Contributions to the Theories of Topological Groups

Dissertation by

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In partial fulfillment of the requirements
for the degree of Doctor of Philosophy

California Institute of Technology
Pasadena, California
October 1944

It is a pleasure to acknowledge my indebtedness to Professor A. D. Michal and to express my profound gratitude to him for his advice and encouragement.

CONDENSATION

The brief abstracts of the papers contained in this thesis are copied from the Bulletin of the American Mathematical Society.

Abstract (48-1-96)

An nth order differential is defined for a function $F(x)$ with arguments and values in topological groups and increments in the central subgroup of the argument space with a relativized topology and a generation postulate. The fundamental theorems on unicity, continuity, linear combinations, and iterative functions are then proved.

Abstract (48-5-158)

After an abstract calculus of finite differences is defined, functional definitions of a monomial and polynomial for elements of the group as increments are given. The theorem on the homogeneity of a polynomial is proved for central and arbitrary differences; for central differences the difference being a function of the increment alone implies the difference is a monomial; the independence of the central difference of polynomials and the unique decomposition for the abelian valued case are made to depend on the product of a Vandermonde determinant and a finite product of binomial coefficients. The theory is essentially a generalization of the work of Van der Lijn on abstract polynomials in abelian groups.

Abstract (50-1-48)

By analyzing an example formulated by A. Tychonoff, the spaces $\mathcal{H}^{\rho}$, $0 \leq \rho < \infty$, are defined in a manner analogous to that for classical Hilbert space; some basic properties such as linearity, necessary and sufficient conditions for normability, separability, and sufficient conditions for local convexness are proved.

Introduction

The doctoral dissertation which is submitted here in partial fulfillment of the requirements for the degree of doctor of philosophy is a collection of the author's published papers on the field of abstract space theory.

The first and second articles, "Differential Calculus in Topological Groups," have been reprinted from the Revista de Ciencias, vol. 44 (1942), p. 485 and vol. 45 (1943), p. 45. These papers represent an extension to topological groups of a topologically invariant theory of differentials.

The third article, "Abstract Polynomials in Non-Abelian Groups," has been reprinted from the "Bulletin of the American Mathematical Society," vol. 49 (1943), p. 253. In this highly condensed note the modern abstract approach is used in analyzing the theory of polynomials in non-abelian groups.

The fourth paper, "A Note on Generalized Hilbert Space," is to be published or has been published in Revista Mathematicas y Fisica Teorica, vol. 5 (1944). This note is a unification by means of the concept of a generalized Hilbert space of many well known, but diverse, results.

Certain other results on powers of matrices, characterization of abstract exponential functions, and an abstraction of the Weierstrass approximation theorem have not been included as they are not, as yet, in a form acceptable for publication.

DIFFERENTIAL CALCULUS IN TOPOLOGICAL GROUPS

by KNOX MILLSAPS

Introduction: Many writers, notably Fréchet[1] and Michal[2], have studied various abstract calculi. This note is a generalization to topological groups of a pure topologico-algebraic theory of differentiation. In § 1, the fundamental definitions of topological group differentials are stated. In § 2, the fundamental theorems on unicity, continuity of differentiable functions, differentiability of iterative functions, and the topological invariance of the definitions are proved. In § 3, the specializations of the argument and value spaces are explicitly discussed.

The author gratefully acknowledges the assistance of Dr. A. D. Michal who gave his time in offering thoughtful suggestions and pertinent criticisms.

1. Let $f(x)$ denote a function on U_{x_0}, where U_{x_0} is a neighborhood[3] of x_0, contained in TG, a topological group[4] with a central subgroup[5] C_1 with a generation postulate of the type used by Michal[6] and a relativized topology, to T, a topological group with a central subgroup C_2. All neighborhoods are neighborhoods of the unit element with respect to the relativized topo-

logy unless otherwise specified. The addition of an A to the notation for a topological group will imply that the group is abelian; the usual group notations will be used[7].

Definition of First Order K_1 Differential

The function $f(x)$ *will be said to be first order* K_1 *differentiable at* $x = x_o$, *and* $f_1(x_o; \delta x)$ *will be called a first order* K_1 *differential of* $f(x)$ *at* $x = x_\sigma$ *if*

(a) $f_1(x_\sigma; \delta x)$ *is linear*[8] *for all* $\delta x \varepsilon C_1$;

(b) *there exist functions* $\epsilon_i(x_o, x_1, x_2)$ *with properties which are obvious generalizations of the* $\epsilon(x_\sigma, x_1, x_2)$ *of Michal*[9];

(c) *there exists a neighborhood* N *such that* (1) *the values of* $\epsilon_i(x_o, \delta x, \delta x)$ *are commutative for* $\delta x \varepsilon N$ (2) $f_1(x_\sigma; \delta x)$ *is a first order approximation to the increment* $f(x_o \delta x) f^{-1}(x_\sigma)$ *in the sense that*

$$(A)\ f(x_o \delta x) f^{-1}(x_o) \overset{-1}{f_1}(x_o; \delta x) = \Pi_{i=1}^{g} \epsilon_i(x_o, \delta x, \delta x) \text{ and } \Pi_{i=1}^{g} \epsilon_i(x_o, \delta x, \delta x) \varepsilon C_2$$

for $\delta x \varepsilon N$.

Definition of First Order K_2 Differential

Denote the first order K_2 *differential of* $f(x)$ *by* $f_2(x_o; \delta x)$. *The only difference between* K_1 *and* K_2 *differentials is the substitution of the following equation for* (A).

(B) $f^{-1}(x_0)\, f(x_0 \delta x)\, f_2^{-1}(x_0;\delta x) = \Pi_{i=1}^{g} \epsilon_i(x_0,\delta x,\delta x)$ *and* $\Pi_{i=1}^{g} \epsilon_i(x_0,\delta x,\delta x) \varepsilon C_2$ *for* $\delta x \varepsilon N$[10].

2.

