y, t, z) \mapsto (x, \zeta y, t, \zeta^{-1} z),$$

where ζ is a choice of a primitive m^{th} root. Then the minimal model resolution of singularities of the GIT quotient $(C \times D)/G$ is a smooth projective k3-type surface. Varying the base point $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N)$ one obtains a family of k3-type surfaces of dimension $N - 3$.

Let $\mathcal{X} = \mathcal{X}(G, \underline{a}, \underline{b}, N)$ be the family associated to the K3 type datum $(G, \underline{a}, \underline{b}, N)$. Then $\mathcal{X}$ admits a smooth model over $\mathbb{Z}[\frac{1}{2m}]$ (see 3). Let $p \nmid 2m$ be a prime and X be a k3-type surface in $\mathcal{X}$. By abuse of notation, we use X to denote the reduction of X at p which is a smooth proper k3-type surface in positive characteristic. Let E_m be the m^{th} cyclotomic field and E_m^0 be its totally real field. We write d for the order of p mod m. Our main theorem is the following.

[2] The invariant $h(X)$ is also the height of the formal Brauer group of X.
[3] Established by the works of several authors.

Theorem (Theorem 4.0.9)**.** *Let $\mathcal{X}$, X, p and d be as above.*

1. *If p is not inert in E_m/E_m^0, then d divides $h(X)$ and*

$$1 \leq \frac{h(X)}{d} \leq N-2.$$

Furthermore, for any $1 \leq i \leq N-2$, there exists a k3-type surface X' in $\mathcal{X}$ such that $h(X') = id$.

2. *If p is inert in E_m/E_m^0, we assume further that p is sufficient large when N is even, then d divides $h(X)$ and*

$$1 \leq \frac{h(X)}{d} \leq \lfloor \frac{N-2}{2} \rfloor \text{ or } h(X) = \infty.$$

In addition, for any $1 \leq i \leq \lfloor \frac{N-2}{2} \rfloor$, there exists a k3-type surface X' in $\mathcal{X}$ such that $h(X') = nd$.

Moreover, if $\mathcal{X}$ corresponds to a family of K3 surfaces, then there exists a supersingular K3 surface X' in $\mathcal{X}$ with $\sigma_0(X') = (N-2-\lfloor \frac{N-2}{2} \rfloor)d$.

Section 4 discusses how to obtain an effective bound for p in the second part of the theorem. Table 5.1 lists all special families for $m \leq 100$ from a brute-force search done by a computer program. Note that there are no examples for $23 \leq m \leq 100$. It is an open question whether Table 5.1 is complete.

While we are not able to prove this for k3-type surfaces in general, we can easily show the following.

Proposition 1.2.1. *There are no other special families of K3 surfaces arising from cyclic covers of the projective line except those listed in Table 5.1.*

We remark that our examples cover all the possible heights $1, \ldots, 10$, and ∞ of a K3 surface.

Example 1.2.1. *The K3 type datum* $(4, (2, 3, 3), (1, 1, 2), 3)$ *gives rise to a single K3 surface X with complex multiplication by $\mathbb{Q}[i]$. Furthermore, X is defined over $\mathbb{Z}$ and has supersingular reduction for all primes $p \equiv 3 \pmod 4$. The first example of this phenomenon goes back to Tate's K3 surface: $x^4 + y^4 + z^4 + t^4 = 0$ which also has supersingular reduction for all primes $p \equiv 3 \pmod 4$.*

Example 1.2.2. *Let $\mathcal{X}$ be the 1-dimensional family of K3 surfaces associated to* $(7, (2, 4, 4, 4), (1, 2, 4), 4)$ *and $X \in \mathcal{X}$. At primes $p \equiv 1 \pmod 7$ we have $h(X) = 1$ or $h(X) = 2$. At primes $p \equiv 2, 4 \pmod 7$ we have $h(X) = 3$ or $h(X) = 6$. At primes $p \equiv 3, 5 \pmod 7$ we have $h(X) = 6$ or $h(X) = \infty$ (and $\sigma_0(X) = 6$). Finally at primes $p \equiv 6 \pmod 7$ we have $h(X) = 2$ or $h(X) = \infty$ (and $\sigma_0(X) = 2$).*

For a fix datum $(G, \underline{a}, \underline{b}, N)$ and a prime p of good reduction, the set of possible Newton polygons that occur on the family $\mathcal{X}$ has a natural "lying above" order. With respect to this order, the lowest polygon is called μ-ordinary and the highest is called basic. Using these notions, the authors show a refined version of Serre's ordinariness conjecture for certain type of abelian varieties [3] and a generalized Elkies's theorem for a specific family of cyclic covers of the projective line [17].

As a consequence of their results and our calculation, we can deduce the following results that are new even for K3 surfaces (see Chapter 5 for details and discussion). For a K3 surface X over a number field L, we write $\mathcal{E}(X)$ for the endomorphism algebra of the transcendental lattice $T(X)$ of X (see 2.4.6) and L^{conn} for the extension of L that maps to the connected component of the l-adic monodromy group (see Section 5, Chapter 5).

Theorem (Theorem 5.0.9)**.** *Let X be a K3 surface over a number field L such that $\mathcal{E}(X)$ is an abelian field F and $L^{conn} \subseteq FL$. If F is totally real, we assume further that $T(X)_{\mathbb{Q}}$ has even dimension over $\mathcal{E}(X)$. Then the set of primes of L of μ-ordinary reduction has density 1.*

Remark 1.2.1. *If $\mathcal{X}(G, \underline{a}, \underline{b}, N)$ corresponds to a family of K3 surfaces, then a generic surface X in the family satisfies the conditions of the theorem.*

Theorem (Theorem 5.0.12)**.** *Let X_α be a K3 surface in the family* $(5, (1, 1, 1, 2), (1, 1, 3), 4)$. *In particular, X_α arises from the pair of cyclic covers*

$$C_\alpha : y^m = x(x-1)(x-\alpha)$$

and

$$D : z^m = t(t-1).$$

Assume in addition that $j := \frac{(\alpha^2-\alpha+1)^3}{\alpha^2(\alpha-1)^2} \in \mathbb{Q} \cap [0, \frac{27}{4}]$ and the reduction of C_α at 5 is singular. Then X_α can be defined over $\mathbb{Q}$ and there exists infinitely many primes at which the reduction of X_α has basic Newton polygon.

Remark 1.2.2. *The two results together imply that a set of density 0, namely the set of basic primes, is infinite.*

1.3 The proof strategy

Theorem 4.0.9

The main idea is to reduce the problem to the calculation of discrete invariants that occur on an intermediate special family of abelian varieties, which we describe next.

Assume that $p \nmid 2m$ is a prime and we work in characteristic p. By abuse of notation, a moduli space $\mathcal{M}$ also denotes its geometric fiber $\mathcal{M} \otimes \overline{\mathbb{F}}_p$. The Hurwitz space $\mathcal{H}(G, \underline{a})$ is the moduli space of G-covers of $\mathbb{P}^1$ with inertia type $\underline{a}$. The K3 type family $\mathcal{X}(G, \underline{a}, \underline{b}, N)$ can be parametrized over $\mathcal{H}(G, \underline{a})$ in the following way. Let $\underline{\lambda} \in \mathcal{H}(G, \underline{a})$ be a basepoint. By $X_{\underline{\lambda}}$ and $C_{\underline{\lambda}}$ we mean the corresponding k3-type surface and cyclic cover of $\mathbb{P}^1$ above $\underline{\lambda}$. The Jacobian $J(C_{\underline{\lambda}})$ of $C_{\underline{\lambda}}$ admits a decomposition up to isogenies into the new part and the old part:

$$J(C_{\underline{\lambda}}) \sim J(C_{\underline{\lambda}})^{new} \oplus J(C_{\underline{\lambda}})^{old}.$$

We can show that the Newton polygon of $J(C_{\underline{\lambda}})^{new}$ determines the Newton polygon of $X_{\underline{\lambda}}$. Hence it remains to determine the variation of Newton polygons for $J(C_{\underline{\lambda}})^{new}$ when $\underline{\lambda}$ varies in $\mathcal{H}(G, \underline{a})$. Let θ denote the map $C_{\underline{\lambda}} \mapsto J(C_{\underline{\lambda}})^{new}$ so we have an induced morphism

$$\theta : \mathcal{H}(G, \underline{a}) \to \mathcal{A}_{g_{new}},$$

where $g_{new} = \frac{\phi(m)(N-2)}{2}$ is the genus of $J(C_{\underline{\lambda}})^{new}$ (see 2.3.2) and $\mathcal{A}_{g_{new}}$ is the moduli space of polarized abelian varieties of genus g_{new}. From 3.0.2, the image $Z(G, \underline{a}) := \theta(\mathcal{H}(G, \underline{a}))$ is open and dense in a certain Shimura variety of PEL type which we denote by $Sh(G, \underline{f}^{new})$[4]. Here $\underline{f}^{new}$ is the signature of the action of G on the tangent space of $J(C_{\underline{\lambda}})^{new}$. It follows that the Zariski closure $\overline{Z(G, \underline{a})}$ is $Sh(G, \underline{f}^{new})$.

The Newton polygons that occur on the Shimura variety side are well-known. From the moduli point of view, they are the Newton polygons of the corresponding universal abelian schemes over $Sh(G, \underline{f}^{new})$. In [14], Kottwitz gives a combinatorial description of the so-called Kottwitz set $B(G, \underline{f}^{new})$[5] of admissible Newton polygons equipped with additional structure from the group-theoretic data at an unramified prime p. In our setting, the signature $\underline{f}^{new}$ is simple so the Kottwitz set admits a total order

$$\nu_{\mathrm{ord}} =: \nu_0 > \nu_1 > \cdots > \nu_l := \nu_{\mathrm{basic}}$$

[4] These varieties were studied by Deligne and Mostow [7].

[5] A more common notation is $B(G, \mu)$ where μ is the local cocharacter describing the polarized Hodge structure at p.

where the maximal element ν_{ord} is called μ-ordinary and minimal ν_{basic} called basic. This order agrees with the "lying above" order of lower convex polygons. Let $Sh[\nu]$ denote the locus of $Sh(G, \underline{f}^{new})$ with Newton polygon ν. For the PEL-type Shimura varieties, Viehman and Wedhorn [34] show that all admissible polygons occur, i.e., $Sh[\nu]$ is nonempty $\forall \nu \in B(G, \underline{f}^{new})$.

Thus, the question can be reformulated as which Newton strata has nonempty intersection with the open image $Z(G, \underline{a})$. Let $Z(G, \underline{a})[\nu] := Z(G, \underline{a}) \cap Sh[\nu]$. Via the geometry of Newton stratification, we can reduce the problem to showing that $Z(G, \underline{a})[\nu_{basic}] \neq \emptyset$. This last step can be done by a detailed comparison between boundary components of the Hurwitz space $\mathcal{H}(G, \underline{a})$ and the basic locus $Sh[\nu_{basic}]$, at least when p is large. A discussion on an effective bound for p is given in 4.

Theorems 5.0.9 and 5.0.12

For Theorem 5.0.9, we borrow the authors' proof from [3] with minor adjustments. For completeness, we summarize the main steps as following:

1. We construct a conjugacy-invariant algebraic function on the l-adic monodromy group $G_{X,l}$ that detects μ-ordinariness and use Serre's results to reduce density question to a trace calculation on the connected components of $G_{X,l}$;

2. We use the conditions F is abelian and $L^{conn} \subseteq FL$ to realize the group of connected components of $G_{X,l}$ as a permutation subgroup the Weyl group of the orthogonal group;

3. A direct calculation with matrices shows that the trace is non-constant on each connected component.

We remark that unlike the case of abelian varieties, for K3 surfaces, the techniques also work when $\mathcal{E}_X$ is totally real and $T(X)$ has even rank over $\mathcal{E}_X$.

Theorem 5.0.12 follows from the main theorem of [17] and the calculations in 4.

1.4 Future work

This section discusses further directions in increasing order of difficulty.

Non-cyclic covers

The group G can be non-cyclic. Because of the K3 type condition, D has to be branched above 3 points. In particular, G has at most two generators. Thus the next cases to consider will be products or semidirect products of two cyclic groups. The calculations of the signature via Hurwitz characters and minimal resolution of singularity, though slightly more complicated, could be done to obtain to new examples of K3 type families.

Complete occurrence of the Artin invariants

Similar to the Newton polygon, if we could show that the lowest Artin invariant occurs, then from purity of the stratification, all admissible Artin invariants occur. The former step would require an analysis of the number of superspecial points in $Sh(G, \underline{f}^{new})$ to verify whether it grows with p.

Density of primes for other invariants such as the Picard rank

As an analogue to Serre's conjecture, an interesting direction is studying the density of primes at which the Picard rank has to be of some "generic type". The strategy we could carry can be: first calculate of admissible Picard ranks for a K3 surface with extra endomorphism, then find a conjugacy-invariant algebraic function that detects genericity of the Picard rank. Since the Picard group consists of Tate classes, a potential characterization can be related to the eigenvalue p of the Frobenius $\mathrm{Fr}_{\mathfrak{p}}$.

Completeness of all K3-type families arising from cyclic covers of $\mathbb{P}^1$

In [20], the author shows that there are precisely 20 special families of Jacobians that arise from cyclic covers of $\mathbb{P}^1$. In a similar light, we can conjecture that there are no further family of k3-type surfaces arising from cyclic covers of $\mathbb{P}^1$ except those listed in Table 5.1. The nonexistence of further families reduces to the nonexistence of further integer solutions to the equation involving fractional functions from 3.0.1. This is perhaps the most open and difficult question.

1.5 Structure of the thesis

In Chapter 2, we introduce general notations and recall some preliminaries. Chapter 3 reviews Moonen's construction of special families of k3-type surfaces and prove relevant features. In Chapter 4, we prove our first main theorem. Chapter 5 presents other theorems as applications.

Chapter 2

BACKGROUND

2.1 General notations

Throughout, let k be a perfect field of positive characteristic p. We write $W(k)$ for ring of Witt vectors over k and $K(k)$ for its faction field $W(k)[\frac{1}{p}]$, or just W and K for short. We use σ to denote the Frobenius automorphism of k as well as its lifts to W and K.

We fix the following notations:

- $\overline{\mathbb{Q}}_p$ is an algebraic closure of $\mathbb{Q}_p$;
- $\mathbb{C}_p$ is the p-adic completion of $\overline{\mathbb{Q}}_p$ and we fix an identification $\mathbb{C}_p \simeq \mathbb{C}$;
- $\mathbb{Q}_p^{un}$ is the maximal unramified extension $\mathbb{Q}_p$ and $\mathbb{Z}_p^{un}$ is its ring of integers;
- For a positive integer $m \geq 2$, let μ_m denote the group scheme of m^{th} roots of unity, E_m denote the m^{th} cyclotomic field of degree $\varphi(m)$, and fix a choice $\zeta_m \in \mathbb{C}$ of a primitive m^{th} root of unity;
- For a ring R, $R^{\times}$ is the subset of nonzero elements and R^* is the subset of invertible elements.
- For a real number x, $\langle x \rangle = x - \lfloor x \rfloor$ is the fractional part of x.

2.2 The group algebra

Let $G \simeq \mathbb{Z}/m\mathbb{Z}$ be the cyclic group of order m. The group algebra

$$\mathbb{Q}[G] \simeq \mathbb{Q}[X]/(X^m - 1)$$

is the same as the coordinate ring of the group scheme μ_m of m^{th} roots of unity.

Let $\mathcal{T} := \hom_{\mathbb{Q}}(\mathbb{Q}[G], \mathbb{C})$. Then $\mathcal{T}$ can be identified with G via

$$n \in \mathbb{Z}/m\mathbb{Z} \mapsto \tau_n \in T$$

where τ_n maps the generator of G to $(\zeta_m)^n \in \mathbb{C}^*$. We write τ_0 for the trivial character and τ^* for the image of τ under complex conjugation in $\mathbb{C}$; in particular, $\tau_n^* = \tau_{-n}$.

Grouping the characters with the same kernel, we have yet another way to describe the group algebra

$$\mathbb{Q}[G] \simeq \prod_{d|m} E_d.$$

Each τ is given by the composition of the projection $\mathbb{Q}[G]$ to E_d with a complex character of E_d of order d. Since $\mathbb{Q}[G] \otimes \mathbb{C} = \prod_{\tau\in\mathcal{T}} \mathbb{C}$, any module $\mathbb{Q}[G] \otimes \mathbb{C}$-module W admits a decomposition

$$W = \oplus_{\tau\in\mathcal{T}} W_\tau$$

where each W_τ is the Eigenspace on which $g \otimes 1 \in \mathbb{Q}[G] \otimes \mathbb{C}$ acts as multiplication by $\tau(g)$.

Assume that p does not divide the order m of G. Then each character $\tau : \mathbb{Q}[G] \to \mathbb{C}_p$ (under the identification $\mathbb{C}_p \simeq \mathbb{C}$) is unramified at p and factors through $\mathbb{Q}_p^{un}$. Let us by abuse of notation also use $\mathcal{T}$ for $\hom_{\mathbb{Q}}(\mathbb{Q}[G], \mathbb{Q}_p^{un})$. The Frobenius σ of $\mathbb{Q}_p^{un}$ induces an automorphism $\sigma : \tau_n \mapsto \tau_{pn}$ of $\mathcal{T}$. We write $\mathfrak{O}$ for the set of σ-orbits $\mathfrak{o}$ in $\mathcal{T}^* := \mathcal{T} - \{\tau_0\}$. The elements in the same orbit $\mathfrak{o}$ have the same kernel $R_\tau = R_{\mathfrak{o}} \subset G$ and order $d_\tau = d(\mathfrak{o}) = [G : R(\mathfrak{o})]$ (if the orbit contains τ_n then this order is the same as the additive order of n in $\mathbb{Z}/m\mathbb{Z}$). Each orbit $\mathfrak{o}$ is in bijection with a prime $p_{\mathfrak{o}}$ of $E_{d_{\mathfrak{o}}}$ above p such that the following decomposition holds

$$\mathbb{Q}[G] \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{\mathfrak{o}\in\mathfrak{O}} E_{d_{\mathfrak{o}}, p_{\mathfrak{o}}}$$

where $E_{d_{\mathfrak{o}}, p_{\mathfrak{o}}}$ is the completion of $E_{d_{\mathfrak{o}}}$ at the prime $p_{\mathfrak{o}}$.

2.3 Cyclic covers of the projective line and their Jacobians

Let G be as in the previous section. We fix the following numerical data: a positive integer $N \geq 3$ and an N-tuple $a = (a_1, \ldots, a_N)$ of elements in G.

Definition 2.3.1. *The datum $(G, \underline{a}, N)$ is called a monodromy datum with inertia type $\underline{a}$ if it satisfies the following conditions:*

1. *$a_1, ..., a_N$ are nontrivial elements that generate the whole group G;*

2. *their product $\prod_j a_j$ is equal to the identity of G.*

We remark that the above conditions mean $a_i \not\equiv 0 \pmod m$ for all i, $\gcd(G, a_1, \ldots, a_N) =$ 1, *and $a_1 + \cdots + a_m \equiv 0 \pmod m$.*

Fix an N-tuple $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N)$ where $\lambda_i \neq \lambda_j$. A monodromy datum $(G, \underline{a}, N)$ defines a smooth irreducible curve $C = C_{\underline{\lambda}}(G, \underline{a}, N)$ by the equation

$$y^m = (x - \lambda_1)^{a_1}(x - \lambda_2)^{a_2} \ldots (x - \lambda_N)^{a_N}.$$

In addition, C has the structure of a Galois cover of $\mathbb{P}^1$ branched above N points with Galois group G (or G-cover for short) via the covering map $(x, y) \mapsto x$ and the group scheme μ_m acting via

$$\zeta_m \cdot (x, y) := (x, \zeta_m y).$$

The monodromy conditions ensure that the μ_m action on C over $\mathbb{P}^1$ is Galois. When $\underline{\lambda}$ is fixed, according to Riemann's Existence Theorem, the isomorphism classes of G-covers are in bijection with branching data up to permutations of the a_i's and automorphisms of G. The later group is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^*$.

Remark 2.3.1. *The action of the group scheme μ_m is étale over $\mathbb{Z}[\frac{1}{m}, \zeta_m]$. For convenience, all schematic constructions will be assumed to be over the base scheme $S := Spec(\mathbb{Z}[\frac{1}{m}, \zeta_m])$.*

Let $J(C) = J(G, \underline{a})$ denote the Jacobian of C. The dimension of $J(C)$ is equal to the genus of C and can be computed from the monodromy datum via Riemann-Hurwitz formula:

$$g(C) = 1 + \frac{(N-2)m - \sum_{i=1}^{N} \gcd(G, a_i)}{2}.$$

The tangent space $V(C) := \mathrm{Lie}(J(C))$ can be identified with the sheaf of holomorphic differentials $H^0(C, \Omega^1_C)$. The μ_m-action on C induces an action of the group algebra $\mathbb{Q}[G]$ on $H^0(C, \Omega^1_C)$. Then V decomposes as a $\mathbb{Q}[G] \otimes \mathbb{C}$-module into Eigenspaces

$$V = \bigoplus_{\tau \in \mathcal{T}} V_\tau$$

where μ_G acts on V via multiplication by $\tau(G)$.

The complex dimension $f(\tau)$ of V_τ, or multiplicity of τ, can be computed by the Hurwitz-Chevalley-Weil formula (see [23, Theorem 3.3]):

$$f(\tau_n) = \begin{cases} 0 & \text{if } n = 0 \\ -1 + \sum_{i=1}^{N} \langle \frac{-na_i}{m} \rangle & \text{otherwise.} \end{cases} \tag{2.1}$$

The tuple $\underline{f} = (f(\tau))_{\tau\in\mathcal{T}}$ is called the signature and only depends on the monodromy datum, not the base point $\underline{\lambda}$.

