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Warren Edward Mathews

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ABSTRACT

An analysis of the one-dimensional flow of vapor between parallel liquid surfaces of identical composition but different temperatures is presented. The low-velocity steady-flow analysis reported previously by Plesset is extended to steady flows in which the Mach number approaches unity, and to low-velocity flows in which the liquid surface spacing and temperatures are allowed to vary slowly with time. More important among the new results obtained are 1) an exact solution in closed form of the (non-linear) steady-flow equations, subject only to the condition $\gamma M^2 \leq 1$, and 2) a relatively simple approximate form of the nonsteady-flow perturbation solution which applies whenever the product of liquid surface spacing and unperturbed current density is not unusually small.

The perturbation technique developed for the one-dimensional problem is extended also to cylindrically symmetric two-dimensional and spherically symmetric three-dimensional flows. In addition, an alternative solution for the pressure perturbation is obtained by a method which, while clearly non-exact, does not explicitly involve a neglect of higher order terms.

TABLE OF CONTENTS

I. INTRODUCTION

The low-velocity one-dimensional flow of vapor between stationary liquid surfaces of identical composition but different constant temperatures has been investigated by Plesset.[1] The purpose of the present thesis is to extend the analysis to steady flows in which the Mach number approaches unity, and to low-velocity flows in which the wall spacing and temperatures are allowed to vary slowly with time.

1.1 Statement of the Problem

The following problem is considered: A liquid surface at $x = x_a$ is maintained at a temperature T_a, and a second surface of the same liquid composition at $x = x_b$ is maintained at the temperature T_b, with $x_b > x_a$ and $T_b < T_a$. The positions x_a, x_b and the temperatures T_a, T_b may be either constant or slowly varying with time. It is supposed that the space between the two liquid surfaces contains only the vapor of the liquid concerned and that the flow of vapor from $x = x_a$ to $x = x_b$ is one-dimensional, i.e., that all quantities are functions of x and possibly t only. It also is assumed that the flow is either strictly steady ($\partial/\partial t = 0$) or steady-state in the sense that no starting transient is involved. Departures of the vapor from perfect gas behavior, dependence of the specific and latent heats and thermal conductivity of the vapor on temperature and density, and viscous effects in the vapor flow are neglected.

1.2 Summary of Results

In Part II, the equations for one-dimensional flow of a perfect gas are developed in a form particularly suitable for application to steady and near-steady flows. The boundary conditions pertinent to the present problem are considered, and an appropriate nondimensionalization of the flow equations and boundary conditions is derived.

In Part III, the equations developed in Part II are applied to the stationary problem considered by Plesset and his results are reobtained by neglecting terms of order u^2/c^2 with respect to unity. In addition, however, an exact solution of the steady-flow equations is obtained in closed form, subject only to the condition that u^2/c^2 at the warmer liquid surface remain less than $\gamma^{-1} = c_v/c_p$. In general, this limit on the square of the Mach number is of the order of 3/4. With the aid of the exact solution, it is shown that the low-velocity approximation to the current density may yield negligible errors even in the limit $\gamma u^2/c^2 = 1$, and that the approximate expression for the temperature distribution remains valid except in the highest velocity flows.

The extension to low-velocity nonsteady flows is made in Part IV, using a reinterpretation of the steady-flow solution as the basis for a perturbation analysis. Development of the complete solution of the first-order perturbation equations is followed by an investigation of the limiting form attained by this solution as the liquid surface temperatures become equal. Most important of the nonsteady flow results, however, is a relatively simple approximate form of the perturbation solution which is found to apply whenever the product of liquid surface spacing and unperturbed current density is not unusually small.

In Part V, the perturbation technique developed in Part IV is extended to cylindrically symmetric two-dimensional and spherically symmetric three-dimensional flows. In addition, an alternative solution for the pressure perturbation is obtained by a method which, while clearly non-exact, does not explicitly involve a neglect of higher order terms.

II. BASIC ONE-DIMENSIONAL FLOW EQUATIONS AND BOUNDARY CONDITIONS

2.1 Basic Flow Equations

In this section, the equations for one-dimensional nonadiabatic flow of a perfect gas are developed in a form particularly suitable for application to steady and near-steady flows. The specific heats and thermal conductivity of the gas are assumed independent of temperature and density, and viscosity is neglected. Partial differentiation is indicated by subscripts, while total differentiation with respect to time ("particle derivative") is indicated by a dot above the quantity differentiated.

The fundamental relations governing the flow under consideration are the conservation of mass,

$$(\rho u)_x + \rho_t = 0 , \tag{1}$$

the conservation of momentum,

$$\rho \dot{u} + p_x = 0 , \tag{2}$$

the conservation of energy,

$$\rho(\dot{e} + u\dot{u}) + (pu)_x - (kT_x)_x = 0 , \tag{3}$$

and the equation of state,

$$p = \rho RT \, . \tag{4}$$

Here ρ is the density, u the velocity, p the pressure, T the temperature, k the thermal conductivity, e the intrinsic energy per unit mass, and R the gas constant per unit mass of the gas.

The inaccessibility of a general solution of this system of non-linear equations is well known. An equivalent set of equations can be derived, however, which is readily integrable for steady flows in the limit of low velocity. Near-steady flows of moderate velocity can then be analyzed by application of an appropriate iterative or perturbation procedure to these derived equations.

It will be observed that Eq. (1) is already of the desired form, since ρ_t is arbitrarily small in a sufficiently near-steady flow. Equation (2), which may be written as

$$\rho(u_t + uu_x) + p_x = 0 \, , \tag{5}$$

is readily brought into the desired form by adding to it Eq. (1) multiplied by u, yielding

$$(p + \rho u^2)_x + (\rho u)_t = 0 \, . \tag{6}$$

On subtraction of Eq. (2) multiplied by u, Eq. (3) reduces to

$$\rho \dot{e} + pu_x - kT_{xx} = 0 \, , \tag{7}$$

where account has been taken of the assumed constancy of k. With the substitution of $c_v\dot{T}$ for $\dot{e}$, where c_v is the specific heat at constant volume, Eq. (7) becomes

$$\rho c_v T_t + \rho u c_v T_x + p u_x - k T_{xx} = 0 . \tag{8}$$

With the aid of Eqs. (4) and (1), the third term of Eq. (8) can be written as

$$\begin{aligned} p u_x &= (pu)_x - u p_x \\ &= (\rho u R T)_x - u p_x \\ &= \rho u R T_x + RT(\rho u)_x - u p_x \\ &= \rho u R T_x - RT\rho_t - u p_x . \end{aligned} \tag{9}$$

Substitution of Eq. (9) into Eq. (8) yields

$$\begin{aligned} kT_{xx} - c_p \rho u T_x &= c_v \rho T_t - RT\rho_t - u p_x \\ &= c_p \rho T_t - p_t - u p_x , \end{aligned} \tag{10}$$

where c_p is the specific heat at constant pressure and use has been made of the relationship[2]

$$R = c_p - c_v . \tag{11}$$

Since ρu is independent of x in steady flow [see Eq. (1)], and if the gas is assumed to be calorically as well as thermally perfect, only the term up_x keeps Eq. (10) from being evidently of the desired form. Equation (6), however, can be written as

$$p_x + (\rho u)_t = - \left[\frac{(\rho u)^2}{\rho}\right]_x$$

$$= - 2u(\rho u)_x + u^2 \rho_x , \tag{12}$$

which on substitution for $(\rho u)_x$ from Eq. (1) and expression of ρ_x by means of Eq. (4) yields

$$p_x + (\rho u)_t = 2u\rho_t + u^2 \frac{p_x - \rho RT_x}{RT} \tag{13}$$

$$up_x = \frac{- u(\rho u)_t + 2u^2\rho_t - \frac{u^2}{RT} \rho uRT_x}{1 - \frac{u^2}{RT}} . \tag{14}$$

The speed of sound, c, in a gas is given by[3]

$$c^2 = \gamma \frac{p}{\rho} , \tag{15}$$

which for a perfect gas becomes

$$c^2 = \gamma RT , \tag{16}$$

where γ is the ratio of specific heats, c_p/c_v. Substitution of Eqs. (14) and (16) into Eq. (10) yields

$$kT_{xx} - c_p \rho u \left[1 + \frac{(\gamma-1)\frac{u^2}{c^2}}{1 - \gamma \frac{u^2}{c^2}} \right] T_x = c_p \rho T_t - p_t + \frac{u(\rho u)_t - 2u^2 \rho_t}{1 - \gamma \frac{u^2}{c^2}} , \tag{17}$$

which form clearly has the desired properties.

Since u^2/c^2 will be neglected with respect to unity in obtaining a first-order solution of Eq. (17), it is worth noting that the final term in this equation is of order u^2/c^2 with respect to the other two terms of the right-hand member. Specifically,

$$\begin{aligned} u(\rho u)_t - 2u^2 \rho_t &= \frac{3}{2} \rho (u^2)_t - (\rho u^2)_t \\ &= \frac{3}{2} \rho (\frac{u^2}{c^2} \gamma RT)_t - (\frac{u^2}{c^2} \gamma p)_t \\ &= \frac{3}{2} \gamma \frac{u^2}{c^2} \rho RT_t - \gamma \frac{u^2}{c^2} p_t + \frac{\gamma}{2} p \, (\frac{u^2}{c^2})_t , \end{aligned} \tag{18}$$

so that Eq. (17) finally can be written as

$$(1 - \gamma \frac{u^2}{c^2})kT_{xx} - (1 - \frac{u^2}{c^2})c_p\rho uT_x$$

$$= (1 - \frac{3-\gamma}{2} \frac{u^2}{c^2})c_p\rho T_t - p_t + \frac{\gamma}{2} p (\frac{u^2}{c^2})_t . \qquad (19)$$

With J written for the current density, ρu, and M for the Mach number, u/c, the derived forms of the four basic flow relations are summarized below:

$$J_x = - \rho_t \qquad (20)$$

$$[p(1 + \gamma M^2)]_x = - J_t \qquad (21)$$

$$(1 - \gamma M^2)kT_{xx} - (1 - M^2)c_pJT_x$$

$$= (1 - \frac{3-\gamma}{2} M^2)c_p\rho T_t - p_t + \frac{\gamma}{2} p (M^2)_t \qquad (22)$$

$$p = \rho RT \qquad (23)$$

2.2 Boundary Conditions

Of concern in the present paper is the flow of vapor between two liquid surfaces whose temperatures are specified functions of time. In this case, the four boundary conditions required for a unique solution of the flow equations arise from the kinetic-theory connection between the temperature, vapor density, and current density at each of the two liquid surfaces, and the assumption that at each boundary the temperature of the vapor is the same as that of the liquid.[4]

The positions of the two liquid surfaces are designated x_a and x_b, respectively. For definiteness, it is assumed that the surface at x_a is the warmer of the two, and that $x_b > x_a$. For stationary liquid surfaces, the current densities across the boundaries are given by Plesset[5] as*

$$J_a = \left(\frac{RT_a}{2\pi}\right)^{1/2} (\alpha\rho_a^{\,e} - \beta\rho_a) \tag{24}$$

$$J_b = \left(\frac{RT_b}{2\pi}\right)^{1/2} (\beta\rho_b - \alpha\rho_b^{\,e}) \,, \tag{25}$$

where α is the evaporation coefficient for vaporization from the liquid surface, β is the accomodation coefficient for condensation on the liquid surface, and $\rho_a^{\,e}$, for example, is the equilibrium or saturation vapor density at the temperature T_a.

Since uniform translation of either surface cannot affect the validity of the corresponding Eq. (24) or (25) in a reference frame moving with the surface, it follows that the proper generalization of these equations for uniformly moving surfaces is

$$J_a = \left(\frac{RT_a}{2\pi}\right)^{1/2} (\alpha\rho_a^{\,e} - \beta\rho_a) + \rho_a x_{at} \tag{26}$$

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* The subscripts a and b in the present development correspond respectively to the subscripts o and l in Plesset's treatment of the steady-flow solution.

$$J_b = \left(\frac{RT_b}{2m}\right)^{1/2} (\beta\rho_b - \alpha\rho_b{}^e) + \rho_b x_{bt} . \tag{27}$$

In the present analysis it is assumed that accelerations of the liquid surfaces are sufficiently small that departures from Eqs. (26) and (27) (due to non-Maxwellian velocity distributions) are negligible. The coefficients α and β are assumed independent of temperature.

Under the assumption that the vapor obeys the perfect-gas law, [Eq. (23)], the generalized expressions for J_a and J_b alternatively can be written in the form

$$J_a = (2\pi RT_a)^{-1/2} (\alpha p_a{}^e - \beta p_a) + \rho_a x_{at} \tag{28}$$

$$J_b = - (2\pi RT_b)^{-1/2} (\alpha p_a{}^e - \beta p_b) + \rho_b x_{bt} . \tag{29}$$

A good approximation to p^e in many cases of practical interest is given by the approximate Clausius-Clapeyron equation[6]

$$\frac{d \log p^e}{dT} = \frac{L}{RT^2} , \tag{30}$$

which on integration becomes

$$p^e = Ke^{-\frac{L}{RT}} , \tag{31}$$

where L is the latent heat of vaporization per unit mass and K is an integration constant depending on the particular substance involved.

The equilibrium vapor pressure for water as given by Eq. (31) (with K evaluated at 100° C) is compared with the actually observed behavior in Fig. 1. Substitution of Eq. (31) into Eqs. (28) and (29) yields

$$J_a = (2\pi RT_a)^{-1/2} (\alpha K e^{-\frac{L}{RT_a}} - \beta p_a) + \rho_a x_{at} \tag{32}$$

$$J_b = -(2\pi RT_b)^{-1/2} (\alpha K e^{-\frac{L}{RT_b}} - \beta p_b) + \rho_b x_{bt} \; . \tag{33}$$

2.3 Normalization of Flow Equations and Boundary Conditions

In order to simplify the notation for further manipulation and to ascertain the minimum number of significant parameters in the problem, it is desirable to nondimensionalize the various variables in such a way as to eliminate (formally) as many as possible of the constants appearing in the basic equations and boundary conditions. The following notation is introduced, the normalizing factors being denoted by asterisks:

$$j = \frac{J}{J^*} \tag{34}$$

$$r = \frac{\rho}{\rho^*} \tag{35}$$

$$\psi = \frac{p}{p^*} \tag{36}$$

$$\theta = \frac{1}{\varphi} = \frac{T}{T^*} \tag{37}$$

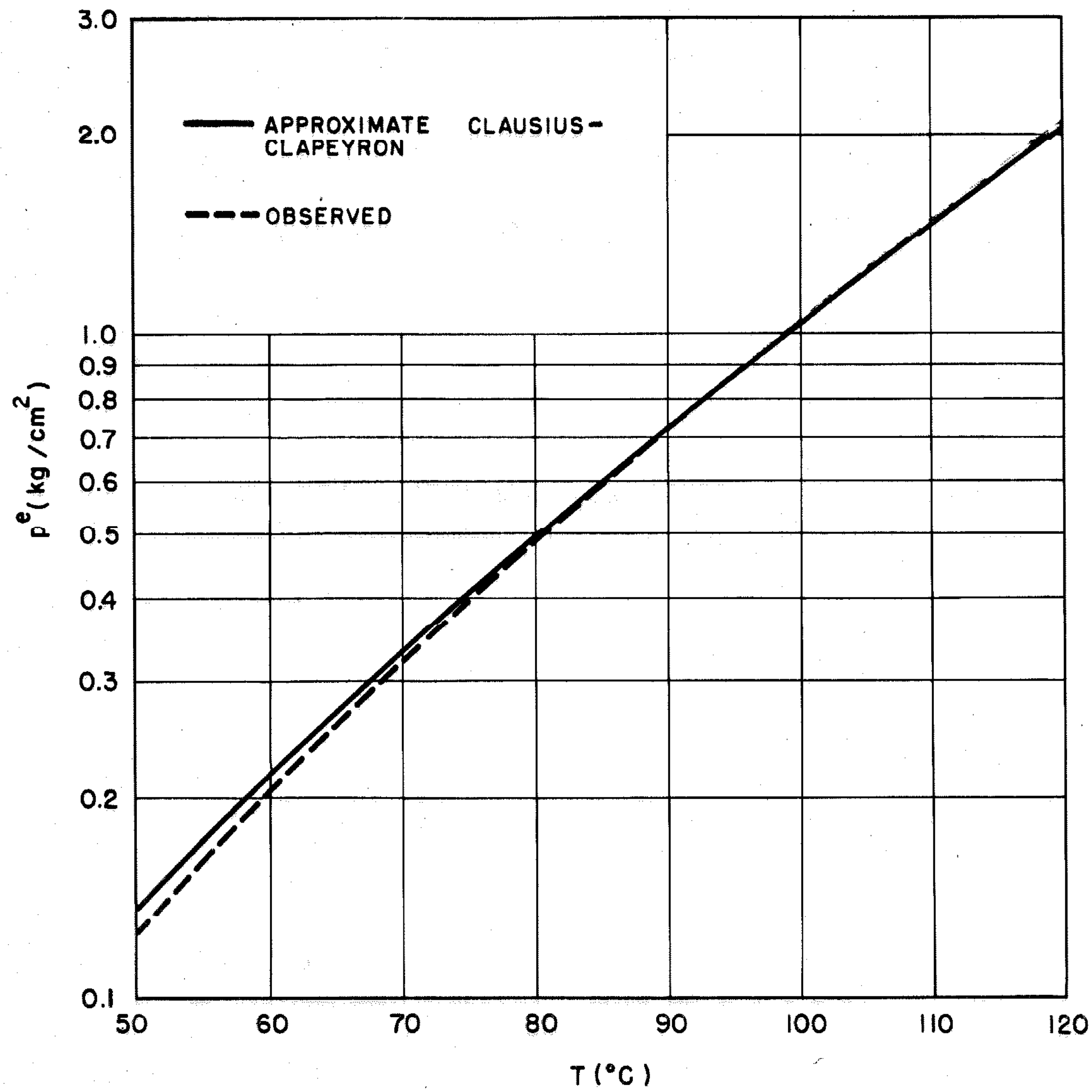

Figure 1. Equilibrium Vapor Pressure for Water

$$\xi = \frac{x}{x^*} \tag{38}$$

$$\tau = \frac{t}{t^*} . \tag{39}$$

Equation (31) suggests that T^* be taken as

$$T^* = \frac{L}{R} , \tag{40}$$

while Eqs. (20) and (23) suggest the relationships

$$J^* = \rho^* \frac{x^*}{t^*} \tag{41}$$

$$p^* = \rho^* R T^* . \tag{42}$$

On acceptance of these suggested relationships, Eq. (32) becomes

$$\begin{aligned} j_a &= \frac{1}{J^*} \left(\frac{\varphi_a}{2\pi R T^*}\right)^{1/2} (\alpha K e^{-\varphi_a} - \beta \rho^* R T^* \psi_a) + r_a \xi_{a\tau} \\ &= \frac{t^*}{x^*} \beta \left(\frac{L}{2\pi}\right)^{1/2} \varphi_a^{1/2} \left(\frac{\alpha K}{\beta L \rho^*} e^{-\varphi_a} - \psi_a\right) + r_a \xi_{a\tau} . \end{aligned} \tag{43}$$

The further specifications

$$u^* = \frac{x^*}{t^*} = \beta \left(\frac{L}{2\pi}\right)^{1/2} \tag{44}$$

$$\rho^* = \frac{\alpha K}{\beta L} \tag{45}$$

reduce Eqs. (31) and (43) to

$$\psi^e = \frac{\beta}{\alpha} e^{-\varphi} \tag{46}$$

$$j_a = \varphi_a^{1/2} (e^{-\varphi_a} - \psi_a) + r_a \xi_{a\tau} \,. \tag{47}$$

Equation (22) suggests that the normalizing length x^* be defined by

$$x^* = \frac{k}{c_p J^*} , \tag{48}$$

in which case Eq. (44) requires that

$$t^* = \frac{x^*}{u^*} = \frac{k}{c_p J^* \beta} (\frac{2\pi}{L})^{1/2} \,. \tag{49}$$

With the introduction of the convenient parameter

$$\sigma = \frac{(u^*)^2}{RT^*} = \frac{(u^*)^2}{L} = \frac{\beta^2}{2\pi} , \tag{50}$$

the flow equations (20) to (23) assume the dimensionless form

$$j_\xi = - r_\tau \tag{51}$$

$$[\psi(1 + \gamma M^2)]_\xi = - \sigma j_\tau \tag{52}$$

$$(1 - \gamma M^2)\theta_{\xi\xi} - (1 - M^2) j \theta_\xi$$

$$= (1 - \frac{3-\gamma}{2} M^2) r \theta_\tau - \frac{\gamma-1}{\gamma} \psi_\tau + \frac{\gamma-1}{2} \psi (M^2)_\tau \tag{53}$$

$$\psi = r\theta \tag{54}$$

$$M^2 = \frac{\sigma}{\gamma} \frac{j^2}{r^2 \theta} = \frac{\sigma}{\gamma} \frac{j^2 \theta}{\psi^2} , \tag{55}$$

while the boundary conditions become

$$j_a = \varphi_a^{1/2} (e^{-\varphi_a} - \psi_a) + r_a \xi_{a\tau} \tag{56}$$

$$j_b = - \varphi_b^{1/2} (e^{-\varphi_b} - \psi_b) + r_b \xi_{b\tau} . \tag{57}$$

It will be observed that the parameter σ occurs only in the flow equations and not in the boundary conditions. The reverse might be expected on physical grounds since β, which completely determines σ, is a property only of the liquid-vapor interface and not of the free vapor. Actually the parameter σ can be made to appear in either of the two sets of equations simply by normalizing the temperature appropriately. The present normalization is generally the more convenient since it yields somewhat simpler algebraic forms and since in most cases it leads to values of the normalized variables j, r, ψ, and θ which are roughly of the same order of magnitude.

III. STEADY-FLOW SOLUTION

For the steady flow ($\partial / \partial \tau = 0$) resulting when the two liquid surfaces are held stationary and at constant temperatures, Eqs. (51) and (52) yield the first integrals

$$j \;=\; \text{constant} \tag{58}$$

$$\psi(1 + \gamma M^2) \;=\; \text{constant}, \tag{59}$$

Eq. (53) reduces to the total differential equation

$$(1 - \gamma M^2)\,\frac{d^2\theta}{d\xi^2} - (1 - M^2)\,j\,\frac{d\theta}{d\xi} \;=\; 0\ , \tag{60}$$

Eqs. (54) and (55) remain unchanged, and Eqs. (56) and (57) reduce to

$$j \;=\; \varphi_a^{\,1/2}\,(e^{-\varphi_a} - \psi_a) \;=\; -\,\varphi_b^{\,1/2}\,(e^{-\varphi_b} - \psi_b)\ . \tag{61}$$

3.1 Low-Velocity Approximation

If M^2 and γM^2 can be neglected with respect to unity, Eqs. (59) and (60) give approximately

$$\psi \;=\; \text{constant} \tag{62}$$

$$\frac{d\theta}{d\xi}\,e^{-j\xi} \;=\; \text{constant}\ . \tag{63}$$

In this case, solution of Eq. (61) first for $\psi = \psi_a = \psi_b$ and then for j and designation of these solutions by the subscript o yields

$$\psi_o = \frac{\varphi_a^{1/2}\, e^{-\varphi_a} + \varphi_b^{1/2}\, e^{-\varphi_b}}{\varphi_a^{1/2} + \varphi_b^{1/2}} \tag{64}$$

$$j_o = \frac{e^{-\varphi_a} - e^{-\varphi_b}}{\theta_a^{1/2} + \theta_b^{1/2}} \,, \tag{65}$$

while integration of Eq. (63) and imposition of the boundary conditions on θ gives

$$\theta_o = \theta_c\left[1 - \mu e^{-j_o(\xi_b - \xi)}\right] , \tag{66}$$

where

$$\theta_c \equiv \theta(\xi = -\infty) = \frac{\theta_a - \theta_b e^{-\Delta}}{1 - e^{-\Delta}} \tag{67}$$

$$\mu \equiv 1 - \frac{\theta_b}{\theta_c} = \frac{\theta_a - \theta_b}{\theta_a - \theta_b e^{-\Delta}} \tag{68}$$

$$\Delta \equiv j(\xi_b - \xi_a) = j_o(\xi_b - \xi_a) \;. \tag{69}$$

It follows from Eq. (54) that the normalized vapor density, r, in the present approximation is given by

$$r_o = \frac{\psi_o}{\theta_o} = \frac{r_c}{1 - \mu e^{-j_o(\xi_b - \xi)}} \ , \tag{70}$$

where

$$r_c \equiv \frac{\psi(\xi = -\infty)}{\theta_c} = \frac{\psi_o}{\theta_c} \ . \tag{71}$$

For future convenience, definitions of θ_c, μ, Δ, and r_c more general than the specific expressions applicable in the low-velocity approximation are indicated in Eqs. (67) to (71). The specification "$\xi = -\infty$" related to the subscript c refers to a purely formal substitution in the expressions for the variables concerned. These expressions have no physical significance for $\xi < \xi_a$, of course.

The results summarized in Eqs. (64) to (71) are essentially identical with those reported previously by Plesset[1], except that approximate analytical representations have been introduced for the equilibrium vapor densities ρ_a^e and ρ_b^e which remain as parameters in his analysis. Boundary values of vapor density, current density, and flow velocity for water vapor, calculated from Eqs. (70), (65), (35), and (34) with $\rho^* = 22.5$ g/cm^3 and $J^* = 5.4 \times 10^4$ g/cm^2 sec, are compared with Plesset's corresponding results in Figs. 2 to 4.

Plesset has pointed out that under ordinary circumstances $\Delta = (c_p/k) J_o (x_b - x_a)$ is a large number so that $e^{-\Delta}$ is extremely small. For water in the neighborhood of 100°C, for example, a temperature difference of only 1°C between liquid surfaces separated 1 mm

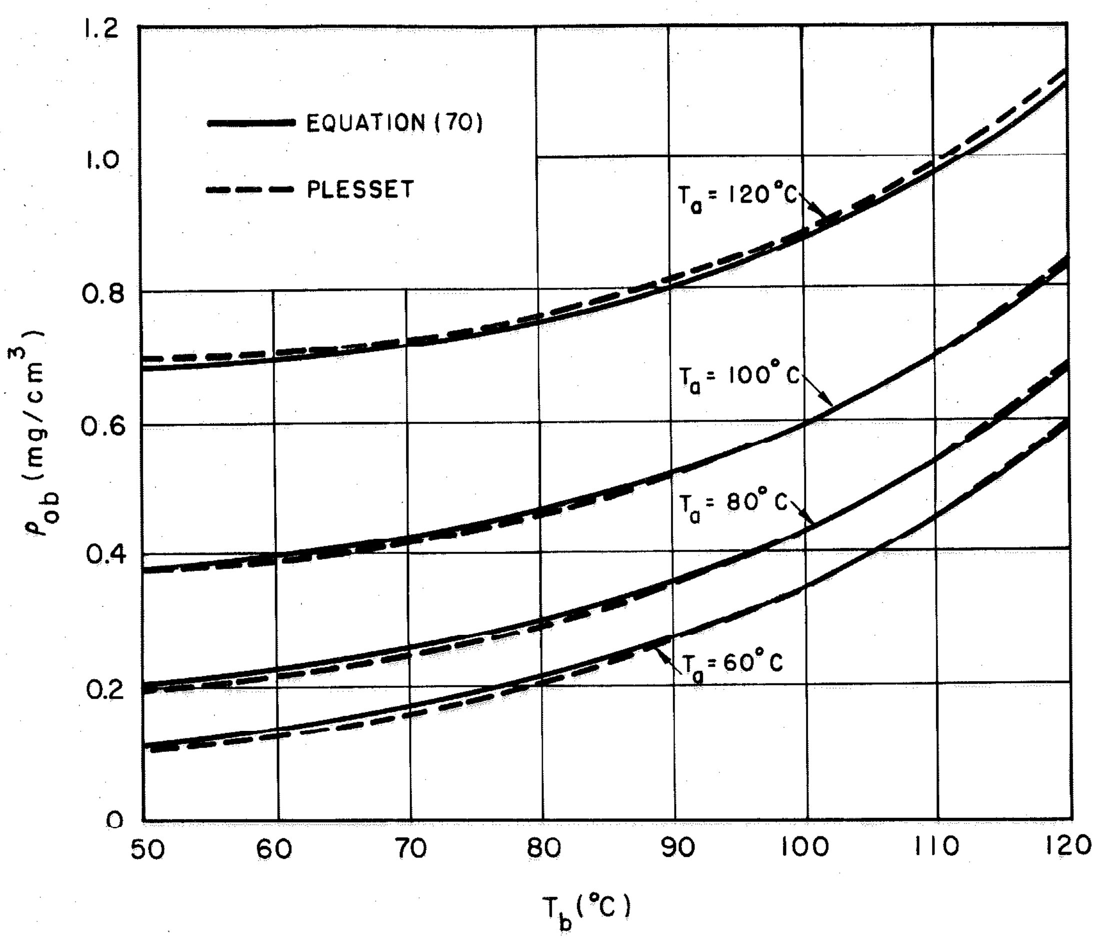

Figure 2. Boundary Values of Steady-Flow Vapor Density for Water Vapor (ρ^* = 22.5 g/cm^3, L = 9720 cal/mol)

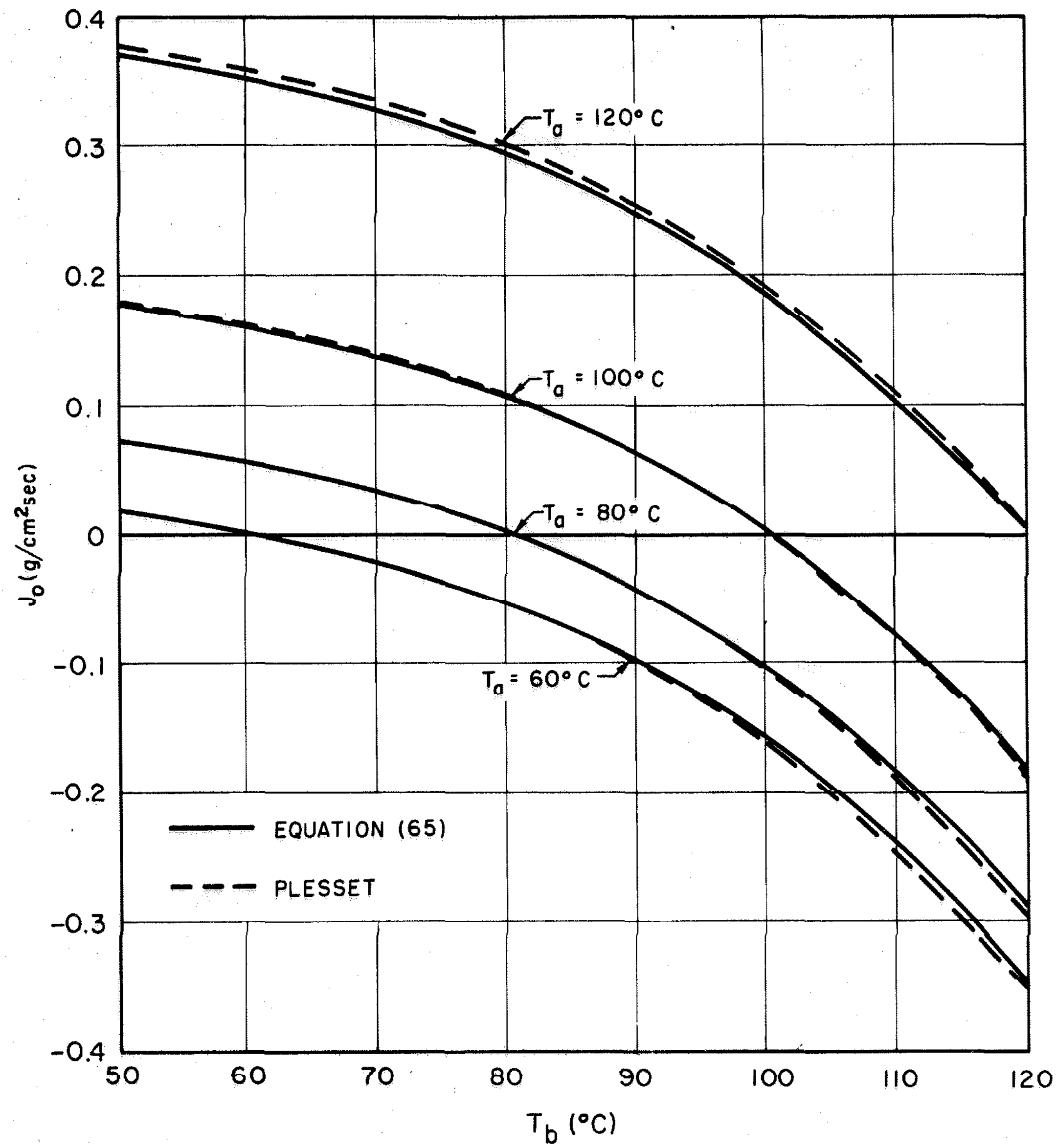

Figure 3. Boundary Values of Steady-Flow Current Density for Water Vapor ($J^* = 5.4 \times 10^4$ g/cm^2 sec, L = 9720 cal/mol)