THEOREM I: *If a first order* K_1 *(or* K_2*) differential of* $f(x)$ *exists at* $x = x_0$, *then it is unique for all* $\delta x \varepsilon C_1$.[11]

PROOF: Assume two distinct differentials; $M(x_0; z) \neq N(x_0; z)$ for $z \varepsilon C_1$. From (A) or (B) it follows that

$$N(x_0;\delta x)\, M^{-1}(x_0;\delta x) = \Pi_{n=g}^{k+1} \epsilon_n^{-1}(x_0,\delta x,\delta x)\, \Pi_{n=1}^{k} \epsilon_n(x_0,\delta x,\delta x) \text{ for } \delta x \varepsilon S .$$

It is not hard to show that $N(x_0;z) M^{-1}(x_0;z) = \Pi_{n=g}^{k+1} \epsilon_n^{-1}(x_0,\delta x,z)\, \Pi_{n=1}^{k} \cdot \epsilon_n(x_0,\delta x,z)$ where $\delta x \varepsilon Q$, $z \equiv \delta x^n$, and $z \varepsilon C_1$. From the uniformity of the epsilon functions $N(x_0;z) M^{-1}(x_0;z) = 1$ for $z \varepsilon P$. Applying the generation postulate again, the above equation becomes $M(x_0;z) = N(x_0;z)$ for $z \varepsilon C_1$. Contradiction.

THEOREM II: *If* $f(x)$ *is* K_1 *(or* K_2*) differentiable at* $x_2 = x_0$, *then* $f(x)$ *is continuous*[12] *at* $x = x_0$.

Let $g(y)$ be defined on $f(U_{x_0}) \subset TG_2$ to T; let $\Psi(x) \equiv g(f(x))$.

THEOREM III: *If* $f(x)$ *and* $g(y)$ *are* K_1 *(or* K_2*) differentiable at* $x = x_0$ *and* $y = y_0 = f(x_0)$ *respectively, then* $\Psi(x)$ *is* K_1 *(or* K_2*) differentiable at* $x = x_0$.

PROOF: It can be shown that for $\delta x \varepsilon W, g[f(x_0\, \delta x)]\, g^{-1}[f(x_0)]$
$g_1^{-1}[f(x_0); f_1(x_0; \delta x)] = \Pi_{i=1}^{b} \epsilon_i\, [f(x_0), \Pi_{k=1}^{a} \epsilon_k\, (x_0, \delta x, \delta x)\, f_1(x_0; \delta x),$

$\Pi_{k=1}^{a} \epsilon_k(x_o, \delta x, \delta x)\, f_1(x_o; \delta x)]\, g_1[f(x_o);\ \Pi_{k=1}^{a} \epsilon_k(x_o, \delta x, \delta x)]$. If the following definitions are made

$$g_1 [f(x_o);\ \Pi_{k=1}^{a} \epsilon_k(x_o, x_1, x_2) \equiv \Pi_{k=1}^{a} \epsilon_k' (x_o, x_1, x_2)$$

$$\Pi_{i=1}^{b} \epsilon_i [f(x_o),\ \Pi_{k=1}^{a} \epsilon_k(x_o, x_1, x_1)\, f_1(x_o; x_1),\ \Pi_{k=1}^{a} \epsilon_k(x_o, x_1, x_2)\, f_1(x_o; x_2)] \equiv$$

$\Pi_{k=1+a}^{a+b+1} \epsilon_k'(x_o, x_1; x_2)$, then $\Psi(x_o \delta x)\ \Psi^{-1}(x_o)\ \Psi_1^{-1}(x_o; \delta x) = \Pi_{k=1}^{a+b+1} \epsilon_k'(x_o, \delta x, \delta x)$ for $\delta x \varepsilon M$. One can see that the $\epsilon_k'(x_o, x_1, x_2)$ are epsilon functions, and condition $(c)-(2)$ is satisfied. The proof is similar for K_2 differentials.

THEOREM IV: *The* K_1 *(or* K_2*) differentiability of a function* $f(x)$ *on* TG_1 *to* TG_2 *is a* K_1 *and* K_2 *differential topological property* [13].

3. Consider functions defined on a TAG to a T. The specializations necessary in the definitions of K_1 and K_2 differentiabilities follow immediately. The kinds of differentials will be exhaustively determined by the order of the factors in equations analogous to (A) and (B). *Denote the differentiabilities of* $f(x)$ *on* TAG *to* T *by* G_1 *and* G_2, *where the values of the subscripts imply the same order as in equations* (A) *and* (B) *respectively.* [14]

If the assumption is made that the elements of the value space are commutative and the elements of the argument space not necessarily so, then the changes in K_1 and K_2 differentiabilities are evident. *Denote the one kind of differentiability by* L. [15]

When the argument and value space are commutative, K_1 and K_2 differentiabilities reduce to M_1 differentiability.[16]

THEOREM V: *If $f(x)$ and $g(y)$ are differentiable as indicated at the left of the table at $x = x_0$ and at the top of the table at $y = y_0 = f(x_0)$ respectively, then $\Psi(x)$ is differentiable as indicated in the table at $x = x_0$.*

	M_1	L	G_1	G_2	K_1	K_2
M_1	M_1	N[17]	G_1	G_2	N	N
L	L	N	K_1	K_2	N	N
G_1	N	M_1	N	N	G_1	G_2
G_2	N	M_1	N	N	G_1	G_2
K_1	N	L	N	N	K_1	K_2
K_2	N	L	N	N	K_1	K_2

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Footnotes

(1) M. Frechet, "La Notion de Différentielle dans L'Analyse Générale", *Annales Scientifiques de L'École Normale Supérieure*, vol. 42 (1925), pp. 293-323.

(2) A. D. Michal, "General Differential Geometries and Related Topics", *Bull. Amer. Math. Soc.*, vol. 45 (1939), pp. 529-563.

(3) Definition 13, L. Pontrjagin, *Topological Groups*, Princeton (1939).

(4) Definition 22, Pontrjagin, *loc. cit.*

(5) Definition 23, Pontrjagin, *loc. cit.*

(6) A. D. Michal, "First Order Differentials of Functions with Arguments and Values in Topological Groups", *Revista de Ciencias*, (in press).

