A SPECIAL-PURPOSE ELECTRIC ANALOG COMPUTER AND ITS APPLICATION TO THE SOLUTION OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS

Thesis by

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In Partial Fulfillment of the Requirements

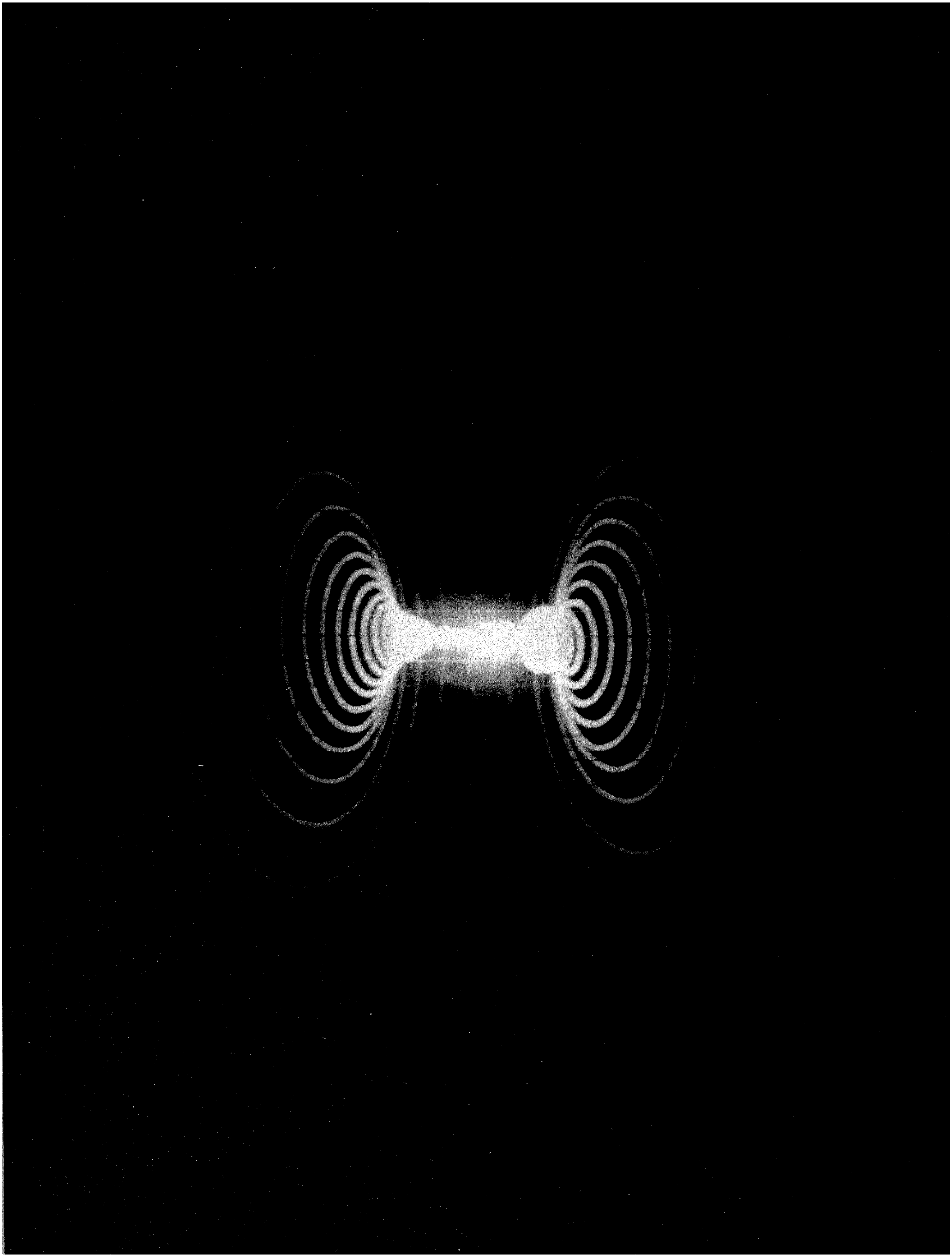
for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1953



PREFACE

The author's first interest in the field of nonlinear mechanics was aroused when he and Robert S. Macmillan undertook an investigation of nonlinear resonance phenomena under the direction of Dr. R. H. MacNeal. A rather complete experimental investigation of the types of solution peculiar to a second-order nonlinear differential equation was made. A straight-line-segment type of nonlinearity was used, and the forcing function was sinusoidal. The large-scale, general-purpose electric analog computer at the California Institute of Technology was used in this project.

Since the nonlinearities existing in nature are seldom as naive as those with which we had worked, the author was glad to have the opportunity at a later date to work with Clement J. Savant and Robert S. Neiswander on the development of a flexible arbitrary function generator. This function generator, which might be called an electronic slide rule, required first the development of a satisfactory linear-to-logarithmic converter. The literature search and the experimental development of a satisfactory converter were done cooperatively, and an article describing this work has been submitted to the magazine Electronics for publication.

After the basic development of the converter was completed, Clement Savant and the author worked together to bring the nonlinear computer, including the forementioned arbitrary function generator

PREFACE (Continued)

to its present state of development. The computer is operated most efficiently by two men, particularly because of the large amount of data taken photographically. For completeness, the theses of both Savant and the author will include descriptions of the computer; however, the attempt has been made to indicate the division of labor by content.

The author wishes to thank his wife particularly, without whose understanding and assistance this thesis would not now be written; also Dr. G. D. McCann, who directed the research; and Dr. C. H. Wilts, Dr. R. H. MacNeal, and B. H. Locanthi, all of whom gave help and guidance.

ABSTRACT

This thesis consists of three parts. (1) An analog computer investigation was made of certain phenomena that can be found among the solutions of a simple nonlinear differential equation. (2) An arbitrary function generator and associated equipment were developed to be used in the solution of more difficult nonlinear equations such as those representing nonlinear servomechanisms. This function generator operates on the logarithmic principle and can be likened to an electronic slide rule. (3) The operation of the function generator was demonstrated by forming a nonlinear computer using the generator and other active and passive circuit elements. This computer was used to investigate the solutions to several modifications of Van der Pol's equation.

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I. INTRODUCTION

The term nonlinear mechanics is misleading, for an investigation in the field of nonlinear mechanics can be completely divorced from the applications of Newtonian principles. As a matter of fact, at the present time most of the systems in which nonlinear elements are used to an advantage are electrical. The common watch and the automotive shock absorber are noteworthy exceptions. The expression nonlinear mechanics could well be replaced by the term nonlinear systems, which would convey a clearer meaning.

Why does one study nonlinear systems in general and nonlinear differential equations in particular? The answer is that one is forced to do so by nature. Whereas engineers and scientists for the most part treat each such problem as though it were linear, this approach is seldom if ever exact. The best spring can be extended so far that it takes a permanent set. Iron-cored inductors, even in their most refined state, have an inductance coefficient that is not a constant but rather a function of the current passing through. We deal daily with vacuum tubes whose most idealized characteristic curves are not linear over the ranges of common operation. The cheapest truck has overload springs. The gas laws are not valid at the extremes of temperature and pressure. The human ear and its electromechanical counterparts have response characteristics that change with sound intensity. To catalog all the nonlinearities that we know would be to catalog all of nature that we understand, for nature seems to be inherently nonlinear. We should realize that there is nothing sacred about linear systems except our ability to analyze them. A magnetic amplifier is only one example of the working of a nonlinear element to help us obtain a desired result. We must strive toward accepting the nonlinearities that we are forced to use, toward intelligently using other nonlinear elements because they are perhaps less expensive or more available, and toward using nonlinear elements in systems to improve their operating characteristics.

II. METHODS OF SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS

The problems containing nonlinearities may possibly be divided into two categories, namely, those capable of analytic solution and those not capable of analytic solution.

A. Analytical methods of solution

The problems most commonly dealt with fall into the first class.

1. Linearization. One method frequently used is merely to ignore the nonlinearity, that is, to linearize the problem. An example of a problem of this type is that provided by a simple pendulum. The differential equation describing the motion can be written in the following form:

$$\ddot{\theta} + \frac{g}{L} \sin \theta = 0 \qquad (1)$$

Now a general form of a linear differential equation of the nth order is as follows:

$$\frac{d^n x}{dt^n} + P_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots \qquad (2)$$

$$+ P_{n-1}(t)\frac{dx}{dt} + P_n(t)\,x = Q(t)$$

The dependent variable and all its derivatives appear raised, at most, only to the first power; this characteristic constitutes the definition of a linear differential equation. Referring to this definition, we see that Equation (1) does not satisfy the requirements, as sin θ cannot be expressed in terms of θ plus a constant. We can, however, express sin θ as a power series in θ, and when this is done, Equation (1) yields

$$\ddot{\theta} + \frac{g}{L}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right) = \theta \qquad (3)$$

The reader is no doubt aware of the step that is, as a rule, taken next. "If we assume that all terms beginning with $\theta^3/3!$ are small compared to θ," the text usually reads, and our problem is suddenly reduced to the familiar

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \qquad (4)$$

This approach is characteristic of that used in many problems involving nonlinearities. We make a linear approximation to the true problem as we know it, though we should realize that our best statement of the problem no doubt merely approximates the actual problem as it exists in the physical world. Having simplified the problem to the point where we can solve it analytically, we obtain a solution. Often the next step is to generalize from the particular solution obtained, all the while ignoring the fact that our base for extrapolation is none too firm. Then, having solved the problem theoretically, we go to the designing board and try to design out the original nonlinearities as much as possible, taking care not to add any new ones. We use ball bearings to cut down on Coulomb friction; we put in stops to make sure that the amplitudes are always small. In general, we try to linearize the system so that our mathematics has some relation to fact. We should not be overcritical of this method, because it has often given excellent results. Furthermore, many of the problems in which the method is not used with success are not easily attacked by any more exact method.

2. Approximate methods, small nonlinearities. More sophisticated methods of solving nonlinear differential equations with small deviations from linearity are available (Ref. 1); however, aside from predicting the existence of certain types of solutions that are unobtainable from any linear analysis, these methods are not of particular practical importance to the engineer. All of them are laborious to use, and some of them do not yield information as to wave shape. These methods have great importance theoretically, however, as their application has resulted in the prediction of effects found only in systems having nonlinear elements. Demonstration of some of these effects will be found in Sections III and V.

3. Approximate methods, certain large nonlinearities. Some extremely nonlinear systems can be analyzed. A common example is that of the thyratron sweep generator. Normally no knowledge of the behavior of thyratron conduction is required except the firing voltage and the extinguishing voltage. In the analysis, one considers a linear system until the voltage on the condenser exceeds the firing voltage. Then the voltage across the condenser is assumed to be instantly changed to the extinguishing voltage, and the cycle starts to repeat. A similar approach is very useful in the study of multivibrators, magnetic amplifiers, gas-tube flipflops, and many other practical circuits. Minorsky (Ref. 2) discusses applications of this method at some length. The gist of this method is that one assumes (1) that the system has two or more stable states and (2) that the transfer from one state to the next occurs instantly whenever the dependent variable has a particular value.

Other phenomena no doubt will be discovered as investigation into more complex nonlinear differential equations progresses.

Reviewing the discussion thus far in this section, one finds that analytical methods either suggested or referred to can be used to solve problems with either small or very large nonlinearities.

B. Methods requiring computing machines

The second category, composed of problems that cannot be dealt with by analytic means, remains to be discussed. A simple example of this type of problem is represented by the equation

$$\dot{x} + ax + \int f(x)\,dt = F\cos\omega t \tag{5}$$

when f(x) is defined in the following manner:

$$f(x) = k_1 \text{ FOR } |x| < x_0$$

$$f(x) = k_2 \text{ FOR } |x| \geqslant x_0$$

and $k_2/k_1 = 9$. The reduced equation

$$\dot{x} + ax + \int f(x)\,dt = 0 \tag{6}$$

is piecewise linear and can be solved for any set of parameters with a particular set of initial conditions. Topological methods are available (Ref. 2) for the solution of this transient problem and yield considerable information without too great an expenditure of labor. However, if one considers the complete Equation (5), none of the methods suggested is of any use except in very special cases.

Practical control systems are almost never represented by equations as simple as those discussed. As a result, the methods of solution suggested are of little more than academic interest to many engineers. Solution, however, is possible by at least three methods.

1. Numerical integration. Step-by-step numerical integration of the equations is always possible, and with the advent of the new high-speed digital computers, this method is practical for some

problems. Programming of a problem as simple as the one proposed is quite a chore, and the parameters in the equations cannot be varied easily. If one is interested only in the steady state, the necessary calculation of the complete transient is expensive and time-consuming. The accuracy of the digital computer can be made as high as is desired; indeed for some problems this device offers the only solutions having sufficient accuracy.

2. Differential analyzer solution. The differential analyzer, either mechanical or electrical, also offers solutions to Equation (5). The principal objections to the use of this analyzer are the difficulty of entering the nonlinearities into the equations and the nearly complete loss of a direct analog. The part of our learning labeled intuition is of no use when the physics is broken down into the operational-equation language of the differential analyzer. The loss of the direct analogy is not inherent in the differential analyzer, but in the nature of men. If the extra computer elements required to maintain the analogy are available, normally they will be used to obtain a closer approximation to the physics. Obtaining steady-state solutions without spending appreciable time waiting for the computer is another difficulty, particularly of the mechanical analyzer, because of its slow speed.

3. Electric analog computer solution. This computer seems to offer great promise for the immediate solution of the equations arising from nonlinear systems. In many practical systems, the governing differential equation is at least of the third order, and frequently more than one

nonlinearity is to be considered. The electric analog computer offers speed of solution, ready variation of parameters, large choice of initial conditions, and retention of the direct analog. One may consider that the last point is overemphasized, but mathematics has not yet been developed for solving problems even as simple as those used for illustrations. The engineer must use every available aid. Some men's minds work more on the analog basis than on the digital, and a man having this type of mind can gain much from the retention of the direct analog with its clear physical interpretation. The author believes that the digital machines are more suitable for other individuals, however, and that the digital and analog computers will in time be working side by side on the same type of problems at about the same rate. Many technological and methodical advances must be made before that time, however, in order to decrease the cost, complexity, and setup time of the digital machine and to increase its reliability.

III. ANALOG COMPUTER INVESTIGATION OF SIMPLE NONLINEAR SYSTEM

The author's experimental investigation in the field of nonlinear mechanics was begun with the idea of gaining some insight into the types of phenomena that occur, rather than with the idea of solving any particular problem.

A. Physical system

A very common differential equation, that representing the physical system shown in Figure 1, was chosen for investigation. The system as shown is usually assumed to be linear and thus offers no difficulty in obtaining a completely general solution for the motion, regardless of the initial conditions of velocity and displacement. The system

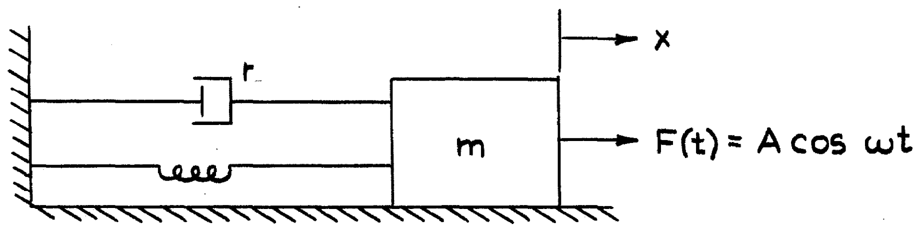


Figure 1. Simple Mechanical System

will be made nonlinear by the use of a spring whose spring constant k becomes a function of the displacement. In this case, the differential equation governing the motion of the system shown in Figure 1 becomes

$$m\ddot{x} + r\dot{x} + k(x)x = A\cos\omega t \tag{7}$$

In order to keep the problem from being unduly complicated, the only functions k(x) used were of the following types:

Type 1

$$\begin{aligned} k(x) &= k_1 \quad \text{for} \quad |x| \leqslant x_o \\ k(x) &= k_2 \quad \text{for} \quad |x| > x_o \end{aligned} \tag{8}$$

Type 2

$$\begin{aligned} k(x) &= k_1 \quad \text{for} \quad x \leqslant x_o \\ k(x) &= k_2 \quad \text{for} \quad x > x_o \end{aligned} \tag{9}$$

Type 3

$$k(x) = k_1 \quad \text{for} \quad x \leqslant 0$$
$$k(x) = k_2 \quad \text{for} \quad x > 0 \qquad (10)$$

Figure 2, whose main object is to illustrate the circuits used in the generation of the analogous electric nonlinearities, shows these functions graphically. In the presentation used in Figure 2, $k(x)x$ is shown as $f(x)$. The functions $f(x)$ all consist of straight-line sections, and the reduced equation corresponding to Equation (7) is thus piecewise linear and can be solved analytically. With a forcing function as simple as that of Equation (7), however, no analytic solution is available. The simplest of the functions $f(x)$ is of type 3, as it can be formed from only two straight-line segments.

When Equation (7) with the type 3 $f(x)$ is first examined, one is inclined to believe that a steady-state solution could be obtained analytically. Two equations representing general forms of the particular solutions can be written, one valid with x 0 and the other valid with x 0. The equating of the two sets of final and initial values of the two "half" cycles would then allow solution of all the unknown constants. Since the dependent variable x controls the change from one solution to the other, determination of the constants involves insuperable difficulties of trigonometric manipulation. Trial-and-error methods could yield approximate steady-state results without undue labor, but a solution of this type would defeat the purpose of the investigation.

The reader should realize by now that a simple differential

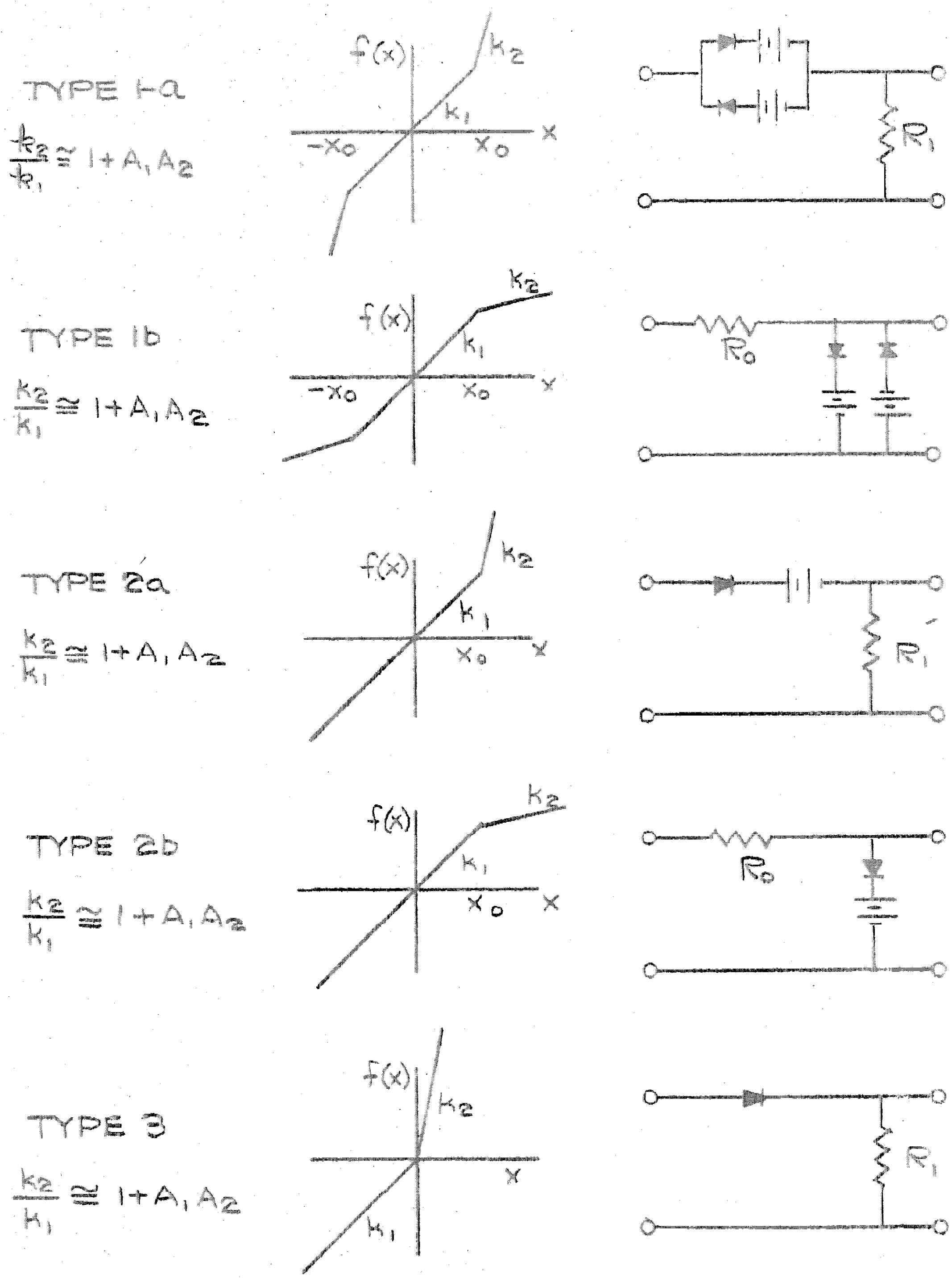


FIGURE 2

TYPES OF NONLINEARITIES AND THE ELECTRICAL CIRCUITS USED IN THEIR PRODUCTION.

equation with a very simple nonlinearity can resist solution by analytical means and that some other means of solution must be used if one is to investigate a large number of cases. As was pointed out at the beginning of this section, the object of the investigation was not the solution of particular problems but rather the development of a "feel" for the type of solutions possible with a simple equation. The electric analog computer cannot be surpassed as an educational tool when applied to this simple equation. Solutions are rapid, the presentation can be graphic, and the ease of varying parameters cannot be equaled. Most important, however, is the fact that one never loses sight of the physics in the solution. With only a small amount of experience, the operator learns to interpret the electrical display with that of the corresponding mechanical (or other) system. A capacitor that changes its capacitance as a function of its change is easily compared to an overload spring that starts to function at a certain displacement.

In order that the results obtained may have some generality, the equations of the mechanical system are reduced to dimensionless form.

Equation (7) can be reduced by making the substitutions

$$\omega_0^2 = k_1/m \tag{11}$$

$$\tau = \omega_0 t \tag{12}$$

$$y = x/x_0 \tag{13}$$

For nonlinearities of the first and second types, Equation (7) becomes

$$y'' + \frac{r}{\sqrt{k_1 m}} y' + y = \frac{A}{k_1 x_0} \cos \frac{\omega}{\omega_0} \tau \quad \text{FOR } |y| \leq 1 \tag{14}$$

$$y'' + \frac{r}{\sqrt{k_1 m}} y' + \frac{k_2}{k_1} y = \frac{A}{k_1 x_0} \cos \frac{\omega}{\omega_0} \tau \quad \text{FOR } |y| > 1 \tag{15}$$

For the type 3 nonlinearity one obtains in a similar manner

$$y'' + \frac{r}{\sqrt{k_1 m}} y' + y = \cos \frac{\omega}{\omega_0} \tau \quad \text{FOR } y \leq 0 \tag{16}$$

$$y'' + \frac{r}{\sqrt{k_1 m}} y' + \frac{k_2}{k_1} y = \cos \frac{\omega}{\omega_0} \tau \quad \text{FOR } y > 0 \tag{17}$$

Equations (16) and (17) are applicable when the nonlinearities are of the third type.

B. Electric analog

The electrical analog to be used depends upon availability of equipment and background of the experimenter. The equations of the mechanical system (Eqs. 14 and 15 or 16 and 17) can be set up electrically using either the nodal or the loop analogy. The loop analogy was used, and the problems sometimes encountered in the use of current generators with reactive loads were thus avoided.

A simplified circuit for the electric analog of the mechanical system is shown in Figure 3. The capacity of the condenser C is a

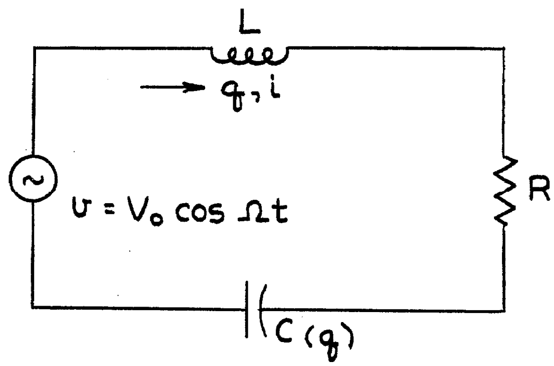


Figure 3. Basic Analog Circuit

function of the charge q.

1. Basic equations. The differential equation describing the circuit is

$$L\ddot{q} + R\dot{q} + \frac{1}{C(q)}\,q = V_0 \cos \Omega t \tag{18}$$

Equation (18) can be reduced to dimensionless form by making substitutions similar to those of Equations (11), (12), and (13).

$$\Omega_0^2 = 1/LC_1 \tag{19}$$

$$\tau = \Omega_0 t \tag{20}$$

$$Q = q/q_0 \tag{21}$$

Thus Equation (18) becomes

$$Q'' + R\sqrt{\frac{C_1}{L}}\,Q' + Q = \frac{V_0 C_1}{q_0} \cos \frac{\Omega}{\Omega_0}\tau \quad \text{FOR} \quad |Q| \leq 1 \tag{22}$$

$$Q'' + R\sqrt{\frac{C_1}{L}}\, Q' + \frac{C_1}{C_2} Q = \frac{V_0 C_1}{q_0} \cos \frac{\Omega}{\Omega_0} \tau \text{ FOR } |Q| > 1 \tag{23}$$

Equations (22) and (23) describe the system for nonlinearities of the first and second types. In a similar manner the following equations are obtained:

$$Q'' + R\sqrt{\frac{C_1}{L}}\, Q' + Q = \frac{V_0 C_1}{q_0} \cos \frac{\Omega}{\Omega_0} \tau \text{ FOR } |Q| \leq 0 \tag{24}$$

$$Q'' + R\sqrt{\frac{C_1}{L}}\, Q' + \frac{C_1}{C_2} Q = \frac{V_0 C_1}{q_0} \cos \frac{\Omega}{\Omega_0} \tau \text{ FOR } |Q| > 0 \tag{25}$$

Equations (24) and (25) are applicable to systems with nonlinearities of the third type.

The independent, dimensionless, and analogous parameters describing the mechanical and electrical systems are

$$a = \frac{r}{\sqrt{k_1 m}} = R\sqrt{\frac{C_1}{L}} \tag{26}$$

$$b = \frac{k_2}{k_1} = C_1/C_2 \tag{27}$$

$$c = \frac{A}{k_1 x_0} = \frac{V_0 C_1}{q_0} \tag{28}$$

$$d = \frac{\omega}{\omega_0} = \frac{\Omega}{\Omega_0} \tag{29}$$

Reviewing the information contained in Equations (26) through (29), we find that

Parameter a = the reciprocal of the Q of the system, or twice the per unit damping.

Parameter b = the inverse ratio of the two capacity values, or the ratio of the two spring constants.

Parameter c = the ratio of the peak driving voltage to the voltage at which the capacitor changes value, or the ratio of the peak applied force to the force required statically just to bring the mass into the nonlinear region.

Parameter d = the ratio of the driving frequency to the small signal resonant frequency, or, when the nonlinearity is of type 3, the ratio of the driving frequency to the resonant frequency corresponding to the smaller spring constant.

2. Generation of nonlinearity. The circuit used to provide the effect of a nonlinear capacitor is shown in Figure 4. The amplifiers A_1 and A_2 have high input impedances, 0.5 megohm and 0.4 megohm, respectively, and low output impedances, 0.9 ohm and 25 ohms, respectively (Ref. 3). $R_o + R_1$ is much greater than the output

impedance of amplifier A_1, and the input impedance of amplifier A_2

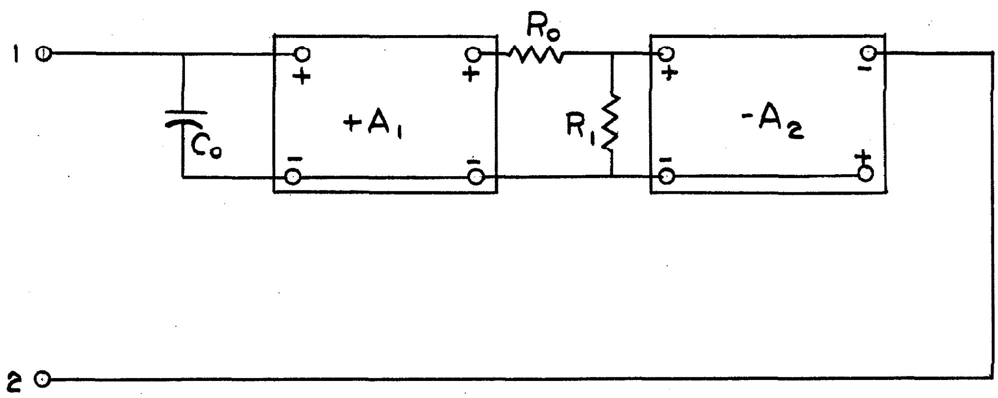


Figure 4. Nonlinear Capacitor Circuit

is much greater than R_1. It can be shown (Ref. 4) that the effective capacity between terminals 1 and 2 is approximately

$$C = \frac{C_o}{1 + \frac{A_1 A_2 R_1}{R_o + R_1}} \tag{30}$$

The capacity is caused to appear nonlinear by making the resistors R_o or R_1 nonlinear. For the types of nonlinearities desired, the circuit of Figure 5 can be used instead of one of the resistors R_o or R_1. The diodes are biased with batteries of voltage E. When the voltage across a and b is less than E, the resistance of the circuit is equal to the back resistance of the diodes (about 1 megohm). When the voltage across a and b is greater than E, the resistance of the circuit is equal to the forward resistance of one diode (about 100 ohms). Thus the circuit is equivalent to a nonlinear resistor whose value changes rather abruptly from 1 megohm to 100 ohms when the voltage across it becomes equal to the biasing voltage E. The value of the capacitor changes when the voltage across C_0 reaches a value

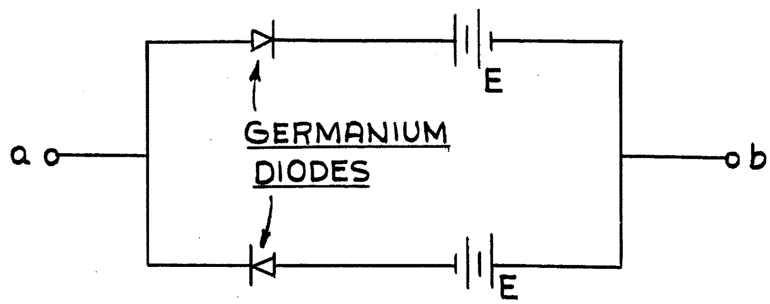


Figure 5. Nonlinear Resistor Circuit

determined by the circuit parameters. The diode circuits used to obtain the three types of nonlinearities are shown in Figure 2.

C. Resonant phenomena

Several phenomena of interest were obtained from analog solution of the system of Figure 1. Three of these are closely related and are known as ultraharmonic, subharmonic, and ultrasubharmonic resonances.

1. Definitions. There exists a good deal of confusion regarding the definition of these terms. Stoker (pp. 7-8 of Ref. 1) starts with the lossless second-order linear differential equation obtained by setting $r = 0$ and $k(x) = k_0$ in Equation (7). From the solution of this equation he purports to arrive at suitable definitions of the three terms. However, his definitions require the concept of a fixed frequency of free oscillations of the system. His failing is that of many persons who work in this field; that is, he thinks in terms of linear systems and tries to define nonlinear phenomena in terms of

linear phenomena. The concept of a free oscillation in a nonlinear system is certainly satisfactory. However, the association of a fixed frequency, independent of amplitude, with this free oscillation is meaningless except perhaps for very special systems or else for systems with extremely small nonlinearities.

This general definition is proposed for the terms under discussion. If the forcing function has a period of T_f, if the forced oscillation has a period T_o, and if continuous oscillations exist, then the following relationship is true.

$$T_o = \frac{m}{n} T_f \tag{31}$$

where m and n are positive integers. Then

If $m = n$, the oscillation is harmonic.

If $m = 1$, $n \neq 1$, the oscillation is subharmonic.

If $m \neq 1$, $n = 1$, the oscillation is ultraharmonic.

If $m \neq 1$, $n \neq 1$, the oscillation is ultrasubharmonic.

2. Properties. Only harmonic oscillations are possible in linear systems. Pure subharmonic, ultraharmonic, and ultrasubharmonic oscillations are infrequent in nature, are generally difficult to produce, and usually have little stability. The class C frequency-multiplying amplifier may seem to disprove the preceding statement, since it certainly does generate frequencies that are higher than the driving frequency, and since the stability of oscillation is excellent.

Where, then, is the flaw? Closer analysis than is usually performed will show that the output wave shape will not be sinusoidal unless the tank circuit is lossless and unloaded. Damping caused by the coil and load losses will cause the amplitude of each cycle of oscillation to decrease exponentially (assuming linear viscous damping) after each plate-current pulse. Thus the stable oscillation is actually a combination of harmonic and ultraharmonic oscillations. The combination of harmonic and ultraharmonic oscillations occurs frequently in nature, and rather frequently the oscillations are stable. The combination of harmonic and either subharmonic or ultrasubharmonic oscillations is not difficult to produce but is nearly always unstable or, at least, very sensitive to changes of the parameters of the system.

D. Experimental results

The electric analog of the mechanical system described in Section III-A was used to obtain examples of the nonlinear oscillations described. The type of nonlinearity used initially is shown as type 1-a in Figure 2. A common mechanical system described in this

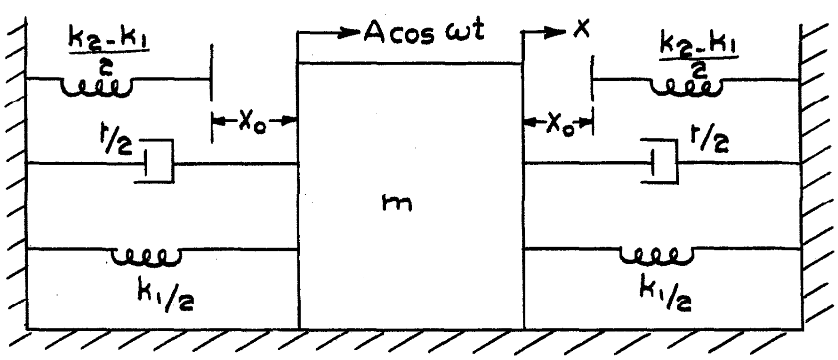


Figure 6. Mechanical System Having Type 1-a Nonlinearity

manner is shown as Figure 6, which represents, at least approximately, many systems having symmetrical overload springs.

Figures 7 through 12 show amplitude-time waveforms and phase trajectories for the nonlinear systems investigated. In most instances, corresponding linear amplitude-time waveforms and phase trajectories are presented. In almost all cases, harmonic oscillations will also exist. The linear waveforms can be used as an aid in identifying the type of oscillation exhibited by the nonlinear system. The four dimensionless parameters, a, b, c, and d of Equations (26) through (29) are given for most amplitude-time waveforms and phase trajectories. In addition, three oscilloscope-gain coefficients, V_A, V_P, and H_P, are given. V and H refer to the vertical and horizontal oscilloscope amplifiers, respectively; A refers to the amplitude-time waveform, and P refers to the phase trajectory. Two sets of coefficients are defined as follows:

For nonlinearities of the first and second types,

$$V_A = \frac{\text{vertical sensitivity}}{\text{diode bias voltage}} \tag{32}$$

$$V_P = \frac{\text{vertical sensitivity}}{2\pi \sqrt{2}\ RC_o\ f_o\ V_o} \tag{33}$$

$$H_P = \frac{\text{horizontal sensitivity}}{\text{diode bias voltage}} \tag{34}$$

For nonlinearities of the third type,

$$V_A = \frac{\text{vertical sensitivity}}{\sqrt{2}\ V_o} \tag{35}$$

$$V_P = \frac{\text{vertical sensitivity}}{2\pi\sqrt{2}\,RC_o\ f_o\ V_o} \tag{36}$$

$$H_P = \frac{\text{horizontal sensitivity}}{\sqrt{2}\ V_o} \tag{37}$$

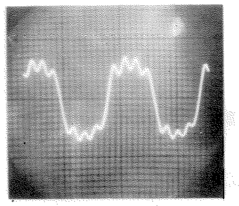
The oscilloscope-amplifier sensitivity is given in volts per inch. The diode bias voltage has been referred to the input of the positive-gain amplifier. V_o is the system driving voltage in rms volts; f_o is the resonant frequency of the linear system.

