Part I

A Formal Expansion Theory for Functions of One or More Variables

Part II

The Number of Representations Function for Binary Quadratic Forms

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Part I

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Introduction

While pursuing an investigation of the general properties of the confluent hypergeometric functions of two variables at the suggestion of Harry Bateman the question of the expansion of arbitrary functions in terms of these arose. Efforts to obtain results by extending the theory of orthogonal Tchebycheff polynomials or by some other integral method proved quite unsuccessful. Finally the formal algebraic method suggested by the Neumann expansions in Bessel functions was devised and developed to solve the problem at hand. We shall give an extended discussion of this method, together with sufficient examples to illustrate its application.

The idea of representing the power series $\sum_{m=0}^{\infty} a_m \frac{x^m}{m!}$ by the exponential e^{ax}, using the umbral notation $a_n = a^n$, is apparently due to Blissard[1] who used it to obtain very elegantly the properties of the Bernoulli and related numbers. In recent years this umbral notation has been extended by a great many investigators. It has been placed on a firm algebraic basis by E. T. Bell[2] to whom the author is greatly indebted for constant counsel and encouragement. I should also like to state my appreciation to Harry Bateman, without whose original guidance the work would never have been begun.

For convenience in writing formulae, we use certain

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[1] John Blissard, Theory of Generic Equations, IV-VI (1861-1864).
[2] E. T. Bell, Algebraic Arithmetic (New York, 1927), pp. 146-159.

notation conventions. In place of a repeated summation:

$$\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_s=0}^{\infty} a_{m_1,\cdots m_s}$$

we shall write merely,

$$\sum_{m_1,\cdots,m_s} a_{m_1,\cdots,m_s}$$

Unless explicitly stated, all summations are to extend over the range 0 to ∞. Often we shall include polynomials under such a connection by using coefficients vanishing for all values of the index larger than a certain value. Such cases will be self-evident and will be left to the observation of the reader.

Since we shall be dealing with hypergeometric functions, a great variety of Gamma functions, factorials, and binomial coefficients might be introduced. Rather, we shall use uniformily the symbol introduced by Appell

$$(\lambda, k) = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$$

Some simple properties of this symbol derived from its relation to the Gamma function will be used.

$$(a,k) = a(a+1)\ \ldots\ (a+k-1),\ k \text{ integral}$$

$$(a,o) = 1,\ (1,k) = k!,\ \ k \text{ integral}$$

$$(a,-k) = \frac{(-)^k}{(1-a,k)},\ k \text{ integral}$$

$$(-n,k) = 0,\ n,\ k \text{ integral}\ \ k > n$$

$$(a,n+k) = (a,k)\ (a+k,n)$$

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = (-)^r \frac{(-n, r)}{(1, r)}$$

$$(a, nk) = m^{nk} \left(\frac{a}{m}, k\right)\left(\frac{a+1}{m}, k\right) \cdots \cdots \left(\frac{a+m-1}{m}, k\right)$$

Chapter I

Formal Expansion Theory

The expansion of known functions in terms of sequences of polynomials or other functions has been considered from many angles. It forms the basis of the theory of orthogonal functions and of polynomial approximations. The extensive work of Neumann and Schlömilch series in Bessel functions is of this nature. Although the methods of analysis are more powerful and more inclusive, yet a purely formal algebraic treatment of this problem exhibits a very attractive elegance. Partial treatments of this nature have been discussed by previous workers,[1] but the completely general method seems to have been overlooked.

In the work to follow we shall uniformly deal with functions defined by the power series

1) $$f(x) = \sum_{n} \frac{a_n}{(1, n)} x^n$$

or equally well by the matrix of coefficients

2) $$a = | a_0, a_1, \ldots, a_n, \ldots |$$

Preliminary to a consideration of the general case,

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[1] N. Nielsen, Fonctions Metaspherique, Chap. IV.
N. Nielsen, Recherches sur le developpement d'une fonction analytiques en series de fonctions hypergeometriques, Ann. Scientifiques d' Ec. Norm., (3) XXX (1913), 12.
S. Pincherle, Alcuni teoremi sopra gli sviluppi en serie per funzioni analitiche, Lombardo Rendiconti, (2) XV (1882) 224.
J. M. Whittaker, Interpalotory Function Theory, (Cambridge, 1937)

let us illustrate the theory by determining in terms of the a_n the coefficients b_n of