(7) Hans Zassenhaus, *Lehrbuch der Gruppentheorie*, Teubner (1937).

(8) S. Banach, *Theorie des Operations Lineaires*, Warsaw (1932), pp. 23. In this definition the substitution of multiplicative distributivity for additivity is the obvious abstraction.

(9) A. D. Michal, *loc. cit.* (*6*), Section 1.

(10) Let A be the totality of all n-rowed square matrices of the form $\alpha \equiv \| a^i_j \|$ whose elements are real numbers and whose determinants are different from zero. Define $\alpha\beta \equiv \gamma$ where $a^i_j b^j_k \equiv c^i_k$.

Define U_m for $m = 1, 2, \ldots\ldots\ldots$ to be the set of all matrices of the form $\alpha + I \equiv \|a^i_j + \delta^i_j\|$ where I is the unit matrix, and α is in the set of all matrices whose elements do not exceed $\frac{1}{m}$ in absolute value. A unique topologization exists for which U_m form a complete system of neighborhoods of the identity of the abstract group A.

Let $f(x) \equiv x'$ where $x^i_j \equiv x'^j_i$; the reader can easily verify that $f_1(x_0; \delta x) = f_2(x_0; \delta x) = (\delta x)'$. Many other differentiable functions of matrices could be given—see J. H. M. Wedderburn, *Lectures on Matrices*, Volume XVII, Colloquium Publications of Amer. Math. Soc.

(11) The generation postulate is needed for the proof of this and only this theorem. The postulate is redundant for linear topological spaces.

(12) A function $f(x)$ on T_1 to T_2 will be called continuous at $x = x_0$; if given V of $f(x_0)$, there exists U of x_0 such that $f(U) \subset V$. As an alternative definition consider the following: if given V of 1, there exists U of 1 such that $f(x_0 y) f^{-1}(x_0) \varepsilon V$ for $y \varepsilon U$. Consider also the definitions that follow by changing the order of the value and argument factors. The equivalences of all of definitions are immediate consequences of a paper by F. Leja, "Sur la Notion du Groupe Abstrait Topologique", *Fundamenta Mathematicae*, vol. 29 (1927), pp. 37–45.

(13) A. D. Michal, *loc. cit.* (*6*), Section 2.

(14) Let TAG be the ad ditive group of real numbers; let T be the group of 2×2 non-singular matrices with real elements under row by column multiplication. Introduce the natural to-

pologies; define $f(t) \equiv \| a_{ij}(t) \|$, where $a_{ij}(t)$ are real functions of a real variable. Consider the set of T for which $a_{11} = a_{12}$, $a_{21} = - a_{12}$, $a_{11}(t_1 + t_2) = a_{11}(t_1)\, a_{11}(t_2) - a_{22}(t_1)\, a_{21}(t_2)$ and $a_{21}(t_1 + t_2) = a_{21}(t_1)\, a_{22}(t_2) - a_{22}(t_1)\, a_{12}(t_2)$. A solution of the above equations is the matrice representation of the one real parameter Euclidean rotations. It follows that $f_{g_1}(t_o; \delta t) = f_{g_2}(t_o; \delta t) = \| a_{ij}(\delta t) \|$ for all $t_o, \delta t \varepsilon$ TAG, where $f_{g_1}(t_o; \delta t)$ and $f_{g_2}(t_o; \delta t)$ denote the G_1 and G_1 differentials of $f(t)$.

(15) As an example, consider TG to be the space of non-zero quaternions — $\alpha \equiv a_i \varepsilon^i$, where $a_i \varepsilon R$ and $(\varepsilon^i)^2 = \Pi\, \varepsilon^i = -1$ for $i = 1,2,3,4$ — with quaternion multiplication as the group operation; take TA to be the space of positive numbers with the group operation as ordinary multiplication. Introduce the natural topologies — cf. Examples 6, 26, and 37 in Pontrjagin, *loc. cit.* If $f(d) \equiv \Sigma (a_i)^2$, then $f_L(d_o; \delta d) = \Sigma (\delta d_i)^2$, where $f_L(d_o; \delta d)$ denotes the L differential of $f(d)$.

(16) A. D. Michal, *loc. cit.* (6), Section 1.

(17) N implies that the iterative functiou is not defined.

DIFFERENTIAL CALCULUS IN TOPOLOGICAL GROUPS II.

by Knox Millsaps

Introduction: In a previous paper first order differentials of functions with arguments and values in topological groups were defined and studied.(1) An extension to n–th order differentials is the purpose of this paper. In § 4, definitions for n–th order K_1 and K_2 differentials are stated; in § 5, the theorems on the unicity of n–th order K_1 and K_2 differentials and the differentiability of iterative functions with an explicit formulation of the differentials of the composite function in terms of the differentials of the composing functions are proved. It should be noted that the results are applicable to n–th order Fréchet and Michal differentials.(2) The notions of K_1 and K_2 differentiabilities are purely topologico-algebraic.

It is a pleasure to acknowledge the help of Dr. A. D. Michal.

4. The notations introduced in the preceding part of the development are preserved; in addition, the value groups are restricted to contain no elements of finite order. It is found convenient to introduce the following notations:

(a) $n!! \equiv n!\,(n-1)! \,\ldots\ldots\, 2!\; 1!$,

(b) $[x]\,E(n) \equiv x^{n}$,

(c) $$P^{n}_{\alpha_1 \ldots \alpha_n} \equiv \frac{n!}{\alpha_1! \ldots\ldots \alpha_n!\; (1!)^{\alpha_1} \cdots (n!)^{\alpha_n}}\,{}^{(3)} ,$$

(d) $x_{(i)} \equiv x$ for $i = 1, 2 \ldots\ldots$,

and (e) $[C^{\alpha_n}_{n}\, C^{\alpha_m}_{m}]\; E(p)$ denotes the product of p (p is obviously bounded) distinct factors in which the K_1 (or K_2) differentials of order n and m occur α_n and α_m times respectively as the arguments in the K_1 (or K_2) differential of order $\alpha_n + \alpha_m$.