1. Ultraharmonic oscillations. The first oscillogram of Figure 7 is typical of the solutions obtained. Notice the large harmonic oscillation with the relatively small ultraharmonic content. The principal ultraharmonic is seven times the frequency of the driving force. A comparison between the frequency of the principal ultraharmonic and the frequency of the system is interesting, assuming that the spring is linear and that its value is the value existing for large displacements. Since the driving frequency is 0.21 times the small signal resonant frequency, and since the ratio of the spring constants is 4.12, the linear approximation yields $\sqrt{4.12}/0.21 = 9.7$ as the ratio of the frequency of the ultraharmonic to the driving frequency. This result is reasonable, since a lower ratio would be given by a better approximation to the spring. One such approximation would be a straight line drawn through the origin with a least-squares fit to the actual spring function <u>over the range of oscillation</u>. The underlined words are important, as they re-emphasize the fact that solutions of nonlinear differential equations are nearly always

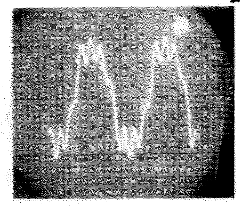
functions of the amplitude.

Another phenomenon typical of nonlinear differential equations is shown by the first two oscillograms of Figure 7. Nothing was changed between the two oscillograms except the driving frequency, and the change was of the order of 5 per cent. Note the change in wave shape. The principal ultraharmonic is now six times the forcing frequency, and the transition between the two oscillograms is abrupt. The author believes that in the future someone will be able to enunciate general laws of quantization of nonlinear phenomena. In all the investigations carried out by the author, these sudden changes were found to exist, sometimes with simple physical explanations and sometimes without. Perhaps the most basic phenomenon observed in this study has been the sudden changes, for at least one nonlinear differential equation yields solutions whose form is independent of amplitude when only a single forcing function is used. This independence of the form of solution upon amplitude is discussed in Section III-D-4.

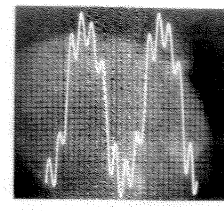
An advantage in the use of the phase trajectory over the more common amplitude-time presentation is shown by the last pair of oscillograms of Figure 7. The amplitude-time plot shows what appears to be an excellent sinusoid; yet the phase plot shows a considerable deviation from an ellipsoidal shape. The phase-trajectory presentation is very sensitive to changes in wave shape and probably should find increased engineering usage as it becomes better known.



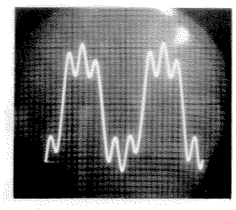
a = .02 V = .233
b = 4.12
c = 2.37/2 V = .153
d = .21 H = .187



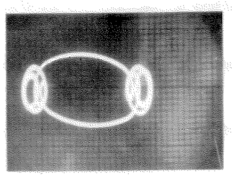
a = .02 V = .180
b = 4.12
c = 2.37/2 V = .133
d = .23 H = .203

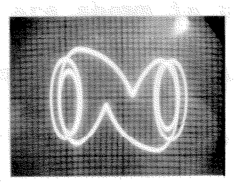


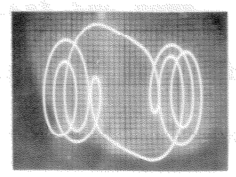
a = .02 V = .220
b = 4.12
c = 7.6√2 V = .212
d = .25 H = .300

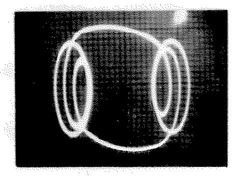


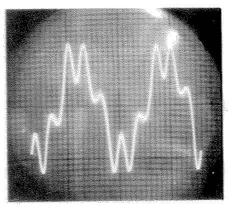
a = .02 V = .173
b = 4.12
c = 2.37/2 V = .133
d = .30 H = .203



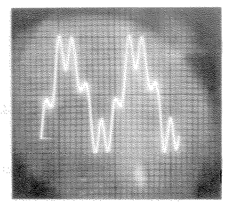




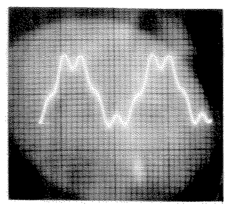




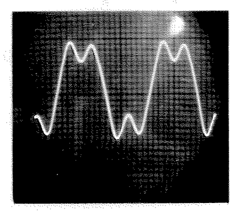
a = .02 V = .173
b = 4.12
c = 2.37/2
d = .31



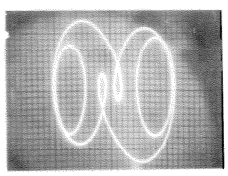
a = .02
b = 4.12
c = 7.6 √2
d = .35

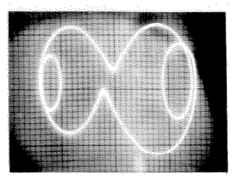


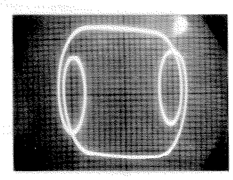
a = .02
b = 4.12
c = 7.6 √2
d = .38

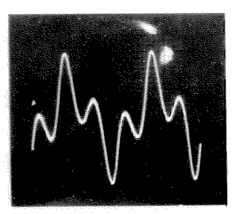


a = .02 V = .210
b = 4.12
c = 2.37/2 V = .133
d = .46 H = .203

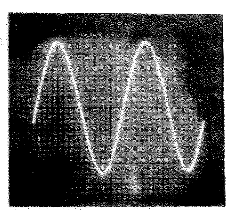




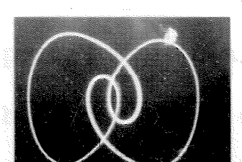


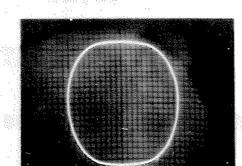


a = .02 V = .210
b = 4.12
c = 2.37/2 V = .213
d = .47 H = .203



a = .02
b = 4.12
c = 7.6 √2
d = 1.42





SOLUTIONS OF THE EQUATION

$$y'' + ay' + k(y) = C \cos d\tau$$

SHOWING HARMONIC OSCILLATIONS AND ULTRAHARMONIC OSCILLATIONS.

$k(y) = y$ FOR $|y| \leq 1$

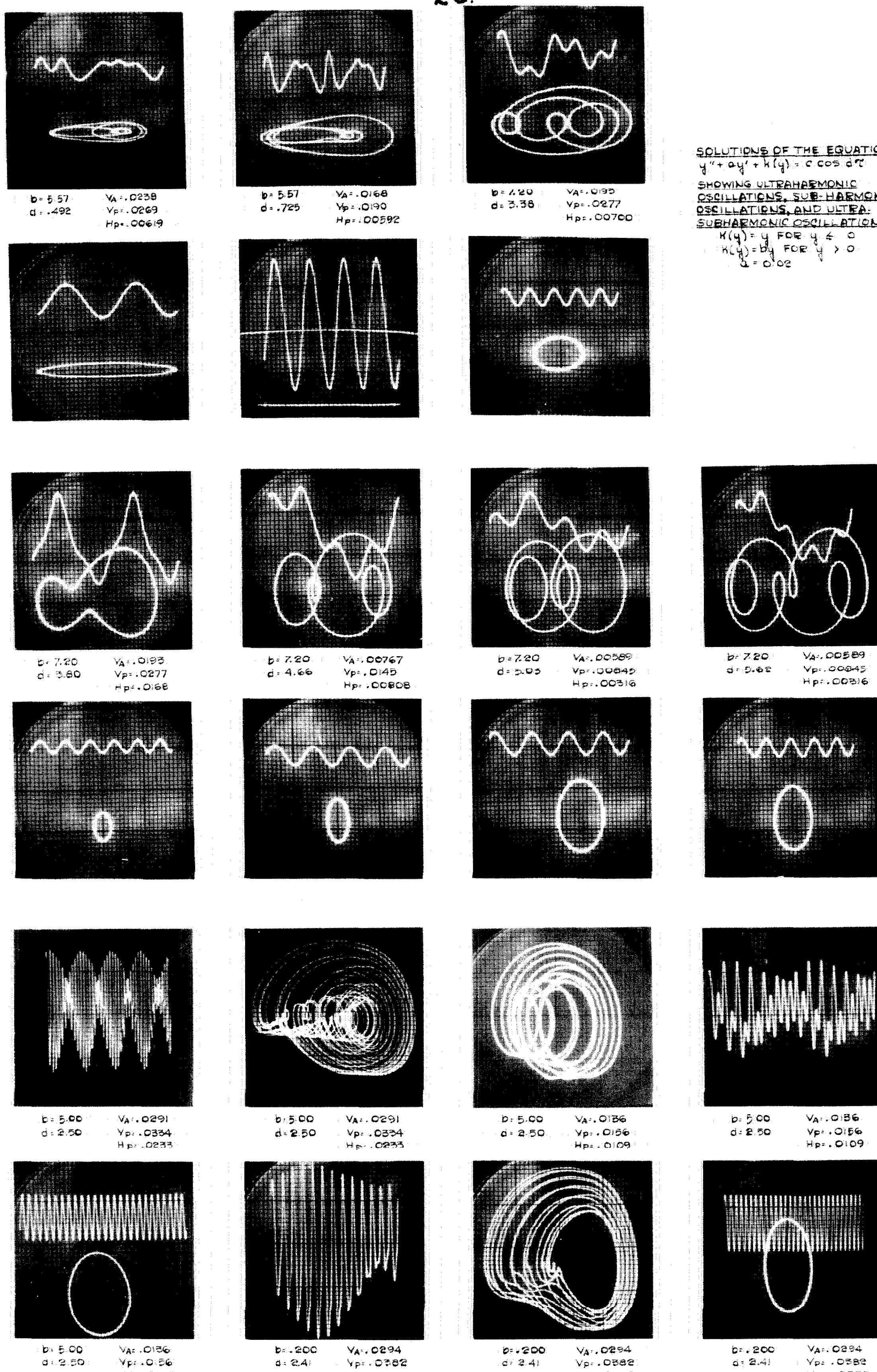
$k(y) = by$ FOR $|y| > 1$

FIG. 7

2. Subharmonic oscillations. Figure 8 shows examples of subharmonic oscillations when the system has the same type of nonlinearity as before; that is, type 1-a of Figure 2. Particularly interesting results are shown in the last two rows. The linear response is shown below, and the nonlinear above. The center oscillogram shows the production of a commercially acceptable sine wave whose frequency is just one-third of the forcing frequency. Considering that no moving parts are involved and that the frequency division is exact, the nonlinear electrical system producing the oscillation might have practical value, if it were only more stable. The last oscillograms show the effect of a small change in driving frequency and the change from division by three to division by four; the resultant loss of a usable wave shape can be seen. Another disadvantage exists from a practical viewpoint. The system must be started with proper initial conditions in order to obtain the subharmonic at all.

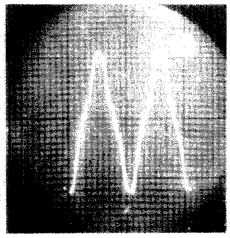
3. Oscillations with other nonlinearities. The first group of oscillograms of Figure 9 shows an assortment of waveforms available with a type 1-b nonlinearity (Figure 2). Notice the small high-frequency oscillations in the stiff-spring region of the first oscillogram and the corresponding "hot-dog" type of phase trajectory.

The second group showing waveforms was obtained with the asymmetric spring, type 2-a. The results are much as one expects if he has studied the symmetrical cases. The asymmetric spring corresponds to a single overload spring such as is used on light trucks.

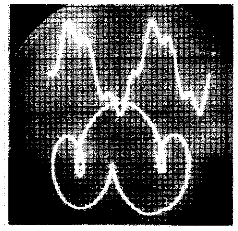


SOLUTIONS OF THE EQUATION
$y'' + ay' + k(y) = c \cos d\tau$
SHOWING ULTRAHARMONIC OSCILLATIONS, SUB-HARMONIC OSCILLATIONS, AND ULTRA-SUBHARMONIC OSCILLATIONS
$k(y) = y$ FOR $y \leq 0$
$k(y) = by$ FOR $y > 0$
$a = 0.02$

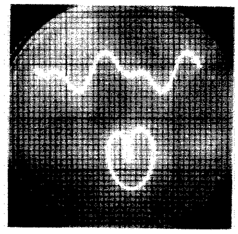
FIG. 8



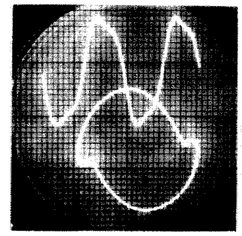
$b = .243$ $V_A = .256$
$c = 7.07$ $V_P = .067$
$d = .11$ $H_P = .333$



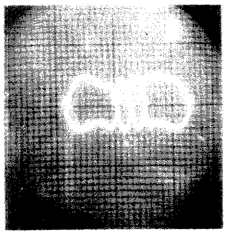
$b = .139$ $V_A = .413$
$c = 65.8$ $V_P = .0113$
$d = 3.11$ $H_P = .320$

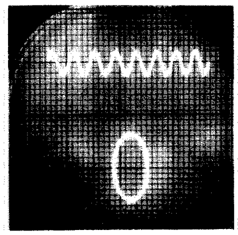


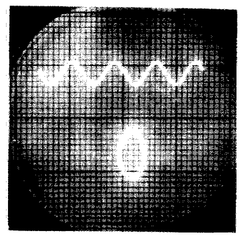
$b = .139$ $V_A = .293$
$c = 17.8$ $V_P = .0267$
$d = 4.16$ $H_P = .280$

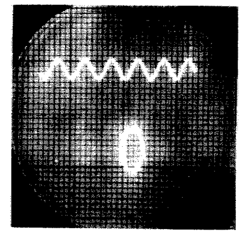


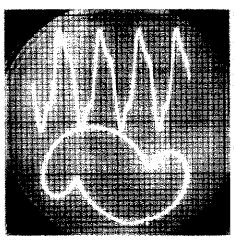
$b = .139$ $V_A = .293$
$c = 17.8$ $V_P = .0267$
$d = 4.34$ $H_P = .280$



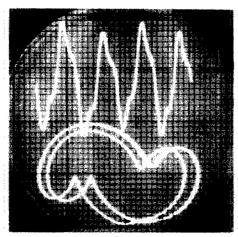




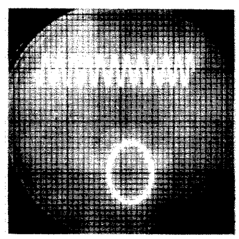




$b = .139$ $V_A = .413$
$c = 34.8$ $V_P = .0214$
$d = 4.11$ $H_P = .320$



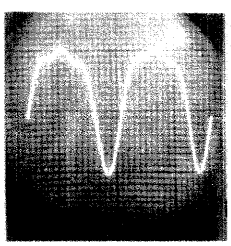
$b = .139$ $V_A = .413$
$c = 34.8$ $V_P = .0214$
$d = 4.11$ $H_P = .320$



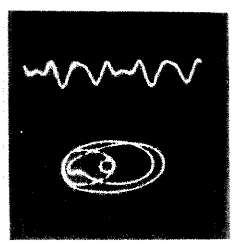
$b = .139$ $V_A = .413$
$c = 34.8$ $V_P = .0232$
$d = 4.11$ $H_P = .320$

SOLUTIONS OF THE EQUATION
$y'' + ay' + k(y) = C \cos d\tau$
SHOWING ULTRAHARMONIC OSCILLATIONS, SUBHARMONIC OSCILLATIONS, AND ULTRA-SUBHARMONIC OSCILLATIONS.
$k(y) = y$ FOR $|y| \leq 1$
$k(y) = by$ FOR $|y| > 1$
$a = 0.02$

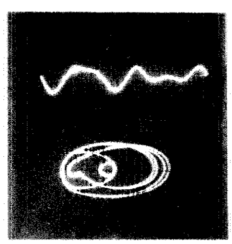
PLATE 3a



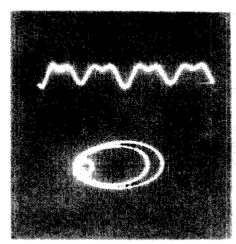
$b = 4.12$ $V_A = .240$
$c = 3.35$
$d = .40$



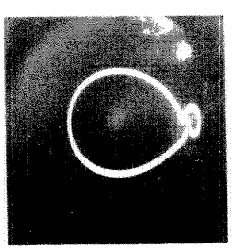
$b = 12.0$ $V_A = .900$
$c = 9.11$ $V_P = .152$
$d = 2.29$ $H_P = .233$



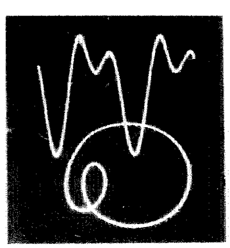
$b = 12.0$ $V_A = .900$
$c = 9.78$ $V_P = .147$
$d = 2.29$ $H_P = .233$



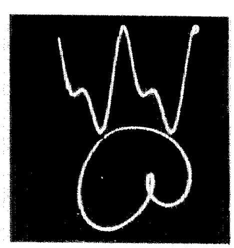
$b = 12.0$ $V_A = .900$
$c = 9.40$ $V_P = .142$
$d = 2.29$ $H_P = .233$



$b = 4.12$
$c = 3.35$ $V_P = .106$
$d = .40$ $H_P = .286$



$b = 5.50$ $V_A = .473$
$c = 48.9$ $V_P = .0406$
$d = 2.68$ $H_P = .447$



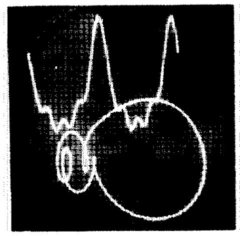
$b = 5.50$ $V_A = .473$
$c = 48.9$ $V_P = .0406$
$d = 3.02$ $H_P = .447$

SOLUTIONS SHOWING ULTRA-HARMONIC OSCILLATIONS, SUB-HARMONIC OSCILLATIONS, AND ULTRA-SUBHARMONIC OSCILLATIONS.
$k(y) = y$ FOR $y \leq 1$
$k(y) = by$ FOR $y > 1$
$a = 0.02$

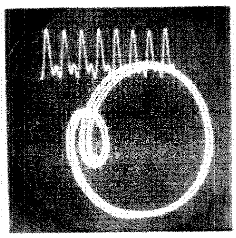
FIG. 9

Figure 10 shows examples of wave forms produced by the same system, except that the asymmetric spring now goes soft (type 2-b in Figure 2). Of particular interest are oscillograms 2 through 5. In the first of these, the frequency of the forced oscillation is one-eighth that of the forcing function. This subharmonic was difficult to obtain and was not stable, changing abruptly to a subharmonic of one-half the driving frequency. The fourth harmonic, shown in the second oscillogram of the group, was also unstable. The third oscillogram of the group shows the stable second subharmonic. Note the strong harmonic content evident in all three oscillograms. The last oscillogram of the group shows the linear harmonic oscillation obtained when the system is not excited enough initially to allow the nonlinearity to have any effect.

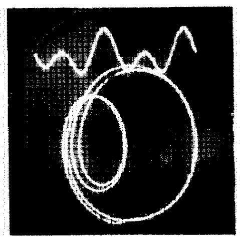
4. Oscillations obtained with simpler system. Figures 11 and 12 show examples of the response yielded by the same system, and the nonlinearity of a more simple sort is shown as type 3 (Figure 2). This type of nonlinearity is of particular interest because the number of parameters required to describe the system completely is found to be reduced from four to three. Only the frequency, damping, and ratio of the spring constants are required, as the response is independent of the forcing amplitude. Thus the parameter c of Equation (35) is unnecessary. The nondependence upon amplitude can be seen from the physics of the system. Consider the mass as its displacement goes through the origin. The velocity is the controlling parameter at this time. In the previous cases, the type of solution



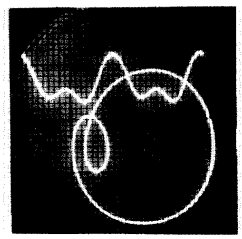
$b = .337$ $V_A = 1.23$
$c = 27.3$ $V_P = .0175$
$d = .278$ $H_P = 1.00$



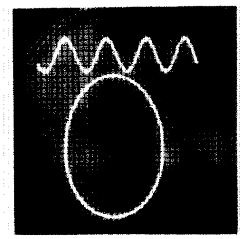
$b = .337$ $V_A = 1.00$
$c = 23.6$ $V_P = .0203$
$d = 2.16$ $H_P = .380$



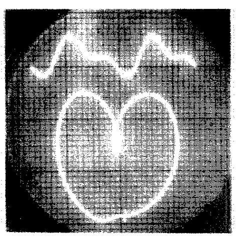
$b = .337$ $V_A = 1.00$
$c = 23.6$ $V_P = .0203$
$d = 2.12$ $H_P = .380$



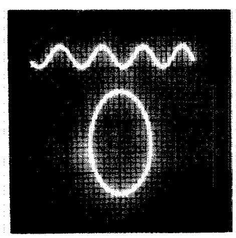
$b = .337$ $V_A = 1.00$
$c = 23.6$ $V_P = .0203$
$d = 2.12$ $H_P = .380$



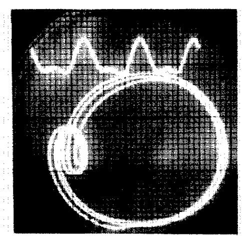
$b = .337$ $V_A = 1.00$
$c = 23.6$ $V_P = .0203$
$d = 2.12$ $H_P = .380$



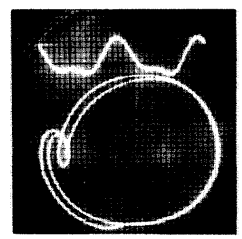
$b = .337$ $V_A = 1.00$
$c = 23.6$ $V_P = .0203$
$d = 2.58$ $H_P = .380$



$b = .337$ $V_A = 1.00$
$c = 23.6$ $V_P = .0203$
$d = 2.58$ $H_P = .380$



$b = .200$ $V_A = 1.23$
$c = 25.5$ $V_P = .0188$
$d = 2.66$ $H_P = .280$



$b = .200$ $V_A = 1.23$
$c = 25.5$ $V_P = .0188$
$d = 2.75$ $H_P = .200$

SOLUTIONS OF THE EQUATION

$y'' + ay' + k(y) = c \cos d\tau$

SHOWING ULTRAHARMONIC OSCILLATIONS, SUBHARMONIC OSCILLATIONS, AND ULTRA-SUBHARMONIC OSCILLATIONS.

$k(y) = y$ FOR $y \leq 1$
$k(y) = 2y$ FOR $y > 1$
$a = 0.02$

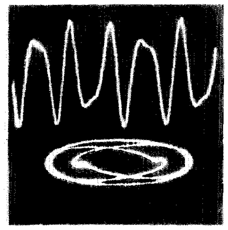
FIG. 10

depended upon whether or not velocity was great enough to carry the mass into the region where the spring constant changed. Solution depended, first, upon the fact that the velocity had the minimum value required to carry the mass into the stiffer-spring region and, second, upon the velocity of the mass when it entered this region. With simpler nonlinearity, only the amplitude of the solution can change upon entering the new region with different velocities. Experimental verification of nondependence upon amplitude was obtained. Some of the phenomena presented are so sensitive to changes in parameters, however, that the slightest zero shift in one of the amplifiers would make the solutions dependent upon the amplitude of the forcing function. All the types of response obtainable with the system having the more complex nonlinearity were duplicated with this simpler system. The author believes that, if further work were to be done with straight-line segment nonlinearities, the simple nonlinearity just mentioned would yield the greatest amount of education for a given invested effort.

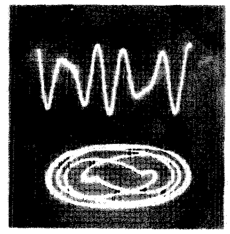
The first few oscillograms of Figure 11 show the extremely complex waveforms that can be produced with this simple system. The third and fourth rows show examples of frequency division similar to those of Figure 8. The last two rows are included because they demonstrate extreme cases of frequency division. The author does not guarantee that the system producing these waveforms was the simple one described. Externally the system looked the same, but the possibility exists that one of the amplifiers was overloaded and that it was thus introducing an unknown nonlinearity into the system. These oscillations were extremely sensitive to a change in any of the parameters.

Figure 12 completes the display of results obtained by the use of the simple system of Figure 1 when the spring-constant function was of type 3 (Figure 2). Except for the obvious dissymmetry of the waveforms resulting from the type of spring function used, the waveforms are comparable to all the others shown in Figures 7 through 11. No comment is made about specific oscillograms except for the second, third, and fourth of the fourth row. Repeated mention has been made of the abrupt changes in waveform that can be caused by a small change in one parameter of the system. These three oscillograms show the two stable waveforms and also the transition between them.

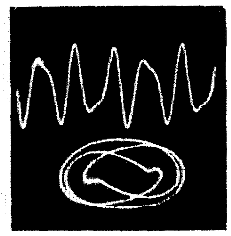
5. Resonant hysteresis. Resonance phenomena are very different in nonlinear systems from those in linear systems, as the possibility of more than one stable steady-state oscillation exists for a given set of parameters. If one recalls that the response of a nonlinear system to any forcing function is usually dependent upon the amplitude (Ref. 1), then he can see that, as a simple example, the resonant frequency of a system can be a function of the amplitude of oscillation. If this is so, then there exists the possibility of having two or more stable states of oscillation. Experimentally we find that sudden transitions from one state to another can result from very small variations in the value of any parameter in a system. If these



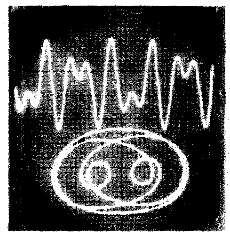
b = 5.50 V_A = .27
c = 8.5 V_p = .036
d = 3.16 H_p = .21



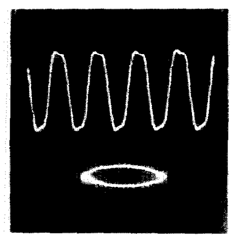
b = 5.50 V_A = .31
c = 4.6 V_p = .17
d = 2.84 H_p = .17



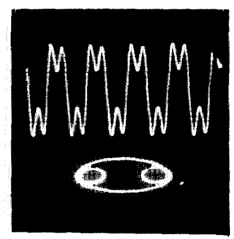
b = 5.50 V_A = .15
c = 4.7 V_p = .17
d = 2.84 H_p = .12



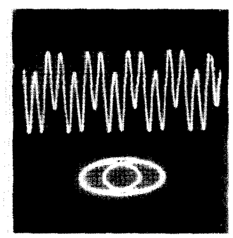
b = 5.50 V_A = .27
c = 14.6 V_p = .056
d = 3.16 H_p = .17



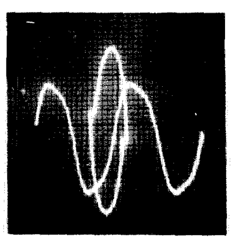
b = 5.50 V_A = .10
c = 3.2 V_p = .25
d = 3.85 H_p = .10



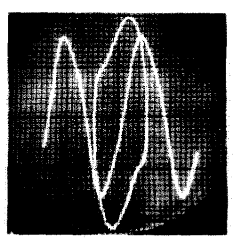
b = 5.60 V_A = .10
c = 9.2 V_p = .25
d = 3.85 H_p = .10



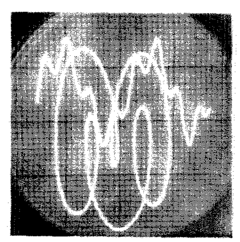
b = 5.50 V_A = .10
c = 4.6 V_p = .25
d = 3.85 H_p = .10



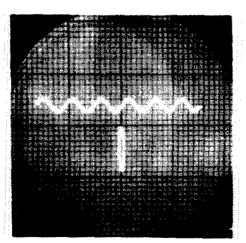
b = 7.20 V_A = .20
c = 19.8 V_p = .015
d = 5.70 H_p = .60

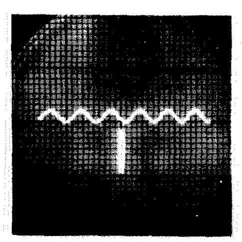


b = 7.20 V_A = .20
c = 19.8 V_p = .015
d = 6.09 H_p = .60



b = 7.20 V_A = .20
c = 19.8 V_p = .015
d = 6.75 H_p = .60





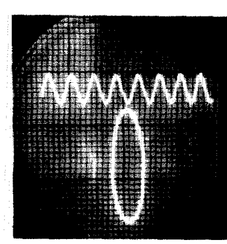
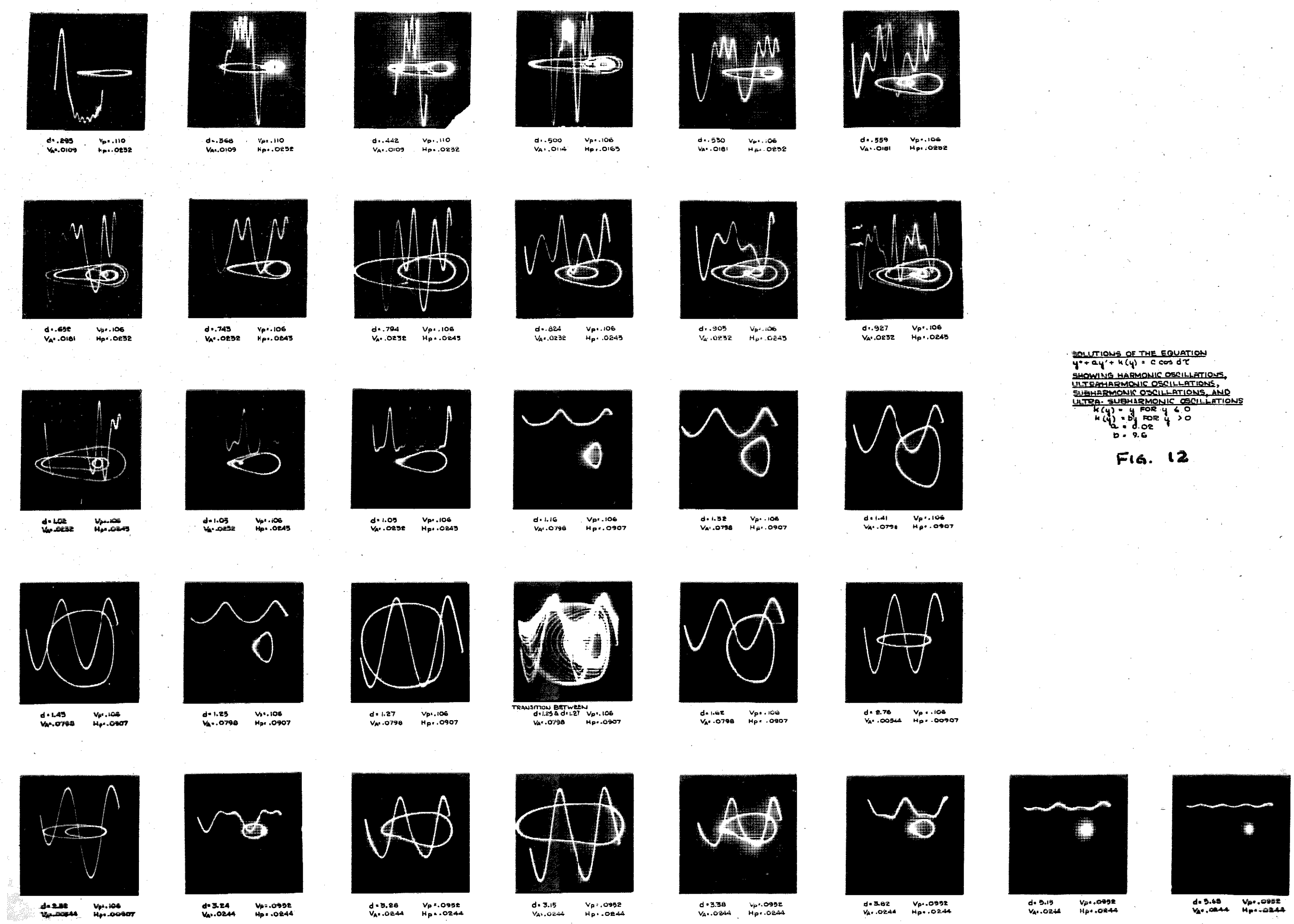


FIG. 11

SOLUTIONS SHOWING SUBHARMONIC OSCILLATIONS
AND ULTRA-SUBHARMONIC OSCILLATIONS

$k(y) = y$ FOR $|y| \leq 1$
$k(y) = by$ FOR $|y| > 1$
$a = 0.02$



SOLUTIONS OF THE EQUATION
$y'' + ay' + k(y) = c \cos d\tau$
SHOWING HARMONIC OSCILLATIONS,
ULTRAHARMONIC OSCILLATIONS,
SUBHARMONIC OSCILLATIONS, AND
ULTRA-SUBHARMONIC OSCILLATIONS
$k(y) = y$ FOR $y < 0$
$k(y) = by$ FOR $y > 0$
$a = 0.02$
$b = 9.6$

FIG. 12

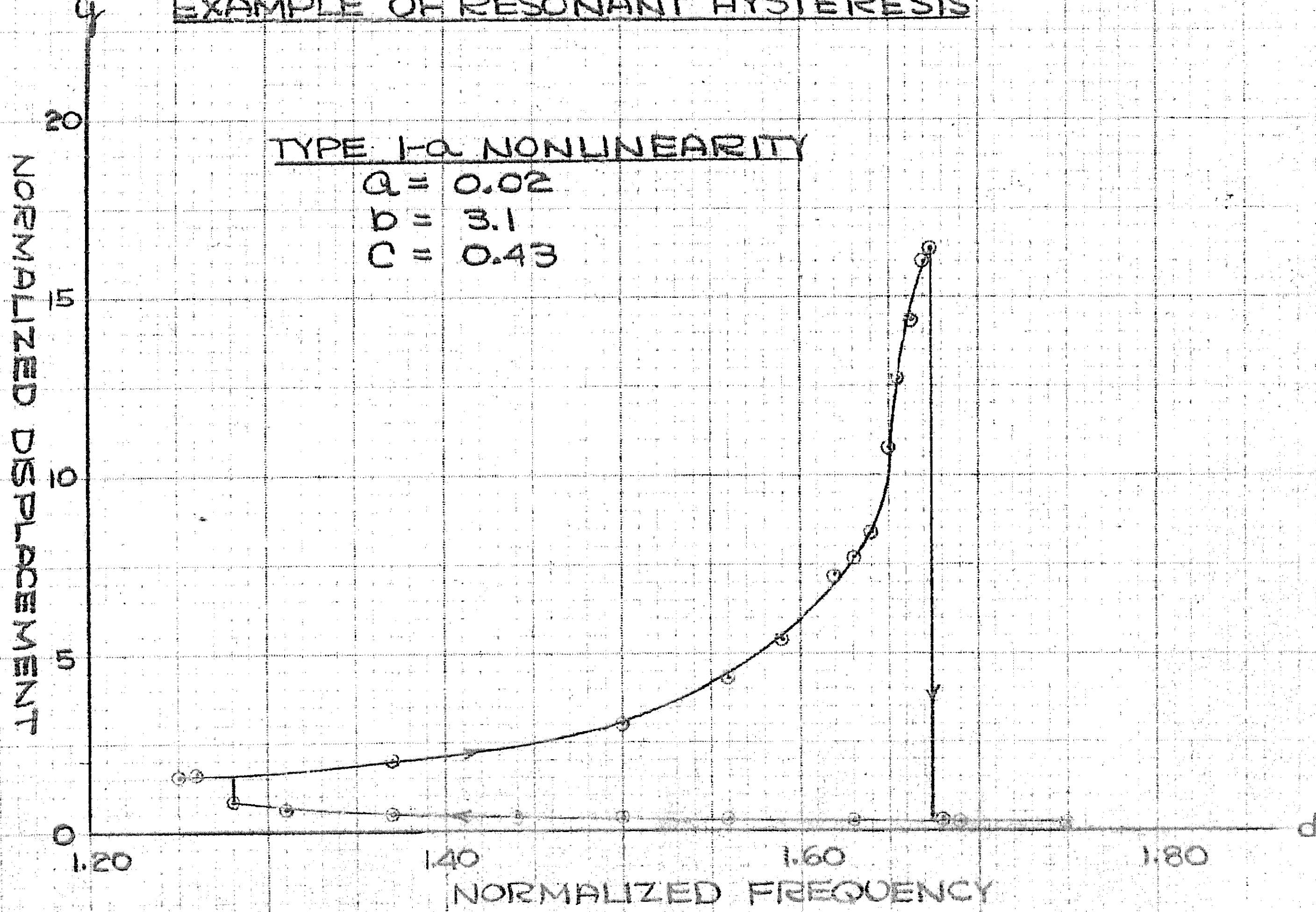
sudden transitions are dependent upon whether the frequency was increasing or decreasing, the system is said to exhibit resonant hysteresis. Resonant curves showing these sudden changes (or jumps) are known as drag loops.

Figure 13 shows an example of resonant hysteresis obtained with the analog computer. Because the waveform is nonsinusoidal, peak measurements were made, and the set of parameters used was chosen so as to yield nearly sinusoidal oscillations.

The name resonant hysteresis, which is in common use, clearly describes the phenomenon shown in Figure 13. However, the variation of any parameter in the system investigated can give abrupt changes, and the variation of nearly any parameter can give hysteresis phenomena. A hysteresis effect similar to the one described also occurs with the ultraharmonics and the subharmonics.