3) $$f(x) = \sum_n \frac{b_n (1-x)^n}{(1,n)}$$

Since

4) $$(1-x)^n = \sum_r \frac{(-n,r)}{(1,r)} x^r$$

we have

5) $$\sum_n \frac{b_n (1-x)^n}{(1,n)} = \sum_n \frac{b_n}{(1,n)} \sum_r \frac{(-n,r)}{(1,r)} x^r$$

$$= \sum_r \frac{x^r}{(1,r)} \sum_n \frac{(-n,r)}{(1,n)} b_n$$

so that

6) $$a_r = \sum_n \frac{(-n,r)}{(1,n)} b_n = (-)^r \sum_n \frac{b_{n+r}}{(1,n)}$$

To obtain b_s, then, we sum over r

7) $$\sum_r \frac{a_{r+s}}{(1,r)} = \sum_{r,n} (-)^{r+s} \frac{b_{n+r+s}}{(1,r)(1,n)}$$

$$= \sum_p (-)^{p+s} b_{p+s} \sum_n (-)^n \frac{1}{(1,n)(1,p-n)} = (-)^s b_s$$

Hence

8) $$f(x) = \sum_n (-)^n \frac{(1-x)^n}{(1,n)} \sum_r \frac{a_{r+n}}{(1,r)}$$

More generally, if we consider the polynomials,

9) $$\Lambda_n(x) = \sum_r \lambda_r \frac{(-n,r)}{(1,r)} x^r$$

Following the above argument we have, if

10) $$f(x) = \sum_n \frac{b_n A_n(x)}{(1,n)} = \sum_{n,r} \frac{(-n,r)\, b_n \lambda_r x^r}{(1,n)(1,r)},$$

11) $$a_r = \lambda_r \sum_n \frac{(-n,r)}{(1,n)} b_n = (-)^r \lambda_r \sum_n \frac{b_{n+r}}{(1,n)}$$

and to obtain b_n we sum as before over r in;

12) $$\sum_r \frac{a_{r+s}}{\lambda_{r+s}(1,r)} = \sum_{r,n} (-)^{r+s} \frac{b_{n+r+s}}{(1,r)(1,n)} = (-)^s b_s$$

So that

13) $$f(x) = \sum_n (-)^n \frac{A_n(x)}{(1,n)} \sum_r \frac{a_{r+n}}{\lambda_{r+n}(1,r)}$$

In particular, if $\lambda_r = \xi^{-r}$, we have the formal Taylor's expansion of a function about an arbitrary point:

14) $$f(x) = \sum_n \frac{(x-\xi)^n}{(1,n)} \sum_r \frac{\xi^r a_{r+n}}{(1,r)}$$

It is thus possible to proceed to develop our entire theory. The awkwardness of handling the algebra, however, would make the labor prohibitive and hide the elegance of the results.

To avoid this, the umbral notation of Blissard[2] is introduced and will be used throughout.

We consider

15) $$f(x) = \sum_n \frac{a_n x^n}{(1,n)}$$

? as defined by the infinite one-rowed matrix

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2 Blissard, loc. cit.

16) $$a = |a_0, a_1, \ldots, a_m, \ldots| = |a_i|$$

and adopt the convention

17) $$a^m = a_m, \qquad a^{-m} = \frac{1}{a_n}$$

so that

18) $$f(x) = e^{ax}$$

Hence, on this basis:

19) $$f(x) = e^{ax} = e^{\frac{a}{\lambda} - \frac{a}{\lambda}(1 - \lambda x)}$$
$$= \sum_n (-)^n \frac{(1-\lambda x)^n}{(1,n)} \frac{a^n}{\lambda^n} e^{\frac{a}{\lambda}}$$
$$= \sum_m (-)^m \frac{A_m(x)}{(1,m)} \sum_r \frac{a_{m+r}}{\lambda_{m+r}(1,r)},$$

giving the previously derived expansion very elegantly. Of course such a derivation is only suggestive, and the other is necessary for completeness.

Such a formal derivation of an expansion by either of the methods used above gives no information regarding the range of validity, nor even whether the original and deduced series converge in the same range of the variable. In the discussion to follow this difficulty will be even more in evidence. Inasmuch as this forms a large part of the theory of any one particular sequence of functions, it would carry us quite off our track to deal with the problem extensively for any large class of functions. Rather, we shall assume that the arbitrary function and the sequence have some common region of convergence and we shall be satisfied with the conclusions to be drawn from Weierstrass'[3].