DEFINITION OF N-th ORDER K_1 DIFFERENTIAL.

A function $f_1(x_0; \delta_1 x; \ldots\ldots; \delta_n x)$ *with arguments* $\delta_i x \varepsilon C_1$ *and values in* T_2 *will be called an n-th order* K_1 *differential of* $f(x)$ *at* $x = x_0$ *with increments* $\delta_1 x, \ldots\ldots, \delta_n x$ *if*

(1) $f_1(x_0; \delta_1 x; \ldots\ldots; \delta_q x)$ *for* $q = 1, 2, \ldots\ldots, n-1$ *exist at* $x = x_0$;

(2) $f_1(x_0; \delta_1 x; \ldots\ldots; \delta_n x)$ *is a completely symmetric* $i-$ *uniform multilinear function in* $\delta_i x$; (4)

(3) $\epsilon(x_0, x, x_1, \ldots\ldots, x_n)$ *on* T_1 *to* C_2 *such that*

(a) $\epsilon(x_0, 1, x_1, \ldots\ldots, x_n) = 1$ *for* $x_i \varepsilon T_1$,

(b) $\epsilon(x_o, x, x_1^{k_1}, \ldots\ldots, x_n^{k_n}) = [\epsilon(x_o, x, x_1, \ldots, x_n)] \, E(\Pi k_i)$ *for* $k_i \in P$ [(5)], $x \in K \subset C_1$, *and* $x_i \in T_1$,

(c) $\epsilon(x_o, x, x_1, \ldots, x_n) \in A$ *for* $x \in B(A)$ *and* $x_i \in N_i$;

(4) $f_1(x_o; \delta_1 x; \ldots; \delta_n x)$ *is an n–th order approximation to* $f(x_o \delta x) f^{-1}(x_o)$ *in the sense that*

(A′) $[f(x_o \delta x) f^{-1}(x_o)] \, E(n!!) \, \Pi_{i=1}^{n} [f_1^{-1}(x_o; \delta_{(1)} x; \ldots; \delta_{(i)} x] \, E\left(\frac{n!!}{i!!}\right) =$

$= \epsilon(x_o, \delta_{(1)} x, \ldots, \delta_{(n+1)} x)$ *for* $\delta x \in Q$.

Definition of N–th Order K_2 Differential.

Denote the n–th order K_2 *differential of* $f(x)$ *by* $f_2(x_o; \delta_1 x; \ldots; \delta_n x)$. *Let condition (3) and the analogues of (1) and (2) continue to hold; substitute the following condition for (4).*

(4′) $f_2(x_o; \delta_1 x; \ldots\ldots; \delta_n x)$ *is an n–th order approximation to* $f^{-1}(x_o) f(x_o \delta x)$ *in the sense that*

(B′) $[f^{-1}(x_o) f(x_o \delta x)] \, E(n!!) \Pi_{i=1}^{n} [f_2^{-1}(x_o; \delta_{(1)} x; \ldots; \delta_{(i)} x)] \, E\left(\frac{n!!}{i!!}\right) =$

$\epsilon(x_o, \delta_{(1)} x, \ldots, \delta_{(n+1)} x)$ *for* $\delta x \in S$.

5.

Theorem VI: *If an n–th order* K_1 *(or* K_2*) differential of* $f(x)$ *exists at* $x = x_o$, *then it is unique for* $\delta_i x \in C_1$.

Proof: For $n=1$ the theorem has been proved.[6] It is assumed that the differentials of order $1, \ldots\ldots, n-1$ are unique; in addition, it is assumed that two differentials of the n–th order $M(x_0; \delta_1 x; \ldots; \delta_n x) \neq N(x_0; \delta_1 x; \ldots; \delta_n x)$ for $\delta_i x \in C_1$. From (A′) or (B′) it can be deduced by algebraic manipulation that

$$M(x_0; \delta_{(1)} x; \ldots; \delta_{(n)} x) N^{-1}(x_0; \delta_{(1)} x; \ldots; \delta_{(n)} x) = \epsilon_M^{-1}(x_0, \delta_{(1)} x, \ldots, \delta_{(n+1)} x) \cdot$$

$$\cdot\ \epsilon_N(x_0, \delta_{(1)} x, \ldots, \delta_{(n+1)} x) \quad \text{for} \quad \delta x \in N .$$

From (2), (3) – (a) (b) (c), and the repeated use of the generation postulate for C_1, one may prove that $N(x_0; \delta_{(1)} x; \ldots; \delta_{(n)} x) = M(x_0; \delta_{(1)} x; \ldots; \delta_{(n)} x)$ for $\delta x \in C_1$. From (2) and the assumption that the value group contains no elements of finite order, it has been shown that $N(x_0; \delta_1 x; \ldots; \delta_n x) = M(x_0; \delta_1 x; \ldots; \delta_n x)$ for $\delta_i x \in C_1$.[7] The induction is complete.

THEOREM VII: *If $f(x)$ and $g(y)$ possess K_1 (or K_2) differentials of order n at $x = x_0$ and $y = y_0 = f(x_0)$ respectively, then $\Psi(x) \equiv g(f(x))$ possesses an $n-th$ order K_1 (or K_2) differential at $x = x_0$, and*

$$\Psi(x_0; \delta_1 x; \ldots; \delta_n x) = \Pi \left[C_1^{\alpha_1} \ldots\ldots C_n^{\alpha_n} \right] E(P^n_{\alpha_1 \ldots \alpha_n}) ,$$

where the product is taken over all non–negative solutions of $\Sigma_{i=1}^n i\, \alpha_i = n$, and C_q^o denotes the absence of the $q-th$ order K_1 (or K_2) differential of $f(x)$.