6. Effect of damping. The effect of changing the damping has not been discussed, but it can be noted that the oscillograms of Figures 7 through 12 were taken with a relatively low-loss system. The reason for this situation is that additional damping caused the system to behave much more as if it were linear. The ultraharmonics were obtainable without difficulty when superimposed upon an harmonic oscillation, but their magnitudes were considerably reduced. Subharmonics became even more difficult to obtain, and their stability decreased. The resonant hysteresis effect was very sensitive to additional damping, as would be expected, and the effect vanished completely with only moderate amounts of damping.

FIGURE 13
EXAMPLE OF RESONANT HYSTERESIS



If the damping had been nonlinear, the effect of varying the damping would have been very different. In Section V a system having nonlinear damping is investigated. The spring-restoring force is linear, and the damping is a function of the displacement. The solutions are found to depart farther and farther from those of the linear system as the damping is increased. This result is to be expected as the system becomes more and more nonlinear.

The author believes that the experimental investigation just described was worth while. A mathematician, after reading the report, might be shocked into the realization that most of the methods he knows for attacking even simple nonlinear equations are of little value. This realization plus some good experimentally obtained examples of solutions might lead the right man to devise new mathematical techniques which would be more applicable. The physicist or engineer might well afford to study the oscillograms and thus improve his understanding of the types of solutions that nonlinear equations can yield. The author achieved several results from the investigation. Important to him were the development of an interest in the field and a desire to explore more fully the types of solutions that can be obtained from nonlinear systems. More fundamental was the development of a "feel" for simple nonlinear effects and the development of an ability to interpret physically the results obtained.

The realization that each practical nonlinear problem at the present time requires a particular solution caused the author to become interested in the development of an electronic function generator. The development of this function generator and some of the associated equipment required for the analog solution of certain nonlinear differential equations is described in the next section.

IV. DEVELOPMENT OF ARBITRARY FUNCTION GENERATOR

As was pointed out in Section III, a nonlinear function generator is needed when quantitative answers are to be obtained to problems involving practical nonlinearities.

A. Review of available function generators

Several arbitrary function generators were available when the work described in this section was started. However, none of the available function generators was thought to be completely satisfactory. If the engineer is willing to operate at a very low speed such as that used by the mechanical differential analyzer, a simple and satisfactory function generator is available. In its simplest form a graph of the function desired is placed on a plotting table, and the dependent variable is used to control the transverse motion of the carriage. A human operator controls the longitudinal motion by means of a hand wheel and keeps the stylus on the curve. Thus the longitudinal displacement will, within the ability of the operator and the accuracy of the graph, represent an arbitrary function of the dependent variable. Photoelectric systems have been designed that obviate the need for a human operator and simultaneously improve the accuracy. For the electric differential analyzers, a similar device is available, except that the function is represented by gluing a piece of conducting wire upon the graph paper.

Both of these systems are satisfactory and can be made as accurate as necessary, but the amount of labor involved in setting up one particular function is seen to be considerable.

If the desired function is square, cube, square root, or cube root of the dependent variable, then the function can be set up by using cascaded precision potentiometers. If the square or cube is desired, two or three closed-loop controllers are required to turn the potentiometers. If roots are desired, then an additional computing amplifier is needed.

For the higher speed of operation required by simulators and analog computers, an electronic multiplier is available. One or more multipliers and a high-speed computing amplifier can be used to generate the same functions obtained with potentiometers in the electric differential analyzer. For more complicated functions, an optical servo system equivalent to the plotting table mentioned for use with the differential analyzer is available. It is known as a photoformer and can yield functions with an accuracy of 1 per cent. By analog-computer standards the frequency response of the device is unlimited. The fundamental shortcoming of the photoformer is the same as that of all the devices mentioned; that is, the amount of effort required to change from one function to another is too great.

The reader probably asks, "Why this sudden interest in changing the function rapidly? Why not set it up and leave it alone?" If one is content merely to analyze nonlinear systems, then no adequate reason can be given for the development of a new function generator.

In the future, however, the problems of stabilizing, of linearizing response, and of improving the speed of response of nonlinear systems will certainly have to be met. Synthesis will replace analysis as the primary problem to be faced, and a flexible, easily changed function generator will be required.

Such a function generator has been developed. Though the author has not had the opportunity to do much more than solve some standard nonlinear problems while testing out the device, he believes that it will find considerable use in the future.

B. Mathematical principles of operation

Many of the nonlinearities found in nature can be represented by an equation of the form $y = cx^n$ where c is a real constant and n is not necessarily an integer. Figure 14 (to be found in a pocket attached to the cover) shows x^n versus x with $20 \geqslant n \geqslant -20$ and $0 \leqslant x \leqslant 2.5$. Familiarity with this simple function apparently has bred neglect, for many functions usually approximated by power series could be better approximated by a single term of the form $y = cx^n$. This large-size figure is included with the thought that it might be used quantitatively when setting up arbitrary functions. A rather obvious way of obtaining such functions is suggested by the principle of operation of the slide rule. Consider the following set of equations:

$$z = \log_a x \tag{38}$$

$$w = nz + \log_a c \tag{39}$$

$$y = \exp_a(w) \tag{40}$$

or $$y = \exp_a\{n\log_a x + \log_a c\} \tag{41}$$

or $$y = cx^n \tag{42}$$

Equation (42) represents the function that we wish to mechanize. Equations (38), (39), and (40) tell how to perform the mechanization and point out the principal elements required. Equation (38) tells that we must have a device that will take the logarithm. Equation (39) states that we must have apparatus that will multiply by constant factors and add constant terms. Equation (40) points out that apparatus which will exponentiate (that is, take the inverse logarithm) must be available. Since an electrical function generator is desired, all the variables in Equations (38) through (42) and elsewhere in the section are assumed to be voltages scaled as numbers. Thus no trouble is encountered, for example, in taking the logarithm of 10 volts and having the result be in volts.

Before proceeding to describe the development of any of this equipment, two restrictions applying to Equations (38) through (42) should be pointed out.

First, the assumption has been made that x is always positive, since the logarithm of a negative number is complex. The arbitrary function generator, however, acts only upon the magnitude of the input, and electronic switches are provided which allow the choice of either an even or an odd function at the output. Section IV-E explains the switching operation in more detail.

Second, the equations are too restrictive for our purposes. If one has several logarithm-taking elements, the addition of amplifiers or attenuators and suitable bias supplies to the equipment already mentioned allows a function of the following type to be generated:

$$y = ax^{\ell}(x + b)^{m}(x + c)^{n} \ldots \tag{43}$$

where a, b, c, ℓ, m, and n are real constants. If $\ell = m = n = 1$, Equation (43) would represent a useful function, but this restriction is unnecessary. In practice, the exponents will fall in the following ranges:

$$0.2 \leqslant \ell, m, \ldots \leqslant 5$$

$$-0.2 \geqslant \ell, m, \ldots \geqslant -5$$

Considering the fact that b, c, . . . can be negative, one sees that the function given by Equation (43) is extremely versatile.

C. Logarithm-taking element

How does one take logarithms electrically? This was the question that was first given attention by the author, as he knew that meters were available commercially which purported to have logarithmic scales. No devices having exponential characteristics were known to him.

1. Commercially available elements. The Ballantine Laboratories, Incorporated, of Boonton, New Jersey, manufacture a line of logarithmic-scaled voltmeters, but one finds that the linear-to-logarithmic conversion is done by the combination of a nonlinear meter movement and a nonlinear meter scale. The Kalbfell Laboratories of San Diego, California, manufacture a device known as the Logaten which consists of eight crystal diodes and an approximately equal number of linear

resistors. The output of a Logaten when feeding a high impedance is the logarithm of the input over an input range of over two decades. The linearity and stability of this device, though sufficient for many measurement applications, are not particularly good. Also the logarithm base is very high. Thus the output-voltage range is very small, and the drift stability requirements imposed upon other apparatus would be difficult to satisfy.

2. Nonlinear passive elements. The simplicity of circuits using crystalline diodes as possible logarithm-taking elements (henceforth called LTE) is appealing. An investigation was made, and the voltage across a selenium rectifier was found to be approximately the logarithm of the current. For voltage-to-current conversion, a large series resistance is added, and the circuit becomes that of Figure 15. A

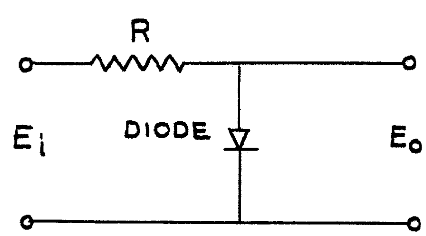


Figure 15. LTE Using Crystalline Diode

typical response curve obtained using the circuit of Figure 15 is shown as Figure 16. Notice the small change in output voltage resulting from a large change in the input. The base to which the logarithm is taken is high, of the order of 12 . If the selenium rectifiers had exhibited

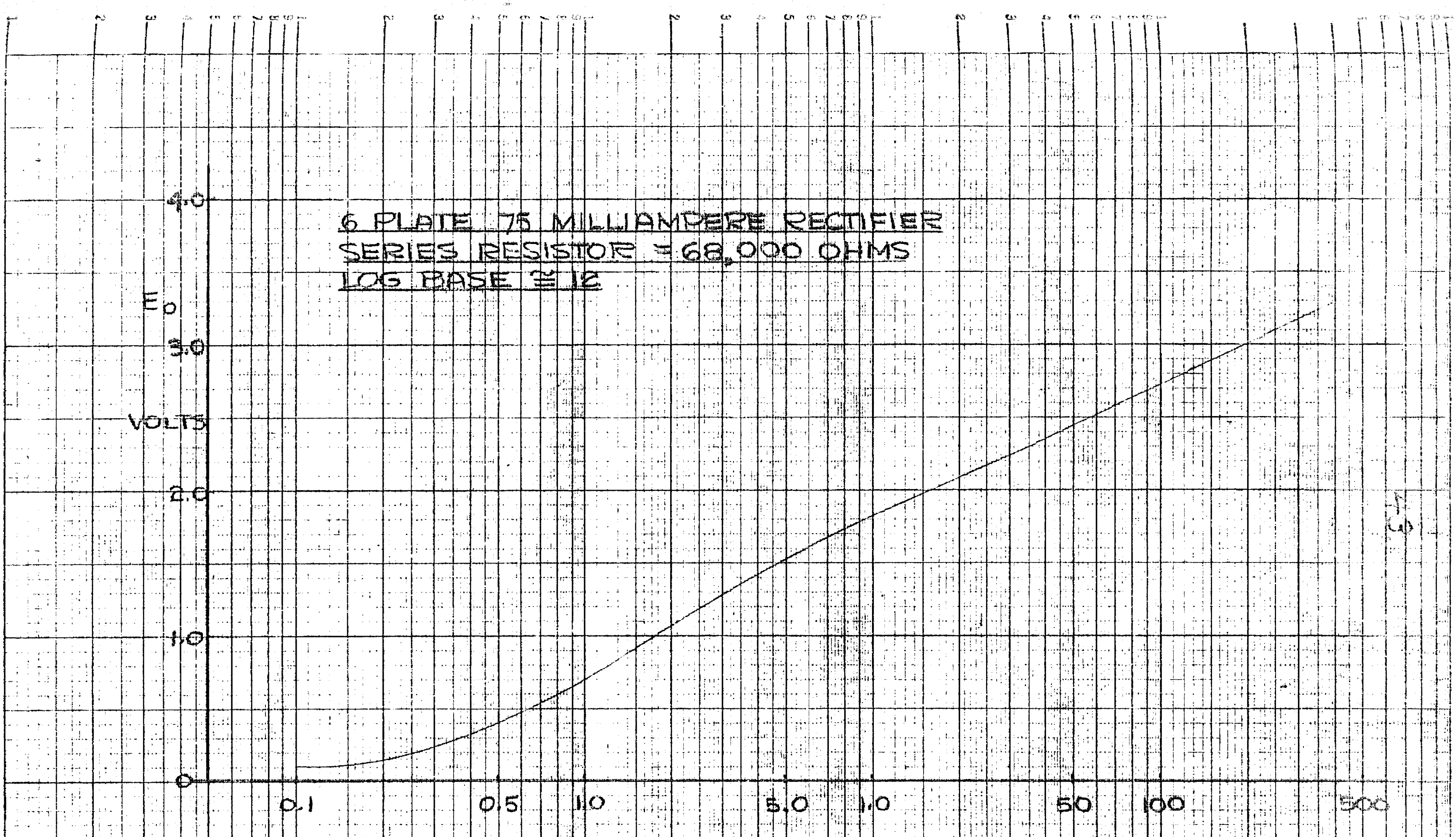


FIGURE 16 TRANSFER FUNCTION OF SELENIUM RECTIFIER

any reasonable drift stability, their low cost would have tempted the author to try to develop selection and aging processes which would make them usable as LTE's. However, in all the rectifiers tested, large drifts in the output voltage occurred which were functions not only of temperature and age but also of other undetermined factors.

Germanium diodes, when used in the circuit of Figure 15, exhibited characteristics similar to those of the selenium rectifiers. The logarithm base was higher; stability, better; and linearity, worse.

3. Nonlinear active elements. Mention has been made in the literature to the use of triode vacuum tubes as LTE. Perhaps the most recent mention (Ref. 5) even suggests the use of its LTE in an arbitrary function generator similar to the one described in this thesis. However, no literature is known in which an LTE is described in sufficient detail to suggest that the writer ever obtained a usable unit. Nadel,* in an unpublished memorandum to R.S. Neiswander, described a triode LTE which took logarithms rather accurately over an input range of 1 to 300

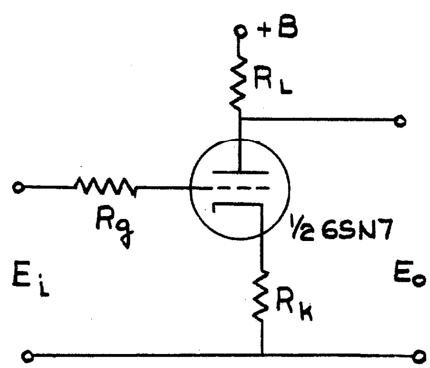


Figure 17. Basic Triode LTE

volts to a base of about 3. The circuit diagram for the triode LTE is

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*Tests at North American Aerophysics Laboratory, 1950, conducted by Albert N. Nadel.

given in Figure 17. The author was unable to duplicate Nadel's results and attempted to find another usable region of operation of the tube.

4. Search for good LTE. The first results were very discouraging, as only over a small range of input voltage would the output be linear (log scale). Varying any one of the parameters seemed to improve the linearity, but the time consumed in plotting one curve of output voltage versus input voltage was at least 5 minutes. A connection was tapped into a vacuum-tube voltmeter just past the input attenuator, and the input to the LTE could then be varied rapidly, though in fairly coarse steps. Using this technique, the author discovered a new LTE having a usable input voltage range, but unfortunately it had a very high base.

The first effort toward speeding up the procedure of testing a possible LTE was to develop an exponentially falling voltage across a large capacitor, to apply this voltage to the LTE, and to observe the output by using a Brush recorder. This system had several disadvantages. Because the input resistance of the LTE is not a constant, the resistor through which the capacitor discharged had to be relatively small; yet the time constant of the circuit had to be kept large so that the recorder could follow the output with no error. The Brush recorder has such a small deflection that judgment of linearity by inspection is difficult, and it is impossible to see the effect of making a small change in one of the parameters without plotting the curves.

The idea of applying an exponentially decreasing voltage was sound, however, and the following system was adopted. A millisecond relay is driven by half-wave rectified 60-cyc/sec voltage producing square waves of voltage of nearly equal "halves." This square wave is partially integrated by an RC circuit and then applied to a cathode follower. The dc bias is removed from the cathode follower's output, which is then applied to the would-be LTE. The diagram of the circuit just described constitutes a part of Figure 18. The complete circuit has other functions necessary for the discovery and alignment of an LTE. All the parameters of the LTE are easily adjustable, and metering is provided where necessary. The slide-back voltmeter incorporated yields the data from which the approximate logarithm base and dc level can be determined.

Since the oscilloscope is not an accurate measuring instrument, the LTE, optimized by the procedure just given, is checked by a point-by-point dc measurement. This step is facilitated by the inclusion of two Helipots with Duodials. The first, whose resistance is negligible compared with series grid resistance, applies the voltage to the LTE. The dial readings of 0 to 1000 in unit steps must be multiplied by 0.3 to give the actual input voltage but can be plotted directly onto the semilog paper used. Since the plate-load resistor can be of the order of 0.5 megohm, some provision must be made for accurately measuring the plate voltage without loading. This measurement is achieved by using a null method. The other Helipot is connected to the 300-volt supply through series resistors so that the

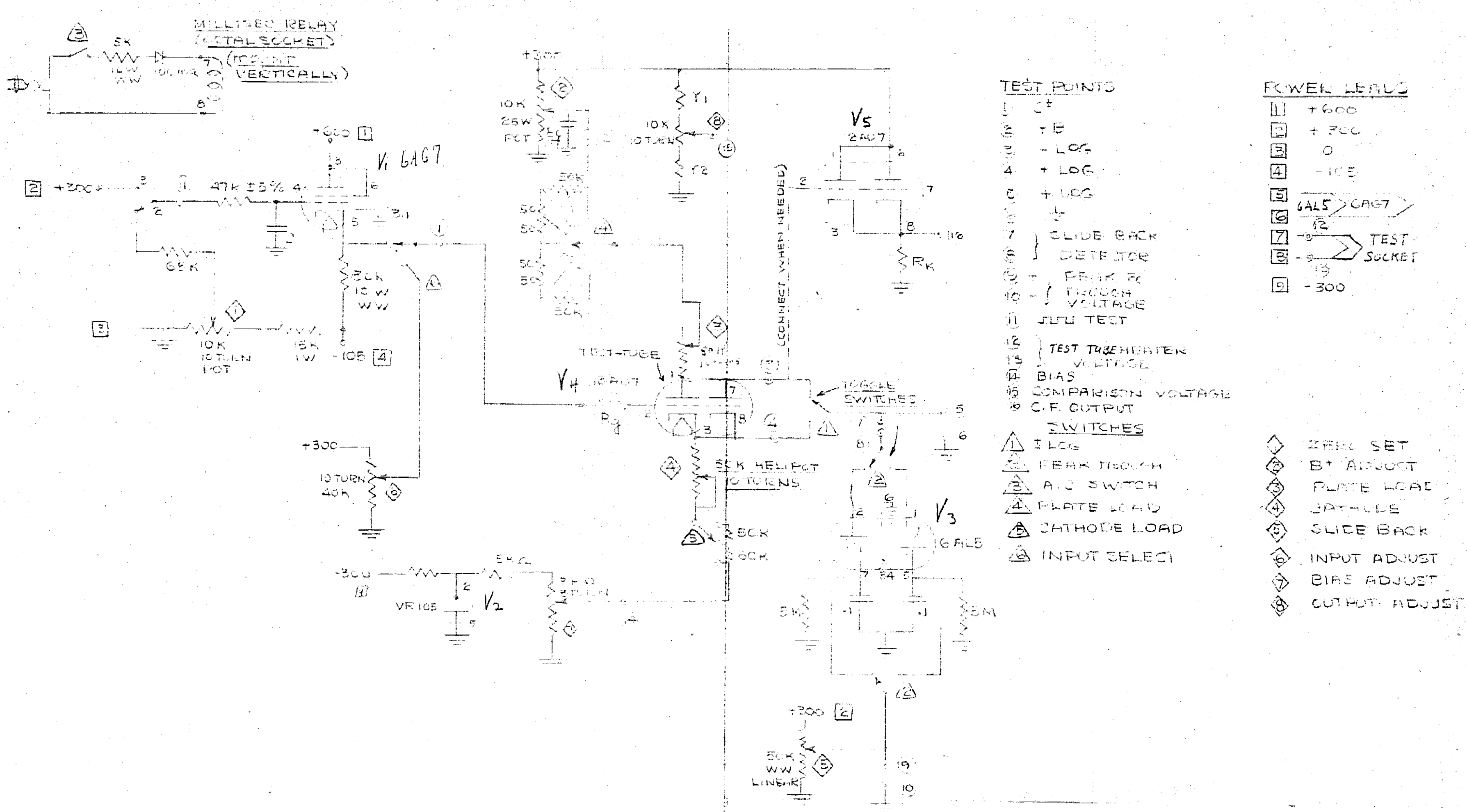


FIGURE 18
LTE SEARCH, TEST, AND ALIGNMENT UNIT

voltage range of its slider approximates that of the LTE. A center-scale meter with ranges of 0 to $\pm$ 50 volts and 0 to $\pm$ 100 μ amps serves as the null detector, and the voltage range of the Helipot is determined by measuring the voltages at its ends with a sensitive voltmeter. The dial readings of the Helipot can be plotted directly as linearity check or may be converted into equivalent voltages.

The LTE search, test, and alignment unit as shown in Figure 18 has proved to be successful. Within an hour, many of the possible parameter combinations of the circuit can be used with a particular triode. After some experience using this device, one can follow the general trends in a systematic manner.

The author found several combinations of tubes and resistors which served to take logarithms. In all cases, the conversion became poor at high input levels. Figure 19 shows the exponentially falling voltage applied to the LTE. Figure 20 shows the response to an exponential input, and it is seen that the LTE yields the negative logarithm. Thus the departure from linearity (log scale) corresponds to a large input voltage. This fault was eliminated almost completely by feeding some of the input signal to the cathode through a resistor. Figure 21 shows the response of the same circuit as that displayed in Figure 20 except that the feed-through resistor was added. Figure 22

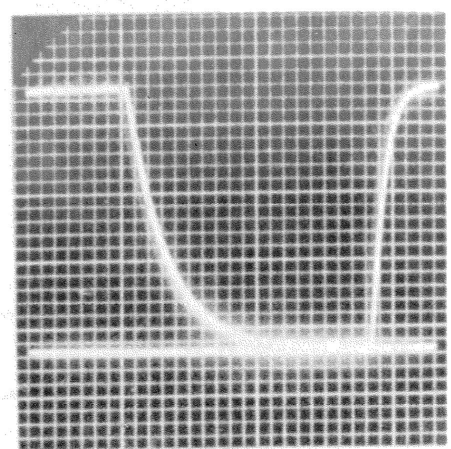


Figure 19. Oscillogram, Output of Exponential Generator

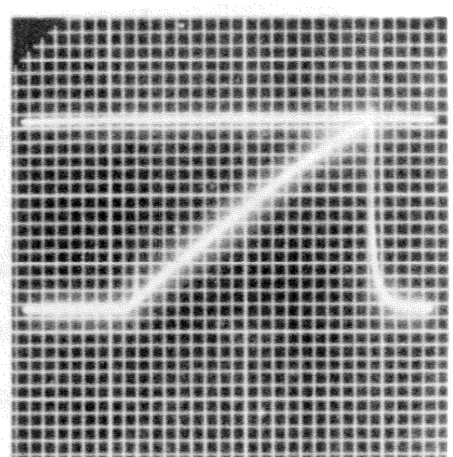


Figure 20. Oscillogram, Basic Triode LTE Response to Exponential Input

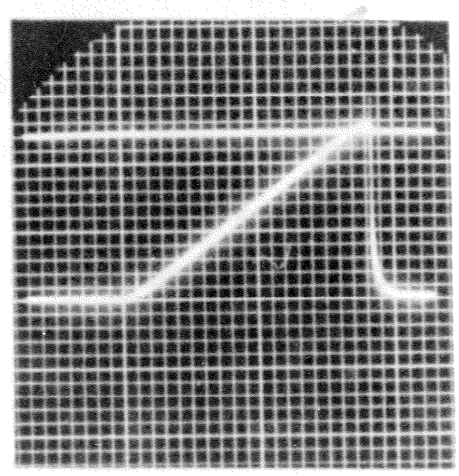


Figure 21. Oscillogram, Modified Triode LTE Response to Exponential Input

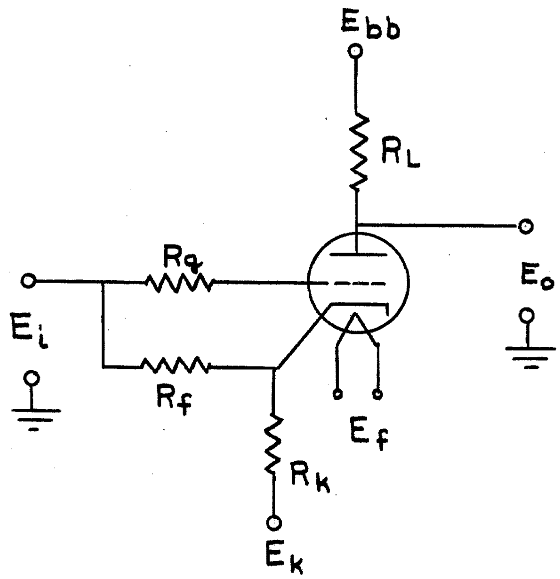


Figure 22. Modified Triode LTE

shows the circuit of the modified LTE.

5. Usable LTE's. Figure 23 gives the data on a number of practical LTE's. The variation in the size of the optimum feed-through resistor is seen to be considerable. The notation of Figure 23 is the same as that of Figure 22. The column labeled Base gives the base to which the logarithm is taken. V_o is the voltage output when the input is 1 volt.

The large variation in V_o should be noted. When used for computing, the LTE should give output with respect to ground, and removal of the constant voltage can be troublesome. The circuit adopted, the last one described in Figure 23, has a rather low dc level which is removed with negligible change in the logarithm base.

The following observations can be made in general regarding triode LTE:

1. The higher the power-supply voltage, the lower is the base

Tube	E_{bb}	E_f	R_L	R_K	R_f	R_g	E_K	Base	V_o
1/2 12AY7	300	5.8	450KΩ	12.4KΩ	470K	2.5MΩ	-3.0	1.30	60
1/2 12AY7	300	5.8	320	14.0	470	2.5	-8.5	1.38	33
1/2 12AU7	280	5.8	100	14.0	5000	2.5	-13.6	1.78	25
1/2 12AU7	280	5.8	27	150	5000	2.5	-3.4	1.78	72
1/2 12AT7	290	5.8	36	5.6	1000	2.5	-24	1.18	113
1/2 12AX7	290	5.8	18	5.3	150	2.5	-19	1.14	231
1/2 12AX7	290	5.8	160	6.3	400	2.5	-5.9	1.13	110
2/2 12AX7	300	6.0	350	3.0	270	2.2	-3.0	1.20	30

Figure 23. Parameters Which Yield Practical LTE's

of the logarithms. To keep the dc level down to a reasonable value, however, one must use large values of the plate-load resistor. This is in general a wise step and tends to keep the plate dissipation down.

2. High mu tubes seem to have the lowest log base.
3. Tubes with high transconductance seem to be unable to give linearity in their log taking.
4. Feed-through to the cathode resistor is a powerful method of improving the linearity for high input voltages.
5. The value of the voltage applied to the cathode return affects the linearity mostly with low input voltages.
6. The value of the cathode resistor affects the shape of the whole curve, but not uniformly.
7. The filament voltage seems to have no effect on the general shape of the curve.
8. Having set up a LTE, one will find that the substitution of different tubes of the same type does not affect the linearity greatly and makes only small changes in the base of the logarithms and the dc level.

6. Complete LTE unit. The LTE unit as used in the arbitrary function generator is pictured in Figure 24, and its schematic diagram is shown in Figure 25. The actual logarithm-taking element is seen to be of the type shown in Figure 22. The cathode follower prevents loading of the triode and supplies a low impedance input. The 10,000-ohm level potentiometer allows the dc level to be adjusted

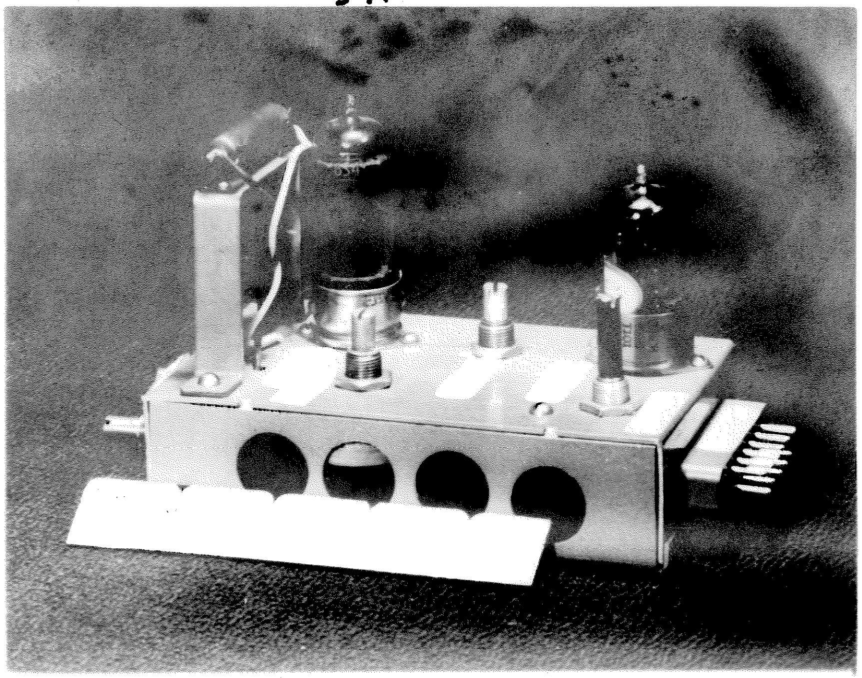
with negligible change of the base of the logarithm. The base potentiometer is essentially a gain control. After the LTE's are initially linearized, they are made completely interchangeable by the adjustment of the base and level controls.

The linearity (log scale) of a typical well-aligned LTE is demonstrated in Figure 26.

The LTE's are operated from well-regulated power supplies, the filaments being heated by a storage battery. Drift is of the order of 75 mv/hr referred to the output after several hours of operation.

Construction of the logarithm-taking elements involves no special precautions. The only lead critical as to length and placement is the one connecting the 2.2-megohm resistor (Figure 25) to the grids of the 12AX7 tube. To prevent thermal drift, the resistors in the plate circuit of the 12AX7 and the grid circuit of the 6S4 should be operated conservatively. These resistors should not be placed where heat from the 6S4 or its 20,000-ohm cathode resistor can affect them.

The nonlinear characteristic of a vacuum tube is used in this device. Can the nonlinear characteristic be depended upon to remain constant? In order to discover the effect of time upon the characteristic, the experimenters built an aging rack. Six new 12AX7 tubes from three different manufacturers were operated for 1000 hours. During this period the tubes were removed and placed in an LTE circuit several times, and their conversion characteristic was measured. Though the results were somewhat random, the conclusion was reached that the tube characteristics stabilize considerably after approximately 500 hours of aging.



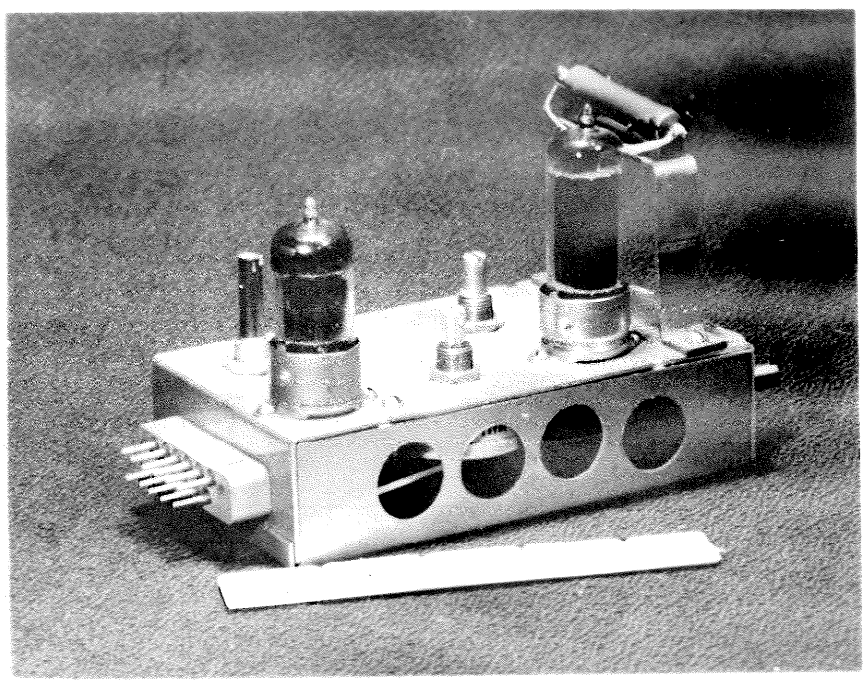
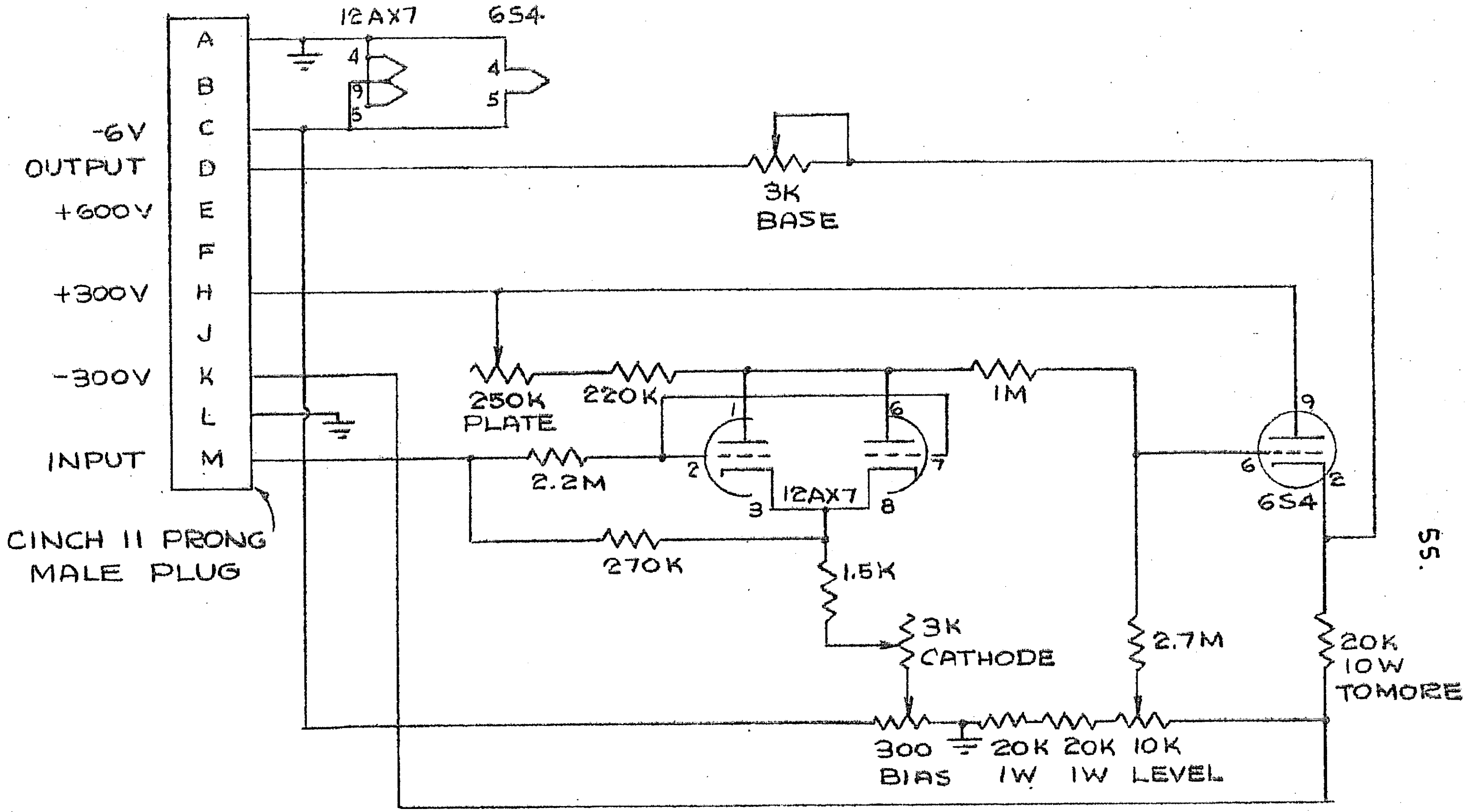


