Black Hole Perturbation Theory beyond General Relativity and Holographic Gravity in Flat Spacetime

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Dongjun Li

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Dongjun Li
ORCID: 0000-0002-1962-680X

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ABSTRACT

In this thesis, we study two topics in using gravitational waves (GWs) to probe fundamental physics. The first topic is using black hole (BH) perturbation theory to model GW emissions by binary BH mergers in gravity theories beyond Einstein's general relativity (GR). The second topic is studying holographic quantum gravity signatures around interferometers in flat spacetime.

For BH perturbation theory beyond GR, we first construct a novel formalism based on Teukolsky's seminal work in the 1970s. Our modified Teukolsky formalism works for BHs with arbitrary spin in a broad class of beyond-GR theories that admit an effective field theory description. We derive this formalism by following Chandrasekhar's prescription to make some convenient gauge choices, under which different degrees of curvature perturbations naturally decouple. In the end, we get two decoupled and potentially separable second-order partial differential equations for the Weyl scalars Ψ_0 and Ψ_4, representing the ingoing and outgoing gravitational radiations of a perturbed BH, respectively. Our formalism works for both linear and nonlinear orders in the beyond-GR couplings. We then apply our formalism to specific examples.

In the first example, we study the isospectrality breaking of quasinormal modes (QNMs) in beyond-GR theories, where the even- and odd-parity QNMs have different frequencies. We apply the modified Teukolsky formalism and the eigenvalue perturbation method to construct a direct connection between the parity features of a theory and its structure of isospectrality breaking. In the second example, we focus on the QNMs of dynamical Chern-Simons gravity up to the first order in the slow-rotation expansion. We first directly compute the scalar field equation and the modified Teukolsky equations for Ψ_0 and Ψ_4 in the ingoing and outgoing radiation gauges, respectively. We then reduce all the equations to radial ordinary differential equations by projection to the spin-weighted spheroidal harmonics. We find that the scalar field is only coupled to the odd-parity perturbations, which is consistent with the previous studies. We then compute the QNM frequencies for the non-rotating case via the eigenvalue perturbation method. The results from the two gauges are self-consistent and agree well with previous results using metric perturbations. Since this is ongoing work, we briefly discuss the strategy for the rotating case at the end. In the third example, we apply a similar analysis to certain parametrized axisymmetric deviations of non-rotating BHs using a Weyl multipole expansion. We compute the

QNM frequencies directly and analyze their connections to the multipole structure of a BH spacetime.

For holographic gravity in flat spacetime, we build an effective model for geometrical spacetime fluctuations driven by entropic fluctuations, or "geontropic fluctuations" for short, in the casual diamond defined by an interferometer. Our model involves a bosonic scalar field with some nontrivial occupation number, called "pixellon." The pixellon field characterizes all the nonlinear holographic quantum gravity fluctuations within a causal diamond in flat spacetime. We then build up a framework for computing the gauge-invariant observables of geontropic fluctuations for an interferometer with equal arms separated by arbitrary angles. We compute both the power spectral density and angular correlation of length fluctuations in such an interferometer for the pixellon model. We then use the existing or predicted noise spectra of LVK, LISA, GEO-600, and Holometer to constrain the pixellon model. In our follow-up study, we further extend the pixellon model to incorporate configurations of multiple interferometers. We then apply this extended pixellon model to calculate the power spectral density of geontropic fluctuations in Cosmic Explorer, Einstein Telescope, NEMO, and optically-levitated sensors. For Cosmic Explorer, Einstein Telescope, and NEMO, we find that the signal of the pixellon model could exceed the detector's predicted sensitivity by one or two orders of magnitudes.

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Chapter 1

INTRODUCTION

The detection of gravitational waves (GWs) emitted by over a hundred binary black hole (BH) mergers [1–3] usher us into a new era of gravitational physics. With the much-improved sensitivity in the fourth observing run of the LIGO-Virgo-KAGRA collaboration, we are now able to detect these binary mergers at a single-detector signal-to-noise ratio threshold of 8 with a distance as far as 1600 Mpc from the Earth [4]. Anticipating next-generation ground-based detectors, such as Cosmic Explorer [5–7] and Einstein Telescope [8, 9], and space-based detectors, such as LISA [10], TianQin [11], and Taiji [12], we will be able to observe mergers of compact objects even in much larger parameter space, including mergers of primordial and supermassive BHs, extreme mass-ratio inspirals (EMRIs), BH-neutron star (NS) mergers, and possibly the first binary NS merger [6, 13]. This extensively expanded scope of targets provides abundant opportunities to scrutinize strong gravity physics, such as probing the near-horizon geometry of BHs, examining modifications to general relativity (GR), discovering exotic compact objects, understanding asymptotic symmetries of spacetime, and unveiling possible quantum gravity signatures [6, 13]. Furthermore, these GW detections also open avenues to investigate numerous questions in astrophysics, cosmology, and particle physics, including but not limited to the formation channels of supermassive BHs in the early universe, the physics of dense matter at high temperatures, the properties of dark matter and dark energy, and the evolution of our galaxy and universe [6, 13].

To reliably study different physics from GW detections, it is crucial to accurately and efficiently model the GW emissions by different compact objects in various gravity theories and diverse astrophysical environments. The first part of this thesis (Chapters 2–6) focuses on a specific direction in the waveform modeling by constructing a new formalism based on Teukolsky's seminal work in [14–16] to model the GWs emitted by BHs in beyond-GR theories and astrophysical environments and applying this formalism to selected examples. Besides generating reliable waveforms, it is also important to understand the physics around interferometers. The second part of this thesis (Chapters 7 and 8) focuses on studying certain quantum gravity effects around the interferometers in flat spacetime, first proposed by Verlinde and Zurek in [17–22], and computing the relevant observables for current and future GW detectors.

1.1 BH perturbation theory beyond GR

Accurate and efficient waveform models of GW emissions are crucial for reliably extracting information from GW detections and examining GR. For the past twenty years, significant progress has been made in waveform modeling in GR, including both numerical relativity (NR) [23–27], where the full nonlinear Einstein equations are evolved on some numerical grid, and analytical or semi-analytical approaches, where the Einstein equations are perturbatively solved by expansion about some small parameters. There are several main analytical or semi-analytical approaches [28]: post-Newtonian expansion [29–33], which expands about the ratio of a system's internal speed v with the speed of light c; post-Minkowskian expansion [34–38], which expands a curved spacetime about the Minkowski background using the gravitational constant G; gravitational self-force theory [39–43], which expands about the small mass ratio of two compact objects; and BH perturbation theory [14–16, 44–50], which considers vacuum perturbations of an isolated BH. Although NR usually produces the most accurate waveforms, where all the nonlinear features of gravity are completely captured, NR is expensive. For example, simulations of a binary BH merger with a highly spinning remnant can take up to several weeks or months [51, 52], which is not suitable for certain jobs in GW data analysis, such as parameter estimation. Many efforts have been taken to shorten the computation time of NR, such as finding better gauges [51], interpolating waveforms across large parameter space (e.g., surrogate models [53–55]), or hybridizing analytical or semi-analytical approaches with NR [54, 56–59]. As NR folds information in a highly nonlinear way, these analytical or semi-analytical approaches also aid in understanding NR results by extracting key features of a binary merger in different limits. Thus, it is crucial to further deepen our understanding of these analytical or semi-analytical approaches while developing NR.

BH perturbation theory, as one of the most important analytical or semi-analytical approaches, focuses on gravitational perturbations of an isolated stationary BH[1], given that the merger of binary BHs always settles down to a stationary geometry in GR. BH perturbation theory provides very accurate descriptions of a binary BH merger's ringdown, where the spacetime can be modeled as vacuum perturbations of a remnant BH, and EMRIs, where a stellar-mass compact object perturbs a supermassive BH.

[1]We do not distinguish BH perturbation theory from gravitational self-force in this thesis since their mathematical formulations are similar

1.1.1 Regge-Wheeler and Zerilli-Moncrief formalism in GR

When the BH being perturbed is non-rotating such that the background metric is static and spherically symmetric, one can follow the prescription by Regge, Wheeler, Zerilli, and Moncrief [44–46] to derive two decoupled radial ordinary differential equations. Specifically, consider the Schwarzschild metric

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi)\,, \quad f(r) = 1 - 2M/r\,, \tag{1.1}$$

where we use the geometrized unit ($G = c = 1$). One can decompose the linear-order perturbation of the Schwarzschild metric into two categories based on their parity, i.e.,

$$h_{\mu\nu} = h^{\mathrm{E}}_{\mu\nu} + h^{\mathrm{O}}_{\mu\nu}\,, \qquad \hat{P}h^{\mathrm{E}}_{\mu\nu} = h^{\mathrm{E}}_{\mu\nu}\,, \quad \hat{P}h^{\mathrm{O}}_{\mu\nu} = -h^{\mathrm{O}}_{\mu\nu}\,, \tag{1.2}$$

where $\hat{P}$ is the parity operator: $\hat{P}f(\theta,\phi) \to f(\pi-\theta,\phi+\pi)$, and $h^{\mathrm{E}}_{\mu\nu}$ and $h^{\mathrm{O}}_{\mu\nu}$ denote the even- and odd-parity modes, respectively. One can further decompose the perturbation into multipoles,

$$h_{\mu\nu} = \sum_{\ell,m}\left(h^{\mathrm{E},\ell m}_{\mu\nu}e^{-i\omega^{\mathrm{E}}_{\ell m}t} + h^{\mathrm{O},\ell m}_{\mu\nu}e^{-i\omega^{\mathrm{O}}_{\ell m}t}\right)\,, \tag{1.3}$$

which in the Regge-Wheeler (RW) gauge becomes [44–46, 60, 61]

$$\begin{aligned} h^{\mathrm{E},\ell m}_{\mu\nu}dx^\mu dx^\nu = &\Big(f(r)H^{\ell m}_0(r)dt^2 + 2H^{\ell m}_1(r)dtdr + H^{\ell m}_2(r)f^{-1}(r)dr^2 \\ &+ r^2K^{\ell m}(r)d\theta^2 + r^2\sin^2\theta K^{\ell m}(r)d\phi^2\Big)Y_{\ell m}(\theta,\phi)\,, \end{aligned} \tag{1.4a}$$

$$\begin{aligned} h^{\mathrm{O},\ell m}_{\mu\nu}dx^\mu dx^\nu = &\, 2h^{\ell m}_0(r)\left[\sin\theta\partial_\theta Y_{\ell m}(\theta,\phi)dtd\phi - \csc\theta\partial_\phi Y_{\ell m}(\theta,\phi)dtd\theta\right] \\ &+ 2h^{\ell m}_1(r)\left[\sin\theta\partial_\theta Y_{\ell m}(\theta,\phi)drd\phi - \csc\theta\partial_\phi Y_{\ell m}(\theta,\phi)drd\theta\right]\,, \end{aligned} \tag{1.4b}$$

where $Y_{\ell m}(\theta,\phi)$ are spherical harmonics. Defining two master variables, the Zerilli-Moncrief (ZM) function $\Psi^{\ell m}_{\mathrm{ZM}}(r)$ and RW function $\Psi^{\ell m}_{\mathrm{RW}}(r)$ [44–46],

$$\Psi^{\ell m}_{\mathrm{ZM}}(r) \equiv \frac{r^2}{\lambda r + 3M}K^{\ell m}(r) - \frac{r-2M}{\lambda r + 3M}\int dt\; H^{\ell m}_1(r)\,, \tag{1.5}$$

$$\Psi^{\ell m}_{\mathrm{RW}}(r) \equiv \frac{r-2M}{r^2}h^{\ell m}_1(r)\,, \tag{1.6}$$

where $\lambda = \frac{1}{2}(\ell-1)(\ell+2)$, one gets from the vacuum Einstein equations that

$$\left[\partial^2_{r^*} + \left(\omega^{\mathrm{E,O}}_{\ell m}\right)^2 - V^{\mathrm{ZM,RW}}_{\ell m}(r)\right]\Psi^{\ell m}_{\mathrm{ZM,RW}}(r) = 0\,,$$

$$
\begin{aligned}
V_{\ell m}^{\mathrm{ZM}}(r) &= \frac{2f(r)}{r^3}\frac{9M^3+3\lambda^2 M r^2+\lambda^2(1+\lambda)r^3+9M^2(\lambda r)}{(3M+\lambda r)^2}\,, \\
V_{\ell m}^{\mathrm{RW}}(r) &= \left(1-\frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2}-\frac{6M}{r^3}\right)\,,
\end{aligned}
\tag{1.7}
$$

where $r^* = r + 2M \ln(r/2M - 1)$ is the tortoise coordinate. In this case, the ZM function $\Psi_{\mathrm{ZM}}^{\ell m}(r)$ and RW function $\Psi_{\mathrm{RW}}^{\ell m}(r)$ represent even- and odd-parity metric perturbations, respectively. All the metric components could be computed from $\Psi_{\mathrm{ZM}}^{\ell m}(r)$ and $\Psi_{\mathrm{RW}}^{\ell m}(r)$ [44–46]. Although $\Psi_{\mathrm{ZM}}^{\ell m}(r)$ and $\Psi_{\mathrm{RW}}^{\ell m}(r)$ satisfy different equations, one can transform the equation of one to the other by the Chandrasekhar transformation [49]. This indicates that the even- and odd-parity modes are redundant for Schwarzschild BHs, so their frequency must be the same, $\omega_{\ell m}^{\mathrm{E}} = \omega_{\ell m}^{\mathrm{O}}$, the so-called isospectrality [49, 60].

1.1.2 BH ringdown and EMRIs in GR

In general, Eq. (1.7) has two linearly independent solutions, which are either purely outgoing at infinity ($r^* \to \infty$) or purely ingoing at the horizon ($r^* \to -\infty$), i.e.,

$$
\Psi_{\mathrm{ZM,RW}}^{\ell m}(r^* \to -\infty) \propto e^{-i\omega_{\ell m} r^*} \quad \text{or} \quad \Psi_{\mathrm{ZM,RW}}^{\ell m}(r^* \to \infty) \propto e^{i\omega_{\ell m} r^*}\,. \tag{1.8}
$$

Usually, the conditions in Eq. (1.8) cannot be simultaneously satisfied. One exception is the quasinormal mode (QNM), which dominates the ringdown phase of GW emissions and carries a quickly decaying and oscillating amplitude. At QNM frequencies, the two solutions become redundant and simultaneously satisfy the boundary conditions in Eq. (1.8). The QNM frequencies $\omega_{\ell m}$ are also the poles of the Green's function for Eq. (1.7), so $\omega_{\ell m}$ can carry an additional quantum number n, labeling the n-th pole of the Green's function [50]. The QNMs with $n = 0$ are called fundamental modes, while the QNMs with $n > 0$ are called overtones. To compute the QNM frequencies, one naive way is to evolve a solution $\propto e^{i\omega_{\ell m} r^*}$ from infinity to the horizon, evolve another solution $\propto e^{-i\omega_{\ell m} r^*}$ from the horizon to infinity, and iteratively find $\omega_{\ell m}$ making these two evolved solutions match at a middle point. However, since $\omega_{\ell m}$ for a QNM always has a negative imaginary part, the solution diverges at both the horizon and infinity, making this naive approach numerically challenging. A more reliable and widely used approach is to turn calculating QNMs into solving continuous fraction equations, first invented by Leaver in [62, 63].

Besides studying BH ringdown, BH perturbation theory has also been widely used to compute the waveforms of EMRIs. One main approach for generating EMRI waveforms is to use an adiabatic expansion and two-timescale analysis, given that the time scale of the inspiral is much longer than the time scale of any orbital motion of the secondary [64–74]. In this case, one decomposes the whole inspiral into sequences of orbits, where each orbit is parametrized by three constants of motion: the orbital energy E, the orbital angular momentum L_z, and the Carter constant Q [71]. One can evolve one orbit to another one by computing the shifts of the constants of motion due to gravitational radiations, which occur at the much longer time scale of inspiral. In leading order, the shifts of the orbital energy E and orbital angular momentum L_z are given by the RW and ZM functions, e.g.,

$$\dot{E}^{\ell m} \propto \omega_{\ell m}^2 |\Psi_{\mathrm{ZM,RW}}^{\ell m}|^2 , \quad \dot{L}_z^{\ell m} \propto \dot{E}^{\ell m}/\omega_{\ell m} . \tag{1.9}$$

Unlike computing QNMs, the metric perturbations are now driven by the stress tensor $T_p^{\alpha\beta}$ of the secondary,

$$T_p^{\alpha\beta} = \mu \int u^\alpha u^\beta \frac{\delta^{(4)}\left(x^\sigma - x_p^\sigma(\tau)\right)}{\sqrt{-g}} d\tau , \tag{1.10}$$

where μ is the secondary's mass, $x_p^\sigma(\tau)$ is the secondary's worldline parametrized by τ, and u^α is the secondary's four-velocity, so one needs to solve the sourced Einstein equations. One can linearize metric perturbations about the small mass ratio $q = \mu/M$ with M being the central supermassive BH's mass. The ratio of the time scales for the orbital motions and the radiation reaction is also proportional to q. From the sourced Einstein equations, one can follow the similar procedures above to derive the sourced RW and ZM equations, with the left-hand side the same as Eq. (1.7). The relation between a generic stress tensor and the source terms in the RW and ZM equations can be found in [45, 61]. The sourced RW and ZM equations can be solved by using Green's function of Eq. (1.7). Using Eq. (1.9), one can then sequentially evolve the orbits and piece them together into a full inspiral. Many studies have followed this approach to generate EMRI waveforms for equatorial circular (or quasi-circular) orbits of non-rotating [64–66, 68–70, 72, 73] and rotating BHs [67, 71], which later got extended to inclined eccentric orbits of rotating BHs [28, 75–78]. Notice that for rotating BHs, instead of solving the RW and ZM equations, one solves the Teukolsky equations instead [14–16], as discussed below. The waveforms calculated by only incorporating the gravitational radiations in Eq. (1.9), or the first-order dissipative self-force (linear in q), are regarded as "adiabatic" or "zero post-adiabatic" [73, 74]. In recent years, significant efforts have been put into computing higher-order contributions in the adiabatic expansion, such as these "first post-adiabatic" EMRI waveforms in [79, 80].

1.1.3 Teukolsky formalism in GR

The study of QNMs and EMRIs for rotating BHs in GR follows a framework similar to the one above for non-rotating BHs. Nevertheless, due to the lack of spherical symmetry, it is very challenging to decouple different metric components directly, find two master equations governing them, and reduce these two equations into two ordinary differential equations in the radial coordinate. An alternative approach is to focus on curvature perturbations instead, which was first developed by Teukolsky in [14–16] and became the foundation of BH perturbation theory. Built on the Newman-Penrose formalism [81], the Teukolsky formalism focuses on perturbations of two radiative Weyl scalars Ψ_0 and Ψ_4,

$$\Psi_0 = C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta \,, \quad \Psi_4 = C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta \,, \tag{1.11}$$

where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor, and $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ are the null tetrad basis vectors in the Newman-Penrose formalism, as we will review in more detail in Chapter 2. Teukolsky found two separable second-order partial differential equations governing $\Psi_{0,4}$, which are decoupled from all the other Newman-Penrose quantities. After a decomposition into harmonics, the angular part of $\Psi_{0,4}$ satisfies the equations of spin-weighted spheroidal harmonics [14–16]. The radial equations of $\Psi_{0,4}$ in the r^* coordinate are Schrödinger-like equations similar to Eq. (1.7) but with more complicated potentials. Furthermore, the metric perturbations associated with the perturbations of $\Psi_{0,4}$ can be reconstructed. For vacuum perturbations, one can follow the standard procedures developed by Chranznowski, Cohen, Kegeles, and Ori [82–89] or alternative procedures in [49, 90]. For non-vacuum perturbations (such as EMRIs), metric reconstruction is more challenging, but several frameworks have been developed [42, 43, 91–93] and applied to the study of higher-order self-force [79, 80]. To study QNMs, one can apply the Leaver's method [62, 63] to compute the frequencies $\omega_{\ell m}$ as well as the angular and radial parts of $\Psi_{0,4}$. For EMRIs, the energy and angular momentum carried away by gravitational radiations can be computed using relations similar to Eq. (1.9) [14–16].

For the past fifty years, BH perturbation theory at linear order in vacuum GR has been extensively studied and is well understood. In recent years, there has been a continuously growing interest in observing nonlinear features of gravity from GW detections. For example, the nonlinearity in GR can cause a pair of inertial GW detectors to be permanently displaced from each other after a GW passes by, the so-called displacement memory effect [94–96], which can be related to the asymptotic symmetries of spacetimes [97–100] and possibly detectable by future

GW detectors [6, 13, 101, 102]. For BH perturbation theory, nonlinear effects have also attracted much attention. The nonlinear extension of the Teukolsky formalism in GR was first done by Campanelli and Lousto in [103]. One important application of the nonlinear Teukolsky formalism is to compute the second-order or higher-order self-force of EMRIs, where one calculates the shifts of E and L_z at nonlinear orders in the small mass ratio q. The second-order self-force will contribute to the first post-adiabatic EMRI waveforms, which are important for data analysis of EMRIs since the phase errors due to adiabatic expansion can accumulate to a sizable amount during EMRIs' long inspiral time [28, 42, 79]. On the other hand, by including higher-order self-force and calibrating to NR, the validity of the waveforms generated by BH perturbation theory can be extended from EMRIs to intermediate mass-ratio inspirals and even comparable-mass mergers [28, 52, 58, 79, 80, 104–106]. For ringdown, nonlinear QNMs have been observed in NR simulations [107–110], where their amplitudes have been both extracted from NR simulations [107–110] and computed directly using BH perturbation theory [111].

1.1.4 Beyond-GR theories and astrophysical environments

Despite passing many tests in the solar system and other weak gravity regimes [112], GR has its limitations and is undergoing tests in strong gravity. On the fundamental side, GR's incompatibility with quantum mechanics has motivated searches for a unified description of quantum gravity, such as string theory [113–118] and loop quantum gravity [119–128], the low-energy effective theories of which can generate modifications to GR. On the observational side, GR on its own is not sufficient to explain, for example, the origin of dark matter [129, 130] and dark energy [131–133], the matter-antimatter asymmetry [134] and late-time acceleration [135, 136] of our universe, and the anomalous galaxy rotation curves [130, 137]. For this reason, modifications to GR are introduced as one resolution of these observational anomalies. Some of widely studied beyond-GR theories include higher-derivative gravity [138–140], $f(R)$ gravity [141, 142], scalar-tensor theories [143], dynamical Chern-Simons (dCS) gravity [144, 145], Einstein dilation Gauss-Bonnet (EdGB) [146–148] or scalar Gauss-Bonnet gravity [149], Horndeski theory [150], Einstein-Aether theory [151], Horava gravity [152], massive gravity [153], and higher dimensional gravity theories [154]. Many of these theories originate from string theory or loop quantum gravity while also explaining certain observational puzzles. For example, the dCS correction, where a pseudoscalar field is coupled to a parity-violating quadratic term in the Riemann tensor and the Riemann dual tensor (the so-called Pontryagin density), appears when one uses the Green-Schwarz mechanism [155] to cancel gravitational anomaly in the heterotic string theory [156, 157]. The dCS terms are also necessary for preserving gauge invariance in loop quantum gravity [158]. On the observational side, the parity-violating terms in dCS gravity can be used to explain matter-antimatter asymmetry [156, 159, 160], while the pseudoscalar field as an axion can be a candidate for dark matter.

These beyond-GR theories can leave imprints on both the ringdown phase of a binary BH merger and the EMRI waveform. For example, one notable feature of the BH QNMs in GR is isospectrality, where the even- and odd-parity QNMs have the same frequencies. However, this isospectrality was observed to be broken in many beyond-GR theories, for example, dCS gravity [161–165], EdGB gravity [166–170], and higher-derivative gravity [171–175]. Furthermore, due to the presence of extra scalar or vector fields in certain theories, beyond-GR QNMs do not only contain gravitational-led modes but also scalar-led [161–163, 166–170] or vector-led modes [176–178], the frequencies of which usually distribute along different branches in the complex plane. For EMRIs, these additional non-metric fields can radiate additional energy and angular momentum, leading to additional phases in the waveforms [163, 167, 179–183], which can be as large as hundreds of GW cycles due to the long inspiral time of EMRIs [13]. Despite many efforts that have been made for the past twenty years to model the ringdown phase or the EMRIs in a variety of beyond-GR theories, most of these studies only focused on non-rotating BHs [161–163, 166–168, 171, 172, 176, 177], and only a handful of calculations were done for slowly rotating BHs [164, 165, 169, 170, 173]. The main challenge was that the widely used RW and ZM equations work only for non-rotating BHs or slowly rotating BHs using the extension in [184]. On the other hand, the remnants of binary BH mergers are usually fast-rotating due to the large angular momentum associated with the inspiral [3, 185], demanding a formalism valid for BHs with large spin. One potential resolution is to extend the Teukolsky formalism from GR to beyond-GR theories. However, this extension could be very challenging since background BH spacetimes are not necessarily Petrov type D under the Petrov classification [186], as required by the Teukolsky formalism, in many beyond-GR theories [145, 187].

Besides corrections to GR, astrophysical environments can also leave imprints on BH QNMs and EMRIs [13, 188, 189]. It is very unlikely that binary BHs are in a vacuum, but rather, they can be embedded in some gaseous accretion disks [190, 191] or plasma [192], surrounded by dark matter clouds or spikes [193–196], or interacting with other compact objects nearby [197, 198]. For example, when the Compton wavelength of ultralight bosonic dark matter is comparable to the radius of a rotating supermassive BH, the dark matter can efficiently drain energy from the BH via superradiance and grow into a macroscopic cloud [199–202]. These clouds can modulate EMRI waveforms by affecting the secondary's motion and exerting backreactions on the BH geometry [196]. Furthermore, the cloud itself can also emit nearly continuous GWs [203], the spectrum of which can be computed in a similar way as the QNM spectrum [204]. The QNM spectrum of a BH can also be affected when there is matter present, which, similar to beyond-GR corrections, can result in the isospectrality breaking [188]. Thus, to test GR, it is important not to misidentify environmental effects as modifications of GR since they might result in similar modifications to GW waveforms. Although many efforts have been made to study BH QNMs and EMRIs in astrophysical environments [190–192, 196, 199, 205–207], most of these examples focus on non-rotating BHs or simple cases for rotating BHs when the Teukolsky formalism in GR still applies.

To conclude, the much better sensitivity of next-generation GW detectors requires more accurate and efficient waveform modeling not only in GR but also in the presence of beyond-GR and astrophysical environmental effects. BH perturbation theory, as one of the most important (semi-)analytical approaches to waveform modeling, is well-developed in GR but still requires significant study in theories beyond GR with environmental effects present, especially for more astrophysically relevant, fast-spinning BHs. In the first part of this thesis (Chapters 2–6), we will present a novel formalism based on Teukolsky's seminal work in [14–16] for studying BH perturbations in the more complicated settings above, applicable to general rotating BHs. In Chapter 2, we construct this modified Teukolsky formalism for a wide class of beyond-GR theories, though this formalism can also be potentially used for studying environmental effects. We then apply the modified Teukolsly formalism to analyze the isospectrality breaking of BH QNMs in Chapter 3, compute the BH QNM frequency shifts in dCS gravity in Chapters 4 and 5, and conduct parametrized ringdown modeling in Chapter 6.

1.2 Holographic gravity in flat spacetime

Since the twentieth century, understanding how gravity interacts with quantum mechanics has been a central theme of fundamental physics. Einstein's GR has been very successful in describing the gravitational interactions of macroscopic objects. Passing many tests under weak gravity [112], GR has been repeatedly examined in the strong gravity regime via detections of GWs generated by drastic coalescences of extremely compact objects [208]. Yet no conclusive signature of that GR is violated has been detected so far. Quantum mechanics, on the other hand, has been very successful in describing the physics of the microscopic world. Integrating quantum mechanics with special relativity, quantum field theory and the resulting standard model of particle physics can accurately describe the electromagnetic interactions among electrons and photons [209–212], the weak interactions among leptons [213–216], and the strong interactions among quarks [217–219]. Many attempts have been made to integrate these two crucial efforts by developing a unified theory of quantum gravity. Perturbative quantization of gravity was found to be nonrenormalizable [220, 221], so non-perturbative descriptions of quantum gravity, such as string theory [113–118] and loop quantum gravity [119–128], have been developed. Nonetheless, one can treat perturbative quantum gravity as an effective field theory (EFT) [222], where the nonrenormalizable effects or ultraviolet divergences are suppressed beyond the Planck scale and not directly accessible by experiments. Nevertheless, new physics, such as string theory and loop quantum gravity, could exist and allow an ultraviolet-complete description. In the low-energy regime, one can still quantize gravity perturbatively without incurring any divergence due to this separation of scales and make predictions. From this EFT viewpoint, it seems that one could never detect quantum gravity in real experiments, at least in the foreseeable future, since the size of Planck length $l_p \sim 10^{-35}$ m is far lower than the sensitivity of the most sensitive detectors measuring spacetime fluctuations, such as the laser interferometers of the LIGO-Virgo-KAGRA collaboration [223–225] and even future detectors (e.g., Cosmic Explorer [6, 7] and Einstein Telescope [9]) with much better sensitivity.

1.2.1 Tests of quantum gravity

Despite the difficulty, numerous efforts have been devoted to searching for signatures of quantum gravity. Many studies focus on probing the spacetime near mergers of compact objects, such as BHs or NSs, where gravity is strong and highly nonlinear, so quantum gravity effects might get amplified. For example, quantum gravity expects Hawking radiations near a BH [226], where pairs of particles and antiparticles are created by vacuum fluctuations. Some particles can escape from the horizon and carry information about the BH to the observers at infinity, while others fall into the BH. As these escaping particles can extract energy and angular momentum from the BH, the BH may evaporate in the end [227–229]. However, the information carried away by particles falling into the BH might be lost during the evaporation of the BH, resulting in the BH information paradox [230, 231]. One resolution is to modify the spacetime near the BH horizon, such as introducing firewalls destroying all the infalling observers [232, 233] or less violent non-local quantum effects around the BH [234, 235]. Other proposals also consider horizonless compact objects, for example, "fuzzballs" [236–238], that can mimic a BH by having the same mass, charge, and angular momentum, but the horizon of a BH is replaced by a fuzzy "cap" with micro-structure, which can encode information of the fuzzball. These fuzzballs can also be the microstates of a BH, which provide a microscopic description of the BH entropy [239]. These modifications, directly at or near the horizon, can result in observable GW signatures. For example, an ordinary classical BH horizon absorbs all the GW radiations, while these near-horizon structures or horizonless objects can partially reflect GWs, generating GW echos [240–243]. The modifications of the horizon can also result in a change in the multipolar moments of a BH, which can be potentially detected during the inspiral stage of a binary merger [244–246]. Other tests of quantum gravity using GWs emitted by binary mergers include the test of the no-hair theorem [247–252], the investigation of the gravitational memory effects and the asymptotic symmetries [6, 13, 94–102], and the examination of beyond-GR theories which are low-energy EFTs of certain quantum gravity theories [13, 138–140, 144–148, 249]. Furthermore, some proposals also consider observing the quantum-gravity decoherence of light emitted by astrophysical objects [253]. As if gravity is quantum, the noisy quantum-mechanical fluctuations of spacetime can interact with light like other environmental effects such that photons lose their information and get dissipated, though the effects might be tiny [253].

Besides testing quantum gravity with astrophysical events, many proposals consider testing the quantum nature of gravity itself using table-top experiments [254]. For example, two massive objects might get entangled by gravitational interactions if gravity is quantum, which can be potentially tested by these matter-wave interferometers [255, 256]. It was argued in [257–259] that even for the entanglement of massive objects mediated by a Newtonian interaction, quantum information is present, so table-top experiments could be valid approaches for testing quantum gravity despite their low energy. Specifically, one cannot distinguish entanglement by a Newtonian field from entanglement by gravitons; the latter's existence will directly verify the quantum nature of gravity [259]. Within each matter-wave interferometer, beams of massive particles are sent along paths separated by finite distance and then recombined. By measuring the phase difference across two finitely separated matter-wave interferometers, one can test whether the particles across different interferometers are entangled using Bell inequality [254]. Recent work [260–262] also compute stochastic fluctuations of the length measured by laser interferometers when gravitons are in different quantum states (e.g., vacuum, coherent, thermal, or squeezed). It was found that although vacuum or coherent states of gravitons have negligible effects, gravitons in thermal or squeezed states might have the possibility to be detected [261].

1.2.2 The holographic principle

Most of the tabletop or ground-based experiments considered above rely on the EFT description of perturbative quantum gravity. On the other hand, full quantum gravity is possibly non-perturbative, as in loop quantum gravity and string theory, where the infrared and ultraviolet scales can actually mix with each other [263]. Therefore, is it possible to extract some of these non-perturbative ultraviolet features of quantum gravity from low-energy laboratory experiments in the infrared? The development of the anti-de Sitter (AdS)/conformal field theory (CFT) correspondence or the holographic principle in general shows some promise along this route. The study of the holographic principle can be traced to the discovery of the Bekenstein-Hawking entropy [226, 264], where the entropy of a BH $S_{\rm BH}$ is given by the surface area A_H of the BH's event horizon, i.e.

$$S_{\rm BH} = \frac{A_H}{4G}\,, \tag{1.12}$$

where G is the gravitational constant. In other words, the bulk information of a BH is contained in its co-dimension one boundary, like the "hologram of a three-dimensional image on a two-dimensional surface" [220]. In this case, both the infrared (i.e., the BH horizon area A_H) and ultraviolet scales (i.e., the Planck length l_p within $G \propto l_p^2$) are present in Eq. (1.12). This idea of holography was further explored by 't Hooft and Susskind [220, 265] and got concretely verified by Maldacena and others [266–268], where they found that the supergravity living in the $d+1$-dimensional AdS space is dual to certain CFT living on the d-dimensional boundary. Specifically, they discovered that the four-dimensional super Yang-Mills theory with four supersymmetries is dual to the Type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$, where S^5 are five compact dimensions [266–268]. This discovery has spurred numerous research in this direction, with more holographic correspondence being proposed or found, such as the de Sitter (dS)/CFT correspondence [269] and Kerr/CFT correspondence [270]. Another important finding was made by Ryu and Takayanagi [271], where they discovered that for a co-dimension two surface B at the boundary of AdS, the entanglement entropy of a CFT in the subsystem B with its complement $\bar{B}$, i.e.,

$$S_{\rm ent} = -\operatorname{Tr}_B \rho_B \log \rho_B \,, \tag{1.13}$$

where ρ_B is the density matrix of B, is given by the area of the extremal surface Σ anchored at the boundary of B, i.e.,

$$S_{\rm ent} = \frac{A(\Sigma)}{4G} \,. \tag{1.14}$$

This relation is very similar to the Bekenstein-Hawking entropy of a BH in Eq. (1.12). In the bulk of AdS, the extremal surface Σ plays a similar role as a BH horizon, which is the entangling surface between the subsystems holographically dual to B and $\bar{B}$ in the bulk. Similarly, the degrees of freedom inside and outside a BH horizon are entangled with each other.

Since its proposal, the AdS/CFT correspondence (and the holographic principle in general) has been an extremely useful theoretical tool to further our understanding of quantum gravity. They also help the study of nuclear physics and condensed matter systems. Despite their theoretical importance, until recently, there were almost no systematic studies prescribing how to examine the AdS/CFT observationally or experimentally. The Fermilab Holometer [272–275] was one of the first few attempts to test the holographic principle using realistic experiments. The apparatus is made up of two independent co-located laser interferometers, each of which has perpendicular arms of length 40 m. The experiment was to test a particular model in [272]. Motivated by the holographic principle, Ref. [272] proposed a particular two-point function for the length fluctuations δL measured by the interferometer, with a scale of

$$\langle \delta L^2 \rangle \sim l_p L \,, \tag{1.15}$$

where L is the system size (i.e., $L = 40$ m for the Holometer). In this particular model, the holographic fluctuations accumulate like a random walk (i.e., their power spectral density is constant in frequency) along the interferometer arms within the light roundtrip time $2L/c$, while the fluctuations with temporal separation larger than $2L/c$ do not correlate. Unlike the scale expected from a naive EFT treatment of perturbative quantum gravity, where the leading-order fluctuations should be at $\delta L \sim l_p$, Eq. (1.15) gains a $\sqrt{L/l_p}$ enhancement in δL. This enhancement can be interpreted as the $\sqrt{N}$ enhancement for a random walk, where N is the number of steps, so L/l_p roughly gives the holographic degrees of freedom probed by an interferometer [22]. References [272, 273, 275] then constrained these fluctuations using the noise spectra of LIGO [276], GEO-600 [276, 277], and Holometer [275] but found no signatures of this particular quantum gravity model.

1.2.3 Verlinde-Zurek effects

Another important proposal for experimentally examining holography was later made by Verlinde, Zurek, and other collaborators [17–22]. In [17], Verlinde and Zurek first considered a set of different two-point functions for the length fluctuations δL of interferometer arms. They found that by treating the transversal and longitudinal directions differently, the two-point function of δL is roughly at the same scale of Eq. (1.15) but suppressed by additional factors of 4π. More specifically, the fluctuations in [17] are correlated along the transversal directions with respect to the interferometers while accumulating like a random walk along the longitudinal directions. They further argued that the angular correlation should satisfy

$$\langle \delta L\left(\tilde{\mathbf{r}}_1\right) \delta L\left(\tilde{\mathbf{r}}_2\right) \rangle \sim l_p L \mathbf{G}\left(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2\right) , \quad \mathbf{G}\left(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2\right) = \sum_{\ell,m} \frac{Y_{\ell,m}\left(\tilde{\mathbf{r}}_1\right) Y^*_{\ell,m}\left(\tilde{\mathbf{r}}_2\right)}{\ell^2 + \ell + 1} , \quad (1.16)$$

where $\tilde{\mathbf{r}}_i$ are coordinates of points on a unit sphere. For an interferometer, $\tilde{\mathbf{r}}_i$ are the angular coordinates of the end mirrors, and the sphere is the spherical volume that can be probed by an interferometer up to rotations around its center. If one removes the additional 1 in the denominator of $\mathbf{G}\left(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2\right)$, it is the same Green's function associated with the Laplacian on a sphere, while this additional 1 comes from integrating out the longitudinal directions along the interferometer arms. Additionally, Eq. (1.16) is the same Green's function satisfied by the angular parts of the spherical shockwave geometry studied by 't Hooft in [278].

To derive these features from the first principle, Verlinde and Zurek considered a more familiar setup in AdS, where a causal diamond is anchored to the AdS boundary [18]. The Rindler horizon Σ of this causal diamond coincides with the Ryu-Takayanagi surface of a subsystem B at the boundary of AdS. Thus, the surface Σ is also the entangling surface for B and its complement $\bar{B}$. Experimentally, this causal diamond can be formed by shooting light beams from the AdS boundary to a mirror at the Rindler horizon Σ, which then reflects the light back to the boundary. Notably, they found that the fluctuations of the light traveling time δT are at the same scale as Eq. (1.15). Moreover, δT is driven by the fluctuations of the modular Hamiltonian K associated with the Killing vector preserving the boost invariance of the Rindler horizon Σ, and K satisfies

$$\langle K \rangle = \langle \Delta K^2 \rangle = \frac{A(\Sigma)}{4\pi G} . \quad (1.17)$$

The same quantities and similar relations have also been studied in [279, 280]. Later in [20], Zurek and Bank showed that the relation in Eq. (1.17) is rather general by looking at near-horizon dynamics, so Eq. (1.17) also works for causal diamonds in flat spacetime up to an overall constant, and Eq. (1.15) is expected to be valid for an interferometer. On the other hand, the angular correlations in Eq. (1.16) can be derived from the shockwave geometry [21]. As found in [21], the relation in Eq. (1.17) can be reproduced from the uncertainty relations between the light ray operators associated with the ingoing and outgoing shockwaves, where the shockwaves are generated by vacuum fluctuations.

Built upon the foundational work in [17, 18, 20, 21], additional efforts on the theoretical side have been devoted to verifying Eqs. (1.15)–(1.17) in different contexts, such as dimensional reduction of Einstein gravity to Jackiw-Teitelboim gravity [281, 282] in AdS [283], relating shockwaves to gravitational memory effects [284] or near-horizon fluid dynamics [285, 286], and more. On the observational side, Zurek in [19] constructed an effective model, the "pixellon" model, to describe these "geontropic" fluctuations (i.e., geometrical fluctuations induced by entropy fluctuations) in terms of some metric perturbations, allowing a direct calculation of an interferometer's observable. These pixellon fields are bosonic scalars with a nonzero occupation number, characterizing the spacetime fluctuations due to modular energy fluctuations associated with the interferometer's causal diamond. Although the pixellon appears as the amplitude of a plane GW in [22], it contains information about nonlinear vacuum fluctuations via its occupation number. Reference [22] then computed the auto-correlation function of the length fluctuation of a single interferometer arm and used the noise spectra of the Holometer in [275] to put a constraint on the model.

One limitation of the model in [22] is that it is only valid for a single interferometer arm. This is insufficient if one wants to seriously constrain the model with the noise spectra of these GW detectors, which usually involve subtraction of length fluctuations from two arms. For the same reason, this model does not capture the unique angular dependence in Eq. (1.16). Another issue is that the method of calculating observables in [22] is not completely valid when the pixellon's characteristic wavelength is about the same as the interferometer's size, which is actually another unique feature of the pixellon model [22]. In the second part of this thesis (Chapters 7 and 8), we will solve these issues by extending the pixellon model in [22]. In Chapter 7, we extend the pixellon model in [22] to describe an interferometer with two arms separated by arbitrary angles. We then develop a framework to compute the gauge-invariant observables of an interferometer under these geontropic fluctuations, apply it to get the power spectral density and angular correlation of a single interferometer, and constrain the pixellon model with current GW detectors. In Chapter 8, we further extend the pixellon model in Chapter 7 to incorporate configurations of multiple interferometers and other interferometer-like experiments. We then investigate the whole landscape of high-frequency GW detectors, discuss their potential for detecting geontropic fluctuations, and show the potential influences of geontropic fluctuations on future GW detectors, such as Cosmic Explorer and Einstein Telescope.

1.3 Organization of the thesis

In this section, we provide a brief summary of each chapter from Chapter 2 to 8.

1.3.1 Chapter 2: Perturbations of spinning black holes beyond General Relativity: Modified Teukolsky equation

We extend the Teukolsky formalism in GR to study gravitational perturbations of BHs in a large class of beyond-GR theories. We first focus on non-Ricci-flat, Petrov type D BH backgrounds in modified gravity and derive the modified Teukolsky equation using two approaches: 1. direct decoupling following Teukolsky and 2. making convenient gauge choices following Chandrasekhar. By generalizing Chandrasekhar's approach, we then extend this analysis to non-Ricci-flat, Petrov type I BH backgrounds in modified gravity, assuming they can be treated as a linear perturbation of Petrov type D BH backgrounds in GR. We derive a modified Teukolsky equation, i.e., a set of linear, decoupled differential equations that describe dynamical perturbations of non-Kerr BHs for the radiative Newman-Penrose scalars Ψ_0 and Ψ_4. We further show that our formalism can be extended beyond linear order in both modified gravity corrections and GW perturbations.

1.3.2 Chapter 3: Isospectrality breaking in the Teukolsky formalism

Parity, as a fundamental symmetry of nature, has been observed to break in many theories beyond GR. One example is that even- and odd-parity gravitational perturbations of non-rotating and rotating BHs can have different QNM frequencies in modified gravity, breaking the isospectrality of GR. For BHs with arbitrary spin in modified gravity, there were no avenues to compute QNMs except NR, until recent extensions of the Teukolsky formalism. We describe how to use the modified Teukolsky formalism in Chapter 2 to study isospectrality breaking in modified gravity.

We first introduce how definite-parity modes are defined through combinations of Weyl scalars in GR, and then, we extend this definition to modified gravity. We then use the eigenvalue perturbation method to show how the degeneracy in QNMs of different parity is broken in modified gravity. We build a direct connection between the parity symmetry of beyond-GR theories and the isospectrality-breaking structure of QNMs. To demonstrate our analysis, we apply it to two specific modified gravity theories, dCS and EdGB gravity, and find consistency with previous analyses using other approaches.

1.3.3 Chapter 4: Perturbations of spinning black holes in dynamical Chern-Simons gravity: Slow rotation equations

We study gravitational perturbations of slowly rotating BHs in dCS gravity by implementing the modified Teukolsky formalism developed in Chapter 2. Specifically, we derive the master equations for the Ψ_0 and Ψ_4 Weyl scalar perturbations that characterize the radiative part of gravitational perturbations, as well as the master equation for the scalar field perturbations. We employ metric reconstruction techniques to obtain explicit expressions for all relevant quantities. Finally, by leveraging the properties of spin-weighted spheroidal harmonics to eliminate the angular dependence from the evolution equations, we derive two radial, second-order, ordinary differential equations for Ψ_0 and Ψ_4, respectively. These two equations are coupled to another radial, second-order, ordinary differential equation for the scalar field perturbations. The master equations we obtain can then be numerically integrated to obtain the QNM spectrum of slowly rotating BHs in dCS gravity,

1.3.4 Chapter 5: Perturbations of spinning black holes in dynamical Chern-Simons gravity: Slow rotation quasinormal modes

We compute the QNM spectrum of slowly rotating BHs in dCS gravity using the radial modified Teukolsky equations and the radial scalar field equations computed in Chapter 4. We first simplify the radial master equations in Chapter 4 by using the properties of the source terms under parity transformation and complex conjugation. Using the analysis of isospectrality breaking in Chapter 3, we find that the solutions to the modified Teukolsky equations in dCS gravity are still of definite parity, and the scalar field only couples to the odd-parity modes, consistent with previous results using other approaches. We then review the eigenvalue perturbation method and discuss how to apply it when there is an extra scalar field coupled to the Weyl scalar perturbation. We demonstrate our approach by computing the QNM frequency shifts for non-rotating BHs in dCS and find good agreement with previous results. The procedures for computing the QNM spectrum of slowly rotating BHs in dCS gravity are similar to those for the non-rotating case. This chapter is based on a work in preparation.

1.3.5 Chapter 6: Spectroscopy of bumpy black holes: Non-rotating case

We study the QNM spectrum of parametrized deviations of a non-rotating BH's geometry in the vacuum. Following Vigeland and Hughes, we model these parametrized deviations as axisymmetric multipole moments in the Weyl coordinates with amplitudes much less than the amplitude of the Schwarzschild BH potential. These tiny "bumps" in the BH geometry satisfy the linearized vacuum Einstein equations and are asymptotically flat. We use the modified Teukolsky formalism in Chapter 2 to derive two decoupled differential equations for the radiative Weyl scalars: Ψ_0 and Ψ_4. We focus on the equation of Ψ_0 and use the procedures developed in Chapter 3 to analyze the isospectrality breaking of QNMs for these bumpy BHs. We use the eigenvalue perturbation method to compute the QNM shifts of both even- and odd-parity modes with $\ell = 2, 3$ and up to the overtone number $n = 2$ for Weyl multipoles with $\ell_W = 2, 3$. We discuss the connection between these multipolar BH deformations and the structure of QNM spectrum shifts.

1.3.6 Chapter 7: Interferometer response to geontropic fluctuations

Motivated by Verlinde and Zurek's prediction of observable quantum gravity effects in flat spacetime, we model vacuum fluctuations in quantum gravity with a scalar field called "pixellon." This pixellon field is characterized by a high occupation number and coupled to the metric. The occupation number of the pixellon field is given by a thermal density matrix, whose form is motivated by fluctuations in the vacuum energy, which have been shown to be conformal near a light-sheet horizon. For the experimental measurement of interest in an interferometer, the size of the energy fluctuations is fixed by the area of a surface bounding the volume of spacetime being interrogated by an interferometer. We compute the interferometer response to these "geontropic" scalar-metric fluctuations, including the power spectral density and the angular correlation, and apply our results to current and future interferometer measurements, such as LIGO and the proposed GQuEST experiment.

1.3.7 Chapter 8: Quantum gravity background in next-generation gravitational wave detectors

We study the effects of geontropic fluctuations in quantum gravity, proposed by Verlinde and Zurek, on next-generation GW detectors. We first examine the detectability of geontropic fluctuations by different types of high-frequency GW detectors and discuss how our investigation fits into a broader interest in studying high-frequency GWs. We find that standard interferometers are the most optimal for detecting geontropic fluctuations. We then extend the pixellon model in Chapter 7 to describe geontropic fluctuations around more complicated configurations of interferometers. In addition to the observables computed in Chapter 7, we compute the power spectral density of the cross-correlation function for length fluctuations of two interferometers. We apply the results to several next-generation GW detectors, including Cosmic Explorer, Einstein Telescope, NEMO, and optically levitated sensors. We show that if geontropic fluctuations appear in the upcoming GQuEST experiment, they will be a large background for astrophysical GW searches in observatories like Cosmic Explorer and Einstein Telescope.

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Part I

Black hole perturbation theory beyond general relativity

Chapter 2

PERTURBATIONS OF SPINNING BLACK HOLES BEYOND GENERAL RELATIVITY: MODIFIED TEUKOLSKY EQUATION

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2.1 Introduction

General relativity (GR) has passed a plethora of experimental tests in the Solar system [1] and in binary pulsars systems [2, 3], making it the most successful theory of gravity to date. With the detection of gravitational waves (GWs) by the LIGO/Virgo/Kagra (LVK) collaboration [4], tests in the extreme gravity regime, where gravity is simultaneously strong, dynamical and non-linear, have gained prominence in the last decade [1, 5–8]. Such tests will become only stronger with the next generation of ground-based [9, 10] and space-based detectors [11], allowing for even more stringent constraints on modifications to GR (see e.g., Refs. [8, 12–17]).

Einstein's theory, although very successful, can be interpreted as having difficulties explaining certain theoretical and observational anomalies, which has motivated the study of modified theories of gravity. For example, the incompatibility between GR and quantum mechanics has motivated efforts in a variety of unified theories, such as loop quantum gravity [18–20] and string theory [21, 22]. Observational anomalies could include the late-time acceleration of the Universe [23, 24] (without the inclusion of an "unnaturally" small cosmological constant [25, 26]), the anomalous galaxy rotation curves [27, 28] (without the inclusion of dark matter [29]), and the matter-antimatter asymmetry of the Universe [30] (without the inclusion of additional sources of parity violation required by the Sakharov conditions [29, 31, 32]). All of these perceived anomalies have resulted in a zoo of modifications to GR, which can be both consistent with all current tests, while still yield deviations in the extreme gravity regime. For this class of theories, GWs may be excellent probes to study and possibly constrain deviations from Einstein's theory.

An important source of GWs is the coalescence of compact objects: the inspiral, merger, and ringdown of a binary system composed of black holes (BHs) and/or neutron stars (NSs). All of these coalescence phases can be used to test GR and constrain deviations. For instance, the presence of extra (scalar or vector) radiative degrees of freedom can be constrained with the inspiral phase of GWs emitted in binary BH coalescence. These fields can increase the rate at which orbital energy is radiated away from the system, thus affecting the orbital dynamics [7, 33–37], which can be modeled with post-Newtonian methods. The GW observations made by the LVK collaboration in the inspiral regime can then be used to determine whether binary BHs spiral in at the expected GR rate or not, thus allowing for constraints on the existence of these additional radiative fields [12–14].

On the other hand, modifications to the exterior BH geometry as well as the dynamics of these modified gravity theories may be constrained with ringdown GWs, emitted as the BH remnant settles to its final, stationary configuration. These waves can be characterized as a sum of quasinormal modes (QNMs), whose complex frequency contains information about the remnant BH background [38–46]. The LVK observation of ringdown GWs and the measurement of the complex frequencies of a set of QNMs can then be used to probe the exterior geometry of the remnant [47]. In particular, these observations can yield tests of the Kerr hypothesis (i.e., that all astrophysical BHs can be described by the Kerr metric) [47–49]. The GWs emitted during ringdown can be studied by considering gravitational perturbations of a background BH spacetime, obtaining their evolution equations, and then solving the latter to find the spectrum of perturbations. Additionally, depending on the theory, there might be additional degrees of freedom present, leading to additional or coupled evolution equations that can be solved to obtain the QNM frequency spectra [50–57]. This forms the basis of BH perturbation theory, which has been used to study QNMs of non-rotating BHs in GR [38, 39, 41–43] and modified gravity [50–53, 58]. When the background spacetime is that of a non-rotating BH, the background metric is static and spherically symmetric, so the time and angular dependence of the evolution equations of the perturbations can be easily separated. In GR, the resulting coupled radial equations can then be further reduced to two decoupled equations, one for odd parity perturbations and another for even parity perturbations [41, 42]. In modified gravity, however, one may not be able to decouple all the radial equations, so there can be more than one equation in each parity besides the equations of extra non-metric fields [50–57].

When considering background spacetimes that represent spinning BHs, however, the situation is much more complicated. This is because such BHs are mathematically represented through a background metric that is stationary and axisymmetric. The lack of spherical symmetry renders the evolution equations for the metric perturbations non-separable. Fortunately, an alternate method, prescribed by Teukolsky in 1973 [44], allows for the separation of the perturbation equations when one works with curvature quantities (instead of metric quantities), characterized in the Newman-Penrose (NP) formalism [59]. The latter arises naturally from the introduction of spinor calculus into GR and is a special type of tetrad calculus. Using the NP formalism, the perturbations of a Schwarzschild BH in GR were studied by Price [60] and extended later in [61]. Combining these results with Teukolsky's [44], a separable decoupled equation for each of the two components of the perturbed Weyl tensor (Ψ_0 and Ψ_4) can be obtained. These decoupled equations paved the way for QNM studies in GR, allowing for the accurate computation of the QNM frequencies of Kerr BHs [62, 63].

The Teukolsky formalism [44], however, is not generally applicable in modified theories of gravity. In particular, this formalism applies only when the Einstein equations hold and when the background spacetime is of Petrov type D [40, 64], i.e., when all Weyl scalars except Ψ_2 vanish on the background spacetime. However, modified theories of gravity do not necessarily satisfy the Einstein equations, and the background BH solutions in these theories need not be of Petrov type D in general. This is the case, for instance, in quadratic theories of gravity (such as dynamical Chern-Simons (dCS) gravity [65, 66] or scalar-Gauss-Bonnet (sGB) gravity [67, 68]), where a dynamical field is non-minimally coupled to a quadratic curvature invariant. In these theories, the field equations are not Einstein's, and isolated, rotating BHs are of the algebraically general Petrov type I [69], i.e., only the Ψ_0 and Ψ_4 background Weyl scalars vanish. Therefore, the Teukolsky formalism cannot be used directly to prescribe master equations for the evolution of curvature perturbations in such beyond GR BH backgrounds.

The study of BH perturbations and their QNMs in modified gravity has gained prominence in the recent decade. However, for the most part, these calculations have been limited to the non-rotating and the slowly rotating case. In the spherically-symmetric, non-rotating case, QNMs have been calculated using metric perturbation theory, e.g., in dCS gravity [50, 51, 58], Einstein-dilaton-Gauss-Bonnet (EdGB) gravity [52, 53], Einstein-Aether theory [70–74], higher-derivative gravity (quadratic [75], cubic [76], and more generically [77, 78]), and Horndeski gravity [79]. In the axisymmetric, rotating case, reducing all the metric perturbations into a single perturbation function

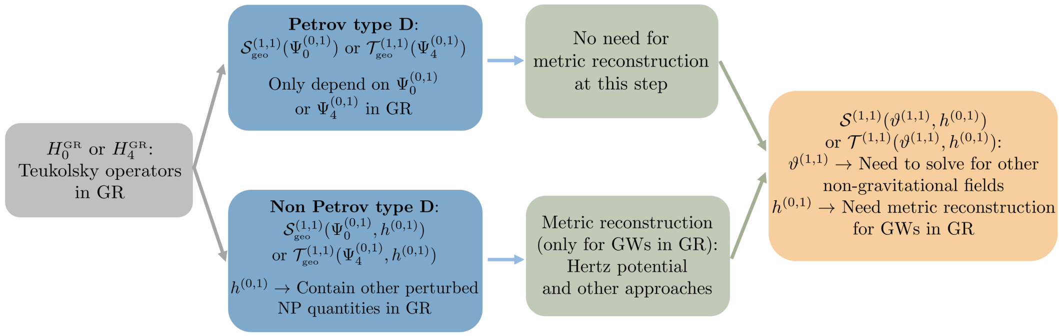

Figure 2.1: Schematic flow chart of the different possible terms that may arise in the modified Teukolsky equation for any Petrov type I spacetime in modified gravity, where the background can be treated as a linear perturbation of a Petrov type D spacetime in GR. The origin of these correction terms and the strategies to evaluate them are outlined here and discussed in detail in Sec. 2.5.3. For comparison, the corresponding procedures for any Petrov type D spacetime in modified gravity theory is also shown.

(e.g., Regge Wheeler function or a Zerilli-Moncrief function) is difficult, so studies have resorted to the slow-rotation approximation at leading order, e.g., in EdGB gravity [56, 57], dCS gravity [54, 55], and higher-derivative gravity [80, 81]. Purely numerical studies of perturbed spinning BHs, resulting from the merger of two other BHs, have also been done in dCS gravity, but they typically suffer from secularly-growing uncontrolled remainders [82, 83].

One can in principle extend the slow-rotation approximation to the QNM spectrum of rotating BHs in modified gravity to higher order in rotation, but this can be a daunting task. This is because the GWs emitted during ringdown are produced by BH remnants that typically spin at about 65% of their maximum or higher [84]. The accurate calculation of the QNM spectrum of such BHs then requires one to go to at least fifth order in a slow-rotation expansion or higher [85]. Nonetheless, it has been shown in [57] (see also [86, 87]) that one can improve the convergence of the slow-rotation expansion using the Padé approximation. In EdGB, one may then only consider up to second order in the slow-rotation expansion to deal with BHs spinning at about 70% of their maximum. Additionally, going to higher order in spin leads to mode coupling between the ℓ modes, the $\ell \pm 1$ modes and higher modes [54, 88, 89], where ℓ is the orbital number of the spherical harmonic decomposition. Therefore, instead of extending the slow-rotation approximation, we here focus on developing a new formalism, motivated from the work of Teukolsky and Chandrasekhar, to understand the evolution of curvature perturbations and therefore the QNM spectrum of rotating BHs of arbitrary spin in modified gravity.

Executive summary

We here develop and apply a method to find the evolution equations of gravitational perturbations around non-Ricci-flat and Petrov type I BH backgrounds in modified gravity, where the BH background can be treated as linear perturbations of a Petrov type D background in GR. We begin by focusing on backgrounds that are still Petrov type D, but are not described by the Kerr metric because they satisfy field equations that are not Einstein's, i.e., the background spacetime is not Ricci flat. In this context, we extend the usual Teukolsky formalism, and also develop a new approach to find the curvature perturbation equations in a particular gauge, following Chandrasekhar [40]. We show that these two approaches yield the same perturbation equations.

Let us describe both of these approaches in more detail, beginning first with a brief refresher of how these approaches are applied in GR. In the traditional Teukolsky's approach, one begins by considering two Bianchi identities and one Ricci identity in the NP formalism. Using these equations along with the GR vacuum field equations and imposing the requirement that the background is Ricci-flat (i.e., the Ricci tensor vanishes on the background) and Petrov type D, one can in principle generate a commutator relation that eliminates the coupling between the perturbed Weyl scalars $\Psi_0^{(1)}$ and $\Psi_1^{(1)}$ and between $\Psi_4^{(1)}$ and $\Psi_3^{(1)}$. However, in the process of obtaining the commutation relation, one has to make use of additional Bianchi identities. This procedure is not tedious in GR because many NP scalars and spin coefficients vanish identically, but it can be non-trivial in modified gravity.

In Chandrasekhar's approach [40], one makes use of suitable gauge conditions to simplify the perturbed equations without the need to use additional Bianchi identities. In this special gauge, the background and perturbed Weyl scalar Ψ_1 and Ψ_3 vanish, so the two Bianchi identities and the Ricci identity mentioned above simplify and depend now only on three unknown quantities. Decoupling these equations, one then obtains a master equation for the perturbed Weyl scalars $\Psi_i^{(1)}$ of the form,

$$H_i^{\text{GR}}\Psi_i^{(1)} = 0\,, \quad i \in \{0, 4\}\,, \tag{2.1}$$

where H_i^{GR} are the Teukolsky differential operators [44].

As mentioned earlier, we begin our analysis by modifying both of these approaches so that they are applicable in modified gravity for curvature perturbations of non-Ricci-flat BHs that are still Petrov type D. In the traditional Teukolsky's approach, we first develop a commutator relation by using additional Bianchi identities. Due to the complicated nature of the field equations in modified gravity, there are more non-vanishing NP quantities, thereby leading to more terms in the perturbation equations. To leading order in the perturbation and in deformations from GR, however, only the Bianchi identities and the commutator relations of GR are required since all additional terms vanish. In the Chandrasekhar's approach, we first show that even in modified gravity, a gauge still exists in which the perturbed $\Psi_1^{(1)}$ and $\Psi_3^{(1)}$ vanish. Using this gauge, the curvature perturbations can be easily decoupled.

To derive the master equation, we find a two-parameter expansion useful. We use ϵ to denote the size of the GW perturbations and ζ the strength of the modified gravity correction. With this at hand, we show that any NP quantities Ψ can be expanded as

$$\Psi = \Psi^{(0,0)} + \zeta\Psi^{(1,0)} + \epsilon\Psi^{(0,1)} + \zeta\epsilon\Psi^{(1,1)} \,. \tag{2.2}$$

We then show that both approaches lead to a modified evolution equation for the curvature perturbations of the form

$$\begin{aligned} H_0^{\mathrm{GR}}\Psi_0^{(1,1)} &= \mathcal{S}_{\mathrm{geo}}^{(1,1)}(\Psi_0^{(0,1)}) + \mathcal{S}^{(1,1)}(\vartheta^{(1,1)}, h^{(0,1)}) \,, \\ H_4^{\mathrm{GR}}\Psi_4^{(1,1)} &= \mathcal{T}_{\mathrm{geo}}^{(1,1)}(\Psi_4^{(0,1)}) + \mathcal{T}^{(1,1)}(\vartheta^{(1,1)}, h^{(0,1)}) \,, \end{aligned} \tag{2.3}$$

where the H_i^{GR} differential operators are the same as the Teukolsky ones in GR [44]. Here, we have listed the dynamical quantities [i.e., $\mathcal{O}(\epsilon)$ terms] inside the parentheses. The source terms $\mathcal{S}^{(1,1)}$ and $\mathcal{T}^{(1,1)}$ arise from the perturbed and modified field equations, and they are functionals of any additional dynamical scalar, vector or tensor field in the theory (denoted as $\vartheta^{(1,1)}$ above) and the GW metric perturbation (denoted as $h^{(0,1)}$). The source terms $\mathcal{S}_{\mathrm{geo}}^{(1,1)}$ and $\mathcal{T}_{\mathrm{geo}}^{(1,1)}$ arise from the homogeneous part of the two Bianchi identities due to the correction to the background spacetime in modified gravity, and they are functionals of the dynamical $\Psi_{0,4}^{(0,1)}$ in GR.

The evaluation of the source terms, which is required to evaluate the curvature perturbation evolution equations, requires knowledge of $h^{(0,1)}$ and $\vartheta^{(1,1)}$. The source terms $\mathcal{S}^{(1,1)}$ and $\mathcal{T}^{(1,1)}$ depend on $h^{(0,1)}$, so the evaluation of the right-hand side of Eq. (2.3) requires the reconstruction of the GW metric perturbation in GR $h^{(0,1)}$. This can be accomplished with the well-developed methods of Chrzanowski [90] and others [40, 91, 92]. Moreover, the source terms $\mathcal{S}^{(1,1)}$ and $\mathcal{T}^{(1,1)}$ also depend on the evolution of the perturbed scalar, vector, or tensor degrees of freedom that the theory may admit ϑ. The evolution of these degrees of freedom has to be solved simultaneously with the solution to the curvature perturbations.

With this at hand, we then apply Chandrasekhar's approach to modified gravity theories for non-Ricci-flat and Petrov type I BH backgrounds. In such spacetimes, the biggest challenge is that many background NP quantities are non-vanishing. Working perturbatively (i.e., treating the BH background as a deformation of the Petrov type D background in GR), one can eliminate the perturbed Ψ_1 and Ψ_3 from the evolution equations and obtain a separated and decoupled equation for Ψ_0 and Ψ_4. Schematically, these equations look a lot like the decoupled equations when dealing with non-Ricci-flat and Petrov type D backgrounds, except that now the source terms $\mathcal{S}_{\rm geo}^{(1,1)}$ and $\mathcal{T}_{\rm geo}^{(1,1)}$ can also be functionals of the GW metric perturbation, namely,

$$\begin{aligned} H_0^{\rm GR}\Psi_0^{(1,1)} &= \mathcal{S}_{\rm geo}^{(1,1)}(\Psi_0^{(0,1)}, h^{(0,1)}) + \mathcal{S}^{(1,1)}(\vartheta^{(1,1)}, h^{(0,1)})\,, \\ H_4^{\rm GR}\Psi_4^{(1,1)} &= \mathcal{T}_{\rm geo}^{(1,1)}(\Psi_4^{(0,1)}, h^{(0,1)}) + \mathcal{T}^{(1,1)}(\vartheta^{(1,1)}, h^{(0,1)})\,, \end{aligned} \tag{2.4}$$

This time we see that both source terms to the curvature perturbation evolution equations require the reconstruction of the GW metric perturbation in GR. As in the Petrov type D case, we also see that the source terms $\mathcal{S}^{(1,1)}$ and $\mathcal{T}^{(1,1)}$ require knowledge of the evolution of the perturbed scalar, vector, or tensor degrees of freedom that the modified theory may admit ϑ. Figure 2.1 shows schematically the structure of the master equations for Ψ_0 and Ψ_4.

In the rest of the paper, we derive and present the results summarized above in detail. In Sec. 2.2, we present a brief review of the NP formalism and relevant NP equations. We also review the analysis presented by Teukolsky (i.e., the Teukolsky formalism) and by Chandrasekhar (using a gauge choice) for Petrov type D spacetimes in GR. In Sec. 2.3, we discuss a subset of modified gravity theories that our work can be applied to and prescribe a perturbation scheme for them. We then extend both Teukolsky's and Chandrasekhar's approaches to Petrov type D spacetimes in these modified gravity theories in Sec. 2.4. In Sec. 2.5, we prescribe and discuss in detail the formalism to study perturbations of an algebraically general: Petrov type I spacetime in modified gravity theories which can be treated as a linear perturbation of a Petrov type D spacetimes in GR. In Sec. 2.6, we discuss the connection of the formalism developed in Sec. 2.5 to the second-order Teukolsky formalism in GR. We further show that our formalism can be generalized to higher order in both ζ and ϵ, which is thus a bGR extension of the higher-order Teukolsky formalism in GR developed in [93]. Finally, in Sec. 2.7, we summarize our work and discuss some avenues for future work. Henceforth, we adopt the following conventions unless stated otherwise: we work in 4-dimensions with metric signature $(-,+,+,+)$ as in [94]. For all NP quantities except the metric signature, we use the notation adapted by Chandrasekhar in [40].

2.2 NP formalism and perturbations of BHs in GR

With the study of GWs using tetrad and spinor calculus gaining prominence in the 1960s, Ezra Newman and Roger Penrose presented a formalism that combines these two techniques to derive a very compact and useful set of equations that are equivalent to the field equations [59]. This set of equations consists of a linear combination of equations for the Riemann tensor in terms of the Ricci rotation coefficients or spinor affine connections [59]. The different possible components of the Riemann tensor or the Weyl tensor in a null tetrad or a null basis were then associated with certain quantities, called the NP coefficients or NP scalars. This formalism provided a new tool to understand GW properties, such as polarizations and ringdown modes in more detail [44, 95–98]. Using the NP framework, Teukolsky presented a formalism to study the ringdown phase of spinning BHs in GR [44, 98, 99] and to study the dynamical perturbations of Kerr BHs, or more generally, Petrov type D spacetimes in GR.

In this section, we provide a quick refresher of the NP formalism and discuss the necessary equations for developing a formalism to obtain master equations for GW perturbations in GR. Using these equations, we present in brief the approach prescribed by Teukolsky [44] and by Chandrasekhar [40] to obtain separable decoupled differential equations for perturbations of BHs in GR. For a reader familiar with these topics, we recommend starting from Sec. 2.3, where we extend the aforementioned formalism to BHs in modified gravity.

2.2.1 NP formalism: A quick review

In this subsection, we present a quick overview of the relevant equations under the NP formalism required for our work. For an in-depth overview, we provide further details of the NP formalism [40, 59] in Appendix 2.8. In the NP formalism, a null tetrad $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ is introduced at every point of a four-dimensional pseudo-Riemannian manifold of signature +2 and metric $g_{\mu\nu}$. The vectors l^μ and n^μ are real, whereas m^μ and $\bar{m}^\mu$ are complex, with a overhead bar denoting complex conjugation. The tetrad 4-vectors must also satisfy the following orthogonality properties:

$$l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = 0\,,$$

$$l_\mu n^\mu = -m_\mu \bar{m}^\mu = -1\,,$$
$$l_\mu m^\mu = l_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0\,. \tag{2.5}$$

Given such a null tetrad, the metric can be expressed as

$$g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu\,. \tag{2.6}$$

Intrinsic derivatives in the NP formalism are defined as

$$D\phi \equiv \phi_{;\mu} l^\mu\,, \quad \Delta\phi \equiv \phi_{;\mu} n^\mu\,, \quad \delta\phi \equiv \phi_{;\mu} m^\mu\,, \quad \delta^*\phi \equiv \phi_{;\mu} \bar{m}^\mu\,. \tag{2.7}$$

For any tetrad, we can also perform Lorentz transformations on it, i.e., three rotations and three boosts. These transformations can be mapped to three types of tetrad rotations, which are characterized by six real variables on the tetrad basis vectors, such that the orthogonality properties in Eq. (2.5) are preserved [40]. These three types of tetrad rotations are discussed in detail in Appendix 2.8.

In the NP formalism, the fundamental variables are 5 Weyl scalars ($\Psi_1, \Psi_2, \ldots$), 12 spin coefficients ($\kappa, \pi, \varepsilon, \ldots$), and 10 NP Ricci scalars ($\Phi_{00}, \Phi_{01}, .., \Lambda$), which are generally complex quantities. The mathematical form of all these quantities is presented in Appendix 2.8. These quantities allow one to construct certain fundamental relations of the NP formalism: 18 complex Ricci identities [Eq. (2.122)] and 9 complex plus 2 real Bianchi identities [Eqs. (2.123)] [40]. The Ricci identities are derived from appropriate linear combinations of Eq. (2.116) and (2.117), while the Bianchi identities come from Eq. (2.118). Some Ricci identities relevant for this work are

$$(D - \rho - \rho^* - 3\varepsilon + \varepsilon^*)\,\sigma - (\delta - \tau + \pi^* - \alpha^* - 3\beta)\kappa - \Psi_0 = 0\,, \tag{2.8a}$$
$$(\Delta + \mu + \mu^* + 3\gamma - \gamma^*)\,\lambda - (\delta^* + 3\alpha + \beta^* + \pi - \tau^*)\nu + \Psi_4 = 0\,, \tag{2.8b}$$
$$(D - \varepsilon + \varepsilon^* - \rho)\,\tau - (\Delta - 3\gamma + \gamma^*)\,\kappa - \pi^*\rho - (\tau^* + \pi)\,\sigma - \Psi_1 - \Phi_{01} = 0\,, \tag{2.8c}$$
$$(D - \rho^* + \varepsilon^*)\,\beta - (\delta + \alpha^* - \pi^*)\,\varepsilon - (\alpha + \pi)\,\sigma - (\mu + \gamma)\,\kappa - \Psi_1 = 0\,, \tag{2.8d}$$
$$(D - \rho^* + \varepsilon^*)\,\beta - (\delta + \alpha^* - \pi^*)\,\varepsilon - (\alpha + \pi)\,\sigma - (\mu + \gamma)\,\kappa - \Psi_1 = 0\,, \tag{2.8e}$$

while the Bianchi identities useful for this work are

$$(\delta^* - 4\alpha + \pi)\,\Psi_0 - (D - 2\varepsilon - 4\rho)\Psi_1 - 3\kappa\Psi_2 = S_1\,, \tag{2.9a}$$
$$(\Delta - 4\gamma + \mu)\Psi_0 - (\delta - 4\tau - 2\beta)\Psi_1 - 3\sigma\Psi_2 = S_2\,, \tag{2.9b}$$
$$(\delta + 4\beta - \tau)\Psi_4 - (\Delta + 2\gamma + 4\mu)\Psi_3 + 3\nu\Psi_2 = S_3\,, \tag{2.9c}$$
$$(D + 4\varepsilon - \rho)\Psi_4 - (\delta^* + 4\pi + 2\alpha)\,\Psi_3 + 3\lambda\Psi_2 = S_4\,, \tag{2.9d}$$

where we have defined

$$
\begin{aligned}
S_1 \equiv\ & (\delta + \pi^* - 2\alpha^* - 2\beta)\, \Phi_{00} - (D - 2\varepsilon - 2\rho^*)\, \Phi_{01} \\
& + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \kappa^*\Phi_{02}\,,
\end{aligned} \tag{2.10a}
$$

$$
\begin{aligned}
S_2 \equiv\ & (\delta + 2\pi^* - 2\beta)\, \Phi_{01} - (D - 2\varepsilon + 2\varepsilon^* - \rho^*)\, \Phi_{02} \\
& - \lambda^*\Phi_{00} + 2\sigma\Phi_{11} - 2\kappa\Phi_{12}\,,
\end{aligned} \tag{2.10b}
$$

$$
\begin{aligned}
S_3 \equiv\ & - (\Delta + 2\mu^* + 2\gamma)\, \Phi_{21} + (\delta^* - \tau^* + 2\alpha + 2\beta^*)\, \Phi_{22} \\
& + 2\nu\Phi_{11} + \nu^*\Phi_{20} - 2\lambda\Phi_{12}\,,
\end{aligned} \tag{2.10c}
$$

$$
\begin{aligned}
S_4 \equiv\ & - (\Delta + \mu^* + 2\gamma - 2\gamma^*)\, \Phi_{20} + (\delta^* + 2\alpha - 2\tau^*)\, \Phi_{21} \\
& + 2\nu\Phi_{10} - 2\lambda\Phi_{11} + \sigma^*\Phi_{22}\,.
\end{aligned} \tag{2.10d}
$$

The remaining equations are presented in Appendix 2.8.

The above equations can be recast in a simpler form if we define the following operators:

$$
\begin{aligned}
& F_1 \equiv \delta^* - 4\alpha + \pi\,, \quad F_2 \equiv \Delta - 4\gamma + \mu\,, \\
& J_1 \equiv D - 2\varepsilon - 4\rho\,, \quad J_2 \equiv \delta - 4\tau - 2\beta\,, \\
& E_1 \equiv \delta - \tau + \pi^* - \alpha^* - 3\beta\,, \quad E_2 \equiv D - \rho - \rho^* - 3\varepsilon + \varepsilon^*\,,
\end{aligned} \tag{2.11}
$$

$$
\begin{aligned}
& F_3 \equiv \delta + 4\beta - \tau\,, \quad F_4 \equiv D + 4\varepsilon - \rho\,, \\
& J_3 \equiv \Delta + 2\gamma + 4\mu\,, \quad J_4 \equiv \delta^* + 4\pi + 2\alpha\,, \\
& E_3 \equiv \delta^* + 3\alpha + \beta^* + \pi - \tau^*\,, \quad E_4 \equiv \Delta + \mu + \mu^* + 3\gamma - \gamma^*\,,
\end{aligned} \tag{2.12}
$$

so we can rewrite Eqs. (2.9a)-(2.9b) and Eq. (2.8a) as

$$F_1\Psi_0 - J_1\Psi_1 - 3\kappa\Psi_2 = S_1\,, \tag{2.13a}$$

$$F_2\Psi_0 - J_2\Psi_1 - 3\sigma\Psi_2 = S_2\,, \tag{2.13b}$$

$$E_2\sigma - E_1\kappa - \Psi_0 = 0\,, \tag{2.13c}$$

while Eqs. (2.9c)-(2.9d) and Eq. (2.8b) can be written as

$$F_3\Psi_4 - J_3\Psi_3 + 3\nu\Psi_2 = S_3\,, \tag{2.14a}$$

$$F_4\Psi_4 - J_4\Psi_3 + 3\lambda\Psi_2 = S_4\,, \tag{2.14b}$$

$$E_4\lambda - E_3\nu + \Psi_4 = 0\,. \tag{2.14c}$$

For this work, we also need a commutator of the intrinsic derivatives introduced in Eq. (2.7), namely,

$$[\delta, D] = (\alpha^* + \beta - \pi^*)\, D + \kappa\Delta - (\rho^* + \varepsilon - \varepsilon^*)\, \delta - \sigma\delta^*\,. \tag{2.15}$$

The other commutators of intrinsic derivatives can be found in Appendix 2.8.

Let us conclude with a brief discussion of the Petrov classification [40, 64]. The Petrov classification is an organizational scheme based on the examination of the algebraic structure of the Weyl curvature tensor. Since the Weyl scalars in the NP formalism depend on the Weyl tensor [see e.g., Eq. (2.120)], one can classify solutions in a given theory based on the vanishing of the Weyl scalars for the given solution. The classification is as follows:

1. Type I: $\Psi_0 = \Psi_4 = 0$.
2. Type II: $\Psi_0 = \Psi_1 = \Psi_4 = 0$.
3. Type D: $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$.
4. Type III: $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0$.
5. Type N: $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$.

Isolated stationary BHs in GR are of Petrov type D, while these BHs in modified gravity theories, such as in dCS gravity or EdGB gravity, are of Petrov type I [66, 69]. Since Petrov type I spacetimes are the most general type of spacetime in the Petrov classification, they are also called algebraically general. The rest of the spacetimes in the Petrov classification, including Petrov type D, are classified as algebraically special.

2.2.2 Teukolsky formalism for Petrov type D spacetimes in GR

In this subsection, we present the formalism first prescribed by Teukolsky in 1972 [44], where using the NP formalism, he obtained a set of separable, decoupled gravitational perturbation equations for Kerr BHs in GR. More specifically, Teukolsky expanded all curvature quantities into a background plus a perturbation; for example, the Weyl scalars are expanded into

$$\Psi_i = \Psi_i^{(0)} + \epsilon\, \Psi_i^{(1)} \tag{2.16}$$

for $i \in (0, 1, 2, 3, 4)$, where the superscript (0) means that these quantities are computed from the background metric, while the superscript (1) stands for a perturbation from this background with ϵ an order-counting parameter. With this in hand, Teukolsky was then able to derive separable and decoupled equations for the curvature perturbations $\Psi_0^{(1)}$ and $\Psi_4^{(1)}$ of a Kerr BH.

The following derivation, which follows closely that of [44], applies to any Petrov type D vacuum background metric in GR, which includes the Schwarzschild and Kerr metrics. Let us then choose the l^μ and n^μ vectors of the unperturbed tetrad along the repeated principal null directions of the Weyl tensor. Thus, for a Petrov type D vacuum GR spacetime, we have

$$\Psi_0^{(0)} = \Psi_1^{(0)} = \Psi_3^{(0)} = \Psi_4^{(0)} = 0\,, \quad \kappa^{(0)} = \sigma^{(0)} = \nu^{(0)} = \lambda^{(0)} = 0\,. \tag{2.17}$$

The result on the second line of Eq. (2.17) can also be seen to come from the Bianchi identities in Eq. (2.123).

The GR field equations in trace-reversed form can be expressed as

$$R^{\mu\nu} = 8\pi\left(T^{\mu\nu} - \frac{1}{2}Tg^{\mu\nu}\right)\,, \tag{2.18}$$

where $T^{\mu\nu}$ is the stress-energy tensor and T is its trace. Since we are working with vacuum spacetimes, $T^{\mu\nu} = 0$, and thus $R^{\mu\nu} = 0$. Using this in Eq. (2.121), we can see that all background and perturbed values of Φ_{ij} for $i, j \in \{0, 1, 2\}$ vanish. For instance,

$$\Phi_{00} \equiv -\frac{1}{2}R_{11} = -\frac{1}{2}R_{\mu\nu}l^\mu l^\nu = 4\pi T_{ll} = 0\,. \tag{2.19}$$

Thus, using Eq. (2.10), we see that S_1, S_2, S_3 and S_4 vanish identically for vacuum GR spacetimes.

To study the perturbations of BHs, we require differential equations for $\Psi_0^{(1)}$ and $\Psi_4^{(1)}$ since these represent curvature perturbations associated with propagating metric perturbations. We first present the formalism to obtain a differential equation for $\Psi_0^{(1)}$, and later, we apply the same to $\Psi_4^{(1)}$. Consider then the vacuum Ricci identity of Eq. (2.13c) and the Bianchi identities in Eqs. (2.13a) and (2.13b). As mentioned previously, in vacuum GR spacetimes, the right-hand side of these equations vanish. Furthermore, using Eq. (2.17), the corresponding perturbation equations to leading order in the perturbation take the form

$$F_1^{(0)}\Psi_0^{(1)} - J_1^{(0)}\Psi_1^{(1)} - 3\kappa^{(1)}\Psi_2^{(0)} = 0\,, \tag{2.20a}$$
$$F_2^{(0)}\Psi_0^{(1)} - J_2^{(0)}\Psi_1^{(1)} - 3\sigma^{(1)}\Psi_2^{(0)} = 0\,, \tag{2.20b}$$
$$E_2^{(0)}\sigma^{(1)} - E_1^{(0)}\kappa^{(1)} - \Psi_0^{(1)} = 0\,. \tag{2.20c}$$

In order to simplify the notation, we will drop the superscript (0) for all background quantities for the remainder of this section. Multiplying Eq. (2.20c) by the background

Ψ_2 Weyl scalar and plugging in for E_1 and E_2 using Eq. (2.11), one finds

$$\begin{aligned} &(D - 4\rho - \rho^* - 3\varepsilon + \varepsilon^*)\left(\Psi_2\sigma^{(1)}\right) \\ &- (\delta - 4\tau + \pi^* - \alpha^* - 3\beta)\left(\Psi_2\kappa^{(1)}\right) - \Psi_2\Psi_0^{(1)} = 0\,, \end{aligned} \tag{2.21}$$

where we have used Eqs. (2.123h) and (2.123g), which for the background Ψ_2 reduce to

$$D\Psi_2 = 3\rho\Psi_2, \qquad \delta\Psi_2 = 3\tau\Psi_2\,. \tag{2.22}$$

In order to be consistent with the simplified notation, we introduce

$$E_1^{\rm GR} = \delta - 4\tau + \pi^* - \alpha^* - 3\beta\,, \tag{2.23a}$$

$$E_2^{\rm GR} = D - 4\rho - \rho^* - 3\varepsilon + \varepsilon^*\,, \tag{2.23b}$$

so Eq. (2.21) can be written more compactly as

$$E_2^{\rm GR}\left(\Psi_2\sigma^{(1)}\right) - E_1^{\rm GR}\left(\Psi_2\kappa^{(1)}\right) = \Psi_2\Psi_0^{(1)}\,. \tag{2.24}$$

To obtain a differential equation for $\Psi_0^{(1)}$, we need to eliminate $\Psi_1^{(1)}$ from Eqs. (2.20a) and (2.20b). This can be done by making use of the following commutation relation:

$$E_2^{\rm GR}J_2 - E_1^{\rm GR}J_1 = 0\,. \tag{2.25}$$

This relation can be shown to hold for any Petrov type D spacetime in GR by using Eqs. (2.8c)-(2.8e) and Eq. (2.15). On operating $E_2^{\rm GR}$ on Eq. (2.20b), $E_1^{\rm GR}$ on Eq. (2.20a), and subtracting one equation from the other, $\Psi_1^{(1)}$ vanishes identically. Using Eq. (2.21), we finally have

$$\left(E_2^{\rm GR}F_2 - E_1^{\rm GR}F_1 - 3\Psi_2\right)\Psi_0^{(1)} = 0\,. \tag{2.26}$$

This is the decoupled equation for $\Psi_0^{(1)}$ for any Petrov type D vacuum spacetime in GR. As shown by Geroch, Held, and Penrose (GHP) [100], the NP equations are invariant under the exchange $l^\mu \leftrightarrow n^\mu$ and $m^\mu \leftrightarrow \bar{m}^\mu$, where the choice of l^μ and n^μ has no effect on this symmetry. Applying this transformation to Eq. (2.26), one finds the decoupled differential equation for $\Psi_4^{(1)}$ for a Petrov type D vacuum spacetime in GR, namely,

$$\left(E_4^{\rm GR}F_4 - E_3^{\rm GR}F_3 - 3\Psi_2\right)\Psi_4^{(1)} = 0\,, \tag{2.27}$$

where we have introduced

$$E_3^{\rm GR} \equiv \delta^* + 3\alpha + \beta^* + 4\pi - \tau^*\,, \tag{2.28a}$$

$$E_4^{\text{GR}} \equiv \Delta + 4\mu + \mu^* + 3\gamma - \gamma^* \,. \tag{2.28b}$$

An alternate derivation using the GHP formalism was provided by Stewart [101]. However, for the purpose of this section, we will stick with the formalism laid down by Teukolsky.

2.2.3 Chandrasekhar's approach for Petrov type D spacetimes in GR

Chandrasekhar introduced another way to derive the Teukolsky equation in [40] by utilizing the gauge freedom of the tetrad. As briefly mentioned in Sec. 2.2.1 and discussed in detail in Appendix 2.8, one is free to rotate the tetrad following Eq. (2.126) such that all the normalization and orthogonality conditions in Eq. (2.5) are preserved.

Let us then consider a type II rotation, which is given by

$$n \to n \,, \quad m \to m + bn \,, \quad \bar{m} \to \bar{m} + b^* n \,, \quad l \to l + b^* m + b\bar{m} + bb^* n \tag{2.29}$$

[see also Eq. (2.126b)], and set the rotation parameter b to be of leading order in the perturbation, i.e., $b = b^{(1)}$. Ignoring all higher-order terms, the perturbed Weyl scalars transform into [see e.g., Eq. (2.127b)]

$$\begin{aligned} &\Psi_0^{(1)} \to \Psi_0^{(1)} + 4b^{(1)}\Psi_1^{(0)} \,, \quad \Psi_1^{(1)} \to \Psi_1^{(1)} + 3b^{(1)}\Psi_2^{(0)} \,, \\ &\Psi_2^{(1)} \to \Psi_2^{(1)} + 2b^{(1)}\Psi_3^{(0)} \,, \quad \Psi_3^{(1)} \to \Psi_3^{(1)} + b^{(1)}\Psi_4^{(0)} \,, \quad \Psi_4^{(1)} \to \Psi_4^{(1)} \,. \end{aligned} \tag{2.30}$$

Since for a Petrov type D spacetime, $\Psi_{i\neq 2}^{(0)} = 0$, all the $\Psi_{i\neq 1}^{(1)}$ remain invariant under such a rotation. By choosing $b^{(1)} = -\Psi_1^{(1)}/\left(3\Psi_2^{(0)}\right)$, the perturbed Weyl scalar $\Psi_1^{(1)}$ can be removed directly without the use of any additional Bianchi identities and commutation relations used in Sec. 2.2.2. Another way to understand this gauge choice is that we have three equations for four unknowns in Eqs. (2.20), so there is one arbitrary function to be determined.

Using this gauge freedom to set $\Psi_1^{(1)} = 0$ through a tetrad rotation, one can now easily derive the Teukolsky equation. First, use this gauge freedom to set $\Psi_1^{(1)} = 0$ in Eqs. (2.20a)-(2.20b), and then solve for $\kappa^{(1)}$ and $\sigma^{(1)}$. Now insert these solutions back into Eq. (2.20c) to find

$$\left(\mathcal{E}_2 F_2 - \mathcal{E}_1 F_1 - 3\Psi_2\right)\Psi_0^{(1)} = 0 \,, \tag{2.31}$$

where we have defined

$$\mathcal{E}_i \equiv \Psi_2 E_i \Psi_2^{-1} \,. \tag{2.32}$$

Here, we have dropped the superscript (0) for all unperturbed quantities. Applying the GHP transformation explained below Eq. (2.26), one finds an equation for $\Psi_4^{(1)}$, namely,

$$(\mathcal{E}_4 F_4 - \mathcal{E}_3 F_3 - 3\Psi_2)\,\Psi_4^{(1)} = 0\,. \tag{2.33}$$

The $\mathcal{E}_i$ operators can be simplified using the product rule. Doing so, one finds

$$\mathcal{E}_1 = \delta - \tau + \pi^* - \alpha^* - 3\beta - \frac{1}{\Psi_2}\delta\Psi_2\,, \tag{2.34a}$$

$$\mathcal{E}_2 = D - \rho - \rho^* - 3\varepsilon + \varepsilon^* - \frac{1}{\Psi_2}D\Psi_2\,, \tag{2.34b}$$

$$\mathcal{E}_3 = \delta^* + 3\alpha + \beta^* + \pi - \tau^* - \frac{1}{\Psi_2}\delta^*\Psi_2\,, \tag{2.34c}$$

$$\mathcal{E}_4 = \Delta + \mu + \mu^* + 3\gamma - \gamma^* - \frac{1}{\Psi_2}\Delta\Psi_2\,, \tag{2.34d}$$

In deriving Eqs. (2.31)-(2.33), we have also multiplied the whole equation by $3\Psi_2$. We will see in Sec. 2.4.2 that this makes Eqs. (2.31)-(2.33) exactly the same as Eqs. (2.26)-(2.27), so $\mathcal{E}_i = E_i^{\mathrm{GR}}$. Note that one can also derive the equation for Ψ_4 in the same way we derived an equation for Ψ_0 (i.e., without the GHP transformation), using the fact that a type I rotation at $\mathcal{O}(\epsilon)$ can be used to set $\Psi_3^{(1)}$ to zero.

It should not be surprising that one obtains the same equation following the traditional Teukolsky's approach and Chandrasekhar's approach. From Eq. (2.30) and other tetrad rotations discussed in Appendix 2.8 that one can perform in Eqs. (2.126), one can see that $\Psi_0^{(1)}$ and $\Psi_4^{(1)}$ are gauge-invariant quantities under linear perturbations. In Chandrasekhar's approach, since one does not need to use any additional Bianchi identities and commutation relations to cancel off $\Psi_1^{(1)}$, there are fewer equations one needs to worry about, and this will be helpful when dealing with the more complicated non-Petrov-type-D spacetime backgrounds of modified gravity theories. However, to convince ourselves that the equivalence between these two approaches is not broken when considering beyond GR theories, in Sec. 2.4 we will find a modified master equation using both approaches and show that the two methods are equivalent in modified gravity theories.

2.3 Framework of perturbation in modified gravity theories

In this section, we discuss a subset of modified gravity theories that the formalism developed in this work can be applied to. We classify these theories into two classes based on the presence of additional non-metric fields in the action that define these theories. For both classes, we provide some examples by explicitly writing down the Lagrangian, the equations of motion for all the fields, and the properties of BH spacetimes, which serve as the background to our perturbation analysis. We then prescribe a perturbation scheme using a two-parameter expansion for both classes of modified gravity theories.

2.3.1 Theories of gravity beyond GR

In this subsection, we provide a quick overview of certain modified theories of gravity relevant for this work and discuss the BH spacetimes in these theories, which serve as a background for our perturbation scheme. Consider then a class of theories defined through the following beyond GR Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\rm GR} + \ell^p \mathcal{L}_{\rm bGR} + \mathcal{L}_{\rm matter} + \mathcal{L}_{\rm field} \,, \tag{2.35}$$

where $\mathcal{L}_{\rm GR}$ is the Einstein-Hilbert Lagrangian, $\mathcal{L}_{\rm matter}$ is the matter Lagrangian, $\mathcal{L}_{\rm field}$ is the Lagrangian for all other (non-metric) dynamical fields (including all kinetic and potential terms of these fields) that the theory may permit, and $\mathcal{L}_{\rm bGR}$ is a Lagrangian that contains non-Einstein-Hilbert curvature terms and can, in principle, include non-minimal couplings to the non-metric dynamical fields of the theory. The quantity ℓ in Eq. (2.35) is a dimension-full scale that characterizes the strength of the GR correction, and p is a number to ensure that $\ell^p \mathcal{L}_{\rm bGR}$ has the right dimensions. We can classify the beyond GR theories described by the Lagrangian in Eq. (2.35) based on the presence or absence of additional non-metric dynamical fields, i.e., based on whether $\mathcal{L}_{\rm field}$ vanishes. Note that we here do not consider theories with non-dynamical, prior or "fixed" fields that couple to the metric tensor. In this work then, we define this classification as

- $\mathcal{L}_{\rm field} \neq 0 \Longrightarrow$ Class A,
- $\mathcal{L}_{\rm field} = 0 \Longrightarrow$ Class B.

An example of beyond GR theories of class A that we will consider is dCS gravity. This theory is defined by the Lagrangian in Eq. (2.35) with the choices

$$\begin{aligned}
\mathcal{L}_{\rm GR} &= (16\pi)^{-1} R \,, \\
\mathcal{L}^{\rm dCS}_{\rm bGR} &= \frac{1}{4}\vartheta\, {}^*R^{\mu}{}_{\nu}{}^{\kappa\delta} R^{\nu}{}_{\mu\kappa\delta} \,, \\
\mathcal{L}^{\rm dCS}_{\rm field} &= -\frac{1}{2} g^{\mu\nu} (\nabla_\mu \vartheta)(\nabla_\nu \vartheta) \,,
\end{aligned} \tag{2.36}$$

and $\ell = \ell_{\rm dCS}$ is the dCS coupling constant with $p = 2$. R is the Ricci scalar, $g_{\mu\nu}$ is the metric, and ϑ is a massless, pseudoscalar, axion-like field that non-minimally couples to the Pontryagin curvature invariant ${}^*R^{\mu}{}_{\nu}{}^{\kappa\delta}R^{\nu}{}_{\mu\kappa\delta}$, where

$$ {}^*R^{\mu}{}_{\nu}{}^{\kappa\delta} = \frac{1}{2}\epsilon^{\mu}{}_{\nu\alpha\beta}R^{\alpha\beta\kappa\delta} \tag{2.37} $$

is the dual of the Riemann tensor. The field equations in dCS gravity are

$$ \begin{aligned} R_{\mu\nu} = 8\pi\Big\{&(T^{\rm M}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\rm M}) + (\nabla_\mu\vartheta)(\nabla_\nu\vartheta) \\ &- 2\alpha_{\rm dCS}\left[(\nabla_\sigma\vartheta)\epsilon^{\sigma\delta\alpha}{}_{(\mu}\nabla_\alpha R_{\nu)\delta} + (\nabla_\sigma\nabla_\delta\vartheta){}^*R^{\delta}{}_{(\mu\nu)}{}^{\sigma}\right]\Big\}\,, \end{aligned} \tag{2.38} $$

$$ \Box\vartheta = -\frac{\alpha_{\rm dCS}}{4}\,{}^*R^{\mu}{}_{\nu}{}^{\kappa\delta}R^{\nu}{}_{\mu\kappa\delta}\,, \tag{2.39} $$

where Eq. (2.38) is the trace-reversed metric field equation, and Eq. (2.39) is the scalar field equation. The dCS coupling constant $\alpha_{\rm dCS} \equiv \ell^2_{\rm dCS}$ determines the strength of the Chern-Simons (CS) modification and has dimensions of $[\text{Length}]^2$. Stationary and vacuum BH solutions in this theory are not Ricci-flat, so they are obviously not represented by the Kerr metric [102–104]. Instead, spinning BHs in dCS gravity have a corrected event horizon location, ergosphere, and different exterior multipole moments [102] to name a few corrected quantities. Moreover, dCS BHs are of non-Ricci-flat Petrov type I spacetimes in the Petrov classification given in Sec. 2.2.1. To leading order in spin, however, the BHs in this theory remain non-Ricci-flat and of Petrov type D [69, 102, 105].

Another example of a class A beyond GR theory is EdGB gravity [106], which is a special case of sGB gravity [107]. Using Eq. (2.35) and the conventions in [56, 107], EdGB theory is defined via

$$ \begin{aligned} \mathcal{L}_{\rm GR} &= (16\pi)^{-1}R\,, \\ \mathcal{L}^{\rm EdGB}_{\rm bGR} &= (64\pi)^{-1}e^{\theta}\mathcal{G}\,, \\ \mathcal{L}^{\rm EdGB}_{\rm field} &= -(32\pi)^{-1}g^{\mu\nu}(\nabla_\mu\theta)(\nabla_\nu\theta)\,, \end{aligned} \tag{2.40} $$

where

$$ \mathcal{G} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2 \tag{2.41} $$

is the Gauss-Bonnet curvature invariant, and $\ell = \ell_{\rm EdGB}$ is the EdGB coupling constant with $p = 2$. The quantity θ is a massless dilaton-like scalar field that non-minimally couples to the Gauss-Bonnet invariant $\mathcal{G}$. The metric field equation for EdGB gravity in trace-reversed form is then given by [56]

$$ R_{\mu\nu} = 8\pi(T^{\rm M}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\rm M}) + \frac{1}{2}(\nabla_\mu\theta)(\nabla_\nu\theta) - \alpha_{\rm EdGB}\left(\mathcal{K}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{K}\right)\,, $$

$$\mathcal{K}_{\mu\nu} = \frac{1}{8}\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right)\epsilon^{\delta\sigma\gamma\alpha}\nabla_{\beta}\left({}^{*}R^{\rho\beta}{}_{\gamma\alpha}e^{\theta}\nabla_{\delta}\theta\right)\,,$$
$$\mathcal{K} = g_{\mu\nu}\mathcal{K}^{\mu\nu}\,, \tag{2.42}$$

whereas the scalar field equation is

$$\Box\theta = -\frac{\alpha_{\rm EdGB}}{4}e^{\theta}\mathcal{G}\,. \tag{2.43}$$

The quantity $\alpha_{\rm EdGB} \equiv l^2_{\rm EdGB}$ is the coupling constant of EdGB theory and has dimensions of $[\text{Length}]^2$. Stationary and vacuum BH solutions in this theory, just like in dCS gravity, are non-Ricci-flat and are not represented by the Kerr metric [108–112]. Rotating BHs in EdGB theory are described by non-Ricci-flat Petrov type I spacetimes in general, but to leading order in spin, they are described by non-Ricci-flat Petrov type D spacetimes [69].

An example of class B beyond GR theories is higher-derivative gravity [81] because this theory contains no non-metric dynamical fields. Following Eq. (2.35), the Lagrangian of this theory can be represented by

$$\mathcal{L}_{\rm GR} = (16\pi)^{-1}R\,,$$
$$\mathcal{L}^{\rm HD}_{\rm bGR} = (16\pi)^{-1}(\lambda_{\rm even}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}R_{\delta\gamma}{}^{\mu\nu} + \lambda_{\rm odd}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}\,{}^{*}R_{\delta\gamma}{}^{\mu\nu})\,,$$
$$\mathcal{L}^{\rm HD}_{\rm field} = 0\,, \tag{2.44}$$

where we have only kept terms with up to six derivatives of the metric (a more general discussion can be found in [81]). $\ell = \ell_{\rm HD}$ is the higher-derivative gravity coupling constant with $p = 4$. The quantities $\lambda_{\rm even}$ and $\lambda_{\rm odd}$ are dimensionless coupling constants that are introduced to distinguish terms that preserve or break parity. The field equation in trace-reversed form is [81]

$$R_{\mu\nu} = 8\pi(T^{\rm M}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\rm M}) - \mathcal{E}^{(6)}_{\mu\nu}\,,$$
$$\mathcal{E}^{(n)}_{\mu\nu} = P^{(n)}{}_{(\mu}{}^{\rho\sigma\gamma}R_{\nu)\rho\sigma\gamma} - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{(n)} + 2\nabla^{\sigma}\nabla^{\rho}P^{(n)}_{(\mu|\sigma|\nu)\rho}\,,$$
$$P^{(6)}_{\mu\nu\rho\sigma} = 3\alpha^{\rm even}_{\rm HD}R^{\alpha\beta}_{\mu\rho}R_{\alpha\beta\rho\sigma} + \frac{3\alpha^{\rm odd}_{\rm HD}}{2}\left(R^{\alpha\beta}_{\mu\rho}\,{}^{*}R_{\alpha\beta\rho\sigma} + R^{\alpha\beta}_{\mu\rho}\,{}^{*}R_{\rho\sigma\alpha\beta}\right)\,, \tag{2.45}$$

where $\alpha^{\rm even}_{\rm HD} \equiv \ell^4_{\rm HD}\lambda_{\rm even}$ and $\alpha^{\rm odd}_{\rm HD} \equiv \ell^4_{\rm HD}\lambda_{\rm odd}$ are coupling constants that determine the strength of the parity-preserving and the parity-breaking higher-derivative gravity corrections. The quantity $\mathcal{L}_{(n)}$ refers to the Lagrangian with n derivatives of the metric in higher-derivative gravity, so $\mathcal{L}_{(6)} = \mathcal{L}^{\rm HD}_{\rm bGR}$. Rotating BHs in higher-derivative gravity are non-Ricci-flat [81], but their Petrov type has not yet been studied in detail.

Theories described by the Lagrangian given in Eq. (2.35) only form a subset of all possible theories. This subset does not just include dCS gravity [65, 66], EdGB gravity [106, 110, 113–115], and higher-derivative theories of gravity [80, 81, 116–119], but it also includes, for example, sGB gravity in general [120], quadratic gravity theories without additionally coupled fields [121, 122], and higher dimensional gravity theories [123, 124] to name a few. These theories can also be classified based on whether their stationary and vacuum (i.e., no matter) BH solutions are Ricci-flat or non-Ricci-flat. For a beyond GR theory that admits Ricci-flat, Petrov type D BH spacetimes, perturbations can be studied within the standard Teukolsky formalism presented in Sec. 2.2.2, so we do not focus on these theories here. In this work, instead, we focus on the dynamical perturbations of BHs that are non-Ricci-flat and either Petrov type D or Petrov type I. Therefore, our work applies to dCS gravity [65, 66, 69, 105], EdGB and sGB gravity [69], and higher-derivative gravity [80, 81, 116–119].

2.3.2 Perturbation scheme

In this subsection, we discuss the perturbation scheme that is applicable to the modified gravity theories discussed in Sec. 2.3.1. To solve for the dynamical gravitational perturbations of a BH background in any such modified gravity theory perturbatively, we need a multi-variable expansion of all NP quantities. Generalizing the discussion in [82] for dCS gravity to any modified gravity theory that can be studied perturbatively (in an effective field theory approach), we need at least two expansion parameters[1]:

(i) ζ, a dimensionless parameter that characterizes the strength of the correction to GR (which typically will depend on the ratio of the scale ℓ to the BH mass), and

(ii) ϵ, a dimensionless parameter that describes the size of the GW perturbations, which also appears in GR.

In this work, we additionally impose that ζ is the leading order at which beyond GR corrections to the metric field $h_{\mu\nu}^{\mathrm{bGR}}$ appear, while the leading-order correction to other non-metric fields may enter with other (possibly lower) powers of ζ.

[1] Note that in [82], ϵ is used for the strength of the correction to GR, and ζ is used for the size of GW perturbations, which is opposite to our choices here. We here choose to remain consistent with previous literature in GR [40, 41] and in dCS gravity [66, 102].

In order to understand the coupling constant ζ better, let us first relate it to the coupling constants of the different modified gravity theories we used as examples in Sec. 2.3.1. For class A beyond GR theories with non-minimal coupling, the extra non-metric fields $\vartheta_{\rm bGR}$, e.g., ϑ in dCS gravity and θ in EdGB gravity, are sourced by the metric field and are proportional to terms of $O(\alpha_{\rm bGR})$, where $\alpha_{\rm bGR}$ is the coupling constant associated with $\mathcal{L}_{\rm bGR}$ in Eq. (2.35), e.g., $\alpha_{\rm dCS}$ in dCS gravity and $\alpha_{\rm EdGB}$ in EdGB gravity. The field $\vartheta_{\rm bGR}$ then back-reacts onto the metric and sources the metric perturbations $h^{\rm bGR}_{\mu\nu}$, which are also multiplied by a factor of $\alpha_{\rm bGR}$. Thus, to leading order, $\vartheta_{\rm bGR} \sim \alpha_{\rm bGR}$ and $h^{\rm bGR}_{\mu\nu} \sim \alpha_{\rm bGR}\vartheta_{\rm bGR}$, so $\zeta \sim \alpha^2_{\rm bGR}$. This is evident from Eqs. (2.38) and (2.42), where $\zeta \sim \alpha^2_{\rm dCS}$ for dCS gravity and $\zeta \sim \alpha^2_{\rm EdGB}$ for EdGB gravity. For class B beyond GR theories, the metric perturbations are driven by the metric fields at lower order and are proportional to $\alpha_{\rm bGR}$, so $\zeta \sim \alpha_{\rm bGR}$. For example, from Eq. (2.45), one can see that $\zeta \sim \alpha^{\rm even,odd}_{\rm HD}$.

By requiring that $h^{\rm bGR}_{\mu\nu}$ enters at $O(\zeta)$, $R_{\mu\nu}$ must also enter at $O(\zeta)$ since we are focusing on background spacetimes that are perturbed from the vacuum solutions in GR. This can be seen in Eqs. (2.38), (2.42), and (2.45). In addition, for both classes of beyond GR theories, since metric perturbations in modified gravity are sourced by the metric field in GR either indirectly via extra non-metric fields (class A) or directly (class B), the leading-order terms of the metric field in $R_{\mu\nu}$ must be of $O(\zeta^0)$. Thus, when computing $R_{\mu\nu}$, we only need the metric at $O(\zeta^0, \epsilon^0)$ or $O(\zeta^0, \epsilon^1)$. The perturbative order of $R_{\mu\nu}$ and the metric field in it will be important when we discuss the decoupling of the modified Teukolsky equation in Sec. 2.4.1 and Sec. 2.5.3.

Besides the metric field, we also have the NP quantities (i.e., tetrad basis vectors, Weyl scalars, spin coefficients, and NP Ricci scalars) generated from it. Although the beyond GR correction to the metric field enters at $O(\zeta)$, the beyond GR correction to the NP quantities does not necessarily enter at $O(\zeta)$ if we make certain gauge choices on some NP quantities, which will be discussed in detail in Sec. 2.5.1. For simplicity, we want all the NP quantities to have the same expansion pattern as the metric field, so here we construct a NP tetrad which is corrected by beyond GR theories at $O(\zeta)$ to leading order. Thus, all the other NP quantities are naturally corrected by modified gravity at $O(\zeta)$ to leading order.

In order to ensure that all the NP quantities will be corrected at $O(\zeta)$, we must find a tetrad that shared this same property, namely,

$$e_{a\mu} = e^{(0,0)}_{a\mu} + \zeta\delta e^{(1,0)}_{a\mu}\,, \tag{2.46}$$

where $\delta e_{a\mu}^{(1,0)}$ is a perturbation of $\mathcal{O}(\zeta^1, \epsilon^0)$ of the Kinnersley tetrad $e_{a\mu}^{(0,0)}$. Here, we have used the superscript (n, m) to denote terms at $\mathcal{O}(\zeta^n, \epsilon^m)$. The only constraint on a NP tetrad is the orthogonality condition in Eq. (2.5). Let us expand the correction to the Kinnersley tetrad $\delta e_{a\mu}^{(1,0)}$ in terms of the original tetrad $e_{a\mu}^{(0,0)}$ in GR,

$$\delta e_{a\mu}^{(1,0)} = A_{ab}^{(1,0)} e_{\mu}^{\ b(0,0)} \,. \tag{2.47}$$

To satisfy Eq. (2.5), we need to have that

$$\left(e_{a\mu}^{(0,0)} + \zeta \delta e_{a\mu}^{(1,0)}\right)\left(e_{b\nu}^{(0,0)} + \zeta \delta e_{b\nu}^{(1,0)}\right)\left(g^{\mu\nu(0,0)} + \zeta h^{\mu\nu(1,0)}\right) = \eta_{ab} \,, \tag{2.48}$$

where η_{ab} is the metric defined in Eq. (2.111), $g^{\mu\nu(0,0)}$ is the metric of the GR background, and $h^{\mu\nu(1,0)}$ represents the modification to the metric due to deviation from GR. Up to $\mathcal{O}(\zeta)$, we can equivalently require that

$$\delta e_{a\mu}^{(1,0)} e_{b\nu}^{(0,0)} g^{\mu\nu(0,0)} + e_{a\mu}^{(0,0)} \delta e_{b\nu}^{(1,0)} g^{\mu\nu(0,0)} - e_{a\mu}^{(0,0)} e_{b\nu}^{(0,0)} h^{\mu\nu(1,0)} \,, \tag{2.49}$$

where we have used the condition $g^{\mu\nu(0,0)} e_{a\mu}^{(0,0)} e_{b\nu}^{(0,0)} = \eta_{ab}$. Inserting the expansion of Eq. (2.47) in the above condition and using the condition $g^{\mu\nu(0,0)} e_{a\mu}^{(0,0)} e_{b\nu}^{(0,0)} = \eta_{ab}$ again, one finds

$$A_{ab}^{(1,0)} + A_{ba}^{(1,0)} = 2A_{(ab)}^{(1,0)} = -h_{ab}^{(1,0)} \,, \tag{2.50}$$

where $h_{ab}^{(1,0)} = e_{a\mu}^{(0,0)} e_{b\nu}^{(0,0)} h^{\mu\nu(1,0)}$, and thus $A_{(ab)}^{(1,0)} = -\frac{1}{2} h_{ab}^{(1,0)}$. In general, $A_{ab}^{(1,0)}$ can have 16 independent components, which can be separated into a symmetric tensor $A_{(ab)}^{(1,0)}$ with 10 independent components and an antisymmetric tensor $A_{[ab]}^{(1,0)}$ with 6 independent components. Since Eq. (2.50) does not impose any constraints on $A_{[ab]}^{(1,0)}$, the components of $A_{[ab]}^{(1,0)}$ correspond to 6 degrees of gauge freedom to further rotate the tetrad. We can choose $A_{[ab]}^{(1,0)} = 0$, so the perturbed tetrad is

$$A_{ab}^{(1,0)} = -\frac{1}{2} h_{ab}^{(1,0)} \,, \quad \delta e_{a\mu}^{(1,0)} = -\frac{1}{2} e_{a\nu}^{(0,0)} h_{\mu}^{\ \nu(1,0)} \,. \tag{2.51}$$

Using the tetrad in Eqs. (2.46) and (2.51), we are able to expand the metric field and all the NP quantities generated from it with the same perturbative scheme. In this paper, we are interested in linear dynamical perturbations of any Petrov type I stationary spacetime, which itself is a linear deformation of the Kerr metric, so all terms beyond $\mathcal{O}(\zeta^1, \epsilon^1)$ will be ignored. Up to $\mathcal{O}(\zeta^1, \epsilon^1)$, if we use the tetrad in Eqs. (2.46) and (2.51), the Weyl scalars can be expanded as

$$\Psi_i = \Psi_i^{(0)} + \epsilon \Psi_i^{(1)}$$

$$= \Psi_i^{(0,0)} + \zeta \Psi_i^{(1,0)} + \epsilon \Psi_i^{(0,1)} + \zeta \epsilon \Psi_i^{(1,1)} \,, \tag{2.52}$$

and the same expansion applies to the metric field and all the other NP quantities.For the beyond GR theories of class A mentioned in Sec. 2.3.1, additional fields may be present. For the examples presented, the pseudoscalar field in dCS gravity can be perturbatively expanded as

$$\vartheta = \vartheta^{(0)} + \epsilon \vartheta^{(1)} = \zeta \vartheta^{(1,0)} + \zeta \epsilon \vartheta^{(1,1)} \,. \tag{2.53}$$

A scalar field θ in EdGB gravity can also be expanded perturbatively in a similar manner. For both ϑ and θ, the background and perturbed GR pieces vanish. Notice that other work sometimes chooses to expand extra fields starting at ζ^0 [82, 83, 125, 126] or $\zeta^{\frac{1}{2}}$ [103], since these extra fields usually enter at lower order than the metric field as explained above. In our case, we choose to absorb the coupling constant into the expansion of the extra fields for convenience in the order counting, so our expansion starts at ζ. In latter sections, we may also rotate the tetrad in Eqs. (2.46) and (2.51) using Eqs. (2.126) such that certain NP quantities vanish on the background. If the expansion in Eq. (2.52) is not broken, we will use the rotated tetrad for the convenience of calculations. In the case that Eq. (2.52) is violated due to those rotations, we will use Eqs. (2.46) and (2.51) as our background tetrad.

Besides ζ and ϵ, one may have to deal with additional expansion parameters, such as the dimensionless spin χ in the slow-rotation expansion, but an expansion in ζ and ϵ is necessary and sufficient to demonstrate how the Teukolsky equation in modified gravity can be derived. Below, we may write some quantities with only one superscript, e.g., $\Psi^{(n)}$, which represents the n-th order term in the expansion of Ψ in ϵ, as shown in the first line of Eq. (2.52), so all the other expansions are hidden for simplicity.

2.4 Perturbations of Petrov type D spacetimes in theories beyond GR

In this section, we present a method to extend the formalism shown in Sec. 2.2 for obtaining the perturbation equations for Petrov type D BHs in modified theories of gravity discussed in Sec. 2.3.1. We particularly focus on spacetimes that are stationary and vacuum solutions to modified gravity theories, and although they may not be Ricci flat, they remain of Petrov type D. As discussed in Sec. 2.3.1, an example of such a spacetime is BH solutions in dCS gravity, expanded to leading order in the dimensionless spin parameter [66, 69] and obtained in an effective field theory (EFT) approach. We will use the perturbation scheme introduced in Sec. 2.3.2.

Extending the formalism developed for Petrov type D spacetimes in GR (either the traditional Teukolsky's approach or the Chandrasekhar's approach) to include Petrov type D spacetimes that are non-Ricci-flat in modified gravity is a stepping stone in developing a formalism that is applicable to algebraically general Petrov type I spacetimes in beyond GR theories.

2.4.1 Extending the Teukolsky formalism beyond GR: non-Ricci-flat and Petrov type D backgrounds

In this subsection, we present an extension to the Teukolsky formalism presented in Sec. 2.2.2 for non-GR non-Ricci-flat Petrov type D spacetimes. We follow a procedure similar to that presented in Sec. 2.2.2 with the aim of developing a formalism to obtain the decoupled differential equation describing the dynamical pieces of Ψ_0 and Ψ_4. This subsection along with the next one form the backbone of the development of a formalism for the algebraically general Petrov type I spacetimes in beyond GR theories.

We begin by considering modified theories of gravity whose isolated (stationary and vacuum) BH solutions are non-Ricci-flat, i.e., the Ricci tensor obtained from trace-reversed vacuum field equations (i.e., no matter present) no longer vanish. For instance, in theories such as dCS or EdGB, where a scalar field is non-minimally coupled to a quadratic term in curvature [52, 54, 65, 66], cubic, or higher-order theories of gravity [80, 81, 116–119, 127], the metric field equations lead to a non-vanishing Ricci tensor and are therefore non-Ricci-flat. This can easily be seen in the dCS gravity example with the trace-reversed field equation (2.38), where the Ricci tensor clearly does not vanish even in vacuum due to the non-vanishing of the Riemann tensor and a non-trivial pseudo-scalar field.

When the background is non-Ricci-flat, the unperturbed Bianchi identities acquire sources. In the NP language, the non-vanishing of the Ricci tensor implies that NP Ricci scalars Φ_{ij} for $i, j \in (0, 1, 2)$ also do not vanish [see e.g., Eq. (2.19)]. Consequently, the source terms of Eqs. (2.13a)-(2.13b) are non-vanishing for non-Ricci-flat, non-GR BH background. But if we require that the non-Ricci-flat background be of Petrov type D, then the background Weyl scalars

$$\Psi_0^{(0)} = \Psi_1^{(0)} = \Psi_3^{(0)} = \Psi_4^{(0)} = 0\,. \tag{2.54}$$

Unlike in the GR case, however, the background spin coefficients no longer vanish in general, as one can verify explicitly by inserting Eq. (2.54) in Eqs. (2.123). Consequently, we still have additional terms that are non-vanishing in the equations presented in Sec. 2.2.2. More specifically, the full Bianchi identities recast in the form of Eqs. (2.13) now take the form

$$F_1\Psi_0 - J_1\Psi_1 - 3\kappa\Psi_2 = S_1\,, \tag{2.55a}$$

$$F_2\Psi_0 - J_2\Psi_1 - 3\sigma\Psi_2 = S_2\,, \tag{2.55b}$$

$$E_2\sigma - E_1\kappa - \Psi_0 = 0\,, \tag{2.55c}$$

where S_1 and S_2 are given in Eq. (2.10), $(E_{1,2}, F_{1,2}, J_{1,2})$ are defined in Eq. (2.11), and (κ, σ) are spin coefficients presented in Appendix 2.8. Notice that we have not yet performed a perturbative expansion to separate the background from the perturbed Weyl scalars.

Adapting a method similar to that presented in Sec. 2.2.2 to obtain a differential equation for Ψ_0, we need to eliminate the Ψ_1 dependence from the above equations by developing an appropriate commutation relation for this type of beyond GR theories. While eliminating the Ψ_1 dependence, we will also naturally decouple Ψ_0 from the κ and σ dependence in the above equations, as shown below. To decouple the equations, we prescribe the following steps:

1. Multiply Eq. (2.55c) by Ψ_2.

2. Use the chain rule such that the intrinsic derivatives act on the product of Ψ_2 with either σ or κ. For instance,

$$\Psi_2(D\sigma) = D(\Psi_2\sigma) - \sigma(D\Psi_2)\,. \tag{2.56}$$

   For modified theories of gravity, the second term above is different from Eq. (2.22) because it is modified due to the non-vanishing of the NP Ricci scalars. For instance, when looking at Eq. (2.123h),

$$D\Psi_2 = 3\rho\Psi_2 - P_1\,, \tag{2.57}$$

   where P_1 are all the non-vanishing terms from the Bianchi identity in Eq. (2.123h). However, when working with this approach, we have more algebraic complications involved in decoupling all curvature perturbations. Therefore, for the purpose of this subsection, we continue to work with Eq. (2.56).

3. Using Eq. (2.56), we can rewrite the operators in Eq. (2.55c) as $\mathcal{E}_1$ and $\mathcal{E}_2$, as defined in Eqs. (2.34).

4. The commutator acting on Ψ_1 is then given by

$$(\mathcal{E}_2 J_2 - \mathcal{E}_1 J_1)\, \Psi_1\,. \tag{2.58}$$

5. Now expand Ψ_1 as shown in Eq. (2.52), i.e.,

$$\Psi_1 = \Psi_1^{(0,0)} + \zeta \Psi_1^{(1,0)} + \epsilon \Psi_1^{(0,1)} + \zeta\epsilon \Psi_1^{(1,1)}\,. \tag{2.59}$$

Since the BH background is Petrov type D, the background $\Psi_1^{(0,0)}$ and $\Psi_1^{(1,0)}$ vanish. The quantity $\Psi_1^{(0,1)}$ is generated by the perturbed (GW) metric in GR, which can be set to zero through a convenient choice of gauge, as we have shown in Sec. 2.2.3. Therefore, to leading order in ζ and ϵ, the terms inside the parenthesis of Eq. (2.58) must be evaluated on the GR BH background as in Eq. (2.25). *Following these arguments, the commutator given by Eq.* (2.58) *vanishes for non-Ricci-flat and Petrov type D BH backgrounds in the class of modified gravity theories we considered.*

Multiplying Eqs. (2.55a) and (2.55b) by $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, subtracting one from the other, and expanding to leading order in ϵ, we find

$$H_0^{(0)} \Psi_0^{(1)} = \mathcal{S}^{(1)}\,, \tag{2.60}$$

where we have defined

$$H_0 = \mathcal{E}_2 F_2 - \mathcal{E}_1 F_1 - 3\Psi_2\,, \tag{2.61a}$$

$$\mathcal{S} = \mathcal{E}_2 S_2 - \mathcal{E}_1 S_1\,. \tag{2.61b}$$

Expanding Eq. (2.60) using the two parameter expansion in Eq. (2.52), at leading orders in ζ and ϵ, we have

$$H_0^{(0,0)} \Psi_0^{(1,1)} + H_0^{(1,0)} \Psi_0^{(0,1)} = \mathcal{S}^{(1,1)}\,. \tag{2.62}$$

Notice, similar to the case in GR, the expansion in ϵ is sufficient to derive Eq. (2.60), and an expansion in ζ is imposed at the end to get the equation at $O(\zeta^1, \epsilon^1)$.

We can now use the GHP transformation to derive an analogous modified Teukolsky equation for the perturbed Ψ_4. Let us then apply the exchange transformation $l^\mu \leftrightarrow n^\mu$, $m^\mu \leftrightarrow \bar{m}^\mu$ to Eq. (2.60) and use the definitions given in Eq. (2.11) to find

$$H_4^{(0)} \Psi_4^{(1)} = \mathcal{T}^{(1)}\,, \tag{2.63}$$

which, expanded in ζ, becomes

$$H_4^{(0,0)}\Psi_4^{(1,1)} + H_4^{(1,0)}\Psi_4^{(0,1)} = \mathcal{T}^{(1,1)}\,, \tag{2.64}$$

where we have defined

$$H_4 = \mathcal{E}_4 F_4 - \mathcal{E}_3 F_3 - 3\Psi_2\,, \tag{2.65a}$$

$$\mathcal{T} = \mathcal{E}_4 S_4 - \mathcal{E}_3 S_3\,. \tag{2.65b}$$

S_3 and S_4 are defined in Eq. (2.10), while $\mathcal{E}_3$ and $\mathcal{E}_4$ are defined in Eq. (2.34).

Equations (2.60) and (2.63) therefore represent a modified Teukolsky equation. The differential operators acting on $\Psi_{0,4}^{(1)}$ are similar in functional form to those of the standard Teukolsky equation in GR. Notice, however, that these operators are not the same as their GR counterparts [i.e., corrected by $H_{0,4}^{(1,0)}$ in Eqs. (2.62) and (2.64)] because the Bianchi identities are modified. In the GR limit, one can of course show that they are equivalent to each other because the Bianchi identities no longer depend on NP Ricci scalars, so they reduce to Eq. (2.22). Note importantly that the left-hand side of Eqs. (2.60) and (2.63) describe all GW perturbations since they are not expanded in power of ζ.

The modified Teukolsky equations (2.60) and (2.63) contain source terms that are of $O(\zeta)$ and thus absent in GR. After an expansion in ζ in Eqs. (2.62) and (2.64), we notice that the source terms $\mathcal{S}^{(1)}$ and $\mathcal{T}^{(1)}$ depend on dynamical NP quantities at $O(\zeta^1, \epsilon^1)$ [i.e., $\mathcal{S}^{(1,1)}$ and $\mathcal{T}^{(1,1)}$]. These sources terms depend on the S_i terms in Eqs. (2.61b) and (2.65b), which are products of differential operators constructed from the tetrad and the NP Ricci scalars Φ_{ij}. As discussed in Sec. 2.3.2, since $R_{\mu\nu}$ is $O(\zeta)$, Φ_{ij} is always of $O(\zeta^1, \epsilon^0)$ or $O(\zeta^1, \epsilon^1)$, which then means the tetrad that is needed to compute the differential operators must be of $O(\zeta^0, \epsilon^0)$ and $O(\zeta^0, \epsilon^1)$. In addition, all the metric fields in $R_{\mu\nu}$ must also be of $O(\zeta^0, \epsilon^0)$ and $O(\zeta^0, \epsilon^1)$. *We therefore conclude that curvature perturbations of a non-Ricci-flat, Petrov type D BH background satisfy a decoupled equation.*

The tetrad at $O(\zeta^0, \epsilon^0)$ is just the Kinnersley tetrad of Eq. (2.26), but the tetrad at $O(\zeta^0, \epsilon^1)$ must be reconstructed from the metric perturbation at $O(\zeta^0, \epsilon^1)$. That is, one needs to first solve the Teukolsky equation in GR for the GR Weyl scalars $\Psi_{0,4}^{(0,1)}$ and then reconstruct the GR GW metric perturbation to build the perturbed tetrad at $O(\zeta^0, \epsilon^1)$. This is in stark contrast to the GR case since for a Ricci-flat Petrov type D BH background in GR, metric reconstruction is not required to study GW perturbations. Metric reconstruction in GR has already been worked out in the vacuum case by Chrzanowski [90] and Cohen and Kegeles [91] (see e.g., [128, 129] for a short review) using Hertz potential. There are also approaches that avoid using Hertz potential by solving the remaining Bianchi identities, Ricci identities, and commutation relations, for example in [40, 92]. Clearly then, such metric reconstruction in GR is possible, and we leave a further analysis of their implementation in our decoupled equations to future work.

2.4.2 Extending Chandrasekhar's approach beyond GR: non-Ricci-Flat and Petrov type D backgrounds

Similar to the Petrov type D vacuum GR case, we can also follow Chandrasekhar's approach to remove $\Psi_1^{(1)}$ directly. By doing the same type II rotation of Sec. 2.2.3 with the rotation parameter $b^{(1)} = -\Psi_1^{(1)}/(3\Psi_2^{(0)})$, we can set $\Psi_1^{(1)} = 0$. Then, from Eqs. (2.55a)-(2.55b), we again solve for κ and σ first. Notice that the κ and σ we have solved for may also contain $O(\epsilon^0)$ terms since they do not necessarily vanish in a non-Ricci-flat Petrov type D background. We then insert the solutions for κ and σ in terms of $\Psi_0^{(1)}$ and $S_i^{(1)}$ back into Eq. (2.55c) to obtain a single equation for $\Psi_0^{(1)}$. We have verified explicitly that this equation is exactly the same as Eq. (2.60). Applying the GHP transformation, one again finds Eq. (2.63) for $\Psi_4^{(1)}$.

As shown above, the final modified Teukolsky equation obtained using the two approaches (i.e., the Teukolsky's approach and Chandrasekhar's approach) are equivalent for both Ricci-flat and non-Ricci-flat, Petrov type D BH backgrounds. A main difference between the two methods is in how the equations for the curvature perturbations Ψ_0 and Ψ_4 are decoupled from Ψ_1 and Ψ_3, respectively. Chandrasekhar's approach has a significant algebraic advantage over Teukolsky's original formalism, as the former utilizes available gauge freedom to make convenient gauge choices to eliminate Ψ_1 and Ψ_3 dependence. For non-Ricci-flat, Petrov type D backgrounds in modified gravity, Teukolsky's approach is not significantly more complicated than in GR, but this is no longer true when considering non-Ricci-flat, Petrov type I backgrounds. In the latter case, Teukolsky's approach is more involved because of the non-vanishing of additional NP quantities leading to more non-vanishing terms in these equations. In Chandrasekhar's approach, however, one can continue to leverage gauge freedom to eliminate certain NP quantities without the need for developing a commutator relation like that of Eqs. (2.25) and (2.58) or using additional Bianchi identities. Because of this, we will employ Chandrasekhar's approach in what follows to develop a formalism to study perturbations of non-Ricci-flat, Petrov type I spacetimes in modified theories of gravity.

2.5 Extension of the Teukolsky formalism beyond GR: non-Ricci-flat and non-Petrov-type-D backgrounds

In this section, we extend Chandrasekhar's approach to non-Ricci-flat backgrounds that are algebraically general. As seen in Sec. 2.2.3 and 2.4.2, choosing a convenient gauge for the background and for the perturbed NP quantities, certain NP quantities can be eliminated from the NP equations when deriving the (modified) Teukolsky equation to obtain a single decoupled equation for Ψ_0 and Ψ_4. In this section, we first explore these gauge choices for background and perturbed NP quantities in more detail while treating the Petrov type I spacetime as a linear perturbation of a Petrov type D spacetime in GR. We then derive the master equations for dynamical Weyl scalars Ψ_0 and Ψ_4, discuss the modifications introduced due to non-GR effects, and provide a brief discussion on how to evaluate this equation for beyond GR theories.

Before proceeding with this section, it is important to distinguish between two background concepts that we introduce in this work. In general, the line element of a BH background spacetime for theories beyond GR discussed in Sec. 2.3.1 can be expressed as

$$ds^2 = ds^2_{\rm GR} + \zeta \tilde{ds}^2_{\rm bGR} \,. \tag{2.66}$$

Here, we have introduced the following symbols:

(i) ds^2 is the line element of the *background spacetime* or the *background* for short, which is the stationary part of the full spacetime.

(ii) $ds^2_{\rm GR}$ is the line element of the *original background*, which is the background *all* the perturbations, including the stationary ones (e.g., $\tilde{ds}^2_{\rm bGR}$), are built on top of.

For instance, the line element of a slowly rotating BH in dCS gravity to leading order in spin takes the form of Eq. (2.66) with [102]

$$\tilde{ds}^2_{\rm dCS} = \frac{5M^4}{4}\frac{a}{r^4}\left(1+\frac{12}{7}\frac{M}{r}+\frac{27}{10}\frac{M^2}{r^2}\right)\sin^2\theta dt d\phi \,, \tag{2.67}$$

$$ds^2_{\rm GR} = -f(r)dt^2 - \frac{4Ma\sin^2\theta}{r}dt d\phi + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \,. \tag{2.68}$$

Here, in our notation, the original background is given by Eq. (2.68) whereas the background spacetime is given by the sum of Eqs. (2.67) and (2.68). This is of course just a simple example of our notation, which holds true for theories that can be described using the Lagrangian given in Eq. (2.35). In general, the background

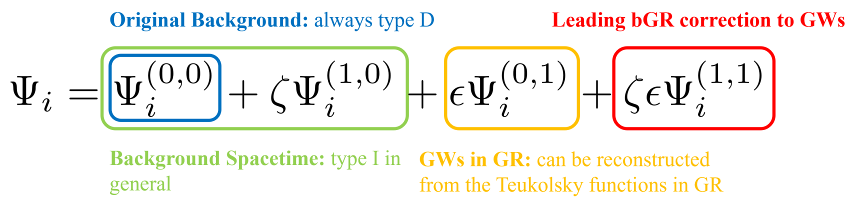

Figure 2.2: A diagram to illustrate the meaning of different terms in the expansion of NP quantities in Eq. (2.52).

spacetime includes $O(\zeta^0, \epsilon^0)$ and $O(\zeta^1, \epsilon^0)$ parts, while the original background is just of $O(\zeta^0, \epsilon^0)$ (i.e., it is the Kerr BH spacetime for arbitrarily spinning BHs).

Although the concepts of a background and an original background spacetime may sometimes correspond to the same thing (e.g., to the Kerr BH spacetime in GR), these concepts can sometimes be different in modified gravity theories. For example, in the theories discussed in Sec. 2.3.1, the Kerr metric is not a solution for all stationary and axisymmetric BHs. Rather these BHs are represented by spacetimes that are non-Ricci-flat and non-Petrov-type-D when not expanded in spin. In such cases, the background of the dynamical gravitational perturbation we study would be such a non-Ricci-flat and non-Petrov-type-D spacetime, but the original background would still be the Kerr spacetime. In Fig. 2.2, we present the relation between these two different background concepts and the terms in the expansion of NP quantities in Eq. (2.52).

2.5.1 Gauge choice for the background spacetime: $O(\zeta^0, \epsilon^0)$ and $O(\zeta^1, \epsilon^0)$

For a non-Petrov-type-D modified background spacetime, the gauge choice in Eq. (2.54) is not possible. For example, as found in [103], the metric describing a rotating BH in dCS gravity need not be of Petrov type D once one incorporates second-order and higher in rotation effects; in that case, the metric is now of Petrov type I, which is the most general type in the Petrov classification. However, we can still set $\Psi_0^{(0)} = \Psi_4^{(0)} = 0$ for a Petrov type I spacetime as discussed in [40] and shown for dCS gravity in [69], so we *could* use a gauge such that

$$\Psi_{0,1,3,4}^{(0,0)} = 0\,, \quad \Psi_0^{(1,0)} = \Psi_4^{(1,0)} = 0\,, \tag{2.69}$$

but we will *not* for the following reasons.

Although the gauge defined by requiring that Eq. (2.69) holds simplifies Eqs. (2.13) and (2.14), it may spoil our assumption that the leading correction to the tetrad enters at $O(\zeta^1)$. As shown in [69], for dCS gravity in the slow-rotation approximation, in order to impose that $\Psi_0^{(1,0)} = \Psi_4^{(1,0)} = 0$ at $O(\chi^2)$, we need to modify the tetrad at $O(\zeta^{\frac{1}{2}}, \chi^2)$, and this induces a nonzero $\Psi_1^{(\frac{1}{2},0)}$ and $\Psi_3^{(\frac{1}{2},0)}$. These $O(\zeta^{\frac{1}{2}})$ terms are not covered by our expansion strategy in Eq. (2.52), which only contains terms of $O(\zeta^0, \epsilon^0)$, $O(\zeta^1, \epsilon^0)$, $O(\zeta^0, \epsilon^1)$, and $O(\zeta^1, \epsilon^1)$ for all quantities. For this reason, we only impose

$$\Psi_{0,1,3,4}^{(0,0)} = 0 \tag{2.70}$$

and leave all $O(\zeta^1, \epsilon^0)$ perturbations general. These properties are summarized on the left two columns of Table 2.1. In this case, we will use the background tetrad in Eqs. (2.46) and (2.51) such that Eq. (2.70) is satisfied, and the expansion in Eq. (2.52) is not broken.

2.5.2 Gauge choice for the dynamical perturbations: $O(\zeta^0, \epsilon^1)$ and $O(\zeta^1, \epsilon^1)$

Different gauge choices can be made separately at different perturbative orders. Sec. 2.5.1 fixed the gauge for the background spacetime at $O(\zeta^0, \epsilon^0)$ and $O(\zeta^1, \epsilon^0)$, but we still have gauge freedom at $O(\zeta^0, \epsilon^1)$ and $O(\zeta^1, \epsilon^1)$. As in Secs. 2.2.3 and 2.4.2, we shall impose

$$\Psi_1^{(0,1)} = \Psi_3^{(0,1)} = \Psi_1^{(1,1)} = \Psi_3^{(1,1)} = 0\,. \tag{2.71}$$

In this gauge, Eqs. (2.13a)-(2.13b) for the dynamical part of Ψ_0 and Ψ_1 decouple directly, and so do Eqs. (2.14a)-(2.14b) for the dynamical part of Ψ_3 and Ψ_4.

As discussed in [40], in a Petrov type D spacetime, we can always make a gauge choice such that the linear perturbations to Ψ_1 and Ψ_3 vanish without affecting Ψ_0 and Ψ_4, so only Ψ_0 and Ψ_4 are gauge invariant quantities in a linear perturbation theory. Since at $O(\zeta^0)$, the background spacetime is the Petrov type D spacetime of GR, it then follows that we can always make the gauge choice in Eq. (2.71) at $O(\zeta^0, \epsilon^1)$.

Next, we need to show that Eq. (2.71) holds at $O(\zeta^1, \epsilon^1)$. If we treat $\Psi_{1,3}^{(1,1)}$ as the $O(\zeta^0, \epsilon^1)$ perturbation to $\Psi_{1,3}^{(1,0)}$, it is not clear that we can make a gauge choice in Eq. (2.71) since the *background spacetime* at $O(\zeta^1, \epsilon^0)$ is not necessarily Petrov type D. However, we can also treat $\Psi_{1,3}^{(1,1)}$ as the $O(\zeta^1, \epsilon^1)$ perturbation to $\Psi_{1,3}^{(0,0)}$ in the *original background*. Since the *original background* is the Petrov type D spacetime in GR, Eq. (2.71) should still hold.

Let us show that, at $O(\zeta^1, \epsilon^1)$, $\Psi_{1,3}^{(1,1)}$ can be eliminated by a tetrad rotation at $O(\zeta^1, \epsilon^1)$. Let us consider $\Psi_1^{(1,1)}$ explicitly and apply a type II rotation [cf. Eq. (2.126b)], with a parameter $b^{(1,1)}$ at $O(\zeta^1, \epsilon^1)$. This leads to, at $O(\zeta^1, \epsilon^1)$,

$$\begin{aligned}
\Psi_0^{(1,1)} &\to \Psi_0^{(1,1)} + 4b^{(1,1)}\Psi_1^{(0,0)}\,,\\
\Psi_1^{(1,1)} &\to \Psi_1^{(1,1)} + 3b^{(1,1)}\Psi_2^{(0,0)}\,,\\
\Psi_2^{(1,1)} &\to \Psi_2^{(1,1)} + 2b^{(1,1)}\Psi_3^{(0,0)}\,,\\
\Psi_3^{(1,1)} &\to \Psi_3^{(1,1)} + b^{(1,1)}\Psi_4^{(0,0)}\,,\\
\Psi_4^{(1,1)} &\to \Psi_4^{(1,1)}\,.
\end{aligned} \tag{2.72}$$

We are motivated to require that $b = O(\zeta^1, \epsilon^1)$ since we want to perturb about the *original background*. By letting $b^{(1,1)} = -\Psi_1^{(1,1)}/(3\Psi_2^{(0,0)})$, we can set $\Psi_1^{(1,1)} = 0$. With the background gauge choice that ensures Eq. (2.70) holds, we can easily see from Eq. (2.72) that all the other Weyl scalars at $O(\zeta^1, \epsilon^1)$ are unaffected such that

$$\Psi_1^{(1,1)} \to 0\,, \quad \Psi_{0,2,3,4}^{(1,1)} \to \Psi_{0,2,3,4}^{(1,1)}\,. \tag{2.73}$$

Similarly, by applying a type I rotation [Cf. Eq. (2.126a)] and choosing the rotation parameter $a^{(1,1)} = [-\Psi_3^{(1,1)}/(3\Psi_2^{(0,0)})]^*$, we can set

$$\Psi_3^{(1,1)} \to 0\,, \quad \Psi_{0,1,2,4}^{(1,1)} \to \Psi_{0,1,2,4}^{(1,1)}\,. \tag{2.74}$$

Properties of the $O(\zeta^0, \epsilon^1)$ and $O(\zeta^1, \epsilon^1)$ contributions to the Weyl scalars are summarized on the right half of Table 2.1.