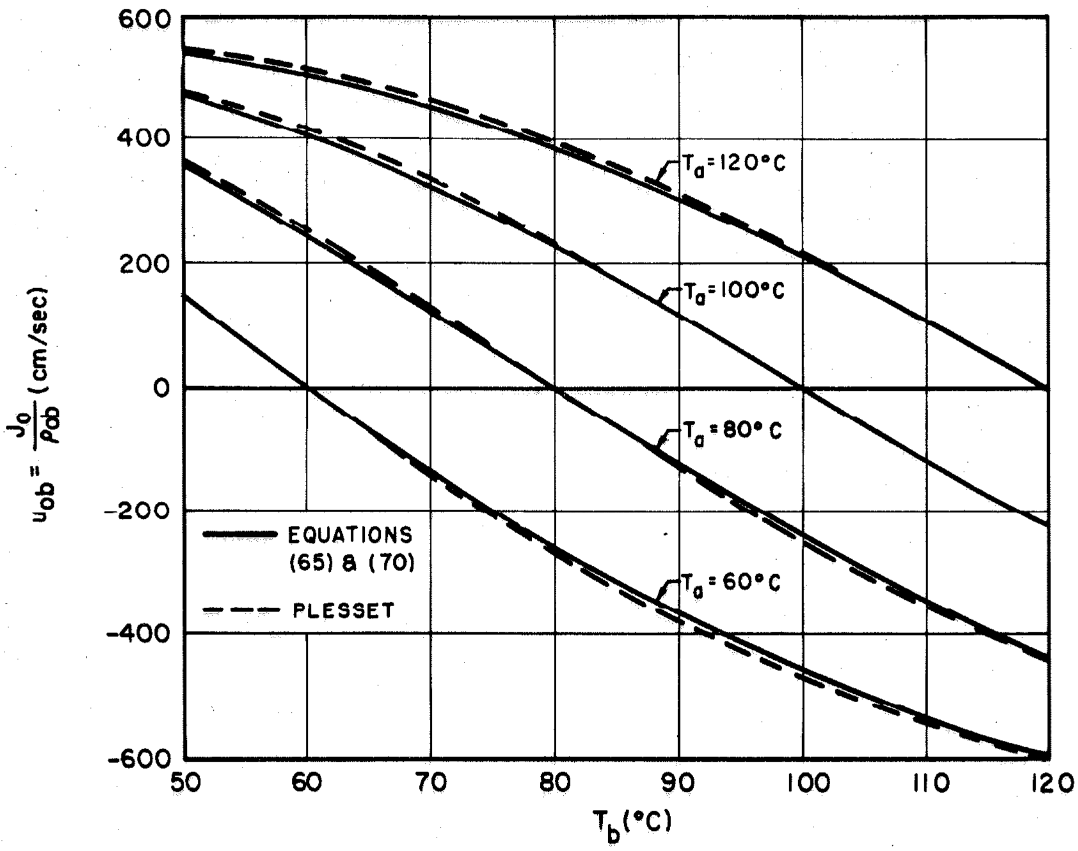

Figure 4. Boundary Values of Steady-Flow Velocity for Water Vapor
(β = 0.04, L = 9720 cal/mol)

still yields a Δ of about 6, for which $e^{-\Delta}$ is of the order of 10^{-3}. In such cases Eqs. (66), (68), (70), and (71) become, to a good approximation,

$$\theta_o \approx \theta_a\left[1 - \mu e^{-j_o(\xi_b - \xi)}\right] \tag{72}$$

$$\mu \approx \frac{\theta_a - \theta_b}{\theta_a} = 1 - \frac{\theta_b}{\theta_a} \tag{73}$$

$$r_o \approx \frac{r_a}{1 - \mu e^{-j_o(\xi_b - \xi)}} \tag{74}$$

$$r_a = \frac{\psi_o}{\theta_a} \ . \tag{75}$$

When these approximate equations apply, the space-dependent terms in Eqs. (72) and (74) are significant only in the immediate vicinity of the cooler surface at $\xi = \xi_b (x = x_b)$.

3.2 Applicability of Low-Velocity Approximation

Since γ is necessarily greater than unity, the condition for applicability of Eqs. (64) to (71) is simply $\gamma M^2 \ll 1$. Except in border-line cases, this condition can be tested with sufficient accuracy by calculating the approximate value of M^2 corresponding to ψ_o, j_o and θ_o in accordance with Eq. (55).

Before making this calculation, it is convenient to introduce the abbreviations

$$\varepsilon = \frac{\theta_a - \theta_b}{\theta_a} = \frac{\varphi_b - \varphi_a}{\varphi_b} \qquad 0 \leqslant \varepsilon < 1 \qquad (76)$$

$$n = \frac{\theta_b^{1/2}}{\theta_a^{1/2}} = \frac{\varphi_a^{1/2}}{\varphi_b^{1/2}} = (1 - \varepsilon)^{1/2} \qquad 0 < n \leqslant 1 \qquad (77)$$

$$s = e^{-(\varphi_b - \varphi_a)} = e^{-\frac{\varepsilon}{1-\varepsilon}\varphi_a} \qquad 0 < s \leqslant 1 \qquad (78)$$

$$z = -j(\xi_b - \xi) \quad , \qquad -\Delta \leqslant z \leqslant 0 \qquad (79)$$

with the use of which Eqs. (64) to (71) assume the forms

$$\psi_o = \frac{n + s}{1 + n} e^{-\varphi_a} \qquad (80)$$

$$j_o = \frac{1 - s}{1 + n} \varphi_a^{1/2} e^{-\varphi_a} \qquad (81)$$

$$\theta_o = \theta_c (1 - \mu e^z) \qquad (82)$$

$$\theta_c = \theta_a \frac{1 - (1-\varepsilon)e^{-\Delta}}{1 - e^{-\Delta}} = \theta_a \frac{1 - \varepsilon}{1 - \mu} = \frac{\theta_a}{1 - \mu e^{-\Delta}} \qquad (83)$$

$$\mu = \frac{\varepsilon}{1 - (1-\varepsilon)e^{-\Delta}} \qquad (84)$$

$$r_o = \frac{r_c}{1 - \mu e^z} \qquad (85)$$

$$r_c = \frac{n + s}{1 + n} (1 - \mu e^{-\Delta}) \varphi_a e^{-\varphi_a} \quad . \qquad (86)$$

Substitution into Eq. (55) now yields

$$\gamma M_o^2 = \sigma\left(\frac{1-s}{n+s}\right)^2 \frac{1-\mu e^z}{1-\mu e^{-\Delta}} \leqslant \sigma\left(\frac{1-s}{n+s}\right)^2$$

$$\underset{n\to 0}{\sim} \frac{\sigma}{n^2} \; . \tag{87}$$

For water, σ is of the order of 10^{-4} (see Appendix A) so that in this case γM_o^2 is still much smaller than unity even for $n^2 = 1 - \varepsilon$ as small as 0.01, implying that $T_b = 0.01\, T_a$. In order to achieve this condition with $T_a = 100^oC$, the cooler "liquid" surface would have to have a temperature of 4 degrees absolute! For the more sensible case of $T_a = 100^oC$ and $T_b = 0^oC$, γM_o^2 differs from σ by less than a factor of 2. It thus appears that for water, and undoubtedly for many other liquids of practical interest as well, the low-velocity approximation is thoroughly justified in all physically achievable situations.

3.3 Exact Solution for Current Density

In spite of the conclusion just stated, it is worth observing that an exact solution of the steady-flow problem, Eqs. (58) to (61), can be obtained. Such a solution is useful not only in evaluating the low-velocity approximation and otherwise in the immediate problem, but also as an indication of techniques which may be generally applicable to the analysis of steady perfect-gas flows.

The starting point is Eq. (59), which can be written as

$$\psi + \sigma \frac{j^2\theta}{\psi} = A \text{ (constant)} \quad . \tag{88}$$

Multiplication through by ψ, solution of the resulting quadratic, and choice of the solution which reduces to $\psi = A$ as $j \to 0$ yields

$$\psi = \frac{A}{2}\left[1 + \left(1 - \frac{4\sigma j^2\theta}{A^2}\right)^{1/2}\right] \quad . \tag{89}$$

Substitution of Eq. (89) into Eq. (61) then leads to the pair of equations

$$\psi_a = \frac{A}{2}\left[1 + \left(1 - \frac{4\sigma j^2}{\varphi_a A^2}\right)^{1/2}\right] = e^{-\varphi_a} - \frac{j}{\varphi_a^{1/2}} \tag{90}$$

$$\psi_b = \frac{A}{2}\left[1 + \left(1 - \frac{4\sigma j^2}{\varphi_b A^2}\right)^{1/2}\right] = e^{-\varphi_b} + \frac{j}{\varphi_b^{1/2}} \quad . \tag{91}$$

On multiplying these equations through by 2/A, subtracting unity from each side, squaring and reducing, one obtains

$$A = \frac{(\varphi_a^{1/2} e^{-\varphi_a} - j)^2 + \sigma j^2}{\varphi_a^{1/2} (\varphi_a^{1/2} e^{-\varphi_a} - j)} \tag{92}$$

$$A = \frac{(\varphi_b^{1/2} e^{-\varphi_b} + j)^2 + \sigma j^2}{\varphi_b^{1/2} (\varphi_b^{1/2} e^{-\varphi_b} + j)} \quad , \tag{93}$$

between which A could be eliminated to give a single cubic equation in j.

It is convenient, however, to observe from Eq. (61) that

$$\varphi_a^{1/2} e^{-\varphi_a} = \varphi_a^{1/2} \psi_o + j_o \tag{94}$$

$$\varphi_b^{1/2} e^{-\varphi_b} = \varphi_b^{1/2} \psi_o - j_o \ , \tag{95}$$

where ψ_o and j_o are the low-velocity solutions given in Section 3.1. In terms of the quantities*

$$\lambda = \frac{j - j_o}{j_o} \tag{96}$$

$$h = \frac{\varphi_b^{1/2} \psi_o}{j_o} = \frac{n + s}{n(1 - s)} \ , \tag{97}$$

Eqs. (92) and (93) then can be written as

$$\frac{A}{\psi_o} = \frac{(nh - \lambda)^2 + \sigma(\lambda + 1)^2}{nh(nh - \lambda)} \tag{98}$$

$$\frac{A}{\psi_o} = \frac{(h + \lambda)^2 + \sigma(\lambda + 1)^2}{h(h + \lambda)} \ , \tag{99}$$

---

* Though used here simply as a convenient abbreviation, the quantity h can be recognized as $(\sigma/\gamma)^{1/2}$ times the inverse of the low-velocity approximation to the Mach number at the cooler liquid surface.

which on elimination of A yields

$$\lambda(\lambda + h)(\lambda - nh) + \sigma(\lambda + 1)^2[\lambda + (1 - n)h] = 0 \quad . \qquad (100)$$

On collection of like powers of λ, Eq. (100) assumes the form

$$\boxed{\begin{aligned} 0 = \; & \lambda^3(1 + \sigma) \\ & + \lambda^2[(1 + \sigma)(1 - n)h + 2\sigma] \\ & + \lambda[\quad 2\sigma(1 - n)h + \sigma - nh^2] \\ & + \quad\quad \sigma(1 - n)h \end{aligned}} \quad , \qquad (101)$$

the solutions of which can be obtained in closed form by standard methods. The physically meaningful solution of Eq. (101) is that which reduces to $\lambda = 0$ as $n \to 1$. Substitution of this solution into Eqs. (96) and (99) yields the values of j and A, respectively.

Although the exact solution of Eq. (101) could be written out in full, it is more instructive first to examine the behavior of this equation under high-velocity flow conditions, when the correction coefficient λ should be most important. It is seen from Eq. (87) that high Mach numbers correspond to small values of n, for which h is very nearly unity. If this approximation is made and if both n and σ are neglected with respect to unity, Eq. (101) becomes approximately

$$\lambda^3 + \lambda^2 + \lambda(3\sigma - n) + \sigma = 0 \quad . \tag{102}$$

Equation (102) clearly has one solution in the neighborhood of - 1 and may or may not have other real solutions depending on the values of n and σ. If the left-hand member of Eq. (102) is denoted by $f(\lambda)$, then

$$\frac{df(\lambda)}{d\lambda} = 3\lambda^2 + 2\lambda + 3\sigma - n \quad , \tag{103}$$

which has zeros at approximately - 2/3 and $(n - 3\sigma)/2$. Evaluation of $f(\lambda)$ at these points yields

$$f(-\frac{2}{3}) \approx \frac{4}{27} \tag{104}$$

$$f(\frac{n - 3\sigma}{2}) \approx \sigma - (\frac{n - 3\sigma}{2})^2 \quad . \tag{105}$$

Thus $(n - 3\sigma)/2$ becomes a zero of $f(\lambda)$ as well as $f'(\lambda)$ when $n \approx 3\sigma + 2\sigma^{1/2} \approx 2\sigma^{1/2}$, the second form following from the assumption that $\sigma \ll 1$. The behavior of Eq. (101) under the assumed conditions is therefore as shown qualitatively in Fig. 5. For sufficiently large n, the physically interesting root of $f(\lambda)$ corresponds to the smaller positive λ intercept, which coalesces with the other positive

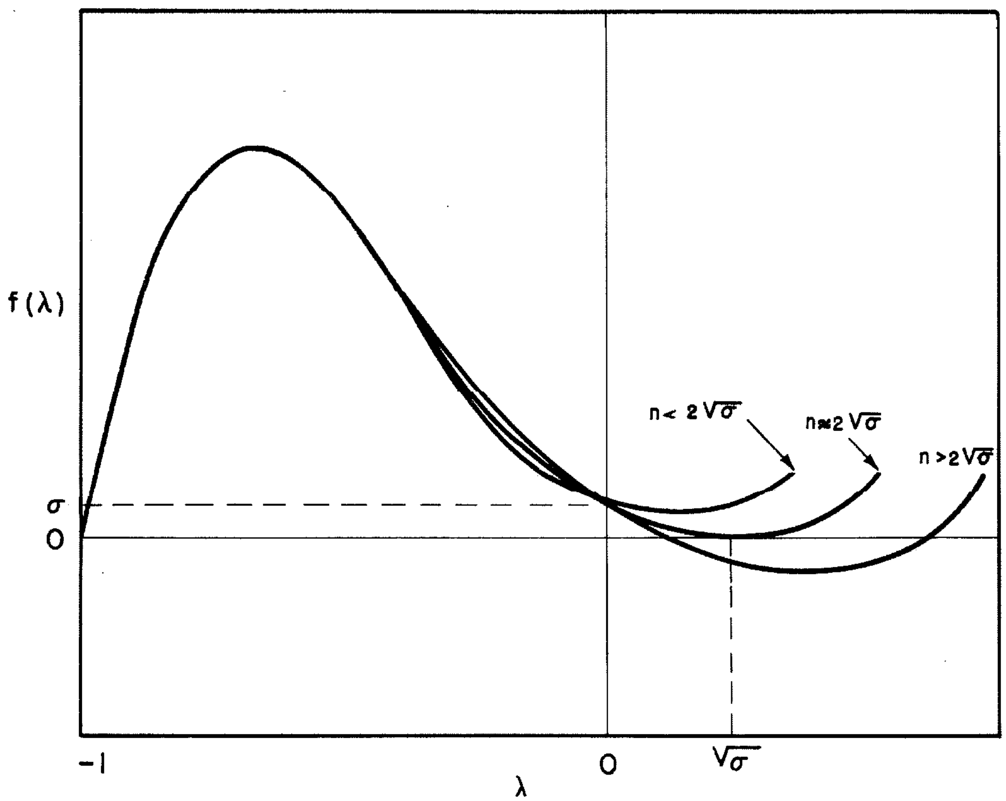

Figure 5. Qualitative Behavior of $f(\lambda)$ for Small σ

intercept as n is reduced to approximately $2\sigma^{1/2}$ and then ceases to exist as n is reduced still further.

The failure of λ to vary continuously with n as the latter decreases through $2\sigma^{1/2}$ is associated with the passing through unity of γM^2 at the warmer liquid surface (x_a).* Thus from Eqs. (55), (61), (94), (96), and (97),

$$\gamma M_a{}^2 = \sigma \frac{j^2}{\psi_a{}^2 \varphi_a} = \sigma \frac{j_o{}^2(1 + \lambda)^2}{[\varphi_a{}^{1/2} \psi_o - (j - j_o)]^2}$$

$$= \sigma(\frac{1 + \lambda}{nh - \lambda})^2 \quad , \qquad (106)$$

which reduces to unity for $h \approx 1$, $n \approx 2\sigma^{1/2}$, and $\lambda \approx n/2 \ll 1$. In the region $\gamma M_a{}^2 < 1$, it is evident from Fig. 5 that n can be no smaller than $2\sigma^{1/2}$ and λ no larger than $\sigma^{1/2}$. Thus for water, for which σ is approximately 3×10^{-4}, the exact j differs from j_o by less than 2 percent in all vapor flows for which $\gamma M_a{}^2 < 1$ (essentially all subsonic flows). The exact solution of Eq. (101) for water vapor is plotted in Fig. 6.

In view of the upper limit on λ just found and the fact that $h \geqslant 1$, it follows that if $\sigma^{1/2} \ll 1$ then Eq. (101) can be approximated for present purposes by simply dropping the λ^3 term.

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* The significance of the critical condition $\gamma M^2 = 1$ is discussed, for example, in Reference 7.

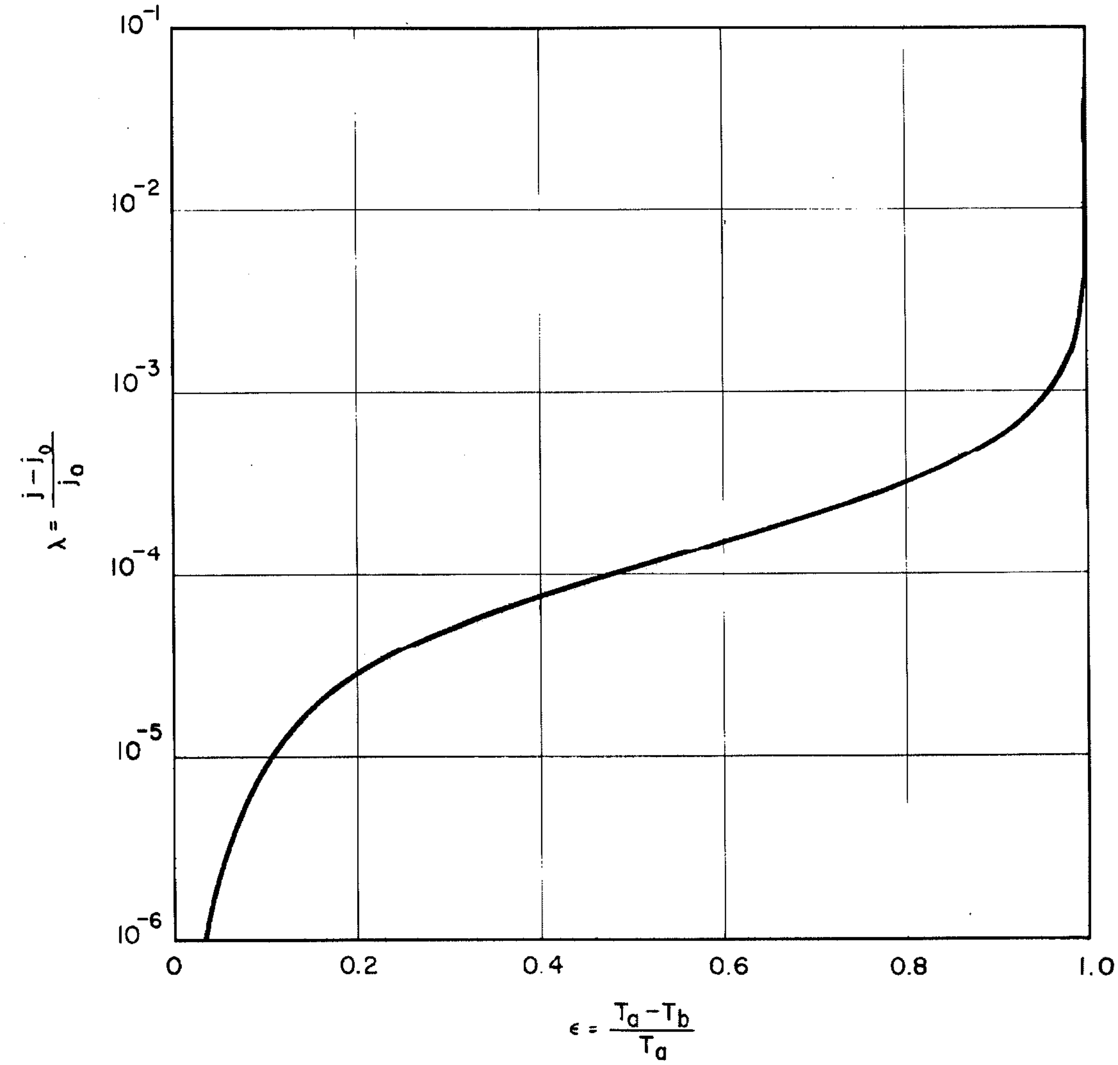

Figure 6. Fractional Difference between Exact Steady-Flow Current Density and Low-Velocity Approximation for Water Vapor (β = 0.04, L = 9720 cal/mol, $T_a \approx 100^\circ$ C)

Continuing to neglect σ with respect to n and unity but making no specializing assumptions as to the magnitudes of n and h, one then obtains

$$\lambda^2[(1 - n)h + 2\sigma] - \lambda nh^2 + \sigma(1 - n)h = 0 . \tag{107}$$

If in addition $2\sigma \ll 1/\varphi_a$, so that $2\sigma \ll (1 - n)h$ for all values of n, then Eq. (107) reduces to

$$\lambda^2 - \lambda \frac{nh}{1 - n} + \sigma = 0 , \tag{108}$$

the significant solution of which is

$$\lambda = \frac{nh}{2(1 - n)} \left\{ 1 - \left[1 - 4\sigma \left(\frac{1 - n}{nh} \right)^2 \right]^{1/2} \right\} . \tag{109}$$

Except in the immediate vicinity of the $\gamma M_a^{\ 2} = 1$ limit at $nh/(1 - n) = 2\sqrt{\sigma}$, the square root in Eq. (109) can be expanded to yield

$$\lambda \approx \sigma \frac{1 - n}{nh} = \sigma \frac{(1 - n)(1 - s)}{n + s} \begin{cases} \underset{n \to 0}{\sim} \dfrac{\sigma}{n} \\ \underset{\varepsilon \to 0}{\sim} \sigma \dfrac{\varepsilon^2 \varphi_a}{4} \end{cases} . \tag{110}$$

3.4 Exact Solution for Temperature Distribution

Let it be assumed that the exact expressions for j and A have been obtained by substitution of the appropriate solution of Eq. (101) into Eqs. (96) and (99), respectively. Equation (89) then serves to express ψ in terms of the single remaining unknown θ. Thus substitution of Eq. (89) into the second form of Eq. (55) and the latter into Eq. (60) yields a differential equation for θ involving no other unknown quantities. Solution of this equation yields the values of θ and thence of ψ, completing the solution of the steady-state flow problem. The procedure just outlined is carried out below, with j and A retained as convenient abbreviations for their presumably known expressions in terms of the liquid surface temperatures. In terms of the characteristic dimensionless space variable z

$$z = -j(\xi_b - \xi) \tag{111}$$

introduced in Eq. (79), with differentiation with respect to z indicated by a prime (') and γM^2 assumed different from unity throughout the flow,* Eq. (60) becomes

---

* The requirement that γM^2 be different from unity throughout the flow actually implies that γM^2 must be less than unity throughout, since this quantity is always less than unity at the cooler surface. This can be seen by writing, in analogy to Eq. (106),

$$\gamma M_b^2 = \sigma \frac{j^2}{\psi_b^2 \varphi_b} = \sigma \left(\frac{1 + \lambda}{h + \lambda}\right)^2 < \sigma < 1 \ .$$

$$\theta'' = \left(1 + \frac{\gamma - 1}{\gamma}\,\frac{\gamma M^2}{1 - \gamma M^2}\right)\theta'$$

$$= \left(1 + \frac{\gamma - 1}{\gamma}\,\frac{\sigma j^2 \theta}{\psi^2 - \sigma j^2 \theta}\right)\theta'$$

$$= \left\{1 + \frac{\gamma - 1}{\gamma}\,\frac{\sigma j^2 \theta}{\frac{A^2}{2}\left[1 + \left(1 - \frac{4\sigma j^2 \theta}{A^2}\right)^{1/2}\right] - 2\sigma j^2 \theta}\right\}\theta' \; . \tag{112}$$

With the abbreviation

$$\eta(z) = \frac{4\sigma j^2}{A^2}\,\theta(z) \quad , \tag{113}$$

Eq. (112) can be written as

$$\eta'' = \left\{1 + \frac{\gamma - 1}{2\gamma}\,\frac{\eta}{1 + (1 - \eta)^{1/2} - \eta}\right\}\eta'$$

$$= \left\{1 + \frac{\gamma - 1}{2\gamma}\,\frac{\eta}{(1 - \eta)^{1/2}[1 + (1 - \eta)^{1/2}]}\right\}\eta'$$

$$= \left\{1 + \frac{\gamma - 1}{2\gamma}[(1 - \eta)^{-1/2} - 1]\right\}\eta' \quad . \tag{114}$$

This second-order nonlinear differential equation can be reduced to an equation of first order with the aid of the substitutions*

$$w(z) = 1 - \eta(z) = 1 - \frac{4\sigma j^2}{A^2}\theta(z) \qquad (115)$$

$$f(w) = w'(z) \quad , \qquad (116)$$

which allow Eq. (114) to be written successively as

$$w'' = \left(\frac{\gamma + 1}{2\gamma} + \frac{\gamma - 1}{2\gamma} w^{-1/2}\right)w' \qquad (117)$$

$$f\,\frac{df}{dw} = \left(\frac{\gamma + 1}{2\gamma} + \frac{\gamma - 1}{2\gamma} w^{-1/2}\right)f \quad . \qquad (118)$$

Apart from the trivial solution $f = w' = 0$, Eq. (118) reduces to

$$\frac{df}{dw} = \frac{\gamma + 1}{2\gamma} + \frac{\gamma - 1}{2\gamma} w^{-1/2} \quad . \qquad (119)$$

Integration of this linear equation for f and resubstitution of w' for f in the result yields the first-order equation

$$w' = \frac{\gamma + 1}{2\gamma} w + \frac{\gamma - 1}{\gamma} w^{1/2} - B \quad . \qquad (120)$$

---

* The use of Eq. (115) is suggested by Kamke[(8)], Eq. 6.54, page 554.

In view of the choice of sign made in Eq. (89), the symbol $w^{1/2}$ must be understood as representing the positive square root of w.

It now is convenient to make the change of variable

$$v = w^{1/2} \quad , \tag{121}$$

transforming Eq. (120) into

$$\begin{aligned} vv' &= \frac{\gamma + 1}{4\gamma} v^2 + \frac{\gamma - 1}{2\gamma} v - \frac{B}{2} \\ &= \frac{\gamma + 1}{4\gamma} \left(v^2 + 2\,\frac{\gamma - 1}{\gamma + 1} v - \frac{2B\gamma}{\gamma + 1}\right) \quad . \end{aligned} \tag{122}$$

Separation of variables then gives

$$\int \frac{v\,dv}{v^2 + 2\,\frac{\gamma-1}{\gamma+1} v - \frac{2B\gamma}{\gamma+1}} = \frac{\gamma + 1}{4\gamma} \int dz = \frac{\gamma + 1}{4\gamma} z + C. \tag{123}$$

Evaluation of the first integral in Eq. (123) with the aid of Dwight,[9] Eqs. 160.11 and 160.01, yields

$$\frac{\gamma+1}{2\gamma} z + 2C = \log\left[(v-v_c)(v+v_c+2b)\right] - \frac{b}{v_c+b} \log \frac{v-v_c}{v+v_c+2b} \,, \tag{124}$$

where

$$b = \frac{\gamma - 1}{\gamma + 1} \tag{125}$$

$$v_c = b\left\{-1 + \left[1 + \frac{2B\gamma(\gamma+1)}{(\gamma-1)^2}\right]^{1/2}\right\} \quad . \tag{126}$$

Multiplication of Eq. (124) by v_c+b and elimination of the logarithms yields finally

$$\boxed{(v - v_c)^{v_c}\ (v + v_c + 2b)^{v_c+2b} = C_1\ d^{\frac{\gamma+1}{2\gamma}(v_c+b)z}} \quad . \tag{127}$$

The constant C_1 is readily evaluated in terms of v_c from the condition $v = v_b$ at $z = 0$, and the quantity v_c can then be determined from the condition $v = v_a$ at $z = -\Delta$.

Unfortunately Eq. (127) cannot in general be resolved to express v, and thus η, explicitly as a function of z. A number of typical curves of η versus z have been plotted in Fig. 7, however, for $\gamma = 4/3$, $\eta_b = 0$ $(v_b = 1)$, and different values of $\eta_c = 1 - v_c^2$.* It should be pointed out that the common boundary value $\eta_b = 0$ maintained in Fig. 7 for simplicity in computing is not strictly possible physically. Actually η_b is given by Eq. (113) as

$$\eta_b = \frac{4\sigma j^2}{A^2}\,\theta_b \quad . \tag{128}$$

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* The solid curves in Figs. 7 and 8, which could have been calculated by numerical substitution in Eq. (127), actually were obtained by solving Eq. (120) on an analog computer.