Figure 24. Photographs of Practical LTE



LOGARITHM TAKING ELEMENT

FIGURE 25

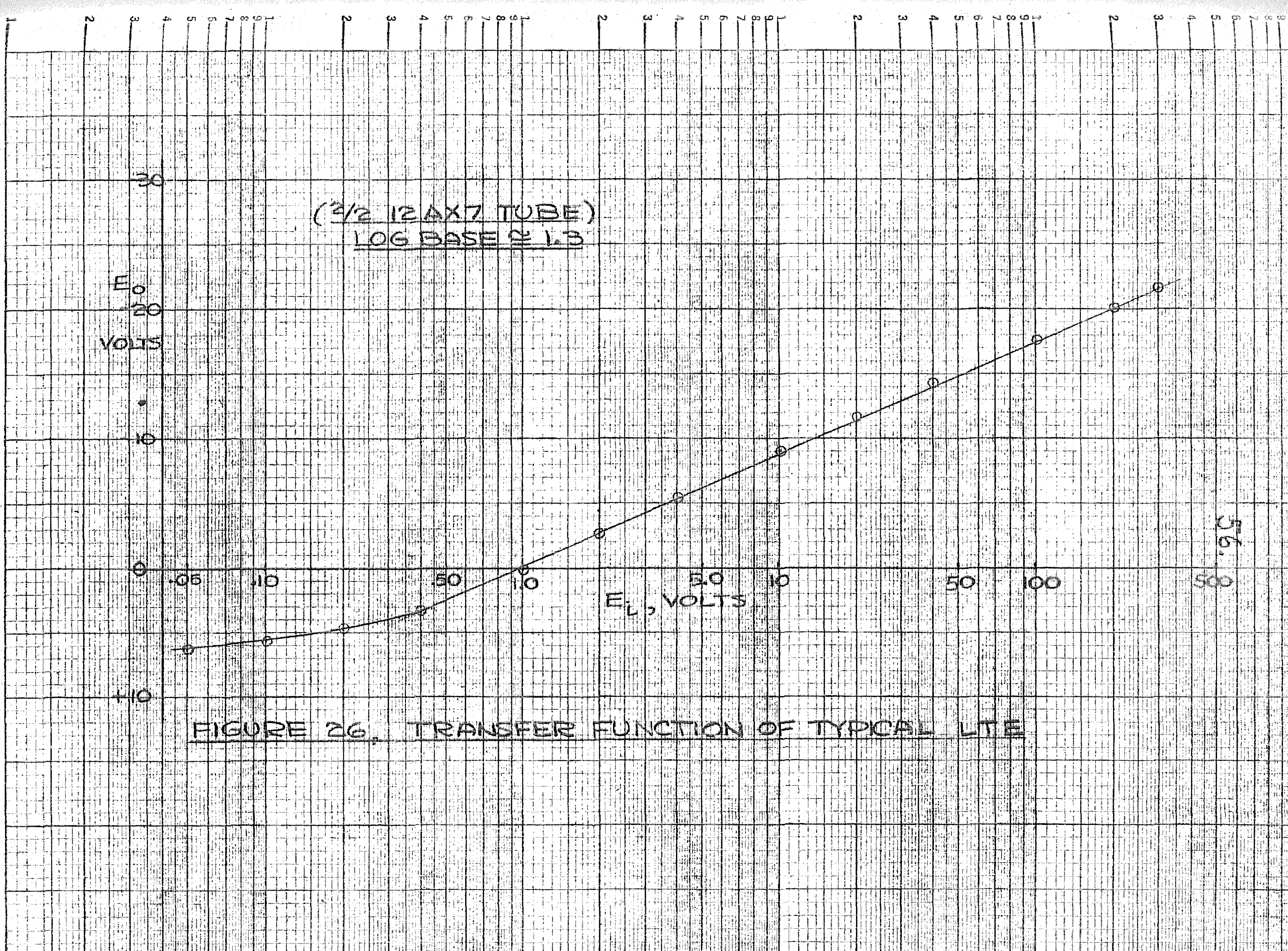
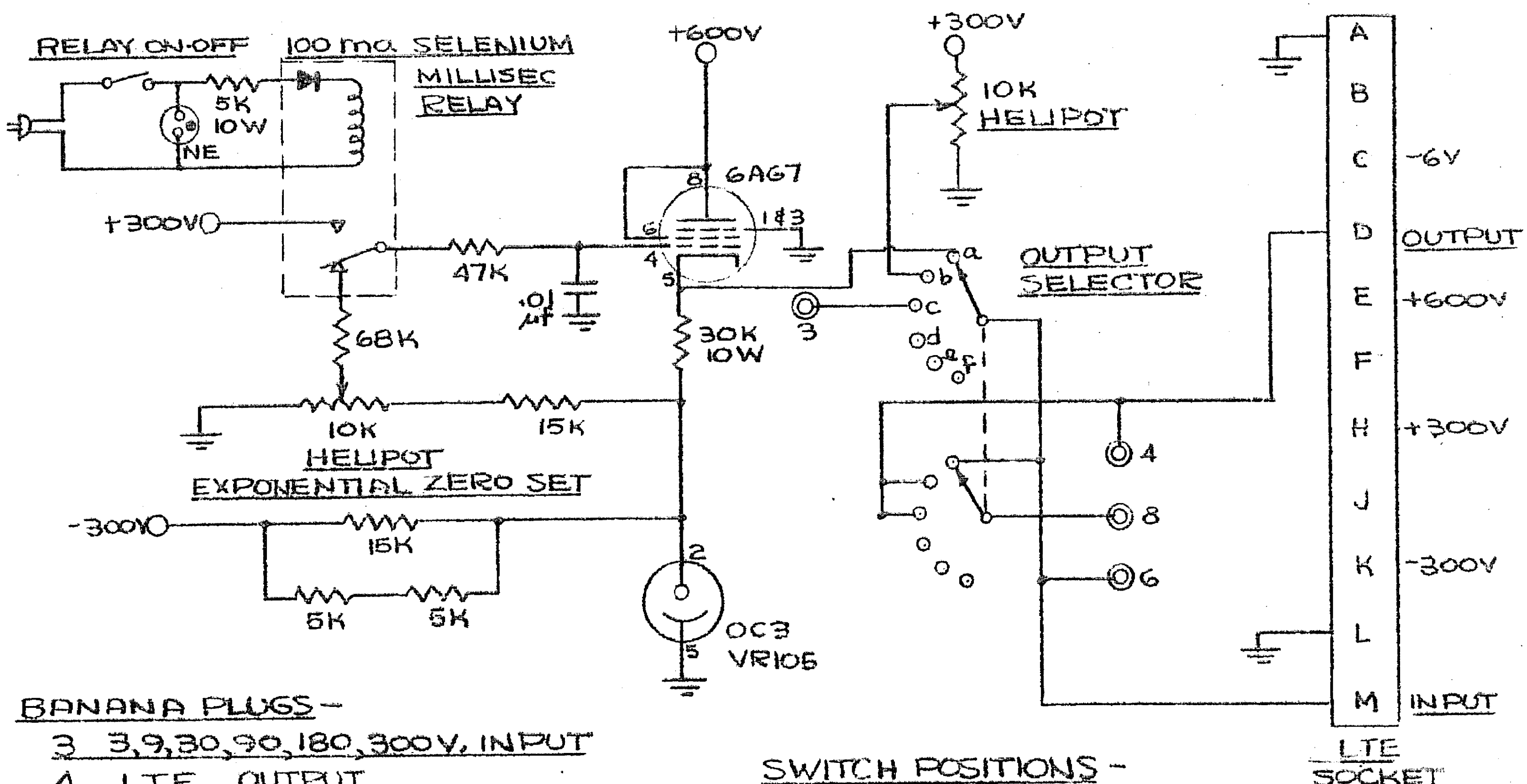


FIGURE 26. TRANSFER FUNCTION OF TYPICAL LITE

All the tubes used in the LTE now have been aged 500 hours. After using the LTE's, the author reached the conclusion that the linearity should be checked every 50 hours of operation and that the level should be set at each period of operation. In the period in which the LTE units were being built and tested, the LTE search, test, and alignment unit (Figure 18) was rebuilt. A socket was mounted that would accept the plug-in LTE units, and other changes were made to permit the adjustment procedure to be simplified. The resulting unit is shown schematically in Figure 27.

The following instructions are given to expedite the use of the unit when aligning an LTE.

1. Plug power cable into power-distribution socket.
2. Plug LTE into socket on unit.
3. Turn on all power supplies and dc filament voltage, and allow at least 15-minute warmup period.
4. Set the zero adjustment on the exponential source by switching to position a, turning off the relay and varying the zero-adjust knob until a voltmeter connected from jack 6 to ground reads zero.
5. Observe the response of the LTE to the exponential input on the oscilloscope; vary the plate, cathode, and bias potentiometers until a straight line is observed.
6. Connect the step-voltage unit input to the + 300-volt supply and ground; then connect the output to jack 3.



BANANA PLUGS –

3 3,9,30,90,180,300V, INPUT
4 LTE OUTPUT
6 LTE INPUT
8 VOLTMETER
9 CHASSIS GND.
10 CHASSIS GND.

SWITCH POSITIONS –

a. EXPONENTIAL
b. 0-300 V. CONTINUOUS
c. 3,9,30,90,180,300 VOLTS.

L.T.E. TEST & SET UP UNIT

FIGURE 27

With output selector switch at position C, apply static voltages of 3, 9, 30, 90, 180, and 300 from the test-voltage unit to the LTE; plot the output (jack 4) on semilogarithmic paper.

7. Repeat the procedure until the final plot produces the desired accuracy.

8. With 1 volt applied, adjust the level knob until the output is zero.

9. Connect the LTE to the load which it will supply, set the step-voltage unit to 300 volts, and adjust the base control until the output voltage is $\log_a 300$. The base a must be chosen so that the LTE with the poorest conversion can be adjusted. No difficulty should be encountered in adjusting all the LTE's to have a working base of 1.30 and perhaps even 1.25.

D. Inverse log-taking element

The development of a satisfactory inverse logarithm-taking element (henceforth ILTE) was initiated as soon as the LTE was an accomplished fact. The basic idea is simple, consisting merely of putting an LTE in the feedback loop of a high-gain amplifier. This theory was given a rough check through use of a pair of analysis laboratory amplifiers with a net gain of approximately 2000, one LTE in the feedback loop and two LTE's as input converters. This crude computer, when stabilized by the addition of a large lumped capacity, multiplied two voltages together with fair accuracy, but the frequency response was uniform to only a few cycles per second.

1. Analysis. The following analysis was then carried out in an attempt to discover the basic design requirements of the LTE. Consider the circuit of Figure 28. The drift voltages of the amplifier

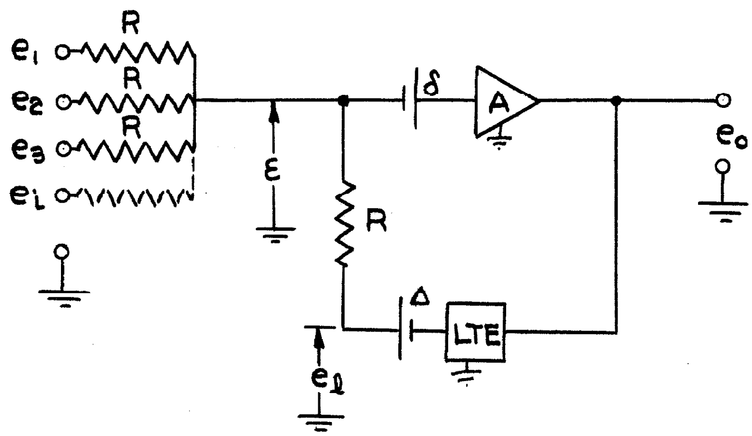


Figure 28. Basic Circuit of ILTE

(referred to the input) and of the LTE (referred to the output) are shown as δ and Δ , respectively. Equation (44) results from the application of the superposition theorem, Equation (45) is the defining equation of the amplifier, and Equation (46) is the defining equation of the LTE. All voltages shown are measured to ground.

$$\frac{e_\ell + \sum^{n} e_i}{n+1} = \varepsilon \tag{44}$$

$$(\varepsilon + \delta) A = e_o \tag{45}$$

$$\Delta - \log_a e_o = e_l \qquad (46)$$

Substituting Equation (44) in Equation (45)

$$\left[\frac{e_l + \sum^n e_i}{n+1} + \delta\right] A = e_o \qquad (47)$$

Substituting Equation (46) in Equation (47)

$$\left[\frac{\Delta - \log_a e_o + \sum^n e_i}{n+1} + \delta\right] A = e_o \qquad (48)$$

Rearrange Equation (48)

$$\log_a e_o = \sum^n e_i + \Delta - \frac{e_o(n+1)}{A} + \delta(n+1) \qquad (49)$$

Take the antilogarithm to the base a

$$e_o = \exp_a\left[\sum^n e_i + \Delta - \frac{n+1}{A} e_o + (n+1)\delta\right] \qquad (50)$$

which may be written

$$e_o = \left\{\exp_a\left(\sum^n e_i\right)\right\}\left\{\exp_a\left(\Delta + (n+1)\delta - \frac{n+1}{A} e_o\right)\right\} \qquad (51)$$

Equation (51) shows that the circuit will produce an output voltage which is the product of two factors. The first factor represents the desired result, and the second factor represents the error inherent in the device. Since we know nothing of the polarity of δ or Δ , consideration of the worst case (i.e., that in which the errors resulting from δ , Δ , and e_o have the same polarity) is reasonable. Equation (51) can then be expressed approximately in the following form, assuming that the drift voltages are small and that the amplifier gain is high:

$$e_o \cong \left\{ \exp_a \sum^{n} e_i \right\} \left\{ \left(1 + [\Delta + (n+1)\delta] \ln a\right)\left(1 + \frac{(n+1)e_o}{A} \ln a\right) \right\} \tag{52}$$

From Equation (52) one can write the two following approximate expressions for the percentage errors resulting from drift $\mathcal{E}_d$ and lack of sufficient gain $\mathcal{E}_g$, respectively:

$$\mathcal{E}_d \cong 100 \left\{ \Delta + (n+1)\delta \right\} \ln a \tag{53}$$

$$\mathcal{E}_g \cong 100 \frac{n+1}{A} e_o \ln a \tag{54}$$

2. Performance. Assuming that the following values of the parameters are valid, the values of $\mathcal{E}_d$ and $\mathcal{E}_g$ are calculated, using Equations (53) and (54):

Δ = 50 mv	A = 8000
δ = 25 mv	n = 2
a = 1.2	$\mathcal{E}_d$ = 2.3%
e_o = 300 volts	$\mathcal{E}_g$ = 2.1%

The assumptions just made are somewhat optimistic; yet the experimental results show that the ILTE is considerably more accurate than is predicted and that it has less drift. Figure 29 shows the linearity (log scale) of the output of an ILTE when plotted versus the input; one can see that the largest error is approximately 1 per cent of the maximum output. Since the LTE in the ILTE was adjusted approximately before the loop was closed, the error resulting from having insufficient gain was balanced out almost completely. If other ILTE's were to be built, the design could be simplified somewhat, and the initial stabilization could be made much easier by a reduction of the gain of the amplifier. The adjustment procedure would become more arduous, however. The drift as observed in operation is of the order of 1 volt per hour with 3.0 volts out and 3 volts per hour with 300 volts out. There must be some cancellation of drift, though this effect was not intentionally designed into the circuit.

Frequency response of a nonlinear device of this type is difficult to define, as neither the input nor the output is sinusoidal. One must remember that both the LTE and the ILTE are essentially unidirectional and that they are of little use without the associated input and output switches. The over-all frequency response of the switching circuits, the LTE, and the ILTE is to be found in Section IV-E.

3. Circuit details. The complete schematic diagram of the inverse logarithm-taking element is shown as Figure 30, and a photograph is given as Figure 31.

There are three inputs to the ILTE, although normal use demands only two. A slight difference in base exists between the operation

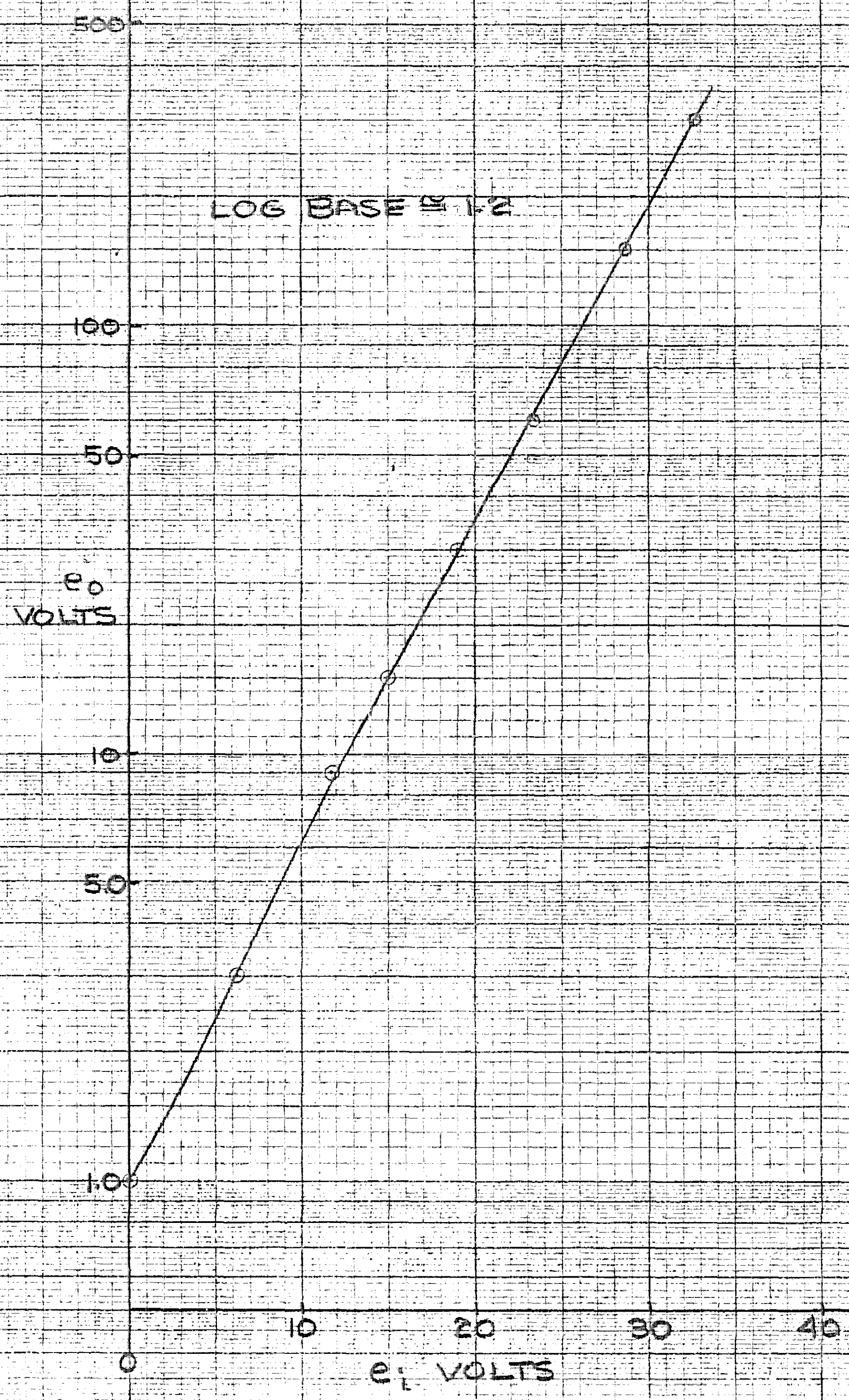
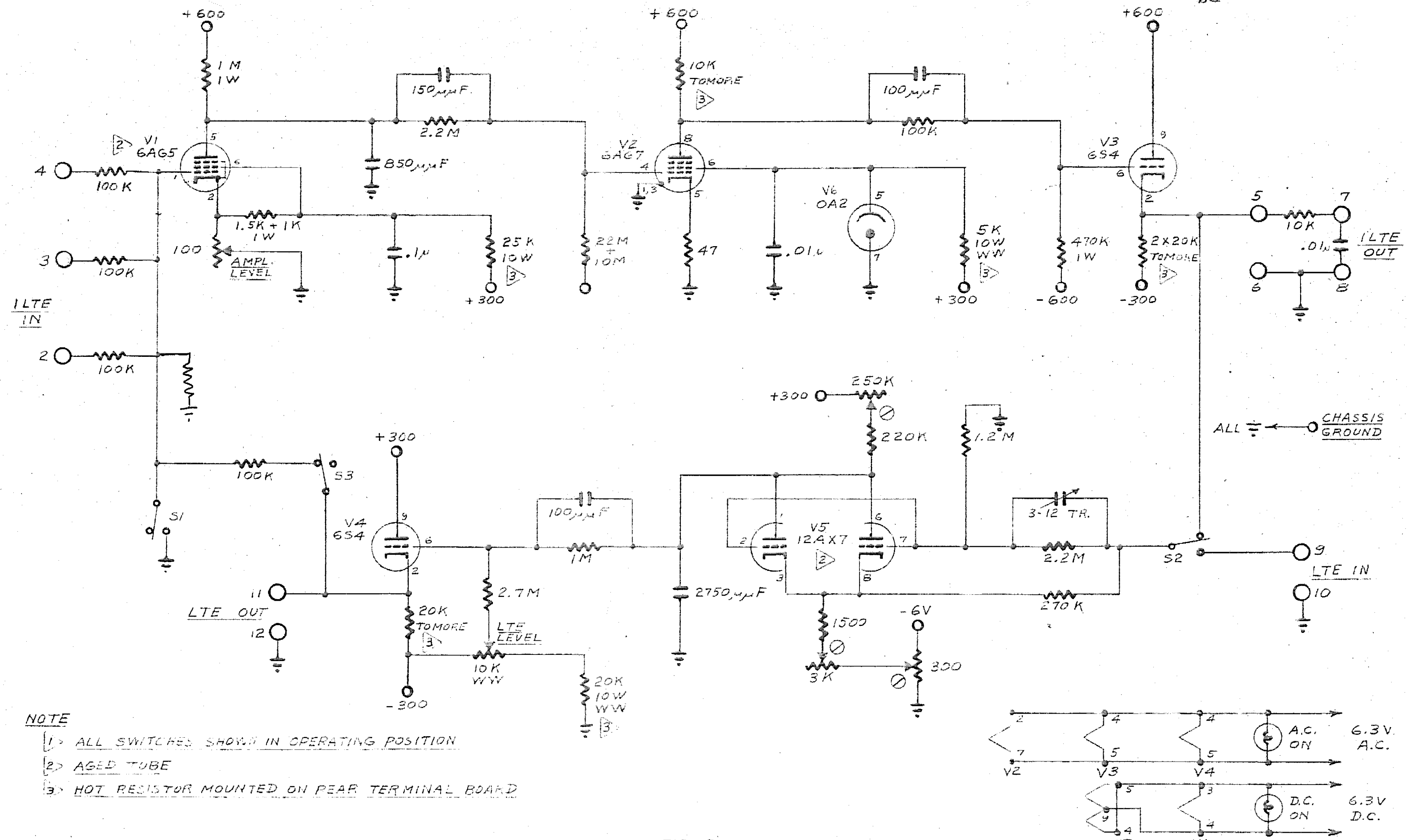


FIGURE 29
TRANSFER FUNCTION OF ILTE

when the unused input is grounded and when it is left unconnected. The 10,000-ohm resistor from the summing point to ground reduces the effective gain of the amplifier and makes the base of the ILTE less sensitive to changes of output impedance of the devices driving the ILTE. The positive-gain amplifier is of nearly standard design. The first stage has very low screen voltage and a large plate resistor. It operates in a high-gain region in which its gain is approximately 1000. A large ratio exists between the cutoff frequencies of the two stages. The first stage cuts off in the low audio range, whereas the second stage cuts off at about 500 kc. Phase-correction networks are necessary between the stages. Gain of the amplifier is 75,000, but the effective gain is reduced to 8000 by the 10,000-ohm resistor mentioned above. The 6S4 cathode follower gives the ILTE a low output impedance, and the phase-lag network which feeds terminal 10 is available when needed to reduce the high-frequency response of the unit.

The LTE part of the circuit is similar to that shown in Figure 23 except for modifications required to make the loop stable. Since the LTE is a nonlinear device, its amplitude and phase response are terms that must be defined. For the purpose of obtaining expressions useful in the attempt to stabilize the loop, one can assume that the signal will consist of a constant voltage plus a smaller sinusoidal voltage. Then the usual linearizing assumptions are made, with the provisos that the gain and phase shift of the

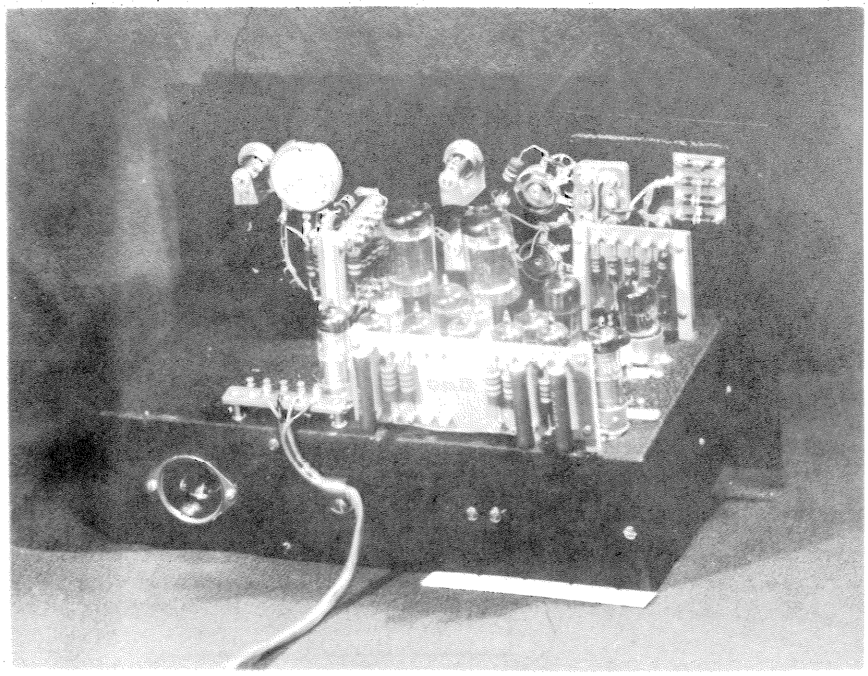


NOTE

1. ALL SWITCHES SHOWN IN OPERATING POSITION
2. AGED TUBE
3. HOT RESISTOR MOUNTED ON REAR TERMINAL BOARD

FIG. 30

INVERSE LOG TAKING ELEMENT



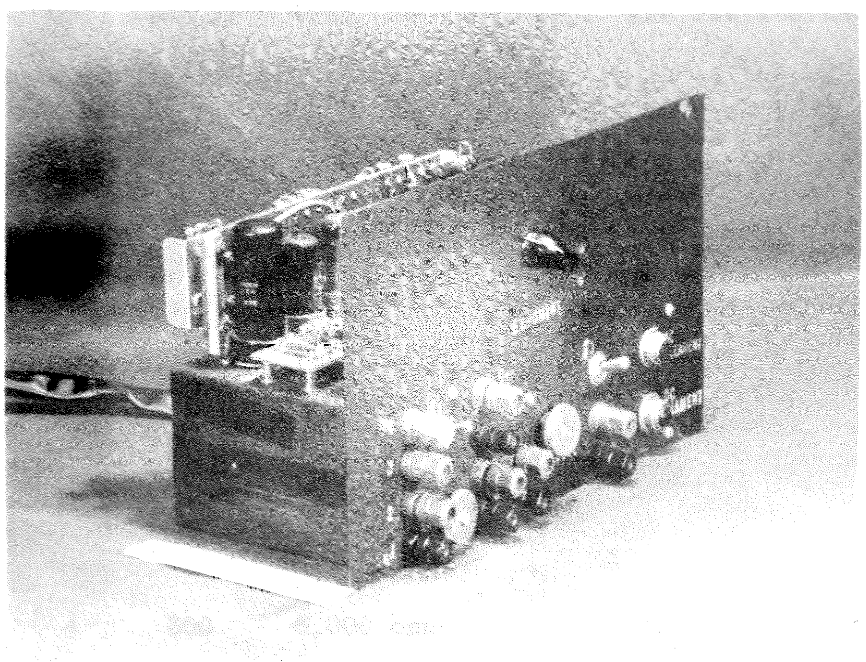


Figure 31. Photographs of ILTE

stage are functions of the constant term of the input voltage. When the system is oscillating, these assumptions are worthless; as criteria for the possibility of oscillation, however, they are valid. The LTE has a gain, then, that is inversely proportional to the amplitude and an input circuit that can be approximated by the circuit of Figure 32. The capacity C will be the Miller input

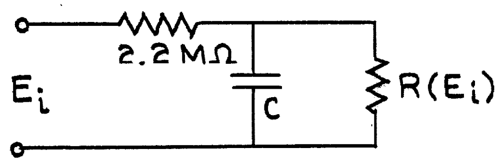


Figure 32. LTE Equivalent Input Circuit

capacity and will thus vary with the frequency. The resistance R was found on a particular LTE to vary with the input voltage E_i in the following manner:

E_i	$R(E_i)$
0	not defined
3	190,000 ohms
9	85,000 ohms
30	46,000 ohms
90	27,000 ohms
300	6,000 ohms

By making some assumptions as to the variation in input capacitance and by adding the 1.2-megohm shunt resistor, one is able to pick a value for a condenser to be placed in parallel with the 2.2-megohm resistor that will never allow phase lag over the frequency range in which oscillations could occur. The design as presented is extremely conservative as far as stability of the loop is concerned, and the frequency range of the ILTE could be extended considerably if required. One point of difficulty might be mentioned. In the original design, a Helipot was included in the output circuit of the LTE. This is a desirable feature, as it allows easy adjustment of the base to which the input is raised. However, the Helipot gave sufficient phase shift at high frequencies to make the stabilization of the loop very difficult, particularly because the phase shift is a function of the Helipot knob setting.

The ILTE is easy to adjust initially and requires no adjustment after that time except to cancel out drift and perhaps, after many hours of operation, to cancel out the effects of further aging of the 12AX7. Switches provided on the panel take the LTE out of the loop and bring its input and output terminals to the panel. After following the normal LTE adjustment procedure on the LTE, one closes the loop; the remaining small adjustments are made by trial and error, direct current being used on the input. After one has aligned the LTE's, alignment of the ILTE proves to be a simple matter.

The base of the LTE will depend upon the tube used and will be determined from the graphs used in the linearity adjustment. The dc level is adjusted by grounding the inputs and setting the output to 1 volt by use of the level control. The variable resistor in the cathode of the 6AG5 also is used to adjust the level, if necessary. Maladjustment of either of these controls gives the result predicted by Equation (52); that is the output will be of the following form:

$$e_{\circ} = K \exp_a e_i \tag{55}$$

Under certain conditions of large-signal input, adjustment of the level controls so as to give a $K > 1$ allows the ILTE to yield a larger undistorted output.

E. Switching problem

Completion of the development of the logarithm-taking element and inverse-logarithm-taking element resulted in the obtaining of basic mathematical elements required for the arbitrary function generator. These components alone cannot be used to any advantage unless the dependent-variable input to the function generator is always positive. In order that the function generator could enjoy more general usage, input and output switching circuits were developed. The input switching device acts very much as the center-tapped transformer and rectifier tube do in a full-wave power supply; that is, for either polarity of input, the output would always have the same polarity. Because the LTE can accept only positive input signals, the input switch always yields a positive output. The problem at the output of the ILTE is greater, for the unidirectional output voltage must be converted into an alternating voltage. The sensing problem would be very simple if only one input signal were used, but often there will be more than one input. If two or more input signals are used, a sensing circuit must be provided which will compare their polarities and determine the polarity that the output should have. The output switching circuit controlled by the sensing circuit will then yield the desired function.

1. Generation of cubic function. Perhaps the easiest way to gain an understanding of the polarity inverters is to consider an example of their use. The over-all operation of the function generator will be demonstrated with the use of oscillograms taken at different points in the circuit shown in block form as Figure 33. For the purpose of illustration, the generation of a function of the type indicated in Equation (43) will be shown. The exponents ℓ, m, and n will be set to unity so that the resulting function will be cubic. The constants b and c will both be negative. The result is given here for reference.

$$y = ax\,(x+b)\,(x+c) \tag{56}$$

Because the example will demonstrate the generation of a cubic function, a restriction is placed upon the range of the independent variable x by lack of equipment. Only two input polarity inverters are available and the output polarity inverter will accept only two sensing signals. The linear sweep generator supplying the independent variable is adjusted so that its output never goes negative, and thus LTE_1 can be operated without an inverter.

Figures 34 through 41 are oscillograms showing the voltage waveforms throughout the function generator when only the (x + c) factor of Equation (56) was used. Figure 34 shows the waveform

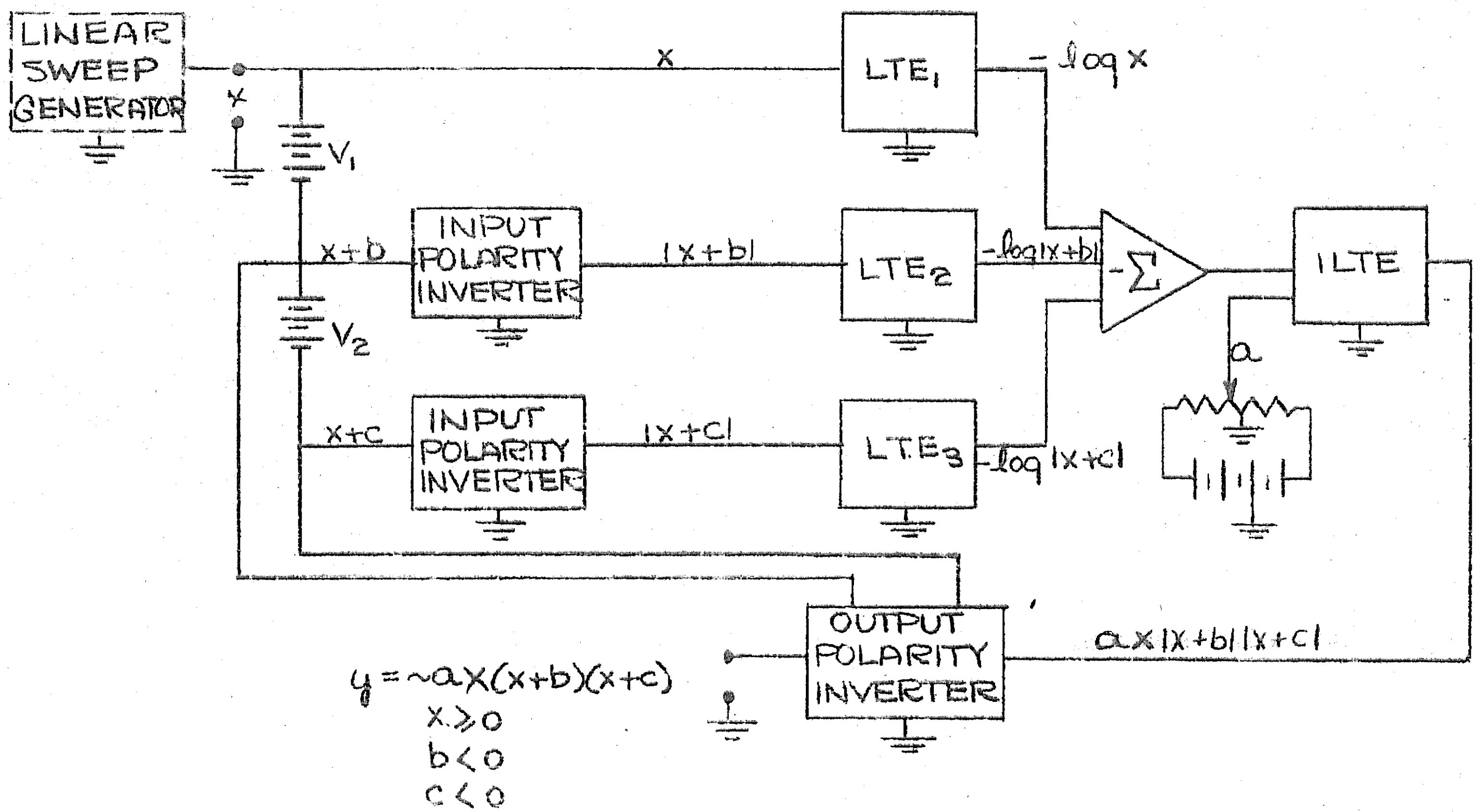


FIGURE 33
GENERATION OF A CUBIC FUNCTION
BLOCK DIAGRAM OF ARBITRARY FUNCTION GENERATOR

entering the input polarity inverter. The output is shown in Figure 35. The output of the LTE_3 is shown before and after inversion by the summing amplifier in Figures 36 and 37. The unidirectional output of the ILTE is shown in Figure 38. The positive- and negative-channel outputs of the output polarity inverter are shown in Figures 39 and 40. The sum of these two outputs is the output of the function generator and is shown in Figure 41. Comparison of Figures 34 and 41 shows the linearity of the function generator.

Figures 42 and 43 show the input and output waveforms for the X factor (Eq. 56). Figures 44 and 45 likewise show the waveforms for the (x + b) factor.

Figure 46 is an oscillogram showing the output of the function generator when only two factors are used.

$$y = a\,(x+b)(x+c) \tag{57}$$

Equation (57) shows the function. The output polarity inverter gives an inverted output when the sensing signals have the same polarity. Thus the a in Equation (57) must be negative for agreement with the oscillogram.