Theorem 2.1

Let $P_0(x), P_1(x), \ldots\ldots$

be an ordered, infinite sequence of power series in x, each containing both positive and negative powers of the variable with arbitrary coefficients. Let it be possible to choose two real quantities R, R^1, with $R^1 > R \geq o$, such that for all values of x satisfying

21) $$R < |x| < R'$$

each of the series $P_\lambda(x)$ converges as well as the sum $\sum_{\lambda=0}^{\infty} P_\lambda(x)$ and further that the latter converges uniformly for all values of x having the same absolute value.

Then, if $A_\mu^{(\nu)}$ is the coefficient of x^μ in $P_\nu(x)$, the sum $\sum_{\nu=0}^{\infty} A_\mu^{(\nu)}$ has a definite finite value for every value of n, designated by A_μ, and it may be shown that for every value of x satisfying 21) the series $\sum_{\mu=-\infty}^{\infty} A_\mu x^\mu$ converges and the equation

22) $$\sum_{\nu=0}^{\infty} P_\nu(x) = \sum_{\mu=-\infty}^{\infty} A_\mu x^\mu$$

holds.

1. Function of a Single Variable

We shall call two sequences of functions $P_n(x)$, $Q_n(y)$, defined by the matrices $p_m = |p_\lambda^n|$, $q_m = |q_\lambda^n|$, associate

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[3] K. Weierstrass, Werke, Bd. 2, s 205.

if they are such that

23)
$$e^{xy} = \sum_n \frac{P_n(x)\, Q_n(y)}{(1,n)}$$

Under this convention and the above assumption as to convergence, our basic theorem on formal expansion can be stated conveniently

Theorem 2.2

If $f(x)$ is given by a matrix $a = |a_i|$, and $P_n(x)$, $Q_n(y)$ are associate sequences of functions, then

24)
$$f(x) = e^{ax} = \sum_n \frac{P_n(x)\, Q_n(a)}{(1,n)}$$

Using the umbral notation, this is immediately evident. However, we shall present a more vigorous derivation since we have not developed sufficiently the properties of the matrices.

Since $P_n(x)$ and $Q_n(y)$ are associate, we have:

25)
$$e^{xy} = \sum_p \frac{x^p y^p}{(1,p)} = \sum_n \frac{P_n(x)\, Q_n(y)}{(1,n)}$$
$$= \sum_{r,s} \frac{x^r y^s}{(1,r)(1,s)} \sum_n \frac{p_r^n q_s^n}{(1,n)}$$

so that

26)
$$\sum_n \frac{p_r^n q_s^n}{(1,n)} = \begin{cases} 0 & , r \neq s \\ (1,r) & , r = s \end{cases}$$

being the essential condition interrelating the coefficients of the associate functions.

Now we assume that there exists for f(x) an expansion:

27)
$$f(x) = \sum_m{}' \frac{\alpha_m}{(1,m)} P_m(x)$$

$$= \sum_{m,r} \frac{\alpha_n \rho^n_r x^r}{(1,n)(1,r)} = \sum_r{}' \frac{x^r}{(1,r)} \sum_n{}' \frac{\alpha_n \rho^n_r}{(1,n)}$$

Hence with

28)
$$f(x) = \sum_r{}' \frac{a_r x^r}{(1,r)},$$

29)
$$a_r = \sum_n{}' \frac{\alpha_n \rho^n_r}{(1,n)}$$

To solve for α_m we multiply by γ^m_r, to be determined, and sum over r in:

30)
$$\sum_r{}' \frac{\gamma^m_r a_r}{(1,r)} = \sum_{n,r}{}' \frac{\alpha_n \gamma^m_r \rho^n_r}{(1,n)(1,r)}$$

$$= \sum_n{}' \frac{\alpha_n}{(1,n)} \sum_r{}' \frac{\gamma^m_r \rho^n_r}{(1,r)} = \alpha_m$$

if γ^m_r is such that

31)
$$\sum_r{}' \frac{\gamma^m_r \rho^n_r}{(1,r)} = \begin{cases} 0, & m \neq n \\ (1,m), & m = n \end{cases}$$