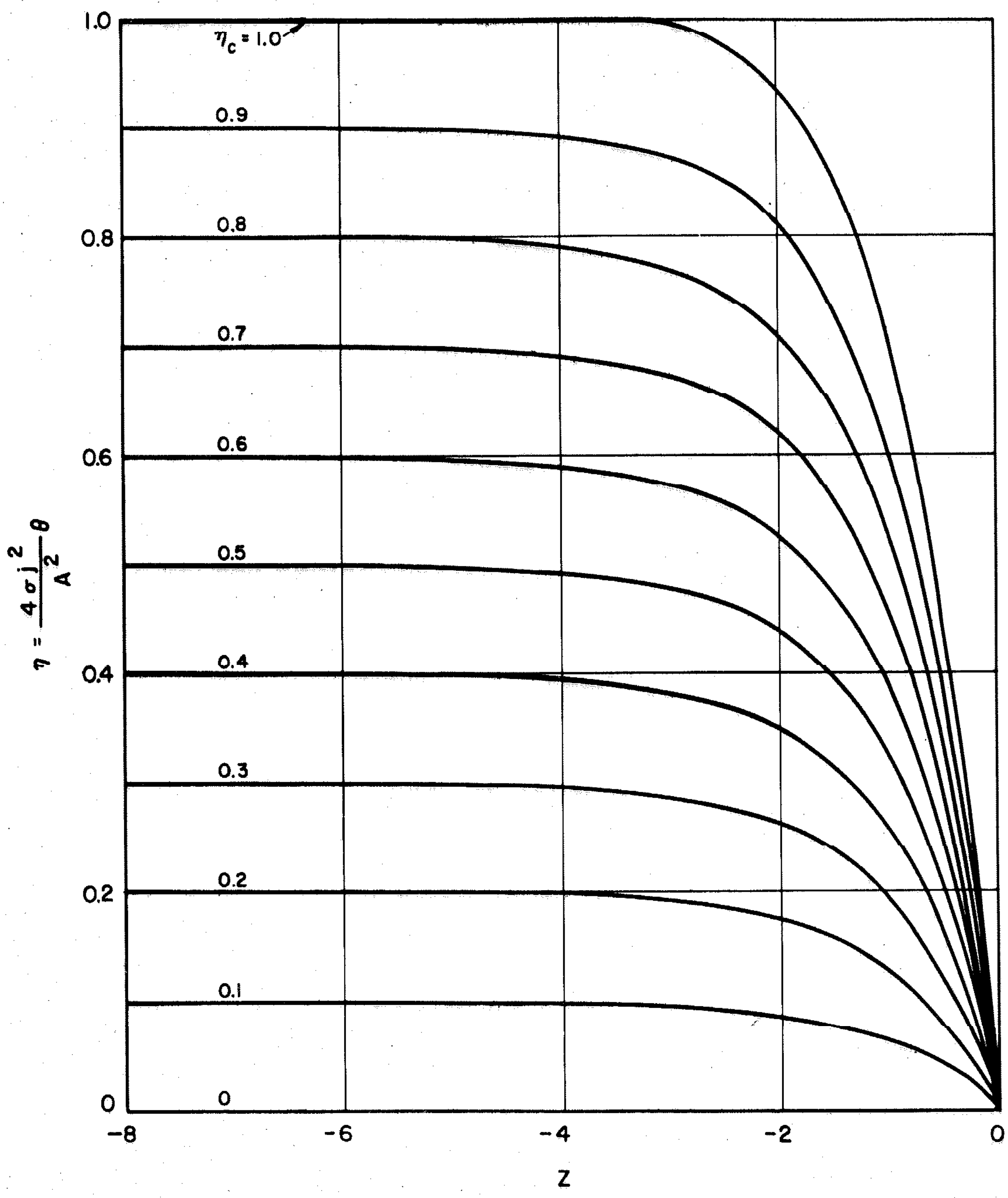

Figure 7. Variation of Temperature in Neighborhood of Cooler Liquid Surface ($\gamma = 4/3$, $\eta_b \approx 0$)

For σ reasonably small, however, this departure of η_b from zero has no appreciable effect on the curves of Fig. 7. It has been shown that j is well approximated by j_o when σ is small and it follows from Eq. (99) that A is equally well approximated by ψ_o. Thus substitution from Eqs. (80), (81), and (77) yields

$$\eta_b \approx 4\sigma\left(\frac{1 - s}{n + s}\right)^2 \varphi_a \theta_b = 4\sigma\left(\frac{1 - s}{1 + s/n}\right)^2 < 4\sigma \quad . \tag{129}$$

For water, for example, 4σ is approximately 0.001, which differs from zero by less than the probable plotting errors in the curves presented.

Special mention should be made of the curve for $\eta_c = 1 (v_c = 0)$, in which case Eq. (127) reduces to

$$\begin{aligned} (v + 2b)^{2b} &= (v_b + 2b)^{2b} \, e^{\frac{\gamma+1}{2\gamma} bz} \\ v &= -2b + (v_b + 2b) \, e^{\frac{\gamma+1}{4\gamma} z} \\ &\approx -2b + (1 + 2b) \, e^{\frac{\gamma+1}{4\gamma} z} \quad . \end{aligned} \tag{130}$$

According to Eq. (130), v reduces to zero at

$$\begin{aligned} z_o &= \frac{4\gamma}{\gamma + 1} \log \frac{2b}{v_b + 2b} \\ &\approx \frac{4\gamma}{\gamma + 1} \log \frac{2b}{1 + 2b} = \frac{4\gamma}{\gamma + 1} \log \frac{2\gamma - 2}{3\gamma - 1} \quad , \end{aligned} \tag{131}$$

and should become negative at more negative values of z. It has been pointed out following Eq. (120), however, that $v = w^{1/2}$ is necessarily non-negative. This seeming paradox is resolved by the alternate solution $v \equiv 0$, which evidently satisfies Eq. (122) for $B = 0$ (corresponding to $v_c = 0$) but which has been lost in the unsophisticated reduction of Eq. (127) to Eq. (130). Thus the solution corresponding to $v_c = 0$ changes abruptly at $z = z_o$ from that given by Eq. (130) to the constant solution $v \equiv 0$. In Fig. 7, where η rather than v is plotted, the function η and first derivative η' remain continuous at $z = z_o$, while the second derivative η'' experiences a jump of magnitude $(\gamma - 1)^2/2\gamma^2$ there. For $\gamma = 4/3$, the approximate ($v_b \approx 1$) value of z_o given by Eq. (131) and corresponding to Fig. 7 is $- 3.44$.

In Fig. 8, representative curves from Fig. 7 are compared with the corresponding curves calculated from the low-velocity approximation

$$\eta_o = \eta_c(1 - \mu e^z), \qquad \mu = 1 - \frac{\eta_b}{\eta_c} \tag{132}$$

obtained from Eqs. (82) and (113) (note $\mu = 1$ in Figs. 7 and 8). For not too large values of η_c, this comparison can be made analytically from Eq. (127) by evaluating C_1 then dividing through by it, raising both sides to the power $1/v_c$, substituting for v, v_b, and v_c in the form

$$v = 1 - \frac{\eta}{2} - \frac{\eta^2}{8} - \ldots , \tag{133}$$

expanding in powers of η and η_c, and neglecting terms of higher than second order. The result is

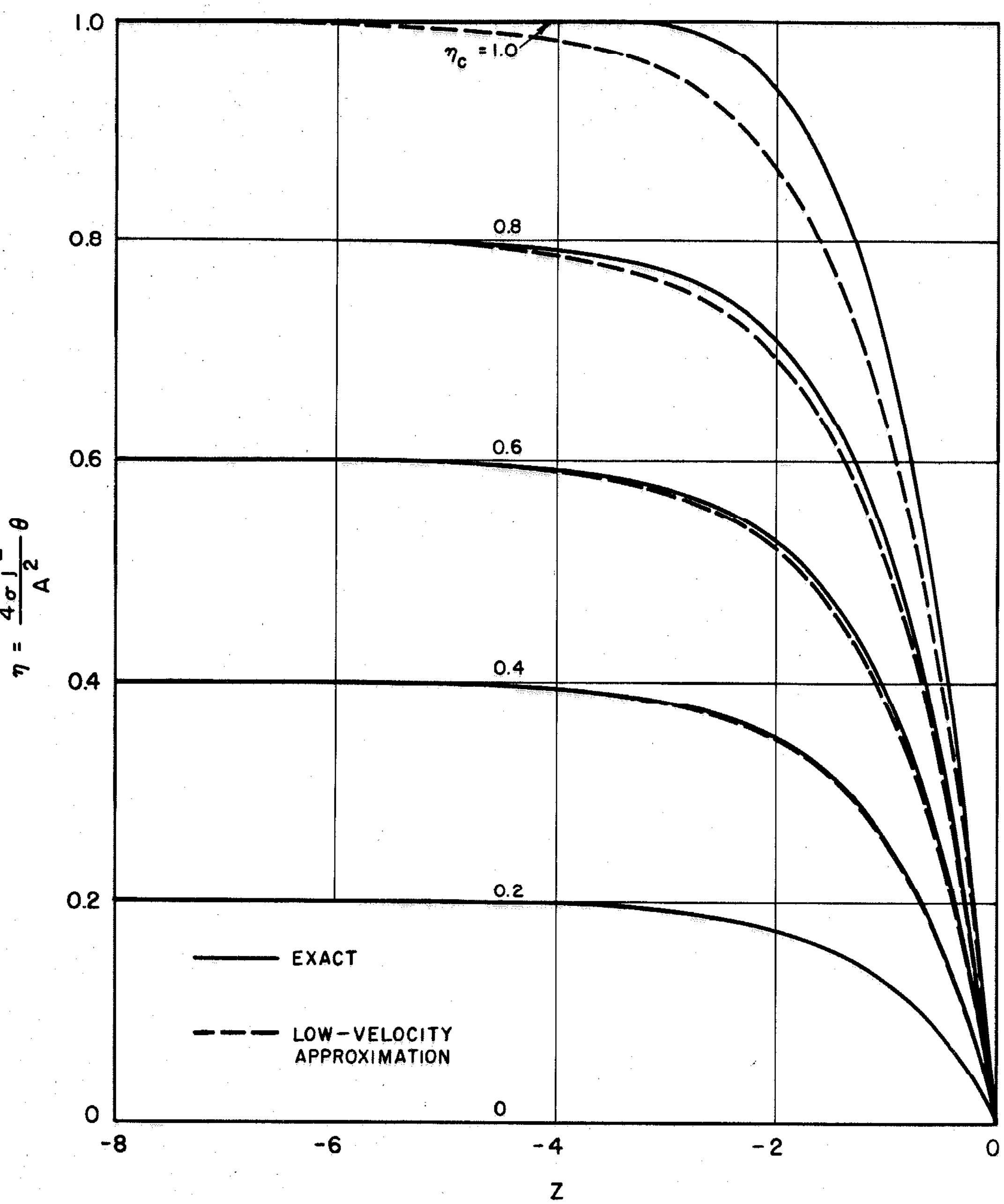

Figure 8. Comparison of Exact Temperature Variation (Equation 127) with Low-Velocity Approximation ($\gamma = 4/3$, $\eta_b \approx 0$)

$$\frac{\eta_c - \eta}{\eta_c - \eta_b}\left[1 - \frac{\gamma - 1}{8\gamma}(\eta - \eta_b)\right] \approx e^{(1 + \frac{\gamma-1}{4\gamma}\eta_c)z}$$

$$\approx e^z \left(1 + \frac{\gamma - 1}{4\gamma}\eta_c z\right) \qquad (134)$$

$$\eta \approx \eta_c\left\{1 - \mu e^z\left[1 + \frac{\gamma - 1}{8\gamma}(\eta - \eta_b + 2\eta_c z)\right]\right\}$$

$$= \eta_c\left[(1 - \mu e^z) - \frac{\gamma - 1}{8\gamma}\mu e^z(\eta - \eta_b + 2\eta_c z)\right] . \qquad (135)$$

Consolidation of the two terms involving η yields

$$\eta \approx \eta_c \frac{(1 - \mu e^z) - \frac{\gamma - 1}{8\gamma}\mu e^z(2\eta_c z - \eta_b)}{1 + \frac{\gamma - 1}{8\gamma}\eta_c \mu e^z}$$

$$\approx \eta_c\left[(1 - \mu e^z) - \frac{\gamma - 1}{8\gamma}\eta_c \mu e^z(\mu - \mu e^z + 2z)\right] . \qquad (136)$$

Thus, to first order, the fractional error in using Eq. (132) to represent η is

$$\frac{\eta - \eta_o}{\eta_o} = \frac{\gamma - 1}{8\gamma}\eta_c \mu e^z \frac{\mu(1 - e^z) + 2z}{\mu e^z - 1}$$

$$\leq \frac{\gamma - 1}{8\gamma}\eta_c . \qquad (137)$$

It follows that when $\gamma = 4/3$, for example, Eq. (132) and the equivalent Eq. (82) are in error by less than 2 percent when η_c is 0.6 or less.

3.5 Exact Solution for Pressure Distribution

In terms of the variables η, w, and v introduced in Section 3.4, Eq. (89) can be written alternatively as

$$\begin{aligned}\psi &= \frac{A}{2}[1 + (1 - \eta)^{1/2}] \\ &= \frac{A}{2}(1 + w^{1/2}) \\ &= \boxed{\frac{A}{2}(1 + v)} \quad .\end{aligned} \tag{138}$$

The coefficient A can be evaluated by substituting the appropriate solution of Eq. (101) into Eq. (99). The values of the normalized pressure, ψ, then follow directly from solution of Eq. (127) for v.

It has been shown in Section 3.3 that if $\sigma^{1/2}$ is small compared to unity, then λ is also small ($\lambda \leqslant \sigma^{1/2}$). Since h $>$ 1, Eq. (99) then becomes, neglecting second-order products of λ and σ,

$$\begin{aligned}\frac{A}{\psi_o} &\approx \frac{h^2 + 2\lambda h + \sigma}{h^2 + \lambda h} \\ &\approx 1 + \frac{\lambda}{h} + \frac{\sigma}{h^2} \quad .\end{aligned} \tag{139}$$

Under the condition stated, the maximum value of λ (for $\gamma M_a^2 \leqslant 1$) is much larger than σ, and is attained when h is very nearly unity. Thus the maximum fractional deviation of A from ψ_o is essentially the same as the maximum fractional deviation of j from j_o. In contrast to j and θ, however, ψ is approximated closely by ψ_o only if η_c is relatively small -- of the order of 0.04 for a maximum $(\psi - \psi_o)/\psi_o$ of 2 percent, for example.

The dependence of ψ/A on η, as given by Eq. (138), is shown graphically in Fig. 9. A series of typical pressure distributions, corresponding to the temperature distributions of Fig. 7, are plotted in Fig. 10.

3.6 Relationship Between Mach Number and Characteristic Temperature Variables

Equations (55) and (113) can be combined in the form

$$\left(\frac{2\psi}{A}\right)^2 \gamma M^2 = \eta = 1 - v^2 = (1 + v)(1 - v) \qquad (140)$$

while Eq. (89) can be written as

$$\frac{2\psi}{A} = 1 + (1 - \eta)^{1/2} = 1 + v \quad . \qquad (141)$$

Substitution of Eq. (141) into Eq. (140) gives

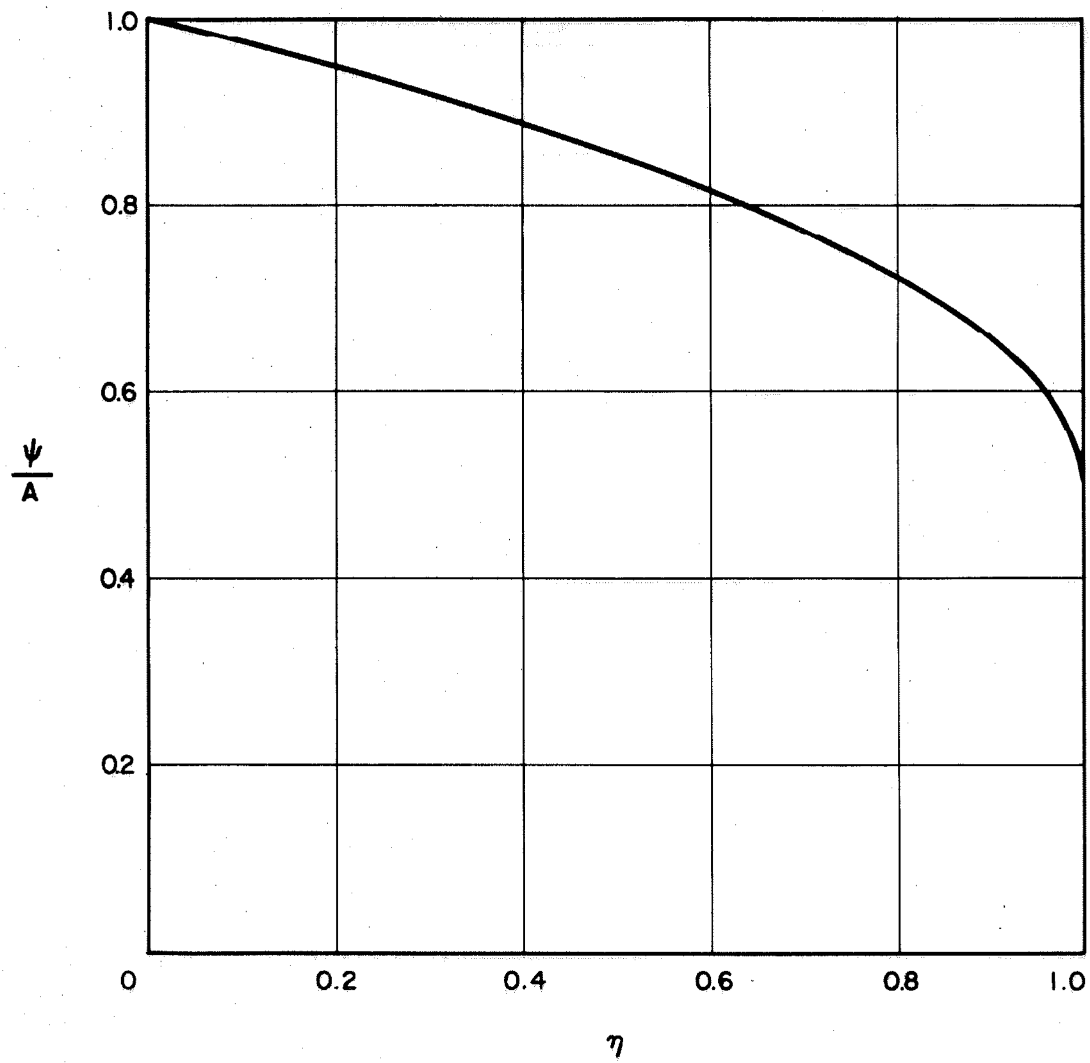

Figure 9. ψ/A versus η

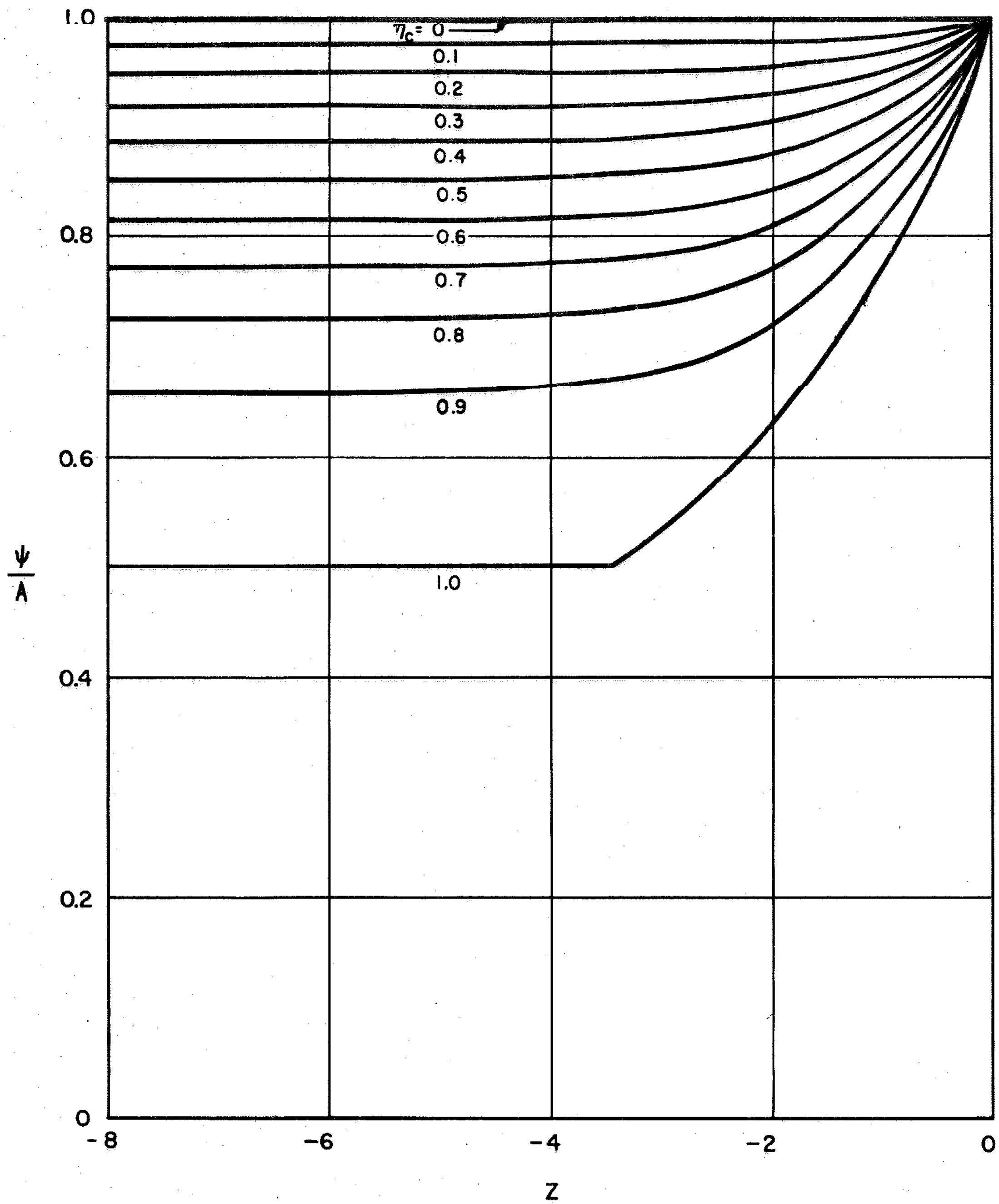

Figure 10. Variation of Pressure in Neighborhood of Cooler Liquid Surface ($\gamma = 4/3$, $\eta_b \approx 0$)

$$\boxed{\gamma M^2 = \frac{1 - v}{1 + v} = \frac{1 - (1 - \eta)^{1/2}}{1 + (1 - \eta)^{1/2}}} , \tag{142}$$

which is readily inverted to yield

$$v = \frac{1 - \gamma M^2}{1 + \gamma M^2} \tag{143}$$

$$\boxed{\eta = 1 - \left(\frac{1 - \gamma M^2}{1 + \gamma M^2}\right)^2} . \tag{144}$$

This relationship between η and M is shown graphically in Fig. 11. A series of typical Mach number distributions, again corresponding to the temperature distributions of Fig. 7, are plotted in Fig. 12.

3.7 Validity of Basic Approach

Fundamental to the analysis which has been carried out so far, is the assumption that the vapor concerned can be treated as a continuum with properties derivable from classical kinetic theory. In order for this to be a reasonable assumption, it is necessary that the separation of the liquid surfaces and any smaller distances in which significant changes are predicted are considerably greater than the mean free path. In the second category, it appears sufficient to consider the characteristic length x^*/j of the essentially exponential variations in the neighborhood of the cooler liquid surface.

The mean free path ℓ is given by[10]

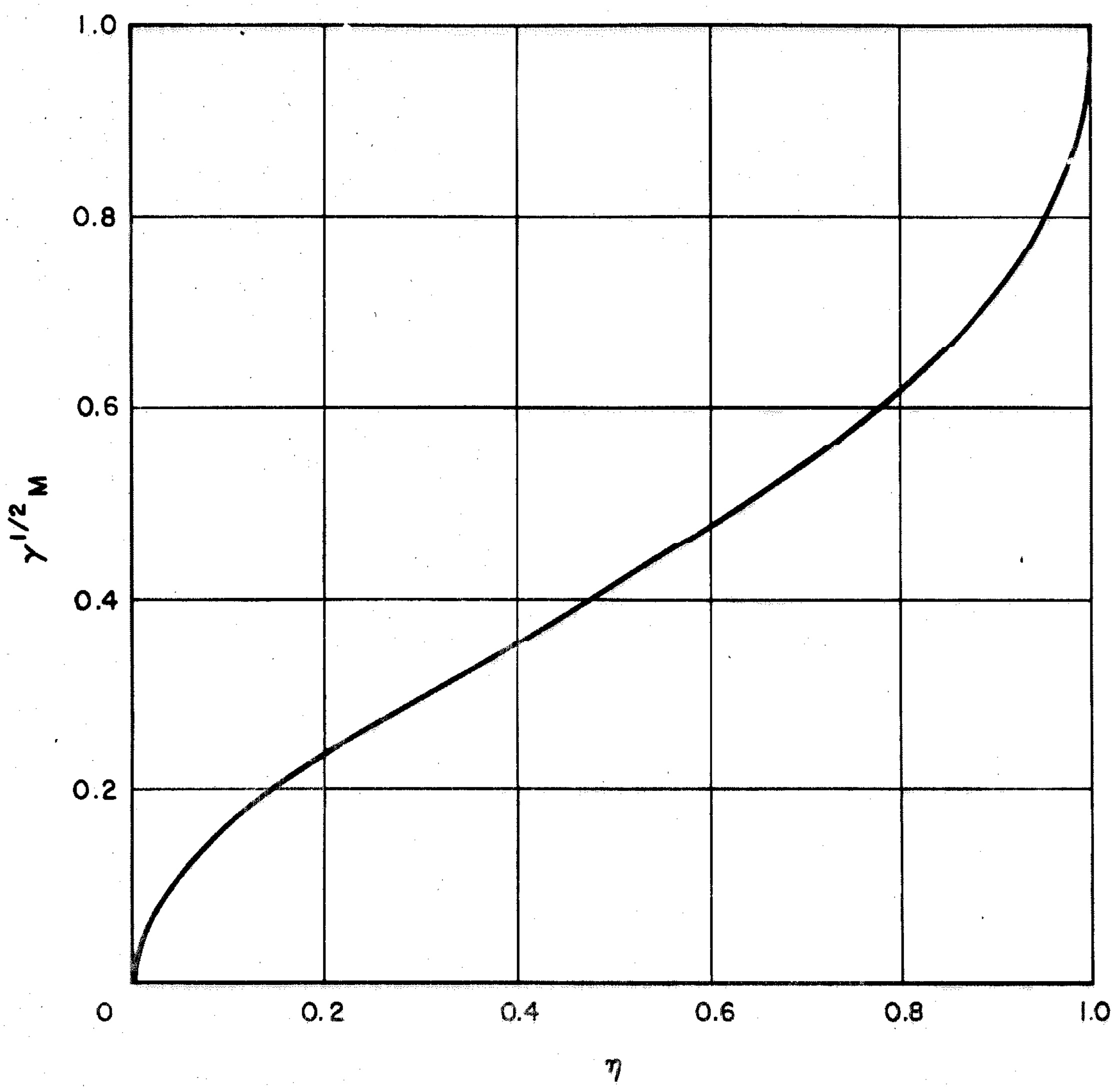

Figure 11. $\gamma^{1/2}M$ versus η

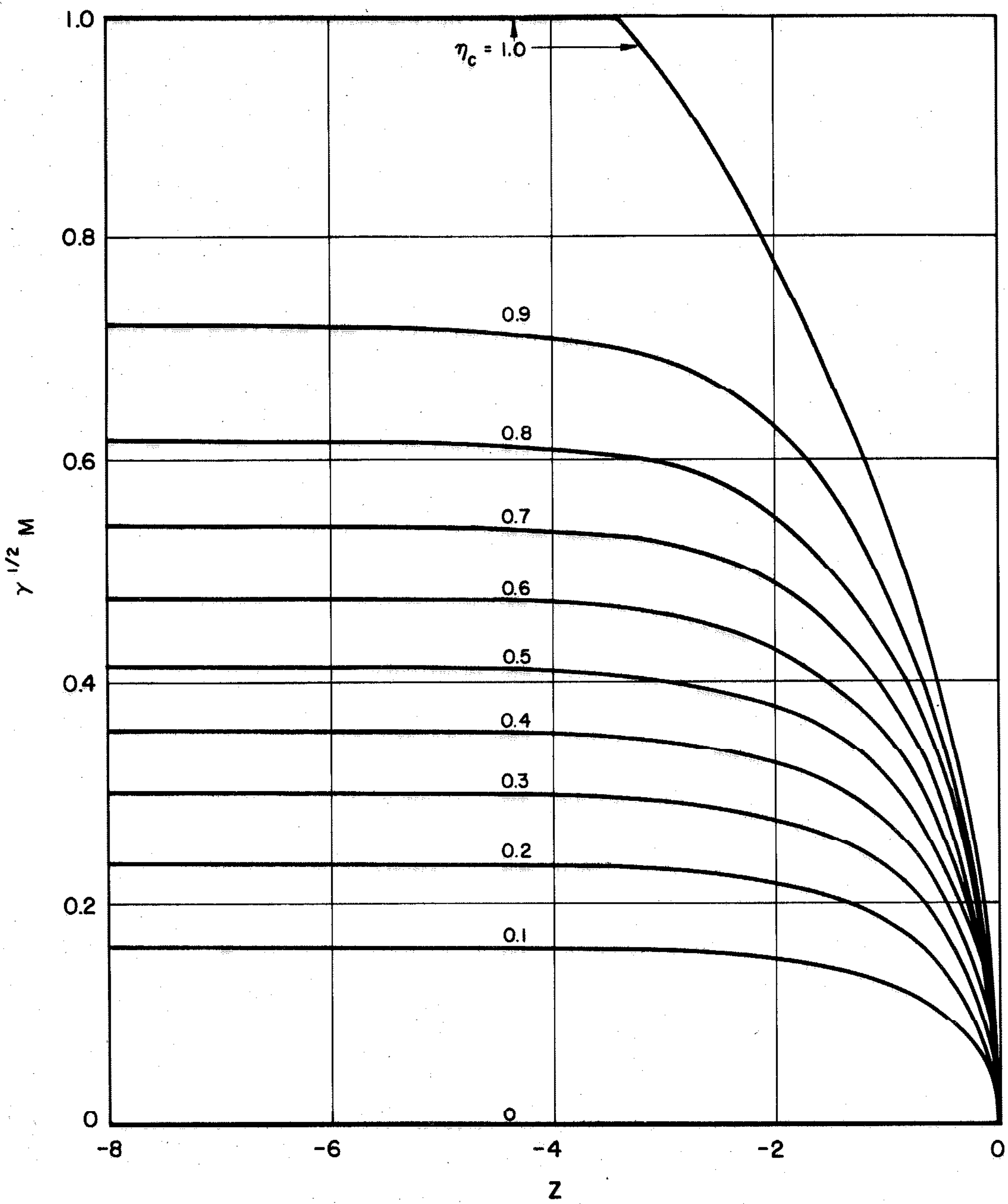

Figure 12. Variation of Mach Number in Neighborhood of Cooler Liquid Surface ($\gamma = 4/3$, $\eta_b \approx 0$)

$$\ell = \frac{1}{\sqrt{2}\,\pi N d^2} , \qquad (145)$$

where N is the number of molecules per unit volume and d is the effective diameter of each molecule (i.e., $\pi d^2/4$ is the cross-section for collision). When the perfect gas law is applicable, the numerical density N can be written as

$$N = \frac{n_A}{v_o} \frac{\rho}{\rho_o} , \qquad (146)$$

where n_A is Avogadro's number, v_o is the molal volume at 0^oC and one atmosphere, and ρ_o is the density under the same standard conditions. The molecular diameter d can be inferred from the van der Waals constant b, which presumably is four times the volume actually occupied by the molecules in one mole of the vapor(11):

$$b = 4 n_A \frac{\pi d^3}{6} \qquad (147)$$

$$d = \left(\frac{3b}{2\pi n_A}\right)^{1/3} . \qquad (148)$$

Equation (146) alternatively can be written in terms of the normalized density, temperature, and pressure variables r, φ, and ψ as

$$N = \frac{n_A r}{v_o \varphi_o \psi_1} , \qquad (149)$$

where $\varphi_o = L/(273.2\ R)$ is the value of φ at 0^oC and ψ_1 is the value of ψ corresponding to a vapor pressure of one atmosphere. The latter quantity is readily evaluated by substituting $\varphi_a = \varphi_b = \varphi_1$, where φ_1 corresponds to the boiling temperature at atmospheric pressure, into Eq. (64). The result is

$$\psi_1 = e^{-\varphi_1} . \tag{150}$$

On substitution of Eqs. (148) to (150) and use of Eq. (48), Eq. (145) can be written as

$$\ell = \frac{\zeta x^*}{r} , \tag{151}$$

where

$$\zeta = \frac{c_p J^* 2^{1/6} \ v_o \varphi_o e^{-\varphi_1}}{k(9\pi n_A b^2)^{1/3}} . \tag{152}$$

The ratio of the characteristic length x^*/j to the mean free path, ℓ, is then given by

$$\frac{x^*}{j\ell} = \zeta^{-1} \frac{r}{j} . \tag{153}$$

The quantity r/j will be recognized as the inverse of the normalized flow velocity u/u^*. Using Eqs. (96) and (138), this quantity can be expressed as

$$\frac{r}{j} = \frac{\varphi\psi}{j} = \frac{\varphi A(1 + v)}{2j_o(1 + \lambda)} \quad . \tag{154}$$

Under the assumption that $\sigma \ll 1$, so that $\lambda \ll 1$ and $A \approx \psi_o$ [see Eq. (139) and sentence following Eq. (106)], Eq. (153) then becomes

$$\frac{x^*}{j\ell} \approx \frac{\varphi\psi_o(1 + v)}{2\zeta j_o} = \frac{1}{2\zeta}\,\frac{n + s}{1 - s}\,\varphi_a^{-1/2}\,\varphi(1 + v) \quad . \tag{155}$$

For water, the value of ζ is about 0.17 (see Appendix A) so that Eq. (155) yields

$$\frac{x^*}{j\ell} \geq \frac{x^*}{j\ell_a} \approx 3\,\frac{n + s}{1 - s}\,\varphi_a^{1/2}\,(1 + v_a) > 3n\varphi_a^{1/2} \quad , \tag{156}$$

the final inequality following from the fact that the values of s and v_a lie between zero and unity. Even in the unrealistically extreme case $T_b = 0^oC$ and $T_a = 250^oC$, φ_a and n are still greater than 9 and 0.7, respectively. It follows from Eq. (156) that in practical cases involving water the characteristic length x^*/j is well over five mean free paths. Thus, at least in the case of water, the implied use of ordinary kinetic theory in the analysis of the present thesis is justified if only the spacing between the two liquid surfaces is at least a few mean free paths.