Figure 47 shows the function given by Equation (56), and the cubic form can be recognized.

In the example just shown, the sensing circuits were required to inspect the voltages corresponding to (x + b) and (x + c) where they entered the input polarity inverters. If these factors had the same

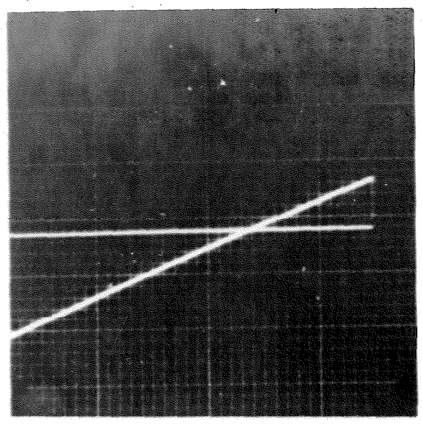


Figure 34

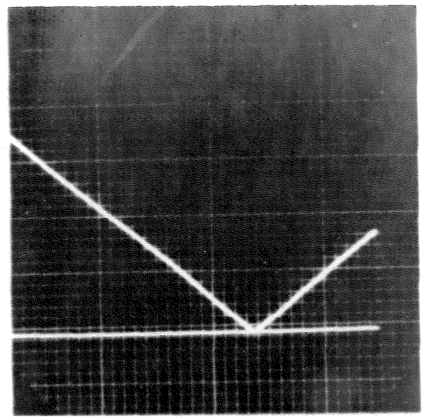


Figure 35

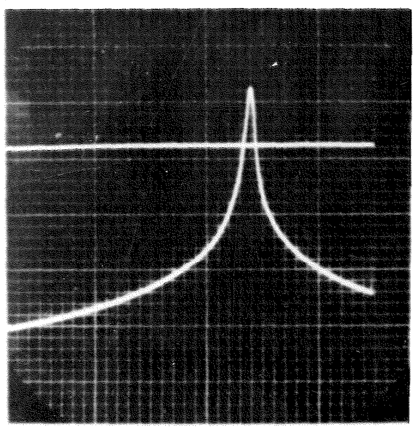


Figure 36

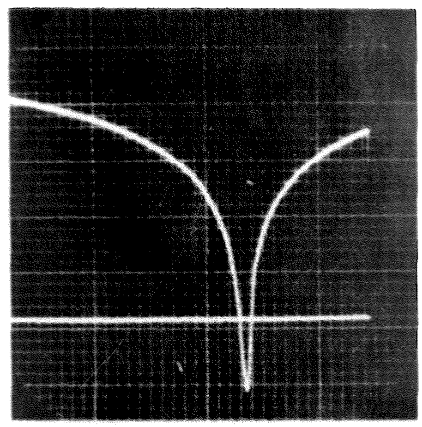


Figure 37

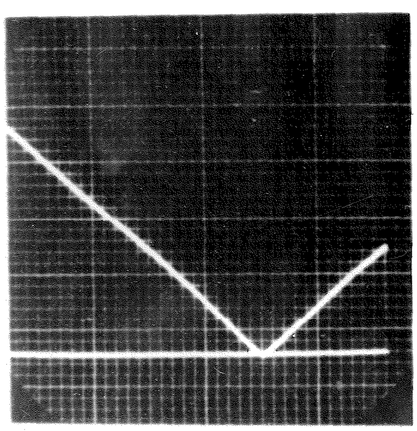


Figure 38

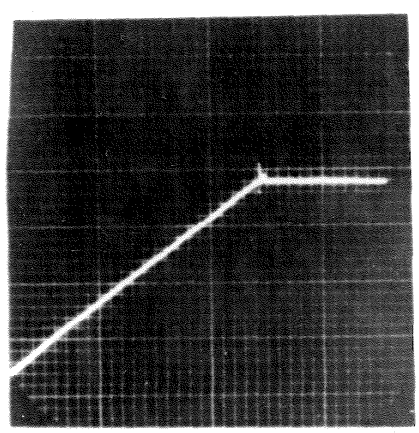


Figure 39

Oscillograms Demonstrating the Operation of the Function Generator

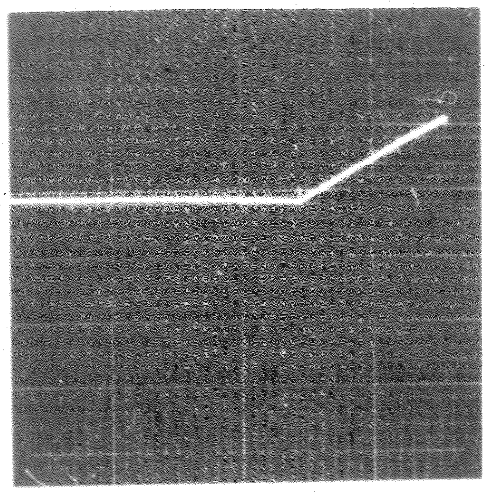


Figure 40

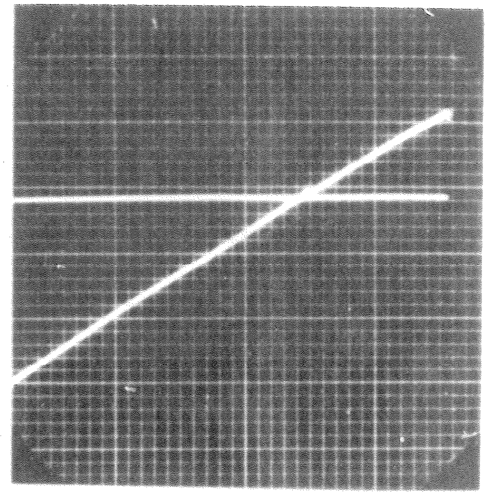


Figure 41

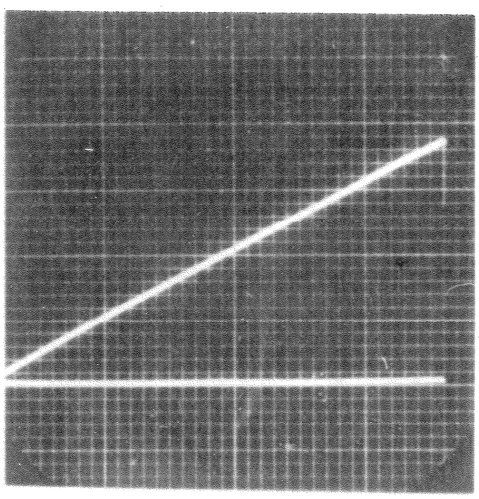


Figure 42

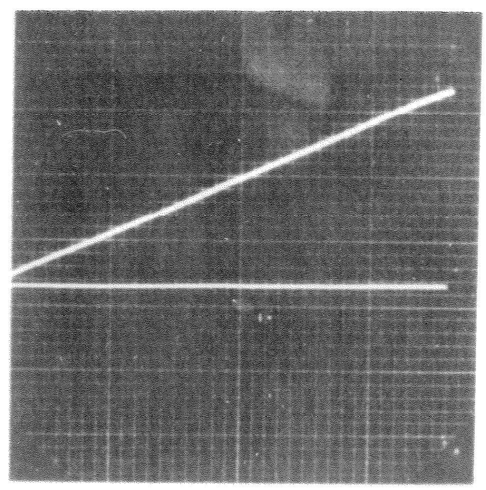


Figure 43

Oscillograms Demonstrating the Operation of the Function Generator

polarity, the output polarity inverter inverted the signal coming from the ILTE. If the factors had opposite polarities, the signal was not inverted.

2. More general functions. Consider now the problem of generating the function of Equation (43) when ℓ , m, and n are not positive integers. Negative integers ℓ , m, and n correspond to division and require only reversal of the input to the ILTE. If any one of the exponents is noninteger, the mathematical problem becomes more difficult, particularly when the input to an input switch goes negative. The previously mentioned similarity of the function generator to the slide rule is evident in this case. Both devices deal only with absolute magnitudes of numbers. Normal operation of the function generator when producing a function of the type shown in Equation (43) is to give the magnitude of y that results if the absolute value of each factor is raised to power ℓ , m, or n. The polarity of y is that of the product of $x(x + b)(x + c)$. Equation (43) should be written in the following manner, if the true operation of the generator is to be shown:

$$y = a|x|^{\ell-1} \cdot |x+b|^{m-1} \cdot |x+c|^{n-1} \cdot x(x+b)(x+c) \qquad (58)$$

3. Input polarity inverter. This inverter is shown schematically in Figure 48, and a photograph of one is given as Figure 49. The circuit is straightforward, consisting of one positive and one negative gain dc amplifier and a double rectifier. Both amplifiers

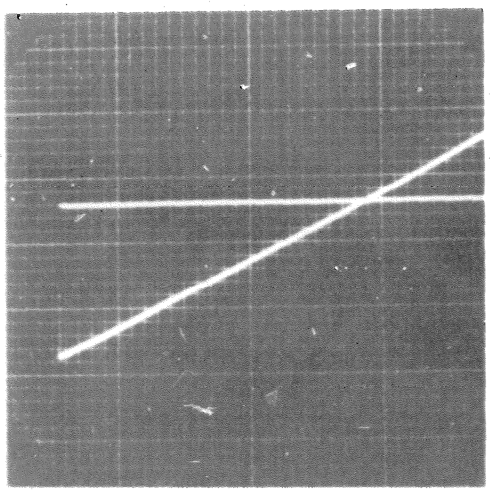


Figure 44

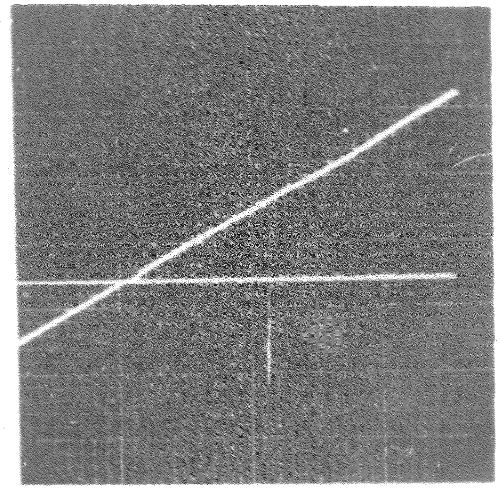


Figure 45

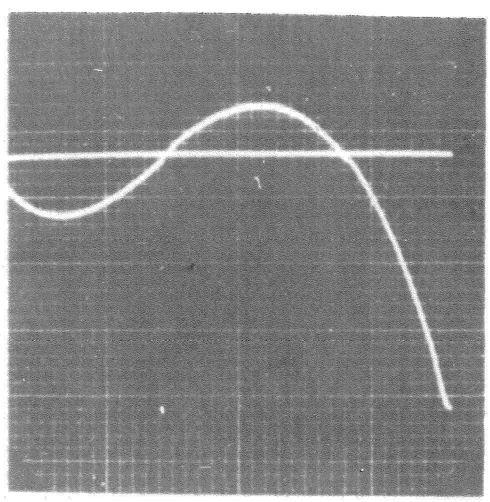


Figure 46

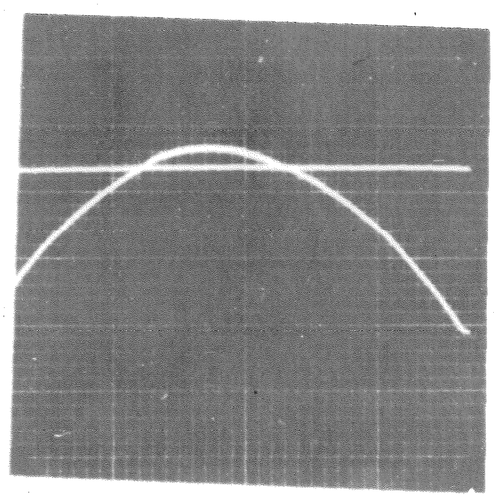


Figure 47

Oscillograms Demonstrating the Operation of the Function Generator

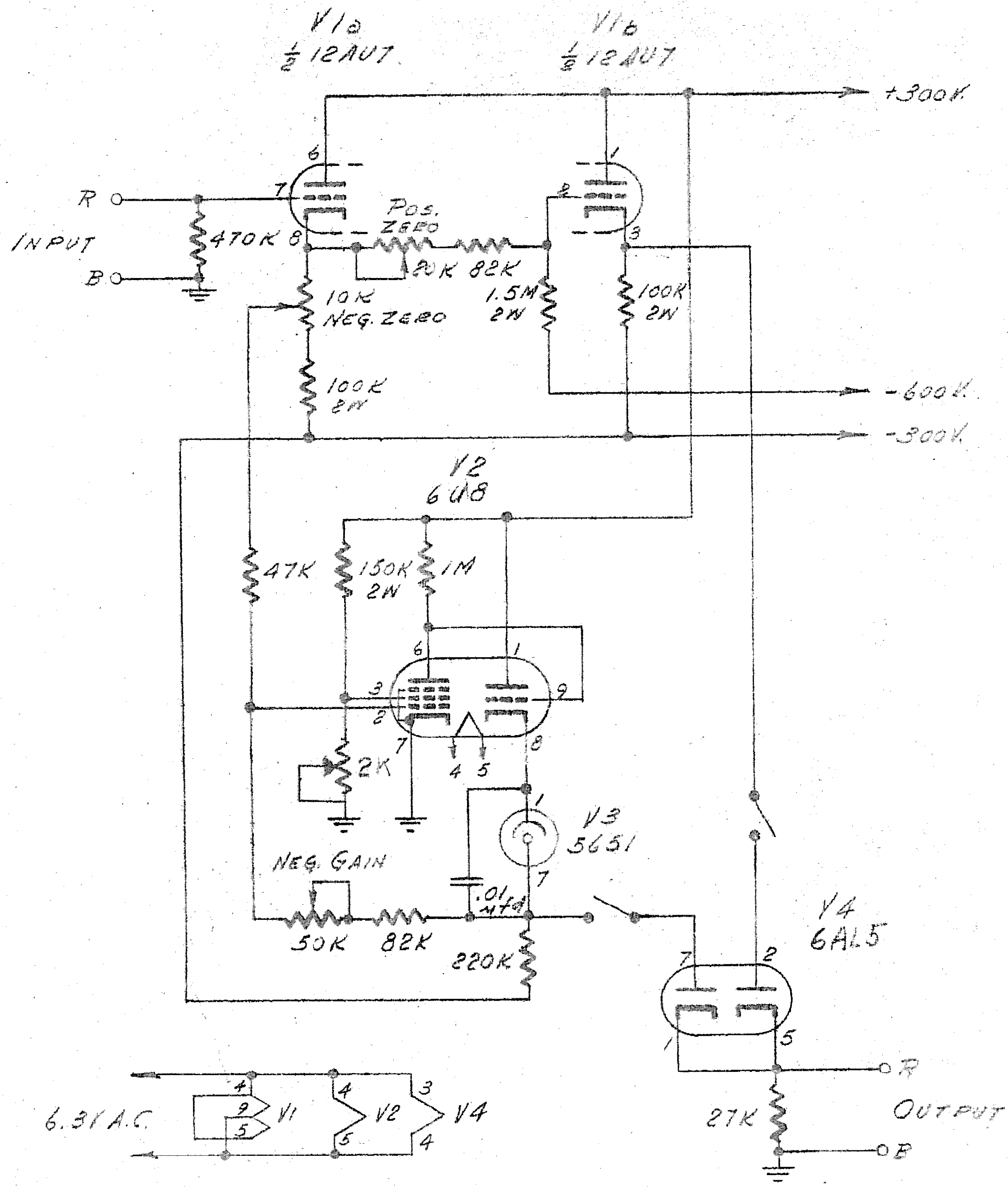
have a large amount of inverse feedback, and as a result the unit is stable and linear. The circuit complications result from the fact that both the input and output must work with respect to ground, and thus several level changes are required. The gain of the input polarity inverter is approximately 3/4, and the peak input voltage is 100. Drift is less than 1/10 volt per hour.

Switches are provided that allow for using either of or both of the input polarities. These switches are convenient when the unit is being set up and are useful when certain types of asymmetrical functions are being generated.

The setup procedure consists of the following steps:

1. Turn on the input polarity inverter, and allow it to warm up for 10 minutes.
2. Short the input.
3. Connect the voltmeter to the output terminal.
4. Put positive input switch in "on" position (up); set output to + 50 mv using the positive zero control.
5. Put in + 45 volts and measure the output voltage.
6. Short the input.
7. Put negative input switch in "on" position; set output to + 50 mv using negative zero control.
8. Put in -45 volts and adjust negative gain control until the output voltage is the same as was attained in step 5.

The input polarity inverter is a stable, dependable device. Its only failing is its inability to yield larger output voltages. Since



SWITCHES SHOWN IN OFF POSITION (DOWN)

FIG. 48

INPUT POLARITY INVERTER

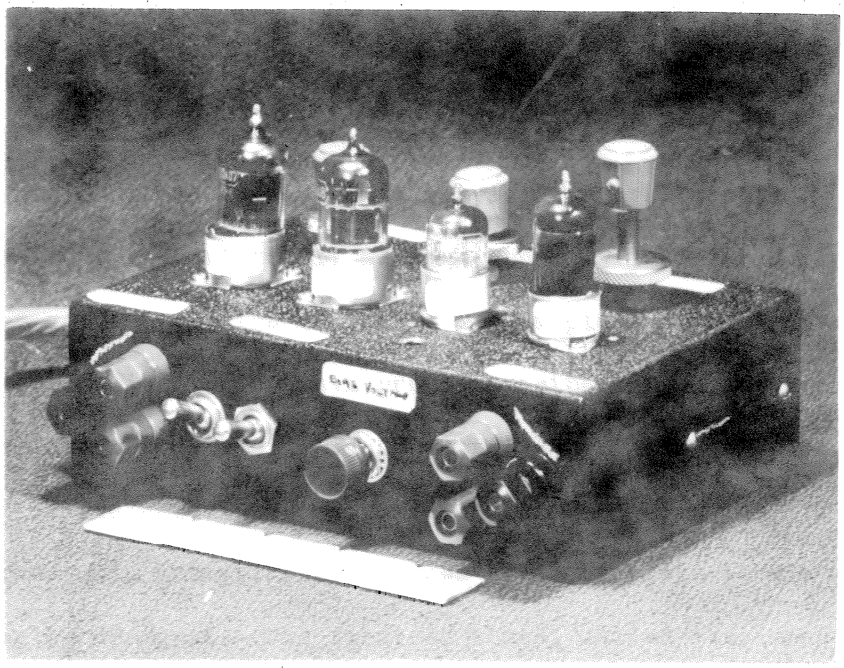


Figure 49. Photograph of Input Polarity Inverter

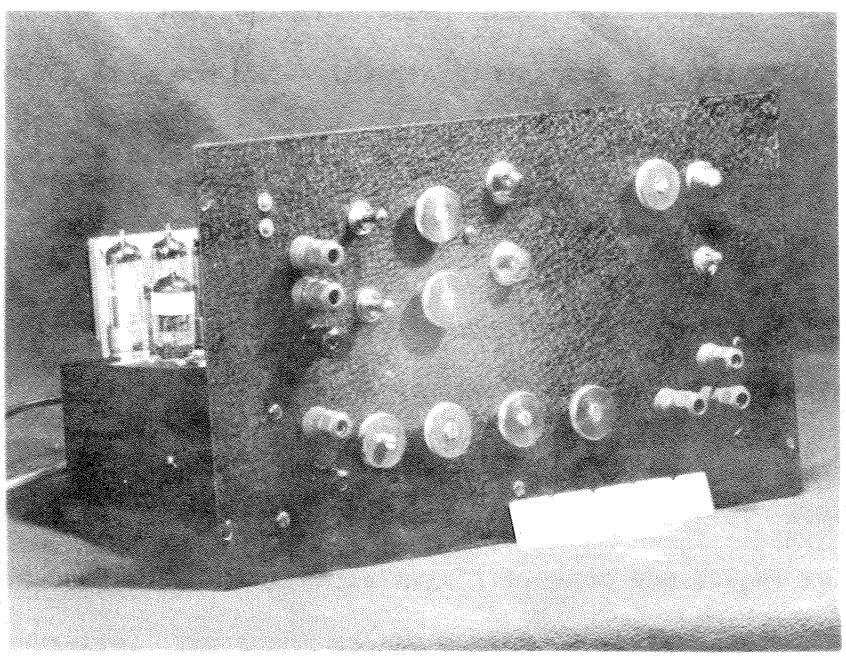


Figure 50. Photograph of Output Polarity Inverter

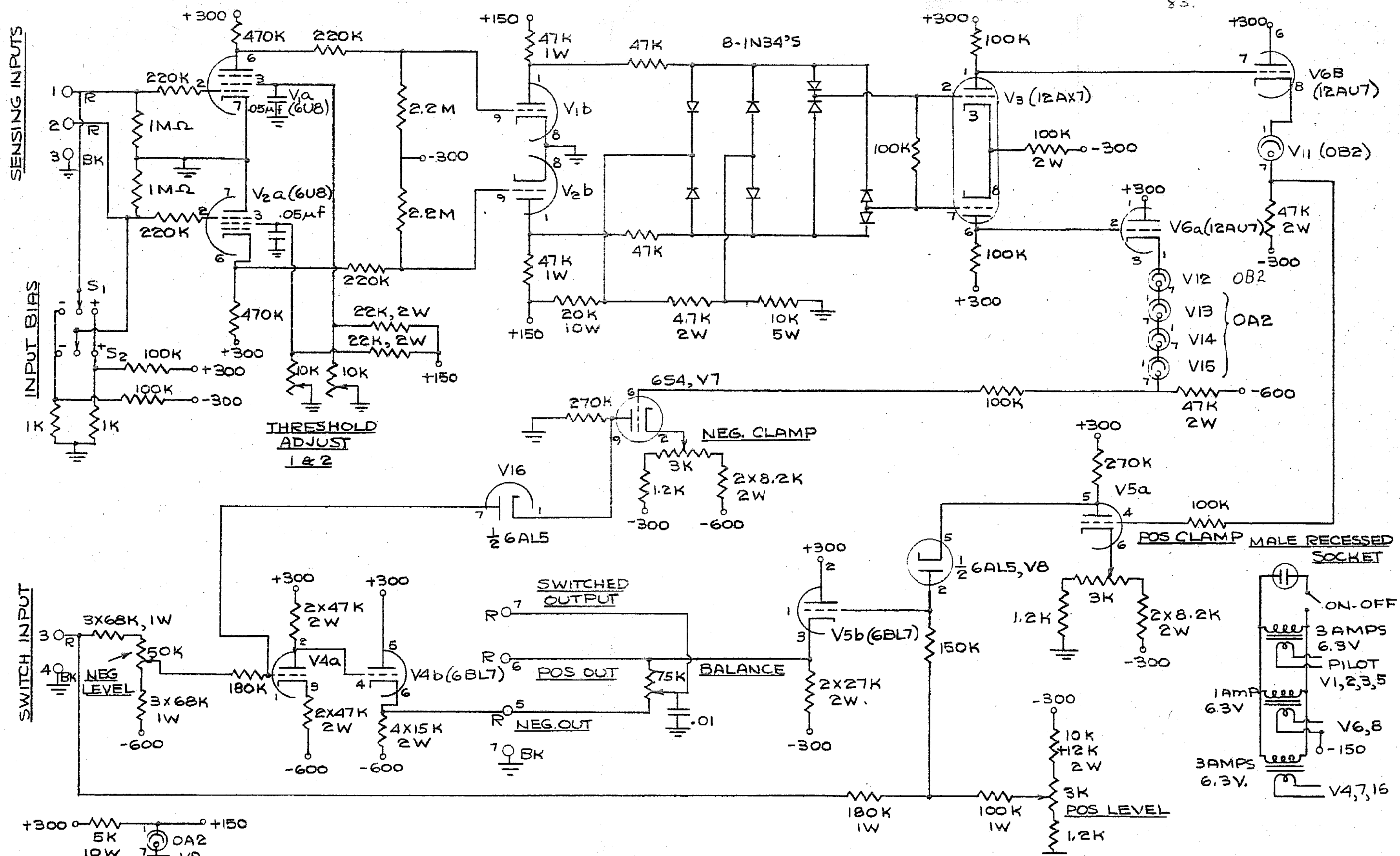
the LTE's are most accurate with large input signals, the input inverters should be redesigned so as to yield 300 volts with an input as small as + 50 volts, if maximum accuracy is desired from the function generator. The design of such a piece of apparatus should not be difficult, the main problems being those of level changing and drift stability. The inverter which has been designed and built is satisfactory for most uses. The fact that the LTE and ILTE are inherently monopolarity devices called for the development of two polarity changers. The first of these, called the input polarity inverter, has been described.

4. Output polarity inverter. This inverter, pictured in Figure 50, is considerably more complicated than the input polarity inverter because of its logical system. The basic need is for two parallel amplifiers, one which inverts the signal and the other which does not. Circuits must be included which will allow only one amplifier to operate at a time, thus effectively controlling the polarity of the output. This control must be a function of the polarities of two other signals applied to the sensing input terminals. Suppose that two voltages are to be multiplied together to obtain their product voltage. If both input voltages have the same polarity, then the output voltage should also be positive. However, if either of the input voltages is negative, then the output voltage should be negative. The input to the output polarity inverter will always be positive because it comes from the output of the ILTE. The sensing input terminals of the output polarity inverter can be

connected elsewhere in the circuit to points having the proper input polarities, and the circuits included in the output polarity inverter will give a signal to the output terminals of the proper polarity. The complete schematic diagram of the output polarity inverter is shown as Figure 51.

Three dual-purpose tubes and eight germanium diodes are required in the logical section of the output polarity inverter. The first two tubes, V1 and V2, constitute two parallel, very-high-gain amplifiers. The outputs of these tubes, after maximum and minimum level clamping, give the information as to whether their respective sensing input terminals were positive or negative. The form of the voltage output is at either one of two levels. The succeeding bridge rectifier and differential amplifier compare these two voltages, and the voltages at the plates of V3 indicate whether or not the sensing inputs are of the same polarity.

The cathode follower V6 and the associated voltage reference tubes V11 through V15 transfer the result of this polarity comparison to the grids of the positive and negative clamping amplifier tubes V5 and V7. The grid-voltage swing available for these tubes is of such magnitude and level that the tubes are either cut off or else operate with their grids positive. From the nature of the circuit, it may be seen that only one of these two triodes would be conducting at any instant. The plates of these clamping triodes are coupled to the two amplifier channels by diodes V8 and V16. The positive-gain amplifier consists merely of an input attenuator, which includes the



POLARITY SENSER AND OUTPUT INVERTER

FIGURE 51

positive level control and the amplifier stage. The amplifier is a cathode follower. The negative-gain channel is similar except for the addition of a highly degenerated inverting amplifier. Both amplifiers are clamped in the same manner. When a channel is open, the clamping tube controlling this channel is cut off; then the coupling diode has no effect. However, when the channel is clamped, the cathode of the diode is lowered to a voltage controlled by the appropriate positive or negative clamp control; then the grid is unable to move with the signal applied. Complete clamping is not obtained in either channel, and the gains of the two channels are not equal; the balance control, however, adjusts the summing ratio to give a symmetrical output. The outputs of both channels are brought out to panel terminals for ease of adjustment and service. The switching transients developed are filtered out by the addition of the 0.01 μf capacitor from output terminal to ground and the resistance of the balance potentiometer.

The output-voltage range of this device was limited to + 50 volts by the building in of an constant attenuation of about 8. Since the ILTE saturates with an output voltage of +300 volts, the output polarity inverter will never saturate, nor will the analysis laboratory amplifier, if its gain is set to -2. The reason why the analysis laboratory amplifier was used is that often a very low output impedance is desirable, and the experimenters considered that the project would be more economical of time and money if these amplifiers were used instead of building a very low impedance output into the output polarity inverter. The circuit is arranged so that the inverter will give a negative output voltage when the two sensing inputs are of the same polarity. Only minor changes in the circuit would be necessary to reverse this output polarity. There are two switches on the front panel labeled input bias. When only one signal is to be sensed for polarity, the switch corresponding to the other sensing channel can be set to connect either a positive or negative reference voltage into this channel. Such an arrangement gives complete flexibility to the output polarity. For convenience in testing, a switch is provided for each input.

The sensing circuits require 3 volts rms for essentially perfect switching of the output. Figure 52 shows a 20-volt rms output when the sensing voltage was 1.4 volts rms. The small deviation from sinusoidal may be noted at the origin. Figure 53 shows the same output when the sensing signal is increased to 7.0 volts rms. The frequency response of the output polarity inverter is demonstrated by Figures 54 and 55. The input to the polarity inverter is a constant voltage. In Figure 54 the sensing input frequency is 20 cyc/sec, and in Figure 55, 200 cyc/sec. The amplitude of the sensing signal is 7.0 volts rms.

Drift in the first model of the output polarity inverter is about 1 volt per hour on each channel; thus it is the worst offender in the function generator. This drift results principally from the heating of resistors, and an improved physical design combined with the use of low-temperature-coefficient resistors could no doubt reduce

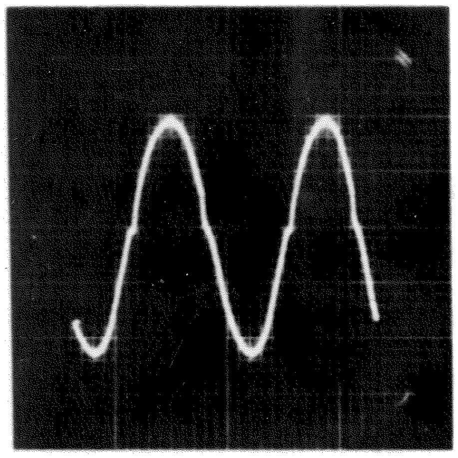


Figure 52

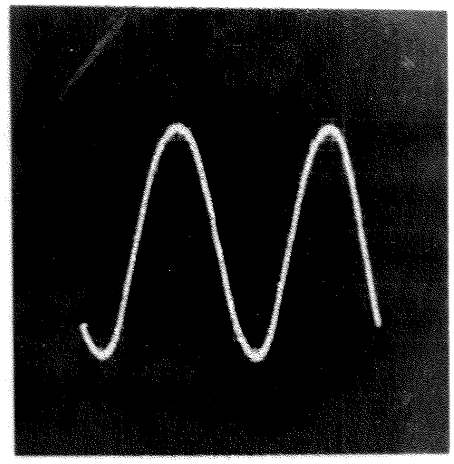


Figure 53

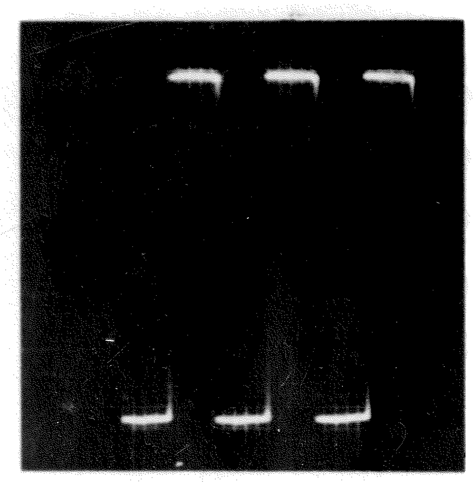


Figure 54

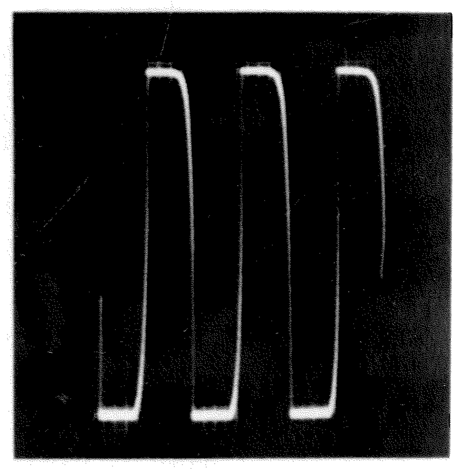


Figure 55

Oscillograms, Output Polarity Inverter

the value by a factor of 2 or 3. The frequency response of the polarity inverters exceeds those of the LTE and ILTE. The over-all frequency response of an input polarity inverter, LTE , ILTE, and output polarity inverter depends slightly upon the amplitude of the signal. When the system is set so as to produce an exponent of 1, the over-all response can be depended upon to be flat from direct current to 200 cyc/sec and down 3 db at a frequency not less than 1500 cyc/sec.

The output polarity inverter may be most efficiently aligned in the following manner:

1. Operate for 1 hour.
2. Short the input.
3. Connect the voltmeter on the positive output terminal.
4. Set the sensing input switches opposite; turn positive clamp full clockwise.
5. Set positive level to zero.
6. Set input sensing switches up.
7. S et positive clamp just barely to zero.
8. Put voltmeter on negative output terminal.
9. Set negative clamp counterclockwise.
10. Set input sensing switches up.
11. Set negative level to zero.
12. Set sensing input switches opposite.
13. Set negative clamp to just barely zero.

14. Connect input to a positive, low-impedance voltage source of 100 to 200 volts. Connect sine wave input to one sensing input of amplitude at least 10 volts.
15. Adjust balance controls until output is symmetrical about zero.
16. Reduce sensing input to 0.5 volt rms and adjust threshold control for symmetry of output.
17. Repeat step 16 on the other sensing channel.

F. Summing-inverting amplifier

The one remaining piece of equipment necessary for the operation of the function generator is described next. The logarithm-taking elements have as their transfer function the following:

$$e_o = -\log e_i \tag{59}$$

If one wishes to generate a function of the form

$$y = (x+b)^m / (x+c)^n \tag{60}$$

the ILTE must be presented with a voltage of the form

$$e = m \log (x+b) - n \log (x+c) \tag{61}$$

Equation (59) shows that the polarity of the term in Equation (61) corresponding to division in Equation (60) is correct but that the polarity of the other term must be reversed. Most functions will consist in part of at least two factors multiplied together, and some convenient means of summing and reversing these voltages is needed.

If the factors going to make the arbitrary function are raised to more than the first power, amplification must be provided between the LTE and the ILTE unless one is willing to sacrifice accuracy. No arrangement has been made to provide for an amplifier to be used when dividing, but the circuit in Figure 56 serves as a summing and inverting amplifier when one is multiplying or raising to a power greater than unity.

A standard CIT analysis laboratory computing amplifier is used in conjunction with a plug-in panel which carries the summing and gain-controlling components. When the gain of the computing amplifier is set at maximum, the following equation gives the transfer function of the resultant amplifier:

$$e_0 = -\left(\frac{1}{m} + \frac{1}{10} + \frac{m}{10}\right)\left(e_1 + e_2 + e_3\right) \qquad (62)$$

where m is the reading of the Duodial on the Helipot (units 0 to 1000), and e_1, e_2, and e_3 are the three input voltages. The source impedances of e_1, e_2, and e_3 must be low for accurate summing, but negligible interaction results between inputs. The function given by Equation (62) is also shown on Figure 56.