Proposition 2.3.1. *$f(\tau) + f(\tau^*) = N - 2 - l(\tau)$ where $l(\tau) = \#\{i : d(\tau) \mid a_i\}$.*

Proof. This follows from Hurwitz-Chevalley-Weil formula and that $\langle\frac{-na_i}{m}\rangle + \langle\frac{na_i}{m}\rangle$ is 0 if $d(\tau) \mid a_i$ and is 1 otherwise. □

Remark 2.3.2. *As a corollary, $f(\tau) + f(\tau^*)$ only depends on the orbit $\mathfrak{o}$ of τ. It is convenient to write $g(\mathfrak{o}) := f(\tau) + f(\tau^*)$ for this number.*

Each character τ gives rise to a cyclic quotient $G/R_\tau \simeq \mathbb{Z}/d_\tau\mathbb{Z}$. We write R for $\ker\tau$ and d for d_τ. The datum $(\mathbb{Z}/d\mathbb{Z}, \underline{a} \pmod d)$ is a monodromy datum for a cyclic $\mathbb{Z}/d\mathbb{Z}$-cover of $\mathbb{P}^1$. There is a surjective map

$$C(G, \underline{a}) \to C(\mathbb{Z}/d\mathbb{Z}, a \pmod d), \quad (x, y) \mapsto (x, y^{m/d})$$

which by functoriality induces an injection

$$J(\mathbb{Z}/d\mathbb{Z}, \underline{a} \pmod d) \hookrightarrow J(G, \underline{a})$$

as an abelian subvariety. We define the new part of $J(G, \underline{a})$ to be

$$J(G, \underline{a})^{new} := J(G, \underline{a})/(\sum_{d|m, d\neq m} J(\mathbb{Z}/d\mathbb{Z}, \underline{a} \pmod d))).$$

Up to isogeny, we have a decomposition

$$J(G, \underline{a}) \sim \bigoplus_{d|m} J(\mathbb{Z}/d\mathbb{Z}, \underline{a} \pmod d)^{new}$$

that is compatible with the decomposition of the group algebra into the product of cyclotomic fields E_d for $d \mid m$.

Proposition 2.3.2. *The genus of the new part $J(G, \underline{a})^{new}$ is $g^{new} = \frac{(N-2)\varphi(m)}{2}$.*

Proof. The genus of $J(G, \underline{a})^{new}$ is the same as the dimension of

$$V^{new} = \bigoplus_{\tau : d(\tau) = m} V_\tau.$$

There are $\frac{\varphi(m)}{2}$ pairs (τ, τ^*) for which $l(\tau) = 0$, and thus, $f(\tau) + f(\tau^*) = N - 2$. □

Example 2.3.1. *The monodromy datum* $(\mathbb{Z}/5\mathbb{Z}, (1,1,1,2), 4)$ *defines a family of curves* C_λ *by the affine equation*

$$y^5 = x(x-1)(x-\lambda).$$

When λ *varies, we get a 1-dimensional family of* G*-covers of* $\mathbb{P}^1$*. A basis for* $H^0(C_\lambda, \Omega^1)$ *(see [20, Lemma 2.7]) is given by*

$$\frac{dx}{y^2}, \quad \frac{dx}{y^3}, \quad \frac{dx}{y^4}, \quad \frac{xdx}{y^4}.$$

The action of G *is defined via* $y \mapsto \zeta_5 y$ *so* G *acts via* τ_2 *on* $\frac{dx}{y^3}$*,* τ_3 *on* $\frac{dx}{y^2}$*, and* τ_1 *on* $\frac{dx}{y^4}$ *and* $\frac{xdx}{y^4}$*. It follows that*

$$f(\tau_2) = f(\tau_3) = 1, f(\tau_1) = 2, f(\tau_4) = 0.$$

These multiplicities of the eigenspaces agree with Hurwitz-Chevalley-Weil formula.

The Hurwitz space $\mathcal{H}(G, \underline{a})$

Fix a monodromy datum $(G, \underline{a})$ with inertia type $\underline{a}$. Let $\mathcal{H}(G)$ be the moduli space of G-covers of $\mathbb{P}^1$ and $\overline{\mathcal{H}(G)}$ denote its Deligne-Mumford compactification. The later is the moduli space of stable G-covers of $\mathbb{P}^1$. The moduli space $\mathcal{H}(G, \underline{a})$ with ordered $\underline{a}$ is an irreducible component of $\mathcal{H}(G)$ and has a model over $\mathbb{Z}[\frac{1}{m}]$. It classifies G-covers with ordered inertia type $\underline{a}$.

Let $U \subset (\mathbb{A}^1)^N$ be the complement of the weak diagonal, i.e., the scheme of N-tuples of distinct points of $\mathbb{A}$. Consider the projective curve $B \subset \mathbb{P}^2_U$ whose affine curve over a point $\underline{\lambda} \in U$ is given by

$$y^m = (x-\lambda_1)^{a_1}(x-\lambda_2)^{a_2} \dots (x-\lambda_N)^{a_N}.$$

There exists an open subscheme $T \subset U$ for which there exists a smooth proper G-cover C_T over T with inertia type $\underline{a}$. Then $\mathcal{H}(G, \underline{a})$ together with its universal family can be constructed as the stable quotient of $C_T \to \mathbb{P}^1_T$ by the action of $\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Sym}(N)$ (see [20], 2.2). From $\dim T = \dim U = N$ and $\dim \mathrm{Aut}(\mathbb{P}^1) = 3$, we deduce that

$$\dim \mathcal{H}(G, \underline{a}) = N - 3. \tag{2.2}$$

If $p \nmid m$ is a prime then both $\mathcal{H}(G, \underline{a})$ and $\overline{\mathcal{H}(G, \underline{a})}$ have good reduction at p.

The Shimura varieties $Sh(G, \underline{f})$ and $Sh(G, \underline{f}^{new})$

Let B be the group algebra $\mathbb{Q}[G]$. Let $\mathcal{G} := \mathrm{GSp}_{2g} \cap \mathrm{GL}_B$ be the connected reductive algebraic group over $\mathbb{Q}$ consisting of the endomorphisms of $H^1(J_C, \mathbb{Z})$ that commute with B. Let X be a $\mathcal{G}(\mathbb{R})$-conjugacy class of the homomorphism $h : \mathbb{S}^1 \to G(\mathbb{R})$ parametrizing the Hodge structure on $H^1(J_C, \mathbb{Z})$ with an action of B prescribed by $\underline{f}$. Then $(\mathcal{G}, X)$ is a Shimura datum in the sense of Deligne. This datum gives rise to a moduli variety denoted by $Sh(G, \underline{f})$ whose reflex field is contained in the cyclotomic field E_m. Furthermore, $Sh(G, \underline{f})$ has a good integral model over $\mathbb{Z}[\frac{1}{m}, \zeta_m]$. For a prime p not dividing m, by abuse of notation, we use $Sh(G, \underline{f})$ to denote the reduction modulo p and $Sh(G, \underline{f})_{\overline{\mathbb{F}}_p}$ to denote its geometric fiber.

When the basepoint $\underline{\lambda}$ varies over $\mathcal{H}(G, \underline{a})$, the Jacobian $J_{\underline{\lambda}}(G, \underline{a})$ is a principally polarized abelian variety equipped with the action of $\mathbb{Q}[G]$ on the tangent space of $J_{\underline{\lambda}}(G, \underline{a})$ specified by $\underline{f}$. There exists a classifying morphism defined over $\mathbb{Z}[\frac{1}{m}, \zeta_m]$

$$\phi : \mathcal{H}(G, \underline{a}) \to Sh(G, \underline{f}), \quad \lambda \mapsto J(C_\lambda).$$

From [20, (3.3.1)], the dimension of $Sh(G, \underline{f})$ is the same as the dimension of $\mathcal{G}$ which can be computed as

$$\dim Sh(G, \underline{f}) = \sum_{(l,-l)} f(\tau_l) f(\tau_{-l}) + \begin{cases} \frac{f(\tau_k)(f(\tau_k)+1)}{2} \text{ if } m = 2k \text{ is even,;} \\ 0 \text{ if } m \text{ is odd} \end{cases} \tag{2.3}$$

where the sum is over all pairs $(l, -l)$ such that $2l \neq m$.

Similarly, we write $Sh(G, \underline{f}^{new})$ for the Shimura variety that is the moduli space of the new part $J(G, \underline{a})^{new}$.

2.4 k3-type surface

In this section we work over $\mathbb{C}$.

Definition 2.4.1. *An K3 surface over $\mathbb{C}$ is a smooth projective surface X such that*

$$h^1(X, \mathcal{O}_X) = 0 \text{ and } \omega_X \simeq \mathcal{O}_X,$$

that is, X is simply connected and has trivial canonical bundle.

Example 2.4.1. *If X is a smooth quartic in $\mathbb{P}^3$, then $\omega_X \simeq \mathcal{O}_X$ via the adjunction formula, and $h^1(X, \mathcal{O}_X) = 0$ from taking cohomology of the short exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0.$$

In particular, X is a K3 surface.

Example 2.4.2. *If A is a abelian surface, then the surface quotient $A/\pm id$ has 16 singularities of type A_1. Its minimal resolution of singularities $Km(A)$ is a K3 surface, called the Kummer surface associated to A.*

Definition 2.4.2. *A Hodge structure of (pure) weight n is a free $\mathbb{Z}$-module $H_{\mathbb{Z}}$ of finite rank together with a (Hodge) decomposition of complex vector spaces*

$$H := H_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$$

such that $\overline{H^{p,q}} = H^{p,q}$.

A rational Hodge structure $H_{\mathbb{Q}}$ is a finite dimensional vector space over $\mathbb{Q}$ with a Hodge decomposition of $H_{\mathbb{Q}} \otimes \mathbb{C}$. If $H_{\mathbb{Z}}$ is a Hodge structure then $H_{\mathbb{Z}} \otimes \mathbb{Q}$ is a rational Hodge structure.

The Hodge number $h^{p,q}$ is the complex dimension $\dim_{\mathbb{C}} H^{p,q}$ and the set $\mathcal{S} \subset \mathbb{Z}^2$ of (p,q) with $h^{p,q} \neq 0$ is called the type of H.

Example 2.4.3. *If X is a compact Kähler manifold, e.g., the analytic variety associated to a smooth projective variety, its n^{th} Betti cohomology group $H^n(X,\mathbb{Z})$ modulo torsion is a pure Hodge structure.*

By Chow's theorem, among the compact Kähler manifolds, the algebraic ones are those that admit an ample line bundle L. The embedding by L into some projective space gives rise to a bilinear form $Q(\mathcal{L})$ on $H^n(X,\mathbb{Z})$ satisfying certain compatibility and positivity conditions. Such linear algebra data are captured in the notion of a polarized Hodge structure.

Definition 2.4.3. *A polarized Hodge structure of weight n is a Hodge structure $H_{\mathbb{Z}}$ together with a non-degenerate integer bilinear pairing Q on $H_{\mathbb{Z}}$ whose linear extension to H satisfies the following conditions:*

1. $Q(x,y) = (-1)^n Q(y,x)$;
2. $Q(x,y) = 0$ *for all* $x \in H^{p,q}, y \in H^{p',q'}$ *and* $p \neq q'$;
3. $i^{p-q} Q(x,\bar{x}) > 0$ *for all* $x \in H^{p,q}$ *and* $x \neq 0$.

Definition 2.4.4. *A K3 type Hodge structure is a polarized Hodge structure of weight* 2 *with Hodge numbers $h^{2,0} = 1$, $h^{1,1} = n$, and $h^{0,2} = 1$ (here n is necessarily even).*

Definition 2.4.5. *A k3-type surface X is a smooth projective surface such that $h^1(X, O_X) = 0$ and its second Betti cohomology $H^2(X, \mathbb{Z})$ is a K3 type Hodge structure.*

Example 2.4.4. *Let $(X, \mathcal{L})$ be an algebraic K3 surface where $\mathcal{L}$ is a primitive ample line bundle (necessarily of degree $2d$ for some positive integer d). The Chern class $c(\mathcal{L})$ is a $(1,1)$-class and determines the isomorphism class of L since $h^{1,0}(X) = 0$. Using $c(\mathcal{L})$, we can change the sign of the intersection pairing to obtain a polarization form $Q(\mathcal{L})$ on $H^2(X, \mathbb{Z})$ so that $(H^2(X, \mathbb{Z}), Q(\mathcal{L}))$ is a polarized K3 type Hodge structure of rank 22. To construct the moduli of algebraic K3 surfaces, one fixes the degree of L and considers the primitive cohomology $H^2_{prim}(X, \mathbb{Z}) := c(\mathcal{L})^{\perp} \subset H^2(X, \mathbb{Z})$ with the restricted bilinear form of signature $(2, 19)$ and rank 21. $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ are isomorphic polarized K3 surfaces if and only if there exists a Hodge isometry between $(H^2(X, \mathbb{Z}), Q(\mathcal{L}))$ and $(H^2(X', \mathbb{Z}), Q(\mathcal{L}'))$ taking $c(\mathcal{L})$ to $c(L')$.*

We recall the definition of complex multiplication, or CM, for K3 surface analogous to CM abelian varieties.

Definition 2.4.6. *Let $T(X)$ denote the orthogonal complement of $NS(X)$ inside $H^2(X(\mathbb{C}), \mathbb{Z})$ with respect to the Poincaré pairing. The $\mathbb{Z}$-module $T(X)$ equipped with the restricted Poincaré pairing is called the transcendental lattice of X.*

From $NS(X) \otimes \mathbb{Q} = H^2(X, \mathbb{Q}) \cap H^{1,1}(X)$, we deduce that $T(X)_{\mathbb{Q}}$ is a rational Hodge structure of K3 type, i.e.,

$$T(X) \otimes \mathbb{C} = T^{2,0} \oplus T^{1,1} \oplus T^{0,2}$$

where $T^{1,1}$ is self-conjugate and $\overline{T^{2,0}} = T^{0,2}$ has complex dimension 1. Let $\mathcal{E}(X)$ denote the endomorphism algebra $\mathrm{End}_{\mathrm{Hg}}(T(X))$ of the Hodge structure $T(X)_{\mathbb{Q}}$. One can show that $\mathcal{E}(X)$ is commutative via a natural embedding $\mathcal{E}(X) \hookrightarrow \mathrm{End}_{\mathbb{C}}(H^{2,0}(X)) \cong \mathbb{C}$. In particular, $\mathcal{E}(X)$ is either a totally real or an imaginary number field. Note that $[E : \mathbb{Q}] \leq 20$ for dimension reason.

Definition 2.4.7. *X is said to have complex multiplication if $\mathcal{E}(X)$ is equal to a CM field F and in addition $[F : \mathbb{Q}] = rank_{\mathbb{Z}} T(X)$. In particular, there exists a unique complex embedding τ of L such that F acts on $T(X)^{2,0}$ via multiplication by τ. The datum of (F, τ) is called the CM-type of X.*

2.5 Newton polygon

We recall the definition F-crystal over k and define its Newton polygon.

F-crystals

Definition 2.5.1. *An F-crystal (G, φ) is a free $W(k)$-module of finite rank together with an injective map $\varphi : M \to M$ that is both σ-linear and additive.*

An F-isocrystal is a finite dimensional vector space over K together with a bijective σ-linear map φ.

We write M for an F-crystal and omit φ from the notation without any confusion.

Example 2.5.1. *For a smooth projective variety X over k, its n^{th} crystalline cohomology group modulo torsion $H := H^n(X/W)/torsion$ is a free W-module of finite rank. The absolute Frobenius morphism of X induces a σ-linear map $\varphi : H \to H$ by functoriality. Poincaré duality induces a perfect pairing*

$$\langle , \rangle : H^i \times H^{2\dim(X)-i} \to H^{2\dim(X)}(X/W) \simeq W$$

satisfying

$$\langle \varphi(x), \varphi(y) \rangle = p^{\dim(X)} \sigma(\langle x, y \rangle).$$

Since $\langle , \rangle$ is perfect and $\sigma : W \to W$ is injective, φ is injective, and thus, (H, φ) is an F-crystal.

Example 2.5.2. *Let $W\langle T \rangle$ be the non-commutative polynomial ring in one variable over $W(k)$ subject to the relations*

$$Tx = \sigma(x)T \text{ for all } x \in W(k).$$

Let $\alpha = \frac{r}{s} \in \mathbb{Q}_{\geq 0}$ be a rational number in reduced form. Then the $W(k)$-module

$$M_\alpha := W\langle T \rangle / (T^s - p^r)$$

equipped with $\varphi : x \mapsto Tx$ is an F-crystal (M_α, φ) of rank s. The rational value α is called the slope of this F-crystal.

The next result classifies F-crystal up to isogeny when k is algebraically closed.

Theorem 2.5.1 (Dieudonné-Manin)**.** *If k is algebraically closed, then the category of F-crystals over k up to isogeny is semi-simple and the simple objects are of form (M_α, φ) for $\alpha \in \mathbb{Q}_{>0}$.*

Definition 2.5.2. *Let M be an F-crystal M over an algebraically closed field k and consider the decomposition of M into the simple objects up to isogeny*

$$M \sim \bigoplus_{\alpha \in \mathbb{Q}_{\geq 0}} M_\alpha^{n_\alpha}.$$

The elements in the set

$$\{\alpha \in \mathbb{Q}_{\geq 0} \mid n_\alpha \neq 0\}$$

are called the Newton slopes of M. For each slope α of M, its multiplicity is defined as

$$m_\alpha := n_\alpha \cdot rank_{W(k)}.(M_\alpha).$$

The slopes in ascending order

$$0 \leq \alpha_1 < \cdots < \alpha_l$$

together with their multiplicities $m_1, \ldots, m_l$ define the Newton polygon of M. It can be equivalently encoded in the lower convex polygon connecting the $l+1$ points with integer coordinates

$$\{(0,0), (m_1, m_1\alpha_1), (m_1 + m_2, m_1\alpha_1 + m_2\alpha_2), \ldots, (\sum_{i=1}^{l} m_i, \sum_{i=1}^{l} m_i\alpha_i)\}.$$

For an F-crystal M over k, its Newton polygon is defined as the Newton polygon of $M \otimes_{W(k)} W(\bar{k})$ which is an F-crystal over $\bar{k}$. Let $\nu(M)$ denote this polygon.

Next, as φ is injective, $M/\varphi(M)$ is an Artinian W-module. Using the structure theorem for Artinian modules over the PID W, there exists s non-negative integers h_i and an isomorphism

$$M/\varphi(M) \cong \bigoplus_{i \geq 1} (W/p^i W)^{h_i}.$$

In addition, we define $h_0 := \mathrm{rank}_W(M) - \sum_{i\geq 1} h_i$. The integers $i \geq 0$ such that $h_i \neq 0$ are called the Hodge slopes of M. The Hodge slopes i in ascending order together with their multiplicities h_i define the Hodge polygon of M. Let $\mu(M)$ denote this polygon.

The Newton polygon always lies on or above the Hodge polygon. Furthermore, for smooth proper varieties X satisfying certain conditions, the Hodge polygon of its crystalline cohomology are equivalently encoded in the Hodge numbers.

Proposition 2.5.2. *Let M be an F-crystal over k. Then $\nu(M)$ lies on or above $\mu(M)$.*

Theorem 2.5.3 (Mazur, Nygaard, Ogus)**.** *Let X be a smooth and proper variety over k. If $H^n_{cris}(X/W)$ is torsion-free and the Frölicher spectral sequence of X degenerates at E_1, then $h_i = \dim H^{n-i}(X, \Omega^i_{X/k})$ for all $0 \leq i \leq n$.*

The varieties of interest in this thesis are good reductions of smooth projective varieties in characteristic 0 and hence satisfy the conditions of the theorem.

Example 2.5.3. *Let A be an abelian variety of dimension g over k. The crystalline cohomology $H^1_{cris}(A/W)$ is an F-crystal of rank $2g$. Its Hodge polygon is $\{(0,0),(g,0),(2g,g)\}$. Its Newton polygon lies above the Hodge polygon, has the same endpoints, and has integer breakpoints such that if α is a slope with multiplicity n, then $1-\alpha$ is also a slope with multiplicity n.*

Definition 2.5.3. *A K3 F-crystal of rank n is an F-crystal (H, φ) of rank n together with a symmetric bilinear form*

$$\langle , \rangle : H \otimes_W H \to W$$

such that

1. *$\varphi \otimes k$ has rank 1;*
2. *$p^2 H \subset \varphi(H)$;*
3. *$\langle , \rangle$ is a perfect pairing;*
4. *$\langle \varphi(x), \varphi(y) \rangle = p^2 \langle x, y \rangle$.*

The K3 crystal (H, φ) is called supersingular if in addition it is purely of slope 1.

Example 2.5.4. *If X is a K3 surface over k, then $H := H^2_{cris}(X/W)$ equipped with the map φ induced by the absolute Frobenius and the bilinear pairing $\langle , \rangle$ from Poincaré duality is a K3 F-crystal of rank 22.*

Proposition 2.5.4. *There exists an integer $h := h(X) \in \{1, 2, \ldots, 10\}$ such that the Newton polygon of $H^2_{cris}(X/W)$ has slopes*

$$(1 - 1/h, 1, 1 + 1/h)$$

with multiplicities

$$(h, 22-2h, h).$$

Or the Newton polygon has slopes 1 only in which case we set $h(X) = \infty$.

Proof. Poincaré duality implies that $\langle F(x), F(y)\rangle = p^2\sigma\langle x, y\rangle$ which means the Newton slopes are closed under the map $\lambda \mapsto 2-\lambda$.

As the Hodge numbers of X are $1, 20, 1$, and the Newton polygon has to lie above or on the Hodge polygon with integer break points, $\nu(X)$ has slopes 1 only or there exists an integer h that is the x-coordinate of the first break point $(h, h-1)$. This point corresponds to the slope $1-1/h$ with multiplicity h and so $1+1/h$ must also be a slope with multiplicity h. The only remaining slope possible is 1 with multiplicity $22-2h$.

A priori, $1 \leq h \leq 11$. One considers an ample line bundle L on X. Its first Chern class $c_1(L) \in H^2_{cris}(X/W)$ satisfies

$$F(c_1(L) = c_1(F^*(L)) = c_1(L^{\otimes p}) = pc_1(L).$$

Hence $H^2_{cris}(X/W)$ has at least one slope 1 and $h < 11$. □

Remark 2.5.1. *We call $h(X)$ the height of X. The terminology stems from the fact that $h(X)$ is also the height of the formal Brauer group of X.*

Example 2.5.5. *If A an abelian surface over k, then $H^2_{cris}(A/W) \simeq \Lambda^2 H^1_{cris}(A/W)$ is a K3 F-crystal of rank 6.*

Definition 2.5.4. *A k3-type surface over k is a smooth proper surface X such that $h^1(X, \mathcal{O}_X) = 0$, the Frölicher spectral sequence of X degenerates at E_1, and the second crystalline cohomology $H^2_{cris}(X/W)$ is a K3 F-crystal.*

Artin invariant

There exists a crystalline Chern map

$$c_1 : \mathrm{Pic}(X) \to H^2_{cris}(X/W)$$

which is a homomorphism of abelian groups. We call the image of this map the Neron-Severi group $\mathrm{NS}(X)$. For any $L \in \mathrm{Pic}(X)$,

$$c_1(F^*(L)) = c_1(L^{\otimes p}) = pc_1(L)$$

where F is the absolute Frobenius morphism of X. This means the image $\mathrm{NS}(X)$ in $H^2_{cris}(X/W)$ is contained in the $\mathbb{Z}_p$-submodule of elements x such that $\varphi(x) = px$.