In the theory developed in preceding sections, the wall separation $(x_b - x_a)$ enters through the dimensionless parameter

$$\Delta = \frac{j}{x^*}(x_b - x_a) = \frac{j\,\ell_a}{x^*}\,\frac{x_b - x_a}{\ell_a} \quad . \tag{157}$$

If the minimum requirement on $(x_b - x_a)$ is arbitrarily taken as $5\,\ell_a$, the corresponding restriction on Δ becomes

$$\Delta > 5\,\frac{j\,\ell_a}{x^*} \approx 10\,\zeta\,\frac{1-s}{n+s}\,\frac{\varphi_a^{-1/2}}{1+v_a} > 5\,\zeta\,\varphi_a^{-1/2}\,\frac{1-s}{n+s} \, . \tag{158}$$

For water [to which Eq. (156) applies] with T_a assumed to be in the neighborhood of $100^{\circ}C$ ($\varphi_a \approx 13$), Eq. (158) is satisfied if

$$\Delta > \frac{1-s}{4(n+s)} \approx \begin{cases} 0.25(1-\varepsilon)^{-1/2} & \varepsilon \to 1 \\ 1.6\,\varepsilon & \varepsilon \to 0 \quad . \end{cases} \tag{159}$$

IV. FIRST-ORDER SOLUTION FOR LOW-VELOCITY NONSTEADY FLOW

A general solution on the nonlinear system of equations (51) to (57) appears to be beyond reach. If the positions and temperatures of the bounding liquid surfaces are sufficiently slowly varying functions of time, however, an approximate solution for the vapor flow should be obtainable as a perturbation from the steady-flow solution. In the following sections an appropriate perturbation procedure is developed and the first-order solution for low-velocity flows carried through.

4.1 Linearization of Flow Equations

In application of the usual perturbation technique to the present problem one would write, assuming that the actual nonsteady flow differs only slightly from a steady flow and indicating all dependences on ξ and τ explicitly,

$$j(\xi, \tau) = j_o + \sum_{n=1}^{\infty} j_n(\xi,\tau) \tag{160}$$

$$\psi(\xi, \tau) = \psi_o + \sum_{n=1}^{\infty} \psi_n(\xi,\tau) \tag{161}$$

$$\theta(\xi,\tau) = \theta_o(\xi) + \sum_{n=1}^{\infty} \theta_n(\xi,\tau) \tag{162}$$

$$r(\xi,\tau) = r_o(\xi) + \sum_{n=1}^{\infty} r_n(\xi,\tau) \quad . \tag{163}$$

Equations for the presumably successively smaller (with increasing n) perturbation terms j_n, ψ_n, θ_n and r_n would then be obtained by substituting Eqs. (160) to (163) into Eqs. (51) to (54), expanding all products, and finally equating terms of like order, the order of a term being reckoned by the sum of the subscripts involved in it. The possible usefulness of this approach lies in the fact that the solution of the (nonlinear) zero-order equations already is known (steady-flow solution), while all of the higher order equations are linear. Unfortunately, in spite of their linearity, the equations obtained in this way are rather intractable since they form inseparable systems of three partial differential equations, two of first order and one of second order, each.

The procedure just outlined, however, fails to take advantage of the particular form in which the flow equations have been written. An alternative approach is to absorb the gross time variation of j, ψ, θ and r in the zero-order terms of Eqs. (160) to (163) by evaluating these zero-order terms not for some actually steady flow but rather for the family of steady flows corresponding to the instantaneous values of $\theta_a(\tau)$, $\theta_b(\tau)$, $\xi_a(\tau)$ and $\xi_b(\tau)$, with time considered as a parameter. For example, when the low-velocity approximation (Sections 3.1 and 3.2) applies, j_o, ψ_o, θ_o and r_o in Eqs. (160) to (163) are to be taken as

$$j_o(\tau) = \frac{1 - s(\tau)}{1 + n(\tau)} \varphi_a^{1/2}(\tau) e^{-\varphi_a(\tau)} \tag{164}$$

$$\psi_o(\tau) = \frac{n(\tau) + s(\tau)}{1 + n(\tau)} e^{-\varphi_a(\tau)} \tag{165}$$

$$\theta_o(\xi,\tau) = \theta_c(\tau)[1 - \mu(\tau)e^{z(\xi,\tau)}] \tag{166}$$

$$r_o(\xi,\tau) = \frac{r_c(\tau)}{1 - \mu(\tau)e^{z(\xi,\tau)}} , \tag{167}$$

where

$$z(\xi,\tau) = - j_o(\tau)[\xi_b(\tau) - \xi] \tag{168}$$

and $n(\tau)$, $s(\tau)$, $\theta_c(\tau)$, $\mu(\tau)$, and $r_c(\tau)$ are given by Eqs. (77), (78), (83), (84), and (86) with similar substitutions. Thus modified, the zero-order terms of Eqs. (160) to (163) represent the instantaneous equilibrium solution of the time-dependent problem.

This quasi-steady-flow solution (not necessarily involving the low-velocity approximation) no longer satisfies the zero-order equations obtained as described previously, since the partial space derivatives (left-hand members) still vanish identically while the time derivatives (right-hand members) no longer do. If the positions and temperatures of the bounding liquid surfaces are sufficiently slowly varying, however, the operator $\partial/\partial\tau$ can be considered to raise by at least unity the order of _any_ quantity upon which it operates. From this point of view, the right-hand members of Eqs. (51) to (53) are _already_ of the first order of smallness because of the partial time derivative involved in each term. Thus the zero-order equations actually should be

$$j_{o\xi} = 0 \tag{169}$$

$$[\psi_o(1 + \gamma M_o^2)]_\xi = 0 \tag{170}$$

$$(1 - \gamma M_o^2)\theta_{o\xi\xi} - j_o(1 - M_o^2)\theta_{o\xi} = 0 \quad , \tag{171}$$

which _are_ satisfied by the quasi-steady-state forms.

When the low-velocity approximation is applicable, the first-order equations obtained in this manner are

$$j_{1\xi} = - r_{o\tau} \tag{172}$$

$$\psi_{1\xi} = - \sigma j_{o\tau} \tag{173}$$

$$\theta_{1\xi\xi} - j_o\theta_{1\xi} = j_1\theta_{o\xi} + r_o\theta_{o\tau} - \frac{\gamma - 1}{\gamma}\psi_{o\tau} \tag{174}$$

$$r_1 = \frac{\psi_1}{\theta_o} - \frac{\psi_o\theta_1}{\theta_o^2} \quad , \tag{175}$$

the second-order equations are

$$j_{2\xi} = -r_{1\tau} \tag{176}$$

$$\psi_{2\xi} = -\sigma j_{1\tau} \tag{177}$$

$$\theta_{2\xi\xi} - j_o\theta_{2\xi} = j_2\theta_{o\xi} + j_1\theta_{1\xi} + r_1\theta_{o\tau} + r_o\theta_{1\tau} - \frac{\gamma-1}{\gamma}\psi_{1\tau} \tag{178}$$

$$r_2 = \frac{\psi_2}{\theta_o} - \frac{\psi_1\theta_1 + \psi_o\theta_2}{\theta_o^{\,2}} + \frac{\psi_o\theta_1^{\,2}}{\theta_o^{\,3}} \quad , \tag{179}$$

and the higher order equations follow similarly. When solved in the order presented, these form a sequence of essentially independent linear total differential equations in ξ for the perturbation quantities j_n, ψ_n, θ_n and r_n.

The modified interpretation of θ_o implies that, for $n \geqslant 1$,

$$\theta_{na} = \theta_{nb} = 0 \quad . \tag{180}$$

It follows from Eqs. (56) and (57) that, again for $n \geqslant 1$,

$$j_{na} = -\varphi_a^{1/2}\,\psi_{na} + r_{n-1,a}\,\xi_{a\tau} \tag{181}$$

$$j_{nb} = \varphi_b^{1/2}\,\psi_{nb} + r_{n-1,b}\,\xi_{b\tau} \quad . \tag{182}$$

The four boundary conditions for each order n stated in Eqs. (180) to (182) are just sufficient to evaluate the four arbitrary functions of time which arise in the solution of the corresponding set of perturbation equations [e.g. for n = 1, Eqs. (172) to (175)].

4.2 Solution of First-Order Equations

Integrating Eqs. (172) and (173) directly and treating Eq. (174) initially as a first-order equation in $\theta_{1\xi}$, one finds the general solutions of these equations to be

$$j_1 = - \int r_{o\tau} d\xi + f_1(\tau)$$

$$= - \frac{\partial}{\partial \tau} \int r_o d\xi + f_1(\tau) \tag{183}$$

$$\psi_1 = - \sigma j_{o\tau} \xi + f_2(\tau) \tag{184}$$

$$\theta_1 = \int e^{j_o \xi} \left[\int e^{-j_o \xi} R(\xi,\tau) d\xi + f_3(\tau) \right] d\xi + f_4(\tau) \quad , \tag{185}$$

where the f's are arbitrary functions of τ and

$$R(\xi,\tau) = j_1 \theta_{o\xi} + r_o \theta_{o\tau} - \frac{\gamma - 1}{\gamma} \psi_{o\tau} \quad . \tag{186}$$

As found previously in connection with the steady-flow solution, it is convenient for calculational purposes to express all space dependences in terms of the variable z, Eq. (168). Doing so in Eqs. (183) to (186), one obtains the equivalent forms (assuming $j_o \neq 0$)

$$j_1 = -\frac{\partial}{\partial \tau} \int \frac{r_o}{j_o} dz + f_1'(\tau) \tag{187}$$

$$\psi_1 = -\sigma \frac{j_{o\tau}}{j_o} z + f_2'(\tau) \tag{188}$$

$$\theta_1 = \frac{1}{j_o^2} \int e^z \int e^{-z} R dz^2 + f_3'(\tau) e^z + f_4'(\tau) \tag{189}$$

$$R = j_1 j_o \theta_{oz} + r_o \theta_{o\tau} - \frac{\gamma - 1}{\gamma} \psi_{o\tau} \ . \tag{190}$$

Explicit expressions for the zero-order quantities r_o, j_o, θ_o and ψ_o are given in Eqs. (164) to (167).

The integrations indicated in Eqs. (187) to (189) are performed, and the arbitrary functions evaluated with the aid of Eqs. (180) to (182), in Appendix B. In carrying out this calculation, it is found convenient to express most of the time derivatives in the form of logarithmic derivatives, indicated by a tilde (~),

$$\widetilde{f(\tau)} = \frac{f_\tau}{f} = (\log f)_\tau \ . \tag{191}$$

The following abbreviations are introduced:

$$\delta = \xi_b - \xi_a = \frac{\Delta}{j_o} \tag{192}$$

$$\omega = \frac{r_c}{j_o} = \frac{n+s}{1-s}(1 - \mu e^{-\Delta})\varphi_a^{1/2} \tag{193}$$

$$G(z) = \tilde{\omega} \log(1 + \mu \frac{1-e^z}{1-\mu}) + (\tilde{\theta}_c - \tilde{\theta}_b)\frac{1-e^z}{1-\mu e^z} - \tilde{j}_o \frac{\mu z e^z}{1-\mu e^z} \tag{194}$$

$$H(z) = \frac{z^2}{2}\mu e^z + \sum_{k=2}^{\infty} \frac{(\mu e^z)^k}{k(k-1)^2} - \sum_{k=2}^{\infty} \frac{\mu^k}{k(k-1)^2} . \tag{195}$$

The final solutions for j_1, ψ_1, and θ_1 as functions of z are given in Eqs. (B-37) to (B-39).

Visualization and interpretation of these solutions is facilitated by introduction of a new space variable y,

$$y = \frac{\xi_b - \xi}{\xi_b - \xi_a} = \frac{\xi_b - \xi}{\delta} = -\frac{z}{\Delta}, \quad 0 \leqslant y \leqslant 1 \tag{195a}$$

which expresses the distance from the cooler liquid surface as a fraction of the total inter-surface spacing. With this substitution, Eqs. (B-37) to (B-39) become

$$\frac{1}{\omega}j_1 = \tilde{r}_c\Delta y + G(-\Delta y) - \frac{1}{1+n}\left[\Delta(\tilde{r}_c + \frac{\tilde{\delta}}{1-\mu e^{-\Delta}} + \frac{\sigma\tilde{j}_o\varphi_a^{1/2}}{\omega} + G(-\Delta)\right] + \frac{r_o\xi_{b\tau}}{\omega} \qquad (196)$$

$$\frac{1}{\omega}\psi_1 = \frac{\sigma\tilde{j}_o}{\omega}\Delta y - \theta_a^{1/2}\frac{n}{1+n}\left[\Delta(\tilde{r}_c + \frac{\tilde{\delta}}{1-\mu e^{-\Delta}} + \frac{\sigma\tilde{j}_o\varphi_a^{1/2}}{\omega} + G(-\Delta)\right] \qquad (197)$$

$$\frac{j_o}{\omega\theta_c}\theta_1 = (\Delta y e^{-\Delta y} - \Delta e^{-\Delta}\frac{1-e^{-\Delta y}}{1-e^{-\Delta}})\Big\{\tilde{\omega}\mu[1 - \log(1-\mu)]$$

$$+ \tilde{\theta}_c - \tilde{\theta}_b - \frac{\mu}{1+n}\left[\Delta(\tilde{r}_c + \frac{\tilde{\delta}}{1-\mu e^{-\Delta}} + \frac{\sigma\tilde{j}_o\varphi_a^{1/2}}{\omega}) + G(-\Delta)\right]\Big\}$$

$$+ \Delta(\frac{\gamma-1}{\gamma}\tilde{\psi}_o - \tilde{\theta}_c)(\frac{1-e^{-\Delta y}}{1-e^{-\Delta}} - y)$$

$$+ \tilde{\omega}\left[H(-\Delta y) - H(-\Delta)\frac{1-e^{-\Delta y}}{1-e^{-\Delta}}\right] \qquad (198)$$

The basic space-dependent functions involved in Eqs. (194) and (198) are plotted for several values of the parameters μ and Δ in Figs. 13 to 18.

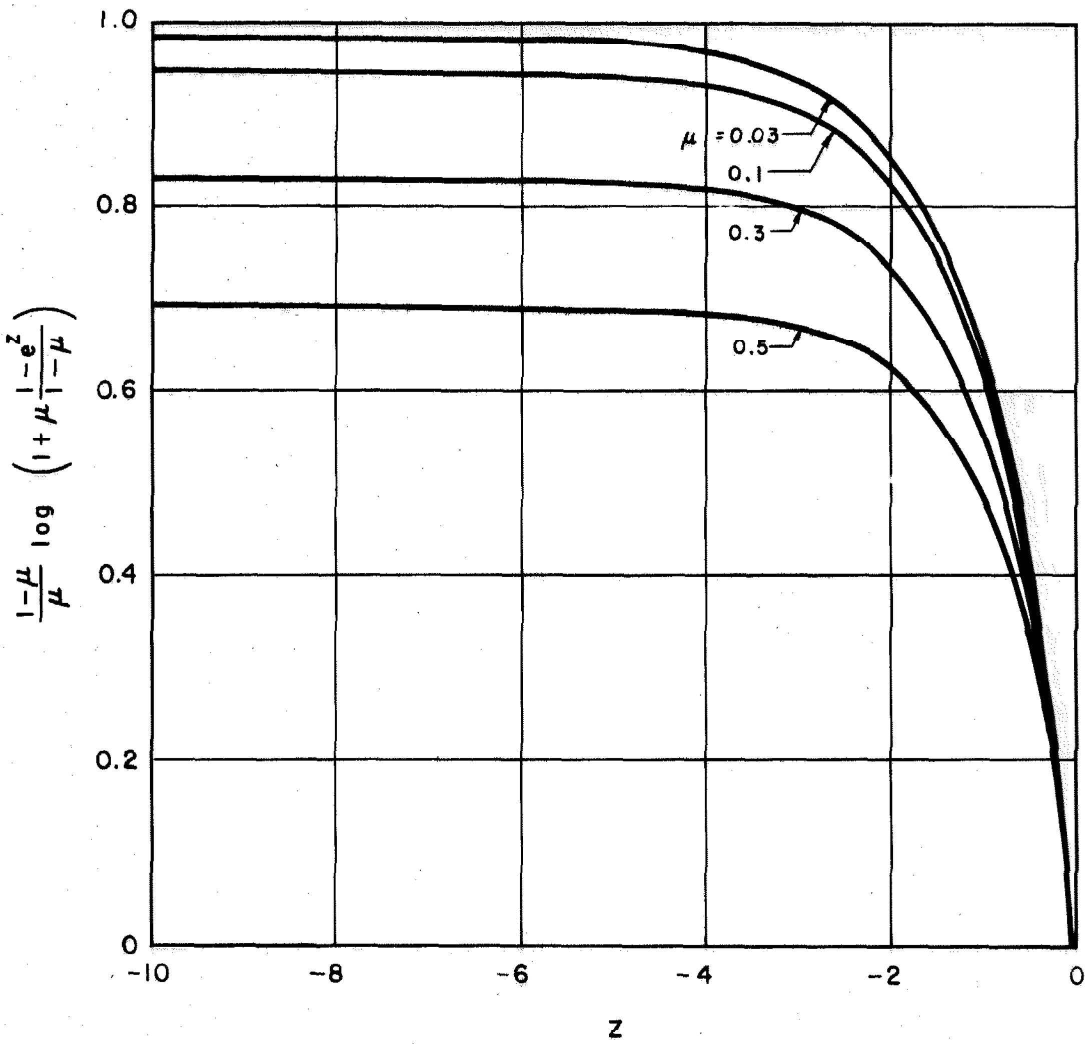
1.0
0.8
0.6
0.4
0.2
0
-10
-8
-6
-4
-2
0
z
$\frac{1-\mu}{\mu} \log \left(1+\mu \frac{1-e^z}{1-\mu}\right)$
μ = 0.03
0.1
0.3
0.5

Figure 13

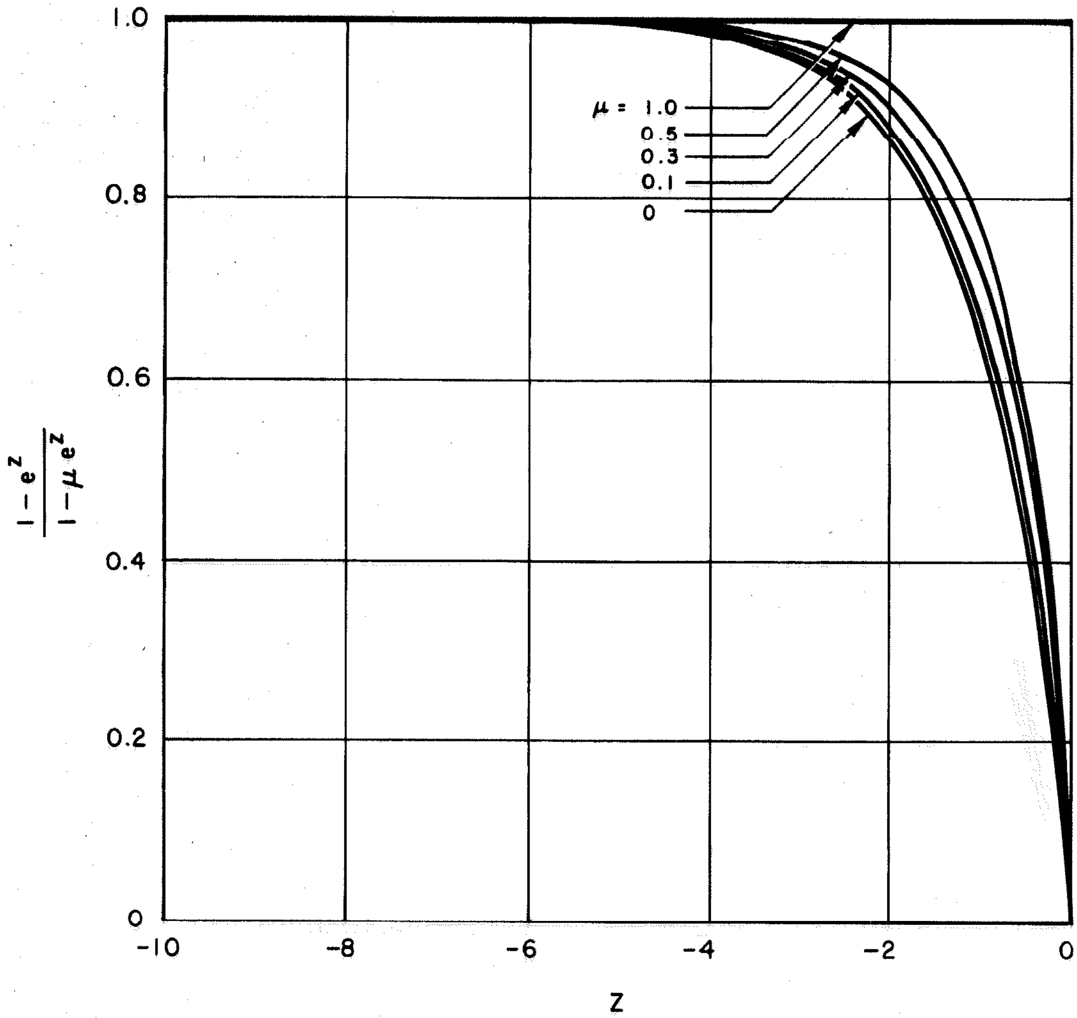

Figure 14

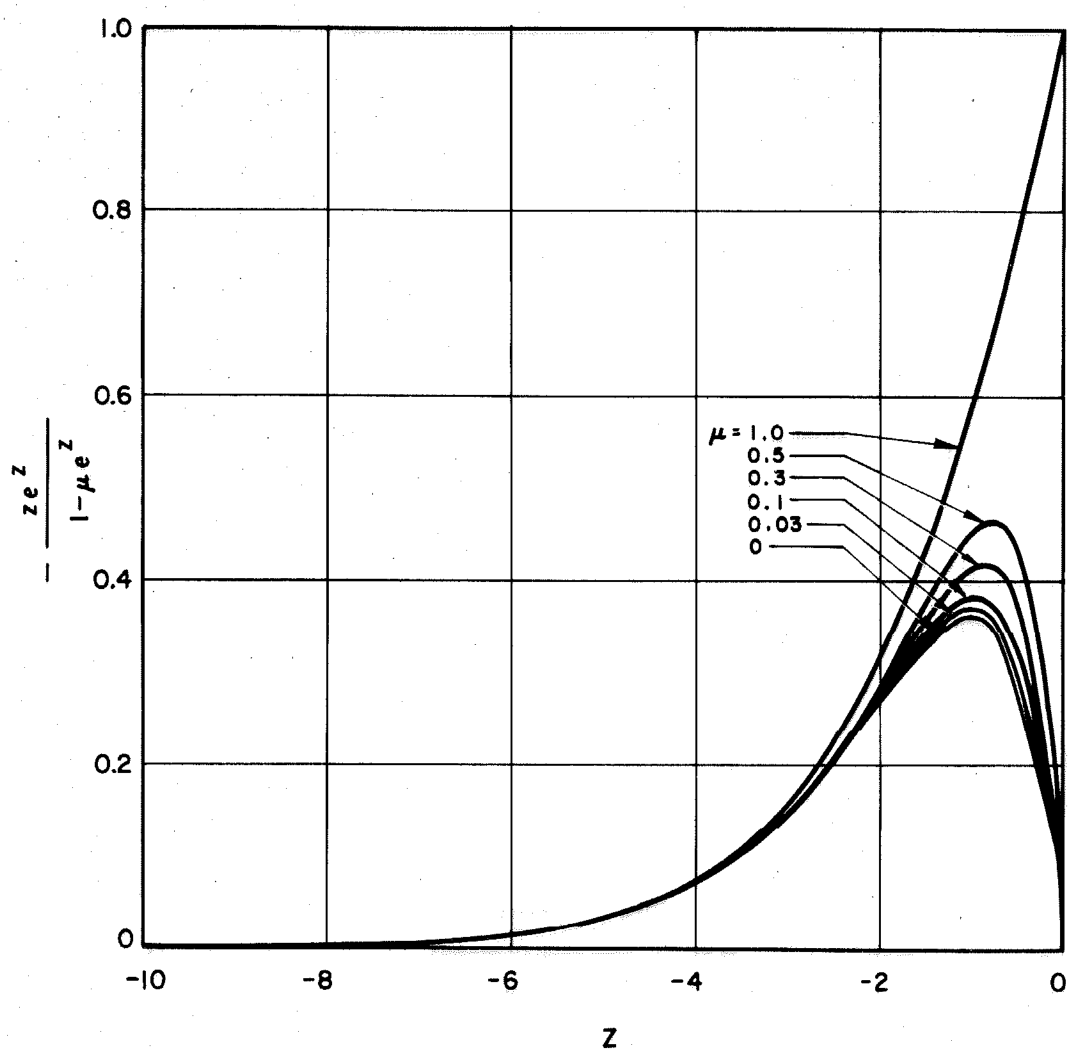

Figure 15

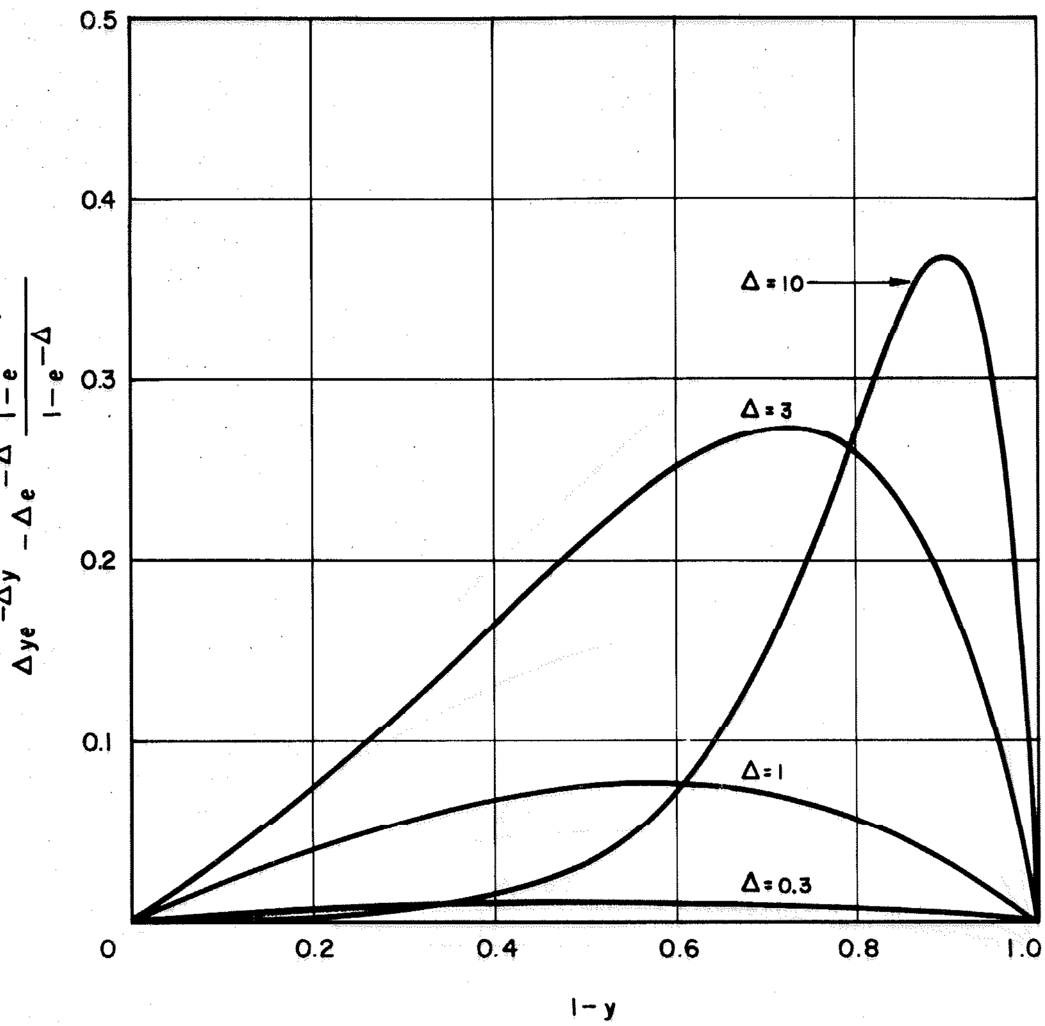
0.5
0.4
0.3
0.2
0.1
0
0.2
0.4
0.6
0.8
1.0
$\Delta y e^{-\Delta y} - \Delta e^{-\Delta} \frac{1-e^{-\Delta y}}{1-e^{-\Delta}}$
$\Delta = 10$
$\Delta = 3$
$\Delta = 1$
$\Delta = 0.3$
$1-y$