But if this is true, we can deduce by multiplying by and summing over n, that

32)
$$\xi^m_s = \sum_n{}' \frac{\xi^n_s}{(1,n)} \sum_r{}' \frac{\gamma^m_r \rho^n_r}{(1,r)}$$

$$= \sum_r{}' \frac{\gamma^m_r}{(1,r)} \sum_n{}' \frac{\xi^n_s \rho^n_r}{(1,n)} = \gamma^m_s$$

by 26) relating ξ^n_s and ρ^n_r

Therefore

33)
$$\alpha_m = \sum_r{}' \frac{\xi^m_r a_r}{(1,r)} = Q_m(a)$$

and

34) $$f(x) = \sum_m \frac{P_m(x)\, Q_m(a)}{(1,m)}$$

The two sets of equivalent conditions:

26) $$\sum_n{}' \frac{P_r^n\, q_s^n}{(1,n)} = \begin{cases} 0, & r \neq s \\ (1,r), & r = s \end{cases}$$

and

35) $$\sum_n{}' \frac{P_n^r\, q_n^s}{(1,n)} = \begin{cases} 0, & r \neq s \\ (1,r), & r = s \end{cases}$$

form a basis on which we can build several important theorems on the respective form of P_n and Q_n.

Let $|p|$ represent the discriminant of the coefficients of the $P_n(x)$:

36) $$|p| = \begin{vmatrix} p_0^0 & p_0^1 & p_0^2 & \cdots \\ p_1^0 & p_1^1 & p_1^2 & \cdots \\ p_2^0 & p_2^1 & p_2^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

In general this will be an infinite order determinant and the question of convergence is best considered in the particular cases which arise. With this notation we have:

Theorem 2.3

Let $P_n(x)$ be any arbitrary sequence of functions

37) $$P_m(x) = \sum_r{}' \frac{p_r^m}{(1,r)} x^r$$

If there exists an associate sequence of functions $Q_n(y)$, the coefficients of the matrix q_n representing $Q_n(x)$, are given by:

38)
$$q_s^n = (-)^{n+s} \frac{n!\,s!}{|p|} \begin{vmatrix} p_0^0 & p_0^1 & \cdots & p_0^{n-1} & p_0^{n+1} & \cdots \\ p_1^0 & p_1^1 & \cdots & p_1^{n-1} & p_1^{n+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{s-1}^0 & p_{s-1}^1 & \cdots & p_{s-1}^{n-1} & p_{s-1}^{n+1} & \cdots \\ p_{s+1}^0 & p_{s+1}^1 & \cdots & p_{s+1}^{n-1} & p_{s+1}^{n+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

In the proof of the previous theorem we showed that when the series $Q_n(y)$ existed the coefficients were related to the matrix P_n by the equations:

26)
$$\sum_n{}' \frac{p_r^n q_s^n}{(1,n)} = \begin{cases} 0, & r \neq s \\ (1,r), & r = s \end{cases}$$

These can be regarded as an infinite system of linear equations in $q_s^n/n!$ for all n. Solving these according to the standard methods, we obtain the result stated in the theorem. The same theory would be secured by starting with the equations

35)
$$\sum_r{}' \frac{p_r^n q_r^m}{(1,r)} = \begin{cases} 0, & n \neq m \\ (1,n), & n = m \end{cases}$$

and solving for $q_r^m/r!$

Utilizing this theorem, we have others as direct consequences.

Theorem 2.4

If $P_n(x)$ and $Q_n(y)$ are associate sequences of functions and $\lambda = /\lambda_0, \lambda_1, \cdots, \lambda_n, /$ is an arbitrary matrix, then $P_n(\lambda x)$ and $Q_n(\frac{y}{\lambda})$ are associate sequences.

If $/P_\lambda/$ is the determinant of the coefficients of the $P_n(\lambda x)$, then, since the matrix of the coefficients is $\rho_n \lambda = |\lambda_i \rho_i^n|$,

39) $$|\rho_\lambda| = \prod_r \lambda_r \, |\rho|$$

Then replacing the ρ_r^n in the expression for q_s^n, by $\lambda_r \rho_r^n$ we obtain the new $\bar{q}_s^n$.