2.5.3 Modified Teukolsky equation in non-Ricci-flat and algebraically general backgrounds

We can now derive the modified Teukolsky equation for non-Ricci-flat and Petrov type I spacetimes. Here, we only show how to obtain the equation for the dynamical perturbation to Ψ_0, but the same procedure can be applied to Ψ_4, or one can perform the GHP transformation $l^\mu \leftrightarrow n^\mu$, $m^\mu \leftrightarrow \bar{m}^\mu$ on the Ψ_0 equation to find the equation for Ψ_4 [100].

2.5.3.1 Elimination of κ and σ

From Eqs. (2.13a)-(2.13b), we can solve for κ and σ in terms of other NP quantities. Inserting κ and σ from Eqs. (2.13a)-(2.13b) into Eq. (2.13c) and multiplying the resulting equation by $3\Psi_2$ to match the form of the original Teukolsky equation [44] when $\zeta = 0$, one finds

$$\Psi_2 E_2\left[\Psi_2^{-1}\left(F_2\Psi_0 - J_2\Psi_1 - S_2\right)\right] - \Psi_2 E_1\left[\Psi_2^{-1}\left(F_1\Psi_0 - J_1\Psi_1 - S_1\right)\right] - 3\Psi_2\Psi_0 = 0\,. \tag{2.75}$$

Types of Terms	Stationary Background		Dynamical GWs	
	Original Background (GR)	Stationary Modification to Original Background	GWs on Original Background	GW Corrections
Orders / Weyl Scalar	$O(\zeta^0,\epsilon^0)$	$O(\zeta^1,\epsilon^0)$	$O(\zeta^0,\epsilon^1)$	$O(\zeta^1,\epsilon^1)$
Ψ_0	0	$\neq 0$	$\neq 0^{(a)}$	$\neq 0^{(a)}$
Ψ_1	0	$\neq 0$	$0^{(b)}$	$0^{(b)}$
Ψ_2	$\neq 0$	$\neq 0$	$\neq 0^{(c)}$	$\neq 0^{(d)}$
Ψ_3	0	$\neq 0$	$0^{(b)}$	$0^{(b)}$
Ψ_4	0	$\neq 0$	$\neq 0^{(a)}$	$\neq 0^{(a)}$

Table 2.1: Properties of Weyl scalars in the Chandrasekhar gauge for non-Petrov-type-D modified BH spacetimes with GWs. Quantities on the *stationary background* columns are already known. For quantities on the *dynamical GWs* columns, items labeled as (a) are scalars that need to be solved for, labeled as (b) are set to zero by gauge, labeled as (c) can be reconstructed from $\Psi_0^{(0,1)}$ or $\Psi_4^{(0,1)}$, while labeled by (d) do not appear in the modified Teukolsky equation.

Re-organizing this equation to extract the operators that act on Ψ_0, Ψ_1, S_1, and S_2, we find

$$H_0\Psi_0 - H_1\Psi_1 = \mathcal{S}\,, \tag{2.76}$$

where H_0 and $\mathcal{S}$ are defined in Eq. (2.61), and we have defined

$$H_1 \equiv \mathcal{E}_2 J_2 - \mathcal{E}_1 J_1\,, \tag{2.77}$$

with $\mathcal{E}_i$ defined in Eq. (2.34).

2.5.3.2 Gauge choice and general strategy

The derivation so far has combined the three equations in Eqs. (2.13a)-(2.13c) into a single equation (2.76). Our next goal is to keep only $\Psi_0^{(1,1)}$ and no other $O(\zeta^1,\epsilon^1)$ contributions of Weyl scalars, spin connection coefficients, or intrinsic derivatives. Note that $O(\zeta^0,\epsilon^0)$ and $O(\zeta^1,\epsilon^0)$ are known background components, while $O(\zeta^0,\epsilon^1)$ can be reconstructed from linear perturbation of Kerr.

For terms on the left-hand side of Eq. (2.76), we will find the following pattern, where an operator O operates on a field ψ, and we are interested in the $O(\zeta^1,\epsilon^1)$ component, with

$$(O\psi)^{(1,1)} = O^{(1,1)}\psi^{(0,0)} + O^{(0,1)}\psi^{(1,0)} + O^{(1,0)}\psi^{(0,1)} + O^{(0,0)}\psi^{(1,1)}\,. \tag{2.78}$$

As we shall see in Sec. 2.5.3.3, because of our gauge choice in Table 2.1, the only non-vanishing $\mathcal{O}(\zeta^1, \epsilon^1)$ quantity we will encounter will be $\Psi_0^{(1,1)}$.

For terms on the right-hand side of Eq. (2.76), we will argue in Sec. 2.5.3.4 that they can all be obtained from the background geometry and the $\mathcal{O}(\zeta^0, \epsilon^1)$ metric perturbation $h^{(0,1)}$ because GWs on the modified background $h^{(1,1)}$ do not contribute to the source term.

2.5.3.3 Analysis of the general modified Teukolsky equation: the $H_0\Psi_0$ and $H_1\Psi_1$ terms

For the first term on the left-hand side of Eq. (2.76), expanding $H_0\Psi_0$ to $\mathcal{O}(\zeta^1, \epsilon^1)$, one finds the following three types of terms:

$$(H_0\Psi_0)^{(1,1)} = H_0^{(0,0)}\Psi_0^{(1,1)} + H_0^{(1,0)}\Psi_0^{(0,1)} + H_0^{(0,1)}\Psi_0^{(1,0)} \,. \tag{2.79}$$

Since at $\mathcal{O}(\zeta^0, \epsilon^0)$, Eq. (2.76) becomes $H_0^{(0,0)}\Psi_0^{(0,0)} = 0$, $H_0^{(0,0)}$ is the Teukolsky differential operator that acts on Ψ_0 in GR, which was discussed in Sec. 2.4.1. Therefore, the first term in Eq. (2.79) is just the Teukolsky equation in GR but for $\Psi_0^{(1,1)}$. The second term vanishes in GR but is generically nonzero in modified gravity. This is because $\Psi_0^{(0,1)}$ is a solution to the Teukolsky equation presented in Eq. (2.26). As discussed in Sec. 2.2, this is a gauge invariant quantity and thus non-vanishing in general. On the other hand, the operator $H_0^{(1,0)}$ can be evaluated using the background metric for the spacetime in the modified theory of gravity under consideration.

The third term only shows up for non-Petrov-type-D spacetime since $\Psi_0^{(1,0)} = 0$ if the modified background spacetime is Petrov type D. The operator $H_0^{(0,1)}$ contains Weyl scalars, spin coefficients, and intrinsic derivatives at $\mathcal{O}(\zeta^0, \epsilon^1)$, so as discussed at the end of Sec. 2.4.1, we need to reconstruct the metric of GW perturbations in GR. By applying one of these metric reconstruction procedures and rotating the reconstructed tetrad to the gauge in Eq. (2.71), one is able to evaluate all the terms in $H_0^{(0,1)}$.

The last two terms in Eq. (2.79) come from the homogeneous part of the Bianchi and Ricci identities. These terms are purely geometrical, and we can interpret them as source terms induced by stationary perturbations contained in the background geometry. We can then rewrite Eq. (2.79) as

$$(H_0\Psi_0)^{(1,1)} = H_0^{\rm GR}\Psi_0^{(1,1)} - \mathcal{S}_{0,\rm D}^{(1,1)} - \mathcal{S}_{0,\rm non\text{-}D}^{(1,1)} \,, \tag{2.80}$$

where we have defined

$$H_0^{\rm GR} \equiv H_0^{(0,0)}\,, \tag{2.81}$$
$$\mathcal{S}_{0,\rm D}^{(1,1)} \equiv -H_0^{(1,0)}\Psi_0^{(0,1)}\,, \tag{2.82}$$
$$\mathcal{S}_{0,\text{non-D}}^{(1,1)} \equiv -H_0^{(0,1)}\Psi_0^{(1,0)}\,. \tag{2.83}$$

Moving on to the second term on the left-hand side of Eq. (2.76) and using properties in Table 2.1, we obtain

$$(H_1\Psi_1)^{(1,1)} = H_1^{(0,1)}\Psi_1^{(1,0)}\,. \tag{2.84}$$

Similar to $H_0^{(0,1)}$, $H_1^{(0,1)}$ is also made up of Weyl scalars, spin coefficients, and intrinsic derivatives at $O(\zeta^0,\epsilon^1)$, so we need metric reconstruction for this term as well. This term vanishes in any Petrov type D spacetime since $\Psi_1 = 0$ with an appropriate choice of gauge at the background level. Similar to $H_0^{(0,1)}\Psi_0^{(1,0)}$, we can effectively treat $H_1^{(0,1)}\Psi_1^{(1,0)}$ as a source term involving $\Psi_1^{(1,0)}$ and induced by the stationary perturbation of background geometry. Let us then define

$$\mathcal{S}_{1,\text{non-D}}^{(1,1)} \equiv H_1^{(0,1)}\Psi_1^{(1,0)}\,. \tag{2.85}$$

The source term $\mathcal{S}_{1,\text{non-D}}^{(1,1)}$ along with the source terms $\mathcal{S}_{0,\rm D}^{(1,1)}$ and $\mathcal{S}_{0,\text{non-D}}^{(1,1)}$ given in Eqs. (2.82)-(2.83) come from the homogeneous part of the Bianchi and Ricci identities. Grouping these source terms together, we define

$$\mathcal{S}_{\rm geo}^{(1,1)} \equiv \mathcal{S}_{0,\rm D}^{(1,1)} + \mathcal{S}_{0,\text{non-D}}^{(1,1)} + \mathcal{S}_{1,\text{non-D}}^{(1,1)}\,. \tag{2.86}$$

2.5.3.4 Analysis of the general modified Teukolsky equation: the $\mathcal{S}$ term

Besides the source terms generated by the correction to the background metric, we also have corrections to the Einstein-Hilbert action due to modified gravity theory, including extra fields not present in GR (i.e., class A beyond GR theories) or higher-order terms in curvature (i.e., class B beyond GR theories) as discussed in detail in Sec. 2.3.1. In a perturbative treatment, all these corrections manifest as some source terms on the right-hand side of the Einstein equations, so we have a non-zero “effective” stress tensor, or in the trace reversed form, a non-zero Ricci tensor, even in the case without ordinary matter (see e.g., the discussion of dCS gravity, EdGB gravity, and higher-derivative gravity cases in Sec. 2.3.1).

Let us first look at class A beyond GR theories, where there are additional fields introduced by modified gravity, such as the pseudo scalar field coupled to the

Pontryagin density in dCS gravity. Let us focus on one of these extra fields, which we represent generically as ϑ. Since this field vanishes in GR, $\vartheta^{(0,0)} = \vartheta^{(0,1)} = 0$ also in general. From Eqs. (2.10a)-(2.10b), we see that the terms in S_i couple Φ_{ij} with either the directional derivatives or the spin coefficients. According to Eq. (2.121), the Φ_{ij} are linear functions of $R_{\mu\nu}$ contracted with the tetrad basis,

$$\Phi_{ij} \propto R_{\mu\nu} e_i{}^{\mu} e_j{}^{\nu}\,, \quad \{i,j\} \in \{0,1,2\}\,. \tag{2.87}$$

Since $\vartheta^{(0,0)} = \vartheta^{(0,1)} = 0$, $\Phi_{ij}^{(1,1)} \sim \vartheta^{(1,0)} h^{(0,1)} + \vartheta^{(1,1)} g^{(0,0)}$, where $g^{(0,0)}$ represents the terms only involving background metric in GR. Then, S_i in $\mathcal{S}$ can only enter at $O(\zeta^1)$, so

$$\begin{aligned} \mathcal{S}^{(1,1)} &= \mathcal{E}_2^{(0,0)} S_2^{(1,1)} - \mathcal{E}_1^{(0,0)} S_1^{(1,1)} + \mathcal{E}_2^{(0,1)} S_2^{(1,0)} - \mathcal{E}_1^{(0,1)} S_1^{(1,0)} \\ &\sim \vartheta^{(1,0)} h^{(0,1)} + \vartheta^{(1,1)} g^{(0,0)}\,. \end{aligned} \tag{2.88}$$

The source $\mathcal{S}$ at $O(\zeta^1, \epsilon^1)$, $\mathcal{S}^{(1,1)}$, couples the GWs in GR and the extra field ϑ, so we need to solve the equations of motions of these non-gravitational fields to find their contributions to the stress tensor and $\mathcal{S}^{(1,1)}$ in the modified Teukolsky equation. In our notation, the modified Teukolsky equation describing the evolution of the GW perturbations due to the modification to GR can then be expressed as

$$H_0^{\rm GR} \Psi_0^{(1,1)} = \mathcal{S}_{0,\rm D}^{(1,1)} + \mathcal{S}_{0,\text{non-D}}^{(1,1)} + \mathcal{S}_{1,\text{non-D}}^{(1,1)} + \mathcal{S}^{(1,1)}\,, \tag{2.89}$$

where all the quantities have been defined in Eqs. (2.82), (2.83), (2.85), and (2.61b). Notice that the differential operator acting on $\Psi_0^{(1,1)}$ is the same as the differential operator that appears in the Teukolsky equation for GR BH spacetimes discussed previously in Sec. 2.2.2.

One can find the solution to these extra fields in different ways. One way is to solve the equations of motions of these extra fields and the modified Teukolsky equation in parallel. Another way is to use the order-reduction scheme introduced in [125], in which one solves the equations of motions of these extra fields first and then insert them into the modified Teukolsky equation. Notice here that we have absorbed the coupling constant multiplying ϑ in $R_{\mu\nu}$ into the perturbative order of ϑ. For example, as discussed in Sec. 2.3.2, ϑ itself is usually of $O(\alpha_{\rm bGR})$, where $\alpha_{\rm bGR}$ is the coupling constant in front of $\mathcal{L}_{\rm bGR}$ in Eq. (2.35). The same coupling constant also shows up in front of these beyond GR corrections in $R_{\mu\nu}$, e.g., Eqs. (2.38) and (2.42), so the contribution of ϑ to $R_{\mu\nu}$ is of $O(\alpha_{\rm bGR}^2)$ or $O(\zeta)$. Thus, the equation of motion of ϑ is at lower order than the gravitational field equation, which allows us to follow the order-reduction scheme in [125], although this procedure is likely to introduce secularly-growing uncontrolled remainders. All these calculations depend on the details of the target modified gravity theory, so we will not discuss them in detail here, and instead provide some examples in Sec. 2.5.3.5 and leave the case-by-case study to future work.

Another way to generate these source terms is due to corrections to the Einstein-Hilbert action that are only made up of gravitational fields, e.g., higher-derivative gravity [116–119], which we classified as class B beyond GR theories in Sec. 2.3.1. In this case, by pure order counting, the kind of terms that can appear are of the form $h^{(1,0)}h^{(0,1)}$. These terms are similar in form to $\mathcal{S}_{\rm geo}^{(1,1)}$, given in Eq. (2.86), and so have that $\mathcal{S}_{\rm geo}^{(1,1)} = \mathcal{O}(h^{(1,0)}h^{(0,1)})$. Therefore, $\mathcal{S}_{\rm geo}^{(1,1)}$ takes the form of a coupling between the GWs in GR and the stationary modification to the background metric. Though $h^{(1,0)}$ can be generated by $\vartheta^{(1,0)}$, if we treat it as an arbitrary stationary correction to the background metric, the way it couples to GWs in GR is independent of the gravity theory, as we have discussed above. In contrast, the source terms coming from the non-vanishing stress tensor and made up of only gravitational fields depend on the details of the modified gravity theory, so they cannot be treated universally when only knowing the correction to the background metric. On the other hand, like these $\mathcal{S}_{\rm geo}^{(1,1)}$ terms, we do not need to solve the equations of motion of other non-gravitational fields, so these terms can be evaluated directly with the background metric and the reconstructed metric for GWs in GR when knowing the stress tensor in the target modified gravity theory.

One of the major successes of Teukolsky's formalism in GR, presented in Sec. 2.2.2, was the separation of the master equation into a radial and an angular equation, when written in a coordinate basis, such as in the Boyer-Lindquist coordinates of the Kerr BH spacetime. Each of these equations need then to be solved independently as an eigenvalue problem. Since the differential operator acting on the beyond GR, leading-order correction to GW perturbations remains unchanged from GR, the left-hand side of the beyond GR master equation in Eq. (2.89) is naturally separable into a radial and an angular part. Furthermore, one can separate the right-hand side of Eq. (2.89) by making use of the orthogonality properties of the spin-weighted spheroidal harmonics (which are the solution to the angular master equation for GR BH Petrov type D spacetimes) to project the source terms onto the original angular basis. Following this trick, the separability of the master equations into a radial and an angular equation must hold for beyond GR, Petrov type I, non-Ricci-flat spacetimes as well. When looking at the example theories presented in Sec. 2.3.1, one may also encounter a mode coupling between different ℓ modes (e.g., between ℓ and $\ell \pm 1$ modes at leading order in the slow rotation expansion [54–56]). This is seen when coupling between different perturbation functions exist, both in GR [88] and beyond GR theories [54–57].

2.5.3.5 Examples of equations of motion of extra (non-metric) fields

In the previous section, we showed that to evaluate $\mathcal{S}^{(1,1)}$, one needs to solve the equations of motion of these non-metric extra fields. In this section, we provide the equations of motion of the pseudoscalar field ϑ in dCS gravity and the scalar field θ in EdGB gravity as a demonstration.

In dCS gravity, expanding the equation of motion of ϑ in Eq. (2.39) using the perturbation scheme in Eq. (2.52), we find, at $O(\zeta^1, \epsilon^1)$,

$$\square^{(0,0)}\vartheta^{(1,1)} = -\frac{1}{16\pi^{\frac{1}{2}}} M^2 \left[R^*R\right]^{(0,1)} - \square^{(0,1)}\vartheta^{(1,0)} \,, \tag{2.90}$$

where R^*R is a shorthand for ${}^*R^{\mu}{}_{\nu}{}^{\kappa\sigma} R^{\nu}{}_{\mu\kappa\sigma}$, and we follow [54] to use $\zeta_{\rm dCS} \equiv 16\pi\alpha_{\rm dCS}^2/M^4$ as the dCS gravity expansion parameter. We have also absorbed a factor of $(\zeta_{\rm dCS})^{1/2}$ into the expansion of ϑ. To solve Eq. (2.90) in the Teukolsky formalism, one first needs to project all quantities onto the NP tetrad. For example, the Pontryagin density and the wave operator decompose into

$$R\,{}^*R = -8i\mathcal{E}(3\Psi_2^2 - 4\Psi_1\Psi_3 + \Psi_0\Psi_4 - c.c.) \,, \tag{2.91}$$

$$\begin{aligned}\square\vartheta = \; & [\{\delta, \delta^*\} - \{D, \Delta\} + (\gamma + \gamma^* - \mu - \mu^*)D \\ & + (\rho + \rho^* - \varepsilon - \varepsilon^*)\Delta + (\pi - \tau^* - \alpha + \beta^*)\delta \\ & + (\pi^* - \tau - \alpha^* + \beta)\delta^*]\,\vartheta \,,\end{aligned} \tag{2.92}$$

where $i\mathcal{E} = \epsilon_{\mu\nu\rho\sigma} l^\mu n^\nu m^\rho \bar{m}^\sigma$, and $\mathcal{E}$ is a real function. These NP projected quantities now need to be expanded in the two-parameter scheme to properly evaluate Eq. (2.90) and then to solve it.

Similarly, in EdGB gravity, using $\zeta_{\rm EdGB} \equiv 16\pi\alpha_{\rm EdGB}^2/M^4$ as the EdGB gravity expansion parameter and expanding Eq. (2.43), we find

$$\square^{(0,0)}\theta^{(1,1)} = -\frac{1}{16\pi^{\frac{1}{2}}} M^2 \mathcal{G}^{(0,1)} - \square^{(0,1)}\theta^{(1,0)} \,. \tag{2.93}$$

Now, the wave operator and the Gauss-Bonnet invariant must be projected onto the NP tetrad to find, once more that $\square\theta$ is given by Eq. (2.92) after replacing ϑ with θ, and the NP projected $\mathcal{G}$ is

$$\mathcal{G} = 8(3\Psi_2^2 - 4\Psi_1\Psi_3 + \Psi_0\Psi_4 + c.c.)\,. \tag{2.94}$$

Here, we also absorbed a factor of $(\zeta_{\text{EdGB}})^{1/2}$ into the expansion of θ. As before, to solve Eq. (2.93), one must now expand these NP projected quantities in our two-parameter scheme.

For both cases, we end up with a usual scalar field equation with source terms that depend on NP quantities at $\mathcal{O}(\zeta^0, \epsilon^1)$. Thus, we can first reconstruct these NP quantities and then use Eqs. (2.91), (2.92), and Eq. (2.94) to express the source terms in terms of $\Psi_0^{(0,1)}$ or $\Psi_4^{(0,1)}$. After this, one can either solve the scalar field equation and the modified Teukolsky equation concurrently [54–57], or use the order-reduction scheme to solve for the scalar field first and plug it into the modified Teukolsky equation.

To summarize, we have found the modified Teukolsky equation of Ψ_0 for any non-Ricci-flat and algebraically general background spacetime that can be treated as a linear perturbation of a Petrov type D spacetime, namely,

$$H_0^{\text{GR}}\Psi_0^{(1,1)} = \mathcal{S}_{\text{geo}}^{(1,1)} + \mathcal{S}^{(1,1)}\,, \tag{2.95}$$

where we have defined

$$\begin{aligned}
\mathcal{S}_{\text{geo}}^{(1,1)} &= \mathcal{S}_{0,\text{D}}^{(1,1)} + \mathcal{S}_{0,\text{non-D}}^{(1,1)} + \mathcal{S}_{1,\text{non-D}}^{(1,1)}\,,\\
\mathcal{S}_{0,\text{D}}^{(1,1)} &= -H_0^{(1,0)}\Psi_0^{(0,1)}\,,\\
\mathcal{S}_{0,\text{non-D}}^{(1,1)} &= -H_0^{(0,1)}\Psi_0^{(1,0)}\,,\\
\mathcal{S}_{1,\text{non-D}}^{(1,1)} &= H_1^{(0,1)}\Psi_1^{(1,0)}\,,
\end{aligned} \tag{2.96}$$

where H_0 and H_1 are defined in Eqs. (2.77), and $\mathcal{S}$ is defined in Eq. (2.61b). The equation for Ψ_4 can be derived by performing a GHP transformation on Eq. (2.95),

$$H_4^{\text{GR}}\Psi_4^{(1,1)} = \mathcal{T}_{\text{geo}}^{(1,1)} + \mathcal{T}^{(1,1)}\,, \tag{2.97}$$

where we have defined

$$\begin{aligned}
\mathcal{T}_{\text{geo}}^{(1,1)} &= \mathcal{T}_{4,\text{D}}^{(1,1)} + \mathcal{T}_{4,\text{non-D}}^{(1,1)} + \mathcal{T}_{3,\text{non-D}}^{(1,1)}\,,\\
\mathcal{T}_{4,\text{D}}^{(1,1)} &= -H_4^{(1,0)}\Psi_4^{(0,1)}\,,\\
\mathcal{T}_{4,\text{non-D}}^{(1,1)} &= -H_4^{(0,1)}\Psi_4^{(1,0)}\,,\\
\mathcal{T}_{3,\text{non-D}}^{(1,1)} &= H_3^{(0,1)}\Psi_3^{(1,0)}\,,
\end{aligned} \tag{2.98}$$

where $H_4^{\rm GR}$ is the Teukolsky operator for Ψ_4 in GR [see Eq. (2.65a)], and

$$H_3 \equiv \mathcal{E}_4 J_4 - \mathcal{E}_3 J_3 \,. \tag{2.99}$$

For the source terms $\mathcal{S}_{\rm geo}^{(1,1)}$ or $\mathcal{T}_{\rm geo}^{(1,1)}$, they can be computed from the modified background metric, the solutions to the Teukolsky equation in GR, and the reconstructed metric for GWs in GR. For $\mathcal{S}^{(1,1)}$ or $\mathcal{T}^{(1,1)}$, we may need to solve the equations of motion of other non-gravitational fields and evaluate the stress tensor. We have collected the full expressions of all the terms in the modified Teukolsky equation above in Appendix 2.9. In addition, the equations above are presented in an abstract form using NP symbols; they can be further simplified when considering perturbations of specific background spacetimes in specific coordinates and tetrads, e.g., Kerr in Boyer-Lindquist coordinates and in the Kinnersley tetrad.

2.6 Extension of Framework to higher order in the coupling

One important observation about Eqs. (2.95) and (2.97) is that they are in a very similar format to the second-order Teukolsky equation in GR [93]. In this section, we discuss the connection between the leading-order modified Teukolsky formalism and the second-order Teukolsky formalism in GR, which demonstrates that many techniques well-developed (in different contexts) in GR can be directly reused in modified gravity. Moreover, we show that our formalism can be generalized to higher orders [i.e., $O(\zeta^m, \epsilon^n)$, $m \geq 0$, $n \geq 1$], which is then a beyond GR extension of the higher-order Teukolsky formalism developed in [93] for GR. For a general discussion of non-linear multiple-parameter perturbation theory in relativity, we refer the reader to [130–133].

2.6.1 Connection to the second-order Teukolsky formalism in GR

Since Teukolsky presented the linear-order perturbation equation in [44], higher-order Teukolsky equations have been of great interest to the community. On the one hand, the inability of the linear-order Teukolsky equation to estimate the errors due to the use of a perturbative expansion makes the study of higher-order Teukolsky equations necessary [93]. On the other hand, higher-order perturbations enable the study of certain physical systems that cannot be studied sufficiently accurately within the linear-order scheme, such as head-on collisions in the close-limit approximation [93, 134, 135], self-force in EMRIs [92, 136–142], etc. On the observational side, recent studies of non-linearities that show up in numerical relativity suggest that second- and higher-order perturbations may be important for the analysis of gravitational wave data [143–145].

In [93], the Teukolsky equation was successfully extended to second- and higher-order, so let us show now that these higher-order equations are very similar to what we obtained in this paper. Comparing our Eq. (2.97) to the vacuum case ($T^{\mu\nu}_{\rm matter} = 0$) of Eqs. (7)-(10) in [93], these equations take a very similar format if we replace all the terms proportional to $h^{(0,1)}h^{(1,0)}$ with $h^{(0,1)}h^{(0,1)}$ and set the source term due to $\mathcal{L}_{\rm bGR}$ in Eq. (2.35) to zero, $\mathcal{T}^{(1,1)} = 0$. More precisely, if we follow the approach in this work to derive the Teukolsky equation at $O(\zeta^0, \epsilon^2)$, we find

$$H_4^{\rm GR}\Psi_4^{(0,2)} = \mathcal{T}_{\rm geo}^{(0,2)}\,, \quad \mathcal{T}_{\rm geo}^{(0,2)} = -H_4^{(0,1)}\Psi_4^{(0,1)}\,. \tag{2.100}$$

These are the equations that ought to be compared to the work in GR at second order in perturbation theory.

Equation (2.100) and Eqs. (7)-(10) from [93] are similar in form, as expected in perturbation theory, where the principal part of the equation remains unchanged at each order and is driven by lower order perturbations. Nonetheless, our Eq. (2.100) is simpler. First, there are no terms in $\Psi_3^{(0,1)}$ since they are removed by our gauge choice in Eq. (2.71). Second, there are no terms that depend on $\lambda^{(0,1)}$ and $\nu^{(0,1)}$, since λ and ν, just like κ and σ, are eliminated from the equations from the beginning, as shown in Sec. 2.5.3.1. To compare Eq. (2.100) with Eqs. (7)-(10) from [93], we choose the same gauge given in Eq. (2.71). In this case, $\Psi_3^{(0,1)} = 0$, and one can solve for $\lambda^{(0,1)}$ and $\nu^{(0,1)}$ in terms of $\Psi_4^{(0,1)}$ [40], so all the $\lambda^{(0,1)}$ and $\nu^{(0,1)}$ related terms become additional operators acting on $\Psi_4^{(0,1)}$ in Eq. (2.100). In Appendix 2.10, we have shown this consistency explicitly following this prescription.

Further, we notice that Eq. (2.100) and Eqs. (2.97)-(2.98), are also similar. When studying Petrov type I spacetimes in modified gravity, we did not make any assumptions about what NP quantities vanish at $O(\zeta^1, \epsilon^0)$ to avoid sabotaging our perturbation scheme, as discussed in Sec. 2.5.1. For the second-order Teukolsky formalism in GR, the stationary Petrov type I background at $O(\zeta^1, \epsilon^0)$ is replaced by the "dynamical background," driven by GW perturbations at $O(\zeta^0, \epsilon^1)$, where most NP quantities also do not vanish. Due to this connection, many challenges shared by these two situations have been solved in the second-order Teukolsky formalism in GR, such as metric reconstruction at $O(\zeta^0, \epsilon^1)$. The success of applying the second-order Teukolsky formalism to the study of self-force in [92, 136–142] strongly suggests that our modified Teukolsky formalism is feasible numerically.

Despite these similarities, there are also differences between these two efforts. One major difference is the presence of extra non-metric fields in class A beyond GR theories. Unlike in GR, even without matter, one needs to evaluate the effective stress-energy tensor driven by these intrinsic extra fields, and thus, solve their equations of motion concurrently. Nonetheless, as discussed in Sec. 2.5.3.5, this issue was already dealt with in the studies of slowly rotating BHs using metric perturbations in dCS [54, 55] and EdGB [56, 57]. Besides the issue of extra fields, one also has to be careful when constructing the background tetrad in these non-Ricci-flat backgrounds, as shown in Sec. 2.3.2.

2.6.2 Modified Teukolsky formalism beyond $O(\zeta^1, \epsilon^1)$

As illustrated in the previous section, second- and higher-order BH perturbation theory in GR has been of great interest due to its importance in constraining the first-order perturbations and its need when dealing with certain physical systems. In the case of modified gravity, one does not just have to deal with non-linear terms in ϵ, but also with non-linear terms in the dimensionless coupling constant ζ. When the beyond GR theory itself is known at higher order, these higher-order corrections due to modified gravity might be interesting, since there might be non-linear phenomena that is not described by the linear theory. For these reasons, we follow [93] to extend our formalism beyond $O(\zeta^1, \epsilon^1)$.

Let us consider some perturbations at $O(\zeta^M, \epsilon^N)$, $M \geq 0$, $N \geq 1$. First, we need to find a tetrad with terms up to $O(\zeta^M, \epsilon^N)$, such that the orthogonality condition in Eq. (2.5) is satisfied while our perturbation scheme is preserved, similar to what we did in Sec. 2.3.2. For $1 \leq m \leq M$, expanding the correction to the tetrad at $O(\zeta^m, \epsilon^0)$, we have

$$\delta e_{a\mu}^{(m,0)} = A_{ab}^{(m,0)} \delta e_{b\mu}^{(0,0)} \,. \tag{2.101}$$

Through induction, one can easily show that we can solve for all $A_{ab}^{(m,0)}$ iteratively, where $1 \leq m \leq M$. Let us assume $\delta e_{b\mu}^{(1,0)}, \cdots, \delta e_{b\mu}^{(M-1,0)}$ are known, and the base case $\delta e_{b\mu}^{(1,0)}$ was shown in Sec. 2.3.2. We also assume that the corrections to the background metric $h_{\mu\nu}^{(1,0)}, \cdots, h_{\mu\nu}^{(M,0)}$ are known. Then, to satisfy Eq. (2.5), we need

$$\left(e_{a\mu}^{(0,0)} + \sum_{m=1}^{M} \zeta^m \delta e_{a\mu}^{(m,0)}\right)\left(e_{b\nu}^{(0,0)} + \sum_{m=1}^{M} \zeta^m \delta e_{b\nu}^{(m,0)}\right)\left(g^{\mu\nu(0,0)} + \sum_{m=1}^{M} \zeta^m h^{\mu\nu(m,0)}\right) = \eta_{ab} \,. \tag{2.102}$$

For convenience, let us introduce

$$\mathcal{U}^{(M,0)} \equiv \sum_{\substack{i+j+k=M,\\ M>i,j,k>0}} \delta e_{a\mu}^{(i,0)} \delta e_{b\nu}^{(j,0)} h^{\mu\nu(k,0)} \,, \tag{2.103}$$

where every term on the right-hand side is assumed to be known, and $\mathcal{U}^{(0,0)} = 0$ when $M = 1$. Then, following the same procedure as in Sec. 2.3.2, at $O(\zeta^M, \epsilon^0)$ we have

$$2A^{(M,0)}_{(ab)} = -h^{(M,0)}_{ab} - \mathcal{U}^{(M,0)}\,, \tag{2.104}$$

where $\mathcal{U}^{(M,0)}$ contains $A^{(m,0)}_{(ab)}$, with $1 \leq m < M$ solved in the previous steps. If we pick the same gauge as in Sec. 2.3.2 to set $A^{(M,0)}_{[ab]} = 0$, then we find

$$A^{(M,0)}_{ab} = -\frac{1}{2}\left(h^{(M,0)}_{ab} + \mathcal{U}^{(M,0)}\right)\,. \tag{2.105}$$

Thus, this proves that one can iteratively find higher-order corrections to the background tetrad, such that the orthogonality condition in Eq. (2.5) is preserved.

Next, let us consider tetrad rotations. Inspecting the rotations we performed in Eqs. (2.30) and (2.72), one can immediately notice that, under any type II rotation [cf. Eq. (2.126b)] with rotation parameter $b^{(m,n)}$ at $O(\zeta^m, \epsilon^n)$ with $m \geq 0$, $n \geq 1$, the Weyl scalars at $O(\zeta^m, \epsilon^n)$ transform as

$$\begin{aligned}
\Psi_0^{(m,n)} &\to \Psi_0^{(m,n)} + 4b^{(m,n)}\Psi_1^{(0,0)}\,,\\
\Psi_1^{(m,n)} &\to \Psi_1^{(m,n)} + 3b^{(m,n)}\Psi_2^{(0,0)}\,,\\
\Psi_2^{(m,n)} &\to \Psi_2^{(m,n)} + 2b^{(m,n)}\Psi_3^{(0,0)}\,,\\
\Psi_3^{(m,n)} &\to \Psi_3^{(m,n)} + b^{(m,n)}\Psi_4^{(0,0)}\,,\\
\Psi_4^{(m,n)} &\to \Psi_4^{(m,n)}\,,
\end{aligned} \tag{2.106}$$

where any terms beyond $O(\zeta^m, \epsilon^n)$ are dropped. Since the background at $O(\zeta^0, \epsilon^0)$ is Petrov type D, where $\Psi^{(0,0)}_{0,1,3,4} = 0$, if we pick $b^{(m,n)} = -\Psi_1^{(m,n)}/(3\Psi_2^{(0,0)})$, then

$$\Psi_1^{(m,n)} \to 0\,, \quad \Psi^{(m,n)}_{0,2,3,4} \to \Psi^{(m,n)}_{0,2,3,4}\,. \tag{2.107}$$

Similarly, by performing a type I rotation with the rotation parameter $a^{(m,n)} = -\left[\Psi_3^{(m,n)}/(3\Psi_2^{(0,0)})\right]^*$, one can remove $\Psi_3^{(m,n)}$.

One may worry that a rotation at $O(\zeta^{m_1}, \epsilon^{n_1})$ will affect the Weyl scalars at $O(\zeta^{m_2}, \epsilon^{n_2})$, where $m_2 > m_1$, $n_2 > n_1$, since many Weyl scalars at $O(\zeta^{m_2-m_1}, \epsilon^{n_2-n_1})$ might be nonzero. However, this problem can be avoided if one performs these rotations systematically from lower order to higher order. For example, one may consider the following procedures:

1. Perform tetrad rotations step by step from $O(\zeta^0, \epsilon^1)$ to $O(\zeta^M, \epsilon^1)$ to remove $(\Psi^{(0,1)}_{1,3}, \cdots, \Psi^{(M,1)}_{1,3})$.

2. Next, perform tetrad rotations step by step from $O(\zeta^0, \epsilon^2)$ to $O(\zeta^M, \epsilon^2)$ to remove $(\Psi_{1,3}^{(0,2)}, \cdots, \Psi_{1,3}^{(M,2)})$.

3. $\cdots$

4. At the N-th step, perform tetrad rotations step by step from $O(\zeta^0, \epsilon^N)$ to $O(\zeta^M, \epsilon^N)$ to remove $(\Psi_{1,3}^{(0,N)}, \cdots, \Psi_{1,3}^{(M,N)})$.

Following this sequence, any higher-order modifications to $\Psi_{1,3}$ due to lower-order rotations are removed at the corresponding step, and higher-order rotations do not affect the lower-order $\Psi_{1,3}$, which have been set to 0. Thus, for any perturbation at $O(\zeta^M, \epsilon^N)$ with $M \geq 0$, $N \geq 1$, we can consistently set

$$\Psi_{1,3}^{(m,n)} = 0\,, \quad 0 \leq m \leq M\,,\ 1 \leq n \leq N\,. \tag{2.108}$$

Now, one can directly make an expansion of Eq. (2.76) similar to what we did at $O(\zeta^1, \epsilon^1)$ in Sec. 2.5.3. One direct consequence of the tetrad rotations above is that we can drop all $\Psi_1^{(m,n)}$, with $m \geq 0$, $n \geq 1$ [e.g., Eq. (2.108)], so there is only the stationary part of Ψ_1 contributing to Eq. (2.76). Then, following the same procedures as in Sec. 2.5.3, for perturbations at $O(\zeta^M, \epsilon^N)$, we find

$$H_0^{\rm GR}\Psi_0^{(M,N)} = \mathcal{S}_{\rm geo}^{(M,N)} + \mathcal{S}^{(M,N)}\,, \tag{2.109}$$

where

$$\begin{aligned}
\mathcal{S}_{\rm geo}^{(M,N)} &= \mathcal{S}_{0,I}^{(M,N)} + \mathcal{S}_{0,II}^{(M,N)} + \mathcal{S}_{1}^{(M,N)}\,,\\
\mathcal{S}_{0,I}^{(M,N)} &= \sum_{(m,n)=(0,1)}^{(m,n)<(M,N)} -H_0^{(M-m,N-n)}\Psi_0^{(m,n)}\,,\\
\mathcal{S}_{0,II}^{(M,N)} &= \sum_{m=1}^{M} -H_0^{(M-m,N)}\Psi_0^{(m,0)}\,,\\
\mathcal{S}_{1}^{(M,N)} &= \sum_{m=1}^{M} -H_1^{(M-m,N)}\Psi_1^{(m,0)}\,,\\
\mathcal{S}^{(M,N)} &= \sum_{m=1,n=0}^{(m,n)\leq(M,N)} \left[\mathcal{E}_2^{(M-m,N-n)}S_2^{(m,n)} - \mathcal{E}_1^{(M-m,N-n)}S_1^{(m,n)}\right]\,,
\end{aligned} \tag{2.110}$$

and where $(m,n) < (M,N)$ means $m \leq M, n < N$ or $m < M, n \leq N$. The equation for Ψ_4 can be found from the GHP transformation of Eqs. (2.109)-(2.110). For the case of higher-order perturbations in GR, $\zeta = 0$, so one can simply set

$S_{0,II}^{(M,N)} = S_{1}^{(M,N)} = \mathcal{S}^{(M,N)} = 0$, where the sum starts from $O(\zeta^1)$. As discussed in Sec. 2.6.1 and shown in detail in Appendix 2.10, if one chooses the gauge in which $\Psi_{1,3}^{(0,n)} = 0$, with $1 \leq n \leq N$, then Eqs. (7)-(10) of [93] are the same as the GHP transformation of Eqs. (2.109)-(2.110). Thus, one can treat this higher-order extension of our formalism as a modified-gravity generalization of the higher-order Teukolsky formalism in [93].

2.6.3 Potential challenges

In the previous subsection, we have successfully extended our formalism to higher order in both ϵ and ζ. In this case, all NP quantities are decoupled at each perturbative order, and Weyl scalars $\Psi_{0,4}$ can be solved, given their solutions at lower orders. This shows that similar to any perturbation theory problem (e.g., solving the hydrogen atom in quantum mechanics), by working out the leading-order perturbation theory, one can iterate it to solve for higher-order perturbations. On the other hand, this procedure also inherits the same challenges of any perturbation theory solution. For example, the source terms made up of lower-order perturbations become complicated at very high order. However, developing a non-perturbative approach is outside the scope of this work, and one may have to rely on numerical relativity in the end. In this subsection, we will discuss other challenges and potential solutions when applying this higher-order modified Teukolsky formalism to the first few orders beyond $O(\zeta^1, \epsilon^1)$ [e.g., $O(\zeta^2, \epsilon^1)$ or $O(\zeta^1, \epsilon^2)$], where perturbation theory is still tractable.

The major challenge of this higher-order modified Teukolsky formalism is the need of metric reconstruction in non-Ricci-flat backgrounds, since we need to evaluate NP quantities at $O(\zeta^m, \epsilon^n)$ with $m > 0, n \geq 1$ in general. For example, at $O(\zeta^2, \epsilon^1)$ or $O(\zeta^1, \epsilon^2)$, one needs to reconstruct the perturbed metric at $O(\zeta^1, \epsilon^1)$. At this order, we have taken advantage of the fact that the metric reconstruction procedure for $O(\zeta^0, \epsilon^1)$ GW perturbations in GR is well developed [40, 90–92, 128, 129]. However, for general perturbations at $O(\zeta^m, \epsilon^n)$, the metric reconstruction procedure is unknown. Moreover, when $m > 0$, the correction to the Einstein-Hilbert action generates some effective stress-energy tensor [see Sec. 2.3.1], so the traceless condition $g^{\mu\nu} h_{\mu\nu} = 0$ in the radiation gauge used in these metric reconstruction procedures with a Hertz potential [90, 91, 128, 129] is violated.

However, this issue is not just present in our modified Teukolsky formalism, but also in the higher-order Teukolsky formalism in GR, since lower-order perturbations become effective sources in the higher-order version of the Teukolsky equation. References [142, 146, 147] have shown that one can extend the Hertz potential approach by adding certain correction fields to the metric perturbation constructed from a usual Hertz potential. These correction fields can be obtained from certain decoupled ordinary differential equations, sourced by the effective stress-energy tensor. These references have proven that this procedure works for any smooth, compactly-supported source, which is unfortunately not satisfied by sources driven by non-linear couplings of gravitational fields. Thus, to apply their formalism to our non-linear Teukolsky formalism, additional work would have to be done. Besides an extension of the Hertz potential approach, there are also methods that do not rely on the radiation gauge, such as the approach of solving the remaining NP equations directly [40, 92, 148]. This approach has been implemented for vacuum Petrov type D spacetimes [92, 148], and it is worth exploring whether one can extend it to non-vacuum backgrounds.

Another challenge is the presence of extra fields. For the class A beyond GR theories mentioned in Sec. 2.3.1, one has to solve the coupled equations of metric fields and extra fields at each perturbed order. In terms of solving the coupled equation itself, this will not be a huge challenge since similar problems have been solved in these approaches using metric perturbations [54, 55]. There might be numerical challenges when going to very high order since the source terms are complicated non-linear couplings of reconstructed NP quantities with extra fields at lower orders, which need to be solved together with the modified Teukolsky equation. Nonetheless, this is merely an unavoidable consequence of perturbation theory.

To summarize, the connection of our work to the second-order Teukolsky formalism in GR demonstrates the feasibility of the approach presented in this work. When applying our formalism to specific modified gravity theories, one should not expect more difficulties than when solving the second-order Teukolsky equation in GR, which has been widely studied. On the other hand, the formalism developed in this work aims to incorporate corrections from modified gravity, so it contains features unique to modified gravity and cannot be directly obtained from the second-order Teukolsky formalism in GR. The extension of our formalism to higher order naturally generalizes the higher-order Teukolsky formalism in [93] from GR to modified gravity. As a consistency check, we have studied the limiting case of $\zeta \to 0$, compared the results to those obtained in [93], and presented these concrete comparisons in Appendix 2.10.

2.7 Discussions

In this work, we extended the Teukolsky formalism to non-Ricci-flat, Petrov type D BH backgrounds, as well as to non-Ricci-flat, Petrov type I BH backgrounds that can be treated as a linear perturbation of a Petrov type D background. We began by presenting a brief review of the derivation of the Teukolsky equation for a Ricci-flat and Petrov type D background in GR via the original approach in Teukolsky's paper [44], as well as using an approach proposed by Chandrasekhar [40]. These two approaches differ in the method adopted to eliminate the Ψ_1 and Ψ_3 dependence from the two Bianchi identities and one Ricci identity [see e.g., Eq. (2.13)]. Teukolsky's approach makes use of additional Bianchi identities to obtain a commutation relation to eliminate Ψ_1 and Ψ_3. Chandrasekhar's approach uses the available gauge freedom to make a convenient gauge choice that eliminates Ψ_1 and Ψ_3 directly. One can then solve these equations to obtain a single decoupled differential equation for the perturbed Weyl scalars Ψ_0 and Ψ_4.

We first extended both approaches to obtain the modified Teukolsky equation in a generic modified gravity theory that allows BH backgrounds to be non-Ricci-flat and Petrov type D backgrounds. Since the background is now non-Ricci-flat, we have additional non-vanishing background NP quantities. We then used the two approaches described above to obtain decoupled differential equations for the perturbed Weyl scalars Ψ_0 and Ψ_4. We found that for non-Ricci-flat, Petrov type D BH backgrounds in modified gravity, the master equations for curvature perturbations acquire a source term [see e.g., Eqs. (2.60) and (2.63)]. In order to evaluate these source terms, we found that one needs to perform metric reconstruction from the GR curvature perturbations [40, 90–92, 128, 129] [i.e., to $O(\zeta^0, \epsilon^1)$, where ζ labels the order of the GR deformation, and ϵ labels the order of the dynamic GW perturbation from the stationary background]. We showed that both the Teukolsky's approach and the Chandrasekhar's approach lead to the same modified Teukolsky equation, but the latter is algebraically simpler and thus more convenient.

The algebraic simplicity of Chandrasekhar's approach makes this method ideal for the study of curvature perturbations of BH backgrounds that are non-Ricci-flat and Petrov type I. We thus extended Chandrasekhar's approach to such BH backgrounds. The non-vanishing of the background NP Ricci scalars, the background NP spin coefficients, and the background Weyl scalars Ψ_1,Ψ_2, and Ψ_3 forces the NP equations [see e.g., Eq. (2.13)] to have additional non-vanishing NP quantities. However, when one requires the BH background to be a perturbation of a non-Ricci-flat, Petrov type D BH background at leading order in the GR deformation, the equations do decouple. This is achieved by rotating the tetrad such that the perturbed Weyl scalars $\Psi_1^{(1,1)}$ and $\Psi_3^{(1,1)}$ (at linear order in both the non-GR expansion parameter and the GW expansion parameter) vanish. With this, we then derived a single decoupled differential equation for $\Psi_0^{(1,1)}$ and $\Psi_4^{(1,1)}$.

The modified Teukolsky equation obtained in this way has the structure of the traditional Teukolsky equation but with certain source terms. The differential operator on the left-hand side of the modified Teukolsky equation acts on the perturbed Weyl scalar $\Psi_{0,4}$, and it has a functional form that is similar to the Teukolsky operators in GR [44]. The source terms on the right-hand side of the modified Teukolsky equation arise either because of either (i) modifications to the stationary BH background spacetime, or (ii) additional stress-tensor terms due to corrections to the Einstein-Hilbert action.

The first type of source terms comes from the homogeneous part of certain Bianchi and Ricci identities [see e.g., Eqs. (2.13)]. Some of these source terms can be directly evaluated using the modified background metric and the solution to the Teukolsky equation in GR. The rest are couplings of $O(\zeta^1, \epsilon^0)$ corrections to the Weyl scalars with the $O(\zeta^0, \epsilon^1)$ corrections to the metric due to GWs in GR. Thus, in order to evaluate these source terms, we need to reconstruct the metric for the curvature perturbations in GR [40, 90–92, 128, 129], just as in the case of non-Ricci-flat, Petrov type D backgrounds.

The second type of source terms comes from the stress tensor due to corrections to the Einstein-Hilbert action. We have classified the modified gravity theories into two classes based on the presence or absence of extra non-gravitational dynamical fields. Class A beyond GR theories can have couplings to other dynamical scalar, vector or tensor fields (as is the case in dCS gravity [65, 66], EdGB gravity [106, 113, 114], Horndeski theory [149], scalar-tensor theories [150], $f(R)$ gravity [121, 127], Einstein-Aether theory [151], and bi-gravity [152]). Class B beyond GR theories depend only on the gravitational field and there are no additional dynamical fields (as is the case in certain effective field theory extensions of GR, such as higher-derivative gravity [116–119]). For class B beyond GR theories, these source terms can be directly evaluated with the background metric and the reconstructed metric. For class A beyond GR theories, one must solve the equations of motion for these extra fields to evaluate the stress tensor, and this can only be done on a theory-per-theory basis. The case-by-case treatment of these extra field equations is left to future work.

The major goal of this work was to simplify the perturbed gravitational equations in general for modified gravity theories that admit non-Ricci-flat and Petrov type I or Petrov type D BH backgrounds such that all the curvature perturbations are packed into two fundamental variables Ψ_0 and Ψ_4. With this at hand, one can now in principle evaluate all source terms and separate the modified Teukolsky equation into radial and angular parts to solve for the QNM frequencies of perturbed BHs in modified gravity. It is important to realize that this was not possible until this work due to the inherently complicated nature of the perturbed field equations when working with metric perturbations. Indeed, up until now, the QNM spectrum of perturbed BHs in modified gravity had only been studied for non-rotating BHs [e.g., in dCS gravity [50, 51, 58], EdGB theory [52, 53], Einstein-Aether theory [70–74], higher-derivative gravity (quadratic [75], cubic [76], and more generic [77, 78]), and Horndeski gravity [79]] or for slowly rotating BHs (e.g., in EdGB theory [56], dCS gravity [54, 55], and higher-derivative gravity [80, 81]). The only study of QNM perturbations of rotating BHs was carried out in dCS gravity from numerical relativity simulations of BH mergers, but these suffer from secularly-growing uncontrolled remainders [82, 83].

Our work creates a new path to directly calculate the corrections to the QNM frequencies of perturbed BHs with arbitrary spin in modified gravity and, more generally, any background spacetime that can be treated as a linear perturbation of a Petrov type D spacetime. One of our next major goals is to do a case-by-case study of all these well-motivated modified theories, using the formalism developed here, to then use GW observations to constrain these theories. For dCS gravity, we would like to compare the QNM frequencies obtained for arbitrarily rotating BHs to those found in the slow-rotation approximation to linear order in spin [54], as well as others that use metric perturbations [50, 51, 54, 55, 58] and numerical relativity [82, 83, 125, 126].

By extending the Teukolsky formalism, we have also laid the foundation for studying gravitational perturbations other than QNMs around BHs in modified gravity. For example, the Teukolsky formalism has been applied to compute gravitational waveforms and energy/angular momentum fluxes sourced by a point particle orbiting around a BH in extreme mass-ratio binary inspirals (EMRI) [153–159]. The same procedure has been applied to a few modified gravity theories, e.g., in scalar-tensor theories [160] and for a spinning horizonless compact object [161], where the Teukolsky formalism in GR can be directly applied. With this extended Teukolsky formalism, we are now able to study EMRIs in a much wider class of modified gravity theories. These results can also be compared with those obtained using post-Newtoninan studies of EMRIs in GR and modified gravity [58, 162–167].

Another example is the break of isospectrality (where even and odd parity modes have the same QNM frequencies) in certain modified gravity theories, e.g., dCS gravity [50, 51, 54, 55, 58], EdGB gravity [52, 53, 56], and higher-derivative gravity [81]. The study of isospectrality is mostly done with metric perturbations since the Zerilli-Moncrief and the Regge-Wheeler functions naturally divide the metric perturbations into even and odd parity sectors [41, 42]. For BHs with arbitrary spin, there are no known extensions of the Zerilli-Moncrief and the Regge-Wheeler functions, so we may have to use NP quantities in this extended Teukolsky formalism to study parity breaking. Since Teukolsky equation does not naturally classify its solutions into different parities, we will first need to understand better what even and odd parity modes mean in the Teukolsky formalism and their connections to the Zerilli-Moncrief and Regge-Wheeler functions even in GR. This, and much more, is now possible thanks to the derivation of a master evolution equation for curvature perturbations in modified gravity.

In this work, we have focused on the formalism up to leading order in modified gravity corrections, i.e., at $O(\zeta)$. This is mainly because the theories we have discussed in Sec. 2.3.1 are only presented to leading order in corrections since these are treated in an effective field theory approach, considering small deformations from GR. However, one can consider a modified theory of gravity different from the examples shown in Sec. 2.3.1, where one can look at higher-order deformations from GR. As discussed in Sec. 2.6, our leading-order formalism can be extended to higher order [$O(\zeta^m, \epsilon^n)$, $m \geq 0$, $n \geq 1$] by iterating the perturbation scheme in Sec. 2.3.2 and the procedure of finding the master equation in Sec. 2.5. However, utmost care needs to be taken when considering theories at higher than leading-order corrections to GR, as such theory may admit ghost modes [103]. Additionally, this formalism relies on the approximation that the theories mentioned in Sec. 2.3.1 are an effective field theory of GR. Therefore, the spacetimes we can probe using this formalism cannot deviate too much from their GR counterparts.

To present the feasibility of our formalism extending the Teukolsky equation to non-Ricci-flat Petrov type D and Petrov type I spacetimes, our collaboration is already working on a series of calculations. The first in this planned series of works is the study of perturbations of a non-Ricci-flat vacuum Petrov type D BH spacetime representing a slowly rotating BH to leading order in spin in dCS gravity [168]. In [168], we will present the calculation of the perturbed field equations. These field equations, as expected from the results of this paper, are sourced equations which we will compute in the null basis. We will then implement the necessary metric reconstruction procedures and tetrad rotations. In the last step, we will convert all NP quantities to a coordinate basis to separate the master equation into radial and angular ordinary differential equations with couplings between the gravitational and scalar sectors. Then, in a follow-up work [169], we will make use of the EVP method to calculate the QNM frequencies of these BH spacetimes and verify our results with previously obtained frequencies computed in the slow-rotation limit [54, 55]. We will then extend these calculations to arbitrarily spinning BHs in dCS gravity, which are described by non-Ricci-flat, vacuum, Petrov type I BH metrics in [170]. This problem is more challenging due to the presence of additional theory-independent source terms (i.e., $\mathcal{S}_{\rm geo}^{(1,1)}$), which need metric reconstruction (e.g., $\mathcal{S}_{0,\text{non-D}}^{(1,1)}$). However, it is much simpler to evaluate these additional terms than the theory-dependent source terms (i.e., $\mathcal{S}^{(1,1)}$) coupled to the pseudoscalar field, which we would have already computed in our previous work [168] on Petrov type D BHs in dCS gravity mentioned above. We expect that through these extensions, we will acquire a deep knowledge of QNMs in modified gravity.

Note added after completion: While writing up our analysis, we became aware of an equivalent and independent analysis of decoupled equations for gravitational perturbations around BHs in modified gravity [171]. Instead of using the NP formalism, Ref. [171] focuses mostly on the Einstein equations and shows how to partially decouple them, following the order-reduction scheme in [125]. To make the equations of gravitational perturbations separable, Ref. [171] uses Wald's formalism to project the Einstein equations onto a (modified) Teukolsky equation [172]. Although our work is independent of that of Ref. [171], there are similarities in the general format of the final master equation. For example, both approaches require metric reconstruction of GWs in GR. Reference [171] also presents a direct derivation of the modified Teukolsky equation following Teukolsky's original approach [44]. Our work greatly simplifies the NP approach through the use of gauge freedom, following Chandrasekhar's approach [40]. These two independent studies can be used to validate results when computing the shift of QNM frequencies in certain modified gravity theories.

2.8 Appendix: NP formalism (continued)

In Sec. 2.2.1, we have presented the orthogonality relations for the tetrad basis vectors in NP formalism. One can further compactly express the relation in Eq. (2.5) as $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, where

$$e^\mu_m = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu) \, ,$$
$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \tag{2.111}$$

where we have used Latin indices to denote the null tetrad indices whereas the Greek indices are the tensor indices. Further, using the metric and the null tetrad, we can define the quantity known as Ricci rotation coefficients, which are similar to Christoffel symbols. These are complex quantities in nature and defined as

$$\gamma_{cab} = e_{a\mu;\nu} e^\mu_c e^\nu_b \tag{2.112}$$

with the symmetry,

$$\gamma_{cab} = -\gamma_{acb} \, . \tag{2.113}$$

The commutation relations of the intrinsic derivatives are related to the Ricci rotation coefficients by

$$\left[e^\mu_a, e^\mu_b\right] = \left(\gamma_{cba} - \gamma_{cab}\right) e^{c\mu} \, . \tag{2.114}$$

The tetrad components of the Riemann tensor can then be defined by

$$R_{abcd} = R_{\alpha\beta\gamma\delta} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d \, . \tag{2.115}$$

Using a form of Eq. (2.112), the Riemann tensor can also be expressed in terms of the Ricci rotation coefficients,

$$R_{abcd} = -\gamma_{abc,d} + \gamma_{abd,c} + \gamma_{abf}\left(\gamma^f{}_{cd} - \gamma^f{}_{dc}\right) + \gamma^f{}_{ac}\gamma_{bfd} - \gamma^f{}_{ad}\gamma_{bfc} \, , \tag{2.116}$$

where $\gamma_{abc,d} \equiv \gamma_{abc,\mu} e^\mu_d$. The relationship among the Riemann tensor, Weyl tensor $C_{\alpha\beta\gamma\delta}$, and Ricci tensor $R_{\alpha\beta}$ remains unchanged in tetrad notation.

$$\begin{aligned} R_{abcd} = C_{abcd} &- \frac{1}{2}\left(\eta_{ac}R_{bd} - \eta_{bc}R_{ad} - \eta_{ad}R_{bc} + \eta_{bd}R_{ac}\right) \\ &+ \frac{1}{6}\left(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}\right) R \, . \end{aligned} \tag{2.117}$$

In tetrad notation, Bianchi identities ($R_{\alpha\beta[\gamma\delta;\mu]} = 0$) take the form,

$$R_{ab[cd;f]} = \frac{1}{6}\sum_{[cdf]}\big[R_{abcd,f} - \eta^{nm}\left(\gamma_{naf}R_{mbcd}\right.$$
$$\left. +\gamma_{nbf}R_{amcd} + \gamma_{ncf}R_{abmd} + \gamma_{ndf}R_{abcm}\right)\big]\ . \tag{2.118}$$

2.8.1 NP quantities

With the formalism developed above, Newman and Penrose defined twelve complex functions known as the spin coefficients which can be defined in terms of the Ricci rotation coefficients (and thus the tetrad). The spin coefficients are as follows:

$$\begin{aligned}
\kappa &= \gamma_{131} = l_{\mu;\nu}m^{\mu}l^{\nu}\,,\\
\pi &= -\gamma_{241} = -n_{\mu;\nu}\bar{m}^{\mu}l^{\nu}\,,\\
\varepsilon &= \frac{1}{2}(\gamma_{121} - \gamma_{341}) = \frac{1}{2}(l_{\mu;\nu}n^{\mu}l^{\nu} - m_{\mu;\nu}\bar{m}^{\mu}l^{\nu})\,,\\
\rho &= \gamma_{134} = l_{\mu;\nu}m^{\mu}\bar{m}^{\nu}\,,\\
\lambda &= -\gamma_{244} = -n_{\mu;\nu}\bar{m}^{\mu}\bar{m}^{\nu}\,,\\
\alpha &= \frac{1}{2}(\gamma_{124} - \gamma_{344}) = \frac{1}{2}(l_{\mu;\nu}n^{\mu}\bar{m}^{\nu} - m_{\mu;\nu}\bar{m}^{\mu}\bar{m}^{\nu})\,,\\
\sigma &= \gamma_{133} = l_{\mu;\nu}m^{\mu}m^{\nu}\,,\\
\mu &= -\gamma_{243} = -n_{\mu;\nu}\bar{m}^{\mu}m^{\nu}\,,\\
\beta &= \frac{1}{2}(\gamma_{123} - \gamma_{343}) = \frac{1}{2}(l_{\mu;\nu}n^{\mu}m^{\nu} - m_{\mu;\nu}\bar{m}^{\mu}m^{\nu})\,,\\
\nu &= -\gamma_{242} = -n_{\mu;\nu}\bar{m}^{\mu}n^{\nu}\,,\\
\gamma &= \frac{1}{2}(\gamma_{122} - \gamma_{342}) = \frac{1}{2}(l_{\mu;\nu}n^{\mu}n^{\nu} - m_{\mu;\nu}\bar{m}^{\mu}n^{\nu})\,,\\
\tau &= \gamma_{132} = l_{\mu;\nu}m^{\mu}n^{\nu}\,.
\end{aligned} \tag{2.119}$$

Using Eq. (2.117), one can decompose the Riemann tensor into the Weyl tensor, completely determined by 5 complex Weyl scalars,

$$\begin{aligned}
\Psi_0 &= C_{1313} = C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}l^{\gamma}m^{\delta}\,,\\
\Psi_1 &= C_{1213} = C_{\alpha\beta\gamma\delta}l^{\alpha}n^{\beta}l^{\gamma}m^{\delta}\,,\\
\Psi_2 &= C_{1342} = C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}\bar{m}^{\gamma}n^{\delta}\,,\\
\Psi_3 &= C_{1242} = C_{\alpha\beta\gamma\delta}l^{\alpha}n^{\beta}\bar{m}^{\gamma}n^{\delta}\,,\\
\Psi_4 &= C_{2424} = C_{\alpha\beta\gamma\delta}n^{\alpha}\bar{m}^{\beta}n^{\gamma}\bar{m}^{\delta}\,,
\end{aligned} \tag{2.120}$$

the Ricci tensor, and the Ricci scalar, characterized by 10 NP Ricci scalars,

$$
\begin{aligned}
\Phi_{00} &= \frac{1}{2}R_{11} = \frac{1}{2}R_{\mu\nu}l^{\mu}l^{\nu}\,, \\
\Phi_{01} &= \frac{1}{2}R_{13} = \frac{1}{2}R_{\mu\nu}l^{\mu}m^{\nu}\,,\ \Phi_{10} = \frac{1}{2}R_{14} = \frac{1}{2}R_{\mu\nu}l^{\mu}\bar{m}^{\nu}\,, \\
\Phi_{11} &= \frac{1}{4}(R_{12}+R_{34}) = \frac{1}{2}R_{\mu\nu}(l^{\mu}n^{\nu}+m^{\mu}\bar{m}^{\nu})\,, \\
\Phi_{02} &= \frac{1}{2}R_{33} = \frac{1}{2}R_{\mu\nu}m^{\mu}m^{\nu}\,,\ \Phi_{12} = \frac{1}{2}R_{23} = \frac{1}{2}R_{\mu\nu}n^{\mu}m^{\nu}\,, \\
\Phi_{20} &= \frac{1}{2}R_{44} = \frac{1}{2}R_{\mu\nu}\bar{m}^{\mu}\bar{m}^{\nu}\,,\ \Phi_{21} = \frac{1}{2}R_{24} = \frac{1}{2}R_{\mu\nu}n^{\mu}\bar{m}^{\nu}\,, \\
\Phi_{22} &= \frac{1}{2}R_{22} = \frac{1}{2}R_{\mu\nu}n^{\mu}n^{\nu}\,,\ \Lambda = R/24\,.
\end{aligned}
\tag{2.121}
$$

2.8.2 NP equations

Using the NP quantities defined above, one can consider appropriate linear combinations of Eq. (2.116) and rewrite the equations in terms of the NP quantities. The resulting equations are called Ricci identities in [40] and given by

$$D\rho - \delta^*\kappa = \left(\rho^2 + \sigma\sigma^*\right) + (\varepsilon+\varepsilon^*)\rho - \kappa^*\tau - \kappa(3\alpha+\beta^*-\pi) + \Phi_{00}\,, \tag{2.122a}$$

$$D\sigma - \delta\kappa = (\rho+\rho^*)\sigma + (3\varepsilon-\varepsilon^*)\sigma - (\tau-\pi^*+\alpha^*+3\beta)\kappa + \Psi_0\,, \tag{2.122b}$$

$$D\tau - \Delta\kappa = (\tau+\pi^*)\rho + (\tau^*+\pi)\sigma + (\varepsilon-\varepsilon^*)\tau - (3\gamma+\gamma^*)\kappa + \Psi_1 + \Phi_{01}\,, \tag{2.122c}$$

$$D\alpha - \delta^*\varepsilon = (\rho+\varepsilon^*-2\varepsilon)\alpha + \beta\sigma^* - \beta^*\varepsilon - \kappa\lambda - \kappa^*\gamma + (\varepsilon+\rho)\pi + \Phi_{10}\,, \tag{2.122d}$$

$$D\beta - \delta\varepsilon = (\alpha+\pi)\sigma + (\rho^*-\varepsilon^*)\beta - (\mu+\gamma)\kappa - (\alpha^*-\pi^*)\varepsilon + \Psi_1\,, \tag{2.122e}$$

$$
\begin{aligned}
D\gamma - \Delta\varepsilon &= (\tau+\pi^*)\alpha + (\tau^*+\pi)\beta - (\varepsilon+\varepsilon^*)\gamma - (\gamma+\gamma^*)\varepsilon \\
&\quad + \tau\pi - \nu\kappa + \Psi_2 - \Lambda + \Phi_{11}\,,
\end{aligned}
\tag{2.122f}
$$

$$D\lambda - \delta^*\pi = (\rho\lambda+\sigma^*\mu) + \pi^2 + (\alpha-\beta^*)\pi - \nu\kappa^* - (3\varepsilon-\varepsilon^*)\lambda + \Phi_{20}\,, \tag{2.122g}$$

$$D\mu - \delta\pi = (\rho^*\mu+\sigma\lambda) + \pi\pi^* - (\varepsilon+\varepsilon^*)\mu - \pi(\alpha^*-\beta) - \nu\kappa + \Psi_2 + 2\Lambda\,, \tag{2.122h}$$

$$D\nu - \Delta\pi = (\pi+\tau^*)\mu + (\pi^*+\tau)\lambda + (\gamma-\gamma^*)\pi - (3\varepsilon+\varepsilon^*)\nu + \Psi_3 + \Phi_{21}\,, \tag{2.122i}$$

$$\Delta\lambda - \delta^*\nu = -(\mu+\mu^*)\lambda - (3\gamma-\gamma^*)\lambda + (3\alpha+\beta^*+\pi-\tau^*)\nu - \Psi_4\,, \tag{2.122j}$$

$$\delta\rho - \delta^*\sigma = \rho(\alpha^*+\beta) - \sigma(3\alpha-\beta^*) + (\rho-\rho^*)\tau + (\mu-\mu^*)\kappa - \Psi_1 + \Phi_{01}\,, \tag{2.122k}$$

$$
\begin{aligned}
\delta\alpha - \delta^*\beta &= (\mu\rho-\lambda\sigma) + \alpha\alpha^* + \beta\beta^* - 2\alpha\beta + \gamma(\rho-\rho^*) + \varepsilon(\mu-\mu^*) \\
&\quad - \Psi_2 + \Lambda + \Phi_{11}\,,
\end{aligned}
\tag{2.122l}
$$

$$\delta\lambda - \delta^*\mu = (\rho - \rho^*)\nu + (\mu - \mu^*)\pi + \mu(\alpha + \beta^*) + \lambda(\alpha^* - 3\beta) - \Psi_3 + \Phi_{21}\,, \tag{2.122m}$$

$$\delta\nu - \Delta\mu = \left(\mu^2 + \lambda\lambda^*\right) + (\gamma + \gamma^*)\mu - \nu^*\pi + (\tau - 3\beta - \alpha^*)\nu + \Phi_{22}\,, \tag{2.122n}$$

$$\delta\gamma - \Delta\beta = (\tau - \alpha^* - \beta)\gamma + \mu\tau - \sigma\nu - \varepsilon\nu^* - \beta(\gamma - \gamma^* - \mu) + \alpha\lambda^* + \Phi_{12}\,, \tag{2.122o}$$

$$\delta\tau - \Delta\sigma = (\mu\sigma + \lambda^*\rho) + (\tau + \beta - \alpha^*)\tau - (3\gamma - \gamma^*)\sigma - \kappa\nu^* + \Phi_{02}\,, \tag{2.122p}$$

$$\Delta\rho - \delta^*\tau = -(\rho\mu^* + \sigma\lambda) + (\beta^* - \alpha - \tau^*)\tau + (\gamma + \gamma^*)\rho + \nu\kappa - \Psi_2 - 2\Lambda\,, \tag{2.122q}$$

$$\Delta\alpha - \delta^*\gamma = (\rho + \varepsilon)\nu - (\tau + \beta)\lambda + (\gamma^* - \mu^*)\alpha + (\beta^* - \tau^*)\gamma - \Psi_3\,. \tag{2.122r}$$

Similarly, rewriting Eq. (2.118) in terms of the NP quantities, one gets a set of equations called Bianchi identities in [40]. These equations are given by

$$(\delta^* - 4\alpha + \pi)\Psi_0 - (D - 4\rho - 2\varepsilon)\Psi_1 - 3\kappa\Psi_2 = S_1\,, \tag{2.123a}$$

$$(\Delta - 4\gamma + \mu)\Psi_0 - (\delta - 4\tau - 2\beta)\Psi_1 - 3\sigma\Psi_2 = S_2\,, \tag{2.123b}$$

$$(\delta + 4\beta - \tau)\Psi_4 - (\Delta + 2\gamma + 4\mu)\Psi_3 + 3\nu\Psi_2 = S_3\,, \tag{2.123c}$$

$$(D + 4\varepsilon - \rho)\Psi_4 - (\delta^* + 4\pi + 2\alpha)\Psi_3 + 3\lambda\Psi_2 = S_4\,, \tag{2.123d}$$

$$(\delta^* + 3\pi)\Psi_2 - (D + 2\varepsilon - 2\rho)\Psi_3 - 2\lambda\Psi_1 - \kappa\Psi_4 = S_5\,, \tag{2.123e}$$

$$(\Delta + 3\mu)\Psi_2 - (\delta + 2\beta - 2\tau)\Psi_3 - 2\nu\Psi_1 - \sigma\Psi_4 = S_6\,, \tag{2.123f}$$

$$(\delta - 3\tau)\Psi_2 - (\Delta - 2\gamma + 2\mu)\Psi_1 + \nu\Psi_0 + 2\sigma\Psi_3 = S_7\,, \tag{2.123g}$$

$$(D - 3\rho)\Psi_2 - (\delta^* + 2\pi - 2\alpha)\Psi_1 + \lambda\Psi_0 + 2\kappa\Psi_3 = S_8\,, \tag{2.123h}$$

$$\begin{aligned}
&\delta^*\Phi_{01} + \delta\Phi_{10} - D(\Phi_{11} + 3\Lambda) - \Delta\Phi_{00} \\
&= \kappa^*\Phi_{12} + \kappa\Phi_{21} + (2\alpha + 2\tau^* - \pi)\Phi_{01} + (2\alpha^* + 2\tau - \pi^*)\Phi_{10} - 2(\rho + \rho^*)\Phi_{11} \\
&- \sigma^*\Phi_{02} - \sigma\Phi_{20} + [\mu + \mu^* - 2(\gamma + \gamma^*)]\,\Phi_{00}\,,
\end{aligned} \tag{2.123i}$$

$$\begin{aligned}
&\delta^*\Phi_{12} + \delta\Phi_{21} - \Delta(\Phi_{11} + 3\Lambda) - D\Phi_{22} \\
&= -\nu\Phi_{01} - \nu^*\Phi_{10} + (\tau^* - 2\beta^* - 2\pi)\Phi_{12} + (\tau - 2\beta - 2\pi^*)\Phi_{21} + 2(\mu + \mu^*)\Phi_{11} \\
&- (\rho + \rho^* - 2\varepsilon - 2\varepsilon^*)\Phi_{22} + \lambda\Phi_{02} + \lambda^*\Phi_{20}\,,
\end{aligned} \tag{2.123j}$$

$$\begin{aligned}
&\delta\,(\Phi_{11} - 3\Lambda) - D\Phi_{12} - \Delta\Phi_{01} + \delta^*\Phi_{02} \\
&= \Phi_{22} - \nu^*\Phi_{00} + (\tau^* - \pi + 2\alpha - 2\beta^*)\Phi_{02} - \sigma\Phi_{21} + \lambda^*\Phi_{10} + 2(\tau - \pi^*)\Phi_{11} \\
&- (2\rho + \rho^* - 2\varepsilon^*)\Phi_{12} + (2\mu^* + \mu - 2\gamma)\Phi_{01}\,,
\end{aligned} \tag{2.123k}$$

where S_i are related to the Ricci tensor and defined to be

$$S_1 \equiv (\delta + \pi^* - 2\alpha^* - 2\beta)\,\Phi_{00} - (D - 2\varepsilon - 2\rho^*)\,\Phi_{01}$$

$$+ 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \kappa^*\Phi_{02}\,, \tag{2.124a}$$

$$\begin{aligned} S_2 \equiv\ & (\delta + 2\pi^* - 2\beta)\,\Phi_{01} - (D - 2\varepsilon + 2\varepsilon^* - \rho^*)\,\Phi_{02} \\ & - \lambda^*\Phi_{00} + 2\sigma\Phi_{11} - 2\kappa\Phi_{12}\,, \end{aligned} \tag{2.124b}$$

$$\begin{aligned} S_3 \equiv\ & -(\Delta + 2\mu^* + 2\gamma)\,\Phi_{21} + (\delta^* - \tau^* + 2\alpha + 2\beta^*)\,\Phi_{22} \\ & + 2\nu\Phi_{11} + \nu^*\Phi_{20} - 2\lambda\Phi_{12}\,, \end{aligned} \tag{2.124c}$$

$$\begin{aligned} S_4 \equiv\ & -(\Delta + \mu^* + 2\gamma - 2\gamma^*)\,\Phi_{20} + (\delta^* + 2\alpha - 2\tau^*)\,\Phi_{21} \\ & + 2\nu\Phi_{10} - 2\lambda\Phi_{11} + \sigma^*\Phi_{22}\,, \end{aligned} \tag{2.124d}$$

$$\begin{aligned} S_5 \equiv\ & (\delta - 2\alpha^* + 2\beta + \pi^*)\Phi_{20} - (D - 2\rho^* + 2\varepsilon)\Phi_{21} \\ & - 2\mu\Phi_{10} + 2\pi\Phi_{11} - \kappa^*\Phi_{22} - 2\delta^*\Lambda\,, \end{aligned} \tag{2.124e}$$

$$\begin{aligned} S_6 \equiv\ & (\delta + 2\pi^* + 2\beta)\Phi_{21} - (D - \rho^* + 2\varepsilon + 2\varepsilon^*)\Phi_{22} \\ & - 2\mu\Phi_{11} - \lambda^*\Phi_{20} + 2\pi\Phi_{12} - 2\Delta\Lambda\,, \end{aligned} \tag{2.124f}$$

$$\begin{aligned} S_7 \equiv\ & -(\Delta + 2\mu^* - 2\gamma)\Phi_{01} + (\delta^* - \tau^* + 2\beta^* - 2\alpha)\Phi_{02} \\ & + 2\rho\Phi_{12} + \nu^*\Phi_{00} - 2\tau\Phi_{11} - 2\delta\Lambda\,, \end{aligned} \tag{2.124g}$$

$$\begin{aligned} S_8 \equiv\ & -(\Delta + \mu^* - 2\gamma - 2\gamma^*)\Phi_{00} + (\delta^* - 2\alpha - 2\tau^*)\Phi_{01} \\ & + 2\rho\Phi_{11} + \sigma^*\Phi_{02} - 2\tau\Phi_{10} - 2D\Lambda\,. \end{aligned} \tag{2.124h}$$

For the Bianchi identities, we have re-organized the terms and shuffled the sequence of equations in comparison to the one in [40], so our equations here are consistent with the equations in Sec. 2.2.1.

Finally, the commutation relation in Eq. (2.114) can be written as

$$[\Delta, D] = (\gamma + \gamma^*)\,D + (\varepsilon + \varepsilon^*)\,\Delta - (\tau^* + \pi)\,\delta - (\tau + \pi^*)\,\delta^*\,, \tag{2.125a}$$

$$[\delta, D] = (\alpha^* + \beta - \pi^*)\,D + \kappa\Delta - (\rho^* + \varepsilon - \varepsilon^*)\,\delta - \sigma\delta^*\,, \tag{2.125b}$$

$$[\delta, \Delta] = -\nu^* D + (\tau - \alpha^* - \beta)\,\Delta + (\mu - \gamma + \gamma^*)\,\delta + \lambda^*\delta^*\,, \tag{2.125c}$$

$$[\delta^*, \delta] = (\mu^* - \mu)\,D + (\rho^* - \rho)\,\Delta + (\alpha - \beta^*)\,\delta + (\beta - \alpha^*)\,\delta^*\,. \tag{2.125d}$$

2.8.3 Tetrad rotations

In Sec. 2.2.1, we have mentioned that the tetrad basis vectors can be rotated in certain ways such that the orthogonality conditions in Eq. (2.5) are still preserved. As discussed in [40], all these tetrad rotations can be classified into three types:

$$\mathrm{I}: l \to l\,,\ m \to m + al\,,\ \bar{m} \to \bar{m} + a^* l\,,\ n \to n + a^* m + a\bar{m} + aa^* l\,. \tag{2.126a}$$

$$\mathrm{II}: n \to n\,,\ m \to m + bn\,,\ \bar{m} \to \bar{m} + b^* n\,,\ l \to l + b^* m + b\bar{m} + bb^* n\,. \tag{2.126b}$$

$$\mathrm{III}: l \to A^{-1} l\,,\ n \to An\,,\ m \to e^{i\theta} m\,,\ \bar{m} \to e^{-i\theta}\bar{m}\,. \tag{2.126c}$$

Here, a, b are complex functions, and A, θ are real functions. Under these rotations, the Weyl scalars transform in the following way:

$$\mathrm{I}: \begin{array}{l} \Psi_0 \to \Psi_0\,,\ \Psi_1 \to \Psi_1 + a^*\Psi_0\,,\ \Psi_2 \to \Psi_2 + 2a^*\Psi_1 + (a^*)^2\,\Psi_0\,, \\ \Psi_3 \to \Psi_3 + 3a^*\Psi_2 + 3\,(a^*)^2\,\Psi_1 + (a^*)^3\,\Psi_0\,, \\ \Psi_4 \to \Psi_4 + 4a^*\Psi_3 + 6\,(a^*)^2\,\Psi_2 + 4\,(a^*)^3\,\Psi_1 + (a^*)^4\,\Psi_4\,. \end{array} \tag{2.127a}$$

$$\mathrm{II}: \begin{array}{l} \Psi_0 \to \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4\,, \\ \Psi_1 \to \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4\,, \\ \Psi_2 \to \Psi_2 + 2b\Psi_3 + b^2\Psi_4\,,\ \Psi_3 \to \Psi_3 + b\Psi_4\,,\ \Psi_4 \to \Psi_4\,. \end{array} \tag{2.127b}$$

$$\mathrm{III}: \begin{array}{l} \Psi_0 \to A^{-2}e^{2i\theta}\Psi_0\,,\ \Psi_1 \to A^{-1}e^{i\theta}\Psi_1\,,\ \Psi_2 \to \Psi_2\,,\ \Psi_3 \to Ae^{-i\theta}\Psi_3\,, \\ \Psi_4 \to A^2e^{-2i\theta}\Psi_4\,. \end{array} \tag{2.127c}$$

For the transformations of the spin-coefficients under the tetrad rotations, since we haven't used them explicitly in our calculations, we refer the readers to [40] for all the details.

2.9 Appendix: Modified Teukolsky equation in one place

For convenience of the reader, we organize the modified Teukolsky equation in one place. For Ψ_0, we have

$$\begin{aligned} & H_0^{(0,0)}\Psi_0^{(1,1)} + H_0^{(1,0)}\Psi_0^{(0,1)} + H_0^{(0,1)}\Psi_0^{(1,0)} - H_1^{(0,1)}\Psi_1^{(1,0)} \\ & = \mathcal{E}_2^{(0,0)}S_2^{(1,1)} + \mathcal{E}_2^{(0,1)}S_2^{(1,0)} - \mathcal{E}_1^{(0,0)}S_1^{(1,1)} - \mathcal{E}_1^{(0,1)}S_1^{(1,0)}\,. \end{aligned} \tag{2.128}$$

Here we have

$$H_0 = \mathcal{E}_2F_2 - \mathcal{E}_1F_1 - 3\Psi_2\,, \quad H_1 = \mathcal{E}_2J_2 - \mathcal{E}_1J_1\,, \tag{2.129}$$

and

$$\begin{aligned} & \mathcal{E}_1 = \delta - \tau + \pi^* - \alpha^* - 3\beta - \Psi_2^{-1}\delta\Psi_2\,, \quad F_1 \equiv \delta^* - 4\alpha + \pi\,, \quad J_1 \equiv D - 2\varepsilon - 4\rho\,, \\ & \mathcal{E}_2 = D - \rho - \rho^* - 3\varepsilon + \varepsilon^* - \Psi_2^{-1}D\Psi_2\,, \quad F_2 \equiv \Delta - 4\gamma + \mu\,, \quad J_2 \equiv \delta - 4\tau - 2\beta\,, \end{aligned} \tag{2.130}$$

with

$$\begin{aligned} & S_1 = (\delta + \pi^* - 2\alpha^* - 2\beta)\,\Phi_{00} - (D - 2\varepsilon - 2\rho^*)\,\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \kappa^*\Phi_{02}\,, \\ & S_2 = (\delta + 2\pi^* - 2\beta)\,\Phi_{01} - (D - 2\varepsilon + 2\varepsilon^* - \rho^*)\,\Phi_{02} - \lambda^*\Phi_{00} + 2\sigma\Phi_{11} - 2\kappa\Phi_{12}\,. \end{aligned} \tag{2.131}$$

For Ψ_4, we have

$$H_4^{(0,0)}\Psi_4^{(1,1)} + H_4^{(1,0)}\Psi_4^{(0,1)} + H_4^{(0,1)}\Psi_4^{(1,0)} - H_3^{(0,1)}\Psi_3^{(1,0)} \tag{2.132}$$
$$= \mathcal{E}_4^{(0,0)}S_4^{(1,1)} + \mathcal{E}_4^{(0,1)}S_4^{(1,0)} - \mathcal{E}_3^{(0,0)}S_3^{(1,1)} - \mathcal{E}_3^{(0,1)}S_3^{(1,0)}\,. \tag{2.133}$$

Here we have

$$H_4 = \mathcal{E}_4 F_4 - \mathcal{E}_3 F_3 - 3\Psi_2\,, \quad H_3 = \mathcal{E}_4 J_4 - \mathcal{E}_3 J_3\,, \tag{2.134}$$

and

$$\begin{aligned}&\mathcal{E}_3 = \delta^* + 3\alpha + \beta^* + \pi - \tau^* - \Psi_2^{-1}\delta^*\Psi_2\,, \quad F_3 \equiv \delta + 4\beta - \tau\,, \quad J_3 \equiv \Delta + 2\gamma + 4\mu\,,\\ &\mathcal{E}_4 = \Delta + \mu + \mu^* + 3\gamma - \gamma^* - \Psi_2^{-1}\Delta\Psi_2\,, \quad F_4 \equiv D + 4\varepsilon - \rho\,, \quad J_4 \equiv \delta^* + 4\pi + 2\alpha\,,\end{aligned} \tag{2.135}$$

with

$$\begin{aligned}S_3 &= -(\Delta + 2\mu^* + 2\gamma)\,\Phi_{21} + (\delta^* - \tau^* + 2\alpha + 2\beta^*)\,\Phi_{22} + 2\nu\Phi_{11} + \nu^*\Phi_{20} - 2\lambda\Phi_{12}\,,\\ S_4 &= -(\Delta + \mu^* + 2\gamma - 2\gamma^*)\,\Phi_{20} + (\delta^* + 2\alpha - 2\tau^*)\,\Phi_{21} + 2\nu\Phi_{10} - 2\lambda\Phi_{11} + \sigma^*\Phi_{22}\,.\end{aligned} \tag{2.136}$$

2.10 Appendix: Consistency check with previous higher-order Teukolsky formalism

In this appendix, we show that the GHP transformation of Eqs. (2.109)-(2.110) when $\zeta = 0$ are consistent with Eqs. (7)-(10) of [93] when we are in the same gauge as in Eq. (2.108).

First, let us write down the GHP transformation of Eqs. (2.109)-(2.110) when $\zeta = 0$,

$$H_4^{\rm GR}\Psi_4^{(N)} = \mathcal{T}_{\rm geo}^{(N)}\,, \quad \mathcal{T}_{\rm geo}^{(N)} = \sum_{n=1}^{N-1} -H_4^{(N-n)}\Psi_4^{(n)}\,, \tag{2.137}$$

where we have used the single superscript notation since there is only one expansion parameter, ϵ. In comparison, Ref. [93] found

$$\begin{aligned}&H_4^{\rm GR}\Psi_4^{(N)} = \mathcal{T}'^{(N)}_{\rm geo}\,,\\ &\mathcal{T}'^{(N)}_{\rm geo} = \sum_{n=1}^{N-1}\Big[\left(\mathcal{E}_3^{(0)}F_3^{(N-n)} - \mathcal{E}_4^{(0)}F_4^{(N-n)}\right)\Psi_4^{(n)}\\ &\qquad + 3\mathcal{E}_3^{(0)}\left(\Psi_2^{(n)}\nu^{(N-n)}\right) - 3\mathcal{E}_4^{(0)}\left(\Psi_2^{(n)}\lambda^{(N-n)}\right)\\ &\qquad -3\Psi_2^{(0)}\left(E_3^{(N-n)}\nu^{(n)} - E_4^{(N-n)}\lambda^{(n)}\right)\Big]\,,\end{aligned} \tag{2.138}$$

where we have set all the terms containing $\Psi_3^{(0,n)}$ for $n > 0$ to zero and replaced the operators $\bar{d}_{3,4}$ in [93] with the operators $E_{3,4}$ by observing that

$$\bar{d}_3 = E_3 + 3\pi\,, \quad \bar{d}_4 = E_4 + 3\mu\,, \tag{2.139}$$
$$\bar{d}_3^{(0)} = E_3^{\mathrm{GR}} = \mathcal{E}_3^{(0)}\,, \quad \bar{d}_4^{(0)} = E_4^{\mathrm{GR}} = \mathcal{E}_4^{(0)}\,. \tag{2.140}$$

As discussed in Sec. 2.6.1, to show that Eq. (2.138) is the same as Eq. (2.137), one needs to use Bianchi identities to express λ and ν in terms of Ψ_4 or vice versa. Since $\Psi_3^{(0)} = 0$ for Petrov type D spacetimes, and we have chosen a gauge in which $\Psi_3^{(n)} = 0$ for all $n \geq 1$, we can set $\Psi_3 = 0$ in Eq. (2.14), such that

$$F_3\Psi_4 = -3\Psi_2\nu\,, \quad F_4\Psi_4 = -3\Psi_2\lambda\,, \tag{2.141}$$

where we have also set $S_3 = S_4 = 0$ since we focus on vacuum spacetimes. Notice that Eq. (2.141) is true at all orders in ϵ.

Expressing Ψ_4 in terms of λ and ν is easier when comparing Eq. (2.138) with Eq. (2.137). Let us first perform this transformation on Eq. (2.137). From the definition in Eqs. (2.12) and (2.34), we know that

$$\mathcal{E}_3 = E_3 - \Psi_2^{-1}\delta^*\Psi_2\,, \quad \mathcal{E}_4 = E_4 - \Psi_2^{-1}\Delta\Psi_2\,. \tag{2.142}$$

Inserting Eqs. (2.141)-(2.142) into Eq. (2.137), we find

$$\begin{aligned} H_4\Psi_4 &= (\mathcal{E}_4F_4 - \mathcal{E}_3F_3 - 3\Psi_2)\Psi_4 \\ &= -3\left[E_4(\Psi_2\lambda) - \Delta\Psi_2 - E_3(\Psi_2\nu) + \delta^*\Psi_2 + \Psi_2\Psi_4\right] \\ &= -3\Psi_2\left(E_4\lambda - E_3\nu + \Psi_4\right)\,, \end{aligned} \tag{2.143}$$

which is simply $-3\Psi_2$ times the Ricci identity in Eq. (2.14c). Since Eq. (2.137) is essentially the N-th order expansion of $H_4\Psi_4$, we find

$$\left[-3\Psi_2\left(E_4\lambda - E_3\nu + \Psi_4\right)\right]^{(N)} = 0\,. \tag{2.144}$$

Equation (2.144) is consistent with our procedures to derive the master equation in Secs. 2.5.3 and 2.6.2. The equation we used is indeed $3\Psi_2$ multiplying the Ricci identity Eq. (2.14c) with λ and ν replaced by the Bianchi identities Eqs. (2.14a)-(2.14b). Since the Teukolsky equations have to be consistent with all the Bianchi identities and Ricci identities, one also expects that starting from a Teukolsky equation and simplifying it using Bianchi identities, one will get back the original Ricci identity.

Now, let us transform Eq. (2.138). We first move the first line of $\mathcal{T}'^{(N)}_{\rm geo}$ in Eq. (2.138) to the left-hand side of the equation, so it becomes

$$\begin{aligned}
&\sum_{n=1}^{N}\left(\mathcal{E}_4^{(0)}F_4^{(N-n)}-\mathcal{E}_3^{(0)}F_3^{(N-n)}\right)\Psi_4^{(n)}-3\Psi_2^{(0)}\Psi_4^{(N)}\\
&=\mathcal{E}_4^{(0)}(F_4\Psi_4)^{(N)}-\mathcal{E}_3^{(0)}(F_3\Psi_4)^{(N)}-3\Psi_2^{(0)}\Psi_4^{(N)}\\
&=-3\left[\mathcal{E}_4^{(0)}(\Psi_2\lambda)^{(N)}-\mathcal{E}_3^{(0)}(\Psi_2\nu)^{(N)}\right]-3\Psi_2^{(0)}\Psi_4^{(N)}\,.
\end{aligned}\tag{2.145}$$

Next, subtracting off the second line of $\mathcal{T}'^{(N)}_{\rm geo}$ in Eq. (2.138) from Eq. (2.145), we find

$$\begin{aligned}
&-3\left[\mathcal{E}_4^{(0)}\left(\Psi_2^{(0)}\lambda^{(N)}\right)-\mathcal{E}_3^{(0)}\left(\Psi_2^{(0)}\nu^{(N)}\right)\right]-3\Psi_2^{(0)}\Psi_4^{(N)}\\
&=-3\Psi_2^{(0)}\left(E_4^{(0)}\lambda^{(N)}-E_3^{(0)}\nu^{(N)}+\Psi_4^{(N)}\right)\,,
\end{aligned}\tag{2.146}$$

which, with the last line of $\mathcal{T}'^{(N)}_{\rm geo}$ in Eq. (2.138), gives us

$$-3\Psi_2^{(0)}\left[(E_4\lambda-E_3\nu+\Psi_4)\right]^{(N)}=0\,.\tag{2.147}$$

As discussed above, Eq. (2.147) is expected since the Teukolsky equations are consistent with the Ricci identities.

Comparing Eq. (2.147) to Eq. (2.144), one can notice that the only difference is the overall normalization factor. In Eq. (2.147), this normalization factor is $-3\Psi_2^{(0)}$, while in Eq. (2.144), a normalization factor of $-3\Psi_2$ appears before the expansion. Then, when expanding Eq. (2.144), we also mix lower-order Ricci identities in the equation. For example, we can get the term $-3\Psi_2^{(1)}(E_4\lambda-E_3\nu+\Psi_4)^{(N-1)}$. Nonetheless, after inserting in all the lower-order NP quantities into the equation, these lower-order Ricci identities vanish, since they are automatically satisfied by the lower-order Teukolsky solutions in the previous steps. On the other hand, before inserting lower-order Teukolsky solutions, Eq. (2.144) might be more complicated than Eq. (2.147) due to these lower-order equations.

One can easily remove this difference by replacing the normalization factor $3\Psi_2$ in Eq. (2.76) with $3\Psi_2^{(0,0)}$. The reason we inserted $3\Psi_2$ in Eq. (2.76) is that the $O(\zeta^0,\epsilon^1)$ expansion of the equation reproduces the original Teukolsky equation in GR [44], which is also true if we instead insert $3\Psi_2^{(0,0)}$. Moreover, we can absorb the factors of Ψ_2 and Ψ_2^{-1} in Eq. (2.76) nicely into the operators $\mathcal{E}_i$. If we instead use $3\Psi_2^{(0,0)}$, we can alternatively define the operators $\mathcal{E}_i$ as

$$\mathcal{E}_i=\Psi_2^{(0,0)}E_i\Psi_2^{-1}\tag{2.148}$$

in comparison to the original definition in Eqs. (2.32) and (2.34). For the goals of this paper, finding the $\mathcal{O}(\zeta^1, \epsilon^1)$ corrections to the Teukolsky equation, both ways of normalizing the equation are fine and make little difference.

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Chapter 3

ISOSPECTRALITY BREAKING IN THE TEUKOLSKY FORMALISM

[1] Dongjun Li, Asad Hussain, Pratik Wagle, Yanbei Chen, Nicolás Yunes, and Aaron Zimmerman. "Isospectrality breaking in the Teukolsky formalism". In: *Phys. Rev. D* 109.10 (2024), p. 104026. DOI: 10.1103/PhysRevD.109.104026. arXiv: 2310.06033 [gr-qc].

3.1 Introduction

The development of current and next-generation gravitational wave (GW) detectors allows us for the first time to study the extreme gravitational events that emit these waves and use them to test theories of gravity. General relativity (GR), as one of the most successful gravity theories, has been widely tested [1], but its incompatibility with quantum mechanics motivated the development of new theories of quantum gravity, such as string theory [2–5] and loop quantum gravity [6–9]. Furthermore, to resolve observational anomalies, such as the asymmetry of matter and antimatter abundance in our universe [10], one can also modify the theory of gravity. One notable feature of many beyond GR (bGR) theories is the breaking of parity symmetry, a fundamental symmetry preserved by GR but observed to be broken in other fundamental interactions [11].

For some bGR theories, parity is already broken at the level of action. One subset of these theories includes effective field theory (EFT) extensions of GR in Lorentzian geometry, such as parity-violating ghost-free scalar-tensor gravity [12–14], certain versions of Horava-Lifshitz gravity [15, 16], and parity-violating corrections in higher-derivative gravity without extra fields [17–20]. Another subset is built instead on non-Riemannian geometry [21], such as parity-violating symmetric teleparallel gravity [22]. For these theories, stationary black hole (BH) solutions also violate parity. This can result in the break of equatorial symmetry of Kerr in GR [19, 23–27], where odd-parity multiple moments (odd mass multipole moments and even current multipoles) of rotating BHs become nonzero. The breaking of equatorial symmetry can be detected via, for example, the GWs emitted during extreme mass-ratio inspirals [26].

Besides these theories with explicit parity-violating terms in the action, some bGR theories preserve parity at the level of action and at the level of axisymmetric BH solutions, but they violate parity at the level of gravitational perturbations, such as in dynamical Chern-Simons (dCS) gravity [28–32]. These theories can also break parity cosmologically, for example, when additional degrees of freedom acquire a non-zero vacuum expectation value [33–35]. In this case, it was observed years ago that the amplitude of left-circular or right-circular polarized GWs decreases or increases with propagation, resulting in amplitude birefringence [14, 21, 36–38]. These left-circular and right-circular polarized modes can also propagate with different velocities, causing velocity birefringence [13, 14, 37, 39]. Both amplitude and velocity birefringences can be detected, in principle, with LIGO [36, 40, 41]. These birefringence effects might also leave imprints at a larger scale, for example, generating chiral primordial GWs, which directly affect the cosmic wave background radiation [42–47], or circularly-polarized stochastic GW background, which can be detected by GW detectors [48–50].

Besides propagation effects, gravitational perturbations of BHs in modified gravity can also have parity asymmetry during generation. One important feature of GWs emitted during the ringdown phase of binary BH mergers in GR, or quasinormal modes (QNMs), is that the modes with the same quantum number, but different parity, have the same frequency [51, 52], a result known as isospectrality. However, in bGR theories, isospectrality is generally broken, similar to the breaking of degeneracies in quantum mechanical perturbation theory. For example, in dCS gravity, it has been found that only odd-parity modes are modified for non-rotating BHs [30, 53, 54]. For spinning BHs, both parities are modified but in different ways [31, 55]. Similar isospectrality breaking of QNMs has been observed in parity-violating corrections of higher-derivative gravity [23, 56–58] and, more interestingly, in certain parity-preserving theories, such as parity-preserving corrections of higher-derivative gravity [23, 56–59] and EdGB theory [60–64]. Such parity asymmetry in the generation of GWs may cause observable effects, depending on whether there is enough signal-to-noise ratio (SNR) to resolve the shifts to the ringdown frequencies (both real and imaginary parts) of the resultant modes [65].

In this work, we focus on the isospectrality breaking of QNMs in these EFT extensions of GR. The study of QNMs has been an important topic in GR and modified gravity because their spectrum allows us to retrieve information on the exterior geometry of BHs and the dynamics of modified gravity theories, which is the idea of BH spectroscopy [66–68]. For these bGR theories, extra non-metric fields (scalar, vector, or tensor) leave imprints on the QNMs. To study QNMs, one major approach is BH perturbation theory, where the gravitational perturbations of an isolated stationary BH are computed, given that the merger of binary BHs always settles down to a stationary geometry in GR. For non-rotating BHs, thanks to spherical symmetry, QNMs can be directly computed from metric perturbations in both GR [69–73] and modified gravity [23, 30, 53, 54, 59–62, 74]. In this case, metric perturbations are separated into two pieces, one with even and one with odd parity. For each parity piece, one can find a single gauge-invariant function that characterizes all degrees of freedom, i.e., the Zerilli-Moncrief (ZM) function for even-parity perturbations and the Regge-Wheeler (RW) function for odd-parity ones, the governing equations of which are decoupled and separable. Since each of the metric perturbation functions has a definite parity, one can easily study isospectrality breaking in this approach.

For rotating BHs, due to the lack of spherical symmetry, it is hard to decouple all the metric fields and find only two functions to represent all the metric components. For this reason, Teukolsky developed another approach for rotating BHs in GR [75–77] within the framework of Newman and Penrose (NP) [78] and using spinor calculus. In the Teukolsky formalism, instead of solving for metric perturbations directly, curvature perturbations, characterized by the Weyl scalars Ψ_0 and Ψ_4, are solved for first, from which the metric can then be reconstructed [51, 79–88]. Both non-rotating and rotating BHs in GR can be mathematically classified as Petrov type D spacetimes [51, 89], the leading-order gravitational perturbations of which are fully described by decoupled and separable Teukolsky equations. However, in the Teukolsky formalism, the modes are not naturally separated into definite parity. To study parity, one then needs to first find combinations of solutions to the Teukolsky equations that generate definite-parity metric perturbations. This work was first done in [80], using metric reconstruction to map definite-parity metric perturbations to Teukolsky functions, and expressed in a simpler form in [90].

In modified gravity, perturbations of spinning BHs were previously studied using metric perturbations in the slow-rotation expansion [31, 55, 56, 63, 64, 91] and using numerical relativity for an arbitrary spin but with secularly-growing errors [92, 93]. However, most of the remnant BHs of binary BH mergers are rapidly rotating (at least 65% of their maximum), as predicted theoretically [94] and confirmed observationally [95]. One can, in principle, extend the approach using metric perturbations in the slow-rotating expansion to higher orders in spin, but to produce reliable results for these fast-spinning BHs, one usually needs to go beyond fifth order in the slow-rotation expansion [96]. Extensions to such a high order will involve complicated couplings between different l modes, so this approach might not be practically feasible. Although in EdGB, Ref. [64] recently found that by resuming the $O(\chi^2)$, slow-rotation expansion of QNMs using Padé approximants [97–99], one might find accurate results for dimensionless spin up to $\chi = a/M \sim 0.7$, it is still worth developing a formalism without explicit reliance on a small spin expansion. An alternative approach, combining metric perturbations with spectral decomposition techniques, was recently developed for Schwarzschild BHs [100] and Kerr BHs (valid up to $\chi \sim 0.95$) [101]. However, it is worth noting that, although promising, such spectral decomposition techniques have only been demonstrated for BHs in GR as of yet.

Recently, Refs. [102, 103] showed that one can extend the Teukolsky formalism in GR to modified gravity for any deformed BHs that do not significantly deviate from their counterparts in GR so that they can be treated through an EFT approach. In this modified Teukolsky formalism, the Weyl scalars Ψ_0 and Ψ_4 are decoupled from other degrees of freedom of curvature perturbations, just like in GR. Their equations are also separable because the homogeneous part of the modified Teukolsky equation is of the same form as in GR, and the source terms can be separated by projection to spin-weighted spheroidal harmonics [102, 103]. Later, Refs. [57, 58] applied the approach of [102, 103] to higher-derivative gravity up to $O(\chi^{14})$. The authors successfully separated the equations into radial and angular parts and computed the QNM frequencies valid up to $\chi \sim 0.7$. Their results also match well with previous calculations using metric perturbations in [56, 91].

Nonetheless, to study isospectrality breaking, one needs to first find out what the definite-parity modes are in these modified Teukolsky equations and derive their equations. In this work, we show that one can extend the definition in [90] to the modified Teukolsky equations in [102, 103]. Furthermore, we derive the equations that govern these definite-parity modes and prescribe how to evaluate the shifts of QNMs using the eigenvalue perturbation (EVP) method of [103–105]. For simplicity, in this work, we only focus on spacetimes that are Petrov type D even in modified gravity, but our results can be easily extended to non-Petrov-type-D spacetimes, where the modified Teukolsky formalism of [102, 103] still applies. We also assume that background spacetimes are parity invariant, which is true for non-rotating and slowly rotating BHs in dCS [106, 107] and EdGB gravity [60, 108, 109], so we can focus on the parity properties of dynamical perturbations. Our work can also be easily extended to BH spacetimes that violate parity, such as those in higher-derivative gravity with parity-violating corrections [23, 56–58], but we leave this for future work.

In the remainder of this paper, we present in more detail our formalism for studying isospectrality breaking of QNMs in modified gravity using the Teukolsky formalism. In Sec. 3.2, we give a quick review of the modified Teukolsky formalism developed by [102, 103]. In Sec. 3.3, we review the construction of definite-parity modes of Teukolsky equations in GR found by [90]. In Sec. 3.4, we show that the same definition of definite-parity modes in GR can be extended to Petrov type D spacetimes in modified gravity. For non-Petrov-type-D spacetimes, we discuss how one might extend our construction and leave details to future work. In Sec. 3.5, we follow the discussion in [103] to derive the shifts of QNM frequencies using the EVP method of [103–105] and show how the degeneracy in QNM frequencies of even- and odd-parity modes is generally broken in modified gravity. We then derive the condition for the modified Teukolsky equation to have definite-parity solutions and present the shifts of their QNM frequencies. In Sec. 3.6, we apply our formalism to two specific bGR theories: dCS and EdGB gravity, and we show that our definite-parity equations agree qualitatively with the equations found by metric perturbations in [30, 31, 53–55, 60–64]. Finally, in Sec. 3.7, we discuss future avenues of this work and conclude.

3.2 Modified Teukolsky equations

In this section, we review the modified Teukolsky formalism in bGR theories developed in [102, 103]. Here, we focus on the equation of Ψ_0, and the equation of Ψ_4 can be found following the same procedure or via the GHP transformation [110].

3.2.1 bGR theories and expansion scheme

As shown in [102], for any modified gravity theory that admits an EFT description and allows perturbation theory, the gravitational perturbations of any non-Ricci-flat, Petrov type I BH can be studied via the curvature perturbation formalism. For this large subset of modified gravity theories, its Lagrangian can be schematically written as

$$\mathcal{L} = \mathcal{L}_{\rm GR} + \ell^p \mathcal{L}_{\rm bGR} + \mathcal{L}_{\rm matter} + \mathcal{L}_{\rm field}\,, \tag{3.1}$$

where $\mathcal{L}_{\rm GR}$ is the Einstein-Hilbert Lagrangian, and $\mathcal{L}_{\rm matter}$ is the Lagrangian of matter. In this work, we focus on vacuum backgrounds, so $\mathcal{L}_{\rm matter} = 0$. $\mathcal{L}_{\rm field}$ is the Lagrangian of extra non-metric fields, including both kinetic and potential terms. The Lagrangian $\mathcal{L}_{\rm bGR}$ describes additional corrections to the Einstein-Hilbert Lagrangian, which may contain non-minimal couplings to the extra non-metric fields. The quantity ℓ with dimensions of length characterizes the strength of the GR correction, with p introduced to ensure that the dimension of $\ell^p \mathcal{L}_{\rm bGR}$ are correct.

Based on whether there are additional non-metric fields, we can divide the subset of modified gravity theories that our modified Teukolsky formalism applies to into the following two classes:

- $\mathcal{L}_{\rm field} \neq 0 \Longrightarrow \text{Class A}\,,$
- $\mathcal{L}_{\rm field} = 0 \Longrightarrow \text{Class B}\,.$

Some examples of class A bGR theories are dCS gravity [28, 29], EdGB gravity [108, 111, 112], Horndeski theory [113], scalar-tensor theories [114], $f(R)$ gravity [115, 116], Einstein-Aether theory [117], and bi-gravity [118]. There are also certain EFT extensions of GR that do not contain extra non-metric fields, so they can be classified as class B bGR theories, such as higher-derivative gravity [17–20].

To study gravitational perturbations in modified gravity in the formalism of [102], we need at least two expansion parameters. In this work, we follow the conventions in [102] and use ζ to denote the strength of bGR corrections and ϵ the size of GW perturbations, a parameter that also appears in the GR case. Both ζ and ϵ are dimensionless, so ζ is usually some power of the ratio of the scale ℓ to the BH mass. We also assume that the leading correction to the metric field due to modified gravity is at least of $O(\zeta)$, so the correction to other non-metric fields enters at lower and non-integer order of ζ [102]. Reference [102] additionally showed that if the background tetrad is carefully chosen, the bGR correction to all NP quantities also enters at $O(\zeta)$. Then, all the NP quantities can be expanded in the following way:

$$\begin{aligned}\Psi_i &= \Psi_i^{(0)} + \epsilon\Psi_i^{(1)} \\ &= \Psi_i^{(0,0)} + \zeta\Psi_i^{(1,0)} + \epsilon\Psi_i^{(0,1)} + \zeta\epsilon\Psi_i^{(1,1)}\,,\end{aligned} \tag{3.2}$$

where we have taken Weyl scalars as an example. In this work, we will hide the expansion in ζ from certain equations to minimize notational clutter, so the superscript will only stand for an expansion in ϵ, as shown in the first line of Eq. (3.2).

In this work, we do not make any assumptions about the relative size of ϵ and ζ. The bGR expansion parameter ζ can be either larger or smaller than ϵ, depending on the details of the binary system and the bGR theory being considered. For example, in dCS gravity, NICER and advanced LIGO observations constrain $\sqrt{\alpha_{\rm dCS}} \leq 8.5$km at 90% confidence [119], where $\alpha_{\rm dCS}$ is the coupling constant of dCS gravity. For an intermediate or extreme mass-ratio inspiral around a BH of mass $M = 10^5 M_\odot$, this bound on $\alpha_{\rm dCS}$ maps to a constraint on $\zeta_{\rm dCS}$ [i.e., Eq. (3.82)] of $\zeta_{\rm dCS} \leq 3.5\times 10^{-5}$. For such an extreme mass-ratio inspiral, however, ϵ is proportional to the mass ratio q of the binary, which can be either larger or smaller than $\zeta_{\rm dCS}$ because $q \in [10^{-3}, 10^{-7}]$. However, corrections with powers of ϵ^n arise from higher-order GR effects, while corrections with powers of ζ^n arise from higher-order bGR effects that generally do not generate GWs at infinity. Therefore, the term $\zeta\epsilon$ that we focus on is the leading bGR contribution to GWs.

3.2.2 Modified Teukolsky equation

Using the expansion scheme in Eq. (3.2), one can then derive the modified Teukolsky equation. For convenience, let us first define the following operators in terms of the NP spin coefficients and tetrad derivatives (see e.g. [78, 120]):

$$D_{[a,b,c,d]} = D + a\varepsilon + b\bar{\varepsilon} + c\rho + d\bar{\rho}\,, \tag{3.3a}$$

$$\mathbf{\Delta}_{[a,b,c,d]} = \mathbf{\Delta} + a\mu + b\bar{\mu} + c\gamma + d\bar{\gamma}\,, \tag{3.3b}$$

$$\delta_{[a,b,c,d]} = \delta + a\bar{\alpha} + b\beta + c\bar{\pi} + d\tau\,, \tag{3.3c}$$

$$\bar{\delta}_{[a,b,c,d]} = \bar{\delta} + a\alpha + b\bar{\beta} + c\pi + d\bar{\tau}\,, \tag{3.3d}$$

where the overhead bar denotes complex conjugation. The equations we start from are two Ricci identities and one Bianchi identity, namely

$$F_1\Psi_0 - J_1\Psi_1 - 3\kappa\Psi_2 = S_1\,, \tag{3.4a}$$

$$F_2\Psi_0 - J_2\Psi_1 - 3\sigma\Psi_2 = S_2\,, \tag{3.4b}$$

$$E_2\sigma - E_1\kappa - \Psi_0 = 0\,, \tag{3.4c}$$

where the operators $F_{1,2}$, $J_{1,2}$, and $E_{1,2}$ are defined via

$$\begin{aligned} &F_1 \equiv \bar{\delta}_{[-4,0,1,0]}\,, \qquad &&F_2 \equiv \mathbf{\Delta}_{[1,0,-4,0]}\,,\\ &J_1 \equiv D_{[-2,0,-4,0]}\,, &&J_2 \equiv \delta_{[0,-2,0,-4]}\,,\\ &E_1 \equiv \delta_{[-1,-3,1,-1]}\,, &&E_2 \equiv D_{[-3,1,-1,-1]}\,, \end{aligned} \tag{3.5}$$

and the source terms $S_{1,2}$ are

$$S_1 \equiv \delta_{[-2,-2,1,0]}\Phi_{00} - D_{[-2,0,0,-2]}\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02}\,, \tag{3.6a}$$

$$S_2 \equiv \delta_{[0,-2,2,0]}\Phi_{01} - D_{[-2,2,0,-1]}\Phi_{02} - \bar{\lambda}\Phi_{00} + 2\sigma\Phi_{11} - 2\kappa\Phi_{12}\,. \tag{3.6b}$$

To derive the modified Teukolsky equation, Ref. [102] made some convenient gauge choices for both the background spacetime and the dynamical perturbations, following Chandrasekhar [51]. Since we care about BH spacetimes that are modifications of Petrov type D spacetimes in GR, one can use the Kinnersley tetrad to set

$$\Psi_{0,1,3,4}^{(0,0)} = 0\,. \tag{3.7}$$

For dynamical perturbations, Ref. [102] showed that one can rotate the $O(\zeta^1, \epsilon^1)$ part of the tetrad, such that

$$\Psi_{1,3}^{(0,1)} = \Psi_{1,3}^{(1,1)} = 0\,. \tag{3.8}$$

In this gauge, one can then easily decouple $\Psi_0^{(1,1)}$ from other NP quantities and derive a single decoupled equation for $\Psi_0^{(1,1)}$, namely [102]

$$H_0^{\mathrm{GR}}\Psi_0^{(1,1)} = \mathcal{S}_{\mathrm{geo}}^{(1,1)} + \mathcal{S}^{(1,1)}\,, \tag{3.9}$$

where H_0^{GR} is the Teukolsky operator in GR, and the source terms $\mathcal{S}_{\mathrm{geo}}^{(1,1)}$ and $\mathcal{S}^{(1,1)}$ are given by

$$\begin{aligned}
\mathcal{S}_{\mathrm{geo}}^{(1,1)} &= \mathcal{S}_{0,\mathrm{D}}^{(1,1)} + \mathcal{S}_{0,\text{non-D}}^{(1,1)} + \mathcal{S}_{1,\text{non-D}}^{(1,1)}\,,\\
\mathcal{S}_{0,\mathrm{D}}^{(1,1)} &= -H_0^{(1,0)}\Psi_0^{(0,1)}\,,\\
\mathcal{S}_{0,\text{non-D}}^{(1,1)} &= -H_0^{(0,1)}\Psi_0^{(1,0)}\,,\\
\mathcal{S}_{1,\text{non-D}}^{(1,1)} &= H_1^{(0,1)}\Psi_1^{(1,0)}\,,
\end{aligned} \tag{3.10}$$

and

$$\mathcal{S}^{(1,1)} = \mathcal{E}_2^{(0,0)}S_2^{(1,1)} + \mathcal{E}_2^{(0,1)}S_2^{(1,0)} - \mathcal{E}_1^{(0,0)}S_1^{(1,1)} - \mathcal{E}_1^{(0,1)}S_1^{(1,0)}\,. \tag{3.11}$$

The operators $H_{0,1}$ and $\mathcal{E}_{1,2}$ are defined via

$$\begin{aligned}
&H_0 = \mathcal{E}_2 F_2 - \mathcal{E}_1 F_1 - 3\Psi_2\,, \quad H_1 = \mathcal{E}_2 J_2 - \mathcal{E}_1 J_1\,,\\
&\mathcal{E}_1 = E_1 - \frac{1}{\Psi_2}\delta\Psi_2\,, \quad \mathcal{E}_2 = E_2 - \frac{1}{\Psi_2}D\Psi_2\,.
\end{aligned} \tag{3.12}$$

The equation for $\Psi_4^{(1,1)}$ can be found in [102]. In Eq. (3.10), the source term $\mathcal{S}_{\mathrm{geo}}$ comes from the homogeneous part of the Bianchi identities and Ricci identities in Eq. (3.4), so it is generated by modifications to the background spacetime and does not involve terms from the effective stress tensor. Within $\mathcal{S}_{\text{geo}}$, the terms $\mathcal{S}_{i,\text{non-D}}$ only appear in non-Petrov-type-D spacetimes, while $\mathcal{S}_{i,\text{D}}$ also appears in Petrov type D spacetimes. On the other hand, the source term $\mathcal{S}$ comes from the effective stress-energy tensor, so it depends on the details of the modified gravity theory and may contain extra non-metric fields.

Inspecting Eqs. (3.9)–(3.11), one notices that every NP quantity in $\mathcal{S}_{\text{geo}}^{(1,1)}$ has lower order than $O(\zeta^1, \epsilon^1)$, and the only terms at $O(\zeta^1, \epsilon^1)$ are $\Psi_0^{(1,1)}$ and $\mathcal{S}^{(1,1)}$. In [102], it was additionally shown that for class B bGR theories, $\mathcal{S}^{(1,1)} \sim h^{(1,0)} h^{(0,1)}$, while for class A bGR theories, $\mathcal{S}^{(1,1)} \sim \vartheta^{(1,0)} h^{(0,1)} + \vartheta^{(1,1)} g^{(0,0)}$, where ϑ represents extra (scalar, vector or tensor fields) non-metric fields. For both cases, there are no factors of $h_{\mu\nu}^{(1,1)}$, so we have fully decoupled $\Psi_0^{(1,1)}$ from all metric fields. For class A bGR theories, we also have $\vartheta^{(1,1)}$, but as shown in [102, 103], these extra non-metric fields can be solved for first by following the order-reduction scheme in [121]. The key idea is that for these non-minimal coupling class A bGR theories, the bGR corrections always drive non-metric fields first before driving GW perturbations [102, 103]. Thus, when writing down $\vartheta^{(1,1)}$, we actually have absorbed the coupling constant into ϑ, while it enters at a lower order. For details of decoupling non-metric fields from Ψ_0, one can refer to [103].

Besides using the gauge freedom of both the background spacetime and the dynamical perturbations, one can also derive the modified Teukolsky equation without making any explicit gauge choices, as done in the original derivation of the Teukolsky equation in GR [75] and in modified gravity in [103]. Reference [103] showed that instead of using the NP language from the beginning, one could follow the idea in [122] and work with the Einstein equations directly to then project to a modified Teukolsky equation at the end. In spite of the many differen approaches to derive the modified Teukolsky equation, the final master equation shares many similarities. One major feature is that the master equation always contains terms at $O(\zeta^0, \epsilon^1)$, which requires metric reconstruction of GW perturbations in GR, as one can observe in Eqs. (3.10) and (3.11). In the next section, we will introduce one of these metric reconstruction procedures. For the terms at $O(\zeta^0, \epsilon^0)$ and $O(\zeta^1, \epsilon^0)$, one can directly compute them using the background metric. To transform the modified Teukolsky equation in Eq. (3.9) to the one for definite-parity modes, the next step is to understand what definite-parity modes of the Teukolsky equation are in both GR and modified gravity.

3.3 Modes with Definite Parity in GR

In this section, we review definite-parity solutions to the Teukolsky equation in GR, following [90]. Our focus is on bGR, beyond-Kerr BH spacetimes, whereas we show below that isospectrality can be broken by dynamical effects. Since this is our primary goal, we further assume that, like in the Kerr solution, the stationary spacetime is invariant under the parity transformation. This assumption holds for known BH solutions in modified gravity theories, such as in dCS gravity [106, 123] and EdGB gravity[60, 108, 109], and seems physically reasonable for a stationary BH. To make this more concrete, we first define what we mean by a parity transformation.

Let the spacetime be a manifold $\mathcal{M}$, with an open set $U \subset \mathcal{M}$ inside it that contains Boyer-Lindquist-like coordinates, i.e., the metric g on $\mathcal{M}$ has the functional form of a modified Kerr metric on U in these coordinates. Define the parity operator $\hat{P}$ as an operator that acts on functions in these Boyer-Lindquist coordinates. The action of the operator is the following:

$$\hat{P}[f(t,r,\theta,\phi)] = f(t,r,\pi-\theta,\phi+\pi)\,. \tag{3.13}$$

Then for metric perturbations [69, 70, 90], the modes with even and odd parity are defined to be

$$\hat{P}h_{\mu\nu}^{\mathrm{E,O}} = \pm(-1)^l h_{\mu\nu}^{\mathrm{E,O}}\,, \tag{3.14}$$

where l is the angular momentum number after decomposing metric perturbations into spheroidal harmonics.[1] To define parity for solutions to the Teukolsky equation, we then need to know the relation between metric perturbations and curvature perturbations.

3.3.1 Metric reconstruction in GR

To find curvature perturbations from metric perturbations, we can directly compute Weyl scalars from the perturbed metric. In contrast, reconstructing metric perturbations from Weyl scalars is a more complicated process. Fortunately, this procedure was developed for Kerr BHs or, more generally, Petrov type D spacetimes in GR either via an intermediate Hertz potential [79–87] or by solving the Bianchi identities, Ricci identities, and commutation relations [51, 88]. In this work, we follow the approach using the Hertz potential, the so-called Chrzanowski-Cohen-Kegeles (CCK) procedure [86], and the conventions in [86, 87] due to more explicit algebraic relations between Weyl scalars and metric perturbations. In this section, we only provide a

[1] We have followed the definition of definite-parity modes in Sec. IC2 of [90].

brief introduction, and more details can be found in Appendix 3.9. Furthermore, since in this section we review metric reconstruction in GR, our expressions hold only to $O(\zeta^0, \epsilon^1)$.

As discussed in [86, 87], the Hertz potential $\Psi_{\rm H}^{(0,1)}$ generates metric perturbations $h_{\mu\nu}^{(0,1)}$ that solve the linearized Einstein equations in GR. For simplicity, we will drop the superscripts of $h_{\mu\nu}^{(0,1)}$ and $\Psi_{\rm H}^{(0,1)}$ for the rest of this section. In the outgoing radiation gauge (ORG) [87], where $n^\mu h_{\mu\nu} = 0$ and $h \equiv g^{\mu\nu} h_{\mu\nu} = 0$. The perturbed metric $h_{\mu\nu}$ is related to $\Psi_{\rm H}$ via

$$\begin{aligned} h_{\mu\nu} = & -\rho^{-4} \left[n_\mu n_\nu \bar{\delta}_{[-3,-1,5,0]} \bar{\delta}_{[-4,0,1,0]} + \bar{m}_\mu \bar{m}_\nu \boldsymbol{\Delta}_{[5,0,-3,1]} \boldsymbol{\Delta}_{[1,0,-4,0]} \right. \\ & \left. -n_{(\mu} \bar{m}_{\nu)} \left(\bar{\delta}_{[-3,1,5,1]} \boldsymbol{\Delta}_{[1,0,-4,0]} + \boldsymbol{\Delta}_{[5,-1,-3,-1]} \bar{\delta}_{[-4,0,1,0]} \right) \right] \Psi_{\rm H} + \text{c.c}\,. \end{aligned} \tag{3.15}$$

In the ingoing radiation gauge (IRG) [86], where $l^\mu h_{\mu\nu} = 0$ and $h = 0$, we have instead that

$$\begin{aligned} h_{\mu\nu} = & \left[l_\mu l_\nu \bar{\delta}_{[1,3,0,-1]} \bar{\delta}_{[0,4,0,3]} + \bar{m}_\mu \bar{m}_\nu D_{[-1,3,0,-1]} D_{[0,4,0,3]} \right. \\ & \left. -l_{(\mu} \bar{m}_{\nu)} \left(D_{[1,3,1,-1]} \bar{\delta}_{[0,4,0,3]} + \bar{\delta}_{[-1,3,-1,-1]} D_{[0,4,0,3]} \right) \right] \bar{\Psi}_{\rm H} + \text{c.c}\,. \end{aligned} \tag{3.16}$$

Notice that, since we have chosen the opposite signature from [86, 87], our $h_{\mu\nu}$ has a different sign.

Using Eqs. (3.15) and (3.16), one can derive the relation between Weyl scalars and $\Psi_{\rm H}$. For example, for perturbations of Kerr in ORG [87],

$$\Psi_4 = \frac{1}{32} \rho^4 \Delta^2 D^{\dagger 4} \Delta^2 \bar{\Psi}_{\rm H}\,, \tag{3.17a}$$

$$\Psi_0 = \frac{1}{8} \left[\mathcal{L}^4 \bar{\Psi}_{\rm H} + 12 M \partial_t \Psi_{\rm H} \right]\,, \tag{3.17b}$$

while in IRG

$$\Psi_0 = -\frac{1}{2} D^4 \bar{\Psi}_{\rm H}\,, \tag{3.18a}$$

$$\Psi_4 = -\frac{1}{8} \rho^4 \left[\mathcal{L}^{\dagger 4} \bar{\Psi}_{\rm H} - 12 M \partial_t \Psi_{\rm H} \right]\,, \tag{3.18b}$$

where

$$\begin{aligned} & \Delta = r^2 - 2Mr + a^2\,, \quad \rho = -\frac{1}{1 - ia\cos\theta}\,, \\ & D = l^\mu \partial_\mu = \frac{r^2 + a^2}{\Delta} \partial_t + \partial_r + \frac{a}{\Delta} \partial_\phi\,, \\ & D^\dagger = -\frac{r^2 + a^2}{\Delta} \partial_t + \partial_r - \frac{a}{\Delta} \partial_\phi\,, \\ & \mathcal{L}_s = -ia \sin\theta \partial_t - \left[\partial_\theta + i \csc\theta \partial_\phi - s\cot\theta \right]\,, \\ & \mathcal{L}_s^\dagger = ia \sin\theta \partial_t - \left[\partial_\theta - i \csc\theta \partial_\phi - s\cot\theta \right]\,, \end{aligned} \tag{3.19}$$

and $\mathcal{L}^4 = \mathcal{L}_1\mathcal{L}_0\mathcal{L}_{-1}\mathcal{L}_{-2}$, $\mathcal{L}^{\dagger 4} = \mathcal{L}^\dagger_{-1}\mathcal{L}^\dagger_0\mathcal{L}^\dagger_1\mathcal{L}^\dagger_2$. Here, we have also dropped the superscript $(0,1)$ of $\Psi_{0,4}$ for simplicity. All the Weyl scalars in this subsection are assumed to be at $O(\zeta^0, \epsilon^1)$.

In this work, we focus on the modified Teukolsky equation for Ψ_0, so it is convenient to work with IRG, where $\Psi_{\rm H}$ can be reconstructed from Ψ_0 by inverting Eq. (3.18a) using the Teukolsky-Starobinsky identities [77, 124, 125], e.g.,

$$\bar{\Psi}_{\rm H} = -2\mathfrak{C}^{-1}\Delta^2\left(D^\dagger\right)^4\left[\Delta^2\Psi_0\right]. \tag{3.20}$$

Here, $\mathfrak{C}$ is the mode-dependent Teukolsky-Starobinsky constant [51, 83],[2]

$$\mathfrak{C} = \lambda^2(\lambda+2)^2 - 8\omega^2\lambda\left[\alpha^2(5\lambda+6) - 12a^2\right] + 144\omega^4\alpha^4 + 144\omega^2M^2, \tag{3.21}$$

where ω is the QNM frequency associated with a specific (l, m, ω) mode of Ψ_0, λ is the separation constant used by Chandrasekhar [51], and $\alpha^2 \equiv a^2 - am/\omega$. The relation between $\Psi_{\rm H}$ and other Weyl scalars in IRG can be found in Appendix 3.9. For the Schwarzschild background, these relations greatly simplify. Since we use the relation between Ψ_2 and $\Psi_{\rm H}$ in the Schwarzschild limit frequently in Sec. 3.6, we present it here for convenience,

$$\Psi_2 = -\frac{1}{2}D^2(\bar{\delta}+2\beta)(\bar{\delta}+4\beta)\bar{\Psi}_{\rm H}. \tag{3.22}$$

3.3.2 Definition of even- and odd-parity modes

With the relation in Eq. (3.16), we can now define the modes with definite parity in GR. For convenience, let us define an operator $\hat{\mathcal{P}} \equiv \hat{C}\hat{P}$, where $\hat{P}$ is the parity transformation, and $\hat{C}$ is the complex conjugation,

$$\hat{\mathcal{P}}f = \hat{C}\hat{P}f = \hat{C}f(\pi-\theta, \phi+\pi) = \bar{f}(\pi-\theta, \phi+\pi). \tag{3.23}$$

In [126], the same $\hat{\mathcal{P}}$ operator was also constructed and used for studying scalar and vector QNMs and their isospectrality in EFT extensions of GR. With the definition above and $\hat{P}^2 = \hat{C}^2 = \hat{\mathcal{I}}$, where $\hat{\mathcal{I}}$ is the identity operator, one can easily show that

$$\hat{\mathcal{P}}^2 = \hat{\mathcal{I}}, \quad \hat{C}\hat{\mathcal{P}} = \hat{P}, \quad \hat{P}\hat{\mathcal{P}} = \hat{C}, \tag{3.24}$$

because $\hat{P}$, $\hat{C}$, and $\hat{\mathcal{P}}$ commute with each other. Using Eq. (3.24), we will replace $\hat{P}$ with $\hat{C}$ and $\hat{\mathcal{P}}$ in most places. When $\hat{\mathcal{P}}$ acts on another operator $\hat{X}$, we have

$$\hat{\mathcal{P}}[\hat{X}] = \hat{\mathcal{P}}\hat{X}\hat{\mathcal{P}}, \tag{3.25}$$

[2] $\mathfrak{C}$ is the constant p in [83].

and similarly for $\hat{C}$. Other useful properties of $\hat{P}$, $\hat{C}$, and $\hat{\mathcal{P}}$ are listed in Appendix 3.8.

As discussed in [90], at $O(\zeta^0, \epsilon^0)$ in the Kinnersley tetrad of Kerr,

$$\begin{aligned} &\hat{\mathcal{P}}\{D, \mathbf{\Delta}\} = \{D, \mathbf{\Delta}\}\,, &&\hat{\mathcal{P}}\left\{\delta, \bar{\delta}\right\} = -\left\{\delta, \bar{\delta}\right\}\,, \\ &\hat{\mathcal{P}}\{\rho, \mu, \gamma\} = \{\rho, \mu, \gamma\}\,, &&\hat{\mathcal{P}}\{\alpha, \beta, \pi, \tau\} = -\{\alpha, \beta, \pi, \tau\}\,, \end{aligned} \tag{3.26}$$

and other spin coefficients are zero. For convenience, let us rewrite Eqs. (3.15) and (3.16) as

$$h_{\mu\nu} = O_{\mu\nu}\bar{\Psi}_{\mathrm{H}} + \bar{O}_{\mu\nu}\Psi_{\mathrm{H}}\,, \tag{3.27}$$

where $O_{\mu\nu}$ denotes the operator converting $\bar{\Psi}_{\mathrm{H}}$ to $h_{\mu\nu}$. Using Eq. (3.26), one can show that

$$\hat{\mathcal{P}}O_{\mu\nu} = O_{\mu\nu}\,. \tag{3.28}$$

Thus,

$$\hat{\mathcal{P}}h_{\mu\nu} = O_{\mu\nu}\hat{\mathcal{P}}\bar{\Psi}_{\mathrm{H}} + \bar{O}_{\mu\nu}\hat{\mathcal{P}}\Psi_{\mathrm{H}}\,. \tag{3.29}$$

For even- and odd-parity metric perturbations, $\hat{\mathcal{P}}h^{\mathrm{E,O}}_{\mu\nu} = \pm(-1)^l h^{\mathrm{E,O}}_{\mu\nu}$. Comparing Eq. (3.29) to Eq. (3.27), we find that the Hertz potentials $\Psi^{\mathrm{E,O}}_{\mathrm{H}}$ generated from $h^{\mathrm{E,O}}_{\mu\nu}$ must transform as

$$\hat{\mathcal{P}}\Psi^{\mathrm{E,O}}_{\mathrm{H}} = \pm(-1)^l\Psi^{\mathrm{E,O}}_{\mathrm{H}}\,. \tag{3.30}$$

Since the operators converting $\bar{\Psi}_{\mathrm{H}}$ to $\Psi_{0,4}$ in Eqs. (3.17a) and (3.18a) are invariant under $\hat{\mathcal{P}}$, the even- and odd-parity modes of $\Psi_{0,4}$ must transform in the same way as Eq. (3.30).

We can then define the definite-parity modes $\Psi^{\mathrm{E,O}}_{lm\omega}$ of $\Psi_{0,4}$ to be

$$\Psi^{\mathrm{E,O}}_{lm\omega} := \Psi_{lm\omega} \pm (-1)^l\hat{\mathcal{P}}\Psi_{lm\omega} \tag{3.31}$$

because then we have that

$$\begin{aligned} \hat{\mathcal{P}}\Psi^{\mathrm{E,O}}_{lm\omega} &= \hat{\mathcal{P}}\Psi_{lm\omega} \pm (-1)^l\Psi_{lm\omega} \\ &= \pm(-1)^l\left[\Psi_{lm\omega} \pm (-1)^l\hat{\mathcal{P}}\Psi_{lm\omega}\right] \\ &= \pm(-1)^l\Psi^{\mathrm{E,O}}_{lm\omega}\,, \end{aligned} \tag{3.32}$$

where $\Psi_{lm\omega}$ is a single (l, m, ω) mode solving the Teukolsky equation of either Ψ_0 or Ψ_4. For simplicity, we shall also write

$$\Psi_{\mathrm{E,O}} := \Psi \pm (-1)^l\hat{\mathcal{P}}\Psi\,, \quad \Psi := \Psi_{lm\omega}\,, \tag{3.33}$$

where we have dropped the mode label $lm\omega$ for all the fields in Eq. (3.31). Henceforth, we always assume that Ψ is a single (l, m, ω) mode. Using Eq. (3.33), we can also express Ψ and $\hat{\mathcal{P}}\Psi$ in terms of $\Psi_{\mathrm{E,O}}$,

$$\Psi = \frac{1}{2}\left(\Psi_{\mathrm{E}} + \Psi_{\mathrm{O}}\right) , \quad \hat{\mathcal{P}}\Psi = \frac{(-1)^l}{2}\left(\Psi_{\mathrm{E}} - \Psi_{\mathrm{O}}\right) , \tag{3.34}$$

which provides the inverse map from definite-parity modes back to the full solutions to the Teukolsky equation.

3.3.3 Relation between definite-parity modes and solutions to the Teukolsky equation

To generate metric perturbations with definite parity, besides the transformation property in Eq. (3.30), we also need $\Psi_{\mathrm{E,O}}$ to solve the Teukolsky equation; otherwise, the metric perturbations generated will not solve the Einstein equations. The modes in Eq. (3.31) are not necessarily solutions to the modified Teukolsky equation since $\hat{\mathcal{P}}\Psi$ is not guaranteed to be a solution except in some special cases.

For Kerr, one can show that Eq. (3.31) are solutions to the Teukolsky equation by using the transformation properties of Teukolsky functions under $\hat{P}$ and $\hat{C}$. According to [90], the radial Teukolsky functions ${}_sR_{lm\omega}(r)$ and the angular Teukolsky functions ${}_sS_{lm\omega}(\theta)$ for Kerr in GR satisfy

$$\begin{aligned} {}_s\bar{R}_{lm\omega} &= (-1)^m {}_sR_{l-m-\bar{\omega}} , \\ {}_sS_{lm\omega}(\pi - \theta) &= (-1)^{m+l} {}_{-s}S_{lm\omega}(\theta) , \\ {}_s\bar{S}_{lm\omega}(\theta) &= (-1)^{m+s} {}_{-s}S_{l-m-\bar{\omega}}(\theta) . \end{aligned} \tag{3.35}$$

Thus, by applying the above relations for $s = \pm 2$, we can rewrite $\hat{\mathcal{P}}\Psi$ as

$$\begin{aligned} \hat{\mathcal{P}}\Psi &= \bar{\Psi}_{lm\omega}(\pi - \theta, \phi + \pi) \\ &= {}_{\pm 2}\bar{R}_{lm\omega} e^{-i(m(\phi+\pi)-\bar{\omega}t)} {}_{\pm 2}\bar{S}_{lm\omega}(\pi - \theta) \\ &= (-1)^l {}_{\pm 2}R_{l-m-\bar{\omega}} e^{-i(m\phi-\bar{\omega}t)} {}_{\pm 2}S_{l-m-\bar{\omega}} , \end{aligned} \tag{3.36}$$

where in the third line we have used the relations in Eq. (3.35). Thus, for Kerr, we also have

$$\Psi^{\mathrm{E,O}}_{lm\omega} = \Psi_{lm\omega} \pm \Psi_{l-m-\bar{\omega}} , \tag{3.37}$$

which is the definition used in [90].

In general, the modes in Eq. (3.37) do not necessarily satisfy the transformation rule in Eq. (3.30) as one can explicitly check. For Kerr, due to the relations in Eq. (3.35), we can transform $\Psi_{l-m-\bar{\omega}}$ to $\hat{\mathcal{P}}\Psi_{lm\omega}$, so Eq. (3.37) becomes modes of definite parity. Generally, we need to define the even and odd modes for $\Psi_{lm\omega}$ and $\Psi_{l-m\bar{\omega}}$ separately using Eq. (3.33). For Kerr, since $\hat{\mathcal{P}}\Psi_{lm\omega} = (-1)^l \Psi_{l-m\bar{\omega}}$, the even and odd modes for $\Psi_{lm\omega}$ and $\Psi_{l-m\bar{\omega}}$ are degenerate, $\Psi^{\mathrm{E,O}}_{lm\omega} = \pm\Psi^{\mathrm{E,O}}_{l-m\bar{\omega}}$.

Another feature of the definition in Eq. (3.33) is that the modes with definite parity are linear combinations of modes with frequency ω and the (negative of its) conjugate $-\bar{\omega}$. One may wonder whether we can define modes with definite parity without mixing modes with different frequencies. The answer is no. For a generic Hertz potential, we can always decompose it into modes with different frequencies, e.g., $\Psi = \sum_\omega A_\omega(r,\theta,\phi)e^{-i\omega t}$. Since modes with definite parity need to transform as in Eq. (3.30), we must have

$$\sum_\omega A_\omega(r,\theta,\phi)e^{-i\omega t} = \pm(-1)^l \sum_\omega \bar{A}_\omega(r,\pi-\theta,\phi+\pi)e^{i\bar{\omega}t}\,, \tag{3.38}$$

so $A_{-\bar{\omega}}(r,\theta,\phi) = \pm(-1)^l \bar{A}_\omega(r,\pi-\theta,\phi+\pi)$. Thus, any mode with frequency ω must be accompanied by a mode with frequency $-\bar{\omega}$. This additionally shows that any solution of the Teukolsky equation in GR has the decomposition in Eq. (3.33) if it is a definite-parity mode.

To summarize, the definition in Eq. (3.31) is a more fundamental definition than the one in Eq. (3.37) since it does not rely on the specific properties of the solutions to the Teukolsky equation in Kerr. As such, we use Eqs. (3.31) and (3.33) for the rest of the work. The major goal of this work is to study the correction to the definite-parity Teukolsky solutions in GR defined in Eq. (3.33). In many cases, we do not expect that the modified Teukolsky equation admits solutions with definite parity for generic systems that generate GWs. We also do not expect the modes defined in Eq. (3.33) to generate metric perturbations with definite parity in modified gravity. Nonetheless, in the next section, we show that the definition in Eqs. (3.31) and (3.33) still work for Petrov type D spacetimes that are perturbations of Kerr in modified gravity.

3.4 Modes with Definite Parity in Modified Gravity

In the previous section, we have shown that the Teukolsky functions defined in Eq. (3.33) generate metric perturbations with definite parity in GR using metric reconstruction. However, for general modified gravity theories, the procedure to reconstruct metric perturbations from the modified Teukolsky functions is not known. On the other hand, one may also wonder whether the definition in Eq. (3.33) still works for modified gravity. Although one can find bGR corrections to the GR, definite-parity QNMs without knowing whether the corrected modes have definite parity, it is still important to understand what definite-parity QNMs are in modified gravity. Along with the results of Sec. 3.5, one can then directly check whether a modified gravity theory admits modes with definite parity as solutions to the modified Teukolsky equations. In this section, we show that the definition in Eq. (3.33) also works for Petrov type D spacetimes in modified gravity.

Since metric reconstruction in modified gravity is generally unknown, we first start from metric perturbations with definite parity and then compute the parity transformation of the Weyl scalars generated from these metric perturbations. In our derivation, we make certain gauge choices such that all the NP quantities have simple transformation properties under $\hat{\mathcal{P}}$. In Sec. 3.4.3, we further show that although our derivation is not manifestly gauge-invariant, the parity properties of Weyl scalars $\Psi_{0,4}$ are gauge-invariant. In this work, we only aim to define definite-parity modes for Petrov type D spacetimes in modified gravity. More specifically, we consider Petrov type D BHs that are modifications of Kerr. For non-Petrov-type-D spacetimes in modified gravity, there are additional complexities, so we discuss a potential strategy and leave further investigations for future work.