Figure 16

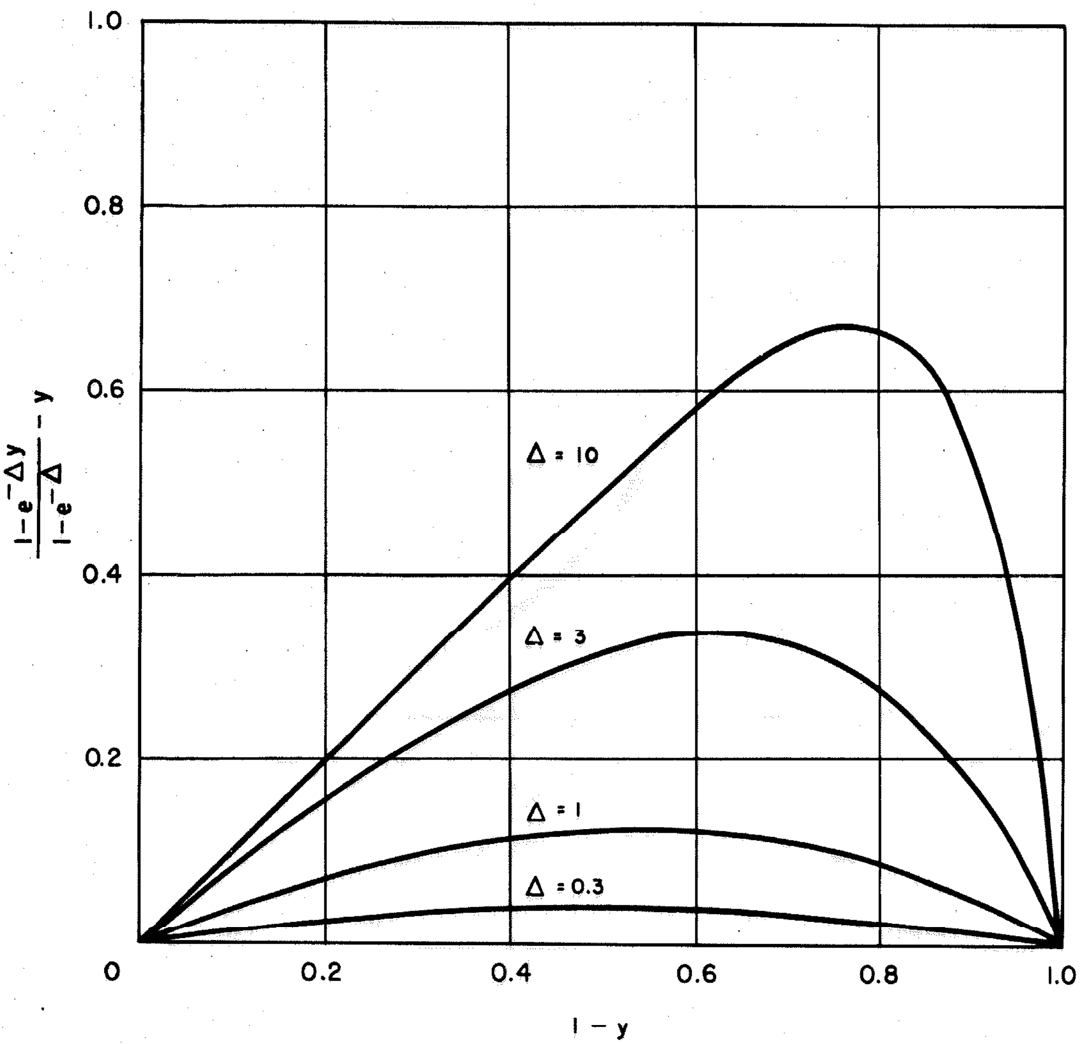
1.0
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1.0
$\frac{1-e^{-\Delta y}}{1-e^{-\Delta}} - y$
1 − y
Δ = 10
Δ = 3
Δ = 1
Δ = 0.3

Figure 17

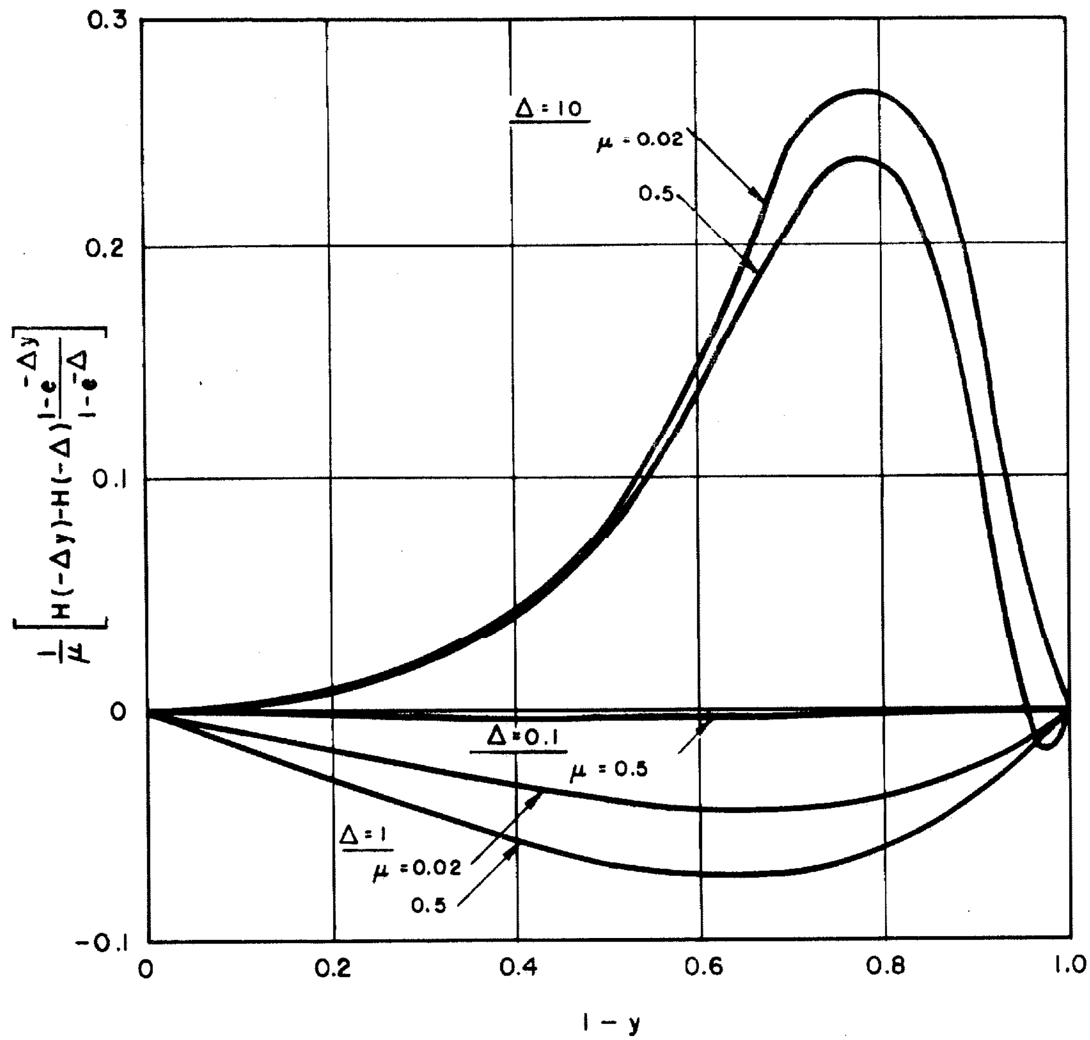
0.3
0.2
0.1
0
-0.1
$\frac{1}{\mu}\left[H(-\Delta y)-H(-\Delta)\frac{1-e^{-\Delta y}}{1-e^{-\Delta}}\right]$
Δ = 10
μ = 0.02
0.5
Δ = 0.1
μ = 0.5
Δ = 1
μ = 0.02
0.5
0
0.2
0.4
0.6
0.8
1.0
1 − y

Figure 18

Since the given data in the present problem are the temperatures and positions of the two liquid surfaces as functions of time, it is desirable to express all of the logarithmic derivatives involved in Eqs. (194) to (198) in terms of $\tilde{\theta}_a$, $\tilde{\theta}_b$ and $\tilde{\delta}$. Such expressions are derived in Appendix B (Section B-5) and are conveniently summarized by introducing the notation

$$\tilde{j}_o = A_j\tilde{\theta}_a + B_j\tilde{\theta}_b \tag{199}$$

$$\tilde{\psi}_o = A_\psi\tilde{\theta}_a + B_\psi\tilde{\theta}_b \tag{200}$$

$$\tilde{\theta}_c = A_\theta\tilde{\theta}_a + B_\theta\tilde{\theta}_b + D_\theta\tilde{\delta} \tag{201}$$

$$\tilde{r}_c = A_r\tilde{\theta}_a + B_r\tilde{\theta}_b + D_r\tilde{\delta} \tag{202}$$

$$\tilde{\omega} = A_\omega\tilde{\theta}_a + B_\omega\tilde{\theta}_b + D_\omega\tilde{\delta} \tag{203}$$

and the abbreviation

$$g = \frac{\Delta e^{-\Delta}}{1 - e^{-\Delta}} \; . \tag{204}$$

It is shown in Appendix B that

$$A_j = \frac{\varphi_a}{1 - s} - \frac{1}{2(1 + n)} \quad (205)$$

$$B_j = -\frac{s\varphi_b}{1-s} - \frac{n}{2(1 + n)} \quad (206)$$

$$A_\psi = \frac{n\varphi_a}{n + s} - \frac{n(1 - s)}{2(1 + n)(n + s)} = \frac{n(1 - s)}{n + s} A_j \quad (207)$$

$$B_\psi = \frac{s\varphi_b}{n + s} + \frac{n(1 - s)}{2(1 + n)(n + s)} = -\frac{1 - s}{n + s} B_j \quad (208)$$

$$A_\theta = \mu(\frac{1}{\varepsilon} - gA_j) \quad (209)$$

$$B_\theta = 1 - \mu(\frac{1}{\varepsilon} + gB_j) \quad (210)$$

$$D_\theta = -\mu g \quad (211)$$

$$A_r = A_\psi - A_\theta \quad (212)$$

$$B_r = B_\psi - B_\theta \quad (213)$$

$$D_r = -D_\theta = \mu g \quad (214)$$

$$A_\omega = A_\psi - A_j - A_\theta \quad (215)$$

$$B_\omega = B_\psi - B_j - B_\theta \quad (216)$$

$$D_\omega = -D_\theta = \mu g \quad (217)$$

In view of the form of Eqs. (199) to (203), it will be observed that ψ_1 and θ_1 depend on $\tilde{\delta} = (\xi_{b\tau} - \xi_{a\tau})/\delta$ but not on $\xi_{b\tau}$ or $\xi_{a\tau}$ individually. This result is to be expected on physical grounds, of course, since uniform translation of the entire system cannot affect the pressure or temperature distributions. On the other hand such translation does affect the current densities which would be measured by a stationary observer, as evidenced by the term involving $\xi_{b\tau}$ in Eq. (196). Of more fundamental interest than the j_1 measured in an arbitrarily stationary reference frame, however, is the intrinsic incremental j given by

$$j_{1i} = j_1 - r_o[\xi_{a\tau}y + \xi_{b\tau}(1 - y)] \quad , \tag{218}$$

the term involving r_o being of purely kinematical origin. Substitution of Eq. (218) into Eq. (196) yields

$$\boxed{\begin{aligned} \frac{1}{\omega} j_{1i} &= (\tilde{r}_c + \frac{\tilde{\delta}}{1 - \mu e^{-\Delta y}}) \Delta y + G(-\Delta y) \\ &\quad - \frac{1}{1 + n}\left[\Delta(\tilde{r}_c + \frac{\tilde{\delta}}{1 - \mu e^{-\Delta}} + \frac{\sigma \tilde{j}_o \varphi_a^{1/2}}{\omega}) + G(-\Delta)\right] \end{aligned}} \; . \tag{219}$$

4.3 Limiting Form of Solution as $j_o \to 0$ ($\theta_a \to \theta_b$)

The solution of the linearized nonsteady-flow problem summarized in Eqs. (197), (198), and (219) assumes an indeterminate form as ε, and thus j_o, tends to zero ($\theta_a \to \theta_b$), since in this limit ω, $\widetilde{\omega}$, and $\widetilde{j}_o$ tend to infinity while Δ tends to zero. The limiting indeterminate form is evaluated in Appendix C. The result is

$$
\begin{aligned}
j_{1i} \xrightarrow[\varepsilon \to 0]{} \; & \widetilde{\theta}_a \frac{\delta\varphi_a e^{-\varphi_a}}{4}\left[\varphi_a(2y - 1 - \sigma) - 2y^2 + 1\right] \\
& + \widetilde{\theta}_b \frac{\delta\varphi_a e^{-\varphi_a}}{4}\left[\varphi_a(2y - 1 + \sigma) + 2y^2 - 4y + 1\right] \\
& + \delta_\tau \frac{\varphi_a e^{-\varphi_a}}{2}(2y - 1)
\end{aligned}
\qquad (220)
$$

$$
\begin{aligned}
\psi_1 \xrightarrow[\varepsilon \to 0]{} & - \widetilde{\theta}_a \frac{\delta\varphi_a^{1/2} e^{-\varphi_a}}{4}\left\{\varphi_a\left[1 - \sigma(2y - 1)\right] - 1\right\} \\
& - \widetilde{\theta}_b \frac{\delta\varphi_a^{1/2} e^{-\varphi_a}}{4}\left\{\varphi_a\left[1 + \sigma(2y - 1)\right] - 1\right\} \\
& - \delta_\tau \frac{\varphi_a^{1/2} e^{-\varphi_a}}{2}
\end{aligned}
\qquad (221)
$$

$$\theta_1 \xrightarrow[\varepsilon\to 0]{} \tilde{\theta}_a \delta^2 e^{-\varphi_a} \left(\frac{\gamma - 1}{\gamma}\frac{\varphi_a}{2} - \frac{1 + y}{3}\right) \frac{y(1 - y)}{2}$$

$$+ \tilde{\theta}_b \delta^2 e^{-\varphi_a} \left(\frac{\gamma - 1}{\gamma}\frac{\varphi_a}{2} - \frac{2 - y}{3}\right) \frac{y(1 - y)}{2} \quad . \qquad (222)$$

In connection with Eqs. (220) to (222), it may be useful to observe that the quantity $\delta\varphi_a e^{-\varphi_a} = \delta r_a$ is of the order of magnitude of the wall spacing measured in mean free path-lengths. This follows from Eqs. (151) and (192), which give

$$\delta r_a = \frac{x_b - x_a}{x^*} r_a = \frac{x_b - x_a}{\ell_a} \frac{\ell_a r_a}{x^*} = \varsigma \frac{x_b - x_a}{\ell_a} . \qquad (223)$$

For water, for example, ς is approximately 0.17. If $(x_b - x_a)/\ell_a$ is denoted by N_ℓ and it is assumed that $\sigma \ll 1$, Eqs. (220) to (222) can be written approximately as

$$\frac{4}{\varsigma N_\ell} j_{1i} \approx \tilde{\theta}_a[\varphi_a(2y - 1) - 2y^2 + 1]$$

$$+ \tilde{\theta}_b[\varphi_a(2y - 1) + 2y^2 - 4y + 1]$$

$$+ 2\tilde{\delta}(2y - 1) \qquad (224)$$

$$\frac{4\varphi_a^{1/2}}{\zeta N_\ell}\psi_1 \approx -\tilde{\theta}_a(\varphi_a - 1)$$

$$-\tilde{\theta}_b(\varphi_a - 1)$$

$$-2\tilde{\delta} \tag{225}$$

$$\frac{2}{\delta\zeta N_\ell}\theta_1 \approx -\tilde{\theta}_a\left(\frac{\gamma - 1}{\gamma} - 2\,\frac{1 + y}{3\varphi_a}\right)\frac{y(y - 1)}{2}$$

$$-\tilde{\theta}_b\left(\frac{\gamma - 1}{\gamma} - 2\,\frac{2 - y}{3\varphi_a}\right)\frac{y(y - 1)}{2} \quad . \tag{226}$$

4.4 Approximate Form of Solution for Large Δ

As pointed out by Plesset(1) and mentioned in Section 3.1, the parameter Δ is in many cases of practical interest a large number. The simplification of the steady-flow solution in such cases has been seen in Eqs. (72) to (75), and an even more striking simplification occurs in the nonsteady-flow solution developed in Section 4.2.

Considering first the function $G(z)$ which appears in all three of the Eqs. (197), (198), and (219), one sees from Figs. 13 to 15 that

$$\left|\frac{1}{\mu}\log\left(1 + \mu\,\frac{1 - e^z}{1 - \mu}\right)\right| \leqslant \frac{1}{1 - \mu} \tag{227}$$

$$\left|\frac{1 - e^z}{1 - \mu e^z}\right| \leqslant 1 \tag{228}$$

$$\left| \frac{ze^z}{1 - \mu e^z} \right| \leq 1 \quad . \tag{229}$$

Furthermore it is shown in Appendix D that, at least for φ_a greater than about 5,* each of the quantities $\mu\tilde{j}_o$, $\mu\tilde{\omega}$, $\tilde{\theta}_c$, and $\tilde{\theta}_b$ is of the same order as or less than $\tilde{r}_c$. It therefore follows that $G(z)$ is of the same order as or less than $\tilde{r}_c/(1 - \mu)$ and thus that $|G(z)| \ll \Delta|\tilde{r}_c|$ if $\Delta \gg (1 - \mu)^{-1}$. With the additional assumption that $\sigma \ll 1$, Eqs. (197) and (219) reduce approximately to

$$\begin{aligned} \psi_1 &\approx -\omega\Delta\theta_b^{1/2}\frac{\tilde{r}_c + \tilde{\delta}}{1 + n} \\ &\approx -r_c\delta\theta_b^{1/2}\frac{\tilde{\psi}_o - \tilde{\theta}_a + \tilde{\delta}}{1 + n} \end{aligned} \tag{230}$$

$$\begin{aligned} j_{1i} &\approx \omega\Delta(\tilde{r}_c y + \tilde{\delta}\frac{j}{1 - \mu e^{-\Delta y}} - \frac{\tilde{r}_c + \tilde{\delta}}{1 + n}) \\ &\approx r_c\delta(\tilde{\psi}_o - \tilde{\theta}_a + \tilde{\delta})(y - \frac{1}{1 + n}) \quad . \end{aligned} \tag{231}$$

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* An order-of-magnitude estimate of the minimum φ_a to be expected for a given substance is φ_1, the value of φ corresponding to the normal (1 atmosphere) boiling temperature. Epstein lists values of the ratio L/T_1 (ℓ_B/T_B in his notation) for a number of the elements, including the rare gases, the primary atmospheric gases, the halogens, and metals ranging from sodium to lead.[12] Division by R yields the corresponding values of φ_1. Except for helium ($\varphi_1 = 2.9$) and hydrogen ($\varphi_1 = 5.4$), the values of φ_1 for all the substances listed are greater than 7, and in most cases are greater than 10. The value of φ_1 for water is 13.1.

The second forms of these equations follow from the facts that $\tilde{\theta}_c \approx \tilde{\theta}_a$ and $\mu \approx \varepsilon$ when $\Delta \gg 1$.

Turning to Eq. (198), one observes from Figs. 16 to 18 that for large Δ the function

$$\frac{1 - e^{-\Delta y}}{1 - e^{-\Delta}} - y \approx 1 - y - e^{-\Delta y} \tag{232}$$

is of the order of or larger than the functions

$$\Delta y e^{-\Delta y} - \Delta e^{-\Delta} \frac{1 - e^{-\Delta y}}{1 - e^{-\Delta}} \approx \Delta y e^{-\Delta y} \tag{233}$$

and

$$\frac{1}{\mu}\left[H(-\Delta y) - H(-\Delta)\frac{1 - e^{-\Delta y}}{1 - e^{-\Delta}}\right] . \tag{234}$$

Again it is shown in Appendix D that, at least for φ_a greater than about 5, the $\tilde{\theta}_a$ and $\tilde{\theta}_b$ components of the quantities $\mu\tilde{\omega}$, $\tilde{\theta}_c$, and $\tilde{\theta}_b$ are all of the same order as or less than the corresponding components of $\tilde{\psi}_o$, while the $\tilde{\delta}$ components of these quantities are no larger than that of $\tilde{\theta}_c$. Thus if $\Delta \gg (1 - \mu)^{-1}$ and $\sigma \ll 1$, as assumed in deriving Eqs. (230) and (231), Eq. (198) reduces approximately to

$$\frac{j_o}{\omega\theta_c}\,\theta_1 \approx \Delta\left(\frac{\gamma-1}{\gamma}\,\tilde{\psi}_o - \tilde{\theta}_c\right)(1-y-e^{-\Delta y}) - \Delta y e^{-\Delta y}\,\frac{\mu\Delta}{1+n}(\tilde{r}_c + \tilde{\delta})$$

$$\approx \Delta\left[\left(\frac{\gamma-1}{\gamma}\,\tilde{\psi}_o - \tilde{\theta}_a\right)(1-y-e^{-\Delta y}) - (\tilde{\psi}_o - \tilde{\theta}_a + \tilde{\delta})(1-n)\,\Delta y e^{-\Delta y}\right] . \tag{235}$$

It will be observed that the dependences of ψ_1, j_{1i}, and θ_1 on the wall temperature variations are easily separated from the dependences on the wall motions in Eqs. (230), (231), and (235), since $\tilde{\psi}_o$ and $\tilde{\theta}_a$ are independent of the wall motions while $\tilde{\delta}$ is independent of the temperature variations. With this separation indicated explicitly, the ratios of the three approximate perturbation solutions to the corresponding zero-order solutions can be written as

$$\frac{\psi_1}{\psi_o} \approx -\varphi_a^{1/2}\,\delta(\tilde{\psi}_o-\tilde{\theta}_a)\frac{n}{1+n} - \varphi_a^{1/2}\,\delta_\tau\,\frac{n}{1+n} \tag{236}$$

$$\frac{j_{1i}}{j_o} \approx \omega\delta(\tilde{\psi}_o-\tilde{\theta}_a)(y-\frac{1}{1+n}) + \omega\delta_\tau(y-\frac{1}{1+n}) \tag{237}$$

$$\frac{\theta_1}{\theta_a} \approx \omega\delta\left[(\frac{\gamma-1}{\gamma}\,\tilde{\psi}_o-\tilde{\theta}_a)(1-y-e^{-\Delta y}) - (\tilde{\psi}_o-\tilde{\theta}_a)(1-n)\,\Delta y e^{-\Delta y}\right] -\omega\delta_\tau(1-n)\,\Delta y e^{-\Delta y} \,. \tag{238}$$

From Eqs. (193), (207) and (208), the quantities ω and $\tilde{\psi}_o$ are given by

$$\omega = \frac{r_c}{j_o} \approx \frac{n+s}{1-s}\,\varphi_a^{1/2} \tag{239}$$

$$\tilde{\psi}_o = \tilde{\theta}_a \left[\frac{n\varphi_a}{n+s} - \frac{n(1-s)}{2(1+n)(n+s)}\right] + \tilde{\theta}_b \left[\frac{s\varphi_b}{n+s} + \frac{n(1-s)}{2(1+n)(n+s)}\right] . \quad (240)$$

Although general expressions for ψ_1, j_{1i}, and θ_1 as functions of y have been found, it frequently is the boundary values ($x = x_a$, x_b) which are of greatest practical interest. Substitution of zero or unity as appropriate for y in Eqs. (236) and (237) yields

$$\frac{\psi_{1a}}{\psi_o} \approx \frac{\psi_{1b}}{\psi_o} \approx - \varphi_a^{1/2} \, \delta \, \frac{n}{1+n}[(\tilde{\psi}_o - \tilde{\theta}_a) + \tilde{\delta}] \quad (241)$$

$$\frac{j_{1ia}}{j_o} \approx - \frac{\omega}{\varphi_a^{1/2}} \frac{\psi_1}{\psi_o} \approx - \frac{\varphi_a^{1/2} \psi_1}{j_o} \quad (242)$$

$$\frac{j_{1ib}}{j_o} \approx \frac{\omega}{n\varphi_a^{1/2}} \frac{\psi_1}{\psi_o} \approx \frac{\varphi_b^{1/2} \psi_1}{j_o} . \quad (243)$$

The last two equations will be recognized as simply restatements of the basic boundary conditions, Eqs. (181) and (182). The boundary values of θ_1 are zero, of course.

The results of sample calculations for water based on Eqs. (241) to (243) are presented in Figs. 19 to 22.* These results are expressed

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* In Figs. 19 to 22, the subscripts a and b refer to the two liquid surfaces at $x = x_a$ and $x = x_b > x_a$, respectively, without regard to which surface is at the higher temperature. In plotting the various curves, however, account has been taken of the fact that in Eqs. (241) to (243) [more directly, in Eqs. (244) to (246)], as in most of the equations throughout this thesis, it is specifically assumed that $T_a > T_b$. Thus the lower curve in Fig. 19, for example, is actually a plot of $p_1^{(a)}/(x_b - x_a)T_{at}$ versus T_a (in the notation of the equations) with the designations a and b reversed for plotting purposes.

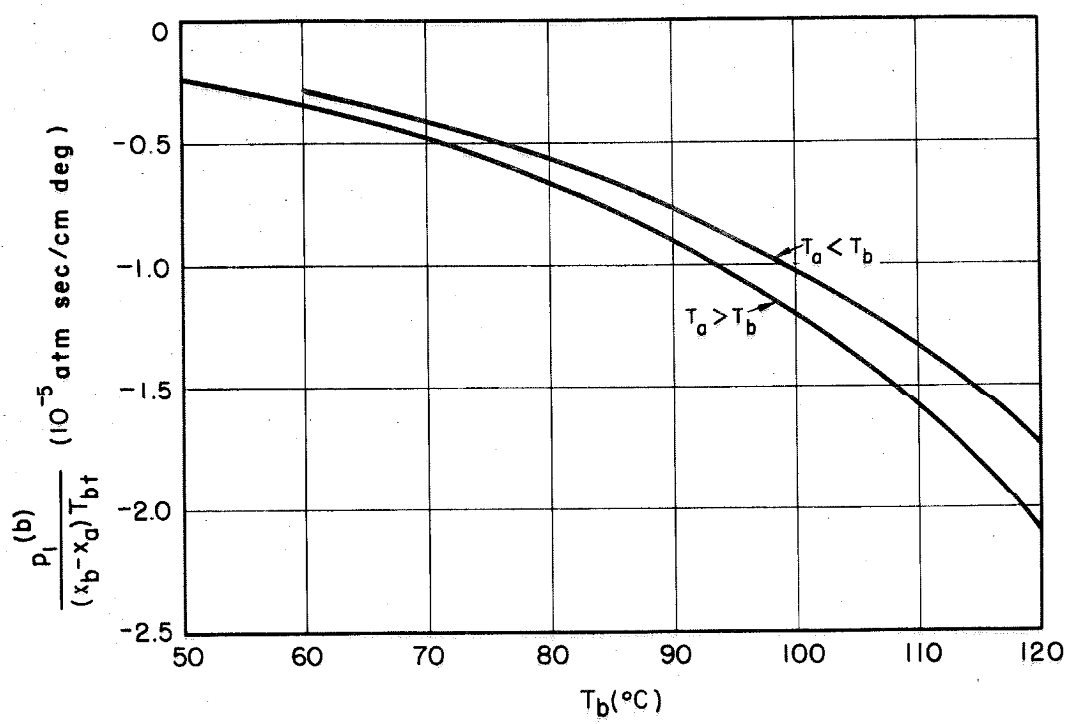

Figure 19. Approximate ($\Delta \gg 1$) Perturbation Pressure (p_1) due to Temperature Variation at One Liquid Surface (T_{bt}) versus Temperature of That Surface (T_b) for Water Vapor ($\rho^* = 22.6$ g/cm^3, L = 9720 cal/mol, $\beta = 0.04$). Curves Apply with Negligible Error for 60° C$<T_a<120^\circ$ C

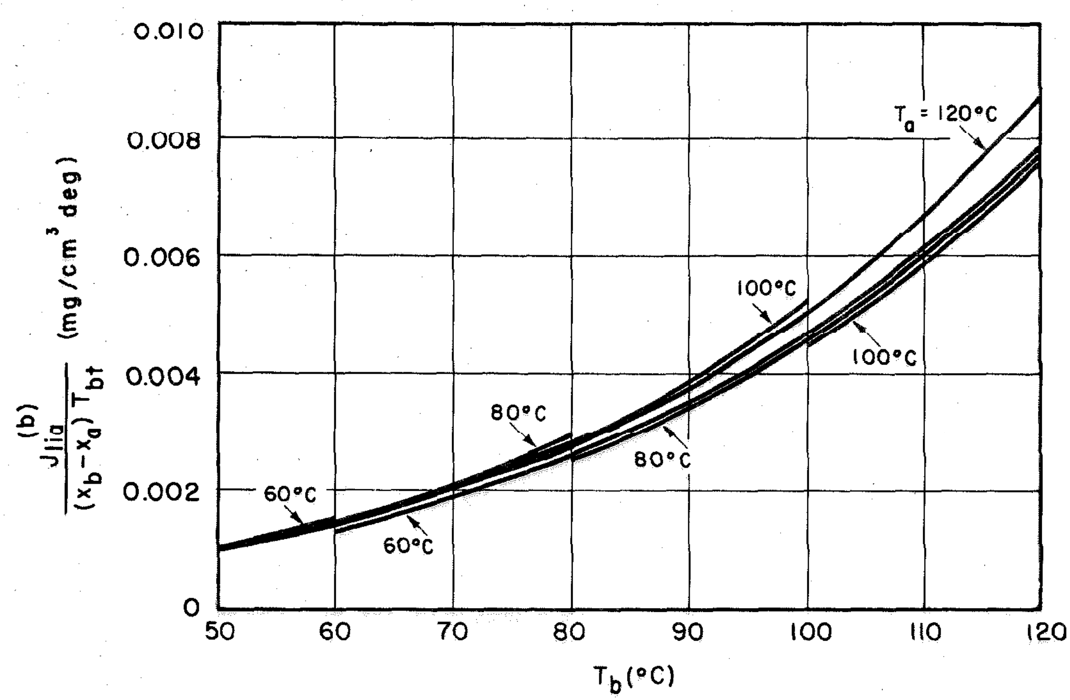

Figure 20. Approximate ($\Delta \gg 1$) Boundary Values of Perturbation Current Density (J_{1i}) due to Temperature Variation at One Liquid Surface (T_{bt}) versus Temperature of That Surface (T_b) for Water Vapor ($\rho^* = 22.6$ g/cm^3, L = 9720 cal/mol). Curves Give Values of $J_{1i}^{(b)}$ at $x = x_a$ Only; Values at $x = x_b$ Are not Sufficiently Different to Be Shown Clearly on Same Plot

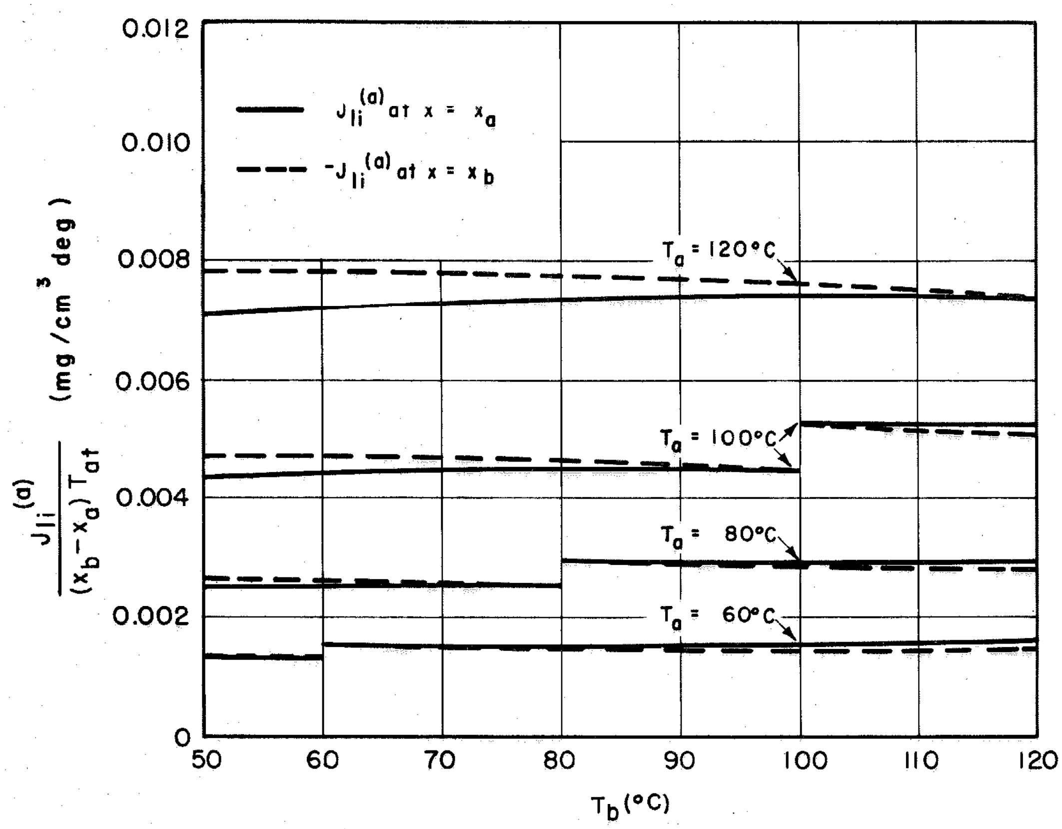

Figure 21. Approximate ($\Delta \gg 1$) Boundary Values of Perturbation Current Density (J_{li}) due to Temperature Variation at One Liquid Surface (T_{at}) versus Temperature of Other Surface (T_o) for Water Vapor ($\rho^* = 22.6$ g/cm^3, L = 9720 cal/mol)