40) $$\bar{q}_s^n = (-)^{n+s} \frac{n!\,s!}{|\rho_\lambda|} \begin{vmatrix} \lambda_0 \rho_0^0 & \cdots \\ \cdots & \cdots \\ \lambda_{s-1} \rho_{s-1}^0 & \cdots \\ \lambda_{s+1} \rho_{s+1}^0 & \cdots \\ \cdots & \cdots \end{vmatrix}$$

$$= (-)^{n+s} \frac{\prod_{r=0}^{s-1} \lambda_r \prod_{s+1}^{\infty} \lambda_r \; n!\,s!}{\prod_r \lambda_r \; |\rho|} \begin{vmatrix} \rho_0^0 & \cdots \\ \cdots & \cdots \\ \rho_{s-1}^0 & \cdots \\ \rho_{s+1}^0 & \cdots \\ \cdots & \cdots \end{vmatrix} = \frac{q_s^n}{\lambda_s}$$

Hence corresponding to $P_n(\lambda x)$ we have

41) $$\sum_s{}' \frac{\bar{q}_s^n y^s}{(1,s)} = \sum_s{}' \frac{q_s^n y^s}{(1,s)\lambda_s} = Q_n\left(\frac{y}{\lambda}\right)$$

This very important theorem enables us to obtain from a given set of associate functions $P_n(x)$, $Q_n(y)$ an almost endless variety of new sets and in our later applications we shall see its usefulness in obtaining some of the classical expansions.

Occasionally it happens that the expansion of an arbitrary function in a series of functions is derived by some other method aside from that discussed here. In these cases we can thereby deduce useful information from

Theorem 2.5

If we have given

42) $$f(x) = e^{ax} = \sum_m{}' \frac{P_m(x)\, Q_m(a)}{(1,m)}$$

then, subject to convergence conditions, we have also

43) $$f(x) = \sum_m{}' \frac{Q_m(x)\, P_m(a)}{(1,m)}$$

When the first expansion is given, the coefficient of P_n and Q_n must certainly satisfy the conditions

26) $$\sum_m{}' \frac{p_s^m\, q_r^m}{(1,m)} = \begin{cases} 0\,, & r \neq s \\ (1,r)\,, & r = s \end{cases}$$

which are perfectly symetrical in the p's and q's. Hence the second expansion follows by our previous argument.

If the $P_n(x)$ have some peculiar form, Theorem 2.3 often enables us to infer directly certain properties of the $Q_n(y)$. In view of its import we state

Theorem 2.6

If $P_n(x)$ and $Q_n(y)$ are associate functions, and if $P_n(x)$ is a polynomial of the n'th degree in x, then $Q_n(y)$ contains y to no power less than the n'th.

Under the assumptions here, $p_r^n = 0$, $r > n$, and without loss of generality, we can set the leading coefficient of $P_n(x)$, $p_n^n = 1$, so that $|p| = \prod p_n^n = 1$. Hence for $s < n$

$$44)\qquad q_s^m = (-)^{m+s} m!\,s! \begin{vmatrix} p_0^0 & p_0^1 & \cdots & p_0^{s-1} & p_0^s & p_0^{s+1} & \cdots & p_0^{m-1} & p_0^{m+1} & \cdots \\ 0 & p_1^1 & \cdots & p_1^{s-1} & p_1^s & p_1^{s+1} & \cdots & p_1^{m-1} & p_1^{m+1} & \cdots \\ - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & \cdots & p_{s-1}^{s-1} & p_{s-1}^s & p_{s-1}^{s+1} & \cdots & p_{s-1}^{m-1} & p_{s-1}^{m+1} & \cdots \\ 0 & 0 & \cdots & 0 & 0 & p_{s+1}^{s+1} & \cdots & p_{s+1}^{m-1} & p_{s+1}^{m+1} & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & p_{s+2}^{m-1} & p_{s+2}^{m+1} & \cdots \\ - & - & - & - & - & - & - & - & - & - \end{vmatrix}$$

$$= 0$$

while for $s > m$

$$45)\qquad q_s^m = (-)^{m+s} m!\,s! \begin{vmatrix} p_0^0 & p_0^1 & \cdots & p_0^{m-1} & p_0^{m+1} & \cdots & p_0^{s-1} & p_0^s & p_0^{s+1} & \cdots \\ 0 & p_1^1 & \cdots & p_1^{m-1} & p_1^{m+1} & \cdots & p_1^{s-1} & p_1^s & p_1^{s+1} & \cdots \\ - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & \cdots & p_{m-1}^{m-1} & p_{m-1}^{m+1} & \cdots & p_{m-1}^{s-1} & p_{m-1}^s & p_{m-1}^{s+1} & \cdots \\ 0 & 0 & \cdots & 0 & p_m^{m+1} & \cdots & p_m^{s-1} & p_m^s & p_m^{s+1} & \cdots \\ 0 & 0 & \cdots & 0 & p_{m+1}^{m+1} & \cdots & p_{m+1}^{s-1} & p_{m+1}^s & p_{m+1}^{s+1} & \cdots \\ - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & \cdots & 0 & p_{s-1}^{m+1} & \cdots & p_{s-1}^{s-1} & p_{s-1}^s & p_{s-1}^{s+1} & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & p_{s+1}^{s+1} & \cdots \\ - & - & - & - & - & - & - & - & - & - \end{vmatrix}$$