Proof: From (A′) and Theorem II the following equation may be derived.

$$[g(f(x_0\delta x))\, g^{-1}(f(x_0))]\, E(n!!)\, \Pi_{i=1}^{n}\, [g_1^{-1}(f(x_0);\ (f(x_0\,\delta x)\, f^{-1}(x_0))_{(1)};\ldots\ldots;$$

$$;\ldots;\ (f(x_0\,\delta x)\, f^{-1}(x_0))_{(i)}]\, E\left(\frac{n!!}{i!!}\right) = \mathfrak{c}_g[f(x_0),\ (f(x_0\,\delta x)\ f^{-1}(x_0))_{(1)},\ldots$$

$$\ldots,\ (f(x_0\,\delta x)\ f^{-1}(x_0))_{(n+1)}] \quad \text{for} \quad \delta x \in Z\,.$$

It is convenient to consider a representative factor when $n > 3$ and to introduce the notation $f(x_0\,\delta x)\ f^{-1}(x_0) \equiv I$

$$[g_1^{-1}(f(x_0);\ I_{(1)};\ldots;I_{(n-3)})]\ E\left(\frac{n!!}{(n-3)!!}\right) =$$

$$[g_1^{-1}(f(x_0);\ [I_{(1)}]\ E(4!!);\ I_{(2)};\ldots;\ I_{(n-3)})]\ E\left(\frac{n!!}{4!!(n-3)!!}\right) =$$

$$[g_1^{-1}(f(x_0);[I_{(1)}]E(3!!);[I_{(2)}]E(2!!);I_{(3)};\ldots;I_{(n-3)})]E\left(\frac{n!!}{3!!\,2!!\,(n-3)!!}\right) =$$

$$[g_1^{-1}(f(x_0);\ [I_{(1)}]\ E(2!!);\ [I_{(2)}]\ E(2!!);\ [I_{(3)}]\ E(2!!);\ I_{(4)};\ldots\ldots$$

$$\ldots\ldots;\ I_{(n-3)})]\ E\left(\frac{n!!}{(2!!)^3\ (n-3)!!}\right)$$

Performing this process for the K_1 differentials of order $1,\ldots,n$ (8), it is intuitively seen that

$$[\Psi(x_o \delta x)\, \Psi^{-1}(x_\sigma)]\, E(n!!)\, \Pi^{n}_{i=1}\, [C_1^{\alpha_1} \ldots C_i^{\alpha_i}]\, E\left(\frac{P^{i}_{\alpha_1 \ldots \alpha_i}\ n!!}{i!!}\right) =$$

$$= \epsilon_{\Psi}\,(x_o,\, \delta_{(1)}\, x, \ldots,\, \delta_{(n+1)}\, x)$$

for $\delta x \varepsilon H$, where $\epsilon_{\Psi}\,(x_\sigma, x, x_1, \ldots, x_n)$ is the epsilon function defined as the product of $\epsilon_g(x_o, x, x_1, \ldots, x_n)$ and the difference of the factors in the first and last equations of the proof.

The proof for K_2 differentials is entirely equivalent. (9)

Footnotes

(1) K. Millsaps, "Differential Calculus in Topological Groups", Revista de Ciencias, Año XLIV (Diciembre, 1942) N.° 442.

(2) A. D. Michal, "Higher Order Differentials of Functions with Arguments and Values in Topological Abelian Groups", *Revista de Ciencias*, vol. 42 (1942), pp. 170-176.

(3) For a discussion of this function see H. S. Wall, "On the N-th Derivative of F(x)" *Bull. Amer. Math. Soc.*, vol. 44 (1938), pp. 395-398.

(4) A. D. Michal, *loc. cit.*, pp. 158-159.

(5) P denotes the set of positive integers.

(6) K. Millsaps, *loc. cit.*, Theorem I.

(7) A. D. Michal, *loc. cit.*, Lemma 1.1.

(8) If $n, j \varepsilon P$ and $n > j$, then

(a) $\Pi_{i=0}^{j-1} (n-i)^{i+1} \geqq \Pi_{i=0}^{j-1} (j+1-i)^{i+1}$

It is readily seen that

(b) $\Pi_{i=1}^{j-1} (j+1-i)^{i+1} \geqq k_1!! \dots k_{n-j}!!$, where $n > j$, $\Sigma_{i=1}^{n-j} k_i \equiv n$,

and $0 < k_i \leqq j+1$ for by reductio ad absurdum one gets

(c) $\max [\Pi_{i=1}^{n-j} k_i!!] = (j+1)!!$.

Changing the notation, one has

(d) $\Pi_{i=0}^{j-1} (n-i)^{i+1} = \dfrac{n!!}{(n-j)!!}$.

Combining (a), (b), (c), and (d), the desired result that

(e) $\dfrac{n!!}{(n-j)!!} \geqq \Pi_{i=1}^{n-j} k_i!!$, where $n > j$, $\Sigma_{i=1}^{n-j} k_i = n$, and

$0 < k_i \leqq j + 1$, follows.

(9) As an example of a second order K_1 and K_2 differentiable function whose second orden K_1 and K_2 differentials differ from unity, one may consider the matric function $\log x$. For the definition of $\log x$ see J. H. M. Wedderburn, *Lectures on Matrices*, Volume XVII, Colloquium Publications Amer. Math. Soc., pp. 122-123.

ABSTRACT POLYNOMIALS IN NON-ABELIAN GROUPS

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Introduction. The aim of this note is to give some generalizations for groups of the theories of abstract polynomials as developed by Fréchet, Gateaux, Martin, Mazur, Michal, Orlicz and more recently, Van der Lijn.[1] Although the theories are equivalent for functions with arguments and values in abelian groups,[2] this equivalence is not the case when the argument and value groups are non-abelian.[3]

In §1, a calculus of finite differences for functions with arguments and values in non-abelian groups which contain no elements of finite order is defined, and the fundamental definitions of polynomials and monomials are stated. In §2, the homogeneity in the increment of the n-difference of a polynomial of degree n is proved, and the theorem on unique pseudo-decomposition is proved after giving some preliminary theorems on the structure of differences of arbitrary functions and polynomials. In §3, a brief discussion of the extensions to non-abelian groups as value spaces is given.