G. Application of function generator

The function generator was demonstrated in Section IV-E-1 when

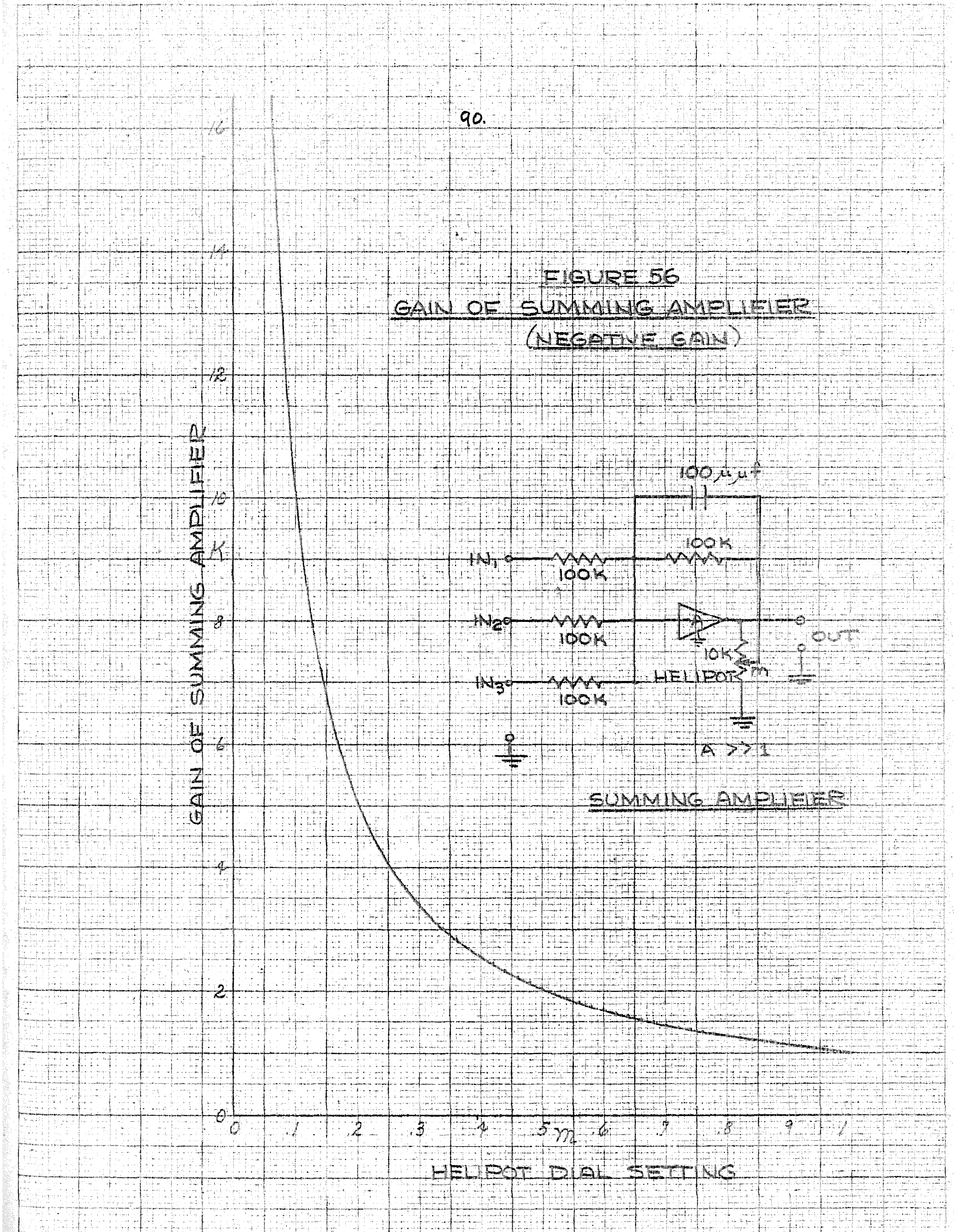


FIGURE 56
GAIN OF SUMMING AMPLIFIER
(NEGATIVE GAIN)
GAIN OF SUMMING AMPLIFIER
K
16
14
12
10
8
6
4
2
0
0
.1
.2
.3
.4
.5
m
.6
.7
.8
.9
1
HELIPOT DIAL SETTING
100 μμf
100K
IN1
100K
IN2
100K
OUT
10K
m
HELIPOT
IN3
100K
A >> 1
SUMMING AMPLIFIER

quadratic and cubic functions were displayed.

1. Product of three cubic factors. Mention was made in Section IV-E-2 of the fact that functions of the form of Equation (58) were normally generated. Figure 57 shows the function represented by the following equation, which is a special case of Equation (58). Note that the first three divisions of the oscilloscope trace are meaningless and that the functions start from zero.

$$y = a\,|x|^2 \cdot |x+b|^2 \cdot |x+c|^2 \cdot x\,(x+b)\,(x+c) \tag{63}$$

Equation (63) can be altered by disconnecting one or more of the sensing-input connections, and the resulting functions can be represented by the following two equations:

$$y = a\,|x|^3 \cdot |x+b|^3 \cdot |x+c|^3 \tag{64}$$

$$y = a\,|x|^3 \cdot |x+b|^3 \cdot |x+c|^2 \cdot (x+c) \tag{65}$$

The function represented by Equation (64) is shown in Figure 58 and results from disconnecting both sensing inputs. Figure 59 shows the function of Equation (65) produced by disconnecting only one sensing input. Figure 60 shows the function resulting when the other sensing input is disconnected. Less rational functions can be generated by interchanging the s ensing-input connections.

The functions demonstrated may seem to be extreme examples of the

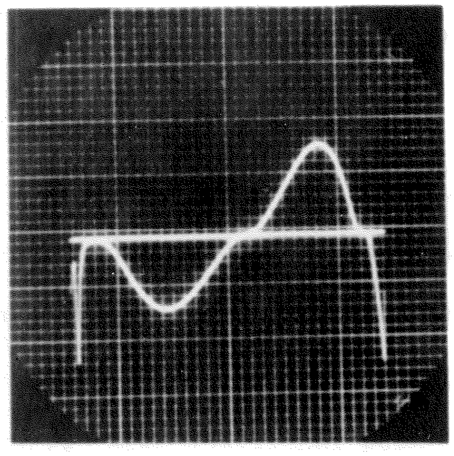


Figure 57

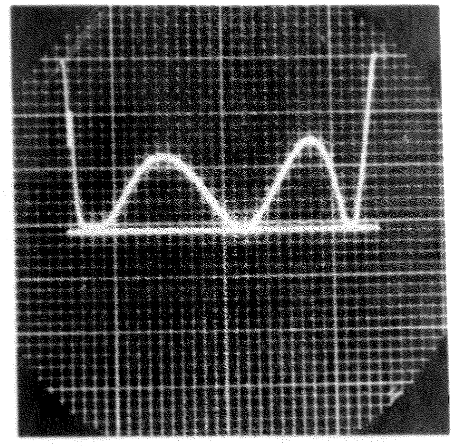


Figure 58

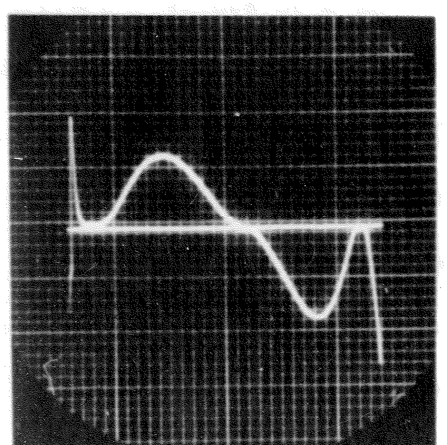


Figure 59

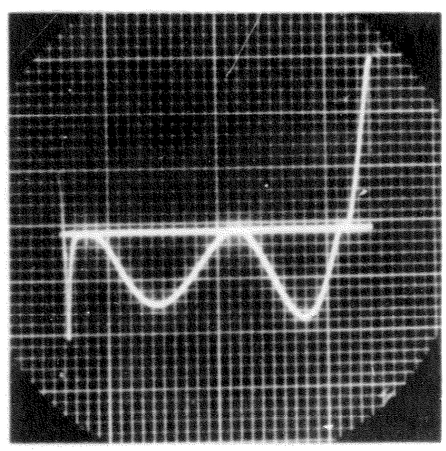


Figure 60

Oscillograms Showing Variations in Arbitrary Function variations that may be obtained from Equation (58). This is not, however, the case. Each factor of Equation (58) can be a separate dependent or independent variable. Each factor can be raised to an arbitrary power. The sign of y can be made to depend upon other dependent or independent variables than those appearing in the factors. With additional polarity inverters and LTE's, the number of factors can be increased so that any practical single-valued function can be represented. Several examples are now cited.

2. Other examples and applications. Figures 61 through 64 show functions of the form

$$y = x^m \sin^n kx \tag{66}$$

for different values of m and n. Figures 61, 62, and 63 show the cases in which $m \cong 1$ and $n \cong 1$, 3, and 1/3, respectively. Figure 64 shows a case in which $m \cong \frac{1}{3}$ and $n \cong 1$. These four oscillograms were obtained by multiplying together two independent variables. The first was the output of the linear sweep generator (See Appendix A), and the second was the output of audio oscillator.

Figures 65 and 66 demonstrate the multiplication of two sinusoids. The following equation

$$y = \sin(kx + \theta) \sin kx \tag{67}$$

represents the functions shown. Figure 65 shows the function when

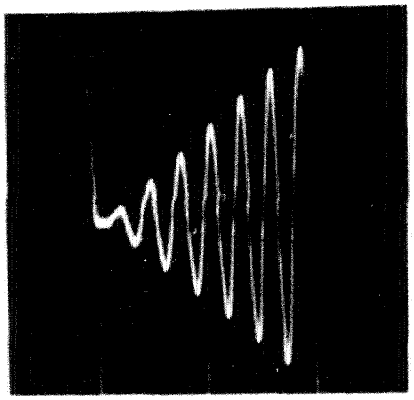


Figure 61

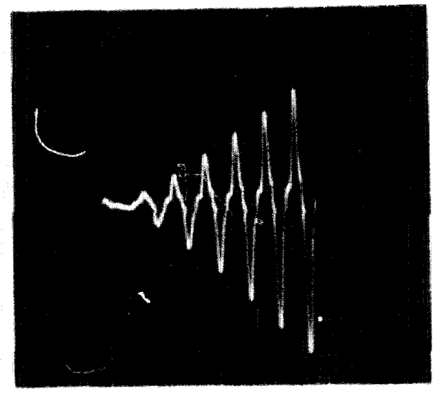


Figure 62

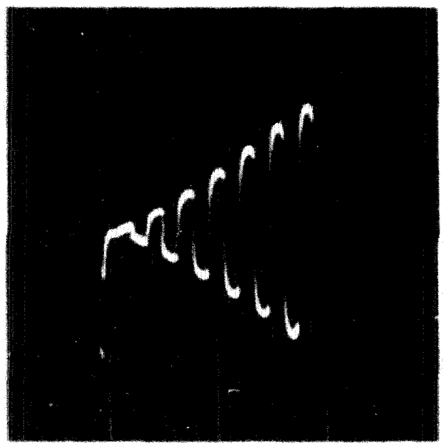


Figure 63

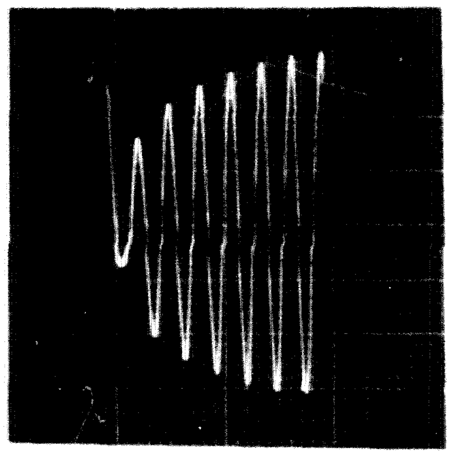


Figure 64

Oscillograms Showing Functions of Form $y = x^m \sin^n kx$

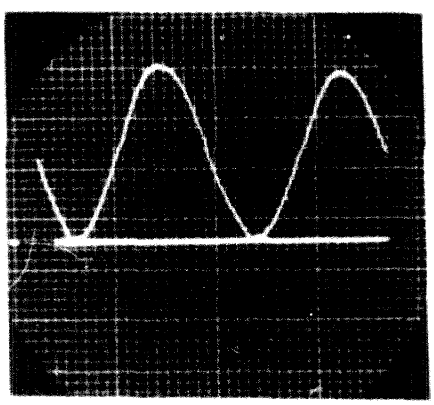


Figure 65

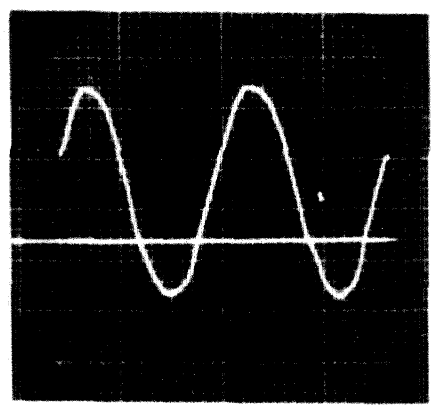


Figure 66

Oscillograms Showing Functions of Form $y = \sin(kx + \phi)\sin kx$

$\theta = 0$ (Eq. 67). Figure 66 shows the function when $\theta \neq 0$. The x axis is shown for reference. Connecting an average-reading voltmeter to the terminals of the output polarity inverter would be the only step necessary to make a single-phase voltmeter. A polyphase voltmeter capable of giving instantaneous wattage during a fault on a power system could be designed, the arbitrary function generator being used as the multiplier.

The correlation functions used in communication theory require the multiplication of factors. If the function generator were used as a multiplier, correlation would be done without the usual intermediate conversion of signal voltage to graphical or tabular form.

The output polarity inverter alone can be used to provide the time-varying term in the analog solution of Hill's equation (Ref. 1). The function generator can be used to provide the time-varying term in the Mathieu equation (Ref. 1). Savant and the author set up the analog for Hill's equation and observed some of the fascinating nonrepetitive waveforms that can be produced. The existence of stable parameter combinations was observed, but no quantitative work was done.

H. Accuracy of function generator

Although the principal uses of the function generator will require flexibility more than accuracy, an over-all accuracy check was made. Figures 67 and 68, oscillograms taken when the function generator was first tested, show functions of the form

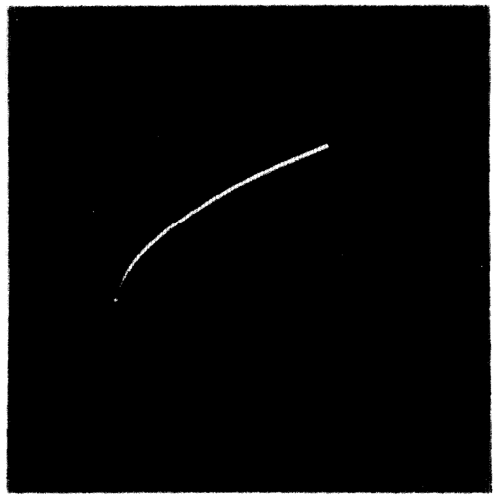


Figure 67

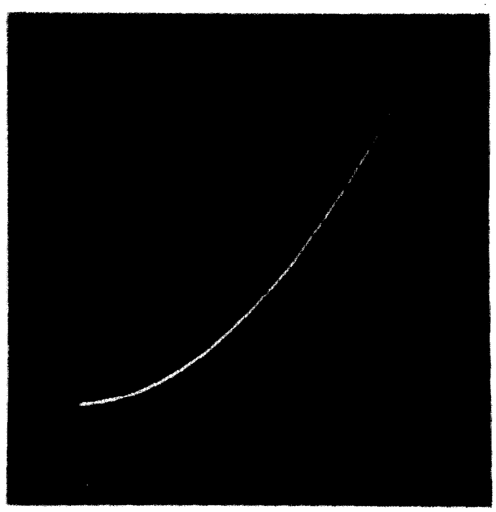


Figure 68

Oscillograms Showing Functions of Form $y = x^n$

$$y = x^n \qquad (68)$$

The negatives were enlarged to make prints that were 8 inches square. These prints were scaled and plotted on log-log graph paper. Figure 69 shows the results. Errors in the functions are seen to be less than 2 per cent of the maximum. The function generator should have an over-all accuracy of closer than 5 per cent when it is properly adjusted and is generating functions of the type shown in Figures 57 through 60.

V. NONLINEAR COMPUTER APPLICATIONS

The arbitrary function generator described in Section IV requires accessory equipment for its operation. Appendix A contains descriptions and schematic diagrams of the other apparatus built for use with the function generator. If the arbitrary function generator, its accessory equipment, and a few passive circuit elements are combined, an analog computer capable of solving certain nonlinear problems results. Savant and the author studied several nonlinear problems with the use of the computer. The results obtained from the study of a system having nonlinear damping are reported in this section.

A. Soft Van der Pol oscillator

Van der Pol's equation (Ref. 2) is a special form of the following equation:

$$\ddot{x} + F(x,\dot{x}) + x = A \sin \omega t \tag{69}$$

If one thinks about the mechanical or electrical system that can be described by Equation (69), he realizes that the second term can correspond to a damping term. In particular, if one writes Equation (69) in the following manner:

$$\ddot{x} + f(x)\dot{x} + x = A \sin \omega t \tag{70}$$

he sees that Equation (70) has nonlinear damping. Van der Pol considered a special case of Equation (70) in which $f(x) = -\mu(1-x^2)$, and $A = 0$. His choice of f(x) was not accidental; he wanted to write a simple differential equation capable of exhibiting self-excited oscillations. Van der Pol's equation,

$$\ddot{x} - \mu(1-x^2)\dot{x} + x = 0 \tag{71}$$

exhibits the general properties of self-excited, self-limiting, soft oscillators. The damping is negative for small displacements, decreasing in magnitude as x increases until, at $x = 1$, the damping term vanishes. For amplitudes of oscillation exceeding $x = 1$, the damping becomes positive; thus the amplitude of oscillation is self-limiting. The equation as written is that of a soft oscillator, that is, one which is unstable when $x = 0$. A hard oscillator, discussed in

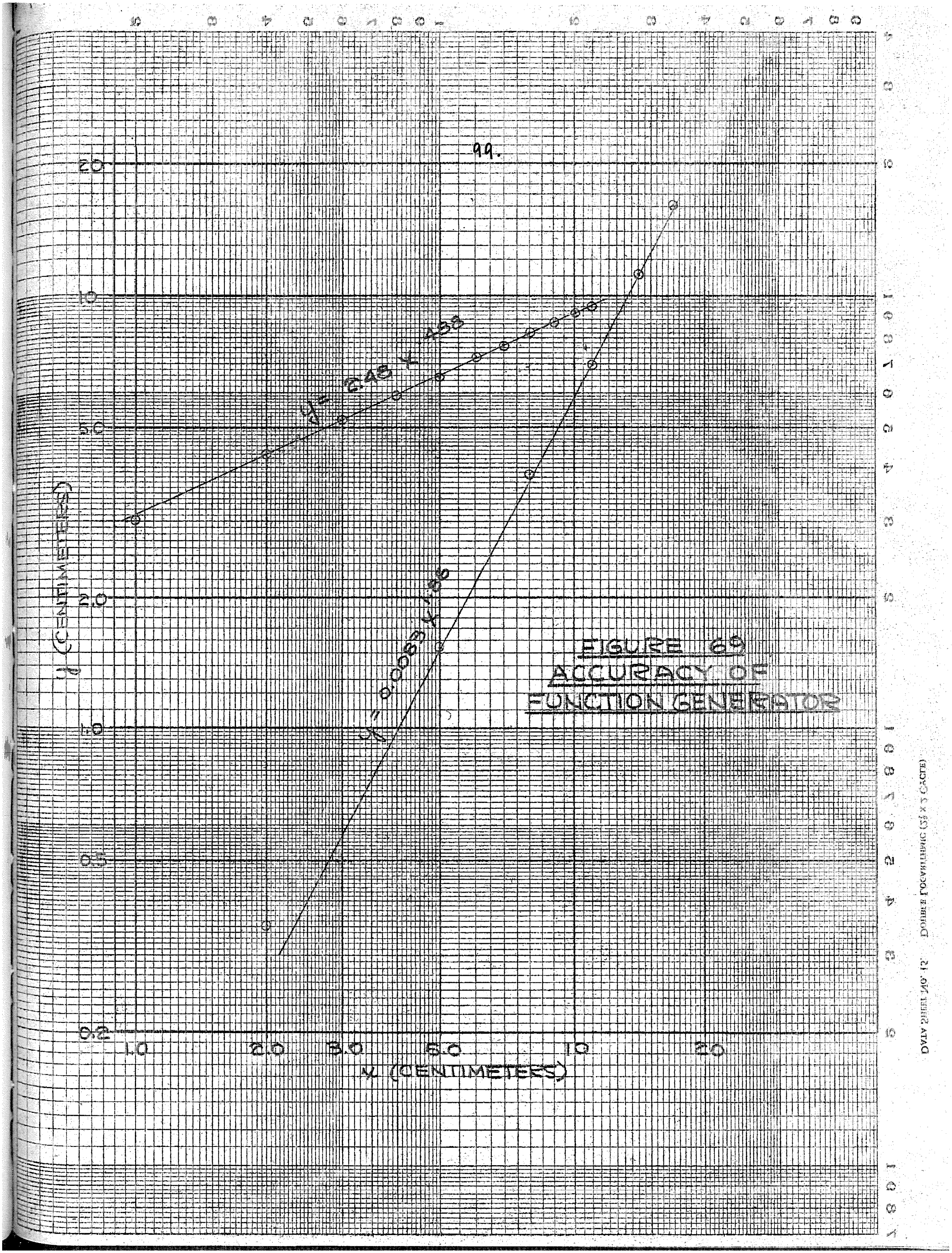


FIGURE 69
ACCURACY OF
FUNCTION GENERATOR
y = 2.48 X .488
y = 0.0083 X 1.86
y (CENTIMETERS)
X (CENTIMETERS)
20
10
5.0
2.0
1.0
0.5
0.2
1.0
2.0
3.0
5.0
10
20
DATA SHEET NO. 12
DOUBLE LOGARITHMIC (3½ x 3 CYCLE)

Section V-D, has positive damping for small displacements and therefore is stable near $x = 0$.

Consider now the circuit shown in Figure 70. The differential

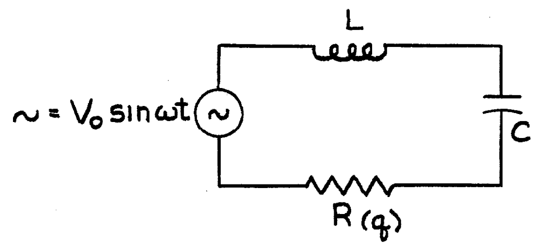


Figure 70. Simple Series Circuit

equation describing this circuit can be expressed

$$L\ddot{q} + R(q)\,\dot{q} + \frac{1}{C}q = V_0 \sin \omega t \tag{72}$$

Comparison of Equation (70) and (72) shows that the circuit of Figure 70 can be described, after suitable scoring by a general form of Van der Pol's equation. The corresponding nonlinear terms are $f(x)$ and $R(q)$, and displacement is analogous to charge.

The analog solution of Equation (71) is undertaken in this section. The more general form of Van der Pol's equation is treated in Section V-B.

The electrical analog of Equation (71) would require two function generators. The first would form the function $(1 - x^2)$; the second, which could be replaced by a multiplier if available, would form the product of $(1 - x^2)$ and $\dot{x}$. However, if Equation (71) is integrated once, the following equation results:

$$\dot{x} - \mu \left(x - \frac{x^3}{3}\right) + \int x\,dt = 0 \tag{73}$$

The analog of Equation (73) requires only one function generator. Figure 71 shows the block diagram of the complete electric analog required. L_p and C_p of Figure 71 correspond to L and C of Figure 70. The other components are required to simulate the variable damping term. R_p of course represents linear positive damping. Its effect is reversed, however, by feeding back a voltage of opposite polarity and greater magnitude through the 25,000-ohm Helipot and the amplifier A_3. Operation of the remaining blocks was described in Section IV. Their use is to generate the function x^3, also fed back into the loop through the amplifier A_3, which has negligible output impedance. In Figure 71, the loop corresponding to that of Figure 70 then goes through C_p, L_p, and R_p, through the ground connection to the amplifier A_3, through the output circuit of A_3, and back to C_p.

Demonstration of the types of solution available from Van der Pol's equation is best shown by the use of oscillograms. The function $x^3/3$ is shown in Figure 72. The complete second term of Equation (73) is shown in Figure 73. Realize that if the damping were linear, the equivalent term would be a straight line lying in the first and third quadrants and passing through the origin. Figures 74 and 75 show the build-up of amplitude and velocity versus time, respectively, with $\mu = 1$. The same information is presented by the phase trajectory

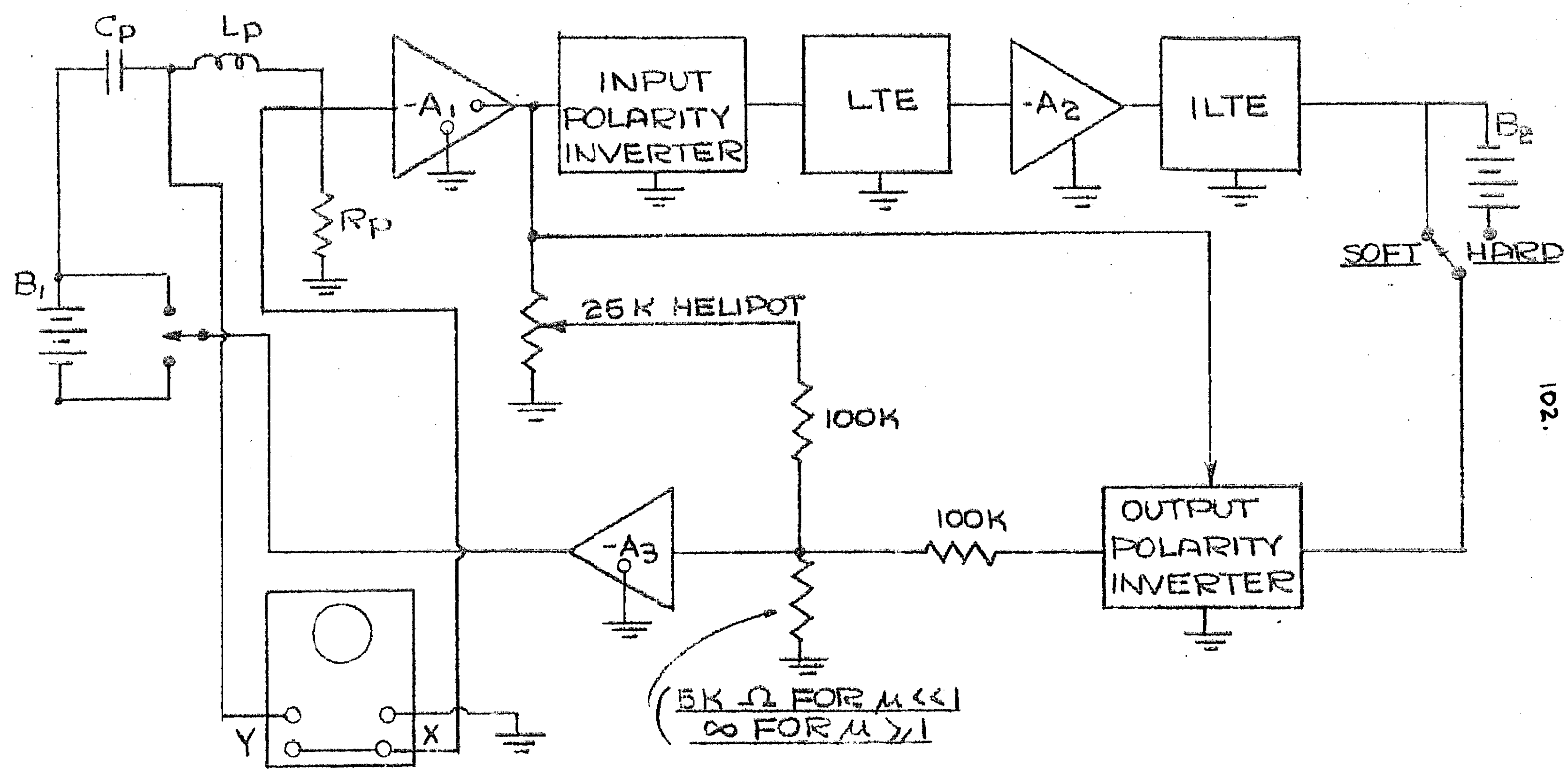


FIGURE 71
SOLUTION OF VAN DER POL'S EQUATION
BLOCK DIAGRAM

shown in Figure 76. Note that the x axis is reversed because the oscilloscope does not have isolated inputs to the x and y channels. Figures 77, 78, and 79 present x, $\dot{x}$, and a phase trajectory, respectively, when $\mu = 8.4$. Likewise, Figures 80, 81, and 82 present x, $\dot{x}$ and a phase trajectory, respectively, when $\mu = 0.1$. In Figure 82 one also sees the amplitude decrease to that of the stable limit cycle when the system is given a large initial displacement.

B. Modified soft Van der Pol oscillator

The form of the nonlinear second term of Equation (73) was shown in Figure 73. Physically it can be seen that the general form of the oscillations is not changed much if the nonlinear damping term is altered slightly. Solutions to a special case of Equation (70), that given by

$$\dot{x} - x + x|x|^{1.4} + \int x dt = 0 \tag{74}$$

are shown. Figures 83, 84, and 85 show x, $\dot{x}$, and several phase trajectories for the build-up of oscillation of Equation (74) when $\mu = 1.0$. Close similarity can be seen between Figures 83, 84, and 85 and Figures 74, 75, and 76.

C. Forced Van der Pol oscillator

Two special cases of Equation (70) have been discussed earlier in this section. Both cases were found to yield oscillations when no forcing function was supplied. If one considers the case represented

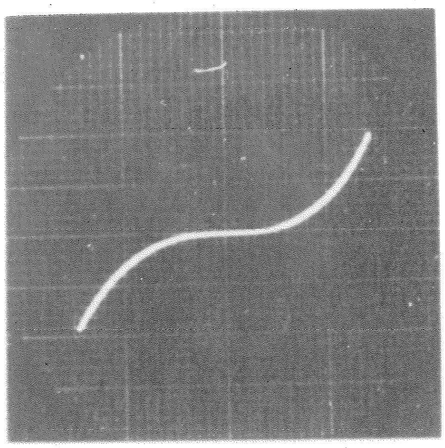


Figure 72

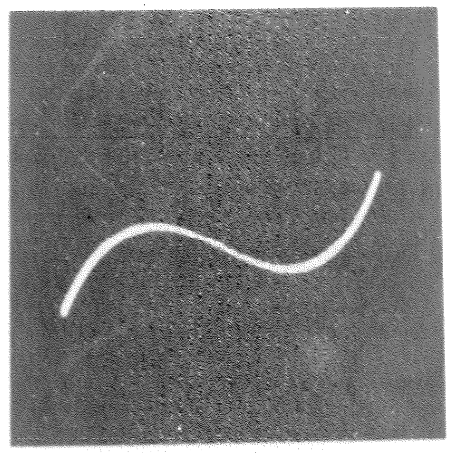


Figure 73

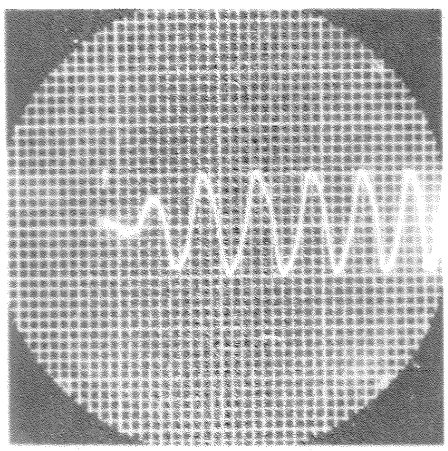


Figure 74

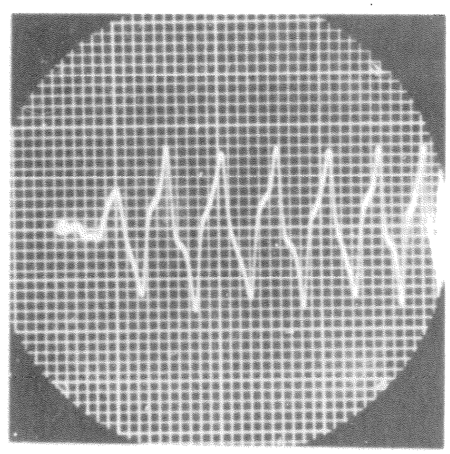


Figure 75

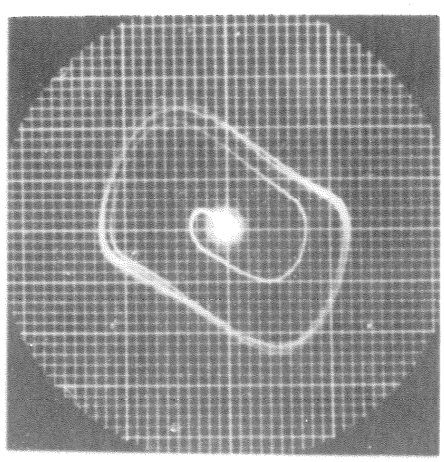


Figure 76

Oscillograms Demonstrating a Soft Van der Pol Oscillator

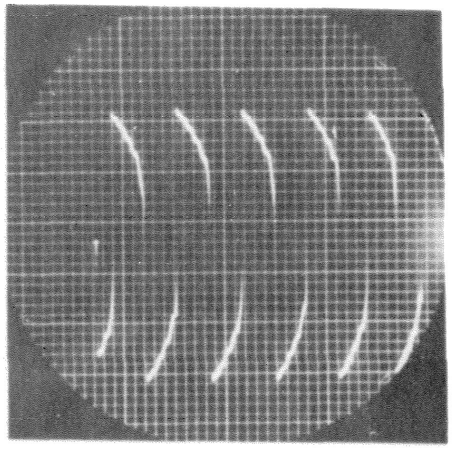


Figure 77

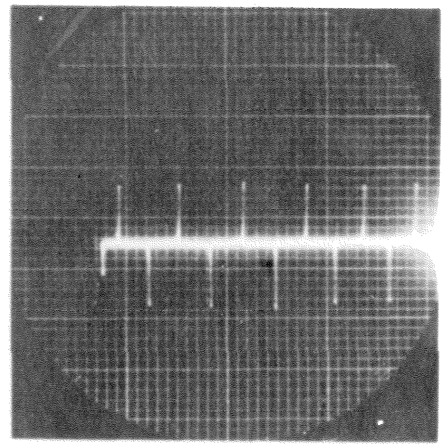


Figure 78

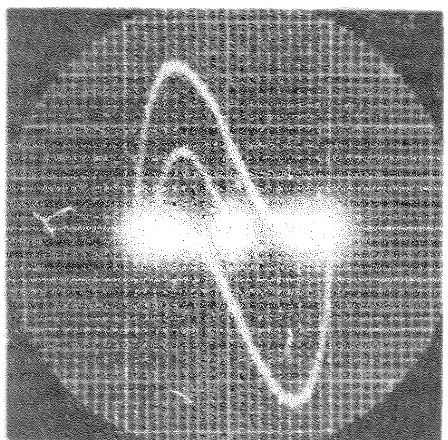


Figure 79

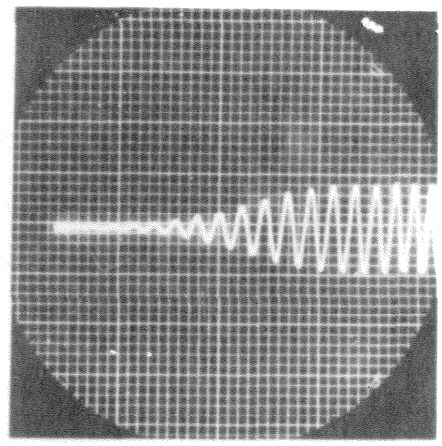


Figure 80

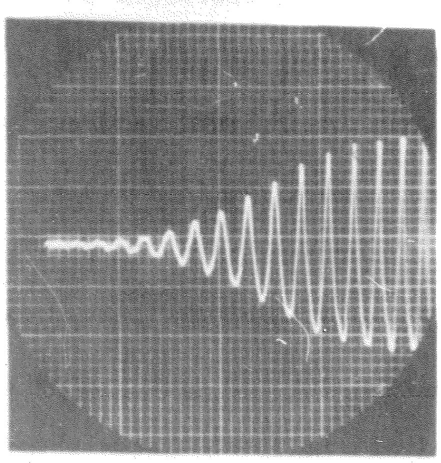


Figure 81

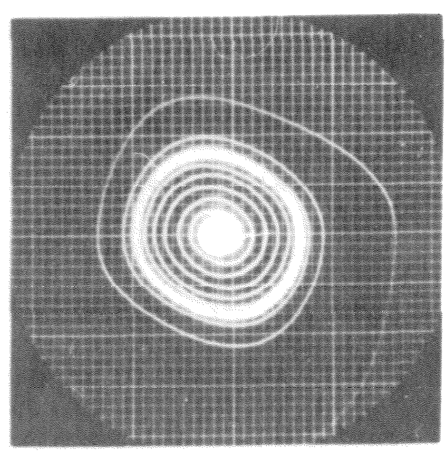


Figure 82

Oscillograms Demonstrating a Soft Van der Pol Oscillator by

$$\ddot{x} - \mu (1 - x^2) \dot{x} + x = A \sin \omega t \tag{75}$$

he finds a new effect, that of synchronization. The basic frequency of the Van der Pol oscillator can change so that the ratio between its frequency and the forcing frequency $\omega/2\pi$ will be a rational fraction. If the value of A in Equation (75) is small, the synchronization occurs only in certain ranges of the forcing frequency. Figures 86 to 93 show oscillograms that demonstrate synchronization. These oscillograms were taken from the circuit shown in Figure 71 except that the battery B_1 was replaced by an audio oscillator. The μ of Equation (75) was approximately unity. Figures 86 and 87 show the amplitude versus time and a phase trajectory, respectively, when no synchronizing signal exists. The frequency of the nonsynchronized Van der Pol oscillator was approximately 20 cyc/sec.