Definition 2.5.5. *Let H be a K3 F-crystal. The Tate module of H is defined as*

$$T_H := \{x \in H : \varphi(x) = px\}.$$

Remark 2.5.2. *The Tate conjecture (established for K3 surfaces over finite fields due to works of several authors) implies that c_1 induces an isomorphism after tensoring with $\mathbb{Q}_p$. Hence the Picard rank of X is given by*

$$\rho(X) := rank_{\mathbb{Z}_p} Pic(X) = rank_{\mathbb{Z}_p} T_H.$$

Proposition 2.5.5 (Proposition 4.7, [18])**.** *Let $(H, \varphi, \langle,\rangle)$ be a supersingular K3 F-crystal. Then*

$$rank_W H = rank_{\mathbb{Z}_p} T_H$$

and the bilinear form $\langle,\rangle$ restricted to T_H is non-degenerate form

$$\langle,\rangle : T_H \otimes_{\mathbb{Z}_p} T_H \longrightarrow \mathbb{Z}_p$$

which is not perfect. In particular,

1. *$ord_p(T_H, \langle,\rangle) = 2\sigma_0 > 0$ for some integer σ_0 called the Artin invariant of H.*
2. *$(T_H, \langle,\rangle)$ is determined up to isometry of polarized $\mathbb{Z}_p$-lattices by σ_0.*
3. *There exists an orthogonal decomposition*

$$(T_H, \langle,\rangle) = (T_0, p\langle,\rangle) \oplus (T_1, \langle,\rangle)$$

where $(T_0, \langle,\rangle)$ and $(T_1, \langle,\rangle)$ are $\mathbb{Z}_p$-lattices with perfect bilinear forms such that $rank_{\mathbb{Z}_p} T_0 = 2\sigma_0$ and $rank_{\mathbb{Z}_p} T_1 = rank_W H - 2\sigma_0$, respectively.

Thus σ_0 is an integer between 1 and 10.

The μ-ordinary polygon and Kottwitz method

The main references of this section are [14, 19, 25].

Let k be a perfect algebraically closed field in characteristic p. We briefly recall Kottwitz method to compute the set of admissible Newton polygons of an F-isocrystals with compatible bilinear form and an action of a split unramified semi-simple algebra B over $\mathbb{Q}_p$. Via Morita equivalence, it suffices to consider unramified extension B of $\mathbb{Q}_p$.

Let B be a degree d unramified extension of $\mathbb{Q}_p$ and O_B be a maximal order of B with residue field $\mathbb{F}_q$.

By an F-crystal with O_B structure we mean an F-crystal $(M, \varphi, \langle . \rangle)$ together with an injection $O_B \hookrightarrow \mathrm{End}(M, \varphi)$ and an involution $* : O_B \to O_B$ such that for all $b \in O_B$

$$\langle b-, - \rangle = \langle -, b^* - \rangle.$$

Let $I := \mathrm{Hom}(O_B, W(\overline{\mathbb{F}}_p))$. The group I can be identified with $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and the cyclic group $\mathbb{Z}/d\mathbb{Z}$. Since M is a module over $O_B \otimes W(k)$ and k contains $\overline{\mathbb{F}}_p$, we have a decomposition into character spaces

$$M = \oplus_{i \in I} M_i.$$

For each i, the map φ restricts to a σ-linear map $\varphi_i : M_i \to M_{i+1}$. Then φ^d restricts to a σ^d-linear map on each M_i and ϕ is a morphism between the F-crystals (M_i, ϕ^d) and (M_{i+1}, ϕ^d). In particular, the (M_i, ϕ^d) are isogenous summands of (M, ϕ) - a slope α appears in $\nu(M, \phi)$ with multiplicity m if and only if the slope $d\alpha$ appears in $\nu(M_i, \phi^d)$ with multiplicity m/d.

Next, we consider the decomposition of Artinian W-modules

$$M/\phi(M) = \oplus_i M_i/\phi_i(M_{i-1}).$$

Let μ_i be the Hodge polygon of $M_i/\phi_i(M_{i-1})$ so that $\mu(M)$ is the amalgamate sum of the μ_i. The μ-ordinary polygon (also called σ-invariant Hodge polygon) of M is defined as

$$\nu_{ord} := \frac{1}{d} \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \sigma^i(\mu(M)).$$

Here the sum is entry-wise where we organize $\mu(M)$ into d blocks corresponding to the μ_i and $\sigma^i(\mu(M))$ is obtained from $\mu(M)$ by replacing μ_j with μ_{j+i} for all j.

Let us compute the μ-ordinary polygons for the Shimura variety $Sh(G, \underline{f})_{\overline{\mathbb{F}}_p}$ from 2.3.

Example 2.5.6. *Let $x \in Sh(G, \underline{f})(\overline{\mathbb{F}}_p)$ and A_x be the corresponding abelian variety over $\overline{\mathbb{F}}_p$. The crystalline cohomolgy $M := H^1_{cris}(A/W(\overline{\mathbb{F}}_p))$ is an F-crystal together with endomorphism by the split unramified semi-simple algebra $\mathbb{Q}[G] \otimes \mathbb{Q}_p$. The action of the maximal order*

$$\mathbb{Z}[G] \otimes \mathbb{Z}_p = \prod_{\mathfrak{o} \in \mathfrak{O}} O_{d_{\mathfrak{o}}, \mathfrak{o}}$$

induces a decomposition

$$M = \oplus_{\mathfrak{o}\in\mathfrak{O}} M_{\mathfrak{o}}$$

into F-crystals with $\mathcal{O}_{d_{\mathfrak{o}},\mathfrak{o}}$ structure $M_{\mathfrak{o}}$. The μ-ordinary polygon of M is defined as the amalgamate sum of the μ-ordinary polygons of the $M_{\mathfrak{o}}$.

Let $s(\mathfrak{o})$ be the number of distinct values of $\{f(\tau) \mid \tau \in \mathfrak{o}\}$ in $[1, g(\mathfrak{o}) - 1]$. Let $E(1), \dots, E(s(\mathfrak{o})$ denote these distinct values such that

$$E(0) := g(\mathfrak{o}) > E(1) > \cdots > E(s(\mathfrak{o})) > E(s(\mathfrak{o}) + 1) := 0.$$

Lemma 2.5.1. *With notations as above, the μ-ordinary polygon $\mu_{\mathfrak{o}}$ of $M_{\mathfrak{o}}$ is given by the $s(\mathfrak{o}) + 1$ distinct slopes*

$$0 \le \alpha(0) < \alpha(1) < \cdots < \alpha(s(\mathfrak{o})) \le 1$$

such that for all $0 \le l \le s(\mathfrak{o})$

$$\alpha(t) = \frac{1}{\#\mathfrak{o}} \sum_{0}^{t} \#\{\tau \in \mathfrak{o} \mid f(\tau) = E(i)\}$$

with multiplicity

$$m(t) = \#\mathfrak{o} \cdot (E(l) - E(l + 1)).$$

Proof. For each orbit $\mathfrak{o}$, we identify $\mathfrak{o}$ with $\mathrm{Hom}(\mathcal{O}_{d_{\mathfrak{o}},\mathfrak{o}}, W(\overline{F}_p))$ and $\mathbb{Z}/\#\mathfrak{o}\mathbb{Z}$. Consider the decomposition into character spaces

$$M_{\mathfrak{o}} = \oplus_{\tau_i \in \mathfrak{o}} M_i.$$

By a comparison theorem between Betti and algebraic de Rham cohomology, μ_i has the form $\{0, \dots, 0, 1, \dots, 1\}$ with multiplicities $g(\mathfrak{o}) - f(\tau_i)$ and $f(\tau_i)$. The formula then follows from averaging over the μ_i. □

By [34, Theorem 1.6], the Newton polygons that occur on $Sh(G, \underline{f})_{\overline{\mathbb{F}}_p}$ are precisely of the forms $\oplus_{\mathfrak{o}\in\mathfrak{O}} \nu_{\mathfrak{o}}$ such that the following conditions hold:

1. If $\mathfrak{o}^* \neq \mathfrak{o}$, then $M_{\mathfrak{o}} \cong M_{\mathfrak{o}^*}^{\vee}(1)$, i.e., if $\nu_{\mathfrak{o}}$ has slopes $\alpha_1, \dots, \alpha_s$ with multiplicities $m_1, \dots, m_s$ then $\nu_{\mathfrak{o}^*}$ has slopes $1 - \alpha_1, \dots, 1 - \alpha_s$ with multiplicities $m_1, \dots, m_s$. If $\mathfrak{o}^* = \mathfrak{o}$, then $\nu_{\mathfrak{o}}$ is symmetric, i.e. if α is a slope with multiplicity m then $1 - \alpha$ is also a slope with multiplicity m.

2. $\nu_{\mathfrak{o}} \geq \mu_{\mathfrak{o}}$.

3. The multiplicities of slopes in $\nu_{\mathfrak{o}}$ are divisible by $\#\mathfrak{o}$.

The polygons satisfying all these conditions are also called admissible. The set of admissible polygons is called the Kottwitz set and denoted by $B(G, \underline{f})$. The highest polygon in this set is called the basic polygon.

Let F be an abelian extension of $\mathbb{Q}$ of degree m and p be an unramified rational prime for F. As F is abelian, we have a well-defined Frobenius element Fr_p in $\mathrm{Gal}(F/\mathbb{Q})$. We identify the set of primes of F above p with the set $\mathfrak{O}$ of Fr_p-orbits in $\mathrm{Gal}(F/\mathbb{Q})$. Let K be the subfield of F over which p splits completely, i.e., $K = F^{\langle \mathrm{Fr}_p \rangle}$, and set $d := [F : K]$. We have

$$F \otimes \mathbb{Q}_p \simeq \prod_{\mathfrak{o} \in \mathfrak{O}} F_{\mathfrak{o}}$$

where each $F_{\mathfrak{o}}$ is an unramified extension of degree d of $\mathbb{Q}_p$. Let σ^* denote the image of σ under the action of complex conjugation in $\mathrm{Gal}(F/\mathbb{Q})$. In particular, $\sigma^* = \sigma$ if F is totally real.

Lemma 2.5.2. *Let $(M, \varphi, \langle,\rangle)$ be a K3 F-crystal over $\mathbb{Z}_p$ of rank n together with endomorphism by a maximal order O of $F \otimes \mathbb{Q}_p$ such that $\langle b-, -\rangle = \langle -, b^*-\rangle$ for all $b \in O$. Then the following hold:*

1. *If $\mathfrak{o} \neq \mathfrak{o}^*$ or $n > m$, then $\mu_{ord}(M)$ has slopes $1 - \frac{1}{d}, 1, 1 + \frac{1}{d}$ with multiplicities $d, n - 2d, d$.*

2. *If $\mathfrak{o} = \mathfrak{o}^*$ and $n = m$, then $\mu_{ord}(M)$ is supersingular, i.e., has slopes 1 only.*

Proof. Consider the decomposition of (M, φ) into the F-crystals with $O_{F_{\mathfrak{o}}}$ structure $M_{\mathfrak{o}}$ of rank $n/\#\mathfrak{O}$

$$M \simeq \oplus_{\mathfrak{o} \in \mathfrak{O}} M_{\mathfrak{o}}.$$

The compatibility condition implies that the Hodge slopes of $M_{\mathfrak{o}}$ and $M_{\mathfrak{o}^*}$ are in bijection under the map $\lambda \mapsto 2 - \lambda$. As the Hodge polygon of M has the form $(0, 1, \ldots, 1, 2)$, there exists a unique $\mathfrak{e} \in \mathfrak{O}$ such that $M_{\mathfrak{e}}$ has Hodge slope 0. Now a routine case by case calculation shows that

1. If $\mathfrak{o} \neq \mathfrak{o}^*$ then $\mu_{ord}(M_{\mathfrak{e}})$ is obtained by averaging over the d Hodge polygons of length n/m, one of the form $\{0, 1, \ldots, 1\}$ and $d-1$ of the form $\{1, 1, \ldots, 1\}$.

Similarly, $\mu_{ord}(M_{\mathfrak{e}}^*)$ is the average of the d Hodge polygons of length n/m: one of the form $\{1, \ldots, 1, 2\}$ and $d-1$ of the form $\{1, 1, \ldots, 1\}$. The remaining $M_{\mathfrak{o}}$ have slopes 1 only.

2. If $\mathfrak{o} = \mathfrak{o}^*$ and $n > m$ then $\mu_{ord}(M_{\mathfrak{e}})$ is obtained by averaging over the d Hodge polygons of length n/m: one of the form $\{0, 1, \ldots, 1\}$, one of the form $\{1, \ldots, 1, 2\}$, and $d-2$ of the form $\{1, 1, \ldots, 1\}$. The remaining $M_{\mathfrak{o}}$ have slopes 1 only.

3. If $\mathfrak{o} = \mathfrak{o}^*$ and $n = m$ then $\mu_{ord}(M_{\mathfrak{e}})$ is obtained by averaging over the d Hodge polygons of length 1: one of the form $\{0\}$, one of the form $\{2\}$, and $d-2$ of the form $\{1\}$, which results in slopes 1 only.

□

The Artin invariant and Weyl group element

The main references for this section are [9, 22].

In [22], the authors define orthogonal F-zips and prove that they are classified by certain elements of the Weyl group of the associated orthogonal group called the final types. The example of interest to us is the primitive de Rham cohomology $H^2_{prim}(X, k)$ of a K3 surface over k which is the reduction mod p of its primitive K3 F-crystal $H^2_{prim}(X/W)$. We describe to state the main result from [9] that describes the correspondence from final types to Newton polygons and Artin invariants.

Let $(M, \varphi, \langle,\rangle)$ be a K3 F-crystal of even rank $2n$. Let $(\overline{M}, \overline{\varphi}, \langle,\rangle)$ denote the reduction mod p which is a vector space over k equipped with an orthogonal filtration . Let $SO(2n)$ denote the standard special orthogonal group. With a suitable choice of basis, the Weyl group $W_{SO(2n)}$ can be identified with the subgroup of $\mathfrak{S}_{2n}$ of permutations π such that $\pi(i) + \pi(2n+1-i) = 2n+1$ and the number of i with $\pi(i) > n+1$ is even. The n simple reflections of $W_{SO(2n)}$ are the permutations $s_i = (i, i+1)(2n-i, 2n+1-i)$ for $i = 1, \ldots, n-1$, and $s_n = (n-1, n+1)(n, n+2)$. The length of a permutation π is defined as

$$l(\pi) = \#\{1 \leq i < j \leq n : \pi(i) > \pi(j)\} + \#\{1 \leq i < j \leq n : \pi(i) + \pi(j) > 2n+1\}.$$

We set $J := \{s_2, \ldots, s_n\}$ so that the set of cosets $W_J \diagdown W_{SO(2n)}$ consists of $2n$ elements $\omega_1 \succ \omega_2 \succ \cdots \succ \omega_{2n}$ with the total Bruhat order.

Proposition 2.5.6 (Theorem 7.1. [9])**.** *With notations as above, we have:*

1. *M has finite height $h \leq n$ if and only if $\overline{M}$ has final type ω_h.*

2. *M is supersingular with Artin invariant $\sigma_0 \leq n$ if and only if $\overline{M}$ has final type $\omega_{2n+1-\sigma_0}$.*

Lemma 2.5.3. *Let B be a degree d unramified extension of $\mathbb{Q}_p$ and $(M, \varphi, \langle,\rangle)$ be K3 crystal with O_B structure of rank m over O_B. Assume that we are in the situation where $b^* \neq b$, then*

1. *If M has finite height h, then d divides h.*

2. *If M is supersingular with Artin invariant σ_0, then d divides σ_0.*

Proof. 1. We recall the decomposition into character spaces that are isogenous F-crystals
$$M = \oplus_{\tau \in I} M_\tau.$$
In particular, in the case of finite height, the multiplicity h is divisible by the cardinality d of I.

2. From $\langle bx, y\rangle = \langle x, b^* y\rangle$ and $b \neq b^*$, we deduce that the bilinear form admits an orthogonal decomposition
$$(M, \langle,\rangle) = \oplus_{\tau \in I} (M_\tau, \langle,\rangle_\tau).$$
By definition, the Artin invariant is the p-ordinal of the discriminant of $\langle,\rangle$ restricted to a basis of Tate classes of M. Let $x = (x_{\tau_i})_{\tau_i \in I}$ and $y = (y_{\tau_i})_{\tau_i \in I}$ be two Tate classes, i.e., $\varphi(x) = px$ and $\varphi(y) = py$. Since $\varphi : M_{\tau_i} \to M_{\tau_{i+1}}$, we nave $\varphi(x_{\tau_i}) = px_{\tau_{i+1}}$ and $\varphi(y_{\tau_i}) = py_{\tau_{i+1}}$.

It suffices to show that the discriminant of each $\langle,\rangle_\tau$ is the same, i.e., for all $\tau_i \in I$, we have
$$\langle x_{\tau_i}, y_{\tau_i}\rangle_{\tau_i} = \langle x_{\tau_{i+1}}, y_{\tau_{i+1}}\rangle_{\tau_{i+1}}.$$
Indeed we can verify that
$$p^2\langle x_{\tau_i}, y_{\tau_i}\rangle_{\tau_i} = \langle \varphi(x_{\tau_i}), \varphi(y_{\tau_i})\rangle_{\tau_i} = \langle px_{\tau_{i+1}}, py_{\tau_{i+1}}\rangle_{\tau_i} = p^2\langle x_{\tau_{i+1}}, y_{\tau_{i+1}}\rangle_{\tau_{i+1}}.$$

□

C h a p t e r 3

CALCULATION OF FAMILIES OF K3-TYPE SURFACES

In this chapter, we review the construction and show important features of the families for which our main theorems apply.

Inventory of examples

Let $C = C(G, \underline{a}, N_C)$ and $D = D(G, \underline{b}, N_D)$ be two monodromy data. The group scheme μ_G acts diagonally on the product $C \times D$ via

$$(x, y, z, t) \mapsto (x, \zeta y, z, \zeta^{-1} t).$$

Let $\mathbb{C}[C]$ and $\mathbb{C}[D]$ denote the affine coordinate rings over $\mathbb{C}$. We consider the GIT quotient $S := (C \times D)/\mu_G$ which has isolated singularities of cyclic-quotient type. After blowing up all these singularities, we get a smooth surface $\tilde{S}$ that may not be minimal in general. One can compute the minimal model $X := \tilde{S}^{\min}$ by searching and blowing down the (-1)-divisors.

Definition 3.0.1. *We call the monodromy datum $(G, \underline{a}, \underline{b}, N_C, N_D)$ a K3 type pair if the resulting surface X is of K3 type.*

Using the birational invariance of $h^{2,0}$ and Kunneth formula, we can show that

$$h^{2,0}(X) = h^{2,0}((C \times D)/\mu_G) = \dim_{\mathbb{C}} H^{2,0}(C \times D)^G = \sum_{(\tau,\tau^*)} f_C(\tau) f_D(\tau^*).$$

Proposition 3.0.1. *X is a K3 type if and only if there exists a unique $\tau \in \mathcal{T}$ such that $f_C(\tau) f_D(\tau^*) = 1$ and $f_C(\tau) f_D(\tau^*) = 0$ otherwise.*

Proof. This follows directly from the definition of k3-type surface and the Kunneth formula for $h^{2,0}(X)$. □

Corollary 3.0.2. *If the monodromy data $C(G, \underline{a}, N_C)$ and $D(G, \underline{b}, N_D)$ is a K3 type pair, then $N_C = 3$ or $N_D = 3$. Assume that $N_D = 3$, then $N_C - 3 = \dim Sh(G, \underline{f}_C^{new})$.*

Proof. From Proposition 2.3.1, $f_C(\tau_l) + f_C(\tau_{-l}) = N_c - 2$ and $f_C(\tau_l) + f_C(\tau_{-l}) = N_D - 2$ for any $l \in (\mathbb{Z}/m\mathbb{Z})^*$. This means the unique index l such that $f_C(\tau_l) f_D(\tau_{-l}) = 1$

is in $(\mathbb{Z}/m\mathbb{Z})^*$. Assume on the contrary that both N_C and N_D are greater than 3 so that $N_c - 2$ and $N_D - 2$ are both greater than 1. Thus, $f_C(\tau_{-l})$ and $f_D(\tau_l)$ are both at least 1 so $f_C(\tau_{-l})f_D(\tau_l)$ is at least 1, which is a contradiction.

Without loss of generality, assume that $N_D = 3$ so $\{f_D(\tau_l), f_D(\tau_{-l})\} = \{0, 1\}$ for every $l \in (\mathbb{Z}/m\mathbb{Z})^*$. Let l_0 be the unique index such that $f_D(\tau_{l_0}) = f_C(\tau_{-l_0}) = 1$ so $f_C(\tau_{l_0})f_C(\tau_{-l_0}) = N_C - 3$. For other $l \in (\mathbb{Z}/m\mathbb{Z})^*$, $f_C(\tau_l)f_C(\tau_{-1}) = 0$ since if both are nonzero then $f_D(\tau_l) = f_D(\tau_{-l}) = 0$ but their sum is 1.

It follows from 2.3 that $\dim Sh(G, \underline{f}_C^{new}) = \sum f_C(\tau_l)f_C(\tau_{-l}) = N_C - 1$. $\square$

The unique τ in 3.0.1 can be chosen to be τ_{-1} without changing the isomorphism class of the monodromy datum and this choice corresponds to normalizing the inertia data such that

$$\sum a_i = 2m \text{ and } \sum b_i = m.$$

To summarize, we shall work with normalized K3 type datum $(G, \underline{a}, \underline{b}, N)$ satisfying the following conditions:

1. $a_1 + \cdots + a_N = 2m$ and $\gcd(a_1, \ldots, a_N, m) = 1$;
2. $b_1 + b_2 + b_3 = m$ and $\gcd(b_1, b_2, b_3, m) = 1$;
3. $f_C(\tau_{-1})f_D(\tau_1) = 1$ and $f_C(\tau_{-i})f_D(\tau_i) = 0$ for $i \neq 1$.