I should like to thank Professor A. D. Michal for his helpful suggestions and constructive criticisms during the preparation of this note.

1. **Definitions.** For the purposes of polynomial theory the value groups are restricted to contain no elements of finite order.

To construct the calculus of finite differences for functions with arguments and values in non-abelian groups, we define

$$
{}_{\omega}\Delta[\alpha_1;\beta_1]f(x) \equiv \begin{cases} f(x\omega)f^{-1}(x) & \text{if} \quad \alpha_1 = 1 \quad \text{and} \quad \beta_1 = 1, \\ f(\omega x)f^{-1}(x) & \text{if} \quad \alpha_1 = 1 \quad \text{and} \quad \beta_1 = 2, \\ f^{-1}(x)f(x\omega) & \text{if} \quad \alpha_1 = 2 \quad \text{and} \quad \beta_1 = 1, \\ f^{-1}(x)f(\omega x) & \text{if} \quad \alpha_1 = 2 \quad \text{and} \quad \beta_1 = 2, \end{cases}
$$

Presented to the Society, April 11, 1942; received by the editors April 21, 1942.

[1] References to these theories will be found in the bibliography. This list will be referred to by numbers in brackets.

[2] The equivalence of some of these definitions was proved by Martin in his California Institute of Technology thesis, 1932, and of the remaining definitions by Van der Lijn. A summary has been given by Van der Lijn [**1**, pp. 78–80].

[3] If the abstraction of additivity is multiplicative distributivity, then the generalizations of the definitions of Mazur and Orlicz [**1**, p. 63] and Van der Lijn, [**1**, pp. 60–61] are not equivalent; this is easily seen by considering $f(x) \equiv x^k$.

and inductively

$$\overset{n}{_\omega\Delta}[\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n]f(x)$$
$$\equiv {_\omega\Delta}[\alpha_n; \beta_n] \overset{n-1}{_\omega\Delta}[\alpha_1, \cdots, \alpha_{n-1}; \beta_1, \cdots, \beta_{n-1}]f(x).$$

DEFINITION OF A MONOMIAL. *A* $[\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n]$ *monomial*[4] *is a function* $f(x)$ *which satisfies for all* x *and* ω *the following functional equation*

$$\overset{n}{_\omega\Delta}[\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n]f(x) = f^{n!}(\omega).$$

The obviously unique value of n is termed the degree of the monomial.

DEFINITION OF A POLYNOMIAL. *A function* $f(x)$ *which satisfies the following functional equation for all* x *and* ω

$$\overset{n+1}{_\omega\Delta}[\alpha_1, \cdots, \alpha_{n+1}; \beta_1, \cdots, \beta_{n+1}]f(x) = 1$$

will be called a $[\alpha_1, \cdots, \alpha_{n+1}; \beta_1, \cdots, \beta_{n+1}]$ *polynomial.*[5]

The least value of n for which the above equation holds will be termed the degree of the polynomial.

The binomial coefficients are denoted in the usual manner; $\Gamma^i_{k,n}$ is defined by

$$\Gamma^i_{k,n} \equiv \sum \frac{n!}{\alpha_1!\alpha_2! \cdots \alpha_{k+1}!}$$

where $\sum_{i=1}^{k+1}\alpha_i = n$, $\sum_{q=2}^{k+1}(q-1)\alpha_q = i$, $i \neq nk$, and $i \neq 0$. An arbitrary element of the central subgroup will be denoted by ϑ. To simplify notation, the dropping of unnecessary indices implies that the value of a product of factors is independent of the order of the particular factors controlled by the dropped indices.

The next equation is a generalization of an identity due to Marchaud[6]

$$\overset{n}{_{\omega^{k+1}}\Delta}[\beta_1, \cdots, \beta_n]f(x) = \sum_{i=0}^{nk} \sum_j \overset{n}{_\omega\Delta}[\beta_1, \cdots, \beta_n]f[P_{ij}],$$

---

[4] The inner automorphisms of a group are interesting examples of a $[1; 2]$ monomial; similarly, the canonical transformations of quantum mechanics.

[5] If the elements of a group are taken to be the $n \times n$ matrices whose elements are in a commutative field and whose determinants do not vanish, and if the group operation is defined as row by column multiplication, then $f[\|x^i_j\|] \equiv \|a^i_k x^k_m x^m_n b^n_j\|$ is a $[1, 1; 2, 1]$ polynomial.

[6] Marchaud [1, p. 368].

where $j=1, \cdots, \Gamma^i_{k,n}$ and P_{ij} denotes a particular permutation of $\omega, \omega, \cdots, \omega$ (i times) and x.

2. **Fundamental theorems.** Let $A(x)$ denote a function with arguments in a non-abelian group and values in an abelian group. The theorems of Van der Lijn for functions with arguments and values in abelian groups can be extended with a few immediate changes to hold for differences of the type $\overset{n}{_y}\Delta A(x)$. This section is devoted to abstractions for differences of the kind $\overset{n}{_\omega}\Delta[\beta_1, \cdots, \beta_n]A(x)$, where ω is an arbitrary element.