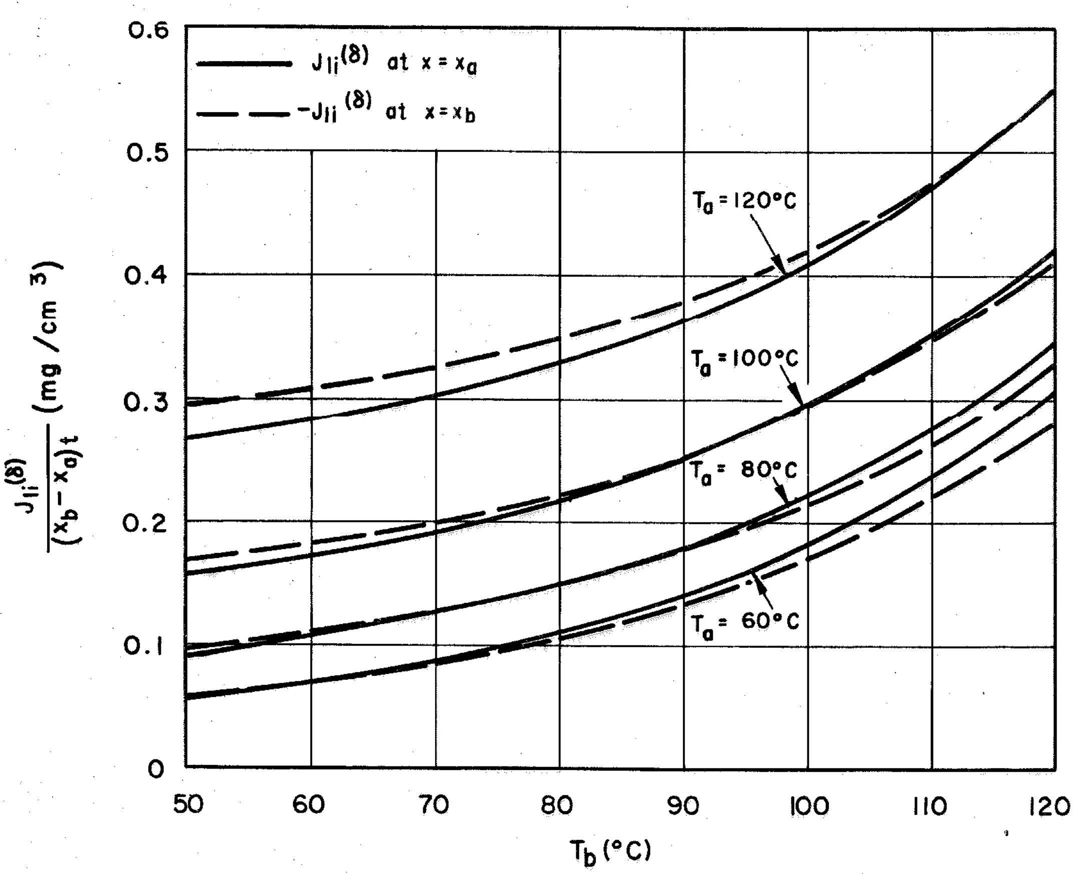

Figure 22. Approximate ($\Delta >> 1$) Boundary Values of Perturbation Current Density (J_{1i}) due to Relative Motion of Liquid Surfaces for Water Vapor ($\rho^* = 22.6$ g/cm^3)

directly in terms of appropriate physical (dimensional) quantities, and the contributions to p_1 and J_{1i} due to T_{at}, T_{bt}, and $(x_b - x_a)_t$ are indicated separately by means of the superscripts (a), (b), and (δ), respectively. From Eqs. (241) to (243), with the aid of Eqs. (239) and (71) and the fact that $\theta_c \approx \theta_a$ under the condition considered ($\Delta >> 1$), it follows that

$$J_{1ia}^{(a)} = - nJ_{1ib}^{(a)} = - \frac{J^*}{p^*} \varphi_a^{1/2} p_1^{(a)}$$

$$= T_{at}(x_b - x_a) \frac{\rho^*}{T^*} (\varphi_a \psi_o \frac{n}{1 + n}) \varphi_a (A_\psi - 1) \qquad (244)$$

$$J_{1ia}^{(b)} = - nJ_{1ib}^{(b)} = - \frac{J^*}{p^*} \varphi_a^{1/2} p_1^{(b)}$$

$$= T_{bt}(x_b - x_a) \frac{\rho^*}{T^*} (\varphi_a \psi_o \frac{n}{1 + n}) \varphi_b B_\psi \qquad (245)$$

$$J_{1ia}^{(\delta)} = - nJ_{1ib}^{(\delta)} = - \frac{J^*}{p^*} \varphi_a^{1/2} p_1^{(\delta)}$$

$$= (x_b - x_a)_t \rho^* (\varphi_a \psi_o \frac{n}{1 + n}) \quad , \qquad (246)$$

where A_ψ and B_ψ are the coefficients of $\widetilde{\theta}_a$ and $\widetilde{\theta}_b$, respectively, in Eq. (240).

The discontinuities in the curves of Figs. 20 and 21 at $T_a = T_b$ are to be expected, since equality of the two surface temperatures is incompatible with the assumed condition $\Delta \gg 1$. These two temperatures can be arbitrarily near equality without invalidating the large Δ assumption, however, if the spacing $x_b - x_a$ is sufficiently great. The condition which must be met in this case is seen from Eq. (C-20) to be

$$\delta = \frac{x_b - x_a}{x^*} \gg \frac{2}{\varphi_a^{3/2} e^{-\varphi_a} \varepsilon} . \tag{247}$$

For water in the neighborhood of 100°C, for example, the condition becomes approximately

$$x_b - x_a \gg \frac{0.017}{T_a - T_b} \text{ cm} , \tag{248}$$

indicating that in this case a spacing of 1 cm is sufficient to yield $\Delta \gg 1$ for temperature differences as small as 0.1 degree.

4.5 Range of Validity of First-Order Solution

The perturbation equations solved in Section 4.2 were obtained formally in Section 4.1 as the first step in an infinite sequence of successive approximations. Thus one way to establish the range of validity of the first-order solution would be to calculate the second-order perturbation quantities and require that they be small compared to the corresponding first-order quantities. A considerably simpler procedure, however, is to estimate the magnitude of neglected terms on the assumption that the first-order perturbation is in fact the entire perturbation.

To carry out this second procedure, Eqs. (160) to (163) with all terms for $n \geq 2$ dropped, that is

$$j = j_o + j_1 \tag{249}$$

$$\psi = \psi_o + \psi_1 \tag{250}$$

$$\theta = \theta_o + \theta_1 \tag{251}$$

$$r = r_o + r_1 \quad , \tag{252}$$

are substituted in Eqs. (51) to (54) with the terms involving M^2 neglected. The result of this substitution is

$$j_{o\xi} + j_{1\xi} = -(r_{o\tau} + r_{1\tau}) \tag{253}$$

$$\psi_{o\xi} + \psi_{1\xi} = -\sigma(j_{o\tau} + j_{1\tau}) \tag{254}$$

$$\theta_{o\xi\xi} - j_o\theta_{o\xi} + \theta_{1\xi\xi} - (j_o + j_1)\theta_{1\xi}$$

$$= j_1\theta_{o\xi} + (r_o+r_1)(\theta_{o\tau} + \theta_{1\tau}) - \frac{\gamma - 1}{\gamma}(\psi_{o\tau} + \psi_{1\tau}) \quad (255)$$

$$\psi_o + \psi_1 = (r_o + r_1)(\theta_o + \theta_1) \quad . \quad (256)$$

In view of the zero-order Eqs. (169) to (171), Eqs. (253) to (256) reduce to

$$j_{1\xi} = -(r_{o\tau} + r_{1\tau}) \quad (257)$$

$$\psi_{1\xi} = -\sigma(j_{o\tau} + j_{1\tau}) \quad (258)$$

$$\theta_{1\xi\xi} - (j_o + j_1)\theta_{1\xi} = j_1\theta_{o\xi} + (r_o + r_1)(\theta_{o\tau} + \theta_{1\tau})$$

$$- \frac{\gamma - 1}{\gamma}(\psi_{o\tau} + \psi_{1\tau}) \quad (259)$$

$$r_1 = \frac{\psi_1 - r_o\theta_1}{\theta_o + \theta_1} = \frac{\psi_1\theta_o - \psi_o\theta_1}{\theta_o(\theta_o + \theta_1)} \quad . \quad (260)$$

Equations (257) to (260) are the exact equations for j_1, ψ_1, θ_1, and r_1 when these quantities are interpreted in the sense of Eqs. (249) to (252). The significance of the first-order solution developed in Section 4.2 can therefore be estimated by comparing Eqs. (172) to (175), from which it was obtained, with the corresponding

Eqs. (257) to (260). It is seen that Eq. (172) approximates Eq. (257), and thus yields a useful expression for j_1, when

$$|r_{1\tau}| \ll |r_{o\tau}| \quad . \tag{261}$$

Similarly Eq. (173) approximates Eq. (258), and thus yields a useful expression for ψ_1, when

$$|j_{1\tau}| \ll |j_{o\tau}| \; . \tag{262}$$

There are two alternative situations under which Eq. (174) approximates Eq. (259) and thus yields a useful expression for θ_1. The first, described by the inequalities

$$|j_1| \ll |j_o| \tag{263}$$

$$|r_1 \theta_{o\tau} + (r_o + r_1)\theta_{1\tau} - \frac{\gamma-1}{\gamma}\psi_{1\tau}| \ll |r_o \theta_{o\tau} - \frac{\gamma-1}{\gamma}\psi_{o\tau}| \quad , \tag{264}$$

is of significance when j_o is large. The second, in which Eq. (263) is replaced by

$$|j_1 \theta_{1\xi}| \ll |r_o \theta_{o\tau} - \frac{\gamma-1}{\gamma}\psi_{o\tau}| = 0(\psi_{o\tau}) \quad , \tag{265}$$

is found useful as j_o tends to zero. The condition (264) which is common to both cases generally is assured if

$$|r_1| \ll |r_o| \tag{266}$$

$$|\theta_{1\tau}| \ll |\theta_{o\tau}| \tag{267}$$

$$|\psi_{1\tau}| \ll |\psi_{o\tau}| \quad . \tag{268}$$

The condition (266), in turn, is assured if

$$|\psi_1| \ll |\psi_o| \tag{269}$$

$$|\theta_1| \ll |\theta_o| \quad . \tag{270}$$

Due to the smoothness of the functions involved, the conditions (261), (262), (267), and (268) can be expected to hold when the corresponding relationships between the undifferentiated quantities apply. It is important to observe, however, that the condition (262) also holds in many cases for which (263) does not apply, since small values of j_o usually arise from a critical balance between large component currents flowing in the two directions, no corresponding cancellation necessarily appearing in the derivative $j_{o\tau}$. Thus, apart from isolated singular situations, the condition (262) can be expected to hold when the intrinsic first-order current density is small compared to either <u>component</u> of the unperturbed current density. It is seen from Eqs. (181), (182), and (218) that the ratios of j_{1i} to the unperturbed condensation current density at the two liquid surfaces are given by

$$\frac{j_{1ia}}{-\varphi_a^{1/2}\psi_o} = \frac{\psi_{1a}}{\psi_o} \; ; \qquad \frac{j_{1ib}}{\varphi_b^{1/2}\psi_o} = \frac{\psi_{1b}}{\psi_o} \; . \tag{271}$$

It thus appears that the condition (262) can be expected to hold when (269) applies.

In summary of the arguments presented so far, it appears that the first-order solution for ψ_1 developed in Section 4.2 is useful when

$$\left|\frac{\psi_1}{\psi_o}\right| \ll 1 \; , \tag{272}$$

that the solution for j_1 is useful when in addition

$$\left|\frac{\theta_1}{\theta_o}\right| \ll 1 \; , \tag{273}$$

and that the solution for θ_1 is useful when either

$$\left|\frac{j_1}{j_o}\right| \ll 1 \tag{274}$$

or

$$|j_1\theta_{1\xi}| \ll |\psi_{o\tau}| \tag{275}$$

holds as well.

It having been established that the first-order solution for ψ_1 can be expected to be significant whenever $|\psi_1/\psi_o| \ll 1$, the next step is to investigate the conditions under which (273) and one or the other of (274) and (275) hold. Comparison of Eqs. (237) and (238), which apply when j_o (and thus Δ) is large, reveals that in this case the maximum value of θ_1/θ_a in the interval $0 \leqslant y \leqslant 1$ is of the same order as or less than the maximum value of j_{1i}/j_o in the interval. Also it is seen from Eqs. (243) and (239) that if s differs appreciably from unity (as it does when j_o is large), the maximum value of j_{1i}/j_o is of the same order as ψ_1/ψ_o. Thus when j_o is large, the entire first-order solution can be expected to be significant when $|\psi_1/\psi_o| \ll 1$.

The situation when j_o is small presumably is typified by the limiting case $j_o = 0$, considered in Section 4.3. In this case the condition (274) can never hold, and attention must be given to (275) instead. Considering for simplicity only the dependences on $\widetilde{\theta}_a$ and $\widetilde{\theta}_b$ (a similar argument holds with respect to the δ_τ dependences) and assuming $\varphi_a \gg 1$, it follows from Eqs. (220) and (222) that*

$$|j_1|_{max} \sim \frac{\delta\varphi_a^{\,2}\, e^{-\varphi_a}}{4} |\widetilde{\theta}_a + \widetilde{\theta}_b| \tag{276}$$

$$|\theta_{1\xi}|_{max} = \frac{1}{\delta}|\theta_{1y}|_{max} \sim \frac{\delta\varphi_a e^{-\varphi_a}}{4} \frac{\gamma-1}{\gamma}|\widetilde{\theta}_a + \widetilde{\theta}_b| \quad . \tag{277}$$

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* The symbol $\sim$ as used here means "is of the order of".

Equation (222) also yields, since $\theta_o = \theta_a$ when $j_o = 0$,

$$\left|\frac{\theta_1}{\theta_o}\right| \sim \frac{\delta^2 \varphi_a^{\;2} e^{-\varphi_a}}{4} \frac{\gamma - 1}{\gamma} \left|\tilde{\theta}_a + \tilde{\theta}_b\right| \quad , \tag{278}$$

while Eqs. (C-17) and (C-19) give

$$|\psi_{o\tau}| = |\psi_o \tilde{\psi}_o| = \frac{\varphi_a e^{-\varphi_a}}{2} \left|\tilde{\theta}_a + \tilde{\theta}_b\right| \quad . \tag{279}$$

It follows immediately from Eqs. (276) to (279) that when $j_o = 0$,

$$|j_1 \theta_{1\xi}|_{max} \sim \frac{1}{2} \left|\frac{\theta_1}{\theta_o}\right| |\psi_{o\tau}| \quad , \tag{280}$$

so that in this case the condition (275) is satisfied if (273) is. It appears, therefore, that the first-order solutions for j_1 and θ_1 are useful whenever $|\psi_1/\psi_o| \ll 1$ and $|\theta_1/\theta_o| \ll 1$, regardless of the magnitudes of j_o and j_1/j_o. It is seen from the different powers of δ appearing in Eqs. (221) and (222) that the conditions on ψ_1/ψ_o and θ_1/θ_o just stated are independent when j_o is small.

4.6 Qualitative Behavior of First-Order Solution

The general features of the first-order solution for low-velocity nonsteady flow developed above are easily seen from the simplified forms obtained for the conditions $j_o = 0$ (Section 4.3) and $j_o \gg \delta^{-1}$ (Section 4.4). These forms can be simplified still further for qualitative purposes by assuming $\varphi_a \gg 1$, whereupon the solution for $j_o = 0$ becomes

$$\psi_1 \approx -\delta \frac{\varphi_a^{1/2} e^{-\varphi_a}}{2} \left[\frac{\varphi_a}{2} (\tilde{\theta}_a + \tilde{\theta}_b) + \tilde{\delta} \right] \tag{281}$$

$$j_{1i} \approx \delta \frac{\varphi_a e^{-\varphi_a}}{2} \left[\frac{\varphi_a}{2} (\tilde{\theta}_a + \tilde{\theta}_b) + \tilde{\delta} \right] (2y - 1) \tag{282}$$

$$\theta_1 \approx \delta^2 \frac{\gamma - 1}{\gamma} e^{-\varphi_a} \left[\frac{\varphi_a}{2} (\tilde{\theta}_a + \tilde{\theta}_b) \right] \frac{y(1 - y)}{2} , \tag{283}$$

and the solution for $j_o \gg \delta^{-1}$ ($\Delta \gg 1$) becomes

$$\psi_1 \approx -\delta \frac{n}{1 + n} \varphi_a^{1/2} e^{-\varphi_a} \left[\tilde{\psi}_o + \tilde{\delta} \right] \tag{284}$$

$$j_{1i} \approx \delta \frac{n + s}{1 + n} \frac{\varphi_a e^{-\varphi_a}}{2} \left[\tilde{\psi}_o + \tilde{\delta} \right] \left(2y - \frac{2}{1 + n}\right) \tag{285}$$

$$\theta_1 \approx \delta^2 \frac{n+s}{1+n} \frac{\gamma-1}{\gamma} e^{-\varphi_a} \left[\tilde{\psi}_o \right] \frac{1}{\Delta} \left(\frac{1-e^{-\Delta y}}{1-e^{-\Delta}} - y \right) + O(\Delta y e^{-\Delta y}). \tag{286}$$

An expression for $\tilde{\psi}_o$ is given in Eq. (240). It is easily verified that Eqs. (284) to (286) reduce to the corresponding Eqs. (281) to (283) as j_o tends to zero.

It is seen from Eqs. (281) and (284) that when the equilibrium vapor pressure $(p^*\psi_o)$ is changing slowly, the actual pressure $[p^*(\psi_o + \psi_1)$ to first order$]$ lags behind by an amount which to first order is uniform throughout the vapor-occupied region and proportional to the liquid surface spacing. The actual vapor pressure also is affected approximately uniformly by slow relative motion of the liquid surfaces, the pressure increasing when the surfaces are approaching each other and decreasing when they are receding, as would be expected.

It is seen from Eqs. (282) and (285) that the normalized perturbation current density is of the same order of magnitude as the normalized perturbation pressure but, unlike the latter, varies approximately linearly over the distance between the two liquid surfaces. This linear variation is easily understood as a response to the uniform pressure perturbation, a negative ψ_1, for example, causing perturbation currents to flow into the vapor region from both liquid surfaces. The rapidity with which pressure differences are equalized (as evidenced by the uniformity of ψ_1) accounts for the linear attenuation of these perturbation currents. The point where the resultant perturbation current is zero moves from the midpoint between the surfaces toward the warmer surface as the fractional temperature difference between the surfaces is increased from zero.

The behavior of the perturbation temperature distribution is indicated by Eqs. (283) and (286). The abrupt exponential variation of θ_o in the immediate neighborhood of the cooler liquid surface when Δ is large is matched by similar exponential behavior of θ_1, while the near constancy of θ_o elsewhere under the same condition is accompanied by an essentially linear variation of θ_1. In general, θ_1 attains its maximum magnitude at a point closer to the cooler than to the warmer liquid surface. In the limit $j_o = 0$ the space variation of θ_1 becomes symmetric with respect to the two surfaces, of course, assuming a parabolic form. It also will be observed that as j_o tends to zero, the perturbation temperatures become proportional to the square of the distance between the liquid surfaces.

From Eqs. (283) and (286), the actual temperatures away from the liquid boundaries [$T^*(\theta_o + \theta_1)$ to first order] are seen to lead the corresponding equilibrium temperatures ($T^*\theta_o$) as the latter are caused to change by slowly changing one or both of the surface temperatures. The reasonableness of this result can be seen by considering the relationship

$$\frac{p_1}{p_o} = \frac{\rho_1}{\rho_o} + \frac{T_1}{T_o} , \tag{287}$$

which holds to first order by virtue of the assumed perfect-gas behavior of the vapor.* Since $T_1 = 0$ at either liquid surface, p_1/p_o

* Equation (287) is equivalent to Eq. (175) divided by r_o.

and ρ_1/ρ_o must be equal there. In other words, the fractional amounts by which $\rho \approx \rho_o + \rho_1$ lags behind ρ_o and by which $p \approx p_o + p_1$ lags behind p_o are approximately equal at the liquid surfaces. Whereas the pressure lag has been seen to be essentially uniform throughout the vapor, however, the density lag would be expected to increase toward the interior of the vapor-occupied region due to the inertia inherent in mass transport. Equation (287), representing the perfect gas law, then indicates that T_1 also should increase in magnitude, but in the opposite algebraic sense from ρ_1 (and thus from p_1), toward the interior of the vapor.

An additional observation from Eqs. (283) and (286) is that the vapor temperature distribution, when expressed in terms of the relative space variable y, is essentially unaffected by slow motions of the liquid surfaces.

A significant effect common to all of the perturbation quantities is a considerably greater dependence on variations of the higher surface temperature ($T^*\theta_a$) than on variations of the lower ($T^*\theta_b$) when the fractional difference, ε, between these two temperatures is appreciable. This effect, which enters mathematically through the coefficients of $\tilde{\theta}_a$ and $\tilde{\theta}_b$ in Eq. (240) for $\tilde{\psi}_o$, follows from the fact that s/n tends rapidly (essentially exponentially) to zero as $\varepsilon\varphi_a$ increases beyond unity. For $\varepsilon\varphi_a$ of the order of 3 or greater, the ratio of the $\tilde{\theta}_a$ sensitivity to the $\tilde{\theta}_b$ sensitivity is between $2\varphi_a$ and $4\varphi_a$, which quantities generally are of the order of 20 or greater.

V. EXTENSION TO TWO- AND THREE-DIMENSIONAL FLOWS

In principle, a perturbation technique similar to that developed in Section 4.1 can be applied to any near-steady flow for which the corresponding steady flow problem can be solved explicitly. If the flow distribution depends significantly on two or more space coordinates, however, the perturbation equations corresponding to Eqs. (172) to (174) will be partial instead of total differential equations in the space variables and in general will not be so readily solvable. Thus the principal two- and three-dimensional flows to which the application of the present technique is fairly direct are those having cylindrical and spherical symmetry, respectively.

5.1 Flow Equations and First-Order Solution

For cylindrically symmetric two-dimensional flow, with x now designating the radial coordinate, the basic flow relations corresponding to Eqs. (20) to (22) are

$$J_x + \frac{1}{x} J = -\rho_t \tag{288}$$

$$[p(1 + \gamma M^2)]_x + \gamma M^2 \frac{p}{x} = -J_t \tag{289}$$

$$(1 - \gamma M^2)kT_{xx} + \left[(1 - \gamma M^2)\frac{k}{x} - (1 - M^2)c_p J\right]T_x$$

$$= (1 - \frac{3-\gamma}{2} M^2)c_p\rho T_t - p_t + \frac{\gamma}{2} p(M^2)_t - \gamma M^2 \frac{pu}{x} . \tag{290}$$

In the spherically symmetric three-dimensional case, the corresponding equations are

$$J_x + \frac{2}{x} J = - \rho_t \tag{291}$$

$$[p(1 + \gamma M^2)]_x + 2\gamma M^2 \frac{p}{x} = - J_t \tag{292}$$

$$(1 - \gamma M^2)kT_{xx} + [2(1 - \gamma M^2) \frac{k}{x} - (1 - M^2)c_p J]T_x$$

$$= (1 - \frac{3-\gamma}{2} M^2)c_p \rho T_t - p_t + \frac{\gamma}{2} p(M^2)_t - 2\gamma M^2 \frac{pu}{x} \quad . \tag{293}$$

It will be observed that the two cases can be considered simultaneously by writing the flow equations in the form

$$J_x + \frac{\nu}{x} J = - \rho_t \tag{294}$$

$$[p(1 + \gamma M^2)]_x + \nu\gamma M^2 \frac{p}{x} = - J_t \tag{295}$$

$$(1 - \gamma M^2)kT_{xx} + [\nu(1 - \gamma M^2) \frac{k}{x} - (1 - M^2)c_p J]T_x$$

$$= (1 - \frac{3-\gamma}{2} M^2)c_p \rho T_t - p_t + \frac{\gamma}{2} p(M^2)_t - \nu\gamma M^2 \frac{pu}{x} \quad , \tag{296}$$

where $\nu = 1$ in the cylindrically symmetric case and $\nu = 2$ in the spherically symmetric case.*

The application of Eqs. (294) to (296) corresponding to the one-dimensional problem treated in previous sections is to a vapor-filled cavity in liquid. As before, the radius and surface temperature of the cavity are considered given data, implying in effect that the mechanical and thermal dynamics of the liquid surrounding the cavity are already known (see, for example, References 13 and 14).

If the cavity radius is designated by x_a, the zero-order solution of Eqs. (294) to (296) (instantaneous equilibrium solution of the time-dependent problem) is clearly

$$J_o = 0 \tag{297}$$

$$T_o = T_a \tag{298}$$

$$p_o = p^e(T_a) \tag{299}$$

$$\rho_o = \rho^e(T_a) \quad . \tag{300}$$

---

* The general form of Eqs. (294) to (296) also includes the one-dimensional case considered at length in previous sections (by taking $\nu = 0$). The subsequent treatment in that case is somewhat different, however, since two boundary temperatures can be specified arbitrarily there while only a single such arbitrary boundary specification can be made in the two- and three-dimensional cases under consideration.

The first-order low-velocity ($\gamma M^2 \ll 1$) equations are then

$$J_{1x} + \frac{\nu}{x} J_1 = - \rho_t^e \tag{301}$$

$$p_{1x} = 0 \tag{302}$$

$$T_{1xx} + \frac{\nu}{x} T_{1x} = \frac{1}{k} (c_p \rho^e T_{at} - p_t^e) \quad , \tag{303}$$

which have the general solutions

$$J_1 = - \frac{\rho_t^e}{\nu + 1} x + \frac{f_1(t)}{x^\nu} \tag{304}$$

$$p_1 = p_1(t) \tag{305}$$

$$T_{1x} = \frac{1}{k(\nu + 1)}(c_p \rho^e T_{at} - p_t^e)x + \frac{f_2(t)}{x^\nu} \quad . \tag{306}$$

In order for J_1 and T_{1x} to remain finite at the origin with $\nu > 0$, $f_1(t)$ and $f_2(t)$ must be identically zero. Thus, since $T_{1a} = 0$,

$$J_1 = - \frac{\rho_t^e}{\nu + 1} x \tag{307}$$

$$p_1 = p_1(t) \tag{308}$$

$$T_1 = \frac{1}{2k(\nu + 1)}(c_p \rho^e T_{at} - p_t^e)(x^2 - x_a{}^2) \quad . \tag{309}$$

Finally, the first-order boundary condition

$$J_{1a} = \frac{\beta p_1}{(2\pi RT_a)^{1/2}} + \rho^e x_{at} \tag{310}$$

yields

$$p_1 = -\frac{(2\pi RT_a)^{1/2}}{\beta}\left[\frac{\rho_t^e}{\nu + 1} x_a + \rho^e x_{at}\right] . \tag{311}$$

On introduction of the approximate Clausius-Clapeyron relationship, Eq. (31), and the nondimensional notation of Section 2.3, Eqs. (307), (309), and (311) become

$$j_1 = -\frac{\xi}{\nu+1}(\varphi_a e^{-\varphi_a})_\tau = -\frac{\xi}{\nu+1}\varphi_a{}^2 e^{-\varphi_a}(\varphi_a - 1)\theta_{a\tau} \tag{312}$$

$$\theta_1 = \frac{\xi^2 - \xi_a{}^2}{2(\nu + 1)}\left[\varphi_a e^{-\varphi_a}\theta_{a\tau} - \frac{\gamma - 1}{\gamma}(e^{-\varphi_a})_\tau\right]$$

$$= \frac{\xi_a{}^2 - \xi^2}{2(\nu + 1)}\varphi_a e^{-\varphi_a}\left(\frac{\gamma - 1}{\gamma}\varphi_a - 1\right)\theta_{a\tau} \tag{313}$$

$$\psi_1 = -\varphi_a{}^{-1/2}\left[\frac{\xi_a}{\nu + 1}(\varphi_a e^{-\varphi_a})_\tau + \varphi_a e^{-\varphi_a}\xi_{a\tau}\right]$$

$$= -\varphi_a{}^{1/2} e^{-\varphi_a}\left[\frac{\xi_a}{\nu + 1}\varphi_a(\varphi_a - 1)\theta_{a\tau} + \xi_{a\tau}\right] . \tag{314}$$

The intrinsic incremental j, neglecting the component of purely kinematical origin [cf. Eq. (218)], is given by

$$j_{1i} = j_1 - r_o \xi_{a\tau} \frac{\xi}{\xi_a}$$

$$= -\xi\varphi_a e^{-\varphi_a}\left[\frac{\varphi_a(\varphi_a - 1)}{\nu + 1}\,\theta_{a\tau} + \frac{\xi_{a\tau}}{\xi_a}\right]. \qquad (315)$$

The results of sample calculations for water based on Eqs. (312) to (315) and using the notation introduced in Eqs. (244) to (246) are presented in Figs. 23 to 25.

Although derived with cylindrically and spherically symmetric flows specifically in mind, Eqs. (312) to (315) hold also for one-dimensional flow in the particular case $T_a = T_b$, if ν be taken equal to zero and distance be measured positively either way from the mid-point between the two liquid surfaces. This is easily verified by making the substitutions $\tilde{\theta}_b = \tilde{\theta}_a$, $\delta = 2\xi_a$, $2y - 1 = -\xi/\xi_a$, and $y(1 - y) = -(\xi^2 - \xi_a^{\,2})/4\xi_a^{\,2}$ in Eqs. (220) to (222). The factor of 4 by which the ordinates of Figs. 23 and 24 exceed those of Figs. 20 and 19 arises from the facts that (1) x_a in Part V corresponds to $(x_b - x_a)/2$ in Part IV, and (2) T_{at} in Part V corresponds to equal T_{at} and T_{bt} in Part IV.