Remark 3.0.1. *Moonen only considered the case $N_C \geq 4$ to obtain examples of positive dimensional families but the construction also works for $N_C = 3$. The following example illustrates how to compute the minimal model of the resolution of singularities.*

Example 3.0.1. *Take $(5, (3, 3, 4), (1, 1, 3), 3)$ to be the datum. This datum defines affine curves*

$$C : y^5 = x^3(x - 1)^3$$

and

$$D : z^5 = t(t - 1).$$

They correspond to a K3 type pair since $f_C = (0, 1, 0, 1)$ and $f_D = (1, 1, 0, 0)$. The action of μ_5 on the product is given by $(x, y, t, z) \mapsto (x, \zeta_5 y, t, \zeta_5^{-1} z)$. The ring of invariants is generated by x, t, and yz.

The surface $S = (C \times D)/\mu_5$ has affine model

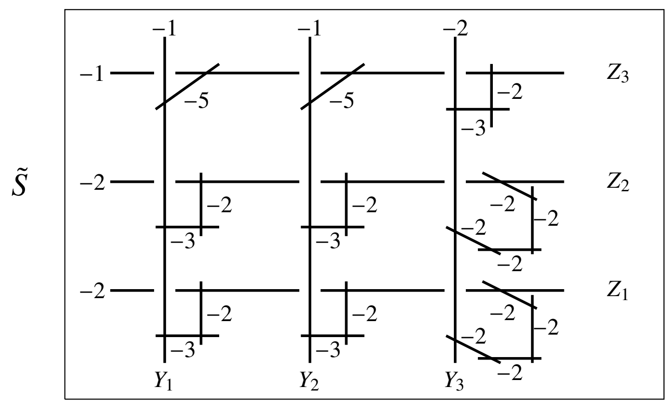
$$(yz)^5 = x^3(x-1)^3 t(t-1).$$

On S the singularities are of the form (P_i, Q_j) where $P_i \in \{0, 1, \infty\}$ and $Q_j \in \{0, 1, \infty\}$.

Let $\bar{Y}_i$ be the image of $\{P_i\} \times D$ in S and $\bar{Z}_j$ be the image of $C \times \{Q_j\}$ under the quotient map $C \times D \to S$. Then $\bar{Y}_i \cap \bar{Z}_j = \{(P_i, Q_j)\}$ and each point (P_i, Q_j) is a cyclic-quotient singularity of type $(n, r) = (n(i, j), r(i, j))$ as in the following table:

$(5,3)$	$(5,3)$	$(5,4)$	$\bar{Z}_1$
$(5,3)$	$(5,3)$	$(5,4)$	$\bar{Z}_2$
$(5,1)$	$(5,1)$	$(5,3)$	$\bar{Z}_3$
$\bar{Y}_1$	$\bar{Y}_2$	$\bar{Y}_3$	

Resolving these singularities we obtain $\tilde{S} \to S$ that looks like:



where Y_i, Z_j are the strict transforms of $\bar{Y}_i, \bar{Z}_j$ to $\tilde{S}$. The continued fraction representation of n/r determines the string of exceptional divisors $E_{i,j}^{(k)}$ connecting Y_i and Z_j together with their self-intersection numbers. We can also compute the self-intersection number of the canonical sheaf $K_{\tilde{S}}^2 = -4$. To get a minimal surface, we first blow down the (-1)-curves Y_1, Y_2, and Z_3. After blowing down Z_3, $E_{3,3}^{(2)}$ becomes a (-1)-curve so we blow down this divisor and arrive at a surface X with $K_X^2 = 0$, or in other words, X is a K3 surface.

Minimal model of resolution of singularities

We now describe the well-known procedure in general which involves 2 main steps: first resolve all singularities on S to get a non-singular resolution $\tilde{S}$; then perform a search for (-1)-exceptional divisors on $\tilde{S}$ and blown them down to get a minimal model X.

Resolving singularities (see [24, Section 1.2]). Let $s : \tilde{S} \to S$ be the total blowup of singularities associated to the K3 type data $(G, \underline{a}, \underline{b}, N)$. Let $P_i \in \{\lambda_1, \dots, \lambda_N\}$ and $Q_j \in \{0, 1, \infty\}$. Let $\bar{Y}_i$ be the image of $\{P_i\} \times D$ in S and $\bar{Z}_j$ be the image of $C \times \{Q_j\}$ under the quotient map $C \times D \to S$. We set

$$\begin{aligned}
h &= lcm(\gcd(G, a_i), \gcd(G, b_j)), \\
n(i, j) &= \frac{m}{h}, \\
c(i) &= (\frac{a_i}{\gcd(G, a_i)})^{-1} \in (\mathbb{Z}/\gcd(G, a_i)\mathbb{Z})^*, \\
d(j) &= (\frac{b_j}{\gcd(G, b_j)})^{-1} \in (\mathbb{Z}/\gcd(G, b_j)\mathbb{Z})^*, \\
r(i, j) &= c(i)d(j)^{-1} \in (\mathbb{Z}/n(i, j)\mathbb{Z})^*.
\end{aligned}$$

Then $\bar{Y}_i$ and $\bar{Z}_j$ intersect in $\gcd(a_i, b_j, m)$ singular points of cyclic-quotient type given by the pair (n, r). Let Y_i, Z_j be the strict transforms of $\bar{Y}_i, \bar{Z}_j$ to $\tilde{S}$, and let $n/q = [r_1, \dots, r_t]$ be the continued fraction expansion. For each $R \in \bar{Y}_i \cap \bar{Z}_j$, the exceptional fiber $s^{-1}(R)$, so-called the Hirzbrunch-Jung string, consists of a chain $E_1, \dots, E_t$ of rational curves such that in the sequence $Z_j, E_1, \dots, E_t, Y_i$ each curve transversally intersects the next and the self-intersection number is given by $E_k^2 = -r_k$ for $k = 1, \dots, t$.

Furthermore, if Z_i contains v singularities of types (n_k, r_k) for $k = 1, \dots, v$ then its self-intersection number $Z_i^2 = \sum_{k=1}^{v} -r_k/n_k$. On the other hand, if Y_j contains v singularities of types (n_k, r_k) for $k = 1, \dots, v$ then the self-intersection number $Y_j^2 = \sum_{k=1}^{v} -r'_k/n_k$ where r'_k is the inverse of $r_k \pmod{n_k}$ (see [24, Proposition 2.8]).

Finally, the self-intersection number of $\tilde{S}$ is given by

$$K_{\tilde{S}}^2 = \frac{8(g_C - 1)(g_D - 1)}{m} + \sum_{R \in \mathrm{Sing}(S)} h_R$$

with

$$h_R = 2 - \frac{2 + r + r'}{n} - \sum_{k=1}^{t} (r_k - 2)$$

where R has singularity type $(n, r) = [r_1, \dots, r_t]$ and r' is the inverse of $r \pmod{n}$ (see [24, Proposition 2.5]).

Getting the minimal model. We first write down the intersection matrix whose rows correspond to all strict transforms and exceptional divisors in the Hirzbrunch-Jung strings. In each iteration, we simply look for the next divisor whose self-intersection number is -1 and whose coefficient in $K_{\tilde{S}}$ is positive, remove the corresponding row, and update our matrix accordingly. It follows that the only curves that may be blown down must come from following types:

1. the (-1)-strict transforms Y_i for which $\frac{m-a_i}{\gcd(G,a_i)} > 1$;
2. the (-1)-strict transforms Z_j for which $\frac{b_j}{\gcd(G,b_j)} > 1$;
3. the exceptional components E_k in the Hirzbrunch-Jung strings such that either Y_i or Z_j got blown down.

Once we have exhausted all such divisors, we arrive at the minimal model X with self-intersection number $K_X^2 = K_{\tilde{S}}^2 + w$ where w is the number of (-1)-curves which were blown down. This process can be efficiently done via a computer program: given as input the data $(G, \underline{a}, \underline{b}, N)$, the output K_X^2 determines the Kodaira dimension of X. Moonen shows that X is a K3 surface if $K_X^2 = 0$ and is of general type otherwise.

Ring of definition.

Let $\mathbb{Z} \subset R \subset \mathbb{C}$ be a ring. We shall show that if C and D are defined over R, so is X.

Proposition 3.0.3. *Assume that C and D are defined over R. Then the affine subring of invariants $\mathbb{C}[C \times D]^G \subset \mathbb{C}[C \times D]$ is defined over R.*

Proof. We have $\mathbb{C}[C \times D] \simeq \mathbb{C}[x, y, z, t]/[y^m = \prod_i (x - \lambda_i)^{a_i}, t^m = \prod_j (z - \mu_j)^{b_j}]$, where λ and $\mu \in R$. Since the G-action does not change the degree of a monomial, it suffices to find the ring of invariant monomials. This set consists of precisely the monomials whose y's degree and t's degree are equal, i.e.,

$$\mathbb{C}[C \times D]^G = \mathbb{C}[x, z, yt]/[(yt)^m = \prod_i (x - \lambda_i)^{a_i} \prod_j (z - \mu_j)^{b_j}],$$

and this descends to $R[x, z, yt]/[(yt)^m = \prod_i (x - \lambda_i)^{a_i} \prod_j (z - \mu_j)^{b_j}]$ which is defined over R. $\square$

As a corollary from GIT, the quotient $S := (C \times D)/G$ is defined over R. Furthermore, the (-1)-divisors are rational, i.e., isomorphic to $\mathbb{P}^1$, so the minimal model X

is defined over R. In particular, if C and D are defined over $\mathbb{Z}$, then X is defined over $\mathbb{Z}$. The reduction mod p of X when $p \nmid m$ is smooth a projective k3-type surface over $\mathbb{F}_p$.

Polarization.

We construct a polarization on X as following. On $C \times D$ we consider the tensor product $L := L_1 \otimes L_2$ of principal ample line bundles on C and D. Then the tensor power L^m is a G-invariant ample line bundle on $C \times D$ where G acts via m^{th} roots of unity. Then we take the pushforward of L^m under the finite map $C \times D \to S$ to obtain an ample line bundle L' on S. By the projection formula and the birational invariance of $h^{2,0}(S, L')$, we get an ample line bundle of degree $2m$ on X. One can extend this construction to a family by considering an ample line bundle over the base $\mathcal{H}(G, \underline{a})$.

Corollary 3.0.4. *For any K3 type datum* $(G, \underline{a}, \underline{b}, N)$*, the family* $\mathcal{X}(G, \underline{a}, \underline{b}, N)$ *has a model over* $\mathbb{Z}[\frac{1}{2m}]$.

Chapter 4

PROOF OF THE FIRST MAIN THEOREM

Let $\mathcal{X} = \mathcal{X}(G, \underline{a}, \underline{b}, N)$ be the family associated to the K3 type datum $(G, \underline{a}, \underline{b}, N)$. Let $p \nmid 2m$ be a prime and X be a k3-type surface in $\mathcal{X}$. By abuse of notation, we use X to denote the reduction of X at p which is a smooth proper k3-type surface in positive characteristic.

We calculate the Newton polygon variation in the family $\mathcal{X}$ when C moves in the family parametrized by $\mathcal{H}(m, \underline{a})$ via two steps:

1. We first show how the Newton polygon of X can be determined from that of C and D. In particular, when D is fixed and C varies in a family, each Newton polygon of X corresponds one-to-one to a Newton polygon of $J(C)^{new}$.
2. We find all possible Newton polygons of $J(C)^{new}$.

The technical tools for our calculation are crystalline cohomology and F-crystals with additional structure. We first recall the following wonderful result due to Ekedahl.

Proposition 4.0.1. *[8, Proposition 4.1] Let S be smooth projective surface over k. Suppose G is a finite group acting on S whose order is prime to p. Let X be the minimal resolution of singularities of S/G. Then $H^2_{cris}(X/W(k)) = H^2(S/W(k))^G \oplus C$ where C is the $W(k)$-submodule generated by the Chern classes of the exceptional divisors of $X \to S/G$. Furthermore, the decomposition is orthogonal with respect to Poincaré pairing product and the pairing restricted to C is perfect.*

As Chern classes all have slope 1, Ekedahl's result reduces the problem to calculating the slopes of $H^2_{cris}(C \times D/W(k))^G$. The Kunneth formula and G acting trivially on H^0_{cris} and H^2_{cris} implies that

$$H^2_{cris}(C\times D)^G = H^0_{cris}(C)\otimes H^2_{cris}(D)\oplus H^0_{cris}(D)\otimes H^2_{cris}(C)\oplus(H^1_{cris}(C)\otimes H^1_{cris}(D))^G.$$

The slope of $H^0_{cris}(C)$ (resp. $H^0_{cris}(D)$) is 0 and $H^2_{cris}(D)$ (resp. $H^2_{cris}(C)$) is 1 (by compatibility of Poincaré duality), and thus, their tensor product has slope 1 and

Artin invariant 0 (coming from the determinant of the Kronecker product of the two pairings). The action of $G \cong \mu_m$ on C and D induces an action of

$$\mathbb{Z}[\mu_m] \otimes \mathbb{Z}_p = \prod_{\mathfrak{o} \in \mathfrak{O}} O_{d_{\mathfrak{o}},\mathfrak{o}}$$

on crystalline cohomology by functoriality and thus decompositions of F-crystals

$$H^1_{cris}(C) = \bigoplus_{\mathfrak{o} \in \mathfrak{O}} M_{\mathfrak{o}}(C) \text{ and } H^1_{cris}(D) = \bigoplus_{\mathfrak{o} \in \mathfrak{O}} M_{\mathfrak{o}}(D).$$

In particular

$$(H^1_{cris}(C) \otimes H^1_{cris}(D))^G = \bigoplus_{\mathfrak{o} \in \mathfrak{O}} M_{\mathfrak{o}}(C) \otimes_{O_{d_{\mathfrak{o}},\mathfrak{o}}} M_{\mathfrak{o}^*}(D),$$

where $\otimes_{O_{d_{\mathfrak{o}},\mathfrak{o}}}$ is the tensor product between F-crystals with $O_{d_{\mathfrak{o}},\mathfrak{o}}$-module structure.

Now let $\nu_{\mathfrak{o}}(C)$ and $\nu_{\mathfrak{o}}(D)$ denote the Newton polygon of $M_{\mathfrak{o}}(C)$ and $M_{\mathfrak{o}}(D)$ respectively. Let $\mathfrak{O}'$ denote the set of orbits $\mathfrak{o}$ such that $d_{\mathfrak{o}} \nmid b_j$ for all j. We show the following:

Proposition 4.0.2. *Each $\nu_{\mathfrak{o}*}(D)$ is isoclinic with slope $\lambda_{o*}(D)$. Furthermore*

1. *if $\mathfrak{o} \in \mathfrak{O}'$, the Newton polygon of $M_{\mathfrak{o}}(C) \otimes_{O_{d_{\mathfrak{o}},\mathfrak{o}}} M_{\mathfrak{o}^*}(D)$ is obtained by adding $\lambda_{o*}(D)$ to the slopes of $\nu_{\mathfrak{o}}(C)$;*
2. *if $\mathfrak{o} \notin \mathfrak{O}'$, the component $M_{\mathfrak{o}}(C) \otimes_{O_{d_{\mathfrak{o}},\mathfrak{o}}} M_{\mathfrak{o}^*}(D)$ is trivial.*

Proof. From the multiplication type of D, $M_{\mathfrak{o}^*}(D) \cong O_{d_{\mathfrak{o}},\mathfrak{o}}$ if $\mathfrak{o} \in \mathfrak{O}'$ and is 0 otherwise, hence the second part. Suppose that we are now in the case $\mathfrak{o} \in \mathfrak{O}'$. Then a calculation using Kottwitz method implies that $M_{\mathfrak{o}^*}(D)$ is isoclinic with slope $\lambda_{\mathfrak{o}^*}(D) = \frac{n_{\mathfrak{o}^*}(D)}{\#\mathfrak{o}^*}$ where $n_{\mathfrak{o}^*}(D) = \#\{i \in \mathfrak{o}^* \mid f_D(i) = 1\}$. Hence the first part follows. □

Proposition 4.0.3. *Suppose that $\mathfrak{o} \in \mathfrak{O}'$ and $1 \notin \mathfrak{o}$, then $M_{\mathfrak{o}}(C) \otimes_{O_{d_{\mathfrak{o}},\mathfrak{o}}} M_{\mathfrak{o}^*}(D)$ is isoclinic with slope 1.*

Proof. We first show that $f_C(\mathfrak{o})$ has a single nonzero multiplicity, i.e., $f_C(i) \in \{0, g_C(\mathfrak{o})\}$ for all $i \in \mathfrak{o}$. Indeed, assume on the contrary that there exists $i \neq j \in \mathfrak{o}$ such that $g_C(\mathfrak{o}) \geq f_C(i) > f_C(j) > 0$. Hence, $f_C(j^*) = g_C(\mathfrak{o}) - f_C(j) > 0$. It follows from $f_C(l) f_D(l^*) = 0$ for all $l \neq 1$ that both $f_D(j)$ and $f_D(j^*)$ must be 0: this cannot happen since $f_D(j) + f_D(j^*) = g_D(\mathfrak{o}) = 1$.

A calculation using Kottwitz method for $\nu_{\mathfrak{o}}(C)$ implies that it has a single slope $\frac{n_{\mathfrak{o}}(C)}{\#\mathfrak{o}}$ where $n_{\mathfrak{o}}(C) = \#\{i \in \mathfrak{o} \mid f(i) = g_C(\mathfrak{o})\}$. It also follows from $f_C(l) f_D(l^*) = 0$ for all $l \neq 1$ that there is a bijection between the set of nonzero multiplicities in $f_C(\mathfrak{o})$ and the set of zero multiplicities in $f_D(\mathfrak{o}^*)$. Hence $n_{\mathfrak{o}}(C) + n_{\mathfrak{o}^*}(D) = \#\mathfrak{o} = \#\mathfrak{o}^*$ and from the previous proposition $\nu_{\mathfrak{o}}(C) + \nu_{\mathfrak{o}^*}(D)$ has slope 1 only. □

Corollary 4.0.4. *When D is fixed and C varies in a family, the Newton polygon of X varies in one-to-one correspondence to the Newton polygon of $J(C)^{new}$.*

Proof. Since the orbit of 1 is contained in the new part of $H^1_{cris}(C)$, it follows from the two previous propositions that the slopes of $H^2_{cris}(X)$ are uniquely determined by the slopes of the new part of $H^1_{cris}(C)$. For a smooth and proper curve, $H^1_{cris}(C) = H^1_{cris}(J_C)$ so their new parts are isomorphic as well. □

Deligne and Mostow [5] show that the classifying morphism

$$\theta : \mathcal{H}(m, \underline{a}) \to Sh(G, \underline{f}^{new})$$

for the family $J(C)^{new}$ is dominant and quasi-finite onto $Sh(G, \underline{f}^{new})$. In particular, the Zariski closure $\overline{Z^{\text{new}}}(G, \underline{a})$ is $Sh(G, \underline{f})$.

On one hand, Kottwitz [14] gives a combinatorial description of a set $B(G, \underline{f}^{new})$ of admissible Newton polygons equipped with the extra structure from E_m and multiplication type $\underline{f}^{new}$. The Kottwitz set in the case of interest to us admits a total order

$$\nu_{\text{ord}} =: \nu_0 \succ \nu_1 \succ \cdots \succ \nu_l := \nu_{\text{basic}}$$

where the maximal element ν_{ord} is called μ-ordinary and minimal ν_{basic} called basic. Let $Sh[\nu_i]$ denote the locus of $Sh(G, \underline{f}^{new})$ with Newton polygon ν_i - from the moduli point of view, the locus of abelian schemes with Newton polygon ν_i. On the other hand, Viehman-Wedhorn [34] show that all admissible polygons occur, i.e., $Sh[\nu_i]$ is nonempty $\forall i$, for the PEL type Shimura variety $Sh(G, \underline{f}^{new})$. Furthermore, these loci give rise to the so-called Newton stratification by locally closed subsets

$$Sh(G, \underline{f}^{new}) = Sh[\nu_0] \sqcup Sh[\nu_1] \sqcup \cdots \sqcup Sh[\nu_l]$$

that satisfies

1. $Sh[\nu_i]$ is equidimensional of codimension i and open in the Zariski closure $\overline{Sh}[\nu_i]$;

2. $\overline{Sh}[\nu_i] = \sqcup_{j\geq i} Sh[\nu_j]$;
3. $\mathrm{codim}(Sh[\nu_{i+1}], \overline{Sh[\nu_i]}) = 1$;
4. (de Jong's purity) for a smooth irreducible subvariety inside $Sh(G, \underline{f}^{new})$, the induced Newton stratification goes up in codimension 1.

Let $Z[\nu] := Z \cap Sh(G, \underline{f})[\nu]$ for any admissible ν. The next result reduces our calculation to showing $Z^{\mathrm{new}}(G, \underline{a})[\nu_{\mathrm{basic}}] \neq \emptyset$.

Proposition 4.0.5. *If $Z^{new}(G, \underline{a})[\nu_{basic}] \neq \emptyset$, then $Z^{new}(G, \underline{a})[\nu_i] \neq \emptyset$ for all i.*

Proof. Firstly, $Z^{\mathrm{new}}(G, \underline{a})[\nu_{\mathrm{ord}}] \neq \emptyset$ since both $Z^{\mathrm{new}}(G, \underline{a})$ and $Sh(G, \underline{f})[\nu_{\mathrm{ord}}]$ are open and dense inside $Sh(G, \underline{f})$. Then de Jong's purity and $Z^{\mathrm{new}}(G, \underline{a})[\nu_{\mathrm{basic}}] \neq \emptyset$ implies that the induced Newton stratification on $Z^{\mathrm{new}}(G, \underline{a})$ goes up exactly l times (otherwise the basic locus would have had incorrect codimension inside $Sh(G, \underline{f})$). In other words, $Z^{\mathrm{new}}(G, \underline{a})[\nu_i] \neq \emptyset$ for all i. □

As $\overline{Z^{\mathrm{new}}}(G, \underline{a}) = Sh(G, \underline{f})$, the basic polygon ν_{basic} occurs on $\overline{Z^{\mathrm{new}}}(G, \underline{a})$. But it is not straightforward to see why ν_{basic} already occurs on the smooth locus $Z^{\mathrm{new}}(G, \underline{a})$. From the moduli point of view, this means there exists a smooth curve $C(G, \underline{a})$ for which J_C^{new} has basic polygon. The next section shows that this is indeed the case, at least when p is sufficiently large.