3.4.1 $\hat{\mathcal{P}}$-transformation of stationary NP quantities

In this section, we compute the parity transformation of NP quantities at $O(\epsilon^0)$. For Kerr BHs in GR, we can use the Kinnersley tetrad such that at $O(\zeta^0, \epsilon^0)$,

$$\begin{aligned} &\hat{\mathcal{P}} D^{(0,0)} = D^{(0,0)}\,, \qquad \hat{\mathcal{P}} \boldsymbol{\Delta}^{(0,0)} = \boldsymbol{\Delta}^{(0,0)}\,, \\ &\hat{\mathcal{P}} \delta^{(0,0)} = -\delta^{(0,0)}\,, \qquad \hat{\mathcal{P}} \bar{\delta}^{(0,0)} = -\bar{\delta}^{(0,0)}\,. \end{aligned} \tag{3.39}$$

For tetrads corrected at $O(\zeta^1, \epsilon^0)$, we can follow [102] to choose

$$\delta e_{a\mu}^{(1,0)} = -\frac{1}{2} e_{a\nu}^{(0,0)} h_{\mu}^{\ \nu(1,0)}\,, \tag{3.40}$$

such that all the orthogonality conditions of NP tetrads are satisfied. Recall that we assume the BH spacetime invariant under parity, so $\hat{\mathcal{P}} h_{\mu\nu}^{(1,0)} = \hat{P} h_{\mu\nu}^{(1,0)} = h_{\mu\nu}^{(1,0)}$, where we have used that $h_{\mu\nu}^{(1,0)}$ is real. Then, using Eq. (3.39), we find

$$\begin{aligned} &\hat{\mathcal{P}} D^{(1,0)} = D^{(1,0)}\,, \qquad \hat{\mathcal{P}} \boldsymbol{\Delta}^{(1,0)} = \boldsymbol{\Delta}^{(1,0)}\,, \\ &\hat{\mathcal{P}} \delta^{(1,0)} = -\delta^{(1,0)}\,, \qquad \hat{\mathcal{P}} \bar{\delta}^{(1,0)} = -\bar{\delta}^{(1,0)}\,. \end{aligned} \tag{3.41}$$

Since the background tetrad in GR and modified gravity have the same transformation properties under $\hat{\mathcal{P}}$, we do not distinguish them below by suppressing the index orders in ζ so that the superscripts give the order in the expansion of ϵ only.

With Eqs. (3.39) and (3.41), one can then compute the $\hat{\mathcal{P}}$-transformation of any spin coefficient or Ricci rotation coefficient using the definition,

$$\gamma_{cab} = e_{a\mu;\nu} e^{\mu}_{c} e^{\nu}_{b} \,. \tag{3.42}$$

From Eqs. (3.39), (3.41), and (3.42), we know that the spin coefficients with even number of m^μ, $\bar{m}^\mu$, and their corresponding directional derivatives are invariant under $\hat{\mathcal{P}}$, while the ones with odd numbers of m^μ, $\bar{m}^\mu$, and their corresponding directional derivatives pick up a minus sign under $\hat{\mathcal{P}}$. For example, $\kappa = \gamma_{131}$ only contains one m^μ, so $\hat{\mathcal{P}}\kappa^{(0)} = -\kappa^{(0)}$. Although $\kappa^{(0)}$ vanishes in Petrov type D spacetimes, we still list its parity transformation property since it only depends on the transformation rules in Eqs. (3.39) and (3.41). Similarly, we find

$$\begin{aligned}
&\hat{\mathcal{P}}\left\{\sigma^{(0)}, \lambda^{(0)}, \varepsilon^{(0)}, \rho^{(0)}, \mu^{(0)}, \gamma^{(0)}\right\} = \left\{\sigma^{(0)}, \lambda^{(0)}, \varepsilon^{(0)}, \rho^{(0)}, \mu^{(0)}, \gamma^{(0)}\right\}\,,\\
&\hat{\mathcal{P}}\left\{\kappa^{(0)}, \nu^{(0)}, \alpha^{(0)}, \beta^{(0)}, \tau^{(0)}, \pi^{(0)}\right\} = -\left\{\kappa^{(0)}, \nu^{(0)}, \alpha^{(0)}, \beta^{(0)}, \tau^{(0)}, \pi^{(0)}\right\}\,,
\end{aligned} \tag{3.43}$$

which are consistent with the results in [90] for GR.

3.4.2 $\hat{\mathcal{P}}$-transformation of dynamical NP quantities

At $O(\epsilon^1)$, the tetrad can be expressed in terms of the tetrad at $O(\epsilon^0)$. As found in [88, 127], one can use the tetrad freedom to choose

$$\begin{aligned}
D^{(1)} &= -\frac{1}{2} h^{(1)}_{ll} \mathbf{\Delta}^{(0)}\,,\\
\mathbf{\Delta}^{(1)} &= -\frac{1}{2} h^{(1)}_{nn} D^{(0)} - h^{(1)}_{ln} \mathbf{\Delta}^{(0)}\,,\\
\delta^{(1)} &= -h^{(1)}_{nm} D^{(0)} - h^{(1)}_{lm} \mathbf{\Delta}^{(0)} + \frac{1}{2} h^{(1)}_{m\bar{m}} \delta^{(0)} + \frac{1}{2} h^{(1)}_{mm} \bar{\delta}^{(0)}\,.
\end{aligned} \tag{3.44}$$

Since this tetrad choice is possible at both $O(\zeta^0, \epsilon^1)$ and $O(\zeta^1, \epsilon^1)$, we also suppress the expansion in ζ here for simplicity. Taking $h^{(1)}_{\mu\nu}$ to be the metric perturbations with definite parity of Eq. (3.14), we find that the dynamical tetrad transforms under $\hat{\mathcal{P}}$ as

$$\begin{aligned}
&\hat{\mathcal{P}} D^{(1)} = \pm(-1)^l D^{(1)}\,, \quad \hat{\mathcal{P}} \mathbf{\Delta}^{(1)} = \pm(-1)^l \mathbf{\Delta}^{(1)}\,,\\
&\hat{\mathcal{P}} \delta^{(1)} = \mp(-1)^l \delta^{(1)}\,, \quad \hat{\mathcal{P}} \bar{\delta}^{(1)} = \mp(-1)^l \bar{\delta}^{(1)}\,,
\end{aligned} \tag{3.45}$$

where the factor $\pm$ of $D^{(1)}$ and $\mathbf{\Delta}^{(1)}$ depends on the parity of $h^{(1)}_{\mu\nu}$ generating these tetrad perturbations, e.g., $+$ for even parity and $-$ for odd parity, and similarly for the factor $\mp$ of $\delta^{(1)}$ and $\bar{\delta}^{(1)}$. The eigenvalues of $\hat{\mathcal{P}}$ in Eq. (3.45) have contributions from both the background tetrad [Eqs. (3.39) and (3.41)] and the perturbed metric $h^{(1)}_{\mu\nu}$ [$\pm(-1)^l$ for even and odd parity, respectively]. For example, since $\hat{\mathcal{P}} h^{(1)}_{ll} = \pm(-1)^l h^{(1)}_{ll}$ for even-or odd-parity $h_{\mu\nu}^{(1)}$, respectively, and $\hat{\mathcal{P}}\mathbf{\Delta}^{(0)} = \mathbf{\Delta}^{(0)}$, we find $\hat{\mathcal{P}}D^{(1)} = \pm(-1)^l D^{(1)}$. In total, $D^{(1)}$ and $\mathbf{\Delta}^{(1)}$ preserve the parity of $h_{\mu\nu}^{(1)}$, while $\delta^{(1)}$ and $\bar{\delta}^{(1)}$ flip its parity.

To find the $\hat{\mathcal{P}}$-transformation of the spin coefficients at $O(\epsilon^1)$, one can first express the spin coefficients in terms of metric perturbations. This can be done by linearizing the commutation relation, e.g. following [51]. We have listed the results in Appendix 3.9, which are consistent with the results in [88]. Then, using Eqs. (3.39), (3.41), (3.43), and (3.45), one can find that

$$\begin{aligned}\hat{\mathcal{P}}\left\{\sigma^{(1)},\lambda^{(1)},\varepsilon^{(1)},\rho^{(1)},\mu^{(1)},\gamma^{(1)}\right\} &= \pm(-1)^l\left\{\sigma^{(1)},\lambda^{(1)},\varepsilon^{(1)},\rho^{(1)},\mu^{(1)},\gamma^{(1)}\right\}\,,\\ \hat{\mathcal{P}}\left\{\kappa^{(1)},\nu^{(1)},\alpha^{(1)},\beta^{(1)},\tau^{(1)},\pi^{(1)}\right\} &= \mp(-1)^l\left\{\kappa^{(1)},\nu^{(1)},\alpha^{(1)},\beta^{(1)},\tau^{(1)},\pi^{(1)}\right\}\,,\end{aligned} \tag{3.46}$$

which have very similar transformation properties as the spin coefficients at $O(\epsilon^0)$ in Eq. (3.43) up to the overall factor $\pm(-1)^l$ of $h_{\mu\nu}^{(1)}$ under $\hat{\mathcal{P}}$.

With this in hand, let us now study the transformation properties of the Weyl scalars. Using the Ricci identity, one has that

$$\Psi_0 = D_{[-3,1,-1,-1]}\sigma - \delta_{[-1,-3,1,-1]}\kappa\,. \tag{3.47}$$

Using Eqs. (3.39), (3.41), (3.43), (3.45), and (3.46), we then find that

$$\hat{\mathcal{P}}\Psi_0^{(1)} = \pm(-1)^l\Psi_0^{(1)}\,, \tag{3.48}$$

which includes both $O(\zeta^0,\epsilon^1)$ and $O(\zeta^1,\epsilon^1)$ contributions and is consistent with Eq. (3.30). The same, of course, is also true for $\Psi_4^{(1)}$. In Sec. 3.3.3, we have shown that for any mode to have the transformation properties of Eq. (3.48), the mode has to have the decomposition of Eq. (3.33). This confirms that the modes defined in Eq. (3.33) are definite-parity Teukolsky solutions for Petrov type D spacetimes in modified gravity. Reference [57] has found the same definite-parity modes of $\Psi_{0,4}$ via metric reconstruction, but they only intended to construct these definite-parity modes in GR for evaluating QNM shifts. Nonetheless, our work extends the definition in GR to Petrov type D spacetimes in modified gravity.

3.4.3 Gauge invariance

In the analysis above, we have chosen a specific tetrad in Eqs. (3.39), (3.41), and (3.45), so one may wonder whether our definition in Eq. (3.33) still works if we choose a different tetrad. Moreover, besides the freedom of rotating the tetrad, one can also perform local coordinate transformations, which one may worry also affect our analysis. In this section, we show that, although the proof above selected a specific tetrad, its argument about definite-parity modes is both tetrad- and coordinate-invariant.

Generally, for perturbations at $O(\zeta^1, \epsilon^1)$, one should consider both coordinate and tetrad transformations at $O(\zeta^1, \epsilon^0)$, $O(\zeta^0, \epsilon^1)$, and $O(\zeta^1, \epsilon^1)$. For simplicity, let us focus on Ψ_0, and an analogous argument can be made for Ψ_4. For a general combination of type I, II, and type III rotations with rotation parameters a, b, A, and ϑ, where a and b are complex functions, and A and ϑ are real functions, the Weyl scalar Ψ_0 transforms as [51],

$$\Psi_0 \to A^{-2}e^{2i\vartheta}\Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4\,, \tag{3.49}$$

where we have kept all orders of the rotation parameters. For tetrad rotations at $O(\zeta^1, \epsilon^1)$, Eq. (3.49) reduces to

$$\Psi_0^{(1,1)} \to \Psi_0^{(1,1)} - 2[\delta A^{(1,1)} - i\vartheta^{(1,1)}]\Psi_0^{(0,0)} + 4b^{(1,1)}\Psi_1^{(0,0)}\,, \tag{3.50}$$

where we have define $\delta A = A - 1$. Since we are interested in background spacetimes that are modifications of Petrov type D spacetimes in GR, $\Psi_0^{(0,0)} = \Psi_1^{(0,0)} = 0$, so $\Psi_0^{(1,1)}$ is invariant under tetrad rotations at $O(\zeta^1, \epsilon^1)$, which is consistent with the result in [102]. Similarly, for tetrad rotations at $O(\zeta^0, \epsilon^1)$,

$$\Psi_0^{(1,1)} \to \Psi_0^{(1,1)} - 2[\delta A^{(0,1)} - i\vartheta^{(0,1)}]\Psi_0^{(1,0)} + 4b^{(0,1)}\Psi_1^{(1,0)}\,. \tag{3.51}$$

For Petrov type D spacetimes in modified gravity, $\Psi_0^{(1,0)} = \Psi_1^{(1,0)} = 0$, so $\Psi_0^{(1,1)}$ is also invariant under tetrad rotations at $O(\zeta^0, \epsilon^1)$.

However, for Petrov type I spacetimes, since $\Psi_{0,1}^{(1,0)}$ are nonzero, $\Psi_0^{(1,1)}$ is not invariant under tetrad rotations at $O(\zeta^0, \epsilon^1)$. Then to justify our arguments in Secs. 3.4.1 and 3.4.2, we need to first construct a tetrad- and coordinate-invariant dynamical curvature perturbation from $\Psi_0^{(1,1)}$. Although such a quantity has not been found in modified gravity yet, there have been similar efforts for second-order GW perturbations in GR. Reference [127] found that $\Psi_{0,4}^{(0,2)}$ are also not invariant under tetrad rotations and coordinate transformations at $O(\zeta^0, \epsilon^1)$. Solutions to this issue include adding correction terms to $\Psi_{0,4}^{(0,2)}$ to construct a tetrad- and coordinate-invariant quantity [127] or studying GW perturbations in an asymptotically flat representation of Kerr [128]. As shown in [102], our modified Teukolsky formalism can be directly mapped to the second-order Teukolsky formalism in GR in [127], so this issue in modified gravity can probably be solved using similar techniques. We leave the construction of a tetrad- and coordinate-invariant dynamical curvature perturbation and the definition of definite-parity modes in Petrov type I spacetimes in modified gravity to our future work.

Besides tetrad rotations at $O(\zeta^1, \epsilon^1)$ and $O(\zeta^0, \epsilon^1)$, we can also rotate the tetrad at $O(\zeta^1, \epsilon^0)$. In this case,

$$\Psi_0^{(1,1)} \to \Psi_0^{(1,1)} - 2[\delta A^{(1,0)} - i\vartheta^{(1,0)}]\Psi_0^{(0,1)} + 4b^{(1,0)}\Psi_1^{(0,1)} . \tag{3.52}$$

Since $\Psi_{0,1}^{(0,1)}$ are GW perturbations in GR, which are nonzero in general, $\Psi_0^{(1,1)}$ is not invariant under tetrad rotations at $O(\zeta^1, \epsilon^0)$. However, this behavior is very similar to tetrad rotations at $O(\zeta^0, \epsilon^0)$ in GR, which also shift $\Psi_{0,4}^{(0,1)}$ and correspond to large Lorentz transformations of the background spacetime. In contrast, gauge transformations enter at the same order as GW perturbations, or $O(\epsilon^1)$ in our notation. For Kerr BHs in GR, the Teukolsky equation is decoupled and separable in Boyer-Lindquist coordinates and in the Kinnersley tetrad, where definite-parity modes can also be easily defined. However, we do not expect the same situation in other coordinates and tetrads in general. Similarly, there are convenient coordinates and tetrads in modified gravity, where parity transformation on $\Psi_{0,4}^{(1,1)}$ is still described by the same operator $\hat{\mathcal{P}}$, with definite-parity modes having eigenvalues ± 1 under $\hat{\mathcal{P}}$. For bGR BHs invariant under parity, one can always use Eq. (3.40) to construct a bGR tetrad which transforms the same way as the Kinnersley tetrad under parity. In this case, the transformation of $\Psi_{0,4}^{(1,1)}$ is still described by $\hat{\mathcal{P}}$, with definite-parity modes having eigenvalues of ± 1. If one further rotates this tetrad at the $O(\zeta^1, \epsilon^0)$ order, then generically the parity transformation for $\Psi_{0,4}^{(1,1)}$ will have to be replaced by an operator different from $\hat{\mathcal{P}}$. Nevertheless, under that operator, definite-parity modes still exist, and will still have eigenvalue ± 1. For this reason, we can constrain the allowed tetrad rotations at $O(\zeta^1, \epsilon^0)$ to the ones preserving Eq. (3.41) (i.e., $\hat{\mathcal{P}}\{\delta A^{(1,0)}, \vartheta^{(1,0)}, b^{(1,0)}\} = \{\delta A^{(1,0)}, -\vartheta^{(1,0)}, -b^{(1,0)}\}$), similar to the choice in [126], so $\Psi_{0,4}^{(1,1)}$ has simple transformation under parity [i.e., Eq. (3.48)].

Finally, we can also perform coordinate transformations at $O(\zeta^1, \epsilon^1)$ and $O(\zeta^0, \epsilon^1)$. For the same reasons as discussed above, we do not care about coordinate transformations at $O(\zeta^0, \epsilon^0)$ and $O(\zeta^1, \epsilon^0)$. Under coordinate transformations $x^\mu \to x^\mu + \epsilon\xi^{\mu(0,1)} + \zeta\epsilon\xi^{\mu(1,1)}$, $\Psi_0^{(1,1)}$ transforms as [127],

$$\Psi_0^{(1,1)} \to \Psi_0^{(1,1)} + \xi^{\mu(0,1)}\partial_\mu\Psi_0^{(1,0)} + \xi^{\mu(1,1)}\partial_\mu\Psi_0^{(0,0)} . \tag{3.53}$$

For Petrov type D spacetimes in modified gravity, $\Psi_0^{(0,0)} = \Psi_0^{(1,0)} = 0$, so $\Psi_0^{(1,1)}$ is also coordinate-invariant. Thus, our arguments in Secs. 3.4.1 and 3.4.2 to find the transformation rule in Eq. (3.48) are both tetrad- and coordinate-invariant. For Petrov type I spacetimes, although one can set $\Psi_0^{(1,0)} \neq 0$ by rotating the background tetrad [51], we may have to work with the tetrad where $\Psi_0^{(1,0)} = 0$ to preserve our perturbation scheme in Sec. 3.2.1, as argued in [102]. Thus, we leave the construction of a tetrad- and coordinate-invariant dynamical curvature perturbation to our future work.

3.5 Isospectrality breaking in modified gravity

In Secs. 3.3 and 3.4, we have found the QNMs of Weyl scalars that generate definite-parity metric perturbations of Petrov type D spacetimes in GR and modified gravity. In this section, we compute the shift of QNM frequencies of these definite-parity modes using the modified Teukolsky equation found in [102, 103]. We first need to extract the source terms having overlaps with the QNMs in GR, which shift the QNM frequencies.

As shown in Sec. 3.3.1, metric reconstruction mixes the modes with frequency ω and $-\bar{\omega}$, so we need to disentangle the source terms with frequency ω from the terms with frequency $-\bar{\omega}$ within the modified Teukolsky equation. After finding the equations with definite-frequency source terms, we then apply the EVP approach of [103–105] to compute the shifts in QNMs. In the rest of this section, we show that the solutions form a two-dimensional subspace, the eigenvectors of which are two linear combinations of (l, m) and $(l, -m)$ modes determined by the source terms. The frequencies of these two linear combinations are generally different, so the degeneracy of each (l, m, ω) mode in GR is broken, like in quantum mechanics, as observed, for example, in dCS gravity [30, 31, 53–55], EdGB gravity [60–64], and higher-derivative gravity [56–58]. Nonetheless, in the special case that the source terms are invariant under the $\hat{\mathcal{P}}$-transformation, these eigenvectors become even- and odd-parity modes. One can thus see this section as a non-trivial extension of Sec. IVC in [103].

3.5.1 Identification of the source terms that shift QNM frequencies

In this subsection, we extract the source terms within the modified Teukolsky equations having overlaps with the QNMs in GR. In other words, we are interested in the terms of Eqs. (3.10) and (3.11) that are driven by $h_{\mu\nu}^{(0,1)}$ or $\Psi_{0,4}^{(0,1)}$.

In Sec. 3.2, we have discussed that there are two types of source terms. First, the source term $\mathcal{S}_{\text{geo}}^{(1,1)}$ [Eq. (3.10)] comes from the correction to the homogeneous part of the Bianchi and Ricci identities, so it is purely geometrical and only depends on corrections to the background geometry. The terms in $\mathcal{S}^{(1,1)}_{\rm geo}$ take the form of either $H_i^{(0,1)}\Psi_i^{(1,0)}$ or $H_i^{(1,0)}\Psi_i^{(0,1)}$, where H_i are operators that involve only the metric. Both $H_i^{(0,1)}$ and $\Psi_i^{(0,1)}$ are driven by $h_{\mu\nu}^{(0,1)}$ and can be reconstructed from the Hertz potential $\Psi_{\rm H}^{(0,1)}$.

The next set of source terms is encoded in $\mathcal{S}^{(1,1)}$ [Eq. (3.11)], which comes from the effective stress-energy tensor and depends on the details of the modified gravity theory. As discussed in Sec. 3.2, this type of source term contains two classes. For bGR theories of class B, there are no extra non-metric fields, so $\mathcal{S}^{(1,1)}$ is driven by $h_{\mu\nu}^{(0,1)}$ directly as shown in detail in [102]. For bGR theories of class A, there are extra non-metric fields, so we need to be more careful. As discussed in detail in [102], for class A bGR theories with non-minimal coupling, extra non-metric fields are first driven by $h_{\mu\nu}^{(0,1)}$, so they can also shift the QNM frequencies. Similar to the order-reduction scheme in [121], one can first solve the master equations of extra non-metric fields, which are also sourced Teukolsky equations when extra fields are of spin 0 or 1. Since the sources are GR metric fields, the equations of extra fields effectively decouple from the modified Teukolsky equation of metric fields, which occur at the next order in the bGR coupling. One can then apply standard techniques, such as Green functions, to solve for these extra fields first and then insert the solutions into the source terms of the modified Teukolsky equation. Nonetheless, the homogeneous part of these extra non-metric fields can oscillate at other frequencies, for example, as observed with the scalar perturbations in dCS gravity [31, 92, 93].

Incorporating only the source terms driven by $h_{\mu\nu}^{(0,1)}$, we can then write the modified Teukolsky equation in Eqs. (3.9)–(3.11) as

$$H\Psi^{(1,1)} = \mathcal{S}^{\mu\nu}h_{\mu\nu}^{(0,1)}\,, \tag{3.54}$$

where $H = H_0^{\rm GR}$ and $\Psi^{(1,1)} = \Psi_0^{(1,1)}$ for Ψ_0. The equation that $\Psi_4^{(1,1)}$ satisfies is of the same form, but with the replacements $H_0^{\rm GR} \to H_4^{\rm GR}$ and $\Psi_0^{(1,1)} \to \Psi_4^{(1,1)}$. In Sec. 3.3.1, we have shown how to reconstruct $h_{\mu\nu}^{(0,1)}$ from the Hertz potential $\Psi_{\rm H}^{(0,1)}$, i.e.,

$$h_{\mu\nu}^{(0,1)} = \left(\mathcal{O}_{\mu\nu} + \bar{\mathcal{O}}_{\mu\nu}\hat{\mathcal{C}}\right)\bar{\Psi}_{\rm H}^{(0,1)}\,. \tag{3.55}$$

Here, we have pulled out the complex conjugation operator $\hat{\mathcal{C}}$ acting on $\bar{\Psi}_{\rm H}^{(0,1)}$ in Eq. (3.27) since it transforms any mode with frequency ω to $-\bar{\omega}$. Furthermore, if one reconstructs $\bar{\Psi}_{\rm H}^{(0,1)}$ from $\Psi_0^{(0,1)}$ in IRG or $\Psi_4^{(0,1)}$ in ORG, the operators acting on $\bar{\Psi}_{\rm H}^{(0,1)}$ do not mix modes with different frequencies [i.e., Eq. (3.20)], as shown in

[83]. Thus, we can absorb these operators into $O_{\mu\nu}$ and $\bar{O}_{\mu\nu}$, so Eq. (3.54) can be further written as

$$H\Psi^{(1,1)} = \mathcal{S}^{\mu\nu}\left(O_{\mu\nu} + \bar{O}_{\mu\nu}\hat{C}\right)\Psi^{(0,1)} \,. \tag{3.56}$$

In principle, one can also have additional operators $\hat{C}$ within $\mathcal{S}^{\mu\nu}$, but since $h_{\mu\nu}^{(0,1)}$ is real, we do not need to pull them out. In the rest of this section, we take Eq. (3.56) as our modified Teukolsky equation.

3.5.2 Degeneracy breaking

In Eq. (3.56), we notice that the modes with frequency ω and $-\bar{\omega}$ are mixed due to the operator $\hat{C}$ acting on $\Psi^{(0,1)}$. Thus, to solve Eq. (3.56), one generally needs to consider the linear combinations

$$\begin{aligned}
\Psi_{\eta}^{(0,1)} &= \Psi^{(0,1)} + \eta\Psi_{\hat{\mathcal{P}}}^{(0,1)} \,, \\
\Psi_{\eta}^{(1,1)} &= \Psi^{(1,1)} + \eta\Psi_{\hat{\mathcal{P}}}^{(1,1)} \,.
\end{aligned} \tag{3.57}$$

Here, $\Psi_{\hat{\mathcal{P}}}^{(0,1)}$ and $\Psi_{\hat{\mathcal{P}}}^{(1,1)}$ are the modes with GR QNM frequency that is the negative complex conjugate of the frequency of $\Psi^{(0,1)}$ and $\Psi^{(1,1)}$ so that we can solve Eq. (3.56) consistently. The constant η is some complex number, though it is not completely defined at this moment since one can in principle absorb it into the definition of $\Psi_{\hat{\mathcal{P}}}^{(0,1)}$ and $\Psi_{\hat{\mathcal{P}}}^{(1,1)}$. In Eq. (3.62), we will fix the relative normalization between $\Psi^{(0,1)}$ and $\Psi_{\hat{\mathcal{P}}}^{(0,1)}$, so η will become well defined.

In Eq. (3.57), the modes $\Psi^{(0,1)}$ and $\Psi^{(1,1)}$ refer to a specific (l, m, ω) mode of $O(\zeta^0, \epsilon^1)$ and $O(\zeta^1, \epsilon^1)$ perturbations of $\Psi_{0,4}$, respectively, i.e.,

$$\begin{aligned}
\Psi^{(0,1)} &= \psi_{lm}^{(0,1)}(r,\theta)\exp\left[-i\left(\omega_{lm}^{(0)} + \zeta\omega_{lm}^{(1)}\right)t + im\phi\right], \\
\Psi^{(1,1)} &= \psi_{lm}^{(1,1)}(r,\theta)\exp\left[-i\left(\omega_{lm}^{(0)} + \zeta\omega_{lm}^{(1)}\right)t + im\phi\right],
\end{aligned} \tag{3.58}$$

where we have suppressed indices corresponding to the spin weight s and the overtone number n in these modes for simplicity. Notice that, in Eq. (3.58), we have perturbed the frequencies of both the GR QNM $\Psi^{(0,1)}$ and the bGR QNM $\Psi^{(1,1)}$, following the EVP method in [103–105]. Moreover, the QNM frequency shifts of $\Psi^{(0,1)}$ and $\Psi^{(1,1)}$ are the same; otherwise, the two sides of Eq. (3.56) cannot balance. This approach is the same, in essence, as the Poincaré-Lindstedt method of solving secular perturbation problems, introducing shifts of the QNM frequency to cancel off secularly growing terms due to the GR QNMs resonantly driving the modified Teukolsky equation. The shift in the QNM frequency plays a similar role to the slow timescale of multiple-scale analysis [129], which has been applied to spin-precessing systems and post-Newtonian dynamics in GR [130–132].

Similarly, the modes $\Psi^{(0,1)}_{\hat{\mathcal{P}}}$ and $\Psi^{(1,1)}_{\hat{\mathcal{P}}}$ correspond to the $(l,-m,-\bar{\omega}^{(0)}_{lm})$ mode of $\Psi^{(0,1)}$ and its perturbations, respectively, i.e.,

$$\begin{aligned} \Psi^{(0,1)}_{\hat{\mathcal{P}}} &= \psi^{(0,1)}_{\hat{\mathcal{P}}\,l-m}(r,\theta)\exp\left[-i\left(-\bar{\omega}^{(0)}_{lm}+\zeta\omega^{(1)}_{l-m}\right)t-im\phi\right], \\ \Psi^{(1,1)}_{\hat{\mathcal{P}}} &= \psi^{(1,1)}_{\hat{\mathcal{P}}\,l-m}(r,\theta)\exp\left[-i\left(-\bar{\omega}^{(0)}_{lm}+\zeta\omega^{(1)}_{l-m}\right)t-im\phi\right], \end{aligned} \tag{3.59}$$

where we have used that in GR, for any $\omega^{(0)}_{lm}$, there exists a $\omega^{(0)}_{l-m}$ such that $\omega^{(0)}_{l-m} = -\bar{\omega}^{(0)}_{lm}$. The modes $\Psi^{(1,1)}$ and $\Psi^{(1,1)}_{\hat{\mathcal{P}}}$ can be directly mapped to the modes $\psi^{\pm(2)}_{s}$ in [103]. In Eq. (3.57), we have distinguished the mode $\Psi^{(0,1)}_{\hat{\mathcal{P}}}$ and $\Psi^{(1,1)}_{\hat{\mathcal{P}}}$ from $\hat{\mathcal{P}}\Psi^{(0,1)}$ and $\hat{\mathcal{P}}\Psi^{(1,1)}$, the $\hat{\mathcal{P}}$-transformation of $\Psi^{(0,1)}$ and $\Psi^{(1,1)}$, since we do not know the relation between $\omega^{(1)}_{l-m}$ and $-\bar{\omega}^{(1)}_{lm}$ at this stage of the calculation in modified gravity. In the case that $\Psi^{(0,1)}_{\mathcal{P}} = \hat{\mathcal{P}}\Psi^{(0,1)}$ and $\Psi^{(1,1)}_{\mathcal{P}} = \hat{\mathcal{P}}\Psi^{(1,1)}$, the modes $\Psi^{(0,1)}_{\eta} + \zeta\Psi^{(1,1)}_{\eta}$ with $\eta = \pm 1$ have definite parity in Petrov type D spacetimes in modified gravity, as we have shown in Sec. 3.4.

Inserting the ansatz in Eq. (3.57) into Eq. (3.56), we find

$$\begin{aligned} & H\left(\Psi^{(1,1)}+\eta\Psi^{(1,1)}_{\hat{\mathcal{P}}}\right) \\ & = \mathcal{S}^{\mu\nu}\left[\left(O_{\mu\nu}\Psi^{(0,1)}+\bar{\eta}\bar{O}_{\mu\nu}\hat{C}\Psi^{(0,1)}_{\hat{\mathcal{P}}}\right)_{A}+\left(\bar{O}_{\mu\nu}\hat{C}\Psi^{(0,1)}+\eta O_{\mu\nu}\Psi^{(0,1)}_{\hat{\mathcal{P}}}\right)_{B}\right], \end{aligned} \tag{3.60}$$

where the first and last term on the right-hand side of Eq. (3.60) come from acting $O_{\mu\nu}$ in Eq. (3.56) on $\Psi^{(0,1)}$ and $\Psi^{(0,1)}_{\hat{\mathcal{P}}}$, respectively. The second and third term come from acting $\bar{O}_{\mu\nu}\hat{C}$ in Eq. (3.56) on $\Psi^{(0,1)}_{\hat{\mathcal{P}}}$ and $\Psi^{(0,1)}$, respectively. We have grouped the first two terms on the right-hand side together (i.e., group A) since they have the same GR QNM frequency $\omega^{(0)}_{\ell m}$. Similarly, the last two terms in group B have the same GR QNM frequency $-\bar{\omega}^{(0)}_{\ell m}$. In addition, we also need to match the bGR phase within group A or group B. Since the bGR QNM frequency of the first and second term in the group A is $\omega^{(1)}_{\ell m}$ and $-\bar{\omega}^{(1)}_{l-m}$, respectively, we have to impose

$$\omega^{(1)}_{l-m} = -\bar{\omega}^{(1)}_{lm}\,. \tag{3.61}$$

The same constraint can also be obtained by requiring the terms in group B to have the same bGR phase. After imposing Eq. (3.61), the phase of $H\Psi^{(1,1)}$ and $H\Psi^{(1,1,)}_{\hat{\mathcal{P}}}$ also match the phase of group A and B, respectively, so Eq. (3.60) is completely balanced and solvable. The frequency of $\Psi^{(0,1)}_{\hat{\mathcal{P}}}$ is now the complex conjugate of the frequency of $\Psi^{(0,1)}$. Since $\Psi^{(0,1)}_{\hat{\mathcal{P}}}$ is a $(l,-m)$ mode of the solution to the Teukolsky equation in GR, and $\hat{\mathcal{P}}\Psi^{(0,1)}_{lm\omega} = (-1)^l \Psi_{l-m-\bar{\omega}}$ [i.e., Eq. (3.36)], we can conveniently choose

$$\Psi^{(0,1)}_{\hat{\mathcal{P}}} = \hat{\mathcal{P}}\Psi^{(0,1)} \tag{3.62}$$

such that Eq. (3.60) becomes

$$\begin{aligned} &H\left(\Psi^{(1,1)} + \eta\Psi^{(1,1)}_{\hat{\mathcal{P}}}\right) \\ &= \mathcal{S}^{\mu\nu}\left[\left(\mathcal{O}_{\mu\nu} + \bar{\eta}\bar{\mathcal{O}}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right)\Psi^{(0,1)} + \left(\bar{\mathcal{O}}_{\mu\nu}\hat{C}\hat{\mathcal{P}} + \eta\mathcal{O}_{\mu\nu}\right)\hat{\mathcal{P}}\Psi^{(0,1)}\right], \end{aligned} \tag{3.63}$$

where we have pulled out a factor of $\hat{\mathcal{P}}$ in the second and third term using $\hat{\mathcal{P}}^2 = 1$. Notice, the operator $\hat{C}\hat{\mathcal{P}} = \hat{P}$ does not change the frequency of the mode it acts on.

Separating the equation into two parts for $\Psi^{(1,1)}$ and $\Psi^{(1,1)}_{\hat{\mathcal{P}}}$, we find

$$H\Psi^{(1,1)} = \mathcal{S}^{\mu\nu}\left(\mathcal{O}_{\mu\nu} + \bar{\eta}\bar{\mathcal{O}}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right)\Psi^{(0,1)}, \tag{3.64a}$$

$$\eta H\Psi^{(1,1)}_{\hat{\mathcal{P}}} = \mathcal{S}^{\mu\nu}\left(\eta\mathcal{O}_{\mu\nu} + \bar{\mathcal{O}}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right)\hat{\mathcal{P}}\Psi^{(0,1)}. \tag{3.64b}$$

Acting $\hat{\mathcal{P}}$ on Eq. (3.64), we also find

$$H\left(\hat{\mathcal{P}}\Psi^{(1,1)}\right) = (\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\left(\mathcal{O}_{\mu\nu} + \eta\bar{\mathcal{O}}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right)\hat{\mathcal{P}}\Psi^{(0,1)}, \tag{3.65a}$$

$$\bar{\eta}H\left(\hat{\mathcal{P}}\Psi^{(1,1)}_{\hat{\mathcal{P}}}\right) = (\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\left(\bar{\eta}\mathcal{O}_{\mu\nu} + \bar{\mathcal{O}}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right)\Psi^{(0,1)}, \tag{3.65b}$$

where we have used that

$$\hat{\mathcal{P}}H = H\,, \quad \hat{\mathcal{P}}\mathcal{O}_{\mu\nu} = \mathcal{O}_{\mu\nu}\,. \tag{3.66}$$

One can notice that Eq. (3.64) is redundant with respect to Eq. (3.65), since the latter is just a $\hat{\mathcal{P}}$-transformation of the former, both of which have to be satisfied simultaneously. Therefore, one can either solve the pair of Eqs. (3.64a) and (3.65b) or the pair of Eqs. (3.64b) and (3.65a). Let us focus on the first pair and apply the EVP method in [103–105] to remove the wave functions at $O(\zeta^1, \epsilon^1)$. References [103–105] have defined an inner product

$$\langle \tilde{H}\psi(r,\theta)|\varphi(r,\theta)\rangle = \langle \psi(r,\theta)|\tilde{H}\varphi(r,\theta)\rangle\,, \tag{3.67}$$

that makes the Teukolsky operator H in GR self-adjoint, where $\tilde{H}$ is the harmonic decomposition of H into modes $e^{-i\omega t + im\phi}$. The functions $\psi(r,\theta)$ and $\varphi(r,\theta)$ are the (r,θ) parts of any modes satisfying the QNM boundary conditions, such as $\psi^{(0,1)}_{lm}(r,\theta)$ and $\psi^{(1,1)}_{lm}(r,\theta)$ in Eq. (3.58). Expanding $\omega_{\ell m}$ about ζ on the left-hand side of Eqs. (3.64a) and (3.65b) and taking the inner product [i.e., Eq. (3.67)] of these two equations with $\Psi^{(0,1)}$, we find

$$\frac{1}{\langle\partial_\omega\tilde{H}\rangle_{lm}}\begin{pmatrix}\langle\mathcal{S}^{\mu\nu}O_{\mu\nu}\rangle_{lm} & \langle\mathcal{S}^{\mu\nu}\bar{O}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\rangle_{lm}\\ \langle(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\bar{O}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\rangle_{lm} & \langle(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})O_{\mu\nu}\rangle_{lm}\end{pmatrix}\begin{pmatrix}1\\ \bar{\eta}\end{pmatrix}=\omega^{(1)}_{lm}\begin{pmatrix}1\\ \bar{\eta}\end{pmatrix}, \tag{3.68}$$

where we have defined the shorthand notation

$$\langle O\rangle_{lm}=\langle\psi^{(0,1)}_{lm}|\tilde{O}\psi^{(0,1)}_{lm}\rangle\,, \tag{3.69}$$

where $\tilde{O}$ is the harmonic decomposition of the operator O into modes $e^{-i\omega t+im\phi}$. In Eq. (3.68), to remove $\psi^{(1,1)}_{lm}$, we have used that $\psi^{(0,1)}_{lm}$ solves the Teukolsky equation in GR, i.e., $\tilde{H}\psi^{(0,1)}_{lm}=0$, such that $\langle\psi^{(0,1)}_{lm}|\tilde{H}\psi^{(1,1)}_{lm}\rangle=0$.

Equation (3.68) is a standard eigenvalue problem in degenerate perturbation theory. One can either calculate the eigenvalues of Eq. (3.68) directly or follow [103] to solve for $\bar{\eta}$ first. Multiplying the first equation in Eq. (3.68) by $\bar{\eta}$ and equating the left-hand side of it to the left-hand side of the second equation, we find a quadratic equation in $\bar{\eta}$,

$$\bar{\eta}^2\left\langle\mathcal{S}^{\mu\nu}\bar{O}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right\rangle_{lm}+\bar{\eta}\left\langle\left[\mathcal{S}^{\mu\nu}-(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\right]O_{\mu\nu}\right\rangle_{lm}-\left\langle(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\bar{O}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right\rangle_{lm}=0\,. \tag{3.70}$$

Since Eq. (3.70) is quadratic, $\bar{\eta}$ has two solutions $\bar{\eta}^{\pm}$,

$$\begin{aligned}\bar{\eta}^{\pm}=\frac{1}{2\left\langle\mathcal{S}^{\mu\nu}\bar{O}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right\rangle_{lm}}&\Bigg\{\left\langle\left[(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})-\mathcal{S}^{\mu\nu}\right]O_{\mu\nu}\right\rangle_{lm}\\ &\pm\sqrt{\left\langle\left[\mathcal{S}^{\mu\nu}-(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\right]O_{\mu\nu}\right\rangle^2_{lm}+4\left\langle\mathcal{S}^{\mu\nu}\bar{O}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right\rangle_{lm}\left\langle(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\bar{O}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right\rangle_{lm}}\Bigg\}\,.\end{aligned} \tag{3.71}$$

For each solution, we find a correction $\omega^{(1)}_{lm}$ to the frequency of the mode $(l,m,\omega^{(0)}_{lm})$ in GR by solving $\omega^{(1)}_{lm}$ from the first equation of Eq. (3.68) in terms of $\bar{\eta}$ and plugging in the solutions of $\bar{\eta}$ in Eq. (3.71),

$$\begin{aligned}\omega^{\pm(1)}_{lm}&=\frac{\left\langle\mathcal{S}^{\mu\nu}\left(O_{\mu\nu}+\bar{\eta}^{\pm}\bar{O}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right)\right\rangle_{lm}}{\langle\partial_\omega\tilde{H}\rangle_{lm}}\\ &=\frac{1}{2\langle\partial_\omega\tilde{H}\rangle_{lm}}\Bigg\{\left\langle\left[\mathcal{S}^{\mu\nu}+(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\right]O_{\mu\nu}\right\rangle_{lm}\\ &\quad\pm\sqrt{\left\langle\left[\mathcal{S}^{\mu\nu}-(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\right]O_{\mu\nu}\right\rangle^2_{lm}+4\left\langle\mathcal{S}^{\mu\nu}\bar{O}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right\rangle_{lm}\left\langle(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\bar{O}_{\mu\nu}\hat{C}\hat{\mathcal{P}}\right\rangle_{lm}}\Bigg\}\,.\end{aligned} \tag{3.72}$$

The QNM frequency shifts $\omega_{lm}^{\pm(1)}$ in Eq. (3.72) can be also found by solving the characteristic equation of the matrix in Eq. (3.68) directly. One can take the difference of $\omega_{lm}^{\pm(1)}$ to characterize the degree of isospectrality breaking, i.e.,

$$\begin{aligned}\delta\omega_{lm}^{(1)} &= \omega_{lm}^{+(1)} - \omega_{lm}^{-(1)} \\ &= \frac{\sqrt{\left\langle\left[\mathcal{S}^{\mu\nu} - (\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\right]\mathcal{O}_{\mu\nu}\right\rangle_{lm}^{2} + 4\left\langle\mathcal{S}^{\mu\nu}\bar{\mathcal{O}}_{\mu\nu}\hat{\mathcal{C}}\hat{\mathcal{P}}\right\rangle_{lm}\left\langle(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\bar{\mathcal{O}}_{\mu\nu}\hat{\mathcal{C}}\hat{\mathcal{P}}\right\rangle_{lm}}}{\langle\partial_\omega\tilde{H}\rangle_{lm}}\,.\end{aligned} \tag{3.73}$$

Thus, for isospectrality to be preserved beyond GR, we need

$$\left\langle\left[\mathcal{S}^{\mu\nu} - (\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\right]\mathcal{O}_{\mu\nu}\right\rangle_{lm}^{2} + 4\left\langle\mathcal{S}^{\mu\nu}\bar{\mathcal{O}}_{\mu\nu}\hat{\mathcal{C}}\hat{\mathcal{P}}\right\rangle_{lm}\left\langle(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\bar{\mathcal{O}}_{\mu\nu}\hat{\mathcal{C}}\hat{\mathcal{P}}\right\rangle_{lm} = 0\,. \tag{3.74}$$

One possibility is that the first and the second term in Eq. (3.74) vanishes independently. Notice that $\left\langle\mathcal{S}^{\mu\nu}\bar{\mathcal{O}}_{\mu\nu}\hat{\mathcal{C}}\hat{\mathcal{P}}\right\rangle_{lm}$ and $\left\langle(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\bar{\mathcal{O}}_{\mu\nu}\hat{\mathcal{C}}\hat{\mathcal{P}}\right\rangle_{lm}$ have to vanish *simultaneously* for the following reason. First, we must simultaneously have $\left\langle\mathcal{S}^{\mu\nu}\bar{\mathcal{O}}_{\mu\nu}\hat{\mathcal{C}}\hat{\mathcal{P}}\right\rangle_{l\pm m} = 0$ since we are solving the (l,m) and $(l,-m)$ modes jointly. More specifically, one can set $\Psi^{(0,1)}$ and $\Psi^{(1,1)}$ to be the $(l,-m)$ mode in Eq. (3.57), repeat the same argument above, and replace $\langle\cdots\rangle_{lm}$ in Eq. (3.74) with $\langle\cdots\rangle_{l-m}$ in the end. Due to $\hat{\mathcal{P}}\Psi_{lm}^{(0,1)} = (-1)^l\Psi_{l-m}^{(0,1)}$ [Eq. (3.36)] and Eq. (3.62), the $(l,-m)$ mode of $\Psi_\eta^{(0,1)}$ in Eq. (3.57) only differs from the (l,m) mode by an overall constant, so $\omega_{lm}^{(1)} = \omega_{l-m}^{(1)}$, and any constraint on isospectrality must be redundant for $(l,\pm m)$. Next, since $\hat{\mathcal{P}}\Psi_{lm}^{(0,1)} = (-1)^l\Psi_{l-m}^{(0,1)}$ and $\hat{\mathcal{P}}\bar{\mathcal{O}}_{\mu\nu} = \bar{\mathcal{O}}_{\mu\nu}$, $\left\langle(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\bar{\mathcal{O}}_{\mu\nu}\hat{\mathcal{C}}\hat{\mathcal{P}}\right\rangle_{l\mp m}$ is a $\hat{\mathcal{P}}$-transformation of $\left\langle\mathcal{S}^{\mu\nu}\bar{\mathcal{O}}_{\mu\nu}\hat{\mathcal{C}}\hat{\mathcal{P}}\right\rangle_{l\pm m}$, and these terms have to vanish simultaneously. The same arguments also work for $\left\langle\mathcal{S}^{\mu\nu}\mathcal{O}_{\mu\nu}\right\rangle_{lm}$ and $\left\langle(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu})\mathcal{O}_{\mu\nu}\right\rangle_{lm}$. In this case, the matrix in Eq. (3.68) becomes diagonal, so its eigenvectors are $(1,0)$ and $(0,1)$. The first eigenvector can be directly found from Eq. (3.70). The second eigenvector is not directly captured by Eq. (3.70) since we have fixed the normalization of $\Psi^{(0,1)}$ and $\Psi^{(1,1)}$ to be unity in Eq. (3.57). Nonetheless, Eqs. (3.68) and (3.70) need to be satisfied for $(l,-m)$, the solution to which corresponds to the second eigenvector. This indicates that the (l,m) and $(l,-m)$ modes decouple in Eq. (3.63).

For the two terms in Eq. (3.74) to vanish, there are several sub-cases. First, let us consider the case when the operators inside the inner products vanish, i.e., $\hat{\mathcal{P}}\mathcal{S}^{\mu\nu} = \mathcal{S}^{\mu\nu}$ and $\mathcal{S}^{\mu\nu}\bar{\mathcal{O}}_{\mu\nu} = 0$, the latter of which indicates that the source terms do not contain any complex conjugation $\hat{\mathcal{C}}$ acting on the GR QNMs according to Eq. (3.56). This condition usually cannot happen since most operators in the source terms $\mathcal{S}^{(1,1)}$ and $\mathcal{S}_{\rm geo}^{(1,1)}$ [i.e., Eqs. (3.4), (3.5), and (3.12)] contain both the NP quantities and their complex conjugates. Furthermore, as we will see in Sec. 3.6 and

Appendix 3.11, the Ricci tensor in the NP basis in many bGR theories also contains both types of terms. At $O(\zeta^0, \epsilon^1)$, this mixing of NP quantities and their complex conjugates results in a mixing of terms proportional to $\Psi_0^{(0,1)}$ and $\bar{\Psi}_0^{(0,1)}$, where $\mathcal{S}^{\mu\nu}$ cannot annihilate $\bar{O}_{\mu\nu}$, as one can observe in the reconstructed quantities in Sec. 3.3.1 and Appendix 3.9.

One exception is when the source terms in Eq. (3.56) only contain $\mathcal{S}_{\rm geo}^{(1,1)}$, and the bGR background spacetime is Petrov type D, so $\mathcal{S}_{\rm geo}^{(1,1)}$ takes the form $\mathcal{S}_{\rm geo}^{(1,1)} = -H_0^{(1,0)}\Psi_0^{(0,1)}$ [Eq. (3.10)]. In this case, one can easily verify that $\hat{\mathcal{P}}H_0^{(1,0)} = H_0^{(1,0)}$ using the transformation properties in Sec. 3.4.1 (we have assumed that the background metric is invariant under $\hat{\mathcal{P}}$), and no complex conjugation $\hat{C}$ acts on Ψ_0, so isospectrality is preserved for this correction, as we will discuss in more detail in Sec. 3.6. Notice, in the case that $\hat{\mathcal{P}}\mathcal{S}^{\mu\nu} = \mathcal{S}^{\mu\nu}$ (the source term is preserved under parity) but the second term in Eq. (3.74) does not vanish, isospectrality can still break, as we will discuss in more detail in Sec. 3.5.3.

Second, let us consider the case in which the operators in Eq. (3.74) annihilate the mode $\psi_{lm}^{(0,1)}$. This can happen, for example, when the operator is proportional to the Teukolsky operator (or with additional operators acting on it). Third, let us consider the case in which the mode $\psi_{lm}^{(0,1)}$ and the one after applying the operators in Eq. (3.74) are orthogonal, so their inner product vanishes. For example, if the operator shifts the l of $\psi_{lm}^{(0,1)}$, the two modes are orthogonal to each other due to the orthogonality of the spin-weighted spheroidal harmonics. However, we do not know any beyond-GR theory where the second or the third case naturally happens unless we design the corrections deliberately.

Another possibility to have Eq. (3.74) satisfied is that the two terms cancel each other, instead of vanishing independently. Since $O_{\mu\nu}$ and $\bar{O}_{\mu\nu}$ are fixed by the metric reconstruction procedures, we can only play with $\mathcal{S}^{\mu\nu}$. However, it is almost impossible to construct such a $\mathcal{S}^{\mu\nu}$ consistently for all (l, m) modes, since $\mathcal{S}^{\mu\nu}$ is symmetric, and thus, it only contains 10 independent components. Moreover, the eigenvectors corresponding to Eq. (3.71) are now degenerate, e.g.,

$$\bar{\eta}^{+} = \bar{\eta}^{-} = \frac{\left\langle \left[(\hat{\mathcal{P}}\mathcal{S}^{\mu\nu} - \mathcal{S}^{\mu\nu}) \right] O_{\mu\nu} \right\rangle_{lm}}{2\left\langle \mathcal{S}^{\mu\nu}\bar{O}_{\mu\nu}\hat{C}\hat{\mathcal{P}} \right\rangle_{lm}} \neq 0\,. \tag{3.75}$$

Unlike the $\bar{\eta}^{\pm} = 0$ case, this degeneracy is not due to our assumption that the modes $\Psi^{(0,1)}$ and $\Psi^{(1,1)}$ have a nonzero amplitude [i.e., Eq. (3.57)]. Thus, the matrix in Eq. (3.68) truly has an incomplete basis of eigenvectors, which indicates that the full solution to Eq. (3.64) may also contain modes that are not harmonic in time, i.e., $\Psi^{(1,1)} \propto t e^{-i(\omega_{lm}^{(0)}+\zeta\omega_{lm}^{(1)})t}$, similar to the critically damping case of a simple harmonic oscillator. Since this linear dependence in t enters at $O(\zeta^0)$, this mode is different from expanding the exponential in Eq. (3.58) about ζ, where one gets $\Psi^{(1,1)} \propto \left[1 - i\zeta\omega_{lm}^{(1)}t + O(\zeta^2)\right] e^{-i\omega_{lm}^{(0)}t}$. Therefore, this term cannot be re-absorbed by perturbing the QNM frequency. Physically, this means that the BH potential generating this mode responds to GW perturbations strongly. Nonetheless, this mode may not be truly problematic since the imaginary part of ω is always negative, so the mode still exponentially decays in time.

Except for these special cases, the isospectrality of even- and odd-parity modes in the QNM frequencies at each (l, m) is broken by modified gravity corrections. In Sec. 3.6, we will see that $\omega_{lm}^{+(1)} \neq \omega_{lm}^{-(1)}$ (after summing up the contribution from both $\mathcal{S}^{(1,1)}$ and $\mathcal{S}_{\rm geo}^{(1,1)}$) in all the examples we have considered.

A similar analysis specifically for higher-derivative gravity was done by [56] using metric perturbations, and by [57, 58] using the modified Teukolsky equation. Here, by following [103], we provide a more general equation of QNM frequency shifts [e.g., Eq. (3.72)], valid for a broad class of modified gravity theories. Our results are consistent with [103], but simplified using the parity properties of H and $\mathcal{O}_{\mu\nu}$. This allows for a systematic study of the relation between modified gravity corrections and the structure of isospectrality breaking.

3.5.3 Solutions with definite parity

One may also want to know when the modified Teukolsky equation still admits definite-parity solutions, i.e., solutions for which $\hat{\mathcal{P}}\Psi_{\rm E,O} = \pm(-1)^l\Psi_{\rm E,O}$ [i.e., Eq. (3.30)]. For this reason, let us consider the special case that $\eta = \pm(-1)^l$ and $\Psi_{\hat{\mathcal{P}}}^{(1,1)} = \hat{\mathcal{P}}\Psi^{(1,1)}$, which corresponds to even- and odd-parity modes for Petrov type D spacetimes in modified gravity, as shown in Sec. 3.4. Inserting $\eta = \pm(-1)^l$ into Eq. (3.64a), we find the definite-parity modified Teukolsky equations to be

$$H\Psi_{\rm E,O}^{(1,1)} = \mathcal{S}^{\mu\nu}\left(\mathcal{O}_{\mu\nu} \pm (-1)^l\bar{\mathcal{O}}_{\mu\nu}\hat{\mathcal{C}}\hat{\mathcal{P}}\right)\Psi^{(0,1)} . \tag{3.76}$$

The solutions to Eq. (3.76) can then be obtained from Eq. (3.72), i.e.,

$$\omega_{lm}^{\rm E,O(1)} = \frac{\left\langle \mathcal{S}^{\mu\nu}\left(\mathcal{O}_{\mu\nu} \pm (-1)^l\bar{\mathcal{O}}_{\mu\nu}\hat{\mathcal{C}}\hat{\mathcal{P}}\right)\right\rangle_{lm}}{\langle\partial_\omega\tilde{H}\rangle_{lm}} . \tag{3.77}$$

On the other hand, Eqs. (3.64a) and (3.65b) need to be satisfied simultaneously, and we have only used Eq. (3.64a) to get Eq. (3.77). From Eq. (3.65b), we similarly find

$$\omega_{lm}^{\mathrm{E,O}(1)} = \frac{\left\langle \hat{\mathcal{P}} \mathcal{S}^{\mu\nu} \left(O_{\mu\nu} \pm (-1)^l \bar{O}_{\mu\nu} \hat{C} \hat{\mathcal{P}} \right) \right\rangle_{lm}}{\langle \partial_\omega \tilde{H} \rangle_{lm}} \,. \tag{3.78}$$

Comparing Eqs. (3.78) to (3.77), we find a constraint on $\mathcal{S}^{\mu\nu}$, i.e.,

$$\hat{\mathcal{P}} \mathcal{S}^{\mu\nu} = \mathcal{S}^{\mu\nu} \,, \tag{3.79}$$

which implies that the source term $\mathcal{S}^{\mu\nu}$ needs to transform in the same way as the Teukolsky operator H in GR under $\hat{\mathcal{P}}$. On the other hand, if one assumes $\hat{\mathcal{P}} \mathcal{S}^{\mu\nu} = \mathcal{S}^{\mu\nu}$, one finds that $\eta = \pm 1$ using Eq. (3.70) and $\Psi_{\hat{\mathcal{P}}}^{(1,1)} = \hat{\mathcal{P}} \Psi^{(1,1)}$ using Eqs. (3.64a) and (3.65b). Thus, for Petrov type D spacetimes in modified gravity, the solutions to the modified Teukolsky equation generate definite-parity perturbations if and only if $\hat{\mathcal{P}} \mathcal{S}^{\mu\nu} = \mathcal{S}^{\mu\nu}$.

The constraint in Eq. (3.79) is closely related to how one diagonalizes the correction to the Hamiltonian for degenerate systems in quantum mechanics. For degenerate perturbation theory in quantum mechanics, the modes that naturally diagonalize the perturbed Hamiltonian are the eigenvectors of a certain Hermitian operator that commutes with both the original Hamiltonian and the perturbation to the Hamiltonian. In our case, the operator $\hat{\mathcal{P}}$ commutes with both the Teukolsky equation in GR and the modified Teukolsky equation, since according to Eqs. (3.66) and (3.79),

$$[\hat{\mathcal{P}}, H] f = (\hat{\mathcal{P}} H)(\hat{\mathcal{P}} f) - H(\hat{\mathcal{P}} f) = 0 \,, \tag{3.80a}$$

$$[\hat{\mathcal{P}}, \mathcal{S}^{\mu\nu}] f = (\hat{\mathcal{P}} \mathcal{S}^{\mu\nu})(\hat{\mathcal{P}} f) - \mathcal{S}^{\mu\nu}(\hat{\mathcal{P}} f) = 0 \,. \tag{3.80b}$$

Thus, the even- and odd-parity modes naturally "diagonalize" the modified Teukolsky equation when $\hat{\mathcal{P}} \mathcal{S}^{\mu\nu} = \mathcal{S}^{\mu\nu}$. In more general cases, when $\hat{\mathcal{P}}$ does not commute with the source terms, one must diagonalize manually as in Sec. 3.5.2. In the next section, we apply the analysis developed in this section to two specific modified gravity theories: dCS and EdGB gravity.

3.6 Application

In this section, we apply the formalism above to specific corrections to the Teukolsky equation for two relatively simple examples. In particular, we consider two widely studied modified gravity theories: dCS and EdGB gravity. We will not present the details of these two theories here, since one can find them in [28–31, 53–55, 106, 107] for dCS and in [60–64, 108, 133, 134] for EdGB gravity. We choose to follow the convention of the action in [106] for dCS and [63] for EdGB theory. The formalism developed in this work also directly applies to other bGR theories with parity-invariant BH solutions that are Petrov type D. For example, all spherically symmetric BH spacetimes in bGR theories are both Petrov type D and parity-invariant [135]. Furthermore, although the definition of definite-parity modes in Sec. 3.3.2 only applies to parity-invariant Petrov type D spacetimes, the analysis of isospectrality breaking in Sec. 3.5 generally applies to any bGR theories admitting an EFT description.

As discussed in Secs. 3.2 and 3.5, modifications to the Teukolsky equation generally originate from two different places:

1. The modification to the background geometry, e.g., $\mathcal{S}^{(1,1)}_{\text{geo}}$.

2. The effective stress-energy tensor specific to each modified gravity theory, e.g., $\mathcal{S}^{(1,1)}$.

For all these examples of modified gravity theories, we discuss the leading contribution to both types of source terms. In this work, we also choose to focus on the Petrov type D backgrounds and leave the generalization to the non-Petrov-type D case for future work. It was found in [136] that rotating BHs in both dCS and EdGB gravity are Petrov type D below or at $O(\chi)$, where $\chi = a/M$ is the dimensionless spin, and become Petrov type I beyond $O(\chi)$. This implies we must focus only on slowly rotating BHs in dCS and EdGB gravity to linear order in rotation. Thus, we must carry out an additional expansion in χ such that all the quantities are expanded as

$$\Psi = \Psi^{(0,0,0)} + \zeta\Psi^{(1,0,0)} + \chi\Psi^{(0,1,0)} + \epsilon\Psi^{(0,0,1)} + \cdots . \tag{3.81}$$

For simplicity, we also focus on the equation that governs the evolution of Ψ_0, while the equation for Ψ_4 can be easily obtained by a GHP transformation [102].

3.6.1 dCS gravity

In this theory, there is no correction to the background geometry for non-rotating BHs, since the Pontryagin density vanishes for spherically-symmetric spacetimes [106]. In this case, the leading order correction to the background geometry enters at $O(\zeta^1, \chi^1, \epsilon^0)$ with

$$\zeta = \zeta_{\text{dCS}} \equiv \frac{\alpha_{\text{dCS}}^2}{\kappa_g M^4}, \quad \kappa_g \equiv \frac{1}{16\pi G}, \tag{3.82}$$

where $\alpha_{\rm dCS}$ is the coupling constant of dCS gravity, and we have chosen the coupling constant of the pseudoscalar field action $\beta = 1$. For simplicity, we will drop the subscript labeling the modified gravity theory. Then, the leading-order contribution to $\mathcal{S}_{\rm geo}^{(1,1)}$ enters at $O(\zeta^1, \chi^1, \epsilon^1)$. Since slowly rotating BHs at $O(\zeta^1, \chi^1, \epsilon^0)$ in dCS gravity are of Petrov type D [106], $\mathcal{S}_{\rm geo}^{(1,1,1)}$ only depends on the correction to the Teukolsky operator $H_0^{(1,1,0)}$. On the other hand, since the pseudoscalar field ϑ is driven by the GW perturbations, there is a nonzero effective stress tensor at $O(\zeta^1, \chi^0, \epsilon^1)$ [30, 53, 54]. Thus, the leading contribution to $\mathcal{S}^{(1,1)}$ is $\mathcal{S}^{(1,0,1)}$.

3.6.1.1 Correction due to $\mathcal{S}_{\rm geo}^{(1,1)}$ in dCS

Expanding the Teukolsky equation in GR to $O(\chi^1)$, we first find

$$
\begin{aligned}
H_0^{(0,0)} &= H_0^{(0,0,0)} + \chi H_0^{(0,1,0)} + \cdots\,, \\
H_0^{(0,0,0)} &= \frac{1}{r-r_s}\left[-6(r-r_s) + 4r(r-3M)\partial_t + r^3\partial_t^2\right] - 6(r-M)\partial_r - r(r-r_s)\partial_r^2 \\
&\quad - \cot\theta\partial_\theta - \partial_\theta^2 + \csc\theta^2\left(4 - 4i\cos\theta\partial_\phi - \partial_\phi^2\right)\,, \\
H_0^{(0,1,0)} &= -4M\left\{\frac{1}{r(r-r_s)}\left[(r-M)\partial_\phi + Mr\partial_t\partial_\phi\right] - i\cos\theta\partial_t\right\}\,,
\end{aligned}
\tag{3.83}
$$

where $r_s = 2M$ is the Schwarzschild radius, $H_0^{(0,0,0)}$ is the Teukolsky operator on the Schwarzschild background in GR, and $H_0^{(0,1,0)}$ is the leading slow-rotation correction to $H_0^{(0,0,0)}$. We have also restored the full coordinate dependence of these operators here. Under the $\hat{\mathcal{P}}$ transformation, we find that $\hat{\mathcal{P}}H_0^{(0,0,0)} = H_0^{(0,0,0)}$, so the Teukolsky equation on the Schwarzschild background in GR admits definite-parity solutions. In addition, $\hat{\mathcal{P}}H_0^{(0,1,0)} = H_0^{(0,1,0)}$, which is expected since the Kerr background admits GW perturbations of definite parity. There is also no isospectrality breaking due to $\mathcal{S}_{\rm geo}^{(0,1,1)} = H_0^{(0,1,0)}\Psi_0^{(0,0,1)}$, since there is no mixing of modes with frequency ω and $-\bar{\omega}$.

Next, let us compute $H_0^{(1,1,0)}$. In this work, we only sketch the key steps, and more details can be found in [137], where the complete modified Teukolsky equation (before separation into definite-parity parts) is found up to $O(\zeta^1, \chi^1, \epsilon^1)$. To compute $H_0^{(1,1,0)}$, one needs to first find a corrected tetrad that satisfies all the orthogonality conditions at $O(\zeta^1, \chi^1, \epsilon^0)$. The background spacetime at $O(\zeta^1, \chi^1, \epsilon^0)$ was found in [106], where all the components of $h_{\mu\nu}^{(1,1,0)}$ vanish except

$$
h_{t\phi}^{(1,1,0)} = \frac{5M^5}{8r^4}\left(1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right)\sin^2\theta\,. \tag{3.84}
$$

One can explicitly check that $\hat{\mathcal{P}} h_{\mu\nu}^{(1,1,0)} = h_{\mu\nu}^{(1,1,0)}$ in dCS gravity, consistent with our assumption. Since the background is still Petrov type D [106], by tetrad rotations, one can find a frame where $\Psi_i^{(1,1,0)} = 0$ for $i = \{0, 1, 3, 4\}$. Notice that in this frame, one does not necessarily have $\kappa^{(1,1,0)} = \sigma^{(1,1,0)} = \lambda^{(1,1,0)} = \nu^{(1,1,0)} = 0$, as implied by the Goldberg-Sachs theorem [51], since we are in a non-Ricci-flat spacetime [137]. One can now use this modified tetrad and Eq. (3.12) to compute $H_0^{(1,1,0)}$,

$$
\begin{aligned}
H_{0,\mathrm{dCS}}^{(1,1,0)} = &\ \frac{M^4}{224 r^7 (r - r_s)} \left(C_1(r) \partial_\phi - 4 r^2 C_2(r) \partial_\phi \partial_t \right) \\
&- \frac{i M^4}{8 r^7} \cos\theta \left(C_3(r) + \frac{r^2 D_4(r)}{2} \partial_t \right) \\
&+ \frac{i M^4}{16 r^6} \left[(r - r_s) C_4(r) \cos\theta \partial_r - \frac{C_5(r)}{2r} \sin\theta \partial_\theta \right] ,
\end{aligned}
\tag{3.85}
$$

where $C_i(r)$ are functions of r found in [137] and listed in Appendix 3.10 for convenience. We can check that $\hat{\mathcal{P}} H_{0,\mathrm{dCS}}^{(1,1,0)} = H_{0,\mathrm{dCS}}^{(1,1,0)}$, so the modified Teukolsky equation up to $O(\zeta^1, \chi^1, \epsilon^1)$ admits definite-parity solutions if we ignore the source term $\mathcal{S}^{(1,1,1)}$ associated with ϑ. Similar to the $O(\zeta^0, \chi^1, \epsilon^1)$ correction, since there is no mixing of modes with different frequencies, $\mathcal{S}_{\mathrm{geo}}^{(1,1,1)}$ preserves isospectrality.

3.6.1.2 Correction due to $\mathcal{S}^{(1,1)}$ in dCS

In this subsection, we compute the correction due to $\mathcal{S}^{(1,1)}$. As discussed above, the leading contribution to $\mathcal{S}^{(1,1)}$ is $\mathcal{S}^{(1,0,1)}$ in dCS gravity. In these previous works [30, 53, 54], they found that only the odd-parity modes are modified for non-rotating BHs in dCS gravity. We now verify this result using our formalism based on the Teukolsky framework.

As found in [106], the trace-reverse Einstein equations takes the form

$$
\begin{aligned}
R_{\mu\nu} = &- \left(\frac{1}{\kappa_g} \right)^{1/2} M^2 \left[(\nabla^\sigma \vartheta)\, \epsilon_{\sigma\delta\alpha(\mu} \nabla^\alpha R_{\nu)}{}^\delta + \left(\nabla^\sigma \nabla^\delta \vartheta \right) {}^*R_{\delta(\mu\nu)\sigma} \right] \\
&+ \frac{1}{2\kappa_g \zeta} \left(\nabla_\mu \vartheta \right) \left(\nabla_\nu \vartheta \right) .
\end{aligned}
\tag{3.86}
$$

To be consistent with [102, 137], we have absorbed an additional factor of $\zeta^{1/2}$ into the expansion of ϑ, so its expansion also follows Eq. (3.2). In other words, we have multiplied the first and second terms in Eq. (3.86) by $\zeta^{-1/2}$ while the third term by ζ^{-1}. The equation of motion of ϑ at $O(\zeta^1, \epsilon^1)$ is then [102]

$$
\square^{(0,0)} \vartheta^{(1,1)} = - \frac{1}{16 \pi^{1/2}} M^2 \left[R\, {}^*R \right]^{(0,1)} - \square^{(0,1)} \vartheta^{(1,0)} .
\tag{3.87}
$$

In this work, we are interested in modified BH spacetimes that are vacuum in GR, so $R_{\mu\nu} = 0$ at $O(\zeta^0)$. As argued in [102], all the metric fields in $R_{\mu\nu}$ have to enter at $O(\zeta^0)$ in Eq. (3.86), so the first term in this equation vanishes. In addition, since there is no correction to the background metric at $O(\zeta^1, \chi^0, \epsilon^0)$, $\vartheta^{(1,0,0)} = 0$ [106]. At $O(\zeta^1, \chi^0, \epsilon^1)$, the last term in Eq. (3.86) is proportional to $\left(\nabla_\mu \vartheta^{(1,0,1)}\right)\left(\nabla_\nu \vartheta^{(1,0,0)}\right)$, but $\vartheta^{(1,0,0)} = 0$, so this term vanishes. In the end, only the second term in Eq. (3.86) contributes at $O(\zeta^1, \chi^0, \epsilon^1)$. Since only $\vartheta^{(1,0,1)}$ is nonzero, the term ${}^*R_{\delta(\mu\nu)\sigma}$ coupled to it has to be stationary, so we do not need metric reconstruction here. Now the only term that can mix modes with different frequencies, and thus break isospectrality, is $\vartheta^{(1,0,1)}$, so we need to focus on Eq. (3.87).

Since $\vartheta^{(1,0,0)} = 0$, the last term in Eq. (3.87) vanishes, and only the first term in Eq. (3.87) is important. Projecting this term into the NP basis, one can find that

$$R\,{}^*R = -8i(3\Psi_2^2 - 4\Psi_1\Psi_3 + \Psi_0\Psi_4 - c.c.)\,, \tag{3.88}$$

which is made up of quadratic terms in Weyl scalars. Since we are interested in $O(\epsilon^1)$ corrections, one of the Weyl scalars in each pair has to be stationary. For Petrov type D spacetimes, all the Weyl scalars vanish except Ψ_2, so

$$[R\,{}^*R]^{(0,1)} = -48i\left(\Psi_2^{(0,0)}\Psi_2^{(0,1)} - \bar{\Psi}_2^{(0,0)}\bar{\Psi}_2^{(0,1)}\right). \tag{3.89}$$

In Schwarzschild, $\Psi_2^{(0,0,0)} = -M/r^3$ is real, so

$$[R\,{}^*R]^{(0,0,1)} = \frac{48iM}{r^3}\left(\Psi_2^{(0,0,1)} - \bar{\Psi}_2^{(0,0,1)}\right) = -\frac{96M}{r^3}\mathcal{I}\left[\Psi_2^{(0,0,1)}\right], \tag{3.90}$$

where $\mathcal{I}[f]$ refers to the imaginary part of f.

One can now naturally ask whether we can remove $[R\,{}^*R]^{(0,0,1)}$ via a tetrad rotation or a coordinate transformation at $O(\zeta^0, \chi^0, \epsilon^1)$. The answer is no. First, one can explicitly check that all the tetrad rotations at $O(\zeta^0, \chi^0, \epsilon^1)$ leave $\Psi_2^{(0,0,1)}$ unchanged since $\Psi_1^{(0,0,0)} = \Psi_3^{(0,0,0)} = 0$. Second, under the coordinate transformation $x^\mu \to x^\mu + \xi^\mu$, where ξ is at $O(\zeta^0, \chi^0, \epsilon^1)$, $\Psi_2^{(0,0,1)}$ transforms as [127]

$$\Psi_2^{(0,0,1)} \to \Psi_2^{(0,0,1)} + \xi^{\mu(0,0,1)}\partial_\mu \Psi_2^{(0,0,0)}\,, \tag{3.91}$$

which implies that

$$\mathcal{I}\left[\Psi_2^{(0,0,1)}\right] \to \mathcal{I}\left[\Psi_2^{(0,0,1)}\right] + \xi^{\mu(0,0,1)}\partial_\mu \mathcal{I}\left[\Psi_2^{(0,0,0)}\right]. \tag{3.92}$$

Since $\Psi_2^{(0,0,0)}$ is real, $\mathcal{I}\left[\Psi_2^{(0,0,0)}\right] = 0$, so $[R\,{}^*R]^{(0,0,1)}$ is invariant under both tetrad and coordinate transformations at $O(\zeta^0, \chi^0, \epsilon^1)$.

More generally, for an arbitrary χ, the source term in Eq. (3.87) is still tetrad- and coordinate-invariant. The tetrad invariance is easy to confirm since $\Psi_2^{(0,1)}$ is invariant under tetrad rotations at $O(\zeta^0, \epsilon^1)$ if the background spacetime is Petrov type D at $O(\zeta^0, \epsilon^0)$. Furthermore, $\Box^{(0,1)}$ is tetrad-invariant. On the other hand, unlike at $O(\chi^0)$, $\Psi_2^{(0,0)}$ is complex, so the coordinate invariance needs to be shown in another way. For convenience, let us denote the source term in Eq. (3.87) as $\mathcal{S}_\vartheta^{(1,1)}$. Then under the coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ at $O(\zeta^0, \epsilon^1)$, we find

$$\mathcal{S}_\vartheta^{(1,1)} \to \mathcal{S}_\vartheta^{(1,1)} + \xi^{\mu(0,1)} \partial_\mu \mathcal{S}_\vartheta^{(0,0)} , \tag{3.93}$$

but notice that

$$\mathcal{S}_\vartheta^{(0,0)} = -\frac{1}{16\pi^{1/2}} M^2 [R\, {}^*R]^{(0,0)} - \Box^{(0,0)} \vartheta^{(1,0)} = 0 \tag{3.94}$$

due to the same equation of Eq. (3.87) at $O(\zeta^1, \epsilon^0)$, i.e.,

$$\Box^{(0,0)} \vartheta^{(1,0)} = -\frac{1}{16\pi^{1/2}} M^2 \, [R\, {}^*R]^{(0,0)} \; . \tag{3.95}$$

Thus, $\mathcal{S}_\vartheta^{(1,1)}$ is both tetrad- and coordinate-invariant.

By pure order counting, one can write the source term $\mathcal{S}$ in Eq. (3.11) as

$$\mathcal{S}_{\rm dCS}^{(1,0,1)} = \mathcal{F}^{\rm dCS} \left(\Psi_2^{(0,0,1)} - \bar{\Psi}_2^{(0,0,1)} \right) , \tag{3.96}$$

where $\mathcal{F}^{\rm dCS}$ is some complicated operator converting the source term driving ϑ in Eq. (3.87) to the source term in Eq. (3.11). The operator $\mathcal{F}^{\rm dCS}$ contains three parts:

1. The inversion of $\Box$ in Eq. (3.87) to solve for ϑ.
2. The NP quantities and derivatives acting on ϑ in Eq. (3.86).
3. The tetrad acting on $R_{\mu\nu}$ to convert it to NP Ricci scalars and the operators in Sec. 3.2 to convert NP Ricci scalars to $\mathcal{S}_{\rm dCS}$.

Despite being a complicated operator, $\mathcal{F}^{\rm dCS}$ only contains stationary terms, so it will not mix modes with different frequencies. Thus, to study isospectrality-breaking properties without computing QNM shifts, we do not need to know the exact form of $\mathcal{F}^{\rm dCS}$.

To show that only the odd-parity modes are modified, we essentially need to show that Eq. (3.72) vanishes for $\eta = (-1)^l$ but not for $\eta = (-1)^{(l+1)}$. Then, we need to use metric reconstruction to rewrite $\Psi_2^{(0,0,1)}$ in terms of $\Psi_0^{(0,0,1)}$ or the Hertz potential, e.g., Eq. (3.22). For simplicity, we can absorb the operator $-D^2/2$ into $\mathcal{F}^{\text{dCS}}$ since D is a real operator. We then rewrite Eq. (3.96) as

$$\mathcal{S}_{\text{dCS}} = \mathcal{F}^{\text{dCS}} \left(\mathcal{O} - \bar{\mathcal{O}}\hat{\mathcal{C}} \right) \bar{\Psi}_{\text{H}}^{(0,0,1)} \,, \;\; \mathcal{O} = (\bar{\delta} + 2\beta)(\bar{\delta} + 4\beta) \,. \tag{3.97}$$

Comparing Eq. (3.97) to Eq. (3.56), one can extract that

$$\mathcal{S}^{\mu\nu}\mathcal{O}_{\mu\nu} = \mathcal{F}^{\text{dCS}}\mathcal{O} \,, \quad \mathcal{S}^{\mu\nu}\bar{\mathcal{O}}_{\mu\nu} = -\mathcal{F}^{\text{dCS}}\bar{\mathcal{O}} \,, \tag{3.98}$$

so Eq. (3.72) becomes

$$\omega_{lm}^{\text{dCS}(1)} = \frac{\left\langle \mathcal{F}^{\text{dCS}} \left(\mathcal{O} \mp (-1)^l \bar{\mathcal{O}}\hat{\mathcal{C}}\hat{\mathcal{P}} \right) \mathscr{D} \right\rangle_{lm}}{\langle \partial_\omega \tilde{H}_0 \rangle_{lm}} \,, \tag{3.99}$$

where $\mathscr{D}$ denotes the operator converting $\Psi_0^{(0,0,1)}$ to $\bar{\Psi}_{\text{H}}^{(0,0,1)}$ in Eq. (3.20).

With this expression at hand, we can now show that $\mathcal{O}\mathscr{D}\bar{\Psi}_0^{(0,0,1)} = (-1)^l \bar{\mathcal{O}}\hat{\mathcal{C}}\hat{\mathcal{P}}\mathscr{D}\bar{\Psi}_0^{(0,0,1)}$ or $\mathcal{O}\bar{\Psi}_{\text{H}}^{(0,0,1)} = (-1)^l \bar{\mathcal{O}}\hat{P}\bar{\Psi}_{\text{H}}^{(0,0,1)}$, and thus, only the odd modes are modified. Using Eqs. (3.26) and (3.97), one can find that

$$\hat{\mathcal{P}}\mathcal{O} = \mathcal{O} \,, \tag{3.100}$$

so $\bar{\mathcal{O}} = \hat{P}\mathcal{O}$. Then, it is equivalent to show that $\hat{P}\left(\mathcal{O}\bar{\Psi}_{\text{H}}^{(0,0,1)}\right) = (-1)^l \mathcal{O}\bar{\Psi}_{\text{H}}^{(0,0,1)}$. In other words, $\mathcal{O}\bar{\Psi}_{\text{H}}^{(0,0,1)}$ transforms in the same way as $Y_{lm}(\theta,\phi)$ under the standard parity transformation $\hat{P}$. The easiest way to show this is to recognize that

$$\left(\bar{\delta} + 2s\beta\right) f = -\frac{1}{\sqrt{2}r}\bar{\eth} f \,, \quad \bar{\eth} f = -\left(\partial_\theta - i\csc\theta\partial_\phi + s\cot\theta\right) f \,, \tag{3.101}$$

where f has spin weight s, and $\bar{\eth}$ is the operator lowering spin weight by 1 [138], e.g.,

$$\bar{\eth}\, {}_sY_{lm} = -[(l+s)(l-s+1)]^{1/2}\, {}_{s-1}Y_{lm} \,. \tag{3.102}$$

From Eq. (3.20), one can notice that $\bar{\Psi}_{\text{H}}$ has the same spin weight as Ψ_0 in IRG, e.g., $\bar{\Psi}_{\text{H}}^{(0,0,1)} \propto {}_2Y_{lm}(\theta,\phi)$. Then,

$$\mathcal{O}\bar{\Psi}_{\text{H}}^{(0,0,1)} = \frac{1}{2r^2}\bar{\eth}^2\bar{\Psi}_{\text{H}}^{(0,0,1)} \propto Y_{lm}(\theta,\phi) \,, \tag{3.103}$$

so $\mathcal{O}\bar{\Psi}_{\text{H}}^{(0,0,1)}$ transforms like $Y_{lm}(\theta,\phi)$ under parity transformations, i.e., $\hat{P}\left(\mathcal{O}\bar{\Psi}_{\text{H}}^{(0,0,1)}\right) = (-1)^l\mathcal{O}\bar{\Psi}_{\text{H}}^{(0,0,1)}$.

To check whether the modified Teukolsky equation has definite-parity solutions, we need to check whether $\hat{\mathcal{P}}\mathcal{S}^{\mu\nu(1,0,1)} = \mathcal{S}^{\mu\nu(1,0,1)}$. Since $\hat{\mathcal{P}}O_{\mu\nu} = O_{\mu\nu}$ and $\hat{\mathcal{P}}O = O$, using Eq. (3.98), one can alternatively check whether $\hat{\mathcal{P}}\mathcal{F}^{\text{dCS}} = \mathcal{F}^{\text{dCS}}$. In this case, one has to know the exact expression of $\mathcal{S}_{\text{dCS}}$. Generally, this is a non-trivial calculation, but for non-rotating BHs in dCS gravity, since only ϑ is dynamical in $\mathcal{S}_{\text{dCS}}$, one can easily evaluate $\mathcal{S}_{\text{dCS}}$ in terms of ϑ and the background metric without doing metric reconstruction. In Appendix 3.11.1, we show how to evaluate $\mathcal{S}_{\text{dCS}}^{(1,0,1)}$ in dCS gravity in detail and provide the result of $\mathcal{F}_{\text{dCS}}$ in Eq. (3.145). One can now easily verify that $\hat{\mathcal{P}}\mathcal{F}^{\text{dCS}} = \mathcal{F}^{\text{dCS}}$, so the modified Teukolsky equation at $O(\zeta^1, \chi^0, \epsilon^1)$ in dCS gravity still admits definite-parity solutions.

In this subsection, we have so far shown successfully that only the odd modes are modified for non-rotating BHs in dCS gravity using the NP language developed in this work, which is consistent with [30, 53, 54]. One can also carry out the same calculation at $O(\zeta^1, \chi^1, \epsilon^0)$ and compare to the results using metric perturbations in [31, 55]. In Sec. 3.6.1.1, we have found the correction due to $\mathcal{S}_{\text{geo}}^{(1,1,1)}$. The correction due to $\mathcal{S}_{\text{dCS}}^{(1,1,1)}$ is much more complicated, and can be found in [137]. In [139], we will apply the formalism developed in this work and the expression found in [137] to compute the correction to QNMs directly and compare to [31, 55]. Another interesting avenue for future work is to find a direct mapping between the modified RW (ZM) equations in [30, 31, 53–55] and the odd (even) modified Teukolsky equations in this work, as we discuss in Sec. 3.7.

3.6.2 EdGB gravity

The structure of the metric shifts to the EdGB BH solution is qualitatively different from that of the dCS case. In EdGB, we follow [140] to define the expansion parameter ζ to be

$$\zeta = \zeta_{\text{EdGB}} \equiv \frac{\alpha_{\text{EdGB}}^2}{\kappa_g M^4}\,, \tag{3.104}$$

where α_{EdGB} is the coupling constant of EdGB gravity in [63], and we have chosen the coupling constant of the scalar field action $\beta = 1$.[3] One finds that the shifts in the background metric at $O(\zeta)$ actually contain a component that is independent of spin. In other words, EdGB gravity perturbs the non-rotating BH solution. Since non-rotating BHs in EdGB gravity are Petrov type D [140], the leading correction to $\mathcal{S}_{\text{geo}}^{(1,1)}$

[3] Note that we choose to follow the convention of the EdGB action in [63], whereas the expressions in Eq. (3.105) are taken from [140]. In this case, the coupling constant α_{EdGB} in [140] is $\frac{1}{64\pi}$ of the one in [63]. We adjust the expressions in Eqs. (3.105) and (3.138) to account for the change in convention.

is $\mathcal{S}_{\rm geo}^{(1,0,1)}$, completely determined by $H_0^{(1,0,0)}$. Similarly, the leading contribution to $\mathcal{S}^{(1,1)}$ is $\mathcal{S}^{(1,0,1)}$.

3.6.2.1 Correction due to $\mathcal{S}_{\rm geo}^{(1,1)}$ in EdGB

The corrections due to $\mathcal{S}_{\rm geo}^{(1,1)}$ are made of several parts as noted in Eq. (3.10). However, note that at the leading order $(\zeta^1, \chi^0, \epsilon^0)$ in EdGB, we know that $\Psi_{0,1,3,4}^{(1,0)} = 0$. Hence, the only contribution to $\mathcal{S}_{\rm geo}^{(1,1)}$ comes from $\mathcal{S}_{0,D}^{(1,1)} = H_0^{(1,0)}\Psi_0^{(0,1)}$. By following similar procedures to what we presented in Sec. 3.6.1.1 for dCS, and using the metric from [140],

$$
\begin{aligned}
h_{tt}^{(1,0,0)} &= -\frac{1}{(64\pi)^2}\frac{M^3}{3r^3}\left[1+\frac{26M}{r}+\frac{66M^2}{5r^2}+\frac{96M^3}{5r^3}-\frac{80M^4}{r^4}\right],\\
h_{rr}^{(1,0,0)} &= -\frac{1}{(64\pi)^2}\frac{M^2}{(r-2M)^2}\left[1+\frac{M}{r}+\frac{52M^2}{3r^2}+\frac{2M^3}{r^3}+\frac{16M^4}{5r^4}-\frac{368M^5}{3r^5}\right],
\end{aligned}
\tag{3.105}
$$

where the other components of $h_{\mu\nu}^{(1,0,0)}$ vanish, we find

$$
H_{0,\rm EdGB}^{(1,0,0)} = D_1(r) + D_2(r)\partial_t + D_3(r)\partial_r + D_4(r)\partial_t^2 + D_5(r)\partial_r^2\,. \tag{3.106}
$$

Note that the factor of $1/(64\pi)^2$ comes from the use of the conventions of $\alpha_{\rm EdGB}$ in [63]. The full form of the radial functions $D_i(r)$ can be found in Appendix 3.10. One can also easily check that $\hat{\mathcal{P}}h_{\mu\nu}^{(1,0,0)} = h_{\mu\nu}^{(1,0,0)}$ for EdGB, consistent with our assumption. Since $H_{0,\rm EdGB}^{(1,0,0)}$ does not depend on the θ or ϕ coordinates, we can use Eq. (3.125) to easily show that

$$
\hat{\mathcal{P}}H_{0,\rm EdGB}^{(1,0,0)} = H_{0,\rm EdGB}^{(1,0,0)}\,. \tag{3.107}
$$

In this case, the modified Teukolsky equation still admits definite-parity solutions up to $\mathcal{O}(\zeta^1, \chi^0, \epsilon^1)$ if we ignore the source terms driven by the nonminimally coupled scalar field φ. Furthermore, in EdGB, $\mathcal{S}_{\rm geo}^{(1,0,1)}$ does not break isospectrality because it does not mix the modes with frequencies ω and $-\bar{\omega}$.

3.6.2.2 Correction due to $\mathcal{S}^{(1,1)}$ in EdGB

In this subsection, we compute $\mathcal{S}^{(1,0,1)}$ in EdGB gravity, where the calculation is similar to the dCS case. First, the trace-reversed Einstein equations in EdGB gravity take the form [63, 134],

$$
R_{\mu\nu} = -\kappa_g^{1/2}M^2\left(\mathcal{K}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{K}\right) + \frac{1}{2\zeta}(\nabla_\mu\varphi)(\nabla_\nu\varphi)\,,
$$

$$\mathcal{K}_{\mu\nu} = \frac{1}{8}\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right)\epsilon^{\delta\sigma\gamma\alpha}\nabla_\beta\left({}^*R^{\rho\beta}{}_{\gamma\alpha}e^{\varphi}\nabla_\delta\varphi\right),$$
$$\mathcal{K} = g_{\mu\nu}\mathcal{K}^{\mu\nu}, \tag{3.108}$$

with the equation of the scalar field φ at $O(\zeta^1, \epsilon^1)$ being [102]

$$\Box^{(0,0)}\varphi^{(1,1)} = -\frac{1}{16\pi^{1/2}}M^2\mathcal{G}^{(0,1)} - \Box^{(0,1)}\varphi^{(1,0)}, \tag{3.109}$$

where the Gauss-Bonnet invariant $\mathcal{G}$ is defined to be

$$\mathcal{G} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2. \tag{3.110}$$

Similar to the dCS case, we have absorbed a factor of ζ into the expansion of φ by multiplying the first term of $R_{\mu\nu}$ in Eq. (3.108) by $\zeta^{-1/2}$ and the second term by ζ^{-1}.

Unlike in dCS gravity, since $\varphi^{(1,0,0)} \neq 0$ for non-rotating BHs in EdGB, we need metric reconstruction to evaluate $h_{\mu\nu}^{(0,0,1)}$ coupled to $\varphi^{(1,0,0)}$ for $O(\zeta^1, \chi^0, \epsilon^1)$ corrections. For simplicity, in this work, we only consider the terms in Eq. (3.108) driven by $\varphi^{(1,0,1)}$ or its derivatives. The evaluation of the terms proportional to $h_{\mu\nu}^{(0,0,1)}$ or its derivatives in Eq. (3.108) is more complicated, while a similar calculation in dCS gravity has been done in [137]. Under this simplification, all the metric fields in Eq. (3.108) can be evaluated at the Schwarzschild background, and we can focus on Eq. (3.109). To evaluate the source terms in Eq. (3.109), we need to compute $\Box$ and $\mathcal{G}$ in the NP basis. At $O(\zeta^1, \chi^0, \epsilon^1)$, since $\varphi^{(1,0,0)} \neq 0$, unlike the pseudoscalar field in dCS, both terms in Eq. (3.109) will contribute.

For the term $\mathcal{G}^{(0,1)}$, we find in the NP basis,

$$\mathcal{G} = 8(3\Psi_2^2 - 4\Psi_1\Psi_3 + \Psi_0\Psi_4 + c.c.). \tag{3.111}$$

We can notice that, in the dCS case, $R\,{}^*R$ is proportional to the imaginary part of $3\Psi_2^2 - 4\Psi_1\Psi_3 + \Psi_0\Psi_4$, while $\mathcal{G}$ is proportional to the real part of the same quantity. Expanding Eq. (3.111) to $O(\zeta^0, \epsilon^1)$, we find

$$\mathcal{G}^{(0,1)} = 48\left(\Psi_2^{(0,0)}\Psi_2^{(0,1)} + \bar{\Psi}_2^{(0,0)}\bar{\Psi}_2^{(0,1)}\right), \tag{3.112}$$

which in Schwarzschild becomes

$$\mathcal{G}^{(0,0,1)} = -\frac{48M}{r^3}\left(\Psi_2^{(0,0,1)} + \bar{\Psi}_2^{(0,0,1)}\right) = -\frac{96M}{r^3}\mathcal{R}\left[\Psi_2^{(0,0,1)}\right], \tag{3.113}$$

where $\mathcal{R}[f]$ refers to the real part of f. Following the same reasoning as in Sec. 3.6.1.2, one can argue that the part of $\mathcal{S}^{(1,0,1)}$ generated by $\mathcal{G}^{(0,0,1)}$ takes the form

$$\mathcal{F}^{\rm EdGB}\left(\Psi_2^{(0,0,1)} + \bar{\Psi}_2^{(0,0,1)}\right), \tag{3.114}$$

where $\mathcal{F}^{\rm EdGB}$ contains pieces similar to $\mathcal{F}^{\rm dCS}$ but with the effective Ricci tensor given by Eq. (3.108). If one only considers the shift of QNM frequencies due to this term, then $\omega^{(1)}_{lm}$ is given by Eq. (3.99), with $\mathcal{F}_{\rm dCS}$ replaced by $\mathcal{F}_{\rm EdGB}$ and the sign $\mp$ between the terms proportional to $\mathcal{O}$ and $\mathcal{O}\hat{\mathcal{C}}\hat{\mathcal{P}}$ replaced by $\pm$, so the QNMs of odd-parity modes are not modified by these terms.

For the contribution from $\square^{(0,0,1)}\varphi^{(1,0,0)}$, we have shown in detail how to reconstruct $\square^{(0,0,1)}$ in [137], so here we just present the results we found,

$$\square^{(0,0,1)}\varphi^{(1,0,0)} = -\frac{1}{2r^3}(r\partial_r^2 + 2\partial_r)\Phi(r)\bar{\eth}^2\left(\Psi_{\rm H}^{(0,0,1)} + \bar{\Psi}_{\rm H}^{(0,0,1)}\right)\,, \tag{3.115}$$

where we have used that $\varphi^{(0,0,1)} = \Phi(r)$ is a pure radial function in EdGB gravity. Similar to the case of dCS gravity, the source term in Eq. (3.109) is tetrad- and coordinate-invariant, following the same argument in Sec. 3.6.1.2.

In total, following the same procedures in Sec. 3.6.1.2, we find

$$\omega_{lm}^{\rm EdGB(1)} = \frac{\left\langle \mathcal{F}^{\rm EdGB}\left(\mathcal{O}' \pm (-1)^l \bar{\mathcal{O}}'\hat{\mathcal{C}}\hat{\mathcal{P}}\right)\mathscr{D}\right\rangle_{lm}}{\langle \partial_\omega \tilde{H}_0\rangle_{lm}}\,, \tag{3.116}$$

where

$$\mathcal{O}'\bar{\Psi}_{\rm H}^{(0,0,1)} = \left[\frac{1}{2r}\left(r\partial_r^2 + 2\partial_r\right)\Phi(r) - \frac{3}{4\pi^{1/2}}\left(\frac{M}{r}\right)^3 D^2\right]\frac{1}{r^2}\bar{\eth}^2\bar{\Psi}_{\rm H}^{(0,0,1)} \propto Y_{lm}(\theta,\phi)\,, \tag{3.117}$$

and the last term of Eq. (3.117) comes from the $\mathcal{O}$ operator defined in Eq. (3.97). Unlike in Sec. 3.6.1.2, we have not absorbed the factor of $-D^2/2$ in $\mathcal{O}$ into the definition of $\mathcal{F}^{\rm EdGB}$. Following the same argument in Sec. 3.6.1.2, one can easily see that for this type of contribution, only the QNM frequencies of even parity are shifted. This result is consistent with [60–62] since what we have essentially shown is that the scalar field φ is only driven by even-parity gravitational perturbations at $\mathcal{O}(\zeta^1, \chi^0, \epsilon^1)$. In Appendix 3.11.2, we further show that the modified Teukolsky equation driven by this contribution still admits solutions of definite parity by showing that $\hat{\mathcal{P}}\mathcal{F}^{\rm EdGB} = \mathcal{F}^{\rm EdGB}$.

3.7 Discussion

In this work, we developed a framework to study the isospectrality breaking of QNMs in modified gravity using the modified Teukolsky formalism developed in [102, 103]. To analyze isospectrality breaking using the Teukolsky formalism, one has to first know how definite-parity modes are defined in terms of Weyl scalars $\Psi_{0,4}$. In GR, we followed [90] to construct definite-parity modes of $\Psi_{0,4}^{(0,1)}$ by using the relation between metric perturbations and the Hertz potential. We found that at each (l, m), the Weyl scalars generating definite-parity metric perturbations are linear combinations of the mode (l, m, ω) and its $\hat{\mathcal{P}}$-transformation, where $\hat{\mathcal{P}}$ is the parity transformation, but with an additional complex conjugation. Due to the transformation properties of Teukolsky functions under $\hat{\mathcal{P}}$, these modes are equal to the sum (difference) of the modes (l, m, ω) and $(l, -m, -\bar{\omega})$ for even (odd) parity, consistent with the definition in [90].

In modified gravity, we showed that the same definition in GR still applies for Petrov type D spacetimes. Since the relation between metric perturbations and the Hertz potential is not known in modified gravity in general, we instead started from definite-parity metric perturbations and derived the parity properties of $\Psi_{0,4}^{(1,1)}$ directly. The entire procedure is closely related to reconstructing NP quantities from metric perturbations. Using tetrad rotations, we first found some convenient gauges where the transformation property of both the background and the dynamical tetrad is simple under $\hat{\mathcal{P}}$. The transformation property of spin coefficients was then determined from commutation relations. Using Ricci identities, we finally obtained the $\hat{\mathcal{P}}$-transformation of $\Psi_{0,4}^{(1,1)}$ generated by definite-parity metric perturbations, which is the same as in GR.

After defining definite-parity modes of $\Psi_{0,4}^{(0,1)}$ and $\Psi_{0,4}^{(1,1)}$, we then proceeded to derive the equations that govern them from the modified Teukolsky equation. Since the source terms that shift QNM frequencies are those having overlaps with QNMs in GR, we first extracted these source terms. To evaluate the latter, one needs to perform metric reconstruction, which mixes the modes with frequency ω and $-\bar{\omega}$. Thus, the solutions to the modified Teukolsky equation are also linear combinations of these two modes in general. Using the EVP method developed in [103–105], we then found that the solutions form a two-dimensional subspace, so the degeneracy in QNM frequencies of even- and odd-parity modes is broken in modified gravity in general, consistent with the finding in [103].

In the special case that the solutions become even- and odd-parity modes, the source terms of the modified Teukolsky equation are constrained to transform in the same way as the Teukolsky operator in GR under $\hat{\mathcal{P}}$. This constraint is closely related to how one solves the degenerate perturbation problem in quantum mechanics. We showed that the invariance of the source operator $\mathcal{S}^{\mu\nu}$ and the Teukolsky operator H

under $\hat{\mathcal{P}}$ implies that they commute with $\hat{\mathcal{P}}$. Similarly, in quantum mechanics, one can diagonalize the perturbed Hamiltonian by using the eigenstates of a Hermitian operator commuting with both the original and the perturbed Hamiltonian.

To demonstrate our framework, we then applied this analysis of isospectrality breaking to two specific cases: dCS and EdGB gravity. For simplicity, we only considered the leading correction to the homogeneous part of the equation and the leading contribution from the effective stress-energy tensor. In dCS gravity, we found that the correction to the homogeneous part does not break isospectrality. For the correction from the effective stress-energy tensor, we showed that only the odd modes are shifted for non-rotating BHs using our modified Teukolsky formalism, consistent with the results found using metric perturbations in [30, 53, 54]. For EdGB gravity, we similarly found no isospectrality breaking in the homogeneous part. For the correction due to the stress-energy tensor, we only focused on the terms driven by the dynamical scalar field. In this case, only the even modes are affected, consistent with the result in [60–62].

There are several future avenues of research that our work enables. First, one can study potential observational signatures of isospectrality breaking in QNM frequencies, the most direct of which would be the branching of the QNM spectrum. One can investigate the extraction of these branching QNMs from real observational data using well tested ringdown analysis frameworks like `ringdown` [141] and `PyRing` [142]. Additionally, an analysis of how the SNR affects this extraction (similar to resolvability arguments in [65]) can aid an understanding of when this effect may be significant. Furthermore, these definite-parity modes of $\Psi_{0,4}$ are also related to other decompositions of gravitational perturbations, such as mass or current quadrupoles, left- or right-circularly polarized modes, and plus or cross polarizations. By studying these relations, one can translate the isospectrality breaking of QNM frequencies to observational effects in other modes or polarizations. For example, different QNM frequencies of even- and odd-parity modes might lead to different frequencies of plus and cross polarizations. In fact, some parity violating theories even feature different propagation speeds for different polarizations — parameterized theory-agnostic tests for such theories were laid out, for example, in [36] and more recently in [41], and connected to specific parity violating theories in [41].

Second, when determining the $\hat{\mathcal{P}}$-transformation of $\Psi_{0,4}^{(1,1)}$, we have made some convenient gauge choices. The choice of certain gauges is fine for Petrov type D spacetimes since $\Psi_{0,4}^{(1,1)}$ are both tetrad- and coordinate-invariant up to $O(\zeta^1, \epsilon^1)$, which is not the case for Petrov type I spacetimes. The same issue was also encountered in the study of the second-order Teukolsky equation in GR [88, 127], where various authors found that $\Psi_{0,4}^{(0,2)}$ is also not invariant under gauge transformations at $O(\epsilon^1)$. By either adding quantities at $O(\epsilon^1)$ to $\Psi_{0,4}^{(0,2)}$ [127], or by choosing some asymptotically flat coordinate system [88], one is able to construct gauge-invariant curvature perturbations at $O(\epsilon^2)$. Due to the connection of our modified Teukolsky formalism to the second-order Teukolsky formalism in GR [102], we can then apply a similar procedure to construct gauge-invariant quantities at $O(\zeta^1, \epsilon^1)$ and extend our definition of definite-parity modes to Petrov type I spacetimes in future work.

Third, in our examples in dCS and EdGB gravity, we have found the shifts of the QNM frequencies in terms of abstract NP quantities. One can then take the equations here to compute the numerical values of the QNM frequencies directly and compare them to previous results using metric perturbations in [30, 31, 53–55, 60–64]. A subset of the authors has already derived the coordinate-based modified Teukolsky equation for slowly rotating BHs in dCS gravity [137] and is currently computing the QNM shifts from it [139]. Furthermore, in our examples above, we assumed that the (pseudo)scalar field equation could be inverted, and the EVP method could be applied to these terms. In previous literature, the EVP method has only been shown to be valid for source terms that are directly driven by $\Psi_{0,4}^{(0,1)}$ [103–105]. It will be worth studying whether the inner product in the EVP method is still well-behaved when integrating the Green's function of the (pseudo)scalar field along with $\Psi_{0,4}^{(0,1)}$ over the contour defined in [103–105]. In the case without extra non-metric fields, such as higher-derivative gravity, Refs. [57, 58] have used this modified Teukolsky formalism to compute QNM shifts and found good agreement with their previous results using metric perturbations in [56].

Besides comparing QNM frequencies, another way of comparing our results to the approach using metric perturbations is to directly map our definite-parity modified Teukolsky equations [i.e., Eq. (3.76)] to the modified ZM and RW equations directly. For non-rotating BHs in GR, this map was found by Chandrasekhar [51, 143]. Due to isospectrality, one can also find a map between RW and ZM equations [51], which was recently extended to slowly rotating BHs at $O(\chi^1)$ by [56]. For Kerr, there has not been a map found between the Teukolsky equation and the RW/ZM equations, since the latter is not known for BHs with arbitrary spin. Some Chandrasekhar-like transformations have been developed, for example, to convert the long-range potential of the Teukolsky equation to a short-range potential [144].

All these transformations are special cases of generalized Darboux transformations [145], which have been widely used in the study of supersymmetry [146]. One future avenue is then extending the Chandrasekhar transformation to our modified Teukolsky equation in the slow-rotation expansion and comparing it to the modified ZM and RW equations found in [30, 31, 53–56, 60–64, 91].

Our framework provides a novel approach to studying isospectrality breaking in modified gravity. This work is an intermediate but imperative step in using the modified Teukolsky equations to compute the shifts of QNM frequencies. For BHs with arbitrary spin, our framework is also the only viable analytical approach to study isospectrality breaking since QNMs can only be computed from the (modified) Teukolsky equation, and no (modified) ZM/RW equations are known in this case. Recent works have used spectral methods to investigate quasinormal modes (QNMs) within the context of Schwarzschild [100] and Kerr [101] BHs in GR. It is conceivable that these methodologies may be extensible to explore the QNMs of rotating BHs within modified theories of gravity. With this work, we look forward to developing a deeper understanding of these isospectrality-breaking theories of gravity using BH spectroscopy.

3.8 Appendix: Properties of operators $\hat{P}$, $\hat{C}$, $\hat{\mathcal{P}}$

In this appendix, we provide some useful relations for the operators $\hat{P}$, $\hat{C}$, and $\hat{\mathcal{P}}$ defined in Sec. 3.3. Let $\alpha, \beta \in \mathbb{C}$, $f, g \in \Lambda^0(\mathcal{U})$, and $\hat{\mathcal{I}}$ be the identity operator. Then $\hat{P}$ defined in Eq. (3.13) satisfies

$$\hat{P}^2 = \hat{\mathcal{I}}\,, \tag{3.118a}$$

$$\hat{P}[\alpha f + \beta g] = \alpha \hat{P}[f] + \beta \hat{P}[g]\,, \tag{3.118b}$$

$$\hat{P}[f \cdot g \cdot h \cdots] = \hat{P}[f] \cdot \hat{P}[g] \cdot \hat{P}[h] \cdots\,. \tag{3.118c}$$

The parity operator $\hat{P}$ commutes with all the derivatives:

$$[\hat{P}, \partial_t] = [\hat{P}, \partial_r] = [\hat{P}, \partial_\phi] = 0 \tag{3.119}$$

except the θ derivative, with which it anti-commutes:

$$\{\hat{P}, \partial_\theta\} = 0\,. \tag{3.120}$$

For the complex conjugate operator $\hat{C}$, we have

$$\hat{C}^2 = \hat{\mathcal{I}}\,, \tag{3.121a}$$

$$\hat{C}[\alpha f + \beta g] = \bar{\alpha}\hat{C}[f] + \bar{\beta}\hat{C}[g]\,, \tag{3.121b}$$

$$\hat{C}[f \cdot g \cdot h \cdots] = \hat{C}[f] \cdot \hat{C}[g] \cdot \hat{C}[h] \cdots\,. \tag{3.121c}$$

Since the coordinates are all real, the complex conjugate operator $\hat{C}$ commutes with all the derivatives:

$$[\hat{C}, \partial_t] = [\hat{C}, \partial_r] = [\hat{C}, \partial_\phi] = [\hat{C}, \partial_\theta] = 0\,. \tag{3.122}$$

In addition, $\hat{P}$ and $\hat{C}$ commute with each other.

In Eq. (3.23), we have combined $\hat{P}$ and $\hat{C}$ to define another operator $\hat{\mathcal{P}}$, where

$$\hat{\mathcal{P}} = \hat{C}\hat{P}\,. \tag{3.123}$$

Using Eqs. (3.118)–(3.122), we find

$$\hat{\mathcal{P}}^2 = \hat{\mathcal{I}}\,, \tag{3.124a}$$

$$\hat{\mathcal{P}}[\alpha f + \beta g] = \bar{\alpha}\hat{\mathcal{P}}[f] + \bar{\beta}\hat{\mathcal{P}}[g]\,, \tag{3.124b}$$

$$\hat{\mathcal{P}}[f \cdot g \cdot h \cdots] = \hat{\mathcal{P}}[f] \cdot \hat{\mathcal{P}}[g] \cdot \hat{\mathcal{P}}[h] \cdots \tag{3.124c}$$

and

$$[\hat{\mathcal{P}}, \partial_t] = [\hat{\mathcal{P}}, \partial_r] = [\hat{\mathcal{P}}, \partial_\phi] = 0\,, \quad \{\hat{\mathcal{P}}, \partial_\theta\} = 0\,. \tag{3.125}$$

3.9 Appendix: Reconstruction of NP quantities

In this appendix, we provide some additional equations of reconstructed NP quantities following [88, 127]. Let us first assume a general reconstructed metric $h_{\mu\nu}$ without going to the specific IRG or ORG. In [137], we have only considered the case that the background spacetime is Petrov type D, so the results here are more general. Then to reconstruct NP quantities, the first step is to reconstruct the tetrad. We can first express the reconstructed tetrad in terms of the background tetrad

$$e_a^{\mu(1)} = A_a{}^{b(1)} e_b^{\mu(0)}\,. \tag{3.126}$$

As shown in [88, 127], one can always use the six degrees of freedom of tetrad rotations to set some of the $A_a^{b(1)}$ coefficients to 0. Then expanding $h_{\mu\nu}$ in terms of $e_\mu^{a(1)}$ and $e_\mu^{a(0)}$ using the completeness relation

$$g_{\mu\nu} = -2l_{(\mu}n_{\nu)} + 2m_{(\mu}\bar{m}_{\nu)} \tag{3.127}$$

and its expansion

$$h_{\mu\nu} = -2\left[l^{(1)}_{(\mu}n^{(0)}_{\nu)} - l^{(0)}_{(\mu}n^{(1)}_{\nu)} + m^{(1)}_{(\mu}\bar{m}^{(0)}_{\nu)} + m^{(0)}_{(\mu}\bar{m}^{(1)}_{\nu)}\right]\,, \tag{3.128}$$

one can find that [88, 127],

$$l^{\mu(1)} = \frac{1}{2} h_{ll} n^{\mu} , \tag{3.129a}$$

$$n^{\mu(1)} = \frac{1}{2} h_{nn} l^{\mu} + h_{ln} n_{\mu} , \tag{3.129b}$$

$$m^{\mu(1)} = h_{nm} l^{\mu} + h_{lm} n^{\mu} - \frac{1}{2} h_{m\bar{m}} m^{\mu} - \frac{1}{2} h_{mm} \bar{m}^{\mu} , \tag{3.129c}$$

where we have dropped the superscripts of $e_a^{\mu(0)}$ and $h_{ab}^{(1)}$ for simplicity. Notice that the perturbed tetrad in Eq. (3.129) has an opposite sign from the one in [88, 127] since we used an opposite signature, as one can see in Eqs. (3.127) and (3.128).

To find the spin coefficients, we follow the idea in [51] to expand the commutation relation defining Ricci rotation coefficients

$$\left[e_a^{\mu}, e_b^{\mu}\right] = \left(\gamma^c{}_{ba} - \gamma^c{}_{ab}\right) e_c^{\mu} = C_{ab}{}^c e_c^{\mu} , \tag{3.130}$$

and spin coefficients are just linear combinations of Ricci rotation coefficients [51],

$$\begin{aligned}
&\kappa = \gamma_{131} , \quad \pi = -\gamma_{241} , \quad \varepsilon = \frac{1}{2}(\gamma_{121} - \gamma_{341}) , \\
&\rho = \gamma_{134} , \quad \lambda = -\gamma_{244} , \quad \alpha = \frac{1}{2}(\gamma_{124} - \gamma_{344}) , \\
&\sigma = \gamma_{133} , \quad \mu = -\gamma_{243} , \quad \beta = \frac{1}{2}(\gamma_{123} - \gamma_{343}) , \\
&\tau = \gamma_{132} , \quad \nu = -\gamma_{242} , \quad \gamma = \frac{1}{2}(\gamma_{122} - \gamma_{342}) .
\end{aligned} \tag{3.131}$$

Expanding Eq. (3.130) using Eq. (3.126), one then finds

$$C_{ab}{}^{c(1)} = \partial_a A_b{}^c - \partial_b A_a{}^c - \left(A_a{}^d C_{bd}{}^c - A_b{}^d C_{ad}{}^c + A_d{}^c C_{ab}{}^d\right) , \tag{3.132}$$

where we have dropped the superscript of $C_{ab}{}^{c(0)}$ at the right-hand side. Inserting Eqs. (3.129) and (3.131) into Eq. (3.132) and using the definition in Eq. (3.131), we find the perturbed spin coefficients to be

$$\begin{aligned}
\kappa^{(1)} = &\frac{1}{2} \delta_{[-2,-2,1,1]} h_{ll} - D_{[-2,0,0,-1]} h_{lm} \\
&- \kappa h_{ln} + \sigma h_{l\bar{m}} - \frac{1}{2} \bar{\kappa} h_{mm} - \frac{1}{2} \kappa h_{m\bar{m}} ,
\end{aligned} \tag{3.133a}$$

$$\sigma^{(1)} = -\frac{1}{2} D_{[-2,2,1,-1]} h_{mm} + (\bar{\pi} + \tau) h_{lm} - \frac{1}{2} \bar{\lambda} h_{ll} , \tag{3.133b}$$

$$\lambda^{(1)} = (\pi + \bar{\tau}) h_{n\bar{m}} + \frac{1}{2} \mathbf{\Delta}_{[-1,1,2,-2]} h_{\bar{m}\bar{m}} + \lambda h_{ln} - \frac{1}{2} \bar{\sigma} h_{nn} , \tag{3.133c}$$

$$\nu^{(1)} = -\frac{1}{2}\bar{\delta}_{[2,2,-1,-1]}h_{nn} + \mathbf{\Delta}_{[0,1,2,0]}h_{n\bar{m}} + \nu h_{ln} + \lambda h_{nm} - \frac{1}{2}\nu h_{m\bar{m}} - \frac{1}{2}\bar{\nu}h_{\bar{m}\bar{m}}\,, \tag{3.133d}$$

$$\begin{aligned}\epsilon^{(1)} = \frac{1}{4}\Big[&\mathbf{\Delta}_{[-1,1,0,-2]}h_{ll} - \bar{\delta}_{[-2,0,-3,-2]}h_{lm} + \delta_{[-2,0,1,2]}h_{l\bar{m}} \\ &- 2D_{[0,0,\frac{1}{2},-\frac{1}{2}]}h_{ln} - (\rho-\bar{\rho})h_{m\bar{m}} - \bar{\kappa}h_{nm} + \kappa h_{n\bar{m}} - \bar{\sigma}h_{mm} + \sigma h_{\bar{m}\bar{m}}\Big]\,,\end{aligned} \tag{3.133e}$$

$$\begin{aligned}\rho^{(1)} = \frac{1}{2}\Big[&-\mu h_{ll} - \bar{\delta}_{[-2,0,-1,0]}h_{lm} + \delta_{[-2,0,1,2]}h_{l\bar{m}} \\ &- (\rho-\bar{\rho})h_{ln} - D_{[0,0,1,-1]}h_{m\bar{m}} - \bar{\kappa}h_{nm} + \kappa h_{n\bar{m}}\Big]\,,\end{aligned} \tag{3.133f}$$

$$\begin{aligned}\mu^{(1)} = \frac{1}{2}\Big[&-\rho h_{nn} - \bar{\delta}_{[0,2,-2,-1]}h_{nm} + \delta_{[0,2,0,1]}h_{n\bar{m}} \\ &+ (\mu+\bar{\mu})h_{ln} + \mathbf{\Delta}_{[-1,1,0,0]}h_{m\bar{m}} + \nu h_{lm} - \bar{\nu}h_{l\bar{m}}\Big]\,,\end{aligned} \tag{3.133g}$$

$$\begin{aligned}\gamma^{(1)} = \frac{1}{4}\Big[&-D_{[0,2,1,-1]}h_{nn} - \bar{\delta}_{[0,2,-2,-1]}h_{nm} + \delta_{[0,2,2,3]}h_{n\bar{m}} \\ &- (\mu-\bar{\mu}-4\gamma)h_{ln} - (\mu-\bar{\mu})h_{m\bar{m}} + \nu h_{lm} - \bar{\nu}h_{l\bar{m}} + \lambda h_{mm} - \bar{\lambda}h_{\bar{m}\bar{m}}\Big]\,,\end{aligned} \tag{3.133h}$$

$$\begin{aligned}\alpha^{(1)} = \frac{1}{4}\Big[&-D_{[-2,0,-1,-2]}h_{n\bar{m}} + \delta_{[-2,0,1,1]}h_{\bar{m}\bar{m}} + \mathbf{\Delta}_{[-2,1,4,-2]}h_{l\bar{m}} \\ &- \bar{\delta}_{[0,0,-1,-1]}h_{ln} - \bar{\delta}_{[2,0,-1,-1]}h_{m\bar{m}} - \nu h_{ll} + 3\lambda h_{lm} - \bar{\kappa}h_{nn} - \bar{\sigma}h_{nm}\Big]\,,\end{aligned} \tag{3.133i}$$

$$\begin{aligned}\beta^{(1)} = \frac{1}{4}\Big[&-D_{[-4,2,2,-1]}h_{nm} - \bar{\delta}_{[0,2,-1,-1]}h_{mm} + \mathbf{\Delta}_{[1,2,2,0]}h_{lm} \\ &- \delta_{[0,0,-1,-1]}h_{ln} + \delta_{[0,-2,1,1]}h_{m\bar{m}} - \bar{\nu}h_{ll} - \bar{\lambda}h_{l\bar{m}} - \kappa h_{nn} + 3\sigma h_{n\bar{m}}\Big]\,,\end{aligned} \tag{3.133j}$$

$$\begin{aligned}\pi^{(1)} = \frac{1}{2}\Big[&D_{[2,0,-1,0]}h_{n\bar{m}} + \tau h_{\bar{m}\bar{m}} + \mathbf{\Delta}_{[0,1,0,-2]}h_{l\bar{m}} \\ &- \delta_{[0,0,-1,-1]}h_{ln} + \bar{\tau}h_{m\bar{m}} + \lambda h_{lm} - \bar{\sigma}h_{nm}\Big]\,,\end{aligned} \tag{3.133k}$$

$$\begin{aligned}\tau^{(1)} = \frac{1}{2}\Big[&-D_{[0,2,0,-1]}h_{nm} + \pi h_{mm} - \mathbf{\Delta}_{[1,0,-2,0]}h_{lm} \\ &+ \delta_{[0,0,1,1]}h_{ln} + \bar{\pi}h_{m\bar{m}} - \bar{\lambda}h_{l\bar{m}} + \sigma h_{n\bar{m}}\Big]\,.\end{aligned} \tag{3.133l}$$

In Eq. (3.133), we do not assume anything about the background spacetime, so the background may be Petrov type I, and all the spin coefficients at the background can be nonzero. Thus, we can use the above equations for our analysis in Sec. 3.4. For Petrov type D spacetimes in GR, where $\kappa^{(0,0)} = \sigma^{(0,0)} = \lambda^{(0,0)} = \nu^{(0,0)} = 0$, our result is the same as the one in [88] up to a minus sign due to the opposite signature we used. However, the result in [127] has some discrepancies with the result here and in [88], which might be due to errors. If we additionally use the IRG, we can then further set $h_{ll} = h_{ln} = h_{lm} = h_{l\bar{m}} = h_{m\bar{m}} = 0$ in Eq. (3.133).

To find the perturbed Weyl scalars, one can use Ricci identities to compute Weyl scalars from spin coefficients,

$$\Psi_0 = D_{[-3,1,-1,-1]}\sigma - \delta_{[-1,-3,1,-1]}\kappa\,, \tag{3.134a}$$

$$\Psi_1 = D_{[0,1,0,-1]}\beta - \delta_{[-1,0,1,0]}\varepsilon - (\alpha+\pi)\sigma + (\gamma+\mu)\kappa\,, \tag{3.134b}$$

$$\begin{aligned}\Psi_2 = \frac{1}{3}\Big[&\bar{\delta}_{[-2,1,-1,-1]}\beta - \delta_{[-1,0,1,1]}\alpha + D_{[1,1,1,-1]}\gamma - \mathbf{\Delta}_{[-1,1,-1,-1]}\varepsilon \\ &+ \bar{\delta}_{[-1,1,-1,-1]}\tau - \mathbf{\Delta}_{[-1,1,-1,-1]}\rho + 2(\nu\kappa - \lambda\sigma)\Big]\,,\end{aligned} \tag{3.134c}$$

$$\Psi_3 = \bar{\delta}_{[0,1,0,-1]}\gamma - \mathbf{\Delta}_{[0,1,0,-1]}\alpha + (\varepsilon+\rho)\nu - (\beta+\tau)\lambda\,, \tag{3.134d}$$

$$\Psi_4 = \bar{\delta}_{[3,1,1,-1]}\nu - \mathbf{\Delta}_{[1,1,3,-1]}\lambda\,. \tag{3.134e}$$

Equation (3.134) works at all order for any spacetime, so we can use them for our analysis in Sec. 3.4. Here, we have also followed [88, 127] to linearly combine certain Ricci identities such that there are no NP Ricci scalars Φ_{ab} in the equations, and the equations work for non-vacuum spacetime. Using the NP quantities on the background with the perturbed tetrad in Eq. (3.129) and the perturbed spin coefficients in Eq. (3.133), one can then write down the perturbed Weyl scalars in terms of metric perturbations directly.

For Petrov type D spacetimes in GR, using Eqs. (3.15) and (3.16), one can further write down the perturbed NP quantities in terms of the Hertz potential. In [81, 86], they computed the perturbed Weyl scalars directly from the Riemann tensor, and they found in the IRG in Eq. (3.16),

$$\Psi_0^{(0,1)} = -\frac{1}{2}D_{[-3,1,0,-1]}D_{[-2,2,0,-1]}h_{mm}\,, \tag{3.135a}$$

$$\begin{aligned}\Psi_1^{(0,1)} = -\frac{1}{8}\Big[&2D_{[-1,1,1,-1]}D_{[0,2,1,-1]}h_{nm} + D_{[-1,1,1,-1]}\delta_{[-2,2,-2,-1]}h_{mm} \\ &+ \bar{\delta}_{[-3,1,-3,-1]}D_{[-2,2,0,-1]}h_{mm}\Big]\,,\end{aligned} \tag{3.135b}$$

$$\begin{aligned}\Psi_2^{(0,1)} = -\frac{1}{12}\Big[&D_{[1,1,2,-1]}D_{[2,2,2,-1]}h^1_{nn} + 2\left(D_{[1,1,2,-1]}\bar{\delta}_{[0,2,-1,-1]}\right. \\ &\left.+\bar{\delta}_{[-1,1,-2,-1]}D_{[0,2,1,-1]}\right)h_{nm} + \bar{\delta}_{[-1,1,-2,-1]}\bar{\delta}_{[-2,2,-2,-1]}h_{mm}\Big]\,,\end{aligned} \tag{3.135c}$$

$$\begin{aligned}\Psi_3^{(0,1)} = -\frac{1}{8}\Big[&\left(D_{[3,1,3,-1]}\bar{\delta}_{[2,2,0,-1]} + \bar{\delta}_{[1,1,-1,-1]}D_{[2,2,,2,-1]}\right)h^1_{nn} \\ &+ \bar{\delta}_{[1,1,-1,-1]}\bar{\delta}_{[0,2,-1,-1]}h_{nm}\Big]\,,\end{aligned} \tag{3.135d}$$

$$\Psi_4^{(0,1)} = -\frac{1}{2}\Big[\bar{\delta}_{[3,1,0,-1]}\bar{\delta}_{[2,2,0,-1]}h^1_{nn} + 3\Psi_2\left(\tau\bar{\delta}_{[4,0,0,0]} - \rho\boldsymbol{\Delta}_{[0,0,4,0]}\right. \\ \left. -\mu D_{[4,0,0,0]} + \pi\delta_{[0,4,0,0]} + 2\Psi_2\right)\bar{\Psi}_{\rm H}\Big]\,, \tag{3.135e}$$

where $\Psi_{0,4}^{(0,1)}$ reduce to Eq. (3.18) in the Boyer-Lindquist coordinates of Kerr. We have also defined h^1_{nn} to be the piece of h_{nn} proportional to $\bar{\Psi}_{\rm H}$ in Eq. (3.16), i.e., $h^1_{nn} = \bar{\delta}_{[1,3,0,-1]}\bar{\delta}_{[0,4,0,3]}\bar{\Psi}_{\rm H}$.

To compare our results with Eq. (3.135), we compute the perturbed Weyl scalars using the Ricci identities in Eq. (3.134) in Kerr such that $\varepsilon^{(0,0)} = 0$. We also perform a direct calculation by linearizing the Riemann tensor first and then projecting it into the NP basis. For both calculations, we use the tetrad in Eq. (3.129) with the IRG, and we find an agreement for $\Psi_{0,1,2,4}^{(0,1)}$. While for $\Psi_3^{(0,1)}$, we find a disagreement. This is not very surprising since $\Psi_3^{(0,1)}$ is not invariant under both tetrad rotations and infinitesimal coordinate changes at $O(\epsilon)$. Since both our calculation and Refs. [81, 86] use the IRG, we have used the same coordinate freedom. This is also manifested by that our $\Psi_2^{(0,1)}$ matches Eq. (3.135), which is invariant under tetrad rotations at $O(\epsilon)$ but not invariant under coordinate transformations at $O(\epsilon)$. Thus, the difference between our result and Eq. (3.135) is due to different tetrad choices, while Refs. [81, 86] did not clearly specify their tetrad at $O(\epsilon)$.

In the case of Schwarzschild, with the tetrad in Eq. (3.129) and the perturbed metric in the IRG in Eq. (3.16), we find

$$\Psi_0^{(0,1)} = -\frac{1}{2}D^4\bar{\Psi}_{\rm H}\,, \tag{3.136a}$$

$$\Psi_1^{(0,1)} = -\frac{1}{2}D^3(\bar{\delta} + 4\beta)\bar{\Psi}_{\rm H}\,, \tag{3.136b}$$

$$\Psi_2^{(0,1)} = -\frac{1}{2}D^2(\bar{\delta} + 2\beta)(\bar{\delta} + 4\beta)\bar{\Psi}_{\rm H}\,, \tag{3.136c}$$

$$\Psi_3^{(0,1)} = -\frac{1}{2}D\bar{\delta}(\bar{\delta} + 2\beta)(\bar{\delta} + 4\beta)\bar{\Psi}_{\rm H} + \frac{3}{2}\Psi_2 h_{n\bar{m}}\,, \tag{3.136d}$$

$$\Psi_4^{(0,1)} = -\frac{1}{2}(\bar{\delta} - 2\beta)\bar{\delta}(\bar{\delta} + 2\beta)(\bar{\delta} + 4\beta)\bar{\Psi}_{\rm H} + \frac{3}{2}\Psi_2\left[\mu D + \rho(\boldsymbol{\Delta} + 4\gamma) - 2\Psi_2\right]\Psi_{\rm H}\,, \tag{3.136e}$$

which is the same as Eq. (3.135) in the Schwarzschild limit, except there is an additional $\frac{3}{2}\Psi_2 h_{n\bar{m}}$ correction to $\Psi_3^{(0,1)}$ due to different tetrad choices.

3.10 Appendix: Radial Functions in $\mathcal{S}_{\mathrm{geo}}^{(1,1)}$

In this appendix, we provide the radial functions $C_i(r)$ in Eq. (3.85) and $D_i(r)$ in Eq. (3.106). The radial functions $C_i(r)$ are found in [137], where

$$C_1(r) = 17640M^4 - 17196M^3r + 6M^2r^2 + 210Mr^3 + 1295r^4\,, \tag{3.137a}$$

$$C_2(r) = 189M^3 + 120M^2r + 70Mr^2\,, \tag{3.137b}$$

$$C_3(r) = 342M^3 - 816M^2r - 385Mr^2 - 165r^3\,, \tag{3.137c}$$

$$C_4(r) = 774M^2 + 360Mr + 145r^2\,, \tag{3.137d}$$

$$C_5(r) = 1800M^3 - 378M^2r - 240Mr^2 - 185r^3\,. \tag{3.137e}$$

The radial functions $D_i(r)$ are given by

$$\begin{aligned}&\frac{1}{(64\pi)^2M}\frac{15}{2}r^7(r-2M)^3D_1(r)\\ &= 168960M^9 + 6720M^8r - 232448M^7r^2 + 129928M^6r^3 - 24108M^5r^4\\ &\quad + 13900M^4r^5 - 8090M^3r^6 + 1530M^2r^7 - 150Mr^8 + 15r^9\,,\end{aligned} \tag{3.138a}$$

$$\begin{aligned}&\frac{1}{(64\pi)^2M}\frac{15}{2}r^5(r-2M)^3D_2(r)\\ &= 253440M^8 - 344992M^7r + 146720M^6r^2 - 28584M^5r^3 + 16872M^4r^4\\ &\quad - 8240M^3r^5 + 1210M^2r^6 - 75Mr^7 + 15r^8\,,\end{aligned} \tag{3.138b}$$

$$\begin{aligned}&\frac{1}{(64\pi)^2M}\frac{15}{2}r^6(r-2M)^2D_3(r)\\ &= 212160M^8 - 310624M^7r + 139352M^6r^2 - 25728M^5r^3 + 14630M^4r^4\\ &\quad - 7720M^3r^5 + 1275M^2r^6 - 120Mr^7 + 15r^8\,,\end{aligned} \tag{3.138c}$$

$$\begin{aligned}&\frac{1}{(64\pi)^2M^3}\frac{15}{2}r^3(r-2M)^2D_4(r)\\ &= 400M^4 - 96M^3r - 66M^2r^2 - 130Mr^3 - 5r^4\,,\end{aligned} \tag{3.138d}$$

$$\begin{aligned}&\frac{1}{(64\pi)^2M^2}\frac{15}{2}r^5D_5(r)\\ &= 1840M^5 + 48M^4r - 30M^3r^2 - 260M^2r^3 - 15Mr^4 - 15r^5\,.\end{aligned} \tag{3.138e}$$

3.11 Appendix: $\mathcal{S}$ in the modified Teukolsky equations

In this appendix, we present the source term $\mathcal{S}$ of the modified Teukolsky equations due to the effective stress tensor for non-rotating BHs in dCS and EdGB gravity. Here, we only briefly summarize the procedure in [137] and apply it to these two simple non-rotating examples. In addition, for EdGB gravity, we only focus on the source terms with dynamical scalar field $\varphi^{(1,0,1)}$ for simplicity. For a more complete prescription of how to evaluate these source terms, one can refer to [137] for slowly rotating BHs in dCS gravity. The procedure in [137] can be extended to BHs with arbitrary spin in dCS and other modified gravity.

3.11.1 dCS

As discussed in Sec. 3.6.1.2, for non-rotating BHs in dCS, the only nonzero contribution of Eq. (3.86) is the term $\left(\nabla^{\sigma}\nabla^{\delta}\vartheta\right)$. In addition, since $\vartheta^{(1,0,0)}$ vanishes, ϑ only has dynamical contribution $\vartheta^{(1,0,1)}$. For the same reason, all the metric fields in $\mathcal{S}_{\rm dCS}^{(1,0,1)}$ are evaluated on the stationary Schwarzschild background, so no metric reconstruction is needed. At $O(\zeta^1,\chi^0,\epsilon^1)$, the only place requiring metric reconstruction is to solve the equation of motion of $\vartheta^{(1,0,1)}$ since it is driven by $\Psi_2^{(0,0,1)}$ [i.e., Eq. (3.90)]. Given $\vartheta^{(1,0,1)}$ is solved, one can then project the term $\left(\nabla^{\sigma}\nabla^{\delta}\vartheta\right)$ onto the NP tetrad and evaluate all the metric fields using their Schwarzschild values.

Let us present this calculation in more detail. First, inspecting the source terms $S_{1,2}$ in Eq. (3.6) of the Bianchi identities in Eqs. (3.4a) and (3.4b), the only nonzero contributions of Φ_{ij} are from Φ_{00}, Φ_{01}, and Φ_{02} since $\kappa^{(1,0,1)}=\sigma^{(1,0,1)}=\lambda^{(1,0,1)}=0$. Then from the definition

$$\Phi_{00}=\frac{1}{2}R_{11}\,,\quad \Phi_{01}=\frac{1}{2}R_{13}\,,\quad \Phi_{02}=\frac{1}{2}R_{33}\,, \tag{3.139}$$

we notice the only relevant components of $R_{\mu\nu}$ are R_{11}, R_{13}, and R_{33}. Projecting the equation of $R_{\mu\nu}$ in Eq. (3.86) onto the NP tetrad, we find

$$\begin{aligned}
R_{11}^{\rm dCS}=i\mathcal{R}_1^{\rm dCS}\Big\{&(D\vartheta)\Big[\lambda\Psi_0-\bar{\lambda}\bar{\Psi}_0-(\alpha+\bar{\beta}+\pi)\Psi_1+(\bar{\alpha}+\beta+\bar{\pi})\bar{\Psi}_1\\
&+(\varepsilon+\bar{\varepsilon})(\Psi_2-\bar{\Psi}_2)\Big]\\
&-(\mathbf{\Delta}\vartheta)\Big[\bar{\sigma}\Psi_0-\sigma\bar{\Psi}_0-\bar{\kappa}\Psi_1+\kappa\bar{\Psi}_1\Big]\\
&+(\delta\vartheta)\Big[(\bar{\alpha}-\beta)\bar{\Psi}_0+\bar{\sigma}\Psi_1+(\varepsilon-\bar{\varepsilon}-\bar{\rho})\bar{\Psi}_1-\bar{\kappa}(\Psi_2-\bar{\Psi}_2)\Big]\\
&-(\bar{\delta}\vartheta)\Big[(\alpha-\bar{\beta})\Psi_0-(\varepsilon-\bar{\varepsilon}+\rho)\Psi_1+\sigma\bar{\Psi}_1+\kappa(\Psi_2-\bar{\Psi}_2)\Big]\\
&-\frac{1}{2}\Psi_0\{\bar{\delta},\bar{\delta}\}\vartheta+\frac{1}{2}\bar{\Psi}_0\{\delta,\delta\}\vartheta+\Psi_1\{D,\bar{\delta}\}\vartheta-\bar{\Psi}_1\{D,\delta\}\vartheta\\
&-\frac{1}{2}(\Psi_2-\bar{\Psi}_2)\{D,D\}\vartheta\Big\}+\mathcal{R}_2^{\rm dCS}(D\vartheta)(D\vartheta)\,,
\end{aligned} \tag{3.140a}$$

$$
\begin{aligned}
R_{13}^{\rm dCS} = \frac{i}{2}\mathcal{R}_1^{\rm dCS}\Big\{&(D\vartheta)\Big[\nu\Psi_0 - (\gamma+\bar{\gamma}+\mu+\bar{\mu})\Psi_1 - 2\bar{\lambda}\bar{\Psi}_1\\
&+ (\bar{\alpha}+\beta+\bar{\pi})(\Psi_2+2\bar{\Psi}_2) - 2(\varepsilon+\bar{\varepsilon})\bar{\Psi}_3\Big]\\
&- (\mathbf{\Delta}\vartheta)\Big[(\alpha+\bar{\beta}+\bar{\tau})\Psi_0 - (\varepsilon+\bar{\varepsilon}+\rho+\bar{\rho})\Psi_1 - 2\sigma\bar{\Psi}_1 + \kappa(\Psi_2+2\bar{\Psi}_2)\Big]\\
&+ (\delta\vartheta)\Big[\lambda\Psi_0 - (\alpha-\bar{\beta}+\pi-\bar{\tau})\Psi_1 + 2(\bar{\alpha}-\beta)\bar{\Psi}_1\\
&+ (\varepsilon-\bar{\varepsilon}-\bar{\rho})(\Psi_2+2\bar{\Psi}_2) + 2\bar{\kappa}\bar{\Psi}_3\Big]\\
&- (\bar{\delta}\vartheta)\Big[(\gamma-\bar{\gamma}-\bar{\mu})\Psi_0 + (\bar{\alpha}-\beta+\bar{\pi}-\tau)\Psi_1 + \sigma(\Psi_2+2\bar{\Psi}_2) - 2\kappa\bar{\Psi}_3\Big]\\
&- \Psi_0\{\mathbf{\Delta},\bar{\delta}\}\vartheta + \Psi_1\Big[\{D,\mathbf{\Delta}\} + \{\delta,\bar{\delta}\}\Big]\vartheta + \bar{\Psi}_1\{\delta,\delta\}\vartheta\\
&- (\Psi_2+2\bar{\Psi}_2)\{D,\delta\}\vartheta + \bar{\Psi}_3\{D,D\}\vartheta\Big\} + \mathcal{R}_2^{\rm dCS}(D\vartheta)(\delta\vartheta)\,,
\end{aligned}
\tag{3.140b}
$$

$$
\begin{aligned}
R_{33}^{\rm dCS} = i\mathcal{R}_1^{\rm dCS}\Big\{&-(D\vartheta)\Big[\bar{\nu}\Psi_1 - \bar{\lambda}(\Psi_2-\bar{\Psi}_2) - (\bar{\alpha}+\beta+\bar{\pi})\bar{\Psi}_3 + (\varepsilon+\bar{\varepsilon})\bar{\Psi}_4\Big]\\
&- (\mathbf{\Delta}\vartheta)\Big[(\gamma+\bar{\gamma})\Psi_0 - (\bar{\alpha}+\beta+\tau)\Psi_1 + \sigma(\Psi_2-\bar{\Psi}_2) + \kappa\bar{\Psi}_3\Big]\\
&+ (\delta\vartheta)\Big[\nu\Psi_0 - (\gamma-\bar{\gamma}+\mu)\Psi_1 - (\bar{\alpha}-\beta)(\Psi_2-\bar{\Psi}_2) + (\varepsilon-\bar{\varepsilon}-\bar{\rho}) + \bar{\kappa}\bar{\Psi}_4\Big]\\
&+ (\bar{\delta}\vartheta)\Big[\bar{\nu}\Psi_0 - \bar{\lambda}\Psi_1 - \sigma\bar{\Psi}_3 + \kappa\bar{\Psi}_4\Big]\\
&- \frac{1}{2}\Psi_0\{\mathbf{\Delta},\mathbf{\Delta}\}\vartheta + \Psi_1\{\mathbf{\Delta},\delta\}\vartheta - \frac{1}{2}(\Psi_2-\bar{\Psi}_2)\{\delta,\delta\}\vartheta\\
&- \bar{\Psi}_3\{D,\delta\}\vartheta + \frac{1}{2}\bar{\Psi}_4\{D,D\}\vartheta\Big\} + \mathcal{R}_2^{\rm dCS}(\delta\vartheta)(\delta\vartheta)\,,
\end{aligned}
$$

$$
\mathcal{R}_1^{\rm dCS} \equiv -\left(\frac{1}{\kappa_g}\right)^{\frac{1}{2}} M^2\,, \quad \mathcal{R}_2^{\rm dCS} \equiv \frac{1}{2\kappa_g\zeta_{\rm dCS}}\,.
\tag{3.140c}
$$

The complete procedure of this projection and the projection of other components of $R_{\mu\nu}$ in dCS gravity can be found in [137]. Notice, following [137], we have absorbed the coupling constant into the expansion of ϑ such that its expansion also follows Eq. (3.2), so we need to insert an ζ^{-1} into $\mathcal{R}_2^{\rm dCS}$ to compensate for this. Although Eq. (3.140) is complicated, its value at $O(\zeta^1,\chi^0,\epsilon^1)$ is simple since many Weyl scalars and spin coefficients vanish on the Schwarzschild background. Using

$$
\begin{aligned}
&\Psi_{0,1,3,4}^{(0,0,0)} = 0\,, \quad \bar{\Psi}_2^{(0,0,0)} = \Psi_2^{(0,0,0)}\,,\\
&\bar{\alpha}^{(0,0,0)} = \alpha^{(0,0,0)} = -\beta^{(0,0,0)}\,, \quad \bar{\rho}^{(0,0,0)} = \rho^{(0,0,0)}\,,\\
&\bar{\mu}^{(0,0,0)} = \mu^{(0,0,0)}\,, \quad \bar{\gamma}^{(0,0,0)} = \gamma^{(0,0,0)}\,,
\end{aligned}
\tag{3.141}
$$

and other spin coefficients in Schwarzschild vanish, we find

$$\begin{aligned}
&\Phi^{(1,0,1)}_{00,\text{dCS}} = \Phi^{(1,0,1)}_{02,\text{dCS}} = 0\,, \\
&\Phi^{(1,0,1)}_{01,\text{dCS}} = -\frac{3i}{4}\mathcal{R}^{\text{dCS}}_1 \Psi_2(\{D,\delta\} + \rho\delta)\vartheta^{(1,0,1)}\,,
\end{aligned} \tag{3.142}$$

where we have dropped the superscripts of terms at $O(\zeta^0, \chi^0, \epsilon^0)$.