5.2 Alternative Solution for Pressure Perturbation

An expression for the pressure perturbation due to motion of the cavity wall (wall temperature assumed constant) can be obtained by

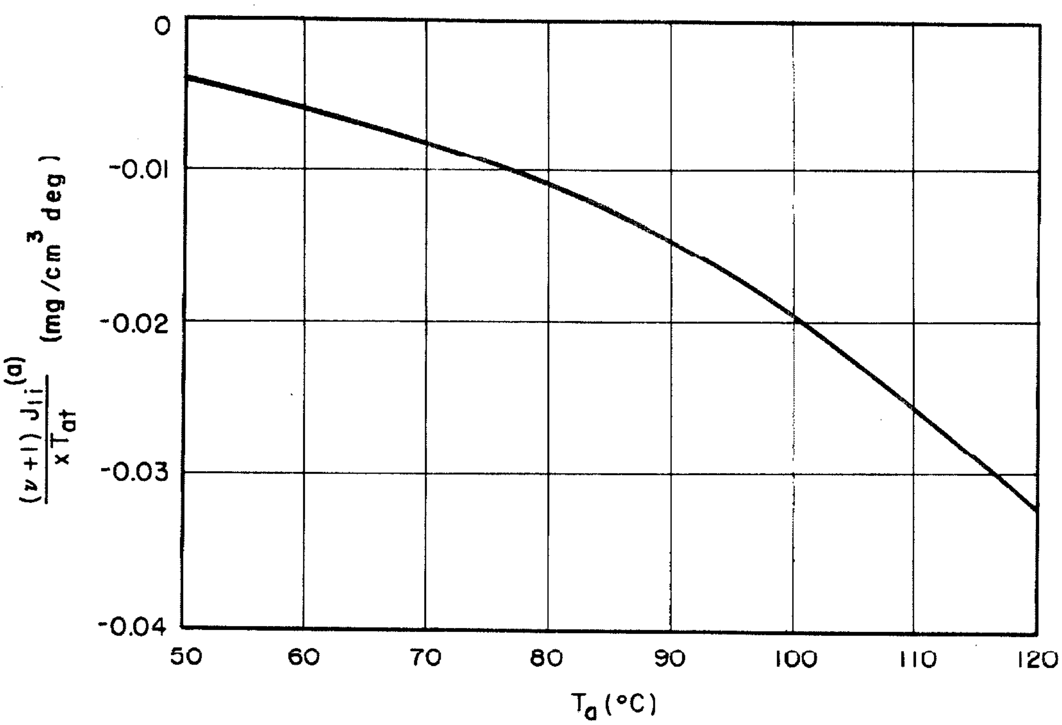

Figure 23. Perturbation Current Density in Vapor-Filled Symmetric Cavity due to Surface Temperature Variation. Substance Water ($\rho^* = 22.6$ g/cm^3, L = 9720 cal/mol)

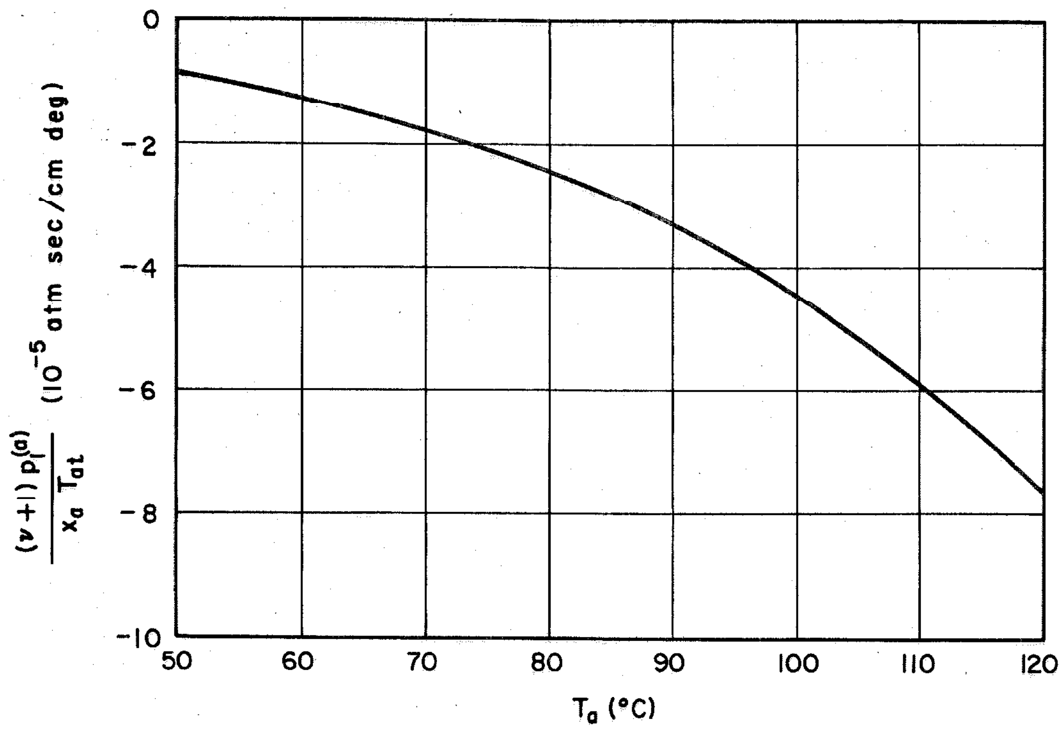

Figure 24. Perturbation Pressure in Vapor-Filled Symmetric Cavity due to Surface Temperature Variation. Substance Water ($\rho^* = 22.6$ g/cm^3, L = 9720 cal/mol, $\beta = 0.04$)

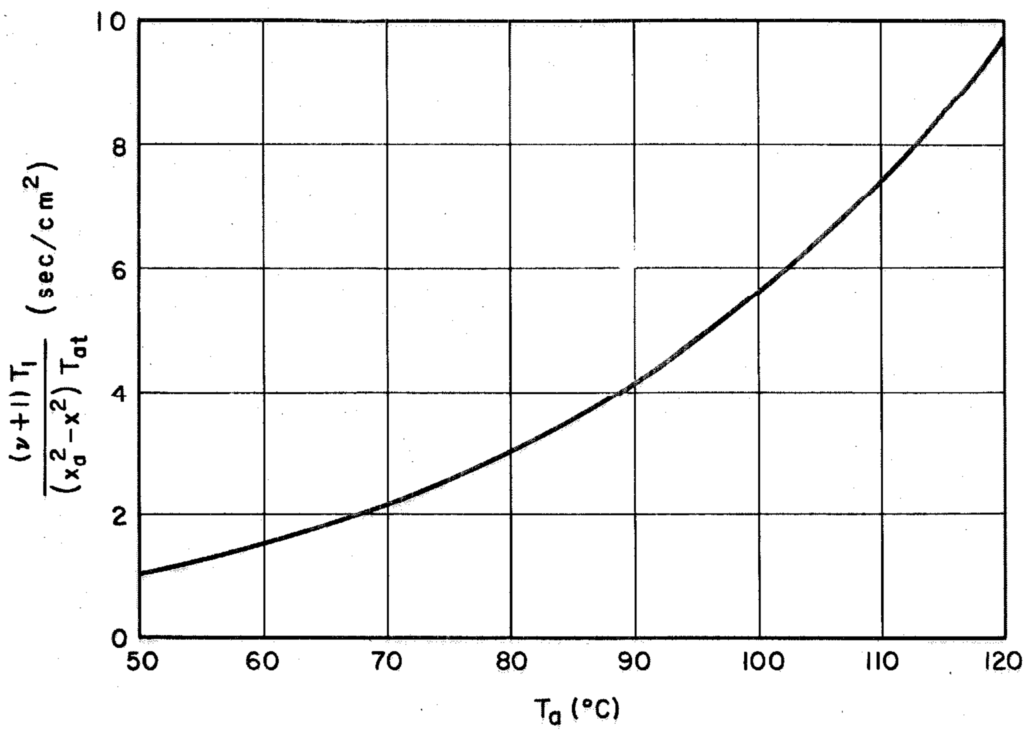

Figure 25. Perturbation Temperature in Vapor-Filled Symmetric Cavity due to Surface Temperature Variation. Substance Water ($k = 2380$ erg/cm sec deg, $c_p = 2.02 \times 10^7$ erg/g deg, $\rho^* = 22.6$ g/cm^3, $\gamma = 1.324$)

somewhat more direct physical reasoning than that employed above. This alternative solution is also of interest because it does not explicitly involve a neglect of higher order terms, although the assumptions on which it is based clearly make it non-exact.

It is assumed that the temperature throughout the cavity remains uniform and constant and that the pressure also remains uniform, though of course not necessarily equal to the instantaneous equilibrium pressure.* The perfect-gas law which the vapor is assumed to obey can then be written in the form

$$pV = mRT \ , \tag{316}$$

where V is the volume of a typical section of the cavity and m is the total mass of vapor in that volume.** Differentiation with respect to time gives

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* The assumption that p is uniform throughout the cavity implies that x_a is essentially constant except possibly over time intervals much greater than the relaxation time in the vapor. This is roughly equivalent to the assumption stated in Section 2.2 following Eq. (27).

** In the spherical case, V can be taken as the volume of the entire cavity. In the cylindrical and one-dimensional cases, V refers to an arbitrary longitudinal section and an arbitrary cylindrical section, respectively.

$$RT\dot{m} = \dot{p}V + p\dot{V}$$

$$= S(\frac{\dot{p}x_a}{\nu+1} + p\dot{x}_a) \quad , \tag{317}$$

where S is the liquid surface area bounding the volume V.

From symmetry, the current density is the same at all points on the liquid surface. Thus $\dot{m}$, the total flow of mass into the cavity, is simply

$$\dot{m} = -S(J_a - \rho_a\dot{x}_a) = S(2\pi RT)^{-1/2}(\alpha p^e - \beta p) \quad . \tag{318}$$

Elimination of $\dot{m}$ between Eqs. (317) and (318) and assumption that $\alpha = \beta$ yields

$$\frac{\dot{p}x_a}{\nu+1} + p\dot{x}_a = \beta(\frac{RT}{2\pi})^{1/2}(p^e - p) = \theta_a^{1/2}u^*(p^e - p) \quad , \tag{319}$$

which can be written in the form

$$\frac{d}{dt}\frac{p-p^e}{p^e} + (\nu+1)\frac{\dot{x}_a + \theta_a^{1/2}u^*}{x_a}\frac{p-p^e}{p^e} = -(\nu+1)\frac{\dot{x}_a}{x_a}. \tag{320}$$

If $x_a(t)$ is approximated locally by $\dot{x}_a t$, in accordance with the uniform-pressure assumption (see footnote on page 107) and an arbitrary choice of time origin, the corresponding solution of Eq. (320) is

$$\frac{p - p^e}{p^e} = -\frac{\dot{x}_a}{\dot{x}_a + \theta_a^{1/2} u^*} + \text{transient term}$$

$$= -\frac{\dot{x}_a}{\dot{x}_a + \beta(2\pi\gamma)^{-1/2} c} + \text{transient term} \qquad (321)$$

where c is the velocity of sound in the vapor. The steady-state term in Eq. (321) is plotted as a function of $\dot{x}_a/c$ for water in Fig. 26.

It is of interest to compare the steady-state term in Eq. (321) with the corresponding first-order pressure solution obtained previously. For $\theta_{a\tau}$ equal to zero, Eq. (314) divided by $\psi_o = e^{-\varphi_a}$ becomes

$$\frac{\psi_1}{\psi_o} = \frac{p_1}{p^e} = -\varphi_a^{1/2} \xi_{a\tau} = -\frac{\dot{x}_a}{\theta_a^{1/2} u^*} . \qquad (322)$$

As should be expected, this result is simply the first term in an expansion of the steady-state component of Eq. (321) in powers of $\dot{x}_a/\theta_a^{1/2}u^*$.

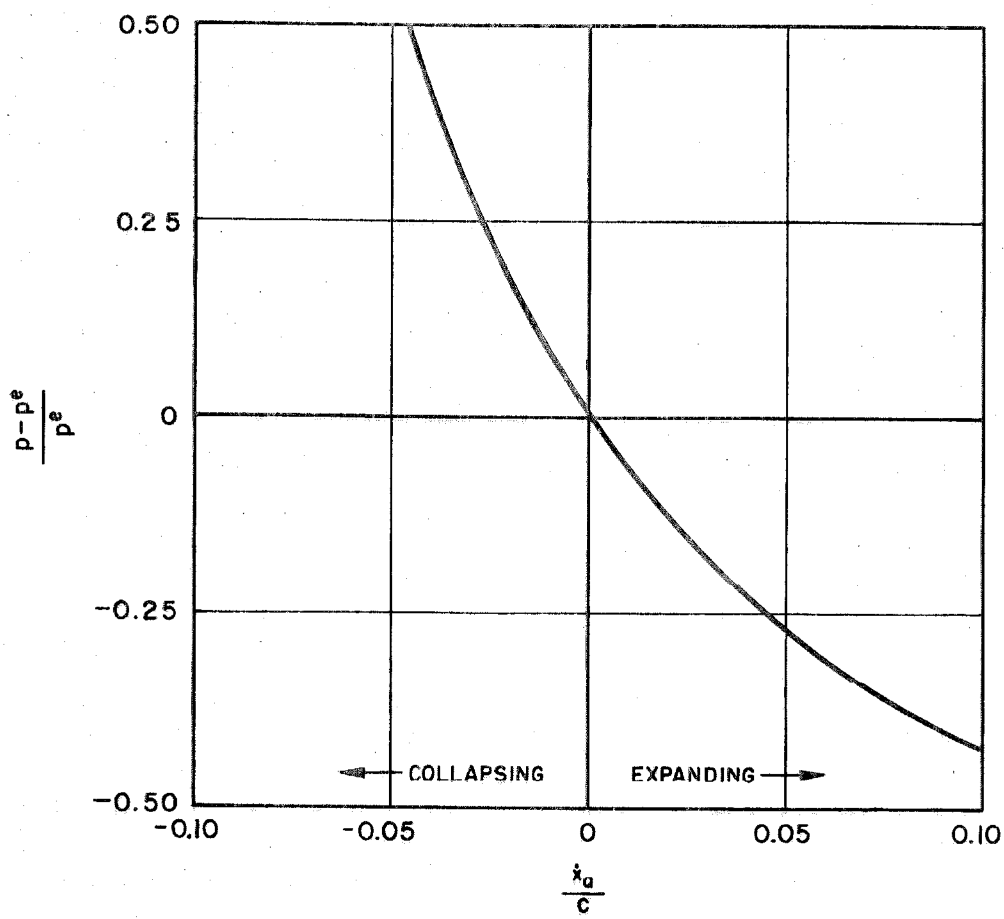

Figure 26. Steady-State Pressure Perturbation in Vapor-Filled Symmetric Cavity due to Radial Motion of Cavity Wall. Substance Water (β = 0.04, γ = 1.324, c = velocity of sound in vapor)

APPENDIX A

Properties of Water Vapor in the Neighborhood of 100°C

For water vapor at 100°C and 1 atmosphere pressure, the following values are given by the sources indicated

c_p	=	0.4820 cal/g deg	(Ref. 15, p. 1821)
	=	8.683 cal/mol deg	
	=	2.017×10^7 erg/g deg	
k	=	2380 erg/cm sec deg	(Ref. 16)
L	=	9720 cal/mol	(Ref. 2, p. 118)
	=	2.258×10^{10} erg/g	
R	=	1.986 cal/deg mol	(Ref. 2, p. 398)
α	=	0.04	(Ref. 17)
γ	=	1.324	(Ref. 15, p. 1821)
ρ^e	=	5.98×10^{-4} g/cm^3	(Ref. 15, p. 1906)

On the basis of the data listed above and the assumption that $\alpha = \beta$, the values of the normalizing factors and the dimensionless parameter σ introduced in Section 2.3 are calculated below for water vapor in the neighborhood of 100°C.

$$T^* = \frac{L}{R} = 4.89 \times 10^3 \ {}^\circ K$$

$$u^* = \beta\left(\frac{L}{2m}\right)^{1/2} = 2.4 \times 10^3 \text{ cm/sec}$$

$$\rho^* = \frac{\alpha T}{\beta T^*} \rho^e e^{T^*/T} = 22.6 \text{ g/cm}^3$$

$$J^* = \rho^* u^* = 5.4 \times 10^4 \text{ g/cm}^2 \text{ sec}$$

$$p^* = \rho^* L = 5.1 \times 10^{11} \text{ dyne/cm}^2$$
$$= 4.5 \times 10^5 \text{ atm}$$

$$x^* = \frac{k}{c_p J^*} = 2.2 \times 10^{-9} \text{ cm}$$

$$t^* = \frac{x^*}{u^*} = 9.1 \times 10^{-13} \text{ sec}$$

$$\sigma = \frac{\beta^2}{2\pi} = 2.6 \times 10^{-4} \quad .$$

It also is desired to evaluate for water vapor the parameter ς, defined in Eq. (152). For this purpose the following additional physical constants are required:

$$b = 30.52 \text{ cm}^3\text{/mol} \qquad \text{(Ref. 2, p. 14)}$$

$$n_A = 6.064 \times 10^{23} \text{ molecules/mol} \qquad \text{(Ref. 2, p. 398)}$$

$$v_o = 2.241 \times 10^4 \text{ cm}^3\text{/mol} \qquad \text{(Ref. 2, p. 398)}$$

In Eq. (152), φ_o and φ_1 represent the values of T^*/T corresponding to 0^oC and the boiling temperature at atmospheric pressure, respectively. In the case of water these quantities assume the values

$$\varphi_o = 17.9$$

$$\varphi_1 = 13.1$$

Substitution in Eq. (152) yields

$$\zeta = \frac{2^{1/6}\, v_o \varphi_o e^{-\varphi_1}}{x^*(9\pi n_A b^2)^{1/3}} = 0.17 \quad .$$

APPENDIX B

Solution of Linearized Equations for Low-Velocity Nonsteady One-Dimensional Flow

The general solution of the first-order linearized flow equations (172) to (175) is given in Eqs. (187) to (189) as

$$j_1 = -\frac{\partial}{\partial\tau}\int\frac{r_o}{j_o}\,dz + f_1'(\tau) \tag{B-1}$$

$$\psi_1 = -\sigma\frac{j_{o\tau}}{j_o}z + f_2'(\tau) \tag{B-2}$$

$$\theta_1 = \frac{1}{j_o^2}\int e^z\int e^{-z}\,R\,dz^2 + f_3'(\tau)e^z + f_4'(\tau) \tag{B-3}$$

where

$$R = j_1 j_o \theta_{oz} + r_o\theta_{o\tau} - \frac{\gamma-1}{\gamma}\psi_{o\tau} \;. \tag{B-4}$$

The quantities r_o, j_o, θ_o, and ψ_o involved in the right-hand members are given by Eqs. (164) to (167).

B-1. Equation for j_1

With the substitution

$$\omega = \frac{r_c}{j_o} , \tag{B-5}$$

Eq. (B-1) can be written as

$$j_1 - f_1' = - \frac{\partial}{\partial \tau}\left[\omega \int \frac{dz}{1 - \mu e^z}\right]$$

$$= - \frac{\partial}{\partial \tau}\left\{\omega\left[z - \log(1 - \mu e^z)\right]\right\} . \tag{B-6}$$

In evaluating and expressing the derivative of a product such as that involved in Eq. (B-6), it is convenient to make use of logarithmic differentiation. A tilde (~) will be used to denote the logarithmic derivative with respect to τ, defined for any function $f(\tau)$ by

$$\tilde{f} = \frac{f_\tau}{f} = (\log f)_\tau . \tag{B-7}$$

With the observation from Eq. (168) that

$$z_\tau = \tilde{j}_o z - j_o \xi_{b\tau} , \tag{B-8}$$

Eq. (B-6) then yields

$$j_1 - f_1{}' = -\omega\Big[\tilde{\omega}z + z_\tau - \tilde{\omega}\log(1 - \mu e^z)$$

$$+ \frac{\mu e^z}{1 - \mu e^z}(\tilde{\mu} + z_\tau)\Big]$$

$$= -\omega\Big[\tilde{r}_c z - \tilde{\omega}\log(1 - \mu e^z)$$

$$+ \frac{\mu e^z}{1 - \mu e^z}(\tilde{\mu} + \tilde{j}_o z) - \frac{j_o \xi_{b\tau}}{1 - \mu e^z}\Big] \quad . \qquad \text{(B-9)}$$

In order to facilitate the introduction of boundary conditions later on, it is convenient to collect those terms of Eq. (B-9) which tend to zero as z becomes negatively large into a function G(z) which is designed to vanish at z = 0:

$$G(z) = \tilde{\omega}\log(1 - \mu e^z) - \tilde{\omega}\log(1 - \mu) - (\tilde{\mu} + \tilde{j}_o z)\frac{\mu e^z}{1 - \mu e^z}$$

$$+ \tilde{\mu}\frac{\mu}{1 - \mu}$$

$$= \tilde{\omega}\log\left(1 + \mu\frac{1 - e^z}{1 - \mu}\right) + \tilde{\mu}\frac{\mu(1 - e^z)}{(1 - \mu)(1 - \mu e^z)}$$

$$- \tilde{j}_o\frac{\mu z e^z}{1 - \mu e^z} \quad . \qquad \text{(B-10)}$$

With this substitution, Eq. (B-9) becomes

$$\boxed{j_1 = \omega[-\tilde{r}_c z + G(z)] + r_o\xi_{b\tau} + [f_1' - \omega G(-\infty)]} \quad . \qquad \text{(B-11)}$$

In view of the relationship

$$\frac{\mu}{1-\mu}\tilde{\mu} = \tilde{\theta}_c - \tilde{\theta}_b \qquad \text{(B-12)}$$

which follows from Eq. (68), the function G(z) alternatively can be written as

$$G(z) = \tilde{\omega}\log(1 + \mu\frac{1-e^z}{1-\mu}) + (\tilde{\theta}_c - \tilde{\theta}_b)\frac{1-e^z}{1-\mu e^z}$$

$$- \tilde{j}_o \frac{\mu z e^z}{1-\mu e^z} \quad . \qquad \text{(B-13)}$$

B-2. Equation for ψ_1

On introduction of the logarithmic derivative notation, Eq. (B-2) becomes

$$\boxed{\psi_1 = -\sigma\tilde{j}_o z + f_2'} \quad . \qquad \text{(B-14)}$$

B-3. Equation for θ_1

On evaluation of θ_{oz} from Eq. (166), Eq. (B-4) becomes

$$R = -j_1 j_o \theta_c \mu e^z + r_o \theta_o \tilde{\theta}_o - \frac{\gamma - 1}{\gamma} \psi_o \tilde{\psi}_o$$

$$= \psi_o\left(-\frac{j_1}{\omega} \mu e^z + \tilde{\theta}_o - \frac{\gamma - 1}{\gamma} \tilde{\psi}_o\right) \quad . \qquad \text{(B-15)}$$

From Eq. (B-6), j_1 can be expressed in the form

$$-\frac{j_1}{\omega} = \tilde{\omega}[z - \log(1 - \mu e^z)] + z_\tau - (\tilde{\theta}_o - \tilde{\theta}_c) - \frac{f_1'}{\omega} , \qquad \text{(B-16)}$$

which on substitution into Eq. (B-15) yields

$$e^{-z}R = \psi_o\left\{\mu\tilde{\omega}[z - \log(1 - \mu e^z)] + \mu z_\tau + (\tilde{\theta}_o - \tilde{\theta}_c)\frac{1 - \mu e^z}{e^z}\right.$$

$$\left. + \frac{\tilde{\theta}_c - \frac{\gamma-1}{\gamma}\tilde{\psi}_o}{e^z} - \frac{\mu f_1'}{\omega}\right\}$$

$$= \psi_o\left\{\mu\tilde{\omega}[z - \log(1 - \mu e^z)] + \frac{\tilde{\theta}_c - \frac{\gamma-1}{\gamma}\tilde{\psi}_o}{e^z}\right.$$

$$\left. - \mu\left(\tilde{\mu} + \frac{f_1'}{\omega}\right)\right\} \qquad \text{(B-17)}$$

$$\int e^{-z} R dz = \psi_o \left\{ \mu\tilde{\omega}\left[\frac{z^2}{2} - \int \log(1-\mu e^z) dz\right] - \frac{\tilde{\theta}_c - \frac{\gamma-1}{\gamma}\tilde{\psi}_o}{e^z} \right.$$

$$\left. - \mu(\tilde{\mu} + \frac{f_1{}'}{\omega}) z \right\} \quad . \qquad \text{(B-18)}$$

On introduction of the convenient abbreviation

$$H(z) = \frac{z^2}{2}\mu e^z - \int_0^z \mu e^z \int_{-\infty}^z \log(1 - \mu e^z) dz^2 \quad , \qquad \text{(B-19)}$$

Eq. (B-18) can be written as

$$\int e^{-z} R dz = \psi_o\left[\tilde{\omega} e^{-z} H_z - \mu(\tilde{\omega} + \tilde{\mu} + \frac{f_1{}'}{\omega}) z + \frac{\frac{\gamma-1}{\gamma}\tilde{\psi}_o - \tilde{\theta}_c}{e^z}\right] . \qquad \text{(B-20)}$$

It follows that

$$\int e^z \int e^{-z} R dz^2 = \psi_o\left[\tilde{\omega} H - \mu(\tilde{\omega} + \tilde{\mu} + \frac{f_1{}'}{\omega})(z - 1)e^z\right.$$

$$\left. + (\frac{\gamma - 1}{\gamma}\tilde{\psi}_o - \tilde{\theta}_c) z\right] \qquad \text{(B-21)}$$

$$\theta_1 = \frac{\psi_o}{j_o^2}\left\{f_4'' + (\frac{\gamma-1}{\gamma}\,\tilde{\psi}_o - \tilde{\theta}_c)z + \left[f_3'' - \mu(\tilde{\omega}+\tilde{\mu}+\frac{f_1'}{\omega})z\right]e^z + \tilde{\omega}H(z)\right\} \quad . \tag{B-22}$$

The function H(z) can be evaluated by expanding the integrand of Eq. (B-19) in powers of μe^z, which is always less than unity. Thus

$$H(z) = \frac{z^2}{2}\mu e^z + \int_\mu^{\mu e^z}\int_0^{\mu e^z}\sum_{k=1}^{\infty}\frac{(\mu e^z)^{k-1}}{k}\left[d(\mu e^z)\right]^2$$

$$= \frac{z^2}{2}\mu e^z + \sum_{k=2}^{\infty}\frac{(\mu e^z)^k}{k(k-1)^2} - \sum_{k=2}^{\infty}\frac{\mu^k}{k(k-1)^2} \quad . \tag{B-23}$$

B-4. Evaluation of Arbitrary Functions from Boundary Conditions

The arbitrary functions f_1', f_2', f_3'', and f_4'' in Eqs. (B-11), (B-14), and (B-22) can be evaluated by substituting these equations into Eqs. (180) to (182). Thus Eqs. (B-22) and (180) yield

$$f_4'' + f_3'' = 0 \tag{B-24}$$

$$f_4'' - (\frac{\gamma-1}{\gamma}\,\tilde{\psi}_o - \tilde{\theta}_c) + \left[f_3'' + \mu(\tilde{\omega} + \tilde{\mu} + \frac{f_1'}{\omega})\Delta\right]e^{-\Delta} + \tilde{\omega}H(-\Delta) = 0 \quad , \tag{B-25}$$

from which it follows that

$$f_3'' = \frac{\Delta\left[-\left(\frac{\gamma-1}{\gamma}\tilde{\psi}_o - \tilde{\theta}_c\right) + \mu\left(\tilde{\omega} + \tilde{\mu} + \frac{f_1'}{\omega}\right)e^{-\Delta}\right] + \tilde{\omega}H(-\Delta)}{1 - e^{-\Delta}} \tag{B-26}$$

$$f_4'' = -f_3'' \ . \tag{B-27}$$

Substitution of Eqs. (B-26) and (B-27) into Eq. (B-22) gives

$$\frac{j_o^2}{\psi_o}\theta_1 = \left(\frac{\gamma-1}{\gamma}\tilde{\psi}_o - \tilde{\theta}_c\right)\left(z + \Delta\frac{1 - e^z}{1 - e^{-\Delta}}\right)$$

$$- \mu\left(\tilde{\omega} + \tilde{\mu} + \frac{f_1'}{\omega}\right)\left(ze^z + \Delta e^{-\Delta}\ \frac{1 - e^z}{1 - e^{-\Delta}}\right)$$

$$+ \tilde{\omega}\left[H(z) - H(-\Delta)\frac{1 - e^z}{1 - e^{-\Delta}}\right] \ . \tag{B-28}$$

Solution of Eqs. (181) and (182) for ψ_{1a} and ψ_{1b}, respectively, and application of Eq. (B-14) yields

$$\theta_a^{1/2}\ (j_{1a} - r_{oa}\xi_{a\tau}) = -(\sigma\tilde{j}_o\Delta + f_2') \tag{B-29}$$

$$\theta_b^{1/2}\ (j_{1b} - r_{ob}\xi_{b\tau}) = f_2' \ , \tag{B-30}$$

while Eq. (B-11) gives

$$j_{1a} = \omega\left[\tilde{r}_c\Delta + G(-\Delta) - G(-\infty)\right] + r_{oa}\xi_{b\tau} + f_1' \tag{B-31}$$

$$j_{1b} = -\omega G(-\infty) + r_{ob}\xi_{b\tau} + f_1' \quad . \tag{B-32}$$

Elimination of f_2' between Eqs. (B-29) and (B-30) and substitution for j_{1a} and j_{1b} from Eqs. (B-31) and (B-32) leaves an equation in f_1' alone:

$$\omega\theta_a^{1/2}\left[\tilde{r}_c\Delta + G(-\Delta) - G(-\infty)\right] + \theta_a^{1/2} r_{oa}(\xi_{b\tau} - \xi_{a\tau})$$

$$+ \theta_a^{1/2} f_1' + \sigma\tilde{j}_o\Delta - \omega\theta_b^{1/2} G(-\infty) + \theta_b^{1/2} f_1' = 0 \ . \tag{B-33}$$

With introduction of the notation

$$\delta = \xi_b - \xi_a = \frac{\Delta}{j_o} \tag{B-34}$$

and use of Eqs. (77), (B-5), and (167), Eq. (B-33) can be solved for f_1' in the form

$$f_1' = \omega G(-\infty) - \frac{\omega}{1+n}\left[\Delta\left(\tilde{r}_c + \frac{\tilde{\delta}}{1-\mu e^{-\Delta}} + \frac{\sigma\tilde{j}_o\varphi_a^{1/2}}{\omega}\right)\right.$$

$$\left. + G(-\Delta)\right] \ . \tag{B-35}$$

The final arbitrary function f_2' can now be evaluated with the aid of the relationship

$$f_2' = \theta_b^{1/2}\left[f_1' - \omega G(-\infty)\right] , \qquad \text{(B-36)}$$

which follows from Eqs. (B-30) and (B-32).

Substitution of Eqs. (B-35) and (B-36) into Eqs. (B-11), (B-14), and (B-28) and use of Eq. (B-13) yields

$$\frac{1}{\omega} j_1 = -\tilde{r}_c z + G(z) - \frac{1}{1+n}\left[\Delta\left(\tilde{r}_c + \frac{\tilde{\delta}}{1-\mu e^{-\Delta}} + \frac{\sigma \tilde{j}_o \varphi_a^{1/2}}{\omega}\right.\right.$$

$$\left.\left. + G(-\Delta)\right] + \frac{r_o \xi_{b\tau}}{\omega} \qquad \text{(B-37)}$$

$$\frac{1}{\omega} \psi_1 = -\frac{\sigma \tilde{j}_o z}{\omega} - \theta_a^{1/2} \frac{n}{1+n}\left[\Delta\left(\tilde{r}_c + \frac{\tilde{\delta}}{1-\mu e^{-\Delta}}\right.\right.$$

$$\left.\left. + \frac{\sigma \tilde{j}_o \varphi_a^{1/2}}{\omega}\right) + G(-\Delta)\right] \qquad \text{(B-38)}$$

$$\frac{j_o}{\omega\theta_c}\,\theta_1 = -\left(ze^z + \Delta e^{-\Delta}\,\frac{1 - e^z}{1 - e^{-\Delta}}\right)\left\{\tilde{\omega}\mu[1 - \log(1 - \mu)]\right.$$

$$+ \tilde{\theta}_c - \tilde{\theta}_b - \frac{\mu}{1 + n}\left[\Delta(\tilde{r}_c + \frac{\tilde{\delta}}{1 - \mu e^{-\Delta}}\right.$$

$$\left.\left. + \frac{\sigma\tilde{j}_o\varphi_a^{1/2}}{\omega} + G(-\Delta)\right]\right\}$$

$$+ \left(\frac{\gamma-1}{\gamma}\,\tilde{\psi}_o - \tilde{\theta}_c\right)\left(z + \Delta\,\frac{1 - e^z}{1 - e^{-\Delta}}\right)$$

$$+ \tilde{\omega}\left[H(z) - H(-\Delta)\,\frac{1 - e^z}{1 - e^{-\Delta}}\right] \quad . \qquad \text{(B-39)}$$

B-5. Evaluation of Logarithmic Derivatives in Terms of $\tilde{\theta}_a$, $\tilde{\theta}_b$, and $\tilde{\delta}$

Since the given data in the present problem are the temperatures and positions of the two liquid surfaces as functions of time, it is desirable to express all of the logarithmic derivatives involved in the above equations in terms of $\tilde{\theta}_a$, $\tilde{\theta}_b$ and $\tilde{\delta}$. As a first step, expressions involving only $\tilde{\theta}_a$, $\tilde{\theta}_b$, $\tilde{\delta}$, $\tilde{\psi}_o$ and $\tilde{j}_o$ are obtained. The latter two of these are then evaluated in terms of $\tilde{\theta}_a$ and $\tilde{\theta}_b$ from Eqs. (164) and (165).