Boundary of Hurwitz space and basic locus

The following relies on the boundary behavior of the Hurwitz space $\mathcal{H}(G, \underline{a})$ (see [35, Chapter 4] and [16, Section 5.2]). Since Z^{new} is the image of $\mathcal{H}(G, \underline{a})$ under the classifying map, a moduli point $[J_C^{\mathrm{new}}]$ on the boundary $\partial Z^{\mathrm{new}} := \overline{Z^{\mathrm{new}}}(G, \underline{a}) - Z^{\mathrm{new}}(G, \underline{a})$ comes from the new part of the Jacobian of a singular curve $C = C(G, \underline{a})$ of compact type. In particular, C is obtained via clutching a pair of compatible curves $C_1 = C(G, \underline{a_1})$ and $C_2 = C(G, \underline{a_2})$. Such compatible $(\underline{a_1}, \underline{a_2})$ is called a degeneration type of $C(G, \underline{a})$. By definition a degeneration type satisfies $\underline{f_1} + \underline{f_2} = \underline{f}$ and the Newton polygon of C is the amalgamate sum of that of C_1 and C_2. For a fixed monodromy datum $(G, \underline{a})$, we denote by $T = T(G, \underline{a})$ the set of all degeneration types. What is important to us is that T is finite and $\underline{f}^{new}$ is of a particularly simple form so the compatibility condition ensures that $\underline{f_1}^{new}$ and $\underline{f_2}^{new}$ must have simple forms. Recall that the new part of the signature records the multiplicities of the action of E_m.

Definition 4.0.1. *Let $n \geq 3$. A signature $\underline{f}$ for an E_m-module is called simple of rank $n-2$ if $\{f(i), f(i^*)\} = \{1, n-3\}$ for a unique $i \in (\mathbb{Z}/m\mathbb{Z})^*$ and $\{f(i), f(i^*)\} = \{0, n-2\}$ otherwise.*

If $\underline{f}$ is simple of rank $n-2$ then $\dim Sh(G, \underline{f}) = n-3$. In our case, $\underline{f}^{new}$ is simple of rank $N-2$ so it follows from 2.3 that $\dim Sh(G, \underline{f}^{new}) = N-3$. By definition, if $(\underline{a_1}, \underline{a_2})$ is a degeneration type of $(G, \underline{a})$ then we can assume that

$$\underline{f_1}^{\text{new}} + \underline{f_2}^{\text{new}} = \underline{f}^{\text{new}} \text{ and } \underline{f_1}^{\text{new}} \text{ is simple.}$$

That is, $\underline{f_1}^{\text{new}}$ is simple of rank $n_1 - 2$, $\underline{f_2}^{\text{new}}$ satisfies $\{f_2(i), f_2(i^*)\} = \{0, n_2 - 2\}$ for all $i \in (\mathbb{Z}/m\mathbb{Z})^*$, and $n = n_1 + n_2 - 2$.

In what follows, we look that the new parts of the corresponding Jacobians and Shimura varieties only so we can drop *new* from the notation of the signature for abbreviation without any confusion.

Proposition 4.0.6. *Suppose that $\underline{f}$ is simple of rank $N-2$ and let $\mathfrak{e} \in \mathfrak{O}$ be the orbit of the unique i where $f(i) = 1$. Let $d := \#\mathfrak{e}$ and $n := \#\{j \in \mathfrak{e} : f(j) = N-2\}$.*

1. *If $\mathfrak{e} \neq \mathfrak{e}^*$, or equivalently, $\mathfrak{p}$ is not inert in E_m/E_m^0, then*

$$\begin{cases} \#B(G, \underline{f}) = N-2; \\ \nu_{basic,\mathfrak{e}} \text{ is isoclinic of slope } \frac{n}{d} + \frac{1}{d(N-2)}; \\ \dim Sh(G, \underline{f})[\nu_{basic}] = 0. \end{cases}$$

2. *If $\mathfrak{e} = \mathfrak{e}^*$, or equivalently, $\mathfrak{p}$ is inert in E_m/E_m^0, then*

$$\begin{cases} \#B(G, \underline{f}) = \lfloor \frac{N}{2} \rfloor; \\ \nu_{basic,\mathfrak{e}} \text{ is isoclinic of slope } \frac{1}{2}; \\ \dim Sh(G, \underline{f})[\nu_{basic}] = N-2-\lfloor \frac{N}{2} \rfloor. \end{cases}$$

Proof. This is a direct calculation from the combinatorial description of $B(G, \underline{f})$. Recall that any $\nu \in B(G, \underline{f})$ has a decomposition $\nu = \oplus_{\mathfrak{o}} \nu_{\mathfrak{o}}$. For any $\mathfrak{o} \neq \mathfrak{e}$, since f is simple, there is a single nonzero multiplication in $f(\mathfrak{o})$. This means $\nu_{\text{ord},\mathfrak{o}} = \nu_{\text{basic},\mathfrak{o}}$. Hence, it suffices to find all possible polygons for $\nu_{\mathfrak{e}}$.

Case 1. $\mathfrak{e} \neq \mathfrak{e}^*$

$\nu_{\mathrm{ord},\mathfrak{e}}$ has slopes: $\frac{n}{d}$ with multiplicity $d(N-3)$ and $\frac{n+1}{d}$ with multiplicity d.

The Newton polygon of $\nu_{\mathrm{ord},\mathfrak{e}}$ can be drawn in the plane with endpoints $\{(0,0),(d(N-2),n(N-2)+1)\}$ and an integer breakpoint $(d(N-3),n(N-3))$. An admissible polygon corresponds to a choice of an integer breakpoint on the segment connecting $(0,0)$ and $(d(N-3),n(N-3))$ whose first coordinate is a multiple of d. There are $N-2$ such choices and the basic one corresponds to a straight line of slope $\frac{n}{d}+\frac{1}{d(N-2)}$.

By de Jong's purity, $\dim Sh(G,\underline{f})[\nu_{\mathrm{basic}}] = N-2-\#B(G,\underline{f}) = 0$.

Case 2. $\mathfrak{e}=\mathfrak{e}^*$ so d has to be even and $n=\frac{d}{2}-1$

$\nu_{\mathrm{ord},\mathfrak{e}}$ has slopes: $\frac{1}{2}-\frac{1}{d}$ with multiplicity d, $\frac{1}{2}$ with multiplicity $d(N-4)$, and $\frac{1}{2}+\frac{1}{d}$ with multiplicity d.

The Newton polygon of $\nu_{\mathrm{ord},\mathfrak{e}}$ is symmetric with endpoints $\{(0,0),(d(N-2),\frac{d}{2}(N-2))\}$ and 2 integer break points $\{(d,\frac{d}{2}-1),(d(N-3),\frac{d}{2}(N-3)-1)\}$. Any admissible polygon corresponds to a choice of an integer breakpoint on the left-half of the segment connecting $\frac{d}{2}(N-2))$ and $(d,\frac{d}{2}-1)$ whose first coordinate is a multiple of d. There are $\lfloor\frac{N}{2}\rfloor$ such choices and the slopes of the basic one are all $\frac{1}{2}$ (a supersingular polygon).

By de Jong's purity, $\dim Sh(G,\underline{f})[\nu_{\mathrm{basic}}]) = N-2-\#B(G,\underline{f}) = N-2-\lfloor\frac{N}{2}\rfloor$.

$\square$

Suppose that $(\underline{f_1},\underline{f_2})$ is a degeneration of $\underline{f}=\underline{f}^{new}$. There is a natural inclusion of Shimura varieties

$$Sh(G,\underline{f_1})\times Sh(G,\underline{f_2}) \hookrightarrow Sh(G,\underline{f})$$

compatible with taking the amalgamate sum of Newton polygons. In what follows, we drop G from the notation for abbreviation.

Theorem 4.0.7. *If N is odd or p is sufficiently large, the basic locus $Sh(\underline{f})[\nu_{basic}]$ is not contained in*

$$\bigcup_{(\underline{a_1},\underline{a_2})\in T} Sh(\underline{f_1})\times Sh(\underline{f_2}).$$

Proof. Assume on the contrary that

$$Sh(\underline{f})[\nu_{\mathrm{basic}}] \subset \bigcup_{(\underline{a_1},\underline{a_2})\in T} Sh(\underline{f_1})\times Sh(\underline{f_2}).$$

Then the compatibility with Newton stratification implies that in fact

$$Sh(\underline{f})[\nu_{\text{basic}}] \subset \bigcup_{(\underline{a_1},\underline{a_2})\in T} Sh(\underline{f_1})[\nu_{1,\text{basic}}] \times Sh(\underline{f_2})[\nu_{2,\text{basic}}]. \tag{4.1}$$

In particular, $\nu_{\text{basic}} = \nu_{1,\text{basic}} \oplus \nu_{2,\text{basic}}$. The following follows directly from Kottwitz method: $\nu_{2,\text{basic}}$ is isoclinic of slope $\frac{n}{d}$ and $\dim Sh(\underline{f_2}) = 0$.

Case 1. $\mathfrak{e} \neq \mathfrak{e}^*$: From the previous proposition, the basic slope $\frac{n}{d} + \frac{1}{d(N-2)}$ of ν_{basic} in $\mathfrak{e}$ is not the same as the basic slope $\frac{n}{d} + \frac{1}{d(N_1-2)}$ of $\nu_{1,\text{basic}}$ in $\mathfrak{e}$: a contradiction.

Case 2. $\mathfrak{e} = \mathfrak{e}^*$: In this case, all basic polygons are supersingular. Comparing dimensions of both sides of the inclusion yields

$$N - 2 - \lfloor \frac{N}{2} \rfloor \leq N_1 - 2 - \lfloor \frac{N_1}{2} \rfloor.$$

This holds only when N is even, $N_1 = N - 1$, and $N_2 = 3$.

On one hand, Li et el ([16, Theorem 8.1]) show that when N is even and f is simple, the number of irreducible components of $Sh(\underline{f})[\nu_{\text{basic}}]$ grows to infinity with p. On the other hand, Xiao and Zhu ([36, Theorem 1.1.4]) show that if N_1 is odd, f_1 is simple, and $\mathfrak{p} \mid p$ is inert in E_m/E_m^0, the number of irreducible components of $Sh(\underline{f_1})[\nu_{1,\text{basic}}]$ is bounded at all unramified primes. Thus, we have arrived at a contradiction when p is sufficiently large. □

Proposition 4.0.8. *Assume that N is odd or $p \gg 0$, then $Z^{new}(G, \underline{a})[\nu_{basic}] \neq \emptyset$.*

Proof. Assume on the contrary that

$$\partial Z^{\text{new}}(G, \underline{a}) \supset Sh(G, \underline{f})[\nu_{\text{basic}}].$$

The discussion on boundary of Hurwitz space implies that

$$\bigcup_{(\underline{a_1},\underline{a_2})\in T} Sh(G, \underline{f_1}) \times Sh(G, \underline{f_2}) \supset \partial Z^{\text{new}}(G, \underline{a}).$$

We arrive at a contradiction to the previous theorem

$$\bigcup_{(\underline{a_1},\underline{a_2})\in T} Sh(G, \underline{f_1}) \times Sh(G, \underline{f_2}) \supset Sh(G, \underline{f})[\nu_{\text{basic}}].$$

□

Stratification of the supersingular locus by Artin invariant

Assume that $\mathcal{X}(G, \underline{a}, \underline{b}, N)$ is a family of K3 surfaces and we are in Case 2 of Proposition 4.0.6 and Theorem 4.0.7. In particular, the basic polygon ν_{basic} is the supersingular polygon. As a unitary Shimura variety with simple signature, $Sh(G, \underline{f})$ is fully Hodge-Newton decomposable, i.e., $\overline{Z^{\text{new}}}(G, \underline{a})[\nu_{ss}] = Sh(G, \underline{f})[\nu_{ss}]$ is a union of strata by the final type of $M_{\mathfrak{e}}(C)$, each of codimension 1 in the previous one ([12, Theorem 3.3]).

From Proposition 4.0.6, there are $N-2-\lfloor\frac{N-2}{2}\rfloor$ such strata. The final type of $M_{\mathfrak{e}}(C)$ determines the final type of X via $M_{\mathfrak{e}}(C) \otimes_{O_{d,\mathfrak{e}}} M_{\mathfrak{e}}(D)$, or equivalently, its Artin invariant from Proposition 2.5.6. Hence, on $\mathcal{X}[\nu_{ss}]$ we have a further stratification into $N-2-\lfloor\frac{N-2}{2}\rfloor$ strata by the Artin invariant for which the inclusion order agrees with the Bruhat order of final type of $M_{\mathfrak{e}}(C) \otimes_{O_{d,\mathfrak{e}}} M_{\mathfrak{e}}(D)$.

Main theorem: proof

Theorem 4.0.9. *Let $\mathcal{X}$, X, p and d be as above.*

1. *If p is not inert in E_m/E_m^0, then d divides $h(X)$ and*
$$1 \leq \frac{h(X)}{d} \leq N-2.$$
Furthermore, for any $1 \leq i \leq N-2$, there exists a k3-type surface X' in $\mathcal{X}$ such that $h(X') = id$.

2. *If p is inert in E_m/E_m^0, we assume further that p is sufficient large when N is even. Then d divides $h(X)$ and*
$$1 \leq \frac{h(X)}{d} \leq \lfloor\frac{N-2}{2}\rfloor \text{ or } h(X) = \infty.$$
In addition, for any $1 \leq i \leq \lfloor\frac{N-2}{2}\rfloor$, there exists a k3-type surface X' in $\mathcal{X}$ such that $h(X') = id$.

   Moreover, if $\mathcal{X}$ corresponds to a family of K3 surfaces, then there exists a supersingular K3 surface X' in $\mathcal{X}$ with $\sigma_0(X') = (N-2-\lfloor\frac{N-2}{2}\rfloor)d$.

Proof. 1. Because p is not inert in E_m/E_n^0 if and only if $\mathfrak{e} \neq \mathfrak{e}^*$, we are in **Case 1** of Proposition 4.0.6 and Theorem 4.0.7. In particular, there are $N-2$ admissible polygons for $M_{\mathfrak{e}}(C)$. For each $i = 1, \ldots, N-2$, the i-th admissible polygon has slopes $\frac{in+1}{id}$ and $\frac{n}{d}$. On the other hand, $M_{\mathfrak{e}}(D)$ has slope $1-\frac{n}{d}$

only. Thus the Newton polygon of X has slopes $\frac{id+1}{id}$ and 1, i.e., $h(X) = id$. The result follows from the fact that all admissible polygons occur for some C in the family of cyclic covers (Proposition 4.0.5 and 4.0.8).

2. We are in **Case 2** of Proposition 4.0.6 and Theorem 4.0.7. In particular, there are $\lfloor\frac{N-2}{2}\rfloor$ admissible polygons for $M_{\mathfrak{e}}(C)$. A similar arguments holds: for each $i = 1, \ldots, \lfloor\frac{N-2}{2}\rfloor$, the i-th admissible polygon has slopes $\frac{1}{2} + \frac{1}{id}$ and $\frac{1}{2}$, and $M_{\mathfrak{e}}(D)$ has slope $\frac{1}{2}$.

   Regarding the Artin invariant, from Lemma 2.5.3, the $\lfloor\frac{N-2}{2}\rfloor$ strata with finite heights correspond to the final elements $\omega_d, \ldots, \omega_{\lfloor\frac{N-2}{2}\rfloor d}$. The supersingular locus $\mathcal{X}[\nu_{ss}]$ is further stratified by the Artin invariant into $N - 2 - \lfloor\frac{N-2}{2}\rfloor$ strata. These correspond to final elements $\omega_{(N-2)d+1-\sigma_0}$ such that d divides σ_0 and
$$\lfloor\frac{N-2}{2}\rfloor d < (N-2)d + 1 - \sigma_0 \leq (N-2)d.$$
   It follows that the open Artin stratum in $\mathcal{X}[\nu_{ss}]$ is the largest admissible one $\sigma_0 = (N - 2 - \lfloor\frac{N-2}{2}\rfloor)d$ and has to occur for some X' in $\mathcal{X}$.

□

Sufficient bound for p

Assume that we are in Case 2 of Theorem 4.0.7. From 4.1, we want p such that the number of irreducible components of $Sh(\underline{f})[\nu_{\text{basic}}]$ is larger than the number of irreducible components of $\bigcup_{(\underline{a_1},\underline{a_2})\in T} Sh(\underline{f_1})[\nu_{1,\text{basic}}] \times Sh(\underline{f_2})[\nu_{2,\text{basic}}]$.

Fix $x \in Sh(\underline{f})[\nu_{\text{basic}}]$ and let A_x denote the corresponding abelian variety endowed with the PEL structure. We write $I = I_x$ for the group of quasi-isogenies of A_x that are compatible with the E_m action and the polarization. From [16, Proposition 8.5] and [11, Proposition 2.13], a lower bound for the number of irreducible components of $Sh(\underline{f})[\nu_{\text{basic}}]$ can be given by

$$2^{-[(N-2)\frac{\phi(m)}{2}+1]} L(M_I)\tau(I)\frac{p^{(N-2)\frac{\phi(m)}{2}} - 1}{p^{\frac{\phi(m)}{2}} + 1}$$

where

- M_I is the Artin-Tate motive attached to I by Gross;
- $L(M_I)$ is the value of the L-function of M_I at 0;

- $\tau(I)$ is the Tamagawa number of I.

The number of degeneration types $|T|$ can be bounded above by $\binom{N}{2}$. For each $(\underline{a_1}, \underline{a_2})$, the Shimura variety $Sh(\underline{f_2})$ is a single special point. From [36, Theorem 1.1.4(1)], the number of irreducible components of $Sh(\underline{f_1})[\nu_{\text{basic}}]$ is equal to the dimension of the middle Borel-Moore homology group $\dim H^{BM}_{N_1-3}(Sh(\underline{f_1}))$ (note that $N_1 = N - 1$). It follows that the number of irreducible components of $\bigcup_{(\underline{a_1},\underline{a_2})\in T} Sh(\underline{f_1})[\nu_{1,\text{basic}}] \times Sh(\underline{f_2})[\nu_{2,\text{basic}}]$ can be bounded above by

$$\binom{N}{2} \max_{\underline{a_1}\in T}(\dim H^{BM}_{N-4}(Sh(\underline{f_1})).$$

The following result is an easy consequence of the discussion above.

Proposition 4.0.10. *Suppose that p is inert in E_m/E_m^0. With the above notations, $Z^{new}(G, \underline{a})[\nu_{basic}] \neq \emptyset$ when p satisfies*

$$2^{-[(N-2)\frac{\phi(m)}{2}+1]} L(M_I)\tau(I) \frac{p^{(N-2)\frac{\phi(m)}{2}} - 1}{p^{\frac{\phi(m)}{2}} + 1} > \binom{N}{2} \max_{\underline{a_1}\in T}(\dim H^{BM}_{N-4}(Sh(\underline{f_1})).$$

Chapter 5

APPLICATIONS

Table 5.1 list all special families together with all the possible Newton polygons. Our examples have all the possible heights in $\{1, \dots, 10, \infty\}$.

Proposition 5.0.1. *All monodromy data $(G, \underline{a}, \underline{b}, N)$ such that $\mathcal{X}(G, \underline{a}, \underline{b}, N)$ is a family of K3 surfaces are from in Table 5.1.*

Proof. If $\mathcal{X}(G, \underline{a}, \underline{b}, N)$ is a family of K3 surface then the transcendental lattice $T(\mathcal{X})$ has rank at most 22. The rank of $T(\mathcal{X})$ is $\phi(m)(N-2)$ so that $\phi(m)(N-2) \leq 22$. From the bound $\phi(m) \geq \sqrt{\frac{m}{2}}$, we deduce that $m \leq 968$ and a brute-force computer search yields all m together with the monodromy data given in Table 5.1. $\square$

Example 5.0.1. *The examples with $N = 3$ are K3 surfaces with complex multiplication by the m-th cyclotomic field E_m. Our theorem implies that they are supersingular with Artin invariant $\sigma_0 = d/2$ iff p is inert in E_m/E_m^0. The first example of this phenomenon is due to Tate: the K3 surface $x^4 + y^4 + z^4 + t^4 = 0$ is supersingular over any field of characteristic $p \equiv 3 \pmod 4$.*

Example 5.0.2. *When $N = 3$ and $d = 2$, or equivalently, $p \equiv -1 \pmod m$, X is a K3 surface with Artin invariant 1. It is known that such surface has a unique isomorphism class over $\overline{\mathbb{F}}_p$ and admits a model over $\mathbb{F}_p$. Here as an application of our computations, we have found an explicit equation for X.*

Remark 5.0.1. *The Manin problem for K3 surfaces can be formulated as which Newton strata are nonempty in the moduli stack $\mathcal{M}_{2l} \otimes \mathbb{F}_p$ of K3 surfaces equipped with a line bundle of degree $2l$ in characteristic p. The most general result due to Rizov [27] asserts that all polygons occur when d and p are sufficiently large. Here our families provide a positive answer to many explicit pairs of invariants $l = m$ and h as listed in Table 5.1.*

A refinement of Serre's conjecture for K3 surfaces

Let X be a K3 surface over a number field L. We assume the following:

(A1) $F := \mathcal{E}(X)$ is an abelian extension of $\mathbb{Q}$;

(A2) X does not have CM and $n = \mathrm{rank}_F(T(X))$ is even if F is totally real.

We consider the S of primes $\mathfrak{p}$ of L such that:

- X has good reduction at $\mathfrak{p}$;
- $\mathfrak{p}$ has degree 1.

Then S has density 1 since the set of primes of degree 1 has density 1 and X has good reduction at all but finitely many primes of L.

Let $\mathfrak{p} \in S$ and denote by $X_\mathfrak{p}$ the K3 surface over the residue field $\mathbb{F}_p$. Let p be the rational prime in $\mathbb{Q}$ below $\mathfrak{p}$. Let K be the largest subfield of F over which p splits completely and set $d := [F : K]$. The Newton polygon of $X_\mathfrak{p}$ lies above a certain polygon called μ-ordinary, denoted by $\mu_\mathfrak{p}$, that only depends on the splitting behavior of p in F.