THEOREM I. *If $A(x)$ is a $[\beta_1, \cdots, \beta_{n+1}]$ polynomial of degree n, and if k is an integer, then*

$$\overset{n}{_{\omega^k}}\Delta[\beta_1, \cdots, \beta_n]A(x) = k^n \, \overset{n}{_\omega}\Delta[\beta_1, \cdots, \beta_n]A(x).$$

PROOF. For $k=0$ or 1, the theorem is trivial. For $k>1$, we hypothetically have

$$\overset{n}{_\omega}\Delta[\beta_1, \cdots, \beta_n]A(x, \overset{i}{\omega}) - \overset{n}{_\omega}\Delta[\beta_1, \cdots, \beta_n]A(x, \overset{i-1}{\omega}) = 0.$$

By a few manipulations and Marchaud's identity we get

$$\begin{aligned}\overset{n}{_{\omega^k}}\Delta[\beta_1, \cdots, \beta_n]A(x) &= \sum_{i=0}^{n(k-1)} \Gamma^i_{k-1,n} \, \overset{n}{_\omega}\Delta[\beta_1, \cdots, \beta_n]A(x) \\ &= k^n \, \overset{n}{_\omega}\Delta[\beta_1, \cdots, \beta_n]A(x).\end{aligned}$$

For $k<0$, we evidently have

$$\begin{aligned}\overset{n}{_{\omega^k}}\Delta[\beta_1, \cdots, \beta_n]A(x) &= \overset{n}{_{(\omega^{-1})^{-k}}}\Delta[\beta_1, \cdots, \beta_n]A(x) \\ &= (-k)^n \, \overset{n}{_{\omega^{-1}}}\Delta[\beta_1, \cdots, \beta_n]A(x) \\ &= (-)^n(-k)^n \, \overset{n}{_\omega}\Delta[\beta_1, \cdots, \beta_n]A(x) \\ &= k^n \, \overset{n}{_\omega}\Delta[\beta_1, \cdots, \beta_n]A(x).\end{aligned}$$

THEOREM II. *If $\overset{n}{_\omega}\Delta[\beta_1, \cdots, \beta_n]A(x) = g(\omega)$, where $g(\omega)$ is independent of x, then $g(\omega)$ is a $[\beta_1, \cdots, \beta_n]$ monomial of degree n in ω.*

THEOREM III. *If $A(x)$ is a $[\beta_1, \cdots, \beta_{n+1}]$ polynomial of degree n, then $\overset{n}{_\omega}\Delta[\beta_1, \cdots, \beta_n]A(x)$ is independent of x.*

PROOF. If $n+2$ elements of the argument group are denoted by x_i, where

$$x_{i+1} = \prod(x_i, \xi), \qquad i = 0, \cdots, n+1,$$

then by steps roughly analogous to the usual proof of similar theorems we can derive

$$\sum_{p=0}^{n}\sum_{i=0}^{n+1} C_{n+1,i}C_{n,p}(-)^{i} i^{n-p} k^{p}\, {}^{n}_{\omega}\Delta[\beta_1, \cdots, \beta_n]A(x_i) = 0.$$

In the last equation we see that $\sum_{p=0}^{n} C_{n,p}k^p$ is a simple polynomial. From this observation we deduce[7]

$$\sum_{i=0}^{n+1} (-)^{i} C_{n+1,i} i^{n-p}\, {}^{n}_{\omega}\Delta[\beta_1, \cdots, \beta_n]A(x_i) = 0, \qquad p = 0 \cdots, n.$$

If we consider one difference as given and the remaining $n+1$ differences as unknown, then the unique solutions are

$${}^{n}_{\omega}\Delta[\beta_1, \cdots, \beta_n]A(x_i) = {}^{n}_{\omega}\Delta[\beta_1, \cdots, \beta_n]A(x_0), \qquad i = 1, \cdots, n+1,$$

for the determinant of the system of equations is given by

$$(-)^{\delta(n)} \prod_{i=1}^{n+1} C_{n+1,i} V \neq 0$$

where V is a Vandermonde determinant and $\delta(n) = 1$, if $n \equiv 1, 2 \bmod 4$; $\delta(n) = 2$, if $n \equiv 0, 3 \bmod 4$.

THEOREM IV. *If $A(x)$ is a $[\beta_1, \cdots, \beta_{n+1}]$ polynomial of degree n, and if $\rho_n = \prod_{i=0}^{n-1}(n-i)!$, then we have the unique pseudo-decomposition*

$$\rho_n A(x) = \sum_{i=0}^{n} M_i(x),$$

where $M_i(x)$ are $[\beta_1, \cdots, \beta_i]$ monomials of degree i.

3. **Further discussion.** With the generalized definition of a polynomial an extension of Theorem I for functions with arguments and values in non-abelian groups can be made. The trivial converse of Theorem III can be proved in an obvious manner. The non-abelian analogue of Theorem IV does not hold, for counterexamples can be exhibited.[8]

In conclusion, it is remarked that the difficulty with the non-abelian valued case is an inability to solve explicitly a system of group equations.

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[7] It can be shown that if a simple polynomial $\sum_{i=0}^{n} a_i x^i$ of degree n vanishes for $n+1$ distinct values of x, then the coefficients vanish.

[8] With the group defined in (5), a trivial, but sufficient, example is $\|x_j^i\| = \|x_j^i\|\,\|\delta_j^i\| = \|\delta_j^i\|\,\|x_j^i\|$.

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A Note on Generalized Hilbert Space

by

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Although most, if not all, of the results of this note are known, this presentation seems to be an omnibus for a few realizations of many different abstract spaces.

Classical Hilbert space $\mathcal{H}^2$ was introduced by David Hilbert [(5)] who used the concept as an aid in formulating a remarkable general theory of linear integral equations. The next important basic result was the elegant treatment by postulational methods due to von Neumann [(9), pp. 14-17; (10), pp. 64-66].

In this note by analyzing an example formulated by Tcyhonoff [(18)], the spaces $\mathcal{H}^\alpha$ are defined, and some basic properties are proved.

The author would like to acknowledge the generous aid of Professor A. D. Michal.

I. Definition of a Generalized Hilbert Space

The space $\mathcal{H}^\alpha$ is defined as the class of all points whose coordinates are given by members of the set of all denumerably infinite sequences of real numbers such that the series formed by the α^{th} power of the absolute values of the elements of a sequence converges; addition of elements of the class is defined by addition of respective components and multiplication of elements by real numbers as the product of components and real numbers. This may be formulated symbolically in terms of postulates as is given below.

$$\mathcal{H}^{\alpha} \equiv \sum X \quad \ni$$

$$(1)\ X \equiv \{x_1, x_2, \cdots\cdots, x_n, \cdots\cdots\} \equiv \{x_i\},\ (x_i) \in \mathcal{R};$$

$$(2)\ \sum_{i=1}^{\infty} |x_i|^{\alpha} \longrightarrow \mathcal{L}im,\ 0 \leq \alpha < \infty.$$

At this point it is advisable to introduce three convenient definitions which will be used throughout this note. These definitions are

$$(\alpha)\quad O \equiv \{0, 0, \cdots\cdots, 0, \cdots\cdots\};$$

$$(\beta)\quad X + Y \equiv \{x_i + y_i\},\ (x_i, y_i) \in \mathcal{R};$$

$$(\gamma)\quad aX \equiv \{a x_i\},\ (a, x_i) \in \mathcal{R}.$$

II. Fundamental Properties

Although many of them may appear to be trivial, all of the important theorems are explicitly stated for the sake of completeness.