Evaluating $S_{1,2}$ using Eqs. (3.6) and (3.142), we find

$$\begin{aligned}
S^{(1,0,1)}_{1,\text{dCS}} &= \frac{3i}{2}\mathcal{R}^{\text{dCS}}_1 \Psi_2 \left[\delta D^2 + 3\rho(\delta D + \rho\delta)\right] \vartheta^{(1,0,1)}\,, \\
S^{(1,0,1)}_{2,\text{dCS}} &= -\frac{3i}{2}\mathcal{R}^{\text{dCS}}_1 \Psi_2 \left[\delta^2 D + 2\alpha\delta D + \rho\delta^2 + 2\alpha\rho\delta\right] \vartheta^{(1,0,1)}\,,
\end{aligned} \tag{3.143}$$

where we have used NP equations to make simplifications. Then, inserting $S^{(1,0,1)}_{1,2}$ into the definition of $\mathcal{S}^{(1,1)}$ in Eq. (3.11), we find

$$\begin{aligned}
\mathcal{S}^{(1,0,1)}_{\text{dCS}} = &-3i\mathcal{R}^{\text{dCS}}_1 \Psi_2 \left[\delta^2 D^2 + 2\alpha\delta D^2 + 2\rho\delta^2 D \right. \\
&\left. +4\alpha\rho\delta D + 2\rho^2\delta^2 + 4\alpha\rho^2\delta\right] \vartheta^{(1,0,1)} \\
\equiv\, & iQ^{\text{dCS}}\vartheta^{(1,0,1)}\,.
\end{aligned} \tag{3.144}$$

Using the transformation properties in Eq. (3.43), one can easily show that $\hat{\mathcal{P}}Q^{\text{dCS}} = Q^{\text{dCS}}$. Following the definition in Eq. (3.97), we can write

$$\mathcal{F}^{\text{dCS}} = \frac{3}{2\pi^{1/2}} Q^{\text{dCS}} \Box^{-1}\left[\left(\frac{M}{r}\right)^3 D^2\right]\,, \tag{3.145}$$

where D^2 comes from converting Hertz potential $\bar{\Psi}^{(0,0,1)}_{\text{H}}$ to $\Psi^{(0,0,1)}_2$, and $\Box^{-1}$ comes from inverting the equation of motion of $\vartheta^{(1,0,1)}$ in Eq. (3.87). One can easily check that $\hat{\mathcal{P}}\mathcal{F}^{\text{dCS}} = \mathcal{F}^{\text{dCS}}$, so non-rotating dCS BHs admit definite-parity modes as expected.

3.11.2 EdGB

For EdGB, as discussed in Sec. 3.6.2.2, we choose to focus on the terms in $\mathcal{S}^{(1,0,1)}$ proportional to $\varphi^{(1,0,1)}$ or its derivatives, so all the metric fields are stationary. To compute the terms in $\mathcal{S}^{(1,0,1)}$ driven by GW perturbations in GR, one can follow similar procedures in [137]. Following the same argument in Appendix 3.11.1, one only needs to evaluate Φ_{00}, Φ_{01}, and Φ_{02}, or alternatively R_{11}, R_{13}, and R_{33} for this contribution. Projecting Eq. (3.108) onto the NP tetrad, we find

$$
\begin{aligned}
R_{11}^{\text{EdGB}} = \frac{1}{2}\mathcal{R}_1^{\text{EdGB}}\Big\{ &- (D\varphi)\Big[\lambda\Psi_0 + \bar{\lambda}\bar{\Psi}_0 - (\alpha + \bar{\beta} + \pi)\Psi_1 - (\bar{\alpha} + \beta + \bar{\pi})\bar{\Psi}_1 \\
&+ (\varepsilon + \bar{\varepsilon})(\Psi_2 + \bar{\Psi}_2)\Big] \\
&+ (\boldsymbol{\Delta}\varphi)\Big[\bar{\sigma}\Psi_0 + \sigma\bar{\Psi}_0 - \bar{\kappa}\Psi_1 - \kappa\bar{\Psi}_1\Big] \\
&+ (\delta\varphi)\Big[(\bar{\alpha} - \beta)\bar{\Psi}_0 + (\varepsilon - \bar{\varepsilon} - \bar{\rho})\bar{\Psi}_1 - \bar{\sigma}\Psi_1 + \bar{\kappa}(\Psi_2 + \bar{\Psi}_2)\Big] \\
&+ (\bar{\delta}\varphi)\Big[(\alpha - \bar{\beta})\Psi_0 - \sigma\bar{\Psi}_1 - (\varepsilon - \bar{\varepsilon} + \rho)\Psi_1 + \kappa(\Psi_2 + \bar{\Psi}_2)\Big] \\
&+ \frac{1}{2}\Psi_0\{\bar{\delta}, \bar{\delta}\}\varphi + \frac{1}{2}\bar{\Psi}_0\{\delta, \delta\}\varphi - \Psi_1\{D, \bar{\delta}\}\varphi - \bar{\Psi}_1\{D, \delta\}\varphi \\
&+ \frac{1}{2}(\Psi_2 + \bar{\Psi}_2)\{D, D\}\varphi\Big\} + \mathcal{R}_2^{\text{EdGB}}(D\varphi)(D\varphi)\,,
\end{aligned}
\tag{3.146a}
$$

$$
\begin{aligned}
R_{13}^{\text{EdGB}} = \frac{1}{4}\mathcal{R}_1^{\text{EdGB}}\Big\{ &- (D\varphi)\Big[\nu\Psi_0 - (\gamma + \bar{\gamma} + \mu + \bar{\mu})\Psi_1 + 2\bar{\lambda}\bar{\Psi}_1 \\
&+ (\bar{\alpha} + \beta + \bar{\pi})(\Psi_2 - 2\bar{\Psi}_2) + 2(\varepsilon + \bar{\varepsilon})\bar{\Psi}_3\Big] \\
&+ (\boldsymbol{\Delta}\varphi)\Big[(\alpha + \bar{\beta} + \bar{\tau})\Psi_0 - (\varepsilon + \bar{\varepsilon} + \rho + \bar{\rho})\Psi_1 + 2\sigma\bar{\Psi}_1 + \kappa(\Psi_2 - 2\bar{\Psi}_2)\Big] \\
&- (\delta\varphi)\Big[\lambda\Psi_0 - (\alpha - \bar{\beta} + \pi - \bar{\tau})\Psi_1 - 2(\bar{\alpha} - \beta)\bar{\Psi}_1 \\
&+ (\varepsilon - \bar{\varepsilon} - \bar{\rho})(\Psi_2 - 2\bar{\Psi}_2) - 2\bar{\kappa}\bar{\Psi}_3\Big] \\
&+ (\bar{\delta}\varphi)\Big[(\gamma - \bar{\gamma} - \bar{\mu})\Psi_0 + (\bar{\alpha} - \beta + \bar{\pi} - \tau)\Psi_1 + \sigma(\Psi_2 - 2\bar{\Psi}_2) + 2\kappa\bar{\Psi}_3\Big] \\
&+ \Psi_0\{\boldsymbol{\Delta}, \bar{\delta}\}\varphi - \Psi_1\left[\{D, \boldsymbol{\Delta}\} + \{\delta, \bar{\delta}\}\right]\varphi + \bar{\Psi}_1\{\delta, \delta\}\varphi \\
&+ (\Psi_2 - 2\bar{\Psi}_2)\{D, \delta\}\varphi + \bar{\Psi}_3\{D, D\}\varphi\Big\} + \mathcal{R}_2^{\text{EdGB}}(D\varphi)(\delta\varphi)\,,
\end{aligned}
\tag{3.146b}
$$

$$
\begin{aligned}
R_{33}^{\rm EdGB} = \frac{1}{2}\mathcal{R}_1^{\rm EdGB}\Big\{&(D\varphi)\Big[\bar{\nu}\Psi_1 - \bar{\lambda}(\Psi_2+\bar{\Psi}_2) + (\bar{\alpha}+\beta+\bar{\pi})\bar{\Psi}_3 - (\varepsilon+\bar{\varepsilon})\bar{\Psi}_4\Big] \\
&+ (\mathbf{\Delta}\varphi)\Big[(\gamma+\bar{\gamma})\Psi_0 - (\bar{\alpha}+\beta+\tau)\Psi_1 + \sigma(\Psi_2+\bar{\Psi}_2) - \kappa\bar{\Psi}_3\Big] \\
&- (\delta\varphi)\Big[\nu\Psi_0 - (\gamma-\bar{\gamma}+\mu)\Psi_1 - (\bar{\alpha}-\beta)(\Psi_2+\bar{\Psi}_2) \\
&- (\varepsilon-\bar{\varepsilon}-\bar{\rho})\bar{\Psi}_3 - \bar{\kappa}\bar{\Psi}_4\Big] \\
&- (\bar{\delta}\varphi)\Big[\bar{\nu}\Psi_0 - \bar{\lambda}\Psi_1 + \sigma\bar{\Psi}_3 - \kappa\bar{\Psi}_4\Big] \\
&+ \frac{1}{2}\Psi_0\{\mathbf{\Delta},\mathbf{\Delta}\}\varphi - \Psi_1\{\mathbf{\Delta},\delta\}\varphi + \frac{1}{2}(\Psi_2+\bar{\Psi}_2)\{\delta,\delta\}\varphi \\
&- \bar{\Psi}_3\{D,\delta\}\varphi + \frac{1}{2}\bar{\Psi}_4\{D,D\}\varphi\Big\} + \mathcal{R}_2^{\rm EdGB}(\delta\varphi)(\delta\varphi)\,,
\end{aligned}
\tag{3.146c}
$$

$$
\mathcal{R}_1^{\rm EdGB} = -\kappa_g^{\frac{1}{2}}M^2\,, \quad \mathcal{R}_2^{\rm EdGB} = \frac{1}{2\zeta_{\rm EdGB}}\,, \tag{3.146d}
$$

where one can refer to [137] for more details of this projection in dCS gravity. Similarly, we have absorbed one coupling constant into the expansion of φ to be consistent with Eq. (3.2), so $\mathcal{R}_2^{\rm EdGB}$ contains an extra factor of ζ^{-1}. Using Eq. (3.141), one can find that the $O(\zeta^1,\chi^0,\epsilon^1)$ contributions to Φ_{ij} with dynamical φ are

$$
\begin{aligned}
\Phi_{00,\rm EdGB}^{(1,0,1)} &= \frac{1}{2}\mathcal{R}_1^{\rm EdGB}\Psi_2 D^2\varphi^{(1,0,1)} + \mathcal{R}_2^{\rm EdGB}D\varphi^{(1,0,0)}D\varphi^{(1,0,1)}\,, \\
\Phi_{01,\rm EdGB}^{(1,0,1)} &= -\frac{1}{8}\mathcal{R}_1^{\rm EdGB}\Psi_2\left(\{D,\delta\}+\rho\delta\right)\varphi^{(1,0,1)} + \frac{1}{2}\mathcal{R}_2^{\rm EdGB}D\varphi^{(1,0,0)}\delta\varphi^{(1,0,1)}\,, \\
\Phi_{01,\rm EdGB}^{(1,0,1)} &= \frac{1}{2}\mathcal{R}_1^{\rm EdGB}\Psi_2\left(\delta^2+2\alpha\delta\right)\varphi^{(1,0,1)}\,,
\end{aligned}
\tag{3.147}
$$

where we have also used that $\delta^{(0,0,0)}\varphi^{(1,0,0)} = \bar{\delta}^{(0,0,0)}\varphi^{(1,0,0)} = 0$ since $\varphi^{(1,0,0)}$ is a pure radial function [147]. For simplicity, we have also dropped the superscripts of terms at $O(\zeta^0,\chi^0,\epsilon^0)$. Using Eqs. (3.6) and (3.147), we find

$$
\begin{aligned}
S_{1,\rm EdGB}^{(1,0,1)} = &\frac{3}{4}\mathcal{R}_1^{\rm EdGB}\Psi_2\left[\delta D^2 + \rho\left(\delta D + \rho\delta\right)\right]\varphi^{(1,0,1)} \\
&+ \frac{1}{2}\mathcal{R}_2^{\rm EdGB}\Big[D\varphi^{(1,0,0)}\delta D - (D^2-\rho D)\varphi^{(1,0,0)}\delta\Big]\varphi^{(1,0,1)}\,,
\end{aligned}
\tag{3.148}
$$

$$
\begin{aligned}
S_{2,\rm EdGB}^{(1,0,1)} = &-\frac{3}{4}\mathcal{R}_1^{\rm EdGB}\Psi_2\left(\delta^2 D + 2\alpha\delta D + 3\rho\delta^2 + 6\alpha\rho\delta\right)\varphi^{(1,0,1)} \\
&+ \frac{1}{2}\mathcal{R}_2^{\rm EdGB}D\varphi^{(1,0,0)}(\delta^2+2\alpha\delta)\varphi^{(1,0,1)}\,,
\end{aligned}
\tag{3.149}
$$

and using Eq. (3.11), we find

$\mathcal{S}_{\rm EdGB}^{(1,0,1)}$

$$
\begin{aligned}
&= -\frac{3}{2}\mathcal{R}_1^{\rm EdGB}\Psi_2\left(\delta^2 D^2 + 2\alpha\delta D^2 + 2\rho\delta^2 D + 4\alpha\rho\delta D + 2\rho^2\delta^2 + 4\alpha\rho^2\delta\right)\varphi^{(1,0,1)} \\
&\quad + \mathcal{R}_2^{\rm EdGB}\left\{\left(D^2 - 2\rho D\right)\varphi^{(1,0,0)}\delta^2 + \left[2\alpha\left(D^2 - 2\rho D\right) + \frac{1}{2}\delta D^2\right]\varphi^{(1,0,0)}\delta\right\}\varphi^{(1,0,1)} \\
&= \mathcal{Q}^{\rm EdGB}\varphi^{(1,0,1)}\,. \qquad (3.150)
\end{aligned}
$$

Using the transformation properties in Eq. (3.43) and that $\varphi^{(1,0,0}$ is purely radial, we find that $\hat{\mathcal{P}}\mathcal{Q}^{\rm EdGB} = \mathcal{Q}^{\rm EdGB}$. Follow the definition in Eq. (3.114), we can write

$$
\mathcal{F}^{\rm EdGB} = \mathcal{Q}^{\rm EdGB}\Box^{-1}\,, \qquad (3.151)
$$

where $\Box^{-1}$ comes from inverting the equation of motion of $\varphi^{(1,0,1)}$ in Eq. (3.109). One can check that $\hat{\mathcal{P}}\mathcal{F}^{\rm EdGB} = \mathcal{F}^{\rm EdGB}$, so non-rotating EdGB BHs also admit definite-parity modes.

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Chapter 4

PERTURBATIONS OF SPINNING BLACK HOLES IN DYNAMICAL CHERN-SIMONS GRAVITY: SLOW ROTATION EQUATIONS

[1] Pratik Wagle, Dongjun Li, Yanbei Chen, and Nicolás Yunes. "Perturbations of spinning black holes in dynamical Chern-Simons gravity: Slow rotation equations". In: *Phys. Rev. D* 109.10 (2024), p. 104029. DOI: 10.1103/PhysRevD.109.104029. arXiv: 2311.07706 [gr-qc].

4.1 Introduction

The discovery of gravitational waves (GWs) has provided a new avenue for scrutinizing the predictions and phenomenology of Einstein's theory of general relativity (GR) in regimes characterized by non-linear and dynamic gravitational effects [1, 2]. GWs offer the opportunity to investigate the properties of astrophysical objects where gravity is notably intense, such as black holes (BHs) and neutron stars (NSs). In particular, GWs are often generated by the coalescence of binary BH systems, wherein two BHs orbit each other, gradually inspiraling due to the emission of GWs, and ultimately merging to produce a final BH that emits GW radiation as it settles down. GWs during this part of the coalescence, known as the ringdown phase, comprise a superposition of numerous quasinormal modes (QNMs), each with a complex eigenfrequency. By analyzing GW observations, it may thus become possible to efficiently explore the distinctive spectrum of QNMs exhibited by BHs as they ring down [3, 4].

QNMs are the characteristic vibrational modes of BHs that are excited when the BH is perturbed. The study of QNMs can provide important information about the fundamental properties of BHs and their surrounding spacetime. In particular, the QNM spectrum of astrophysical (i.e., uncharged) BHs in GR is fully determined by just two parameters: the mass and spin of the remnant BH. One promising application of QNMs is to test modified gravity theories and alternative models of compact objects [5]. The predictions of modified gravity theories may deviate from GR's and can manifest as modifications of the QNM spectrum of BHs [6–12]. By observing the QNM spectrum of merging BHs with GW detectors, it may be possible to place constraints on these modifications to GR and its models given two or more detections of QNM frequencies with a strong signal-to-noise ratio. Recent GW detections of binary BH mergers have provided some of the most precise tests of GR in the strong-field regime [13]. With improvements in detector technology and advancements in computational techniques, using ringdown to test modified gravity theories seems possible in the near future [14].

To study QNMs in GR, there are two well-established approaches within BH perturbation theory. One approach, proposed by Regge and Wheeler [15], perturbs the metric directly and it has been successfully applied to non-rotating and slowly rotating BHs in GR [15–18], as well as in modified theories of gravity [6–12]. However, this approach has not been successfully applied to BHs with a general spin. An alternative approach uses curvature perturbations, and it was presented by Teukolsky [19] to study rotating Kerr BHs in GR, including their QNM spectrum and dynamical stability [20]. This framework uses the Newman:1961qr (NP) formalism [21] and considers curvature perturbations characterized by quantities known as Weyl scalars. The success of the Teukolsky formalism lies in its ability to provide a decoupled evolution equation for each of the Ψ_0 and Ψ_4 Weyl scalars, which describe transverse gravitational perturbations, and physically represent ingoing and outgoing GWs, respectively. Not only are these quantities decoupled from other gravitational degrees of freedom, their evolution equations are also separable into a radial and an angular equation [4, 20, 22].

The Teukolsky formalism in GR requires the background geometry to be algebraically of type D under the Petrov classification [23], as is the case for Schwarzschild and Kerr spacetimes in GR. This Petrov type D property implies that four of the five Weyl scalars vanish on the background. However, when considering beyond-GR (bGR) theories, the deviations introduced may lead to BHs described by non-Petrov-type-D spacetimes. For instance, rotating BHs in dynamical Chern-Simons (dCS) gravity and Einstein-dilaton Gauss-Bonnet gravity are algebraically general and classified as Petrov type I. As a consequence, the Teukolsky formalism cannot be directly applied to bGR BH spacetimes. Therefore, calculating the QNM spectra of spinning BHs in bGR theories has been an open problem for a long time and warrants new approaches. One potential resolution to this problem became available recently with the development of the modified Teukolsky formalism [24–27]. This formalism, in theory, enables studying perturbations of spinning BHs in bGR theories and calculating the QNM spectra of such non-Ricci-flat, matter vacuum Petrov type I

BH spacetimes. Yet for tests with GW data, a key theoretical challenge remains to calculate the QNM spectra of BHs in modified theories, and then compare them with observations. This work aims at computing the QNM spectrum of one type of BH spacetimes.

In this work, we restrict our attention to modifications to GR where a scalar field is non-minimally coupled to topological invariant quadratic terms in curvature. A subset of this class of theories, known as dCS gravity [28, 29], was proposed to explain the matter-antimatter asymmetry of the universe. This is achieved by the introduction of additional parity-violating gravitational interactions, challenging a fundamental pillar of GR. Due to the quadratic nature of this theory, it is not strongly constrained using weak field tests and evades binary pulsar tests [30] and GW polarization tests [31]. However, early work using the metric perturbation approach shows promise that ringdown tests can be useful in constraining this modified theory of gravity [9].

We focus on slowly rotating BHs in dCS gravity to leading order in spin in this work. This calculation serves as a validation of the newly developed formalism [24], as the results herein can then be confirmed with the results obtained using the metric perturbation approach [9, 32]. We first use the formalism prescribed in [24] to obtain the modified Teukolsky equation for a slowly rotating BH in dCS gravity. To leading order in spin, these BH backgrounds are described by a non-Ricci-flat, matter vacuum Petrov type D spacetime [24, 33]. Solving the Bianchi identities, we obtain the modified Teukolsky equation first in the null NP basis. We then rewrite this equation in the coordinate basis by defining a tetrad (similar to the Kinnersly tetrad in GR). Finally, we make use of the properties of spin-weighted spheroidal harmonics to eliminate the angular dependence and obtain a radial second-order differential equation. Due to the non-minimal coupling between the scalar field and the curvature, the perturbed master equations of the scalar field and the Weyl scalars (Ψ_0 or Ψ_4) are coupled in dCS gravity. Moreover, some of these quantities also require metric reconstruction within GR [34–38], making the problem more challenging, yet still solvable, as we demonstrate here. Having obtained the master equations in this work, we will use the eigenvalue perturbation method [25, 39, 40] in future work to calculate the QNM spectrum and compare it with the results using metric perturbations in dCS gravity [9, 32].

The remainder of this paper is organized as follows. We first present in brief the action and the field equations of dCS gravity in Sec. 4.2 along with the slowly rotating BH solutions in this theory. In Sec. 4.3, we present an overview of the modified Teukolsky formalism in [24], define a three-parameter expansion under the slow-rotation approximation, and calculate the NP quantities on the dCS BH background. In Sec. 4.4, we provide a concise review of the metric reconstruction procedures in GR. In Secs. 4.5, 4.6, and 4.7, we calculate the source terms of the scalar field equation and the modified Teukolsky equation of Ψ_0 and Ψ_4 in the null NP basis, the results of which are summarized in Sec. 4.8. In Sec. 4.9, we project the equations into the coordinate basis and extract their radial parts using the properties of spin-weighted spheroidal harmonics. Finally, in Sec. 4.10, we summarize our work and discuss some future avenues. Henceforth, we adopt the following conventions unless stated otherwise: we work in 4-dimensions with metric signature $(-,+,+,+)$ as in [41]. For all NP quantities except the metric signature, we use the notation adapted by Chandrasekhar in [42].

4.2 BHs in dCS Gravity

In this section, we will present the details of the theory and the background BH spacetime used in this work.

4.2.1 dCS gravity

In this subsection, we briefly review the dCS gravity following the discussion and the convention in [43]. A more detailed review of dCS gravity can be found in [28, 29]. The action of dCS gravity is

$$S = \int d^4x\sqrt{-g}\left\{\kappa_g R + \frac{\alpha}{4}\vartheta R_{\nu\mu\rho\sigma}{}^*R^{\mu\nu\rho\sigma} - \frac{1}{2}\left[\nabla_\mu\vartheta\nabla^\mu\vartheta + 2V(\vartheta)\right] + \mathcal{L}_{\text{matter}}\right\}, \tag{4.1}$$

where $\kappa_g = \frac{1}{16\pi}$, ${}^*R^{\mu\nu\rho\sigma}$ is the dual of the Riemann tensor,

$${}^*R^{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon^{\rho\sigma\alpha\beta}R^{\mu\nu}{}_{\alpha\beta}\,, \tag{4.2}$$

and ϑ is the pseudoscalar field coupled to the Pontryagin density $P := R_{\nu\mu\rho\sigma}{}^*R^{\mu\nu\rho\sigma}$ via the dCS coupling constant α. The quantity $V(\vartheta)$ is a potential for ϑ, which we set to zero for the reasons explained in [44–46], along with any matter contribution $\mathcal{L}_{\text{matter}}$ (since we will work with matter vacuum BH spacetimes) for the remainder of this work. From Eq. (4.1), we find that $[\vartheta] = 1$ and $[\alpha] = L^2$ in geometric units. Using these coupling constants, we can define a dimensionless parameter ζ characterizing the strength of the dCS correction to GR, where ζ is defined in [43] to be

$$\zeta \equiv \frac{\alpha^2}{\kappa_g M^4}\,, \tag{4.3}$$

with M the typical mass of the system. When the system under consideration contains a single black hole, then M is its mass. When considering a binary system, then different corrections to the solutions of the field equations will scale with different (dimension-4) combinations of the two masses.

Varying the action in Eq. (4.1) with respect to the metric and the scalar fields, respectively, we obtain

$$R_{\mu\nu} = -\frac{\alpha}{\kappa_g} C_{\mu\nu} + \frac{1}{2\kappa_g} \bar{T}^{\vartheta}_{\mu\nu} , \tag{4.4}$$

$$\Box\vartheta = -\frac{\alpha}{4} R_{\nu\mu\rho\sigma} {}^{*}R^{\mu\nu\rho\sigma} , \tag{4.5}$$

where $\Box = \nabla_\mu \nabla^\mu$ is the D'Alembertian operator, and

$$C^{\mu\nu} \equiv (\nabla_\sigma \vartheta) \, \epsilon^{\sigma\delta\alpha(\mu} \nabla_\alpha R^{\nu)\delta} + (\nabla_\sigma \nabla_\delta \vartheta) {}^{*}R^{\delta(\mu\nu)\sigma} , \tag{4.6}$$

$$\bar{T}^{\vartheta}_{\mu\nu} \equiv \left(\nabla_\mu \vartheta\right) \left(\nabla_\nu \vartheta\right) . \tag{4.7}$$

We have here adopted the trace-reversed form of the field equations, using the fact that the C-tensor is traceless, as it will render future calculations simpler.

The dCS action presented above is an effective theory that includes only linear in α and quadratic in curvature corrections to the Einstein-Hilbert action, thus ignoring higher order terms in α and in curvature. Therefore, the resulting field equations are also similarly effective, and their solutions ought to be truncated at leading order in α and considered only for systems (and regimes of spacetime) with small curvatures. Various previous work [43, 47] have studied the regime of validity of this effective action and its curvature cutoff. In essence, the effective theory remains valid provided $(\alpha^2/\kappa_g)P^2 \ll 1$, where recall that P has been defined as the Pontryagin density. When this is the case, the higher order in α and in curvature terms neglected in the action above can continue to be ignored. The systems studied in this paper involve the exterior spacetime of remnant BHs with masses in the range $3M_\odot < M < 10^7 M_\odot$. For such systems, the theory remains effective, and $\zeta \ll 1$ provided $\sqrt{\alpha} \ll 10^7$km, which will be assumed henceforth; the best current constraints on α come from NICER and advanced LIGO observations, and they require that $\sqrt{\alpha} \leq 8.5$km at 90% confidence [48]. In this range of α and for these BH masses, the quadratic curvature corrections to the Einstein Hilbert action will remain perturbative, and the higher-order in α and curvature terms will remain controlled relative to the quadratic term included in the dCS action.

4.2.2 Slowly rotating BHs in dCS gravity

Solutions to BHs in dCS gravity have been found both numerically [49] and analytically [43, 50–52]. For this work, we consider the analytical solution found in [50, 51], which were obtained by perturbatively solving the field equations (4.4) and (4.5) to linear order in both the dimensionless spin parameter χ, where $\chi \equiv a/M$ (with the dimensional spin parameter $a = S/M$, where S is the spin angular moment, and M is the BH mass), and the dCS expansion parameter ζ, defined in Eq. (4.3). This analytical solution casts the line element of a slowly rotating BH in dCS gravity as

$$ds^2 = ds^2_{\text{slow}} + ds^2_{\text{dCS}}\,, \tag{4.8}$$

where, following the convention in [42], the line element for the slowly rotating Kerr metric in Boyer-Lindquist coordinates (t, r, θ, ϕ) is given by

$$\begin{aligned} ds^2_{\text{slow}} = g^{\text{slow}}_{\alpha\beta} dx^\alpha dx^\beta - f(r)dt^2 - \frac{4Ma\sin^2\theta}{r} dt d\phi + f(r)^{-1} dr^2 \\ + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\,, \end{aligned} \tag{4.9}$$

where $f(r) = 1 - 2M/r$ is the Schwarzschild factor. The line element in Eq. (4.9) is obtained by keeping only the leading-order terms in a of the Kerr metric in Boyer-Lindquist coordinates, i.e.,

$$\begin{aligned} ds^2_{\text{Kerr}} &= g^{\text{Kerr}}_{\alpha\beta} dx^\alpha dx^\beta \\ &= -\left(1 - \frac{2Mr}{\tilde{\rho}^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\tilde{\rho}^2} dt d\phi + \frac{\tilde{\rho}^2}{\Delta} dr^2 \\ &\quad + \tilde{\rho}^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2\theta}{\tilde{\rho}^2}\right) \sin^2\theta d\phi^2\,, \end{aligned} \tag{4.10}$$

with $\tilde{\rho}^2 \equiv r^2 + a^2\cos^2\theta$ and $\Delta \equiv r^2 - 2Mr + a^2$. To leading order in χ and ζ, the dCS modification to the Kerr line element is given by

$$ds^2_{\text{dCS}} = -\zeta\chi \frac{\tilde{G}(r)}{2} \sin^2\theta dt d\phi\,, \tag{4.11}$$

where

$$\tilde{G}(r) = -\frac{5M^5}{4r^4}\left(1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right)\,. \tag{4.12}$$

The above dCS correction to the line element is of $O(\zeta^1, \chi^1, \epsilon^0)$, and thus, we can use the tri-variate notation that we will introduce in Sec. 4.3.3 to write

$$h^{(1,1,0)}_{t\phi} = -\frac{\tilde{G}(r)}{2} \sin^2\theta\,. \tag{4.13}$$

The dCS metric of a slowly rotating BH is then identical to the Kerr metric, except for the (t, ϕ) component, which acquires the correction presented above. Similarly, the background scalar field at leading order in spin is given by [50]

$$\vartheta^{(0)} = \frac{5}{8}\frac{a\alpha}{M}\frac{\cos\theta}{r^2}\left(1 + \frac{2M}{r} + \frac{18M^2}{5r^2}\right), \tag{4.14}$$

where the superscript denotes the scalar field is evaluated on the background. Although the scalar field ϑ enters at $O(\zeta^{1/2})$, as shown in Eqs. (4.5) and (4.14), we follow [24] to multiply ϑ by an additional factor of $\zeta^{1/2}$ for the simplicity of order counting. In this case, the expansion of ϑ starts at $O(\zeta)$, so at $O(\zeta^1, \chi^1, \epsilon^0)$, we have

$$\begin{aligned} \zeta^{1/2}\vartheta^{(0)} &= \zeta\chi\vartheta^{(1,1,0)} \\ &= \zeta\chi\frac{5M^2}{32\sqrt{\pi}r^2}\left(1 + \frac{2M}{r} + \frac{18M^2}{5r^2}\right)\cos\theta, \end{aligned} \tag{4.15}$$

where we have used Eq. (4.3). The three-parameter expansion in $\{\zeta, \chi, \epsilon\}$ is discussed in more detail later in Sec. 4.3.3.

4.3 BH Perturbations in Teukolsky formalism

In this section, we review the modified Teukolsky formalism developed in [24]. In this paper, we extend the two-parameter expansion scheme in [24] to a three-parameter expansion discussed in Sec. 4.3.3 to incorporate the slow-rotation approximation, following [9, 32].

4.3.1 Modified Teukolsky Equation

As discussed previously in Sec. 4.1, for studying perturbations of non-rotating BHs, we obtain the perturbed field equations and decompose these into master equations by making use of the metric perturbations [15–17, 53]. These metric perturbations are separated into two sectors, depending on their behavior under a parity transformation. For each parity, all the metric degrees of freedom are then packed into one master function: the Regge-Wheeler function for odd parity [15] and the Zerilli-Moncrief function for even parity [16, 17]. The master equations governing these master functions are decoupled from other dynamical degrees of freedom of the metric fields and are separable into radial and angular equations.

However, for rotating BHs in GR, due to the lack of spherical symmetry, to obtain the decoupled and separable perturbed field equations, one has to use the Teukolsky equations [19, 20, 54], where the fundamental variables to solve for are the Weyl scalars $\Psi_{0,4}$ characterizing curvature perturbations. In this case, the master equations of $\Psi_{0,4}$ are decoupled from other NP quantities and are separable into purely radial and purely angular equations. For a quick review of the NP formalism and the Teukolsky formalism in GR, one can refer to the original papers [19–21, 54], the book [42], or more recent papers that work in the Teukolsky formalism [24, 38].

In modified gravity, most calculations for rotating BHs have so far been done using metric perturbations and the slow-rotation expansion, e.g., [9, 32] in dCS gravity, [10, 11] in EdGB theory, and [12, 55] in higher-derivative gravity. However, these approaches cannot deal with BHs with arbitrary spin, which motivated the development of the modified Teukolsky formalism in [24, 25]. Following the formalism in [24, 25], one can find separable and decoupled equations for $\Psi_{0,4}$ of BHs with arbitrary spin in a wide class of modified gravity theories, such as in dCS gravity, which can be treated as an EFT extension of GR. In this paper, we will use the modified Teukolsky equations of $\Psi_{0,4}$ in [24]. For an alternative approach following [56] by projecting the Einstein equations to the Teukolsky equations, one can refer to [25].

In [24], the authors introduced a two-parameter expansion, in terms of ζ and ϵ where

1. ζ is the parameter characterizing the strength of modifications to GR. In the case of dCS gravity, ζ is given by Eq. (4.3).
2. ϵ is the parameter characterizing the strength of gravitational perturbations, which also appears in GR.

We do not assume any hierarchy between ζ and ϵ. The dimensionless coupling constant ζ can be smaller or larger than the perturbative expansion parameter ϵ depending on the BH system under consideration. In this way, one can expand all the NP quantities as

$$\begin{aligned}\Psi_i &= \Psi_i^{(0)} + \epsilon\Psi_i^{(1)} \\ &= \Psi_i^{(0,0)} + \zeta\Psi_i^{(1,0)} + \epsilon\Psi_i^{(0,1)} + \zeta\epsilon\Psi_i^{(1,1)}\end{aligned} \tag{4.16}$$

and the extra non-metric fields, such as the pseudoscalar field ϑ in dCS gravity, as

$$\vartheta = \vartheta^{(0)} + \epsilon\vartheta^{(1)} = \zeta\vartheta^{(1,0)} + \zeta\epsilon\vartheta^{(1,1)}\,, \tag{4.17}$$

where we have reserved the single superscript notation for only an expansion in ϵ. When using the double-superscript notation, however, the first superscript will also refer to contributions proportional to ζ to a given power. In contrast, the second superscript will refer to terms proportional to ϵ to a given power.

Using the expansion in Eqs. (4.16) and (4.17), the authors in [24] found that for a rotating BH described by a matter vacuum, non-Ricci-flat, Petrov type I spacetime that perturbatively deviates from a Petrov type D spacetime in GR, the gravitational wave perturbation Ψ_0 satisfies

$$H_0^{(0,0)}\Psi_0^{(1,1)} = \mathcal{S}_{\text{geo}}^{(1,1)} + \mathcal{S}^{(1,1)}\,, \tag{4.18}$$

where $H_0^{(0,0)}$ is the Teukolsky operator for Ψ_0 in GR [19], and the source terms are divided into a "geometric piece",

$$\mathcal{S}_{\text{geo}}^{(1,1)} = \mathcal{S}_{0,\text{D}}^{(1,1)} + \mathcal{S}_{0,\text{non-D}}^{(1,1)} + \mathcal{S}_{1,\text{non-D}}^{(1,1)}\,, \tag{4.19}$$

with

$$\mathcal{S}_{0,\text{D}}^{(1,1)} = -H_0^{(1,0)}\Psi_0^{(0,1)}\,, \tag{4.20a}$$
$$\mathcal{S}_{0,\text{non-D}}^{(1,1)} = -H_0^{(0,1)}\Psi_0^{(1,0)}\,, \tag{4.20b}$$
$$\mathcal{S}_{1,\text{non-D}}^{(1,1)} = H_1^{(0,1)}\Psi_1^{(1,0)}\,, \tag{4.20c}$$

and a "Ricci piece",

$$\mathcal{S}^{(1,1)} = \mathcal{E}_2^{(0,0)}S_2^{(1,1)} + \mathcal{E}_2^{(0,1)}S_2^{(1,0)} - \mathcal{E}_1^{(0,0)}S_1^{(1,1)} - \mathcal{E}_1^{(0,1)}S_1^{(1,0)}\,, \tag{4.21}$$

with $S_{1,2}$ given by

$$S_1 \equiv \delta_{[-2,-2,1,0]}\Phi_{00} - D_{[-2,0,0,-2]}\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02}\,, \tag{4.22a}$$
$$S_2 \equiv \delta_{[0,-2,2,0]}\Phi_{01} - D_{[-2,2,0,-1]}\Phi_{02} - \bar{\lambda}\Phi_{00} + 2\sigma\Phi_{11} - 2\kappa\Phi_{12}\,. \tag{4.22b}$$

The operators $H_{0,1}$, $\mathcal{E}_{0,1}$ are defined as

$$\begin{aligned} &H_0 = \mathcal{E}_2F_2 - \mathcal{E}_1F_1 - 3\Psi_2\,, \quad H_1 = \mathcal{E}_2J_2 - \mathcal{E}_1J_1\,, \\ &\mathcal{E}_1 = E_1 - \frac{1}{\Psi_2}\delta\Psi_2\,, \quad \mathcal{E}_2 = E_2 - \frac{1}{\Psi_2}D\Psi_2\,, \end{aligned} \tag{4.23}$$

where Ψ_2 is a NP scalar, and we have also defined

$$\begin{aligned} F_1 &\equiv \bar{\delta}_{[-4,0,1,0]}\,, \qquad & F_2 &\equiv \boldsymbol{\Delta}_{[1,0,-4,0]}\,, \\ J_1 &\equiv D_{[-2,0,-4,0]}\,, \qquad & J_2 &\equiv \delta_{[0,-2,0,-4]}\,, \\ E_1 &\equiv \delta_{[-1,-3,1,-1]}\,, \qquad & E_2 &\equiv D_{[-3,1,-1,-1]}\,. \end{aligned} \tag{4.24}$$

The operators that appear in the above definitions are defined for convenience to be

$$\begin{aligned} D_{[a,b,c,d]} &= D + a\varepsilon + b\bar{\varepsilon} + c\rho + d\bar{\rho}\,, && (4.25a)\\ \mathbf{\Delta}_{[a,b,c,d]} &= \mathbf{\Delta} + a\mu + b\bar{\mu} + c\gamma + d\bar{\gamma}\,, && (4.25b)\\ \delta_{[a,b,c,d]} &= \delta + a\bar{\alpha} + b\beta + c\bar{\pi} + d\tau\,, && (4.25c)\\ \bar{\delta}_{[a,b,c,d]} &= \bar{\delta} + a\alpha + b\bar{\beta} + c\pi + d\bar{\tau}\,, && (4.25d) \end{aligned}$$

where $(D, \mathbf{\Delta}, \delta, \bar{\delta})$ are the usual NP differential operators (constructed by contracting the tetrad with partial derivatives), while $(\varepsilon, \rho, \mu, \gamma, \alpha, \beta, \pi, \tau)$ are spin coefficients, with the overhead bar denoting complex conjugation, and (a, b, c, d) are certain constants. For a complete derivation of the equations above and the definition of Weyl scalars, spin coefficients, directional derivatives, and Ricci NP scalars, one can refer to [21, 24].

For this work, we are only interested in studying slowly rotating BHs in dCS gravity up to $O(\chi)$. Such BHs are described by vacuum non-Ricci-flat Petrov type D spacetimes [33]. The modified Teukolsky equations for these BHs hold the same form as given in Eq. (4.18) with the source terms $\mathcal{S}^{(1,1)}_{0,\text{non-D}}$ and $\mathcal{S}^{(1,1)}_{1,\text{non-D}}$ in Eqs. (4.20b)–(4.20c) vanishing.

The equation of $\Psi_4^{(0,1)}$ can be obtained from Eq. (4.18) by the Geroch-Held-Penrose (GHP) transformation [57] and is given in [24]:

$$H_4^{(0,0)}\Psi_4^{(1,1)} = \mathcal{T}^{(1,1)}_{\text{geo}} + \mathcal{T}^{(1,1)}\,, \tag{4.26}$$

where $H_4^{(0,0)}$ is the Teukolsky operator in GR for Ψ_4, and the "geometric piece" of the source terms is defined as

$$\mathcal{T}^{(1,1)}_{\text{geo}} = \mathcal{T}^{(1,1)}_{4,\text{D}} + \mathcal{T}^{(1,1)}_{4,\text{non-D}} + \mathcal{T}^{(1,1)}_{3,\text{non-D}}\,, \tag{4.27}$$

with

$$\begin{aligned} \mathcal{T}^{(1,1)}_{4,\text{D}} &= -H_4^{(1,0)}\Psi_4^{(0,1)}\,, && (4.28a)\\ \mathcal{T}^{(1,1)}_{4,\text{non-D}} &= -H_4^{(0,1)}\Psi_4^{(1,0)}\,, && (4.28b)\\ \mathcal{T}^{(1,1)}_{3,\text{non-D}} &= H_3^{(0,1)}\Psi_3^{(1,0)}\,, && (4.28c) \end{aligned}$$

whereas the "Ricci piece" is defined as

$$\mathcal{T}^{(1,1)} = \mathcal{E}_4^{(0,0)}S_4^{(1,1)} + \mathcal{E}_4^{(0,1)}S_4^{(1,0)} - \mathcal{E}_3^{(0,0)}S_3^{(1,1)} - \mathcal{E}_3^{(0,1)}S_3^{(1,0)}\,, \tag{4.29}$$

with

$$S_3 \equiv -\mathbf{\Delta}_{[0,2,2,0]}\Phi_{21} + \bar{\delta}_{[2,2,0,-1]}\Phi_{22} + 2\nu\Phi_{11} + \bar{\nu}\Phi_{20} - 2\lambda\Phi_{12}\,, \tag{4.30a}$$

$$S_4 \equiv -\mathbf{\Delta}_{[0,1,2,-2]}\Phi_{20} + \bar{\delta}_{[2,0,0,-2]}\Phi_{21} + 2\nu\Phi_{10} - 2\lambda\Phi_{11} + \bar{\sigma}\Phi_{22}\,. \tag{4.30b}$$

The operators $H_{3,4}$ and $\mathcal{E}_{3,4}$ are defined as

$$\begin{aligned} &H_4 = \mathcal{E}_4 F_4 - \mathcal{E}_3 F_3 - 3\Psi_2\,, \quad H_3 = \mathcal{E}_4 J_4 - \mathcal{E}_3 J_3\,, \\ &\mathcal{E}_3 = E_3 - \frac{1}{\Psi_2}\bar{\delta}\Psi_2\,, \quad \mathcal{E}_4 = E_4 - \frac{1}{\Psi_2}\mathbf{\Delta}\Psi_2\,, \end{aligned} \tag{4.31}$$

with

$$\begin{aligned} F_3 &\equiv \delta_{[0,4,0,-1]}\,, \qquad F_4 \equiv D_{[4,0,-1,0]}\,, \\ J_3 &\equiv \mathbf{\Delta}_{[4,0,2,0]}\,, \qquad J_4 \equiv \bar{\delta}_{[2,0,4,0]}\,, \\ E_3 &\equiv \bar{\delta}_{[3,1,1,-1]}\,, \qquad E_4 \equiv \mathbf{\Delta}_{[1,1,3,-1]}\,. \end{aligned} \tag{4.32}$$

Although the formalism above works for BHs with arbitrary spin in dCS gravity, we choose to use the slow-rotation expansion in this paper, so we can check the consistency of our results with prior work using metric perturbations [9, 32] in our next paper [58]. We implement a slow-rotation expansion of the above equations in Sec. 4.3.3.

4.3.2 Structure of the source terms

Here, we further discuss the structure of the source terms in Eq. (4.18) presented in [24]. In particular, we will focus on the source terms that are non-vanishing for a non-Ricci-flat, Petrov type D BH in dCS gravity, given in Eqs. (4.20a) and (4.21). The source term in Eq. (4.20a) only depends on the perturbed Weyl scalar $\Psi_0^{(0,1)}$ in GR and the dCS corrections to the stationary NP quantities at $O(\zeta^1, \epsilon^0)$. One can solve the Teukolsky equation in GR [19, 20] to calculate $\Psi_0^{(0,1)}$ directly. The NP quantities at $O(\zeta^1, \epsilon^0)$ can be computed from the dCS metric in Eq. (4.11), as shown in more detail in Sec. 4.3.4.

Due to the non-minimal coupling between the scalar field and the metric, the source term $\mathcal{S}^{(1,1)}$ in Eq. (4.21) depends on both the scalar field perturbations and the metric perturbations. To compute $\mathcal{S}^{(1,1)}$, we first need to calculate the NP Ricci scalars Φ_{ij} using the stress tensor or the Ricci tensor, i.e.,

$$\Phi_{00} = -\frac{1}{2}R_{11} = -\frac{1}{2}R_{\mu\nu}l^\mu l^\nu\,, \tag{4.33}$$

where l^μ is one of the NP tetrad basis vectors. Since the background and perturbed scalar field in GR vanish, we have that $\vartheta^{(0,0)} = \vartheta^{(0,1)} = 0$. Therefore, the NP Ricci scalars $\Phi_{ij}^{(1,1)}$ can be expressed as a function of the scalar field perturbation $\vartheta^{(1,1)}$ and the metric perturbation $h^{(0,1)}$ as

$$\Phi_{ij}^{(1,1)} = \mathcal{O}(\vartheta^{(1,0)}h^{(0,1)}) + \mathcal{O}(\vartheta^{(1,1)}g^{(0,0)})\,, \tag{4.34}$$

where $g^{(0,0)}$ and $h^{(0,1)}$ are shorthand for terms that depend on the metric tensor of the GR background and of the metric perturbation due to GWs reconstructed in GR, respectively. From Eqs. (4.21), (4.22), and (4.34), we notice that $\mathcal{S}^{(1,1)}$ couples the GWs in GR and the scalar field ϑ, so we need to solve the equations of motions of these non-gravitational fields to find their contributions to the stress tensor and $\mathcal{S}^{(1,1)}$. Morevover, from Eqs. (4.23)–(4.25), we see that $\mathcal{S}^{(1,1)}$ in Eq. (4.21) depends on $\Psi_2^{(0,1)}$, the directional derivatives at $\mathcal{O}(\zeta^0, \epsilon^1)$, and the perturbed spin coefficients at $\mathcal{O}(\zeta^0, \epsilon^1)$, which need to be retrieved from the reconstructed metric perturbation $h_{\mu\nu}^{(0,1)}$ for GR GWs. One can either follow the metric reconstruction approach in [34–37, 59–63], the so-called Chrzanowski-Cohen-Kegeles (CCK) procedures, which involves defining an intermediate quantity called the Hertz potential, or the approach in [38, 42], which solves the remaining NP equation directly. In this paper, we choose to follow the more widely used CCK procedures and apply them to compute the source term $\mathcal{S}^{(1,1)}$.

4.3.3 Slow-rotation expansion

When considering a slow-rotation expansion, in addition to the quantities given in Eqs. (4.16) and (4.17), one needs to consider an additional expansion in the dimensionless spin parameter $\chi = a/M$. As an extension of Eqs. (4.16) and (4.17), all the NP quantities now admit a three-parameter expansion in ζ, ϵ, and χ, where

$$\Psi = \sum_{l,m,n} \zeta^l \chi^m \epsilon^n \Psi^{(l,m,n)}\,, \tag{4.35}$$

as well as the pseudoscalar field,

$$\vartheta = \sum_{l,m,n} \zeta^l \chi^m \epsilon^n \vartheta^{(l,m,n)}\,. \tag{4.36}$$

In what follows, it will sometimes be convenient to hide the χ expansion in more compact notation, such as $\psi^{(1,1)} = \psi^{(1,0,1)} + \chi\psi^{(1,1,1)}$. When only a two-parameter expansion is denoted, the χ expansion will be assumed.

In this paper, we will focus only on linear perturbations in ϵ, along with the small-coupling approximation and the slow-rotation approximation, i.e., up to linear order terms in ζ and χ, respectively. Therefore, our Eqs. (4.35) and (4.36) can be expanded as

$$\begin{aligned}\Psi = {} & \Psi^{(0,0,0)} + \chi\left(\Psi^{(0,1,0)} + \zeta\Psi^{(1,1,0)}\right) \\ & + \epsilon\left(\Psi^{(0,0,1)} + \chi\Psi^{(0,1,1)}\right) + \zeta\epsilon\left(\Psi^{(1,0,1)} + \chi\Psi^{(1,1,1)}\right) ,\end{aligned} \tag{4.37a}$$

$$\vartheta = \zeta\left(\chi\vartheta^{(1,1,0)} + \epsilon\vartheta^{(1,0,1)} + \chi\epsilon\vartheta^{(1,1,1)}\right) . \tag{4.37b}$$

Equation (4.37a) groups the corrections to $\Psi^{(0,0,0)}$ into three sets organized by parenthesis. The first set of terms are stationary corrections to the Schwarzschild metric due to the slow-rotation approximation in GR, $\Psi^{(0,1,0)}$, and in dCS gravity, $\Psi^{(1,1,0)}$. The term $\Psi^{(0,1,0)}$ can be retrieved from the slow-rotation expansion of the Kerr metric in Eq. (4.9), and $\Psi^{(1,1,0)}$ can be evaluated with the $O(\zeta^1, \chi^1, \epsilon^0)$ correction to the metric in Eq. (4.11) found by [50]. Since the Pontryagin density vanishes for any spherically symmetric spacetime, and the Schwarzschild metric is the unique stationary spherically symmetric solution to the Einstein equations, there is no correction to the metric at $O(\zeta^1, \chi^0, \epsilon^0)$ [43]. Thus, we have dropped the term $\Psi^{(1,0,0)}$ in Eq. (4.37a). For the same reason, $\vartheta^{(1,0,0)} = 0$.

The second set of terms are the GW perturbations to the Kerr metric in GR up to $O(\zeta^0, \chi^1, \epsilon^1)$. These terms include perturbed Weyl scalars, NP spin coefficients, and directional derivatives, all of which need to be evaluated in GR but include spin perturbations. To evaluate this type of terms, we need metric reconstruction of GW perturbations at $O(\zeta^0, \epsilon^1)$, the procedures of which are discussed in detail in Sec. 4.4.

The third set of terms are the one we want to solve for, which are corrections to GW perturbations in dCS gravity. The term $\Psi^{(1,0,1)}$ corresponds to gravitational perturbations sourced by non-rotating BHs in dCS gravity. Since $\vartheta^{(1,0,0)} = 0$, $\Psi^{(1,0,1)}$ is purely sourced by the leading contribution to the dynamical pseudoscalar field $\vartheta^{(1,0,1)}$, so only $\mathcal{S}^{(1,0,1)}$ contributes to Eq. (4.18), and no metric reconstruction is needed. The term $\Psi^{(1,1,1)}$ corresponds to leading-order corrections to gravitational perturbations of slowly rotating BHs in dCS gravity. Unlike the non-rotating case, since both the metric and ϑ are corrected at $O(\zeta^1, \chi^1, \epsilon^0)$, $\Psi_{0,4}^{(1,1,1)}$ can either be driven by dynamical GW perturbations in GR or dynamical ϑ.

For the first type of correction, the driving terms can come from $\mathcal{S}_{\text{geo}}^{(1,1)}$ in the form of terms proportional to the product $h^{(1,0)}h^{(0,1)}$. As discussed in Sec. 4.3.1 and [24], this kind of terms is due to the correction to the background geometry, so they are independent of bGR theories. Up to $O(\zeta^1, \chi^1, \epsilon^1)$, the background spacetime is still Petrov type D [43], so $\mathcal{S}^{(1,1,1)}_{0,\text{non-D}} = \mathcal{S}^{(1,1,1)}_{1,\text{non-D}} = 0$ in Eq. (4.19), and one does not need metric construction to evaluate these terms [24]. In Sec. 4.6.1, we will compute $\mathcal{S}^{(1,1)}_{\text{geo}}$ in detail. Besides $\mathcal{S}^{(1,1)}_{\text{geo}}$, there is also contribution from $\mathcal{S}^{(1,1)}$ in the form of terms proportional to the product $\vartheta^{(1,0)} h^{(0,1)}$ due to the effective stress tensor. In this case, metric reconstruction is needed, and we will compute $\mathcal{S}^{(1,1)}$ in Sec. 4.6.2.

For the second type of correction, the driving terms only come from $\mathcal{S}^{(1,1)}$. Since both $\vartheta^{(1,0,1)}$ and $\vartheta^{(1,1,1)}$ are nonzero, the metric field in these terms needs to be evaluated on the Kerr background, expanded to $O(\chi)$. To find $\vartheta^{(1,1)}$, one needs to solve Eq. (4.5) at $O(\zeta^1, \epsilon^1)$, i.e.,

$$\Box^{(0,0)}\vartheta^{(1,1)} = -\frac{M^2}{16\pi^{\frac{1}{2}}}\,[R^*R]^{(0,1)} - \Box^{(0,1)}\vartheta^{(1,0)}\,, \tag{4.38}$$

where we have used Eq. (4.3) to rewrite α in terms of ζ, which then cancels. Notice that the first term on the right-hand side has an implicit factor of ζ. However, this factor cancels out throughout Eq. (4.38) using the perturbative expansion in Eq. (4.37b). In Sec. 4.5, we will compute the source terms of Eq. (4.38). In Sec. 4.6.2, we will compute the source terms driven by $\vartheta^{(1,1)}$ but leave $\vartheta^{(1,1)}$ unevaluated. In our follow-up work [58], we will solve both the modified Teukolsky equation and the scalar field equation jointly to find the QNM shifts. Since metric reconstruction at $O(\zeta^0, \epsilon^1)$ is required for both the modified Teukolsky equation and the scalar field equation, we present a review of the procedures in Sec. 4.4.

4.3.4 NP quantities on background

In this subsection, we will present the background tetrad for a non-Ricci-flat, Petrov type D, slowly rotating, dCS gravity BH spacetime. To obtain the null tetrad for the metric given by the line element in Eq. (4.8), one can follow the general procedures prescribed in [24] or the standard procedures for finding the Kinnersley tetrad in GR given in [42]. In [64], such a tetrad was found by following the second approach. Nonetheless, for completeness, let us re-derive the tetrad following the prescription in [42, 64]. Our result is consistent with the one in [64], but with additional tetrad rotations to set $\Psi^{(1,0)}_{0,1,3,4} = 0$.

To begin, we first find the null geodesics in the equatorial plane for a dCS BH to be

$$
\begin{aligned}
\frac{dt}{d\tau} &= \left[r^2 + a^2 - \frac{aG(r)}{2}\right]\frac{E}{\tilde{\Delta}(r)}\,, &\qquad \frac{dr}{d\tau} &= \pm\sqrt{\Delta(r)\tilde{\Delta}(r)\left(1 - \frac{aG(r)}{r^2}\right)}\frac{E}{\tilde{\Delta}(r)}\,,\\
\frac{d\theta}{d\tau} &= 0\,, & \frac{d\phi}{d\tau} &= \left(a + \frac{G(r)}{2}\right)\frac{E}{\tilde{\Delta}(r)}\,,
\end{aligned}
\tag{4.39}
$$

where $E = -\partial\mathscr{L}/\partial t$ is a constant of motion, $\mathscr{L}$ is the Lagrangian for Kerr in [42], $\tilde{\Delta}(r) = \Delta(r) + 2aMG(r)/r + G(r)^2/4$, and $G(r) = \zeta\chi\tilde{G}(r)$. Following the procedures outlined in [42], we align the tetrad basis vectors l^μ and n^μ along the outgoing and ingoing null geodesics respectively at the equilateral plane with $E = 1$ such that

$$
l^\mu = \frac{1}{\tilde{\Delta}(r)}\left(r^2 + a^2 - \frac{aG(r)}{2}\,,\ \sqrt{\Delta(r)\tilde{\Delta}(r)\left(1 - \frac{aG(r)}{r^2}\right)},\ 0\,,\ a + \frac{G(r)}{2}\right), \tag{4.40}
$$

$$
n^\mu = N\left(r^2 + a^2 - \frac{aG(r)}{2}\,,\ -\sqrt{\Delta(r)\tilde{\Delta}(r)\left(1 - \frac{aG(r)}{r^2}\right)},\ 0\,,\ a + \frac{G(r)}{2}\right), \tag{4.41}
$$

where N is the normalization factor introduced to impose $l^\mu n_\mu = -1$. Since l^μ and n^μ are along null geodesics, $l^\mu l_\mu = n^\mu n_\mu = 0$ is satisfied automatically. Expanding Eqs. (4.40) and (4.41) up to $O(\zeta^1, \chi^1, \epsilon^0)$, we find

$$
l^\mu = \left(\frac{r}{r - r_s}\,,\ 1\,,\ 0\,,\ \frac{\chi M}{r(r - r_s)} + \frac{\zeta\chi\tilde{G}(r)}{2r(r - r_s)}\right), \tag{4.42}
$$

$$
n^\mu = \tilde{N}(r)\left(\frac{r}{r - r_s}\,,\ -1\,,\ 0\,,\ \frac{\chi M}{r(r - r_s)} + \frac{\zeta\chi\tilde{G}(r)}{2r(r - r_s)}\right), \tag{4.43}
$$

where r_s is the Schwarzschild radius given by $r_s = 2M$, and $\tilde{N}(r) = (r - r_s)/2r$. When $\zeta = 0$, l^μ and n^μ reduce to the Kinnersley tetrad of Kerr BHs expanded to $O(\chi^1)$. The tetrad basis vectors l^μ and n^μ in Eqs. (4.42) and (4.43) are the same as the principal null directions in Eq. (31) of [64].

To obtain the remaining components of the null tetrad, notice that the correction to the Kerr metric due to dCS gravity enters at $O(\zeta^1, \chi^1, \epsilon^0)$ only in the $t\phi$-component. Therefore, it can be expected that the corrections to the Kinnersley tetrad are only along the ∂_t and ∂_ϕ directions, which is seen to be true for l^μ and n^μ. Thus, at $O(\zeta^1, \chi^1, \epsilon^0)$, the corrections to the remaining null tetrad components, m^μ and $\bar{m}^\mu$, take the form

$$
m^{\mu(1,1,0)} = \left(m_t(r,\theta),\ 0,\ 0,\ m_\phi(r,\theta)\right)\,, \tag{4.44}
$$

where $\bar{m}^\mu$ can be obtained by taking the complex conjugation of Eq. (4.44). Imposing the remaining orthogonality conditions to $\mathcal{O}(\zeta^1, \chi^1, \epsilon^0)$, we find $m_t(r,\theta) = m_\phi(r,\theta) = 0$. Therefore,

$$m^\mu = \frac{1}{\sqrt{2}r}\left(i\chi M\sin\theta\,,\; 0\,,\; 1 - \frac{i\chi M\cos\theta}{r}\,,\; i\left(1 - \frac{i\chi M\cos\theta}{r}\right)\csc\theta\right). \tag{4.45}$$

Notice that m^μ (and therefore $\bar{m}^\mu$) holds the same form as the Kinnersley tetrad of Kerr BH expanded to $\mathcal{O}(\chi)$. Using Eqs. (4.42), (4.43), and (4.45), to $\mathcal{O}(\zeta^1, \chi^1, \epsilon^0)$, we obtain

$$\begin{aligned} &\Psi_0 = \Psi_4 = 0\,,\;\; \Psi_1 = -\frac{3\sqrt{2}i\zeta\chi A_1(r)}{32r^9}\sin\theta\,,\\ &\Psi_2 = -\frac{M}{r^3} - \frac{3i\chi}{r^4}\left(M^2 - \frac{\zeta A_2(r)}{8r^5}\right)\cos\theta\,,\;\; \Psi_3 = -\frac{3\sqrt{2}i\zeta\chi A_3(r)}{64r^{10}}\sin\theta\,, \end{aligned} \tag{4.46}$$

where $A_i(r)$ are listed in Appendix 4.11.

Such a calculation contradicts the claim that BHs in dCS gravity are Petrov type D spacetimes to $\mathcal{O}(\chi)$. We see that the seeming contradiction arises due to the non-vanishing Ψ_1 and Ψ_3 on the background. However, we can perform tetrad rotations to eliminate $\Psi_1^{(1,1,0)}$ and $\Psi_3^{(1,1,0)}$, as described in detail in Appendix 4.11. This can be achieved by a type II rotation [i.e., Eq. (4.155b)] with $b^{(1,1,0)} = -\sqrt{2}iA_1(r)\sin\theta/(32Mr^6)$ and a type I rotation [i.e., Eq. (4.155a)] with $a^{(1,1,0)} = \sqrt{2}iA_3(r)\sin\theta/(64Mr^7)$, respectively. Following this, we finally obtain

$$\begin{aligned} &\Psi_i = 0 \quad \forall\, i \in \{0,1,3,4\}\,,\\ &\Psi_2 = -\frac{M}{r^3} - \frac{3i\chi}{r^4}\left(M^2 - \frac{\zeta A_2(r)}{8r^5}\right)\cos\theta\,. \end{aligned} \tag{4.47}$$

The explicit expression for the rotated tetrad is listed in Eq. (4.158). Different from [64], we will call the tetrad in Eq. (4.158) the "principal tetrad." Notice that, in [33], a Kinnersley-like tetrad for the dCS metric expanded up to $\mathcal{O}(\zeta^1, \chi^2, \epsilon^0)$ was also found. To impose that $\Psi_{0,4}^{(1,0)}$ vanish at $\mathcal{O}(\chi^2)$, when the background spacetime is Petrov type I, Ref. [33] added terms at $\mathcal{O}(\zeta^{1/2}, \chi^1, \epsilon^0)$ to the tetrad. In this paper, however, we only want to impose $\Psi_{0,1,3,4}^{(1,0,0)} = \Psi_{0,1,3,4}^{(1,1,0)} = 0$, and prefer to keep the expansion scheme in Eq. (4.16), so the tetrad in Eq. (4.158) is more suitable for our purposes.

We have also listed all the spin coefficients up to $\mathcal{O}(\zeta^1, \chi^1, \epsilon^0)$ in the principal tetrad in Appendix 4.11. For any vacuum Petrov type D spacetimes in GR, the

Goldberg-Sachs theorem requires that in the tetrad where $\kappa = \sigma = \lambda = \nu = 0$, the Weyl scalars $\Psi_{0,1,3,4} = 0$ and vice versa. However, in dCS gravity, since the effective stress tensor is nonzero, the background spacetime is non-Ricci-flat. Thus, $\Psi^{(1,0)}_{0,1,3,4}$ do not necessarily vanish in the tetrad where $\kappa^{(1,0)} = \sigma^{(1,0)} = \lambda^{(1,0)} = \nu^{(1,0)} = 0$ and vice versa. For the tetrad in Eqs. (4.42), (4.43), and (4.45), we found that $\kappa^{(1,0)} = \sigma^{(1,0)} = \lambda^{(1,0)} = \nu^{(1,0)} = 0$ while $\Psi^{(1,0)}_{1,3} \neq 0$ up to $O(\zeta^1, \chi^1, \epsilon^0)$. This tetrad is along the principal null directions found in [64]. Nonetheless, the master equation Eq. (4.18) is more simplified when $\Psi^{(1,0)}_{0,1,3,4} = 0$, so we will use the principal tetrad in Eq. (4.158) for the remaining calculations even if the spin coefficients mentioned above do not vanish along the principal tetrad.

4.4 Metric reconstruction

This section reviews how to reconstruct the perturbed metric and the corresponding NP quantities from solutions to the Teukolsky equation for Kerr BHs in GR. There are two approaches to metric reconstruction in general: the first approach involves systematically solving the Bianchi identities, Ricci identities, and commutation relations [38, 42], whereas the second approach, or the CCK procedures, utilizes an intermediate Hertz potential to reconstruct the metric [34–37, 59–63]. In this work, the second approach is employed to perform metric reconstruction.

4.4.1 Metric perturbations

In this subsection, we present the reconstructed metric perturbation $h^{(0,1)}_{\mu\nu}$ for GR GWs. For convenience, in this section, we will drop the superscript $(0, 1)$ of $h^{(0,1)}_{\mu\nu}$ and always assume that $h_{\mu\nu}$ is at $O(\zeta^0, \epsilon^1)$. The CCK procedures can be carried out in two different gauge choices:

$$\text{Ingoing radiation gauge (IRG):} \quad h_{\alpha\beta} l^{\beta} = 0\,, \;\; h = 0\,, \tag{4.48}$$

$$\text{Outgoing radiation gauge (ORG):} \quad h_{\alpha\beta} n^{\beta} = 0\,, \;\; h = 0\,, \tag{4.49}$$

where h is the trace of $h_{\alpha\beta}$ with respect to the background metric. The reconstructed metric $h_{\alpha\beta}$ in the IRG and ORG are given in Eqs. (4.50) and (4.51), respectively [36, 37],

$$\begin{aligned}(h_{\alpha\beta})_{\text{IRG}} = \Big[& l_\alpha l_\beta \left(\bar{\delta}_{[1,3,0,-1]} \bar{\delta}_{[0,4,0,3]} - \lambda D_{[0,4,0,3]} \right) + \bar{m}_\alpha \bar{m}_\beta \left(D_{[-1,3,0,-1]} D_{[0,4,0,3]} \right) \\ & - l_{(\alpha} \bar{m}_{\beta)} \left(D_{[1,3,1,-1]} \bar{\delta}_{[0,4,0,3]} \right) + \bar{\delta}_{[-1,3,-1,-1]} D_{[0,4,0,3]} \Big] \bar{\Psi}_{\text{H}} + \text{c.c.}\,,\end{aligned} \tag{4.50}$$

$$(h_{\alpha\beta})_{\text{ORG}} = -\rho^{-4} \left[n_\alpha n_\beta \left(\delta_{[-3,-1,5,0]} \delta_{[-4,0,1,0]} \right) + m_\alpha m_\beta \left(\Delta_{[0,5,1,-3]} \Delta_{[0,1,0,-4]} \right) \right.$$

$$-n_{(\alpha}m_{\beta)}\left(\delta_{[-3,1,5,1]}\Delta_{[0,1,0,-4]}\right)+\Delta_{[-1,5,-1,-3]}\delta_{[-4,0,1,0]}\Big]\bar{\Psi}_{\rm H}+\text{c.c.}\,, \tag{4.51}$$

where the notation for the derivatives is given by Eq. (4.25), and $\bar{\Psi}_{\rm H}$ is the complex conjugate of the Hertz potential. We have also dropped the superscript $(0,1)$ of the Hertz potential for simplicity. Notice, since we use an opposite signature from [36, 37], our Eqs. (4.50) and (4.51) have an opposite sign from the results in [36, 37].

Although the metric and the NP quantities at $O(\zeta^0,\epsilon^1)$ are reconstructed in certain gauges (i.e., IRG or ORG), the Weyl scalars $\Psi_{0,4}^{(1,1)}$ are gauge invariant for any Petrov type D BH spacetimes, as shown in detail in [65]. Since slowly rotating dCS BHs up to $O(\chi)$ are Petrov type D, the equations of $\Psi_{0,4}^{(1,1)}$ we will derive are gauge invariant. However, for Petrov type I BH spacetimes, $\Psi_{0,4}^{(1,1)}$ are not invariant under gauge transformations at $O(\zeta^0,\epsilon^1)$ [65], so the master equations of $\Psi_{0,4}^{(1,1)}$ will depend on the gauge where we reconstruct the metric. Nonetheless, the QNM frequencies should still be gauge invariant.

4.4.2 Hertz potential

The Hertz potential $\Psi_{\rm H}$ that appears in Eq. (4.50) in the IRG and Eq. (4.51) in the ORG satisfies the Teukolsky equation for $\rho^{-4}\Psi_4^{(0,1)}$ and $\Psi_0^{(0,1)}$, respectively [34, 35, 59, 62, 66]. For convenience, let us drop the superscript $(0,1)$ of $\Psi_{0,4}^{(1,1)}$ and always assume that $\Psi_{0,4}$ are at $O(\zeta^0,\epsilon^1)$ in this subsection. Using the perturbed metric in Eq. (4.50), the relation between the Hertz potential $\Psi_{\rm H}$ and $\Psi_{0,4}$ can be found by directly evaluating the Riemann tensor or by using the Ricci identities. In the IRG, the perturbed Weyl scalars can then be expressed in terms of the Hertz potential using

$$\Psi_0=-\frac{1}{2}D^4\bar{\Psi}_{\rm H}\,, \tag{4.52a}$$
$$\Psi_4=-\frac{1}{8}\rho^4\left[\mathcal{L}^{\dagger 4}\bar{\Psi}_{\rm H}-12M\partial_t\Psi_{\rm H}\right]\,, \tag{4.52b}$$

and in the ORG,

$$\Psi_4=\frac{1}{32}\rho^4\Delta^2D^{\dagger 4}\Delta^2\bar{\Psi}_{\rm H}\,, \tag{4.53a}$$
$$\Psi_0=\frac{1}{8}\left[\mathcal{L}^4\bar{\Psi}_{\rm H}+12M\partial_t\Psi_{\rm H}\right]\,, \tag{4.53b}$$

where $\rho = -1/(1 - ia\cos\theta)$ and

$$\begin{aligned} &D = l^\mu \partial_\mu = \frac{r^2+a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi\,, \quad D^\dagger = -\frac{r^2+a^2}{\Delta}\partial_t + \partial_r - \frac{a}{\Delta}\partial_\phi\,, \\ &\mathcal{L}_s = -ia\sin\theta\partial_t - \left[\partial_\theta + i\csc\theta\partial_\phi - s\cot\theta\right]\,, \quad \mathcal{L}^4 = \mathcal{L}_1\mathcal{L}_0\mathcal{L}_{-1}\mathcal{L}_{-2}\,, \\ &\mathcal{L}_s^\dagger = ia\sin\theta\partial_t - \left[\partial_\theta - i\csc\theta\partial_\phi - s\cot\theta\right]\,, \quad \mathcal{L}^{\dagger 4} = \mathcal{L}_{-1}^\dagger\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger\mathcal{L}_2^\dagger\,, \end{aligned} \tag{4.54}$$

where we only keep terms up to $O(\chi)$ when applying these operators. Note that the D operator was introduced before in Eq. (4.25), but here we provide its expression using the Kinnersely tetrad. We point out that these are the same operators that appear in the Teukolsky-Starobinsky identity [20, 42]. Notice, Eqs. (4.52) and (4.53) follow [36, 37], which corrected a factor of one-half in earlier papers [34, 35, 59].[1]

Similar to the perturbations of Ψ_0 and Ψ_4, the Hertz potential can be defined in the coordinate basis (t, r, θ, ϕ) as

$$\text{IRG}: \quad \bar{\Psi}_{\mathrm{H}} = {}_2\hat{R}_{\ell m}(r)\,{}_2\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega t}\,, \tag{4.55a}$$

$$\text{ORG}: \quad \bar{\Psi}_{\mathrm{H}} = {}_{-2}\hat{R}_{\ell m}(r)\,{}_{-2}\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega t}\,, \tag{4.55b}$$

where ${}_{\pm2}\mathcal{Y}_{\ell m}(\theta,\phi) = {}_{\pm2}S_{\ell m}(\theta)e^{im\phi}$ are spin-weighted spheroidal harmonics of spin weight ±2 solving the angular Teukolsky equation in GR. ${}_{\pm2}\hat{R}_{\ell m}(r)$ are radial functions that can be expressed in terms of the radial Teukolsky functions ${}_2R_{\ell m}^{(0,1)}(r)$ and ${}_{-2}R_{\ell m}^{(0,1)}(r)$ of $\Psi_0^{(0,1)}$ and $\rho^{-4}\Psi_0^{(0,1)}$, respectively, by inverting Eqs. (4.52a) and (4.53a) using the Teukolsky-Starobinsky identity [61],

$${}_2\hat{R}_{\ell m}(r) = -\frac{2}{\mathfrak{C}}\Delta^2 (D_{m\omega}^\dagger)^4\left[\Delta^2\,{}_2R_{\ell m}^{(0,1)}(r)\right]\,, \tag{4.56a}$$

$${}_{-2}\hat{R}_{\ell m}(r) = \frac{32}{\mathfrak{C}}(D_{m\omega})^4\,{}_{-2}R_{\ell m}^{(0,1)}(r)\,. \tag{4.56b}$$

Here, the operators $D_{m\omega}$ and $D_{m\omega}^\dagger$ are mode decomposition of D and $D^\dagger$, respectively [61],

$$D_{m\omega} = \partial_r + i\frac{am-(r^2+a^2)\omega}{\Delta}\,, \quad D_{m\omega}^\dagger = \partial_r - i\frac{am-(r^2+a^2)\omega}{\Delta}\,. \tag{4.57}$$

$\mathfrak{C}$ is the mode-dependent Teukolsky-Starobinsky constant [20, 61, 67, 68]

$$\mathfrak{C} = \lambda^2(\lambda+2)^2 - 8\omega^2\lambda\left[\tilde{\alpha}^2(5\lambda+6) - 12a^2\right] + 144\omega^4\tilde{\alpha}^4 + 144\omega^2M^2\,, \tag{4.58}$$

where $\tilde{\alpha}^2 = a^2 - am/\omega$ and $\lambda = {}_sA_{\ell m} + s + |s|$ with ${}_sA_{\ell m}$ being the Teukolsky's angular separation constant [19]. For a Schwarzschild BH, ${}_sA_{\ell m} = (\ell - s)(\ell + s + 1)$.

[1] Recall that we follow the notation in [42]; the signs of Eq. (4.52) are opposite to the ones in [36, 37].

Furthermore, one can notice from Eq. (4.52) that any (ℓ, m, ω) mode of $\Psi_0^{(0,1)}$ in the IRG generates a mixture of (ℓ, m, ω) and $(\ell, -m, -\bar{\omega})$ modes of $\Psi_4^{(0,1)}$. Thus, it is more convenient to use the ORG when solving the modified Teukolsky equation of $\Psi_4^{(1,1)}$. For a similar reason, when solving the modified Teukolsky equation of $\Psi_0^{(1,1)}$, we will use the IRG.

Substituting the differential operators $D^\dagger_{m\omega}$ and $D_{m\omega}$ in Eq. (4.57) into the expression for the radial part of the Hertz potential in Eq. (4.56), we have

$$ {}_s\hat{R}_{\ell m}(r) = {}_sf_1^{\ell m}(r,\omega,M)\,{}_sR^{(0,1)}_{\ell m}(r) + {}_sf_2^{\ell m}(r,\omega,M)\,{}_sR'^{(0,1)}_{\ell m}(r)\,, \tag{4.59a} $$

$$ {}_s\hat{R}'_{\ell m}(r) = {}_sf_3^{\ell m}(r,\omega,M)\,{}_sR^{(0,1)}_{\ell m}(r) + {}_sf_4^{\ell m}(r,\omega,M)\,{}_sR'^{(0,1)}_{\ell m}(r)\,, \tag{4.59b} $$

where we have made use of the radial Teukolsky equation to reduce all second- and higher-order derivatives of ${}_sR^{(0,1)}_{\ell m}(r)$. In Eq. (4.59), the prime denotes the first derivative with respect to the radial coordinate r. The functions f_i are spin weight s and mode dependent. These functions are lengthy and non-illuminating, so they have been presented in a separate Mathematica notebook [69].

4.4.3 Spin-weighted spheroidal harmonics

Spin-weighted spheroidal harmonics that appear in Eq. (4.55) are solutions to the angular Teukolsky equation in GR [19, 20]. In general, these are eigenfunctions of an equation of the form [70]

$$ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dz}{d\theta}\right) - \left(\frac{m^2+s^2+2ms\cos\theta}{\sin^2\theta} - \gamma^2\cos^2\theta + 2s\gamma\cos\theta - {}_sE^\gamma_{\ell m}\right)z = 0\,, \tag{4.60} $$

where s represents the spin weight, ${}_sE^\gamma_{\ell m}$ is the eigenvalue of the equation, which has been numerically calculated in the literature [20]. Comparing Eq. (4.60) with the angular Teukolsky equation in GR [19], we see that $\gamma = \chi M\omega$. In the slow-rotation expansion, spin-weighted spheroidal harmonics ${}_s\mathcal{Y}^\gamma_{\ell m}$ can be expanded as [20, 70]

$$ {}_s\mathcal{Y}^\gamma_{\ell m} = {}_sY_{\ell m} + \gamma\left({}_sb^m_{\ell,\ell+1}\,{}_sY_{\ell+1\,m} + {}_sb^m_{\ell,\ell-1}\,{}_sY_{\ell-1\,m}\right) + O(\gamma^2)\,, \tag{4.61} $$

where ${}_sY_{\ell m}$ are spin-weighted spherical harmonics with spin weight s. The factors ${}_sb^m_{\ell,\ell\pm1}$ in Eq. (4.61) hold the form [70]

$$ {}_sb^m_{\ell,\ell+1} = -\frac{s\sqrt{\left[(\ell+1)^2-m^2\right]\left[(\ell+1)^2-s^2\right]}}{(\ell+1)^2\sqrt{(2\ell+1)(2\ell+3)}}\,, \tag{4.62a} $$

$$
{}_s b^m_{\ell,\ell-1} = \frac{s\sqrt{\left(\ell^2 - m^2\right)\left(\ell^2 - s^2\right)}}{\ell^2\sqrt{(2\ell-1)(2\ell+1)}} \,. \tag{4.62b}
$$

To evaluate spin-weighted spherical harmonics, one can use

$$
\begin{aligned}
{}_sY_{\ell m}(\theta,\phi) = (-1)^{\ell+m-s}&\sqrt{\frac{(\ell+m)!(\ell-m)!(2\ell+1)}{4\pi(\ell+s)!(\ell-s)!}}\sin^{2\ell}\left(\frac{\theta}{2}\right)e^{im\phi} \\
&\times\sum_{q=0}^{\ell-s}(-1)^q\begin{pmatrix}\ell-s\\ q\end{pmatrix}\begin{pmatrix}\ell+s\\ q+s-m\end{pmatrix}\cot^{2q+s-m}\left(\frac{\theta}{2}\right) \,.
\end{aligned} \tag{4.63}
$$

In the special case that $s=0$, the spin-weighted spherical harmonics become the standard spherical harmonics, i.e.,

$$
{}_0Y_{\ell m} = Y_{\ell m} \,. \tag{4.64}
$$

Spin-weighted spherical harmonics and spin-weighted spheroidal harmonics also obey the following orthogonality relations,

$$
\int_{S^2} dS \; {}_s\mathcal{Y}^{\gamma}_{\ell m} \; {}_s\bar{\mathcal{Y}}^{\gamma}_{\ell' m'} = \delta_{\ell\ell'}\delta_{mm'} \,, \tag{4.65}
$$

$$
\int_{S^2} dS \; {}_sY_{\ell m} \; {}_s\bar{Y}_{\ell' m'} = \delta_{\ell\ell'}\delta_{mm'} \,, \tag{4.66}
$$

where dS is the solid angle element, and the integration is over the entire 2-sphere. At certain places, we might drop the superscript γ of ${}_s\mathcal{Y}^{\gamma}_{\ell m}$ denoting its eigenvalue for simplicity.

A spin-weighted spherical harmonic ${}_sY_{\ell m}$ with spin weight s can also be raised to ${}_{s+1}Y_{\ell m}$ of spin weight $s+1$ via the raising operator ð or lowered to ${}_{s-1}Y_{\ell m}$ of spin weight s via the lowering operator $\bar{\text{ð}}$. The operators ð and $\bar{\text{ð}}$ are defined to be [71]

$$
\eth\, {}_s\mathcal{F} = -\left(\partial_\theta + i\csc\theta\partial_\phi - s\cot\theta\right){}_s\mathcal{F} \,, \tag{4.67a}
$$

$$
\bar{\eth}\, {}_s\mathcal{F} = -\left(\partial_\theta - i\csc\theta\partial_\phi + s\cot\theta\right){}_s\mathcal{F} \,, \tag{4.67b}
$$

where ${}_s\mathcal{F}$ is some angular function in (θ,ϕ) of spin weight s, such that

$$
\eth\, {}_sY_{\ell m} = \left[(l-s)(l+s+1)\right]^{1/2}\, {}_{s+1}Y_{\ell m} \,, \tag{4.68a}
$$

$$
\bar{\eth}\, {}_sY_{\ell m} = -\left[(l+s)(l-s+1)\right]^{1/2}\, {}_{s-1}Y_{\ell m} \,. \tag{4.68b}
$$

One can further rewrite the directional derivatives $\delta^{(0,0,0)}$ and $\bar{\delta}^{(0,0,0)}$ on the Schwarzschild background in terms of ð and $\bar{\text{ð}}$, respectively, i.e.,

$$
\delta^{(0,0,0)}\,{}_s\mathcal{F} = -\frac{1}{\sqrt{2}r}\left(\bar{\eth} - s\cot\theta\right){}_s\mathcal{F} \,, \tag{4.69a}
$$

$$\bar{\delta}^{(0,0,0)}{}_s\mathcal{F} = -\frac{1}{\sqrt{2}r}\left(\bar{\eth} + s\cot\theta\right){}_s\mathcal{F}\,. \tag{4.69b}$$

By expanding $\delta^{(0,0)}$ and $\bar{\delta}^{(0,0)}$ in terms of $\delta^{(0,0,0)}$ and $\bar{\delta}^{(0,0,0)}$ in the slow-rotation limit, one can also replace $\delta^{(0,0)}$ and $\bar{\delta}^{(0,0)}$ with $\eth$ and $\bar{\eth}$. In Secs. 4.5–4.7, we will use Eq. (4.69) to simplify the terms with $\delta^{(0,0)}$ and $\bar{\delta}^{(0,0)}$ acting on spin-weighted spherical harmonics.

4.4.4 Perturbed NP quantities

As we are working within the NP basis, in addition to the perturbed metric given by Eq. (4.50), we also require the reconstruction of the perturbed NP quantities, such as the perturbed tetrad, Weyl scalars, and spin coefficients. We adopt the methodology outlined in [38, 66] to perform this reconstruction. The first step involves expressing the perturbed tetrad in terms of the background tetrad. This is accomplished by expanding the perturbed tetrad in terms of the background tetrad, and then utilizing the transformation properties of the tetrad to obtain the perturbed tetrad components in terms of the background tetrad components such that

$$e_a^{\mu(0,1)} = A_a{}^{b(0,1)} e_b^{\mu(0,0)}\,, \tag{4.70}$$

where e_a^μ represents a null tetrad such that

$$e_a^\mu = \{l^\mu\,,\ n^\mu\,,\ m^\mu\,,\ \bar{m}^\mu\}\,, \tag{4.71}$$

and $A_a{}^b$ are coefficients that map the background tetrad to the perturbed tetrad.

As shown in [38, 66], one can always perform tetrad rotations to set six real parameters of the $A_a{}^{b(0,1)}$ coefficients to zero. Then, expanding $h_{\mu\nu}$ in terms of $e_\mu^{a(0,1)}$ and $e_\mu^{a(0,0)}$ and using the linearized completeness relation, we find

$$h_{\mu\nu}^{(0,1)} = -2\left[l_{(\mu}^{(0,1)} n_{\nu)}^{(0,0)} + l_{(\mu}^{(0,0)} n_{\nu)}^{(0,1)} - m_{(\mu}^{(0,1)} \bar{m}_{\nu)}^{(0,0)} - m_{(\mu}^{(0,0)} \bar{m}_{\nu)}^{(0,1)}\right]\,. \tag{4.72}$$

Comparing Eq. (4.72) to Eq. (4.70), we find [38, 66],

$$\begin{aligned}
l^{\mu(0,1)} &= \frac{1}{2}h_{ll}^{(0,1)} n^\mu\,,\\
n^{\mu(0,1)} &= \frac{1}{2}h_{nn}^{(0,1)} l^\mu + h_{ln}^{(0,1)} n_\mu\,,\\
m^{\mu(0,1)} &= h_{nm}^{(0,1)} l^\mu + h_{lm}^{(0,1)} n^\mu - \frac{1}{2}h_{m\bar{m}}^{(0,1)} m^\mu - \frac{1}{2}h_{mm}^{(0,1)} \bar{m}^\mu\,,
\end{aligned} \tag{4.73}$$

where we have dropped the superscripts of $e_a^{\mu(0,0)}$ for simplicity. Since we have adopted the sign convention in [42], our signature is opposite to that used in [38, 66].

Therefore, the perturbed tetrad in Eq. (4.73) has an opposite sign from the results of [38, 66], as seen in Eq. (4.72). Equation (4.73) works for both the IRG and ORG. In the IRG or ORG, we can further set $h_{la}^{(0,1)} = h_{m\bar{m}}^{(0,1)} = 0$ or $h_{na}^{(0,1)} = h_{m\bar{m}}^{(0,1)} = 0$ in Eq. (4.167), respectively, where a is any index in the NP basis.

For the spin coefficients, one can linearize the commutation relation following [42],

$$\left[e_a^\mu, e_b^\mu\right] = (\gamma^c{}_{ba} - \gamma^c{}_{ab})\, e_c^\mu = C_{ab}{}^c e_c^\mu\,, \tag{4.74}$$

where $\gamma^a{}_{bc}$ is the Ricci rotation coefficients. Using the relation between spin coefficients and Ricci rotation coefficients in Eq. (4.164), one can write $C_{ab}{}^c$ in terms of spin coefficients, as listed in Eq. (4.165) [21, 42]. From Eq. (4.165), one can also solve for spin coefficients in terms of $C_{ab}{}^c$, and the results are in Eq. (4.166). Expanding Eq. (4.74) using Eq. (4.70), one finds

$$C_{ab}{}^{c(0,1)} = \partial_a A_b{}^{c(0,1)} - \partial_b A_a{}^{c(0,1)} - \left(A_a{}^{d(0,1)} C_{bd}{}^c - A_b{}^{d(0,1)} C_{ad}{}^c + A_d{}^{c(0,1)} C_{ab}{}^d\right)\,, \tag{4.75}$$

where the superscripts of $C_{ab}{}^{c(0,0)}$ are dropped for convenience. The coefficients $A_a{}^{b(0,1)}$ can be retrieved from Eq. (4.73). The GR structure constants $C_{ab}{}^{c(0,0)}$ are directly given by Eq. (4.165) and the spin coefficients in GR. With all the quantities in Eq. (4.75), one can then use Eq. (4.75) and (4.166) to evaluate the spin coefficients at $O(\zeta^0, \epsilon^1)$. We have listed our result in Eq. (4.167), which works for both the ORG and IRG. Our result is consistent with [38] up to the overall minus sign due to different signatures, which corrects some errors in [66].

To reconstruct Weyl scalars, one can either directly linearize the Riemann tensor and project it onto the NP basis to find Weyl scalars or use the Ricci identities in Eq. (4.168). For both approaches, we use the perturbed tetrad in Eq. (4.73), and we check that the results are consistent. We also compare our results in the IRG to the equations in [36], which corrected a factor of $1/2$ missed in [59] and are listed in Eq. (4.169). After expressing everything in terms of the Hertz potential, our results of $\Psi_{0,1,2,4}^{(0,1)}$ in the IRG agree perfectly with Eq. (4.169) but not for $\Psi_3^{(0,1)}$. Since $\Psi_3^{(0,1)}$ is not invariant either under tetrad rotations or coordinate transformations at $O(\zeta^0, \epsilon^1)$, this difference indicates that we might have a $O(\zeta^0, \epsilon^1)$ difference in the choices of coordinate or tetrad.

For coordinate- and tetrad-invariant quantities $\Psi_{0,4}^{(0,1)}$, our results are consistent with [36, 59]. In addition, since $\Psi_2^{(0,1)}$ is invariant under tetrad rotations but not coordinate transformations at $O(\zeta^0, \epsilon^1)$ [i.e., Eqs. (4.172) and (4.174)], we are in the same coordinate as [36, 59], consistent with that we all use the IRG. Thus, the difference in $\Psi_3^{(0,1)}$ is only due to tetrad choices at $O(\zeta^0, \epsilon^1)$, where we explicitly follow the convention in [38, 66], but Refs. [36, 59] were not explicit about their tetrad choices at $O(\zeta^0, \epsilon^1)$. More specifically, we find that the tetrad in Eq. (4.73) after setting $h_{la}^{(0,1)} = h_{m\bar{m}}^{(0,1)} = 0$ differs from the tetrad in [36, 59] by a type I rotation. In Schwarzschild, this difference in $\Psi_3^{(0,1)}$ can be compactly written as

$$\Psi_3^{(0,1)} = \Psi_{3,\mathrm{CCK}}^{(0,1)} + \frac{3}{2}\Psi_2 h_{n\bar{m}}^{(0,1)}\,, \tag{4.76}$$

where $\Psi_{3,\mathrm{CCK}}^{(0,1)}$ is the result in [36, 59]. The results of other Weyl scalars at $O(\zeta^0, \epsilon^1)$ in Schwarzschild are listed in Eq. (4.170). For Kerr, we do not have such a simple correction to $\Psi_3^{(0,1)}$, so we will just use the Ricci identity in Eq. (4.168d). Similarly, in the ORG, no previous literature provided results of all the Weyl scalars in terms of $\Psi_{\mathrm{H}}^{(0,1)}$ directly, so we also use the Ricci identity to evaluate them.

When deriving the modified Teukolsky equations, we made the gauge choice that $\Psi_{1,3}^{(0,1)} = 0$, but this is not the case for the tetrad in Eq. (4.73), as one can see in Eqs. (4.168)–(4.170). Thus, to be consistent with the gauge we chose for the modified Teukolsky equations, we need to perform additional type I and type II rotations to remove $\Psi_{1,3}^{(0,1)}$. From Eq. (4.172), we find the rotation parameters to be

$$a^{(0,1)} = -\frac{\bar{\Psi}_3^{(0,1)}}{3\Psi_2}\,, \quad b^{(0,1)} = -\frac{\Psi_1^{(0,1)}}{3\Psi_2}\,. \tag{4.77}$$

Since $\Psi_{0,2,4}^{(0,1)} = 0$ are invariant under tetrad rotations at $O(\zeta^0, \epsilon^1)$, one can continue using Eqs. (4.168)–(4.170) by just setting $\Psi_{1,3}^{(0,1)} = 0$. For spin coefficients, their values after the rotation are listed in Eq. (4.173) following [42]. With these reconstructed quantities, we are now ready to evaluate the source terms in the equation of $\vartheta^{(1,1)}$ in Eq. (4.38).

4.5 The evolution equation for $\vartheta^{(1,1)}$ in the IRG

In this section, we project the equation governing $\vartheta^{(1,1)}$ [Eq. (4.38)] onto the NP basis using the IRG. For convenience, we define the right-hand side of Eq. (4.38) as

$$\mathcal{S}_{\vartheta}^{(1,1)} \equiv -\frac{M^2}{16\pi^{\frac{1}{2}}}\,[R^*R]^{(0,1)} - \Box^{(0,1)}\vartheta^{(1,0)} \tag{4.78}$$

so that the evolution equation for $\vartheta^{(1,1)}$ becomes

$$\Box^{(0,0)}\vartheta^{(1,1)} = \mathcal{S}_{\vartheta}^{(1,1)}\,. \tag{4.79}$$

This equation is first expressed in terms of the NP quantities, following which we evaluate its left-hand side using the background NP quantities in Sec. 4.3.4 and Appendix 4.11 and its right-hand side using the reconstructed NP quantities at $O(\zeta^0, \epsilon^1)$ in Sec. 4.4 and Appendix 4.12. The same methodology demonstrated in this section is applied to computing the modified Teukolsky equation in Sec. 4.6. Figure 4.1 presents a schematic illustration of the steps involved in calculating a completely separated radial evolution equation for the scalar field perturbation in the IRG.

4.5.1 Projection onto the NP basis

In this subsection, we project Eq. (4.38) onto the NP basis. This projection involves the projection of two fundamental quantities: the D'Alembertian operator $\Box$ and the Pontryagin density R^*R onto the NP basis. In particular, our goal is to express these quantities in terms of NP quantities, particularly the below-mentioned quantities in the NP basis, namely

$$\begin{aligned} {}^*R_{abcd} &= \frac{1}{2}\epsilon_{cd}{}^{ef}R_{abef} \\ \nabla_b\nabla_a\vartheta &= \nabla_b\left(\vartheta_{,a}\right) = \vartheta_{,ab} - \gamma^d{}_{ab}\vartheta_{,d} \,. \end{aligned} \tag{4.80}$$

Here, η_{ab} is the metric in the NP basis. The notation $f_{,a}$ denotes the directional derivative of f defined by the tetrad basis e_a^μ. The quantities γ_{abc} are Ricci rotation coefficients, which can be expressed in terms of spin coefficients using Eq. (4.175) [21]. The tensor R_{abcd} can be expressed in terms of Weyl scalars using Eq. (4.177) [21]. Therefore, the Pontryagin density and the $\Box$ operator can be expressed in the NP basis as

$$R^*R = -8i(3\Psi_2^2 - 4\Psi_1\Psi_3 + \Psi_0\Psi_4 - c.c.)\,, \tag{4.81a}$$

$$\begin{aligned} \Box = &-\Big[\{D,\boldsymbol{\Delta}\} - \{\delta,\bar{\delta}\} + (\mu+\bar{\mu}-\gamma-\bar{\gamma})D + (\epsilon+\bar{\epsilon}-\rho-\bar{\rho})\boldsymbol{\Delta} \\ &+(\alpha-\bar{\beta}-\pi+\bar{\tau})\delta + (\bar{\alpha}-\beta-\bar{\pi}+\tau)\bar{\delta}\Big]\,. \end{aligned} \tag{4.81b}$$

The factor of i in Eq. (4.81a) arises from the normalization of the Levi-Civita tensor ϵ_{abcd} in the NP basis. In the literature, such as in [72], the covariant Levi-Civita tensor is typically defined as $\epsilon_{\mu\nu\cdots\gamma} = \sqrt{|g|}\tilde{\epsilon}_{\mu\nu\cdots\gamma}$, where $\tilde{\epsilon}_{\mu\nu\cdots\gamma}$ denotes the Levi-Civita symbol. However, this definition encounters issues when attempting to convert a Levi-Civita tensor from Boyer-Lindquist coordinates to the NP basis, due to the determinant of the Jacobian relating these two bases often being complex. Thus, to convert the tensor density $\tilde{\epsilon}_{\mu\nu\cdots\gamma}$ to a tensor, we instead need to define

$$\epsilon_{\mu\nu\cdots\gamma} = \sqrt{-g}\,\tilde{\epsilon}_{\mu\nu\cdots\gamma}\,, \tag{4.82}$$

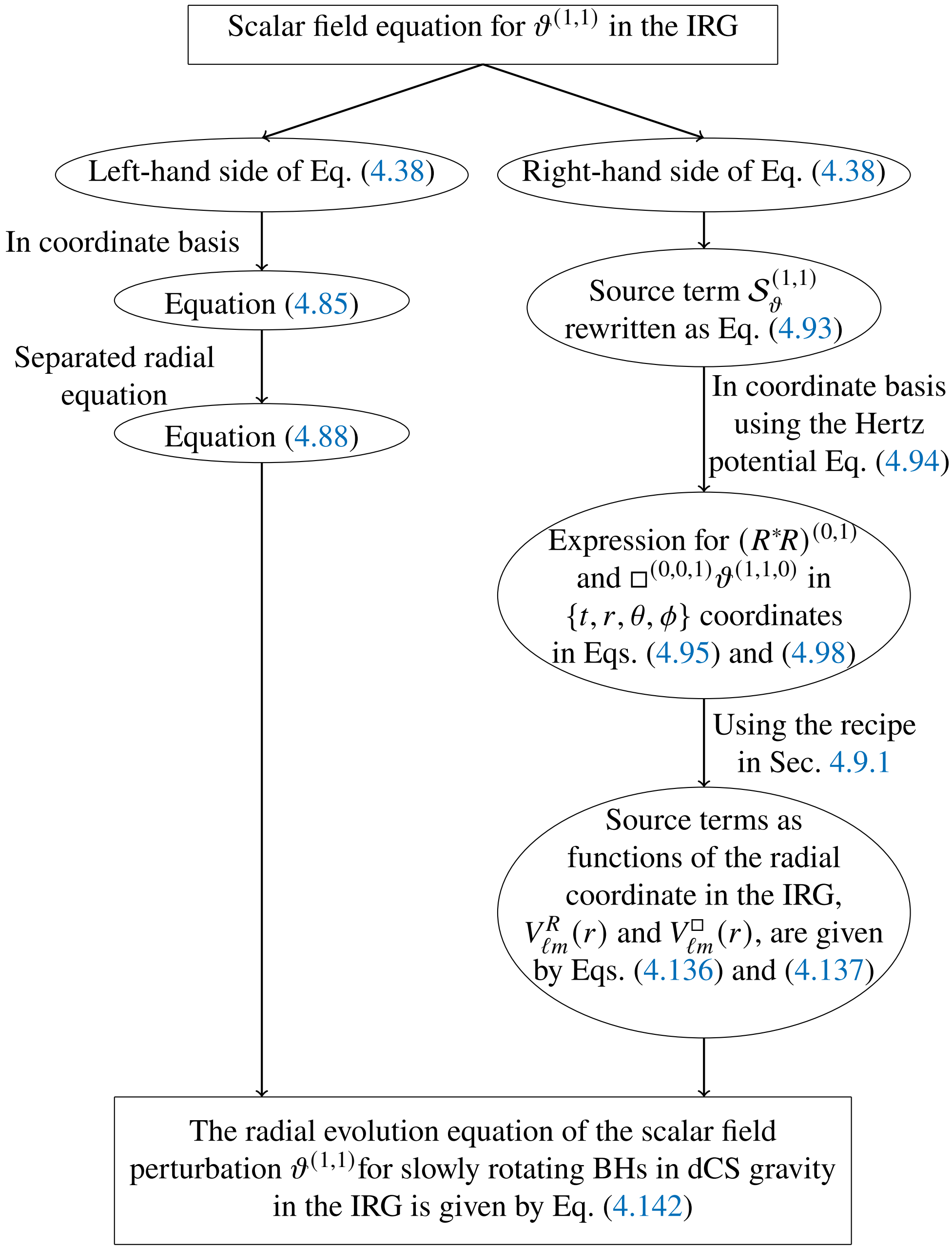

Figure 4.1: A schematic flowchart prescribing the steps involved in obtaining a separated radial evolution equation for the scalar field perturbation $\vartheta^{(1,1)}$ for slowly rotating BHs in dCS gravity. This flowchart summarizes the details of the calculations described in Sec. 4.5 and Sec. 4.9.2. Initial and final outcomes are represented by rectangular boxes, while intermediate results are symbolized by encapsulating bubbles. The directional arrows are meant to seamlessly guide the reader through the logical flow of the calculations.

which has the same normalization factor as the Einstein-Hilbert action. The absolute value in the usual definition is to impose that the Levi-Civita tensor is a real tensor in the Lorentzian signature, but the minus sign of $\sqrt{-g}$ will do the same trick. Since $\eta = 1$, we find the normalization factor to be i rather than 1 from Eq. (4.82). This is also consistent with that

$$\epsilon_{lnm\bar{m}} = \frac{1}{2}\left(\epsilon_{lnm\bar{m}} - \overline{\epsilon_{lnm\bar{m}}}\right) , \tag{4.83}$$

which shows that $\epsilon_{lnm\bar{m}}$ is an imaginary number. We have now expressed all the terms in Eq. (4.38) in the NP basis.

4.5.2 Left-hand side of Eq. (4.38)

In this subsection, we compute the operator $\square^{(0,0)}$ acting on $\vartheta^{(1,1)}$ to obtain the homogeneous component of Eq. (4.38). The operator $\square^{(0,0)}$ can be evaluated directly using the slowly rotating Kerr metric presented in Eq. (4.9). Alternatively, one can use Eq. (4.81b) and the NP quantities of Kerr, expanded up to $O(\chi)$, as given by Eqs. (4.158)–(4.160), and then setting ζ to zero. We therefore find

$$\square^{(0,0)} = -\frac{1}{r^2} H_{\vartheta}^{(0,0)} , \tag{4.84}$$

where $H_{\vartheta}^{(0,0)}$ is the Teukolsky operator for particles with spin weight $s = 0$ in [19],

$$\begin{aligned} H_{\vartheta}^{(0,0)} = & -r(r-r_s)\partial_r^2 - 2(r-M)\partial_r - \frac{\omega^2 r^3 - 4m\chi\omega M^2}{r-r_s} \\ & -\partial_\theta^2 - \cot\theta\partial_\theta + m^2\csc^2\theta , \end{aligned} \tag{4.85}$$

where we have only kept the terms up to $O(\chi)$ and separated $\vartheta^{(1,1)}$ as

$$\vartheta^{(1,1)} = \Theta_{\ell m}(r)_0\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega t} , \tag{4.86}$$

or in the slow-rotation expansion

$$\begin{aligned} \vartheta^{(1,1)} = \Theta_{\ell m}(r) \Big[& {}_0Y_{\ell m}(\theta,\phi) + \chi M\omega \Big({}_0b^m_{\ell,\ell+1}\, {}_0Y_{\ell+1\, m} \\ & + {}_0b^m_{\ell,\ell-1}\, {}_0Y_{\ell-1\, m} \Big) + O(\chi^2) \Big] e^{-i\omega t} . \end{aligned} \tag{4.87}$$

Thus, in the absence of sources, $\Theta_{\ell m}(r)$ satisfies

$$\left[r(r-r_s)\partial_r^2 + 2(r-M)\partial_r + \frac{\omega^2 r^3 - 4\chi m M^2\omega}{r-r_s} - {}_0A_{\ell m} \right] \Theta_{\ell m}(r) = 0 , \tag{4.88}$$

where ${}_0A_{\ell m}$ is the Teukolsky's separation constant for $s = 0$ [19]. We therefore see that the left-hand side of Eq. (4.38) is separable in the radial and angular coordinates. Further, in Sec. 4.9, we show that the complete expression in Eq. (4.38) can be separated into radial and angular parts using spin-weighted spheroidal harmonics of spin weight zero.

4.5.3 $\mathcal{S}_{\vartheta}^{(1,1)}$ in terms of $h^{(1,1)}$ and $\vartheta^{(1,1)}$

To systematically calculate $\mathcal{S}_{\vartheta}^{(1,1)}$ in Eq. (4.78), we can partition it into three parts based on the terms that necessitate metric reconstruction, namely those at $O(\zeta^0, \epsilon^1)$:

1. Weyl scalars at $O(\zeta^0, \epsilon^1)$: These terms are solely determined by the Pontryagin density $(R^*R)^{(0,1)}$. For slowly rotating BHs in dCS gravity, these Weyl scalars in Eq. (4.78) receive contributions from both $h_{\mu\nu}^{(0,0,1)}$ and $h_{\mu\nu}^{(0,1,1)}$. We expand Eq. (4.81a) up to $O(\zeta^0, \epsilon^1)$. Since in Petrov type D spacetimes $\Psi_{0,1,3,4}^{(0,0)} = 0$, the only terms that survive are proportional to Ψ_2, such that

$$(R^*R)^{(0,1)} = -48i\left(\Psi_2^{(0,0)}\Psi_2^{(0,1)} - \bar{\Psi}_2^{(0,0)}\bar{\Psi}_2^{(0,1)}\right), \tag{4.89}$$

   where $\Psi_2^{(0,0)}$ is given by Eq. (4.47) by setting $\zeta = 0$, and $\Psi_2^{(0,1)}$ is given by Eq. (4.169c) using the metric reconstruction procedures.

2. Spin coefficients at $O(\zeta^0, \epsilon^1)$: This dependence arises from $-\Box^{(0,1)}\vartheta^{(1,0)}$. Since we are only interested in the terms up to $O(\zeta^1, \chi^1, \epsilon^1)$ in this work, and $\vartheta^{(1,0,0)} = 0$ as explained in Sec. 4.3.3, the metric fields in $-\Box^{(0,1)}\vartheta^{(1,0)}$ only have the contribution from $h_{\mu\nu}^{(0,0,1)}$. Thus, we only need metric reconstruction at $O(\zeta^0, \chi^0, \epsilon^1)$ for these terms. The first two terms in Eq. (4.81b) will not contribute directly, although one can find additional spin coefficients by using the commutation relations to combine the anti-commutators. For the rest of the terms, we find at $O(\zeta^0, \epsilon^1)$,

$$\begin{aligned}
&\Big[\left(\mu^{(0,1)} + \bar{\mu}^{(0,1)} - \gamma^{(0,1)} - \bar{\gamma}^{(0,1)}\right) D \\
&+ \left(\epsilon^{(0,1)} + \bar{\epsilon}^{(0,1)} - \rho^{(0,1)} - \bar{\rho}^{(0,1)}\right) \boldsymbol{\Delta} \\
&+ \left(\alpha^{(0,1)} - \bar{\beta}^{(0,1)} - \pi^{(0,1)} + \bar{\tau}^{(0,1)}\right) \delta \\
&+ \left(\bar{\alpha}^{(0,1)} - \beta^{(0,1)} - \bar{\pi}^{(0,1)} + \tau^{(0,1)}\right) \bar{\delta}\Big]\, \vartheta^{(1,0)} ,
\end{aligned} \tag{4.90}$$

   where the spin coefficients at $O(\zeta^0, \epsilon^1)$ are given by Eqs. (4.167) and (4.173) using metric reconstruction procedures.

3. Tetrad/directional derivatives are at $O(\zeta^0, \epsilon^1)$. Similar to the second situation, these types of terms also arise from $-\Box^{(0,1)}\vartheta^{(1,0)}$ and vanish at $O(\zeta^1, \chi^0, \epsilon^1)$. Thus, we must only reconstruct the NP quantities at $O(\zeta^0, \chi^0, \epsilon^1)$. Using the

Schwarzschild properties of all the spin coefficients in Eq. (4.184), we find

$$\begin{aligned}&\Big[D^{(0,1)}\boldsymbol{\Delta}+D\boldsymbol{\Delta}^{(0,1)}+\boldsymbol{\Delta}^{(0,1)}D+\boldsymbol{\Delta}D^{(0,1)}\\&-\delta^{(0,1)}\bar{\delta}-\delta\bar{\delta}^{(0,1)}-\bar{\delta}^{(0,1)}\delta-\bar{\delta}\delta^{(0,1)}\\&-2(\gamma-\mu)D^{(0,1)}-2\rho\boldsymbol{\Delta}^{(0,1)}+2\alpha\left(\delta^{(0,1)}+\bar{\delta}^{(0,1)}\right)\Big]\vartheta^{(1,0)}\,,\end{aligned}\tag{4.91}$$

where the tetrad at $O(\zeta^0,\epsilon^1)$ is given by Eqs. (4.73) and (4.171). Notice, the terms at $O(\zeta^0,\epsilon^1)$ in Eqs. (4.90) and (4.91) are all at $O(\zeta^0,\chi^0,\epsilon^1)$ as $\vartheta^{(1,0)}$ is non-vanishing only at $O(\zeta^1,\chi^1,\epsilon^0)$, but we choose to hide the expansion in χ for simplicity.

A similar classification will be used when we compute the source terms in the modified Teukolsky equation for $\Psi_0^{(1,1)}$ and $\Psi_4^{(1,1)}$.

One can further combine the second and the third type of source terms and express them as functionals of the metric components in the NP basis (e.g., $h_{nn}^{(0,1)}$, $h_{nm}^{(0,1)}$, $h_{mm}^{(0,1)}$ in the IRG) and the rotation coefficients (e.g., $a^{(0,1)}$ and $b^{(0,1)}$) such that the separation of variables can be more easily carried out in Sec. 4.9. In this case, we find

$$\begin{aligned}\square^{(0,0,1)}\vartheta^{(1,0,0)}&=\square^{(0,1,1)}\vartheta^{(1,0,0)}=0\,,\\\square^{(0,0,1)}\vartheta^{(1,1,0)}&=-\Big\{h_{nn}^{(0,0,1)}D^2-h_{nm}^{(0,0,1)}\{D,\bar{\delta}\}+h_{mm}^{(0,0,1)}\bar{\delta}^2\\&\quad+\Big[(D-2\rho)h_{nn}^{(0,0,1)}-(\bar{\delta}-2\alpha)h_{nm}^{(0,0,1)}\Big]D\\&\quad-\Big[(D-2\rho)h_{nm}^{(0,0,1)}-(\bar{\delta}-2\alpha)h_{mm}^{(0,0,1)}\Big]\bar{\delta}+c.c.\Big\}\vartheta^{(1,1,0)}\,.\end{aligned}\tag{4.92}$$

Finally, we have

$$\begin{aligned}\mathcal{S}_\vartheta^{(1,0,1)}&=-\frac{M^2}{16\pi^{\frac{1}{2}}}(R^*R)^{(0,0,1)}\,,\\\mathcal{S}_\vartheta^{(1,1,1)}&=-\frac{M^2}{16\pi^{\frac{1}{2}}}(R^*R)^{(0,1,1)}-\square^{(0,0,1)}\vartheta^{(1,1,0)}\,,\end{aligned}\tag{4.93}$$

where $(R^*R)^{(0,0,1)}$ and $(R^*R)^{(0,1,1)}$ are given by Eq. (4.89), and $\square^{(0,0,1)}\vartheta^{(1,1,0)}$ is given by Eq. (4.92).

4.5.4 $\mathcal{S}_\vartheta^{(1,1)}$ in the coordinate basis

We now rewrite $\mathcal{S}_\vartheta^{(1,1)}$ [Eqs. (4.78) and (4.93)] in the coordinate basis (t,r,θ,ϕ) using the perturbed NP quantities found in Sec. 4.4 and Appendix 4.11. From Eqs. (4.78) and (4.93), we notice that $\mathcal{S}_{\vartheta}^{(1,1)}$ contains two pieces: the term proportional to $(R^*R)^{(0,1)}$ and the term proportional to $\square^{(0,1)}\vartheta^{(1,0)}$.

For the first piece, according to Eq. (4.89), we essentially need to evaluate $\Psi_2^{(0,1)}$ up to $O(\chi)$. The value of $\Psi_2^{(0,1)}$ in terms of the Hertz potential $\Psi_{\rm H}^{(0,1)}$ is given by Eq. (4.169c), and $\Psi_{\rm H}^{(0,1)}$ has the expansion in Eq. (4.55). Since we use the slow-rotation approximation in this work, we can further decompose spin-weighted spheroidal harmonics in terms of spin-weighted spherical harmonics using Eqs. (4.61) and (4.62) such that Eq. (4.55) becomes

$$\begin{aligned}\bar{\Psi}_{\rm H} = {}_2\hat{R}_{\ell m}(r)\Big[{}_2Y_{\ell m}(\theta,\phi) + \chi M\omega\Big({}_2b^m_{\ell,\ell+1}\,{}_2Y_{\ell+1\,m} \\ + {}_2b^m_{\ell,\ell-1}\,{}_2Y_{\ell-1\,m}\Big) + O(\chi^2)\Big]\,e^{-i\omega t}\,.\end{aligned} \tag{4.94}$$

Now, one can insert into Eqs. (4.169c) and (4.89) the decomposition in Eq. (4.94) and the background NP quantities at $O(\zeta^0,\epsilon^0)$ in Eqs. (4.158)–(4.160) after setting $\zeta = 0$. After using Eqs. (4.69a) and (4.69b) to simplify the terms with $\delta^{(0,0)}$ and $\bar{\delta}^{(0,0)}$ acting on ${}_sY_{\ell m}(\theta,\phi)$, we find

$$(R^*R)^{(0,1)} \tag{4.95}$$

$$\begin{aligned}
&= e^{-i\omega t}\Big[\Big(g_1^{\ell m}(r,\omega,M){}_2\hat{R}_{\ell m}(r) + g_2^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\Big)\,{}_0Y_{\ell m}(\theta,\phi) \\
&\quad+\chi\Big(g_3^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + g_4^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\Big)\sin\theta\,{}_1Y_{\ell m}(\theta,\phi) \\
&\quad+\chi\Big(g_5^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + g_6^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\Big)\cos\theta\,{}_0Y_{\ell m}(\theta,\phi) \\
&\quad+\chi\,{}_2b^m_{\ell,\ell+1}\Big(g_7^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + g_8^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\Big)\,{}_0Y_{\ell+1\,m}(\theta,\phi) \\
&\quad+\chi\,{}_2b^m_{\ell,\ell-1}\Big(g_9^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + g_{10}^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\Big)\,{}_0Y_{\ell-1\,m}(\theta,\phi)\Big] + \text{c.c.}\,,
\end{aligned} \tag{4.96}$$

where $f'(r)$ denotes the derivative of f with respect to the r coordinate for any function $f(r)$. Here, ${}_2\hat{R}_{\ell m}(r)$ is the radial function of the Hertz potential for slowly rotating Kerr BHs in GR, which can be computed from the radial function of $\Psi_0^{(0,1)}$ using Eq. (4.56a). One can, in principle, expand ${}_2\hat{R}_{\ell m}(r)$ further in χ and drop additional terms above $O(\chi)$ in Eq. (4.95). For simplicity, we choose not to do this additional expansion here but implement it when computing QNMs in [58], where we need to explicitly evaluate the radial functions of $\Psi_{0,4}^{(0,1)}$. We have also used the radial Teukolsky equation to reduce any n-th radial derivative of ${}_2\hat{R}_{\ell m}(r)$ with $n \geq 1$ to ${}_2\hat{R}_{\ell m}$ and ${}_2\hat{R}'_{\ell m}(r)$. The explicit forms of $g_i^{\ell m}(r,\omega,M)$ are long and therefore presented through a separate Mathematica notebook as Supplementary Material [69].

For the second piece, according to Eq. (4.93), there is only a contribution from $\square^{(0,0,1)}\vartheta^{(1,1,0)}$ since the scalar field at $O(\zeta^1, \chi^0, \epsilon^0)$ vanishes in dCS gravity. Thus, we only need the reconstructed metric at $(\zeta^0, \chi^0, \epsilon^1)$. Using Eqs. (4.94), (4.160), and (4.50), we find

$$h_{nn}^{(0,0,1)} = \frac{1}{2r^2}\left(\sqrt{\frac{(\ell+2)!}{(\ell-1)!}}\,{}_2\hat{R}_{\ell m}(r)\,{}_0Y_{\ell m}(\theta,\phi)e^{-i\omega t} + \text{c.c.}\right), \tag{4.97a}$$

$$h_{nm}^{(0,0,1)} = \sqrt{\frac{\ell^2+\ell-2}{2}}\frac{1}{r^2(r-r_s)}\left\{[4M - r(2+i\omega r)]\,{}_2\hat{R}_{\ell m}(r)\right.$$
$$\left. + r(r-r_s)\,{}_2\hat{R}'_{\ell m}(r)\right\}{}_1Y_{\ell m}(\theta,\phi)e^{-i\omega t}, \tag{4.97b}$$

$$h_{mm}^{(0,0,1)} = \frac{1}{r(r-r_s)^2}\left\{\left[(r-r_s)(\ell^2+\ell-2+7i\omega r) - \omega r(ir+2\omega r^2)\right]{}_2\hat{R}_{\ell m}(r)\right.$$
$$\left. + 2(r-r_s)\left(M - i\omega r^2\right){}_2\hat{R}'_{\ell m}(r)\right\}{}_2Y_{\ell m}(\theta,\phi)e^{-i\omega t}. \tag{4.97c}$$

Now, we can evaluate all the directional derivatives and spin coefficients at $O(\zeta^0, \chi^0, \epsilon^1)$ using Eqs. (4.167), (4.171), and (4.173). In the end, using Eq. (4.92), we find

$$\square^{(0,0,1)}\vartheta^{(1,1,0)}$$
$$= e^{-i\omega t}\left[\left(h_1^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + h_2^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\right)\sin\theta\,{}_1Y_{\ell m}(\theta,\phi)\right.$$
$$\left. + h_3^{\ell m}(r,\omega,M)\left(2\vartheta'_R(r) + r\vartheta''_R(r)\right){}_2\hat{R}_{\ell m}(r)\cos\theta\,{}_0Y_{\ell m}(\theta,\phi)\right] + \text{c.c.}, \tag{4.98}$$

where $\vartheta_R(r)$ is the radial part of the background scalar field in Eq. (4.15). In Eq. (4.98), the reconstructed metric only has contribution at $O(\chi^0)$, so the radial function ${}_2\hat{R}_{\ell m}(r)$ is evaluated on the Schwarzschild background. Since we choose not to expand $\hat{R}(r)$ in χ here, we do not distinguish ${}_2\hat{R}_{\ell m}(r)$ evaluated on the Schwarzschild or slowly rotating Kerr background. Combining Eqs. (4.95) and (4.98), we have the source terms in the equation of $\vartheta^{(1,1)}$ up to $O(\chi)$. The master equation of $\vartheta^{(1,1)}$ in the IRG in the coordinate $\{t, r, \theta, \phi\}$ is presented in Eq. (4.128) of Sec. 4.8.

In Sec. 4.9, we will show that Eq. (4.78) up to $O(\chi)$ can be separated into a radial and an angular equation. In the following section, we apply the same procedures to evaluate the source terms in the modified Teukolsky equation of $\Psi_0^{(1,1)}$ in terms of the reconstructed NP quantities and project the equation into the coordinate basis.

4.6 The modified Teukolsky equation of $\Psi_0^{(1,1)}$ in the IRG

In this section, we evaluate the modified Teukolsky equation of $\Psi_0^{(1,1)}$ in Eq. (4.18) following the similar procedures in Sec. 4.5. We first evaluate the left-hand side of

Eq. (4.18) and the source term $\mathcal{S}_{\text{geo}}^{(1,1)}$ due to the correction to the background geometry using the background NP quantities in Sec. 4.3.4 and Appendix 4.11. We then project the source term $\mathcal{S}^{(1,1)}$ onto the NP basis and compute its coordinate-based value using the reconstructed NP quantities in the IRG given in Sec. 4.4 and Appendix 4.12. Figure 4.2 presents a schematic illustration of the steps involved in calculating a completely separated radial evolution equation for the $\Psi_0^{(1,1)}$ Weyl scalar perturbation in the IRG.

4.6.1 Right-hand side of Eq. (4.18) and $\mathcal{S}_{\text{geo}}^{(1,1)}$

Since slowly rotating BHs in dCS gravity are Petrov type D up to $O(\zeta^1, \chi^1, \epsilon^0)$, we only need to compute the Teukolsky operator $H_0^{(0,0)}$ in GR and its stationary correction $H_0^{(1,0)}$ in Eq. (4.20a). Thus, we do not need metric reconstruction in this subsection.

Using the Weyl scalars and spin coefficients found in Sec. 4.3.4 and Eq. (4.23), we find that

$$H_0^{(0,0,0)} = \frac{1}{2r^2} H_{0,\text{TK}}^{(0,0,0)} \,, \tag{4.99a}$$

$$H_0^{(0,1,0)} = \frac{1}{2r^2} H_{0,\text{TK}}^{(0,1,0)} \,, \tag{4.99b}$$

where $H_{0,\text{TK}}^{(0,0,0)}$ and $H_{0,\text{TK}}^{(0,1,0)}$ are $O(\chi^0)$ and $O(\chi^1)$ terms of the Teukolsky operator $H_{0,\text{TK}}$ for Ψ_0 [Eq. (4.7) in [19]],

$$\begin{aligned} H_{0,\text{TK}}^{(0,0,0)} &= -r(r-r_s)\partial_r^2 - 6(r-M)\partial_r - \frac{C(r)}{r-r_s} \\ &\quad - \partial_\theta^2 - \cot\theta\partial_\theta + \left(4 + m^2 + 4m\cos\theta\right)\csc^2\theta - 6 \,, \end{aligned} \tag{4.100a}$$

$$H_{0,\text{TK}}^{(0,1,0)} = -4M\left[\frac{m(i(r-M) - M\omega r)}{r(r-r_s)} - \omega\cos\theta\right], \tag{4.100b}$$

$$C(r) = 4i\omega r(r-3M) + \omega^2 r^3 \,, \tag{4.100c}$$

where we have decomposed the Weyl scalar $\Psi_0^{(1)}$ at $O(\epsilon)$ as

$$\Psi_0^{(1)} = \left[{}_2R_{\ell m}^{(0,1)}(r) + \zeta\, {}_2R_{\ell m}^{(1,1)}(r) + O(\zeta^2)\right] {}_2S_{\ell m}(\theta) e^{-i\omega t + im\phi} \,. \tag{4.101}$$

The Teukolsky equation corresponding to Eqs. (4.100a) and (4.100b) is separable with ${}_2R_{\ell m}^{(0,1)}(r)$ satisfying

$$\begin{aligned} &\left[r(r-r_s)\partial_r^2 + 6(r-M)\partial_r + \frac{C(r)}{r-r_s}\right. \\ &\left. + \frac{4m\chi M(i(r-M) - M\omega r)}{r(r-r_s)} - {}_2A_{\ell m}\right] {}_2R_{\ell m}^{(0,1)}(r) = 0 \,. \end{aligned} \tag{4.102}$$

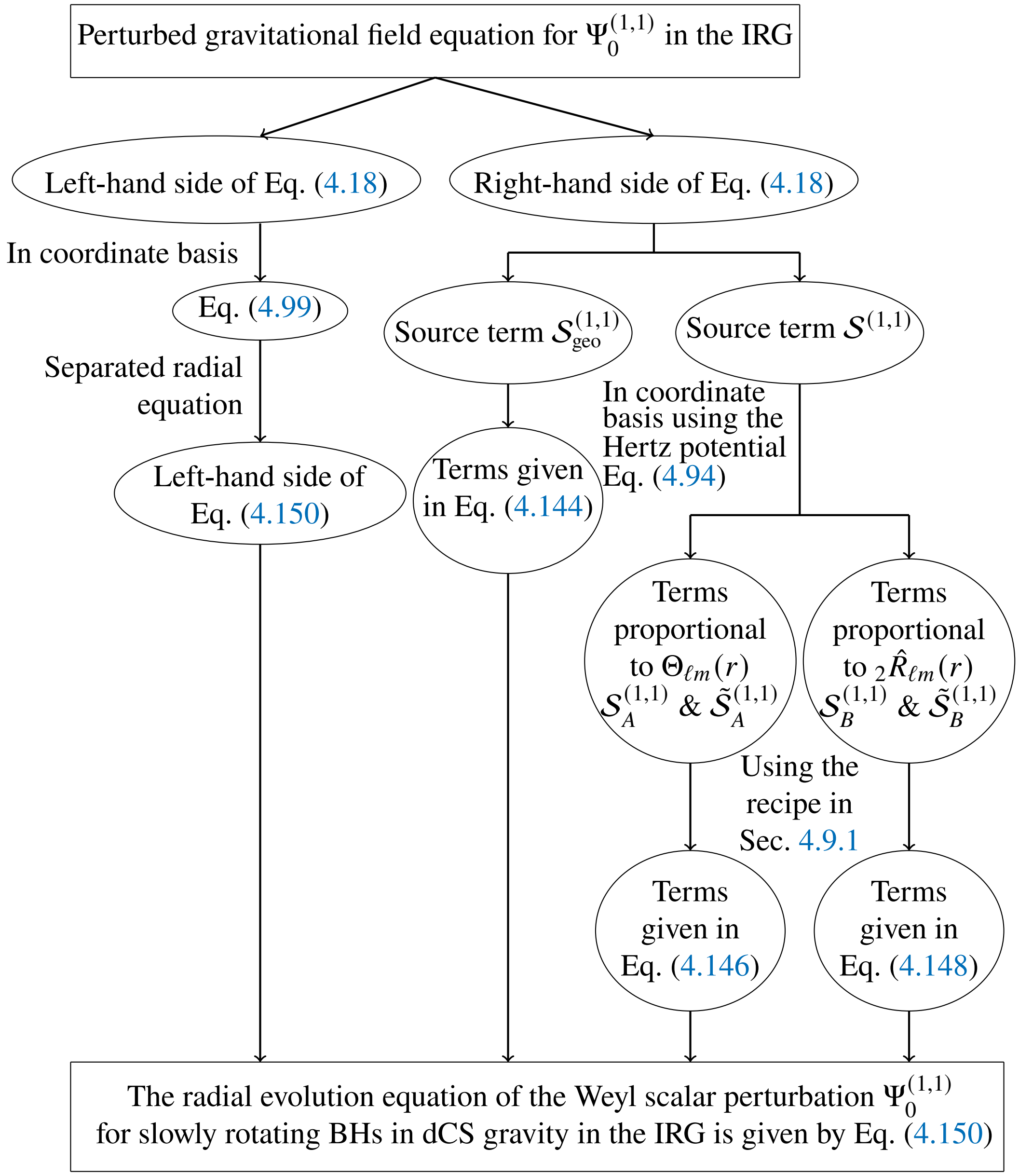

Figure 4.2: A schematic flowchart prescribing the steps involved in obtaining a separated radial evolution equation for the gravitational field perturbation described by the $\Psi_0^{(1,1)}$ Weyl scalar in the IRG for slowly rotating BHs in dCS gravity. This flowchart summarizes the details of the calculations described in detail in Sec. 4.6 and Sec. 4.9.3. Initial and final outcomes are represented by rectangular boxes, while intermediate results are symbolized by encapsulating bubbles. The directional arrows are meant to seamlessly guide the reader through the logical flow of the calculations.

Since there is no correction to the background geometry at $O(\zeta^1, \chi^0, \epsilon^0)$, $H_0^{(1,0,0)} = 0$. For $H_0^{(1,1,0)}$, we find

$$
\begin{aligned}
H_0^{(1,1,0)} = & \frac{imM^4}{448r^9(r-r_s)}\left(C_1(r) + 4i\omega r^2 C_2(r)\right) \\
& - \frac{iM^4}{16r^9}\cos\theta\left(C_3(r) - \frac{i\omega r^2 C_4(r)}{2}\right) \\
& + \frac{iM^4}{32r^8}\left[(r-r_s)C_4(r)\cos\theta\partial_r - \frac{C_5(r)}{2r}\sin\theta\partial_\theta\right],
\end{aligned} \tag{4.103}
$$

where all $C_i(r)$ are listed in Appendix 4.11. The source term $\mathcal{S}_{\rm geo}^{(1,1)}$ is then given by

$$
\mathcal{S}_{\rm geo}^{(1,0,1)} = 0\,, \quad \mathcal{S}_{\rm geo}^{(1,1,1)} = -H_0^{(1,1,0)}\Psi_0^{(0,0,1)}\,. \tag{4.104}
$$

$\mathcal{S}_{\rm geo}^{(1,1,1)}$ can be evaluated in terms of the coordinates using Eqs. (4.101) and (4.103).

4.6.2 Source term $\mathcal{S}^{(1,1)}$

In this subsection, we evaluate the source term $\mathcal{S}^{(1,1)}$ from the effective stress tensor or the source term of the trace-reversed Einstein equation in Eq. (4.4). The first step is to project the Ricci tensor $R_{\mu\nu}$ onto the NP basis such that we can express the NP Ricci scalars Φ_{ij}, where

$$
\Phi_{ij} \sim R_{ab} \sim R_{\mu\nu}e_a{}^\mu e_b{}^\nu\,, \tag{4.105}
$$

in terms of NP quantities (Weyl scalars, spin coefficients, and tetrad) and the scalar field ϑ. The precise relation between Φ_{ij} and R_{ab} is given in Eq. (4.180). Using Eq. (4.4), we find

$$
\begin{aligned}
R_{\mu\nu} = & -\left(\frac{1}{\kappa_g}\right)^{\frac{1}{2}} M^2\left[\left(\nabla^\sigma\vartheta\right)\epsilon_{\sigma\delta\alpha(\mu}\nabla^\alpha R_{\nu)}{}^\delta + \left(\nabla^\sigma\nabla^\delta\vartheta\right)^* R_{\delta(\mu\nu)\sigma}\right] \\
& + \frac{1}{2\kappa_g\zeta}\left(\nabla_\mu\vartheta\right)\left(\nabla_\nu\vartheta\right)\,,
\end{aligned} \tag{4.106}
$$

where we have inserted an additional $\zeta^{-\frac{1}{2}}$ to the term linear in ϑ and an additional ζ^{-1} to the term quadratic in ϑ to compensate the factor of $\zeta^{\frac{1}{2}}$ we have absorbed into the expansion of ϑ in Eq. (4.17). Since ϑ enters at least at $O(\zeta)$, all the metric fields at the right-hand side of Eq. (4.106) are at $O(\zeta^0)$, which can be expressed in terms of NP quantities in GR.

$$
\begin{aligned}
R_{ab} &= \eta^{cd}R_{cadb}\,, \\
\nabla_c R_{ab} &= R_{ab,c} - \gamma^d{}_{ac}R_{db} - \gamma^d{}_{bc}R_{ad}\,.
\end{aligned} \tag{4.107}
$$

Since we are interested in gravitational perturbations of vacuum spacetime, $R^{(0,0)}_{\mu\nu} = R^{(0,1)}_{\mu\nu} = 0$, and all the metric fields in Eq. (4.106) enter at $O(\zeta^0)$, the first term in Eq. (4.106) vanishes. Evaluating the rest of the terms, we find the seven independent components of R_{ab} in terms of Weyl scalars, spin coefficients, directional derivatives, and the scalar field ϑ in Eq. (4.178).

We can now evaluate the source terms in the modified Teukolsky equations. Inspecting the source term $\mathcal{S}^{(1,1)}$ in Eq. (4.21), we can divide it into two parts based on whether $S_{1,2}$ are dynamical,

$$\begin{aligned}
\mathcal{S}^{(1,1)} &= \mathcal{S}^{(1,1)}_{I} + \mathcal{S}^{(1,1)}_{II}\,, \\
\mathcal{S}^{(1,1)}_{I} &= \mathcal{E}^{(0,1)}_{2} S^{(1,0)}_{2} - \mathcal{E}^{(0,1)}_{1} S^{(1,0)}_{1}\,, \\
\mathcal{S}^{(1,1)}_{II} &= \mathcal{E}^{(0,0)}_{2} S^{(1,1)}_{2} - \mathcal{E}^{(0,0)}_{1} S^{(1,1)}_{1}\,.
\end{aligned} \tag{4.108}$$

For $\mathcal{S}^{(1,1)}_{I}$, one can directly evaluate $S^{(1,0)}_{1,2}$ in terms of the stationary scalar field $\vartheta^{(1,0)}$ and the metric in GR using Eqs. (4.22), (4.178), and (4.180). Then we only need to reconstruct the operators $\mathcal{E}^{(0,1)}_{1,2}$ using our results in Sec. 4.4 and Appendix 4.12. The results of $S^{(1,0)}_{1,2}$ are provided in Appendix 4.13.

For $\mathcal{S}^{(1,1)}_{II}$, the only pieces involving metric reconstruction are $S^{(1,1)}_{1,2}$. For $S^{(1,1)}_{1,2}$, we can further divide them into two parts based on whether Φ_{ij} are dynamical,

$$\begin{aligned}
S^{(1,1)}_{1,A} &= \delta^{(0,1)}_{[-2,-2,1,0]}\Phi^{(1,0)}_{00} - D^{(0,1)}_{[-2,0,0,-2]}\Phi^{(1,0)}_{01} \\
&\quad + 2\sigma^{(0,1)}\Phi^{(1,0)}_{10} - 2\kappa^{(0,1)}\Phi^{(1,0)}_{11} - \bar{\kappa}^{(0,1)}\Phi^{(1,0)}_{02}\,, \\
S^{(1,1)}_{2,A} &= \delta^{(0,1)}_{[0,-2,2,0]}\Phi^{(1,0)}_{01} - D^{(0,1)}_{[-2,2,0,-1]}\Phi^{(1,0)}_{02} \\
&\quad - \bar{\lambda}^{(0,1)}\Phi^{(1,0)}_{00} + 2\sigma^{(0,1)}\Phi^{(1,0)}_{11} - 2\kappa^{(0,1)}\Phi^{(1,0)}_{12}\,, \\
S^{(1,1)}_{1,B} &= \delta^{(0,0)}_{[-2,-2,1,0]}\Phi^{(1,1)}_{00} - D^{(0,0)}_{[-2,0,0,-2]}\Phi^{(1,1)}_{01}\,, \\
S^{(1,1)}_{2,B} &= \delta^{(0,0)}_{[0,-2,2,0]}\Phi^{(1,1)}_{01} - D^{(0,0)}_{[-2,2,0,-1]}\Phi^{(1,1)}_{02}\,,
\end{aligned} \tag{4.109}$$

where we used that $\kappa^{(0,0)} = \sigma^{(0,0)} = \lambda^{(0,0)} = 0$. Based on this classification, we can then additionally separate $\mathcal{S}^{(1,1)}_{II}$ into two parts

$$\begin{aligned}
\mathcal{S}^{(1,1)}_{II} &= \mathcal{S}^{(1,1)}_{IIA} + \mathcal{S}^{(1,1)}_{IIB}\,, \\
\mathcal{S}^{(1,1)}_{IIA} &= \mathcal{E}^{(0,0)}_{2} S^{(1,1)}_{2,A} - \mathcal{E}^{(0,0)}_{1} S^{(1,1)}_{1,A}\,, \\
\mathcal{S}^{(1,1)}_{IIB} &= \mathcal{E}^{(0,0)}_{2} S^{(1,1)}_{2,B} - \mathcal{E}^{(0,0)}_{1} S^{(1,1)}_{1,B}\,.
\end{aligned} \tag{4.110}$$

For $\mathcal{S}^{(1,1)}_{IIA}$, similar to $\mathcal{S}^{(1,1)}_{I}$, we only need to evaluate $\Phi^{(1,0)}_{ij}$ using the background metric and the stationary scalar field $\vartheta^{(1,0)}$. The only quantities need metric reconstruction are these additional $O(\zeta^0, \epsilon^1)$ operators acting on $\Phi^{(1,0)}_{ij}$, where $\Phi^{(1,0)}_{ij}$ are listed in Appendix 4.13.

The most complicated piece of $\mathcal{S}^{(1,1)}$ is $\mathcal{S}_{IIA}^{(1,1)}$, which needs metric reconstruction for $\Phi_{ij}^{(1,1)}$. Fortunately, from Eq. (4.109), we notice that we only need to evaluate $\Phi_{00}^{(1,1)}$, $\Phi_{01}^{(1,1)}$, and $\Phi_{02}^{(1,1)}$. To organize our calculations, we can divide $\Phi_{ij}^{(1,1)}$ into four parts based on which kind of terms need metric reconstruction, similar to what we have done in Sec. 4.5. Inspecting Eq. (4.178), we notice that all the terms are some coupling of a Weyl scalar, a scalar field, a spin coefficient, and directional derivatives, which is due to the structure of $R_{\mu\nu}$ in Eq. (4.106). In this case, we make the following classification:

1. Weyl scalars are at $O(\zeta^0, \epsilon^1)$. In this case, the scalar field ϑ is at $O(\zeta^1, \epsilon^0)$. As discussed in Sec. 4.3.3, the leading contribution to $\vartheta^{(1,0)}$ is at $O(\zeta^1, \chi^1, \epsilon^0)$ since non-rotating BHs in dCS gravity are still Schwarzschild. Then since we are interested in $O(\zeta^1, \chi^1, \epsilon^1)$ corrections, all the spin coefficients and directional derivatives are at $O(\zeta^0, \chi^0, \epsilon^0)$, the order of the Schwarzschild background.

2. Spin coefficients are at $O(\zeta^0, \epsilon^1)$. Similar to the first situation, the scalar field ϑ is at $O(\zeta^1, \chi^1, \epsilon^0)$, so all the Weyl scalars and directional derivatives are at $O(\zeta^0, \chi^0, \epsilon^0)$, which are evaluated on the Schwarzschild background in GR.

3. Tetrad/directional derivatives are at $O(\zeta^0, \epsilon^1)$. Similar to the first two cases, since ϑ is at $O(\zeta^1, \chi^1, \epsilon^0)$, so all the Weyl scalars and spin coefficients can be evaluated on the Schwarzschild background.

4. The scalar field ϑ is at $O(\zeta^1, \epsilon^1)$, which has contributions from both $O(\zeta^1, \chi^0, \epsilon^1)$ and $O(\zeta^1, \chi^1, \epsilon^1)$ terms. Then, all the NP quantities generally need to be evaluated on the Kerr background expanded in the slow-rotation expansion to $O(\chi^1)$. Since $\vartheta^{(1,1)}$ also requires us to solve the scalar field equation in Eqs. (4.38), we choose not to compute $\vartheta^{(1,1)}$ in this work but only list the source terms in terms of it. We will solve the scalar field equation together with the modified Teukolsky equation in our follow-up work [58].

At $O(\zeta^1, \chi^1, \epsilon^1)$, using the classification above, we can set many terms to 0 since they are evaluated on the Schwarzschild background (i.e., when ϑ is stationary). Similar to Sec. 4.5, the results of $\Phi_{00}^{(1,1)}$, $\Phi_{01}^{(1,1)}$, and $\Phi_{02}^{(1,1)}$ up to $O(\zeta^1, \chi^1, \epsilon^1)$ are expressed in terms of the perturbed Weyl scalars, metric perturbations, and dynamical ϑ in Appendix 4.13. Due to the complication of $\mathcal{S}^{(1,1)}$, we will not present the results here but provide its coordinate-based values directly in the next subsection.

4.6.3 $\mathcal{S}^{(1,1)}$ in the coordinate basis

In this subsection, we evaluate the coordinate-based values of $\mathcal{S}^{(1,1)}$ using the decomposition of $\vartheta^{(1,1)}$ in Eq. (4.87) and the Hertz potential $\Psi_{\rm H}^{(1,1)}$ in Eq. (4.94). Following Sec. 4.3.1, we separate $\mathcal{S}^{(1,1)}$ into two parts: the terms coupled to the dynamical scalar field $\vartheta^{(1,1)}$ and the terms coupled to the background scalar field $\vartheta^{(1,0)}$.