The amplitude of the synchronizing signal was such that if (1) the input to A_3 (Figure 71) were shorted and (2) the frequency of the synchronizing signal were 60 cyc/sec, then the peak amplitude of the sinusoidal voltage corresponding to displacement (Eq. 75) would be one-third the peak amplitude of the nonsynchronized oscillator. Figures 88 and 89 show the displacement and a phase trajectory when the synchronizing frequency is 60 cyc/sec. The forcing amplitude is not large enough to synchronize the oscillator when its frequency is 62 cyc/sec. Figures 90 and 91 show the

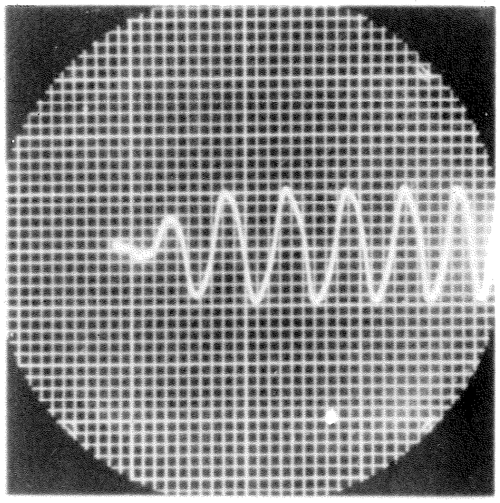


Figure 83

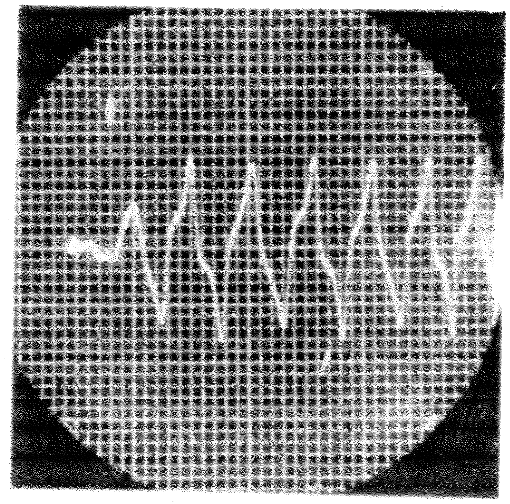


Figure 84

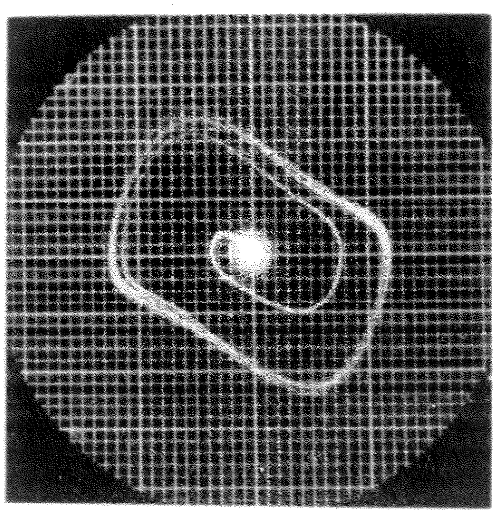


Figure 85

Oscillograms Showing Oscillations Produced
by a Modified Van der Pol Oscillator

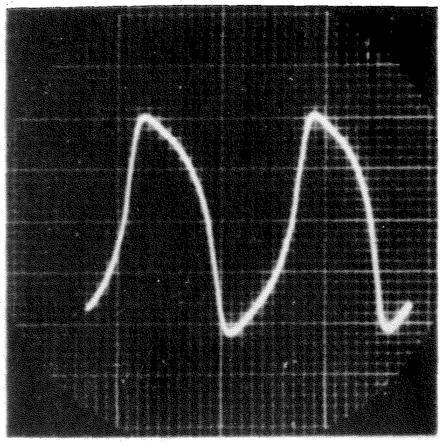


Figure 86

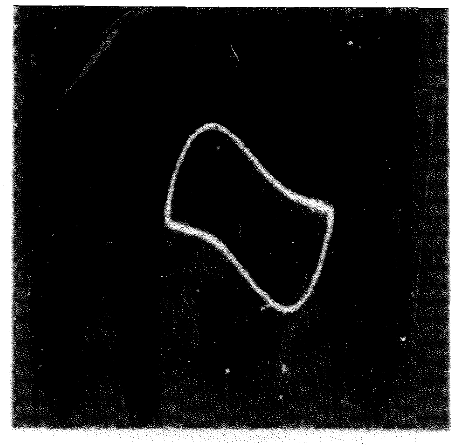


Figure 87

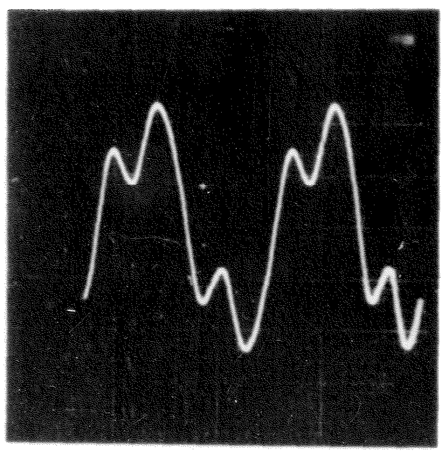


Figure 88

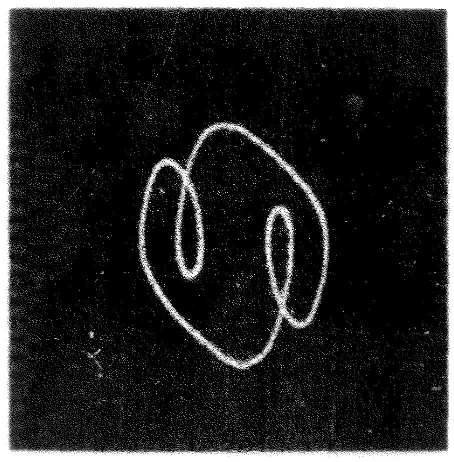


Figure 89

Oscillograms Showing Synchronization
of a Van der Pol Oscillator

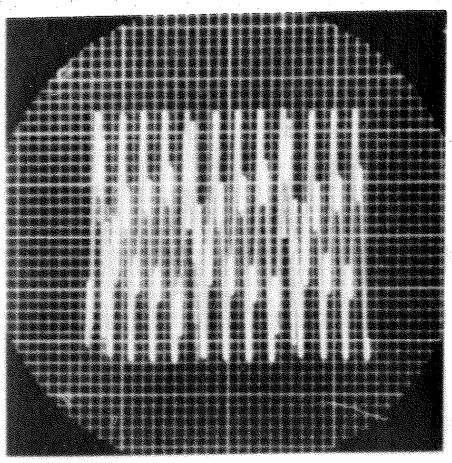


Figure 90

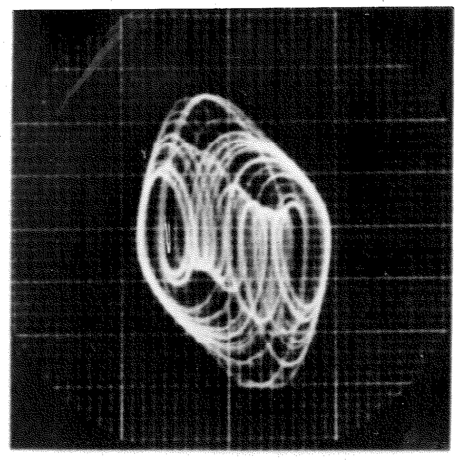


Figure 91

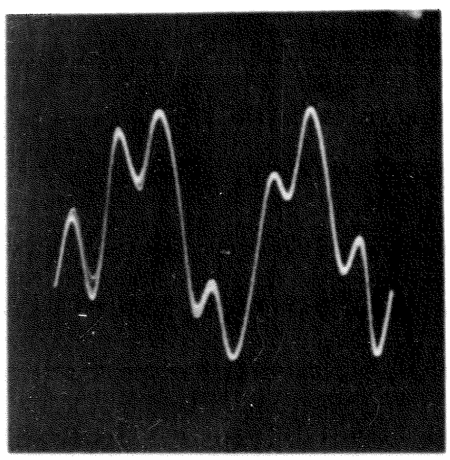


Figure 92

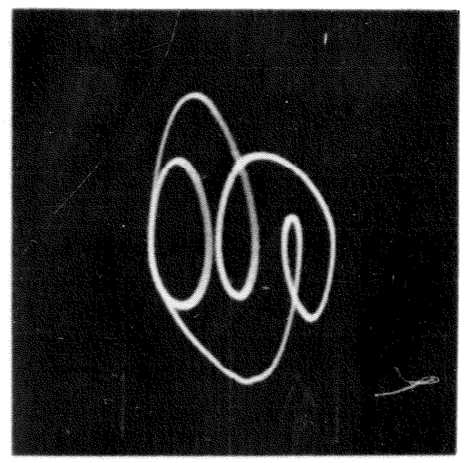


Figure 93

Oscillograms Showing Synchronization
of a Van der Pol Oscillator

resultant oscillation. The effect is seen to be similar to that which would exist if the free oscillation and the forced oscillation were superimposed in some linear circuit. (The oscilloscope is being swept horizontally at approximately 10 cyc/sec in Figures 86, 88, and 92. In Figure 90, the sweep frequency has been reduced to less than 1 cyc/sec to show the complex waveform.) Figures 92 and 93 show oscillograms taken when the forcing frequency is raised to 63.6 cyc/sec. The oscillation is now synchronized at two-sevenths of the forcing frequency.

No general study of synchronization was attempted. The equipment is available, however, and the author suggests that the regions of synchronization of a Van der Pol oscillator be studied. With a fixed value of μ, a chart similar to those used to indicate the stability of the Mathieu equation should result if regions of synchronization were plotted in the A-ω plane. Another possibility for investigation that should yield the same general type of result would be that in which A were held constant, and μ and ω were varied.

D. Hard Van der Pol oscillator

The final application of the computer is to simulate a hard oscillator. Equation (70) can represent a hard oscillator if the damping function f(x) is positive for small displacements and negative for larger displacements. For the existence of a stable limit cycle, the damping must again become positive for even larger displacements.

As was the case in Section V-A, the analog is more simply set up if Equation (70) is integrated once. The actual equation used

$$\dot{x} - \mu\left(x - x^3/a - b\frac{x}{|x|}\right) + \int x\,dt = 0 \tag{76}$$

is seen to be the same as Equation (73) except for the additional term. Figure 71 shows the analog circuit which is the same as that used for the soft oscillator except that the battery B_2 adds a constant voltage to the input of the output polarity inverter. This constant voltage, after switching, is the equivalent of the $(bx/|x|)$ term in Equation (76). Figure 94 shows the value of the damping (second) term if Equation (76) is plotted versus x.

The voltage applied to the principal loop of the analog circuit was obtained from the batteries contained in the stability voltmeter (See Appendix A). Figures 95, 96, and 97 are double exposures showing phase trajectories. In Figure 95, the applied voltage was increased 3 volts between the two exposures. Notice that the smaller voltage is not sufficient to shock the oscillator into the self-excited region of operation. The decrement of the positively damped oscillation is large because of the relatively large effect of the nonlinear friction when the oscillations are small.

Figure 96 is the same as Figure 95 except that the smaller applied voltage was only 1.5 volts less than the larger. The small space separating the two traces shows the position of the barrier cycle (Ref. 1). Intensity modulation of the oscilloscope beam was used in Figures 95 and 96 so that the phase trajectories would contain

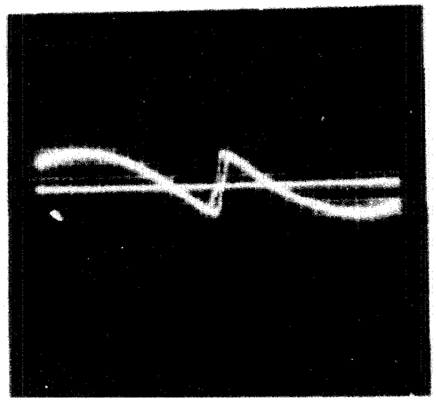


Figure 94

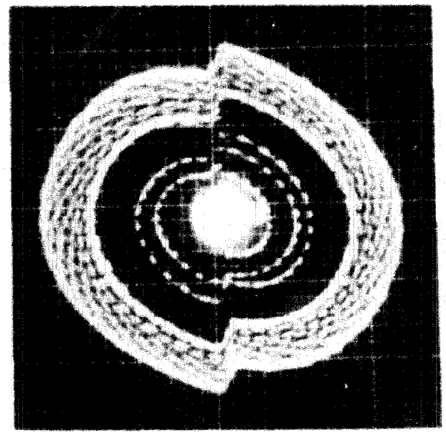


Figure 95

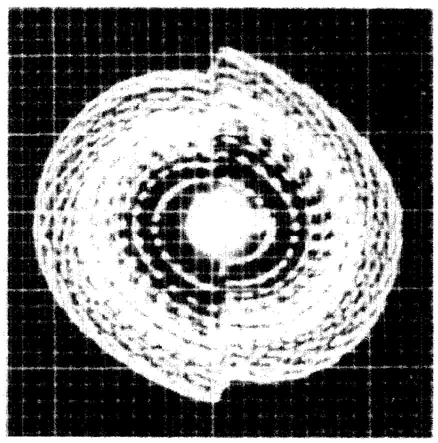


Figure 96

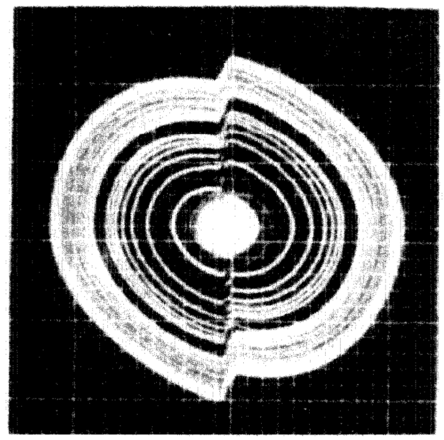


Figure 97

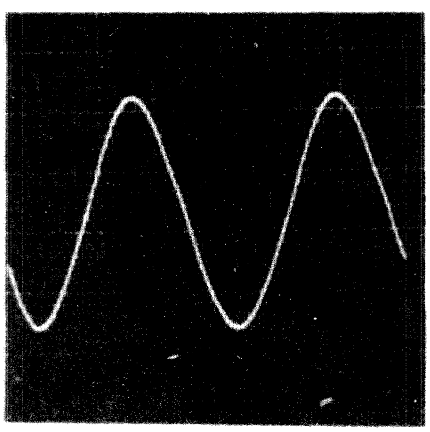


Figure 98

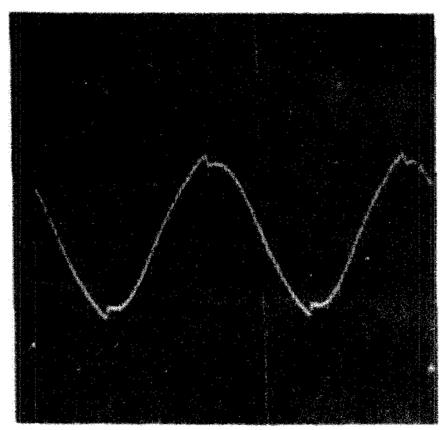


Figure 99

Oscillograms Demonstrating a Hard Van der Pol Oscillator all the information about the wave shape. No amplitude presentation is necessary to complete the presentation. The intensity modulation was not completely isolated from the oscillator circuit, and a slight difference in the phase trajectory appears between Figures 96 and 97. Figure 97 should be the same as Figure 96 except for the intensity modulation. Figures 98 and 99 show the steady-state x and $\dot{x}$, respectively, plotted versus time.

VI. CONCLUSION

Nonlinear systems present some of the most difficult problems and some of the most interesting phenomena that engineers of today must face. The problems will become more pressing as systems become more complex and the space and weight limitations become more severe. The arbitrary function generator described in Section IV, though not completely satisfactory, offers users of electric analog computers a powerful tool to use in their attack upon nonlinear problems.

The simple nonlinear computer that can be constructed from the elements described in Section IV and Appendix A with the addition of computing amplifiers and passive circuit elements can be used to investigate quantitatively a large range of nonlinear systems. Section V demonstrated the use of the computer in only a qualitative manner, as the author was more interested in testing the computer's operation and in improving his intuition than in making an exhaustive study of self-excited oscillators. The computer could be used to good advantage, however, in a study of this type.

The applied or theoretical mathematician could well afford to use the computer to investigate the solutions of the standard nonlinear differential equations more fully in an effort to advance his knowledge of the subject. Practically no theoretical work has been done with third-order nonlinear differential equations, and the author believes that the mathematician might advance more rapidly in the solution of these equations if he were to have prior knowledge as to some of the phenomena that he might expect to find in the solutions.

Further examples of the applications of the computer will be found in Savant's thesis (Ref. 6).

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3. G. D. McCann, C. H. Wilts, B. N. Locanthi, "Electronic Techniques Applied to Analogue Methods of Computation,"Proc. of the I. R. E. (August 1949), Vol. 37, No. 8, 954-961.
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5. Ivan A. Greenwood, Jr., J. Vance Holdam, Jr., and Duncan Macrae, Jr., "Electronic Instruments," Vol. 21 of Radiation Laboratory Series, McGraw-Hill Book Company, Inc., New York, (1948).
6. Clement J. Savant, Jr., "A Nonlinear Computer for the Solution of Servomechanism Problems," (thesis), California Institute of Technology, Pasadena, California, (1953).

APPENDIX A

Accessory Equipment

The LTE, ILTE, and polarity inverters constitute the principal elements of a versatile function generator. Other pieces of equipment are either necessary or convenient when the function generator is used in a nonlinear computer. A description of this equipment follows.

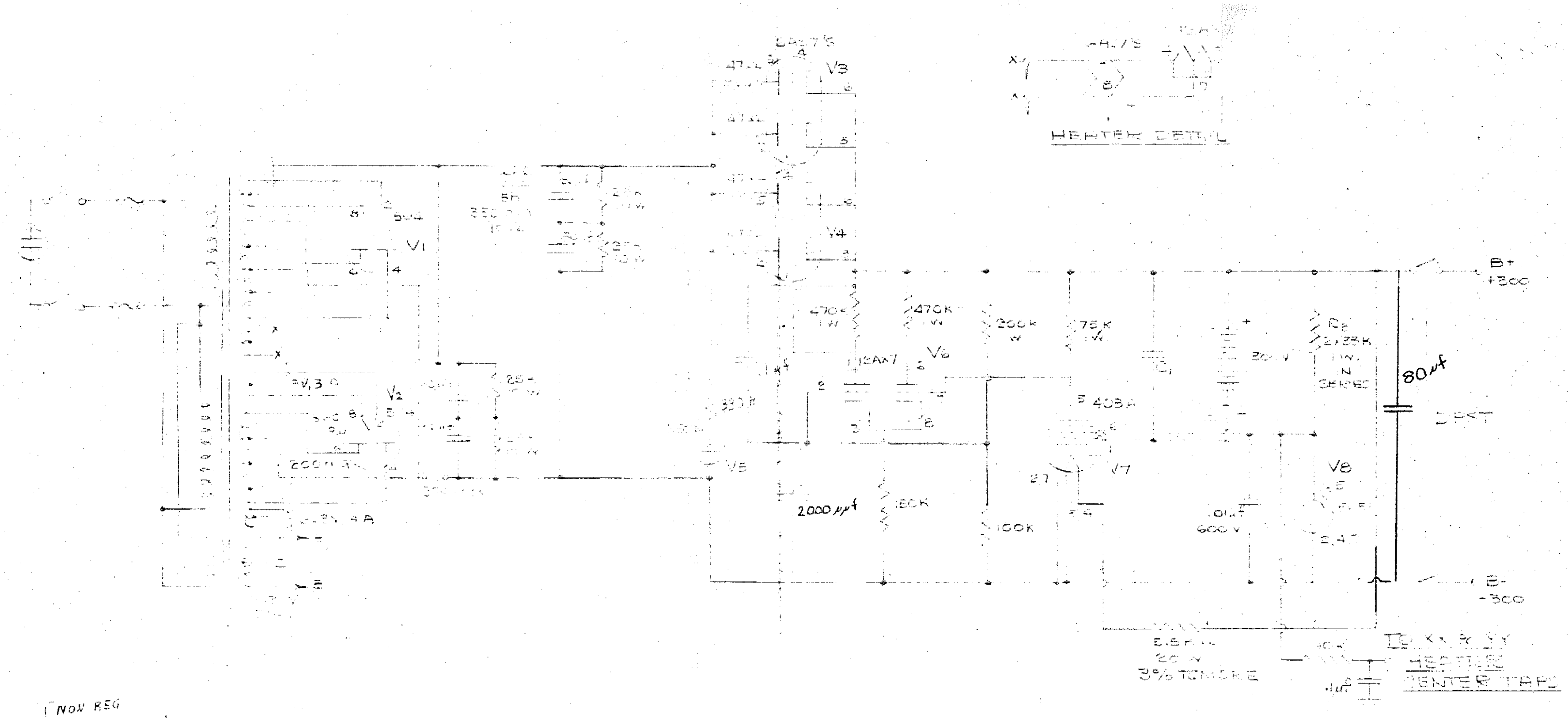
1. Power supplies. When the investigation was started, the experimenters did not know what the final voltage and CURRENT requirements would be. Because the function generator includes cascaded elements, all of which work down to direct current, the assumption was made that the power supply should be well regulated. In order to meet the final voltage and current requirements, the experimenters decided to design and build standard 300-volt, 300-ma supplies that could be operated with both output terminals above ground potential. By suitable series connections, voltages ranging from -900 to +900 would then be available.

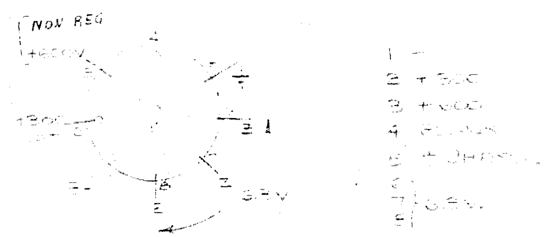
Figure 100 gives the schematic diagram of the basic power supply, one of which is pictured in Figure 101. Inspection of the diagram indicates that the rectifier section is not standard. Two separate power transformers and rectifier tubes feed a common filter section. The reason for this design is practical. Two standard television replacement transformers sell for considerably less than the single larger transformer that they replace. The rectifiers are connected so that any unbalance in the transformers will yield no circulating current.

The filtering is done by a single pi section, but in the interests of economy electrolytic condensers are used. The parallel resistors insure even division of voltage between the condensers.

The regulator consists of the 300-volt dry battery, the 408A pentode comparator, the 12AX7 amplifier, 6AS7 regulator tubes, and associated small components. By using a large dry-battery reference voltage and a regulated heater supply on the 408A, drift of the supply is minimized. The 5651 voltage reference tube maintains the screen potential of the 408A. The 408A is a Western Electric tube equivalent to the 6AK5, except that the heater requires 20 volts at 50 ma. The 5500-ohm Tomore resistor in the heater circuit of the 408A has a temperature coefficient near zero. The time constant of the regulator results principally from the 2000-$\mu\mu$f stabilizing capacitor from the plate of the 12AX7 and is of the order of 1 millisec. The output impedance of the supply is of the order of 10^{-2} ohm, and the 80-μf output capacitor maintains a low output impedance for transient leads.

The drift of the basic 300-volt power supply is less than 25 mv/hr with aged tubes after a warmup period. The repeatability of the output voltage as a supply is turned off and on is not as good as the value given because the 408A and 5651 tubes are overloaded during the first few seconds of operation. If a better supply were required, some arrangement for heating the 408A before application of the high voltage could be made.





OUTPUT CONNECTOR DETAIL

THREE, CONNECTED (IN PARALLEL)

NOTES - POWER SUPPLIES NOS. 1-4 -

1- CHASSIS IS NOT CONNECTED TO B-, BUT IS BROUGHT OUT TO A TERMINAL.

NOTES - POWER SUPPLY NO. 5 -

1- COMMON TRANSFORMER

2- [illegible] TRIODE CONNECTED REGULATOR TUBES

3- 6AK5 COMPARISON TUBE, AC HEATED

4- MAXIMUM CURRENT IS 175 MILLIAMPERES.

300V., 300 MILLIAMPERE REGULATED POWER SUPPLY

FIGURE 100

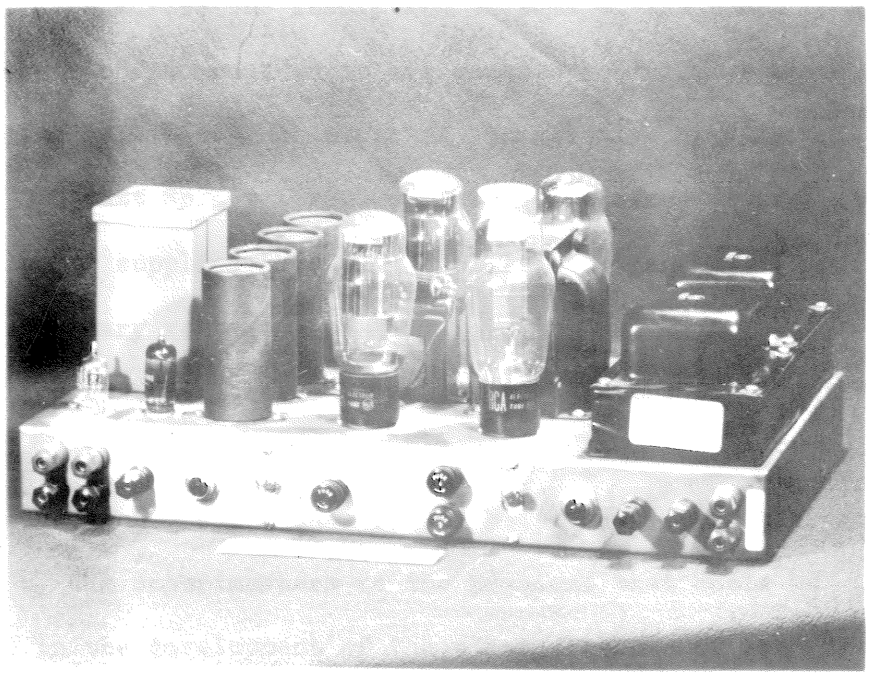


Figure 101. Picture of Standard Power Supply

The dc voltages used in the final version of the arbitrary function generator are -600, -300, + 300, and + 600 with respect to ground. To obtain these voltages, the experimenters connected four power supplies in series. With this type of operation, considerable stress is placed across the insulation of some of the components from the chassis, which is grounded in all cases. No failures have occurred, however, in over 6 months of operation.

Three 500-watt Sola voltage-regulating transformers are used to supply the power supplies, heaters, oscilloscope, and other instruments critical to primary voltage.

A storage battery is used to supply the heaters of certain tubes in the LTE and ILTE.

2. Test bench. While the LTE was being developed, thought was also given by the experimenters to the problems that would be encountered in the development of the other elements of the function generator. The conclusion was reached that, in the long run, time would be saved if a suitable electronic workbench were built. Figure 102 shows a photograph of the bench while being used during part of the later experimental work. On each side of the bench are power outlets where the output of the four power supplies, the ac heater supplies, and the 6-volt dc supply are brought out both to binding posts and sockets. Figure 103 shows the arrangement of the power outlets and gives the connection of the eleven-prong power plugs. A ground bus is conveniently available with jacks and binding posts. Regulated and unregulated 117-volt ac are brought out on strip

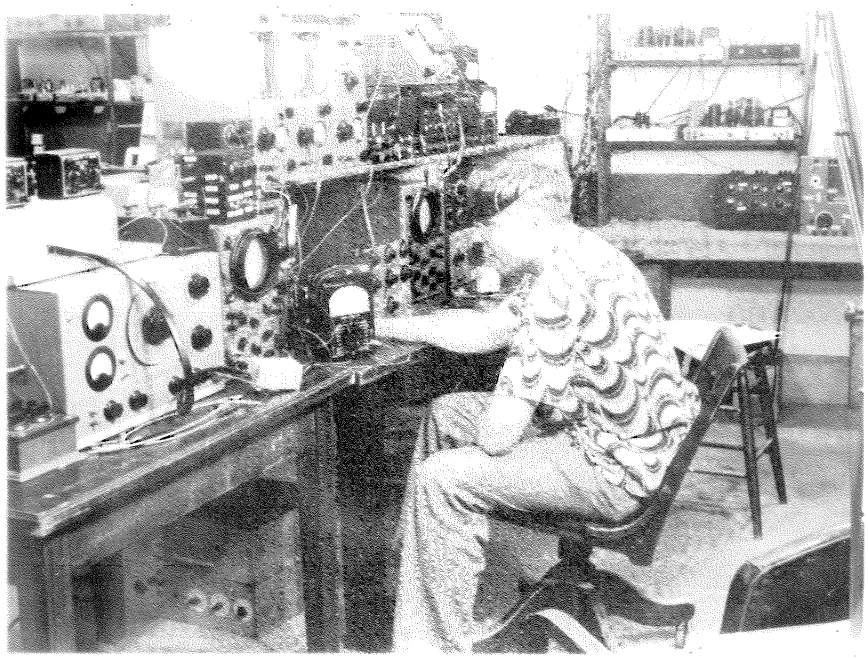
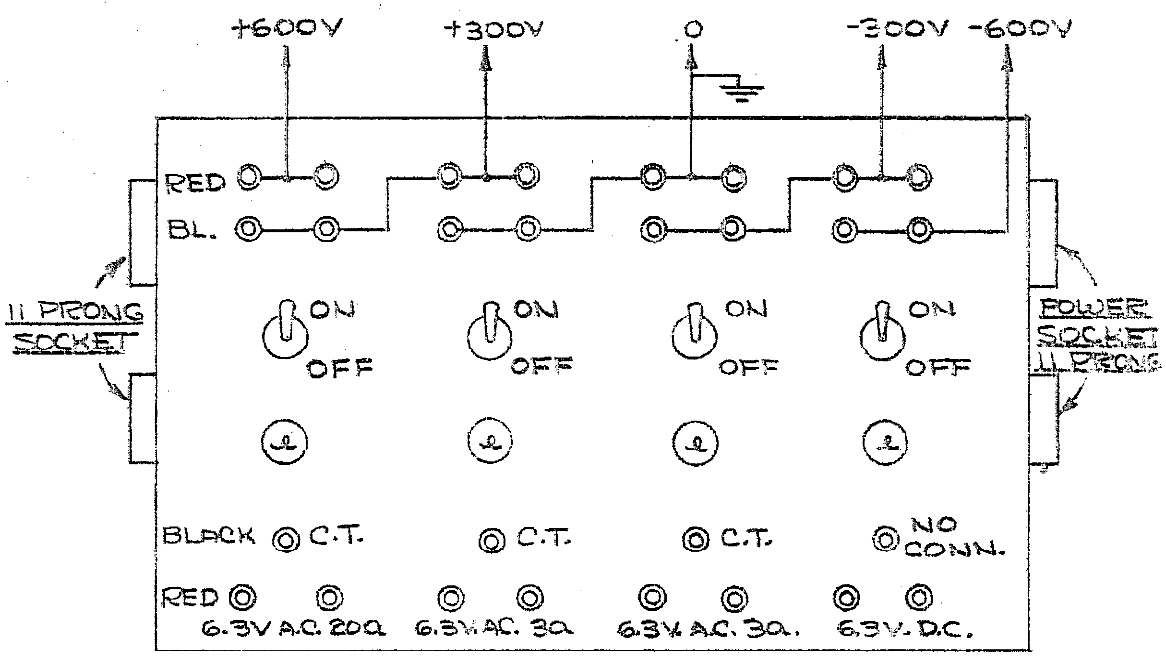
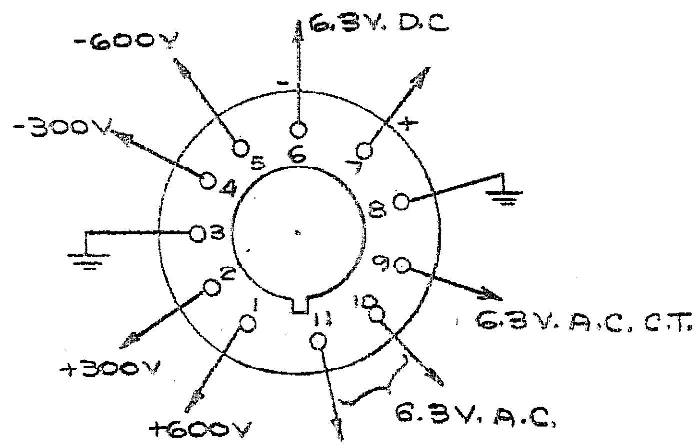


Figure 102. Overall Picture of Work Bench



(BOTTOM POWER SUPPLIES ON SHELVES TOP)

NOTE:
SWITCHES CONTROL BOTH BINDING POSTS AND POWER SOCKETS



POWER PLUG
(FOR 11 PRONG SOCKET)

POWER PANELS
FIGURE 103

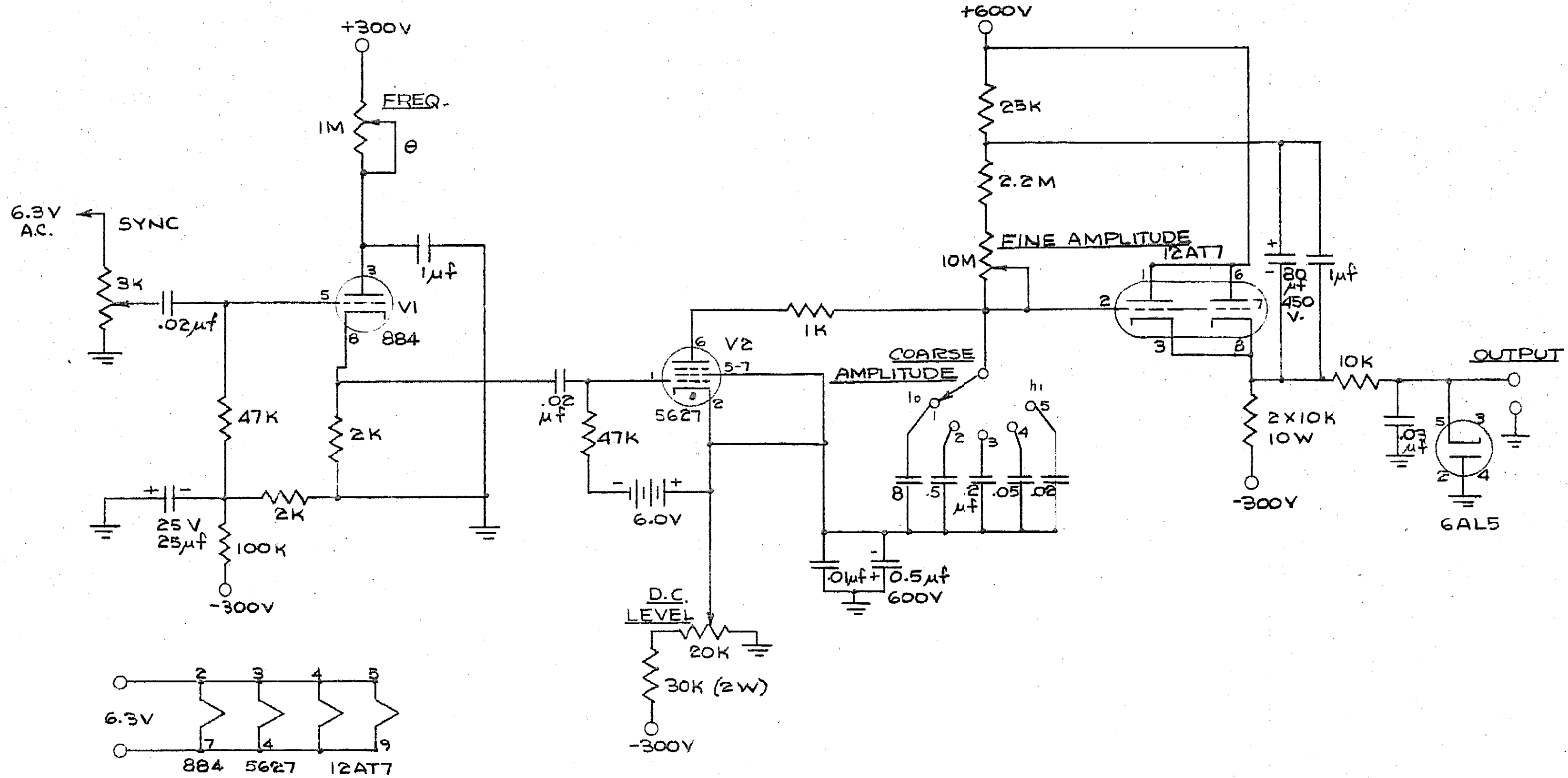
outlets.

Because the bench proved to be very useful, it is suggested that the power supplies and the bench be kept together for use by other students who have problems of applied electronics.

3. Linear sweep generator. A linear-voltage sweep generator was believed to be useful in the testing of the function generator, and a satisfactory unit was developed. When the function generator is used quantitatively, the linear-sweep generator is useful in setting up and checking the arbitrary function being generated.

The problem of generating a large sweep voltage is common, but the requirements often do not call for good drift stability. The circuit shown as Figure 104 and pictured in Figure 105 will deliver a saw-tooth of voltage starting always at zero and increasing to its maximum with a linearity of 1 per cent. The amplitude of the voltage wave is continuously adjustable from about 30 volts to 300 volts, and the repetition rate is set at 30 cyc/sec. Other submultiples of 60 cyc/sec are available by altering the adjustments, and the repetition frequency is synchronized with the ac line. The drift stability is essentially that of the output cathode follower, that is, of the order of 200 mv/hr when operated from regulated power supplies.