THEOREM I: If $\alpha \leq \beta$, then $\mathcal{H}^{\alpha} \subseteq \mathcal{H}^{\beta}$

THEOREM II: The spaces $\mathcal{H}^{\alpha}$ are closed under the addition of elements.

PROOF: The well known principles concerning the binomial series, rearrangement of the terms of an absolutely convergent series, and the results of Bromwich [(3), pp. 29-32] are sufficient to construct an elementary proof.

The quasi-norm $\|X\|_\beta$ is defined in $\mathcal{H}^\alpha$ by the following equation

$$(\delta) \qquad \|X\|_\beta \equiv \left[\sum_{i=1}^{\infty} |X_i|^\alpha\right]^\beta \quad \text{for} \quad 0 \leq \beta < \infty.$$

THEOREM III: The spaces $\mathcal{H}^\alpha$ are linear spaces.

THEOREM IV: The spaces $\mathcal{H}^\alpha$ $(0 \leq \alpha < \infty)$ are complete with respect to $\|X\|_\beta$ for $0 \leq \beta < \infty$.

PROOF: The proof is identical with the verification of Postulate E in Theorem 1.15 of Stone [(7), pp. 15]. The inclusion of Theorem II is necessary to demonstrate the last step.

THEOREM V: A quasi-norm $\|X\|_\beta$ of $\mathcal{H}^\alpha$ is a Banach norm if and only if $\alpha \geqslant 1$ and $\beta = 1/\alpha$.

PROOF: After Minkowski [(6), pp. 115-117] it follows that if then for all $\sum \alpha$ and $\sum \beta$

$$\left[\sum(\alpha+\beta)^\rho\right]^\eta \leqslant \left[\sum \alpha^\rho\right]^\eta + \left[\sum \beta^\rho\right]^\eta \qquad \text{if and only if}$$

$\eta = 1$ for $0 < \rho < 1$ and $\eta = 1/\rho$ for $\rho \geqslant 1$. For the definition of a Banach norm the reader should see Banach [(12), pp. 53]. The verification of the first condition is obvious; the second follows from the above; the third condition holds if and only if $\alpha = 1/\beta$ The requirements for the second and third conditions to hold produce the theorem.

COROLLARY: A Generalized Hilbert Space $\mathcal{H}^{\alpha}$ is a Banach space if and only if $1 \leqslant \alpha < \infty$ and $|X| \equiv \|X\|_{1/\alpha}$.

THEOREM VI: The spaces $\mathcal{H}^{\alpha}$ are separable with respect to $\|X\|_{\beta}$.

PROOF: The theorem follows from the fact that the class of all finite sequences of rational real numbers is denumerably infinite and everywhere dense in $\mathcal{H}^{\alpha}$. If one is given X and $\mathcal{E}^{\alpha\beta} > 0$ there exists $\mathcal{R}$ such that

$$\|X - \mathcal{R}\|_{\beta}^{1/\beta} \leqslant \sum_{i=1}^{k} |x_i - r_i|^{\alpha} + \sum_{i=k+1}^{\infty} |x_i|^{\alpha} < \mathcal{E}^{\alpha}$$

where $\mathcal{R} \equiv \{r_1, r_2, \cdots, r_k, 0, 0, \cdots\}$, and r_i are members of the class of all rational numbers.

COROLLARY: A space $\mathcal{H}^{\alpha}$ $(1 \leq \alpha < \infty)$ with $\|X\| \equiv \|X\|_{1/\alpha}$ is homeomorphic with a compact set in $\mathcal{H}^{2}$.

THEROEM VII: For every n, $n = 1, 2, \cdots$, there exists a set of linearly independent elements.

PROOF: $X_n \equiv \{\delta_n^k\}$, $k = 1, 2, \cdots$, are just such elements.

THEOREM VIII: Sufficient conditions for $\mathcal{H}^{\alpha}$ being locally non-convex with respect to $\|X\|_{\beta}$ are (1) $\beta > 1$, $0 < \alpha\beta \leq 1$ and (2) $\beta < \alpha\beta \leq 1$.

PROOF: By considering an example due to Tcyhonoff [(8), pp. 768-719], one arrives at the expression

$$\varepsilon^{\alpha\beta}\left[\sum_{i=1}^{\infty}\frac{1}{i^{\alpha\beta}}\right]^{\beta}\left[\sum_{j=1}^{\infty}\frac{1}{j^{\beta}}\right]^{-\alpha\beta}$$

The theorem follows upon observing the intervals of convergence of the p-series of Bromwich [(3), p. 34].

III. Realizations of Generalized Hilbert Space

The classical space of Hilbert is the space $\mathcal{H}^{2}$; the abstract realizations of Hilbert space quoted by Stone [(7), pp. 23-32] are further examples; the spaces l^{p} for $p \geq 1$ given by Banach [(2), p. 12] are identical with the spaces $\mathcal{H}^{p}$ for $p \geq 1$; the space $\square$ categorized by Frechet [(4), p. 86] is the space $\mathcal{H}^{1}$. It is to be noted that the spaces $\mathcal{N}^{p}$ for $p \geq 1$ of Ascoli [(1), pp. 63-66] are equivalent to the spaces $\mathcal{H}^{p}$ for $p \geq 1$ if $\|X\| \equiv \|X\|_{1/p}$.

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