For the first part, we find its coordinate-based value to be

$$\begin{aligned}
\mathcal{S}_A^{(1,1)} = e^{-i\omega t} \Big[& \Big(p_1^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + p_2^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
& + p_3^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r) \Big) \, {}_2Y_{\ell m}(\theta,\phi) \\
& + \chi \Big(p_4^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + p_5^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
& + p_6^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r) \Big) \sin\theta \, {}_1Y_{\ell m}(\theta,\phi) \\
& + \chi \Big(p_7^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + p_8^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
& + p_9^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r) \Big) \cos\theta \, {}_2Y_{\ell m}(\theta,\phi) \\
& + \chi \, {}_0b^m_{\ell,\ell+1} \Big(p_{10}^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + p_{11}^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
& + p_{12}^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r) \Big) \, {}_2Y_{\ell+1\, m}(\theta,\phi) \\
& + \chi \, {}_0b^m_{\ell,\ell-1} \Big(p_{13}^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + p_{14}^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
& + p_{15}^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r) \Big) \, {}_2Y_{\ell-1\, m}(\theta,\phi) \Big] \, ,
\end{aligned} \tag{4.111a}$$

$$\begin{aligned}
\tilde{\mathcal{S}}_A^{(1,1)} = e^{i\omega t} \Big[& - \Big(\bar{p}_1^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{p}_2^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) \\
& + \bar{p}_3^{\ell m}(r,\omega,M)\bar{\Theta}''_{\ell m}(r) \Big) \, {}_{-2}\bar{Y}_{\ell m}(\theta,\phi) \\
& - \chi \Big(\bar{p}_4^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{p}_5^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) \\
& + \bar{p}_6^{\ell m}(r,\omega,M)\bar{\Theta}''_{\ell m}(r) \Big) \sin\theta \, {}_{-1}\bar{Y}_{\ell m}(\theta,\phi) \\
& + \chi \Big(\bar{p}_7^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{p}_8^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) \\
& + \bar{p}_9^{\ell m}(r,\omega,M)\bar{\Theta}''_{\ell m}(r) \Big) \cos\theta \, {}_{-2}\bar{Y}_{\ell m}(\theta,\phi) \\
& - \chi \, {}_0b^m_{\ell,\ell+1} \Big(\bar{p}_{10}^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{p}_{11}^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) \\
& + \bar{p}_{12}^{\ell m}(r,\omega,M)\bar{\Theta}''_{\ell m}(r) \Big) \, {}_{-2}\bar{Y}_{\ell+1\, m}(\theta,\phi) \\
& - \chi \, {}_0b^m_{\ell,\ell+1} \Big(\bar{p}_{13}^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{p}_{14}^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) \\
& + \bar{p}_{15}^{\ell m}(r,\omega,M)\bar{\Theta}''_{\ell m}(r) \Big) \, {}_{-2}\bar{Y}_{\ell-1\, m}(\theta,\phi) \Big] \, .
\end{aligned} \tag{4.111b}$$

In Sec. 4.9.3, after getting the radial part of the equation of $\vartheta^{(1,1)}$, we will further express $\Theta''_R(r)$ in terms of $\Theta_{\ell m}(r)$, $\Theta'_{\ell m}(r)$, ${}_2\hat{R}_{\ell m}(r)$, and ${}_2\hat{R}'_{\ell m}(r)$.

Similarly, we find the second part to take the form

$$\begin{aligned}\mathcal{S}_B^{(1,1)} = \chi e^{-i\omega t}\Big[&\Big(q_1^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + q_2^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\Big)\sin\theta\,{}_1Y_{\ell m}(\theta,\phi)\\ &+\Big(q_3^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + q_4^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\Big)\cos\theta\,{}_2Y_{\ell m}(\theta,\phi)\\ &+\Big(q_5^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + q_6^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\Big)\sin\theta\,{}_3Y_{\ell m}(\theta,\phi)\Big]\,,\end{aligned} \tag{4.112a}$$

$$\begin{aligned}\tilde{\mathcal{S}}_B^{(1,1)} = \chi e^{i\omega t}\Big[&\Big(\tilde{q}_1^{\ell m}(r,\omega,M)\,{}_2\bar{\hat{R}}_{\ell m}(r) + \tilde{q}_2^{\ell m}(r,\omega,M)\,{}_2\bar{\hat{R}}'_{\ell m}(r)\Big)\sin\theta\,{}_{-1}\bar{Y}_{\ell m}(\theta,\phi)\\ &+\tilde{q}_3^{\ell m}(r,\omega,M)\,{}_2\bar{\hat{R}}_{\ell m}(r)\cos\theta\,{}_{-2}\bar{Y}_{\ell m}(\theta,\phi)\Big]\,,\end{aligned} \tag{4.112b}$$

where ${}_2\hat{R}_{\ell m}(r)$ is the radial function of the Hertz potential given by Eq. (4.56a), and ${}_2\bar{\hat{R}}_{\ell m}(r)$ is the complex conjugate of ${}_2\hat{R}_{\ell m}(r)$. The radial functions $q_i^{\ell m}(r,\omega,M)$ and $\tilde{q}_i^{\ell m}(r,\omega,M)$ are presented in a Mathematica notebook as Supplementary Material [69]. Note that we have used the radial Teukolsky equation to eliminate any beyond-first-order derivatives of the Hertz potential to obtain a simplified expression in Eqs. (4.112a) and (4.112b). The master equation of $\Psi_0^{(1,1)}$ in the IRG in the coordinate $\{t,r,\theta,\phi\}$ is presented in Eq. (4.128) of Sec. 4.8.

4.7 The evolution equation for $\vartheta^{(1,1)}$ and the modified Teukolsky equation for $\Psi_4^{(1,1)}$ in the ORG

In this section, we evaluate the equations governing the evolution of the perturbed scalar field $\vartheta^{(1,1)}$ and the perturbed Weyl scalar $\Psi_4^{(1,1)}$. Although one can in principle evaluate the evolution equation of $\Psi_4^{(1,1)}$in the IRG, as briefly discussed in Sec. 4.4.2, the evaluation is more convenient in the ORG. We follow a set of steps similar to those in the IRG for $\vartheta^{(1,1)}$ in Sec. 4.5 and for $\Psi_0^{(1,1)}$ in Sec. 4.6. Below, we present the master equations of $\vartheta^{(1,1)}$ and $\Psi_4^{(1,1)}$ in the ORG.

4.7.1 The equation of $\vartheta^{(1,1)}$

The scalar field perturbations are governed by Eq. (4.38). We now represent the right-hand side of Eq. (4.38) as

$$\mathcal{T}_\vartheta^{(1,1)} \equiv -\frac{M^2}{16\pi^{\frac{1}{2}}}\,[R^*R]^{(0,1)} - \Box^{(0,1)}\vartheta^{(1,0)}\,. \tag{4.113}$$

Projecting the Pontryagin density onto the NP basis leads to the same set of equations as described in Sec. 4.5.1 since our choice of gauge does not affect the quantities shown in Eqs. (4.81).

The master equation for the scalar field perturbations in the ORG are same as that shown in the IRG

$$H_\vartheta^{(0,0)} \vartheta^{(1,1)} = \mathcal{T}_\vartheta^{(1,1)} , \tag{4.114}$$

with $H_\vartheta^{(0,0)}$ and $\vartheta^{(1,1)}$ both given in Eqs. (4.85) and (4.86) respectively, whereas $\mathcal{T}_\vartheta^{(1,1)}$ is given in Eq.(4.113). The left-hand side of Eq. (4.114) in the ORG remains unchanged from the IRG since the operator acting on the scalar field perturbations is evaluated on the background. On the other hand, since the source term in Eq. (4.114) depends on perturbed quantities, the value of these quantities is gauge dependent. In the ORG, the Pontryagin density given in Eq. (4.89) holds the following form in the coordinate basis:

$$\begin{aligned}
&({}^*RR)^{(0,1)} \\
&= e^{-i\omega t} \Big[\Big(\mathfrak{g}_1^{\ell m}(r,\omega,M) {}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{g}_2^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}'_{\ell m}(r) \Big)\, {}_0Y_{\ell m}(\theta,\phi) \\
&\quad + \chi \Big(\mathfrak{g}_3^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{g}_4^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}'_{\ell m}(r) \Big) \sin\theta\, {}_{-1}Y_{\ell m}(\theta,\phi) \\
&\quad + \chi \Big(\mathfrak{g}_5^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{g}_6^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}'_{\ell m}(r) \Big) \cos\theta\, {}_0Y_{\ell m}(\theta,\phi) \\
&\quad + \chi\, {}_{-2}b^m_{\ell,\ell+1} \Big(\mathfrak{g}_7^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{g}_8^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}'_{\ell m}(r) \Big)\, {}_0Y_{\ell+1 m}(\theta,\phi) \\
&\quad + \chi\, {}_{-2}b^m_{\ell,\ell-1} \Big(\mathfrak{g}_9^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{g}_{10}^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}'_{\ell m}(r) \Big)\, {}_0Y_{\ell-1 m}(\theta,\phi) \Big] \\
&\quad + \text{c.c.} ,
\end{aligned} \tag{4.115}$$

where functions $\mathfrak{g}_i(r,\omega,M)$ are presented in a separate Mathematica notebook as Supplementary Material [69], and ${}_{-2}\hat{R}'_{\ell m}(r)$ is the radial function of the Hertz potential for a slowly rotating Kerr BHs in GR computed from the radial function of the $\Psi_4^{(0,1)}$ using Eq. (4.56b).

To evaluate the remaining part of the source term $\mathcal{T}_\vartheta^{(1,1)}$, we use the perturbed spin coefficients given in Eq. (4.166) and metric perturbations given in Eq. (4.51). We obtain

$$\begin{aligned}
&\Box^{(0,0,1)} \vartheta^{(1,1,0)} \\
&= e^{-i\omega t} \Big[\Big(\mathfrak{h}_1^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{h}_2^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}'_{\ell m}(r) \Big) \sin\theta\, {}_{-1}Y_{\ell m}(\theta,\phi) \\
&\quad + \mathfrak{h}_3^{\ell m}(r,\omega,M) \left(2\vartheta'_R(r) + r\vartheta''_R(r) \right) {}_{-2}\hat{R}_{\ell m}(r) \cos\theta\, {}_0Y_{\ell m}(\theta,\phi) \Big] + \text{c.c.} ,
\end{aligned} \tag{4.116}$$

where the functions $\mathfrak{h}_i(r)$ are presented in a separate Mathematica notebook as Supplementary Material [69], and $\vartheta_R(r)$ is the radial part of the background scalar field given in Eq. (4.15). Similar to the case evaluated in the IRG in Sec. 4.5, the radial function ${}_{-2}\hat{R}_{\ell m}(r)$ is evaluated on the Schwarzschild background. Combining Eqs. (4.116) and (4.115) gives us the complete source term $\mathcal{T}_{\vartheta}^{(1,1)}$. The master equation of $\vartheta^{(1,1)}$ in the ORG in the coordinate $\{t, r, \theta, \phi\}$ is presented in Eq. (4.128) of Sec. 4.8.

4.7.2 The equation of $\Psi_4^{(1,1)}$

In this subsection, we present the modified Teukolsky equation for the Weyl scalar perturbation $\Psi_4^{(1,1)}$ given in Eq. (4.26) in the coordinate basis. Following steps similar to Sec. 4.6, we separate the source terms into two categories: $\mathcal{T}_{\rm geo}^{(1,1)}$ and $\mathcal{T}^{(1,1)}$ whose forms in the NP basis have been given in Eqs. (4.27)–(4.29).

4.7.2.1 Homogeneous part and $\mathcal{T}_{\rm geo}^{(1,1)}$

Similar to Sec. 4.6.1, by using the Weyl scalars and spin coefficients in Sec. 4.3.4 and Appendix 4.11 along with Eq. (4.31), we find

$$\mathcal{H}_4^{(0,0,0)} = \frac{1}{2r^6} H_{4,\rm TK}^{(0,0,0)}\,, \tag{4.117a}$$

$$\mathcal{H}_4^{(0,1,0)} = \frac{1}{2r^6} H_{4,\rm TK}^{(0,1,0)}\,, \tag{4.117b}$$

where we define

$$\left(H_4^{(0,0,0)} + \chi H_4^{(0,1,0)}\right) \Psi_4^{(1)} \equiv \left(\mathcal{H}_4^{(0,0,0)} + \chi \mathcal{H}_4^{(0,1,0)}\right) \psi_4^{(1)} \tag{4.118}$$

by extracting a factor of ρ^4 from the Weyl scalar $\Psi_4^{(1)}$ following [19], i.e.,

$$\Psi_4^{(1)} \equiv \rho^4 \psi_4^{(1)} = \rho^4 \left[{}_{-2}R_{\ell m}^{(0,1)}(r) + \zeta\, {}_{-2}R_{\ell m}^{(1,1)}(r) + O(\zeta^2)\right] {}_{-2}S_{\ell m}(\theta) e^{-i\omega t + im\phi}\,. \tag{4.119}$$

The operators $H_{4,\rm TK}^{(0,0,0)}$ and $H_{4,\rm TK}^{(0,1,0)}$ are $O(\chi^0)$ and $O(\chi^1)$ terms of the Teukolsky operator $H_{4,\rm TK}$ for ψ_4 [Eq. (4.7) in [19]], respectively,

$$\begin{aligned} H_{4,\rm TK}^{(0,0,0)} &= -r(r-r_s)\partial_r^2 + 2(r-M)\partial_r - \frac{D(r)}{r-r_s} \\ &\quad - \partial_\theta^2 - \cot\theta\partial_\theta + (-2\cot\theta + m\csc\theta)^2 - 2\,, \end{aligned} \tag{4.120a}$$

$$H_{4,\rm TK}^{(0,1,0)} = 4M\left[\frac{m(i(r-M) + M\omega r)}{r(r-r_s)} - \omega\cos\theta\right], \tag{4.120b}$$

$$D(r) = -4i\omega r(r-3M) + \omega^2 r^3\,.$$

The Teukolsky equation corresponding to Eqs. (4.120a) and (4.120b) is separable with ${}_{-2}R^{(0,1)}_{\ell m}(r)$ satisfying

$$\begin{aligned}&\left[r(r-r_s)\partial_r^2-2(r-M)\partial_r+\frac{D(r)}{r-r_s}\right.\\&\left.-\frac{4m\chi M(i(r-M)+M\omega r)}{r(r-r_s)}-{}_{-2}A_{\ell m}\right]{}_{-2}R^{(0,1)}_{\ell m}(r)=0\,.\end{aligned}\tag{4.121}$$

For the same reason in Sec. 4.6.1, we do not need metric reconstruction to compute $\mathcal{T}^{(1,1)}_{\rm geo}$ since the background spacetime is still Petrov type D. The source terms $\mathcal{T}^{(1,1)}_{\rm geo}$ hold the form

$$\mathcal{T}^{(1,0,1)}_{\rm geo}=0\,,\;\mathcal{T}^{(1,1,1)}_{\rm geo}=-H_4^{(1,1,0)}\Psi_4^{(0,0,1)}=-\mathcal{H}_4^{(1,1,0)}\psi_4^{(0,0,1)}\tag{4.122}$$

with

$$\begin{aligned}\mathcal{H}_4^{(1,1,0)}=&\frac{-imM^4}{448r^{13}(r-r_s)}\Big(D_1(r)-4i\omega r^2D_2(r)\Big)\\&+\frac{iM^4}{16r^{13}}\cos\theta\left(D_3(r)-\frac{i\omega r^2D_4(r)}{2}\right)\\&+\frac{iM^4}{32r^{12}}\left[(r-r_s)D_4(r)\cos\theta\partial_r-\frac{D_5(r)}{2r}\sin\theta\partial_\theta\right],\end{aligned}\tag{4.123}$$

where we have absorbed the factor of ρ^4 into $\mathcal{H}_4^{(1,1,0)}$, and the functions $D_i(r)$ are presented in Appendix 4.11.

4.7.2.2 $\mathcal{T}^{(1,1)}$

Using the expression for the metric perturbation in the ORG given in Eq. (4.51), one can evaluate the perturbed spin coefficients and perturbed Weyl scalars at $O(\zeta^0,\epsilon^1)$. These can then be used to evaluate the following source terms, which can be divided into two parts based on whether $S_{3,4}$ are dynamical.

$$\begin{aligned}\mathcal{T}^{(1,1)}&=\mathcal{T}_I^{(1,1)}+\mathcal{T}_{II}^{(1,1)}\,,\\\mathcal{T}_I^{(1,1)}&=\mathcal{E}_4^{(0,1)}S_4^{(1,0)}-\mathcal{E}_3^{(0,1)}S_3^{(1,0)}\,,\\\mathcal{T}_{II}^{(1,1)}&=\mathcal{E}_4^{(0,0)}S_4^{(1,1)}-\mathcal{E}_3^{(0,0)}S_3^{(1,1)}\,.\end{aligned}\tag{4.124}$$

Analogous to Sec. 4.6, $\mathcal{T}_I^{(1,1)}$ consists of terms dependent on the stationary scalar field, the background, and the perturbed metric in GR.

Similarly, we can further divide $\mathcal{T}_{II}^{(1,1)}$ into two categories based on whether terms are proportional to $\Phi_{ij}^{(1,0)}$ or $\Phi_{ij}^{(1,1)}$, which we denote as $\mathcal{T}_{IIA}^{(1,1)}$ and $\mathcal{T}_{IIB}^{(1,1)}$ respectively.

$$\begin{aligned}
\mathcal{T}_{II}^{(1,1)} &= \mathcal{T}_{IIA}^{(1,1)} + \mathcal{T}_{IIB}^{(1,1)} \,, \\
\mathcal{T}_{IIA}^{(1,1)} &= \mathcal{E}_4^{(0,0)} S_{4,A}^{(1,1)} - \mathcal{E}_3^{(0,0)} S_{3,A}^{(1,1)} \,, \\
\mathcal{T}_{IIB}^{(1,1)} &= \mathcal{E}_4^{(0,0)} S_{4,B}^{(1,1)} - \mathcal{E}_3^{(0,0)} S_{3,B}^{(1,1)} \,.
\end{aligned} \tag{4.125}$$

Here $S_{3,B}^{(1,1)}$ and $S_{4,B}^{(1,1)}$ have terms proportional to $\Phi_{ij}^{(1,1)}$, whereas $S_{3,A}^{(1,1)}$ and $S_{4,A}^{(1,1)}$ have terms proportional to $\Phi_{ij}^{(1,0)}$.

Expressing these source terms in the coordinate basis, the terms proportional to the scalar field perturbation are given by

$$\begin{aligned}
\mathcal{T}_{A}^{(1,1)} = e^{-i\omega t} \Big[& \Big(\mathfrak{p}_1^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + \mathfrak{p}_2^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
& + \mathfrak{p}_3^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r)\Big) \, {}_{-2}Y_{\ell m}(\theta,\phi) \\
& + \chi \Big(\mathfrak{p}_4^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + \mathfrak{p}_5^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
& + \mathfrak{p}_6^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r)\Big) \sin\theta \, {}_{-1}Y_{\ell m}(\theta,\phi) \\
& + \chi \Big(\mathfrak{p}_7^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + \mathfrak{p}_8^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
& + \mathfrak{p}_9^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r)\Big) \cos\theta \, {}_{-2}Y_{\ell m}(\theta,\phi) \\
& + \chi \, {}_0b^m_{\ell,\ell+1} \Big(\mathfrak{p}_{10}^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + \mathfrak{p}_{11}^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
& + \mathfrak{p}_{12}^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r)\Big) \, {}_{-2}Y_{\ell+1\, m}(\theta,\phi) \\
& + \chi \, {}_0b^m_{\ell,\ell-1} \Big(\mathfrak{p}_{13}^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + \mathfrak{p}_{14}^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
& + \mathfrak{p}_{15}^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r)\Big) \, {}_{-2}Y_{\ell-1\, m}(\theta,\phi) \Big] \,,
\end{aligned} \tag{4.126a}$$

$$\begin{aligned}
\tilde{\mathcal{T}}_{A}^{(1,1)} = e^{i\omega t} \Big[& - \Big(\bar{\mathfrak{p}}_1^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{\mathfrak{p}}_2^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) \\
& + \bar{\mathfrak{p}}_3^{\ell m}(r,\omega,M)\bar{\Theta}''_{\ell m}(r)\Big) \, {}_{2}\bar{Y}_{\ell m}(\theta,\phi) \\
& - \chi \Big(\bar{\mathfrak{p}}_4^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{\mathfrak{p}}_5^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) \\
& + \bar{\mathfrak{p}}_6^{\ell m}(r,\omega,M)\bar{\Theta}''_{\ell m}(r)\Big) \sin\theta \, {}_{1}\bar{Y}_{\ell m}(\theta,\phi) \\
& + \chi \Big(\bar{\mathfrak{p}}_7^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{\mathfrak{p}}_8^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) \\
& + \bar{\mathfrak{p}}_9^{\ell m}(r,\omega,M)\bar{\Theta}''_{\ell m}(r)\Big) \cos\theta \, {}_{2}\bar{Y}_{\ell m}(\theta,\phi) \\
& - \chi \, {}_0b^m_{\ell,\ell+1} \Big(\bar{\mathfrak{p}}_{10}^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{\mathfrak{p}}_{11}^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) \\
& + \bar{\mathfrak{p}}_{12}^{\ell m}(r,\omega,M)\bar{\Theta}''_{\ell m}(r)\Big) \, {}_{2}\bar{Y}_{\ell+1 m}(\theta,\phi)
\end{aligned}$$

$$-\chi\,{}_0b^m_{\ell,\ell-1}\left(\bar{\mathfrak{p}}^{\ell m}_{13}(r,\omega,M)\bar{\Theta}_{\ell m}(r)+\bar{\mathfrak{p}}^{\ell m}_{14}(r,\omega,M)\bar{\Theta}'_{\ell m}(r)\right.$$
$$\left.+\bar{\mathfrak{p}}^{\ell m}_{15}(r,\omega,M)\bar{\Theta}''_{\ell m}(r)\right){}_2\bar{Y}_{\ell-1m}(\theta,\phi)\Big]\,. \tag{4.126b}$$

The source terms proportional to the background scalar field can be expressed in the coordinate basis as

$$\begin{aligned}\mathcal{T}_B^{(1,1)}=\chi e^{-i\omega t}\Big[&\left(\mathfrak{q}^{\ell m}_1(r,\omega,M)\,{}_{-2}\hat{R}_{\ell m}(r)+\mathfrak{q}^{\ell m}_2(r,\omega,M)\,{}_{-2}\hat{R}'_{\ell m}(r)\right)\sin\theta\,{}_{-1}Y_{\ell m}(\theta,\phi)\\&+\left(\mathfrak{q}^{\ell m}_3(r,\omega,M)\,{}_{-2}\hat{R}_{\ell m}(r)+\mathfrak{q}^{\ell m}_4(r,\omega,M)\,{}_{-2}\hat{R}'_{\ell m}(r)\right)\cos\theta\,{}_{-2}Y_{\ell m}(\theta,\phi)\\&+\left(\mathfrak{q}^{\ell m}_5(r,\omega,M)\,{}_{-2}\hat{R}_{\ell m}(r)+\mathfrak{q}^{\ell m}_6(r,\omega,M)\,{}_{-2}\hat{R}'_{\ell m}(r)\right)\sin\theta\,{}_{-3}Y_{\ell m}(\theta,\phi)\Big]\,,\end{aligned} \tag{4.127a}$$

$$\begin{aligned}\tilde{\mathcal{T}}_B^{(1,1)}=\chi e^{i\omega t}\Big[&\left(\tilde{\mathfrak{q}}^{\ell m}_1(r,\omega,M)\,{}_{-2}\bar{\hat{R}}_{\ell m}(r)+\tilde{\mathfrak{q}}^{\ell m}_2(r,\omega,M)\,{}_{-2}\bar{\hat{R}}'_{\ell m}(r)\right)\sin\theta\,{}_1\bar{Y}_{\ell m}(\theta,\phi)\\&+\tilde{\mathfrak{q}}^{\ell m}_3(r,\omega,M)\,{}_{-2}\bar{\hat{R}}_{\ell m}(r)\cos\theta\,{}_2\bar{Y}_{\ell m}(\theta,\phi)\Big]\,,\end{aligned} \tag{4.127b}$$

where ${}_{-2}\hat{R}_{\ell m}(r)$ is the radial function of the Hertz potential given by Eq. (4.56b). The functions $\mathfrak{q}^{\ell m}_i(r,\omega,M)$ and $\tilde{\mathfrak{q}}^{\ell m}_i(r,\omega,M)$ (not to be confused with $\bar{\mathfrak{q}}^{\ell m}_i(r,\omega,M)$) are functions presented in a Mathematica notebook as Supplementary Material [69]. The master equation of $\Psi_4^{(1,1)}$ in the ORG in the coordinate $\{t,r,\theta,\phi\}$ is presented in Eq. (4.128) of Sec. 4.8.

4.8 Executive Summary of all Master Equations

This section presents an executive summary of the main results of this paper, whose derivation was presented in Secs. 4.5, 4.6, and 4.7. Through Secs. 4.5, 4.6, and 4.7, we used the tetrad defined in Sec. 4.3.4 to rewrite Eqs. (4.18) and (4.38) using the IRG and Eqs. (4.26) and (4.38) using the ORG. In this section, we summarize these results and condense them into a single master equation for convenience. In later sections, we will present and apply a procedure to decouple the master equations for the scalar field perturbation $\vartheta^{(1,1)}$ and the Weyl scalar perturbations $\Psi_{0,4}^{(1,1)}$ (or $\Psi_4^{(1,1)}$) into a set of coupled radial ordinary differential equations in the IRG (or ORG).

With this in mind, the master equations for $\vartheta^{(1,1)}$ [Eq. (4.38)], $\Psi_0^{(1,1)}$ [Eq. (4.18)],

ψ	$\xi(r)$	Ingoing radiation gauge		Outgoing radiation gauge	
		s	$\mathfrak{S}$ (equations)	s	$\mathfrak{S}$ (equations)
$\vartheta^{(1,1)}$	$-r^2$	0	$-\pi^{-\frac{1}{2}}M^2(R^*R)^{(0,1)} - \chi\square^{(0,0,1)}\vartheta^{(1,1,0)}$ Eqs. (4.95) & (4.98)	0	$-\pi^{-\frac{1}{2}}M^2(R^*R)^{(0,1)} - \chi\square^{(0,0,1)}\vartheta^{(1,1,0)}$ Eqs. (4.115) & (4.116)
$\Psi_0^{(1,1)}$	$2r^2$	+2	$\mathcal{S}_{\rm geo}^{(1,1)}+\mathcal{S}_A^{(1,1)}+\tilde{\mathcal{S}}_A^{(1,1)}+\mathcal{S}_B^{(1,1)}+\tilde{\mathcal{S}}_B^{(1,1)}$ Eqs. (4.104), (4.111), &(4.112)	–	–
$\rho^{-4}\Psi_4^{(1,1)}$	$2r^6$	–	–	−2	$\mathcal{T}_{\rm geo}^{(1,1)}+\mathcal{T}_A^{(1,1)}+\tilde{\mathcal{T}}_A^{(1,1)}+\mathcal{T}_B^{(1,1)}+\tilde{\mathcal{T}}_B^{(1,1)}$ Eqs. (4.122), (4.126), &(4.127)

Table 4.1: In this table, we present the quantities Ψ, the spin weight s and the source terms $\mathfrak{S}$ that appear in Eq. (4.128).

and $\Psi_4^{(1,1)}$ [Eq. (4.26)] can *all* be expressed as

$$\begin{aligned} H\psi &= \frac{r^3}{r-r_s}\frac{\partial^2\psi}{\partial t^2} + \frac{4\chi M^2}{r-r_s}\frac{\partial^2\psi}{\partial t\partial\phi} - \csc^2\theta\frac{\partial^2\psi}{\partial\phi^2} - r(r-r_s)\frac{\partial^2\psi}{\partial r^2} - 2(s+1)(r-M)\frac{\partial\psi}{\partial r} \\ &- \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + s\left[\chi M\left(\frac{1}{r}-\frac{1}{r-r_s}\right) - 2i\cot\theta\csc\theta\right]\frac{\partial\psi}{\partial\phi} \\ &+ 2s\left[\frac{r(r-3M)}{r-r_s} + i\chi M\cos\theta\right]\frac{\partial\psi}{\partial t} + \left(s^2\cot^2\theta - s\right)\psi = \xi(r)\mathfrak{S}\,, \end{aligned} \tag{4.128}$$

where H represents the Teukolsky operator for a spin s field in GR given in [19]. Recall that χ is the dimensionless spin parameter, $r_s = 2M$ is the Schwarzschild radius, and M is the BH mass. In Table 4.1, we present the field quantities ψ which satisfy this equation and the source terms, $\xi(r)$ and $\mathfrak{S}$ on the right-hand side of Eq. (4.128), dependent on the gauge and the spin weight s of these fields. Observe that, clearly, the differential operator H in Eq. (4.128) acting on the field quantity ψ is exactly the same as the one derived by Teukolsky in [19] in GR for Kerr BH perturbations [c.f. Eq. (4.7) therein], expanded to leading order in spin. In addition, from Table 4.1 and the source terms in Eqs. (4.104), (4.111), (4.112), (4.122), (4.126), and (4.127), we notice that the (l,m) and (l,m) modes of $\Psi_{0,4}^{(1,1)}$ need to be solved jointly, as we will discuss in more detail in Sec. 4.9.

4.9 Separation of variables and extraction of the radial master equation

In this section, we extract the radial parts of the master equations for $\vartheta^{(1,1)}$ [Eq. (4.38)] (both in the IRG and ORG), for $\Psi_0^{(1,1)}$ [Eq. (4.18)], and for $\Psi_4^{(1,1)}$ [Eq. (4.26)]. Let us first present our procedures for eliminating the angular dependence in these equations and then apply them to specific cases.

4.9.1 Eiminating the angular dependence

To eliminate the angular dependence of these master equations, we utilize the properties of spin-weighted spheroidal harmonics in Sec. 4.4.3 and go through the following procedures:

1. From Secs. 4.5–4.7, we first observe that the master equations of $\vartheta^{(1,1)}$ and $\Psi_{0,4}^{(1,1)}$ after decomposition into spin-weighted spheroidal harmonics [i.e., Eqs. (4.86), (4.101), and (4.119)] all take the form

$$\begin{aligned}
&{}_sH_{\ell m}\left[{}_s\psi_{\ell m}(r)\,{}_sS_{\ell m}(\theta)\right]e^{-i\omega_{\ell m}t+im\phi} \\
&\quad=\sum_k {}_sP^k_{\ell m}(r)\,{}_sf^k_{\ell m}(\theta)e^{-i\omega_{\ell m}t+im\phi}+{}_sQ^k_{\ell m}(r)\,{}_s\bar{f}^k_{\ell m}(\theta)e^{i\bar{\omega}_{\ell m}t-im\phi}\,, \qquad (4.129\text{a})
\end{aligned}$$

$$\begin{aligned}
&{}_sH_{\ell\, -m}\left[{}_s\psi_{\ell\, -m}(r)\,{}_sS_{\ell\, -m}(\theta)\right]e^{-i\omega_{\ell\, -m}t-im\phi} \\
&\quad=\sum_k {}_sP^k_{\ell\, -m}(r)\,{}_sf^k_{\ell\, -m}(\theta)e^{-i\omega_{\ell\, -m}t-im\phi}+{}_sQ^k_{\ell\, -m}(r)\,{}_s\bar{f}^k_{\ell\, -m}(\theta)e^{i\bar{\omega}_{\ell\, -m}t+im\phi}\,,
\end{aligned}$$
$$(4.129\text{b})$$

   where ${}_sH_{\ell m}$ is the (ℓ, m) mode of the Teukolsky operator in GR for particles of spin 0 [i.e., $H_\vartheta^{(0,0)}$ in Eq. (4.85)], spin 2 [i.e., $H_0^{(0,0)}$ in Eqs. (4.99) and (4.100)], and spin -2 [i.e., $\mathcal{H}_4^{(0,0)}$ in Eqs. (4.117) and (4.120)]. The radial function ${}_s\psi_{\ell m}(r)$ is the radial part of $\vartheta^{(1,1)}$ [i.e., $\Theta_{\ell m}(r)$], $\Psi_0^{(1,1)}$ [i.e., ${}_2R^{(1,1)}_{\ell m}(r)$], or $\rho^{-4}\Psi_4^{(1,1)}$ [i.e., ${}_{-2}R^{(1,1)}_{\ell m}(r)$] to be solved for. The angular function ${}_sS_{\ell m}(\theta)$ is the θ-dependent part of spin-weighted spheroidal harmonics ${}_s\mathcal{Y}(\theta,\phi)$. The radial functions ${}_sP^k_{\ell m}(r)$, ${}_sQ^k_{\ell m}(r)$ and angular functions ${}_sf^k_{\ell m}(\theta)$ can be extracted from the source terms found in Secs. 4.5–4.7. In the equation for $\vartheta^{(1,1)}$, we can observe from Eqs. (4.95), (4.98), (4.115), and (4.116) that ${}_0P^k_{\ell m}(r)={}_0\bar{Q}^k_{\ell m}(r)$ since the scalar field is real, while there is no such a constraint for $\Psi_{0,4}^{(1,1)}$ since they are complex in general.

2. Equation (4.129) assumes that a single (ℓ, m) mode [Eq. (4.129a)] or a single $(\ell, -m)$ mode [Eq. (4.129b)] can solve the modified Teukolsky equation. However, this is in general not true since the source term in Eq. (4.129a)

is a mixture of modes proportional to $e^{-i\omega_{\ell m}t}$ and $e^{i\bar{\omega}_{\ell m}t}$, and similarly for Eq. (4.129b). On the other hand, in GR, one has the well-known symmetry [22]

$$\omega^{(0)}_{\ell m} = -\bar{\omega}^{(0)}_{\ell -m}\,, \tag{4.130}$$

so both Eqs. (4.129a) and (4.129b) contain source terms proportional to $e^{-i\omega_{\ell m}t}$ and $e^{-i\omega_{\ell -m}t}$. Thus, one has to consider Eqs. (4.129a) and (4.129b) jointly or solve the linear combination , i.e.,

$$\Psi^{(0,1)}_{\ell m} = {}_sR^{(0,1)}_{\ell m}(r)\,{}_sS_{\ell m}(\theta)e^{-i\omega_{\ell m}t+im\phi} + \eta_{\ell m}\;{}_sR^{(0,1)}_{\ell -m}(r)\,{}_sS_{\ell\,-m}(\theta)e^{i\omega_{\ell m}t-im\phi}\,, \tag{4.131a}$$

$$\Psi^{(1,1)}_{\ell m} = {}_s\psi_{\ell m}(r)\,{}_sS_{\ell m}(\theta)e^{-i\omega_{\ell m}t+im\phi} + \eta_{\ell m}\;{}_s\psi_{\ell -m}(r)\,{}_sS_{\ell\,-m}(\theta)e^{i\omega_{\ell m}t-im\phi}\,, \tag{4.131b}$$

where we have absorbed an overall factor into the normalization of ${}_sR^{(0,1)}_{\ell m}(r)$ and ${}_s\psi_{\ell m}(r)$. Furthermore, we have also taken $\omega^{(0)}_{\ell m} = -\bar{\omega}^{(0)}_{\ell -m}$ not just in GR but also in dCS gravity due to the structure of Eq (4.129). As discussed and shown in more detail in [65], one can solve for both the complex constant $\eta_{\ell m}$ and the QNM frequencies $\omega_{\ell m}$ using the eigenvalue perturbation method in [25, 39, 40]. The combination $(\eta_{\ell m}, \omega_{\ell m})$ have two independent solutions [25, 26, 65], resulting in the breaking of isospectrality. In this case, by plugging the ansatz in Eq. (4.131) into the scalar field equation or the modified Teukolsky equations, using Eq. (4.129), and matching the phase of the terms, we find

$${}_sH_{\ell m}\left[{}_s\psi_{\ell m}(r)\,{}_sS_{\ell m}(\theta)\right] = \sum_k {}_sP^k_{\ell m}(r)\,{}_sf^k_{\ell m}(\theta) + \bar{\eta}_{\ell m}\;{}_sQ^k_{\ell\,-m}(r)\,{}_s\bar{f}^k_{\ell\,-m}(\theta)\,, \tag{4.132a}$$

$${}_sH_{\ell\,-m}\left[{}_s\psi_{\ell\,-m}(r)\,{}_sS_{\ell\,-m}(\theta)\right] = \sum_k {}_sP^k_{\ell\,-m}(r)\,{}_sf^k_{\ell\,-m}(\theta) + \frac{1}{\eta_{\ell m}}{}_sQ^k_{\ell\,m}(r)\,{}_s\bar{f}^k_{\ell\,m}(\theta)\,, \tag{4.132b}$$

where we have divided a factor of $\eta_{\ell m}$ in Eq. (4.132b), and ${}_s\psi_{\ell\pm m}(r)$ are radial solutions tied to $(\eta_{\ell m}, \omega_{\ell m})$. Similar procedures for generic modified gravity theories can be found in [65].

3. For the evolution equation for $\vartheta^{(1,1)}$, we observe that ${}_0f^k_{\ell m}(\theta)$ consists of the following terms:

   - ${}_0Y_{\ell m}(\theta,\phi)$ and ${}_0Y_{\ell\pm1\,m}(\theta,\phi)$,
   - $\cos\theta\,{}_0Y_{\ell m}(\theta,\phi)$, and

- $\sin\theta\,{}_{\pm1}Y_{\ell m}(\theta,\phi)$.

For the master equation for $\Psi_{0,4}^{(1,1)}$, ${}_{\pm2}f_{\ell m}^{k}(\theta)$ contains

- ${}_{\pm2}Y_{\ell m}(\theta,\phi)$ and ${}_{\pm2}Y_{\ell\pm1\,m}(\theta,\phi)$,
- $\cos\theta\,{}_{\pm2}Y_{\ell m}(\theta,\phi)$,
- $\sin\theta\,{}_{\pm1}Y_{\ell m}(\theta,\phi)$ and $\sin\theta_{\pm3}Y_{\ell m}(\theta,\phi)$.

Notice that ${}_{s}f_{\ell m}^{k}(\theta)$ are angular functions in the modified Teukolsky equation for the particle of spin weight s and mode (ℓ,m). The subscript s and subscripts (ℓ,m) do not indicate the mode number of the angular function itself. For example, ${}_{0}f_{\ell m}^{k}(\theta)$ contains terms proportional to $\sin\theta\,{}_{\pm1}Y_{\ell m}(\theta,\phi)$.

4. As shown in Sec. 4.5.2, the homogeneous part of Eq. (4.38) for $\vartheta^{(1,1)}$ is separable in r and θ if one decomposes $\vartheta^{(1,1)}$ into ${}_{0}\mathcal{Y}_{\ell m}(\theta,\phi)$. Thus, to extract the radial part of Eq. (4.38), we multiply Eq. (4.78) by ${}_{0}\bar{\mathcal{Y}}_{\ell m}(\theta,\phi)$ and integrate it over the 2-sphere, utilizing the orthogonality properties of spin-weighted spheroidal harmonics in Eq. (4.65).

5. Similarly, as shown in Sec. 4.6.1, the homogeneous part of the modified Teukolsky equation for $\Psi_0^{(1,1)}$ and $\rho^{-4}\Psi_4^{(1,1)}$ (i.e., $H_0^{(0,0)}$ and $\mathcal{H}_4^{(0,0)}$) are separable in r and θ if one decomposes $\Psi_0^{(1,1)}$ and $\rho^{-4}\Psi_4^{(1,1)}$ into ${}_{2}\mathcal{Y}_{\ell m}(\theta,\phi)$ and ${}_{-2}\mathcal{Y}_{\ell m}(\theta,\phi)$, respectively. Thus, to extract the radial part of Eq. (4.18) and its GHP transformation, we multiply $\mathcal{S}^{(1,1)}$ and $\mathcal{T}^{(1,1)}$ by ${}_{2}\bar{\mathcal{Y}}_{\ell m}(\theta,\phi)$ and ${}_{-2}\bar{\mathcal{Y}}_{\ell m}(\theta,\phi)$, respectively, and integrate them over the 2-sphere.

6. Since we use the slow-rotation approximation in this work, when computing the integrals involving ${}_{s}\mathcal{Y}_{\ell m}(\theta,\phi)$, one can further expand ${}_{s}\mathcal{Y}_{\ell m}(\theta,\phi)$ in terms of ${}_{s}Y_{\ell m}(\theta,\phi)$ using Eq. (4.61). Thus, there are only spin-weighted spherical harmonics in these integrals.

7. After the angular integration, the angular functions ${}_{s}f_{\ell m}(\theta)e^{im\phi}$ in Step 3 become coefficients of the form

$$\Lambda_{s_1 s_2}^{\ell_1\ell_2 m} \equiv \int_{S^2} dS\ {}_{s_1}Y_{\ell_1 m}\ {}_{s_2}\bar{Y}_{\ell_2 m}\,, \tag{4.133a}$$

$$\Lambda_{s_1 s_2 c}^{\ell_1\ell_2 m} \equiv \int_{S^2} dS\ \cos\theta\ {}_{s_1}Y_{\ell_1 m}\ {}_{s_2}\bar{Y}_{\ell_2 m}\,, \tag{4.133b}$$

$$\Lambda_{s_1 s_2 s}^{\ell_1\ell_2 m} \equiv \int_{S^2} dS\ \sin\theta\ {}_{s_1}Y_{\ell_1 m}\ {}_{s_2}\bar{Y}_{\ell_2 m}\,, \tag{4.133c}$$

and the ${}_s\bar{f}_{\ell\, -m}(\theta)e^{im\phi}$ angular functions become coefficients of the form

$$\Lambda^{\dagger \ell_1 \ell_2 -m}_{s_1 s_2} \equiv \int_{S^2} dS \;{}_{s_1}\bar{Y}_{\ell_1 m}\;{}_{s_2}\bar{Y}_{\ell_2 -m}\,, \tag{4.134a}$$

$$\Lambda^{\dagger \ell_1 \ell_2 -m}_{s_1 s_2 c} \equiv \int_{S^2} dS \;\cos\theta\;{}_{s_1}\bar{Y}_{\ell_1 m}\;{}_{s_2}\bar{Y}_{\ell_2 -m}\,, \tag{4.134b}$$

$$\Lambda^{\dagger \ell_1 \ell_2 -m}_{s_1 s_2 s} \equiv \int_{S^2} dS \;\sin\theta\;{}_{s_1}\bar{Y}_{\ell_1 m}\;{}_{s_2}\bar{Y}_{\ell_2 -m}\,. \tag{4.134c}$$

Since spin-weighted spherical harmonics are not orthogonal across different spins over the 2-sphere, one has to calculate these coefficients in general directly. Besides evaluating the integrals in Eqs. (4.133) and (4.134) for different (s_1, ℓ_1, m) and (s_2, ℓ_2, m) every time, there are also other approaches. One approach is to use the series-sum representation of ${}_sY_{\ell m}(\theta, \phi)$ in Eq. (4.63), as discussed in Appendix 4.14 with the results stored in a Mathematica notebook in [69]. Now, the master equations of $\vartheta^{(1,1)}$ and $\Psi_{0,4}^{(1,1)}$ become completely radial, i.e.,

$${}_s\tilde{H}_{\ell m}\left[{}_s\psi_{\ell m}(r)\right] = \sum_k {}_s\mathfrak{t}^k_{\ell m}\,{}_sP^k_{\ell m}(r) + \bar{\eta}_{\ell m}\,{}_s\bar{\mathfrak{t}}^k_{\ell -m}\,{}_sQ^k_{\ell\, -m}(r)\,, \tag{4.135a}$$

$${}_s\tilde{H}_{\ell\, -m}\left[{}_s\psi_{\ell\, -m}(r)\right] = \sum_k {}_s\mathfrak{t}^k_{\ell -m}\,P^k_{\ell\, -m}(r) + \frac{1}{\eta_{\ell m}}{}_s\bar{\mathfrak{t}}^k_{\ell m}\,{}_sQ^k_{\ell\, m}(r)\,, \tag{4.135b}$$

where ${}_s\tilde{H}_{\ell m}$ is the radial Teukolsky operator for a spin s field in GR and given in [19]. The coefficient ${}_s\mathfrak{t}^k_{\ell m}$ comes from the integral of ${}_sS_{\ell m}(\theta)$ and ${}_sf^k_{\ell m}(\theta)$ over the 2-sphere, and similarly for its complex conjugate ${}_s\bar{\mathfrak{t}}^k_{\ell m}$. ${}_s\mathfrak{t}^k_{\ell m}$ and ${}_s\bar{\mathfrak{t}}^k_{\ell m}$ will be given by Eqs. (4.133) and (4.134), respectively.

4.9.2 Radial part of the equation of $\vartheta^{(1,1)}$

In this subsection, we present the radial part of Eq. (4.38) in both the IRG and ORG found by following the procedures in Sec. 4.9.1. In the IRG, we find the radial parts of terms proportional to $e^{-i\omega t}$ in Eqs. (4.95) and (4.98), respectively, to be

$$\begin{aligned} V^R_{\ell m}(r) = &\left(g_1^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + g_2^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\right) \\ &+ \chi\left(g_3^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + g_4^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{10s}\,, \end{aligned} \tag{4.136}$$

$$V^{\square}_{\ell m}(r) = \chi\left(h_1^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + h_2^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{10s}\,, \tag{4.137}$$

where the terms proportional to $b^m_{\ell,\ell\pm1}$ or $\cos\theta\,{}_0Y_{\ell m}(\theta,\phi)$ in Eq. (4.95) are at $O(\chi^2)$ after the angular integration. In the ORG, we find

$$U^R_{\ell m}(r) = \left(\mathfrak{g}_1^{\ell m}(r,\omega,M)\,{}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{g}_2^{\ell m}(r,\omega,M)\,{}_{-2}\hat{R}'_{\ell m}(r)\right)$$

$$+\chi\left(\mathfrak{g}_3^{\ell m}(r,\omega,M)\,{}_{-2}\hat{R}_{\ell m}(r)+\mathfrak{g}_4^{\ell m}(r,\omega,M)\,{}_{-2}\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{-10s}\,, \quad (4.138)$$

$$U^{\square}_{\ell m}(r)=\chi\left(\mathfrak{h}_1^{\ell m}(r,\omega,M)\,{}_{-2}\hat{R}_{\ell m}(r)+\mathfrak{h}_2^{\ell m}(r,\omega,M)\,{}_{-2}\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{-10s}\,, \quad (4.139)$$

where ${}_s\hat{R}_{\ell m}(r)$ is the radial part of the Hertz potential given in Eq. (4.55), $\Lambda^{\ell\ell m}_{10s}$ and $\Lambda^{\ell\ell m}_{-10s}$ are given by Eq. (4.133), and the prime denotes a derivative with respect to the radial coordinate r. The functions

$$\left\{g_i^{\ell m}(r,\omega,M)\,,\ h_j^{\ell m}(r,\omega,M)\,,\ \mathfrak{g}_i^{\ell m}(r,\omega,M)\,,\ \mathfrak{h}_j^{\ell m}(r,\omega,M)\right\}\,, \quad (4.140)$$

where $i\in[1,4]$ and $j\in[1,2]$, are the same functions in Eqs. (4.95), (4.98), (4.115) and (4.116) and presented in a separate Mathematica notebook [69].

Using Eq. (4.59), we can replace the radial Hertz potential ${}_s\hat{R}_{\ell m}(r)$ and its derivative in Eqs. (4.136)–(4.139) with ${}_sR^{(0,1)}_{\ell m}(r)$ and ${}_sR'^{(0,1)}_{\ell m}(r)$. Notice that the form of the equations remains similar with ${}_s\hat{R}_{\ell m}(r)$ now replaced by ${}_sR^{(0,1)}_{\ell m}(r)$ and the prefactors now new functions of $\{r,\omega,M\}$. For instance, in the first parenthesis of Eq. (4.136), one finds that the prefactor of ${}_2R^{(0,1)}_{\ell m}(r)$ is

$$g_1^{\ell m}(r,\omega,M)_2f_1^{\ell m}(r,\omega,M)+g_2^{\ell m}(r,\omega,M)_2f_3^{\ell m}(r,\omega,M)\,. \quad (4.141)$$

Each of the functions that would appear in $V^R_{\ell m}(r)$, $V^{\square}_{\ell m}(r)$, $U^R_{\ell m}(r)$, and $U^{\square}_{\ell m}(r)$ are separately presented in the supplementary Mathematica notebook due to their lengthy nature [69].

Combining Eq. (4.88) with Eqs. (4.136) and (4.137) [or Eqs. (4.138) and (4.139)], we now have a completely radial equation that describes the evolution of the scalar field perturbations,

$$\begin{aligned}\text{IRG:}\quad &\left[r(r-r_s)\partial_r^2+2(r-M)\partial_r+\frac{\omega^2r^3-4\chi mM^2\omega}{r-r_s}-{}_0A_{\ell m}\right]\Theta_{\ell m}(r)\\&=-\frac{M^2}{16\pi^{\frac{1}{2}}}r^2\left(V^R_{\ell m}(r)+\bar{\eta}_{\ell m}V^{\dagger R}_{\ell\, -m}(r)\right)-r^2\left(V^{\square}_{\ell m}(r)+\bar{\eta}_{\ell m}V^{\dagger\square}_{\ell\, -m}(r)\right)\,,\end{aligned} \quad (4.142)$$

$$\begin{aligned}\text{ORG:}\quad &\left[r(r-r_s)\partial_r^2+2(r-M)\partial_r+\frac{\omega^2r^3-4\chi mM^2\omega}{r-r_s}-{}_0A_{\ell m}\right]\Theta_{\ell m}(r)\\&=-\frac{M^2}{16\pi^{\frac{1}{2}}}r^2\left(U^R_{\ell m}(r)+\bar{\eta}_{\ell m}U^{\dagger R}_{\ell\, -m}(r)\right)-r^2\left(U^{\square}_{\ell m}(r)+\bar{\eta}_{\ell m}U^{\dagger\square}_{\ell\, -m}(r)\right)\,.\end{aligned} \quad (4.143)$$

Recall that r_s is the Schwarzschild radius, M is the mass of the BH, χ is the dimensionless spin parameter such that $\chi=a/M$ with a being the spin, ${}_0A_{\ell m}$ is the separation constant for a spin-0 field [19], and $\left\{V^R_{\ell m}(r), V^{\Box}_{\ell m}(r), U^R_{\ell m}(r), U^{\Box}_{\ell m}(r)\right\}$ are radial functions given in Eqs. (4.136)–(4.139). The constant $\bar{\eta}_{\ell m}$ is the relative coefficient between the (ℓ, m) and $(\ell, -m)$ modes of $\Psi_{0,4}^{(1,1)}$ in Eq. (4.131), of which only certain values can solve Eqs. (4.132) and (4.135) consistently. To obtain this coefficient, one has to solve Eq. (4.150) for the (ℓ, m) and $(\ell, -m)$ modes of $\Psi_0^{(1,1)}$ [or Eq. (4.154) for $\Psi_4^{(1,1)}$ in the ORG] jointly. In [65], it was shown that one can turn Eq. (4.150) [or Eq. (4.154)] into an eigenvalue problem, following [25, 39, 40], such that the solutions of $\bar{\eta}_{\ell m}$ correspond to the eigenvectors of the system, and the QNM frequencies $\omega_{\ell m}$ are eigenvalues.

In the above equation, $V^{\dagger R}_{\ell -m}$ refers to taking the complex conjugate of all the radial functions in $V^R_{\ell -m}$ but replacing $\{\Lambda^{\ell_1 \ell_2 m}_{s_1 s_2}, \Lambda^{\ell_1 \ell_2 m}_{s_1 s_2 c}, \Lambda^{\ell_1 \ell_2 m}_{s_1 s_2 s}\}$ with $\{\Lambda^{\dagger \ell_1 \ell_2 m}_{s_1 s_2}, \Lambda^{\dagger \ell_1 \ell_2 m}_{s_1 s_2 c}, \Lambda^{\dagger \ell_1 \ell_2 m}_{s_1 s_2 s}\}$, and similarly for $V^{\dagger \Box}_{\ell -m}$, $U^{\dagger R}_{\ell -m}$, and $U^{\dagger \Box}_{\ell -m}$. Equations (4.142) and (4.143) can now be solved for to obtain the scalar-led QNM frequencies. Notice that there is a coupling between the scalar field perturbations and the gravitational perturbations in GR, which appear in the form of the Hertz potential radial function ${}_{\pm 2}\hat{R}_{\ell m}(r)$ in the IRG or ORG, respectively.

4.9.3 Radial part of the equation of $\Psi_0^{(1,1)}$

In this subsection, we present the radial part of the modified Teukolsky equation for $\Psi_0^{(1,1)}$. Just like in the case of the ϑ field, the left-hand side of the modified Teukolsky equation Eq. (4.18) is the same as the Teukolsky equation in GR and is separable under the decomposition in Eq. (4.101), with its radial part given by Eqs. (4.99) and (4.100). To extract the radial part of the right-hand side, we follow the recipe provided in Sec. 4.9.1 to eliminate all angular dependence.

First, integrating $H_0^{(1,0)} \Psi_0^{(0,1)}$, where $H_0^{(1,0)}$ is given by Eq. (4.103), with ${}_2\bar{\mathcal{Y}}_{\ell m}(\theta, \phi)$, we find the radial part $\mathcal{S}^{\text{geo}}_{\ell m}(r)$ of $\mathcal{S}^{(1,1)}_{\text{geo}}$ to be

$$
\begin{aligned}
&\mathcal{S}^{\text{geo}}_{\ell m}(r) = \\
&\frac{i\chi m M^4}{448 r^9 (r - r_s)} \left(C_1(r) + 4i\omega r^2 C_2(r)\right) {}_2R^{(0,1)}_{\ell m}(r) \\
&- \frac{i\chi M^4}{16 r^9} \left[C_3(r) - C_4(r) \left(\frac{i\omega r^2}{2} + \frac{r(r - r_s)}{2}\partial_r\right)\right] {}_2R^{(0,1)}_{\ell m}(r) \Lambda^{\ell \ell m}_{22c} \\
&- \frac{i\chi M^4}{128 r^9} C_5(r)\, {}_2R^{(0,1)}_{\ell m}(r) \left(\sqrt{(\ell+2)(\ell-1)}\Lambda^{\ell\ell m}_{12s} - \sqrt{(\ell+3)(\ell-2)}\Lambda^{\ell\ell m}_{32s}\right),
\end{aligned}
\tag{4.144}
$$

where we have used Eqs. (4.67) and (4.68) to replace $\partial_\theta \left({}_2Y_{\ell m}(\theta, \phi)\right)$ with ${}_1Y_{\ell m}(\theta, \phi)$ and ${}_3Y_{\ell m}(\theta, \phi)$. In Eq. (4.144), recall once more that r_s is the Schwarzschild radius,

M is the mass of the BH, χ is the dimensionless spin parameter, $\Lambda^{\ell\ell m}_{22c}$ and $\Lambda^{\ell\ell m}_{32s}$ are given by Eqs. (4.133), and the functions C_i with $i \in [1,5]$ are presented in Eqs. (4.162).

Next, due to the structure of Eq. (4.132), multiplying Eq. (4.111a) by ${}_2\bar{\mathcal{Y}}_{\ell m}(\theta,\phi)$ and Eq. (4.111b) by ${}_2\bar{\mathcal{Y}}_{\ell -m}(\theta,\phi)$ and integrating over the 2-sphere, we find

$$\begin{aligned}
&\mathcal{S}^A_{\ell m}(r) = \\
&\left(p_1^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + p_2^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) + p_3^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r)\right) \\
&+\chi\left(p_4^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + p_5^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) + p_6^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{12s} \\
&+\chi\left(p_7^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + p_8^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) + p_9^{\ell m}(r,\omega,M)\Theta''_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{22c}\,,
\end{aligned} \tag{4.145a}$$

$$\begin{aligned}
&\tilde{\mathcal{S}}^A_{\ell m}(r) = \\
&-\left(\bar{p}_1^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{p}_2^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) + \bar{p}_3^{\ell m}(r,\omega,M)\bar{\Theta}''_{\ell m}(r)\right)(-1)^m \\
&-\chi\left(\bar{p}_4^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{p}_5^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) + \bar{p}_6^{\ell m}(r,\omega,M)\bar{\Theta}''_{\ell m}(r)\right)\Lambda^{\dagger\ell\ell -m}_{-12s} \\
&+\chi\left(\bar{p}_7^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{p}_8^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) + \bar{p}_9^{\ell m}(r,\omega,M)\bar{\Theta}''_{\ell m}(r)\right)\Lambda^{\dagger\ell\ell -m}_{-22c}\,.
\end{aligned} \tag{4.145b}$$

Here, $\mathcal{S}^A_{\ell m}(r)$ and $\tilde{\mathcal{S}}^A_{\ell m}(r)$ denote the radial part of the (ℓ,m) mode of $\mathcal{S}^{(1,1)}_A(r)$ and $\tilde{\mathcal{S}}^{(1,1)}_A(r)$, respectively. Notice that both $\mathcal{S}^A_{\ell m}(r)$ and $\tilde{\mathcal{S}}^A_{\ell -m}(r)$ contribute to the (ℓ,m) mode of the radial modified Teukolsky equation in Eq. (4.132a). The coefficient $(-1)^m$ in Eq. (4.145b) comes from that $\Lambda^{\dagger\ell\ell m}_{-s\,s} = (-1)^{m+s}$ since ${}_{-s}\bar{Y}_{\ell -m}(\theta,\phi) = (-1)^{m+s}{}_sY_{\ell m}(\theta,\phi)$. The terms proportional to ${}_0b^m_{\ell,\ell\pm1}$ in Eqs. (4.111a) and (4.111b) are at $O(\chi^2)$ after the angular integration. We can further use Eq. (4.142) to rewrite $\Theta''_{\ell m}(r)$ in term of $\Theta_{\ell m}(r)$, $\Theta'_{\ell m}(r)$, ${}_2\hat{R}_{\ell m}(r)$, and ${}_2\hat{R}'_{\ell m}(r)$ such that

$$\begin{aligned}
\mathcal{S}^A_{\ell m}(r) = &\left(k_1^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + k_2^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r)\right. \\
&\left.+k_3^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + k_4^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\right) \\
&+\chi\left(k_5^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + k_6^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r)\right. \\
&\left.+k_7^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + k_8^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{12s} \\
&+\chi\left(k_9^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + k_{10}^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r)\right. \\
&\left.+k_{11}^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r) + k_{12}^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{22c}\,,
\end{aligned} \tag{4.146a}$$

$$\tilde{\mathcal{S}}^A_{\ell m}(r) = -\left(\bar{k}_1^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{k}_2^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r)\right.$$

$$
\begin{aligned}
&+\bar{k}_3^{\ell m}(r,\omega,M)\,{}_2\bar{\hat{R}}_{\ell m}(r)+\bar{k}_4^{\ell m}(r,\omega,M)\,{}_2\bar{\hat{R}}'_{\ell m}(r)\Big)\,(-1)^m\\
&-\chi\left(\bar{k}_5^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r)+\bar{k}_6^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r)\right.\\
&+\bar{k}_7^{\ell m}(r,\omega,M)\,{}_2\bar{\hat{R}}_{\ell m}(r)+\bar{k}_8^{\ell m}(r,\omega,M)\,{}_2\bar{\hat{R}}'_{\ell m}(r)\Big)\,\Lambda^{\dagger\ell\ell-m}_{-12s}\\
&+\chi\left(\bar{k}_9^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r)+\bar{k}_{10}^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r)\right.\\
&+\bar{k}_{11}^{\ell m}(r,\omega,M)\,{}_2\bar{\hat{R}}_{\ell m}(r)+\bar{k}_{12}^{\ell m}(r,\omega,M)\,{}_2\bar{\hat{R}}'_{\ell m}(r)\Big)\,\Lambda^{\dagger\ell\ell-m}_{-22c}\,,
\end{aligned}
\tag{4.146b}
$$

where some of the radial functions $k_i^{\ell m}(r,\omega,M)$ are

$$
\begin{aligned}
k_1^{\ell m}(r,\omega,M)=&-\frac{1}{r^7(r-r_s)^2}6i\sqrt{\pi(\ell+2)!/(\ell-1)!}M^3\left[M^2(8+2m\chi(3i+2\omega r))\right.\\
&\left.-\,2Mr(4+{}_0A_{\ell m}+i\omega r+m\chi(2i-\omega r))+r^2\left(2+{}_0A_{\ell m}+2ir\omega-2r^2\omega^2\right)\right],
\end{aligned}
\tag{4.147a}
$$

$$
k_2^{\ell m}(r,\omega,M)=\frac{12\sqrt{\pi(\ell+2)!/(\ell-1)!}M^3[M(m\chi-3i)+r(2i-\omega r)]}{r^6(r-r_s)},
\tag{4.147b}
$$

while the remaining functions $k_i^{\ell m}(r,\omega,M)$ for $i\in[3,12]$ are provided in [69]. Recall that $\bar{k}_i^{\ell m}(r,\omega,M)$ are the complex conjugates of $k_i^{\ell m}(r,\omega,M)$.

Similarly, projecting Eqs. (4.112a) and (4.112b) into the radial direction, we find

$$
\begin{aligned}
\mathcal{S}^B_{\ell m}(r)=\chi\Big[&\left(q_1^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r)+q_2^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{12s}\\
&+\left(q_3^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r)+q_4^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{22c}\\
&+\left(q_5^{\ell m}(r,\omega,M)\,{}_2\hat{R}_{\ell m}(r)+q_6^{\ell m}(r,\omega,M)\,{}_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{32s}\Big]\,,
\end{aligned}
\tag{4.148a}
$$

$$
\begin{aligned}
\tilde{\mathcal{S}}^B_{\ell m}(r)=\chi\Big[&\left(\tilde{q}_1^{\ell m}(r,\omega,M)\bar{\hat{R}}_{\ell m}(r)+\tilde{q}_2^{\ell m}(r,\omega,M)\,{}_2\bar{\hat{R}}'_{\ell m}(r)\right)\Lambda^{\dagger\ell\ell-m}_{-12s}\\
&+\tilde{q}_3^{\ell m}(r,\omega,M)\,{}_2\bar{\hat{R}}_{\ell m}(r)\Lambda^{\dagger\ell\ell-m}_{-22c}\Big]\,.
\end{aligned}
\tag{4.148b}
$$

The functions ${}_s\hat{R}_{\ell m}(r)$ and ${}_s\bar{\hat{R}}_{\ell m}(r)$ are the radial parts of the Hertz potential [i.e., Eqs. (4.55)] and its complex conjugate, respectively. Prime denotes a derivative with respect to the radial coordinate r. The coefficients $\left\{\Lambda^{\ell\ell m}_{12s},\Lambda^{\ell\ell m}_{22c},\Lambda^{\ell\ell m}_{32s}\right\}$ and $\left\{\Lambda^{\dagger\ell\ell-m}_{-12s},\Lambda^{\dagger\ell\ell-m}_{-22c}\right\}$ are given by Eqs. (4.133) and (4.134), respectively. Due to the complicated functional form of $q_i^{\ell m}(r,\omega,M)$ with $i\in[1,6]$, we have presented them in a separate Mathematica notebook [69]. The radial functions $\tilde{q}_i^{\ell m}(r,\omega,M)$ are given by

$$
\tilde{q}_1^{\ell m}(r,\omega,M)=\frac{15i\ell(\ell+1)\sqrt{\ell^2+\ell-2}M^4\left(18M^2+5Mr+r^2\right)(6M+r(-3+ir\omega))}{4r^{12}(r-r_s)},
\tag{4.149a}
$$

$$\tilde{q}_2^{\ell m}(r,\omega,M) = \frac{15i\ell(\ell+1)\sqrt{\ell^2+\ell-2}M^4\left(18M^2+5Mr+r^2\right)}{4r^{11}}\,, \tag{4.149b}$$

$$\tilde{q}_3^{\ell m}(r,\omega,M) = -\frac{15i(\ell-1)\ell(\ell+1)(\ell+2)M^4\left(54M^2+10Mr+r^2\right)}{16r^{12}}\,. \tag{4.149c}$$

Combining Eqs. (4.18), (4.99), (4.100), (4.144), (4.146), and (4.148), the modified master equation for the radial part of the $\Psi_0^{(1,1)}$ Weyl scalar is

$$\begin{aligned}&\left[r(r-r_s)\partial_r^2+6(r-M)\partial_r+\frac{C(r)}{r-r_s}+\frac{4m\chi M(i(r-M)-M\omega r)}{r(r-r_s)}-{}_2A_{\ell m}\right]{}_2R_{\ell m}^{(1,1)}(r)\\&=-2r^2\left[\mathcal{S}_{\ell m}^{\text{geo}}(r)+\left(\mathcal{S}_{\ell m}^{A}(r)+\tilde{\mathcal{S}}_{\ell -m}^{A}(r)\right)+\left(\mathcal{S}_{\ell m}^{B}(r)+\bar{\eta}_{\ell m}\tilde{\mathcal{S}}_{\ell -m}^{B}(r)\right)\right]\,,\end{aligned} \tag{4.150}$$

where $C(r)$ is given by Eq. (4.100c), $\mathcal{S}_{\ell m}^{\text{geo}}(r)$ is given in Eq. (4.144), $\mathcal{S}_{\ell m}^{A}(r)$ and $\tilde{\mathcal{S}}_{\ell -m}^{A}(r)$ are given by Eqs. (4.146), whereas $\mathcal{S}_{\ell m}^{B}(r)$ and $\tilde{\mathcal{S}}_{\ell -m}^{B}(r)$ are given by Eqs. (4.148). Notice that there is no $\bar{\eta}_{\ell m}$ in front of $\tilde{\mathcal{S}}_{\ell -m}^{A}(r)$ since $\mathcal{S}_{\ell m}^{A}(r)$ and $\tilde{\mathcal{S}}_{\ell -m}^{A}(r)$ implicitly depend on $\bar{\eta}_{\ell m}$ via ϑ [i.e., Eq. (4.142)]. Furthermore, one needs to solve the (ℓ, m) and $(\ell, -m)$ modes of Eq. (4.150) jointly, from which one can then obtain the coefficient $\bar{\eta}_{\ell m}$ between these two modes defined in Eq. (4.131) and the QNM frequency $\omega_{\ell m}$[2], as we will work out in [58] following the procedures in [65].

Using Eq. (4.59), the radial part ${}_2\hat{R}_{\ell m}(r)$ of the Hertz potential $\Psi_{\rm H}$ in Eqs. (4.146) and (4.148) can be further expressed as functions of the radial Teukolsky function ${}_2R_{\ell m}^{(0,1)}(r)$ for the perturbed Ψ_0 in GR [19], as discussed in Sec. 4.9.2. All necessary functions have been provided in a supplementary Mathematica notebook due to their lengthy nature [69]. One can readily use existing wavefunction ansatz in the literature to evaluate the radial Teukolsky function [22, 73].

Notice that the modified Teukolsky equation for the Weyl scalar perturbation $\Psi_0^{(1,1)}$ exhibits coupling to the scalar field perturbation at $O(\zeta^1,\epsilon^1)$, but no such coupling is seen for the scalar field perturbation at the same order, unlike the case involving metric perturbations [9].

4.9.4 Radial part of the equations of $\Psi_4^{(1,1)}$

In this subsection, we present the radial part of the modified Teukolsky equation for the Weyl scalar perturbation $\Psi_4^{(1,1)}$ for a slowly rotating BH in dCS gravity. Similar

[2]The coefficient $\bar{\eta}_{\ell -m}$ and the QNM frequency $\omega_{\ell -m}$ are redundant with $\bar{\eta}_{\ell m}$ and $\omega_{\ell m}$, respectively, since we solve the (ℓ, m) and $(\ell, -m)$ modes jointly. More specifically, from Eq. (4.132) and a more detailed discussion in [65], one can find that $\bar{\eta}_{\ell -m} = 1/(\bar{\eta}_{\ell m})$ when $\bar{\eta}_{\ell m} \neq 0$ and $\omega_{\ell -m} = -\bar{\omega}_{\ell m}$.

to the case studied in above Sec. 4.9.3, the left-hand side of the modified Teukolsky equation for $\Psi_4^{(1,1)}$ in Eq. (4.26) holds the same form as the left-hand side of the Teukolsky equation for $\Psi_4^{(0,1)}$ [19]. First, multiply $H_4^{(1,0)}\Psi_4^{(0,1)} = \mathcal{H}_4^{(1,0)}\psi_4^{(0,1)}$ by ${}_{-2}\bar{\mathcal{Y}}_{\ell m}(\theta,\phi)$, with $\mathcal{H}_4^{(1,0)}$ given in Eq. (4.123), and integrate over the 2-sphere

$$\begin{aligned}
&\mathcal{T}^{\rm geo}_{\ell m}(r) = \\
&\frac{-i\chi m M^4}{448 r^{13}(r-r_s)}\left(D_1(r) - 4i\omega r^2 D_2(r)\right){}_{-2}R^{(0,1)}_{\ell m}(r) \\
&+ \frac{i\chi M^4}{16 r^{13}}\left[D_3(r) - D_4(r)\left(\frac{i\omega r^2}{2} - \frac{r(r-r_s)}{2}\partial_r\right)\right]{}_{-2}R^{(0,1)}_{\ell m}(r)\Lambda^{\ell\ell m}_{-2-2c} \\
&+ \frac{i\chi M^4}{128 r^{13}} D_5(r)\,{}_{-2}R^{(0,1)}_{\ell m}(r)\left(\sqrt{(\ell+2)(\ell-1)}\Lambda^{\ell\ell m}_{-1-2s} - \sqrt{(\ell+3)(\ell-2)}\Lambda^{\ell\ell m}_{-3-2s}\right),
\end{aligned} \tag{4.151}$$

where ${}_{-2}R^{(0,1)}_{\ell m}(r)$ is the radial function of $\rho^{-4}\Psi_4^{(0,1)}$ presented in Eq. (4.119), $\{\Lambda^{\ell\ell m}_{-1-2s}, \Lambda^{\ell\ell m}_{-3-2s}, \Lambda^{\ell\ell m}_{-2-2c}\}$ are given in Eqs. (4.133), and $D_i(r)$ for $i \in [1,5]$ are presented in Eqs. (4.163).

Next, we multiply Eq. (4.126a) by ${}_{-2}\bar{\mathcal{Y}}_{\ell m}(\theta,\phi)$ and Eq. (4.126b) by ${}_{-2}\bar{\mathcal{Y}}_{\ell -m}(\theta,\phi)$ and integrate over the 2-sphere. We also make use of Eq. (4.143) to decompose the $\Theta''_{\ell m}(r)$ dependence in terms of $\Theta_{\ell m}(r)$, $\Theta'_{\ell m}(r)$, ${}_{-2}\hat{R}_{\ell m}(r)$, and ${}_{-2}\hat{R}'_{\ell m}(r)$ such that

$$\begin{aligned}
\mathcal{T}^A_{\ell m} = &\Big(\mathfrak{k}_1^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + \mathfrak{k}_2^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
&+\mathfrak{k}_3^{\ell m}(r,\omega,M){}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{k}_4^{\ell m}(r,\omega,M){}_{-2}\hat{R}'_{\ell m}(r)\Big) \\
&+\chi\Big(\mathfrak{k}_5^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + \mathfrak{k}_6^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
&+\mathfrak{k}_7^{\ell m}(r,\omega,M){}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{k}_8^{\ell m}(r,\omega,M){}_{-2}\hat{R}'_{\ell m}(r)\Big)\Lambda^{\ell\ell m}_{-1-2s} \\
&+\chi\Big(\mathfrak{k}_9^{\ell m}(r,\omega,M)\Theta_{\ell m}(r) + \mathfrak{k}_{10}^{\ell m}(r,\omega,M)\Theta'_{\ell m}(r) \\
&+\mathfrak{k}_{11}^{\ell m}(r,\omega,M){}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{k}_{12}^{\ell m}(r,\omega,M){}_{-2}\hat{R}'_{\ell m}(r)\Big)\Lambda^{\ell\ell m}_{-2-2c}\,,
\end{aligned} \tag{4.152a}$$

$$\begin{aligned}
\tilde{\mathcal{T}}^A_{\ell m} = &-\Big(\bar{\mathfrak{k}}_1^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{\mathfrak{k}}_2^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) \\
&+\bar{\mathfrak{k}}_3^{\ell m}(r,\omega,M){}_{-2}\bar{\hat{R}}_{\ell m}(r) + \bar{\mathfrak{k}}_4^{\ell m}(r,\omega,M){}_{-2}\bar{\hat{R}}'_{\ell m}(r)\Big)(-1)^m \\
&-\chi\Big(\bar{\mathfrak{k}}_5^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{\mathfrak{k}}_6^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) \\
&+\bar{\mathfrak{k}}_7^{\ell m}(r,\omega,M){}_{-2}\bar{\hat{R}}_{\ell m}(r) + \bar{\mathfrak{k}}_8^{\ell m}(r,\omega,M){}_{-2}\bar{\hat{R}}'_{\ell m}(r)\Big)\Lambda^{\dagger\ell\ell -m}_{1-2s} \\
&+\chi\Big(\bar{\mathfrak{k}}_9^{\ell m}(r,\omega,M)\bar{\Theta}_{\ell m}(r) + \bar{\mathfrak{k}}_{10}^{\ell m}(r,\omega,M)\bar{\Theta}'_{\ell m}(r) \\
&+\bar{\mathfrak{k}}_{11}^{\ell m}(r,\omega,M){}_{-2}\bar{\hat{R}}_{\ell m}(r) + \bar{\mathfrak{k}}_{12}^{\ell m}(r,\omega,M){}_{-2}\bar{\hat{R}}'_{\ell m}(r)\Big)\Lambda^{\dagger\ell\ell -m}_{2-2c}\,,
\end{aligned} \tag{4.152b}$$

where we recall that $\Theta_{\ell m}(r)$ is the radial part of the scalar field perturbation, ${}_{-2}\hat{R}_{\ell m}(r)$ is the radial part of the Hertz potential in the ORG, prime denotes a derivative with respect to the radial coordinate r, and an overhead bar denotes complex conjugation. The constants Λ and $\Lambda^{\dagger}$ are given by Eqs. (4.133) and Eqs. (4.134), respectively, with the relevant subscripts and superscripts. The functions $\mathfrak{k}_i^{\ell m}(r,\omega,M)$ and $\bar{\mathfrak{k}}_i^{\ell m}(r,\omega,M)$ are given in a Mathematica notebook as supplementary material [69]. Similarly, the source terms in Eq. (4.127) can be decomposed into a radial equation as

$$\begin{aligned}
\mathcal{T}^{B}_{\ell m} = \chi \Big[& \Big(\mathfrak{q}_1^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{q}_2^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}'_{\ell m}(r)\Big)\, \Lambda^{\ell\ell m}_{-1-2s} \\
& + \Big(\mathfrak{q}_3^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{q}_4^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}'_{\ell m}(r)\Big)\, \Lambda^{\ell\ell m}_{-2-2c} \\
& + \Big(\mathfrak{q}_5^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{q}_6^{\ell m}(r,\omega,M)\, {}_{-2}\hat{R}'_{\ell m}(r)\Big)\, \Lambda^{\ell\ell m}_{-3-2s}\Big]\,, \qquad (4.153a)
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{T}}^{B}_{\ell m} = \chi \Big[& \Big(\tilde{\mathfrak{q}}_1^{\ell m}(r,\omega,M)\, {}_{-2}\bar{\hat{R}}_{\ell m}(r) + \tilde{\mathfrak{q}}_2^{\ell m}(r,\omega,M)\, {}_{-2}\bar{\hat{R}}'_{\ell m}(r)\Big)\, \Lambda^{\dagger\ell\ell -m}_{1-2s} \\
& + \tilde{\mathfrak{q}}_3^{\ell m}(r,\omega,M)\, {}_{-2}\bar{\hat{R}}_{\ell m}(r)\Lambda^{\dagger\ell\ell -m}_{2-2c}\Big]\,. \qquad (4.153b)
\end{aligned}$$

Using Eq. (4.59), the radial function ${}_{-2}\hat{R}_{\ell m}(r)$ of the Hertz potential $\Psi_{\rm H}$ in Eqs. (4.152) and (4.153) can be expressed in terms of the radial Teukolsky function ${}_{-2}R^{(0,1)}_{\ell m}$ of $\rho^{-4}\Psi_4^{(0,1)}$ in GR, as described in Sec. 4.9.2. All necessary functions have been provided in a supplementary Mathematica notebook [69]. Combining Eqs. (4.26), (4.117), (4.120), and (4.151)–(4.153), we find

$$\begin{aligned}
&\left[r(r-r_s)\partial_r^2 - 2(r-M)\partial_r + \frac{D(r)}{r-r_s} - \frac{4m\chi M(i(r-M)+M\omega r)}{r(r-r_s)} - {}_{-2}A_{\ell m}\right] {}_{-2}R^{(1,1)}_{\ell m}(r) \\
&= -2r^6\left[\mathcal{T}^{\rm geo}_{\ell m}(r) + \Big(\mathcal{T}^{A}_{\ell m}(r) + \tilde{\mathcal{T}}^{A}_{\ell -m}(r)\Big) + \Big(\mathcal{T}^{B}_{\ell m}(r) + \bar{\eta}_{\ell m}\tilde{\mathcal{T}}^{B}_{\ell -m}(r)\Big)\right]\,. \qquad (4.154)
\end{aligned}$$

As before, one has to solve the (ℓ, m) and $(\ell, -m)$ modes of Eq. (4.154) jointly to obtain the coefficient $\bar{\eta}_{\ell m}$ in Eq. (4.131) and the QNM frequency $\omega_{\ell m}$. This analysis shows that the modified Teukolsky equations for the Weyl scalar perturbations $\Psi^{(1,1)}_{0,4}$ and the scalar field perturbation $\vartheta^{(1,1)}$ can be separated into a radial and angular piece. The radial piece can then be integrated numerically to obtain the QNM frequencies.

4.10 Discussion

In this paper, we have employed the modified Teukolsky formalism in [24] and laid down the foundations to investigate the perturbations of slowly rotating BHs in dCS gravity at leading order in spin, where the BH spacetime is non-Ricci-flat, but remains of Petrov type D. In this work, we have calculated the master equations that describe the perturbations of gravitational and scalar fields. To incorporate the slow-rotation approximation, we first extended the two-parameter expansion in [24] to a three-parameter expansion. Following [42, 64], we then re-derived the null geodesics on the equatorial plane, from which we found the NP tetrad for slowly rotating BHs in dCS gravity up to $O(\chi)$. The resulting tetrad is the Kinnersly tetrad expanded to $O(\chi)$, with an additional adjustment accounting for the dCS correction. This tetrad is the same as the one in [64]. Since BHs in dCS gravity are non-Ricci-flat, this direct extension of the Kinnersly tetrad leads to some nonzero background Weyl scalars Ψ_1 and Ψ_3, so we performed additional tetrad rotations to remove them and computed all the background NP quantities in this rotated tetrad.

The source terms of the modified Teukolsky equation for $\Psi_{0,4}$ arise from two distinct contributions. Some originate from the homogeneous component of certain Bianchi and Ricci identities, so they only rely on the corrections to the background geometry. For Petrov type D spacetimes, these stationary corrections only couple to the perturbations of $\Psi_{0,4}$, so we evaluated them using the NP quantities in the dCS background and the solutions to the Teukolsky equation in GR. The other source terms stem from the stress tensor associated with corrections to the Einstein-Hilbert action. In dCS gravity, these source terms couple the scalar field with the metric in GR. Thus, to completely evaluate them, we need to solve for the dynamical scalar field. In this case, we first evaluated the scalar field equation and used the same methodology to guide our calculations for $\Psi_{0,4}$.

Since the scalar field is driven by dynamical metric perturbations in GR, one needs to first reconstruct the metric associated with curvature perturbations in GR. In this work, we chose to follow the CCK procedures developed in [34–37, 59–63], where the perturbed metric is obtained from the Hertz potential, though other procedures in [38, 42] may also apply. Using the reconstructed metric in [34–37, 59–63], we then computed all the perturbed NP quantities in GR following the approach in [38, 66]. Since we also chose the gauge that the perturbations of $\Psi_{1,3}$ vanish in both GR and dCS gravity [24], we performed additional tetrad rotations to transform all the perturbed NP quantities into this gauge. In the end, projecting the scalar field equation onto the NP basis, we used the reconstructed NP quantities to express all the source terms as differential operators acting on the Hertz potential.

The Hertz potential can be obtained from the perturbations of $\Psi_{0,4}$ in GR, which are solutions to the Teukolsky equations in GR. Decomposing the Hertz potential into spin-weighted spheroidal harmonics, we presented the source terms of the scalar field equation in Boyer-Lindquist coordinates explicitly. The radial function of the

Hertz potential was then determined from the radial function of the perturbed $\Psi_{0,4}$ in GR following [61].

In the IRG, the above steps led to three coupled, partial differential equations for the (ℓ, m) and $(\ell, -m)$ modes of the Weyl scalar perturbation in dCS $\Psi_0^{(1,1)}$ and the (ℓ, m) mode of the scalar field perturbation $\vartheta^{(1,1)}$ [the (ℓ, m) and $(\ell, -m)$ modes of $\vartheta^{(1,1)}$ are redundant since $\vartheta^{(1,1)}$ is real], that we refer to as master equations. Similarly, in the ORG, we obtained three coupled, partial differential equations for the (ℓ, m) and $(\ell, -m)$ modes of the Weyl scalar perturbation in dCS $\Psi_4^{(1,1)}$ and the (ℓ, m) mode of $\vartheta^{(1,1)}$. More explicitly, the master equation of $\Psi_0^{(1,1)}$ (or $\Psi_4^{(1,1)}$) consists of the GR Teukolsky operator for a spin 2 (or spin -2) field acting on $\Psi_0^{(1,1)}$ (or $\Psi_4^{(1,1)}$), as well as a source term that depends on $\vartheta^{(1,1)}$ and the Weyl scalar perturbation in GR $\Psi_0^{(0,1)}$ (or $\Psi_4^{(0,1)}$). Similarly, the master equation of $\vartheta^{(1,1)}$ consists of the GR Teukolsky operator for a scalar field acting on $\vartheta^{(1,1)}$ and a source term that depends on $\Psi_0^{(0,1)}$ (or $\Psi_4^{(0,1)}$).

To separate these master equations into radial and angular *ordinary* differential equations, we exploited the orthogonality properties of spin-weighted spheroidal harmonics and performed a harmonic decomposition to eliminate all angular dependence of the source terms. The homogeneous part of the scalar field equation naturally separates, so we obtain a purely radial differential equation [i.e., Eq. (4.142) in the IRG and Eq. (4.143) in the ORG]. Similar procedures were then implemented for the modified Teukolsky equations of $\Psi_{0,4}$. The source terms of the modified Teukolsky equations were expressed in terms of the Hertz potential and the dynamical scalar field. We then projected the source terms into the radial direction by integrating them over spin-weighted spheroidal harmonics. The homogeneous part of these equations separates in the same way as the Teukolsky equations in GR, so we also obtained two radial differential equations for Ψ_0 [i.e., Eq. (4.150)] and Ψ_4 [i.e., Eq. (4.154)], respectively.

Through these procedures, we obtained three coupled, ordinary (radial) differential equations for $\left\{{}_2R_{\ell m}^{(1,1)}(r),\ {}_2R_{\ell -m}^{(1,1)}(r),\ \Theta_{\ell m}^{(1,1)}(r)\right\}$ (or $\left\{{}_{-2}R_{\ell m}^{(1,1)}(r),\ {}_{-2}R_{\ell -m}^{(1,1)}(r),\ \Theta_{\ell m}^{(1,1)}(r)\right\}$ in the ORG), where the first two are radials functions of $\Psi_0^{(1,1)}$ (or $\rho^{-4}\Psi_4^{(1,1)}$), and the last one is the radial function of $\vartheta^{(1,1)}$. All of these equations have the same structure. The left-hand side is the radial Teukolsky operator for particles of spin 2 ($\Psi_0^{(1,1)}$), spin -2 ($\Psi_4^{(1,1)}$), or spin 0 ($\vartheta^{(1,1)}$). For the radial master equation of $\Psi_0^{(1,1)}$ (or $\Psi_4^{(1,1)}$), the right-hand side contains source terms that depend on $\left\{\Theta_{\ell m}^{(1,1)}(r),\ {}_2R_{\ell m}^{(0,1)}(r),\ {}_2R_{\ell -m}^{(0,1)}(r)\right\}$ (or $\left\{\Theta_{\ell m}^{(1,1)}(r),\ {}_{-2}R_{\ell m}^{(0,1)}(r),\ {}_{-2}R_{\ell -m}^{(0,1)}(r)\right\}$), where the last two are radial functions of $\Psi_0^{(0,1)}$ (or $\rho^{-4}\Psi_4^{(0,1)}$). For the radial master equation of $\vartheta^{(1,1)}$, the right-hand side contains source terms that depend on $\left\{{}_2R^{(0,1)}_{\ell m}(r),\ {}_2R^{(0,1)}_{\ell -m}(r)\right\}$ (or $\left\{{}_{-2}R^{(0,1)}_{\ell m}(r),\ {}_{-2}R^{(0,1)}_{\ell -m}(r)\right\}$). The coupled system therefore forms a (Sturm-Liouville) eigenvalue problem that should be amenable to standard procedures to find the eigenvectors and eigenvalues, i.e., the QNM and scalar frequencies.

The primary objective of this study was to apply the modified Teukolsky formalism in [24] to investigate perturbations of BHs in some specific modified theories of gravity. To illustrate this, we considered the case of slowly rotating BHs to leading order in spin within the framework of dCS gravity. Although the slow rotation approximation may not provide highly accurate results for more realistic BHs (with spins $\chi \sim 0.6$), it is a simplified problem for testing the newly developed formalism. Incorporating additional degrees of freedom associated with dCS gravity, coupled with the intricacies introduced by the metric reconstruction procedures, renders this calculation complex. Yet in this work, we successfully demonstrated that the modified Teukolsky equation in [24] does not only decouple Weyl scalars $\Psi_{0,4}$ from other NP quantities but also admits a separation into radial and angular parts, a key advantage of the Teukolsky equation in GR, especially for rapidly rotating BHs. Although this paper focused on the first order in the slow rotation expansion, the separation of the modified Teukolsky equation should hold for any spin since the orthogonality properties of spin-weighted spheroidal harmonics we have used to separate the equation apply for a general spin. Thus, this calculation is an ideal initial step toward determining the QNM spectra for BHs with general spin in modified gravity.

This work creates a new path to directly calculate the corrections to the QNM frequencies for slowly rotating perturbed BHs in dCS gravity. Having obtained the master equations for the perturbed Weyl scalars $\Psi_{0,4}$ and the perturbed scalar field, we can now integrate these equations using numerical integration schemes, such as the eigenvalue perturbation method in [25, 39, 40] to find the QNM spectra. Moreover, the QNM spectra obtained using the modified Teukolsky formalism can then be compared to the results from the metric perturbation approach [6, 7, 9, 32, 74] and numerical relativity [75–78]. Notice that, in higher-derivative gravity, Refs. [26, 27] have followed our formalism to compute the QNMs in the slow-rotation expansion of BHs in that theory, and they obtained results valid for $\chi \lesssim 0.7$. Nonetheless, the dCS case we have focused on is more complicated due to the coupling to the scalar field equation. As discussed above, we have also presented in detail the angular dependence of the master equations of the perturbed $\Psi_{0,4}$ and ϑ and showed explicitly that they are separable, while Refs. [26, 27] only briefly discussed using the orthogonality of spin-weighted spheroidal harmonics to extract the radial equations.

An additional aspect worth exploring is the phenomenon of isospectrality breaking in the QNM spectra. In GR, odd and even parity modes oscillate and decay at the same rate. However, certain modified theories of gravity have been shown to exhibit a breaking of isospectrality, e.g., dCS gravity [6, 7, 9, 32, 74], EdGB gravity [8, 10, 79], and higher-derivative gravity [55]. The investigation of isospectrality breaking has, so far, primarily focused on metric perturbations, as the Zerilli-Moncrief and Regge-Wheeler equations naturally separate metric perturbations into even- and odd-parity sectors [15, 16]. However, for BHs with arbitrary spin, there are no known extensions of the Zerilli-Moncrief and the Regge-Wheeler equations, so we need to use the modified Teukolsky equation to study isospectrality breaking. In another study [65] involving all the authors, the definite-parity modes of curvature perturbations in modified gravity were found, and the features in these bGR theories that result in isospectrality breaking were revealed and demonstrated in several simple cases. Nonetheless, a direct mapping from the Zerilli-Moncrief and Regge-Wheeler functions to the modified Teukolsy equations of these definite-parity modes still remains unknown. The implementation of the modified Teukolsky equation in a concrete bGR theory has opened up possibilities for addressing these questions and more.

Building upon the insights gained from the present study, further investigations can be pursued involving more complex systems within various gravitational theories. As part of our collaborative effort, we are currently engaged in extending this calculation to derive the master equations and QNM spectra for BHs with arbitrary spin in dCS gravity, where the BH spacetime is Petrov type I. In addition, we are also actively involved in computing the master equations for rotating Petrov type I BHs within the framework of EdGB gravity. For the first time, we can explore the QNM spectra for BHs with general spin in a wide range of gravitational theories and spacetime geometries, which can then be compared with real observation data to scrutinize these possible deviations from GR.

4.11 Appendix: Principal tetrad, spin coefficients, and some auxiliary functions

In Sec. 4.3.4, we performed tetrad rotations to set $\Psi_{1,3}^{(1,0)} = 0$. As discussed in [42], these tetrad rotations preserving the orthogonality conditions of the NP tetrad can be divided into three types:

$$\mathrm{I}: l \to l\,,\ m \to m + al\,,\ \bar{m} \to \bar{m} + \bar{a}l\,,\ n \to n + \bar{a}m + a\bar{m} + a\bar{a}l\,. \tag{4.155a}$$

$$\mathrm{II}: n \to n\,,\ m \to m + bn\,,\ \bar{m} \to \bar{m} + \bar{b}n\,,\ l \to l + \bar{b}m + b\bar{m} + b\bar{b}n\,. \tag{4.155b}$$

$$\mathrm{III}: l \to A^{-1}l\,,\ n \to An\,,\ m \to e^{i\varphi}m\,,\ \bar{m} \to e^{-i\varphi}\bar{m}\,, \tag{4.155c}$$

where a and b are complex functions while A and φ are real functions. The transformations of Weyl scalars and spin coefficients under the tetrad rotations in Eq. (4.155) can be found in [42]. The tetrad rotations above are precise, but when these rotation parameters are small, for example at $O(\zeta^1, \epsilon^0)$, the rotations of the tetrad at $O(\zeta^1, \epsilon^0)$ simplify into

$$\begin{aligned}
l^{(1,0)} &\to l^{(1,0)} + \bar{b}^{(1,0)}m + b^{(1,0)}\bar{m} - \delta A^{(1,0)}l\,,\\
n^{(1,0)} &\to n^{(1,0)} + \bar{a}^{(1,0)}m + a^{(1,0)}\bar{m} + \delta A^{(1,0)}n\,,\\
m^{(1,0)} &\to m^{(1,0)} + a^{(1,0)}l + b^{(1,0)}n + i\varphi^{(1,0)}m\,,
\end{aligned} \tag{4.156}$$

where we defined $\delta A = A - 1$ and combined the three types of tetrad rotations. Then, the Weyl scalars at $O(\zeta^1, \epsilon^0)$ transform as

$$\begin{aligned}
\Psi_{0,2,4}^{(1,0)} &\to 0\,,\\
\Psi_1^{(1,0)} &\to \Psi_1^{(1,0)} + 3b^{(1,0)}\Psi_2\,,\\
\Psi_3^{(1,0)} &\to \Psi_3^{(1,0)} + 3\bar{a}^{(1,0)}\Psi_2\,,
\end{aligned} \tag{4.157}$$

where we used that the background at $O(\zeta^0, \epsilon^0)$ is Petrov type D, so $\Psi_{0,1,3,4}^{(0,0)} = 0$. Since the spin coefficients at $O(\zeta^1, \epsilon^0)$ after the rotations can be easily computed from the rotated tetrad, e.g., Eq. (4.158) in this work, we do not provide their general transformations under the tetrad rotations here.

Using the tetrad rotations in Eq. (4.155) and the results in Eq. (4.157), we set $\Psi_{1,3}^{(1,0)} = 0$ and found the principal tetrad to be

$$l^\mu = \left(\frac{r}{r - r_s},\ 1,\ 0,\ \frac{\chi M}{r(r - r_s)} + \frac{\zeta\chi\tilde{G}(r)}{2r(r - r_s)} - \frac{\zeta\chi A_1(r)}{16Mr^7}\right), \tag{4.158a}$$

$$n^\mu = \tilde{N}(r)\left(\frac{r}{r - r_s},\ -1,\ 0,\ \frac{\chi M}{r(r - r_s)} + \frac{\zeta\chi\tilde{G}(r)}{2r(r - r_s)} + \frac{\zeta\chi A_3(r)}{16Mr^7(r - r_s)}\right), \tag{4.158b}$$

$$m^\mu = \frac{1}{\sqrt{2}r}\Bigg(i\chi M\left(1+\zeta\frac{A_3(r)-A_1(r)(r-r_s)}{32M^2r^5(r-r_s)}\right)\sin\theta\,,$$
$$i\zeta\chi\left(\frac{A_3(r)+A_1(r)(r-r_s)}{32Mr^6}\right)\sin\theta\,,\ 1-\frac{i\chi M\cos\theta}{r}\,,\ i\left(1-\frac{i\chi M\cos\theta}{r}\right)\csc\theta\Bigg)\,, \tag{4.158c}$$

where

$$A_1(r) = 126M^7 + 60M^6r + 25M^5r^2\,, \tag{4.159a}$$
$$A_2(r) = 18M^7 + 10M^6r + 5M^5r^2\,, \tag{4.159b}$$
$$A_3(r) = 252M^8 - 6M^7r - 10M^6r^2 - 25M^5r^3\,. \tag{4.159c}$$

In the principal tetrad Eq. (4.158), the spin coefficients at background are

$$\begin{aligned}
\sigma &= \lambda = 0\,,\\
\kappa &= -\frac{4r^2}{(r-r_s)^2}\nu = -\frac{15i\zeta\chi B_1(r)}{16\sqrt{2}r^7}\sin\theta\,,\\
\rho &= \frac{2r}{r-r_s}\mu = -\frac{1}{r}-\frac{iM\chi}{r^2}\left(1-\frac{\zeta B_2(r)}{16r^5}\right)\cos\theta\,,\\
\tau &= -\pi = -\frac{iM\chi}{\sqrt{2}r^2}\left(1+\frac{\zeta B_3(r)}{32r^6}\right)\sin\theta\,,\\
\varepsilon &= \frac{iM\zeta\chi B_2(r)}{32r^7}\cos\theta\,,\\
\gamma &= \frac{\mu}{2}+\frac{1}{4r}\left(1-iM\chi\frac{r-r_s}{r^2}\cos\theta\right)\,,\\
\alpha &= \bar{\beta}-\frac{(r\cos\theta+iM\chi\cos 2\theta)\csc\theta}{\sqrt{2}r^2}\\
&= -\frac{1}{2\sqrt{2}r}\left[\cot\theta-\frac{iM\chi}{r}\left(\left(3+\frac{\zeta B_4(r)}{16r^6}\right)\sin\theta-\csc\theta\right)\right]\,,
\end{aligned} \tag{4.160}$$

with

$$B_1(r) = 42M^6 + 16M^5r + 5M^4r^2\,, \tag{4.161a}$$
$$B_2(r) = 126M^5 + 60M^4r + 25M^3r^2\,, \tag{4.161b}$$
$$B_3(r) = 1800M^6 - 162M^5r - 120M^4r^2 - 125M^3r^3\,, \tag{4.161c}$$
$$B_4(r) = 270M^6 - 6M^5r - 15M^4r^2 - 25M^3r^3\,. \tag{4.161d}$$

Using the above NP quantities at $\mathcal{O}(\zeta^1,\epsilon^0)$, we computed the correction to the Teukolsky operators in Sec. 4.6.1. In Eq. (4.103), these radial functions are defined to be

$$C_1(r) = 17640M^4 - 17196M^3r + 6M^2r^2 + 210Mr^3 + 1295r^4\,, \tag{4.162a}$$

$$C_2(r) = 189M^3 + 120M^2r + 70Mr^2\,, \tag{4.162b}$$

$$C_3(r) = 342M^3 - 816M^2r - 385Mr^2 - 165r^3\,, \tag{4.162c}$$

$$C_4(r) = 774M^2 + 360Mr + 145r^2\,, \tag{4.162d}$$

$$C_5(r) = 1800M^3 - 378M^2r - 240Mr^2 - 185r^3\,. \tag{4.162e}$$

The radial functions in Eq. (4.123) are given by

$$D_i(r) = C_i(r) \quad i \neq 3\,, \tag{4.163a}$$

$$D_3(r) = 1206M^3 - 12M^2r - 45Mr^2 - 125r^3\,. \tag{4.163b}$$

4.12 Appendix: Reconstructed NP quantities

In this appendix, we provide the explicit expressions of these reconstructed NP quantities in Sec. 4.4. As discussed in Sec. 4.4, one can write these structure constants ${C_{ab}}^c$ in terms of spin coefficients using Eq. (4.74) and the definition of spin coefficients in terms of Ricci rotation coefficients

$$\begin{aligned}
&\kappa = \gamma_{131}\,,\ \lambda = -\gamma_{244}\,,\ \sigma = \gamma_{133}\,,\ \mu = -\gamma_{243}\,,\\
&\pi = -\gamma_{241}\,,\ \tau = \gamma_{132}\,,\ \varepsilon = \frac{1}{2}(\gamma_{121} - \gamma_{341})\,,\ \rho = \gamma_{134}\,,\\
&\alpha = \frac{1}{2}(\gamma_{124} - \gamma_{344})\,,\ \beta = \frac{1}{2}(\gamma_{123} - \gamma_{343})\,,\ \nu = -\gamma_{242}\,,\ \gamma = \frac{1}{2}(\gamma_{122} - \gamma_{342})\,.
\end{aligned} \tag{4.164}$$

It was found in [42] that

$$\begin{aligned}
&{C_{12}}^1 = -(\gamma + \bar{\gamma})\,,\ {C_{12}}^2 = -(\varepsilon + \bar{\varepsilon})\,,\ {C_{12}}^3 = \bar{\tau} + \pi\,,\\
&{C_{13}}^1 = -\bar{\alpha} - \beta + \bar{\pi}\,,\ {C_{13}}^2 = -\kappa\,,\ {C_{13}}^3 = \bar{\rho} + \varepsilon - \bar{\varepsilon}\,,\ {C_{13}}^4 = \sigma\,,\\
&{C_{23}}^1 = \bar{\nu}\,,\ {C_{23}}^2 = -\tau + \bar{\alpha} + \beta\,,\ {C_{23}}^3 = -\mu + \gamma - \bar{\gamma}\,,\ {C_{23}}^4 = -\bar{\lambda}\,,\\
&{C_{34}}^1 = \mu - \bar{\mu}\,,\ {C_{34}}^2 = \rho - \bar{\rho}\,,\ {C_{34}}^3 = \bar{\beta} - \alpha\,,
\end{aligned} \tag{4.165}$$

and the other components can be found by using complex conjugation and that ${C_{ab}}^c$ is antisymmetric in its first two indices. Solving the above equation, one can also express spin coefficients in terms of $C_{ab}{}^{c}$,

$$\begin{aligned}
&\kappa = C^{31}{}_{2}\,,\ \sigma = -C^{31}_{4}\,,\ \lambda = C^{42}{}_{3}\,,\ \nu = -C^{42}{}_{1}\,,\\
&\rho = -\frac{1}{2}\left(C_{31}{}^{3} + C_{41}{}^{4} + C_{43}{}^{2}\right)\,,\ \mu = \frac{1}{2}\left(C_{32}{}^{3} + C_{42}{}^{4} - C_{43}{}^{1}\right)\,,\\
&\pi = -\frac{1}{2}\left(C_{41}{}^{1} + C^{42}_{2} + C_{21}{}^{3}\right)\,,\ \tau = \frac{1}{2}\left(C_{31}{}^{1} + C_{32}{}^{2} - C_{21}{}^{4}\right)\,,\\
&\varepsilon = \frac{1}{4}\left(C_{41}{}^{4} - C_{31}{}^{3} + 2C_{21}{}^{2} + \rho - \bar{\rho}\right)\,,\ \gamma = \frac{1}{4}\left(C_{42}{}^{4} - C_{32}{}^{3} + 2C_{21}{}^{1} + \mu - \bar{\mu}\right)\,,\\
&\alpha = \frac{1}{4}\left(C_{41}{}^{1} - C_{42}{}^{2} + 2C_{43}{}^{3} + \bar{\tau} - \pi\right)\,,\ \beta = \frac{1}{4}\left(C_{31}{}^{1} - C_{32}{}^{2} + 2C_{43}{}^{4} + \tau - \bar{\pi}\right)\,.
\end{aligned}\tag{4.166}$$

Then following the procedures in Sec. 4.4, one finds the spin coefficients at $O(\zeta^0, \epsilon^1)$ to be

$$\kappa^{(0,1)} = \frac{1}{2}\delta_{[-2,-2,1,1]}h^{(0,1)}_{ll} - D_{[-2,0,0,-1]}h^{(0,1)}_{lm}\,,\tag{4.167a}$$

$$\sigma^{(0,1)} = -\frac{1}{2}D_{[-2,2,1,-1]}h^{(0,1)}_{mm} + (\bar{\pi} + \tau)h^{(0,1)}_{lm}\,,\tag{4.167b}$$

$$\lambda^{(0,1)} = (\pi + \bar{\tau})h^{(0,1)}_{n\bar{m}} + \frac{1}{2}\Delta_{[-1,1,2,-2]}h^{(0,1)}_{\bar{m}\bar{m}}\,,\tag{4.167c}$$

$$\nu^{(0,1)} = -\frac{1}{2}\bar{\delta}_{[2,2,-1,-1]}h^{(0,1)}_{nn} + \mathbf{\Delta}_{[0,1,2,0]}h^{(0,1)}_{n\bar{m}}\,,\tag{4.167d}$$

$$\begin{aligned}\epsilon^{(0,1)} = \frac{1}{4}\Big[&\mathbf{\Delta}_{[-1,1,0,-2]}h^{(0,1)}_{ll} - 2D_{[0,0,\frac{1}{2},-\frac{1}{2}]}h^{(0,1)}_{ln} - \bar{\delta}_{[-2,0,-3,-2]}h^{(0,1)}_{lm}\\
&+ \delta_{[-2,0,1,2]}h^{(0,1)}_{l\bar{m}} - (\rho - \bar{\rho})h^{(0,1)}_{m\bar{m}}\Big]\,,\end{aligned}\tag{4.167e}$$

$$\begin{aligned}\rho^{(0,1)} = \frac{1}{2}\Big[&-\mu h^{(0,1)}_{ll} - (\rho - \bar{\rho})h^{(0,1)}_{ln} - \bar{\delta}_{[-2,0,-1,0]}h^{(0,1)}_{lm}\\
&+ \delta_{[-2,0,1,2]}h^{(0,1)}_{l\bar{m}} - D_{[0,0,1,-1]}h^{(0,1)}_{m\bar{m}}\Big]\,,\end{aligned}\tag{4.167f}$$

$$\begin{aligned}\mu^{(0,1)} = \frac{1}{2}\Big[&-\rho h^{(0,1)}_{nn} - \bar{\delta}_{[0,2,-2,-1]}h^{(0,1)}_{nm} + \delta_{[0,2,0,1]}h^{(0,1)}_{n\bar{m}}\\
&+ (\mu + \bar{\mu})h^{(0,1)}_{ln} + \mathbf{\Delta}_{[-1,1,0,0]}h^{(0,1)}_{m\bar{m}}\Big]\,,\end{aligned}\tag{4.167g}$$

$$\begin{aligned}\gamma^{(0,1)} = \frac{1}{4}\Big[&-D_{[0,2,1,-1]}h^{(0,1)}_{nn} - \bar{\delta}_{[0,2,-2,-1]}h^{(0,1)}_{nm} + \delta_{[0,2,2,3]}h^{(0,1)}_{n\bar{m}}\\
&- (\mu - \bar{\mu} - 4\gamma)h^{(0,1)}_{ln} - (\mu - \bar{\mu})h^{(0,1)}_{m\bar{m}}\Big]\,,\end{aligned}\tag{4.167h}$$

$$\begin{aligned}\alpha^{(0,1)} = \frac{1}{4}\Big[&-D_{[-2,0,-1,-2]}h^{(0,1)}_{n\bar{m}} + \delta_{[-2,0,1,1]}h^{(0,1)}_{\bar{m}\bar{m}} - \bar{\delta}_{[0,0,-1,-1]}h^{(0,1)}_{ln}\\
&+ \mathbf{\Delta}_{[-2,1,4,-2]}h^{(0,1)}_{l\bar{m}} - \bar{\delta}_{[2,0,-1,-1]}h^{(0,1)}_{m\bar{m}}\Big]\,,\end{aligned}\tag{4.167i}$$

$$\begin{aligned}\beta^{(0,1)} = \frac{1}{4}\Big[&-D_{[-4,2,2,-1]}h^{(0,1)}_{nm} - \bar{\delta}_{[0,2,-1,-1]}h^{(0,1)}_{mm} - \delta_{[0,0,-1,-1]}h^{(0,1)}_{ln}\\
&+ \mathbf{\Delta}_{[1,2,2,0]}h^{(0,1)}_{lm} + \delta_{[0,-2,1,1]}h^{(0,1)}_{m\bar{m}}\Big]\,,\end{aligned}\tag{4.167j}$$

$$\pi^{(0,1)} = \frac{1}{2}\Big[D_{[2,0,-1,0]}h^{(0,1)}_{n\bar{m}} + \tau h^{(0,1)}_{\bar{m}\bar{m}} - \delta_{[0,0,-1,-1]}h^{(0,1)}_{ln} + \mathbf{\Delta}_{[0,1,0,-2]}h^{(0,1)}_{l\bar{m}} + \bar{\tau}h^{(0,1)}_{m\bar{m}}\Big]\,, \tag{4.167k}$$

$$\tau^{(0,1)} = \frac{1}{2}\Big[-D_{[0,2,0,-1]}h^{(0,1)}_{nm} + \pi h^{(0,1)}_{mm} + \delta_{[0,0,1,1]}h^{(0,1)}_{ln} - \mathbf{\Delta}_{[1,0,-2,0]}h^{(0,1)}_{lm} + \bar{\pi}h^{(0,1)}_{m\bar{m}}\Big]\,. \tag{4.167l}$$

For Weyl scalars at $O(\zeta^0, \epsilon^1)$, one can use Ricci identities to retrieve them from spin coefficients. The equations below work for both vacuum and non-vacuum spacetimes since we have linearly combined Ricci identities to remove NP Ricci scalars Φ_{ab} following [38, 66].

$$\Psi_0 = D_{[-3,1,-1,-1]}\sigma - \delta_{[-1,-3,1,-1]}\kappa\,, \tag{4.168a}$$

$$\Psi_1 = D_{[0,1,0,-1]}\beta - \delta_{[-1,0,1,0]}\varepsilon - (\alpha+\pi)\sigma + (\gamma+\mu)\kappa\,, \tag{4.168b}$$

$$\Psi_2 = \frac{1}{3}\Big[\bar{\delta}_{[-2,1,-1,-1]}\beta - \delta_{[-1,0,1,1]}\alpha + D_{[1,1,1,-1]}\gamma - \mathbf{\Delta}_{[-1,1,-1,-1]}\varepsilon + \bar{\delta}_{[-1,1,-1,-1]}\tau - \mathbf{\Delta}_{[-1,1,-1,-1]}\rho + 2(\nu\kappa - \lambda\sigma)\Big]\,, \tag{4.168c}$$

$$\Psi_3 = \bar{\delta}_{[0,1,0,-1]}\gamma - \mathbf{\Delta}_{[0,1,0,-1]}\alpha + (\varepsilon+\rho)\nu - (\beta+\tau)\lambda\,, \tag{4.168d}$$

$$\Psi_4 = \bar{\delta}_{[3,1,1,-1]}\nu - \mathbf{\Delta}_{[1,1,3,-1]}\lambda\,. \tag{4.168e}$$

The equations above are precise, so one needs to linearize them when extracting the Weyl scalars at $O(\zeta^0, \epsilon^1)$ using the perturbed tetrad and spin coefficients at $O(\zeta^0, \epsilon^1)$.

In Refs. [36, 59], they also computed the perturbed Weyl scalars in the IRG and expressed them in terms of the Hertz potential,

$$\Psi_0^{(0,1)} = -\frac{1}{2}D_{[-3,1,0,-1]}D_{[-2,2,0,-1]}h^{(0,1)}_{mm}\,, \tag{4.169a}$$

$$\Psi_1^{(0,1)} = -\frac{1}{8}\Big[2D_{[-1,1,1,-1]}D_{[0,2,1,-1]}h^{(0,1)}_{nm} + D_{[-1,1,1,-1]}\delta_{[-2,2,-2,-1]}h^{(0,1)}_{mm} + \bar{\delta}_{[-3,1,-3,-1]}D_{[-2,2,0,-1]}h^{(0,1)}_{mm}\Big]\,, \tag{4.169b}$$

$$\Psi_2^{(0,1)} = -\frac{1}{12}\Big[D_{[1,1,2,-1]}D_{[2,2,2,-1]}h^{(0,1)}_{nn,1} + 2\left(D_{[1,1,2,-1]}\bar{\delta}_{[0,2,-1,-1]} + \bar{\delta}_{[-1,1,-2,-1]}D_{[0,2,1,-1]}\right)h^{(0,1)}_{nm} + \bar{\delta}_{[-1,1,-2,-1]}\bar{\delta}_{[-2,2,-2,-1]}h^{(0,1)}_{mm}\Big]\,, \tag{4.169c}$$

$$\Psi_3^{(0,1)} = -\frac{1}{8}\Big[\left(D_{[3,1,3,-1]}\bar{\delta}_{[2,2,0,-1]} + \bar{\delta}_{[1,1,-1,-1]}D_{[2,2,,2,-1]}\right)h^{(0,1)}_{nn,1} + \bar{\delta}_{[1,1,-1,-1]}\bar{\delta}_{[0,2,-1,-1]}h^{(0,1)}_{nm}\Big]\,, \tag{4.169d}$$

$$\begin{aligned}\Psi_4^{(0,1)} = & -\frac{1}{2}\Big[\bar{\delta}_{[3,1,0,-1]}\bar{\delta}_{[2,2,0,-1]}h^{(0,1)}_{nn,1} + 3\Psi_2\left(\tau\bar{\delta}_{[4,0,0,0]} - \rho\boldsymbol{\Delta}_{[0,0,4,0]}\right. \\ & \left. -\mu D_{[4,0,0,0]} + \pi\delta_{[0,4,0,0]} + 2\Psi_2\right)\bar{\Psi}_{\mathrm{H}}\Big]\,,\end{aligned} \tag{4.169e}$$

where $h^{(0,1)}_{nn,1}$ is the piece of $h^{(0,1)}_{nn}$ proportional to $\bar{\Psi}_{\mathrm{H}}$ in Eq. (4.50), i.e., $h^{(0,1)}_{nn,1} = \bar{\delta}_{[1,3,0,-1]}\bar{\delta}_{[0,4,0,3]}\bar{\Psi}_{\mathrm{H}}$. As we discussed in Sec. 4.4, our results using either the Ricci identities or direct computation from the linearized Riemann tensor agree for $\Psi^{(0,1)}_{0,1,2,4}$ but not for $\Psi^{(0,1)}_3$ due to different choices of the perturbed tetrad. Thus, we can use Eq. (4.169) for $\Psi^{(0,1)}_{0,1,2,4}$ and Eq. (4.168d) for $\Psi^{(0,1)}_3$.

For Schwarzschild, the equations above simplify into

$$\Psi_0^{(0,1)} = -\frac{1}{2}D^4\bar{\Psi}_{\mathrm{H}}\,, \tag{4.170a}$$

$$\Psi_1^{(0,1)} = -\frac{1}{2}D^3(\bar{\delta} - 4\alpha)\bar{\Psi}_{\mathrm{H}}\,, \tag{4.170b}$$

$$\Psi_2^{(0,1)} = -\frac{1}{2}D^2(\bar{\delta} - 2\alpha)(\bar{\delta} - 4\alpha)\bar{\Psi}_{\mathrm{H}}\,, \tag{4.170c}$$

$$\Psi_3^{(0,1)} = -\frac{1}{2}D\bar{\delta}(\bar{\delta} - 2\alpha)(\bar{\delta} - 4\alpha)\bar{\Psi}_{\mathrm{H}} + \frac{3}{2}\Psi_2 h_{n\bar{m}}\,, \tag{4.170d}$$

$$\Psi_4^{(0,1)} = -\frac{1}{2}(\bar{\delta} + 2\alpha)\bar{\delta}(\bar{\delta} - 2\alpha)(\bar{\delta} - 4\alpha)\bar{\Psi}_{\mathrm{H}} + \frac{3}{2}\Psi_2\left[\mu D + \rho(\boldsymbol{\Delta} + 4\gamma) - 2\Psi_2\right]\Psi_{\mathrm{H}}\,, \tag{4.170e}$$

where we have added the correction term $\frac{3}{2}\Psi_2 h^{(0,1)}_{n\bar{m}}$ to $\Psi^{(0,1)}_3$.

To use a consistent gauge with the one in Sec. 4.3.1, we need to rotate the tetrad to remove $\Psi^{(0,1)}_{1,3}$. Under type I and II tetrad rotations at $O(\zeta^0, \epsilon^1)$, the tetrad becomes

$$\begin{aligned} l^{(0,1)} &\to l^{(0,1)} + \bar{b}^{(0,1)}m + b^{(0,1)}\bar{m} - \delta A^{(0,1)}l\,, \\ n^{(0,1)} &\to n^{(0,1)} + \bar{a}^{(0,1)}m + a^{(0,1)}\bar{m} + \delta A^{(0,1)}n\,, \\ m^{(0,1)} &\to m^{(0,1)} + a^{(0,1)}l + b^{(0,1)}n + i\varphi^{(0,1)}m\,, \end{aligned} \tag{4.171}$$

and the Weyl scalars transform as

$$\Psi^{(0,1)}_{0,2,4} \to 0\,, \quad \Psi_1^{(0,1)} \to \Psi_1^{(0,1)} + 3b^{(0,1)}\Psi_2\,, \quad \Psi_3^{(0,1)} \to \Psi_3^{(0,1)} + 3\bar{a}^{(0,1)}\Psi_2\,. \tag{4.172}$$

The rotation coefficients $a^{(0,1)}$ and $b^{(0,1)}$ are given by Eq. (4.77). For spin coefficients, due to the complication of the reconstructed tetrad, instead of computing the spin coefficients from the rotated tetrad directly, we chose to use the transformation of spin coefficients under tetrad rotations in [42]. In this case, the spin coefficients transform as

$$\kappa^{(0,1)} \to \kappa^{(0,1)} + b^{(0,1)}\rho - Db^{(0,1)}\,,$$

$$
\begin{aligned}
&\sigma^{(0,1)} \to \sigma^{(0,1)} + b^{(0,1)}(2\beta + \tau) - \delta b^{(0,1)}\,,\\
&\lambda^{(0,1)} \to \lambda^{(0,1)} + \bar{a}^{(0,1)}(2\alpha + \pi) + \bar{\delta}\bar{a}\,,\\
&\nu^{(0,1)} \to \nu^{(0,1)} + \bar{a}^{(0,1)}(\mu + 2\gamma) + \boldsymbol{\Delta}\bar{a}^{(0,1)}\,,\\
&\varepsilon^{(0,1)} \to \varepsilon^{(0,1)} + \bar{b}^{(0,1)}\beta + b^{(0,1)}(\alpha + \pi)\,,\\
&\rho^{(0,1)} \to \rho^{(0,1)} + \bar{b}^{(0,1)}\tau + 2b^{(0,1)}\alpha - \bar{\delta}b^{(0,1)}\,,\\
&\mu^{(0,1)} \to \mu^{(0,1)} + a^{(0,1)}\pi + 2\bar{a}^{(0,1)}\beta + \delta\bar{a}^{(0,1)}\,,\\
&\gamma^{(0,1)} \to \gamma^{(0,1)} + a^{(0,1)}\alpha + \bar{a}^{(0,1)}(\beta + \tau)\,,\\
&\alpha^{(0,1)} \to \alpha^{(0,1)} + \bar{a}^{(0,1)}\rho + \bar{b}^{(0,1)}\gamma\,,\\
&\beta^{(0,1)} \to \beta^{(0,1)} + b^{(0,1)}(\mu + \gamma)\,,\\
&\pi^{(0,1)} \to \pi^{(0,1)} + \bar{b}^{(0,1)}\mu + D\bar{a}^{(0,1)}\,,\\
&\tau^{(0,1)} \to \tau^{(0,1)} + a^{(0,1)}\rho + 2b^{(0,1)}\gamma - \boldsymbol{\Delta}b^{(0,1)}\,.
\end{aligned}
\tag{4.173}
$$

Furthermore, besides the tetrad rotations, when one performs coordinate transformations $x^\mu \to x^\mu + \xi^\mu$ at $\mathcal{O}(\zeta^0, \epsilon^1)$, one finds

$$
\Psi^{(0,1)} \to \Psi^{(0,1)} + \xi^{\mu(0,1)}\partial_\mu\Psi^{(0,0)} + \xi^{\mu(0,1)}\partial_\mu\Psi^{(0,0)}\,. \tag{4.174}
$$

for the scalar-type NP quantities such as Weyl scalars and spin coefficients [66].

4.13 Appendix: Expression of Φ_{ij}

In Sec. 4.6.2, we want to rewrite the Ricci tensor in Eqs. (4.106) and (4.107) in terms of Weyl scalars, spin coefficients, and directional derivatives. Since γ_{abc} is antisymmetric in the first two indices, it has 24 independent components. These 24 components can be further reduced to 14 components using complex conjugation, which can then be expressed in terms of spin coefficients using the definition in Eq. (4.164),

$$
\begin{aligned}
&\gamma_{121} = \varepsilon + \bar{\varepsilon}\,,\ \gamma_{122} = \gamma + \bar{\gamma}\,,\ \gamma_{123} = \bar{\alpha} + \beta\,,\ \gamma_{131} = \kappa\,,\ \gamma_{132} = \tau\,,\\
&\gamma_{133} = \sigma\,,\ \gamma_{134} = \rho\,, \gamma_{231} = -\bar{\pi}\,,\ \gamma_{232} = -\bar{\nu}\,,\ \gamma_{233} = -\bar{\lambda}\,,\\
&\gamma_{234} = -\bar{\mu}\,,\ \gamma_{341} = \bar{\varepsilon} - \varepsilon\,,\ \gamma_{342} = \bar{\gamma} - \gamma\,,\ \gamma_{343} = \bar{\alpha} - \beta\,.
\end{aligned}
\tag{4.175}
$$

For Riemann tensor or Weyl tensor, it is antisymmetric within its first pair and second pair of indices and symmetric under the exchange of the first and the second pair of indices, so the total number of independent components reduce to 21 using these symmetries. Besides these symmetries, $C_{a[bcd]} = 0$ and C_{abcd} is traceless in the vacuum, which further give us the following relations in [42]:

$$\begin{aligned}
&C_{1314} = C_{1323} = C_{1424} = C_{2324} = 0\,,\\
&C_{1334} = -C_{1213}\,,\; C_{2334} = C_{1223}\,,\; C_{3434} = C_{1212}\,,\; C_{1342} = \frac{1}{2}\left(C_{1212} - C_{1234}\right)\,,
\end{aligned} \tag{4.176}$$

which reduce the number of independent components of C_{abcd} to 10. These 10 independent components can be represented by 5 Weyl scalars and their conjugates. With complex conjugation, we can find all the components of the Weyl tensor using the symmetries above and the components below,

$$\begin{aligned}
&C_{1212} = \Psi_2 + \bar{\Psi}_2\,,\; C_{1213} = \Psi_1\,,\; C_{1223} = -\bar{\Psi}_3\,,\\
&C_{1234} = \bar{\Psi}_2 - \Psi_2\,,\; C_{1313} = \Psi_0\,,\; C_{2323} = \bar{\Psi}_4\,.
\end{aligned} \tag{4.177}$$

With Eqs. (4.106) and (4.107) and (4.175)–(4.177), we find Eq. (4.106) in the NP basis to be

$$\begin{aligned}
R_{11} = i\mathcal{R}_1\Big\{&(D\vartheta)\Big[\lambda\Psi_0 - \bar{\lambda}\bar{\Psi}_0 - (\alpha+\bar{\beta}+\pi)\Psi_1 + (\bar{\alpha}+\beta+\bar{\pi})\bar{\Psi}_1\\
&+(\varepsilon+\bar{\varepsilon})(\Psi_2-\bar{\Psi}_2)\Big]\\
&-(\Delta\vartheta)\Big[\bar{\sigma}\Psi_0 - \sigma\bar{\Psi}_0 - \bar{\kappa}\Psi_1 + \kappa\bar{\Psi}_1\Big]\\
&+(\delta\vartheta)\Big[(\bar{\alpha}-\beta)\bar{\Psi}_0 + \bar{\sigma}\Psi_1 + (\varepsilon-\bar{\varepsilon}-\bar{\rho})\bar{\Psi}_1 - \bar{\kappa}(\Psi_2-\bar{\Psi}_2)\Big]\\
&-(\bar{\delta}\vartheta)\Big[(\alpha-\bar{\beta})\Psi_0 - (\varepsilon-\bar{\varepsilon}+\rho)\Psi_1 + \sigma\bar{\Psi}_1 + \kappa(\Psi_2-\bar{\Psi}_2)\Big]\\
&-\frac{1}{2}\Psi_0\{\bar{\delta},\bar{\delta}\}\vartheta + \frac{1}{2}\bar{\Psi}_0\{\delta,\delta\}\vartheta + \Psi_1\{D,\bar{\delta}\}\vartheta - \bar{\Psi}_1\{D,\delta\}\vartheta\\
&-(\Psi_2-\bar{\Psi}_2)\frac{1}{2}\{D,D\}\vartheta\Big\} + \mathcal{R}_2(D\vartheta)(D\vartheta)\,,
\end{aligned} \tag{4.178a}$$

$$\begin{aligned}
R_{12} = \frac{i}{2}\mathcal{R}_1\Big\{&(D\vartheta)\Big[\nu\Psi_1 - \bar{\nu}\bar{\Psi}_1 - (\gamma+\bar{\gamma}+\mu+\bar{\mu})(\Psi_2-\bar{\Psi}_2)\\
&+(\bar{\alpha}+\beta+\bar{\pi})\,\Psi_3 - \left(\alpha+\bar{\beta}+\pi\right)\bar{\Psi}_3\Big]\\
&-(\Delta\vartheta)\Big[(\alpha+\bar{\beta}+\bar{\tau})\Psi_1 - (\bar{\alpha}+\beta+\tau)\bar{\Psi}_1\\
&-(\varepsilon+\bar{\varepsilon}+\rho+\bar{\rho})(\Psi_2-\bar{\Psi}_2) + \kappa\Psi_3 - \bar{\kappa}\bar{\Psi}_3\Big]\\
&+(\delta\vartheta)\Big[\lambda\Psi_1 - (\gamma-\bar{\gamma}+\mu)\bar{\Psi}_1 - (\alpha-\bar{\beta}+\pi-\bar{\tau})(\Psi_2-\bar{\Psi}_2)\\
&+(\varepsilon-\bar{\varepsilon}-\bar{\rho})\Psi_3 + \bar{\sigma}\bar{\Psi}_3\Big]\\
&-(\bar{\delta}\vartheta)\Big[(\gamma-\bar{\gamma}-\bar{\mu})\Psi_1 + \bar{\lambda}\bar{\Psi}_1 + (\bar{\alpha}-\beta+\bar{\pi}-\tau)(\Psi_2-\bar{\Psi}_2)
\end{aligned}$$

$$
\begin{aligned}
&+ \sigma\Psi_3 - (\varepsilon - \bar{\varepsilon} + \rho)\bar{\Psi}_3\Big] \\
&- \Psi_1\{\boldsymbol{\Delta}, \bar{\delta}\}\vartheta + \bar{\Psi}_1\{\boldsymbol{\Delta}, \delta\}\vartheta + (\Psi_2 - \bar{\Psi}_2)\Big[\{D, \boldsymbol{\Delta}\} + \{\delta, \bar{\delta}\}\Big]\vartheta \\
&- \Psi_3\{D, \delta\}\vartheta + \bar{\Psi}_3\{D, \bar{\delta}\}\vartheta\Big\} + \mathcal{R}_2(D\vartheta)(\boldsymbol{\Delta}\vartheta)\,,
\end{aligned} \tag{4.178b}
$$

$$
\begin{aligned}
R_{13} = \frac{i}{2}\mathcal{R}_1\Big\{&(D\vartheta)\Big[\nu\Psi_0 - (\gamma + \bar{\gamma} + \mu + \bar{\mu})\Psi_1 \\
&- 2\bar{\lambda}\bar{\Psi}_1 + (\bar{\alpha} + \beta + \bar{\pi})(\Psi_2 + 2\bar{\Psi}_2) - 2(\varepsilon + \bar{\varepsilon})\bar{\Psi}_3\Big] \\
&- (\boldsymbol{\Delta}\vartheta)\Big[(\alpha + \bar{\beta} + \bar{\tau})\Psi_0 - (\varepsilon + \bar{\varepsilon} + \rho + \bar{\rho})\Psi_1 - 2\sigma\bar{\Psi}_1 + \kappa(\Psi_2 + 2\bar{\Psi}_2)\Big] \\
&+ (\delta\vartheta)\Big[\lambda\Psi_0 - (\alpha - \bar{\beta} + \pi - \bar{\tau})\Psi_1 + 2(\bar{\alpha} - \beta)\bar{\Psi}_1 \\
&+ (\varepsilon - \bar{\varepsilon} - \bar{\rho})(\Psi_2 + 2\bar{\Psi}_2) + 2\bar{\kappa}\bar{\Psi}_3\Big] \\
&- (\bar{\delta}\vartheta)\Big[(\gamma - \bar{\gamma} - \bar{\mu})\Psi_0 + (\bar{\alpha} - \beta + \bar{\pi} - \tau)\Psi_1 + \sigma(\Psi_2 + 2\bar{\Psi}_2) - 2\kappa\bar{\Psi}_3\Big] \\
&- \Psi_0\{\boldsymbol{\Delta}, \bar{\delta}\}\vartheta + \Psi_1\Big[\{D, \boldsymbol{\Delta}\} + \{\delta, \bar{\delta}\}\Big]\vartheta + \bar{\Psi}_1\{\delta, \delta\}\vartheta \\
&- (\Psi_2 + 2\bar{\Psi}_2)\{D, \delta\}\vartheta + \bar{\Psi}_3\{D, D\}\vartheta\Big\} + \mathcal{R}_2(D\vartheta)(\delta\vartheta)\,,
\end{aligned} \tag{4.178c}
$$

$$
\begin{aligned}
R_{22} = i\mathcal{R}_1\Big\{&- (D\vartheta)\Big[\bar{\nu}\Psi_3 - \nu\bar{\Psi}_3 - \bar{\lambda}\Psi_4 + \lambda\bar{\Psi}_4\Big] \\
&- (\boldsymbol{\Delta}\vartheta)\Big[(\gamma + \bar{\gamma})(\Psi_2 - \bar{\Psi}_2) - (\bar{\alpha} + \beta + \tau)\Psi_3 + (\alpha + \bar{\beta} + \bar{\tau})\bar{\Psi}_3 + \sigma\Psi_4 - \bar{\sigma}\bar{\Psi}_4\Big] \\
&+ (\delta\vartheta)\Big[\nu(\Psi_2 - \bar{\Psi}_2) - (\gamma - \bar{\gamma} + \mu)\Psi_3 + \lambda\bar{\Psi}_3 + (\bar{\alpha} - \beta)\Psi_4\Big] \\
&+ (\bar{\delta}\vartheta)\Big[\bar{\nu}(\Psi_2 - \bar{\Psi}_2) - \bar{\lambda}\Psi_3 - (\gamma - \bar{\gamma} - \bar{\mu})\bar{\Psi}_3 + (\alpha - \bar{\beta})\bar{\Psi}_4\Big] \\
&- \frac{1}{2}(\Psi_2 - \bar{\Psi}_2)\{\boldsymbol{\Delta}, \boldsymbol{\Delta}\}\vartheta + \Psi_3\{\boldsymbol{\Delta}, \delta\}\vartheta - \bar{\Psi}_3\{\boldsymbol{\Delta}, \bar{\delta}\}\vartheta \\
&- \frac{1}{2}\Psi_4\{\delta, \delta\}\vartheta + \frac{1}{2}\bar{\Psi}_4\{\bar{\delta}, \bar{\delta}\}\vartheta\Big\} + \mathcal{R}_2(\boldsymbol{\Delta}\vartheta)(\boldsymbol{\Delta}\vartheta)\,,
\end{aligned} \tag{4.178d}
$$

$$
\begin{aligned}
R_{23} = \frac{i}{2}\mathcal{R}_1\Big\{&- (D\vartheta)\Big[\bar{\nu}(2\Psi_2 + \bar{\Psi}_2) - 2\bar{\lambda}\Psi_3 - (\gamma + \bar{\gamma} + \mu + \bar{\mu})\bar{\Psi}_3 + (\alpha + \bar{\beta} + \pi)\bar{\Psi}_4\Big] \\
&- (\boldsymbol{\Delta}\vartheta)\Big[2(\gamma + \bar{\gamma})\Psi_1 - (\bar{\alpha} + \beta + \tau)(2\Psi_2 + \bar{\Psi}_2) + 2\sigma\Psi_3 \\
&+ (\varepsilon + \bar{\varepsilon} + \rho + \bar{\rho})\bar{\Psi}_3 - \bar{\kappa}\bar{\Psi}_4\Big] \\
&+ (\delta\vartheta)\Big[2\nu\Psi_1 - (\gamma - \bar{\gamma} + \mu)(2\Psi_2 + \bar{\Psi}_2) - 2(\bar{\alpha} - \beta)\Psi_3 \\
&+ (\alpha - \bar{\beta} + \pi - \bar{\tau})\bar{\Psi}_3 + \bar{\sigma}\bar{\Psi}_4\Big] \\
&+ (\bar{\delta}\vartheta)\Big[2\bar{\nu}\Psi_1 - \bar{\lambda}(2\Psi_2 + \bar{\Psi}_2) + (\bar{\alpha} - \beta + \bar{\pi} - \tau)\bar{\Psi}_3 + (\varepsilon - \bar{\varepsilon} + \rho)\bar{\Psi}_4\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- \Psi_1\{\boldsymbol{\Delta},\boldsymbol{\Delta}\}\vartheta + (2\Psi_2+\bar{\Psi}_2)\{\boldsymbol{\Delta},\delta\}\vartheta - \Psi_3\{\delta,\delta\}\vartheta \\
&- \bar{\Psi}_3\Big[\{D,\boldsymbol{\Delta}\}+\{\delta,\bar{\delta}\}\Big]\vartheta + \bar{\Psi}_4\{D,\bar{\delta}\}\vartheta\Big\} + \mathcal{R}_2(\boldsymbol{\Delta}\vartheta)(\delta\vartheta)\,, \qquad (4.178\text{e})
\end{aligned}
$$

$$
\begin{aligned}
R_{33} = i\mathcal{R}_1\Big\{ &- (D\vartheta)\Big[\bar{\nu}\Psi_1 - \bar{\lambda}(\Psi_2-\bar{\Psi}_2) - (\bar{\alpha}+\beta+\bar{\pi})\bar{\Psi}_3 + (\varepsilon+\bar{\varepsilon})\bar{\Psi}_4\Big] \\
&- (\boldsymbol{\Delta}\vartheta)\Big[(\gamma+\bar{\gamma})\Psi_0 - (\bar{\alpha}+\beta+\tau)\Psi_1 + \sigma(\Psi_2-\bar{\Psi}_2) + \kappa\bar{\Psi}_3\Big] \\
&+ (\delta\vartheta)\Big[\nu\Psi_0 - (\gamma-\bar{\gamma}+\mu)\Psi_1 - (\bar{\alpha}-\beta)(\Psi_2-\bar{\Psi}_2) + (\varepsilon-\bar{\varepsilon}-\bar{\rho}) + \bar{\kappa}\bar{\Psi}_4\Big] \\
&+ (\bar{\delta}\vartheta)\Big[\bar{\nu}\Psi_0 - \bar{\lambda}\Psi_1 - \sigma\bar{\Psi}_3 + \kappa\bar{\Psi}_4\Big] \\
&- \frac{1}{2}\Psi_0\{\boldsymbol{\Delta},\boldsymbol{\Delta}\}\vartheta + \Psi_1\{\boldsymbol{\Delta},\delta\}\vartheta - \frac{1}{2}(\Psi_2-\bar{\Psi}_2)\{\delta,\delta\}\vartheta - \bar{\Psi}_3\{D,\delta\}\vartheta \\
&+ \frac{1}{2}\bar{\Psi}_4\{D,D\}\vartheta\Big\} + \mathcal{R}_2(\delta\vartheta)(\delta\vartheta)\,, \qquad (4.178\text{f})
\end{aligned}
$$

$$
\begin{aligned}
R_{34} = \frac{i}{2}\mathcal{R}_1\Big\{ &(D\vartheta)\Big[\nu\Psi_1 - \bar{\nu}\bar{\Psi}_1 - (\gamma+\bar{\gamma}+\mu+\bar{\mu})(\Psi_2-\bar{\Psi}_2) \\
&+ (\bar{\alpha}+\beta+\bar{\pi})\Psi_3 - (\alpha+\bar{\beta}+\pi)\bar{\Psi}_3\Big] \\
&- (\boldsymbol{\Delta}\vartheta)\Big[(\alpha+\bar{\beta}+\bar{\tau})\Psi_1 - (\bar{\alpha}+\beta+\tau)\bar{\Psi}_1 \\
&- (\varepsilon+\bar{\varepsilon}+\rho+\bar{\rho})(\Psi_2-\bar{\Psi}_2) + \kappa\Psi_3 - \bar{\kappa}\bar{\Psi}_3\Big] \\
&+ (\delta\vartheta)\Big[\lambda\Psi_1 - (\gamma-\bar{\gamma}+\mu)\bar{\Psi}_1 - (\alpha-\bar{\beta}+\pi-\bar{\tau})(\Psi_2-\bar{\Psi}_2) \\
&+ (\varepsilon-\bar{\varepsilon}-\bar{\rho})\Psi_3 + \bar{\sigma}\bar{\Psi}_3\Big] \\
&- (\bar{\delta}\vartheta)\Big[(\gamma-\bar{\gamma}-\bar{\mu})\Psi_1 + \bar{\lambda}\bar{\Psi}_1 + (\bar{\alpha}-\beta+\bar{\pi}-\tau)(\Psi_2-\bar{\Psi}_2) \\
&+ \sigma\Psi_3 - (\varepsilon-\bar{\varepsilon}+\rho)\bar{\Psi}_3\Big] \\
&- \Psi_1\{\boldsymbol{\Delta},\bar{\delta}\}\vartheta + \bar{\Psi}_1\{\boldsymbol{\Delta},\delta\}\vartheta + (\Psi_2-\bar{\Psi}_2)\Big[\{D,\boldsymbol{\Delta}\}+\{\delta,\bar{\delta}\}\Big]\vartheta \\
&- \Psi_3\{D,\delta\}\vartheta + \bar{\Psi}_3\{D,\bar{\delta}\}\vartheta\Big\} + \mathcal{R}_2(\delta\vartheta)(\bar{\delta}\vartheta)\,, \qquad (4.178\text{g})
\end{aligned}
$$

where

$$
\mathcal{R}_1 \equiv -\left(\frac{1}{\kappa_g}\right)^{\frac{1}{2}} M^2\,, \quad \mathcal{R}_2 \equiv \frac{1}{2\kappa_g\zeta}\,, \qquad (4.179)
$$

and the remaining components of R_{ab} can be found by exchanging the indices or complex conjugation.