$$= (-)^{m+s} m!\,s! \begin{vmatrix} p_m^{m+1} & \cdots & p_m^s \\ - & - & - \\ p_{s-1}^{m+1} & \cdots & p_{s-1}^s \end{vmatrix}$$

and clearly $q_m^m = 1$

Similar theorems can be stated if $P_n(x)$ is a polynomial of degree depending in some other manner upon n. The method is obvious and we shall not consider them. Also, if the sequence $P_n(x)$ has any regular properties, corresponding ones can be inferred for the $Q_n(x)$ by Theorem 2.3.

2. Functions of Several Variables

The formal theory here developed admits an immediate extension to functions of several variables in a manner more elegantly direct than similar extensions of other methods of expansions in functions. In order to provide a convenient way of stating our theorems, we introduce a rather obvious generalization of our matrix notation. Let $f(x_1, \cdots, x_k) = f(x_i)$ be a function of k variables whose series representation is:

46)
$$f(x_1, \cdots, x_k) = \sum_m \frac{a_{m_1, m_2 \cdots m_k}}{m_1! m_2! \cdots m_k!} x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$$

Then said function will be represented by the k-dimensional matrix which we denote:

47)
$$a' a'' \cdots a^{(k)} = | a_{i_1, \cdots, i_k} |$$

and we adopt the convention

48)
$$a'^{m_1} a''^{m_2} \cdots a^{(k)^{m_k}} = a_{m_1, m_2, \cdots, m_k}$$

The concept of associate sequences of functions needs to be extended to several variables. We say that two sequences of functions

49)
$$P_m(x_i) \equiv P_{m_1, \cdots, m_k}(x_1, \cdots, x_k)$$
$$Q_m(y_i) \equiv Q_{m_1, \cdots, m_k}(x_1, \cdots, y_k)$$

defined by the matrices

50)
$$p'_{m_1} p''_{m_2} \cdots p^{(k)}_{m_k} = | p^{m_1, m_2, \cdots, m_k}_{i_1, i_2, \cdots, i_k} |$$

51)
$$q'_{m_1} q''_{m_2} \cdots q^{(k)}_{m_k} = | q^{m_1, \cdots, m_k}_{i_1, \cdots, i_k} |$$

are associate if

52) $$e^{\sum_1^k x_i y_i} = \sum_m \frac{P_m(x_i)\, Q_m(y_i)}{\prod_1^k (1, m_i)}$$

With this notation the general expansion theorem for k variables is:

Theorem 2.10

If $f(x_i)$ is a function of k variables represented by the matrix $a' \cdots a^{(k)} = |a_{i_1, \cdots, i_k}|$, and $P_m(x_i)$ and $Q_m(y_i)$ are associate sequences of functions,

Then

48) $$f(x_1, \cdots, x_k) = e^{a' x_1 + \cdots + a^{(k)} x_k} = \sum_m \frac{P_m(x_i)\, Q_m(a^{(i)})}{\prod_1^k (1, m_i)}$$

Again using the matrices and the umbral notation, the theorem is obvious and for the general case we shall be satisfied with this.

For two variables the statement is essentially the same.

Theorem 2.11

If f(x,y) is a function of two variables represented by the matrix $a a^* = |a_{ij}|$, and $P_{m,n}(x,y)$ and $Q_{m,n}(\xi, \eta)$ are associate sequences of functions represented by the matrices $|p_{i,j}^{m,n}|$, $|q_{i,j}^{m,n}|$,

then

49) $$f(x, y) = e^{ax + a^* y} = \sum_{m,n} \frac{P_{m,n}(x,y)\, Q_{m,n}(a, a^*)}{(1,m)\,(1,n)}$$

The proof is a direct extension of that given for one variable. From the definition of $P_{m,n}$ and $Q_{m,n}$ we have:

50)
$$e^{x\xi + y\eta} = \sum_{r,s}' \frac{x^r y^s \xi^r \eta^s}{(1,r)(1,s)} = \sum_{m,n}' \frac{P_{m,n}(x,y)\, Q_{m,n}(\xi,\eta)}{(1,m)(1,n)}$$
$$= \sum_{r,s,u,v}' \frac{x^r y^s \xi^u \eta^v}{(1,r)(1,s)(1,u)(1,v)} \sum_{m,n}' \frac{p^{m,n}_{r,s}\, q^{m,n}_{u,v}}{(1,m)(1,n)}$$

so that

51)
$$\sum_{m,n}' \frac{p^{m,n}_{r,s}\, q^{m,n}_{u,v}}{(1,m)(1,n)} = \begin{cases} 0, & r \neq u \text{ and/or } s \neq v \\ (1,r)(1,s), & r = u,\ s = v \end{cases}$$

We assume there exists the expansion for $f(x,y)$:

52)
$$f(x,y) = \sum_{m,n}' \frac{\alpha_{m,n}}{(1,m)(1,n)} P_{m,n}(x,y)$$
$$= \sum_{m,n,r,s}' \frac{\alpha_{m,n}\, p^{m,n}_{r,s}\, x^r y^s}{(1,m)(1,n)(1,r)(1,s)} = \sum_{r,s}' \frac{x^r y^s}{(1,r)(1,s)} \sum_{m,n}' \frac{\alpha_{m,n}\, p^{m,n}_{r,s}}{(1,m)(1,n)}$$

but

53)
$$f(x,y) = \sum_{r,s}' \frac{a_{r,s}\, x^r y^s}{(1,r)(1,s)}$$

Therefore

54)
$$a_{r,s} = \sum_{m,n}' \frac{\alpha_{m,n}\, p^{m,n}_{r,s}}{(1,m)(1,n)}$$

To solve for $\alpha_{h,k}$ we multiply by $\gamma^{h,k}_{r,s}$, to be determined, and sum over r and s in

55)
$$\sum_{r,s}' \frac{\gamma^{h,k}_{r,s}\, a_{r,s}}{(1,r)(1,s)} = \sum_{m,n,r,s}' \frac{\gamma^{h,k}_{r,s}\, \alpha_{m,n}\, p^{m,n}_{r,s}}{(1,m)(1,n)(1,r)(1,s)}$$
$$= \sum_{m,n}' \frac{\alpha_{m,n}}{(1,m)(1,n)} \sum_{r,s}' \frac{\gamma^{h,k}_{r,s}\, p^{m,n}_{r,s}}{(1,r)(1,s)} = \alpha_{h,k}$$

if $\gamma^{h,k}_{r,s}$ is such that

56)
$$\sum_{r,s}' \frac{\gamma^{h,k}_{r,s}\, p^{m,n}_{r,s}}{(1,r)(1,s)} = \begin{cases} 0, & h \neq m \text{ and/or } k \neq n \\ (1,m)(1,n), & h = m,\ k = n \end{cases}$$

If this is multiplied by $\frac{\mathcal{Q}^{m,n}_{i,j}}{(1,m)(1,n)}$ and summed over m,n we obtain:

57)
$$\mathcal{Q}^{h,k}_{i,j} = \sum_{m,n}{}' \frac{\mathcal{Q}^{m,n}_{i,j}}{(1,m)(1,n)} \sum_{r,s}{}' \frac{\gamma^{h,k}_{r,s}}{(1,r)} \frac{P^{m,n}_{r,s}}{(1,s)}$$
$$= \sum_{r,s}{}' \frac{\gamma^{h,k}_{r,s}}{(1,r)(1,s)} \sum_{m,n}{}' \frac{\mathcal{Q}^{m,n}_{i,j} P^{m,n}_{r,s}}{(1,m)(1,n)} = \gamma^{h,k}_{i,j}$$

by 51). Therefore

58)
$$\alpha_{h,k} = \sum_{r,s}{}' \frac{\mathcal{Q}^{h,k}_{r,s}\, a_{r,s}}{(1,r)(1,s)} = Q_{h,k}(a, a^*)$$

For three or more variables the proof would be similar and it is unnecessary to give the details here.

The analogy of Theorem 2.4 is extremely important in this case.