Remark 5.0.2. *(A1) allows us to realize the group connected components of the l-adic monodromy group as a subgroup of certain Weyl group. On (A2), if X has complex multiplication by F, then an analogue of the Shimura-Taniyama formula for CM abelian varieties shows that the Newton polygon of $X_\mathfrak{p}$ is uniquely determined by the splitting of p in F (see [13, Theorem 1.1]) and coincides with $\mu_\mathfrak{p}$. So in this case, every good prime of L is μ-ordinary.*

We fix an embedding of L into $\mathbb{C}$ and write $X_\mathbb{C}$ for the corresponding complex K3 surface. Let $X_{\overline{L}}$ denote the base change of X to $\overline{L}$.

Proposition 5.0.2. *$\mu_\mathfrak{p}$ has slopes $1 - \frac{1}{d}, 1, 1 + \frac{1}{d}$ with multiplicities $d, 22 - 2d, d$.*

Proof. We first consider the comparison theorem between Betti and p-adic cohomologies

$$H^2(X_\mathbb{C}, \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Q}_p \simeq H^2(X_{\overline{L}}, \mathbb{Q}_p)$$

compatiple with Poincare pairing and Chern class map. In particular

$$H^2(X_{\overline{L}}, \mathbb{Q}_p) \simeq (T(X_\mathbb{C}) \oplus NS(X_\mathbb{C})) \otimes \mathbb{Q}_p.$$

Next, let B_{cris} denote Fontaine's crystalline period field - an extension of $\mathbb{Q}_p$ with an action of $\mathrm{Gal}(\overline{L}/L)$ such that $B_{cris}^{\mathrm{Gal}(\overline{L}/L)} = \mathbb{Q}_p$. There exists a functor

$$D_{cris}(H^2(X_{\overline{L}}, \mathbb{Q}_p)) := (H^2(X_{\overline{L}}, \mathbb{Q}_p) \otimes B_{cris})^{\mathrm{Gal}(\overline{L}/L)}$$

such that there exists an isomorphism

$$D_{cris}(H^2(X_{\overline{L}}, \mathbb{Q}_p)) \simeq H^2_{cris}(X_{\mathfrak{p}}/\mathbb{Q}_p)$$

compatible with the Frobenius action, Poincaré pairing, and Chern class map.

Since $\mathrm{Gal}(\overline{L}/L)$ acting on $X_{\overline{L}}$ preserves the space spanned by algebraic classes $NS(X_{\mathbb{C}})$ and the Frobenius acting on the Chern class of a line bundle is multiplication by p, the image $D_{cris}(NS(X_{\mathbb{C}}) \otimes \mathbb{Q}_p)$ lies in the Tate module of $H^2_{cris}(X_{\mathfrak{p}}/\mathbb{Q}_p)$. As these classes have slope 1 only, the complement $D_{cris}(T(X_{\mathbb{C}}) \otimes \mathbb{Q}_p) \subset H^2_{cris}(X_{\mathfrak{p}}/\mathbb{Q}_p)$ is a K3 F-isocrystal with an action of the endomorphism algebra $F \otimes \mathbb{Q}_p$ and determines the Newton polygon of $H^2_{cris}(X_{\mathfrak{p}}/\mathbb{Q}_p)$. Thus, the slopes of $\mu_{\mathfrak{p}}$ can be computed to be of the desired form following Proposition 2.5.2. □

The condition (A2) also ensures that we are in the case $n \neq m$ of Proposition 2.5.2.

Fix a prime l and consider the l-adic representation on the étale cohomology

$$\rho_{X,l} : \mathrm{Gal}(\overline{L}/L) \to GL(H^2(X_{\overline{L}}, \mathbb{Q}_l)).$$

The l-adic monodromy group, denoted by $G_{X,l}$, is defined as the Zariski closure of the image of $\rho_{X,l}$ in $GL(H^2(X_{\overline{L}}, \mathbb{Q}_l))$. Let $G^{\circ}_{X,l}$ be the identity component of $G_{X,l}$ viewed as an algebraic group over $\mathbb{Q}_l$. Let $\pi_0(G_{X,l})$ be the group of connected components of $G_{X,l}$ which is a finite group. By Galois theory, there exists a finite field extension L^{conn} of L such that $\mathrm{Gal}(L^{\mathrm{conn}}/L) \simeq \pi_0(G_{X,l})$.

We assume one further condition:

(A3) $L^{conn} \subseteq FL$.

Next, let us choose a polarization of degree $2m$ on X, or equivalently, a polarization form ψ on $H^2(X_{\mathbb{C}}, \mathbb{Z})$. Let $\mathbb{S}$ denote the Deligne torus and let $h : \mathbb{S} \to GO(\psi)(\mathbb{R})$ be the structure homomorphism of the rational polarized K3 Hodge structure on $(H^2(X_{\mathbb{C}}, \mathbb{Q}), \psi)$ where $GO(\psi)$ denotes the group of orthogonal similitudes of $(H^2(X_{\mathbb{C}}, \mathbb{Q}), \psi)$. The Mumford-Tate group of X is the smallest algebraic subgroup MT_X of $GO(\psi)$ over $\mathbb{Q}$ such that $MT_X(\mathbb{R})$ contains the image of h in $GO(\psi)(\mathbb{R})$.

The Mumford-Tate conjecture asserts that there exists a canonical isomorphism

$$G^{\circ}_{X,l} \simeq MT_X \otimes \mathbb{Q}_l.$$

The Mumford-Tate conjecture holds for K3 from the works of Tankeev [32, 33].

On the other hand, considering the decomposition

$$H^2(X, \mathbb{Q}) = NS(X)_{\mathbb{Q}} \oplus T(X)_{\mathbb{Q}},$$

one has MT_X fix $NS(X)$ as these are precisely the $\mathbb{Q}$-span of Hodge classes. Since $T(X)_{\mathbb{Q}}$ with the restricted polarization ψ is again a K3 polarized Hodge structure, we can identify MT_X with the Mumford-Tate group of $(T(X)_{\mathbb{Q}}, \psi \mid_{T(X)})$ which is contained in $GO(T(X), \psi)$. From the definition of $\mathcal{E}(X) = F$, we deduce that the Mumford-Tate group of $T(X)$ is contained in the centralizer of F in $GO(T(X), \psi)$.

Theorem 5.0.3 (Zarhin, [37])**.** *With notations as above, the Mumford-Tate group of the polarized Hodge structure $(T(X), \psi)$ is the centralizer of F in $GO(T(X), \psi)$.*

Let $T_l := T(X) \otimes \mathbb{Q}_l = NS(X)^{\perp} \otimes \mathbb{Q}_l \subset H^2(X_{\overline{L}}, \mathbb{Q}_l)$ together with the bilinear form $\psi_l := \psi \otimes \mathbb{Q}_l$. Because Hodge classes of K3 surfaces are absolute Hodge, the Galois action of $\mathrm{Gal}(\overline{L}/L)$ leaves $NS(X) \otimes \mathbb{Q}_l$ invariant. It follows that $\rho_{X,l}$ factors through $GO(T_l, \psi_l)$ and the identity component $G^{\circ}_{X,l}$ commutes with $F \otimes \mathbb{Q}_l$.

The discussion above implies the following:

Corollary 5.0.4. *With notations as above, the identity component $G^{\circ}_{X,l}$ can be identified with the centralizer of $F \otimes \mathbb{Q}_l$ in $GO(T_l, \psi_l)$.*

Let us consider the orthogonal similitude χ which is a 1-dimensional representation of $G_{X,l}$. For each prime $\mathfrak{p} \in S$ above a rational prime p, the Frobenius $\mathrm{Fr}_{\mathfrak{p}} \in \mathrm{Gal}(\overline{L}/L)$ is well defined up to conjugacy. By abuse of notation, let us also use $\mathrm{Fr}_{\mathfrak{p}}$ for its image under $\rho_{X,l}$. Recall that $d = [F : K]$ where K is the subfield of F over which p splits completely. The trace on the algebraic representation $\wedge^d H^2(X_{\overline{L}}, \mathbb{Q}_l) \otimes \chi^{-d}$ gives a well-defined invariant

$$a_{\mathfrak{p}} := \mathrm{Tr}(\mathrm{Fr}_{\mathfrak{p}} \mid \wedge^d H^2(X_{\overline{L}}, \mathbb{Q}_l) \otimes \chi^{-d}).$$

The following proposition is similar to [3, Proposition 3.11].

Proposition 5.0.5. *With notations as above, $\mathfrak{p}$ is not μ-ordinary if and only if $a_{\mathfrak{p}}$ is an integer in $[-C, C]$ where $C = \binom{22}{d}$.*

Proof. The characteristic polynomial of $\mathrm{Fr}_{\mathfrak{p}}$ acting on $H^2(X_{\overline{L}}, \mathbb{Q}_l)$ has the form

$$P(T) := T^{22} + c_1 T^{21} + \cdots + c_{21} T + c_{22}.$$

It is known from Deligne's works on the Weil's conjecture for K3 surfaces [5, 6] that the action of $\mathrm{Fr}_{\mathfrak{p}}$ is semi-simple and the c_i are integers, independent of l. Let us split $P(T)$ over $\overline{\mathbb{Q}}$ into

$$P(T) = (T - \alpha_1) \dots (T - \alpha_{22}).$$

From the Weil's conjecture, the α_i are algebraic integers in O_L with Archimedean absolute value p and $\mathfrak{q}$-adic units for any prime $\mathfrak{q}$ of L away from p.

It is also known that the p-adic valuations of α_i are encoded in the slopes with multiplicities of the Newton polygon of $H^2_{cris}(X_{\mathfrak{p}}/W)$ (see [1, Chapter 8]). The trace of $\mathrm{Fr}_{\mathfrak{p}}$ on $\wedge^d H^2(X_{\overline{L}}, \mathbb{Q}_l)$ is given by

$$b := \mathrm{Tr}(\mathrm{Fr}_{\mathfrak{p}} \mid \wedge^d H^2(X_{\overline{L}}, \mathbb{Q}_l)) = \sum_{I \subset \{1,\dots,22\}, \#I = d} \Pi_{i \in I} \alpha_i$$

which is an integer as b can be written as a polynomial with integer coefficients in the c_i. Since $\mathfrak{p}$ is not μ-ordinary, from Proposition 5.0.2, the smallest slope of $H^2_{cris}(X_{\mathfrak{p}}/W)$ is greater than $1 - \frac{1}{d}$. Thus, the sum of any d slopes of $H^2_{cris}(X_{\mathfrak{p}}/W)$ is greater than $d - 1$, i.e., the p-valuation of $\Pi_{i \in I} \alpha_i$ is greater than $d - 1$. Hence, the p-valuation of b must be at least d or p^d divides b. In addition, as each α_i has absolute value p, we deduce that

$$-p^d \binom{22}{d} \leq b \leq p^d \binom{22}{d}.$$

The result then follows from the fact that $\mathrm{Fr}_{\mathfrak{p}}$ acts on χ via multiplication by p. □

The next discussion follows [3, Section 4.1].

Let Γ denote the image of $\mathrm{Gal}(\overline{L}/L)$ under $\rho_{X,l}$. We have the following commutative diagram of groups

$$\begin{tikzcd}
G^{\circ}_{X,l} \simeq \mathcal{G}^F \arrow[d, hook] & \mathrm{Gal}(\overline{L}/L) \ni \mathrm{Fr}_{\mathfrak{p}} \arrow[dl, "\rho_{X,l}"] \arrow[d, two heads] & \\
G_{X,l} \simeq \overline{\Gamma} \arrow[d, two heads, "\pi_0"] & \mathrm{Gal}(FL/L) \ni \sigma \arrow[r, hook, "\iota"] \arrow[d, two heads, "\kappa"] & \mathrm{Gal}(F/\mathbb{Q}) \\
\pi_0(G_{X,l}) & \mathrm{Gal}(L^{conn}/L). \arrow[l, "\rho_{X,l}", "\simeq"'] &
\end{tikzcd}$$

Our assumptions ensure that κ is a surjection and ι is an injection between abelian groups.

Let $\sigma \in \mathrm{Gal}(FL/L)$ be the element that corresponds to $\mathrm{Fr}_{\mathfrak{p}} \in \mathrm{Gal}(\overline{L}/L)$ restricted to FL. Let $S_\sigma \subset S$ be the set of primes of L whose Frobeni restrict to σ. The subfield K of F over which p splits completely is the same as $F^{\langle\iota(\sigma)\rangle}$. Let $G_{X,l}^{(\sigma)}$ be the connected component of $G_{X,l}$ corresponding to $\kappa(\sigma)$. We define the following conjugacy-invariant algebraic function that only depends on σ.

$$b_\sigma := \mathrm{Tr}(- \mid \wedge^d H^2(X, \mathbb{Q}_l) \otimes \chi^{-d}) : G_{X,l}^{(\sigma)} \to \mathbb{Q}_l.$$

The next result is similar to [3, Proposition 4.9] and [28, Theorem 1].

Proposition 5.0.6. *The density of the set of non μ-ordinary prime is equal to the number of $\sigma \in Gal(FL/L)$ for which b_σ is a constant, divided by $[FL : L]$.*

Proof. Let us fix $\sigma \in \sigma \in \mathrm{Gal}(FL/L)$. Since Γ is a closed and compact subgroup of $GL(H^2(X_{\overline{L}}, \mathbb{Q}_l))$, it is an l-adic analytic subgroup and admits a normalized Haar measure of 1. Let $\sigma\Gamma$ denote the image of the coset $\sigma\mathrm{Gal}(\overline{L}/FL) \subset \mathrm{Gal}(\overline{L}/L)$ under $\rho_{X,l}$ so that $\sigma\Gamma$ has measure $\frac{1}{[FL:L]}$.

For each integer $n \in [-C, C]$ where C is as in Proposition 5.0.5, let T_n be the subset of $G_{X,l}^{(\sigma)}$ where b_σ is equal to n. Then T_n is a conjugacy-invariant closed subset of $G_{X,l}$ as $\pi_0(G_{X,l})$ is abelian and b_σ is algebraic. Let Z be the union of all T_n which is again conjugacy-invariant and closed.

Let $Z' := Z \cap \sigma\Gamma$. The boundary of Z' has measure 0 since it is an l-adic analytic subset of Γ. From Proposition 5.0.5, any non μ-ordinary prime $\mathfrak{p} \in S_\sigma$ has $\mathrm{Fr}_{\mathfrak{p}}$ in Z'. By Chebotarev's density and [29, Corollary 6.10], the density of these primes is the same as the Haar measure of Z'. Since $\sigma\Gamma$ is Zariski dense in $G_{X,l}^{(\sigma)}$, Serre's results [29, Proposition 5.12 and 5.2.1.2] imply that the measure of Z' is 1 if Z contains $G_{X,l}^{(\sigma)}$ and 0 otherwise.

Next, we prove that Z contains $G_{X,l}^{(\sigma)}$ if and only if b_σ is a constant on $G_{X,l}^{(\sigma)}$. As $G_{X,l}^{(\sigma)}$ is connected and Z is the preimage of finitely many points, if Z contains $G_{X,l}^{(\sigma)}$, then b_σ is a constant. On the other hand, if b_σ is a constant c on $G_{X,l}^{(\sigma)}$, it suffices to prove that c is an integer in $[-C, C]$. By Chebotarev's density, there exist infinitely many primes in S_σ above distinct rational primes p and p' whose Frobeni lie in $\sigma\Gamma$. At these primes c is equal to b_p/p^d and $b_{p'}/p'^d$, respectively, where b_p and $b_{p'}$ are integers. We deduce that c is a rational number whose denominator is a divisor of both p and p', hence an integer. Because $-p^d C \leq b_p \leq p^d C$, we have $c \in [-C, C]$.

The proposition follows from summing over all $\sigma \in \mathrm{Gal}(FL/L)$. ∎

Because the density in Proposition 5.0.6 is preserved under matrix scaling and restriction to the kernel of the orthogonal similitude, we may replace $G^{\circ}_{X,l}$ with the group of F-linear special isometries $SO_F(T_l, \psi_l)$. For abbreviation, let us write $\mathcal{G}$ for the l-adic algebraic group $SO(T_l, \psi_l)$ and $\mathcal{G}^F$ for $SO_F(T_l, \psi_l)$ as a subgroup.

The next discussion follows [3, Section 4.2, 4.3] in order to pick a basis to realize $\mathcal{G}^F$ in block diagonal form inside $SO(T_l, \psi_l)$ after a sufficiently large extension of $\mathbb{Q}_l$. The trace b_σ remains the same after passing to extensions of $\mathbb{Q}_l$.

If F is CM field, we assume that F has degree $2m$ and $T(X)$ has rank $2mn$. Let F^0 be the maximal totally real subfield of F of degree m. We have the following decomposition into character spaces as an $F^0 \otimes \overline{\mathbb{Q}}_l$-module

$$T_l \otimes \overline{\mathbb{Q}}_l = \bigoplus_{\tau \in \mathrm{Gal}(F^0/\mathbb{Q})} (T_l^\tau, \psi_l^\tau),$$

where each T_l^τ has dimension $2n$. From Hodge symmetry, the bilinear symmetric form ψ_l^τ on $T_l^\sigma \oplus T_l^{\sigma^*}$ has the form

$$\begin{pmatrix} 0 & J_\tau \\ J_\tau & 0 \end{pmatrix}$$

for some diagonal matrix J_τ. So there is a further decomposition

$$T_l \otimes \overline{\mathbb{Q}}_l = \bigoplus_{\substack{(\sigma,\sigma^*) \in \mathrm{Gal}(F/\mathbb{Q}) \\ \sigma|_{F_0} = \tau}} (T_l^\sigma \oplus T_l^{\sigma^*}, \psi_l^\tau). \tag{5.1}$$

If F is totally real, we assume that F has degree m and T_l has rank $2n$ over $F \otimes \overline{\mathbb{Q}}_l$. In this case, complex conjugation is the identity map on $\mathrm{Gal}(F/\mathbb{Q})$ and we have the following decomposition into character spaces as an $F \otimes \overline{\mathbb{Q}}_l$-module:

$$T_l \otimes \overline{\mathbb{Q}}_l = \bigoplus_{\tau \in \mathrm{Gal}(F/\mathbb{Q})} (T_l^\tau, \psi_l^\tau), \tag{5.2}$$

where again T_l^τ has dimension $2n$ and ψ_l^τ is a symmetric form.

The next lemma is an adaptation of [3, Lemma 4.12].

Lemma 5.0.1. *With notations as above:*

1. *If F is CM, we have*

$$\prod_{i=1}^{m} SL_n \simeq \mathcal{G}^F \hookrightarrow \mathcal{G}$$

given by

$$(M_1, \dots, M_m) \mapsto diag(M_1, M_1^*, \dots, M_m, M_m^*),$$

where $M_i^* := J_{\tau_i}^{-1} M_i^\top J_{\tau_i}$ *for all* $\tau_i \in Gal(F_0/\mathbb{Q})$.

2. *If* F *is totally real, we have*

$$\prod_{i=1}^{m} SO_{2n} \simeq \mathcal{G}^F \hookrightarrow \mathcal{G}$$

given by

$$(M_1, \dots, M_m) \mapsto diag(M_1, \dots, M_m),$$

where $M_i^\top \psi_l^{\tau_i} M_i = \psi_l^{\tau_i}$ *for all* $\tau_i \in Gal(F/\mathbb{Q})$.

Proof. Over $\overline{\mathbb{Q}}_l$, the action of $F \otimes \mathbb{Q}_l$ can be diagonalized with respect to the eigenspaces T_l^σ of dimension n for $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$.

1. If F is CM, a matrix $A \in \mathcal{G}$ that commutes with such diagonal matrix must be of block diagonal form, say

$$A = \mathrm{diag}(M_1, M_1', \dots, M_m, M_m'),$$

where (M_i, M_i') corresponds to conjugate pair (τ_i, τ_i^*). From (5.1), the orthogonal condition $A^\top \psi_l A = \psi_l$ is satisfied if and only if $M_i' = M_i^*$ where $M_i^* := J_{\tau_i}^{-1} M_i^\top J_{\tau_i}$ for all τ_i.

2. The totally real case follows similarly from (5.2).

□

Let T be a maximal torus of $\mathcal{G}^F$ given by $\mathrm{diag}(T_1, T_1^{-1}, \dots, T_m, T_m^{-1})$ where each T_i is diagonal of size n. Because T has rank nm, it is also a maximal torus of $\mathcal{G}$. The Weyl group $W_{\mathcal{G}} := N_{\mathcal{G}}(T)/T$ of $\mathcal{G}$ can be realized as a subgroup of the symmetric group $\mathfrak{S}_{2mn}$. We consider the set of permutations of $\{\pm 1, \pm 2, \dots, \pm mn\}$ where the index set $\{\pm(in - n + 1), \dots, \pm in\}$ corresponds to the pair (T_i, T_i^{-1}). Then $W_{\mathcal{G}}$ consists of permutations π for which $\pi(i) + \pi(-i) = 0$ and such that there is an even number of $1 \leq i \leq mn$ for which $\pi(i) > 0$.

We consider two subgroups of $W_{\mathcal{G}}$:

$$H := N_{\mathcal{G}}(\mathcal{G}^F) \cap N_{\mathcal{G}}(T)/T,$$

and the Weyl group of $\mathcal{G}^F$:

$$W^F := N_{\mathcal{G}^F}(T)/T.$$

If F is CM, the group W^F can be identified with the product of symmetric groups $\prod_{i=1}^m \mathfrak{S}_n$ and embedded in $W_{\mathcal{G}}$ via $(\pi_1, \ldots, \pi_m) \mapsto (\pi_1, \pi_1, \ldots, \pi_m, \pi_m)$.

If F is totally real, the group W^F can be identified with the product of orthogonal Weyl groups $\prod_{i=1}^m W_{SO_{2n}}$ via $(\pi_1, \ldots, \pi_m) \mapsto (\pi_1, \ldots, \pi_m)$.

The next proposition is similar to [3, Proposition 4.14].