The Ricci NP scalars Φ_{ij} are related to the Ricci tensor via

$$\begin{aligned} &\Phi_{00} = \frac{1}{2}R_{11}\,,\ \Phi_{01} = \frac{1}{2}R_{13}\,,\ \Phi_{02} = \frac{1}{2}R_{33}\,,\ \Phi_{10} = \frac{1}{2}R_{14}\,,\ \Phi_{11} = \frac{1}{4}(R_{12}+R_{34})\,,\\ &\Phi_{12} = \frac{1}{2}R_{23}\,,\ \Phi_{20} = \frac{1}{2}R_{44}\,,\ \Phi_{21} = \frac{1}{2}R_{24}\,,\ \Phi_{22} = \frac{1}{2}R_{22}\,,\ \Lambda = R/24\,. \end{aligned} \tag{4.180}$$

Using the projection of the Ricci tensor onto the NP basis in Eq. (4.178), the stationary scalar field in Eq. (4.15), and the NP quantities in Schwarzschild [setting $\zeta = \chi = 0$ in Eqs. (4.158) and (4.160)], we find the $O(\zeta^1, \chi^1, \epsilon^0)$ contributions to Φ_{ij}

$$\Phi_{00}^{(1,1,0)} = \Phi_{02}^{(1,1,0)} = \Phi_{11}^{(1,1,0)} = \Phi_{20}^{(1,1,0)} = \Phi_{22}^{(1,1,0)} = 0\,, \tag{4.181a}$$

$$\Phi_{01}^{(1,1,0)} = \bar{\Phi}_{10}^{(1,1,0)} = -\frac{15iM^5\left(18M^2+8Mr+3r^2\right)\sin\theta}{16\sqrt{2}r^9}\,, \tag{4.181b}$$

$$\Phi_{12}^{(1,1,0)} = \bar{\Phi}_{21}^{(1,1,0)} = -\frac{15iM^5\,(r-r_s)\left(18M^2+8Mr+3r^2\right)\sin\theta}{32\sqrt{2}r^{10}}\,, \tag{4.181c}$$

which are used in Sec. 4.6 to evaluate $S_{IA}^{(1,1,1)}$ and $S_{IIA}^{(1,1,1)}$. From Eq. (4.181), one can also find $S_{1,2}^{(1,0)}$ to be

$$S_1^{(1,1,0)} = -\frac{45iM^5(42M^2+16Mr+5r^2)\sin\theta}{16\sqrt{2}r^{10}}\,,\quad S_2^{(1,1,0)} = 0\,. \tag{4.182}$$

As discussed in Sec. 4.6, to compute $\mathcal{S}_{IIB}$, one needs to evaluate $\Phi_{00}^{(1,1)}$, $\Phi_{01}^{(1,1)}$, and $\Phi_{00}^{(1,1)}$. When ϑ is stationary, since $\vartheta^{(1,0,0)} = 0$, $\Phi_{ij}^{(1,0,1)} = 0$, and we have only contributions from $\Phi_{ij}^{(1,1,1)}$ at $O(\zeta^1, \chi^1, \epsilon^1)$. Based on our classifications in in Sec. 4.6, we find the first type of contributions to be

$$\begin{aligned} \Phi_{00,A}^{(1,1,1)} = &-\frac{i\mathcal{R}_1}{2}\Big[\Psi_0^{(0,0,1)}\left(\bar{\delta}^2+2\alpha\bar{\delta}\right)\vartheta^{(1,1,0)} - \bar{\Psi}_0^{(0,0,1)}\left(\delta^2+2\alpha\delta\right)\vartheta^{(1,1,0)}\\ &+(\Psi_2^{(0,0,1)}-\bar{\Psi}_2^{(0,0,1)})D^2\vartheta^{(1,1,0)}\Big]\,, \end{aligned} \tag{4.183a}$$

$$\begin{aligned} \Phi_{01,A}^{(1,1,1)} = &-\frac{i\mathcal{R}_1}{4}\Big[\Psi_0^{(0,0,1)}\left(\{\mathbf{\Delta},\bar{\delta}\}-\mu\bar{\delta}\right)\vartheta^{(1,1,0)}\\ &+(\Psi_2^{(0,0,1)}+2\bar{\Psi}_2^{(0,0,1)})\left(\{D,\delta\}+\rho\delta\right)\vartheta^{(1,1,0)}\Big]\,, \end{aligned} \tag{4.183b}$$

$$\begin{aligned} \Phi_{02,A}^{(1,1,1)} = &-\frac{i\mathcal{R}_1}{2}\Big[\Psi_0^{(0,0,1)}(\mathbf{\Delta}^2+2\gamma\mathbf{\Delta})\vartheta^{(1,1,0)} - \bar{\Psi}_4^{(0,0,1)}D^2\vartheta^{(1,1,0)}\\ &+(\Psi_2^{(0,0,1)}-\bar{\Psi}_2^{(0,0,1)})(\delta^2+2\alpha\delta)\vartheta^{(1,1,0)}\Big]\,, \end{aligned} \tag{4.183c}$$

where we used that in Schwarzschild,

$$\begin{aligned} &\bar{\alpha}^{(0,0,0)} = \alpha^{(0,0,0)} = -\beta^{(0,0,0)}\,,\quad \bar{\rho}^{(0,0,0)} = \rho^{(0,0,0)}\,,\\ &\bar{\mu}^{(0,0,0)} = \mu^{(0,0,0)}\,,\quad \bar{\gamma}^{(0,0,0)} = \gamma^{(0,0,0)}\,, \end{aligned} \tag{4.184}$$

and other spin coefficients at $O(\zeta^0, \chi^0, \epsilon^0)$ vanish with the gauge choice $\Psi_1^{(0,1)} = \Psi_3^{(0,1)} = 0$. For simplicity, we have also dropped the superscripts of all the terms at $O(\zeta^0, \epsilon^0)$. For the second type of contributions, we have

$$\Phi_{00,B}^{(1,1,1)} = 0\,, \tag{4.185a}$$

$$\begin{aligned}\Phi_{01,B}^{(1,1,1)} = \frac{3i\mathcal{R}_1}{8}\Big\{&2\left[(D+\rho)a^{(0,0,1)} + 2(\gamma+\mu)b^{(0,0,1)}\right]D\vartheta^{(1,1,0)} \\ &+2(D-\rho)b^{(0,0,1)}\mathbf{\Delta}\vartheta^{(1,1,0)} + 2\left[2\alpha b^{(0,0,1)} + (\delta-4\alpha)\bar{b}^{(0,0,1)}\right]\delta\vartheta^{(1,1,0)} \\ &+\left[2(\delta+2\alpha)b^{(0,0,1)} + Dh_{mm}^{(0,0,1)}\right]\bar{\delta}\vartheta^{(1,1,0)}\Big\}\,,\end{aligned} \tag{4.185b}$$

$$\Phi_{02,B}^{(1,1,1)} = 0\,, \tag{4.185c}$$

where we used that in Schwarzschild,

$$\Psi_{0,1,3,4}^{(0,0,0)} = 0\,, \quad \bar{\Psi}_2^{(0,0,0)} = \Psi_2^{(0,0,0)}\,. \tag{4.186}$$

The parameters $a^{(0,0,1)}$ and $b^{(0,0,1)}$ are rotation parameters given by Eq. (4.77). For the third type of contributions, we find

$$\Phi_{00,C}^{(1,1,1)} = 0\,, \tag{4.187a}$$

$$\begin{aligned}\Phi_{01,C}^{(1,1,1)} = -\frac{3i\mathcal{R}_1\Psi_2}{8}\Big\{&2\left[(D+\rho)h_{nm}^{(0,0,1)} + (D+\rho)a^{(0,0,1)}\right]D\vartheta^{(1,1,0)} \\ &+2(D+\rho)b^{(0,0,1)}\mathbf{\Delta}\vartheta^{(1,1,0)} \\ &+2\delta\bar{b}^{(0,0,1)}\delta\vartheta^{(1,1,0)} - \left[(D+\rho)h_{mm}^{(0,0,1)} - 2\delta b^{(0,0,1)}\right]\bar{\delta}\vartheta^{(1,1,0)} \\ &+4\left(h_{nm}^{(0,0,1)} + a^{(0,0,1)}\right)D^2\vartheta^{(1,1,0)} + 4\bar{b}^{(0,0,1)}\delta^2\vartheta^{(1,1,0)} \\ &+2b^{(0,0,1)}\left(\{D,\mathbf{\Delta}\} + \{\delta,\bar{\delta}\}\right)\vartheta^{(1,1,0)} - h_{mm}^{(0,0,1)}\{D,\bar{\delta}\}\vartheta^{(1,1,0)}\Big\}\,,\end{aligned} \tag{4.187b}$$

$$\Phi_{02,C}^{(1,1,1)} = 0\,, \tag{4.187c}$$

where we used Eqs. (4.184) and (4.186).

For the last type of contributions, since $\vartheta^{(1,1)}$ can have both contributions from $\vartheta^{(1,0,1)}$ and $\vartheta^{(1,1,1)}$, the background metric we need is generally up to $O(\zeta^0, \chi^1, \epsilon^0)$. This is also true for the operators converting Φ_{ij} to $\mathcal{S}$. For this reason, we will not expand the expression below explicitly in χ but do the expansion at the end when plugging in the coordinate-based values of the NP quantities. Then, at $O(\zeta^1, \epsilon^1)$, we find

$$\Phi_{00,D}^{(1,1)} = \mathcal{R}_2 D\vartheta^{(1,0)}D\vartheta^{(1,1)} - \frac{i\mathcal{R}_1}{2}(\Psi_2 - \bar{\Psi}_2)D^2\vartheta^{(1,1)}\,, \tag{4.188a}$$

$$\Phi^{(1,1)}_{01,D} = \frac{1}{4}\Big[2\mathcal{R}_2(\delta\vartheta^{(1,0)}D + D\vartheta^{(1,0)}\delta)\vartheta^{(1,1)} + i\mathcal{R}_1(\bar{\alpha}+\beta+\bar{\pi})(\Psi_2+2\bar{\Psi}_2)D\vartheta^{(1,1)}$$
$$-i\mathcal{R}_1(\Psi_2+2\bar{\Psi}_2)(\{D,\delta\}+\bar{\rho}\delta)\vartheta^{(1,1)}\Big]\,, \tag{4.188b}$$

$$\Phi^{(1,1)}_{02,D} = \frac{1}{2}\Big[2\mathcal{R}_2\delta\vartheta^{(1,0)} + i\mathcal{R}_1(\beta-\bar{\alpha})(\Psi_2-\bar{\Psi}_2)\Big]\delta\vartheta^{(1,1)} - \mathcal{R}_1(\Psi_2-\bar{\Psi}_2)\delta^2\vartheta^{(1,1)}\,. \tag{4.188c}$$

Notice, since we do not do an expansion in χ above, Eq. (4.188) works for fully rotating BHs in dCS gravity. In addition, due to the expansion convention we used, where we have absorbed an additional $\zeta^{\frac{1}{2}}$ into the expansion of ϑ, there are terms taking the form $\sim \mathcal{R}_2\vartheta^{(1,0)}\vartheta^{(1,1)}$ at $O(\zeta^1,\epsilon^1)$. These terms come from the usual pseudoscalar action with minimal coupling. In our expansion convention, we have inserted ζ^{-1} before these terms [i.e., Eqs. (4.106) and (4.179)], so their contribution is still at $O(\zeta^1,\epsilon^1)$. In the equations above, we have also dropped the superscript for terms at $O(\zeta^0,\epsilon^0)$ for simplicity.

4.14 Appendix: An approach to compute projection coefficients in Eqs. (4.133) and (4.134)

In this section, we present an approach to compute the projection coefficients in Eqs. (4.133) and (4.134) using the series representation of spin-weighted spherical harmonics ${}_sY_{\ell m}(\theta,\phi)$ in Eq. (4.63). From Eq. (4.63), we can see that the integrals in Eqs. (4.133) and (4.134) become a series sum over q_1 and q_2 of

$$Q(a,b,c,d) \equiv \int d\theta\, \sin^a\left(\frac{\theta}{2}\right)\cot^b\left(\frac{\theta}{2}\right)\sin^{1+c}\theta\cos^d\theta\,, \tag{4.189}$$
$$a = 2(\ell_1+\ell_2)\,,$$
$$b = 2(q_1+q_2)+s_1+s_2-m_1-m_2\,,$$
$$c,d \in \{0,1\}\,,$$

multiplied by the remaining constants dependent on (s_1,ℓ_1,m_1) and (s_2,ℓ_2,m_2) in Eq. (4.63). The integral in Eq. (4.189) can be evaluated analytically in terms of Gamma functions, i.e.,

$$Q(a,b,0,0) = \frac{2\Gamma\left(1+\frac{a-b}{2}\right)\Gamma\left(1+\frac{b}{2}\right)}{\Gamma\left(2+\frac{a}{2}\right)}\,, \tag{4.190a}$$
$$Q(a,b,0,1) = \frac{(2b-a)\Gamma\left(1+\frac{a-b}{2}\right)\Gamma\left(1+\frac{b}{2}\right)}{\Gamma\left(3+\frac{a}{2}\right)}\,, \tag{4.190b}$$

$$Q(a,b,1,0) = \frac{4\Gamma\left(\frac{3+a-b}{2}\right)\Gamma\left(\frac{3+b}{2}\right)}{\Gamma\left(3+\frac{a}{2}\right)}\,, \tag{4.190c}$$

so we can express the coefficients in Eqs. (4.133) and (4.134) as a series sum of Gamma functions, which are much faster to evaluate than direct integration for large $\ell_{1,2}$. More specifically, we get

$$\begin{aligned}
&\boldsymbol{\Lambda}^{\ell_1\ell_2 m}_{s_1 s_2}(\alpha,\beta) \\
&= \frac{1}{2}\sqrt{\frac{(\ell_1+m)!(\ell_1-m)!(2\ell_1+1)}{(\ell_1+s_1)!(\ell_1-s_1)!}}\sqrt{\frac{(\ell_2+m)!(\ell_2-m)!(2\ell_2+1)}{(\ell_2+s_2)!(\ell_2-s_2)!}} \\
&\quad\times\sum_{q=0}^{\ell-s}\Bigg[\begin{pmatrix}\ell_1-s_1\\ q_1\end{pmatrix}\begin{pmatrix}\ell_1+s_1\\ q_1+s_1-m\end{pmatrix}\begin{pmatrix}\ell_2-s_2\\ q_2\end{pmatrix}\begin{pmatrix}\ell_2+s_2\\ q_2+s_2-m\end{pmatrix} \\
&\quad(-1)^{(\ell_1+\ell_2-s_1-s_2+q_1+q_2)}Q(2\ell_1+2\ell_2, 2q_1+2q_2+s_1+s_2-2m,\alpha,\beta)\Bigg]\,,
\end{aligned} \tag{4.191a}$$

$$\begin{aligned}
&\boldsymbol{\Lambda}^{\dagger\ell_1\ell_2 -m}_{s_1 s_2}(\alpha,\beta) \\
&= \frac{1}{2}\sqrt{\frac{(\ell_1+m)!(\ell_1-m)!(2\ell_1+1)}{(\ell_1+s_1)!(\ell_1-s_1)!}}\sqrt{\frac{(\ell_2-m)!(\ell_2+m)!(2\ell_2+1)}{(\ell_2+s_2)!(\ell_2-s_2)!}} \\
&\quad\times\sum_{q=0}^{\ell-s}\Bigg[\begin{pmatrix}\ell_1-s_1\\ q_1\end{pmatrix}\begin{pmatrix}\ell_1+s_1\\ q_1+s_1-m\end{pmatrix}\begin{pmatrix}\ell_2-s_2\\ q_2\end{pmatrix}\begin{pmatrix}\ell_2+s_2\\ q_2+s_2+m\end{pmatrix} \\
&\quad(-1)^{(\ell_1+\ell_2-s_1-s_2+q_1+q_2)}Q(2\ell_1+2\ell_2, 2q_1+2q_2+s_1+s_2,\alpha,\beta)\Bigg]\,,
\end{aligned} \tag{4.191b}$$

where

$$\boldsymbol{\Lambda}^{\ell_1\ell_2 m}_{s_1 s_2}(0,0) = \Lambda^{\ell_1\ell_2 m}_{s_1 s_2}\,,\quad \boldsymbol{\Lambda}^{\ell_1\ell_2 m}_{s_1 s_2}(0,1) = \Lambda^{\ell_1\ell_2 m}_{s_1 s_2 c}\,,\quad \boldsymbol{\Lambda}^{\ell_1\ell_2 m}_{s_1 s_2}(1,0) = \Lambda^{\ell_1\ell_2 m}_{s_1 s_2 s}\,, \tag{4.192a}$$

$$\boldsymbol{\Lambda}^{\dagger\ell_1\ell_2 m}_{s_1 s_2}(0,0) = \Lambda^{\dagger\ell_1\ell_2 m}_{s_1 s_2}\,,\quad \boldsymbol{\Lambda}^{\dagger\ell_1\ell_2 m}_{s_1 s_2}(0,1) = \Lambda^{\dagger\ell_1\ell_2 m}_{s_1 s_2 c}\,,\quad \boldsymbol{\Lambda}^{\dagger\ell_1\ell_2 m}_{s_1 s_2}(1,0) = \Lambda^{\dagger\ell_1\ell_2 m}_{s_1 s_2 s}\,. \tag{4.192b}$$

We have provided this series-sum representation of the coefficients in Eqs. (4.133) and (4.134) in a Mathematica notebook as Supplementary Material [69].

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Chapter 5

PERTURBATIONS OF SPINNING BLACK HOLES IN DYNAMICAL CHERN-SIMONS GRAVITY: SLOW ROTATION QUASINORMAL MODES

[1] Dongjun Li, Pratik Wagle, Nicolás Yunes, and Yanbei Chen. *Perturbations of spinning black holes in dynamical Chern-Simons gravity: Slow rotation quasinormal modes*. in preparation. 2024.

5.1 Introduction

The detection of gravitational waves (GWs) emitted by over a hundred binary black hole (BH) mergers [1] opens an avenue to studying gravity in the strong regime. Einstein's general relativity (GR), although has passed numerous tests in our solar system [2], is still undergoing tests in strong gravity. One important test one could make with GW detections is to look at the ringdown part of the signal, where the GWs are dominated by these quickly decaying and oscillating quasinormal modes (QNMs) when the remnant BH settles down [3]. In GR, QNMs encode the mass, spin, and charge of the remnant BH, which completely determine a BH spacetime as required by the "no-hair" theorem [4, 5], which has been tested with the observational data from the LIGO-Virgo-KAGRA collaboration [6–10].

Nonetheless, GR is unlikely to be the ultimate theory of gravity, as it is incompatible with quantum mechanics and unable to explain certain observational puzzles, such as the asymmetry of matter and antimatter in our universe [11]. For these reasons, modifications to GR are introduced either by constructing a unified theory for quantum gravity, such as string theory [12–17] and loop quantum gravity [18–23], or by explaining certain observational puzzles. In beyond-GR theories, BH spacetimes can carry additional hair due to the presence of other scalar or vector fields [24–30]. The spectrum of QNMs, in principle, allows us to extract these additional hairs as well as the length scale or the coupling constant associated with these beyond-GR interactions, which is the core idea of BH spectroscopy [31–33].

To compute the QNM spectrum for non-rotating BHs in GR, the standard approach was developed by Regge, Wheeler, Zerilli, and Moncrief in [34–36], where one separates the metric perturbations into even and odd parity. For each parity, one can construct a master function, i.e., the Regge-Wheeler (RW) and Zerilli-Moncrief (ZM) functions for the even- and odd-parity metric perturbations, respectively, such that its equation of motion can be reduced to a second-order ordinary differential equation in the radial coordinate. The QNM spectrum of a Schwarzschild BH was initially computed in [37] by numerical integration and can be more systematically calculated using the continuous fraction method developed by Leaver in [38].

For rotating BHs in GR, directly solving metric perturbations is much more challenging due to the lack of spherical symmetry. In this case, one cannot easily decouple the even- and odd-parity modes, find two master functions characterizing all the metric components, and reduce their equations of motion into purely radial ordinary differential equations. Built upon the Newman-Penrose (NP) formalism, Teukolsky found a solution to this problem in [39–41] by instead solving for curvature perturbations represented by two radiative Weyl scalars Ψ_0 and Ψ_4. In this case, Teukolsky was able to reduce the NP equations describing $\Psi_{0,4}$ into two decoupled and separable partial differential equations. The solutions to the angular equations are called spin-weighted spheroidal harmonics, and the radial equations are generalized spheroidal wave equations [38]. Both the angular and radial parts of $\Psi_{0,4}$, as well as the QNM frequencies, can be calculated using the Leaver's method in [38] and extensions of it, such as the Mano-Suzuki-Takasugi method [42–45].

Many efforts have been made over the past ten years to study QNMs in beyond-GR theories. For non-rotating BHs, one can still apply the standard metric perturbation approach developed by [34–36]. Beyond-GR QNMs in the non-rotating case were computed in, for example, dynamical Chern-Simons (dCS) gravity [46–48], Einstein dilaton Gauss-Bonnet theory [49–51], higher-derivative gravity without extra fields [52–54], and Einstein-Aether theory [55, 56]. For rotating BHs, the calculations of QNMs are much more challenging. There are only a few examples available, which all rely on the slow-rotation expansion, such as in dCS gravity [57, 58], EdGB gravity [59, 60], and higher-derivative gravity [61, 62]. Although QNMs for a general rotating BH could be extracted from full numerical simulations, for example, in dCS gravity [63, 64], this approach has the issue of secularly growing terms. Until recently, Refs. [65–67] made an important extension of the Teukolsky formalism from GR to a wide class of beyond-GR theories admitting an effective-field-theory description. This modified Teukolsky formalism, for the first time, allows a systematic semi-analytical calculation of QNMs for BHs with a general spin, including these fast rotating ones, in beyond-GR theories. As most of the remnant BHs of binary mergers are fast rotating [68, 69], applying the modified Teukolsky formalism to specific beyond-GR theories is crucial for conducting BH spectroscopy.

The first attempt was made by [67, 70] on computing the QNM frequency shifts in higher-derivative gravity without extra fields. They considered both the parity-preserving and parity-violating corrections to the Einstein-Hilbert action up to the sixth derivative of the metric. Although the modified Teukolsky formalism works for BHs with a general spin, Refs. [67, 70] performed a slow-rotation expansion up to $O(\chi^{14})$, where $\chi = a/M$ is the dimensionless spin, since the background BH metric is known analytically only under the slow-rotation expansion. Their results are valid for the spin up to $\chi \sim 0.7$. Another attempt was recently made by [71] on a more complicated case, dCS gravity, where a pseudoscalar field ϑ is coupled to a quadratic term in the Riemann tensor and its dual, the so-called Pontryagin density. In [71], we successfully implemented the Chrzanowski-Cohen-Kegeles (CCK) [72–74] procedure for metric reconstruction to compute all the source terms in the modified Teukolsky equation derived in [65]. After a projection to the spin-weighted spheroidal harmonics, we obtained two sets of radial ordinary differential equations in two different gauges, the ingoing radiation gauge (IRG) and the outgoing radiation gauge (ORG). For each set, we got an equation for the curvature perturbation (Ψ_0 in the IRG and Ψ_4 in the ORG) and another equation for the (pseudo)scalar field perturbation. In [71], we performed a slow-rotation expansion up to $O(\chi)$ since we would like to compare our results with the one in [46–48, 57, 58].

In this work, we directly compute the QNM frequency shifts for a slowly rotating BH up to $O(\chi)$ in dCS gravity using the radial equations found in [71]. We apply the analysis of isospectrality breaking in [75] to simplify the radial master equations first and then use the eigenvalue perturbation (EVP) method in [66, 76, 77] to compute the QNM frequency shifts. Specifically, in Sec. 5.2, we briefly review the procedures for computing the modified Teukolsky equations in dCS gravity in [71]. In Secs. 5.3 and 5.4, we apply the procedures in [75] to simplify the master equations for the scalar field ϑ and for the Weyl scalars $\Psi_{0,4}$, respectively. We also discuss the parity features of these equations and their implications on the structure of isospectrality breaking. In Sec. 5.5, we review the EVP method in [66, 76, 77], show how to apply it to the specific case of dCS gravity when an extra scalar field presents, and then compute the QNM frequency shifts. We present our results in Sec. 5.5 and compare them to the previous calculations in [46–48, 57, 58]. In the end, we discuss future avenues in this direction.

5.2 Review of the master equations

In our previous work [71], we applied the modified Teukolsky formalism developed in [65] to compute the master equations governing the perturbations of the Weyl scalars $\Psi_{0,4}$ and the (pseudo)scalar field ϑ for slowly rotating BHs in dCS gravity. We successfully found four radial ordinary differential equations governing the perturbations of Ψ_0 and ϑ in the IRG and the perturbations of Ψ_4 and ϑ in the ORG. In this section, we will briefly review the derivation of these equations.

In dCS, the equations of motion for both the metric field and the scalar field were found to be [28, 30, 78, 79]

$$\Box\vartheta = -\frac{\alpha}{4}R_{\nu\mu\rho\sigma}{}^{*}R^{\mu\nu\rho\sigma}\,, \tag{5.1}$$

$$R_{\mu\nu} = -\frac{\alpha}{\kappa_g}C_{\mu\nu} + \frac{1}{2\kappa_g}\bar{T}^{\vartheta}_{\mu\nu}\,, \tag{5.2}$$

where $\kappa_g = \frac{1}{16\pi}$, α is the dCS coupling constant, $\Box = \nabla_\mu\nabla^\mu$ is the D'Alembertian operator, ${}^{*}R^{\mu\nu\rho\sigma}$ is the dual of the Riemann tensor,

$${}^{*}R^{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon^{\rho\sigma\alpha\beta}R^{\mu\nu}{}_{\alpha\beta}\,, \tag{5.3}$$

and

$$C^{\mu\nu} \equiv (\nabla_\sigma\vartheta)\,\epsilon^{\sigma\delta\alpha(\mu}\nabla_\alpha R^{\nu)\delta} + (\nabla_\sigma\nabla_\delta\vartheta)^{*}R^{\delta(\mu\nu)\sigma}\,, \tag{5.4}$$

$$\bar{T}^{\vartheta}_{\mu\nu} \equiv \left(\nabla_\mu\vartheta\right)\left(\nabla_\nu\vartheta\right)\,. \tag{5.5}$$

For a detailed review of dCS gravity and how Eqs. (5.1) and (5.2) are derived, one can refer to [28, 30, 78, 79]. To solve these two equations, Ref. [71] took an effective field theory approach by performing a two-parameter expansion, i.e.,

$$\vartheta = \zeta\vartheta^{(1,0)} + \zeta\epsilon\vartheta^{(1,1)}\,, \tag{5.6}$$

$$\Psi_i = \Psi_i^{(0,0)} + \zeta\Psi_i^{(1,0)} + \epsilon\Psi_i^{(0,1)} + \zeta\epsilon\Psi_i^{(1,1)}\,, \tag{5.7}$$

where we take the Weyl scalars Ψ_i as an example, and the other NP quantities follow the same expansion scheme. The dimensionless constant ζ is related to the dCS coupling constant by

$$\zeta \equiv \frac{\alpha^2}{\kappa_g M^4} \tag{5.8}$$

with M being the typical mass of the system. The other expansion parameter ϵ represents the strength of the GW perturbations of a binary merger's remnant BH. For example, for an EMRI, ϵ is proportional to the mass ratio between the stellar-mass object and the supermassive BH. As discussed in more detail in [71, 75], we do not assume any hierarchy between ζ and ϵ since their relative size depends on the details of the bGR theory and the binary system. In this case, the terms at $O(\zeta^0, \epsilon^0)$ and $O(\zeta^1, \epsilon^0)$ are computed using the slowly rotating Kerr metric and the dCS correction to it. The terms at $O(\zeta^0, \epsilon^1)$ correspond to GWs in GR, while the terms at $O(\zeta^1, \epsilon^1)$ are additional GWs driven by the dCS corrections, which we are interested in.

Using the expansion in Eq. (5.6), the scalar field equation Eq. (5.1) becomes

$$\square^{(0,0)}\vartheta^{(1,1)} = -\frac{M^2}{16\pi^{\frac{1}{2}}}\,[R^*R]^{(0,1)} - \square^{(0,1)}\vartheta^{(1,0)} \,. \tag{5.9}$$

For the metric fields, instead of solving the trace-reversed Einstein equations in Eq. (5.2), we will solve the modified Teukolsy equations found by [65, 71], i.e.,

$$H_0^{(0,0)}\Psi_0^{(1,1)} = \mathcal{S}_{\text{geo}}^{(1,1)} + \mathcal{S}^{(1,1)} \,, \tag{5.10}$$

$$H_4^{(0,0)}\Psi_4^{(1,1)} = \mathcal{T}_{\text{geo}}^{(1,1)} + \mathcal{T}^{(1,1)} \,. \tag{5.11}$$

Here, $H_{0,4}^{(0,0)}$ are the Teukolsky operators in GR for $\Psi_{0,4}$, respectively. The source terms $\mathcal{S}_{\text{geo}}^{(1,1)}$ and $\mathcal{T}_{\text{geo}}^{(1,1)}$ only depend on the dCS correction to the background spacetime $h_{\mu\nu}^{(1,0)}$ as well as the gravitational perturbation $h_{\mu\nu}^{(0,1)}$ in GR, so they are regarded as purely "geometrical." In contrast, $\mathcal{S}^{(1,1)}$ and $\mathcal{T}^{(1,1)}$ directly depend on the effective stress tensor [i.e., Eq. (5.2)], so they are driven by both the GR GW $h_{\mu\nu}^{(0,1)}$ and the scalar field perturbation $\vartheta^{(1,1)}$. Notice that we absorbed a factor of $\zeta^{1/2}$ into ϑ in [71], so $\vartheta^{(1,1)}$ actually enters at $O(\zeta^{1/2}, \epsilon^1)$. Thus, one can solve for $\vartheta^{(1,1)}$ first using Eq. (5.9) and then plug it into Eqs. (5.10) and (5.11) to solve for $\Psi_{0,4}^{(1,1)}$. For complete expressions of Eqs. (5.9)–(5.11) in terms of NP quantities, one can refer to [65, 71].

To evaluate the source terms in Eqs. (5.9)–(5.11), one needs to know the metric perturbation $h_{\mu\nu}^{(0,1)}$ in GR. This was achieved by implementing the CCK procedures [72–74, 80–85] to reconstruct the perturbed metric $h_{\mu\nu}^{(0,1)}$ from the perturbed Weyl scalars $\Psi_{0,4}^{(0,1)}$. This procedure relies on the IRG or the ORG, i.e.,

$$\text{IRG:} \quad h_{\mu\nu}l^\nu = 0\,,\ h = 0\,, \tag{5.12a}$$

$$\text{ORG:} \quad h_{\mu\nu}n^\nu = 0\,,\ h = 0\,, \tag{5.12b}$$

where l^μ and n^ν are two tetrad basis vectors in the NP formalism. Under these gauges, $h_{\mu\nu}^{(0,1)}$ can be expressed in terms of the so-called Hertz potential Ψ_{H},

$$h_{\mu\nu}^{(0,1)} = \mathcal{O}_{\mu\nu}\bar{\Psi}_{\text{H}} + \bar{\mathcal{O}}_{\mu\nu}\Psi_{\text{H}} \,, \tag{5.13}$$

where we use $\bar{f}$ to denote the complex conjugate of f. The operator $O_{\mu\nu}$ in the IRG and ORG can be found in [71, 84, 85]. Expanding $\Psi_{0,4}^{(0,1)}$ using spin-weighted spheroidal harmonics ${}_s\mathcal{Y}_{\ell m}(\theta,\phi) = {}_sS_{\ell m}(\theta)e^{im\phi}$,

$$\Psi_0^{(0,1)} = \sum_{\ell,m} {}_2R_{\ell m}^{(0,1)}(r)\,{}_2\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}\,, \tag{5.14a}$$

$$\Psi_4^{(0,1)} = \sum_{\ell,m} {}_{-2}R_{\ell m}^{(0,1)}(r)\,{}_{-2}\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}\,, \tag{5.14b}$$

and similarly for $\bar{\Psi}_{\rm H}$,

$$\text{IRG}: \quad \bar{\Psi}_{\rm H} = \sum_{\ell,m} {}_2\hat{R}_{\ell m}(r)\,{}_2\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}\,, \tag{5.15a}$$

$$\text{ORG}: \quad \bar{\Psi}_{\rm H} = \sum_{\ell,m} {}_{-2}\hat{R}_{\ell m}(r)\,{}_{-2}\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}\,, \tag{5.15b}$$

one can compute the radial parts ${}_s\hat{R}_{\ell m}(r)$ of $\Psi_{\rm H}$ from ${}_sR_{\ell m}^{(0,1)}(r)$ following [81],

$${}_2\hat{R}_{\ell m}(r) = -\frac{2}{\mathfrak{C}}\Delta^2(r)(D_{\ell m}^\dagger)^4\left[\Delta^2(r)\,{}_2R_{\ell m}^{(0,1)}(r)\right]\,, \tag{5.16a}$$

$${}_{-2}\hat{R}_{\ell m}(r) = \frac{32}{\mathfrak{C}}(D_{\ell m})^4{}_{-2}R_{\ell m}^{(0,1)}(r)\,. \tag{5.16b}$$

The operators $D_{\ell m}$ and $D_{\ell m}^\dagger$ are defined to be

$$D_{\ell m} = \partial_r + i\frac{am-(r^2+a^2)\omega_{\ell m}}{\Delta(r)}\,, \quad D_{\ell m}^\dagger = \partial_r - i\frac{am-(r^2+a^2)\omega_{\ell m}}{\Delta(r)}\,, \tag{5.17}$$

where a is the spin of the BH, and $\Delta(r) = r^2 - 2Mr + a^2$. The coefficient $\mathfrak{C}$ is the mode-dependent Teukolsky-Starobinsky constant [41, 67, 70, 81, 86, 87],

$$\begin{aligned}
\mathfrak{C} = {} & 144M^2\omega_{\ell m}^2 + \left(8 + 6{}_sB_{\ell m} + {}_sB_{\ell m}^2\right)^2 - 8\left[-8 + {}_sB_{\ell m}^2\left(4 + {}_sB_{\ell m}\right)\right]m\gamma_{\ell m} \\
& + 4\left[8 - 2{}_sB_{\ell m} - {}_sB_{\ell m}^2 + {}_sB_{\ell m}^3 + 2\left(-2 + {}_sB_{\ell m}\right)\left(4 + 3{}_sB_{\ell m}\right)m^2\right]\gamma_{\ell m}^2 \\
& - 8m\left(8 - 12{}_sB_{\ell m} + 3{}_sB_{\ell m}^2 + 4\left(-2 + {}_sB_{\ell m}\right)m^2\right)\gamma_{\ell m}^3 \\
& + 2\left(42 - 22{}_sB_{\ell m} + 3{}_sB_{\ell m}^2 + 8\left(-11 + 3{}_sB_{\ell m}\right)m^2 + 8m^4\right)\gamma_{\ell m}^4 \\
& - 8m\left[3{}_sB_{\ell m} + 4\left(-4 + m^2\right)\right]\gamma_{\ell m}^5 + 4\left(-7 + {}_sB_{\ell m} + 6m^2\right)\gamma_{\ell m}^6 - 8m\gamma_{\ell m}^7 + \gamma_{\ell m}^8\,,
\end{aligned} \tag{5.18}$$

where $\tilde{\alpha}^2 = a^2 - am/\omega_{\ell m}$, $\gamma_{\ell m} = \chi M\omega_{\ell m}$, ${}_sB_{\ell m} = {}_sA_{\ell m} + s$, s is the spin weight, and ${}_sA_{\ell m}$ is the angular separation constant in the Teukolsky equations [39]. Here, we have used the Teukolsky-Starobinsky coefficient found in [67, 70] instead of the original one in [41, 81, 86, 87], the latter of which was derived for real frequencies. For convenience, let us also define

$$\mathcal{D}^{\dagger}_{\ell m} \equiv -\frac{2}{\mathfrak{C}}\Delta^2(r)(D^{\dagger}_{\ell m})^4\Delta^2(r)\,, \quad \mathcal{D}_{\ell m} \equiv \frac{32}{\mathfrak{C}}(D_{\ell m})^4\,. \tag{5.19}$$

In [71], we used the CCK procedures to evaluate all the source terms in Eqs. (5.9)–(5.11). We found that the (ℓ, m) and $(\ell, -m)$ modes of ${}_2\hat{R}_{\ell m}(r)$ (or ${}_{-2}\hat{R}_{\ell m}(r)$) in the IRG (or ORG) are coupled to each other in the source terms. Thus, to solve Eqs. (5.9)–(5.11) consistently, we need to solve the (ℓ, m) and $(\ell, -m)$ modes of Ψ_0 (or Ψ_4) jointly by using the following ansatz:

$${}_s\Psi^{(0,1)}_{\ell m} = \sum_{\ell m} {}_sR^{(0,1)}_{\ell m}(r)\,{}_s\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t} + \eta_{\ell m}\;{}_sR^{(0,1)}_{\ell -m}(r)\,{}_s\mathcal{Y}_{\ell\, -m}(\theta,\phi)e^{i\bar{\omega}_{\ell m}t}\,, \tag{5.20a}$$

$${}_s\Psi^{(1,1)}_{\ell m} = \sum_{\ell m} {}_sR^{(1,1)}_{\ell m}(r)\,{}_s\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t} + \eta_{\ell m}\;{}_sR^{(1,1)}_{\ell -m}(r)\,{}_s\mathcal{Y}_{\ell\, -m}(\theta,\phi)e^{i\bar{\omega}_{\ell m}t}\,, \tag{5.20b}$$

where ${}_2\Psi_{\ell m} \equiv \Psi_{0,\ell m}$ and ${}_{-2}\Psi_{\ell m} \equiv \Psi_{4,\ell m}$. Following [71, 75], to solve the (ℓ, m) and $(\ell, -m)$ modes consistently, we have imposed in Eq. (5.20) that

$$\omega_{\ell -m} = -\bar{\omega}_{\ell m}\,, \tag{5.21}$$

which is the same symmetry satisfied by the QNM frequencies in GR. Notice that the coefficient $\eta_{\ell m}$ is well-defined once we fix the normalization of ${}_sR^{(0,1)}_{\ell m}(r)$ to be

$${}_sR^{(0,1)}_{\ell -m}(r) = (-1)^m\,{}_s\bar{R}^{(0,1)}_{\ell m}(r)\,, \tag{5.22}$$

and the same symmetry can be imposed on ${}_s\hat{R}_{\ell m}(r)$. Similarly, we expand the scalar field perturbation $\vartheta^{(1,1)}$ as

$$\vartheta^{(1,1)} = \sum_{\ell,m} \frac{\Theta_{\ell m}(r)}{r}\,{}_0\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}\,. \tag{5.23}$$

Notice that we include an extra factor of $1/r$ in Eq. (5.23) following [46, 47, 57], so we will multiply a factor of r to the scalar field equation found in [71], as discussed in Sec. 5.3. Since ϑ is a real scalar field and ${}_0\mathcal{Y}_{\ell -m}(\theta,\phi) = (-1)^m\,{}_0\bar{\mathcal{Y}}_{\ell m}(\theta,\phi)$, we need

$$\bar{\Theta}_{\ell -m}(r) = (-1)^m\Theta_{\ell m}(r)\,. \tag{5.24}$$

Using the ansatz in Eqs. (5.20) and (5.23) and projecting Eqs. (5.9), (5.10), and (5.11) to ${}_0\mathcal{Y}_{\ell m}(\theta,\phi)$, ${}_2\mathcal{Y}_{\ell m}(\theta,\phi)$, and ${}_{-2}\mathcal{Y}_{\ell m}(\theta,\phi)$, respectively, we got these purely radial ordinary differential equations governing $\left\{\vartheta^{(1,1)}, \Psi_0^{(1,1)}, \Psi_4^{(1,1)}\right\}$ in [71]. In Secs. 5.3 and 5.4, we will closely look at and simplify these equations for the convenience of calculating QNM frequencies in this work. We will focus on the equations of $\vartheta^{(1,1)}$ and $\Psi_0^{(1,1)}$ in the IRG as a demonstration, and the results in the ORG will be presented at the end of each section.

5.3 Scalar field equation

In this section, we review and simplify the master equation governing the scalar field perturbation $\vartheta^{(1,1)}$ in [71]. In [71], we found that up to $O(\zeta^1, \chi^1, \epsilon^1)$, the radial part $\Theta_{\ell m}(r)$ of $\vartheta^{(1,1)}$ in the IRG satisfies

$$\begin{aligned}
&\left[r(r-r_s)\partial_r^2 + r_s\partial_r + \frac{\omega^2 r^3 - 4\chi m M^2\omega}{r-r_s} - \frac{r_s}{r} - {}_0A_{\ell m}\right]\Theta_{\ell m}(r) \\
&= V^R_{\ell m}(r) + V^{\square}_{\ell m}(r) + \bar{\eta}_{\ell m}\left(V^{\dagger R}_{\ell -m}(r) + V^{\dagger\square}_{\ell -m}(r)\right), \qquad (5.25a)\\
&\left[r(r-r_s)\partial_r^2 + r_s\partial_r + \frac{\omega^2 r^3 + 4\chi m M^2\omega}{r-r_s} - \frac{r_s}{r} - {}_0A_{\ell -m}\right]\Theta_{\ell -m}(r) \\
&= \eta_{\ell m}\left(V^R_{\ell -m}(r) + V^{\square}_{\ell -m}(r)\right) + V^{\dagger R}_{\ell m}(r) + V^{\dagger\square}_{\ell m}(r), \qquad (5.25b)
\end{aligned}$$

where $r_s = 2M$ is the Schwarzschild radius, and

$$\begin{aligned}
V^R_{\ell m}(r) &= i\left(g_1^{\ell m}(r)\,{}_2\hat{R}_{\ell m}(r) + g_2^{\ell m}(r)\,{}_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{00} \\
&\quad + \chi\left(g_3^{\ell m}(r)\,{}_2\hat{R}_{\ell m}(r) + g_4^{\ell m}(r)\,{}_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{10s}, \qquad (5.26a)\\
V^{\square}_{\ell m}(r) &= \chi\left(h_1^{\ell m}(r)\,{}_2\hat{R}_{\ell m}(r) + h_2^{\ell m}(r)\,{}_2\hat{R}'_{\ell m}(r)\right)\Lambda^{\ell\ell m}_{10s}. \qquad (5.26b)
\end{aligned}$$

The functions $V^{\dagger R}_{\ell -m}$ and $V^{\dagger\square}_{\ell -m}$ refer to taking the complex conjugate of all the radial functions in $V^R_{\ell m}$ and $V^{\square}_{\ell m}$ but replacing $\{\Lambda^{\ell_1\ell_2 m}_{s_1 s_2}, \Lambda^{\ell_1\ell_2 m}_{s_1 s_2 c}, \Lambda^{\ell_1\ell_2 m}_{s_1 s_2 s}\}$ with $\{\Lambda^{\dagger\ell_1\ell_2 m}_{s_1 s_2}, \Lambda^{\dagger\ell_1\ell_2 m}_{s_1 s_2 c}, \Lambda^{\dagger\ell_1\ell_2 m}_{s_1 s_2 s}\}$. Here, $f'(r)$ denotes the partial derivative of $f(r)$ in r. Notice that Eq. (5.25b) comes from the terms generated by the $(\ell, -m)$ mode of the ansatz in Eq. (5.20). It is different from taking m to $-m$ in Eq. (5.25a) directly, which corresponds to another solution that is a factor $1/\eta_{\ell m}$ of Eq. (5.20). We also redefine all the radial functions $g_i^{\ell m}(r) = g_i^{\ell m}(r, \omega, M)$ and $h_i^{\ell m}(r) = h_i^{\ell m}(r, \omega, M)$ in [71] as following:

$$\begin{aligned}
g_1^{\ell m}(r) &\to i\frac{M^2 r^3}{16\pi^{\frac{1}{2}}}g_1^{\ell m}(r), & g_2^{\ell m}(r) &\to i\frac{M^2 r^3}{16\pi^{\frac{1}{2}}}g_2^{\ell m}(r), \\
g_3^{\ell m}(r) &\to -\frac{M^2 r^3}{16\pi^{\frac{1}{2}}}g_3^{\ell m}(r), & g_4^{\ell m}(r) &\to -\frac{M^2 r^3}{16\pi^{\frac{1}{2}}}g_4^{\ell m}(r), \\
h_1^{\ell m}(r) &\to -r^3 h_1^{\ell m}(r), & h_2^{\ell m}(r) &\to -r^3 h_2^{\ell m}(r). \qquad (5.27)
\end{aligned}$$

In this case, $g_i^{\ell m}(r)$ and $h_i^{\ell m}(r)$ become real functions in $(r, -i\omega, im)$, where the imaginary unit i only comes from the combination $-i\omega$ or im. Using $\omega_{\ell m} = -\bar{\omega}_{\ell -m}$ in Eq. (5.21), we get

$$\bar{g}_i^{\ell -m}(r) = g_i^{\ell m}(r)\,, \quad \bar{h}_i^{\ell -m}(r) = h_i^{\ell m}(r)\,, \tag{5.28}$$

where the expression of $g_i^{\ell m}(r)$ and $h_i^{\ell m}(r)$ before the redefinition can be found in [71] and the supplementary Mathematica notebook [88]. In addition, $g_1^{\ell m}(r)$ and $g_2^{\ell m}(r)$ contain terms up to $O(\chi^1)$, and χ only shows up in the combination χm. The other radial functions $g_i^{\ell m}(r)$ and $h_i^{\ell m}(r)$ in Eq. (5.26) do not contain any factor of χ or m. The coefficients $\{\Lambda_{s_1 s_2}^{\ell_1 \ell_2 m}, \Lambda_{s_1 s_2 c}^{\ell_1 \ell_2 m}, \Lambda_{s_1 s_2 s}^{\ell_1 \ell_2 m}\}$ and $\{\Lambda_{s_1 s_2}^{\dagger \ell_1 \ell_2 m}, \Lambda_{s_1 s_2 c}^{\dagger \ell_1 \ell_2 m}, \Lambda_{s_1 s_2 s}^{\dagger \ell_1 \ell_2 m}\}$ come from projecting the angular functions in the source terms to ${}_0\mathcal{Y}_{\ell m}(\theta, \phi)$, where their definition can be found in [71]. Using Eqs. (5.96), (5.105), and (5.108) in Appendix 5.6, we can show that

$$\Lambda_{10s}^{\ell \ell m} = m \Lambda_{10s}^{\ell \ell 1}\,, \quad \Lambda_{10s}^{\dagger \ell \ell m} = (-1)^{m+1} m \Lambda_{10s}^{\ell \ell 1}\,. \tag{5.29}$$

Since both the radial Teukolsky function ${}_sR_{\ell m}^{(0,1)}(r)$ and the radial part ${}_s\hat{R}_{\ell m}(r)$ of the Hertz potential satisfy Eq. (5.22), Eq. (5.26) reduces to

$$\begin{aligned} V_{\ell m}^{R}(r) = {} & i \left(g_1^{\ell m}(r)\, {}_2\hat{R}_{\ell m}(r) + g_2^{\ell m}(r)\, {}_2\hat{R}'_{\ell m}(r) \right) \\ & + \chi m \left(g_3^{\ell m}(r)\, {}_2\hat{R}_{\ell m}(r) + g_4^{\ell m}(r)\, {}_2\hat{R}'_{\ell m}(r) \right) \Lambda_{10s}^{\ell \ell 1}\,, \end{aligned} \tag{5.30a}$$

$$V_{\ell m}^{\square}(r) = \chi m \left(h_1^{\ell m}(r)\, {}_2\hat{R}_{\ell m}(r) + h_2^{\ell m}(r)\, {}_2\hat{R}'_{\ell m}(r) \right) \Lambda_{10s}^{\ell \ell 1}\,, \tag{5.30b}$$

$$V_{\ell -m}^{\dagger R}(r) = -V_{\ell m}^{R}(r)\,, \quad V_{\ell -m}^{\dagger \square}(r) = -V_{\ell m}^{\square}(r)\,. \tag{5.30c}$$

Therefore, Eq. (5.25) becomes

$$\begin{aligned} & \left[r(r - r_s)\partial_r^2 + r_s \partial_r + \frac{\omega^2 r^3 - 4\chi m M^2 \omega}{r - r_s} - \frac{r_s}{r} - {}_0A_{\ell m} \right] \Theta_{\ell m}(r) \\ & = (1 - \bar{\eta}_{\ell m}) \left(V_{\ell m}^{R}(r) + V_{\ell m}^{\square}(r) \right)\,, \end{aligned} \tag{5.31a}$$

$$\begin{aligned} & \left[r(r - r_s)\partial_r^2 + r_s \partial_r + \frac{\omega^2 r^3 + 4\chi m M^2 \omega}{r - r_s} - \frac{r_s}{r} - {}_0A_{\ell -m} \right] \Theta_{\ell -m}(r) \\ & = (\eta_{\ell m} - 1) \left(V_{\ell -m}^{R}(r) + V_{\ell -m}^{\square}(r) \right)\,. \end{aligned} \tag{5.31b}$$

As shown in [75], the metric generated by the modes with $\bar{\eta}_{\ell m} = 1$ is of even parity, so the even-parity metric perturbations are not coupled to the scalar field up to $O(\zeta^1, \chi^1, \epsilon^1)$ in dCS gravity, consistent with the result in [57]. Furthermore, using Eqs. (5.22), (5.28), and (5.31), one can show that

$$\left[r(r - r_s)\partial_r^2 + r_s \partial_r + \frac{\omega^2 r^3 - 4\chi m M^2 \omega}{r - r_s} - \frac{r_s}{r} - {}_0A_{\ell m} \right] \bar{\Theta}_{\ell -m}(r)$$

$$= (-1)^m (1-\bar{\eta}_{\ell m}) \left(V^R_{\ell m}(r) + V^{\square}_{\ell m}(r) \right) , \tag{5.32}$$

consistent with our assumption in Eq. (5.24).

Similarly, in the ORG, we get

$$\begin{aligned}
&\left[r(r-r_s)\partial_r^2 + r_s\partial_r + \frac{\omega^2 r^3 - 4\chi m M^2 \omega}{r-r_s} - \frac{r_s}{r} - {}_0A_{\ell m} \right] \Theta_{\ell m}(r) \\
&= (1-\bar{\eta}_{\ell m}) \left(U^R_{\ell m}(r) + U^{\square}_{\ell m}(r) \right) , \qquad (5.33a) \\
&\left[r(r-r_s)\partial_r^2 + r_s\partial_r + \frac{\omega^2 r^3 + 4\chi m M^2 \omega}{r-r_s} - \frac{r_s}{r} - {}_0A_{\ell -m} \right] \Theta_{\ell -m}(r) \\
&= (\eta_{\ell m} - 1) \left(U^R_{\ell -m}(r) + U^{\square}_{\ell -m}(r) \right) , \qquad (5.33b)
\end{aligned}$$

where

$$\begin{aligned}
U^R_{\ell m}(r) &= i \left(\mathfrak{g}_1^{\ell m}(r)\, {}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{g}_2^{\ell m}(r)\, {}_{-2}\hat{R}'_{\ell m}(r) \right) \\
&\quad + \chi m \left(\mathfrak{g}_3^{\ell m}(r)\, {}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{g}_4^{\ell m}(r)\, {}_{-2}\hat{R}'_{\ell m}(r) \right) \Lambda^{\ell\ell 1}_{10s} , \qquad (5.34a) \\
U^{\square}_{\ell m}(r) &= \chi m \left(\mathfrak{h}_1^{\ell m}(r)\, {}_{-2}\hat{R}_{\ell m}(r) + \mathfrak{h}_2^{\ell m}(r)\, {}_{-2}\hat{R}'_{\ell m}(r) \right) \Lambda^{\ell\ell 1}_{10s} , \qquad (5.34b) \\
U^{\dagger R}_{\ell -m}(r) &= -U^R_{\ell m}(r) , \quad U^{\dagger \square}_{\ell -m}(r) = -U^{\square}_{\ell m}(r) , \qquad (5.34c)
\end{aligned}$$

and we have used Eq. (5.105) to replace $\Lambda^{\ell\ell 1}_{-10s}$ by $\Lambda^{\ell\ell 1}_{10s}$. Similar to the IRG case, we have redefined $\mathfrak{g}_i^{\ell m}(r)$ and $\mathfrak{h}_i^{\ell m}(r)$ in [71] such that

$$\bar{\mathfrak{g}}_i^{\ell -m}(r) = \mathfrak{g}_i^{\ell m}(r) , \quad \bar{\mathfrak{h}}_i^{\ell -m}(r) = \mathfrak{h}_i^{\ell m}(r) . \tag{5.35}$$

The redefinition of $\mathfrak{g}_i^{\ell m}(r)$ and $\mathfrak{h}_i^{\ell m}(r)$ can be obtained by replacing g and h with $\mathfrak{g}$ and $\mathfrak{h}$ in Eq. (5.27), respectively. In addition, $\mathfrak{g}_1^{\ell m}(r)$ and $\mathfrak{g}_2^{\ell m}(r)$ contain terms up to $O(\chi^1)$, and χ only shows up in the combination χm. The other radial functions $\mathfrak{g}_i^{\ell m}(r)$ and $\mathfrak{h}_i^{\ell m}(r)$ in Eq. (5.34) do not contain any factor of χ or m. The original $\mathfrak{g}_i^{\ell m}(r)$ and $\mathfrak{h}_i^{\ell m}(r)$ of [71] are listed in the Mathematica notebook [88].

5.4 Modified Teukolsky equations

In this section, we review and simplify the modified Teukolsky equation of Ψ_0 in dCS gravity up to $O(\zeta^1, \chi^1, \epsilon^1)$ found by [71]. It was shown in [71] that $\Psi_0^{(1,1)}$ satisfies

$$H_0^{(0,0)} \Psi_0^{(1,1)} = 2r^2 \left(\mathcal{S}^{(1,1)}_{\text{geo}} + \mathcal{S}^{(1,1)}_A + \tilde{\mathcal{S}}^{(1,1)}_A + \mathcal{S}^{(1,1)}_B + \tilde{\mathcal{S}}^{(1,1)}_B \right) , \tag{5.36}$$

where $H_0^{(0,0)}$ is the Teukolsky operator for Ψ_0 in GR and expanded to $O(\chi)$,

$$H_0^{(0,0)} = H_0^{(0,0,0)} + \chi H_0^{(0,1,0)} , \tag{5.37a}$$

$$
\begin{aligned}
H_0^{(0,0,0)} &= -r(r-r_s)\partial_r^2 - 6(r-M)\partial_r - \frac{C(r)}{r-r_s} \\
&\quad - \partial_\theta^2 - \cot\theta\partial_\theta + \left(4+m^2+4m\cos\theta\right)\csc^2\theta - 6\,,
\end{aligned} \tag{5.37b}
$$

$$
H_0^{(0,1,0)} = -4M\left[\frac{m(i(r-M)-M\omega r)}{r(r-r_s)} - \omega\cos\theta\right], \tag{5.37c}
$$

$$
C(r) = 4i\omega r(r-3M) + \omega^2 r^3\,.
$$

In Eq. (5.36), $\mathcal{S}_{\rm geo}^{(1,1)}$ is the source term due to the dCS correction to the geometry of rotating BHs. The terms $\mathcal{S}_A^{(1,1)}$ and $\tilde{\mathcal{S}}_A^{(1,1)}$ are sources associated with the effective stress tensor and driven by the scalar field perturbation $\vartheta^{(1,1)}$. The source terms $\mathcal{S}_B^{(1,1)}$ and $\tilde{\mathcal{S}}_B^{(1,1)}$ are also associated with the effective stress tensor but driven by the Weyl scalar perturbation $\Psi_0^{(0,1)}$ in GR.

As discussed earlier in Sec. 5.2, we need to solve the (ℓ, m) and $(\ell, -m)$ modes of Ψ_0 jointly since they are coupled in the source terms. In [75], it was found that the Teukolsky operator in GR satisfies the symmetry:

$$
\hat{\mathcal{P}} H_0^{(0,0)} = H_0^{(0,0)}\,, \tag{5.38}
$$

where $\hat{\mathcal{P}}$ is an operator combining complex conjugate with parity transformation, i.e.,

$$
\hat{\mathcal{P}} f = \hat{C}\hat{P} f = \hat{C} f(\pi-\theta, \phi+\pi) = \bar{f}(\pi-\theta, \phi+\pi)\,. \tag{5.39}
$$

For this reason, Ref. [75] showed that one could solve Eq. (5.36) jointly with its $\hat{\mathcal{P}}$ transformation such that the latter will generate a consistency relation for the (ℓ, m) mode from the equation of the $(\ell, -m)$ mode. We can then reduce the modified Teukolsky equation of $\Psi_0^{(1,1)}$ to a two-dimensional eigenvalue problem and compute the QNM frequencies [75], as shown in more detail at the end of this subsection and in Sec. 5.5.1. In the following subsections, we will sequentially simplify these three groups of source terms $\mathcal{S}_{\rm geo}^{(1,1)}$, $\left\{\mathcal{S}_A^{(1,1)}, \tilde{\mathcal{S}}_A^{(1,1)}\right\}$, and $\left\{\mathcal{S}_B^{(1,1)}, \tilde{\mathcal{S}}_B^{(1,1)}\right\}$. We will derive their transformation under $\hat{\mathcal{P}}$ and show that the solutions to the modified Teukolsky equation are of definite parity with $\eta_{\ell m} = \pm 1$.

5.4.1 $\mathcal{S}_{\rm geo}^{(1,1)}$

First, we found in [71] that the "geometrical" source term $\mathcal{S}_{\rm geo}^{(1,1)}$ in dCS is

$$
\mathcal{S}_{\rm geo}^{(1,1)} = -\sum_{\ell, m\geq 0} e^{-i\omega_{\ell m} t} H_0^{\ell m(1,0)}\left[{}_2R_{\ell m}^{(0,1)}(r)\,{}_2Y_{\ell m}(\theta,\phi)\right] + \eta_{\ell m}\times(m\to -m)\,, \tag{5.40}
$$

where ${}_sY_{\ell m}(\theta,\phi)$ are spin-weighted spherical harmonics due to the expansion of ${}_s\mathcal{Y}_{\ell m}(\theta,\phi)$ over χ, and

$$\begin{aligned}H_0^{\ell m(1,0)} &= \frac{i\chi m M^4}{448r^9(r-r_s)}\left(C_1(r)+4i\omega_{\ell m}r^2C_2(r)\right)\\ &\quad-\frac{i\chi M^4}{16r^9}\cos\theta\left(C_3(r)-\frac{i\omega_{\ell m}r^2C_4(r)}{2}\right)\\ &\quad+\frac{i\chi M^4}{32r^8}\left[(r-r_s)C_4(r)\cos\theta\partial_r-\frac{C_5(r)}{2r}\sin\theta\partial_\theta\right]. \end{aligned}\tag{5.41}$$

Notice that when taking $m\to -m$ in Eq. (5.40) for $m=0$, we get the mode with frequency $-\bar{\omega}_{\ell 0}$. The radial functions $C_i(r)$ are real in r and listed in [71]. Using that [71]

$$\partial_\theta\left({}_2Y_{\ell m}(\theta,\phi)\right)=\frac{1}{2}\left(\sqrt{(\ell+2)(\ell-1)}\,{}_1Y_{\ell m}(\theta,\phi)-\sqrt{(\ell+3)(\ell-2)}\,{}_3Y_{\ell m}(\theta,\phi)\right), \tag{5.42}$$

we can rewrite $\mathcal{S}_{\rm geo}^{(1,1)}$ as

$$\mathcal{S}_{\rm geo}^{(1,1)}=\sum_{\ell,m\geq 0}e^{-i\omega_{\ell m}t}\mathcal{O}^{\ell m}_{\rm geo}\,{}_2R^{(0,1)}_{\ell m}+\eta_{\ell m}\times(m\to -m)\,, \tag{5.43}$$

with the operator $\mathcal{O}^{\ell m}_{\rm geo}$ defined to be

$$\begin{aligned}\mathcal{O}^{\ell m}_{\rm geo} &= -\frac{i\chi m M^4}{448r^9(r-r_s)}\left(C_1(r)+4i\omega_{\ell m}r^2C_2(r)\right)\\ &\quad+\frac{i\chi M^4}{16r^9}\cos\theta\,{}_2Y_{\ell m}(\theta,\phi)\left[C_3(r)-C_4(r)\left(\frac{i\omega_{\ell m}r^2}{2}+\frac{r(r-r_s)}{2}\partial_r\right)\right]\\ &\quad+\frac{i\chi M^4}{128r^9}C_5(r)\left(\sqrt{(\ell+2)(\ell-1)}\sin\theta\,{}_1Y_{\ell m}(\theta,\phi)\right.\\ &\quad\left.-\sqrt{(\ell+3)(\ell-2)}\sin\theta\,{}_3Y_{\ell m}(\theta,\phi)\right). \end{aligned}\tag{5.44}$$

To perform the $\hat{\mathcal{P}}$ transformation [i.e., Eq. (5.39)] on $\mathcal{S}_{\rm geo}^{(1,1)}$, we also need the following properties of spin-weighted spherical harmonics ${}_sY_{\ell m}(\theta,\phi)$ [89]:

$$\begin{aligned}{}_sY_{\ell m}(\pi-\theta,\phi+\pi)&=(-1)^{\ell}\,{}_{-s}Y_{\ell m}(\theta,\phi)\,,\\ {}_s\bar{Y}_{\ell m}(\theta,\phi)&=(-1)^{m+s}\,{}_{-s}Y_{\ell -m}(\theta,\phi)\,,\end{aligned}\tag{5.45}$$

so we get

$$\begin{aligned}\hat{\mathcal{P}}\left[{}_{\pm 2}Y_{\ell m}(\theta,\phi)\right]&=(-1)^{\ell+m}\,{}_{\pm 2}Y_{\ell -m}(\theta,\phi)\,,\\ \hat{\mathcal{P}}\left[\sin\theta\,{}_{\pm 1}Y_{\ell m}(\theta,\phi)\right]&=(-1)^{\ell+m+1}\sin\theta\,{}_{\pm 1}Y_{\ell -m}(\theta,\phi)\,,\end{aligned}$$

$$\hat{\mathcal{P}}\left[\cos\theta\,{}_{\pm 2}Y_{\ell m}(\theta,\phi)\right] = (-1)^{\ell+m+1}\cos\theta\,{}_{\pm 2}Y_{\ell -m}(\theta,\phi)\,,$$
$$\hat{\mathcal{P}}\left[\sin\theta\,{}_{\pm 3}Y_{\ell m}(\theta,\phi)\right] = (-1)^{\ell+m+1}\sin\theta\,{}_{\pm 3}Y_{\ell -m}(\theta,\phi)\,. \tag{5.46}$$

Using Eq. (5.46) and that $C_i(r)$ are real functions in r, we get

$$\hat{\mathcal{P}}\left[O_{\text{geo}}^{\ell m}\,{}_2R_{\ell m}^{(0,1)}(r)\right] = (-1)^{\ell} O_{\text{geo}}^{\ell -m}\,{}_2R_{\ell -m}^{(0,1)}(r)\,. \tag{5.47}$$

5.4.2 $\mathcal{S}_A^{(1,1)}$ and $\tilde{\mathcal{S}}_A^{(1,1)}$

Next, we know from [71] that the source terms $\mathcal{S}_A^{(1,1)}$ and $\tilde{\mathcal{S}}_A^{(1,1)}$ take the form

$$\begin{aligned}\mathcal{S}_A^{(1,1)} &= \sum_{\ell,m} e^{-i\omega_{\ell m}t}\left[\mathcal{A}_1^{\ell m}(r)\,{}_2Y_{\ell m}(\theta,\phi) + i\chi\mathcal{A}_2^{\ell m}(r)\sin\theta\,{}_1Y_{\ell m}(\theta,\phi)\right.\\ &\quad\left.+ i\chi\mathcal{A}_3^{\ell m}(r)\cos\theta\,{}_2Y_{\ell m}(\theta,\phi)\right]\,,\end{aligned} \tag{5.48a}$$

$$\begin{aligned}\tilde{\mathcal{S}}_A^{(1,1)} &= \sum_{\ell,m} e^{+i\bar{\omega}_{\ell m}t}\left[-\bar{\mathcal{A}}_1^{\ell m}(r)\,{}_{-2}\bar{Y}_{\ell m}(\theta,\phi) + i\chi\bar{\mathcal{A}}_2^{\ell m}(r)\sin\theta\,{}_{-1}\bar{Y}_{\ell m}(\theta,\phi)\right.\\ &\quad\left.- i\chi\bar{\mathcal{A}}_3^{\ell m}(r)\cos\theta\,{}_{-2}\bar{Y}_{\ell m}(\theta,\phi)\right]\,,\end{aligned} \tag{5.48b}$$

where ${}_sY_{\ell m}(\theta,\phi)$ comes from expanding ${}_s\mathcal{Y}_{\ell m}(\theta,\phi)$ over χ. We have also dropped the terms proportional to ${}_0b^m_{\ell,\ell\pm 1}$ in [71] since they contribute at $O(\chi^2)$ after the angular projection, as discussed in [71]. The radial functions $\mathcal{A}_i^{\ell m}(r)$ and $\bar{\mathcal{A}}_i^{\ell m}(r)$ take the form

$$\mathcal{A}_i^{\ell m}(r) = iO_i^{\ell m}\Theta_{\ell m}(r) + \alpha_{\ell m}Q_i^{\ell m}\,{}_2\hat{R}_{\ell m}(r)\,, \tag{5.49}$$

where $O_i^{\ell m}$ and $Q_i^{\ell m}$ are differential operators in r and containing up to first derivative in r. The coefficient $\alpha_{\ell m} = 1 - \bar{\eta}_{\ell m}$ for $m > 0$ and $\alpha_{\ell m} = \eta_{\ell -m} - 1$ for $m < 0$. When $m = 0$, $\alpha_{\ell m}$ is $1 - \bar{\eta}_{\ell 0}$ and $\eta_{\ell 0} - 1$ for the modes with the frequency $\omega_{\ell 0}$ and $-\bar{\omega}_{\ell 0}$, respectively. Furthermore, $O_i^{\ell m}$ and $Q_i^{\ell m}$ satisfy

$$\bar{O}_i^{\ell -m} = O_i^{\ell m}\,, \quad \bar{Q}_i^{\ell -m} = Q_i^{\ell m}\,. \tag{5.50}$$

Notice that the term $Q_i^{\ell m}\,{}_2\hat{R}_{\ell m}(r)$ comes from replacing $\Theta''_{\ell m}(r)$ with $\Theta'_{\ell m}(r)$ and $\Theta_{\ell m}(r)$ using Eq. (5.31), though $\mathcal{S}_A^{(1,1)}$ and $\tilde{\mathcal{S}}_A^{(1,1)}$ are driven by the scalar field perturbation $\vartheta^{(1,1)}$. Similarly, using Eqs. (5.24) and (5.50), one can get

$$\bar{\mathcal{A}}_i^{\ell -m}(r) = (-1)^{m+1}\left[iO_i^{\ell m}\Theta_{\ell m}(r) + \alpha_{\ell m}Q_i^{\ell m}\,{}_2\hat{R}_{\ell m}(r)\right]\,. \tag{5.51}$$

Notice that $\mathcal{A}_i^{\ell -m}(r)$ comes from the terms generated by the $(\ell, -m)$ mode of the solution ansatz in Eq. (5.20) but not taking m to $-m$ in Eq. (5.49) directly.

To analyze the structure of isospectrality breaking, we need to rewrite all the terms in Eq. (5.49) in terms of ${}_2\hat{R}_{\ell m}(r)$ such that we can focus on the vector space of ${}_2\hat{R}_{\ell m}(r)$

following [75]. Denoting the operator acting on $\Theta_{\ell m}(r)$ on the left-hand side of Eq. (5.31) as $\mathcal{H}_\vartheta$, so

$$\begin{aligned}\Theta_{\ell m}(r) &= \alpha_{\ell m}\mathcal{H}_\vartheta^{-1}\left[V^R_{\ell m}(r) + V^{\square}_{\ell m}(r)\right] \\ &\equiv i\alpha_{\ell m}\mathcal{H}_\vartheta^{-1}\mathcal{V}^{\ell m}{}_2\hat{R}_{\ell m}(r)\,, \end{aligned} \tag{5.52}$$

where we have defined an operator $\mathcal{V}^{\ell m}$ according to that $V^R_{\ell m}(r)$ and $\mathcal{V}^{\square}_{\ell m}(r)$ are driven by ${}_2\hat{R}_{\ell m}(r)$ in Eq. (5.30). From Eq. (5.30), we observe that $\mathcal{V}^{\ell m}$ contains up to the first derivative in r. We have also extracted a factor of i such that

$$\bar{\mathcal{V}}^{\ell -m} = \mathcal{V}^{\ell m}\,. \tag{5.53}$$

The operator $\mathcal{H}_\vartheta$ also satisfies the same symmetry in Eq. (5.53). Thus, we can rewrite Eqs. (5.49) and (5.51) as

$$\mathcal{A}_i^{\ell m}(r) = \alpha_{\ell m}\hat{A}_i^{\ell m}{}_2\hat{R}_{\ell m}(r)\,, \tag{5.54a}$$

$$\bar{\mathcal{A}}_i^{\ell -m}(r) = (-1)^{m+1}\alpha_{\ell m}\hat{A}_i^{\ell m}{}_2\hat{R}_{\ell m}(r)\,, \tag{5.54b}$$

with

$$\hat{A}_i^{\ell m} \equiv Q_i^{\ell m} - O_i^{\ell m}\mathcal{H}_\vartheta^{-1}\mathcal{V}^{\ell m}\,. \tag{5.55}$$

Using Eqs. (5.50), (5.53), and (5.55), we get

$$\hat{\mathcal{P}}\hat{A}_i^{\ell m} = \bar{\hat{A}}_i^{\ell m} = \hat{A}_i^{\ell -m}\,. \tag{5.56}$$

Leveraging Eqs. (5.45) and (5.54), we can rewrite $\mathcal{S}_A^{(1,1)}$ and $\tilde{\mathcal{S}}_A^{(1,1)}$ in Eq. (5.48) as

$$\begin{aligned}\mathcal{S}_A^{(1,1)} = \tilde{\mathcal{S}}_A^{(1,1)} = \sum_{\ell, m\geq 0} &(1-\bar{\eta}_{\ell m})e^{-i\omega_{\ell m}t}O_A^{\ell m}{}_2R_{\ell m}^{(0,1)}(r) \\ &+ (\eta_{\ell m}-1)e^{i\bar{\omega}_{\ell m}t}O_A^{\ell -m}{}_2R_{\ell -m}^{(0,1)}(r)\,, \end{aligned} \tag{5.57}$$

such that

$$O_A^{\ell m} = \left[{}_2Y_{\ell m}(\theta,\phi)\hat{A}_1^{\ell m} + i\chi\sin\theta\,{}_1Y_{\ell m}(\theta,\phi)\hat{A}_2^{\ell m} + i\chi\cos\theta\,{}_2Y_{\ell m}(\theta,\phi)\hat{A}_3^{\ell m}\right]\mathcal{D}^\dagger_{\ell m}\,, \tag{5.58}$$

where we have used Eq. (5.45) to replace ${}_{-s}\bar{Y}_{\ell -m}(\theta,\phi)$ with ${}_sY_{\ell m}(\theta,\phi)$ in Eq. (5.48b). We have also used that $\omega_{\ell m} = -\bar{\omega}_{\ell -m}$ to reduce $\tilde{\mathcal{S}}_A^{(1,1)}$ to $\mathcal{S}_A^{(1,1)}$. It is not surprising that $\mathcal{S}_A^{(1,1)} = \tilde{\mathcal{S}}_A^{(1,1)}$ since ϑ is real, $\bar{\Theta}_{\ell m}(r)$ is related to $\Theta_{\ell m}$ via Eq. (5.24), and we sum all the (ℓ, m) modes. For this reason, we will combine $\mathcal{S}_A^{(1,1)}$ with $\tilde{\mathcal{S}}_A^{(1,1)}$ and solve the radial Teukolsky equation jointly with $\Theta_{\ell m}(r)$ only. Using Eqs. (5.22), (5.46), and (5.56), we can show that

$$\hat{\mathcal{P}}\left[O_A^{\ell m}{}_2R_{\ell m}^{(0,1)}(r)\right] = (-1)^\ell O_A^{\ell -m}{}_2R_{\ell -m}^{(0,1)}(r)\,. \tag{5.59}$$

5.4.3 $\mathcal{S}_B^{(1,1)}$ and $\tilde{\mathcal{S}}_B^{(1,1)}$

Lastly, in [71], we got the source terms $\mathcal{S}_B^{(1,1)}$ and $\tilde{\mathcal{S}}_B^{(1,1)}$ to be in the form of

$$\begin{aligned}\mathcal{S}_B^{(1,1)} &= i\chi \sum_{\ell,m\geq 0} e^{-i\omega_{\ell m}t}\left[\mathcal{B}_1^{\ell m}(r)\sin\theta\,{}_1Y_{\ell m}(\theta,\phi) + \mathcal{B}_2^{\ell m}(r)\cos\theta\,{}_2Y_{\ell m}(\theta,\phi)\right.\\ &\quad\left.+\mathcal{B}_3^{\ell m}(r)\sin\theta\,{}_3Y_{\ell m}(\theta,\phi)\right] + \eta_{\ell m}\times(m\to -m)\,, \qquad (5.60a)\\ \tilde{\mathcal{S}}_B^{(1,1)} &= i\chi \sum_{\ell,m\geq 0} e^{+i\bar{\omega}_{\ell m}t}\left[\tilde{\mathcal{B}}_1^{\ell m}(r)\sin\theta\,{}_{-1}\bar{Y}_{\ell m}(\theta,\phi) + \tilde{\mathcal{B}}_2^{\ell m}(r)\cos\theta\,{}_{-2}\bar{Y}_{\ell m}(\theta,\phi)\right]\\ &\quad+\bar{\eta}_{\ell m}\times(m\to -m)\,, \qquad (5.60b)\end{aligned}$$

where we have extracted an overall factor of i from all the radial functions $\mathcal{B}_i^{\ell m}(r)$ and $\tilde{\mathcal{B}}_i^{\ell m}(r)$ such that

$$\bar{\mathcal{B}}_i^{\ell -m}(r) = \mathcal{B}_i^{\ell m}(r)\,,\quad \bar{\tilde{\mathcal{B}}}_i^{\ell -m}(r) = \tilde{\mathcal{B}}_i^{\ell m}(r)\,. \qquad (5.61)$$

The functions $\mathcal{B}_i^{\ell m}(r)$ and $\tilde{\mathcal{B}}_i^{\ell m}(r)$ contain up to the first derivative in r of ${}_2\hat{R}_{\ell m}(r)$ and ${}_2\bar{\hat{R}}_{\ell m}(r)$, respectively. Following Sec. 5.4.2, let us define

$$\mathcal{B}_i^{\ell m}(r) \equiv \hat{B}_i^{\ell m}\,{}_2\hat{R}_{\ell m}(r)\,,\quad \tilde{\mathcal{B}}_i^{\ell m} \equiv \hat{\tilde{B}}_i^{\ell m}\,{}_2\bar{\hat{R}}_{\ell m}(r)\,, \qquad (5.62)$$

where $\hat{B}_i^{\ell m}$ and $\hat{\tilde{B}}_i^{\ell m}$ contain up to the first derivative in r and satisfy the same symmetry as Eq. (5.61), so

$$\hat{\mathcal{P}}\hat{B}_i^{\ell m} = \hat{B}_i^{\ell -m}\,,\quad \hat{\mathcal{P}}\hat{\tilde{B}}_i^{\ell m} = \hat{\tilde{B}}_i^{\ell -m}\,. \qquad (5.63)$$

Thus, we can rewrite $\mathcal{B}_i^{\ell m}(r)$ and $\tilde{\mathcal{B}}_i^{\ell m}(r)$ in Eq. (5.60) as

$$\begin{aligned}\mathcal{S}_B^{(1,1)} &= \sum_{\ell,m\geq 0} e^{-i\omega_{\ell m}t}O_B^{\ell m}\,{}_2\hat{R}_{\ell m}(r) + \eta_{\ell m}\times(m\to -m)\,, \qquad (5.64a)\\ \tilde{\mathcal{S}}_B^{(1,1)} &= \sum_{\ell,m\geq 0} \bar{\eta}_{\ell m}e^{-i\omega_{\ell m}t}\tilde{O}_B^{\ell m}\,{}_2R_{\ell m}^{(0,1)}(r) + (m\to -m)\,, \qquad (5.64b)\end{aligned}$$

with

$$O_B^{\ell m} = i\chi\left[\sin\theta\,{}_1Y_{\ell m}(\theta,\phi)\hat{B}_1^{\ell m} + \cos\theta\,{}_2Y_{\ell m}(\theta,\phi)\hat{B}_2^{\ell m} + \sin\theta\,{}_3Y_{\ell m}(\theta,\phi)\hat{B}_3^{\ell m}\right]\mathcal{D}_{\ell m}^{\dagger}\,, \qquad (5.65a)$$

$$\tilde{O}_B^{\ell m} = i\chi\left[\sin\theta\,{}_1Y_{\ell m}(\theta,\phi)\hat{\bar{B}}_1^{\ell -m} + \cos\theta\,{}_2Y_{\ell m}(\theta,\phi)\hat{\bar{B}}_2^{\ell -m}\right]\mathcal{D}_{\ell m}^{\dagger}\,, \qquad (5.65b)$$

where we have used Eqs. (5.22) and (5.45) to replace $\left\{{}_s\bar{\hat{R}}_{\ell -m}(r), {}_{-s}\bar{Y}_{\ell -m}(\theta,\phi)\right\}$ with $\left\{{}_s\hat{R}_{\ell m}(r), {}_sY_{\ell m}(\theta,\phi)\right\}$ in Eq. (5.64) and that $\omega_{\ell m} = -\bar{\omega}_{\ell -m}$. With Eqs. (5.46) and (5.63), we can show that

$$\begin{aligned}\hat{\mathcal{P}}\left[O_B^{\ell m}\,{}_2R_{\ell m}^{(0,1)}(r)\right] &= (-1)^{\ell}O_B^{\ell -m}\,{}_2R_{\ell -m}^{(0,1)}(r)\,, \qquad (5.66a)\\ \hat{\mathcal{P}}\left[\tilde{O}_B^{\ell m}\,{}_2R_{\ell m}^{(0,1)}(r)\right] &= (-1)^{\ell}\tilde{O}_B^{\ell -m}\,{}_2R_{\ell -m}^{(0,1)}(r)\,. \qquad (5.66b)\end{aligned}$$

5.4.4 Simplification of the radial master equations

In this subsection, we will use the results in Secs. 5.4.1–5.4.3 to simplify the modified Teukolsky equation of $\Psi_0^{(1,1)}$ in Eq. (5.36) and extract its radial part. Combining Eqs. (5.43), (5.57), and (5.64), we can reduce Eq. (5.36) to

$$\begin{aligned}
&H_0^{\ell m(0,0)}\left[{}_2R_{\ell m}^{(1,1)}(r){}_2Y_{\ell m}(\theta,\phi)\right] \\
&= 2r^2\left[\left(O_{\rm geo}^{\ell m}+O_A^{\ell m}+O_B^{\ell m}\right)-\bar{\eta}_{\ell m}\left(O_A^{\ell m}-\tilde{O}_B^{\ell m}\right)\right]{}_2R_{\ell m}^{(0,1)}(r)\,, \qquad (5.67\text{a}) \\
&\eta_{\ell m}H_0^{\ell -m(0,0)}\left[{}_2R_{\ell -m}^{(1,1)}(r){}_2Y_{\ell -m}(\theta,\phi)\right] \\
&= 2r^2\left[\eta_{\ell m}\left(O_{\rm geo}^{\ell -m}+O_A^{\ell -m}+O_B^{\ell -m}\right)-\left(O_A^{\ell -m}-\tilde{O}_B^{\ell -m}\right)\right]{}_2R_{\ell -m}^{(0,1)}(r)\,. \qquad (5.67\text{b})
\end{aligned}$$

One can perform a further $\hat{\mathcal{P}}$ transformation on Eq. (5.67b) and use Eqs. (5.47), (5.59), and (5.66), so Eq. (5.67b) becomes

$$\begin{aligned}
&\bar{\eta}_{\ell m}H_0^{\ell m(0,0)}\left[{}_2R_{\ell m}^{(1,1)}(r){}_2Y_{\ell m}(\theta,\phi)\right] \\
&= 2r^2\left[\bar{\eta}_{\ell m}\left(O_{\rm geo}^{\ell m}+O_A^{\ell m}+O_B^{\ell m}\right)-\left(O_A^{\ell m}-\tilde{O}_B^{\ell m}\right)\right]{}_2R_{\ell m}^{(0,1)}(r)\,, \qquad (5.68)
\end{aligned}$$

where we have chosen the normalization that ${}_2\bar{R}_{\ell -m}^{(1,1)}(r)=(-1)^m{}_2R_{\ell m}^{(1,1)}(r)$. Comparing Eqs. (5.67a) and (5.68), we notice that only $\eta_{\ell m}=\pm 1$ can make these two equations consistent. As found in [75], $\eta_{\ell m}=\pm 1$ correspond to even- and odd-parity metric perturbations, respectively. Furthermore, for even-parity modes ($\eta_{\ell m}=1$), the terms driven by the scalar field perturbation $\vartheta^{(1,1)}$ (i.e., terms with $O_A^{\ell m}$), cancel out with each other. This is due to that the scalar field equation is not driven by the even-parity metric perturbations, as shown in Sec. 5.3. In total, our modified Teukolsky equation has the same structure of isospectrality breaking as the modified RW and ZM equations in [57], where the even- and odd-parity modes decouple, and the scalar field couples to the odd-parity modes only. For this reason, the (ℓ,m) and $(\ell,-m)$ modes of the modified Teukolsky equation contain redundant information, as shown in Eqs. (5.67a) and (5.68) for dCS gravity and in [75] more generally. Thus, we only need to solve the (ℓ,m) mode of the modified Teukolsky equation [Eq. (5.67a)] with the scalar field equation [Eq. (5.31a)].

Now, let us follow the same procedures in [71] to reduce the simplified modified Teukolsky equation in Eq. (5.67a) into a purely radial equation and express it in terms of the radial functions defined in [71]. Integrating Eq. (5.67a) over ${}_2\mathcal{Y}_{\ell m}(\theta,\phi)$, we get that

$$\left[r(r-r_s)\partial_r^2+6(r-M)\partial_r+\frac{4i\omega_{\ell m}r(r-3M)+\omega_{\ell m}^2r^3}{r-r_s}\right.$$

$$+\frac{4i\chi mM((r-M)+iM\omega_{\ell m}r)}{r(r-r_s)}-{}_2A_{\ell m}\Bigg]\,{}_2R^{(1,1)}_{\ell m}(r)$$
$$= -2r^2\left[\left(\mathscr{O}^{\ell m}_{\rm geo}+\mathscr{O}^{\ell m}_{A}+\mathscr{O}^{\ell m}_{B}\right)\mp\left(\mathscr{O}^{\ell m}_{A}-\tilde{\mathscr{O}}^{\ell m}_{B}\right)\right]{}_2R^{(0,1)}_{\ell m}(r)\,, \tag{5.69}$$

where the $\pm$ sign correspond to the solutions with $\eta_{\ell m}=\pm 1$, respectively, and

$$\begin{aligned}
\mathscr{O}^{\ell m}_{\rm geo} = &-\frac{i\chi mM^4}{448r^9(r-r_s)}\left(C_1(r)+4i\omega_{\ell m}r^2C_2(r)\right)\\
&+\frac{i\chi mM^4}{16r^9}\left[C_3(r)-C_4(r)\left(\frac{i\omega_{\ell m}r^2}{2}+\frac{r(r-r_s)}{2}\partial_r\right)\right]\Lambda^{\ell\ell 1}_{22c}\\
&+\frac{i\chi mM^4}{128r^9}C_5(r)\left(\sqrt{(\ell+2)(\ell-1)}\Lambda^{\ell\ell 1}_{12s}-\sqrt{(\ell+3)(\ell-2)}\Lambda^{\ell\ell 1}_{32s}\right)\,, \qquad (5.70\text{a})\\
\mathscr{O}^{\ell m}_{A} = &\left[\hat{A}^{\ell m}_1+i\chi m\Lambda^{\ell\ell 1}_{12s}\hat{A}^{\ell m}_2+i\chi m\Lambda^{\ell\ell 1}_{22c}\hat{A}^{\ell m}_3\right]\mathcal{D}^{\dagger}_{\ell m}\,, \qquad (5.70\text{b})\\
\mathscr{O}^{\ell m}_{B} = &\,i\chi m\left[\Lambda^{\ell\ell 1}_{12s}\hat{B}^{\ell m}_1+\Lambda^{\ell\ell 1}_{22c}\hat{B}^{\ell m}_2+\Lambda^{\ell\ell 1}_{32s}\hat{B}^{\ell m}_3\right]\mathcal{D}^{\dagger}_{\ell m}\,, \qquad (5.70\text{c})\\
\tilde{\mathscr{O}}^{\ell m}_{B} = &-i\chi m\left[\Lambda^{\ell\ell 1}_{12s}\hat{\hat{B}}^{\ell -m}_1-\Lambda^{\ell\ell 1}_{22c}\hat{\hat{B}}^{\ell -m}_2\right]\mathcal{D}^{\dagger}_{\ell m}\,. \qquad (5.70\text{d})
\end{aligned}$$

We have used that

$$\begin{aligned}
\Lambda^{\ell\ell m}_{12s} &= m\Lambda^{\ell\ell 1}_{12s}\,, & \Lambda^{\dagger\ell\ell m}_{-12s} &= (-1)^{m+1}m\Lambda^{\ell\ell 1}_{12s}\,,\\
\Lambda^{\ell\ell m}_{22c} &= m\Lambda^{\ell\ell 1}_{22c}\,, & \Lambda^{\dagger\ell\ell\ell m}_{-22c} &= (-1)^{m}m\Lambda^{\ell\ell 1}_{22c}\,,\\
\Lambda^{\ell\ell m}_{32s} &= m\Lambda^{\ell\ell 1}_{32s}\,, & \Lambda^{\dagger\ell\ell\ell m}_{-32s} &= (-1)^{m+1}m\Lambda^{\ell\ell 1}_{32s}\,,
\end{aligned} \tag{5.71}$$

which can be derived from Eqs. (5.96) and (5.108) in Appendix 5.6. The radial functions $C_i(r)$ are given in [71], and the radial operators $\{\hat{A}^{\ell m}_i, \hat{B}^{\ell m}_i, \hat{\hat{B}}^{\ell m}_i\}$ are given explicitly in Appendix 5.7.

One can notice that all the source terms in Eq. (5.70) are proportional to χm except $\hat{A}^{\ell m}_1$, so they are evaluated on a Schwarzschild background. The operator $\hat{A}^{\ell m}_1$ contains terms at both $O(\chi^0)$ and $O(\chi^1)$, the latter of which only depends on the combination χm. Thus, the dCS correction $\omega^{(1,0)}_{\ell m}$ to the QNM frequency of a slowly rotating BH should be expanded as

$$\omega^{(1,0)}_{\ell m}=\omega^{(1,0,0)}_{\ell m}+m\chi\omega^{(1,1,0)}_{\ell m}+O(\chi^2)\,, \tag{5.72}$$

consistent with the result in [57]. Although the master equations in [57] contain terms proportional to χ only, these terms do not contribute to the QNM frequencies since the equations are invariant under the simultaneous transformation of $\chi\to-\chi$ and $m\to -m$, as shown in more detail in [57]. Furthermore, only after removing these terms did the even- and odd-parity decouple in [57]. In contrast, all the source terms at $O(\chi^1)$ here are proportional to χm, so we do not need to manually drop out any term. The even- and odd-parity modes also naturally decouple in our case.

Similarly, in the ORG, we get the following equation for $\Psi_4^{(1,1)}$:

$$\begin{aligned}&\left[r(r-r_s)\partial_r^2-2(r-M)\partial_r-\frac{4i\omega_{\ell m}r(r-3M)-\omega_{\ell m}^2r^3}{r-r_s}\right.\\&\left.-\frac{4i\chi mM((r-M)-iM\omega_{\ell m}r)}{r(r-r_s)}-{}_{-2}A_{\ell m}\right]{}_{-2}R_{\ell m}^{(1,1)}(r)\\=&-2r^6\left[\left(\mathcal{Q}_{\text{geo}}^{\ell m}+\mathcal{Q}_A^{\ell m}+\mathcal{Q}_B^{\ell m}\right)\mp\left(\mathcal{Q}_A^{\ell m}-\tilde{\mathcal{Q}}_B^{\ell m}\right)\right]{}_{-2}R_{\ell m}^{(0,1)}(r)\,,\end{aligned}\tag{5.73}$$

where

$$\begin{aligned}\mathcal{Q}_{\text{geo}}^{\ell m}=&\frac{i\chi mM^4}{448r^{13}(r-r_s)}\left(D_1(r)-4i\omega_{\ell m}r^2D_2(r)\right)\\&+\frac{i\chi mM^4}{16r^{13}}\left[D_3(r)-D_4(r)\left(\frac{i\omega_{\ell m}r^2}{2}-\frac{r(r-r_s)}{2}\partial_r\right)\right]\Lambda_{22c}^{\ell\ell 1}\\&-\frac{i\chi mM^4}{128r^{13}}D_5(r)\left(\sqrt{(\ell+2)(\ell-1)}\Lambda_{12s}^{\ell\ell 1}-\sqrt{(\ell+3)(\ell-2)}\Lambda_{32s}^{\ell\ell 1}\right)\,,\end{aligned}\tag{5.74a}$$

$$\mathcal{Q}_A^{\ell m}=\left[\hat{\mathscr{A}}_1^{\ell m}+i\chi m\Lambda_{12s}^{\ell\ell 1}\hat{\mathscr{A}}_2^{\ell m}-i\chi m\Lambda_{22c}^{\ell\ell 1}\hat{\mathscr{A}}_3^{\ell m}\right]\mathcal{D}_{\ell m}\,,\tag{5.74b}$$

$$\mathcal{Q}_B^{\ell m}=i\chi m\left[\Lambda_{12s}^{\ell\ell 1}\hat{\mathscr{B}}_1^{\ell m}-\Lambda_{22c}^{\ell\ell 1}\hat{\mathscr{B}}_2^{\ell m}+\Lambda_{32s}^{\ell\ell 1}\hat{\mathscr{B}}_3^{\ell m}\right]\mathcal{D}_{\ell m}\,,\tag{5.74c}$$

$$\tilde{\mathcal{Q}}_B^{\ell m}=-i\chi m\left[\Lambda_{12s}^{\ell\ell 1}\hat{\tilde{\mathscr{B}}}_1^{\ell -m}+\Lambda_{22c}^{\ell\ell 1}\hat{\tilde{\mathscr{B}}}_2^{\ell -m}\right]\mathcal{D}_{\ell m}\,.\tag{5.74d}$$

We have used that

$$\begin{aligned}\Lambda_{-1-2s}^{\ell\ell m}&=m\Lambda_{12s}^{\ell\ell 1}\,,&\Lambda_{1-2s}^{\dagger\ell\ell m}&=(-1)^{m+1}m\Lambda_{12s}^{\ell\ell 1}\,,\\\Lambda_{-2-2c}^{\ell\ell m}&=-m\Lambda_{22c}^{\ell\ell 1}\,,&\Lambda_{2-2c}^{\dagger\ell\ell\ell m}&=(-1)^{m+1}m\Lambda_{22c}^{\ell\ell 1}\,,\\\Lambda_{-3-2s}^{\ell\ell m}&=m\Lambda_{32s}^{\ell\ell 1}\,,&\Lambda_{3-2s}^{\dagger\ell\ell\ell m}&=(-1)^{m+1}m\Lambda_{32s}^{\ell\ell 1}\,,\end{aligned}\tag{5.75}$$

which can be derived from Eqs. (5.96), (5.105), and (5.108) in Appendix 5.6. The radial functions $D_i(r)$ are given in [71], and the radial operators $\{\hat{\mathscr{A}}_i^{\ell m},\hat{\mathscr{B}}_i^{\ell m},\hat{\tilde{\mathscr{B}}}_i^{\ell m}\}$ are given explicitly in Appendix 5.7. Since Eqs. (5.33a) and (5.73) are derived in a different gauge from the one used by Eqs. (5.31a) and (5.69), we will compute the QNM frequency shifts from both pairs and use the results from Eqs. (5.33a) and (5.73) as a consistency check of the results from Eqs. (5.31a) and (5.69) in the next section.

5.5 Calculation of the QNM frequency shifts

In this section, we will evaluate the QNM frequencies for a slowly rotating BH in dCS gravity up to $O(\zeta^1,\chi^1,\epsilon^1)$ using Eqs. (5.31a), (5.33a), (5.69), and (5.73). We

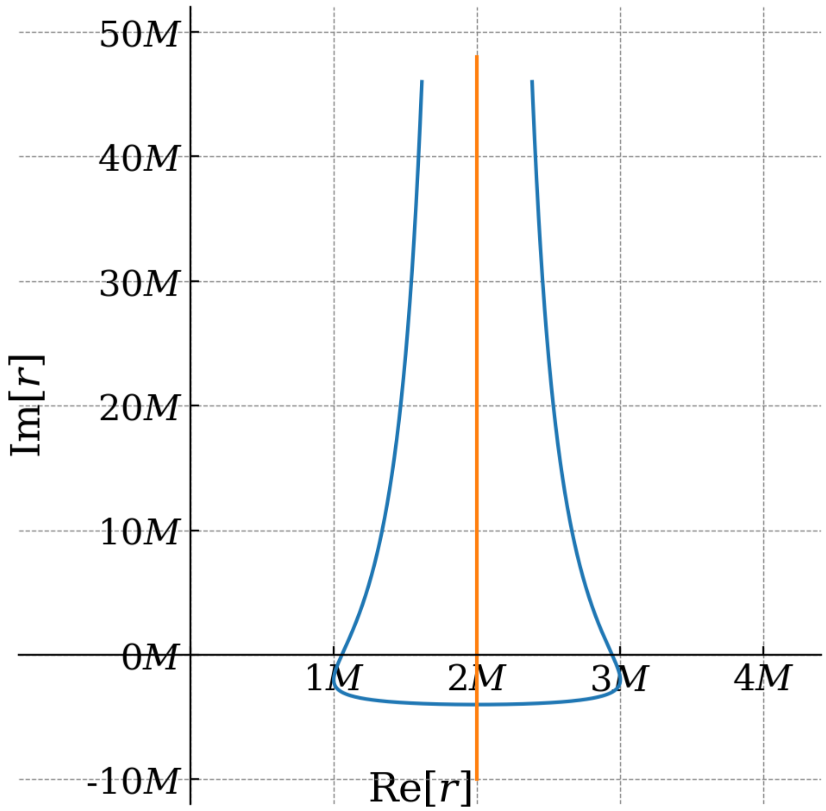

Figure 5.1: The contour $\mathscr{C}$ in Eq. (5.81) with $\xi_{\max} = 10M$. The blue line is the contour $\mathscr{C}$, while the orange line is the imaginary axis at $r = 2M$.

will first review the EVP method in [66, 75–77] and discuss how to apply it to the dCS gravity case. We will then present the results for non-rotating BHs in dCS gravity and discuss the strategy for rotating BHs.

5.5.1 The EVP method

To compute the QNM frequency shifts $\omega_{\ell m}^{(1,0)}$, we choose to follow the EVP approach developed in [66, 75–77]. As one can notice in Eqs. (5.69) and (5.73), the solutions to the homogeneous part of the equation (i.e., ${}_{\pm 2}R_{\ell m}^{(0,1)}(r)$), or the Teukolsky equation in GR, also drive the source terms, potentially leading to secularly growing solutions. To avoid this issue, Refs. [76, 77] developed the EVP method, following the Poincaré-Lindstedt method of solving the secular perturbation problem, by introducing an additional expansion in the QNM frequency to cancel off secularly growing terms. More specifically, consider a system in the form of

$$ {}_s\mathcal{H}_{\ell m}^{(0,0)}\,{}_sR_{\ell m}^{(0,1)} = 0\,, \tag{5.76a} $$

$$ {}_s\mathcal{H}_{\ell m}^{(0,0)}\,{}_sR_{\ell m}^{(1,1)} = {}_s\mathscr{V}_{\ell m}^{(1,0)}\,{}_sR_{\ell m}^{(0,1)}\,, \tag{5.76b} $$

where ${}_s\mathcal{H}_{\ell m}^{(0,0)}$ is the radial Teukolsky operator for a spin-s field in GR, and ${}_s\mathscr{V}_{\ell m}^{(1,0)}$ is some differential operator containing up to first derivative in r. One can expand the QNM frequency $\omega_{\ell m}$ associated with ${}_sR_{\ell m}^{(0,1)}$ and ${}_sR_{\ell m}^{(1,1)}$ as

$$\begin{aligned}\omega_{\ell m} &= \omega_{\ell m}^{(0,0)} + \zeta\omega_{\ell m}^{(1,0)} + O(\zeta^2) \\ &= \left(\omega_{\ell m}^{(0,0,0)} + m\chi\omega_{\ell m}^{(0,1,0)}\right) + \zeta\left(\omega_{\ell m}^{(1,0,0)} + m\chi\omega_{\ell m}^{(1,1,0)}\right) + O(\zeta^2, \chi^2)\,, \end{aligned} \tag{5.77}$$

where we include an additional expansion in χ in the second line, considering the slow-rotation approximation used in [71] and this work. Since ${}_s\mathcal{H}_{\ell m}^{(0,0)}$ depends on $\omega_{\ell m}$, Eq. (5.76) expands to

$${}_s\mathcal{H}_{\ell m}^{(0,0)}\,{}_sR_{\ell m}^{(1,1)} + \omega_{\ell m}^{(1,0)}\partial_\omega\left({}_s\mathcal{H}_{\ell m}^{(0,0)}\right){}_sR_{\ell m}^{(0,1)} = {}_s\mathscr{V}_{\ell m}^{(1,0)}\,{}_sR_{\ell m}^{(0,1)}\,, \tag{5.78}$$

where all the operators are evaluated at the GR QNM frequency $\omega_{\ell m}^{(0,0)}$. The second term on the left-hand side of Eq. (5.78) comes from the expansion of $\omega_{\ell m}$ in Eq. (5.76a).

The key step in the EVP method is to construct an inner product such that the Teukolsky operator in GR is self-adjoint, i.e.,

$$\left\langle\varphi_1(r)\middle|{}_s\mathcal{H}_{\ell m}^{(0,0)}\varphi_2(r)\right\rangle = \left\langle{}_s\mathcal{H}_{\ell m}^{(0,0)}\varphi_1(r)\middle|\varphi_2(r)\right\rangle\,, \tag{5.79}$$

where $\varphi_{1,2}(r)$ are radial functions with the same asymptotic behavior as the radial Teukolsky function of spin weight s in GR. This inner product can be defined as a contour integral over complex r [76, 77],

$$\langle\varphi_1(r)|\varphi_2(r)\rangle = \int_{\mathscr{C}}\Delta^s(r)\varphi_1(r)\varphi_2(r)dr\,, \tag{5.80}$$

where the contour $\mathscr{C}$ is around the positive imaginary axis at the outer horizon r_+. In our case, since $r_+ = 2M$ up to $O(\chi)$, we parametrize the contour as

$$r_{\mathscr{C}}(\xi) = 2M + \frac{4M^2\xi}{4M^2+\xi^2} + i\left(\frac{\xi^2}{2M} - 4M\right)\,, \quad \xi \in [-\xi_{\max}, \xi_{\max}] \tag{5.81}$$

such that $r(\pm\xi_{\max})$ are the right and left ends of the contour, respectively. In Fig. 5.1, we plot the contour for $\xi_{\max} = 10M$. Now conducting an inner product of ${}_sR_{\ell m}^{(0,1)}$ with Eq. (5.78) and using Eqs. (5.76a) and (5.79), we get the first term in Eq. (5.78) to vanish, i.e.,

$$\left\langle{}_sR_{\ell m}^{(0,1)}\middle|{}_s\mathcal{H}_{\ell m}^{(0,0)}\,{}_sR_{\ell m}^{(1,1)}\right\rangle = \left\langle{}_s\mathcal{H}_{\ell m}^{(0,0)}\,{}_sR_{\ell m}^{(0,1)}\middle|{}_sR_{\ell m}^{(1,1)}\right\rangle = 0\,. \tag{5.82}$$

Thus, the QNM frequency shift $\omega_{\ell m}^{(1,0)}$ satisfies

$$\omega_{\ell m}^{(1,0)} = \left\langle {}_s\mathscr{V}_{\ell m}^{(1,0)} \right\rangle \Big/ \left\langle \partial_\omega \left({}_s\mathcal{H}_{\ell m}^{(0,0)} \right) \right\rangle , \tag{5.83}$$

where we use a simplified notation

$$\left\langle \hat{O} \right\rangle = \left\langle {}_sR_{\ell m}^{(0,1)} \middle| \hat{O}\, {}_sR_{\ell m}^{(0,1)} \right\rangle . \tag{5.84}$$

Notice, all the terms within the inner product are evaluated at $\omega_{\ell m}^{(0,0)}$.

For beyond-GR theories, although one generally has a coupled system of $(\ell, \pm m)$ modes or even- and odd-parity modes, Refs. [62, 66, 67, 75] showed how to reduce the system to a two-dimensional eigenvalue problem, so one can still apply the EVP method in [76, 77]. Nonetheless, up to $O(\zeta^1, \chi^1, \epsilon^1)$ in dCS, the modified Teukolsky equation is invariant under $\hat{\mathcal{P}}$ transformation, as shown in Sec. 5.4, so the equations for the (ℓ, m) and $(\ell, -m)$ modes contain the same information. Thus, we can simply apply the one-dimensional EVP method to Eqs. (5.69) and (5.73) and get

$$\omega_{0,\ell m}^{\pm(1,0)} = \frac{\left\langle -2r^2 \left[\left(\mathcal{O}_{\text{geo}}^{\ell m} + \mathcal{O}_A^{\ell m} + \mathcal{O}_B^{\ell m} \right) \mp \left(\mathcal{O}_A^{\ell m} - \tilde{\mathcal{O}}_B^{\ell m} \right) \right] \right\rangle}{\left\langle \partial_\omega \mathcal{H}_0^{\ell m} \right\rangle} , \tag{5.85a}$$

$$\omega_{4,\ell m}^{\pm(1,0)} = \frac{\left\langle -2r^6 \left[\left(\mathcal{Q}_{\text{geo}}^{\ell m} + \mathcal{Q}_A^{\ell m} + \mathcal{Q}_B^{\ell m} \right) \mp \left(\mathcal{Q}_A^{\ell m} - \tilde{\mathcal{Q}}_B^{\ell m} \right) \right] \right\rangle}{\left\langle \partial_\omega \mathcal{H}_4^{\ell m} \right\rangle} , \tag{5.85b}$$

where $\omega_{0,\ell m}^{\pm(1,0)}$ and $\omega_{4,\ell m}^{\pm(1,0)}$ refer to the QNM frequency shifts computed from the equation of $\Psi_0^{(1,1)}$ in the IRG [i.e., Eq. (5.69)] and $\Psi_4^{(1,1)}$ in the ORG [i.e., Eq. (5.73)], respectively. $\mathcal{H}_0^{\ell m}$ and $\mathcal{H}_4^{\ell m}$ are the radial Teukolsky operators acting on ${}_2R_{\ell m}^{(1,1)}$ and ${}_{-2}R_{\ell m}^{(1,1)}$ in Eqs. (5.69) and (5.73), respectively. Their derivatives in ω are

$$\partial_\omega \mathcal{H}_0^{\ell m} = \frac{4ir(r-3M) + 2\omega_{\ell m} r^3 - 4\chi m M^2}{r - r_s} + 2\chi m M \left(\frac{4}{\ell(\ell+1)} + 1 \right) , \tag{5.86a}$$

$$\partial_\omega \mathcal{H}_4^{\ell m} = \frac{-4ir(r-3M) + 2\omega_{\ell m} r^3 - 4\chi m M^2}{r - r_s} + 2\chi m M \left(\frac{4}{\ell(\ell+1)} + 1 \right) . \tag{5.86b}$$

For non-rotating BHs in dCS gravity (i.e., $\chi = 0$), Eq. (5.85) reduces to

$$\omega_{0,\ell m}^{\pm(1,0)} = \frac{\left\langle -2r^2 \left(\hat{A}_1^{\ell m} \mp \hat{A}_1^{\ell m} \right) \mathcal{D}_{\ell m}^\dagger \right\rangle}{\left\langle \left[4ir(r-3M) + 2\omega_{\ell m}^{(0,0)} r^3 \right] / (r - r_s) \right\rangle} , \tag{5.87a}$$

$$\omega_{4,\ell m}^{\pm(1,0)} = \frac{\left\langle -2r^6 \left(\hat{\mathscr{A}}_1^{\ell m} \mp \hat{\mathscr{A}}_1^{\ell m} \right) \mathcal{D}_{\ell m} \right\rangle}{\left\langle \left[-4ir(r-3M) + 2\omega_{\ell m}^{(0,0)} r^3 \right] / (r - r_s) \right\rangle} . \tag{5.87b}$$

In the next subsection, we will evaluate Eq. (5.87) for non-rotating BHs in dCS gravity.

5.5.2 The QNMs of non-rotating BHs in dCS gravity

In this subsection, we evaluate the QNM frequency shifts for a non-rotating BH in dCS gravity using the EVP approach discussed in Sec. 5.5.1. As discussed in Sec. 5.4.2, to evaluate the source terms associated with $\hat{A}_1^{\ell m}$ or $\hat{\mathscr{A}}_1^{\ell m}$ in Eq. (5.87), one needs to first invert or solve the scalar field equation in Eq. (5.31a) or (5.33a) first. For non-rotating BHs in the IRG, Eq. (5.31a) becomes

$$\begin{aligned}&\left[r(r-r_s)\partial_r^2+r_s\partial_r+\frac{\omega_{\ell m}^2 r^3}{r-r_s}-\frac{r_s}{r}-{}_0A_{\ell m}\right]\Theta_{\ell m}(r)\\&=-\frac{2i}{C_2}(1-\eta_{\ell m})\left(g_1^{\ell m}(r)+g_2^{\ell m}(r)\partial_r\right){}_{-2}R_{\ell m}^{(0,1)}(r)\,,\end{aligned}\tag{5.88}$$

where the radial functions $g_1^{\ell m}(r)$ and $g_2^{\ell m}(r)$ are

$$\begin{aligned}g_1^{\ell m}(r)=&-3\sqrt{\Lambda_\ell}M^3\frac{1}{4\sqrt{\pi}r^4(r-r_s)^2}\left[\left(2\omega_{\ell m}^2r^2-8i\omega_{\ell m}r-\ell^2-\ell-4\right)r^2\right.\\&\left.+2M\left(9i\omega_{\ell m}r+\ell^2+\ell+10\right)r-24M^2\right]\,,\\g_2^{\ell m}(r)=&-3\sqrt{\Lambda_\ell}M^3\frac{i\omega_{\ell m}r^2+r-3M}{2\sqrt{\pi}r^3(r-r_s)}\,,\\\Lambda_\ell=&\,(\ell+2)(\ell+1)\ell(\ell-1)\,,\end{aligned}\tag{5.89}$$

and C_2 is a complex constant in the Teukolsky-Starobinsky identity [41, 86, 87],

$$(D_{m\omega})^4{}_{-2}R_{\ell m}^{(0,1)}(r)=C_2\,{}_2\hat{R}_{\ell m}(r)\,,\tag{5.90a}$$

$$\Delta^2(D_{m\omega}^\dagger)^4\left[\Delta^2\,{}_2R_{\ell m}^{(0,1)}(r)\right]=C_{-2}\,{}_{-2}\hat{R}_{\ell m}(r)\,.\tag{5.90b}$$

The complex constants C_2 and C_{-2} satisfy $C_2C_{-2}=\mathfrak{C}$, with $\mathfrak{C}$ being the Teukolsky-Starobinsky constant in Eq. (5.18). Here, we have used Eq. (5.90) to reduce Eq. (5.16) to

$${}_2\hat{R}_{\ell m}(r)=-\frac{2}{C_2}{}_{-2}R_{\ell m}^{(0,1)}(r)\,,\tag{5.91a}$$

$${}_{-2}\hat{R}_{\ell m}(r)=\frac{32}{C_{-2}}{}_2R_{\ell m}^{(0,1)}(r)\,,\tag{5.91b}$$

so we can replace the radial functions ${}_{\pm2}\hat{R}_{\ell m}(r)$ of the Hertz potential with the radial functions ${}_{\mp2}R_{\ell m}^{(0,1)}(r)$ of Ψ_4 or Ψ_0, respectively. Fixing the normalization of ${}_{\pm2}R_{\ell m}^{(0,1)}(r)$, one can also set $C_{-2}=\bar{C}_2$. In contrast, the Teukolsky-Starobinsky constant $\mathfrak{C}$ is normalization independent.

In this work, we compute the radial Teukolsly functions ${}_{\pm2}R_{\ell m}^{(0,1)}(r)$ using the Leaver's method in [38] and evaluate the coefficients $C_{\pm2}$ directly. Specifically, we use the radial Teukolsky equations to reduce the maximum number of derivatives in Eq. (5.90) to one, apply the simplified operators on ${}_{\pm2}R^{(0,1)}_{\ell m}(r)$, and evaluate the ratio between the resulting wave functions and ${}_{\mp2}R^{(0,1)}_{\ell m}$ along the contour $\mathscr{C}$ in Eq. (5.81) (or Fig. 5.1). We find that this ratio is rather stable until very large $|\xi|$, corresponding to r with a large imaginary part. In this case, we simply use the ratio at $r(\xi=0)=2(1-2i)M$ as the value of $C_{\pm2}$.

Using the Chandrasekhar transformation in [90], we can also transform the radial Teukolsky functions ${}_{\pm2}R^{(0,1)}_{\ell m}(r)$ to the RW function $Z^{(0,1)}_{\ell m}(r)$, i.e.,

$$\begin{aligned}
&{}_{\pm2}R^{(0,1)}_{\ell m} = f_{\pm2}(r)\left[V_Z(r)+\left(\frac{2}{r^2}(r-3M)\mp 2i\omega_{\ell m}\right)\Lambda_{\pm}\right]Z^{(0,1)}_{\ell m}(r)\,,\\
&V_Z(r)=\left(1-\frac{r_s}{r}\right)\left(\frac{\ell(\ell+1)}{r^2}-\frac{6M}{r^3}\right),\\
&f_2(r)=r^3\Delta^{-2}(r)\,,\quad f_{-2}(r)=r^3\,,\quad \Lambda_{\pm}=\frac{d}{dr^*}\mp i\omega_{\ell m}\,,
\end{aligned}\tag{5.92}$$

where r^* is the tortoise coordinate, i.e., $dr^*/dr=(r^2+a^2)/\Delta(r)$, and $V_Z(r)$ is determined by the potential in the RW equation. We can then directly compare Eq. (5.88) to the results in [46, 47, 57]. Under the transformation in Eq. (5.92), Eq. (5.88) becomes

$$\left[r(r-r_s)\partial_r^2+r_s\partial_r+\frac{\omega^2_{\ell m}r^3}{r-r_s}-\frac{r_s}{r}-{}_0A_{\ell m}\right]\Theta_{\ell m}(r)=-\frac{3i\sqrt{\Lambda_\ell}M^3}{\sqrt{\pi}r^3}Z^{(0,1)}_{\ell m}(r)\,,\tag{5.93}$$

where we assume $C_{-2}=\bar{C}_2$. Equation (5.93) is consistent with the result in [91] up to an overall constant, which can be compensated when evaluating the modified Teukolsky equation. We have also tried applying the same Chandrasekhar transformation in Eq. (5.92) to the modified Teukolsky equations, but the results do not match the one in [46, 47, 57]. This is not surprising since ${}_{\pm2}R^{(1,1)}_{\ell m}(r)$ satisfy the modified Teukolsky equations in dCS gravity but not the Teukolsky equations in GR. We first need to extend the Chandrasekhar transformation in Eq. (5.92) to the dCS case by including more terms, which we will work out in the future. On the other hand, the source terms in Eq. (5.88) are driven by the GR Teukolsky functions ${}_{\pm2}R^{(0,1)}_{\ell m}(r)$, so we can still apply the Chandrasekhar transformation in GR.

To solve the scalar field from Eq. (5.88), we can use Green's function. Rewriting the left-hand side of Eq. (5.88) in the tortoise coordinate r^*, we get

$$\left[\partial^2_{r^*}+\frac{r-r_s}{r^3}\left(\frac{\omega^2_{\ell m}r^3}{r-r_s}-\frac{r_s}{r}-{}_0A_{\ell m}\right)\right]\Theta_{\ell m}(r)$$

$$= -\frac{4i}{C_2}\frac{r-r_s}{r^3}\left(g_1^{\ell m}(r)+g_2^{\ell m}(r)\partial_r\right){}_{-2}R_{\ell m}^{(0,1)}(r)\,, \tag{5.94}$$

where we have set $\eta_{\ell m}=-1$ since the even-parity modes with $\eta_{\ell m}=1$ do not couple to the scalar field. In this case, the solution to Eq. (5.94) along the contour $\mathscr{C}$ is

$$\Theta_{\ell m}(\xi) = \frac{\Theta^R_{\ell m}(\xi)\int_{-\infty}^{\xi}\Theta^L_{\ell m}(\xi')\mathcal{S}^{\ell m}_{\vartheta}(\xi')\partial_{\xi'}r^*\,d\xi' + \Theta^L_{\ell m}(\xi)\int_{\xi}^{\infty}\Theta^R_{\ell m}(\xi')\mathcal{S}^{\ell m}_{\vartheta}(\xi')\partial_{\xi'}r^*\,d\xi'}{\left(\Theta^R_{\ell m}(\xi)\partial_\xi\Theta^L_{\ell m}(\xi)-\Theta^L_{\ell m}(\xi)\partial_\xi\Theta^R_{\ell m}(\xi)\right)\partial_{r^*}\xi}\,, \tag{5.95}$$

where $f(\xi)$ and $f(\xi')$ mean evaluating $f(r)$ at $r=r_{\mathscr{C}}(\xi)$ and $r=r_{\mathscr{C}}(\xi')$, respectively. Numerically, we evaluate $\xi=\pm\infty$ at $\xi=\pm\xi_{\max}$, respectively. The function $\mathcal{S}^{\ell m}_{\vartheta}(\xi)$ is the source term at the right-hand side of Eq. (5.94). The functions $\Theta^R_{\ell m}(\xi)$ and $\Theta^L_{\ell m}(\xi)$ are solutions to the left-hand side of Eq. (5.94) with asymptotic behaviors $\Theta^R_{\ell m}(\xi\to\infty)\propto e^{i\omega_{\ell m}r^*(\xi)}$ and $\Theta^L_{\ell m}(\xi\to-\infty)\propto e^{-i\omega_{\ell m}r^*(\xi)}$, respectively. We compute $\Theta^R_{\ell m}(\xi)$ and $\Theta^L_{\ell m}(\xi)$ by numerically integrating their asymptotic expansion from $\xi=\xi_{\max}$ and $\xi=-\xi_{\max}$ along the contour to $\xi=-\xi_{\max}$ and $\xi=\xi_{\max}$, respectively. After getting $\Theta_{\ell m}(r)$ along the contour $\mathscr{C}$, we then plug it back into Eq. (5.87a). Notice, by solving $\Theta_{\ell m}(r)$, we effectively compute the piece $\mathcal{H}_{\vartheta}^{-1}\mathcal{V}^{\ell m}{}_2\mathcal{D}^{\dagger}_{\ell m}{}_2R^{(0,1)}_{\ell m}$ of $\hat{A}_1^{\ell m}\mathcal{D}^{\dagger}_{\ell m}{}_2R^{(0,1)}_{\ell m}$ [see Eq. (5.55)]. We can then compute the inner product in Eq. (5.87a) using the solution of $\Theta_{\ell m}(r)$ and ${}_2R^{(0,1)}_{\ell m}(r)$. A similar calculation can be also done in the ORG for $\Psi_4^{(1,1)}$ using Eq. (5.87b).

In Table 5.1, we present the results of $\omega^{(1,0)}_{\ell m}$ for $\ell=2,3$ and the overtones $n=0,1,2$ in this work using either the IRG or the ORG, in [92], and in [71], respectively. Both this work and [92] use the EVP method to compute $\omega^{(1,0)}_{\ell m}$, while Ref. [92] uses the RW and ZM equations of dCS gravity in [46–48, 57, 58] instead of the modified Teukolsky equations here. Ref. [57] also uses the RW and ZM equations but computes $\omega^{(1,0)}_{\ell m}$ via the shooting method. They evolve two independent solutions from the horizon and infinity to a middle point and find the frequency to get these two evolved solutions to match. We notice that all the results using the EVP method are consistent with each other, where the relative differences in both the real and imaginary part of $\omega^{(1,0)}_{\ell m}$ among these results are $\lesssim 10^{-5}$ for the fundamental mode $n=0$. The relative differences between the IRG results here and the results in [92] are $\lesssim 10^{-4}$ for all the modes computed. Since [92] uses the RW and ZM equations, this indicates that the modified Teukolsky equations found in [71] and this work are consistent with the RW and ZM equations in [46–48, 57, 58] for a non-rotating

ℓ	Overtones	MTF+EVP (IRG)	MTF+EVP (ORG)	RW/ZM+EVP	RW/ZM+shooting
	$n = 0$	$0.030768 + 0.015700i$	$0.030768 + 0.015700i$	$0.030768 + 0.015700i$	$0.031831 + 0.015916i$
$\ell = 2$	$n = 1$	$0.063740 + 0.051148i$	$0.063744 + 0.051148i$	$0.063740 + 0.051148i$	–
	$n = 2$	$0.146449 + 0.101397i$	$0.146528 + 0.101362i$	$0.146449 + 0.101398i$	–
	$n = 0$	$0.114273 + 0.020649i$	$0.114265 + 0.020648i$	$0.114273 + 0.020649i$	$0.133690 + 0.025464i$
$\ell = 3$	$n = 1$	$0.144647 + 0.063923i$	$0.145134 + 0.064434i$	$0.144647 + 0.063923i$	–
	$n = 2$	$0.207858 + 0.114091i$	$0.207176 + 0.109616i$	$0.207877 + 0.114103i$	–

Table 5.1: The QNM frequency shifts $\omega_{\ell m}^{(1,0)}$ of a non-rotating BH in dCS gravity for $\ell = 2, 3$ and the overtones $n = 0, 1, 2$. Due to spherical symmetry, $\omega_{\ell m}^{(1,0)}$ with the same ℓ but different m are the same. The word "MTF" is an acronym for the modified Teukolsky formalism. The columns "MTF+EVP (IRG)" and "MTF+EVP (ORG)" contain the results in this work. The results in the column "RW/ZM+EVP" use the EVP method to solve for $\omega_{\ell m}^{(1,0)}$ from the RW and ZM equations of dCS gravity in [46–48, 57, 58], as discussed in detail in [92]. The results in the column "RW/ZM+shooting" are retrieved from [57] directly, which uses the shooting method to solve for $\omega_{\ell m}^{(1,0)}$ from the RW and ZM equations. Since Ref. [57] does not calculate $\omega_{\ell m}^{(1,0)}$ for overtones, we leave these cells blank.

BH in dCS gravity. In our future work, we will prove this consistency analytically by developing a modified Chandrasekhar transformation to relate the RW and ZM equations to the modified Teukolsky equations of definite parity.

Furthermore, the results in the IRG and ORG of this work are also consistent, where the relative differences between the results in these two gauges are $< 1\%$ for all the modes except $\ell = 3$, $n = 2$, which has a relative difference $\sim 4\%$ in $\mathrm{Im}\left[\omega_{\ell m}^{(1,0)}\right]$. We also observe that the relative differences among different methods increase with the overtone number n. This is probably because $\mathrm{Im}\left[\omega_{\ell m}^{(0,0)}\right]$ decreases with n, which is always negative, so the source terms, which are driven by the GR QNMs, decay much faster along the contour $\mathscr{C}$ for a larger n, resulting in larger numerical inaccuracy.

Finally, comparing our results here to the ones in [57], which only contain the fundamental mode $n = 0$, we notice that the relative difference in $\omega_{\ell m}^{(1,0)}$ for $\ell = 2$, $n = 0$ is $< 4\%$. However, for $\ell = 3$, $n = 0$, the relative difference is as large as $\sim 20\%$. Since we previously confirmed that our modified Teukolsky equations are consistent with the RW and ZM equations in [57] for a non-rotating BH in dCS gravity, this large difference is likely due to numerical inaccuracy in using either the EVP method or the shooting method to compute the QNM frequencies. As the amplitude of QNMs diverges at the horizon and infinity but converges to 0 at the ends of the contour $\mathscr{C}$, it is more possible that the shooting method used by [57] has a larger numerical inaccuracy.

5.5.3 Outlook for computing the QNMs of rotating BHs in dCS gravity

In the previous subsection, we computed the QNM frequency shifts for a non-rotating BH in dCS gravity. The next natural step is to apply the same procedures to Eqs. (5.69) and (5.73), which are valid up to $O(\zeta^1, \chi^1, \epsilon^1)$. In this case, since the source terms in Eqs. (5.69) and (5.73) are not only coupled to the scalar field but also the GR QNMs directly, the QNM spectrum of both parity will get shifted, as discussed in Sec. 5.5.1 and confirmed in [57, 58]. The calculation for the odd-parity modes follows the same structure as the non-rotating case, where one needs to solve for the scalar field first. The calculation for the even-parity modes is even simpler since there is no coupling to the scalar field, and one can apply the standard EVP method for a single field in [66, 76, 77]. In addition, besides the gravitational-led QNMs, there are additional scalar-led modes in dCS gravity due to the coupling to the scalar field [57]. In this case, one can reverse the procedures for the odd-parity gravitational-led modes by first solving the modified Teukolsky equations in Eqs. (5.69) and (5.73) at the scalar QNM frequencies. Then, one feeds the solutions as the driving terms to the scalar field equations in Eqs. (5.31a) and (5.33a) and computes the QNM frequency shifts of the scalar-led modes using the EVP method. Since this is still ongoing work, we will present the results at $O(\zeta^1, \chi^1, \epsilon^1)$, including the scalar-led QNMs, in the full paper.

In [71] and this follow-up work, we only considered the case for a slowly rotating BH in dCS gravity up to $O(\chi^1)$. We did this to demonstrate this new approach in computing beyond-GR QNMs with the modified Teukolsky formalism so we can directly compare our results to the ones in [46–48, 57, 58]. Nevertheless, this new approach, in principle, works for BHs with a general spin in dCS gravity, including these fast rotating ones, as demonstrated in [67, 70]. Since the source terms of the modified Teukolsky equations in terms of the NP quantities computed in [71] work for a general spin, and the background metric of a fast rotating BH in dCS gravity can be computed with the method in [93], the only challenge is to implement the metric reconstruction procedures for the full Kerr case. However, the CCK procedures for metric reconstruction have been widely implemented for the full Kerr case in different situations [81, 83–85, 94–96]. Thus, we do not expect metric reconstruction to be an obstacle preventing us from obtaining the QNMs of fast rotating BHs in dCS gravity, as we will demonstrate in our follow-up work.

5.6 Appendix: Properties of the angular projection coefficients

In this appendix, we show some relations for the angular projection coefficients $\{\Lambda^{\ell_1\ell_2 m}_{s_1 s_2}, \Lambda^{\ell_1\ell_2 m}_{s_1 s_2 c}, \Lambda^{\ell_1\ell_2 m}_{s_1 s_2 s}\}$ and $\{\Lambda^{\dagger\ell_1\ell_2 m}_{s_1 s_2}, \Lambda^{\dagger\ell_1\ell_2 m}_{s_1 s_2 c}, \Lambda^{\dagger\ell_1\ell_2 m}_{s_1 s_2 s}\}$ that will be used to simplify the equations in Secs. 5.3 and 5.4.

Following [97], let us first show that

$$\Lambda^{\ell\ell m}_{s_1 s_1 c} = m\Lambda^{\ell\ell 1}_{s_1 s_1 c}\,, \tag{5.96a}$$
$$\Lambda^{\ell\ell m}_{s_1 s_1\pm 1 s} = m\Lambda^{\ell\ell 1}_{s_1 s_1\pm 1 s}\,. \tag{5.96b}$$

In [71], we have defined that

$$\Lambda^{\ell_1\ell_2 m}_{s_1 s_2 c} \equiv \int_{S^2} dS\ \cos\theta\ {}_{s_1}Y_{\ell_1 m}\ {}_{s_2}\bar{Y}_{\ell_2 m}\,, \tag{5.97a}$$
$$\Lambda^{\ell_1\ell_2 m}_{s_1 s_2 s} \equiv \int_{S^2} dS\ \sin\theta\ {}_{s_1}Y_{\ell_1 m}\ {}_{s_2}\bar{Y}_{\ell_2 m}\,, \tag{5.97b}$$

where dS is the solid angle element, and the integration is over the entire 2-sphere. Using that

$${}_0Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta\,, \quad {}_{\pm 1}Y_{10}(\theta,\phi) = \pm\sqrt{\frac{3}{8\pi}}\sin\theta\,, \tag{5.98}$$

we can write $\Lambda^{\ell\ell m}_{s_1 s_1 c}$ and $\Lambda^{\ell\ell m}_{s_1 s_1\pm 1 s}$ as

$$\Lambda^{\ell\ell m}_{s_1 s_1 c} = \sqrt{\frac{4\pi}{3}}\int_{S^2} dS\ {}_{s_1}\bar{Y}_{\ell m}\ {}_{s_1}Y_{\ell m}\ {}_0Y_{10}\,, \tag{5.99a}$$
$$\Lambda^{\ell\ell m}_{s_1 s_1\pm 1 s} = \pm\sqrt{\frac{8\pi}{3}}\int_{S^2} dS\ {}_{s_1\pm 1}\bar{Y}_{\ell m}\ {}_{s_1}Y_{\ell m}\ {}_{\pm 1}Y_{10}\,. \tag{5.99b}$$

Using the relation between the spin-weighted spherical harmonics ${}_sY_{\ell m}(\theta,\phi)$ and the Wigner rotation matrices $D^{\ell}_{sm}(\phi,\theta,\gamma)$ in [98],

$${}_sY_{\ell m}(\theta,\phi)e^{-is\gamma} = \sqrt{\frac{2\ell+1}{4\pi}}D^{\ell}_{-sm}(\phi,\theta,\gamma)\,, \tag{5.100}$$

we can reduce Eq. (5.99) to

$$\Lambda^{\ell\ell m}_{s_1 s_1 c} = \frac{2\ell+1}{8\pi^2}\int_0^{2\pi} d\gamma\int_{S^2} dS\ \bar{D}^{\ell}_{-s_1 m}\,D^{\ell}_{-s_1 m}\,D^{1}_{00}\,, \tag{5.101a}$$
$$\Lambda^{\ell\ell m}_{s_1 s_1\pm 1 s} = \pm\frac{2\ell+1}{4\sqrt{2}\pi^2}\int_0^{2\pi} d\gamma\int_{S^2} dS\ \bar{D}^{\ell}_{-s_1\mp 1 m}\,D^{\ell}_{-s_1 m}\,D^{1}_{\mp 10}\,, \tag{5.101b}$$

where we have added an additional integral in γ and a factor of $1/(2\pi)$. This trick uses that the integrands in Eq. (5.101) do not really depend on γ since all the factors of $e^{-is\gamma}$ cancel out after using Eq. (5.100). Now, using that [97]

$$\int_0^{2\pi} d\gamma \int_{S^2} dS\ \bar{D}^{\ell_3}_{s_3m_3} D^{\ell_2}_{s_2m_2} D^{\ell_1}_{s_1m_1} = \frac{8\pi^2}{2\ell_3+1} \delta_{s_1+s_2,s_3} \delta_{m_1+m_2,m_3} C^{\ell_3m_3}_{\ell_1m_1\ell_2m_2} C^{\ell_3s_3}_{\ell_1s_1\ell_2s_2}\,, \tag{5.102}$$

where $C^{\ell_3m_3}_{\ell_1m_1\ell_2m_2}$ is the Clebsch-Gordan coefficient, we get that

$$\Lambda^{\ell\ell m}_{s_1s_1c} = C^{\ell m}_{10\ell m} C^{\ell(-s_1)}_{10\ell(-s_1)}\,, \tag{5.103a}$$

$$\Lambda^{\ell\ell m}_{s_1s_1\pm 1s} = \pm\sqrt{2} C^{\ell m}_{10\ell m} C^{\ell(-s_1\mp 1)}_{1(\mp 1)\ell(-s_1)}\,, \tag{5.103b}$$

Using the relation in [99] that

$$C^{\ell m}_{10\ell m} = m C^{\ell 1}_{10\ell 1}\,, \tag{5.104}$$

we get Eq. (5.96).

Next, let us show that

$$\Lambda^{\ell\ell m}_{-s_1-s_1c} = -\Lambda^{\ell\ell m}_{s_1s_1c}\,, \tag{5.105a}$$

$$\Lambda^{\ell\ell m}_{-s_1-s_1\mp 1s} = \Lambda^{\ell\ell m}_{s_1s_1\pm 1s}\,. \tag{5.105b}$$

First, using Eqs. (5.45) and (5.97), we get

$$\Lambda^{\ell_1\ell_2m}_{-s_1-s_2c} = (-1)^{s_1+s_2} \int_{S^2} dS\ \cos\theta\ {}_{s_1}\bar{Y}_{\ell_1-m}\ {}_{s_2}Y_{\ell_2-m}\,, \tag{5.106a}$$

$$\Lambda^{\ell_1\ell_2m}_{-s_1-s_2s} = (-1)^{s_1+s_2} \int_{S^2} dS\ \sin\theta\ {}_{s_1}\bar{Y}_{\ell_1-m}\ {}_{s_2}Y_{\ell_2-m}\,. \tag{5.106b}$$

Since the integrands in Eq. (5.106) are real, we can move the complex conjugate of ${}_{s_1}\bar{Y}_{\ell_1-m}$ to ${}_{s_2}Y_{\ell_2-m}$. Thus, we get

$$\Lambda^{\ell_1\ell_2m}_{-s_1-s_2c} = (-1)^{s_1+s_2} \Lambda^{\ell_1\ell_2-m}_{s_1s_2c}\,, \tag{5.107a}$$

$$\Lambda^{\ell_1\ell_2m}_{-s_1-s_2s} = (-1)^{s_1+s_2} \Lambda^{\ell_1\ell_2-m}_{s_1s_2s}\,. \tag{5.107b}$$

Using Eq. (5.96) with Eq. (5.107), we get Eq. (5.105).

Finally, let us show that

$$\Lambda^{\dagger\ell\ell m}_{-s_1s_2c} = (-1)^{m+s_1} \Lambda^{\ell\ell m}_{s_1s_2c}\,, \tag{5.108a}$$

$$\Lambda^{\dagger\ell\ell m}_{-s_1s_2s} = (-1)^{m+s_1} \Lambda^{\ell\ell m}_{s_1s_2s}\,. \tag{5.108b}$$

In [71], we have defined that

$$\Lambda^{\dagger \ell_1 \ell_2 m}_{s_1 s_2 c} \equiv \int_{S^2} dS \ \cos\theta \ {}_{s_1}\bar{Y}_{\ell_1 -m} \ {}_{s_2}\bar{Y}_{\ell_2 m} \,, \tag{5.109a}$$

$$\Lambda^{\dagger \ell_1 \ell_2 m}_{s_1 s_2 s} \equiv \int_{S^2} dS \ \sin\theta \ {}_{s_1}\bar{Y}_{\ell_1 -m} \ {}_{s_2}\bar{Y}_{\ell_2 m} \,. \tag{5.109b}$$

Using Eq. (5.45), we get

$$\Lambda^{\dagger \ell_1 \ell_2 m}_{-s_1 s_2 c} = (-1)^{m+s_1} \int_{S^2} dS \ \cos\theta \ {}_{s_1}Y_{\ell_1 m} \ {}_{s_2}\bar{Y}_{\ell_2 m} \,, \tag{5.110a}$$

$$\Lambda^{\dagger \ell_1 \ell_2 m}_{-s_1 s_2 s} = (-1)^{m+s_1} \int_{S^2} dS \ \sin\theta \ {}_{s_1}Y_{\ell_1 m} \ {}_{s_2}\bar{Y}_{\ell_2 m} \,, \tag{5.110b}$$

which gives us Eq. (5.108) when comparing Eq. (5.110) to Eq. (5.97).

5.7 Appendix: List of radial operators

In this appendix, we provide the explicit expressions of the radial operators $\{\hat{A}^{\ell m}_i, \hat{B}^{\ell m}_i, \hat{\bar{B}}^{\ell m}_i\}$ and $\{\hat{\mathscr{A}}^{\ell m}_i, \hat{\mathscr{B}}^{\ell m}_i, \hat{\bar{\mathscr{B}}}^{\ell m}_i\}$ in terms of the auxiliary radial functions in [71].

First, the radial operators $\{\hat{A}^{\ell m}_i, \hat{B}^{\ell m}_i, \hat{\bar{B}}^{\ell m}_i\}$ used in the equation of $\Psi_0^{(1,1)}$ in the IRG [Eqs. (5.69) and (5.70)] are defined as

$$\hat{A}^{\ell m}_1 = i\left(k^{\ell m}_1(r) + k^{\ell m}_2(r)\partial_r\right)\mathcal{H}^{-1}_{\vartheta}\mathcal{V}^{\ell m} + \frac{1}{1-\bar{\eta}_{\ell m}}\left(k^{\ell m}_3(r) + k^{\ell m}_4(r)\partial_r\right), \tag{5.111a}$$

$$\hat{A}^{\ell m}_2 = \left(k^{\ell m}_5(r) + k^{\ell m}_6(r)\partial_r\right)\mathcal{H}^{-1}_{\vartheta}\mathcal{V}^{\ell m} - \frac{i}{1-\bar{\eta}_{\ell m}}\left(k^{\ell m}_7(r) + k^{\ell m}_8(r)\partial_r\right), \tag{5.111b}$$

$$\hat{A}^{\ell m}_3 = \left(k^{\ell m}_9(r) + k^{\ell m}_{10}(r)\partial_r\right)\mathcal{H}^{-1}_{\vartheta}\mathcal{V}^{\ell m} - \frac{i}{1-\bar{\eta}_{\ell m}}\left(k^{\ell m}_{11}(r) + k^{\ell m}_{12}(r)\partial_r\right), \tag{5.111c}$$

$$\hat{B}^{\ell m}_1 = -i\left(q^{\ell m}_1(r) + q^{\ell m}_2(r)\partial_r\right), \tag{5.111d}$$

$$\hat{B}^{\ell m}_2 = -i\left(q^{\ell m}_3(r) + q^{\ell m}_4(r)\partial_r\right), \tag{5.111e}$$

$$\hat{B}^{\ell m}_3 = -i\left(q^{\ell m}_5(r) + q^{\ell m}_6(r)\partial_r\right), \tag{5.111f}$$

$$\hat{\bar{B}}^{\ell m}_1 = -i\left(\tilde{q}^{\ell m}_1(r) + \tilde{q}^{\ell m}_2(r)\partial_r\right), \tag{5.111g}$$

$$\hat{\bar{B}}^{\ell m}_2 = -i\tilde{q}^{\ell m}_3(r)\,, \tag{5.111h}$$

where $\mathcal{H}^{-1}_{\vartheta}$ is the Green's function corresponding to the left-hand side of Eq. (5.31a), which can be retrieved from Eq. (5.95), and the operator $\mathcal{V}^{\ell m}$ is

$$\mathcal{V}^{\ell m} = g^{\ell m}_1(r) + g^{\ell m}_2(r)\partial_r - i\chi m \Lambda^{\ell\ell 1}_{10s}\left[g^{\ell m}_3(r) + h^{\ell m}_1(r) + \left(g^{\ell m}_4(r) + h^{\ell m}_2(r)\right)\partial_r\right]. \tag{5.112}$$

Notice that there is an extra factor of $1/(1-\bar{\eta}_{\ell m})$ in several terms of Eq. (5.111) since we have extracted a factor of $1-\bar{\eta}_{\ell m}$ in Eq. (5.54) when defining $\hat{A}_i^{\ell m}$. The definition of all the radial functions $k_i^{\ell m}(r) = k_i^{\ell m}(r,\omega,M)$, $q_i^{\ell m}(r) = q_i^{\ell m}(r,\omega,M)$, and $\tilde{q}_i^{\ell m}(r) = \tilde{q}_i^{\ell m}(r,\omega,M)$ can be found in [71] and the supplementary Mathematica notebook [88]. The radial functions $g_i^{\ell m}(r)$ and $h_i^{\ell m}(r)$ follow the redefinition in Eq. (5.27).

Next, the radial operators $\{\hat{\mathscr{A}}_i^{\ell m}, \hat{\mathscr{B}}_i^{\ell m}, \hat{\tilde{\mathscr{B}}}_i^{\ell m}\}$ used in the equation of $\Psi_4^{(1,1)}$ in the ORG [Eqs. (5.73) and (5.74)] are defined as

$$\hat{\mathscr{A}}_1^{\ell m} = i\left(\mathfrak{k}_1^{\ell m}(r) + \mathfrak{k}_2^{\ell m}(r)\partial_r\right)\mathcal{H}_\vartheta^{-1}\mathcal{U}^{\ell m} + \frac{1}{1-\bar{\eta}_{\ell m}}\left(\mathfrak{k}_3^{\ell m}(r) + \mathfrak{k}_4^{\ell m}(r)\partial_r\right), \tag{5.113a}$$

$$\hat{\mathscr{A}}_2^{\ell m} = \left(\mathfrak{k}_5^{\ell m}(r) + \mathfrak{k}_6^{\ell m}(r)\partial_r\right)\mathcal{H}_\vartheta^{-1}\mathcal{U}^{\ell m} - \frac{i}{1-\bar{\eta}_{\ell m}}\left(\mathfrak{k}_7^{\ell m}(r) + \mathfrak{k}_8^{\ell m}(r)\partial_r\right), \tag{5.113b}$$

$$\hat{\mathscr{A}}_3^{\ell m} = \left(\mathfrak{k}_9^{\ell m}(r) + \mathfrak{k}_{10}^{\ell m}(r)\partial_r\right)\mathcal{H}_\vartheta^{-1}\mathcal{U}^{\ell m} - \frac{i}{1-\bar{\eta}_{\ell m}}\left(\mathfrak{k}_{11}^{\ell m}(r) + \mathfrak{k}_{12}^{\ell m}(r)\partial_r\right), \tag{5.113c}$$

$$\hat{\mathscr{B}}_1^{\ell m} = -i\left(\mathfrak{q}_1^{\ell m}(r) + \mathfrak{q}_2^{\ell m}(r)\partial_r\right), \tag{5.113d}$$

$$\hat{\mathscr{B}}_2^{\ell m} = -i\left(\mathfrak{q}_3^{\ell m}(r) + \mathfrak{q}_4^{\ell m}(r)\partial_r\right), \tag{5.113e}$$

$$\hat{\mathscr{B}}_3^{\ell m} = -i\left(\mathfrak{q}_5^{\ell m}(r) + \mathfrak{q}_6^{\ell m}(r)\partial_r\right), \tag{5.113f}$$

$$\hat{\tilde{\mathscr{B}}}_1^{\ell m} = -i\left(\tilde{\mathfrak{q}}_1^{\ell m}(r) + \tilde{\mathfrak{q}}_2^{\ell m}(r)\partial_r\right), \tag{5.113g}$$

$$\hat{\tilde{\mathscr{B}}}_2^{\ell m} = -i\tilde{\mathfrak{q}}_3^{\ell m}(r), \tag{5.113h}$$

where

$$\mathcal{U}^{\ell m} = \mathfrak{g}_1^{\ell m}(r) + \mathfrak{g}_2^{\ell m}(r)\partial_r - i\chi m\Lambda_{10s}^{\ell\ell 1}\left[\mathfrak{g}_3^{\ell m}(r) + \mathfrak{h}_1^{\ell m}(r) + \left(\mathfrak{g}_4^{\ell m}(r) + \mathfrak{h}_2^{\ell m}(r)\right)\partial_r\right]. \tag{5.114}$$

The definition of all the radial functions $\mathfrak{k}_i^{\ell m}(r) = \mathfrak{k}_i^{\ell m}(r,\omega,M)$, $\mathfrak{q}_i^{\ell m}(r) = \mathfrak{q}_i^{\ell m}(r,\omega,M)$, and $\tilde{\mathfrak{q}}_i^{\ell m}(r) = \tilde{\mathfrak{q}}_i^{\ell m}(r,\omega,M)$ can be found in [71] and the supplementary Mathematica notebook [88]. The radial functions $\mathfrak{g}_i^{\ell m}(r)$ and $\mathfrak{h}_i^{\ell m}(r)$ follow the redefinition in Eq. (5.27).