It follows directly from the defining equations (71) and (B-5) that

$$\tilde{r}_c = \tilde{\psi}_o - \tilde{\theta}_c \tag{B-40}$$

$$\tilde{\omega} = \tilde{r}_c - \tilde{j}_o = \tilde{\psi}_o - \tilde{\theta}_c - \tilde{j}_o \ . \tag{B-41}$$

From Eq. (83), with θ_a, θ_b and δ considered functions of time as discussed in Section 4.1, it can be shown that

$$\tilde{\theta}_c = \frac{\theta_a}{\theta_a - \theta_b e^{-\Delta}} \tilde{\theta}_a + \frac{-\theta_b e^{-\Delta}}{\theta_a - \theta_b e^{-\Delta}} \tilde{\theta}_b$$

$$+ \left(\frac{\theta_b e^{-\Delta}}{\theta_a - \theta_b e^{-\Delta}} - \frac{e^{-\Delta}}{1 - e^{-\Delta}}\right) \Delta \tilde{\Delta}$$

$$= \frac{\mu}{\varepsilon} \tilde{\theta}_a + \left(1 - \frac{\mu}{\varepsilon}\right)\tilde{\theta}_b - \mu \frac{\Delta e^{-\Delta}}{1 - e^{-\Delta}} \tilde{\Delta} \ . \tag{B-42}$$

On introduction of the abbreviation

$$g = \frac{\Delta e^{-\Delta}}{1 - e^{-\Delta}} \tag{B-43}$$

and observation from Eq. (B-34) that

$$\tilde{\Delta} = \tilde{j}_o + \tilde{\delta} \ , \tag{B-44}$$

Eq. (B-42) becomes

$$\tilde{\theta}_c = \frac{\mu}{\varepsilon}\tilde{\theta}_a + (1 - \frac{\mu}{\varepsilon})\tilde{\theta}_b - \mu g(\tilde{j}_o + \tilde{\delta}) \quad . \tag{B-45}$$

It remains only to evaluate $\tilde{j}_o$ and $\tilde{\psi}_o$. From Eq. (164), the former is given by

$$\tilde{j}_o = -\frac{s}{1-s}\tilde{s} - \frac{n}{1+n}\tilde{n} + (\varphi_a - \frac{1}{2})\tilde{\theta}_a$$

$$= -\frac{s}{1-s}(\varphi_b\tilde{\theta}_b - \varphi_a\tilde{\theta}_a) - \frac{n}{1+n}\,\frac{\tilde{\theta}_b - \tilde{\theta}_a}{2} + (\varphi_a - \frac{1}{2})\tilde{\theta}_a$$

$$= \left[\frac{\varphi_a}{1-s} - \frac{1}{2(1+n)}\right]\tilde{\theta}_a - \left[\frac{s\varphi_b}{1-s} + \frac{n}{2(1+n)}\right]\tilde{\theta}_b \quad . \tag{B-46}$$

Similarly Eq. (165) gives

$$\tilde{\psi}_o = \frac{1}{n+s}(n\tilde{n} + s\tilde{s}) - \frac{n}{1+n}\tilde{n} + \varphi_a\tilde{\theta}_a$$

$$= \frac{1}{n+s}\left[(\frac{n}{2} + s\varphi_b)\tilde{\theta}_b - (\frac{n}{2} + s\varphi_a)\tilde{\theta}_a\right] - \frac{n}{1+n}\,\frac{\tilde{\theta}_b - \tilde{\theta}_a}{2} + \varphi_a\tilde{\theta}_a$$

$$= \left[\frac{n\varphi_a}{n+s} - \frac{n(1-s)}{2(1+n)(n+s)}\right]\tilde{\theta}_a + \left[\frac{s\varphi_b}{n+s} + \frac{n(1-s)}{2(1+n)(n+s)}\right]\tilde{\theta}_b \quad . \tag{B-47}$$

The final expressions for the various logarithmic derivatives in terms of $\widetilde{\theta}_a$, $\widetilde{\theta}_b$ and $\widetilde{\delta}$ are conveniently summarized by introducing the notation

$$\widetilde{j}_o = A_j\widetilde{\theta}_a + B_j\widetilde{\theta}_b \tag{B-48}$$

$$\widetilde{\psi}_o = A_\psi\widetilde{\theta}_a + B_\psi\widetilde{\theta}_b \tag{B-49}$$

$$\widetilde{\theta}_c = A_\theta\widetilde{\theta}_a + B_\theta\widetilde{\theta}_b + D_\theta\widetilde{\delta} \tag{B-50}$$

$$\widetilde{r}_c = A_r\widetilde{\theta}_a + B_r\widetilde{\theta}_b + D_r\widetilde{\delta} \tag{B-51}$$

$$\widetilde{\omega} = A_\omega\widetilde{\theta}_a + B_\omega\widetilde{\theta}_b + D_\omega\widetilde{\delta} \ . \tag{B-52}$$

Equations (B-40) to (B-47) then yield

$$A_j = \frac{\varphi_a}{1 - s} - \frac{1}{2(1 + n)} \tag{B-53}$$

$$B_j = -\frac{s\varphi_b}{1 - s} - \frac{n}{2(1 + n)} \tag{B-54}$$

$$A_\psi = \frac{n(1 - s)}{n + s} A_j \tag{B-55}$$

$$B_\psi = -\frac{1 - s}{n + s} B_j \tag{B-56}$$

$$A_\theta = \mu(\frac{1}{\varepsilon} - gA_j) \quad \text{(B-57)}$$

$$B_\theta = 1 - \mu(\frac{1}{\varepsilon} - gB_j) \quad \text{(B-58)}$$

$$D_\theta = -\mu g \quad \text{(B-59)}$$

$$A_r = A_\psi - A_\theta \quad \text{(B-60)}$$

$$B_r = B_\psi - B_\theta \quad \text{(B-61)}$$

$$D_r = -D_\theta = \mu g \quad \text{(B-62)}$$

$$A_\omega = A_\psi - A_j - A_\theta \quad \text{(B-63)}$$

$$B_\omega = B_\psi - B_j - B_\theta \quad \text{(B-64)}$$

$$D_\omega = -D_\theta = \mu g \quad . \quad \text{(B-65)}$$

APPENDIX C

Form of Linearized Nonsteady-Flow Solution When Equilibrium Flow Velocity is Zero ($\theta_a = \theta_b$)

The solution of the linearized nonsteady-flow problem derived in Appendix B and summarized in Eqs. (197), (198) and (219) assumes an indeterminate form as ε, and thus j_o, tends to zero ($\theta_a \rightarrow \theta_b$), since in this limit ω, $\tilde{\omega}$, and $\tilde{j}_o$ tend to infinity while Δ tends to zero. The limiting indeterminate form is evaluated in the present appendix.

Of the various quantities involved in Eqs. (194) to (219), all but j_o, ω, $\tilde{j}_o$, $\tilde{\omega}$, z, and Δ tend to finite non-zero limits as $\varepsilon \rightarrow 0$. The dependence on ε of these remaining six quantities arises primarily from their dependence on j_o and is conveniently expressed in terms of the parameter $\Delta = j_o\delta$:

$$j_o = \frac{\Delta}{\delta} \tag{C-1}$$

$$\omega = \frac{r_c}{j_o} = \frac{\delta r_c}{\Delta} \tag{C-2}$$

$$\tilde{j}_o = \frac{j_{o\tau}}{j_o} = \frac{\delta j_{o\tau}}{\Delta} \tag{C-3}$$

$$\tilde{\omega} = \tilde{r}_c - \tilde{j}_o = \tilde{r}_c - \frac{\delta j_{o\tau}}{\Delta} \tag{C-4}$$

$$z = -\Delta y \; . \tag{C-5}$$

The next step is to expand the various basic space-dependent functions involved in powers of Δ. If μ_o and μ_o' are written for the limiting values as $\varepsilon \to 0$ of μ and $\partial\mu/\partial\Delta\Big|_{\varphi_a,\delta}$, respectively, and use is made of Eq. (C-5), the required expansions are found to be

$$\log\left(1 + \mu \frac{1-e^z}{1-\mu}\right) = \frac{\mu_o}{1-\mu_o} y\Delta - \frac{\mu_o}{(1-\mu_o)^2}\left(\frac{y^2}{2} - \frac{\mu_o'}{\mu_o} y\right)\Delta^2 + \ldots \tag{C-6}$$

$$\frac{1 - e^z}{1 - \mu e^z} = \frac{1}{1-\mu_o} y\Delta - \frac{1+\mu_o}{(1-\mu_o)^2}\left(\frac{y^2}{2} - \frac{\mu_o'}{1+\mu_o} y\right)\Delta^2 + \ldots \tag{C-7}$$

$$\frac{\mu z e^z}{1-\mu e^z} = -\frac{\mu_o}{1-\mu_o} y\Delta + \frac{\mu_o}{(1-\mu_o)^2}\left(y^2 - \frac{\mu_o'}{\mu_o} y\right)\Delta^2 + \ldots \tag{C-8}$$

$$\Delta y e^{-\Delta y} - \Delta e^{-\Delta} \frac{1 - e^{-\Delta y}}{1 - e^{-\Delta}} = -\frac{y(y-1)}{2}\Delta^2 + \frac{y(y-1)(4y+1)}{12}\Delta^3 + \ldots \tag{C-9}$$

$$\frac{1 - e^{-\Delta y}}{1 - e^{-\Delta}} - y = -\frac{y(y-1)}{2}\Delta + \frac{y(y-1)(2y-1)}{12}\Delta^2 + \ldots \tag{C-10}$$

$$H(-\Delta y) - H(-\Delta)\frac{1 - e^{z}}{1 - e^{-\Delta}} = \mu_o[1 - \log(1 - \mu_o)]\frac{y(y-1)}{2}\Delta^2$$

$$- \left\{\mu_o[1 - \log(1 - \mu_o)](4y + 1 - 6\frac{\mu_o'}{\mu_o})\right.$$

$$\left. + 2\frac{\mu_o}{1-\mu_o}(y + 1 - 3\mu_o')\right\}\frac{y(y-1)}{12}\Delta^3 + \ldots \qquad \text{(C-11)}$$

Substitution of Eqs. (C-3), (C-4), and (C-6) to (C-8) into Eq. (194) and neglect of higher order terms in Δ yields

$$\frac{G(-\Delta y)}{\Delta} = \frac{\tilde{r}_c}{\Delta}\log(1 + \mu\frac{1-e^{z}}{1-\mu}) + \frac{\tilde{\theta}_c - \tilde{\theta}_b}{\Delta}\frac{1-e^{z}}{1-\mu e^{z}}$$

$$- \frac{\tilde{j}_o}{\Delta}\left[\log(1 + \mu\frac{1-e^{z}}{1-\mu}) + \frac{\mu z e^{z}}{1-\mu e^{z}}\right]$$

$$\xrightarrow[\varepsilon\to 0]{} \frac{\tilde{r}_c\mu_o + \tilde{\theta}_c - \tilde{\theta}_b}{1 - \mu_o}y - \frac{\delta j_{o\tau}\mu_o}{2(1-\mu_o)^2}y^2 \quad . \qquad \text{(C-12)}$$

Since n = 1 in the limit under consideration, Eqs. (219) and (197) then yield similarly

$$\frac{j_{1i}}{\delta r_c} \xrightarrow[\varepsilon\to 0]{} \frac{\tilde{r}_c + \tilde{\theta}_c - \tilde{\theta}_b + \tilde{\delta}}{1 - \mu_o}(y - \tfrac{1}{2}) - \frac{\delta j_{o\tau}\mu_o}{2(1-\mu_o)^2}(y^2 - \tfrac{1}{2}) - \frac{\sigma j_{o\tau}\varphi_a^{1/2}}{2r_c} \qquad \text{(C-13)}$$

$$\frac{\psi_1}{\delta r_c} \xrightarrow[\varepsilon\to 0]{} \frac{\sigma j_{o\tau}}{r_c}(y - \tfrac{1}{2}) - \frac{\theta_a^{1/2}}{2(1-\mu_o)}\left[\tilde{r}_c + \tilde{\theta}_c - \tilde{\theta}_b + \tilde{\delta} - \frac{\delta j_{o\tau}\mu_o}{2(1-\mu_o)}\right] \quad . \qquad \text{(C-14)}$$

In evaluating the limiting form of Eq. (198), it is convenient to consider first the combined coefficient of $\widetilde{\omega}$. With the aid of Eqs. (C-9) and (C-11) (and a reminder to expand the function of μ in powers of Δ), this coefficient is found to have the form

$$H(-\Delta y) - H(-\Delta)\frac{1 - e^{-\Delta y}}{1 - e^{-\Delta}}$$

$$+ \mu[1 - \log(1 - \mu)](\Delta y e^{-\Delta y} - \Delta e^{-\Delta}\frac{1 - e^{-\Delta y}}{1 - e^{-\Delta}})$$

$$= -\frac{\mu_o}{1-\mu_o}\frac{y(y^2-1)}{6}\Delta^3 + O(\Delta^4). \quad \text{(C-15)}$$

It follows from Eqs. (198), (C-3), (C-9), (C-10), and (C-15) that

$$\frac{\theta_1}{\delta^2 r_c \theta_c} = \frac{\widetilde{r}_c - \widetilde{j}_o}{\Delta^2}\left[-\frac{\mu_o}{1-\mu_o}\frac{y(y^2-1)}{6}\Delta^3 + O(\Delta^4)\right]$$

$$+ \frac{\widetilde{\theta}_c - \widetilde{\theta}_b + O(\Delta)}{\Delta^2}\left[-\frac{y(y-1)}{2}\Delta^2 + O(\Delta^3)\right]$$

$$+ \frac{\frac{\gamma-1}{\gamma}\widetilde{\psi}_o - \widetilde{\theta}_c}{\Delta}\left[-\frac{y(y-1)}{2}\Delta + O(\Delta^2)\right]$$

$$\xrightarrow[\epsilon \to 0]{} \left[\widetilde{\theta}_b - \frac{\gamma-1}{\gamma}\widetilde{\psi}_o + \frac{\delta j_{o\tau}\mu_o}{3(1-\mu_o)}(y+1)\right]\frac{y(y-1)}{2} . \quad \text{(C-16)}$$

The limiting forms of $r_c = \psi_o/\theta_c$, $r_c\theta_c = \psi_o$, and $\widetilde{r}_c+\widetilde{\theta}_c = \widetilde{\psi}_o$ involved in Eqs. (C-13), (C-14), and (C-16) are readily evaluated by substituting $\varphi_a = \varphi_b$, $\Delta = 0$, $n = 1$, and $s = 1$ in Eqs. (80), (83), (200), (207), and (208). The results are

$$\psi_o \xrightarrow[\varepsilon\to 0]{} e^{-\varphi_a} \tag{C-17}$$

$$r_c \xrightarrow[\varepsilon\to 0]{} \varphi_a e^{-\varphi_a} (1 - \mu_o) \tag{C-18}$$

$$\widetilde{\psi}_o \xrightarrow[\varepsilon\to 0]{} \frac{\varphi_a}{2} (\widetilde{\theta}_a + \widetilde{\theta}_b) \quad . \tag{C-19}$$

The value of μ_o is found by expanding Δ in powers of ε,

$$\Delta = j_o\delta = \frac{1 - (1 - \varepsilon\varphi_a + \ldots)}{1 + (1 - \frac{\varepsilon}{2} + \ldots)} \varphi_a^{1/2} e^{-\varphi_a} \delta$$

$$= \frac{1}{2} \varphi_a^{3/2} e^{-\varphi_a} \delta\varepsilon + O(\varepsilon^2) \quad , \tag{C-20}$$

and substituting in Eq. (84) to yield

$$\mu_o \equiv \lim_{\varepsilon\to 0} \mu = \lim_{\varepsilon\to 0} \frac{\varepsilon}{1-(1-\varepsilon)[1- \frac{1}{2} \varphi_a^{3/2} e^{-\varphi_a} \delta\varepsilon + O(\varepsilon^2)]}$$

$$= \frac{1}{1 + \frac{1}{2} \varphi_a^{3/2} e^{-\varphi_a} \delta} \quad . \tag{C-21}$$

Finally the limiting form of $j_{o\tau}$, found by substituting Eqs. (205) and (206) in Eq. (199), multiplying by Eq. (164), then passing to the limit, is

$$j_{o\tau} \xrightarrow[\varepsilon \to 0]{} \lim_{\varepsilon \to 0} \left[\frac{\varphi_a^{1/2} e^{-\varphi_a}}{1 + n} (\varphi_a \tilde{\theta}_a - \varphi_b \tilde{\theta}_b) + O(\varepsilon) \right]$$

$$= \frac{1}{2} \varphi_a^{3/2} e^{-\varphi_a} (\tilde{\theta}_a - \tilde{\theta}_b) \quad . \tag{C-22}$$

From Eqs. (C-18), (C-21), and (C-22) it follows that

$$\frac{\delta j_{o\tau} \mu_o}{1 - \mu_o} \xrightarrow[\varepsilon \to 0]{} \tilde{\theta}_a - \tilde{\theta}_b \tag{C-23}$$

$$\frac{j_{o\tau} \varphi_a^{1/2} (1 - \mu_o)}{r_c} \xrightarrow[\varepsilon \to 0]{} \frac{\varphi_a}{2} (\tilde{\theta}_a - \tilde{\theta}_b) \quad . \tag{C-24}$$

On substitution for r_c, $r_c \theta_c = \psi_o$, $\tilde{r}_c + \tilde{\theta}_c = \tilde{\psi}_o$, μ_o, and $j_{o\tau}$ in accordance with Eqs. (C-17) to (C-24), Eqs. (C-13), (C-14), and (C-16) become

$$j_{1i} \xrightarrow[\varepsilon \to 0]{} \tilde{\theta}_a \frac{\delta\varphi_a e^{-\varphi_a}}{4}\left[\varphi_a(2y - 1 - \sigma) - 2y^2 + 1\right]$$

$$+ \tilde{\theta}_b \frac{\delta\varphi_a e^{-\varphi_a}}{4}\left[\varphi_a(2y-1+\sigma) + 2y^2 - 4y + 1\right]$$

$$+ \delta_\tau \frac{\varphi_a e^{-\varphi_a}}{2}(2y - 1) \qquad \text{(C-25)}$$

$$\psi_1 \xrightarrow[\varepsilon \to 0]{} - \tilde{\theta}_a \frac{\delta\varphi_a^{1/2} e^{-\varphi_a}}{4} \left\{\varphi_a\left[1 - \sigma(2y-1)\right] - 1\right\}$$

$$- \tilde{\theta}_b \frac{\delta\varphi_a^{1/2} e^{-\varphi_a}}{4} \left\{\varphi_a\left[1 + \sigma(2y-1)\right] - 1\right\}$$

$$- \delta_\tau \frac{\varphi_a^{1/2} e^{-\varphi_a}}{2} \qquad \text{(C-26)}$$

$$\theta_1 \xrightarrow[\varepsilon \to 0]{} - \tilde{\theta}_a \delta^2 e^{-\varphi_a}\left(\frac{\gamma-1}{\gamma}\,\frac{\varphi_a}{2} - \frac{1+y}{3}\right)\frac{y(y-1)}{2}$$

$$- \tilde{\theta}_b \delta^2 e^{-\varphi_a}\left(\frac{\gamma-1}{\gamma}\,\frac{\varphi_a}{2} - \frac{2-y}{3}\right)\frac{y(y-1)}{2} \quad . \qquad \text{(C-27)}$$

APPENDIX D

Estimates of Magnitudes of Logarithmic Derivative Coefficients When $\Delta \gg 1$

It is the purpose of this appendix to show, at least when φ_a is not too small, 1) that $\mu\tilde{j}_o$, $\mu\tilde{\omega}$, $\tilde{\theta}_c$, and $\tilde{\theta}_b$ are of the order of or less than $\tilde{r}_c$, 2) that the $\tilde{\theta}_a$ and $\tilde{\theta}_b$ components of $\mu\tilde{\omega}$, $\tilde{\theta}_c$, and $\tilde{\theta}_b$ are also of the order of or less than the corresponding components of $\tilde{\psi}_o$, and 3) that the $\tilde{\delta}$ components of $\mu\tilde{\omega}$, $\tilde{\theta}_c$, and $\tilde{\theta}_b$ are no larger than that of $\tilde{\theta}_c$.

For $\Delta \gg 1$, it follows from Eqs. (84) and (204) that

$$\mu \approx \varepsilon \tag{D-1}$$

$$g \ll 1 \; . \tag{D-2}$$

Since ε, μ, n, and s are all restricted to the interval zero to unity, the following estimates can then be made from Eqs. (207) to (217):

$$\left|\frac{\mu A_j}{A_\psi}\right| = \frac{\mu(n+s)}{n(1-s)} < 2\,\frac{\mu}{1-s} \approx 2\,\frac{\varepsilon}{1-s}$$

$$< 2\varphi_a^{-1}\,\frac{\varepsilon\varphi_a}{1-e^{-\varepsilon\varphi_a}} < 2\varphi_a^{-1}(1+\varepsilon\varphi_a) < 2(\varphi_a^{-1}+1) \tag{D-3}$$

$$\left|\frac{\mu B_j}{B_\psi}\right| = \mu \frac{n + s}{1 - s} < 2 \frac{\mu}{1 - s} < 2(\varphi_a^{-1} + 1) \tag{D-4}$$

$$\left|\frac{\mu (A_\omega + A_\theta)}{A_\psi}\right| = \mu \frac{s(1 + n)}{n(1 - s)} < 2 \frac{\mu}{1 - s} < 2(\varphi_a^{-1} + 1) \tag{D-5}$$

$$\left|\frac{\mu (B_\omega + B_\theta)}{B_\psi}\right| = \mu \frac{1 + n}{1 - s} < 2 \frac{\mu}{1 - s} < 2(\varphi_a^{-1} + 1) \tag{D-6}$$

$$B_\psi = \frac{s}{n + s}\left(\frac{\varphi_a}{n^2} - \frac{1}{2}\right) + \frac{1}{2(1 + n)} > \frac{1}{2(1 + n)} > \frac{1}{4} \tag{D-7}$$

$$A_\theta \approx 1 \tag{D-8}$$

$$|B_\theta| \ll 1 \tag{D-9}$$

$$|D_\theta| = |D_r| = |D_\omega| \ll 1 \quad . \tag{D-10}$$

It is seen immediately from Eq. (D-10) that the $\tilde{\delta}$ components of $\mu\tilde{\omega}$, $\tilde{\theta}_c$, and $\tilde{\theta}_b$ (μD_ω, D_θ, and 0, respectively) are equal to or less than D_θ, which is the $\tilde{\delta}$ component of θ_c.

From Eq. (207), A_ψ can be written as

$$A_\psi = \frac{n}{n + s}\left[\varphi_a - \frac{1 - s}{2(1 + n)}\right] \quad . \tag{D-11}$$

If φ_a is considerably greater than unity, then n remains essentially equal to unity for values of ε such that s differs appreciably from zero, so that in this region A_ψ is approximated by

$$A_\psi \approx \frac{\varphi_a}{1+s} - \frac{1-s}{4(1+s)} \tag{D-12}$$

$$\frac{dA_\psi}{d\varepsilon} = \frac{dA_\psi}{ds}\frac{ds}{d\varepsilon} \approx \frac{s\varphi_a(\varphi_a - \frac{1}{2})}{(1+s)^2(1-\varepsilon)^2} > 0 \ . \tag{D-13}$$

On the other hand, when s becomes negligible Eq. (D-11) reduces to

$$A_\psi \approx \varphi_a - \frac{1}{2(1+n)} \tag{D-14}$$

$$\frac{dA_\psi}{d\varepsilon} = \frac{dA_\psi}{dn}\frac{dn}{d\varepsilon} \approx -\frac{1}{4n(1+n)^2} < 0 \ . \tag{D-15}$$

It follows from Eqs. (D-13) and (D-15) that, at least when $\varphi_a \gg 1$, A_ψ has a single maximum and thus its minimum in the interval $0 \leq \varepsilon < 1$ is at one or the other of the end points. In other words,

$$A_\psi \geq \min\left\{\begin{matrix}\frac{\varphi_a}{2} \\ \\ \varphi_a - \frac{1}{2}\end{matrix}\right\} = \frac{\varphi_a}{2} \ . \tag{D-16}$$

In most cases of practical interest φ_a is of the order of ten or greater. Under the arbitrary assumption that φ_a is at least 5,* Eq. (D-16) indicates that $A_\psi \geqslant 2.5$ and Eqs. (D-5) to (D-9) yield the numerical limits

$$\left|\frac{\mu A_\omega}{A_\psi}\right| < \left|\frac{\mu(A_\omega + A_\theta)}{A_\psi}\right| + \left|\frac{\mu A_\theta}{A_\psi}\right| < 2.8 \qquad \text{(D-17)}$$

$$\left|\frac{\mu B_\omega}{B_\psi}\right| \approx \left|\frac{\mu(B_\omega + B_\theta)}{B_\psi}\right| < 2.4 \qquad \text{(D-18)}$$

$$\left|\frac{A_\theta}{A_\psi}\right| \leqslant 0.4 \qquad \text{(D-19)}$$

$$\left|\frac{B_\theta}{B_\psi}\right| \ll 1 \qquad \text{(D-20)}$$

$$\left|\frac{1}{B_\psi}\right| < 4 \ . \qquad \text{(D-21)}$$

Equations (D-17) to (D-21) demonstrate that, at least when $\varphi_a \geqslant 5$, the $\tilde{\theta}_a$ and $\tilde{\theta}_b$ components of $\mu\tilde{\omega}$, $\tilde{\theta}_c$, and $\tilde{\theta}_b$ are of the order of or less than the corresponding components of $\tilde{\psi}_o$.

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* See footnote on page 77.

In view of Eqs. (D-7) and (D-9), Eq. (213) becomes

$$B_r \approx B_\psi > \frac{1}{4} , \tag{D-22}$$

while substitution of Eqs. (D-8) and (D-16) into Eq. (212) gives

$$A_r \approx A_\psi - 1 \geqslant \frac{\varphi_a}{2} - 1 . \tag{D-23}$$

For $\varphi_a \geqslant 5$, Eq. (D-23) implies that

$$A_r \geqslant \frac{3}{2} \tag{D-24}$$

$$\frac{A_\psi}{A_r} \leqslant \frac{5}{3} \tag{D-25}$$

and Eqs. (D-3), (D-4), (D-8), (D-9), and (D-22) yield

$$\left|\frac{\mu A_j}{A_r}\right| = \frac{A_\psi}{A_r}\left|\frac{\mu A_j}{A_\psi}\right| < 4 \tag{D-26}$$

$$\left|\frac{\mu B_j}{B_r}\right| \approx \left|\frac{\mu B_j}{B_\psi}\right| < 2.4 \tag{D-27}$$

$$\left|\frac{\mu A_\omega}{A_r}\right| \leqslant \mu + \left|\frac{\mu A_j}{A_r}\right| < 5 \tag{D-28}$$

$$\left|\frac{\mu B_\omega}{B_r}\right| \leqslant \mu + \left|\frac{\mu B_j}{B_r}\right| < 3.4 \qquad \text{(D-29)}$$

$$\left|\frac{A_\theta}{A_r}\right| \approx \frac{1}{A_r} \leqslant \frac{2}{3} \qquad \text{(D-30)}$$

$$\left|\frac{B_\theta}{B_r}\right| \ll 1 \qquad \text{(D-31)}$$

$$\left|\frac{1}{B_r}\right| < 4 \quad . \qquad \text{(D-32)}$$

Equations (D-10) and (D-26) to (D-32) demonstrate that, at least when $\varphi_a \geqslant 5$, the quantities $\mu\tilde{j}_o$, $\mu\tilde{\omega}$, $\tilde{\theta}_c$, and $\tilde{\theta}_b$ are all of the order of or less than $\tilde{r}_c$.

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LIST OF SYMBOLS

Common subscripts which may be appended as appropriate to the basic symbols listed below are:

x, y, z, ξ, t, τ	Partial derivative with respect to subscript variable
a, b, c	Value at $x = x_a$, $x = x_b$, and $x = -\infty$, respectively
0, 1, 2, . . .	Order of term in perturbation expansion

A superscript asterisk (*) indicates a normalizing value (dependent on particular substance considered). A tilde ($\sim$) over a symbol indicates the logarithmic derivative with respect to τ.

Roman Letters

A	Constant value of $(p + \rho u^2)/p^*$ in steady flow; Coefficient of $\tilde{\theta}_a$
b	van der Waals constant
B	Integration constant in solution for temperature distribution; Coefficient of $\tilde{\theta}_b$
c	Velocity of sound
c_p	Specific heat of vapor at constant pressure
c_v	Specific heat of vapor at constant volume
d	Effective diameter of vapor molecules
D	Coefficient of $\tilde{\delta}$
e	Intrinsic energy of vapor per unit mass; Base of Naperian logarithms
f	Arbitrary function of τ

g	$\Delta e^{-\Delta}/(1 - e^{-\Delta})$
G(z)	Function involved in solutions for j_1, ψ_1, and θ_1
h	$\varphi_b^{1/2} \psi_o/j_o$
H(z)	Function involved in solution for θ_1
j	Normalized current density (J/J^*)
j_{1i}	Intrinsic first-order normalized current density
J	Current density
k	Thermal conductivity of vapor
K	Integration constant in expression for p^e
ℓ	Mean free path in vapor
L	Latent heat of vapor per unit mass
m	Mass of vapor in cavity
M	Mach number (u/c)
n	Square root of temperature ratio $[(T_b/T_a)^{1/2} = (1 - \varepsilon)^{1/2}]$
n_A	Avogadro's number (6.064×10^{23} molecules/mol)
N	Numerical density of vapor molecules
N_ℓ	Spacing between liquid surfaces in mean free path lengths $[(x_b - x_a)/\ell]$
p	Pressure of vapor
p^e	Equilibrium vapor pressure
r	Normalized density of vapor (ρ/ρ^*)
R	Gas constant per unit mass; Right-hand member in differential equation for θ
s	$e^{-\varepsilon\varphi_a/(1-\varepsilon)}$

S	Liquid surface area of cavity
t	Time
T	Temperature
u	Velocity of vapor
v	$w^{1/2}$
v_o	Molal volume at 0°C and 1 atm (2.2414×10^4 cm^3/mol)
V	Volume of cavity
w	$1 - \eta$
x	Distance
y	Fractional distance from cooler liquid surface $[(x_b - x)/(x_b - x_a) = (\xi_b - \xi)/\delta]$
z	Characteristic space variable $[- j(\xi_b - \xi)]$

Greek Letters

α	Evaporation coefficient
β	Accommodation coefficient
γ	Ratio of specific heats (c_p/c_v)
δ	Normalized spacing between liquid surfaces $(\xi_b - \xi_a)$
Δ	Characteristic surface-spacing parameter $(j\delta)$
ε	Fractional temperature difference $[(\theta_a - \theta_b)/\theta_a]$
ζ	Coefficient in expression for mean free path $[2^{1/6} v_o\varphi_o e^{-\varphi_1} (x^*)^{-1} (9\pi n_A b^2)^{-1/3}]$
η	Characteristic temperature variable $(\sigma j^2\theta/A^2)$
θ	Normalized temperature (T/T^*)
λ	Fractional current-density difference $[(j - j_o)/j_o]$

μ	Dimensionless temperature-difference parameter $(1 - \theta_b/\theta_c)$
ν	Parameter depending on dimensionality of flow
ξ	Normalized distance (x/x^*)
ρ	Density of vapor
ρ^e	Equilibrium vapor density
ρ_o	Vapor density at 0^oC and 1 atm
σ	$(u^*)^2/RT^* = \beta^2/2\pi$
τ	Normalized time (t/t^*)
φ	Reciprocal of normalized temperature $(\theta^{-1} = T^*/T)$
φ_o	Value of φ corresponding to 0^oC
φ_1	Value of φ corresponding to boiling temperature at atmospheric pressure
ψ	Normalized pressure (p/p^*)
ω	Reciprocal of asymptotic normalized flow velocity $(r_c/j_o = u^*/u_c)$