Proposition 5.0.7. *We have a natural isomorphism*

$$N_{\mathcal{G}}(\mathcal{G}^F)/\mathcal{G}^F \simeq H/W^F.$$

Proof. We define a map from $N_{\mathcal{G}}(\mathcal{G}^F)$ to H/W^F whose kernel is $\mathcal{G}^F$. Let $g \in N_{\mathcal{G}}(\mathcal{G}^F)$ and consider gTg^{-1} which is a maximal torus of $\mathcal{G}^F$. Since maximal tori are conjugates, there exists $g_0 \in \mathcal{G}^F$ up to $N_{\mathcal{G}^F}(T)$ such that $gTg^{-1} = g_0Tg_0^{-1}$. It follows that $g_0^{-1}g$ is in $N_{\mathcal{G}}(\mathcal{G}^F) \cap N_{\mathcal{G}}(T)$ and the map

$$g \mapsto g_0^{-1}gW^F$$

is well defined. The kernel of this map consists of g such that $g_0^{-1}g \in N_{\mathcal{G}^F}(T)$ or $g \in \mathcal{G}^F$. $\square$

Let $\mathcal{G}_X := \mathcal{G} \cap G_{X,l}$ so that $\pi_0(\mathcal{G}_X) = \pi_o(G_{X,l})$. Because this group is abelian, $\mathcal{G}_X$ is contained in $N_{\mathcal{G}}(\mathcal{G}^F)$. The previous proposition yields a well-defined injective homomorphism

$$\phi : \pi_0(\mathcal{G}_X) \hookrightarrow N_{\mathcal{G}}(\mathcal{G}^F)/\mathcal{G}^F \simeq H/W^F.$$

The next lemma is an adaptation of [3, Lemma 4.15].

Lemma 5.0.2. *With notations as above, there is an isomorphism*

$$\epsilon : H/W^F \simeq S \rtimes \mathfrak{S}_m,$$

where $S \subseteq \{\pm 1\}^m$ is the subgroup with an even number of minuses.

Proof. Because H normalizes $\mathcal{G}^F$ which is in block diagonal form, H consists of block permutation matrices of block size $n \times n$ in the CM case and $2n \times 2n$ in the totally real case. Since W^F consists of products of permutations within $n \times n$ blocks in the CM case and within $2n \times 2n$ blocks in the real case, the quotient coset H/W^F is specified by a permutation of the m blocks together with an interchange within a block. The result then follows from the fact that H as a subgroup of the Weyl group $W_{SO_{2mn}}$ has an even number of $1 \le i \le mn$ such that $\pi(i) < 0$.

□

We adapt the proof of the main theorem [3, Theorem 4.20] to our setting.

Recall that for a $\sigma \in \mathrm{Gal}(FL/L)$, its imagine in $\pi_0(\mathcal{G}_X)$ is $\rho(\kappa(\sigma))$ and while its image in $\mathrm{Gal}(F/\mathbb{Q})$ is $\iota(\sigma)$. In addition, $K = F^{\langle\sigma\rangle}$ and d is equal to the order of σ in $\mathrm{Gal}(F/\mathbb{Q})$. It follows that the order of $\rho(\kappa(\sigma))$ in $\pi_0(\mathcal{G}_X)$ is a divisor e of d.

Next we pick a coset representative B_σ in $N_{\mathcal{G}}(\mathcal{G}^F)$ of $\mathcal{G}^F$ such that

$$B_\sigma \mathcal{G}^F = \mathcal{G}_X^{(\sigma)}.$$

Let $q : N_{\mathcal{G}}(\mathcal{G}^F) \to H/W^F$ denote the quotient from Proposition 5.0.7. Then we need to pick B such that

$$\epsilon(q(B_\sigma)) = \phi(\rho(\kappa(\sigma))).$$

From the discussion above, we have a decomposition

$$B_\sigma = P_\sigma T_\sigma \in N_{\mathcal{G}}(\mathcal{G}^F)$$

where P_σ is the block permutation matrix that corresponds to $\phi(\rho(\kappa(\sigma)))$ and T_σ is a diagonal matrix in T. Then as a permutation P_σ has order e that divides d.

Proposition 5.0.8. *With notations as above, for any $\sigma \in Gal(FL/L)$, the function*

$$b_\sigma = \mathrm{Tr}(- \mid \wedge^d H^2(X, \mathbb{Q}_l) \otimes \chi^{-d}) : B_\sigma \mathcal{G}^F \to \mathbb{Q}_l$$

is not a constant function.

Proof. It suffices to show that $\mathrm{Tr}(P_\sigma T_\sigma A \mid \wedge^d H^2(X, \mathbb{Q}_l) \otimes \chi^{-d})$ is not a constant when A varies in $\mathcal{G}^F$. Assume that F is CM. Let $T_\sigma = (T_1, T_1^{-1}, \dots, T_m, T_m^{-1})$, $A = (A_1, A_1^*, \dots, A_m, A_m^*)$, and set

$$C := T_\sigma A = (C_1, E_1, \dots, C_m, E_m).$$

Let Ω denote the set of cycle types τ in $\mathfrak{S}_d$, $\#\tau$ denote the number of permutations in $\mathfrak{S}_d$ with cycle type τ, and define $\mathrm{Tr}_\tau(M) := \prod_{l\in\tau}\mathrm{Tr}(M^l)$. The permutation form of the exponential formula ([31], Chap 7) yields

$$\mathrm{Tr}(P_\sigma C \mid \wedge^d H^2(X, \mathbb{Q}_l)) = \frac{1}{d!}\sum_{\tau\in\Omega}\#\tau\,\mathrm{Tr}_\tau(P_\sigma C).$$

Because P_σ is a block permutation matrix of order e and C is block diagonal, for l divisible by e, the matrix $(P_\sigma C)^l$ has block diagonal form, i.e., it has non-vanishing trace. Taking A to be the identity matrix I or a matrix of the form $(A_1, A_1^*, \ldots, I_m, I_m)$, one sees that b_σ is not a constant.

The totally real case is similar except that we work with block permutation matrices of block size $2n$.

□

Propositions 5.0.8 and 5.0.11 imply the following theorem:

Theorem 5.0.9. *Let X be a K3 surface over a number field L such that $\mathcal{E}(X)$ is an abelian field F and $L^{conn} \subset FL$. If F is totally real then we assume further that $T(X)$ has even rank as an $\mathcal{E}(X)$-module. Then the set of primes of L of μ-ordinary reduction has density 1.*

Corollary 5.0.10. *The set of primes of L with ordinary reduction has density $\frac{1}{[FL:L]}$.*

Proof. Because the μ-ordinary polygon is ordinary if and only if p is totally split in $F/\mathbb{Q}$, the set of ordinary primes of L is the set of primes $\mathfrak{p}$ such that $\iota(\sigma)$ is the identity element in $\mathrm{Gal}(F/\mathbb{Q})$. This set has density $\frac{1}{[FL:L]}$ by Chebotarev's density.

□

Proposition 5.0.11. *If $\mathcal{X}(G, \underline{a}, \underline{b}, N)$ corresponds to a family of K3 surfaces, then a generic surface X in the family satisfies the conditions of Theorem 5.0.12.*

Proof. Let X be a generic point that corresponds to the cyclic cover C_α. This means X is defined over $L = \mathbb{Q}(\alpha_1, \ldots, \alpha_N)$ and $F = E_m$ the m-th cyclotomic field. From the definition of L^{conn}, it suffices to show that the image of $\mathrm{Gal}(FL)$ under $\rho_{X,l}$ is contained in $G^\circ_{X,l} = \mathcal{G}^F$, i.e., $\mathrm{Gal}(FL)$ commutes with the F-action. This follows from the fact that the F-action is multiplication to the coordinates y and z in the defining equations of C_α and D.

□

An analogue of Elkies's supersingular prime theorem

Theorem 5.0.12. *Let X_α be a K3 surface in the family* $(5, (1, 1, 1, 2), (1, 1, 3), 4)$. *In particular, X_α arises from the pair of cyclic covers*

$$C_\alpha : y^5 = x(x-1)(x-\alpha)$$

and

$$D : z^5 = t(t-1).$$

Assume in addition that $j := \frac{(\alpha^2-\alpha+1)^3}{\alpha^2(\alpha-1)^2} \in \mathbb{Q} \cap [0, \frac{27}{4}]$ and the reduction of C_α at 5 is singular. Then X_α can be defined over $\mathbb{Q}$ and there exists infinitely many primes at which the reduction of X_α has basic Newton polygon.

Proof. In [17], the authors show that C_α is defined over $\mathbb{Q}$ and there exist infinitely many basic rational primes for C_α. From Corollary 3.0.4, X_α is defined over $\mathbb{Q}$. From Corollary 4.0.4, X_α has basic polygon at p if and only if C_α has basic polygon at p. Hence X_α has infinitely rational primes of basic reduction. □

Corollary 5.0.13. *With notations as above, there exist infinitely many primes p at which the Picard rank of X_α jumps up.*

Proof. The family of K3 surfaces has $m = 5$ and $N = 4$ so the Picard rank in characteristic zero is $22 - \phi(m)(N-2) = 14$. At basic primes congruent to $2, 3, 4$ mod 5, the basic polygon coincides with the supersingular polygon. Because the Picard rank is 22 at a supersingular prime (Tate's conjecture), it follows that there exist infinitely many primes at which the Picard rank of X_α jumps up by 8. □

Remark 5.0.3. *This phenomenon is recently proven in general for any K3 surface over a number field in [30]. Our examples confirm their result and further calculate the explicit jump.*

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APPENDIX

With the help of a computer program, we can search for all K3 type monodromy data with $m \leq 100$. The following table lists the families with $m \leq 22$. Note that there are no examples when $23 \leq m \leq 100$. The two last columns list all admissible heights and Artin invariants that occur for our families at all the primes.

m	N	a	b	K^2	h	σ_0
3	3	(2, 2, 2)	(1, 1, 1)	0	1, ∞	1
4	3	(2, 3, 3)	(1, 1, 2)	0	1, ∞	1
5	3	(3, 3, 4)	(1, 1, 3)	0	1, ∞	1, 2
6	3	(3, 4, 5)	(1, 1, 4)	0	1, ∞	1
6	3	(2, 5, 5)	(1, 2, 3)	0	1, ∞	1
7	3	(4, 5, 5)	(1, 1, 5)	0	1, 3, ∞	1, 3
7	3	(4, 4, 6)	(1, 2, 4)	0	1, 3, ∞	1, 3
7	3	(3, 5, 6)	(1, 3, 3)	0	1, 3, ∞	1, 3
7	3	(2, 6, 6)	(2, 2, 3)	0	1, 3, ∞	1, 3
8	3	(4, 5, 7)	(1, 2, 5)	0	1, 2, ∞	1
8	3	(5, 5, 6)	(1, 2, 5)	0	1, 2, ∞	1
8	3	(3, 6, 7)	(1, 3, 4)	0	1, 2, ∞	1
8	3	(3, 6, 7)	(2, 3, 3)	0	1, 2, ∞	1
9	3	(5, 6, 7)	(1, 1, 7)	0	1, 3, ∞	1, 3
9	3	(5, 5, 8)	(1, 3, 5)	0	1, 3, ∞	1, 3
9	3	(5, 6, 7)	(1, 3, 5)	0	1, 3, ∞	1, 3
9	3	(4, 6, 8)	(1, 4, 4)	0	1, 3, ∞	1, 3
9	3	(2, 8, 8)	(2, 3, 4)	0	1, 3, ∞	1, 3
9	3	(4, 6, 8)	(2, 3, 4)	0	1, 3, ∞	1, 3
10	3	(5, 7, 8)	(1, 1, 8)	0	1, ∞	1, 2
10	3	(5, 7, 8)	(1, 2, 7)	0	1, ∞	1, 2
10	3	(6, 7, 7)	(1, 2, 7)	0	1, ∞	1, 2
10	3	(5, 6, 9)	(1, 3, 6)	0	1, ∞	1, 2
10	3	(4, 7, 9)	(1, 4, 5)	0	1, ∞	1, 2
10	3	(5, 7, 8)	(1, 4, 5)	0	1, ∞	1, 2
10	3	(6, 7, 7)	(1, 4, 5)	0	1, ∞	1, 2
10	3	(2, 9, 9)	(2, 3, 5)	0	1, ∞	1, 2
10	3	(3, 8, 9)	(2, 3, 5)	0	1, ∞	1, 2
10	3	(5, 6, 9)	(2, 3, 5)	0	1, ∞	1, 2

10	3	(3, 8, 9)	(3, 3, 4)	0	1, ∞	1, 2
10	3	(5, 6, 9)	(3, 3, 4)	0	1, ∞	1, 2
11	3	(6, 8, 8)	(1, 2, 8)	0	1, 5, ∞	1, 5
11	3	(6, 8, 8)	(1, 2, 8)	0	1, 5, ∞	1, 5
11	3	(6, 7, 9)	(1, 3, 7)	0	1, 5, ∞	1, 5
11	3	(5, 8, 9)	(1, 5, 5)	0	1, 5, ∞	1
11	3	(6, 6, 10)	(2, 3, 6)	0	1, 5, ∞	1
11	3	(6, 6, 10)	(2, 3, 6)	0	1, 5, ∞	1
11	3	(4, 8, 10)	(2, 4, 5)	0	1, 5, ∞	1
11	3	(6, 6, 10)	(2, 3, 6)	0	1, 5, ∞	1
11	3	(3, 9, 10)	(3, 3, 5)	0	1, 5, ∞	1
12	3	(7, 8, 9)	(1, 2, 9)	0	1, 2, ∞	1
12	3	(6, 7, 11)	(1, 4, 7)	0	1, 2, ∞	1
12	3	(7, 7, 10)	(1, 4, 7)	0	1, 2, ∞	1
12	3	(7, 8, 9)	(1, 4, 7)	0	1, 2, ∞	1
12	3	(5, 8, 11)	(1, 5, 6)	0	1, 2, ∞	1
12	3	(5, 9, 10)	(1, 5, 6)	0	1, 2, ∞	1
12	3	(6, 7, 11)	(2, 3, 7)	0	1, 2, ∞	1
12	3	(7, 7, 10)	(2, 3, 7)	0	1, 2, ∞	1
12	3	(7, 7, 10)	(2, 3, 7)	0	1, 2, ∞	1
12	3	(7, 8, 9)	(2, 3, 7)	0	1, 2, ∞	1
12	3	(5, 8, 11)	(2, 5, 5)	0	1, 2, ∞	1
12	3	(5, 9, 10)	(2, 5, 5)	0	1, 2, ∞	1
12	3	(3, 10, 11)	(3, 4, 5)	0	1, 2, ∞	1
12	3	(5, 8, 11)	(2, 5, 5)	0	1, 2, ∞	1
12	3	(5, 8, 11)	(3, 4, 5)	0	1, 2, ∞	1
12	3	(5, 9, 10)	(3, 4, 5)	0	1, 2, ∞	1
13	3	(8, 9, 9)	(1, 3, 9)	0	1, 3, ∞	1, 2, 3
13	3	(8, 8, 10)	(1, 4, 8)	0	1, 3, ∞	1, 2, 3
13	3	(7, 9, 10)	(1, 5, 7)	0	1, 3, ∞	1, 2, 3
13	3	(8, 9, 9)	(1, 3, 9)	0	1, 3, ∞	1, 2, 3
13	3	(6, 8, 12)	(3, 4, 6)	0	1, 3, ∞	1, 2, 3
13	3	(5, 9, 12)	(3, 5, 5)	0	1, 3, ∞	1, 2, 3
13	3	(4, 10, 12)	(4, 4, 5)	0	1, 3, ∞	1, 2, 3
14	3	(7, 10, 11)	(1, 2, 11)	0	1, 3, ∞	1, 3
14	3	(7, 9, 12)	(1, 4, 9)	0	1, 3, ∞	1, 3
14	3	(7, 10, 11)	(1, 5, 8)	0	1, 3, ∞	1, 3
14	3	(7, 9, 12)	(2, 3, 9)	0	1, 3, ∞	1, 3

14	3	(9, 9, 10)	(2, 3, 9)	0	1, 3, ∞	1, 3
14	3	(5, 10, 13)	(2, 5, 7)	0	1, 3, ∞	1, 3
14	3	(5, 11, 12)	(2, 5, 7)	0	1, 3, ∞	1, 3
14	3	(6, 11, 11)	(2, 5, 7)	0	1, 3, ∞	1, 3
14	3	(7, 10, 11)	(2, 5, 7)	0	1, 3, ∞	1, 3
14	3	(7, 9, 12)	(3, 3, 8)	0	1, 3, ∞	1, 3
14	3	(3, 12, 13)	(3, 4, 7)	0	1, 3, ∞	1, 3
14	3	(6, 9, 13)	(3, 4, 7)	0	1, 3, ∞	1, 3
14	3	(7, 9, 12)	(3, 4, 7)	0	1, 3, ∞	1, 3
14	3	(9, 9, 10)	(3, 4, 7)	0	1, 3, ∞	1, 3
14	3	(5, 11, 12)	(4, 5, 5)	0	1, 3, ∞	1, 3
14	3	(7, 10, 11)	(4, 5, 5)	0	1, 3, ∞	1, 3
15	3	(8, 10, 12)	(1, 2, 12)	0	1, 2, 4, ∞	1, 2
15	3	(8, 11, 11)	(1, 3, 11)	0	1, 2, 4, ∞	1, 2
15	3	(9, 10, 11)	(1, 3, 11)	0	1, 2, 4, ∞	1, 2
15	3	(8, 10, 12)	(1, 4, 10)	0	1, 2, 4, ∞	1, 2
15	3	(8, 9, 13)	(1, 5, 9)	0	1, 2, 4, ∞	1, 2
15	3	(8, 10, 12)	(1, 6, 8)	0	1, 2, 4, ∞	1, 2
15	3	(7, 10, 13)	(1, 7, 7)	0	1, 2, 4, ∞	1, 2
15	3	(7, 11, 12)	(1, 7, 7)	0	1, 2, 4, ∞	1, 2
15	3	(8, 10, 12)	(2, 4, 9)	0	1, 2, 4, ∞	1, 2
15	3	(8, 8, 14)	(2, 5, 8)	0	1, 2, 4, ∞	1, 2
15	3	(8, 10, 12)	(2, 5, 8)	0	1, 2, 4, ∞	1, 2
15	3	(6, 10, 14)	(2, 6, 7)	0	1, 2, 4, ∞	1, 2
15	3	(8, 8, 14)	(3, 4, 8)	0	1, 2, 4, ∞	1, 2
15	3	(8, 10, 12)	(3, 4, 8)	0	1, 2, 4, ∞	1, 2
15	3	(3, 13, 14)	(3, 5, 7)	0	1, 2, 4, ∞	1, 2
15	3	(5, 11, 14)	(3, 5, 7)	0	1, 2, 4, ∞	1, 2
15	3	(6, 11, 13)	(3, 5, 7)	0	1, 2, 4, ∞	1, 2
15	3	(7, 9, 14)	(3, 5, 7)	0	1, 2, 4, ∞	1, 2
15	3	(7, 10, 13)	(3, 5, 7)	0	1, 2, 4, ∞	1, 2
15	3	(7, 11, 12)	(3, 5, 7)	0	1, 2, 4, ∞	1, 2
15	3	(4, 12, 14)	(4, 4, 7)	0	1, 2, 4, ∞	1, 2
15	3	(4, 12, 14)	(4, 5, 6)	0	1, 2, 4, ∞	1, 2
16	3	(8, 11, 13)	(1, 2, 13)	0	1, 2, 4, ∞	1, 2
16	3	(8, 11, 13)	(1, 4, 11)	0	1, 2, 4, ∞	1, 2
16	3	(10, 11, 11)	(1, 4, 11)	0	1, 2, 4, ∞	1, 2
16	3	(9, 11, 12)	(1, 6, 9)	0	1, 2, 4, ∞	1, 2

16	3	(7, 11, 14)	(1, 7, 8)	0	1, 2, 4, ∞	1, 2
16	3	(7, 12, 13)	(1, 7, 8)	0	1, 2, 4, ∞	1, 2
16	3	(9, 11, 12)	(2, 3, 11)	0	1, 2, 4, ∞	1, 2
16	3	(8, 9, 15)	(2, 5, 9)	0	1, 2, 4, ∞	1, 2
16	3	(7, 12, 13)	(2, 7, 7)	0	1, 2, 4, ∞	1, 2
16	3	(8, 9, 15)	(3, 4, 9)	0	1, 2, 4, ∞	1, 2
16	3	(9, 9, 14)	(3, 4, 9)	0	1, 2, 4, ∞	1, 2
16	3	(3, 14, 15)	(3, 5, 8)	0	1, 2, 4, ∞	1, 2
16	3	(5, 12, 15)	(3, 5, 8)	0	1, 2, 4, ∞	1, 2
16	3	(5, 13, 14)	(4, 5, 7)	0	1, 2, 4, ∞	1, 2
16	3	(7, 10, 15)	(4, 5, 7)	0	1, 2, 4, ∞	1, 2
16	3	(5, 12, 15)	(5, 5, 6)	0	1, 2, 4, ∞	1, 2
17	3	(10, 12, 12)	(2, 3, 12)	0	1, ∞	1, 2, 4, 8
17	3	(10, 10, 14)	(2, 5, 10)	0	1, ∞	1, 2, 4, 8
17	3	(8, 12, 14)	(2, 7, 8)	0	1, ∞	1, 2, 4, 8
17	3	(9, 10, 15)	(3, 5, 9)	0	1, ∞	1, 2, 4, 8
17	3	(7, 12, 15)	(3, 7, 7)	0	1, ∞	1, 2, 4, 8
17	3	(5, 14, 15)	(5, 5, 7)	0	1, ∞	1, 2, 4, 8
18	3	(9, 13, 14)	(1, 3, 14)	0	1, 3, ∞	1, 3
18	3	(11, 12, 13)	(1, 4, 13)	0	1, 3, ∞	1, 3
18	3	(9, 11, 16)	(1, 6, 11)	0	1, 3, ∞	1, 3
18	3	(11, 11, 14)	(1, 6, 11)	0	1, 3, ∞	1, 3
18	3	(8, 13, 15)	(1, 8, 9)	0	1, 3, ∞	1, 3
18	3	(11, 12, 13)	(1, 8, 9)	0	1, 3, ∞	1, 3
18	3	(9, 13, 14)	(2, 3, 13)	0	1, 3, ∞	1, 3
18	3	(10, 13, 13)	(2, 3, 13)	0	1, 3, ∞	1, 3
18	3	(10, 11, 15)	(2, 5, 11)	0	1, 3, ∞	1, 3
18	3	(7, 12, 17)	(2, 7, 9)	0	1, 3, ∞	1, 3
18	3	(7, 14, 15)	(2, 7, 9)	0	1, 3, ∞	1, 3
18	3	(9, 11, 16)	(3, 4, 11)	0	1, 3, ∞	1, 3
18	3	(11, 12, 13)	(3, 4, 11)	0	1, 3, ∞	1, 3
18	3	(9, 10, 17)	(3, 5, 10)	0	1, 3, ∞	1, 3
18	3	(7, 13, 16)	(3, 7, 8)	0	1, 3, ∞	1, 3
18	3	(9, 13, 14)	(3, 7, 8)	0	1, 3, ∞	1, 3
18	3	(4, 15, 17)	(4, 5, 9)	0	1, 3, ∞	1, 3
18	3	(10, 11, 15)	(4, 5, 9)	0	1, 3, ∞	1, 3
18	3	(7, 12, 17)	(4, 7, 7)	0	1, 3, ∞	1, 3
18	3	(5, 15, 16)	(5, 5, 8)	0	1, 3, ∞	1, 3