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Chapter 6

SPECTROSCOPY OF BUMPY BLACK HOLES: NON-ROTATING CASE

[1] Colin Weller, Dongjun Li, and Yanbei Chen. “Spectroscopy of bumpy BHs: non-rotating case”. In: (May 2024). arXiv: 2405.20934 [gr-qc].

6.1 Introduction

Recent observations from ground-based laser interferometers [1, 2], pulsar timing arrays [3], and very-long-baseline interferometry [4] have spawned a new era of testing general relativity (GR) [5, 6]. While GR has agreed with numerous tests, modifications may emerge beyond some scale [7–10]. In particular, both GR and the standard model of particle physics fail to explain the existence of dark matter [11], the accelerating expansion of the universe [12–14], the present matter-antimatter asymmetry [15], motivating bottom-up modifications to GR to provide explanation. At a more fundamental level, GR and quantum mechanics do not reconcile beyond the Planck scale, which has prompted many to search for a unified theory from the top-down, such as loop quantum gravity [16, 17], string theory [18], and other quantum structure programs [19]. Both the bottom-up and top-down efforts have resulted in a plethora of beyond-GR (bGR) theories, such as scalar-tensor theories [20], Einstein dilaton Gauss-Bonnet gravity [21–23], Horndeski theory [24], dynamical Chern-Simons gravity [25, 26], $f(R)$ gravity [27, 28], and higher-derivative gravity without extra fields [29–31]. Given the breadth of bGR theories, it remains critical to test their predictions, especially when gravity is strong and highly dynamic.

One avenue that may constrain bGR theories is examining the gravitational wave (GW) signal of a perturbed, remnant black hole (BH) produced by a binary BH merger. The GWs emitted in the ringdown phase are characterized by the so-called quasinormal modes (QNMs) with complex frequencies. The real part of the QNM frequencies is related to the orbital and precessional frequencies of null geodesics near the light ring, while the imaginary part encodes the Lyapunov exponent of the orbit [32–34]. Each QNM can be labeled by three integers, (ℓ, m, n), where ℓ and m are angular momentum quantum numbers and n denotes the overtone [35, 36]. In GR, by detecting these QNMs, one can infer the mass and spin of the remnant BH. In bGR theories, these QNMs can carry even more information, such as the length scale of bGR physics and the possible existence of other non-metric fields. Examining the correspondence between QNM spectra and fundamental physics around BHs is generally known as BH spectroscopy [37–40].

Using the linear perturbation theory of a single BH, one can precisely calculate its QNMs in GR. For Schwarzschild BHs obeying spherical symmetry, one can directly separate metric perturbations into even- and odd-parity components, which are well described by the Regge-Wheeler and Zerilli-Moncreif equations [41–43]. These two equations are one-dimensional Schrodinger-type wave equations in the radial coordinate and are decoupled from each other. The resulting QNMs can be found by imposing the gravitational perturbations to be purely outgoing at infinity and purely ingoing at the horizon [44]. For spinning BHs with only axisymmetry, one cannot easily decouple perturbations of different metric components, even in linear order, which makes reducing the Einstein equations into purely radial equations much more challenging. One alternative approach built on the Newman-Penrose (NP) formalism [45] was developed by Teukolsky and Press [46–48], which instead focuses on curvature perturbations represented by perturbations of the Weyl scalars Ψ_0 and Ψ_4. In this case, one can also find two decoupled ordinary differential equations of Ψ_0 and Ψ_4 in the radial coordinate. Since then, QNMs in GR have been studied widely in the literature (see Refs. [39, 49, 50] for reviews). Many semi-analytic [32, 35, 51, 52] and numerical methods [53, 54] have been developed to compute QNMs accurately within GR.

Importantly, QNMs in GR are governed by the "no-hair" theorem, which requires that a BH is completely determined by its mass, angular momentum, and electric charge [55–58]. While it was shown a *real* scalar field cannot source scalar hair [59], it is possible to have "hairy" solutions when GR is coupled to other fields and in modified gravity (e.g., see [60, 61]), which has been tested using GW detections and QNMs [62].

An equivalent statement of the no-hair theorem can be given in terms of field multipole moments. Typically, multipole moments describe the expansion of a field that satisfies a linear differential equation, such as Laplace's equation. Nonetheless, Thorne found how to use multipole moments to describe solutions of the Einstein equations by making a post-Newtonian expansion [63]. Later, Geroch and Hansen (GH) extended this description to the strong gravity regime [64–66], which is equivalent to Thorne's description when the correct approximation is taken [67].

Similar to the multipoles discovered by Thorne, GH multipoles have two types: mass moments M_l and current moments S_l. In particular, for the Kerr geometry, they satisfy (in geometrized units),

$$M_l + iS_l = M(ia)^l\,, \quad a = J/M\,, \tag{6.1}$$

where J and M are the angular momentum and mass of a BH, respectively. These moments were only well-defined for vacuum spacetimes originally, but some recent work has extended the original definition to include sources [68] and bGR theories [69, 70]. Additionally, it has been shown that given all the GH multipoles for a spacetime, the full metric can be reconstructed [71–73].

The intricate connection between a BH's geometry and its multipole moments led Thorne to advocate GW signals as a way to examine it [74]. Later, Ryan calculated the GWs emitted by an extreme mass-ratio inspiral (EMRI) around an axisymmetric central BH with arbitrary GH multipole moments [75]. Following these previous endeavors, Refs. [76–78] further developed the program of *bumpy* BHs, which deviate from the Kerr geometry by multipole moments (i.e., the "bumps"), and computed the modifications to EMRI waveforms. However, none of these analyses calculated the shifts in QNM frequencies due to these additional multipole moments.

On the other hand, there has been a rising interest in performing parametrized ringdown tests [79]. Most of the previous studies either focus on non-rotating BHs [80, 81] or make the eikonal approximation for rotating BHs [40, 82–84]. However, recent GW detections indicate that most remnant BHs are fast rotating [85], while the eikonal approximation only works properly for QNMs with large ℓ, which are subdominant in the ringdown signal and may not even work for certain bGR theories [40]. Some other efforts consider modifying the potential in the radial Teukolsky equations directly [86], while the map from the QNM frequency shifts to the geometric deformations of a BH becomes not transparent.

All these challenges in studying the multipole moments of a BH spacetime and performing parametrized ringdown tests motivate this work. In this work, we will apply the modified Teukolsky formalism developed in [87–89] to directly compute the QNM frequency shifts generated by the multipole moments of the bumpy BHs considered in [77]. This new approach of doing parametrized ringdown tests works in general for Kerr BHs with perturbative axisymmetric deviations. It also does not rely on any eikonal approximation. Furthermore, it allows one to map the deformations of the BH geometry, described by multipole moments, to the QNM frequency shifts directly. Previous works [89–91] have applied the modified Teukolsky formalism to compute the QNMs in specific bGR theories, while we apply it to this theory-agnostic ringdown study. To demonstrate our approach, we will focus on the simplest case of non-rotating bumpy BHs in this work. Although our BHs are non-rotating, the bumps added to them are axisymmetric [77], so the procedures we implement here work for the rotating case in principle.

The manuscript is organized as follows. In Sec. 6.2, we introduce the specific model of bumpy BHs developed in [77], explain their motivation for testing GR, and detail their derivation. In Sec. 6.3, we review the modified Teukolsky formalism and implement the Chrzanowski-Cohen-Kegeles (CCK) metric reconstruction procedure [92–96] to compute all the necessary source terms in the equation. In Sec. 6.4, we review the eigenvalue perturbation (EVP) method in [88, 97–99] and apply it to compute the QNM frequency shifts from the modified Teukolsky equation. We then present the results in Sec. 6.5 and discuss future avenues in Sec. 6.6. For convenience, we provide a flow chart in Fig. 6.1 to summarize our procedures for computing the QNM frequency shifts of the bumpy BHs.

6.2 Bumpy BHs

Bumpy BHs were first introduced in [76] as a way to conveniently parametrize multipole deviations away from the Kerr vacuum. In GR, the no-hair theorem requires a Kerr solution to satisfy

$$M_l + iS_l = M(ia)^l \,, \tag{6.2}$$

where (M_l, S_l) are the mass and current multipoles of the BH, respectively. Thus, one can form a null experiment of GR by considering deviations of the form

$$M_l + iS_l = M(ia)^l + \delta M_l + \delta S_l \,. \tag{6.3}$$

In [77], Vingeland and Hughes further extended the results in [76] to rectify the nonsmooth nature of the bumps and include spinning bumpy BHs. More specifically, they considerred the Weyl metric [100]

$$ds^2 = -e^{2\psi} dt^2 + e^{2\gamma-2\psi} \left(d\rho^2 + dz^2\right) + e^{-2\psi} \rho^2 d\phi^2 \,, \tag{6.4}$$

where (ρ, z) are related to the Boyer-Lindquist coordinates (r, θ) by

$$\rho = r \sin\theta \sqrt{1 - \frac{2M}{r}} \,, \quad z = (r - M)\cos\theta \,. \tag{6.5}$$

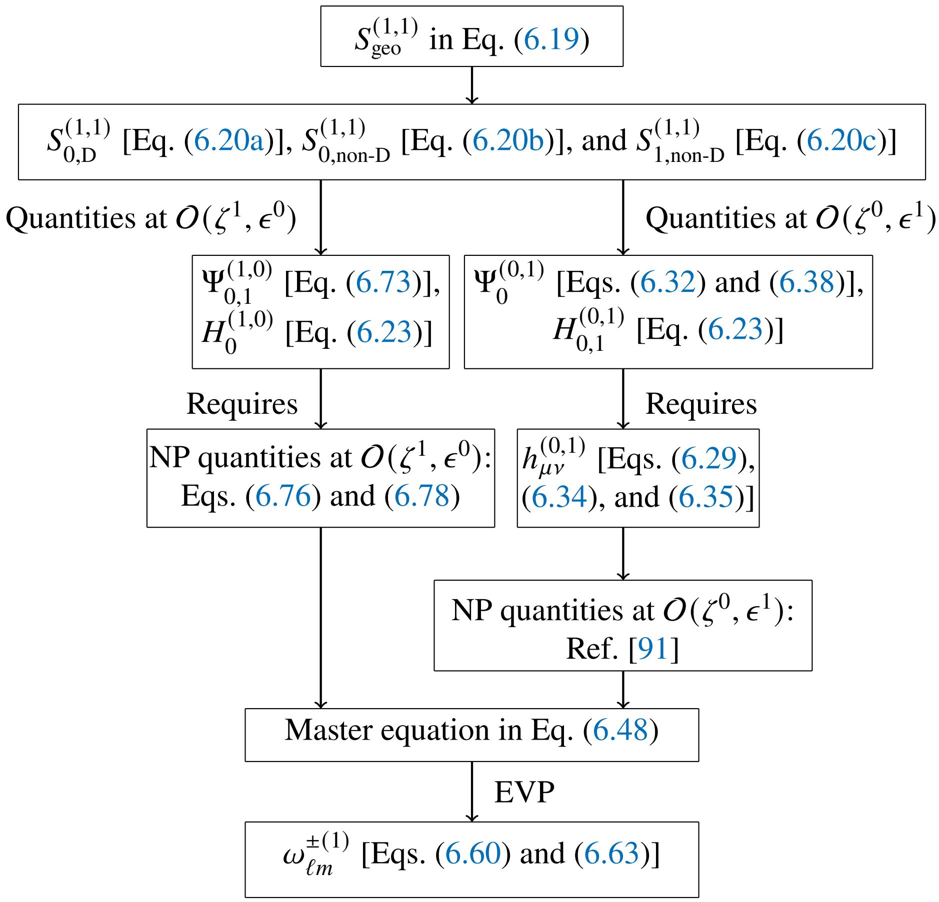

Figure 6.1: A flow chart describing the procedures and the main equations used for computing the QNM frequency shifts $\omega_{\ell m}^{\pm(1)}$ for bumpy BHs.

The terms ψ and γ are functions of ρ and z, i.e., $\psi = \psi(\rho, z)$ and $\gamma = \gamma(\rho, z)$. Imposing the spacetime to be Ricci flat, one gets three equations for γ and ψ:

$$0 = \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2}\,, \tag{6.6a}$$

$$\frac{\partial \gamma}{\partial \rho} = \rho\left[\left(\frac{\partial \psi}{\partial \rho}\right)^2 - \left(\frac{\partial \psi}{\partial z}\right)^2\right], \tag{6.6b}$$

$$\frac{\partial \gamma}{\partial z} = 2\rho \frac{\partial \psi}{\partial \rho}\frac{\partial \psi}{\partial z}\,. \tag{6.6c}$$

Since Eq. (6.6a) is simply Laplace's equation, ψ can be conveniently chosen as harmonic functions [77]. Reference [77] further perturbatively solved Eq. (6.6) by expanding the bumpy BH spacetime around the BHs in GR, e.g.,

$$\psi = \psi_0 + \psi_1\,, \qquad\qquad \psi_1/\psi_0 \ll 1\,,$$

$$\gamma = \gamma_0 + \gamma_1 \,, \qquad\qquad \gamma_1/\gamma_0 \ll 1 \,. \tag{6.7}$$

For non-rotating BHs, ψ_0 and γ_0 correspond to the Schwarzschild metric in the Weyl coordinates with

$$\psi_0 = \ln\tanh(u/2) \,, \quad \gamma_0 = -\frac{1}{2}\ln\left(1 + \frac{\sin^2 v}{\sinh^2 u}\right) \,, \tag{6.8}$$

where (u, v) are prolate spheroidal coordinates (u, v) and are related to the Weyl coordinates (ρ, z) by

$$\rho = M \sinh u \sin v \,, \tag{6.9a}$$

$$z = M \cosh u \cos v \,. \tag{6.9b}$$

Since ψ_1 is a harmonic function when the spacetime is Ricci flat, it can be decomposed into multipoles in Weyl coordinates, i.e.,

$$\psi_1(\rho, z) = \sum_{\ell_W} B_{\ell_W} M^{\ell_W+1} \frac{Y_{\ell_W 0}(\theta_{\rm Weyl})}{(\rho^2 + z^2)^{\frac{\ell_W+1}{2}}} \,, \tag{6.10}$$

where $\cos(\theta_{\rm Weyl}) = z/\sqrt{\rho^2 + z^2}$, and B_{ℓ_W} is a dimensionless coupling constant that parametrizes the magnitude of the bump with multipole ℓ_W. We can assume that $B_{\ell_W} \ll 1$ for our purposes. Using the coordinate transformation in Eq. (6.5), one can transform Eq. (6.4) in the Weyl coordinates to the Boyer-Lindquist coordinates such that

$$\begin{aligned} ds^2 = & -e^{2\psi_1}\left(1 - \frac{2M}{r}\right) dt^2 + e^{2\gamma_1 - 2\psi_1}\left(1 - \frac{2M}{r}\right)^{-1} dr^2 \\ & + r^2 e^{2\gamma_1 - 2\psi_1} d\theta^2 + r^2 \sin^2\theta e^{-2\psi_1} d\phi^2. \end{aligned} \tag{6.11}$$

To solve for the function γ_1, we need to use the remaining equations of motion in Eq. (6.6). Let us focus on $\ell_W = 2$ as an example, and the procedures for higher ℓ_W are similar. At $\ell_W = 2$, Eq. (6.10) gives

$$\psi_1^{\ell_W=2}(\rho, z) = \frac{B_2 M^3}{4}\sqrt{\frac{5}{\pi}} \frac{3\cos^2\theta_{\rm Weyl} - 1}{(\rho^2 + z^2)^{3/2}} \,, \tag{6.12}$$

which in the Boyer-Lindquist coordinates becomes

$$\psi_1^{\ell_W=2}(r, \theta) = \frac{B_2 M^3}{4}\sqrt{\frac{5}{\pi}} \frac{1}{d(r,\theta)^3}\left[\frac{3(r - M)^2\cos^2\theta}{d(r,\theta)^2} - 1\right] \,, \tag{6.13}$$

where

$$d(r,\theta) \equiv \left(r^2 - 2Mr + M^2\cos^2\theta\right)^{1/2} . \tag{6.14}$$

By using Eq. (6.6) and imposing that $\gamma_1 \to 0$ as $r \to \infty$, one obtains

$$\begin{aligned}
&\gamma_1^{\ell_W=2}(r,\theta) = B_2\sqrt{\frac{5}{\pi}}\left[\frac{r-M}{2}\frac{c_{20}(r)+c_{22}(r)\cos^2\theta}{d(r,\theta)^5} - 1\right] , \\
&c_{20}(r) = 2(r-M)^4 - 5M^2(r-M)^2 + 3M^4 , \\
&c_{22}(r) = 5M^2(r-M)^2 - 3M^4 .
\end{aligned} \tag{6.15}$$

Similarly, at $\ell_W = 3$, one can find [77]

$$\psi_1^{\ell_W=3}(r,\theta) = \frac{B_3M^4}{4}\sqrt{\frac{7}{\pi}}\frac{1}{d(r,\theta)^4}\left[\frac{5(r-M)^3\cos^3\theta}{d(r,\theta)^3} - \frac{3(r-M)\cos\theta}{d(r,\theta)}\right] , \tag{6.16a}$$

$$\begin{aligned}
&\gamma_1^{\ell_W=3}(r,\theta) \\
&= \frac{B_3M^5}{2}\sqrt{\frac{7}{\pi}}\cos\theta\left[\frac{c_{30}(r)+c_{32}(r)\cos^2\theta+c_{34}(r)\cos^4\theta+c_{36}(r)\cos^6\theta}{d(r,\theta)^7}\right] ,
\end{aligned} \tag{6.16b}$$

$$\begin{aligned}
&c_{30}(r) = -3r(r-2M) , \quad c_{32}(r) = 10r(r-2M) + 2M^2 , \\
&c_{34}(r) = -7r(r-2M) , \quad c_{36}(r) = -2M^2 .
\end{aligned}$$

In this work, we will only focus on the bumps with $\ell_W = 2, 3$ and compute the corresponding QNM frequency shifts. As one can directly check using Eqs. (6.12), (6.15), and (6.16), the bumps at $\ell_W = 2$ and $\ell_W = 3$ are of even- and odd-parity, respectively. As we will see later, bumps with different parity will generate QNM spectra with different characteristic structures. Thus, studying the bumps at $\ell_W = 2, 3$ already allows us to investigate those characteristic features of QNM spectra in the bumpy BH spacetime, while the procedures for computing the QNM spectra of bumps with higher ℓ_W are similar.

6.3 BH Perturbations in the Modified Teukolsky Formalism

In this section, we briefly review the modified Teukolsky formalism in [87] and detail how to implement it for the bumpy BHs considered in this work. At the end of this section, we provide the form of the master equations governing the QNMs of these bumpy BHs with the complete expressions provided in the supplementary notebook [101].

6.3.1 The modified Teukolsky equation

To compute the QNM frequency shifts driven by the bumps in [77], we choose to apply the modified Teukolsky formalism in [87, 91, 99]. Built upon the seminal work by Teukolsky [46], Ref. [87] has taken an effective field theory extension of the Teukolsky formalism by using a two-parameter expansion, e.g.,

$$\Psi_i = \Psi_i^{(0,0)} + \zeta\Psi_i^{(1,0)} + \epsilon\Psi_i^{(0,1)} + \zeta\epsilon\Psi_i^{(1,1)}\,, \tag{6.17}$$

where we have taken the Weyl scalars Ψ_i as an example. The coefficient ζ is a dimensionless parameter that parametrizes the strength of the deviation from GR. In our case, $\zeta = B_{\ell_W}$, where B_{ℓ_W} are the amplitudes of the bumps in Eq. (6.10). The coefficient ϵ characterizes the magnitude of GW perturbations of certain background spacetime. In this case, the quantities at $O(\zeta^0, \epsilon^0)$ are evaluated in some GR BH spacetimes of Petrov-type-D. For non-rotating BHs considered in this study, the terms $\Psi_i^{(0,0)}$ are evaluated in the Schwarzschild spacetime. The terms at $O(\zeta^1, \epsilon^0)$ are driven by bGR modifications to the background spacetime. In this work, these bGR modifications are bumps described by the Weyl multipole potentials $(\psi_1^{\ell_W}, \gamma_1^{\ell_W})$ in Sec. 6.2. The terms at $O(\zeta^0, \epsilon^1)$ correspond to GWs in GR, while the terms at $O(\zeta^1, \epsilon^1)$ are bGR GWs we are interested in.

Utilizing this expansion, Ref. [87] found a set of decoupled differential equations for $\Psi_0^{(1,1)}$ and $\Psi_4^{(1,1)}$, where

$$H_0^{(0,0)}\Psi_0^{(1,1)} = \mathcal{S}_{\text{geo}}^{(1,1)} + \mathcal{S}^{(1,1)}\,. \tag{6.18}$$

In this work, we choose to focus on $\Psi_0^{(1,1)}$, and the equations for $\Psi_4^{(1,1)}$ can be found in [87]. Here, $H_0^{(0,0)}$ is the Teukolsky operator for Ψ_0 in GR [46]. Reference [87] has divided up the source terms into two categories. The terms in $\mathcal{S}_{\text{geo}}^{(1,1)}$ only depend on the corrections to the background geometry and the GWs in GR, so they are considered purely "geometrical," i.e.,

$$\mathcal{S}_{\text{geo}}^{(1,1)} = \mathcal{S}_{0,\text{D}}^{(1,1)} + \mathcal{S}_{0,\text{non-D}}^{(1,1)} + \mathcal{S}_{1,\text{non-D}}^{(1,1)}\,, \tag{6.19}$$

with

$$\mathcal{S}_{0,\text{D}}^{(1,1)} = -H_0^{(1,0)}\Psi_0^{(0,1)}\,, \tag{6.20a}$$
$$\mathcal{S}_{0,\text{non-D}}^{(1,1)} = -H_0^{(0,1)}\Psi_0^{(1,0)}\,, \tag{6.20b}$$
$$\mathcal{S}_{1,\text{non-D}}^{(1,1)} = H_1^{(0,1)}\Psi_1^{(1,0)}\,. \tag{6.20c}$$

The other set of source terms is directly driven by the effective stress tensor due to corrections to the Einstein-Hilbert action, i.e.,

$$\mathcal{S}^{(1,1)} = \mathcal{E}_2^{(0,0)} S_2^{(1,1)} + \mathcal{E}_2^{(0,1)} S_2^{(1,0)} - \mathcal{E}_1^{(0,0)} S_1^{(1,1)} - \mathcal{E}_1^{(0,1)} S_1^{(1,0)} , \tag{6.21}$$

where $S_{1,2}$ are given by

$$S_1 \equiv (\delta - 2\bar{\alpha} - 2\beta + \bar{\pi})\,\Phi_{00} - (D - 2\varepsilon - 2\bar{\rho})\,\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02} , \tag{6.22a}$$

$$S_2 \equiv (\delta - 2\beta + 2\bar{\pi})\,\Phi_{01} - (D - 2\varepsilon + 2\bar{\varepsilon} - \bar{\rho})\,\Phi_{02} - \bar{\lambda}\Phi_{00} + 2\sigma\Phi_{11} - 2\kappa\Phi_{12} . \tag{6.22b}$$

The terms Φ_{ab} are the NP Ricci scalars, which are projections of the Ricci tensor. The operators $H_{0,1}$, $\mathcal{E}_{0,1}$ are defined as

$$\begin{aligned} &H_0 = \mathcal{E}_2 F_2 - \mathcal{E}_1 F_1 - 3\Psi_2 , \quad H_1 = \mathcal{E}_2 J_2 - \mathcal{E}_1 J_1 , \\ &\mathcal{E}_1 = E_1 - \frac{1}{\Psi_2}\delta\Psi_2 , \quad \mathcal{E}_2 = E_2 - \frac{1}{\Psi_2} D\Psi_2 , \end{aligned} \tag{6.23}$$

where Ψ_2 is an NP scalar, and we have also defined

$$\begin{aligned} F_1 &\equiv \bar{\delta} - 4\alpha + \pi , & F_2 &\equiv \boldsymbol{\Delta} + \mu - 4\gamma , \\ J_1 &\equiv D - 2\varepsilon - 4\rho , & J_2 &\equiv \delta - 2\beta - 4\tau , \\ E_1 &\equiv \delta - \bar{\alpha} - 3\beta + \bar{\pi} - \tau , & E_2 &\equiv D - 3\varepsilon + \bar{\varepsilon} - \rho - \bar{\rho} . \end{aligned} \tag{6.24}$$

Since the non-rotating bumpy BHs considered in this work are Ricci flat [77], and we do not consider any modifications to the Einstein-Hilbert action, we drop the source term $\mathcal{S}^{(1,1)}$ such that Eq. (6.18) reduces to

$$H_0^{(0,0)} \Psi_0^{(1,1)} = \mathcal{S}_{\text{geo}}^{(1,1)} . \tag{6.25}$$

To derive Eq. (6.18), Ref. [87] has chosen a gauge following Chandrasekhar [102] by setting

$$\Psi_1^{(0,1)} = \Psi_3^{(0,1)} = \Psi_1^{(1,1)} = \Psi_3^{(1,1)} = 0 . \tag{6.26}$$

Similar extensions of the Teukolsky equation have also been developed by [88, 89] without choosing the gauge in Eq. (6.26) and by [88] via projecting the Einstein equations following the prescription by Wald in [95]. For a review of all the NP equations and quantities used in this work as well as the Teukolsky formalism, one can refer to [45–48, 87, 102, 103]. Before computing the QNMs as solutions of Eq. (6.25), we must first evaluate the operators $H_0^{(0,1)}$ and $H_1^{(0,1)}$, which are driven by the metric of GR GWs.

6.3.2 Metric reconstruction

To solve for the operators $H_0^{(0,1)}$ and $H_1^{(0,1)}$ at $O(\zeta^0, \epsilon^1)$, we must compute all the spin coefficients and directional derivatives at this order. All these NP quantities depend on the perturbed metric $h_{\mu\nu}^{(0,1)}$ in GR, so we need to reconstruct $h_{\mu\nu}^{(0,1)}$ from the perturbed Weyl scalar $\Psi_0^{(0,1)}$, which satisfies the GR Teukolsky equation $H_0^{(0,0)}\Psi_0^{(0,1)} = 0$. There are multiple approaches for metric reconstruction. For example, one can solve the remaining NP equations systematically after deriving the Teukolsky equation [102–104]. Another more widely used approach is the CCK or CCK-Ori procedure [92–96]. The CCK procedure relies on the radiation gauges. Since Ψ_0 characterizes ingoing gravitational radiations of a perturbed BH, it is more convenient to work with the ingoing radiation gauge (IRG), where

$$h_{ll}^{(0,1)} = h_{lm}^{(0,1)} = h_{ln}^{(0,1)} = h_{l\bar{m}}^{(0,1)} = h_{m\bar{m}}^{(0,1)} = 0\,. \tag{6.27}$$

It was shown in [105] that this gauge can always be imposed on vacuum Petrov-type-D spacetimes. In this case, Refs. [92–95] found that the metric perturbation solving the linearized Einstein equation in vacuum can be expressed as a smooth second-order differential operator acting on the so-called Hertz potential, $\Psi_{\rm H}$,

$$\begin{aligned} h_{\mu\nu}^{(0,1)} = &-l_\mu l_\nu(\bar{\delta}+\alpha+3\bar{\beta}-\bar{\tau})(\bar{\delta}+4\bar{\beta}+3\bar{\tau}) - \bar{m}_\mu\bar{m}_\nu(D-\bar{\rho})(D+3\bar{\rho}) \\ &+ l_{(\mu}\bar{m}_{\nu)}\left[(D-\bar{\rho}+\rho)(\bar{\delta}+4\bar{\beta}+3\bar{\tau}) + (\bar{\delta}-\alpha+3\bar{\beta}-\pi-\bar{\tau})(D+3\bar{\rho})\right]\bar{\Psi}_{\rm H} \\ &+ \text{c.c.}\,, \end{aligned} \tag{6.28}$$

where we have dropped the order-counting superscripts of terms at $O(\zeta^0, \epsilon^0)$. In the IRG, $\Psi_{\rm H}$ satisfies the vacuum Teukolsky equation of $\rho^{-4}\Psi_4^{(0,1)}$ [92–96]. On a Schwarzschild background, Eq. (6.28) simplifies to

$$\begin{aligned} h_{\mu\nu}^{(0,1)} = &-l_\mu l_\nu(\bar{\delta}+2\beta)(\bar{\delta}+4\beta) - \bar{m}_\mu\bar{m}_\nu(D-\rho)(D+3\rho) \\ &+ l_{(\mu}\bar{m}_{\nu)}\left[D(\bar{\delta}+4\beta) + (\bar{\delta}+4\beta)(D+3\rho)\right]\bar{\Psi}_{\rm H} + \text{c.c.}\,. \end{aligned} \tag{6.29}$$

For convenience, let us define

$$h_{\mu\nu}^{(0,1)} = \hat{O}_{\mu\nu}\bar{\Psi}_{\rm H} + \hat{\bar{O}}_{\mu\nu}\Psi_{\rm H}\,. \tag{6.30}$$

To reconstruct NP quantities at $O(\zeta^0, \epsilon^1)$, we start from the perturbed tetrad at $O(\zeta^0, \epsilon^1)$. It was found in [103, 106] that in the IRG, one can pick the tetrad at $O(\zeta^0, \epsilon^1)$ to be

$$l^{\mu(0,1)} = 0\,, \quad n^{\mu(0,1)} = \frac{1}{2}h_{nn}^{(0,1)}l^\mu\,, \quad m^{\mu(0,1)} = h_{nm}^{(0,1)}l^\mu - \frac{1}{2}h_{mm}^{(0,1)}\bar{m}^\mu\,. \tag{6.31}$$

To compute spin coefficients from Eq. (6.31), we can use the linearized commutation relations. For Weyl scalars, one can directly evaluate the Riemann tensor and project it onto the NP basis. A better approach is to use the Ricci identities [103, 106], which express Weyl scalars in terms of derivatives of spin coefficients, so we can reuse the reconstructed spin coefficients. More detailed procedures and the corresponding results of the reconstructed NP quantities in GR can be found in [91, 103, 106]. We will directly use these results in [91, 103, 106] and not present them here for simplicity.

After expressing all the NP quantities in terms of the Hertz potential $\Psi_{\rm H}$, the next step is to calculate $\Psi_{\rm H}$ from the perturbed Weyl scalar $\Psi_0^{(0,1)}$. One approach, developed by Ori in [96], is using the Teukolsky-Starobinsky identity [48, 107, 108] to invert the equation expressing $\Psi_0^{(0,1)}$ in terms of $\bar{\Psi}_{\rm H}$. Decomposing the (ℓ, m) mode of $\Psi_0^{(0,1)}$ and $\bar{\Psi}_{\rm H}^{(0,1)}$ as

$$\Psi_{0,\ell m}^{(0,1)} = {}_2R_{\ell m}^{(0,1)}(r)\,{}_2\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}\,, \tag{6.32}$$

$$\bar{\Psi}_{{\rm H},\ell m} = {}_2\hat{R}_{\ell m}(r)\,{}_2\mathcal{Y}_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}\,, \tag{6.33}$$

Ref. [96] found that

$${}_2\hat{R}_{\ell m}(r) = -\frac{2}{\mathfrak{C}}\Delta^2(r)(D_{\ell m}^\dagger)^4\left[\Delta^2(r)\,{}_2R_{\ell m}^{(0,1)}(r)\right]\,, \tag{6.34}$$

where $\Delta(r) \equiv r^2 - 2Mr + \chi^2M^2$ and

$$D_{\ell m} = \partial_r + i\frac{am - (r^2+a^2)\omega_{\ell m}}{\Delta(r)}\,, \quad D_{\ell m}^\dagger = \partial_r - i\frac{am - (r^2+a^2)\omega_{\ell m}}{\Delta(r)}\,. \tag{6.35}$$

Here, ${}_s\mathcal{Y}_{\ell m}(\theta,\phi) \equiv {}_sS_{\ell m}(\theta)e^{im\phi}$ and ${}_sR_{\ell m}^{(0,1)}(r)$ are solutions to the angular and radial Teukolsky equations, respectively,

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \gamma_{\ell m}^2\sin^2\theta - \frac{(m+s\cos\theta)^2}{\sin^2\theta} - 2\gamma_{\ell m}s\cos\theta + s + 2m\gamma_{\ell m}\right.$$
$$\left. + {}_s\lambda_{\ell m}\right]{}_sS_{\ell m}(\theta) = 0\,, \tag{6.36a}$$

$$\left[\Delta(r)^{-s}\frac{d}{dr}\left(\Delta(r)^{s+1}\frac{d}{dr}\right) + \frac{K^2(r) - 2is(r-M)K(r)}{\Delta(r)} + 4is\omega_{\ell m}r\right.$$
$$\left. - {}_s\lambda_{\ell m}\right]{}_sR_{\ell m}^{(0,1)}(r) = 0\,, \tag{6.36b}$$

where

$$\gamma_{\ell m} \equiv \chi M\omega_{\ell m}\,, \quad {}_s\lambda_{\ell m} \equiv {}_sA_{\ell m} + s + |s|\,, \quad K(r) \equiv \left(r^2 + \chi^2M^2\right)\omega_{\ell m} - \chi Mm\,, \tag{6.37}$$

with ${}_sA_{\ell m}$ being the Teukolsky's angular separation constant [46]. For non-rotating BHs we are considering in this paper, Eq. (6.36) reduces to

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)-\frac{(m+s\cos\theta)^2}{\sin^2\theta}+s+(\ell-1)(\ell+s+1)\right]{}_sS_{\ell m}(\theta)=0\,, \tag{6.38a}$$

$$\begin{aligned}&\left[\Delta(r)^{-s}\frac{d}{dr}\left(\Delta(r)^{s+1}\frac{d}{dr}\right)+\frac{\omega_{\ell m}r(\omega_{\ell m}r^2-2is(r-M))}{r-2M}+4is\omega_{\ell m}r\right.\\&\left.-(\ell-1)(\ell+s+1)\right]{}_sR^{(0,1)}_{\ell m}(r)=0\,.\end{aligned} \tag{6.38b}$$

The constant $\mathfrak{C}$ is the Teukolsky-Starobinsky coefficient [48, 89, 90, 96, 107, 108],

$$\begin{aligned}\mathfrak{C}=\;&144M^2\omega_{\ell m}^2+\left(8+6{}_sB_{\ell m}+{}_sB_{\ell m}^2\right)^2-8\left[-8+{}_sB_{\ell m}^2\left(4+{}_sB_{\ell m}\right)\right]m\gamma_{\ell m}\\&+4\left[8-2{}_sB_{\ell m}-{}_sB_{\ell m}^2+{}_sB_{\ell m}^3+2\left(-2+{}_sB_{\ell m}\right)\left(4+3{}_sB_{\ell m}\right)m^2\right]\gamma_{\ell m}^2\\&-8m\left(8-12{}_sB_{\ell m}+3{}_sB_{\ell m}^2+4\left(-2+{}_sB_{\ell m}\right)m^2\right)\gamma_{\ell m}^3\\&+2\left(42-22{}_sB_{\ell m}+3{}_sB_{\ell m}^2+8\left(-11+3{}_sB_{\ell m}\right)m^2+8m^4\right)\gamma_{\ell m}^4\\&-8m\left[3{}_sB_{\ell m}+4\left(-4+m^2\right)\right]\gamma_{\ell m}^5+4\left(-7+{}_sB_{\ell m}+6m^2\right)\gamma_{\ell m}^6-8m\gamma_{\ell m}^7+\gamma_{\ell m}^8\,,\end{aligned} \tag{6.39}$$

where $\tilde{\alpha}=a^2-am/\omega_{\ell m}$ and ${}_sB_{\ell m}={}_sA_{\ell m}+s$. Notice that we have used the Teukolsky-Starobinsky coefficient found in [89, 90] instead of the original one in [48, 96, 107, 108], the latter of which was derived for real frequencies.

Finally, since we have chosen the gauge in Eq. (6.26), where $\Psi_1^{(0,1)}=\Psi_3^{(0,1)}=0$, we need to perform additional tetrad rotations to remove $\Psi_{1,3}^{(0,1)}$. The required tetrad rotations and the transformation of NP quantities under tetrad rotations are provided in [91]. Using the procedures above, we compute the NP quantities at $\mathcal{O}(\zeta^0,\epsilon^1)$ for Schwarzschild BHs and evaluate the operators $H_0^{(0,1)}$ and $H_1^{(0,1)}$ in the "geometrical" source term $\mathcal{S}_{\rm geo}^{(1,1)}$. The results can be found in the supplemental material [101], and we present the perturbed metric $h_{ab}^{(0,1)}$ in the NP basis here as an example,

$$h_{nn}^{(0,1)}=\frac{\sqrt{\ell(\ell+1)(\ell^2+\ell-2)}}{2r^2}{}_2\hat{R}_{\ell m}(r){}_0Y_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}+\text{c.c.}\,, \tag{6.40a}$$

$$\begin{aligned}h_{nm}^{(0,1)}=\;&\frac{\sqrt{\ell^2+\ell-2}}{\sqrt{2}r^2(r-2M)}\left[r(r-2M)\partial_r-r(2+i\omega_{\ell m}r)+4M\right]\\&{}_2\hat{R}_{\ell m}(r){}_1Y_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}\,,\end{aligned} \tag{6.40b}$$

$$h_{mm}^{(0,1)}=\frac{1}{r(r-2M)^2}\left[2(r-2M)(M-i\omega_{\ell m}r^2)\partial_r\right.$$

$$+(r-2M)(\ell^2+\ell-2+6i\omega_{\ell m}r)-2i\omega_{\ell m}r(M-i\omega_{\ell m}r^2)\big]$$
$$_2\hat{R}_{\ell m}(r)_2Y_{\ell m}(\theta,\phi)e^{-i\omega_{\ell m}t}\,. \tag{6.40c}$$

In the next subsection, we will compute the NP quantities at $O(\zeta^1,\epsilon^0)$ so we can evaluate the modified Teukolsky equation of $\Psi_0^{(1,1)}$ in Eq. (6.25).

6.3.3 The modified Teukolsky equation for non-rotating bumpy BHs

To evaluate Eq. (6.25), we also need to compute the NP quantities at $O(\zeta^0,\epsilon^0)$ and $O(\zeta^1,\epsilon^0)$. For NP quantities at $O(\zeta^0,\epsilon^0)$, we evaluate them on the Schwarzschild background and use the Kinnersely tetrad,

$$l^{\mu(0,0)}=\left(\frac{r^2}{r(r-2M)},1,0,0\right)\,, \tag{6.41a}$$
$$n^{\mu(0,0)}=\left(\frac{1}{2},-\frac{r(r-2M)}{2r^2},0,0\right)\,, \tag{6.41b}$$
$$m^{\mu(0,0)}=\frac{1}{\sqrt{2}r}\left(0,0,1,i\csc\theta\right)\,, \tag{6.41c}$$

where all the NP quantities at $O(\zeta^0,\epsilon^0)$ can be found in [46, 102]. At $O(\zeta^1,\epsilon^0)$, let us first linearize Eq. (6.11) since we assume the dimensionless amplitude B_{ℓ_W} of each bump satisfies $B_{\ell_W}\ll 1$. In this case, Eq. (6.11) becomes

$$\begin{aligned}h^{(1,0)}_{\mu\nu}dx^\mu dx^\nu=&-2\psi_1\left(1-\frac{2M}{r}\right)dt^2+(2\gamma_1-2\psi_1)\left(1-\frac{2M}{r}\right)^{-1}dr^2\\&+r^2(2\gamma_1-2\psi_1)d\theta^2-2\psi_1r^2\sin^2\theta d\phi^2\,.\end{aligned} \tag{6.42}$$

Using Eq. (51) of [87], we find the following tetrad satisfies all the orthogonality conditions,

$$l^{\mu(1,0)}=\left(-\frac{r\psi_1}{r-2M},\psi_1-\gamma_1,0,0\right)\,, \tag{6.43a}$$
$$n^{\mu(1,0)}=\left(-\frac{1}{2}\psi_1,\frac{(r-2M)\,(\gamma_1-\psi_1)}{2r},0,0\right)\,, \tag{6.43b}$$
$$m^{\mu(1,0)}=\left(0,0,\frac{\psi_1-\gamma_1}{\sqrt{2}r},\frac{i\csc\theta\psi_1}{\sqrt{2}r}\right)\,. \tag{6.43c}$$

Using this tetrad, we can compute Weyl scalars and spin coefficients at $O(\zeta^1,\epsilon^0)$. These quantities are summarized in Appendix 6.7. The differential operator $H_0^{(1,0)}$ is provided in the supplemental material [101].

Now, we have all the ingredients to evaluate the modified Teukolsky equation in Eq. (6.25). Due to the complication of the full equation, we choose to provide it in the supplemental material [101], and we present here the schematic form of the equation for the convenience of computing the QNM frequencies. First, since all the terms in $\mathcal{S}_{\rm geo}^{(1,1)}$ depend on the perturbed metric $h_{\mu\nu}^{(0,1)}$ linearly via $\Psi_0^{(0,1)}$, $H_0^{(0,1)}$, or $H_1^{(0,1)}$, we can write $\mathcal{S}_{\rm geo}^{(1,1)}$ as

$$\mathcal{S}_{\rm geo}^{(1,1)} = \hat{\Sigma}^{\mu\nu(1,0)} h_{\mu\nu}^{(0,1)}\,, \tag{6.44}$$

where $\hat{\Sigma}^{\mu\nu(1,0)}$ is determined by the reconstructed NP quantities found in Sec. 6.3.2 and the modifications of NP quantities due to the bumps. For convenience, we will drop the superscript for order counting of $\hat{\Sigma}^{\mu\nu(1,0)}$. Using Eqs. (6.28) and (6.34), we can write (6.44) as

$$H_0^{(0,0)}\Psi_0^{(1,1)} = \hat{\Sigma}^{\mu\nu}\left(\hat{O}_{\mu\nu} + \hat{\bar{O}}_{\mu\nu}\hat{C}\right)\hat{\mathcal{D}}\Psi_0^{(0,1)}\,, \tag{6.45}$$

where $\hat{C}$ is the complex conjugate operator, and $\hat{\mathcal{D}}$ is an operator that satisfies $\hat{\mathcal{D}}\Psi_0^{(0,1)} = \bar{\Psi}_{\rm H}$, which can be determined from Eq. (6.34). Expanding $\Psi_0^{(0,1)}$ and $\Psi_0^{(1,1)}$ as

$$\Psi_0^{(0,1)} = \sum_{\ell,m} {}_2\psi_{\ell m}^{(0,1)}(r,\theta)e^{-i\omega_{\ell m}t+im\phi}\,, \tag{6.46}$$

$$\Psi_0^{(1,1)} = \sum_{\ell,m} {}_2\psi_{\ell m}^{(1,1)}(r,\theta)e^{-i\omega_{\ell m}t+im\phi}\,, \tag{6.47}$$

we get the mode decomposition of Eq. (6.45) to be

$$\begin{aligned}&\sum_{\ell,m} H_{0,\ell m}\left[{}_2\psi_{\ell m}^{(1,1)}(r,\theta)\right]e^{-i\omega_{\ell m}t+im\phi}\\ &= \sum_{\ell,m} P_{\ell m}\left[{}_2\psi_{\ell m}^{(0,1)}(r,\theta)\right]e^{-i\omega_{\ell m}t+im\phi} + Q_{\ell m}\left[{}_2\bar{\psi}_{\ell m}^{(0,1)}(r,\theta)\right]e^{i\bar{\omega}_{\ell m}t-im\phi}\,,\end{aligned} \tag{6.48}$$

where $H_{0,\ell m}$ is the (ℓ, m) mode of the Teukolsky operator H_0 for Ψ_0 in GR. $P_{\ell m}$ and $Q_{\ell m}$ are operators depending on the coordinates (r,θ) and acting on the Weyl scalar perturbation $\Psi_0^{(0,1)}$ in GR. In particular,

$$P_{\ell m} = \left(\hat{\Sigma}^{\mu\nu}\hat{O}_{\mu\nu}\hat{\mathcal{D}}\right)_{\ell m}\,, \quad Q_{\ell m} = \left(\hat{\Sigma}^{\mu\nu}\hat{\bar{O}}_{\mu\nu}\hat{\mathcal{D}}\right)_{\ell m}\,. \tag{6.49}$$

One may further decompose ${}_2\psi_{\ell m}^{(0,1)}(r,\theta)$ and ${}_2\psi_{\ell m}^{(1,1)}(r,\theta)$ into spin-weighted spheroidal harmonics

$${}_2\psi_{\ell m}^{(0,1)}(r,\theta) = {}_2R_{\ell m}^{(0,1)}(r)\,{}_2S_{\ell m}(\theta)\,, \tag{6.50a}$$

$${}_2\psi_{\ell m}^{(1,1)}(r,\theta) = {}_2R_{\ell m}^{(1,1)}(r)\,{}_2S_{\ell m}(\theta)\,. \tag{6.50b}$$

where ${}_sR^{(0,1)}_{\ell m}(r)$ and ${}_sS_{\ell m}(\theta)$ are radial and angular Teukolsky functions for a spin s particle in GR, respectively, and they satisfy Eq. (6.36). One thing we notice in Eq. (6.48) is that the (ℓ, m) and $(\ell, -m)$ modes are coupled to each other in the source terms on the right-hand side, which means we actually need to solve these two modes jointly. In the next section, we will show how to solve these two modes jointly and compute the QNM frequency shifts following the prescription in [99].

6.4 EVP

In this section, we prescribe the procedures to calculate the QNM frequency shifts due to those bumps using Eq. (6.48). From Eq. (6.48), we notice that the GR QNMs are resonantly driving the modified Teukolsky equation since the homogeneous part of Eq. (6.48) is the same as the one in GR, potentially leading to secularly growing terms. One solution is to perform a multiple-scale analysis [109], which was employed to study spin-precessing systems and post-Newtonian dynamics in GR [110–112]. Another solution is the Poincaré-Lindstedt method, which leverages shifts of the eigenfrequency to cancel off secularly growing terms. In this case, the shift in the eigenfrequency plays a similar role as the slow timescale in the multiple-scale analysis. Following a similar idea, Refs. [97, 98] developed the EVP method by perturbing the QNM frequency in GR, i.e.,

$$\omega_{\ell m} = \omega^{(0)}_{\ell m} + \zeta \omega^{(1)}_{\ell m} \,. \tag{6.51}$$

In GR, the QNM frequencies $\omega^{(0)}_{lm}$ satisfy the following symmetry [35]

$$\omega^{(0)}_{\ell m} = -\bar{\omega}^{(0)}_{\ell -m} \,. \tag{6.52}$$

As shown in detail in [99], to solve Eq. (6.48) consistently, one has to assume that the same symmetry still holds at the bGR level, i.e.,

$$\omega^{(1)}_{\ell m} = -\bar{\omega}^{(1)}_{\ell -m} \,. \tag{6.53}$$

In this case, all the source terms in Eq. (6.48) are either proportional to $e^{-i\omega_{\ell m} t}$ or $e^{-i\omega_{\ell -m} t}$. Since the (ℓ, m) and $(\ell, -m)$ modes are coupled to each other, we need to solve these two modes jointly by focusing on the linear combination [88, 91, 99],

$$\begin{aligned}
\Psi^{\eta(0,1)}_{0,\ell m} &= \Psi^{(0,1)}_{0,\ell m} + \eta_{\ell m} \Psi^{(0,1)}_{0,\ell -m} \,, \\
\Psi^{\eta(1,1)}_{0,\ell m} &= \Psi^{(1,1)}_{0,\ell m} + \eta_{\ell m} \Psi^{(1,1)}_{0,\ell -m} \,.
\end{aligned} \tag{6.54}$$

The constant $\eta_{\ell m}$ is the relative coefficient between the (ℓ, m) and $(\ell, -m)$ modes, which is well-defined if we fix the normalization of ${}_2\psi^{(0,1)}_{\ell m}(r, \theta)$ to satisfy

$${}_s\bar{R}^{(0,1)}_{\ell m \omega}(r) = (-1)^m {}_sR^{(0,1)}_{\ell -m -\bar{\omega}}(r) \,,$$

$$
\begin{aligned}
{}_sS_{\ell m\omega}(\pi-\theta) &= (-1)^{m+\ell}{}_{-s}S_{\ell m\omega}(\theta)\,, \\
{}_s\bar{S}_{\ell m\omega}(\theta) &= (-1)^{m+s}{}_{-s}S_{\ell -m-\bar{\omega}}(\theta)\,.
\end{aligned} \tag{6.55}
$$

After plugging the ansatz in Eq. (6.54) into the Teukolsky equation of Ψ_0 in GR [Eq. (6.38)] and the modified Teukolsky equation [Eq. (6.48)], matching the phase of the terms, and expanding the equations over ζ, we get

$$
\begin{aligned}
&H_{0,\ell m}\left[{}_2\psi^{(1,1)}_{\ell m}(r,\theta)\right]+\omega^{(1)}_{\ell m}\partial_\omega H_{0,\ell m}\left[{}_2\psi^{(0,1)}_{\ell m}(r,\theta)\right] \\
&= P_{\ell m}\left[{}_2\psi^{(0,1)}_{\ell m}(r,\theta)\right]+\bar{\eta}_{\ell m}Q_{\ell -m}\left[{}_2\bar{\psi}^{(0,1)}_{\ell -m}(r,\theta)\right]\,,
\end{aligned} \tag{6.56a}
$$

$$
\begin{aligned}
&\eta_{\ell m}H_{0,\ell -m}\left[{}_2\psi^{(1,1)}_{\ell -m}(r,\theta)\right]+\eta_{\ell m}\omega^{(1)}_{\ell -m}\partial_\omega H_{0,\ell -m}\left[{}_2\psi^{(0,1)}_{\ell -m}(r,\theta)\right] \\
&= \eta_{\ell m}P_{\ell -m}\left[{}_2\psi^{(0,1)}_{\ell -m}(r,\theta)\right]+Q_{\ell m}\left[{}_2\bar{\psi}^{(0,1)}_{\ell m}(r,\theta)\right]\,,
\end{aligned} \tag{6.56b}
$$

where the second term on the left-hand side of Eqs. (6.56a) and (6.56b) comes from expanding $\omega_{\ell m}$ about ζ [i.e., Eq. (6.51)] in the GR Teukolsky equation $H_{0,\ell m}\left[{}_2\psi^{(0,1)}_{\ell m}(r,\theta)\right]=0$. All the $\omega_{\ell m}$ and $\omega_{\ell -m}$ terms in Eq. (6.56) are evaluated on the GR QNM frequencies $\omega^{(0)}_{\ell m}$ and $\omega^{(0)}_{\ell -m}$, respectively. Furthermore, following [99], one can apply a parity-complex conjugate transformation $\hat{\mathcal{P}}$,

$$
\hat{\mathcal{P}}f(\theta,\phi)\equiv\bar{f}(\pi-\theta,\phi+\pi)\,, \tag{6.57}
$$

on Eq. (6.56b) such that it becomes

$$
\begin{aligned}
&(-1)^m\bar{\eta}_{\ell m}H_{0,\ell m}\left[{}_2\bar{\psi}^{(1,1)}_{\ell -m}(r,\theta)\right]+\bar{\eta}_{\ell m}\omega^{(1)}_{\ell m}\partial_\omega H_{0,\ell m}\left[{}_2\psi^{(0,1)}_{\ell m}(r,\theta)\right] \\
&= \bar{\eta}_{\ell m}\bar{P}_{\ell -m}(\pi-\theta)\left[{}_2\psi^{(0,1)}_{\ell m}(r,\theta)\right]+\bar{Q}_{\ell m}(\pi-\theta)\left[{}_2\bar{\psi}^{(0,1)}_{\ell -m}(r,\theta)\right]\,,
\end{aligned} \tag{6.58}
$$

where we have used Eqs. (6.53), (6.55), and that $\hat{\mathcal{P}}H_{0,\ell -m}=H_{0,\ell m}$ [99]. Notice that an additional factor of $(-1)^m$ is added to Eq. (6.58) due to $\hat{\mathcal{P}}e^{im\phi}=(-1)^m e^{im\phi}$. Although $e^{im\phi}$ does not show up in Eq. (6.56b), it is necessary to keep track of this factor for self-consistency in the angular part of all the terms.

To solve for $\omega^{(1)}_{\ell m}$, one can construct an inner product, following [88, 97, 98], that makes the Teukolsky operator in GR self-adjoint, i.e.,

$$
\langle H_{0,\ell m}\varsigma(r,\theta)|\varphi(r,\theta)\rangle=\langle\varsigma(r,\theta)|H_{0,\ell m}\varphi(r,\theta)\rangle\,, \tag{6.59}
$$

where $\varsigma(r,\theta)$ and $\varphi(r,\theta)$ are some general functions in (r,θ) with the same asymptotic behaviors as the GR QNMs. As shown in [97, 98], the inner product in Eq. (6.59) can be defined as an integral along certain contour $\mathscr{C}$, where

$$
\langle\varsigma(r,\theta)|\varphi(r,\theta)\rangle=\int_{\mathscr{C}}\Delta^2(r)dr\int\sin\theta\,\varsigma(r,\theta)\varphi(r,\theta)d\theta\,, \tag{6.60}
$$

and we will show below how to construct a contour $\mathscr{C}$ for a bumpy BH. Since $H_{0,\ell m}\left[{}_2\psi^{(0,1)}_{\ell m}(r,\theta)\right]=0$, if we take the inner product of ${}_2\psi^{(0,1)}_{\ell m}(r,\theta)$ with Eqs. (6.56a) and (6.58) and use the property in Eq. (6.59), we can remove the first term on the left-hand side of Eqs. (6.56a) and (6.58) such that this system of equations becomes a standard eigenvalue problem

$$\frac{1}{\langle\partial_\omega H_{0,\ell m}\rangle}\begin{pmatrix}\langle P_{\ell m}\rangle & (-1)^\ell\langle Q_{\ell-m}\hat{C}\hat{\mathcal{P}}\rangle \\ (-1)^\ell\langle\bar{Q}_{\ell m}(\pi-\theta)\hat{C}\hat{\mathcal{P}}\rangle & \langle\bar{P}_{\ell-m}(\pi-\theta)\rangle\end{pmatrix}\begin{pmatrix}1\\ \bar{\eta}_{\ell m}\end{pmatrix}=\omega^{(1)}_{lm}\begin{pmatrix}1\\ \bar{\eta}_{\ell m}\end{pmatrix}, \tag{6.61}$$

where we have defined the shorthand notation

$$\langle O\rangle=\langle\psi^{(0,1)}_{\ell m}|O\psi^{(0,1)}_{\ell m}\rangle\,. \tag{6.62}$$

The matrix in Eq. (6.61) is the same as the one in Eq. (68) of [99], where one can directly map $P_{\ell m}$ and $Q_{\ell m}$ to the (ℓ,m) mode of $\mathcal{S}^{\mu\nu}O_{\mu\nu}$ and $\mathcal{S}^{\mu\nu}\bar{O}_{\mu\nu}$, respectively. The additional factor of $(-1)^\ell$ in the off-diagonal terms comes from that we choose to solve the pair of $\Psi_{0,\ell m}$ and $\Psi_{0,\ell-m}$ instead of $\Psi_{0,\ell m}$ and $\hat{\mathcal{P}}\Psi_{0,\ell m}$ in [99], where $\hat{\mathcal{P}}\Psi^{(0,1)}_{0,\ell m}=(-1)^\ell\Psi_{0,\ell-m}$. Nonetheless, this choice and the resulting factor of $(-1)^\ell$ will not affect $\omega^{(1)}_{\ell m}$, as shown below. The solutions to Eq. (6.61) can be found in [99], where the QNM frequencies are

$$\begin{aligned}\omega^{\pm(1)}_{\ell m}=&\frac{1}{2\langle\partial_\omega H_{0,\ell m}\rangle}\Bigg(\Big\langle P_{\ell m}+\bar{P}_{\ell-m}(\pi-\theta)\Big\rangle\\ &\pm\sqrt{\Big\langle P_{\ell m}-\bar{P}_{\ell-m}(\pi-\theta)\Big\rangle^2+4\Big\langle Q_{\ell-m}\hat{C}\hat{\mathcal{P}}\Big\rangle\Big\langle\bar{Q}_{\ell m}(\pi-\theta)\hat{C}\hat{\mathcal{P}}\Big\rangle}\Bigg),\end{aligned} \tag{6.63}$$

and the coefficients $\eta_{\ell m}$ are

$$\begin{aligned}\bar{\eta}^{\pm}_{\ell m}=&\frac{(-1)^\ell}{2\left\langle Q_{\ell-m}\hat{C}\hat{\mathcal{P}}\right\rangle}\Bigg(\Big\langle\bar{P}_{\ell-m}(\pi-\theta)-P_{\ell m}\Big\rangle\\ &\pm\sqrt{\Big\langle P_{\ell m}-\bar{P}_{\ell-m}(\pi-\theta)\Big\rangle^2+4\Big\langle Q_{\ell-m}\hat{C}\hat{\mathcal{P}}\Big\rangle\Big\langle\bar{Q}_{\ell m}(\pi-\theta)\hat{C}\hat{\mathcal{P}}\Big\rangle}\Bigg).\end{aligned} \tag{6.64}$$

Notice, the factor of $(-1)^\ell$ in Eq. (6.64) disappears if one solve the pair of $\Psi_{0,\ell m}$ and $\hat{\mathcal{P}}\Psi_{0,\ell m}$ instead. As shown in [99], in the special case that $\eta^{\pm}_{\ell m}=\pm1$, the modified QNMs $\Psi^{\eta(1,1)}_{0,\ell m}$ are even- and odd-parity modes, respectively.

The contour for the bumpy BH

To evaluate the inner product in Eq. (6.60), we must choose a contour $\mathscr{C}$ and impose correct boundary conditions so that the Teukolsky operator is self-adjoint. The

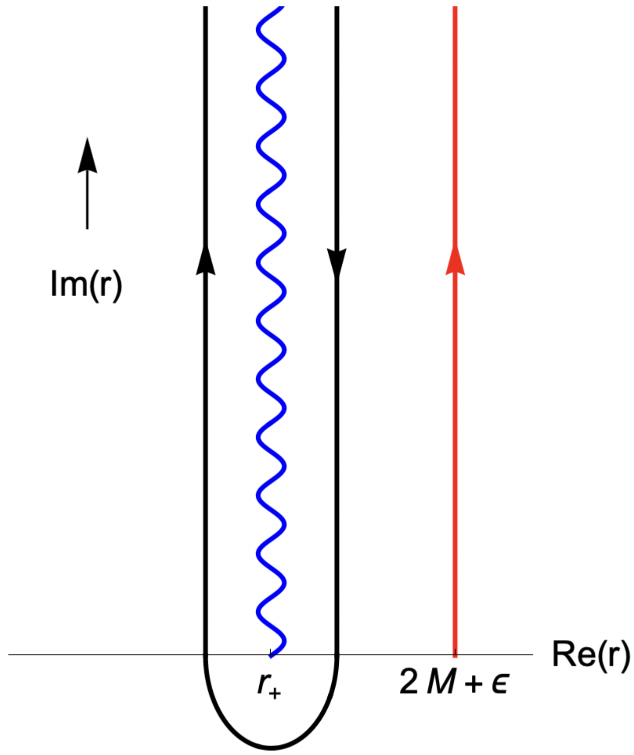

Figure 6.2: The contour $\mathscr{C}_1$ that wraps around the QNM wavefunction branch cut parallel to the imaginary axis at the horizon $r_+ = 2M$. The modified contour $\mathscr{C}_2$ is also shown, which extends from $r = 2M + \epsilon$ to $r = 2M + \epsilon + i\infty$.

boundary conditions of the QNMs require

$$
\begin{aligned}
{}_2R^{(0,1)}_{\ell m}(r) &\sim r^{-5} e^{i\omega_{\ell m} r^*}\,, && r^* \to \infty\,, \\
{}_2R^{(0,1)}_{\ell m}(r) &\sim \Delta^{-2}(r) e^{-i[\omega_{\ell m} - am/(2Mr_+)]r^*}\,, && r^* \to -\infty\,,
\end{aligned}
\tag{6.65}
$$

Here, r^* is the tortoise coordinate, where $r^* = \infty$ and $r^* = -\infty$ correspond to $r = \infty$ and $r = r_+$, respectively, with r_+ being the outer horizon of the Kerr spacetime. For Schwarzschild BHs, $r_+ = 2M$. Since $\mathrm{Im}(\omega_{\ell m}) < 0$, ${}_2R^{(0,1)}_{\ell m}(r)$ diverges at both the horizon and the real infinity. However, a finite inner product can still be constructed by considering a contour $\mathscr{C}$ in the complex plane and analytically continuing the radial Teukolsky functions. Consider a contour with endpoints at $r = a$ and $r = b$ in the complex plane. We need to ensure that the Teukolsky operator is self-adjoint. Evaluating Eq. (6.59) with Eq. (6.60), one can first carry out the angular integral by projecting both $\varsigma(r,\theta)$ and $\varsigma(r,\theta)$ to spin-weighted spheroidal harmonics, i.e.,

$$
\varsigma(r,\theta) = \sum_{\ell m} \varsigma_{\ell m}(r) {}_2S_{\ell m}(\theta)\,,
\tag{6.66}
$$

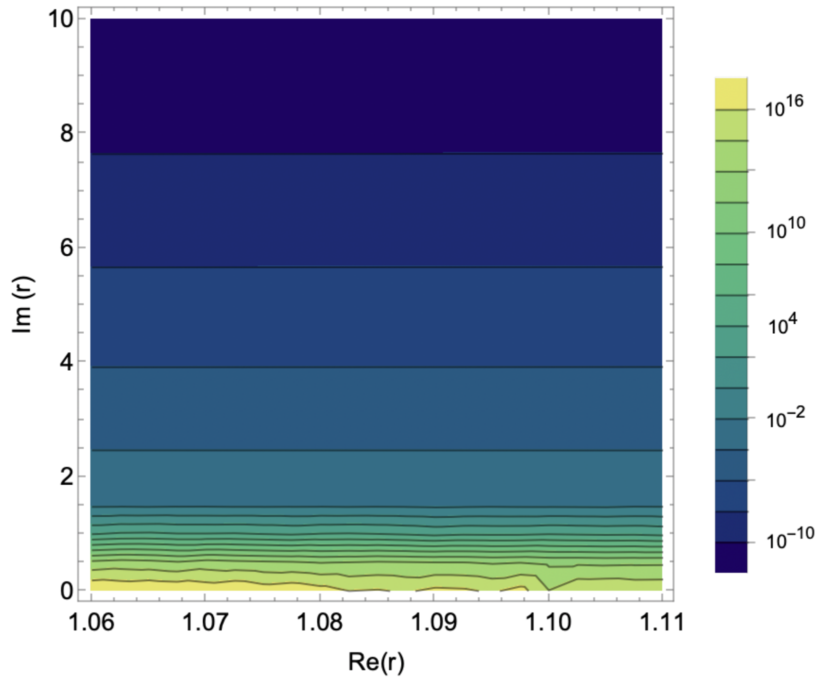

Figure 6.3: The absolute value of $\langle P_{\ell m}\rangle$ after the angular integral in Eq. (6.60) is shown for the $(\ell, m, n) = (2, 1, 0)$ mode and the $\ell_W = 2$ Weyl multipole. We use the convention that $M = 1/2$ in [35] in this plot.

and similarly for $\varphi(r, \theta)$. Then one can use the radial Teukolsky operator given by Eq. (6.36b) and perform the radial integral in Eq. (6.60). After integrating by parts, one is left with

$$\Delta^3(r)[\partial_r \varsigma_{\ell m}(r)\varphi_{\ell m}(r) - \partial_r \varphi_{\ell m}(r)\varsigma_{\ell m}(r)]\Big|_a^b . \tag{6.67}$$

If one follows [88, 97, 98] to choose the contour $\mathscr{C}_1$ in Fig. 6.2, which surrounds the imaginary axis at r_+, the boundary terms in Eq. (6.67) vanish, so Eq. (6.59) is satisfied. It is because we have imposed that $\varsigma_{\ell m}(r)$ and $\varphi_{\ell m}(r)$ satisfy the same boundary conditions as ${}_2R^{(0,1)}_{\ell m}(r)$, while ${}_2R^{(0,1)}_{\ell m}(r) \to 0$ as $r \to r_+ \pm \epsilon + i\infty$, where $\epsilon \ll M$ and ϵ is real.

Although the contour $\mathscr{C}_1$ works for many cases, it does not really apply to the modified Teukolsky equations for the bumpy BHs here. The main reason is that certain NP quantities at $O(\zeta^1, \epsilon^0)$ diverge when $\mathrm{Re}(r) \le 2M$ due to the distance function $d(r, \theta)$ defined in Eq. (6.14). In this case, the radial integral along the half of the contour $\mathscr{C}_1$ inside the horizon diverges. To avoid this issue, we consider an alternative contour $\mathscr{C}_2$, which starts from $r = 2M + \epsilon$, where $\epsilon < M$ and ϵ is real, and ends at $r = 2M + \epsilon + i\infty$, as depicted in Fig. 6.2. To ensure that all the terms in

Eq. (6.61) are well-behaved along $\mathscr{C}_2$, we perform the angular integral in Eq. (6.60) of the source terms and inspect their absolute value for complex-valued r around $\mathscr{C}_2$. As an example, we show in Fig. 6.3 that $\langle P_{\ell m}\rangle$ is non-singular along $\mathscr{C}_2$.

According to Eq. (6.67), to ensure that the Teukolsky operator is still self-adjoint, or at least that we can remove the first term on the left-hand side of Eqs. (6.56a) and (6.58), we need to impose

$$\frac{{}_2R^{(1,1)}_{\ell m}(r)}{\partial_r\left({}_2R^{(1,1)}_{\ell m}(r)\right)} = \frac{{}_2R^{(0,1)}_{\ell m}(r)}{\partial_r\left({}_2R^{(0,1)}_{\ell m}(r)\right)} \tag{6.68}$$

at both $r = 2M + \epsilon$ and $r = 2M + \epsilon + i\infty$. The condition at $r = 2M + \epsilon + i\infty$ is easily satisfied since the source terms in Eqs. (6.56a) and (6.58) vanish as $r \to 2M + \epsilon + i\infty$. However, the condition at $r = 2M + \epsilon$ is not naturally satisfied since the source terms do not vanish near the horizon, resulting in nonzero ${}_2R^{(0,1)}_{\ell m}(r)$ and ${}_2R^{(1,1)}_{\ell m}(r)$.

One resolution is to consider the "membrane paradigm" in [113–116]. This formalism elucidates the physical nature of the BH horizon by modeling it as a fictitious fluid membrane. The dynamics of the membrane are parametrized by a dissipative stress-energy tensor that sets the fluid's velocity, density, pressure, shear viscosity, and bulk viscosity. For example, it was found for a Schwarzschild BH that it has bulk viscosity $\zeta = -1/16\pi$ and shear viscosity $\eta = 1/16\pi$. Altering these transport coefficients would alter the near horizon geometry and, therefore, modify the boundary conditions of the resulting QNMs [117, 118]. For example, one can have some nonzero reflectivity at the membrane, generating GW echos [119]. One may then deliberately pick some fluid stress tensor such that the spacetime is a bumpy BH for $r > 2M + \epsilon$, while the spacetime is still a Schwarzschild spacetime for $2M \leq r \leq 2M + \epsilon$. In this case, we can impose the condition in Eq. (6.68) at $r = 2M + \epsilon$ since ${}_2R^{(1,1)}_{\ell m}$ satisfies the same boundary condition of a GR QNM [i.e., Eq. (6.65)]. In this work, we do not explicitly provide a stress tensor giving rise to Eq. (6.68) but assume its possible existence. We will strictly derive the relation between the fluid stress tensor and the boundary condition of the QNMs at the membrane in our future work. In general, different fluids can result in different boundary conditions other than Eq. (6.65), for example, a nonzero reflectivity at the membrane. This nonzero reflectivity can make additional modifications to the QNM spectrum [119]. Since the boundary condition in Eq. (6.68) is the most natural to use, we will stick to it for the rest of this work.

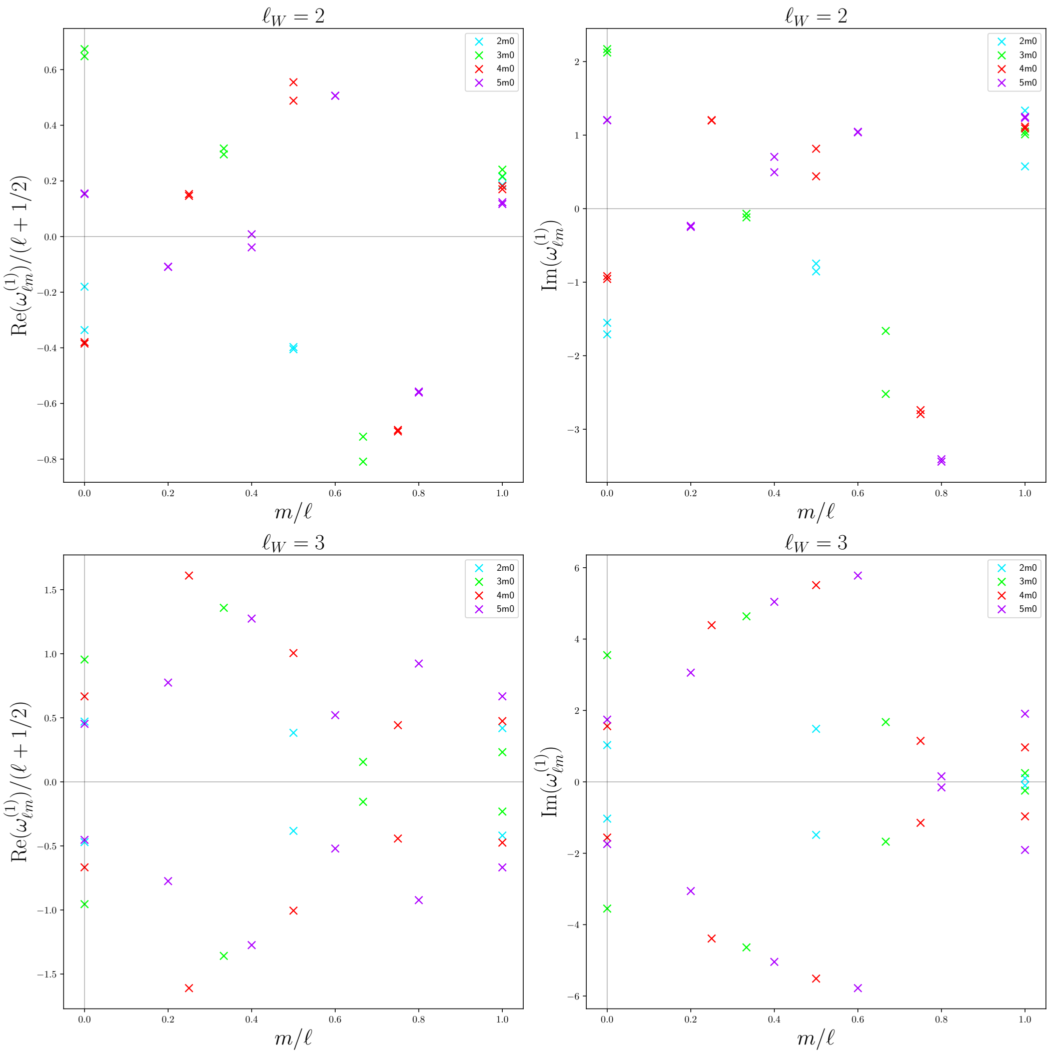

Figure 6.4: The real (left column) and imaginary (right column) parts of the QNM frequency shifts $\omega^{\pm(1)}_{\ell m}$ generated by the $\ell_W = 2$ bump (top row) and $\ell_W = 3$ bump (bottom row) are presented. For each frequency, the fundamental mode $n = 0$ is shown. For simplicity, we use the same marker for $\omega^{+(1)}_{\ell m}$ and $\omega^{-(1)}_{\ell m}$ for each (ℓ, m) mode.

6.5 Results

Using the contour $\mathscr{C}_2$ in Sec. 6.4 and Fig. 6.2, we can now compute the QNM frequency shifts generated by a bumpy BH. The contour $\mathscr{C}_2$ was chosen by setting $\epsilon = 0.2M$. All the $O(\zeta^0, \epsilon^1)$ quantities were computed using the Leaver's method [35]. The results of the QNM frequency shifts are shown in Fig. 6.4, where both the real and imaginary parts of the QNM frequency shifts $\omega^{\pm(1)}_{\ell m}$ are plotted. Specifically, for both the Weyl bumps $\ell_W = 2$ and $\ell_W = 3$, we plot all the modes (ℓ, m, n) with

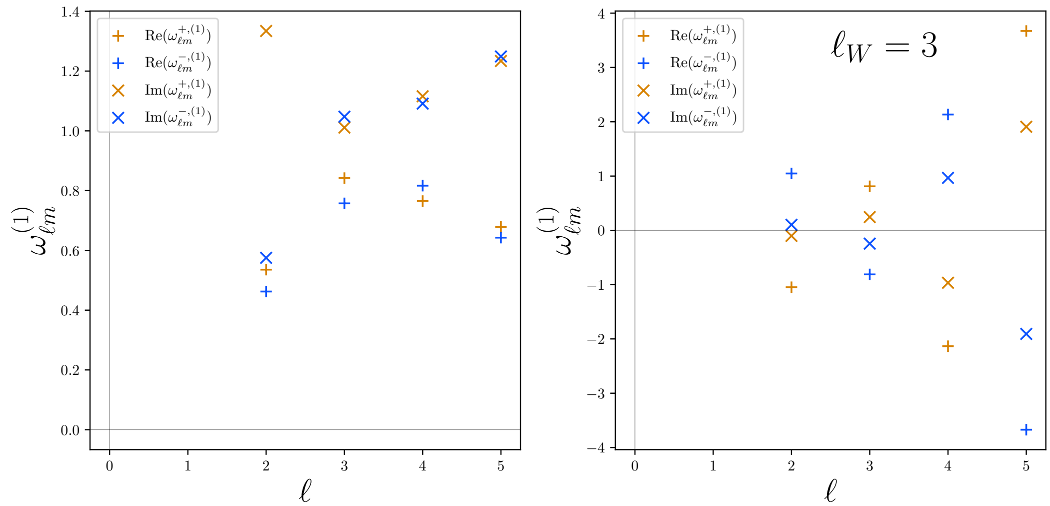

Figure 6.5: The real and imaginary parts of the QNM frequency shifts $\omega^{\pm(1)}_{\ell m}$ for $n = 0$, $\ell = m$ up to $\ell = 5$ for both the $\ell_W = 2$ (left panel) and $\ell_W = 3$ (right panel) Weyl multipole corrections.

$\ell = 2, 3$, $m \geq 0$, and $n = 0$. The frequencies of the modes with $m < 0$ can be found using the relation in Eq. (6.53). The QNM frequency shifts $\omega^{\pm(1)}_{\ell m}$ of any additional overtones we have calculated are listed in the tables in Appendix 6.8.

In Fig. 6.4, one important feature is that for each (ℓ, m) mode, the frequency shift is degenerate for the Weyl bump $\ell_W = 3$, i.e., $\omega^{+(1)}_{\ell m} = -\omega^{-(1)}_{\ell m}$. In contrast, there are two independent shifts for the Weyl bump $\ell_W = 2$. This is a natural consequence of bumps' parity. Notice that the metric correction $h^{(1,0)}_{\mu\nu}$ in (6.42) for each set of potentials $(\psi_1^{\ell_W}, \gamma_1^{\ell_W})$ obeys

$$\hat{\mathcal{P}} h^{(1,0)}_{\mu\nu} = (-1)^{\ell_W} h^{(1,0)}_{\mu\nu} \,, \tag{6.69}$$

as one can explicitly check using Eqs. (6.12), (6.15), and (6.16) for $\ell_W = 2, 3$. We can then derive the $\hat{\mathcal{P}}$ transformation of all the NP quantities at $O(\zeta^1, \epsilon^0)$ following [99]. In the end, we get that $\hat{\mathcal{P}} H_0^{(1,0)} = (-1)^{\ell_W} H_0^{(1,0)}$, which implies

$$\begin{aligned} \langle P_{\ell m} \rangle &= (-1)^{\ell_W} \langle \bar{P}_{\ell -m}(\pi - \theta) \rangle \,, \\ \langle Q_{\ell -m} \hat{C}\hat{\mathcal{P}} \rangle &= (-1)^{\ell_W} \langle \bar{Q}_{\ell m}(\pi - \theta) \hat{C}\hat{\mathcal{P}} \rangle \,. \end{aligned} \tag{6.70}$$

Using Eq. (6.63), we can derive two salient relations governing the QNM frequency shifts $\omega^{\pm(1)}_{\ell m}$. First, for the odd-parity Weyl multipoles, there is an additional symmetry of isospectrality breaking, where $\omega^{-(1)}_{\ell m} = -\omega^{+(1)}_{\ell m}$. Even though we use the notation +

and $-$ to label the frequency shifts, the resulting wavefunctions do not have definite parity since $\eta^{\pm}_{\ell m} \neq \pm 1$ [99]. Second, for the even-parity Weyl multipoles, Eq. (6.64) implies that the resulting wavefunction has definite parity, i.e., $\eta^{\pm}_{\ell m} = \pm 1$ [99]. These features were also observed in [89, 90, 120] when studying the QNM frequency shifts of higher-derivative gravity for both the parity-preserving and parity-violating corrections to the Einstein-Hilbert action. In [99] and this work, we more directly show the origin of these features. Nonetheless, isospectrality is broken for all the modes we have calculated regardless of the parity of the bumps.

Furthermore, it was first shown in [32] that for a Schwarzschild BH, the real part of the QNM frequencies is related to the orbital frequencies of null geodesics near the light ring, while the imaginary part encodes the Lyapunov exponent of the orbit. Specifically, when $m = \ell$, the QNM frequencies $\omega_{\ell mn}$ have the following eikonal approximation,

$$\omega_{\ell mn} \approx (\ell + 1/2)\Omega - i\gamma_L(n + 1/2)\,, \tag{6.71}$$

where Ω is the Keplerian frequency of a circular null geodesic, and γ_L is the Lyapunov exponent. For an axisymmetric spacetime without spherical symmetry, one would naively expect that the real part of $\omega_{\ell mn}$ for fixed low values of n will depend linearly on ℓ for modes with the same values of m/ℓ, which corresponds to the inclination angle of the orbit the mode is associated with. The imaginary part of $\omega_{\ell mn}$ for low values of n are expected to stay roughly constant as ℓ increases while fixing m/ℓ and n, since these values are related to the Lyapunov exponents of the orbits. However, as shown in Fig. 6.5, the real part $\mathrm{Re}(\omega^{(1)}_{\ell m})$ does not depend on ℓ linearly, which is inconsistent with the prediction in Eq. (6.71). Moreover, the imaginary part $\mathrm{Im}(\omega^{(1)}_{\ell m})$ is not constant for the mode $n = 0$. These inconsistencies suggest that the relationship predicted by the eikonal approximation in Eq. (6.71) may need further exploration for these bumpy BHs and BHs in bGR theories in general.

In general, one may not be able to use observational data to examine the QNM frequency shifts for each Weyl bump independently. In this case, we need to sum the contributions to $\omega^{(1)}_{\ell m}$ from bumps with different ℓ_W. As we previously discussed, the bumps with odd ℓ_W and even ℓ_W have different parity, so one needs to use the recombination rule found in [120] in general, i.e.,

$$\omega^{\pm(1)}_{\mathrm{total},\ell m} = \frac{\omega^{+,(1)}_{\mathrm{even},\ell m} + \omega^{-(1)}_{\mathrm{even},\ell m}}{2} \pm \sqrt{\left(\frac{\omega^{+(1)}_{\mathrm{even},\ell m} - \omega^{-(1)}_{\mathrm{even},\ell m}}{2}\right)^2 + \left(\omega^{+(1)}_{\mathrm{odd},\ell m}\right)^2}\,, \tag{6.72}$$

where $\omega^{\pm(1)}_{\text{even},\ell m}$ and $\omega^{\pm(1)}_{\text{odd},\ell m}$ are the QNM frequency shifts generated by the even-parity (i.e., even ℓ_W) and odd-parity (i.e., odd ℓ_W) Weyl multipoles, respectively.

6.6 Conclusion and Outlook

In this work, we used the modified Teukolsky formalism in [87, 99] to compute the QNM frequency shifts for a non-rotating BH with axisymmetric deviations parametrized by Weyl multipoles. Since these bumpy BHs are Ricci flat in the non-rotating case, and we did not consider any corrections to the Einstein-Hilbert action, the only term in the modified Teukolsky equation that contributes is $\mathcal{S}^{(1,1)}_{\text{geo}}$, which only depends on the modifications to the background geometry [i.e., terms at $O(\zeta^1, \epsilon^0)$] and the QNMs in GR [i.e., terms at $O(\zeta^0, \epsilon^1)$]. The terms at $O(\zeta^1, \epsilon^0)$ were directly evaluated using the bumpy BH metric found in [77]. To calculate the $O(\zeta^0, \epsilon^1)$ quantities, we implemented the CCK-Ori metric reconstruction procedure in the IRG. After obtaining the modified Teukolsky equations, we noticed that the source terms mix the (ℓ, m) and $(\ell, -m)$ modes, which is one main cause for isospectrality breaking [99]. Following [88, 99], we solved the (ℓ, m) and $(\ell, -m)$ modes jointly and used the EVP method in [88, 97, 98] to compute the QNM frequencies.

We obtained the QNM frequency shifts for the modes $\ell = 2, 3$ up to the second overtone for both the bumps with multipole $\ell_W = 2$ and $\ell_W = 3$. Some qualitative features were found. Our results showed that isospectrality is broken for both the $\ell_W = 2$ and $\ell_W = 3$ bumps. Specifically, we noticed that the isospectrality breaking structure is related to the parity of the bumps. For odd-parity bumps, we found that the two frequency shifts, due to isospectrality breaking, are opposite to each other. These features are consistent with the ones discovered by [89, 90, 120] in higher-derivative gravity. Furthermore, we notice that the eikonal approximation in GR [32–34] is no longer valid here, so further investigation is needed in these bumpy BH spacetimes. We suspect this breakdown can be related to the chaotic nature of photon orbits near bumpy BHs, as found by Refs. [121, 122].

Since we expect to observe the GW ringdown with a much higher signal-to-noise ratio during the fourth LVK observing run and in the future with third-generation detectors, such as Einstein Telescope [123] and Cosmic Explorer [124], it is critical to accurately model the GWs emitted during the ringdown phase not only in GR but also bGR theories. Our work has made a crucial attempt in this direction by developing a framework to study the ringdown of a BH spacetime with parametrized deviations, which is valid for rotating BHs in general. Through our work, one can directly connect the multipole structure of a BH spacetime to its QNM spectra.

Our efforts can be extended in several directions. In [77], Hughes and Vigeland also derived a spinning bumpy BH spacetime. Since the bumps in the non-rotating case are already axisymmetric, the procedures in this work should naturally extend to spinning bumpy BHs. One subtlety is that the bumpy spinning BH in [77] is not Ricci flat, unlike the non-spinning case. This introduces extra source terms driven by the Ricci tensor and not present in our current implementation. Even though these source terms decay as $r^{-(\ell_W+1)}$ for the Weyl multipole ℓ_W, we still need to investigate whether they significantly contribute to the QNM frequency shifts. Fortunately, the modified Teukolsky formalism can still deal with these extra source terms, as demonstrated in [89–91]. Another theory-agnostic approach could instead use the GH multipole moments, which can parametrize Ricci-flat solutions to the Einstein equations even for rotating BHs. A potential drawback to this strategy is that one cannot uniquely describe a non-vacuum spacetime with multipoles [125], limiting the ability to model more general BH environments [126]. Nonetheless, multipoles can still be a valuable probe of possible bGR corrections and a clean way to formulate no-hair tests of GR.

Since isospectrality breaking is likely a common feature in bGR theories [99], understanding its physical origins and signatures on GW observables is another important direction. In this work, we have made a direct connection from the parity of the bumps to the isospectrality breaking structure of the QNMs. Since the same relations also appeared in higher-derivative gravity [89, 90, 120] and other parametrized ringdown studies of non-rotating BHs [127], it will be worth to derive these relations from the parity properties of the action directly following [99]. Furthermore, we should also investigate how these generic isospectrality-breaking features of QNMs impact GW waveforms. We should study whether we can extract any characteristic features associated with isospectrality breaking from the waveforms. These features are potentially useful for testing the class of parity-preserving or parity-violating theories in a generic way.

6.7 Appendix: $\mathcal{O}(\zeta^1, \epsilon^0)$ quantities

In this section, we list the Weyl scalars and spin coefficients at $\mathcal{O}(\zeta^1, \epsilon^0)$. The Weyl scalars at $\mathcal{O}(\zeta^1, \epsilon^0)$ are given by

$$\Psi_0^{(1,0)} = -\frac{\partial_\theta^2\gamma_1 + (r-2M)\left(2\partial_r + r\partial_r^2\right)\gamma_1 - \cot\theta\partial_\theta\gamma_1 - 2\partial_\theta^2\psi_1 + 2\cot\theta\partial_\theta\psi_1}{2r(r-2M)}, \tag{6.73}$$

$$\Psi_1^{(1,0)} = \frac{2r(r-2M)\left(\cot\theta\partial_r\gamma_1 - 2\partial_r\partial_\theta\psi_1\right) + 2(r+M)\partial_\theta\gamma_1 + 4(r-5M)\partial_\theta\psi_1}{4\sqrt{2}r^2(r-2M)}, \tag{6.74}$$

$$\begin{aligned}\Psi_2^{(1,0)} &= \frac{r(r-2M)\partial_r^2\left(8\psi_1 - 5\gamma_1\right) - 2(2M+r)\partial_r\gamma_1 + 4(M+r)\partial_r\psi_1}{12r^2} \\ &+ \frac{r\left[\cot\theta\left(2\partial_\theta\psi_1 - 3\partial_\theta\gamma_1\right) - 5\partial_\theta^2\gamma_1 + 2\partial_\theta^2\psi_1\right] + 24M\gamma_1 - 24M\psi_1}{12r^3},\end{aligned} \tag{6.75}$$

$$\Psi_3^{(1,0)} = \frac{-2r(r-2M)\partial_r\left(\cot\theta\gamma_1 - 2\partial_\theta\psi_1\right) - 2(r+M)\partial_\theta\gamma_1 + 4(5M-r)\partial_\theta\psi_1}{8\sqrt{2}r^3}, \tag{6.76}$$

$$\Psi_4^{(1,0)} = \frac{(r-2M)\left[\cot\theta\partial_\theta\left(\gamma_1 - 2\psi_1\right) - \partial_\theta^2\gamma_1 - (r-2M)\left(2\partial_r\gamma_1 + r\partial_r^2\gamma_1\right) + 2\partial_\theta^2\psi_1\right]}{8r^3}. \tag{6.77}$$

The spin coefficients at $\mathcal{O}(\zeta^1, \epsilon^0)$ are given by

$$\kappa^{(1,0)} = \frac{\partial_\theta\gamma_1 - 2\partial_\theta\psi_1}{\sqrt{2}(r-2M)}, \tag{6.78a}$$

$$\pi^{(1,0)} = \frac{\partial_\theta\gamma_1}{2\sqrt{2}r}, \tag{6.78b}$$

$$\epsilon^{(1,0)} = \frac{1}{2}\partial_r\psi_1, \tag{6.78c}$$

$$\rho^{(1,0)} = \frac{-r\partial_r\gamma_1 + 2\gamma_1 + 2r\partial_r\psi_1 - 2\psi_1}{2r}, \tag{6.78d}$$

$$\lambda^{(1,0)} = -\frac{(r-2M)\partial_r\gamma_1}{4r}, \tag{6.78e}$$

$$\alpha^{(1,0)} = \frac{\cot\theta(\gamma_1 - \psi_1) + \psi_1}{2\sqrt{2}r}, \tag{6.78f}$$

$$\sigma^{(1,0)} = -\frac{1}{2}\partial_r\gamma_1, \tag{6.78g}$$

$$\mu^{(1,0)} = -\frac{(r-2M)\left[r\left(\partial_r\gamma_1 - 2\partial_r\psi_1\right) - 2\gamma_1 + 2\psi_1\right]}{4r^2}, \tag{6.78h}$$

$$\beta^{(1,0)} = -\frac{\cot\theta\left(\gamma_1 - \psi_1\right) + \partial_\theta\psi_1}{2\sqrt{2}r}, \tag{6.78i}$$

$$\nu^{(1,0)} = -\frac{(r-2M)\left(\partial_\theta\gamma_1 - 2\partial_\theta\psi_1\right)}{4\sqrt{2}r^2}\,, \tag{6.78j}$$

$$\gamma^{(1,0)} = \frac{-2M\gamma + r(r-2M)\partial_r\psi_1 + 2M\psi_1}{4r^2}\,, \tag{6.78k}$$

$$\tau^{(1,0)} = -\frac{\partial_\theta\gamma_1}{2\sqrt{2}r}\,. \tag{6.78l}$$

6.8 Appendix: QNM Frequency Shifts

In this appendix, we explicitly tabulate the QNM frequency shifts $\omega_{\ell m}^{\pm(1)}$ generated by the Weyl multipoles $\ell_W = 2, 3$. All the results in this appendix were calculated using the modified contour $\mathscr{C}_2$ in Fig. 6.2. For the even-parity bump $\ell_W = 2$, the top row of each cell refers to $\omega_{\ell m}^{+(1)}$ with $\eta_{\ell m} = (-1)^\ell$, while the bottom row of each cell refers to $\omega_{\ell m}^{-(1)}$ with $\eta_{\ell m} = (-1)^{\ell+1}$. For the odd-parity bump $\ell_W = 3$, since $\omega_{\ell m}^{-(1)} = -\omega_{\ell m}^{+(1)}$ as discussed in Sec. 6.5, we only list $\omega_{\ell m}^{+(1)}$.

Table 6.1: $\ell = 2\,, \ell_W = 2$

	$m=2$	$m=1$	$m=0$
$n=0$	$0.5356+1.334i$ $0.4624+0.5751i$	$-0.9934-0.7472i$ $-1.013-0.8530i$	$-0.8402-1.552i$ $-0.4508-1.709i$
$n=1$	$-0.2501+1.209i$ $0.5693+1.013i$	$-0.6474-1.133i$ $-0.5678-1.158i$	$-0.08632-1.464i$ $-0.04923-1.735i$
$n=2$	$-0.1020+1.114i$ $0.1645+1.208i$	$-0.3401-1.208i$ $-0.3330-1.162i$	$0.03257-1.560i$ $0.08164-1.433i$

Table 6.2: $\ell = 2\,, \ell_W = 3$

	$m=2$	$m=1$	$m=0$
$n=0$	$-1.048-0.1024i$	$0.9566+1.485i$	$1.176+1.030i$
$n=1$	$0.05390+0.6246i$	$-0.3629-1.547i$	$-0.6855-1.265i$
$n=2$	$0.08806-0.3659i$	$-0.1525-1.440i$	$-0.3984-1.171i$

Table 6.3: $\ell = 3\,, \ell_W = 2$

	$m=3$	$m=2$	$m=1$	$m=0$
$n=0$	$0.8419+1.011i$ $0.7574+1.047i$	$-2.832-2.520i$ $-2.518-1.663i$	$1.108-0.1184i$ $1.034-0.06974i$	$2.358+2.124i$ $2.268+2.170i$
$n=1$	$0.4823+1.185i$ $0.4446+1.118i$	$-1.862-3.176i$ $-2.244-2.753i$	$1.124+0.5143i$ $1.093+0.4545i$	$1.638+2.907i$ $1.602+2.840i$
$n=2$	$0.3154+1.023i$ $0.2872+1.043i$	$-1.323-2.965i$ $-1.562-2.898i$	$0.8396+0.6965i$ $0.8119+0.6996i$	$1.099+2.720i$ $1.072+2.724i$

Table 6.4: $\ell = 3\,, \ell_W = 3$

$n = 0$	$0.8113 + 0.2449i$	$-0.5446 - 1.675i$	$-4.754 - 4.637i$	$3.775 + 2.201i$
$n = 1$	$0.7898 + 0.8195i$	$-0.0214 - 1.425i$	$3.442 + 6.196i$	$3.341 + 3.550i$
$n = 2$	$0.5539 + 1.071i$	$-0.05605 + 0.9151i$	$2.324 + 6.015i$	$2.443 + 3.666i$

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Part II

Holographic gravity in flat spacetime

Chapter 7

INTERFEROMETER RESPONSE TO GEONTROPIC FLUCTUATIONS

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7.1 Introduction

Traditional wisdom in effective field theory (EFT) suggests that quantum fluctuations in the fabric of spacetime should be of the order of $\sim l_p = \sqrt{8\pi G\hbar/c^3} \sim 10^{-34}$ m, where G, $\hbar$, c, and l_p are the gravitational constant, reduced Planck constant, speed of light, and Planck length respectively. Fluctuations on such small time and length scales are experimentally undetectable.

It has, however, been recently argued in multiple different contexts that the length scale L of the physical system itself may enter into the observable [1–6] (see Ref. [7] for a summary)

$$\left\langle \left(\frac{\Delta L}{L}\right)^2 \right\rangle \sim \frac{l_p}{L}, \tag{7.1}$$

where ΔL is the quantum fluctuation of L. For example, in Refs. [1, 4], L is the length of interferometer arm in flat spacetime. More generally, L can be the size of a causal diamond in dS, AdS, and flat spacetime [2, 3], where the causal diamond associated with a volume V consists of points which have the property that all causal curves going through the point must intersect V [8, 9]. These works argued that the naive EFT reasoning is corrected by long-range correlations in the metric fluctuations-such as are known to occur in holography-which allow the UV fluctuations to accumulate into the infrared. A physical analogue is Brownian motion (discussed in Ref. [7]) where the interactions occur at very short distances but become observable on long timescales as the UV effects accumulate.

While the calculations presented in Refs. [1–5] are firmly grounded in standard theoretical techniques, such as AdS/CFT, they have not yet provided important, detailed experimental information, such as the power spectral density. This was the motivation behind the model of Ref. [4]: to provide a framework that reproduces important behaviors of the UV-complete theory while also allowing to calculate detailed signatures in the infrared. In the language of the Brownian motion model, while the fluctuations arise from local interactions, the observable is only defined globally. In the language of an interferometer experiment, one cannot measure spacetime fluctuation within a portion of an interferometer arm length, but must wait for a photon to complete a round trip before making a measurement of the global length fluctuation across the entire arm.

In this work, we continue to develop the model proposed in Ref. [4], utilizing a scalar field coupled to the metric to model the behavior of the spacetime fluctuations proposed in Refs. [1–5]. We call spacetime fluctuations modeled by the scalar field "geontropic fluctuations" since they are geometric fluctuations induced by entropic fluctuations within a finite spatial volume, as we discuss in the next section. In particular, we propose a model in four dimensions, where the metric appears as a breathing mode of a sphere controlled by a scalar field ϕ:

$$ds^2 = -dt^2 + (1 - \phi)(dr^2 + r^2 d\Omega^2)\,. \tag{7.2}$$

Since ϕ effectively controls the area of a spherical surface, it is thus proportional to the entropy of a causal diamond, and may be identified with the dilaton mode studied in Refs. [3, 5], which induces fluctuations in the spherical entangling surface shown in Fig. 7.1 and is modeled by the metric in Eq. (7.2). In the model we consider, ϕ is a scalar field whose quantum fluctuations will be characterized by its occupation number, which we label as σ_{pix}. The subscript denotes "pixellon" following the proposal of Ref. [4], referring to the pixels of spacetime whose fluctuations the scalar field is modeling. While we do not derive the form of the metric in Eq. (7.2), we reproduce the angular correlation proposed in Ref. [1], a non-trivial result (not typical of most metrics) which we take as further evidence that this Ansatz is a good starting point. In addition, the power spectral density has no pathologies in the ultraviolet or infrared, another non-trivial result.

In particular, the quantum fluctuations of the scalar, since they couple to the metric, will give rise to fluctuations in the round-trip time for a photon to traverse from mirror to mirror in an interferometer, as depicted in Fig. 7.1. Similar to Ref. [4], our main goal is to compute the gauge invariant interferometer observable arising from the metric Eq. (7.2), with ϕ being a scalar field having a high occupation number. In contrast to Ref. [4], which calculated length fluctuations utilizing the

Feynman-Vernon influence functional in a single interferometer arm, we will use only linearized gravity and the QFT of a scalar field with a given occupation number. We will thus be able to extend the previous work in Ref. [4], calculating both the power spectral density and angular correlations in the interferometer arms in a manifestly gauge invariant way, checking previous claims made in Ref. [1], as well as making new predictions. Note that while the model is not yet uniquely derived from first principles in the ultraviolet (utilizing for example shockwave geometry [6], i.e., the gravitational field of fast-moving particles with negligible rest mass [10]), we will argue below that it is nevertheless well-motivated from first principles.

More specifically, we consider an interferometer with two arms of equal length L, i.e., with spherical symmetry, and separated by angle θ, as depicted in Fig. 7.1. We assume that the first arm as the reference beam points in the direction $\mathbf{n}_1$, and the second arm as the signal beam points in the direction $\mathbf{n}_2$. We will find that the observable takes the form:

$$\left\langle \frac{\delta T(t_1, \mathbf{n}_1)\delta T(t_2, \mathbf{n}_2)}{4L^2} \right\rangle = \frac{l_p^2}{4L^2} \int_0^L dr_1 \int_0^L dr_2 \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\sigma_{\text{pix}}(\mathbf{p})}{2\omega(\mathbf{p})} \mathcal{F}(r_1, r_2, p, \Delta x)\,, \tag{7.3}$$

where $\delta T(t, \mathbf{n})$ denotes the fluctuation of time delay of light beam sent at time $t - L$ along the direction $\mathbf{n}$, and $p = (\omega, \mathbf{p})$, $\Delta x = (\Delta t, \Delta \mathbf{x})$ are four-vectors. The main object of interest in this paper is $\mathcal{F}(r_1, r_2, p, \Delta x)$, which encapsulates the response of the interferometer gravitationally coupled to the scalar field ϕ.

The rest of the paper is organized around deriving Eq. (7.3). In Sec. 7.2, we review the pixellon scalar field model, with an occupation number σ_{pix} motivated in particular by [4], but also by work demonstrating that the effect of interest is a breathing mode of the horizon [3, 5]. We then couple this scalar field to the Einstein-Hilbert action and derive its equation of motion. In Sec. 7.3, we perform a linearized gravity calculation and derive the observable. In particular, we compute the interferometer response function $\mathcal{F}(r_1, r_2, p, \Delta x)$ from our specific model. In Sec. 7.4, we compute the relevant power spectral density and angular correlation from Eq. (7.3). We then discuss various existing experimental constraints. Finally, in Sec. 7.5, we conclude. Throughout the paper we will work in units $\hbar = c = k_B = 1$ while keeping the gravitational constant $G = l_p^2/(8\pi)$ explicit.

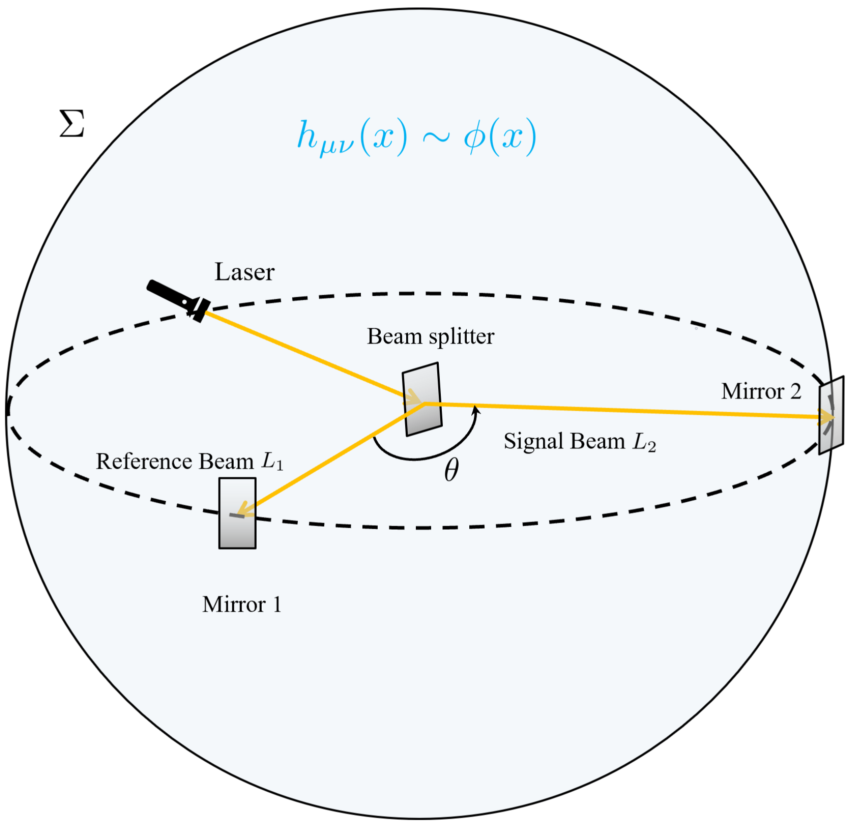

Figure 7.1: Setup of the interferometer.

7.2 Scalar Field Quantum Fluctuations in a Causal Diamond

The main goal of this section is to motivate the form of the scalar occupation number, σ_{pix}, that will be coupled to the metric. Our discussion here is mostly based on Ref. [4], though, as mentioned previously, it is also broadly consistent with the dilaton model presented in Ref. [3, 5]. We first review the pixellon model developed in Refs. [1–6] but use the notation in this work, and the rest of this section directly applies the pixellon model to the specific metric in Eq. (7.2).

The effect of interest, as presented in Refs. [1, 2] is based on fluctuations in the modular Hamiltonian K

$$K = \int_B T_{\mu\nu} \zeta_K^\mu dB^\nu \,, \tag{7.4}$$

where B is some spatial region with a stress tensor $T_{\mu\nu}$, dB^ν is the volume element of B (with dB^ν pointing in the time direction), and ζ_K^μ is the conformal Killing vector of the boost symmetry of Σ, the entangling surface between B and its complement $\bar{B}$ [2, 8]. One can map B to Rindler space, so Σ is also a Rindler horizon. In the context of AdS/CFT, where $T_{\mu\nu}$ is the stress tensor of the boundary CFT, both the vacuum expectation value and the fluctuations of the modular Hamiltonian are known to obey an area law in vacuum [2, 11, 12]

$$\langle K \rangle = \langle \Delta K^2 \rangle = \frac{A(\Sigma)}{4G}\,, \tag{7.5}$$

where $A(\Sigma)$ is the area of Σ. One tempting interpretation of this relation is that $\langle K \rangle \equiv \mathcal{N}$ counts the number of gravitational bits, or pixels, in the system, which is further motivated by the fact that the entanglement entropy $S_{\rm ent} = \langle K \rangle$ is known to hold in a CFT. The fluctuations of those $\mathcal{N}$ bits then satisfy "root-$\mathcal{N}$" statistics:

$$\frac{|\Delta K|}{\langle K \rangle} = \frac{1}{\sqrt{\mathcal{N}}}\,, \tag{7.6}$$

where $|\Delta K| = \sqrt{\langle \Delta K^2 \rangle}$ represents the amplitude of the modular fluctuation.

While the precise relation $\langle K \rangle = \langle \Delta K^2 \rangle$ is demonstrated only in the context of AdS/CFT, one can place a Randall-Sundrum brane in the (5-d) bulk of AdS, inducing gravity on the (flat 4-d) RS brane, and show that Eq. (7.5) holds on the 4-d brane [3]. The measuring apparatus can then be placed on the flat 4-d brane. Further, as shown in [3, 13, 14], gravity is approximately conformal near the horizon. For an interferometer, the light beams are probing the near-horizon geometry of the spherical entangling surface Σ bounding it (shown in Fig. 7.1), so Ref. [3] argued that the correlator of stress tensor takes the same form as any CFT. Thus, $\langle \Delta K^2 \rangle$ follows Eq. (7.5), i.e.,

$$\begin{aligned} \langle \Delta K^2 \rangle \;&\sim\; \int d^2\mathbf{y}\, d^2\mathbf{y}' \frac{dr\, dr' r\, r'}{((r-r')^2 + (\mathbf{y}-\mathbf{y}')^2)^4} \\ &\sim\; A \int \frac{dr\, dr' r\, r'}{(r-r')^6} \sim \frac{A}{\delta^2} \sim \frac{A}{l_p^2}\,, \end{aligned} \tag{7.7}$$

where $\mathbf{y}$ denotes the transverse directions (corresponding to the coordinates on Σ), and $G \sim \delta^2$ corresponds to a UV cut-off in the theory at a distance scale $\delta \sim l_p$. In our case, $r - r' \sim \delta$ corresponds to the distance to the (unperturbed) spherical entangling surface Σ in our setup shown in Fig. 7.1. A similar relation holds for $\langle K \rangle$. More generally, as found in [15], an area law for entanglement entropy does not hold only for a CFT but also any massless scalar QFT, which also motivates the scalar model of geontropic fluctuations in [4] and this work.

The idea of Ref. [4] was thus to model the gravitational effects of modular fluctuations with a massless scalar field, dubbed a "pixellon." Since pixellons are bosonic scalars, their creation and annihilation operators $(a, a^\dagger)$ satisfy the usual commutation relation

$$\left[a_{\mathbf{p}_1}, a^\dagger_{\mathbf{p}_2}\right] = (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2)\,. \tag{7.8}$$

We are interested in modeling the impact of the (fluctuating) effective stress tensor in Eq. (7.13). We will do this by allowing for a non-zero occupation number $\sigma_{\rm pix}(\mathbf{p})$,

$$\mathrm{Tr}\left(\rho_{\rm pix} a^\dagger_{\mathbf{p}_1} a_{\mathbf{p}_2}\right) = (2\pi)^3 \sigma_{\rm pix}(\mathbf{p}_1) \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2) \tag{7.9}$$

such that

$$\mathrm{Tr}\left(\rho_{\rm pix} \{a_{\mathbf{p}_1}, a^\dagger_{\mathbf{p}_2}\}\right) = (2\pi)^3 \left[1 + 2\sigma_{\rm pix}(\mathbf{p}_1)\right] \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2)\,. \tag{7.10}$$

The occupation number should be consistent with the modular energy fluctuation, Eq. (7.6), as we will check explicitly at the end of this section.

The pixellon couples to the metric and sources the stress tensor at second order in perturbations. In general, we can consider a metric of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} + \epsilon^2 H_{\mu\nu} + \dots\,, \tag{7.11}$$

where ϵ is a dimensionless parameter that denotes the order in perturbation theory. The vacuum Einstein Equation (EE) is, parametrically[1],

$$G_{\mu\nu} = \epsilon \left[\nabla^2 h\right]_{\mu\nu} + \epsilon^2 \left(\left[\nabla^2 H\right]_{\mu\nu} - l_p^2 T_{\mu\nu}\right) + \dots = 0\,, \tag{7.12}$$

where the precise form of the equations of motion (e.g., numerical prefactors in the time and spatial derivatives) will depend on the precise form of the metric that we consider below, and where the effective stress tensor is given by

$$T_{\mu\nu} \sim \frac{1}{l_p^2} \left[(\nabla h)^2\right]_{\mu\nu}\,. \tag{7.13}$$

At leading order in perturbation theory, the metric perturbation $h_{\mu\nu}$ satisfies the vacuum EE having a form

$$\left[\nabla^2 h\right]_{\mu\nu} = 0\,. \tag{7.14}$$

However, at second order, the effective stress tensor of $h_{\mu\nu}$ will source a non-zero metric perturbation $H_{\mu\nu}$, i.e.,

$$\left[\nabla^2 H\right]_{\mu\nu} = l_p^2 T_{\mu\nu}\,. \tag{7.15}$$

[1]This argument was formulated in private communication with E. Verlinde in the work leading to Ref. [2].

One can compute $\langle K \rangle$ from $\langle T_{\mu\nu} \rangle$, but as shown in [2], $\langle K \rangle$ does not gravitate and should be subtracted in the metric equation of motion (similar to a tadpole diagram in QFT). Thus, the vacuum expectation value of this stress tensor vanishes, $\langle T_{\mu\nu} \rangle = 0$, consistent with Eqs. (7.13)-(7.14). In contrast, it is expected to have nonzero fluctuations $\langle \Delta K^2 \rangle \sim \langle T_{\alpha\beta} T_{\mu\nu} \rangle \neq 0$, which gravitate and lead to physical observables.

Although $\langle \Delta K^2 \rangle$ is directly related to the *vacuum* two-point function of $H_{\mu\nu}$ or four-point function of $h_{\mu\nu}$, the physical observable can be directly computed from the two-point function of $h_{\mu\nu}$ with a nontrivial density-of-states σ_{pix}. That is, we are using the language of linearized gravity in this work, while our result captures the nonlinearity in Eq. (7.15) and higher orders via σ_{pix}. To compute the fluctuations, we quantize the metric perturbations via the scalar field ϕ, which, to second order in perturbation theory, leads to a nonzero $\langle \Delta K^2 \rangle$, as shown at the end of this section. The major goal of this work is to compute the effects of such quantized metric perturbations on the interferometer depicted in Fig. 7.1.

More specifically, following Ref. [4], we model these energy fluctuations, in the volume of spacetime interrogated with an interferometer, with a thermal density matrix ρ_{pix}, as shown in Eqs. (7.9)-(7.10). The motivation for this choice is based on formal work [8] showing that the reduced density matrix ρ_V of the system V bounded by a sphere S^{d-1} or its causal diamond $\mathcal{D}$ can be mapped to the thermal density matrix ρ_β of the hyperbolic spacetime $R \times H^{d-1}$, which foliates AdS_{d+1}, in the asymptotic limit. A similar argument relating the vacuum state of any QFT in a causal diamond to a thermal density matrix can be found in [16].

Thus, following [4], we are motivated to define a thermal density matrix ρ_{pix} of pixellons using the definition in [17],

$$\rho_{\text{pix}} = \frac{1}{\mathcal{Z}} \exp\left[-\beta \int \frac{d^3\mathbf{p}}{(2\pi)^3} (\epsilon_{\mathbf{p}} - \mu) a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \right], \tag{7.16}$$

$$\mathcal{Z} = \prod_{\mathbf{p}} \frac{1}{1 - e^{-\beta(\epsilon_{\mathbf{p}} - \mu)}}, \tag{7.17}$$

where $\epsilon_{\mathbf{p}}$ is the energy of pixellons with momentum $\mathbf{p}$, and μ is the chemical potential counting background degrees of freedom associated with $\langle K \rangle$ [4].

Furthermore, as in Ref. [4], we identify the energy per degree-of-freedom as

$$\beta(\epsilon_{\mathbf{p}} - \mu) \equiv \beta\omega(\mathbf{p}) \sim \frac{|\Delta K|}{\langle K \rangle}. \tag{7.18}$$

In four dimensions, according to Eq. (7.5),

$$\frac{|\Delta K|}{\langle K\rangle} = \frac{1}{\sqrt{\mathcal{N}}} \sim \frac{l_p}{L}\,, \tag{7.19}$$

suggesting that the energy fluctuation per degree-of-freedom is set by a ratio of UV and IR length scales. Since $\frac{l_p}{L} \ll 1$, we approximate the occupation number $\sigma(\mathbf{p})$ by

$$\sigma_{\text{pix}}(\mathbf{p}) = \frac{1}{e^{\beta\omega(\mathbf{p})} - 1} \approx \frac{1}{\beta\omega(\mathbf{p})}\,. \tag{7.20}$$

More specifically, we identify the IR length scale $1/L \sim \omega(\mathbf{p})$, so we take

$$\sigma_{\text{pix}}(\mathbf{p}) = \frac{a}{l_p\omega(\mathbf{p})}\,, \tag{7.21}$$

where a is the dimensionless number to be measured in an experiment, or fixed in a UV-complete theory. Here $a = 1/(2\pi)$ corresponds to an inverse temperature $\beta = 2\pi l_p$, giving a result most closely mirroring Refs. [1, 2, 4] in amplitude.

Note that $\sigma_{\text{pix}}(\mathbf{p})$ is not Lorentz invariant, but this is to be expected because the measurement of interest via a causal diamond picks out a frame. This is also not contradictory to our statement that we have computed a gauge invariant observable. It is because Lorentz transformations of $\sigma_{\text{pix}}(\mathbf{p})$ are global transformations of background Minkowski spacetime. After the interferometer picks a frame, the interferometer response is independent of how we describe metric perturbations, i.e., independent of local coordinate transformations at scale of metric perturbations, which is what gauge invariance usually means in linearized gravity.

We now apply this pixellon model to the metric in Eq. (7.2) and derive the dispersion relation of ϕ. We start from the linearized Einstein Hilbert action or Fierz-Pauli action [18]

$$\begin{aligned} S_{\text{FP}} &= \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\;h_{\mu\nu}\left(G^{\mu\nu}[h_{\mu\nu}] - \kappa T^{\mu\nu}\right) \\ &= \frac{1}{4\kappa}\int d^4x\,\sqrt{-g}\;h_{\mu\nu}(\eta^{\mu\nu}\Box h - \Box h^{\mu\nu} \\ &\quad - 2\nabla^\mu\nabla^\nu h + 2\nabla_\rho\nabla^\mu h^{\nu\rho} - 2\kappa T^{\mu\nu}) + O(h^3)\,, \end{aligned} \tag{7.22}$$

where $\kappa = 8\pi G$. The Fierz-Pauli action can be derived by expanding the full metric $g_{\mu\nu}$ about the Minkowski metric $\eta_{\mu\nu}$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and keeping the terms quadratic in $h_{\mu\nu}$ in the Einstein Hilbert action [18, 19]. Here, $h_{\mu\nu}$ is the metric perturbation associated with the pixellon ϕ. The terms linear in $h_{\mu\nu}$ are discarded because they can be written as a total derivative [19].

Instead of a functional of a general $h_{\mu\nu}$, $S_{\rm FP}$ in our model is a functional of the metric in Eq. (7.2) and thus a functional of ϕ, so the pixellon's action $S_{\rm pix}[\phi]$ is

$$S_{\rm pix}[\phi] \equiv S_{\rm FP}[h^{\rm pix}_{\mu\nu}[\phi]]\,, \quad h^{\rm pix}_{\mu\nu}\, dx^\mu dx^\nu = ds^2_{\rm pix}\,, \tag{7.23}$$

which after plugging in Eq. (7.2) becomes

$$\begin{aligned} S_{\rm pix}[\phi] &= \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,\phi\left[\nabla^2 - 3\partial_t^2\right]\phi + \kappa\mathcal{L}_{\rm int}[\phi]\,, \\ \mathcal{L}_{\rm int}[\phi] &\equiv -h^{\rm pix}_{\mu\nu}[\phi]T^{\mu\nu}\,. \end{aligned} \tag{7.24}$$

Then the equation of motion (EOM) of ϕ is derived by varying $\mathcal{L}_{\rm pix}$ with respect to ϕ.

$$\left(\partial_t^2 - c_s^2\nabla^2\right)\phi = \frac{\kappa}{c_s^2}\frac{\delta\mathcal{L}_{\rm int}[\phi]}{\delta\phi}\,, \quad c_s \equiv \sqrt{\frac{1}{3}}\,. \tag{7.25}$$

Following the logic of Eqs. (7.12)-(7.13), to leading order in ϕ, the right-hand side of Eq. (7.25) vanishes. Although Eq. (7.25) is source-free, one may find that the effective stress tensor contains linear term in ϕ, which is a tadpole due to imposing the form of metric in Eq. (7.2) and can be subtracted off. Eq. (7.25) also implies that for the metric in Eq. (7.2), ϕ needs to have the dispersion relation

$$\omega = c_s|\mathbf{p}|\,, \quad c_s = \sqrt{\frac{1}{3}} \tag{7.26}$$

using the expansion $\phi = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\phi(\mathbf{p})e^{-i\omega t + i\mathbf{p}\cdot\mathbf{x}}$. It is clear that ϕ is a sound mode with the sound speed $c_s = \sqrt{\frac{1}{3}}$. This sound mode can be related to the hydrodynamical sound mode in fluid gravity and the butterfly velocity of out-of-time-correlators, (e.g., see Refs. [20, 21]), which we will explore in our future work. From Eq. (7.24), we also notice that to canonically normalize ϕ, we can define $\bar\phi$ such that

$$\phi = \sqrt{\kappa}\bar\phi = l_p\bar\phi\,. \tag{7.27}$$

As a consistency check, one can use the metric in Eq. (7.2) and the occupation number in Eq. (7.21) to confirm that $\langle\Delta K^2\rangle$ has the same scaling in Eq. (7.5). Although the physical observable is driven by the two-point function of ϕ as we will discuss in Sec. 7.3, $\langle\Delta K^2\rangle$ is driven by the four-point function of ϕ. One can see this by noting that $K^2 \sim (T_{\mu\nu})^2$ according to Eq. (7.4), while $T_{\mu\nu} \sim \frac{1}{l_p^2}\left[(\nabla\phi)^2\right]_{\mu\nu}$ according to Eq. (7.13). In Sec. 7.3, we find, utilizing the Ansatz Eq. (7.21) for the density of states, $\langle\phi^2\rangle \sim \frac{l_p}{L}$ [see Eq. (7.39)]. Thus, if we identify spatial gradients with the IR length scale $1/L$, we obtain $\langle\Delta K^2\rangle \sim \frac{L^2}{l_p^2} \sim \frac{A}{4G}$, as expected.

7.3 Time Delay in Pixellon Model

The major goal of this work is to compute an interferometer response to fluctuations in the pixellon model. Instead of using the Feynman-Vernon influence functional approach to compute the mirror's motion, e.g., in [4, 19, 22], we compute the time delay of a light beam traveling a round trip directly.

In general, for a metric in the form

$$ds^2 = -(1 - \mathcal{H}_0)dt^2 + (1 + \mathcal{H}_2)dr^2 + 2\mathcal{H}_1 dt dr + \cdots , \tag{7.28}$$

we need to consider three effects: the shift in the clock rate, mirror motion, and light propagation. As discussed in detail in Appendix 7.6, the shift in the clock's rate only depends on $\mathcal{H}_0$, the mirror motion in the radial direction is affected by $\mathcal{H}_{0,1}$, and the light propagation is determined by all three components $\mathcal{H}_{0,1,2}$.

In Appendix 7.7, we further show that if we take all of these three effects into consideration and sum up the resulting time delay for both outbound and inbound light, the total time delay T of a round trip is gauge invariant, so T is a physical quantity to measure. In this section, we compute the shift of T due to geontropic fluctuations and its correlation function using the metric of the pixellon model in Eq. (7.2). To calculate time delay in a generic metric like Eq. (7.28), one can refer to Appendix 7.6.

For the metric in Eq. (7.2), the only nonzero component in the $t - r$ sector of the metric is $\mathcal{H}_2$, so we only need to consider light propagation. Then for a light beam sent at time $t - L$ along the direction $\mathbf{n}$, its total time delay $T(t, \mathbf{n})$ of a round trip is completely determined by the pixellon field ϕ, e.g.,

$$\begin{aligned} T(t, \mathbf{n}) &= 2L - \frac{1}{2}\int_0^L dr\ [\phi(x) + \phi(x')] , \\ x &\equiv (t - L + r, r\mathbf{n}) ,\ x' \equiv (t + L - r, r\mathbf{n}) . \end{aligned} \tag{7.29}$$

We have chosen the start time to be at $t - L$ such that the time coordinate of x and x' are symmetric about t.

Since ϕ satisfies the massless free scalar wave equation with the sound speed $c_s = \frac{1}{\sqrt{3}}$ [i.e., Eqs. (7.25) and (7.26)], the quantization for $\phi(x)$ should be

$$\phi(x) = l_p \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\mathbf{p})}} \left(a_{\mathbf{p}} e^{ip\cdot x} + a_{\mathbf{p}}^\dagger e^{-ip\cdot x} \right) , \tag{7.30}$$

where l_p is to make $\bar{\phi}(x)$ canonically normalized, as discussed in Eq. (7.27). Creation and annihilation operators $a_{\mathbf{p}}$, $a_{\mathbf{p}}^\dagger$ satisfy the commutation relation in Eq. (7.8) with a thermal density matrix ρ_{pix} defined in Eqs. (7.16) and (7.21).

Let us define $\delta T(t, \mathbf{n})$ to be the correction to the total time delay $T(t, \mathbf{n})$. We write the auto-correlation of $\delta T(t, \mathbf{n})$ as

$$\begin{aligned} C(\Delta t, \theta) &\equiv \left\langle \frac{\delta T(t_1, \mathbf{n}_1)\delta T(t_2, \mathbf{n}_2)}{4L^2} \right\rangle , \\ \Delta t &\equiv t_1 - t_2 \, , \quad \theta = \cos^{-1}(\mathbf{n}_1 \cdot \mathbf{n}_2) \, , \end{aligned} \tag{7.31}$$

and using Eq. (7.29), we obtain

$$C(\Delta t, \theta) = \frac{1}{16L^2} \int_0^L dr_1 \int_0^L dr_2 \langle \left(\phi(x_1) + \phi(x_1')\right) \left(\phi(x_2) + \phi(x_2')\right) \rangle \, , \tag{7.32}$$

where $\langle O \rangle$ is a shorthand notation for

$$\langle O \rangle = \text{Tr}(\rho_{\text{pix}} O) \, . \tag{7.33}$$

We have assumed that $C(\Delta t, \theta)$ only depends on Δt, the difference of the time when the two beams are sent, and θ, the angular separation of two arms. We will see that this assumption is true.

Besides the correlation function in Eq. (7.31), a more physical correlation function is to first subtract the time delay of the first arm $T(t, \mathbf{n}_1)$ from the time delay of the second arm $T(t, \mathbf{n}_2)$, where two beams are sent at the same time t, and then correlate this difference of time delay at different beam-sent time:

$$\begin{aligned} C_{\mathcal{T}}(\Delta t, \theta) &\equiv \left\langle \frac{\mathcal{T}(t_1, \theta)\mathcal{T}(t_2, \theta)}{4L^2} \right\rangle , \\ \mathcal{T}(t, \theta) &\equiv T(t, \mathbf{n}_2) - T(t, \mathbf{n}_1) = \delta T(t, \mathbf{n}_2) - \delta T(t, \mathbf{n}_1) \, , \end{aligned} \tag{7.34}$$

such that

$$C_{\mathcal{T}}(\Delta t, \theta) = 2 \left[C(\Delta t, 0) - C(\Delta t, \theta) \right] \, . \tag{7.35}$$

Here, we treat the first arm as the reference beam and the second arm as the signal beam. Since the relation between $C(\Delta t, \theta)$ and $C_{\mathcal{T}}(\Delta t, \theta)$ is directly given by Eq. (7.35), we will focus on $C(\Delta t, \theta)$ in our calculations below. To compute $C(\Delta t, \theta)$ in Eq. (7.32), we need to first compute the correlation function of ϕ. Using Eq. (7.30), we obtain

$$\phi(x) + \phi(x') = l_p \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\mathbf{p})}} 2\cos\left[\omega(L-r)\right] \left(a_{\mathbf{p}} e^{-i\omega t + i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{i\omega t - i\mathbf{p}\cdot\mathbf{x}} \right) . \tag{7.36}$$

Then we have

$$
\begin{aligned}
&\langle\left(\phi(x_1)+\phi(x_1')\right)\left(\phi(x_2)+\phi(x_2')\right)\rangle \\
&=4l_p^2\int\frac{d^3\mathbf{p}_1}{(2\pi)^3}\int\frac{d^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{\sqrt{4\omega_1(\mathbf{p}_1)\omega_2(\mathbf{p}_2)}} \\
&\quad\cos\left[\omega_1(L-r_1)\right]\cos\left[\omega_2(L-r_2)\right]\left[\langle a_{\mathbf{p}_1}a_{\mathbf{p}_2}^\dagger\rangle e^{-i(\omega_1t_1-\omega_2t_2-\mathbf{p}_1\cdot\mathbf{x}_1+\mathbf{p}_2\cdot\mathbf{x}_2)}+c.c.\right],
\end{aligned}
\tag{7.37}
$$

where we have only kept the term proportional to $a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}$ and $a_{\mathbf{p}_1}a_{\mathbf{p}_2}^\dagger$ since the other terms are zero.

To evaluate Eq. (7.37), we need to calculate $\langle a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}\rangle$ and $\langle a_{\mathbf{p}_1}a_{\mathbf{p}_2}^\dagger\rangle$. The former is given directly by Eq. (7.9), $\langle a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}\rangle = \mathrm{Tr}\,(\rho_{\mathrm{pix}}a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}) = (2\pi)^3\sigma_{\mathrm{pix}}(\mathbf{p}_1)\delta^{(3)}(\mathbf{p}_1-\mathbf{p}_2)$. Using both Eq. (7.9) and the commutation relation in Eq. (7.8), we find the latter to be

$$
\begin{aligned}
\langle a_{\mathbf{p}_1}a_{\mathbf{p}_2}^\dagger\rangle &= (2\pi)^3\left[1+\sigma_{\mathrm{pix}}(\mathbf{p}_1)\right]\delta^{(3)}(\mathbf{p}_1-\mathbf{p}_2) \\
&\approx (2\pi)^3\sigma_{\mathrm{pix}}(\mathbf{p}_1)\delta^{(3)}(\mathbf{p}_1-\mathbf{p}_2)\,,
\end{aligned}
\tag{7.38}
$$

where we have used $\sigma_{\mathrm{pix}}(\mathbf{p})\gg 1$ at the last line. Then,

$$
\begin{aligned}
&\langle\left(\phi(x_1)+\phi(x_1')\right)\left(\phi(x_2)+\phi(x_2')\right)\rangle \\
&=4l_p^2\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\sigma_{\mathrm{pix}}(\mathbf{p})}{2\omega(\mathbf{p})}\cos\left[\omega(L-r_1)\right]\cos\left[\omega(L-r_2)\right]\left[e^{-i\omega\Delta t+i\mathbf{p}\cdot\Delta\mathbf{x}}+c.c.\right],
\end{aligned}
\tag{7.39}
$$

where we have defined $\Delta\mathbf{x}\equiv\mathbf{x}_1-\mathbf{x}_2$. Notice that Eq. (7.37) is a complex function in general, so we usually need to symmetrize it over $\mathbf{x}_{1,2}$. Due to our approximation in Eq. (7.38), Eq. (7.39) is a real function, so the one after symmetrization over $\mathbf{x}_{1,2}$ is the same as Eq. (7.39). For simplicity, we will drop the term $c.c.$ and always assume that a complex conjugate is taken.

Finally, plugging Eq. (7.39) into Eq. (7.32), we obtain

$$
\begin{aligned}
C(\Delta t,\theta) = \frac{l_p^2}{4L^2}\int_0^L dr_1\int_0^L dr_2\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\sigma_{\mathrm{pix}}(\mathbf{p})}{2\omega(\mathbf{p})} \\
\cos\left[\omega(L-r_1)\right]\cos\left[\omega(L-r_2)\right]e^{-i\omega\Delta t+i\mathbf{p}\cdot\Delta\mathbf{x}}\,.
\end{aligned}
\tag{7.40}
$$

This is our main result, and we will work on applying it to existing interferometer configurations next.

7.4 Observational Signatures and Constraints

After plugging $\sigma_{\rm pix}(\mathbf{p})$ in Eq. (7.21), Eq. (7.40) is reduced to

$$C(\Delta t,\theta)=\frac{al_p}{8L^2}\int_0^L dr_1\int_0^L dr_2\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\omega^2(\mathbf{p})}$$
$$\cos\left[\omega(L-r_1)\right]\cos\left[\omega(L-r_2)\right]e^{-i\omega\Delta t+i\mathbf{p}\cdot\Delta\mathbf{x}}\,. \tag{7.41}$$

In the next two subsections, we will study the power spectral density and angular correlation of Eq. (7.41) in more detail.

7.4.1 Power spectral density

We first study the power spectral density implied by Eq. (7.41). Carrying out the angular part of the momentum integral in Eq. (7.41), we have

$$\begin{aligned}C(\Delta t,\theta)&=\frac{al_p}{32\pi^2c_s^2L^2}\int_0^L dr_1\int_0^L dr_2\int_0^\infty d|\mathbf{p}|\\&\quad\cos\left[\omega(L-r_1)\right]\cos\left[\omega(L-r_2)\right]e^{-i\omega\Delta t}\int_0^\pi d\vartheta\ \sin\vartheta e^{i|\mathbf{p}||\Delta\mathbf{x}|\cos\vartheta}\\&=\frac{al_p}{16\pi^2c_s^2L^2}\int_0^L dr_1\int_0^L dr_2\int_0^\infty d\omega\\&\quad\cos\left[\omega(L-r_1)\right]\cos\left[\omega(L-r_2)\right]\frac{\sin\left[\omega\mathcal{D}(r_1,r_2,\theta)/c_s\right]}{\omega\mathcal{D}(r_1,r_2,\theta)}e^{-i\omega\Delta t}\,,\end{aligned} \tag{7.42}$$

where we have defined

$$\mathcal{D}(r_1,r_2,\theta)\equiv|\Delta\mathbf{x}|=\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}\,. \tag{7.43}$$

The additional factor of $\frac{1}{c_s^2}$ in Eq. (7.42) comes from using the dispersion relation in Eq. (7.26). $C_{\mathcal{T}}(\Delta t,\theta)$ is directly given by plugging Eq. (7.42) into Eq. (7.35). One thing to notice here is that $C(0,0)$ has a log divergence when integrating ω to infinity. A similar log divergence also shows up in $C(0,0)$ of [1] [see Eq. (7.57)] if we sum all the (ℓ,m) modes without a cutoff in ℓ. Nonetheless, the log divergence in both cases can be regulated by noticing that there is a natural UV cutoff $\ell_{\rm max}$ in the number of observable ℓ modes of $C_{\mathcal{T}}(0,\theta)$, beyond which light is diffracted significantly and thus cannot probe these $\ell>\ell_{\rm max}$ fluctuations. The UV cutoff $\ell_{\rm max}$ can also be translated to a UV cutoff $\omega_{\rm max}$ of frequency, which is usually much higher than the experimental cutoff $\omega_{\rm max}\lesssim 10\,\text{rad GHz}$ of the photodetector [2]. Thus, when showing $C_{\mathcal{T}}(\Delta t,\theta)$ for $\Delta t=0$ in Fig. 7.2, we have imposed a UV cutoff $\omega_{\rm max}=10\,\text{rad GHz}$. Since $\omega_{\rm max}$ enters into Eq. (7.42) via the combination $\omega_{\rm max}L$,

[2]This experimental cutoff is gotten from private communication with Lee McCuller.

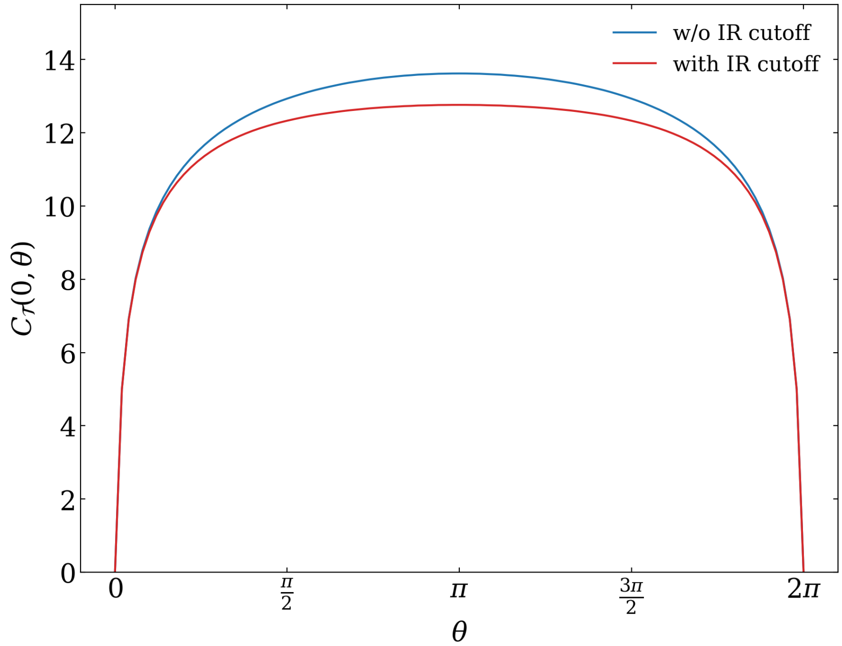

Figure 7.2: Equal-time correlation function $C_{\mathcal{T}}(0,\theta)$ [i.e., Eq. (7.34)] of the pixellon model without IR cutoff in Eq. (7.41) (blue) and with an IR cutoff in Eq. (7.60) (red), where both curves are normalized by $\frac{8\pi^2 c_s^2 L}{a l_p}$. An UV cutoff $\omega_{\max} = 10\,\mathrm{rad\,GHz}$ and arm length $L = 5\,\mathrm{m}$ is used as demonstration.

we have used an arm length $L = 5\,\mathrm{m}$ as demonstration. Notice the signal is maximal when the interferometer arms are back-to-back.

Performing a Fourier transform of $C(\Delta t, \theta)$ with respect to Δt, we obtain the two-sided power spectral density $\tilde{C}(\omega, \theta)$ to be

$$\begin{aligned}\tilde{C}(\omega,\theta) &= \int_{-\infty}^{\infty} dt\; e^{-i\omega t} C(t,\theta) \\ &= \frac{a l_p}{8\pi c_s^2 L^2} \int_0^L dr_1 \int_0^L dr_2 \frac{\sin\left[\omega \mathcal{D}(r_1, r_2, \theta)/c_s\right]}{\omega \mathcal{D}(r_1, r_2, \theta)} \\ &\quad \cos\left[\omega(L - r_1)\right] \cos\left[\omega(L - r_2)\right].\end{aligned} \tag{7.44}$$

To evaluate the power spectral density of $C_{\mathcal{T}}(\Delta t, \theta)$, we can put Eq. (7.44) into Eq. (7.35) such that its power spectral density $\tilde{C}_{\mathcal{T}}(\omega, \theta)$ is

$$\tilde{C}_{\mathcal{T}}(\omega,\theta) = 2[\tilde{C}(\omega,0) - \tilde{C}(\omega,\theta)]. \tag{7.45}$$

In Fig. 7.3, we have plotted Eq. (7.45) over ωL for several different separation angles θ of the interferometer.

In the limit $\omega \to 0$, Eqs. (7.44) and (7.45) reduce to

$$\tilde{C}(\omega, \theta) = \frac{al_p}{8\pi c_s^3} + O(\omega^2 L^2)\,, \tag{7.46}$$

$$\tilde{C}_{\mathcal{T}}(\omega, \theta) = \frac{al_p}{48\pi c_s^5}\omega^2 L^2 (1 - \cos\theta) + O(\omega^4 L^4)\,. \tag{7.47}$$

A major feature of $\tilde{C}(\omega, \theta)$ at low frequencies is that it is flat in frequency, corresponding to the spectrum of white noise. This feature is consistent with the "random walk intuition" of holographic effects in [7], as well as the random walk models in [23, 24]. On the other hand, although $\tilde{C}(\omega, \theta)$ is independent of ω at low frequency, $\tilde{C}_{\mathcal{T}}(\omega, \theta)$ is quadratic in ω. It is because, as one can directly observe from Eq. (7.46), the leading order term of $\tilde{C}(\omega, \theta)$ at low frequency is angle-independent. Thus, when subtracting the time delay of the first arm from the second arm, this leading order term cancels out, and the next order term, which is quadratic in ω and has a nontrivial angular dependence, contributes to $\tilde{C}_{\mathcal{T}}(\omega, \theta)$.

In Eqs. (7.46)-(7.47), there are also additional factors of $\frac{1}{c_s}$ from the expansion of $\sin\left[\omega\mathcal{D}(r_1, r_2, \theta)/c_s\right]$ in Eq. (7.44). Since the leading order term in the expansion of $\sin\left[\omega\mathcal{D}(r_1, r_2, \theta)/c_s\right]$ is linear in its argument, it contributes an additional factor of $\frac{1}{c_s}$ to $\tilde{C}(\omega, \theta)$ in Eq. (7.46). On the other hand, as we explained above, this leading order term is angle-independent, so the next order term, which is cubic in its argument, contributes an additional factor of $\frac{1}{c_s^3}$ to $\tilde{C}_{\mathcal{T}}(\omega, \theta)$ in Eq. (7.47).

One last observation from Eqs. (7.46)-(7.47) is that both $\tilde{C}(\omega, \theta)$ and $\tilde{C}_{\mathcal{T}}(\omega, \theta)$ are regular in low frequency. In [1], an IR regulator at the scale of $\sim \frac{1}{L^2}$ was added to the 2D Laplacian on the sphere to regulate the angular correlation function as we will discuss in Sec. 7.4.2. To perform an analogous calculation and take into account other IR effects, such as information loss due to soft graviton loss, we will apply the procedures in this section to the pixellon model with an IR cutoff at the same scale as in [1] in Sec. 7.4.3.