The sweep generator can be divided into several parts. The 884 thyratron in a standard sweep-generating circuit yields voltage pulses across its 2000-ohm cathode resistor. These pulses can be synchronized by the insertion of a 60-cyc/sec voltage in the grid circuit of the



LINEAR SWEEP GENERATOR
FIGURE 104

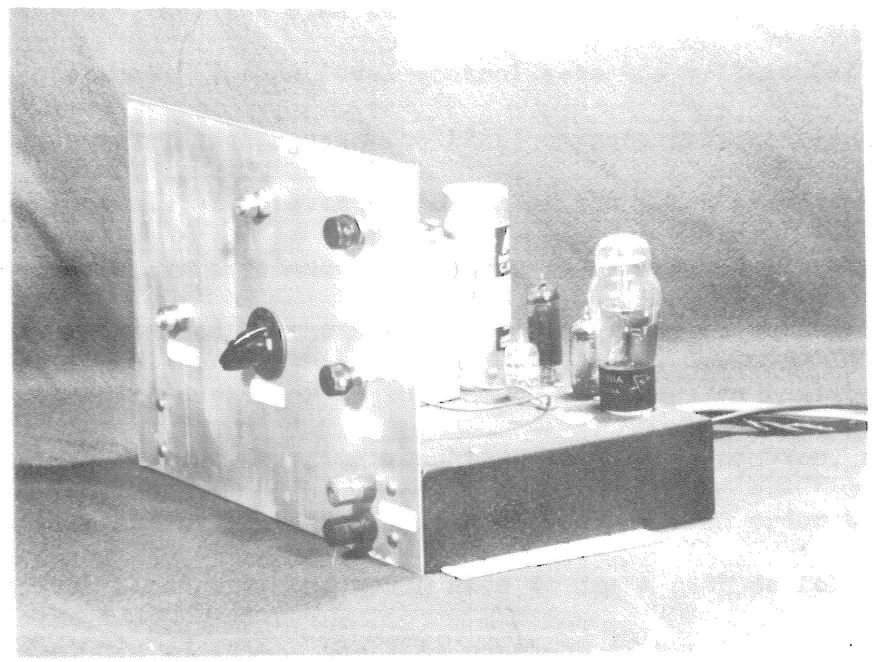


Figure 105. Photograph of Linear Sweep Generator

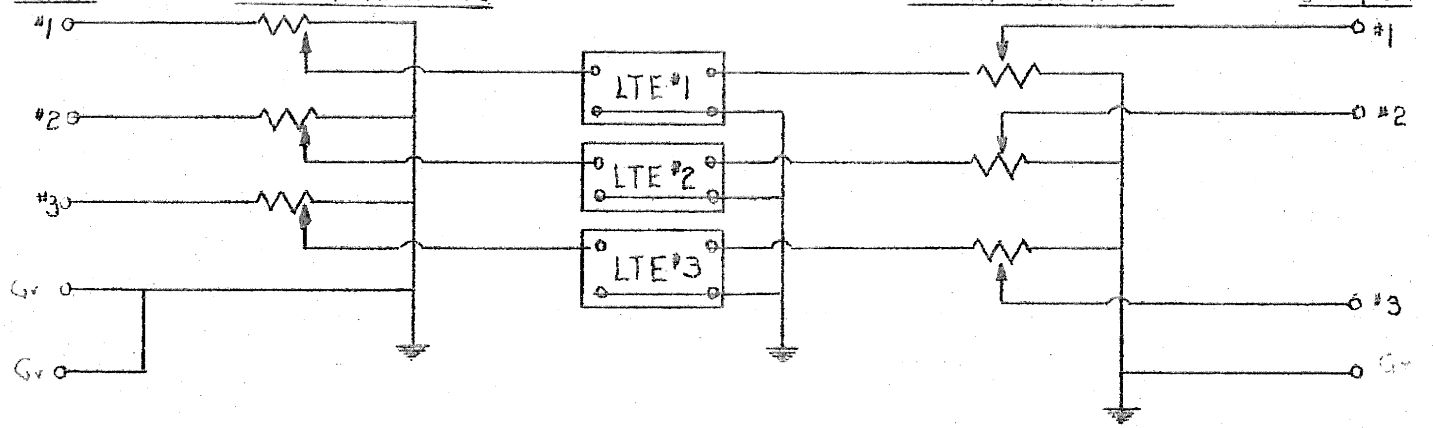
thyratron. The sweep generator will operate satisfactorily with pulse repetition rates between 10 and 60 cyc/sec. The pulses cause the 5627 thryatron to fire and to discharge the sweep capacitor. The coarse amplitude control selects the capacitor to be used in the sweep circuit, and the fine amplitude control varies the charging resistance. The dc level control sets the voltage to which the capacitor discharges. The 12AT7 cathode follower serves two purposes; namely, it prevents loading of the sweep capacitor, and it provides a low impedance source for the current feedback. Linearity is maintained by the use of feedback from the voltage across the charging capacitor to the end of the charging resistor. Nearly constant current charging results from this connection, and the result is a nearly linear saw-tooth of voltage. In order to attain complete linearity, one would have to use a cathode follower of unity gain. In addition, the time constant of the feedback circuit would need to be infinite. Analysis has shown, however, that with the values of the components given and with a repetition rate of 30 cyc/sec, the deviation of the output will not be greater than 1 per cent from linear. However, at low output voltages the nonlinearity of the cathode follower and drift of the 5627, 12AT7, and 6AL5 increase the percentage error. The 6AL5 is used in a clipping circuit in order to give a zero reference when required.

The linear-sweep generator has given trouble-free operation except that the 80-μf, 450-volt feedback capacitor, which is overvolted, has had to be replaced. Electrolytic condensers of 600-volt rating are available, but their size might prohibit their use in this piece of equipment.

4. Computer connector unit. For convenience of operation, a computer connector was built. This unit, diagrammed in Figure 106 and pictured in Figure 107, is almost necessary if more than one LTE is used. The unit has a cable that plugs into the power distribution board and supplies the three LTE's with the necessary potentials. All other necessary connections to the LTE's are made through sockets. The input and output Helipots are useful when setting up a function. The center-tapped Helipot supplying a binding post labeled bias is used to adjust the output level of the ILTE.

5. Voltage control box. The voltage control box shown in the center of Figure 108 and diagrammed in Figure 109 is a general-purpose unit. It was found to be useful whenever an accurately calibrated potentiometer was required in an analog.

6. Oscilloscope calibrator and function selector. The oscilloscope calibrator and function selector shown schematically in Figure 110 was constructed when a nonlinear spring problem was being investigated. The services supplied by this unit should be of value to future users of the computer. The terminals connecting to the analog circuit are at the left of Figure 110. If a series analogy is used and if one plots phase trajectories, the oscilloscope is connected between two components with its chassis above ground.



NOTE · STANDARD POWER CABLE SUPPLIES POWER TO LTE'S & BIAS POTENTIOMETER.

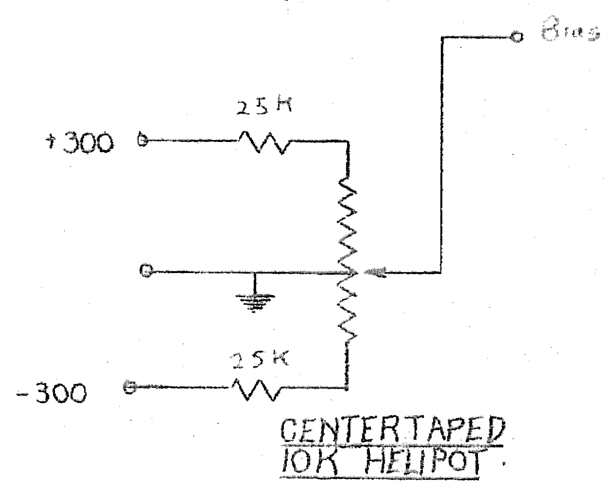


FIG. 106 COMPUTER CONNECTOR UNIT

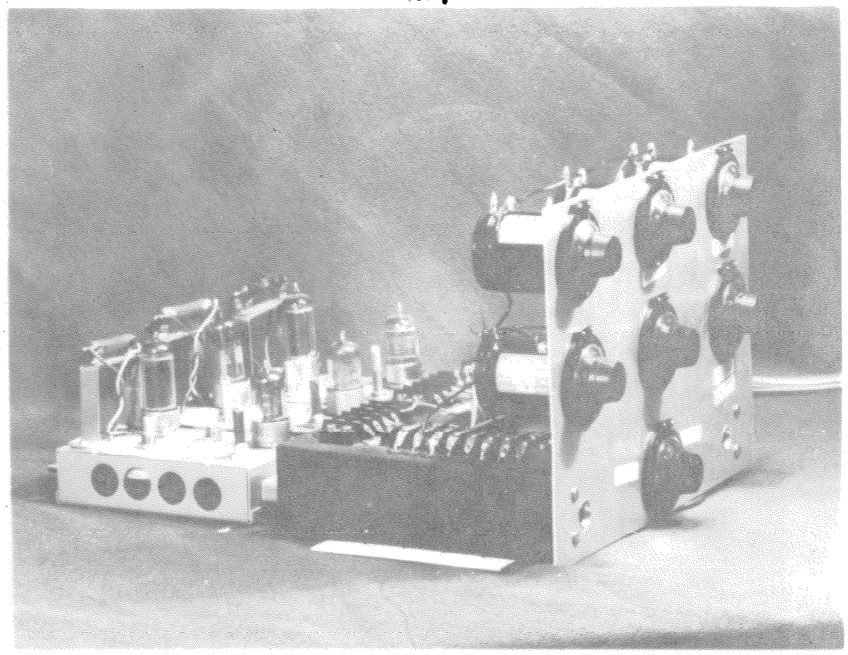


Figure 107. Photograph of Computer Connector Unit

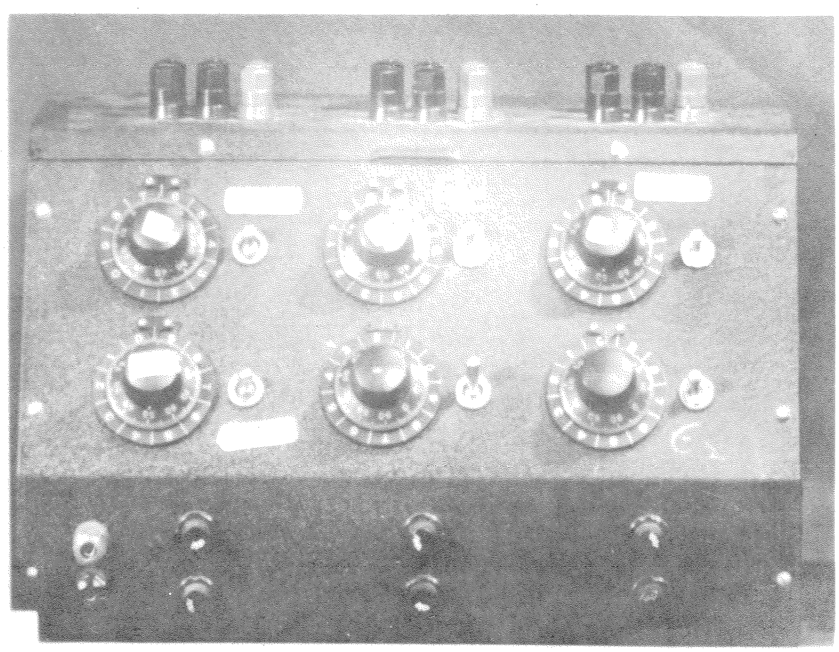
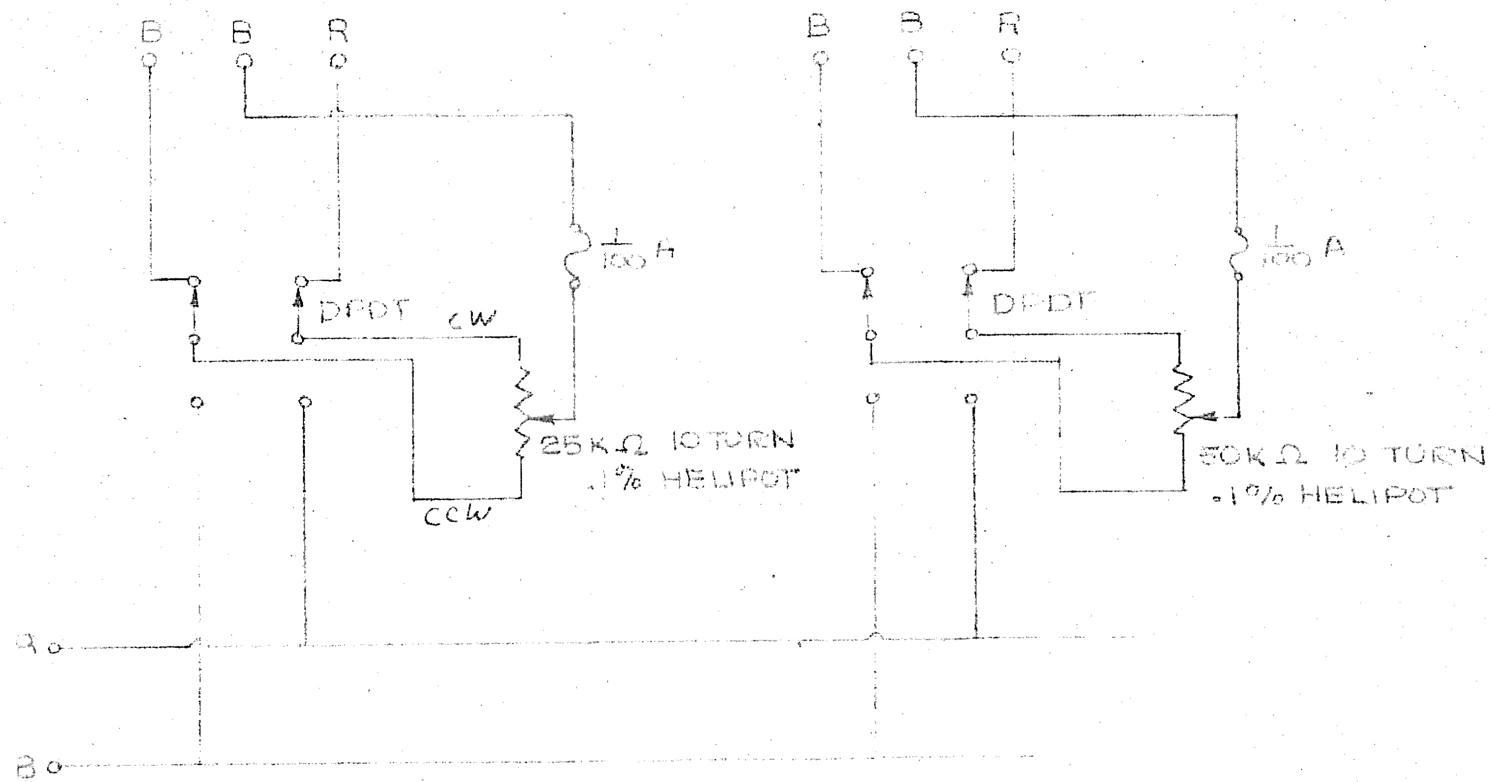


Figure 108. Photograph of Tube Ager, Voltage Control Box, and Small Power Supply



SWITCHES SHOWN WITH HANDLES IN "UP" POSITION

25 KΩ HELIPOTS IN BOTTOM ROW

THREE SETS AS SHOWN ABOVE

FIGURE 109 - VOLTAGE CONTROL BOX

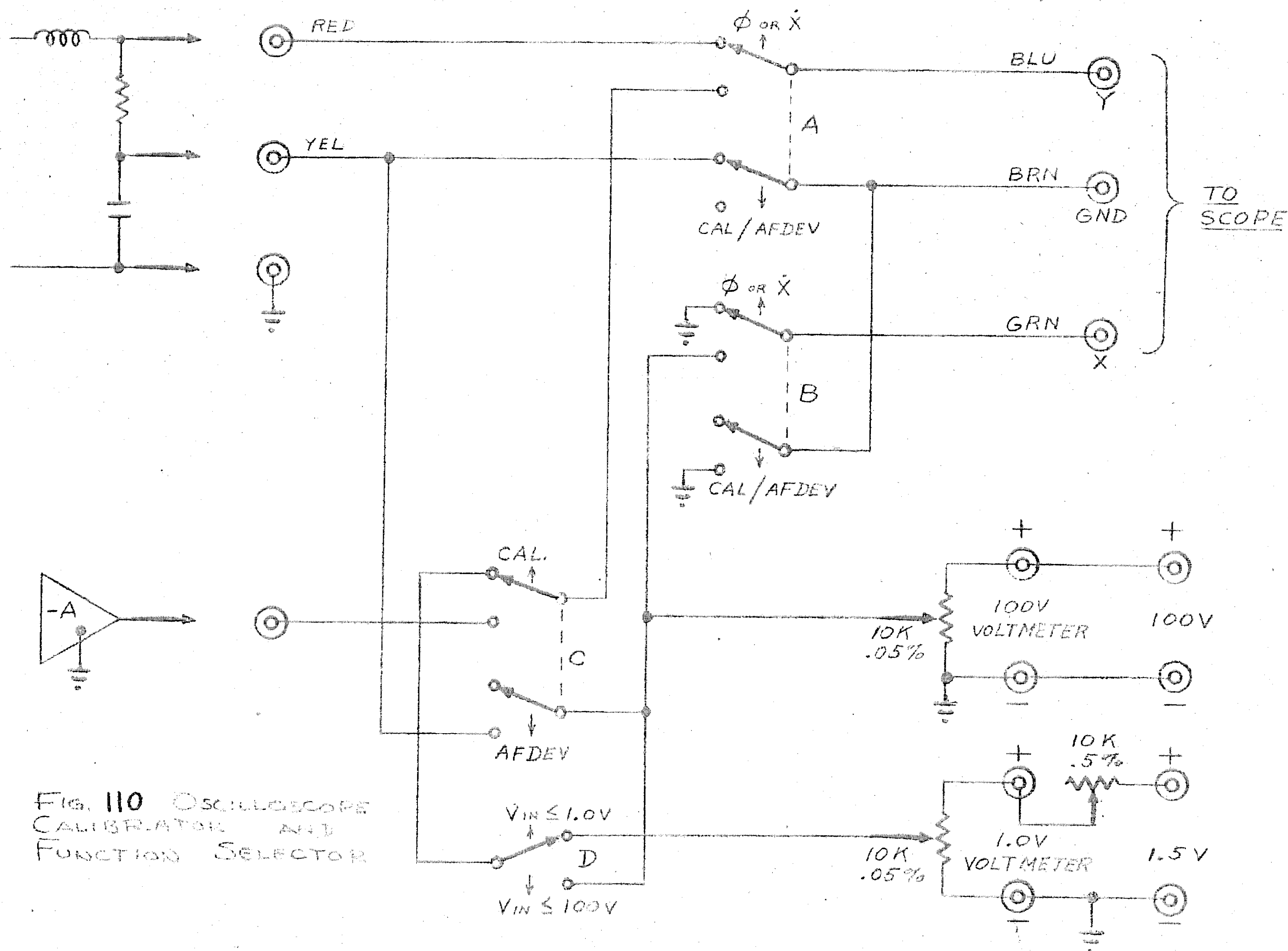


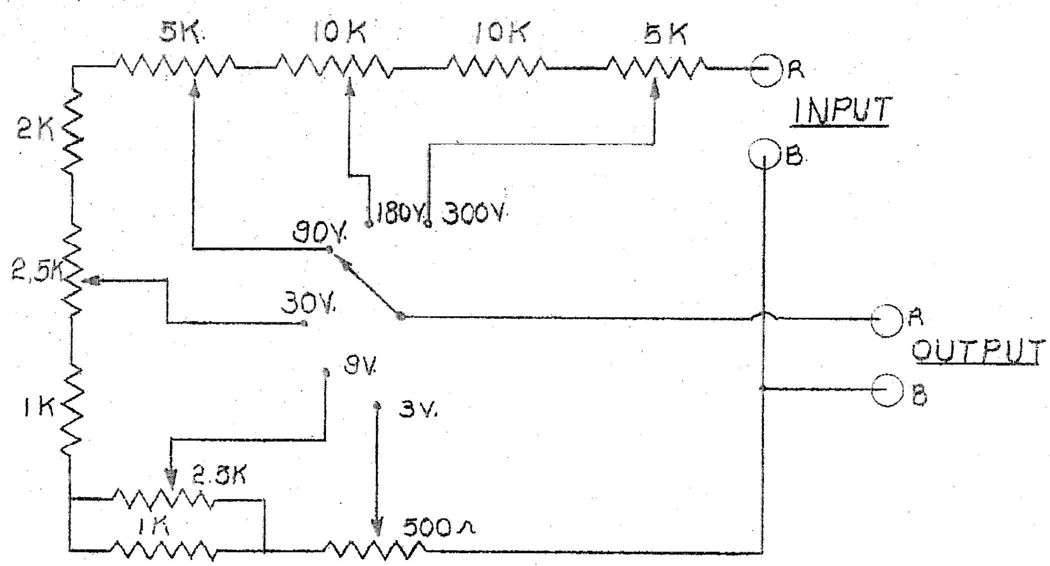
Fig. 110 Oscilloscope Calibrator and Function Selector

The x and y input channels are connected at the opposite ends of the components. If switches A and B are in the "up" position, the oscilloscope is properly connected to show phase trajectories. Reversing these two switches and setting switch C in the "up" position will connect calibration potentiometers to the oscilloscope. If the input voltages to the potentiometers are set properly, the Duodials become direct-reading. When used with a long-persistence screen cathode-ray tube, this calibrator proved to be extremely convenient. The one remaining function of this unit is the display of nonlinearity, obtained by throwing switch C down. This channel could also be used to inspect some other waveform in the analog.

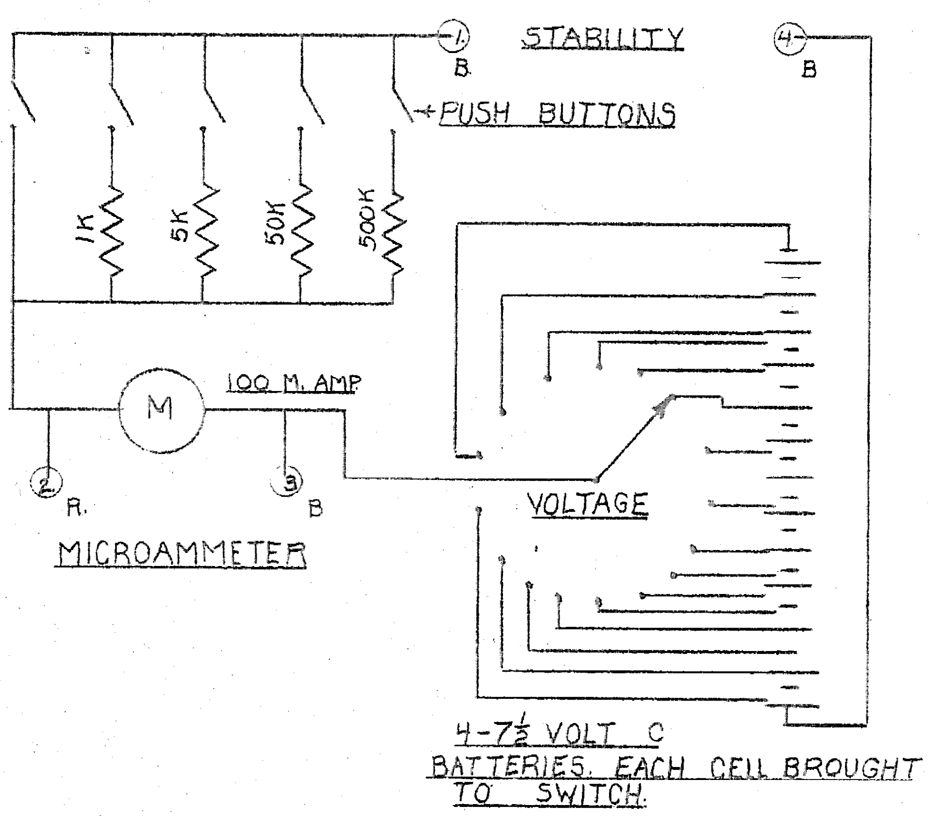
7. Test voltage unit. Figure 111 shows the diagram of the test voltage unit. The final adjustment of the LTE must be done manually, and this unit, once adjusted, reduces the labor considerably.

8. Stability voltmeter. Figure 112 shows the stability voltmeter. The original purpose of this device was measurement of the drift of the LTE. Provision is made for balancing out most of the output from the LTE, and the drift is read directly on the meter. The unit has found most use, however, as a variable low-impedance dc source.

9. Other equipment. The tube ager and battery charger are shown schematically in Figure 113. The tube ager is seen to the left in the photograph of Figure 108. Several other small units were constructed for which diagrams are not available.



TEST VOLTAGE UNIT
FIG. 111



STABILITY VOLTMETER
FIG. 112

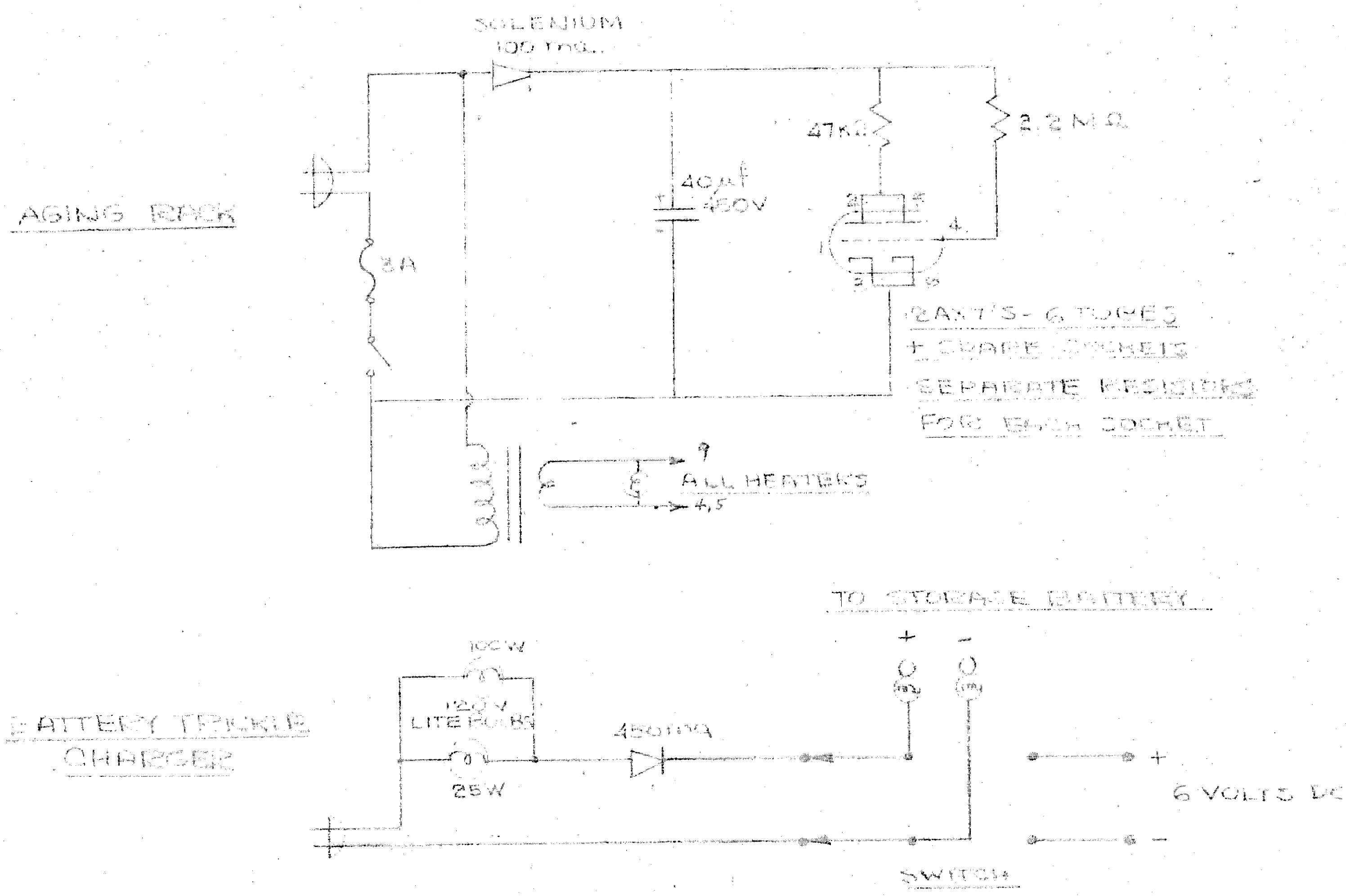

AGING RACK
SOLENIUM
100 ma.
2.2 MΩ
47KΩ
40μf
450V
3A
12AX7'S - 6 TUBES
FOR EACH SOCKET
ALL HEATERS
9
4,5
BATTERY TRICKLE
CHARGER
100W
120V
LITE BULBS
25W
450ma
TO STORAGE BATTERY
+
-
6 VOLTS DC
SWITCH

FIGURE 113

APPENDIX B

Photography

Much of the data presented in this thesis was obtained photographically.

1. Equipment and materials used. The oscillograms were all taken with a single-lens reflex camera using 35-mm film. The lens used was of long focal length, 135 mm, so that (1) the distortion of the image resulting from the curved face of the cathode-ray tube would be minimized and (2) the camera could be placed far enough from the oscilloscope so that no interference with the use of the oscilloscope would result. A 0 to 9999 nonresettable counter was mounted on the front panel of the oscilloscope so that an identifying number was recorded on every negative. The fast panchromatic film used was tank-processed using an active developer. The camera was mounted on a tripod.

The photographic method used would be satisfactory if one did not take many oscillograms. Several shortcomings are evident, however, after taking approximately 500 oscillograms.

1. The camera was placed at varying distances from the oscilloscope screen unless special precautions were taken.
2. The counter was not changed each time a picture was taken; thus the numbering system was not perfect.
3. The 135-mm lens was too slow (f:4.0) to give good

records of fast transients.

4. The tripod, even though nearly 3 feet from the oscilloscope, was always in the way.

5. The room lights had to be turned off to prevent reflections from being recorded upon the film.

If these difficulties could be removed, a very satisfactory setup would result. The virtues of such a system would be:

1. Low cost of film.
2. Positive negative identification.
3. Ease of film storage and identification.
4. Ease of processing in large quantities.
5. Ability to record traces with high writing rates.
6. Consistant object-image scaling.
7. Nearly foolproof operation.

2. Suggested oscillograph setup. The camera should be a 35-mm automatic single-lens reflex camera such as the Kine Exakta or the Alpa Reflex. The lens should be the Zeiss 75-mm, f:1.5 Biotar or its equivalent. The 75-mm lens gives a good compromise between high speed, low distortion, and reasonable cost. The camera should be mounted on a frame hinged to the oscilloscope so that the camera can be pushed up and back when it is not being used and yet will always return to the proper position when desired. When the camera is in operating position, the lens and the oscilloscope screen should be shielded from the room lights. The counter should, when viewed by

the camera lens through a prism or mirror, appear to be in the same plane as the oscilloscope screen. The counter should be magnetically operated by the flash-synchronizing switch in the camera. (The circuit of the counter would have to be arranged so that the number advanced when the synchronizing switch was opened, that is, when the film was advanced.)

The darkroom equipment would include a large Nikor tank that would allow up to ten rolls of film to be processed simultaneously. A special fixed-focus enlarger would be constructed. This enlarger would make prints $3\frac{1}{4}$ by $4\frac{1}{4}$ inches in size. A mask would be built in so that only the round oscilloscope screen and the negative number would be printed. Figure 114 illustrates the form of the resulting

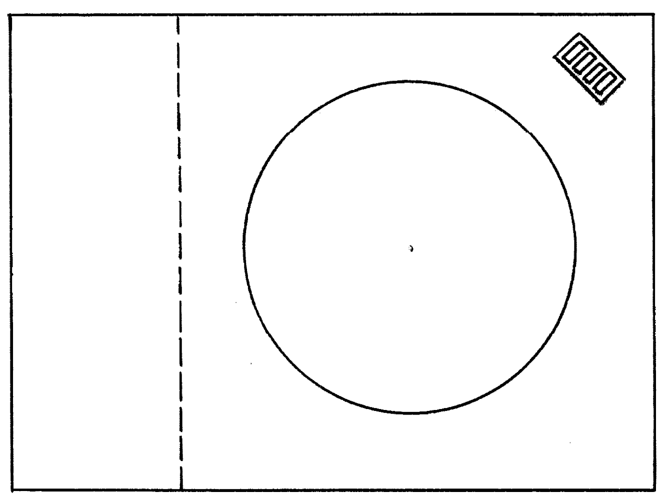


Figure 114. Suggested Format for Oscillograms

print. The left margin of the prints could be punched, and the prints bound into volumes for reference use. If the prints were to be mounted, the inch on the left side could be cut off yielding a square print that would still be permanently identified by number.

The setup described could be assembled for about $350 and would be very useful to any analog computing facility.

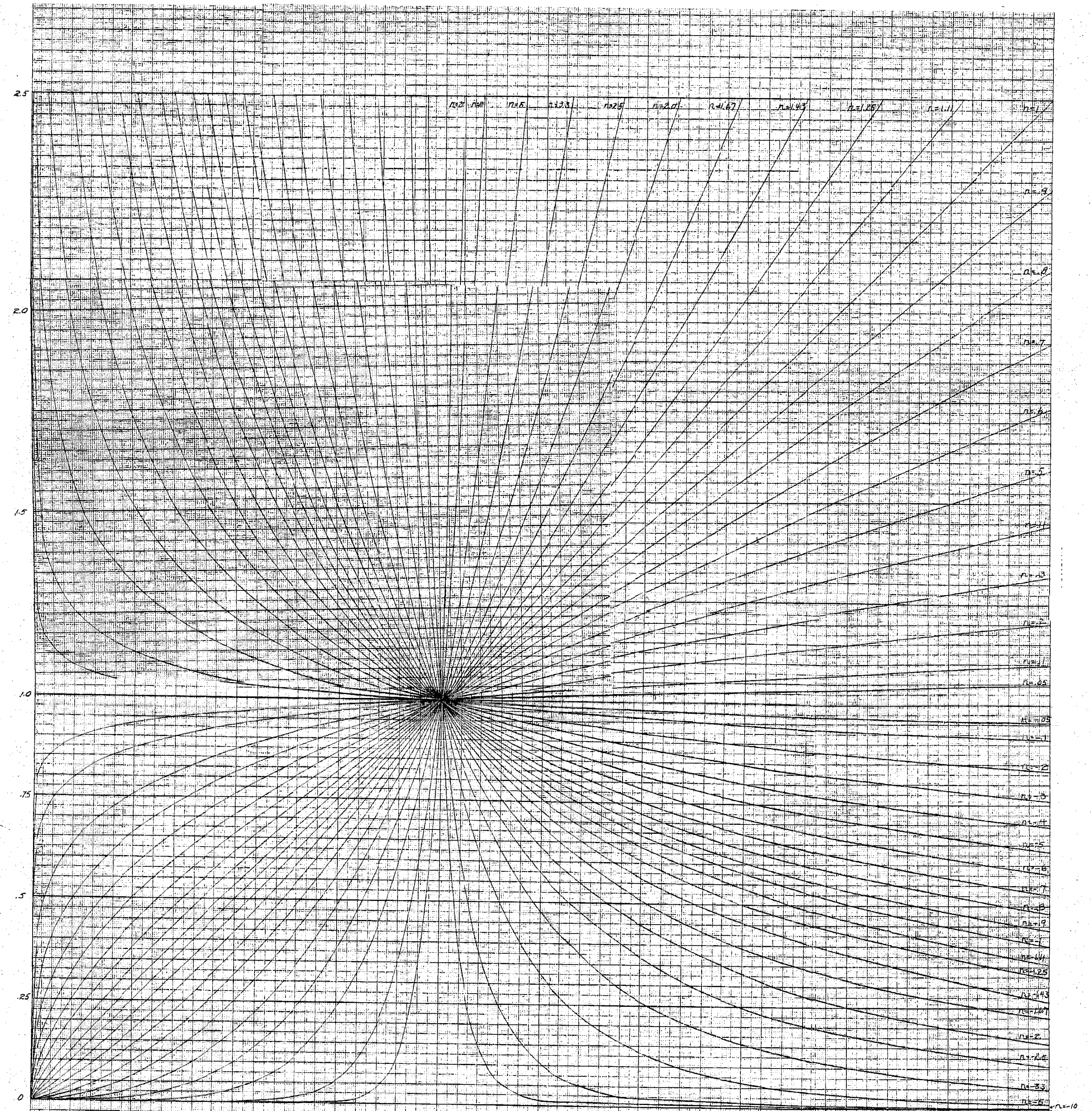


FIGURE 14
K^n VERSUS K