18	3	(5, 14, 17)	(5, 6, 7)	0	1, 3, ∞	1, 3
18	3	(7, 14, 15)	(5, 6, 7)	0	1, 3, ∞	1, 3
18	3	(9, 10, 17)	(5, 6, 7)	0	1, 3, ∞	1, 3
19	3	(12, 12, 14)	(3, 4, 12)	0	1, 3, 9, ∞	1, 3, 9
19	3	(11, 12, 15)	(3, 5, 11)	0	1, 3, 9, ∞	1, 3, 9
19	3	(9, 14, 15)	(3, 7, 9)	0	1, 3, 9, ∞	1, 3, 9
19	3	(10, 12, 16)	(4, 5, 10)	0	1, 3, 9, ∞	1, 3, 9
19	3	(8, 14, 16)	(4, 7, 8)	0	1, 3, 9, ∞	1, 3, 9
19	3	(7, 15, 16)	(5, 7, 7)	0	1, 3, 9, ∞	1, 3, 9
20	3	(12, 13, 15)	(1, 6, 13)	0	1, 2, 4 ∞	1, 2
20	3	(11, 14, 15)	(1, 8, 11)	0	1, 2, 4 ∞	1, 2
20	3	(9, 15, 16)	(1, 9, 10)	0	1, 2, 4 ∞	1, 2
20	3	(10, 13, 17)	(2, 5, 13)	0	1, 2, 4 ∞	1, 2
20	3	(11, 13, 16)	(2, 5, 13)	0	1, 2, 4 ∞	1, 2
20	3	(13, 13, 14)	(2, 5, 13)	0	1, 2, 4 ∞	1, 2
20	3	(11, 14, 15)	(2, 7, 11)	0	1, 2, 4 ∞	1, 2
20	3	(9, 15, 16)	(2, 9, 9)	0	1, 2, 4 ∞	1, 2
20	3	(12, 13, 15)	(3, 4, 13)	0	1, 2, 4 ∞	1, 2
20	3	(12, 13, 15)	(3, 6, 11)	0	1, 2, 4 ∞	1, 2
20	3	(7, 15, 18)	(3, 7, 10)	0	1, 2, 4 ∞	1, 2
20	3	(9, 15, 16)	(3, 7, 10)	0	1, 2, 4 ∞	1, 2
20	3	(9, 15, 16)	(3, 8, 9)	0	1, 2, 4 ∞	1, 2
20	3	(10, 11, 19)	(4, 5, 11)	0	1, 2, 4 ∞	1, 2
20	3	(10, 13, 17)	(4, 5, 11)	0	1, 2, 4 ∞	1, 2
20	3	(11, 11, 18)	(4, 5, 11)	0	1, 2, 4 ∞	1, 2
20	3	(11, 12, 17)	(4, 5, 11)	0	1, 2, 4 ∞	1, 2
20	3	(7, 15, 18)	(4, 7, 9)	0	1, 2, 4 ∞	1, 2
20	3	(9, 12, 19)	(5, 6, 9)	0	1, 2, 4 ∞	1, 2
20	3	(9, 13, 18)	(5, 6, 9)	0	1, 2, 4 ∞	1, 2
20	3	(7, 14, 19)	(5, 7, 8)	0	1, 2, 4 ∞	1, 2
20	3	(7, 16, 17)	(5, 7, 8)	0	1, 2, 4 ∞	1, 2
20	3	(9, 14, 17)	(5, 7, 8)	0	1, 2, 4 ∞	1, 2
20	3	(7, 15, 18)	(6, 7, 7)	0	1, 2, 4 ∞	1, 2
3	4	(1, 1, 2, 2)	(1, 1, 1)	0	1, 2, ∞	1
4	4	(1, 1, 3, 3)	(1, 1, 2)	0	1, 2, ∞	1
4	4	(1, 2, 2, 3)	(1, 1, 2)	0	1, 2, ∞	1
5	4	(1, 3, 3, 3)	(1, 1, 3)	0	1, 2, 4, ∞	1, 2
5	4	(2, 2, 2, 4)	(1, 2, 2)	0	1, 2, 4, ∞	1, 2

6	4	(1, 1, 5, 5)	(1, 2, 3)	0	1, 2, ∞	1
6	4	(1, 2, 4, 5)	(1, 2, 3)	0	1, 2, ∞	1
6	4	(1, 3, 3, 5)	(1, 1, 4)	0	1, 2, ∞	1
6	4	(1, 3, 3, 5)	(1, 2, 3)	0	1, 2, ∞	1
6	4	(1, 3, 4, 4)	(1, 1, 4)	0	1, 2, ∞	1
6	4	(1, 3, 4, 4)	(1, 2, 3)	0	1, 2, ∞	1
6	4	(2, 2, 3, 5)	(1, 2, 3)	0	1, 2, ∞	1
6	4	(2, 3, 3, 4)	(1, 1, 4)	0	1, 2, ∞	1
6	4	(2, 3, 3, 4)	(1, 2, 3)	0	1, 2, ∞	1
7	4	(2, 4, 4, 4)	(1, 2, 4)	0	1, 2, 3, 6, ∞	1,3
7	4	(3, 3, 3, 5)	(1, 3, 3)	0	1, 2, 3, 6, ∞	1,3
8	4	(1, 3, 6, 6)	(1, 3, 4)	0	1, 2, 4, ∞	1
8	4	(1, 5, 5, 5)	(1, 2, 5)	0	1, 2, 4, ∞	1
8	4	(2, 4, 5, 5)	(1, 2, 5)	0	1, 2, 4, ∞	1
8	4	(3, 3, 3, 7)	(1, 3, 4)	0	1, 2, 4, ∞	1
8	4	(3, 3, 3, 7)	(2, 3, 3)	0	1, 2, 4, ∞	1
8	4	(3, 3, 4, 6)	(1, 3, 4)	0	1, 2, 4, ∞	1
8	4	(3, 3, 4, 6)	(2, 3, 3)	0	1, 2, 4, ∞	1
9	4	(1, 5, 5, 7)	(1, 3, 5)	0	1, 2, 3, 6, ∞	1, 3
9	4	(2, 4, 4, 8)	(2, 3, 4)	0	1, 2, 3, 6, ∞	1, 3
9	4	(3, 5, 5, 5)	(1, 3, 5)	0	1, 2, 3, 6, ∞	1, 3
9	4	(4, 4, 4, 6)	(1, 4, 4)	0	1, 2, 3, 6, ∞	1, 3
9	4	(4, 4, 4, 6)	(2, 3, 4)	0	1, 2, 3, 6, ∞	1, 3
10	4	(1, 4, 7, 8)	(1, 4, 5)	0	1, 2, 4, ∞	1, 2
10	4	(1, 5, 7, 7)	(1, 2, 7)	0	1, 2, 4, ∞	1, 2
10	4	(1, 5, 7, 7)	(1, 4, 5)	0	1, 2, 4, ∞	1, 2
10	4	(2, 3, 6, 9)	(2, 3, 5)	0	1, 2, 4, ∞	1, 2
10	4	(2, 4, 7, 7)	(1, 4, 5)	0	1, 2, 4, ∞	1, 2
10	4	(3, 3, 5, 9)	(2, 3, 5)	0	1, 2, 4, ∞	1, 2
10	4	(3, 3, 5, 9)	(3, 3, 4)	0	1, 2, 4, ∞	1, 2
10	4	(3, 3, 6, 8)	(2, 3, 5)	0	1, 2, 4, ∞	1, 2
10	4	(3, 5, 6, 6)	(1, 3, 6)	0	1, 2, 4, ∞	1, 2
10	4	(3, 5, 6, 6)	(2, 3, 5)	0	1, 2, 4, ∞	1, 2
10	4	(4, 4, 5, 7)	(1, 4, 5)	0	1, 2, 4, ∞	1, 2
12	4	(1, 5, 8, 10)	(1, 5, 6)	0	1, 2, 4, ∞	1
12	4	(1, 7, 7, 9)	(1, 4, 7)	0	1, 2, 4, ∞	1
12	4	(2, 6, 7, 9)	(1, 4, 7)	1	1, 2, 4, ∞	1
12	4	(2, 7, 7, 8)	(2, 3, 7)	0	1, 2, 4, ∞	1

12	4	(3, 5, 5, 11)	(3, 4, 5)	0	1, 2, 4, ∞	1
12	4	(3, 5, 6, 10)	(3, 4, 5)	0	1, 2, 4, ∞	1
12	4	(3, 5, 8, 8)	(1, 5, 6)	1	1, 2, 4, ∞	1
12	4	(3, 7, 7, 7)	(1, 4, 7)	0	1, 2, 4, ∞	1
12	4	(3, 7, 7, 7)	(2, 3, 7)	0	1, 2, 4, ∞	1
12	4	(4, 5, 5, 10)	(1, 5, 6)	0	1, 2, 4, ∞	1
12	4	(4, 5, 5, 10)	(3, 4, 5)	0	1, 2, 4, ∞	1
12	4	(4, 6, 7, 7)	(1, 4, 7)	0	1, 2, 4, ∞	1
12	4	(4, 6, 7, 7)	(2, 3, 7)	0	1, 2, 4, ∞	1
12	4	(5, 5, 5, 9)	(1, 5, 6)	0	1, 2, 4, ∞	1
12	4	(5, 5, 5, 9)	(2, 5, 5)	0	1, 2, 4, ∞	1
12	4	(5, 5, 5, 9)	(3, 4, 5)	0	1, 2, 4, ∞	1
12	4	(5, 5, 6, 8)	(1, 5, 6)	0	1, 2, 4, ∞	1
12	4	(5, 5, 6, 8)	(2, 5, 5)	0	1, 2, 4, ∞	1
12	4	(5, 5, 6, 8)	(3, 4, 5)	0	1, 2, 4, ∞	1
14	4	(1, 5, 11, 11)	(2, 5, 7)	2	1, 2, 3, 6, ∞	
14	4	(1, 9, 9, 9)	(3, 4, 7)	3	1, 2, 3, 6, ∞	
14	4	(2, 5, 10, 11)	(2, 5, 7)	0	1, 2, 3, 6, ∞	1, 3
14	4	(3, 3, 9, 13)	(3, 4, 7)	0	1, 2, 3, 6, ∞	1, 3
14	4	(3, 4, 9, 12)	(3, 4, 7)	0	1, 2, 3, 6, ∞	1, 3
14	4	(3, 7, 9, 9)	(2, 3, 9)	0	1, 2, 3, 6, ∞	1, 3
14	4	(3, 7, 9, 9)	(3, 4, 7)	0	1, 2, 3, 6, ∞	1, 3
14	4	(4, 6, 9, 9)	(3, 4, 7)	0	1, 2, 3, 6, ∞	1, 3
14	4	(5, 5, 5, 13)	(2, 5, 7)	0	1, 2, 3, 6, ∞	1, 3
14	4	(5, 5, 7, 11)	(2, 5, 7)	0	1, 2, 3, 6, ∞	1, 3
14	4	(5, 5, 7, 11)	(4, 5, 5)	0	1, 2, 3, 6, ∞	1, 3
14	4	(5, 5, 8, 10)	(2, 5, 7)	0	1, 2, 3, 6, ∞	1, 3
15	4	(2, 8, 8, 12)	(2, 5, 8)	0	1, 2, 4, 8, ∞	1, 2
15	4	(3, 7, 7, 13)	(3, 5, 7)	0	1, 2, 4, 8, ∞	1, 2
15	4	(4, 8, 8, 10)	(3, 4, 8)	0	1, 2, 4, 8, ∞	1, 2
15	4	(5, 7, 7, 11)	(3, 5, 7)	0	1, 2, 4, 8, ∞	1, 2
15	4	(6, 8, 8, 8)	(2, 5, 8)	0	1, 2, 4, 8, ∞	1, 2
15	4	(7, 7, 7, 9)	(3, 5, 7)	0	1, 2, 4, 8, ∞	1, 2
16	4	(7, 7, 7, 11)	(1, 7, 8)	0	1, 2, 4, 8, ∞	1, 2
18	4	(4, 5, 12, 15)	(4, 5, 9)	0	1, 2, 3, 6, ∞	1, 3
18	4	(5, 5, 11, 15)	(4, 5, 9)	0	1, 2, 3, 6, ∞	1, 3
18	4	(7, 7, 7, 15)	(2, 7, 9)	0	1, 2, 3, 6, ∞	1, 3
18	4	(7, 7, 10, 12)	(2, 7, 9)	0	1, 2, 3, 6, ∞	1, 3

22	4	(7, 7, 13, 17)	(4, 7, 11)	0	1, 2, 5, 10, ∞	1, 5
22	4	(9, 9, 9, 17)	(2, 9, 11)	0	1, 2, 5, 10, ∞	1, 5
3	5	(1, 1, 1, 1, 2)	(1, 1, 1)	0	1, 2, 3, ∞	1, 2
4	5	(1, 1, 1, 2, 3)	(1, 1, 2)	0	1, 2, 3, ∞	1, 2
4	5	(1, 1, 2, 2, 2)	(1, 1, 2)	0	1, 2, 3, ∞	1, 2
5	5	(2, 2, 2, 2, 2)	(1, 2, 2)	0	1 - 4, ∞	1, 2, 4
6	5	(1, 1, 1, 4, 5)	(1, 2, 3)	0	1, 2, 3, ∞	1, 2
6	5	(1, 1, 2, 3, 5)	(1, 2, 3)	0	1, 2, 3, ∞	1, 2
6	5	(1, 1, 2, 4, 4)	(1, 2, 3)	0	1, 2, 3, ∞	1, 2
6	5	(1, 1, 3, 3, 4)	(1, 1, 4)	0	1, 2, 3, ∞	1, 2
6	5	(1, 1, 3, 3, 4)	(1, 2, 3)	0	1, 2, 3, ∞	1, 2
6	5	(1, 2, 2, 2, 5)	(1, 2, 3)	0	1, 2, 3, ∞	1, 2
6	5	(1, 2, 2, 3, 4)	(1, 2, 3)	0	1, 2, 3, ∞	1, 2
6	5	(1, 2, 3, 3, 3)	(1, 1, 4)	1	1, 2, 3, ∞	1, 2
6	5	(1, 2, 3, 3, 3)	(1, 2, 3)	0	1, 2, 3, ∞	1, 2
6	5	(2, 2, 2, 3, 3)	(1, 2, 3)	0	1, 2, 3, ∞	1, 2
8	5	(1, 3, 3, 3, 6)	(1, 3, 4)	0	1 - 4, 6, ∞	1, 2
8	5	(3, 3, 3, 3, 4)	(1, 3, 4)	0	1 - 4, 6, ∞	1, 2
8	5	(3, 3, 3, 3, 4)	(2, 3, 3)	0	1 - 4, 6, ∞	1, 2
9	5	(2, 4, 4, 4, 4)	(2, 3, 4)	0	1, 2, 3, 6, 9, ∞	1, 2, 3, 6
10	5	(1, 1, 4, 7, 7)	(1, 4, 5)	0	1 - 4, ∞	1, 2, 4
10	5	(1, 4, 4, 4, 7)	(1, 4, 5)	0	1 - 4, ∞	1, 2, 4
10	5	(2, 3, 3, 3, 9)	(2, 3, 5)	0	1 - 4, ∞	1, 2, 4
10	5	(2, 3, 3, 6, 6)	(2, 3, 5)	0	1 - 4, ∞	1, 2, 4
10	5	(3, 3, 3, 3, 8)	(2, 3, 5)	0	1 - 4, ∞	1, 2, 4
10	5	(3, 3, 3, 5, 6)	(2, 3, 5)	0	1 - 4, ∞	1, 2, 4
12	5	(1, 5, 5, 5, 8)	(1, 5, 6)	0	1 - 4, 6, ∞	1, 2
12	5	(3, 5, 5, 5, 6)	(3, 4, 5)	0	1 - 4, 6, ∞	1, 2
12	5	(4, 5, 5, 5, 5)	(1, 5, 6)	0	1 - 4, 6, ∞	1, 2
12	5	(4, 5, 5, 5, 5)	(3, 4, 5)	0	1 - 4, 6, ∞	1, 2
14	5	(2, 5, 5, 5, 11)	(2, 5, 7)	0	1, 2, 3, 6, 9, ∞	1, 2, 3, 6
14	5	(3, 3, 4, 9, 9)	(3, 4, 7)	0	1, 2, 3, 6, 9, ∞	1, 2, 3, 6
14	5	(5, 5, 5, 5, 8)	(2, 5, 7)	0	1, 2, 3, 6, 9, ∞	1, 2, 3, 6
3	6	(1, 1, 1, 1, 1, 1)	(1, 1, 1)	0	1 - 4, ∞	1, 2
4	6	(1, 1, 1, 1, 1, 3)	(1, 1, 2)	0	1 - 4, ∞	1, 2
4	6	(1, 1, 1, 1, 2, 2)	(1, 1, 2)	0	1 - 4, ∞	1, 2
6	6	(1, 1, 1, 1, 3, 5)	(1, 2, 3)	0	1 - 4, ∞	1, 2
6	6	(1, 1, 1, 1, 4, 4)	(1, 2, 3)	0	1 - 4, ∞	1, 2

6	6	(1, 1, 1, 2, 2, 5)	(1, 2, 3)	0	1 - 4, ∞	1, 2
6	6	(1, 1, 1, 2, 3, 4)	(1, 2, 3)	0	1 - 4, ∞	1, 2
6	6	(1, 1, 1, 3, 3, 3)	(1, 1, 4)	1	1 - 4, ∞	
6	6	(1, 1, 1, 3, 3, 3)	(1, 2, 3)	0	1 - 4, ∞	1, 2
6	6	(1, 1, 2, 2, 2, 4)	(1, 2, 3)	0	1 - 4, ∞	1, 2
6	6	(1, 1, 2, 2, 3, 3)	(1, 2, 3)	0	1 - 4, ∞	1, 2
6	6	(1, 2, 2, 2, 2, 3)	(1, 2, 3)	0	1 - 4, ∞	1, 2
8	6	(1, 3, 3, 3, 3, 3)	(1, 3, 4)	0	1 - 4, 6, 8, ∞	1, 2
10	6	(2, 3, 3, 3, 3, 6)	(2, 3, 5)	0	1 - 4, 8, ∞	1, 2, 4
10	6	(3, 3, 3, 3, 3, 5)	(2, 3, 5)	0	1 - 4, 8, ∞	1, 2, 4
4	7	(1, 1, 1, 1, 1, 1, 2)	(1, 1, 2)	0	1 - 5, ∞	1, 2, 3
6	7	(1, 1, 1, 1, 1, 2, 5)	(1, 2, 3)	0	1 - 5, ∞	1, 2, 3
6	7	(1, 1, 1, 1, 1, 3, 4)	(1, 2, 3)	0	1 - 5, ∞	1, 2, 3
6	7	(1, 1, 1, 1, 2, 2, 4)	(1, 2, 3)	0	1 - 5, ∞	1, 2, 3
6	7	(1, 1, 1, 1, 2, 3, 3)	(1, 2, 3)	0	1 - 5, ∞	1, 2, 3
6	7	(1, 1, 1, 2, 2, 2, 3)	(1, 2, 3)	0	1 - 5, ∞	1, 2, 3
6	7	(1, 1, 2, 2, 2, 2, 2)	(1, 2, 3)	0	1 - 5, ∞	1, 2, 3
10	7	(2, 3, 3, 3, 3, 3, 3)	(2, 3, 5)	0	1 - 5, 8, ∞	1 - 4, 6
4	8	(1, 1, 1, 1, 1, 1, 1, 1)	(1, 1, 2)	0	1 - 6, ∞	1, 2, 3
6	8	(1, 1, 1, 1, 1, 1, 1, 5)	(1, 2, 3)	0	1 - 6, ∞	1, 2, 3
6	8	(1, 1, 1, 1, 1, 1, 2, 4)	(1, 2, 3)	0	1 - 6, ∞	1, 2, 3
6	8	(1, 1, 1, 1, 1, 2, 2, 3)	(1, 2, 3)	0	1 - 6, ∞	1, 2, 3
6	8	(1, 1, 1, 1, 1, 1, 3, 3)	(1, 2, 3)	0	1 - 6, ∞	1, 2, 3
6	8	(1, 1, 1, 1, 2, 2, 2, 2)	(1, 2, 3)	0	1 - 6, ∞	1, 2, 3
6	9	(1, 1, 1, 1, 1, 1, 1, 1, 4)	(1, 2, 3)	0	1 - 7, ∞	1 - 4
6	9	(1, 1, 1, 1, 1, 1, 1, 2, 3)	(1, 2, 3)	0	1 - 7, ∞	1 - 4
6	9	(1, 1, 1, 1, 1, 1, 2, 2, 2)	(1, 2, 3)	0	1 - 7, ∞	1 - 4
6	10	(1, 1, 1, 1, 1, 1, 1, 1, 1, 3)	(1, 2, 3)	0	1 - 8, ∞	1 - 4
6	10	(1, 1, 1, 1, 1, 1, 1, 1, 2, 2)	(1, 2, 3)	0	1 - 8, ∞	1 - 4
6	11	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2)	(1, 2, 3)	0	1 - 9, ∞	1 - 5
6	12	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)	(1, 2, 3)	0	1 - 10, ∞	1 - 5

Table 5.1: Special families of k3-type surfaces