7.4.2 Angular correlation

We now study the angular correlation implied by Eq. (7.41). It will be convenient to first decompose Eq. (7.41) into spherical harmonics and spherical Bessel functions. Using

$$e^{i\mathbf{p}\cdot\mathbf{r}} = \sum_{\ell=0}^{\infty} i^l (2\ell + 1) j_\ell(|\mathbf{p}|r) P_\ell(\cos\theta)\,, \quad \theta = \hat{\mathbf{p}} \cdot \hat{\mathbf{r}}\,, \tag{7.48}$$

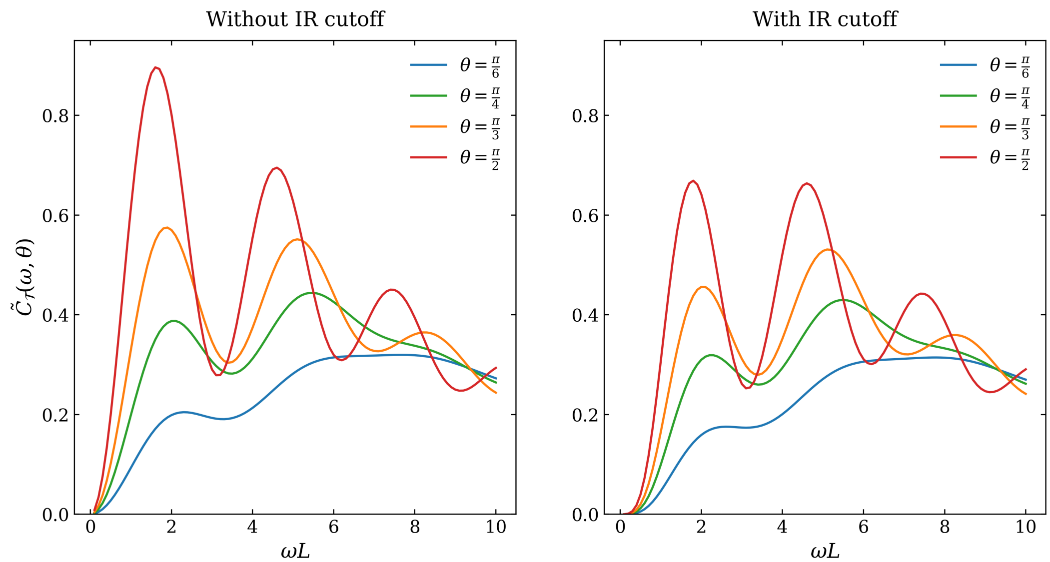

Figure 7.3: Power spectral density $\tilde{C}_{\mathcal{T}}(\omega, \theta)$ [i.e., Eq. (7.45)] of the pixellon model without IR cutoff in Eq. (7.44) (left) and with an IR cutoff in Eq. (7.61) (right), where all the curves are normalized by $\frac{8\pi c_s^2}{a l_p}$.

and the addition theorem

$$P_\ell(\hat{\mathbf{p}} \cdot \mathbf{n}) = \frac{4\pi}{2\ell+1} \sum_m Y^{\ell m*}(\hat{\mathbf{p}}) Y^{\ell m}(\mathbf{n}) \,, \tag{7.49}$$

we obtain

$$\begin{aligned} e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)} = &\sum_{\ell_1,m_1,\ell_2,m_2} 16\pi^2 i^{\ell_1}(-i)^{\ell_2} j_{\ell_1}(|\mathbf{p}|r_1) j_{\ell_2}(|\mathbf{p}|r_2) \\ &Y^{\ell_1 m_1*}(\hat{\mathbf{p}}) Y^{\ell_2 m_2}(\hat{\mathbf{p}}) Y^{\ell_1 m_1}(\mathbf{n}_1) Y^{\ell_2 m_2*}(\mathbf{n}_2) \,. \end{aligned} \tag{7.50}$$

Using $\int d\Omega\ Y^{\ell_1 m_1*}(\hat{\mathbf{p}}) Y^{\ell_2 m_2}(\hat{\mathbf{p}}) = \delta^{\ell_1\ell_2}\delta^{m_1 m_2}$, we can integrate out all the angular dependence of $\mathbf{p}$, so

$$\begin{aligned} C(\Delta t, \theta) = \frac{a l_p}{4\pi c_s^3 L^2} \sum_{\ell,m} \int_0^L dr_1 \int_0^L dr_2 \int_0^\infty d\omega\ \cos\left[\omega(L-r_1)\right] \cos\left[\omega(L-r_2)\right] \\ j_\ell(\omega r_1/c_s) j_\ell(\omega r_2/c_s) Y^{\ell m}(\vartheta_1, \varphi_1) Y^{\ell m*}(\vartheta_2, \varphi_2) e^{-i\omega\Delta t} \,, \end{aligned} \tag{7.51}$$

where we have an additional factor of $\frac{1}{c_s^3}$ from replacing $\mathbf{p}$ with ω using Eq. (7.26). If we define the amplitude of each (ℓ, m) mode of the integrand to be

$$A_{\ell m}(\Delta t, \omega, r_1, r_2) \equiv \cos\left[\omega(L-r_1)\right] \cos\left[\omega(L-r_2)\right] j_\ell(\omega r_1/c_s) j_\ell(\omega r_2/c_s) e^{-i\omega\Delta t} \,, \tag{7.52}$$

Eq. (7.51) can be more compactly written as

$$\begin{aligned} C(\Delta t,\theta) = \frac{a l_p}{4\pi c_s^3 L^2} \sum_{\ell,m} \int_0^L dr_1 \int_0^L dr_2 \int_0^\infty d\omega \\ A_{\ell m}(\Delta t,\omega,r_1,r_2) Y^{\ell m}(\vartheta_1,\varphi_1) Y^{\ell m *}(\vartheta_2,\varphi_2)\,. \end{aligned} \tag{7.53}$$

Let us first look at the equal-time correlator by setting $\Delta t = 0$. The amplitude $c_{\ell m}$ of each (ℓ,m) mode of $C(0,\theta)$ is then given by integrating $A_{\ell m}(0,\omega,r_1,r_2)$ over ω and $r_{1,2}$ as indicated by Eq. (7.53), i.e.,

$$c_{\ell m} = \frac{a l_p}{4\pi c_s^3 L^2} \int_0^L dr_1 \int_0^L dr_2 \int_0^\infty d\omega\; A_{\ell m}(0,\omega,r_1,r_2)\,. \tag{7.54}$$

Since these integrals are hard to evaluate analytically, we have plotted the numerical result in Fig. 7.4. In Fig. 7.4, we have only plotted the modes starting from $\ell = 1$ since the $\ell = 0$ mode, which is angle-independent, is cancelled out in $C_{\mathcal{T}}(\Delta t,\theta)$ as explained in the previous section.

In Fig. 7.4, we have also shown the amplitude of each (ℓ,m) mode found in Ref. [1]. They argued that the angular part of $C(0,\theta)$ should be described by the Green's function $\mathbf{G}(\tilde{\mathbf{r}}_1,\tilde{\mathbf{r}}_2)$ of the 2D Laplacian on the sphere with an additional IR regulator at the scale of $\frac{1}{L^2}$, i.e.,

$$\begin{aligned} &(-\nabla^2_{\tilde{\mathbf{r}}_1} + 1/L^2)\mathbf{G}(\tilde{\mathbf{r}}_1,\tilde{\mathbf{r}}_2) = \delta^{(2)}(\tilde{\mathbf{r}}_1,\tilde{\mathbf{r}}_2)\,, \\ &\delta^{(2)}(\tilde{\mathbf{r}}_1,\tilde{\mathbf{r}}_2) = \frac{1}{L^2}\delta(\cos\theta_1 - \cos\theta_2)\delta(\phi_1 - \phi_2)\,, \end{aligned} \tag{7.55}$$

where $\tilde{\mathbf{r}}_i$ are coordinates on the sphere of radius L. $\mathbf{G}(\tilde{\mathbf{r}}_1,\tilde{\mathbf{r}}_2)$ is scale invariant if we extract the overall factor of $\frac{1}{L^2}$ by defining $\hat{\mathbf{r}}_i = \frac{\tilde{\mathbf{r}}_i}{L}$, so

$$(-\nabla^2_{\hat{\mathbf{r}}_1} + 1)\mathbf{G}(\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2) = \delta^{(2)}(\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2)\,. \tag{7.56}$$

After decomposing $\mathbf{G}(\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2)$ into spherical harmonics, one obtains

$$C(0,\theta) \propto \sum_{\ell,m} \frac{Y^{\ell m}(\vartheta_1,\varphi_1) Y^{\ell m *}(\vartheta_2,\varphi_2)}{\ell(\ell+1)+1}\,. \tag{7.57}$$

Excellent agreement between the pixellon model and the expectation of Ref. [1] is observed.

As mentioned in Sec. 7.4.1, both $\tilde{C}(\omega,\theta)$ and $\tilde{C}_{\mathcal{T}}(\omega,\theta)$ in this work are regular when $\omega \to 0$, even without an IR regulator, e.g., Eqs. (7.46)-(7.47). However, it will still be interesting to study the pixellon model with an IR cutoff due to IR effects from the physical size of the interferometer. We will consider the case with an IR cutoff in Sec. 7.4.3, but in this section, we first consider only the model without an IR cutoff. Thus, when comparing Eq. (7.54) to Ref. [1], we drop the additional 1 in the denominator of Eq. (7.57), which appears due to the insertion of an IR regulator. In this case, the amplitude of each (ℓ, m) mode becomes $\frac{1}{\ell(\ell+1)}$. In Fig. 7.4, one can observe that the angular correlation in this work is very close to the one in [1] without the IR regulator. Note that one also observes the same angular dependence in the shockwave geometry (e.g., see Refs. [6, 10, 25, 26]), a connection we would like to study further in our future work.

One might also be interested in the amplitude $\tilde{c}_{\ell m}(\omega)$ of each (ℓ, m) mode of the power spectral density $\tilde{C}(\omega, \theta)$. Performing a Fourier transform of $C(\Delta t, \theta)$ in Eq. (7.53) and thus a Fourier transform of $A_{\ell m}(\Delta t, \omega, r_1, r_2)$ in Eq. (7.52), we obtain

$$\tilde{c}_{\ell m}(\omega) = \frac{a l_p}{2 c_s^3 L^2} \int_0^L dr_1 \int_0^L dr_2 \; A_{\ell m}(0, \omega, r_1, r_2) \,. \tag{7.58}$$

We have plotted $\tilde{c}_{\ell m}(\omega)$ starting from $\ell = 1$ in Fig. 7.5. A normalization factor of $\ell(\ell+1)$ is multiplied to each curve such that each curve represents the relative power spectra density with respect to the total amplitude $c_{\ell m}$.

To determine an analytical representation of the amplitude of each (ℓ, m) mode, one can also look at $A_{\ell m}(0, \omega, r_1, r_2)$ at the end points $r_1 = r_2 = L$. If we integrate $A_{\ell m}(0, \omega, L, L)$ over ω, we find the amplitude of each (ℓ, m) mode at end points to be

$$L \int_0^\infty d\omega \; A_{\ell m}(0, \omega, L, L) = \frac{\pi c_s}{2(2\ell + 1)} \,, \tag{7.59}$$

which is the major contribution to $c_{\ell m}$ plotted in Fig. 7.4. Although Eq. (7.59) decreases more slowly than Eq. (7.57) over ℓ, we have additional suppression due to, for example, the factors of $\cos\left[\omega(L - r_{1,2})\right]$ in Eq. (7.52) when integrating $A_{\ell m}(0, \omega, r_1, r_2)$ over ω and $r_{1,2}$, so the total amplitude in Eq. (7.54) is very close to Eq. (7.57) without the IR regulator.

7.4.3 IR cutoff

In this section, we apply the calculations in the previous two sections to the pixellon model with an IR cutoff. As discussed above, although both $\tilde{C}(\omega, \theta)$ and $\tilde{C}_{\mathcal{T}}(\omega, \theta)$ are regular in the IR, we still expect an explicit IR cut-off to enter the calculation because of the finite size of the interferometer. We will also find that adding an IR

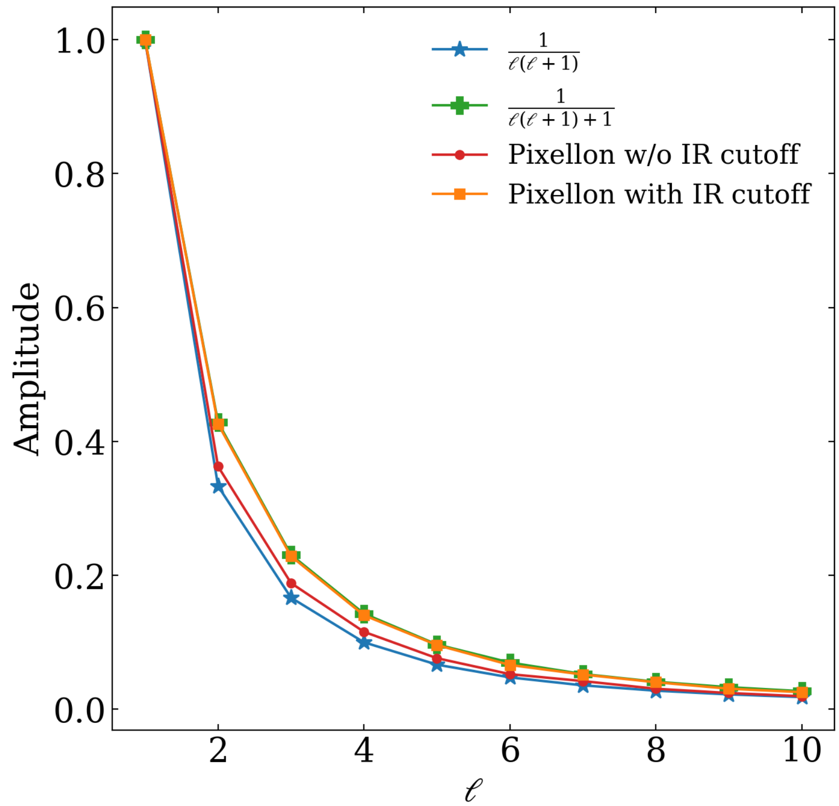

Figure 7.4: The amplitude of each (ℓ, m) mode of the equal-time correlation function $C(0, \theta)$ decomposed into spherical harmonics. The blue and green lines correspond to the amplitude in [1] [i.e., Eq. (7.57)] without and with an IR regulator, respectively. The red and orange lines correspond to $c_{\ell m}$ [i.e., Eq. (7.54)] of the pixellon model without IR cutoff in Eq. (7.52) and with an IR cutoff in Eq. (7.64), respectively. We have normalized the amplitude of each mode by the amplitude of the mode $\ell = 1$.

cut-off gives a better agreement with the angular correlation of Eq. (7.57). For this reason, we place an IR cutoff at a scale $\sim \frac{1}{L^2}$, similar to [1], into Eq. (7.41), e.g.,

$$C(\Delta t, \theta) = \frac{a l_p}{8L^2} \int_0^L dr_1 \int_0^L dr_2 \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\omega^2(\mathbf{p}) + \frac{1}{L^2}} \cos\left[\omega(L - r_1)\right] \cos\left[\omega(L - r_2)\right] e^{-i\omega\Delta t + i\mathbf{p}\cdot\Delta\mathbf{x}} . \tag{7.60}$$

Following the same procedure in Sec. 7.4.1, we find that the power spectral density $\tilde{C}(\omega, \theta)$ in Eq. (7.44) is modulated by an additional factor in ω and L, i.e.,

$$\tilde{C}(\omega, \theta) \to \left(\frac{\omega^2}{\omega^2 + \frac{1}{L^2}} \right) \tilde{C}(\omega, \theta) , \tag{7.61}$$

while $\tilde{C}_{\mathcal{T}}(\omega, \theta)$ is still given by Eq. (7.45). $C_{\mathcal{T}}(0, \theta)$ and $\tilde{C}_{\mathcal{T}}(\omega, \theta)$ with this IR cutoff are shown in Figs. 7.2 and 7.3, respectively.

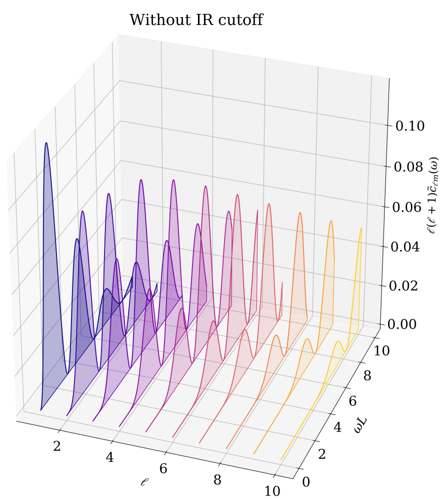

Figure 7.5: The amplitude $\tilde{c}_{\ell m}(\omega)$ [i.e., Eq. (7.58)] of each (ℓ, m) mode of the power spectral density $\tilde{C}(\omega, \theta)$ decomposed into spherical harmonics. The left and right panels are for the pixellon model without IR cutoff in Eq. (7.52) and with an IR cutoff in Eq. (7.64), respectively. We have dropped the overall factor $\frac{al_p}{2c_s^3}$ in both plots and normalized each curve by $\ell(\ell+1)$.

One major effect of the IR cutoff is that the amplitude of $\tilde{C}(\omega, \theta)$ is suppressed at low frequency due to the modulation factor in Eq. (7.61), as one can directly observe in Fig. 7.3. For the same reason, the overall amplitude of $C_{\mathcal{T}}(\Delta t, \theta)$ in the case with an IR cutoff is smaller than the one without IR cutoff as depicted in Fig. 7.2. As frequency increases, the modulation factor goes to 1, so the amplitude of $\tilde{C}(\omega, \theta)$ in these two cases becomes nearly identical. In addition, as the separation angle θ decreases, the difference between these two cases also becomes smaller since interferometers with smaller θ are more sensitive to higher ℓ modes, which have higher characteristic frequency, and thus are less sensitive to the IR cutoff.

One can also determine the suppression factor due to the IR cutoff as $\omega \to 0$ by expanding Eq. (7.61), e.g.,

$$\tilde{C}(\omega, \theta) = \frac{al_p}{8\pi c_s^3}\omega^2 L^2 + O(\omega^4 L^4)\,, \tag{7.62}$$

$$\tilde{C}_{\mathcal{T}}(\omega, \theta) = \frac{al_p}{48\pi c_s^5}\omega^4 L^4 (1 - \cos\theta) + O(\omega^6 L^6)\,. \tag{7.63}$$

The IR behaviors of both $\tilde{C}(\omega, \theta)$ and $\tilde{C}_{\mathcal{T}}(\omega, \theta)$ above are very different from the case without an IR cutoff in Eq. (7.46)-(7.47) due to the additional factor of $\omega^2 L^2$

contributed by the modulation factor in Eq. (7.61). For this reason, one has to be cautious when constraining our model using detectors with peak sensitivity at low frequency, such as LIGO, as discussed in Sec. 7.4.4.

For the angular correlation, after decomposing Eq. (7.60) into spherical harmonics, we find that the amplitudes $c_{\ell m}$ and $\tilde{c}_{\ell m}(\omega)$ of each (ℓ, m) mode of $C(0,\theta)$ and $\tilde{C}(\omega,\theta)$ are given by Eqs. (7.54) and (7.58), respectively, but $A_{\ell m}(\Delta t, \omega, r_1, r_2)$ is modulated by the same factor in Eq. (7.61), i.e.,

$$A_{\ell m}(\Delta t, \omega, r_1, r_2) \to \left(\frac{\omega^2}{\omega^2 + \frac{1}{L^2}}\right) A_{\ell m}(\Delta t, \omega, r_1, r_2)\,. \tag{7.64}$$

We show both $c_{\ell m}$ and $\tilde{c}_{\ell m}(\omega)$ with the IR cutoff in Figs. 7.4 and 7.5, respectively.

Since the overall amplitude of $\tilde{C}(\omega,\theta)$ is suppressed at low frequency, the amplitude $\tilde{c}_{\ell m}(\omega)$ of different (ℓ, m) modes is also suppressed as shown in Fig. 7.5. In Fig. 7.4, one can also observe that the amplitude $c_{\ell m}$ falls off more slowly with ℓ in the case with an IR cutoff since low ℓ modes are more sensitive to this IR cutoff and hence are more suppressed. As noted previously, our model with the IR cutoff better agrees with the results in [1], though one should remain cautious until our model has been fully mapped to a UV-complete theory.

7.4.4 Existing constraints and future projections

In an effort to detect high frequency gravitational waves and quantum gravity signatures, several laboratory-sized interferometer experiments have been implemented to accurately detect tiny spacetime perturbations. The constraints from these experiments are often reported as upper limits on the one-sided noise strain $\sqrt{S_h(f)}$ of the photon round-trip time, obtained by analysing interference patterns. For stationary signals, the strain is defined as [27, 28]

$$\sqrt{S_h^{(n)}(f)} = \sqrt{2\int_{-\infty}^{\infty}\left\langle \frac{\Delta L(\tau)}{L}\frac{\Delta L(0)}{L}\right\rangle e^{-2\pi i f\tau}d\tau}\,, \tag{7.65}$$

which has units of $\mathrm{Hz}^{-1/2}$. This is related to Eq. (7.44) by Eq. (7.45), i.e.,

$$\sqrt{S_h(f)} = \sqrt{2\tilde{C}_{\mathcal{T}}(\omega,\theta)}\,, \tag{7.66}$$

where $\omega = 2\pi f$ and θ is the angle between the two interferometer arms, which is taken to be $\pi/2$ for Holometer, GEO-600 and LIGO, and $\pi/3$ for LISA to account for its triangular configuration. Here we only focus on two of the three arms of LISA as a demonstration. Our power spectrum in Eq. (7.44) can be parameterized more conventionally by defining

$$\alpha \equiv \frac{2\pi}{c_s^2} a\,, \tag{7.67}$$

leading to the peak strain $\sqrt{S_h(f_{\rm peak})} \approx \sqrt{2\alpha l_p}/(4\pi) = \sqrt{\alpha}(2.62 \times 10^{-23})\ \mathrm{Hz}^{-1/2}$ [3]. Here $\alpha \sim 1$ gives the amplitude of the effect computed in [1, 2], and should be considered the natural benchmark [4].

We now compare our predicted strain to the experimental constraints from Holometer [27], GEO-600 [29], LIGO [30], and the projected sensitivity from LISA [31]. Since the four interferometers have different arm lengths, the predicted strain from our models will also differ between these experiments. The result assuming $\alpha = 1$ with or without the IR cutoff using Eqs. (7.44), (7.45), (7.61), (7.66), and (7.67) is plotted in Fig. 7.6. Due to the better peak sensitivity of our predicted strain (i.e., at $\omega L \sim 1$ as shown in Fig. 7.3), the tightest experimental limit comes from LIGO and Holometer measurements, which at 3σ significance, are roughly $\alpha \lesssim 3$ and $\alpha \lesssim 0.7$ (with IR cutoff), and $\alpha \lesssim 0.1$ and $\alpha \lesssim 0.6$ (w/o IR cutoff), respectively. On the other hand, our model is out of reach for GEO-600 and LISA.

Caltech and Fermilab are commissioning a joint theoretical and experimental initiative called Gravity from Quantum Entanglement of Space-Time (GQuEST) [32, 33], dedicated to probing the VZ effect proposed in Ref. [1]. This includes the construction of a tabletop optical Michelson interferometer with arm-length $L = 5$ m, with a novel read-out scheme with single photons rather than the usual interference effect. The advantage of this scheme is that sensitivity beats the standard quantum limit, with signal-to-noise ratio increasing linearly with integration time, rather than the usual square-root dependence. The experiment is projected to be able to constrain $\alpha \lesssim 1$ after 1000 s of background-free integration time, corresponding to a dark count rate of 10^{-3} Hz. We expect the constraint on α to tighten *linearly* with lower dark count rate and longer integration time.

Some previous works on quantifying spacetime fluctuations (motivated by theories other than the VZ effect) argued that the predicted strain should not be directly compared against experimental constraints such as GEO-600 and LIGO [34], since transitional interferometer experiments often utilize Fabry-Perot cavities (e.g., LIGO

[3] This is related to the one-sided displacement spectrum by $S_L(f) = 2L^2\tilde{C}(f)$, which is peaked at $S_L(f_{\rm peak}) = \alpha l_p L^2/(8\pi^2)$.

[4] Since $\alpha = 1$ corresponds to $a = c_s^2/(2\pi)$, the finite propagation speed c_s has led us to make a corrected prescription of $\beta = l_p/a = 2\pi l_p/c_s^2$ in Eqs. (7.20) and (7.21).

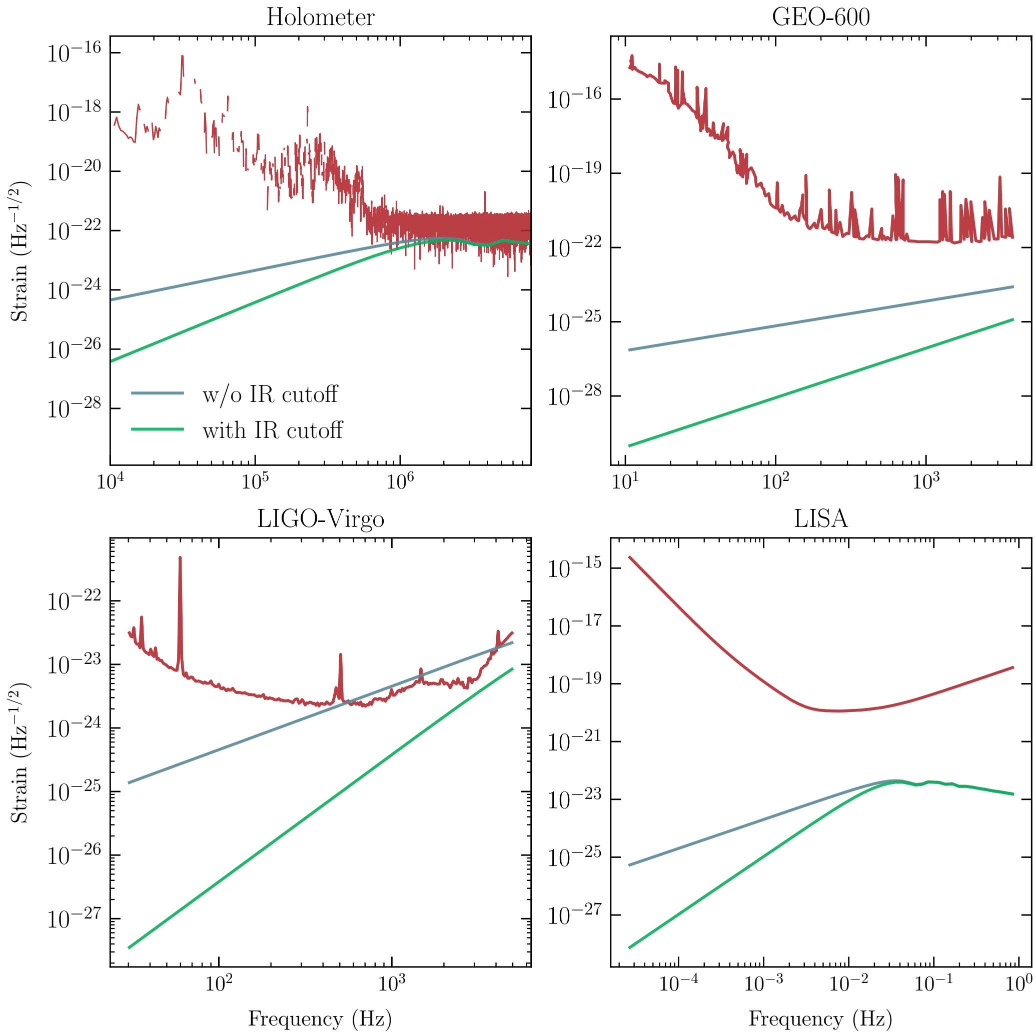

Figure 7.6: Strain comparison between model predictions (blue and green) and experimental / projection constraints (red). The model curves are computed using Eqs. (7.44), (7.45), (7.61), (7.66) and (7.67) assuming $\alpha = 1$, while the experimental curves are extracted from Refs. [27, 29–31]. The LIGO data shown here are obtained by the Livingston detector, but we note that the Hanford detector yields similar constraints.

uses Fabry-Perot cavities within each arm, where the average light storage equals to 35.6 light round trips [35]) to boost the signal-to-noise ratio from astrophysical gravitational waves, while it is unclear whether quantum gravity signals, which are fundamental to spacetime itself, will benefit from additional light-crossings. In Appendix 7.8, we show that spacetime fluctuations based on Eq. (7.2) do accumulate over a Fabry-Perot cavity, thus justifying our direct strain comparison with gravitational experiments.

7.5 Conclusions

In this paper we have investigated fluctuations in the time-of-arrival of a photon in an interferometer, due a scalar field coupled to the metric as in Eq. (7.2) with an occupation number given by Eq. (7.21). This simple scalar field is designed to model the behavior of vacuum fluctuations of the modular energy (e.g., Ref. [2]) from shockwave geometries [6].

We showed that the interferometer observable had a power spectral density quadratically suppressed $\propto \omega^2$ or $\propto \omega^4$, depending on the IR regulator, at low frequency, and an angular correlation between the interferometer arms consistent with that proposed in Ref. [1], as expected from shockwave geometries.

In future work, we plan to more explicitly demonstrate the connection between shockwave geometries and interferometer observables, completing the bridge between the model presented here and the UV-complete theory.

7.6 Appendix: Time Delay in General Metric

In this appendix, we derive the time delay of a generic metric in Eq. (7.28). There are three effects, from the clock rate, the mirror motion, and the light propagation. Only when summing all three do we obtain the gauge invariant observable.

We start by computing the clock's rate. Since $g_{tt} = -(1 - \mathcal{H}_0)$, to the leading order, the proper time differs from the coordinate time by

$$\frac{d\tau}{dt} \approx 1 - \frac{1}{2}\mathcal{H}_0 \, . \tag{7.68}$$

Thus, for a clock with radial position r when there is no metric fluctuation, the difference $\delta\tau$ between the proper time and the coordinate time from $t = t_1$ to $t = t_2$ is

$$\delta\tau(t_1, t_2, r) = -\frac{1}{2}\int_{t_1}^{t_2} dt' \; \mathcal{H}_0(t', r) \, . \tag{7.69}$$

To account for the mirror's motion, we consider the geodesic equation of the mirror

$$0 = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \approx \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{tt} + \Gamma^\mu_{ti}v^i + \cdots \, . \tag{7.70}$$

Since the velocity of the mirror $v^i \ll 1$, to the leading order, $\frac{d^2r}{dt^2} \approx -\Gamma^r_{tt}$. Using $\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta})$, we get

$$\Gamma^r_{tt} = \partial_t h_{tr} - \frac{1}{2}\partial_r h_{tt} = \partial_t \mathcal{H}_1 - \frac{1}{2}\partial_r \mathcal{H}_0 \, , \tag{7.71}$$

so for a mirror at radius r when there is no metric fluctuation, its radial position r_{M} at coordinate time t is

$$r_{\mathrm{M}}(t,r) \approx \int^{t} dt' \int^{t'} dt'' \left[\frac{1}{2}\partial_r \mathcal{H}_0(t'',r) - \partial_{t''}\mathcal{H}_1(t'',r)\right] . \tag{7.72}$$

For the light propagation, the geodesic equation of outgoing light is

$$\frac{dt^{\mathrm{out}}}{dr} \approx 1 + \frac{1}{2}\left(\mathcal{H}_0 + \mathcal{H}_2 + 2\mathcal{H}_1\right) \equiv 1 + \frac{1}{2}\mathcal{H}^{\mathrm{out}} , \tag{7.73}$$

and for ingoing light,

$$\frac{dt^{\mathrm{in}}}{dr} \approx -1 - \frac{1}{2}\left(\mathcal{H}_0 + \mathcal{H}_2 - 2\mathcal{H}_1\right) \equiv -1 - \frac{1}{2}\mathcal{H}^{\mathrm{in}} . \tag{7.74}$$

In total, the proper time T^{out} the light beam takes to reach the mirror is

$$\begin{aligned} T^{\mathrm{out}} &\approx \int_{0+r_{\mathrm{M}}(0,0)}^{L+r_{\mathrm{M}}(L,L)} dr \left[1 + \frac{1}{2}\mathcal{H}^{\mathrm{out}}(r,r)\right] + \delta\tau(0,L,0) \\ &\approx L + r_{\mathrm{M}}(L,L) - r_{\mathrm{M}}(0,0) + \delta\tau(0,L,0) + \frac{1}{2}\int_0^L dr\, \mathcal{H}^{\mathrm{out}}(r,r) . \end{aligned} \tag{7.75}$$

Similarly, for the ingoing light beam,

$$\begin{aligned} T^{\mathrm{in}} &\approx \int_{L+r_{\mathrm{M}}(L,L)}^{0+r_{\mathrm{M}}(2L,0)} dr \left[-1 - \frac{1}{2}\mathcal{H}^{\mathrm{in}}(2L-r,r)\right] + \delta\tau(L,2L,0) \\ &\approx L + r_{\mathrm{M}}(L,L) - r_{\mathrm{M}}(2L,0) + \delta\tau(L,2L,0) + \frac{1}{2}\int_0^L dr\, \mathcal{H}^{\mathrm{in}}(2L-r,r) . \end{aligned} \tag{7.76}$$

Then the total time delay T is given by summing up Eqs. (7.75) and (7.76), $T = T^{\mathrm{out}} + T^{\mathrm{in}}$.

7.7 Appendix: Gauge Invariance of Time Delay

In this appendix, we show that the total time delay $T = T^{\mathrm{out}} + T^{\mathrm{in}}$, where T^{out} and T^{in} are defined in Eqs. (7.75) and (7.76), of the light beam traveling a round trip is a gauge invariant quantity. Since the $t-r$ sector of any metric, e.g., Eq. (7.28), will only be affected by the gauge transformations of coordinate t or r, we will show that T is invariant under these two types of gauge transformations.

7.7.1 Gauge transformations of coordinate t

First, let's consider gauge transformations $x^\mu \longrightarrow x^\mu + \xi^\mu$, where only $\xi_t \neq 0$, so the metric becomes

$$ds^2 = -(1 - \mathcal{H}_0 + 2\partial_t\xi_t)dt^2 + (1+\mathcal{H}_2)dr^2 + 2(\mathcal{H}_1 - \partial_r\xi_t)dtdr + \cdots . \tag{7.77}$$

Since h_{tt} is modified, $\frac{d\tau}{dt} \longrightarrow \frac{d\tau}{dt} + \frac{1}{2}\partial_t \xi_t$, the difference between the proper time and the coordinate time becomes

$$\delta\tau(t_1, t_2, r) \longrightarrow \delta\tau(t_1, t_2, r) + \xi_t(t_2, r) - \xi_t(t_1, r)\,. \tag{7.78}$$

The geodesics equations of light beam are modified into

$$\frac{dt^{\text{out}}}{dr} \approx 1 + \frac{1}{2}\left(\mathcal{H}^{\text{out}} - 2\partial_t\xi_t - 2\partial_r\xi_t\right)\,, \tag{7.79}$$

$$\frac{dt^{\text{in}}}{dr} \approx -1 - \frac{1}{2}\left(\mathcal{H}^{\text{in}} - 2\partial_t\xi_t + 2\partial_r\xi_t\right)\,. \tag{7.80}$$

For mirror's motion, let's define

$$\delta r^{\text{out}} \equiv r_{\text{M}}(L, L) - r_{\text{M}}(0, 0)\,, \tag{7.81}$$

$$\delta r^{\text{in}} \equiv r_{\text{M}}(L, L) - r_{\text{M}}(2L, 0)\,. \tag{7.82}$$

Since $\Gamma^r_{tt} \longrightarrow \Gamma^r_{tt} - \partial_t\partial_r\xi_t + \partial_r\partial_t\xi_t = \Gamma^r_{tt}$ remains unchanged, $\delta r^{\text{out}}_{\text{M}} \longrightarrow \delta r^{\text{out}}_{\text{M}}$ and $\delta r^{\text{in}}_{\text{M}} \longrightarrow \delta r^{\text{in}}_{\text{M}}$. In total,

$$\begin{aligned} T^{\text{out}} &\longrightarrow T^{\text{out}} + \xi_t(L, 0) - \xi_t(0, 0) - \int_0^L dr\ \left(\partial_t\xi_t + \partial_r\xi_t\right)|_{t=r} \\ &= T^{\text{out}} + \xi_t(L, 0) - \xi_t(L, L)\,, \end{aligned} \tag{7.83}$$

$$\begin{aligned} T^{\text{in}} &\longrightarrow T^{\text{in}} + \xi_t(2L, 0) - \xi_t(L, 0) + \int_0^L dr\ \left(\partial_r\xi_t - \partial_t\xi_t\right)|_{t=2L-r} \\ &= T^{\text{in}} - \xi_t(L, 0) + \xi_t(L, L)\,, \end{aligned} \tag{7.84}$$

so the total time delay of a round trip $T \longrightarrow T$ under the gauge transformation of coordinate t.

7.7.2 Gauge transformations of coordinate r

Next, let's consider gauge transformations $x^\mu \longrightarrow x^\mu + \xi^\mu$ with $\xi_r \neq 0$ only. The metric then becomes

$$ds^2 = -(1 - \mathcal{H}_0)dt^2 + (1 + \mathcal{H}_2 - 2\partial_r\xi_r)dr^2 + 2(\mathcal{H}_1 - \partial_t\xi_r)dtdr + \cdots\,. \tag{7.85}$$

The relation between the proper time and the coordinate time remains unchanged. The ingoing and outgoing light's geodesics are modified to be

$$\frac{dt^{\text{out}}}{dr} \approx 1 + \frac{1}{2}\left(\mathcal{H}^{\text{out}} - 2\partial_r\xi_r - 2\partial_t\xi_r\right)\,, \tag{7.86}$$

$$\frac{dt^{\text{in}}}{dr} \approx -1 - \frac{1}{2}\left(\mathcal{H}^{\text{in}} - 2\partial_r\xi_r + 2\partial_t\xi_r\right)\,. \tag{7.87}$$

Γ^r_{tt} now becomes $\Gamma^r_{tt} \to \Gamma^r_{tt} - \partial_t^2 \xi_r$, so

$$\delta r_{\mathrm{M}}^{\mathrm{out}} \to \delta r_{\mathrm{M}}^{\mathrm{out}} + \xi_r(L,L) - \xi_r(0,0)\,, \tag{7.88}$$

$$\delta r_{\mathrm{M}}^{\mathrm{in}} \to \delta r_{\mathrm{M}}^{\mathrm{in}} + \xi_r(L,L) - \xi_r(2L,0)\,. \tag{7.89}$$

Then, in total,

$$T^{\mathrm{out}} \to T^{\mathrm{out}} + \xi_r(L,L) - \xi_r(0,0) - \int_0^L dr\ \ (\partial_r \xi_r + \partial_t \xi_r)\,|_{t=r} = T^{\mathrm{out}}\,, \tag{7.90}$$

$$T^{\mathrm{in}} \to T^{\mathrm{in}} + \xi_r(L,L) - \xi_r(2L,0) - \int_0^L dr\ \ (\partial_r \xi_r - \partial_t \xi_r)\,|_{t=2L-r} = T^{\mathrm{in}}\,, \tag{7.91}$$

so T also remains invariant under the gauge transformation of coordinate r. Thus, we have shown that T is a gauge invariant quantity.

7.8 Appendix: Phase Accumulation in Fabry-Perot Cavity

In this appendix, we show that the spacetime fluctuations in Eq. (7.2) accumulate in a Fabry-Perot cavity, so it is reasonable to compare our predicted strain to the experiments utilizing Fabry-Perot cavities, such as GEO-600 and LIGO, in Sec. 7.5.

A Fabry-Perot Michelson interferometer can be viewed as a linear device that measures the differential single-round-trip phase, $\Delta\Phi = \Phi_1 - \Phi_2$ between the two arms — regardless of whether this phase arises from gravitational waves, displacement of mirrors, or space-time fluctuations. This $\Delta\Phi$ is linearly transferred to the output field z, with noise N added:

$$z(f) = \mathcal{M}(f)\Delta\Phi(f) + N(f)\,. \tag{7.92}$$

In particular, $\mathcal{M}(f)$ contains the build-up (or suppression) of signal due to the Fabry-Perot cavity.

We now convert the strain-referred noise spectrum S_h published by LIGO to a spectrum for $\mathcal{T}$. In obtaining S_h (below 5 kHz, as shown in Fig. 7.6), LIGO used a long-wave-length approximation, and assumed that the wave has a + polarization (stretching along the x and squeezing along the y direction), and propagating along z — perpendicular to the detector plane (e.g., adopted by Chapter 27.6 of [36]). In this case, in the Local Lorentz frame of the beam splitter, the first and second mirrors are going to be displaced by $\pm Lh/2$, leading to phase shifts of

$$\Phi_{1,2} = \pm\omega_0 Lh/c \tag{7.93}$$

and

$$\Delta\Phi = 2\omega_0 Lh/c\,. \tag{7.94}$$

In this way, the $\Delta\Phi$-referred spectrum is related to S_h published by LIGO via

$$\sqrt{S_{\Delta\Phi}} = \frac{2\omega_0 L}{c}\sqrt{S_h}\,. \tag{7.95}$$

We note that at higher frequencies, and/or for interferometers with longer arms, the conversion from h to Φ becomes less trivial. In our case, we have

$$\Delta\Phi(t) = \omega_0[\delta T(t,\mathbf{n}_1) - \delta T(t,\mathbf{n}_2)] = \omega_0\mathcal{T}(t,\theta)\,. \tag{7.96}$$

We therefore have $\sqrt{S_{\Delta\Phi}} = \omega_0\sqrt{S_{\mathcal{T}}}$ and thus

$$\sqrt{S_{\mathcal{T}}} = \frac{2L}{c}\sqrt{S_h}\,. \tag{7.97}$$

This allows us to straightforwardly relate our observable defined in Eqs. (7.34) and (7.45) to the quantity S_h constrained by LIGO. In LIGO, S_h is usually reported as a one-sided spectrum, so we need another factor of 2 when converting the two-sided spectrum $\tilde{C}_{\mathcal{T}}$ in Eq. (7.45) to the one-sided spectrum S_h, i.e.,

$$\sqrt{S_h} = \sqrt{S_{\mathcal{T}}}\Big/\left(\frac{2L}{c}\right) = \sqrt{2\tilde{C}_{\mathcal{T}}\left(\omega,\theta=\frac{\pi}{2}\right)}\,, \tag{7.98}$$

which is consistent with the conversion in Eq. (7.66).

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Chapter 8

QUANTUM GRAVITY BACKGROUND IN NEXT-GENERATION GRAVITATIONAL WAVE DETECTORS

[1] Mathew W. Bub, Yanbei Chen, Yufeng Du, Dongjun Li, Yiwen Zhang, and Kathryn M. Zurek. "Quantum gravity background in next-generation gravitational wave detectors". In: *Phys. Rev. D* 108.6 (2023), p. 064038. DOI: 10.1103/PhysRevD.108.064038. arXiv: 2305.11224 [gr-qc].

8.1 Introduction

Bridging the gap between theory and experiment in the study of quantum gravity is at the forefront of research in physics. Although the effects of quantum gravity are ordinarily expected to appear on unobservably small scales of order the Planck length, $l_p = \sqrt{8\pi G\hbar/c^3} \sim 10^{-34}$ m, recent works [1–7] have shown that this naive effective field theory (EFT) reasoning may not capture the complete physical picture. Instead, Refs. [1, 2] showed, using standard holographic techniques, that spacetime fluctuations accumulate from the UV into the IR to produce an effect that scales with the size L of the physical system. In particular, in flat spacetime, the trajectories of photons in an interferometer of length L enclose a finite spacetime region known as a causal diamond.[1] The interferometer only probes the finite degrees of freedom inside the causal diamond, which are entangled with degrees of freedom outside the causal diamond bounding the interferometer, leading to nonzero entropic fluctuations. The region of spacetime that can be probed by the interferometer arms, emanating radially from an origin located at the beamsplitter, is separated by a spherical entangling surface, i.e., the surface Σ in Fig. 8.1. The geometric fluctuations induced by entropic fluctuations within the causal diamond, or "geontropic fluctuations," manifest as uncertainty in the arm length of the interferometer, as measured by the photon travel time, with a variance that scales as

$$\langle \Delta L^2 \rangle \sim l_p L. \tag{8.1}$$

Additionally, these fluctuations exhibit long-range transverse correlations which enable observation. This result has proven to be theoretically robust, having been

[1] The causal diamond $\mathcal{D}$ of a spatial ball B is defined such that for all points $p \in \mathcal{D}$, every timelike curve passing through p must intersect B [8, 9].

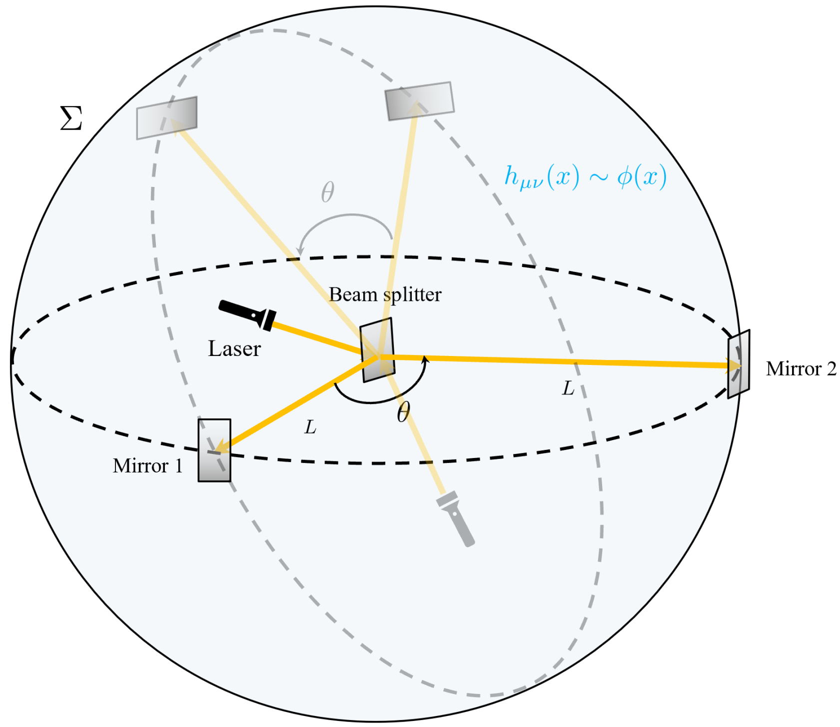


Figure 8.1: The spherical entangling surface of an interferometer with arms of equal length L and separated by angle θ. The faint mirrors and light beams indicate that one can rotate the interferometer about its origin, so the boundary of the spatial volume that can be probed by such rotations defines the spherical entangling surface Σ.

confirmed with several distinct theoretical approaches in Refs. [3–7], such that the geontropic fluctuations are observed in flat Minkowski, dS, and AdS spacetimes. For a summary of all of these works, see Ref. [10].

More recently, Ref. [11], building upon the work of Ref. [3], developed a *model* of these geontropic fluctuations in terms of bosonic degrees of freedom coupled to the metric. The model is designed to capture the most prominent features of the theory developed in Refs. [1, 2, 4–7], while being local and allowing for the explicit computation of the gauge-invariant interferometer observable. It features a scalar field ϕ, the "pixellon", a breathing mode corresponding to spacetime fluctuations of the (spherically symmetric) volume of spacetime under observation. This model allows for the calculation of the power spectral density (PSD) of geontropic fluctuations in spherically-symmetric configurations, in particular for traditional L-shaped interferometers such as LIGO [12] and LISA [13].

Ref. [11] also compared the PSD of the pixellon model to the strain sensitivities of several current and future gravitational wave (GW) detectors, namely LIGO/Virgo [12], Holometer [14], GEO600 [15], and LISA [13]. These experiments either produced modest constraints on the pixellon model (in the cases of LIGO and Holometer) or were not sensitive to the model (in the cases of GEO600 and LISA). There are several general reasons for this. For large instruments such as LISA, we expect a reduced signal as the geontropic strain scales parametrically as $h = \frac{\Delta L}{L} \sim \sqrt{\frac{l_p}{L}}$. On the other hand, existing terrestrial experiments typically have poorer strain sensitivities near the relatively high frequency $\omega_{\text{peak}} \sim \frac{1}{L}$ at which the pixellon signal achieves its peak. In this paper, we build upon this previous work and survey the landscape of next-generation GW detectors, characterizing their sensitivity to geontropic fluctuations as modeled by the pixellon. We also consider these experiments in the context of the upcoming GQuEST experiment [16], which explicitly seeks to measure the geontropic signal. Note that in this paper we assume the pixellon is a good *physically equivalent* description of the geontropic fluctuations predicted by the VZ effect [1, 2, 4–7, 10]. As discussed above, while it has been shown that the pixellon model reproduces important features of the VZ effect (such as the angular correlations), the physical equivalence in all aspects of the interferometer observable has not been shown, and is the subject of ongoing, first-principles calculations. We plan to update observational signatures as the theoretical modeling captures more aspects of the first-principles calculations.

With this caveat in mind, the paper is organized as follows. In Sec. 8.2, we briefly summarize the pixellon model of Refs. [3, 11]. In Sec. 8.3, we review a variety of proposed GW detectors following Ref. [17], and discuss their potential sensitivity to the geontropic signal. In Sec. 8.4, we extend the calculation of the pixellon PSD in Ref. [11] to more general interferometer-like experiments, particularly for those with geometries other than the traditional L-shape, and for optically-levitated sensors. In Sec. 8.5, we then apply the results to specific experiments and compare the geontropic signal to the expected strain sensitivities of these experiments. Finally, in Sec. 8.6, we collect our results and discuss their implications for the future of GW observation.

In anticipation of our main result, in Fig. 8.2, we plot the predicted pixellon signal alongside the strain sensitivities of two prominent next-generation GW detectors: Cosmic Explorer (CE) [18, 19] and the Einstein Telescope (ET) [20]. From these plots, we find a typical geontropic signal exceeds the strain sensitivities of these detectors by two orders of magnitude over a wide range of frequencies. As such, the signal represents a large stochastic background which, if present, would imply a reevaluation of the future of GW astronomy. Moreover, we will show that of the experiments considered in this paper, only CE and ET will have better sensitivity to the geontropic signal than GQuEST, which is a nearer-term apparatus than CE and ET.

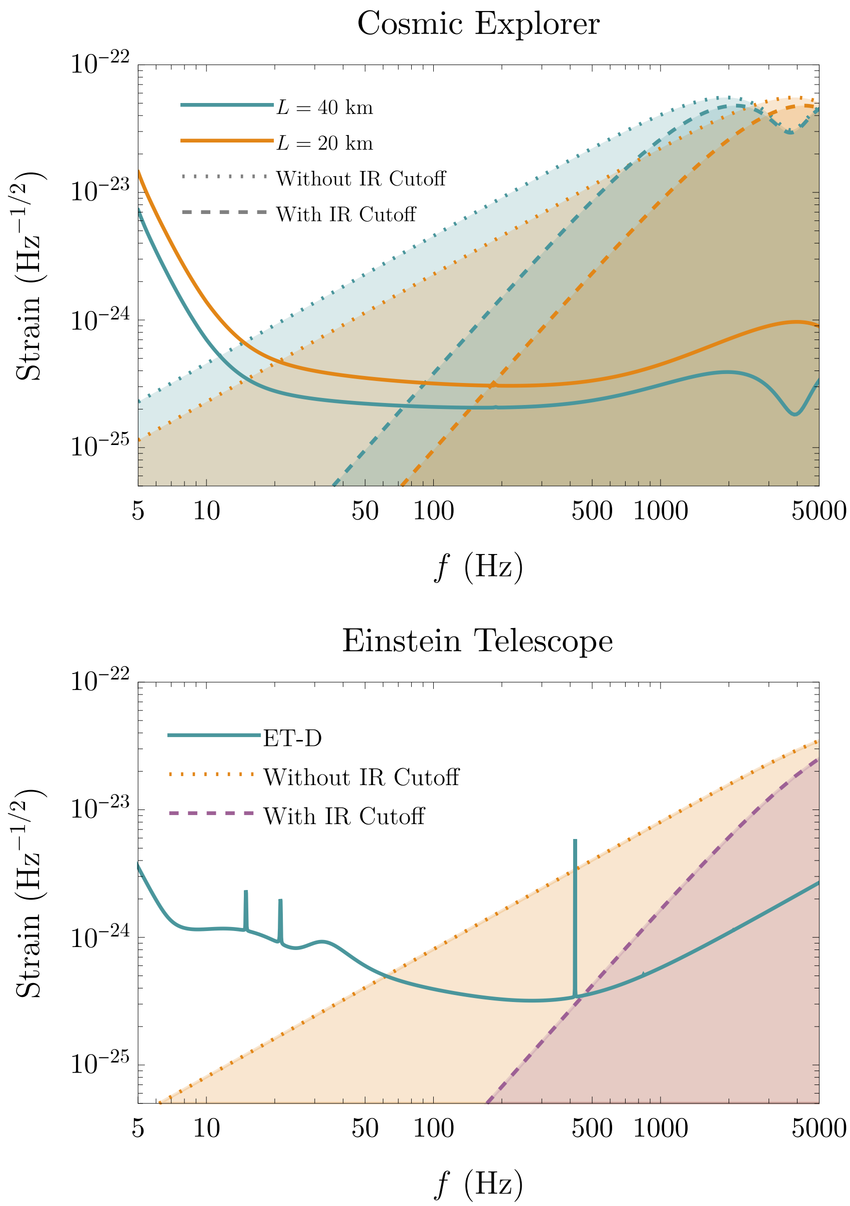


Figure 8.2: Pixellon strain (dashed and dotted lines) overlaid with the strain sensitivities for CE [19] and ET [20] (solid lines). For CE, we have included both designs with arm lengths $L = 20$ km (orange lines) and $L = 40$ km (blue lines). The dotted lines give the pixellon strain from Eq. (8.35) computed without an IR cutoff, and the dashed lines give the same quantity including the IR cutoff from Eq. (8.28). The pixellon strain is computed with the benchmark value $\alpha = 1$.

8.2 Pixellon Model

In this section, we review the pixellon model proposed in Refs. [3, 11] to model the geontropic fluctuations of the spherical entangling surface bounding an interferometer, which is also a specialization of the dilaton model studied in Refs. [4, 5] to causal diamonds in 4-d flat spacetime. Before proceeding, we emphasize that while we expect the pixellon model to reproduce a number of the salient features of the effect proposed in Refs. [1, 2, 4, 6], the physical equivalence between the model and the complete theory remains to be shown. Demonstrating this physical equivalence will be crucial for claiming a decisive test of the VZ effect. More specifically, Ref. [11] considered a breathing mode of the metric associated with the spacetime volume probed by the interferometer,

$$ds^2 = -dt^2 + (1-\phi)(dr^2 + r^2 d\Omega^2)\,, \tag{8.2}$$

where ϕ is a bosonic scalar field,

$$\phi(x) = l_p \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\mathbf{p})}} \left(a_{\mathbf{p}} e^{ip\cdot x} + a^\dagger_{\mathbf{p}} e^{-ip\cdot x} \right)\,, \tag{8.3}$$

and satisfies the dispersion relation of a sound mode,

$$\omega = c_s |\mathbf{p}|\,, \quad c_s = \sqrt{\frac{1}{3}}\,. \tag{8.4}$$

The dispersion relation in Eq. (8.4) and the normalization factor l_p in Eq. (8.3) were derived from plugging the metric in Eq. (8.2) into the linearized Einstein-Hilbert action and varying the action with respect to ϕ [11]. The resulting equation, which gives rise to Eq. (8.4), is not a vacuum linearized Einstein equation but rather includes an additional constraint due to the form of the metric in Eq. (8.2) and the corresponding nonlinear field interactions. Although we have used the language of linearized gravity, the nonlinear interactions are captured by the nonzero occupation number σ_{pix} in Eq. (8.12) [11]. The creation and annihilation operators $(a^\dagger_{\mathbf{p}}, a_{\mathbf{p}})$ satisfy the standard commutation relation,

$$\left[a_{\mathbf{p}_1}, a^\dagger_{\mathbf{p}_2} \right] = (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2)\,. \tag{8.5}$$

Instead of being a vacuum state, ϕ is thermal with a nontrivial thermal density matrix ρ_{pix} [3, 11]:

$$\rho_{\text{pix}} = \frac{1}{\mathcal{Z}} \exp\left[-\beta \int \frac{d^3\mathbf{p}}{(2\pi)^3} (\epsilon_{\mathbf{p}} - \mu) a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \right]\,, \tag{8.6}$$

$$\mathcal{Z} = \prod_{\mathbf{p}} \frac{1}{1 - e^{-\beta(\epsilon_{\mathbf{p}}-\mu)}}, \tag{8.7}$$

where $\epsilon_{\mathbf{p}}$ is the energy of the pixellon mode of momentum $\mathbf{p}$, and μ is the chemical potential counting the background degrees of freedom. In this case, the pixellon modes ϕ have an occupation number given by the standard bosonic statistics, i.e.,

$$\begin{aligned} \mathrm{Tr}\left(\rho_{\text{pix}} a^{\dagger}_{\mathbf{p}_1} a_{\mathbf{p}_2}\right) &= (2\pi)^3 \sigma_{\text{pix}}(\mathbf{p}_1)\delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2), \\ \sigma_{\text{pix}}(\mathbf{p}) &= \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\mu)} - 1}. \end{aligned} \tag{8.8}$$

To further simplify the occupation number $\sigma_{\text{pix}}(\mathbf{p})$, Refs. [3, 11] used that in flat spacetime, the modular Hamiltonian K inside a causal diamond satisfies [1, 4]

$$\langle K \rangle \sim \langle \Delta K^2 \rangle \sim \frac{A(\Sigma)}{l_p^2}, \tag{8.9}$$

and similar results in AdS were found in Refs. [2, 21, 22]. Since the number of gravitational degrees of freedom $\mathcal{N}$ inside the causal diamond is given by

$$\mathcal{N} \equiv \langle K \rangle, \tag{8.10}$$

the energy fluctuation per degree of freedom is given by [3, 11]

$$\beta(\epsilon_{\mathbf{p}} - \mu) \equiv \beta\omega(\mathbf{p}) \sim \frac{\sqrt{\langle \Delta K^2 \rangle}}{\langle K \rangle} \sim \frac{l_p}{L}. \tag{8.11}$$

If one uses Eq. (8.11), identifies $\omega(\mathbf{p}) \sim \frac{1}{L}$, and expands $\sigma_{\text{pix}}(\mathbf{p})$ in Eq. (8.8) to leading order in $\frac{l_p}{L}$, one finds

$$\sigma_{\text{pix}}(\mathbf{p}) = \frac{a}{l_p \omega(\mathbf{p})}, \tag{8.12}$$

where a is a dimensionless number, to be fixed by experiment. In Eq. (8.11), $\beta \sim l_p$ corresponds to the local temperature of the near-horizon region probed by the light beams. Comparing the pixellon model here to Refs. [1–3] and incorporating ϕ as a sound mode [i.e., Eq. (8.4)], Ref. [11] fixed $a = c_s^2/(2\pi)$, which corresponds to $\beta = 2\pi l_p/c_s^2$. Defining

$$\alpha \equiv \frac{2\pi}{c_s^2} a, \tag{8.13}$$

we obtain the theory-motivated benchmark for detection $\alpha \sim 1$.

In Ref. [11], the pixellon model was used to compute the auto-correlation function of length fluctuations of a single Michelson interferometer with length L and separation angle θ. It was found that the peak of the signal is at $\omega L \sim 1$ with an overall amplitude $\sqrt{\langle \Delta L^2 \rangle} \sim \sqrt{l_p L}$. Moreover, the angular correlations from the pixellon model match well with the predictions of Refs. [1, 6] from shockwave geometry. The peak frequency $\omega_{\text{peak}} \sim \frac{1}{L}$ is consistent with both the identification $\omega(\mathbf{p}) \sim \frac{1}{L}$ made by Eq. (8.12) and the pixellon mode being a breathing mode controlling the size of the spherical entangling surface bounding the interferometer. From this typical frequency and the strain's amplitude, one can directly see that for a general detector probing a causal diamond of size L, we need a strain sensitivity $\sqrt{S_h(f)} \lesssim \sqrt{\omega_{\text{peak}} \langle \Delta L^2 \rangle} \sim \sqrt{l_p} \sim 10^{-23}\ \text{Hz}^{-1/2}$ near the frequency $\omega_{\text{peak}} \sim \frac{1}{L}$, where $S_h(f)$ is the one-sided noise strain defined in Eq. (8.34). Most current interferometers, especially those aiming for GW detection, do not have such good strain sensitivity near the free spectral range, which is a higher frequency than is probed by many interferometers. Thus, we would first like to investigate whether other types of high-frequency GW detectors, besides the next-generation interferometers, can potentially detect geontropic signals.

8.3 High-Frequency GW Detectors

This section follows the review in Ref. [17] to investigate a broad class of high-frequency GW detectors with various operating principles. To understand how the detection of geontropic fluctuations fits in this landscape, we first discuss the proposed scientific goals of these high-frequency GW detectors. Most current proposals intend to probe astrophysical objects in unexplored limits, or test quantum gravity near highly curved spacetime. In contrast, the effect considered in Refs. [1–7, 10, 11] and this work fills the gap of examining quantum gravity in flat spacetime. Moreover, the necessary sensitivity and frequency range are within the same regime as other science cases, so utilizing these detectors for geontropic signals is natural. In the second half of this section, we examine these detectors' suitabilities for measuring geontropic fluctuations and argue that interferometer-like experiments are the most optimal.

8.3.1 Sources of high-frequency GWs

Since the successful detection of GWs by the LIGO-Virgo collaboration [23], there have been continuous efforts to improve the sensitivity of GW detectors at higher frequencies. One direct motivation for this is to study extreme astrophysical objects in limits or environments which cannot be reached by current GW detectors. For example, the merger of sub-solar mass primordial BHs of 10^{-9}–$10^{-1}\ M_\odot$ can emit

GWs with frequencies of 10–10^9 kHz [17]. For neutron stars (NSs), the remnant hot, high-density matter after their merger can generate GWs at either $\sim$ 1–4 kHz [24] for a BH remnant or $\gtrsim$ 6 kHz [25, 26] for an NS remnant [27]. These high-frequency GWs provide opportunities to study different phases of matter predicted by quantum chromodynamics in a high-density finite-temperature environment [28]. At larger scales, high-frequency GW detectors will assist in learning about GWs emitted by the thermal plasma of the early universe [29] (1–100 GHz), the stochastic GW background generated by primordial BHs [30] (10–10^{10} THz), cosmic strings [31] (1–10^6 kHz), and other events at cosmological scales [17].

One vital application of these high-frequency GW detectors is to explore quantum gravity, the central focus of this work. Standard tests of quantum gravity using GWs focus on examining the properties of quantum BHs against their classical counterparts. For example, GW detections have been used to test the no-hair theorem [32], stating that any classical stationary BH (a solution to the Einstein-Maxwell equation) is characterized only by its mass, charge, and angular momentum [33]. Still, quantum gravity might dress BHs with hair [34, 35]. The spectrum of GWs can also serve as a test of the horizon's existence [36, 37], where quantum gravity can modify the structure of the near-horizon geometry [38], either drastically via a "firewall" hiding all quantum effects [39], or smoothly with the quantum effects extending over some distance around the BH [40].

Unlike these standard tests, the series of works in Refs. [1–7, 10, 11] instead focus on perturbations of the near-horizon geometry of causal diamonds in flat spacetime due to quantum gravity, which the pixellon models as an effective description. As introduced in Secs. 8.1 and 8.2 and shown in detail in Sec. 8.5.1, the length fluctuations induced by the pixellon in an L-shaped interferometer of length L have a size of $\sqrt{\langle \Delta L^2 \rangle} \sim \sqrt{l_p L}$ and a peak frequency at $\omega L \sim 1$, corresponding to a PSD with an amplitude of $\sim \sqrt{c l_p}$. For an interferometer, or, more generally, a causal diamond with characteristic size $L \sim$ 10 m–10 km, we need a strain sensitivity of $\sim 10^{-23}$ Hz$^{-1/2}$ at the peak frequencies of $\frac{1}{L} \sim$ kHz–MHz, which is within the target sensitivity of many high-frequency GW detectors. Thus, these high-frequency GW detectors planned for various purposes can also be used to test quantum gravity in flat spacetime, which motivates our following investigation.

8.3.2 Detectors for high-frequency GWs

8.3.2.1 Interferometers

The most natural GW detectors to consider are the next-generation interferometers, such as CE [18, 19], ET [41], and NEMO [27], for which the pixellon model was designed to describe the geontropic fluctuations. Although CE and ET are not usually considered high-frequency detectors but instead broadband detectors, they can access frequencies of a few kHz, which are near their free spectral range. For a single interferometer, the causal diamond is naturally defined by the light beams traveling between the mirrors, with its size equal to the interferometer's arm length. Perturbations to the spherical entangling surface bounding the interferometer are then controlled by the pixellon mode. Although the metric in Eq. (8.2) is not spherically symmetric due to the nontrivial angular dependence of $\phi(x)$, its spatial part is conformal to the metric of a 3-ball, adapting to the spherical symmetry of an interferometer.

The pixellon model and the procedure to compute length fluctuations can be extended to alternative configurations of Michelson interferometers, such as the triangular configuration of ET. In Ref. [11], the PSD and the angular correlations of a single L-shaped interferometer with an arbitrary separation angle were computed. In Sec. 8.4, we further show that the previous results can be extended to multiple interferometers if we consistently correlate pixellons in different causal diamonds. The cross-correlations of different interferometers can then be studied, becoming a smoking gun signature of geontropic signals. Another advantage of studying cross-correlations between detectors is that the cross spectrum of a correlated noise background between different detectors can be detected at a level much lower than their individual independent noise spectra [42].

One fundamental barrier for an interferometer to reach the high-frequency regime is the quantum shot noise of lasers (or the high uncertainty of the laser's phase quadrature). The most direct solution to this limitation is to increase the laser power $P_{\rm arm}$, since the PSD of the quantum noise at high frequencies is proportional to $P_{\rm arm}^{-1/2}$ [43], which is the approach adopted by NEMO [27]. However, increasing laser power is technically challenging, with issues such as the parametric instability of the mirrors' motion due to energy transfer from the light beams [44] or the thermal deformation of the mirrors [18, 45].

Besides increasing laser power, one can also inject squeezed vacuum into the dark port of the interferometer, leading to a reduced phase uncertainty at the cost of sacrificing the sensitivity at low frequencies [18]. Nonetheless, the degradation in the low-frequency sensitivity due to quantum radiation pressure noise can be avoided by frequency-dependent squeezing [46–48]. In addition, Refs. [49, 50] recently proposed that one can connect a quantum parametric amplifier to the interferometer to stabilize the "white-light cavity" design in Ref. [51], such that the sensitivity at kHz frequencies can be increased without sacrificing the bandwidth.

In addition, for detecting a stochastic background like the geontropic signal, which is spatially correlated for two physically overlapping interferometers, a cross-correlation method can be established for each individual detector to dig under shot noise [52]. This allows us to achieve a better sensitivity than each detector's noise budget for detecting gravitational waves.

Another way to circumvent quantum shot noise is using photon counting instead of the standard homodyne readout [53]. Such a readout will be implemented in a proposed 5 m tabletop interferometer being commissioned by Caltech and Fermilab under the Gravity from the Quantum Entanglement of Space-Time (GQuEST) collaboration [16], which will explicitly target geontropic fluctuations. By employing photon counting and thereby beating the standard quantum limit, GQuEST will be able to place constraints on α substantially more efficiently in terms of integration time than it would with only a homodyne readout. For a detailed examination of the advantages of photon counting, see Ref. [53]. As GQuEST is a tabletop-sized experiment, it will also be capable of probing the angular correlations of the geontropic fluctuations by adjusting its arm angle. Moreover, it is conceived to be a nearer-term instrument than third generation GW detectors such as CE and ET. As such, should the geontropic signal be detected with GQuEST, this information can be incorporated into the design and planning of future GW detectors, whose strain sensitivities to astrophysical signals might be limited by a geontropic background.

8.3.2.2 Optically-levitated sensors

Besides interferometers, there are other high-frequency GW detectors that operate like an interferometer, such as the optically-levitated sensor described in Refs. [54, 55]. The optically-levitated sensor functions by trapping a dielectric sphere or microdisk in an anti-node of an optical cavity (see Fig. 8.7) [54]. One can also build a Michelson interferometer from optically-levitated sensors by inserting the sensors in each arm's cavity (see Fig. 8.8) [55]. As illustrated in Sec. 8.5.3, one optically-levitated sensor can be effectively treated as two aligned interferometer arms, where the longer arm has the same length ℓ_m as the cavity. The shorter arm has length x_s, the distance to a chosen anti-node of the trapping field. The optically-levitated sensor measures the differential distance change $\delta\ell_m - \delta x_s$, the correlations of which are similar to an interferometer of length $L = \ell_m - x_s$, but not identical since the two arms have to be treated separately. Moreover, as depicted in Fig. 8.8, there are two causal diamonds enclosing the shorter and longer arms, respectively. In Sec. 8.4, we show how to consistently correlate these multiple causal diamonds.

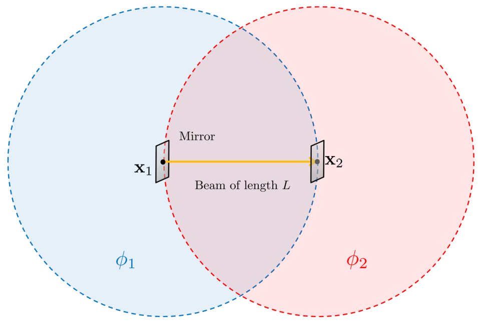
Levitated sensors achieve their gain in sensitivity by making the test masses respond resonantly to gravitational waves whose frequencies match the test masses' natural oscillation frequency in the trapping potential. In the devices considered by Refs. [54, 55], sensitivities are mainly constrained by the thermal noise due to heating of the sensor by the scattering light [55]. The development of techniques to reduce the thermal noise of an optically-trapped object in many other contexts thus allows a better strain sensitivity for the optically-levitated sensor at high frequencies compared to an interferometer [54]. It was further found in Ref. [55] that by using stacked disks as the sensor, the thermal noise due to photon recoiling can be further reduced. In addition, the high-frequency performance of the levitated sensor is further enhanced by its tunability. Indeed, the experiment achieves its peak strain sensitivity when the trapped object is resonantly excited at the trapping frequency, which is widely tunable via laser intensity [55]. In Sec. 8.5.3, we will compare the PSD of length fluctuations measured by the optically-levitated sensor to its predicted strain sensitivity from Ref. [55].

8.3.2.3 Inverse-Gertsenshtein effect and other experiments

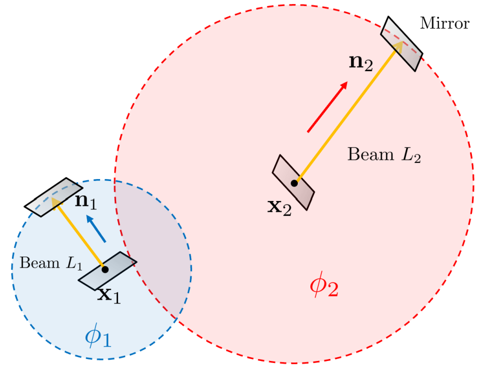
Apart from interferometer-like experiments, there are other high-frequency GW detectors with different working principles. One major class of such experiments uses the inverse-Gertsenshtein effect, which converts gravitons to electromagnetic (EM) waves [56]. For most of these experiments, strong static magnetic fields of several Tesla are used to convert gravitons into photons [17]. Many of these experiments have been designed to detect ultralight axion dark matter, which can also couple to the EM fields, such as the ones using microwave cavities (e.g., ADMX [57], HAYSTAC [58], and SQMS [59]) or pickup circuits (e.g., ABRACADABRA [60] and SHAFT [61]) to receive the signal. Refs. [62–64] found that some of these experiments might be sensitive to high-frequency GWs, especially when the geometry of the detector reflects the spin-2 nature of gravitons. For example, Ref. [64] found that a figure-8 pickup circuit has a much larger sensitivity than a circular loop. For microwave cavities, if the resonant cavity modes have the same spatial profile as the effective current generated by the inverse-Gertsenshtein effect, there is also a boost of the signal [63].

The pixellon model considered in Refs. [3, 11] and this work can be, in principle, used to compute the inverse-Gertsenshtein effect since geontropic fluctuations manifest themselves as metric fluctuations, i.e., Eq. (8.2). As recently shown in Ref. [63], for usual gravitational waves, if one incorporates all physical effects (such as circuits' motion due to coordinate transformation), one can find gauge invariant observables such as current density. However, one fundamental question is whether the pixellon model is appropriate for describing this type of experiment, especially those using microwave cavities. The pixellon model was designed to effectively describe gravitational perturbations of the spherical entangling surface bounding the interferometer, the bifurcate horizon of the causal diamond defined by the light beams. Within the cavity, there is no freely propagating photon, so the detector doesn't probe the near-horizon geometry of any causal diamond. In this case, the pixellon model might not be a good effective description, and geontropic fluctuations might be negligible since they are driven by near-horizon dynamics [4]. Note that the photon counting technique in Ref. [53] also detects the excess photons generated by gravitational perturbations. However, this readout is still embedded in a Michelson interferometer, so there is a well-defined causal diamond.

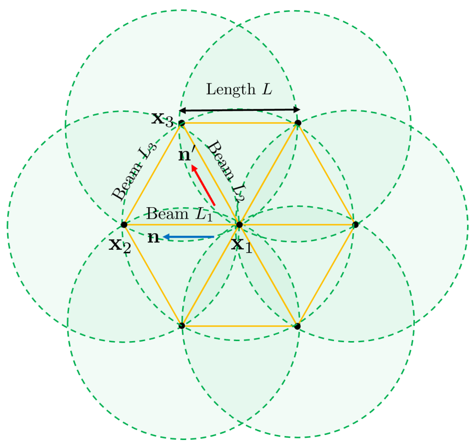
Besides the experiments above, other types of high-frequency GW detectors are discussed in Ref. [17], such as the bulk acoustic wave devices [65], which operate like a resonant mass bar [66] and measure the vibration of piezoelectric materials due to passing GWs. Similarly, GWs can also deform microwave cavities, which couple different resonant cavity modes and can be detected [67]. There are also experiments utilizing the coupling between GWs and electron spin, where the collective electron spin excitations or magnons of ferromagnetic crystals due to GWs are measured [68, 69]. Since no causal diamond is being probed in all of these experiments, geontropic signals might be minimal. For this reason, for the rest of this work, we focus on these interferometer-like experiments and calculate their sensitivity to the pixellon model.



(a) A single light beam. The beam of length L is sent from $\mathbf{x}_1$ at $t = -L$ to $\mathbf{x}_2$ and then reflected by the end mirror.



(b) Two light beams. The beam of length L_1 is sent from $\mathbf{x}_1$ at $t_1 - L_1$ along the direction $\mathbf{n}_1$ and reflected by the end mirror. Similarly, the beam of length L_2 is sent from $\mathbf{x}_2$ at $t_2 - L_2$ along the direction $\mathbf{n}_2$ and then gets reflected.



(c) A web of light beams tiling the entire space-time.

Figure 8.3: Plots of spherical entangling surfaces or spatial slices of the causal diamonds bounding different configurations of light beams. The dashed circles represent entangling surfaces, each of which is associated with a pixellon model. The spatial region inside the entangling surface is shaded. For all the figures above, we have projected the spherical entangling surface to the plane of the light beams.

8.4 Extension of the pixellon model to multiple interferometers

In this section, we extend the calculation in Ref. [11] of the auto-correlation of a single interferometer's length fluctuations to the cross-correlation of two interferometer-like detectors, which may have different arm lengths and origins.

As shown in Ref. [11], for the metric in Eq. (8.2), the only nonzero component in the $t - r$ sector of the metric is h_{rr}, so we only need to consider light propagation when computing length fluctuations. For a light beam sent at time $t - L$ from the origin $\mathbf{x}$ along the direction $\mathbf{n}$, the total time delay $T(t, \mathbf{n})$ of a round trip is given by[2]

$$\begin{aligned} T(t, \mathbf{x}, \mathbf{n}) &= 2L - \frac{1}{2}\int_0^L dr\ [\phi(x) + \phi(x')]\,, \\ x &\equiv (t - L + r, \mathbf{x} + r\mathbf{n})\,,\ x' \equiv (t + L - r, \mathbf{x} + r\mathbf{n})\,. \end{aligned} \tag{8.14}$$

Notice that although Eq. (8.14) has an explicit dependence on the origin $\mathbf{x}$, the auto-correlation function of T or its fluctuations doesn't depend on $\mathbf{x}$, as shown in Ref. [11] and Eq. (8.32). This indicates that geontropic fluctuations are physical, since they don't depend on the choice of coordinates.

Next, let us consider two light beams sent at times $t_1 - L_1$ and $t_2 - L_2$ from positions $\mathbf{x}_1$ and $\mathbf{x}_2$ along directions $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively, as depicted in Fig. 8.3b. We also assume the lengths of the two beams without any geontropic fluctuations to be L_1 and L_2, respectively. Then the correlation function of the length fluctuations δT of these two beams is

$$C(\Delta t, \Delta\mathbf{x}, \mathbf{n}_{1,2}) \equiv \left\langle \frac{\delta T(t_1, \mathbf{x}_1, \mathbf{n}_1)\delta T(t_2, \mathbf{x}_2, \mathbf{n}_2)}{4L_1L_2} \right\rangle\,,\ \Delta t \equiv t_1 - t_2\,,\ \Delta\mathbf{x} \equiv \mathbf{x}_1 - \mathbf{x}_2\,, \tag{8.15}$$

where we have defined $\delta T(t, \mathbf{x}, \mathbf{n}) = T(t, \mathbf{x}, \mathbf{n}) - 2L$ with $T(t, \mathbf{x}, \mathbf{n})$ given in Eq. (8.14). Here, we have assumed that the origins of the light beams enter the cross-correlation function only via their difference $\Delta\mathbf{x}$, so it is independent of the choice of coordinates. We will see this assumption is true in Eq. (8.27).

Since these two light beams are enclosed by two different causal diamonds as shown in Fig. 8.3b, their length fluctuations are separately described by two pixellon models with the metric in Eq. (8.2) centered at $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. To distinguish these two pixellon models, we assign $\phi_1(x)$ and $\phi_2(x)$ to the first and the second beams, respectively. Within each pixellon model, the length fluctuations are still described by Eq. (8.14), so

$$\begin{aligned} C(\Delta t, \Delta\mathbf{x}, \mathbf{n}_{1,2}) = \frac{1}{16L_1L_2}\int_0^{L_1} dr_1 \int_0^{L_2} dr_2 \\ \langle\left(\phi_1(x_1) + \phi_1(x_1')\right)\left(\phi_2(x_2) + \phi_2(x_2')\right)\rangle\,, \end{aligned} \tag{8.16}$$

which is in a similar form as Eq. (32) of Ref. [11]. For convenience, let us define

$$C(x_1, x_2) = \langle(\phi_1(x_1) + \phi_1(x_1'))(\phi_2(x_2) + \phi_2(x_2'))\rangle\,. \tag{8.17}$$

[2]We have corrected a typo in Ref. [11], where the sign before the integral should be minus.

To evaluate $C(x_1, x_2)$, we first need to compute $\langle \phi_1(x_1)\phi_2(x_2)\rangle$, where x_1 and x_2 are in two different causal diamonds. From Eqs. (8.3) and (8.4), we notice that both ϕ_1 and ϕ_2 satisfy the wave equation, as constrained by the linearized Einstein-Hilbert action [11]. Thus, ϕ_1 has translational symmetry, i.e., $\phi_1(y) = e^{-ip\cdot(x-y)}\phi_1(x)$ classically, and similarly for ϕ_2. This implies that although the metric in Eq. (8.2) effectively describes the length fluctuations of a finite-size interferometer, nothing prevents us from propagating the pixellon field $\phi(x)$ to places outside the interferometer. This is also consistent with the fact that ϕ has modes with long wavelengths, as imposed by Eq. (8.12). Thus, ϕ_1 is well-defined in the causal diamond of ϕ_2, and vice versa.

To derive a precise relation between ϕ_1 and ϕ_2, let us consider a single light beam sent from $\mathbf{x}_1$ at $t = -L$ to $\mathbf{x}_2$, as depicted in Fig. 8.3a. To compute the round-trip time delay, one can either use the pixellon model centered at $\mathbf{x}_1$ with the pixellon ϕ_1, or the one centered at $\mathbf{x}_2$ with the pixellon ϕ_2. For the former case, we set the origin of the coordinates at $\mathbf{x}_1$ and align the x-axis with the outgoing light beam, so the shift of the round-trip time delay δT_1 is given by Eq. (8.14),

$$\delta T_1 = -\frac{1}{2}\int_0^L dr\ \ [\phi_1(x) + \phi_1(x')]\ ,\ x_1 = (-L + r, r\hat{\mathbf{x}})\ ,\ x_1' = (L - r, r\hat{\mathbf{x}})\ , \quad (8.18)$$

where the first and second terms correspond to the time delay of the outgoing and ingoing light beams, respectively.

For the latter case, we set the origin at $\mathbf{x}_2$ and align the x-axis with the ingoing light beam. Notice the ingoing beam here is the outgoing beam for the pixellon model at $\mathbf{x}_1$, and vice versa. Then, δT_2 is given by

$$\delta T_2 = -\frac{1}{2}\int_{-L}^0 dr\ \ [\phi_2(x) + \phi_2(x')]\ ,\ x_2 = (r, r\hat{\mathbf{x}})\ ,\ x_2' = (-r, r\hat{\mathbf{x}})\ , \quad (8.19)$$

where the first and second terms correspond to the time delay of the ingoing and outgoing light beams, respectively. One can further make a change of variables $\tilde{r} = r + L$ and shift the coordinates, $\mathbf{x} \to \mathbf{x} + L\hat{\mathbf{x}}$, such that

$$\delta T_2 = -\frac{1}{2}\int_0^L dr\ \ [\phi_2(x) + \phi_2(x)]\ ,\ x_2 = (-L + r, r\hat{\mathbf{x}})\ ,\ x_2' = (L - r, r\hat{\mathbf{x}})\ , \quad (8.20)$$

where we have replaced the symbol $\tilde{r}$ with r at the end. Since $\delta T_1 = \delta T_2$, Eqs. (8.18) and (8.20) indicate that $\phi_1 = \phi_2$.

This relation between $\phi_{1,2}$ does not hold only for these two causal diamonds, but rather the entire spacetime. One can easily see this by tiling the entire spacetime with light beams of the same length L as depicted in Fig. 8.3c. One can repeat the same argument above for every segment of this web of null rays to relate the pixellon models centered at any two adjacent endpoints. Since all of these null rays are connected, one can easily show a universal ϕ across the entire spacetime within the pixellon model. Thus, there is no need to distinguish ϕ in different causal diamonds.

On the other hand, this does not indicate that we can avoid using separate pixellon models for different light beams. The metric in Eq. (8.2) is designed to effectively describe the geontropic fluctuations of any causal diamond located at the origin of the local coordinates picked out by the metric. Thus, the light beams not propagating in the radial direction in these local coordinates cannot be described by the associated pixellon model. Furthermore, the argument of gauge invariance of the calculations in Ref. [11] does not hold for these non-radial light beams, since the angular directions of the metric were ignored in the proof. Nonetheless, one can always find another causal diamond in which the originally non-radial light beam becomes radial, e.g., the causal diamond located at the endpoints of this beam. For example, in Fig. 8.3c, the beams L_1 and L_2 can be described by the pixellon model centered at $\mathbf{x}_1$, but not the beam L_3, although it is in the same causal diamond of the beams $L_{1,2}$. Instead, one should compute the length fluctuations of the beam L_3 using the pixellon models at $\mathbf{x}_2$ or $\mathbf{x}_3$.

One might also worry, in this case, whether the length fluctuations at $\mathbf{x}_1$ have multiple inconsistent descriptions dependent on the causal diamond we choose, particularly with respect to their angular correlations. For example, since the dominant modes of pixellons are low-l modes [11], the pixellon model at $\mathbf{x}_2$ constrains the fluctuations at $\mathbf{x}_1$ to be mostly along $\mathbf{n}$. However, if one uses the pixellon model at $\mathbf{x}_3$, the fluctuations at $\mathbf{x}_1$ are mainly along $\mathbf{n}'$. This is not a contradiction in the pixellon model since light beams in different directions are probing different "polarizations" of pixellons, which control different local entangling surfaces. If one goes to the causal diamond at $\mathbf{x}_1$, the pixellon model consistently predicts that most fluctuations are along the radial direction, so fluctuations along both $\mathbf{n}$ and $\mathbf{n}'$ can potentially be excited. When the light beam is sent along one of these directions, the spherical symmetry is broken by exciting fluctuations mainly in this specific direction.

In this case, to compute the correlation of any two beams as depicted in Fig. 8.3b, we use the metric in Eq. (8.2) centered at $\mathbf{x}_1$ for beam L_1 and the one at $\mathbf{x}_2$ for beam L_2, but do not distinguish ϕ in these two metrics. Thus, Eq. (8.17) becomes

$$C(x_1, x_2) = \langle(\phi(x_1) + \phi(x_1'))(\phi(x_2) + \phi(x_2'))\rangle \,. \tag{8.21}$$

Using Eq. (8.3), we get

$$\begin{aligned}
&C(x_1,x_2)\\
&=4l_p^2\int\frac{d^3\mathbf{p}_1}{(2\pi)^3}\int\frac{d^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{\sqrt{4\omega_1(\mathbf{p}_1)\omega_2(\mathbf{p}_2)}}\cos[\omega_1(L_1-r_1)]\cos[\omega_2(L_2-r_2)]\\
&\left[\langle a_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_2}\rangle e^{-i[\omega_1 t_1-\omega_2 t_2-\mathbf{p}_1\cdot(\mathbf{x}_1+r_1\mathbf{n}_1)+\mathbf{p}_2\cdot(\mathbf{x}_2+r_2\mathbf{n}_2)]}+c.c.\right],\\
&=4l_p^2\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\sigma_{\text{pix}}(\mathbf{p})}{2\omega(\mathbf{p})}\cos[\omega(L_1-r_1)]\cos[\omega(L_2-r_2)]\left[e^{-i\omega\Delta t+i\mathbf{p}\cdot\delta\mathbf{x}}+c.c.\right],
\end{aligned}\tag{8.22}$$

where we have defined

$$\delta\mathbf{x}\equiv\Delta\mathbf{x}+r_1\mathbf{n}_1-r_2\mathbf{n}_2\,.\tag{8.23}$$

Plugging the occupation number in Eq. (8.12), the correlation function of the length fluctuations is given by

$$\begin{aligned}C(\Delta t,\Delta\mathbf{x},\mathbf{n}_{1,2})=\frac{al_p}{8L_1L_2}\int_0^{L_1}dr_1\int_0^{L_2}dr_2\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\omega^2(\mathbf{p})}\\
\cos\left[\omega(L_1-r_1)\right]\cos\left[\omega(L_2-r_2)\right]e^{-i\omega\Delta t+i\mathbf{p}\cdot\delta\mathbf{x}},\end{aligned}\tag{8.24}$$

where we have dropped the $c.c.$ term and hereafter assume for simplicity that the complex conjugate is included implicitly.

Eq. (8.24) is very similar to Eq. (41) of Ref. [11], except that $\delta\mathbf{x}$ also contains the difference between the origins of the two light beams. Evaluating the angular part of the momentum integral, we have

$$\begin{aligned}C(\Delta t,\Delta\mathbf{x},\mathbf{n}_{1,2})=\frac{al_p}{16\pi^2c_s^3L_1L_2}\int_0^{L_1}dr_1\int_0^{L_2}dr_2\int_0^{\infty}d\omega\;\cos\left[\omega(L_1-r_1)\right]\\
\cos\left[\omega(L_2-r_2)\right]\operatorname{sinc}\left[\omega\mathcal{D}(r_{1,2},\Delta\mathbf{x},\mathbf{n}_{1,2})/c_s\right]e^{-i\omega\Delta t},\end{aligned}\tag{8.25}$$

with

$$\mathcal{D}(r_{1,2},\Delta\mathbf{x},\mathbf{n}_{1,2})=|\delta\mathbf{x}|\,.\tag{8.26}$$

The PSD $\tilde{C}(\omega,\Delta\mathbf{x},\mathbf{n}_{1,2})$ is then given by

$$\begin{aligned}\tilde{C}(\omega,\Delta\mathbf{x},\mathbf{n}_{1,2})=\frac{al_p}{8\pi c_s^3N}\int_0^{L_1}dr_1\int_0^{L_2}dr_2\;\cos\left[\omega(L_1-r_1)\right]\\
\cos\left[\omega(L_2-r_2)\right]\operatorname{sinc}\left[\omega\mathcal{D}(r_{1,2},\Delta\mathbf{x},\mathbf{n}_{1,2})/c_s\right],\end{aligned}\tag{8.27}$$

where we have absorbed the normalization L_1L_2 into N. We make this redefinition for convenience since in certain experiments discussed later, PSDs similar to Eq. (8.27) appear but with $N \neq L_1 L_2$. Ref. [11] also considered inserting an IR cutoff $\omega^2(\mathbf{p}) \to \omega^2(\mathbf{p}) + \omega_{\mathrm{IR}}^2$ in Eq. (8.24), similar to Ref. [1], due to the interferometer's finite size and a better agreement of the resulting angular correlation with the prediction in Ref. [1]. In this case, the PSD becomes

$$\tilde{C}(\omega, \Delta\mathbf{x}, \mathbf{n}_{1,2}) \to \frac{\omega^2}{\omega^2 + \omega_{\mathrm{IR}}^2} \tilde{C}(\omega, \Delta\mathbf{x}, \mathbf{n}_{1,2}) \,. \tag{8.28}$$

In the case that the two arms have the same length L, Ref. [11] fixed $\omega_{\mathrm{IR}} = \frac{1}{L}$, which gave a better agreement with the angular correlations predicted in Refs. [1, 6].

One direct application of the results above is to compute the cross-correlation of length fluctuations across two different interferometers. Let the origins of two interferometers be at $\mathbf{x}_{I,II}$, respectively. For the interferometer at $\mathbf{x}_I$, let its two arms be along the directions $\mathbf{n}_{1,2}$ with length L_I. Similarly, let the two arms of the interferometer at $\mathbf{x}_{II}$ be along the directions $\mathbf{n}_{3,4}$ with length L_{II}. Define $\mathcal{T}(\mathbf{x}, t)$ to be the difference of length fluctuations of two arms within a single interferometer at position $\mathbf{x}$, the light beams of which are sent at time t. Then the cross-correlation of the time difference across two arms is

$$\begin{aligned} C_{\mathcal{T}}(\Delta t, \Delta\mathbf{x}, \mathbf{n}_{I,II}) &\equiv \left\langle \frac{\mathcal{T}_I(\mathbf{x}_I, t_1)\mathcal{T}_{II}(\mathbf{x}_2, t_2)}{4 L_I L_{II}} \right\rangle , \\ \mathcal{T}_I(\mathbf{x}_I, t_1) &= \delta T(t_I, \mathbf{x}_I, \mathbf{n}_2) - \delta T(t_I, \mathbf{x}_I, \mathbf{n}_1) \,, \\ \mathcal{T}_{II}(\mathbf{x}_{II}, t_2) &= \delta T(t_{II}, \mathbf{x}_{II}, \mathbf{n}_4) - \delta T(t_{II}, \mathbf{x}_{II}, \mathbf{n}_3) \,, \end{aligned} \tag{8.29}$$

where $\mathbf{n}_I = (\mathbf{n}_1, \mathbf{n}_2)$, $\mathbf{n}_{II} = (\mathbf{n}_3, \mathbf{n}_4)$, and $\Delta\mathbf{x} = \mathbf{x}_I - \mathbf{x}_{II}$ such that

$$\tilde{C}_{\mathcal{T}}(\omega, \Delta\mathbf{x}, \mathbf{n}_{I,II}) = \tilde{C}(\omega, \Delta\mathbf{x}, \mathbf{n}_{1,3}) + \tilde{C}(\omega, \Delta\mathbf{x}, \mathbf{n}_{2,4}) - \tilde{C}(\omega, \Delta\mathbf{x}, \mathbf{n}_{1,4}) - \tilde{C}(\omega, \Delta\mathbf{x}, \mathbf{n}_{2,3}) \,. \tag{8.30}$$

The equation above generally contains complicated geometric factors, and the integral within Eq. (8.25) cannot be easily evaluated for a generic geometry. Thus, we consider several specific configurations in the next section.

8.5 Interferometer-like experiments

In this section, we apply the results of Sec. 8.4 to several types of interferometer-like experiments: a single L-shaped interferometer (e.g., LIGO [12], CE [18, 19], NEMO [27]), the equilateral triangle configuration of multiple interferometers (e.g., LISA [13], ET [20]), and optically-levitated sensors [54, 55].

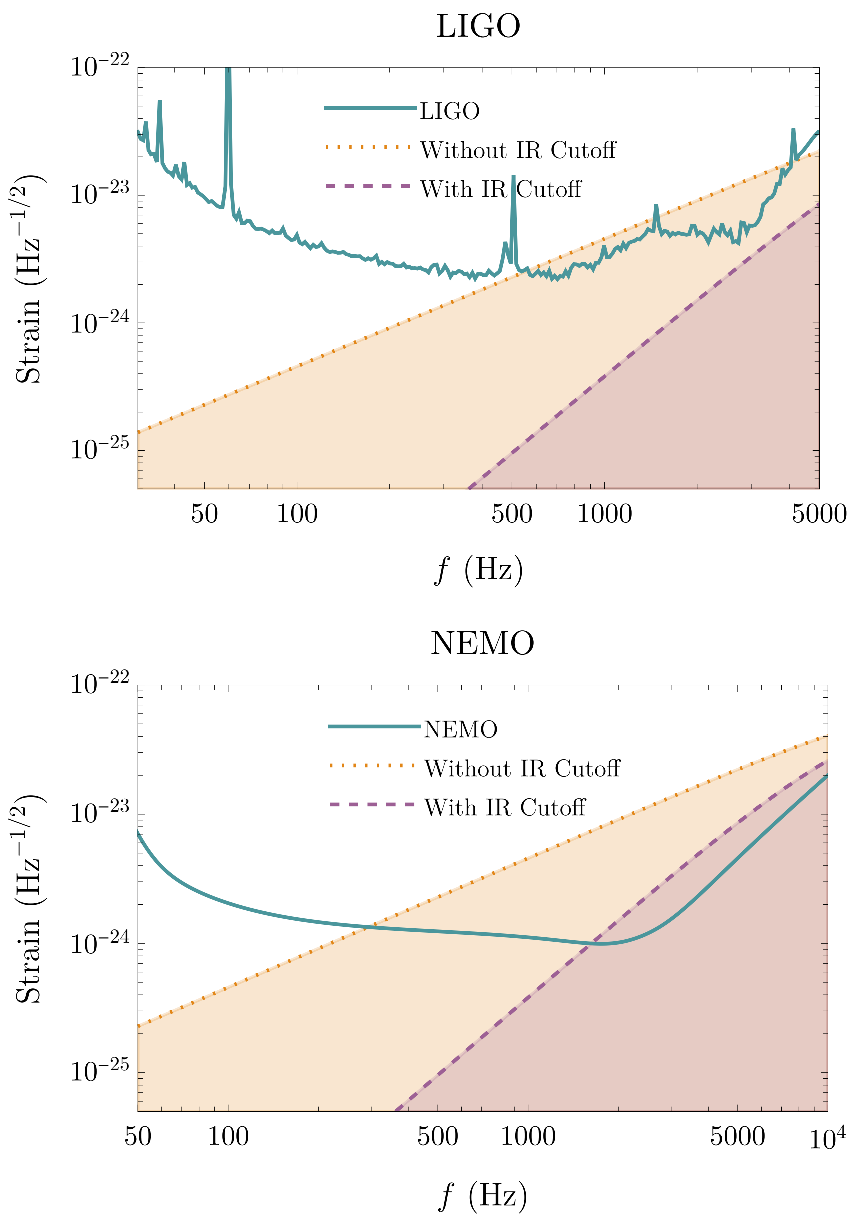


Figure 8.4: Pixellon strain (dashed and dotted lines) overlaid with the strain sensitivities for LIGO [12] and NEMO [70] (solid lines). The LIGO data was obtained from the Livingston detector, and the NEMO data omits suspension thermal noise. The dotted lines give the pixellon strain from Eq. (8.35) computed without an IR cutoff, and the dashed lines give the same quantity including the IR cutoff from Eq. (8.28). We again compute the pixellon strain with $\alpha = 1$.

8.5.1 Single L-shaped interferometer

Ref. [11] calculated the auto-correlation of length fluctuations in an L-shaped interferometer due to geontropic fluctuations. In this case, we have $\mathbf{x}_I = \mathbf{x}_{II}$ and $\mathbf{n}_I = \mathbf{n}_{II}$, so we can set the origin of the coordinates to coincide with the beam splitter of the interferometer. Furthermore, we can align the x–y plane with the plane of the interferometer and choose the x-axis to be along the first arm of the interferometer. Then the whole configuration is determined by the separation angle θ between two arms. In this case, the first two terms are the same in Eq. (8.30) and similarly for the last two terms, so Eq. (8.30) reduces to

$$\tilde{C}_{\mathcal{T}}(\omega, \theta) = 2\tilde{C}(\omega, 0) - 2\tilde{C}(\omega, \theta)\,, \tag{8.31}$$

which is consistent with Eq. (45) of Ref. [11]. The spectrum $\tilde{C}(\omega, \theta)$ is given by Eq. (8.27) after setting $L_1 = L_2 = L$, where L is the length of the interferometer, i.e.,

$$\begin{aligned}\tilde{C}(\omega, \theta) = \frac{a l_p}{8\pi c_s^3 L^2} \int_0^L dr_1 \int_0^L dr_2 \ \operatorname{sinc}\left[\omega \mathcal{D}(r_1, r_2, \theta)/c_s\right] \\ \cos\left[\omega(L - r_1)\right] \cos\left[\omega(L - r_2)\right]\,,\end{aligned} \tag{8.32}$$

where the distance factor $\mathcal{D}$ is now completely determined by r_1, r_2, and θ,

$$\mathcal{D}(r_1, r_2, \theta) = \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta}\,, \tag{8.33}$$

To compare against the strain sensitivity of real experiments, one needs to first convert Eq. (8.32) to the one-sided noise strain S_h defined by Refs. [14, 71]

$$\sqrt{S_h(f)} = \sqrt{2 \int_{-\infty}^{\infty} \left\langle \frac{\Delta L(\tau)}{L} \frac{\Delta L(0)}{L} \right\rangle e^{-2\pi i f \tau} d\tau}\,, \tag{8.34}$$

which has units of $\mathrm{Hz}^{-1/2}$. In many of these interferometers, Fabry-Pérot cavities are used to increase the sensitivity, in which light travels multiple round trips. By converting the strain sensitivity to the phase sensitivity, Ref. [11] showed that the geontropic signal does accumulate in Fabry-Pérot cavities since the output is linear in the phase shift of the light. Thus, it is legitimate to compare our PSD to the strain sensitivity of these experiments. From Eqs. (8.29) and (8.34), Ref. [11] found that

$$\sqrt{S_h(f)} = \sqrt{2\tilde{C}_{\mathcal{T}}(\omega, \theta)}\,. \tag{8.35}$$

Nonetheless, the signal's shape is determined by the geometry of one light-crossing. For example, we expect that the signal peak is at $\omega L \sim 1$, where L is the length of the interferometer instead of the total distance traveled across multiple light-crossings.

Using Eqs. (8.31)–(8.32), Ref. [11] computed the PSD of the pixellon model in several L-shaped interferometers (Holometer [14], GEO-600 [15], and LIGO [12]) and one set of interferometers in LISA [13], and compared the signal to their strain sensitivities. It was found that GEO-600 and LISA are unlikely to detect geontropic fluctuations due to their relatively low peak sensitivity (at $\omega L \sim 1$), while LIGO and Holometer respectively constrain the α-parameter to be $\alpha \lesssim 3$ and $\alpha \lesssim 0.7$ (with an IR cutoff), and $\alpha \lesssim 0.1$ and $\alpha \lesssim 0.6$ (without an IR cutoff) at 3σ significance. Note that the LIGO sensitivity data that we have used here and in Ref. [11] is that from Ref. [12] with the quantum shot noise removed (i.e., the gray curve in Fig. 2 of Ref. [12]) by the quantum-correlation technique in Ref. [52]. Nonetheless, this technique only removes the expectation value of the shot noise but not its variance [72], limiting the extent to which we can dig under the shot noise. More specifically, with a frequency band of Γ and an integration time of T, we expect the noise suppression factor to be $\sim (\Gamma T)^{1/4}$ in amplitude — or until the next underlying noise is revealed. In the particular case of LIGO, that underlying noise includes coating and suspension thermal noise at low frequencies, and laser noise at high frequencies. Further studying these underlying noise sources in LIGO can in principle put more stringent upper limits on the geontropic noise.

Besides the GW detectors above, there are other future L-shaped interferometers to be considered but not included in Ref. [11]. The most important ones are the third-generation GW detectors: CE [18, 19] and ET [20]. CE is a ground-based broadband GW detector using dual-recycled Fabry-Pérot Michelson interferometers with perpendicular arms. CE will have two sites with several potential designs: a 20 km interferometer paired with a 40 km interferometer, or a pair of 20 km or 40 km interferometers. As largely a scale-up of Advanced LIGO [18], CE will operate at room temperature with a fused-silica coating of mirrors to reduce thermal noise, and degenerate optical parametric amplifiers injecting squeezed light with low phase uncertainty to reduce quantum noise (shot noise) at high frequency [73].

ET is an equilateral triangle configuration of three independent nested detectors, each of which contains two dual-recycled Fabry-Pérot Michelson interferometers with arms of length 10 km (plotted in Fig. 8.5) for low- and high-frequency detections, respectively. ET will be built underground to reduce seismic noise. Cryogenic systems are used to reduce thermal noise by cooling the optical systems to 10–20 K at low frequency, while squeezed light (frequency-dependent) is also inserted to reduce quantum noise at high frequency [41].

As briefly discussed in Sec. 8.1 and shown in Fig. 8.2, for the benchmark value $\alpha = 1$, the PSD of the geontropic signal overwhelms the strain sensitivity of CE and

ET by about two orders of magnitude for $f \sim 1$ kHz. For CE, we have considered both the interferometers of length 20 km and 40 km. For ET, we have computed the auto-correlation of a single interferometer within the entire configuration. A study of the cross-correlation of different interferometers is carried out in Sec. 8.5.2.

Besides ET and CE, another next-generation GW detector is NEMO [27], a Michelson interferometer with perpendicular Fabry-Pérot arms of length 4 km. Although with less sensitivity than the full third-generation detectors in general, NEMO is important for testing technological developments to be used in the third-generation detectors while making interesting scientific discoveries, such as understanding the compositions of NSs. Due to its interest in binary NS mergers, NEMO specializes in high-frequency events with its optimal sensitivity at $f \sim 1$–4 kHz [27]. As plotted in Fig. 8.4, within the optimal sensitivity of NEMO, the geontropic signal exceeds the strain sensitivity by about one order of magnitude. Thus, the geontropic signal must be constrained before these next-generation GW detectors can detect other high-frequency events. For future detectors, we have compared the geontropic signal with their design sensitivities, without considering removal of shot noise via the quantum-correlation approach — even though at high frequencies, where the constraints for geontropic noise are the best, these detectors are limited by shot noise. It can be anticipated that at these frequencies, these detectors' shot noise dominates over other types of noise by a significant factor. In this way, these detectors are capable of putting much more stringent bounds on the geontropic α parameter.

8.5.2 Equilateral triangle configurations

In this subsection, we consider configurations of multiple interferometers with certain geometries. For GW detections, these different geometries are helpful in retrieving the polarization of GWs. One important configuration is the equilateral triangle configuration of three interferometer arms, such as LISA [13], or three partially overlapping independent detectors, such as ET [20], as shown in Fig. 8.5. For LISA, the signals of different arms can be time shifted and linearly combined to form virtual Michelson interferometers [74, 75]. Nonetheless, as found in Ref. [11] and discussed in Sec. 8.5.1, LISA is not promising for detecting geontropic signals, so we will focus on the specific configuration of ET.

In Sec. 8.5.1, we computed the auto-correlation of a single interferometer within ET. Although the single-detector quantum-correlation technique discussed in Sec. 8.5.1 allows us to dig under the shot noise, we are still limited by non-quantum noises. On

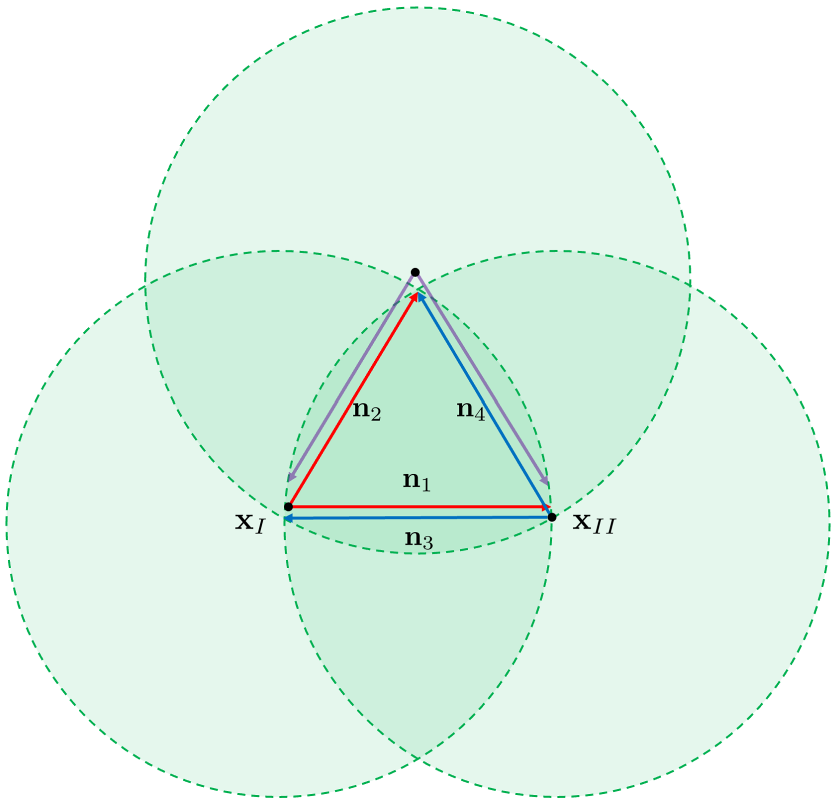


Figure 8.5: Setup of ET. The red, blue, and purple rays correspond to the three detectors in ET, where we have only shown one of the two interferometers within each detector. We choose not to plot the mirrors at the endpoints of the light beams for simplicity.

the other hand, geontropic fluctuations modeled by the pixellon are correlated across different ET detectors. For those uncorrelated non-quantum noises, cross-correlating multiple ET detectors allows us to dig under them with a suppression factor of $\sim (\Gamma T)^{1/4}$. This motivates the calculation of the cross-correlation of different detectors within interferometer configurations such as ET.

Let us consider one set of two interferometers across different detectors within ET, e.g., the red and blue detectors in Fig. 8.5, and pick the origin of coordinates at the origin of the red detector $\mathbf{x}_1$. Let us also pick the x–y plane to be the plane of the interferometers, with the x-axis along $\mathbf{n}_1$. In this case,

$$\begin{aligned} &\mathbf{x}_I = 0\,,\ \mathbf{x}_{II} = L\hat{\mathbf{x}}\,,\ \mathbf{n}_1 = \hat{\mathbf{x}}\,,\ \mathbf{n}_2 = \frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{y}}\,, \\ &\mathbf{n}_3 = -\hat{\mathbf{x}}\,,\ \mathbf{n}_4 = -\frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{y}}\,. \end{aligned} \tag{8.36}$$

Here, we have assumed that the arms along the same line completely overlap with

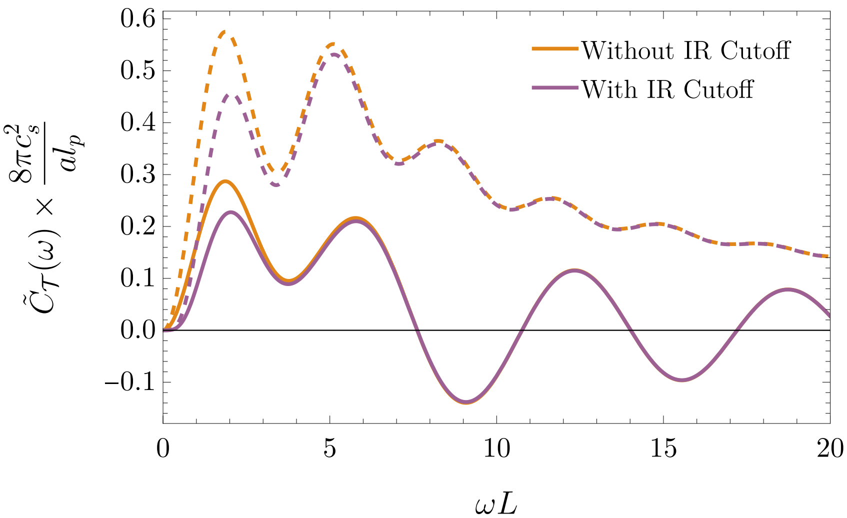


Figure 8.6: The PSD $\tilde{C}_{\mathcal{T}}(\omega)$ of the cross-correlation function of two sets of interferometers within a triangular configuration like ET [Eq. (8.38), solid lines], together with the corresponding auto-correlation $\tilde{C}_{\mathcal{T}}(\omega, \theta = \frac{\pi}{3})$ of a single interferometer within this configuration [Eq. (8.31), dashed lines].

each other (i.e., the arms along $\mathbf{n}_1$ and $\mathbf{n}_3$). In reality, there is a finite separation between these arms, which can be dealt with via the general procedure in Sec. 8.4. Then one can compute $\mathcal{D}(r_{i,j}, \Delta\mathbf{x}, \mathbf{n}_{i,j})$ for all the combinations in Eq. (8.30), i.e.,

$$\begin{aligned}
\mathcal{D}_{13}(r_1, r_2) &= |r_1 + r_2 - L| \,, \\
\mathcal{D}_{24}(r_1, r_2) &= \frac{1}{2}\sqrt{(2L - r_1 - r_2)^2 + 3(r_1 - r_2)^2} \,, \\
\mathcal{D}_{14}(r_1, r_2) &= \frac{1}{2}\sqrt{(2L - 2r_1 - r_2)^2 + 3r_2^2} \,, \\
\mathcal{D}_{32}(r_1, r_2) &= \mathcal{D}_{14}(r_1, r_2) \,.
\end{aligned} \tag{8.37}$$

Here, we have defined $\mathcal{D}_{ij}(r_1, r_2)$ such that r_1 is the integration variable along the arm with direction $\mathbf{n}_i$, and r_2 is the integration variable along the arm with direction $\mathbf{n}_j$. Plugging Eq. (8.37) into Eq. (8.30), we get

$$\begin{aligned}
\tilde{C}_{\mathcal{T}}(\omega) = \frac{a l_p}{8\pi c_s^3 L} \int_0^L dr_1 \int_0^L dr_2 \; & \cos\left[\omega(L - r_1)\right] \cos\left[\omega(L - r_2)\right] \\
& \{\operatorname{sinc}\left[\omega \mathcal{D}_{13}(r_1, r_2)/c_s\right] + \operatorname{sinc}\left[\omega \mathcal{D}_{24}(r_1, r_2)/c_s\right] \\
& -2 \operatorname{sinc}\left[\omega \mathcal{D}_{14}(r_1, r_2)/c_s\right]\} \,,
\end{aligned} \tag{8.38}$$

the result of which is plotted in Fig. 8.6.

Besides the equilateral triangle configuration of ET, one can compute the response of other geometries of interferometers to the pixellon model following the procedure in Sec. 8.4. For example, one can consider two or multiple interferometers with the same length located at the same origin but rotated from each other by certain angles as depicted in Ref. [74]. There are even more complicated geometries, such as the twin 3-d interferometers that will be built at Cardiff University [76]. The authors in Ref. [76] claimed that the angular correlations of geontropic fluctuations, as discussed in detail in Refs. [1, 3, 6, 11], especially the transverse correlations due to the low-ℓ modes, can be probed by this geometry. While, in principle, the geontropic signal can be computed for such a complicated interferometer geometry, the pixellon model may not adequately encapsulate the underlying physics of the VZ effect. Further, first-principles calculations of geontropic fluctuations assume a simple causal diamond radiating outward from a beam splitter. One major feature of the twin 3-d interferometers in Ref. [76] is that the interferometer arms are bent at mirrors $\mathrm{MM_A}$ and $\mathrm{MM_B}$ (see Fig. 1 of Ref. [76]), so the causal diamond of the whole apparatus is distorted. The bent-arm configuration explicitly breaks spherical symmetry, which the previous calculations [3, 11] relied on. Specifically, the pixellon metric in Eq. (8.2) captures metric fluctuations only along interferometer arms that extend radially outward from a beam splitter. One can decompose the bent interferometer arms into segments of straight arms, and, assuming the pixellon model pertains to such a causal diamond, attempt to apply the pixellon model to each segment by choosing *local* coordinates centered at the beam splitter, $\mathrm{MM_A}$, and $\mathrm{MM_B}$, respectively. However, the major obstacle for this procedure is that at $\mathrm{MM_A}$ (or $\mathrm{MM_B}$) there does not exist a closed causal diamond, because light continues to traverse past $\mathrm{MM_A}$ (or $\mathrm{MM_B}$) until it reaches $\mathrm{EM_A}$ (or $\mathrm{EM_B}$) or the beam splitter. Since the calculations in Refs. [3, 11] require a closed causal diamond such that the observable computed is manifestly gauge invariant, one first needs to ascertain whether the procedures in Ref. [11] for computing gauge-invariant quantities are still valid when piecing together these non-closed causal diamonds. Due to these complications, we do not attempt to apply the pixellon model to the Cardiff experiment, as we believe that an accurate prediction for such bent-arm configurations will require a more direct, first-principles calculation requiring better theoretical control than current technology allows. In the next subsection, we focus on another interferometer-like experiment, the optically-levitated sensor.

8.5.3 Optically-levitated sensor

In this subsection, we study the response of the optically-levitated sensor in Refs. [54, 55] to geontropic fluctuations described by the pixellon model. To understand the

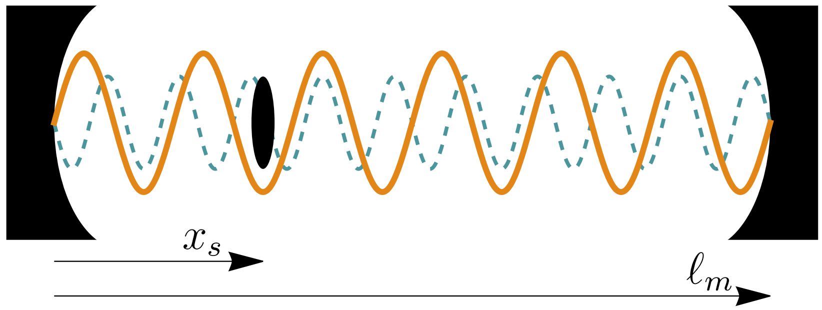


Figure 8.7: Schematic diagram of the optically-levitated sensor as described in Refs. [54, 55]. A dielectric sphere or microdisk is trapped in an anti-node of an optical cavity (solid orange). A second laser (dashed blue) is used to cool the sensor and read out its position. Transverse motion is cooled by additional lasers (not shown).

working principle of the optically-levitated sensor, let us first consider its response to GWs following Ref. [54], working in the local Lorentz frame with origin at the input mirror. Let the unperturbed distance between the optical cavity mirrors be ℓ_m, and the unperturbed distance from the input mirror to the sensor in its trap minimum be x_s. Under a passing GW perpendicular to the cavity with strain h, the proper distances to the mirror and sensor are both shifted,

$$\delta x_s = \frac{1}{2} h x_s \,, \quad \delta \ell_m = \frac{1}{2} h \ell_m \,. \tag{8.39}$$

The new position of the trap minimum can be found from the condition

$$k_t(\ell'_m - x'_{\text{min}}) = k_t(\ell_m - x_{\text{min}}) = \left(n + \frac{1}{2}\right)\pi \,, \tag{8.40}$$

where n is an integer, and k_t is the wavenumber of the trapping laser. The shift of the trap minimum is then given by $\delta x_{\text{min}} = \ell'_m - \ell_m = \delta \ell_m$. Here, we have assumed that the trapping laser has a constant frequency inside the cavity. Thus, the sensor is displaced from its trap minimum by an amount given in Ref. [54] as

$$\Delta X \equiv \delta x_s - \delta x_{\text{min}} = \frac{1}{2} h (x_s - \ell_m) + O(h^2) \,. \tag{8.41}$$

This displacement will result in an oscillatory driving force on the sensor. If the GW frequency matches the trapping frequency ω_0 of the sensor, the driving force will resonantly excite the sensor. The corresponding oscillations can then be measured. When $x_s \ll \ell_m$, the effect of the GW is maximized.

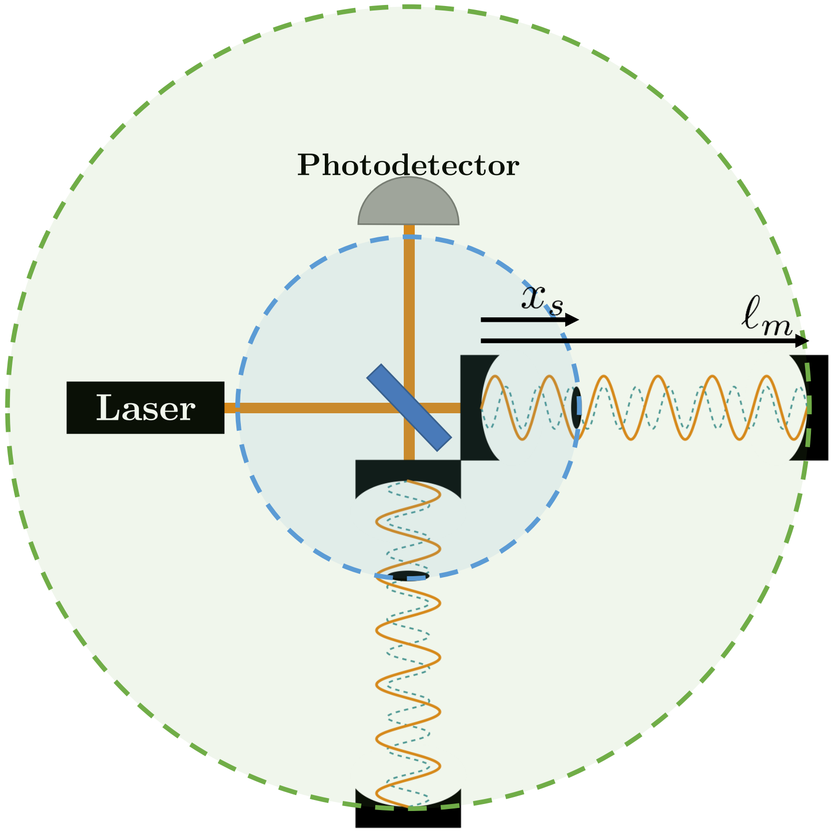


Figure 8.8: Two levitated sensors inserted into the Fabry-Pérot cavities of a Michelson interferometer, as described in Ref. [55]. The entangling surfaces corresponding to the two arms of length x_s and ℓ_m are marked by the blue and green shaded circles, respectively. Note that this diagram ignores the distances between the beam splitter and the input mirrors of the two cavities.

For the pixellon model, the response of the optically-levitated sensor can be calculated similarly. In our case, δx_s and $\delta\ell_m$ are given by

$$\delta x_s = -\frac{1}{4}\int_0^{x_s} dr\,[\phi(x) + \phi(x')] \ , \tag{8.42}$$

$$\delta\ell_m = -\frac{1}{4}\int_0^{\ell_m} dr\,[\phi(y) + \phi(y')] \ , \tag{8.43}$$

where

$$\begin{aligned} x &= (t_x - x_s + r, r\mathbf{n}) \,, \quad & x' &= (t_x + x_s - r, r\mathbf{n}) \,, \\ y &= (t_\ell - \ell_m + r, r\mathbf{n}) \,, \quad & y' &= (t_\ell + \ell_m - r, r\mathbf{n}) \,, \end{aligned} \tag{8.44}$$

and the start times of each beam are chosen to be $t_x - x_s$ and $t_\ell - \ell_m$. Note the additional factor of $\frac{1}{2}$ as compared to Eq. (8.14), since the lengths ℓ_m and x_s are one-half of the corresponding round-trip time delays when there are no geontropic fluctuations. Within a single arm, since there is only a single beam measuring the position of the sensor, we can choose

$$t_x = t + x_s\,, \quad t_\ell = t + \ell_m \tag{8.45}$$

such that the start times of the beam probing the sensor and the end mirror are the same. Notice that, in general, two independent pixellon models should be used for the shorter and longer arms. Nevertheless, since both spherical entangling surfaces are located at the same origin, as depicted in Fig. 8.8, and the pixellon fields ϕ are universal across these two causal diamonds as discussed in Sec. 8.4, the forms of Eqs. (8.42) and (8.43) are very similar. This is consistent with the fact that the metric in Eq. (8.2) is spatially conformal.

The displacement of the levitated sensor from its trap minimum is then given by

$$\Delta X = -\frac{1}{4}\int_0^{x_s} dr\,[\phi(x)+\phi(x')] + \frac{1}{4}\int_0^{\ell_m} dr\,[\phi(y)+\phi(y')]\,. \tag{8.46}$$

Note that Eq. (8.46) is similar, but not identical to, the round-trip time of a photon traveling from position x_s to ℓ_m, i.e.,

$$\begin{aligned}\Delta X|_{x_s\leftrightarrow\ell_m} &= \frac{1}{4}\int_{x_s}^{\ell_m} dr\,[\phi(y)+\phi(y')]\,,\\ y = (t-\ell_m+r, r\mathbf{n})\,, &\quad y' = (t+\ell_m-r, r\mathbf{n}).\end{aligned} \tag{8.47}$$

Using Eq. (8.47) instead of Eq. (8.46) would give a PSD identical to Eq. (8.32) with length $L = \ell_m - x_s$.

We can then define the correlation function of ΔX as

$$C^{\Delta X}(\Delta t,\theta) \equiv \left\langle \frac{\Delta X(t_1,\mathbf{n}_1)\Delta X(t_2,\mathbf{n}_2)}{(\ell_m - x_s)^2}\right\rangle\,, \tag{8.48}$$

where the unit vectors $\mathbf{n}_i$ parameterize the orientations of the two levitated sensor arms, and the angle θ between them is given by $\cos(\theta) = \mathbf{n}_1\cdot\mathbf{n}_2$. The difference between the beam start times is $\Delta t \equiv t_1 - t_2$. Note that the normalization of $C^{\Delta X}$ assumes that the characteristic length of the system is $\ell_m - x_s$, as per the above discussion. Using Eq. (8.46), we find that

$$\begin{aligned}&C^{\Delta X}(\Delta t,\theta)\\ &= \frac{1}{16(\ell_m-x_s)^2}\Bigg[\int_0^{x_s} dr_1\int_0^{x_s} dr_2\, C(x_1,x_2) - \int_0^{x_s} dr_1\int_0^{\ell_m} dr_2\, C(x_1,y_2)\\ &\quad - \int_0^{\ell_m} dr_1\int_0^{x_s} dr_2\, C(y_1,x_2) + \int_0^{\ell_m} dr_1\int_0^{\ell_m} dr_2\, C(y_1,y_2)\Bigg]\,,\end{aligned} \tag{8.49}$$

where $C(x, y)$ is defined in Eq. (8.21). The first and last terms above are correlations between the arms with the same length (either $L = x_s$ or $L = \ell_m$). In contrast, the second and third terms correlate arms with different lengths, i.e., the arm of $L = x_s$ with the arm of $L = \ell_m$.

Following a similar calculation as the one to obtain Eq. (8.27), we find the two-sided PSD $\tilde{C}^{\Delta X}(\omega, \theta)$ as

$$\tilde{C}^{\Delta X}(\omega, \theta) = \left[\tilde{C}^{\Delta X}(\omega, x_1, x_2) + \tilde{C}^{\Delta X}(\omega, y_1, y_2) - 2\tilde{C}^{\Delta X}(\omega, x_1, y_2)\right] , \tag{8.50}$$

where the first two terms are given by Eq. (8.27) with $N = (\ell_m - x_s)^2$ and $\mathcal{D}(r_1, r_2, \theta) = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta)}$. The last term, which corresponds to the correlation between the arms of length $L = x_s$ and $L = \ell_m$, carries an additional geometrical factor of $\cos\left[\omega(\ell_m - x_s)\right]$ due to the difference in the sizes of the causal diamonds, i.e.,

$$\begin{aligned} \tilde{C}^{\Delta X}(\omega, x_1, y_2) = \frac{a l_p}{8\pi c_s^3 (\ell_m - x_s)^2} \int_0^{x_s} dr_1 \int_0^{\ell_m} dr_2 \; \cos\left[\omega(x_s - r_1)\right] \\ \cos\left[\omega(\ell_m - r_2)\right] \cos\left[\omega(\ell_m - x_s)\right] \operatorname{sinc}\left[\omega \mathcal{D}(r_1, r_2, \theta)/c_s\right] . \end{aligned} \tag{8.51}$$

We can also define $\tilde{C}_{\mathcal{T}}^{\Delta X}(\omega, \theta)$ as in Eq. (8.31) via

$$\tilde{C}_{\mathcal{T}}^{\Delta X}(\omega, \theta) = 2\left[\tilde{C}^{\Delta X}(\omega, 0) - \tilde{C}^{\Delta X}(\omega, \theta)\right] . \tag{8.52}$$

In the limit $x_s \to 0$, only the second term in Eq. (8.50) is nonzero, corresponding to the length fluctuations of an interferometer of size $L = \ell_m$. Thus, the levitated sensor can be treated as an ordinary interferometer when x_s is sufficiently small. This is confirmed by Fig. 8.9a, where we plot the interferometer PSD from Eq. (8.32) against the levitated sensor PSD from Eq. (8.50), setting $x_s = \ell_m/50$ and neglecting the IR cutoff for the purpose of demonstration. The interferometer PSD is given by the dashed lines, whereas the levitated sensor PSD is given by the solid lines. We can see that, as expected, the PSDs of these two different types of experiments are very similar in the limit of small x_s. In Fig. 8.9b, we show a similar comparison but instead pessimistically set $x_s = \ell_m/10$. For this larger value of x_s, the PSD for the levitated sensor becomes somewhat larger in magnitude compared to that of the ordinary interferometer, but retains a similar shape. In the limit of $\omega \to 0$, we have

$$\tilde{C}_{\mathcal{T}}^{\Delta X}(\omega, \theta) = \frac{a l_p}{48\pi c_s^5} \omega^2 (\ell_m + x_s)^2 (1 - \cos\theta) + O(\omega^4). \tag{8.53}$$

From the scaling $\tilde{C}_{\mathcal{T}}^{\Delta X}(\omega, \theta) \propto (\ell_m + x_s)^2$, one can see that the signal increases as x_s increases, which is a result of treating the system as two sets of causal diamonds.

However, we expect the above treatment to break down beyond the limit of $x_s \ll \ell_m$. We emphasize that this calculation is not intended to be fully rigorous, but rather seeks to provide a heuristic description of the pixellon model in a levitated sensor experiment. Nevertheless, we continue to expect that the levitated sensor will behave similarly to an L-shaped interferometer in the limit of small x_s.

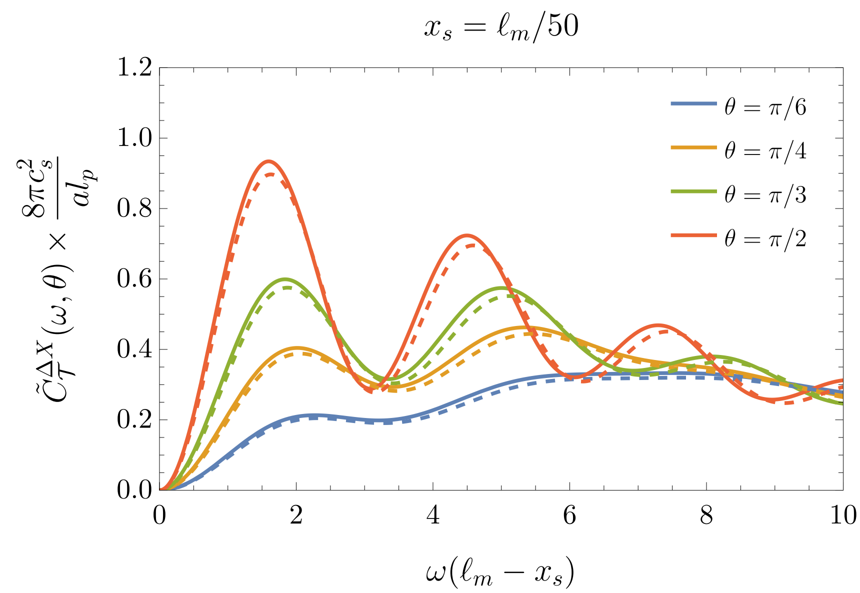
Next, let us compare the PSD found above to the predicted strain sensitivity of optically-levitated sensor experiments. The thermal-noise-limited minimum detectable strain of the optically-levitated sensor at temperature $T_{\rm CM}$ is given by Refs. [54, 55] as

$$h_{\rm limit} = \frac{4}{\omega_0^2 \ell_m} \sqrt{\frac{k_B T_{\rm CM} \gamma_g b}{M} \left[1 + \frac{\gamma_{\rm sc} + R_+}{N_i \gamma_g}\right]} H(\omega_0)\,, \tag{8.54}$$

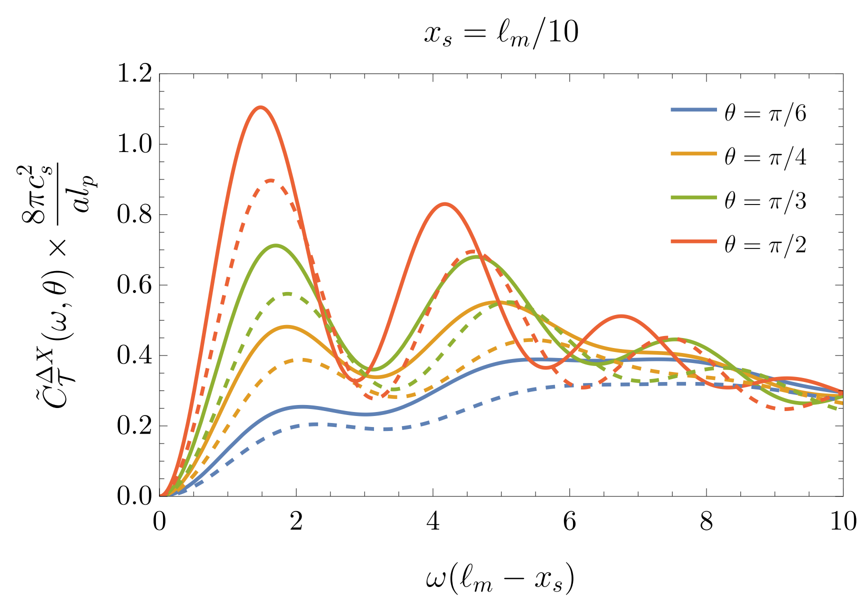
where ω_0 is the trapping frequency, γ_g is the gas-damping coefficient, $\gamma_{\rm sc}$ is the scattered photon-recoil heating rate, b is the bandwidth, M is the mass of the sensor, and $N_i = k_B T_{\rm CM}/\hbar\omega_0$ is the mean initial phonon occupation number. The cavity response function is $H(\omega) = \sqrt{1 + (2\mathcal{F}/\pi)^2 \sin^2(\omega \ell_m/c)}$, where $\mathcal{F}$ is the finesse of the cavity. Detailed expressions for all of these quantities can be found in Refs. [54, 55].

The peak frequency response of the experiment occurs at the trapping frequency ω_0, at which oscillations of the levitated sensor are resonantly enhanced. The trapping frequency can be widely tuned via the laser intensity [55]. Thus, the sensitivity curve for the levitated sensor can be obtained by continuously varying the locus of the sensitivity curve for each fixed value of ω_0, as given by Eq. (8.54).

In Fig. 8.10, we plot the strain sensitivity of the levitated sensor experiment from Ref. [55] (with a sensor consisting of a stack of dielectric disks) against the PSD of the pixellon model from Eqs. (8.50)–(8.52). In Fig. 8.10b, we additionally include an IR cutoff $\omega_{\rm IR} = 1/L$ as in Eq. (8.28), where we take the characteristic length of the system to be $L = \ell_m - x_s$. This choice comes from the comparison of the displacement ΔX with the length fluctuations of an interferometer of size $\ell_m - x_s$, as discussed with relation to Eq. (8.47). Note that Ref. [55] uses a 300 kHz upper bound for their sensitivity curves, citing limitations of power absorption by the suspended sensor. From these plots, we observe that the levitated sensor would only be competitive for detecting the geontropic signal at $\ell_m \gtrsim 100$ m. At the time of writing, a 1 m prototype of this experiment is under construction, and a 100 m device is at the concept stage [17, 55]. That these proposed levitated sensor experiments



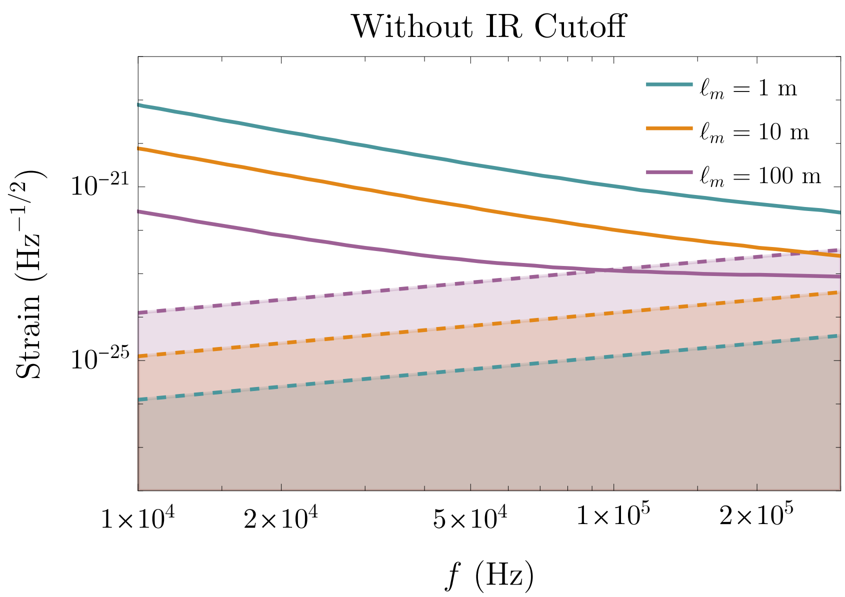
(a) Pixellon PSD with $x_s = \ell_m/50$.



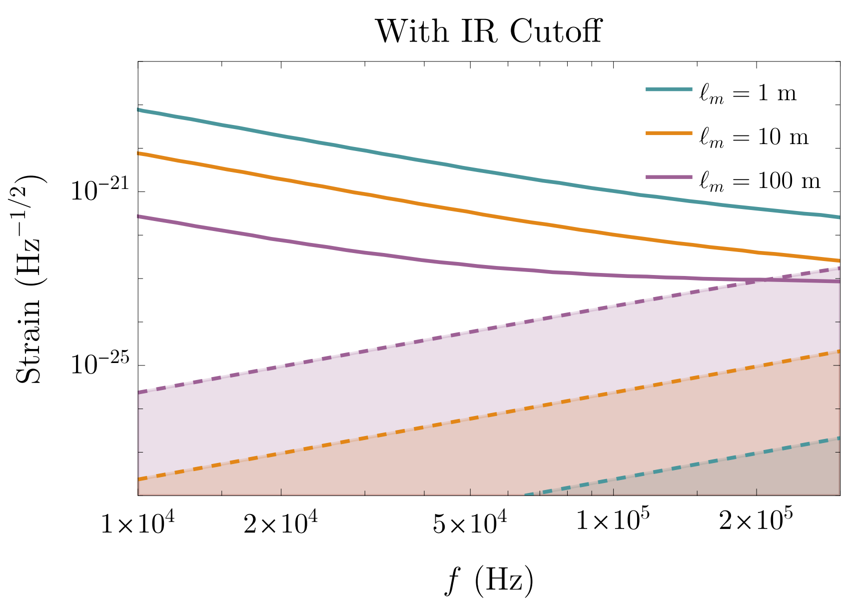
(b) Pixellon PSD with $x_s = \ell_m/10$.

Figure 8.9: Pixellon PSD $\tilde{C}_{\mathcal{T}}^{\Delta X}(\omega,\theta)$ as it would appear in an optically-levitated sensor [Eq. (8.52), solid lines] shown alongside the PSD of an ordinary L-shaped interferometer $\tilde{C}_{\mathcal{T}}(\omega,\theta)$ [Eq. (8.31), dashed lines]. We take the length of the L-shaped interferometer to be $L = \ell_m - x_s$. All PSDs are computed without an IR cutoff.

are not competitive for constraining the pixellon model is expected: their reach in frequency is such that $\omega\ell_m \ll 1$, whereas the pixellon signal is expected to peak at $\omega\ell_m \sim 1$. Finally, let us note that, although the levitated sensors do not move



(a) Strain without an IR cutoff.



(b) Strain with an IR cutoff $\omega_{\mathrm{IR}} = 1/(\ell_m - x_s)$.

Figure 8.10: The pixellon strain (dashed lines) overlaid with the predicted strain sensitivity for a stacked-disk levitated sensor (solid lines), as given by Fig. 3 of Ref. [55]. The color coding corresponds to the size ℓ_m of the levitated sensor. The pixellon strain is computed from Eq. (8.52), and we set $x_s = \ell_m/10$ throughout.

along geodesics, but instead have amplified non-geodesic movements, the same amplification factors are applied to motion induced by the noisy thermal force. In this way, because the device is limited by thermal noise [17, 55], comparing the displacement (8.46) and the thermal strain (8.54), as if there were no trapping, still leads to the correct thermal-noise-limited sensitivity.

8.6 Conclusions

We have considered the effect of the geontropic signal, from the VZ effect proposed in Refs. [1, 2, 4–6], specifically as modeled in Refs. [3, 11], on next-generation terrestrial GW detectors. We have found that if GQuEST observes spacetime fluctuations from the pixellon, Cosmic Explorer and the Einstein Telescope will have a large background to astrophysical sources from vacuum fluctuations in quantum gravity with which to contend. On the other hand, LISA and other lower-frequency devices are insensitive to this signal. Note that in making these predictions we have assumed the physical equivalence of the pixellon model with the VZ effect for interferometer observables, the proof of which is still the subject of ongoing first-principles calculations. Even so, given how large the geontropic signal is expected to be in future GW observatories, our results may inform optimal designs for GW observatories, whether searching for quantum or classical sources of GWs.

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