<u>Theorem 2.12</u>

<u>If $P_{m,n}(x,y)$ and $Q_{m,n}(\xi,\eta)$ are associate sequences of functions and $\lambda\lambda^* = /\lambda_{i,j}/$ is an arbitrary matrix, then $P_{m,n}(\lambda x, \lambda^* y)$ and $Q_{m,n}(\frac{x}{\lambda}, \frac{y}{\lambda^*})$ are associate sequences.</u>

In the proof of Theorem 2.11 it was shown that the conditions that two sequences $P_{m,n}$ and $Q_{m,n}$ be associate, namely

51)
$$\sum_{m,n}{}' \frac{P^{m,n}_{r,s}}{(1,m)} \frac{\mathcal{Q}^{m,n}_{u,v}}{(1,n)} = \begin{cases} 0, & r \neq u \text{ and/or } s \neq v \\ (1,r)(1,s), & r = u,\ s = v \end{cases}$$

was equivalent to the similar

59)
$$\sum_{r,s}{}' \frac{P^{m,n}_{r,s}}{(1,r)} \frac{\mathcal{Q}^{h,k}_{r,s}}{(1,s)} = \begin{cases} 0, & h \neq m \text{ and/or } k \neq n \\ (1,m)(1,n), & h = m,\ k = n \end{cases}$$

Now let the matrices of $P_{m,n}(\lambda x, \lambda^* y)$ and $Q_{m,n}(\frac{\xi}{\lambda}, \frac{\eta}{\lambda^*})$ be designated for the moment by $\bar{P}_m \bar{P}^*_n$ and $\bar{\mathcal{Q}}_m \bar{\mathcal{Q}}^*_n$

Then

60) $$\sum_{r,s} \frac{\bar{P}^{m,n}_{r,s}\, q^{h,k}_{r,s}}{(1,r)(1,s)} = \sum_{r,s} \frac{\lambda_{r,s} P^{m,n}_{r,s} \cdot \frac{1}{\lambda_{r,s}}\, q^{h,k}_{r,s}}{(1,r)\,(1,s)}$$

$$= \sum_{r,s} \frac{P^{m,n}_{r,s}\, q^{h,k}_{r,s}}{(1,r)(1,s)} = \begin{cases} 0, & h \neq m \text{ and/or } k \neq n \\ (1,m)(1,n), & h = m,\ k = n \end{cases}$$

by the original assumption. Hence we show that the second pair satisfies the necessary conditions and are associate.

This theorem has its greatest use in obtaining associate functions for given two-variable functions by building them up from known one-variable associates. Thus, if $P_m(x), Q_m(\xi)$ and $\bar{P}_n(y), Q_n(\eta)$ are two known sets of associate functions, then we have immediately that

$$P_{m,n}(x,y) = P_m(\lambda x)\,\bar{P}_n(\lambda^* y), \quad Q_{m,n}(\xi,\eta) = Q_m\left(\frac{\xi}{\lambda}\right) \bar{Q}_n\left(\frac{\eta}{\lambda^*}\right)$$

are associate sequences. And if $\lambda\lambda^* = |\lambda_{i,j}|$ is perfectly general, the two variable functions can certainly not be actually factored into one-variable functions. We shall use this in all its power in later applications.

Evidently it would be possible to state theorems for special types of two-variable associate functions as we did for the one-variable cases. Those needed are such obvious extensions that we shall mention them only when they are used.

We have indicated in several places the direction of generalization to any number of variables. On the formal basis which we are building these extensions will clearly introduce no new complications. We have restricted ourselves from treating the general case in order to have the results in a form ready to apply in our later discussions.

3. Relation to Tchebycheff Polynomials

Since the Tchebycheff polynomials form a very large class of functions to which this theory is applicable, it is significant to inquire concerning the possible extensions of their theory to this more general case.

Let $P_n(x)$ and $Q_n(y)$ be a set of associated functions, and we shall assume the desired expansions to be in $P_n(x)$. We search for a weight function $W_n(x)$ such that

61) $$\int_{-\infty}^{\infty} P_n(x)\, P_m(x)\, W_n(x)\, dx = \begin{cases} 0, & m \neq n \\ (1,n), & m = n \end{cases}$$

If such a function exists it will certainly be unique and since P_n and Q_n are associate, we would infer directly that

62) $$Q_n(y) = \int_{-\infty}^{\infty} e^{xy}\, W_n(x)\, P_n(x)\, dx$$

In case we have

63) $$W_n(x) = W(x) \cdot W_n$$

where $W(x)$ is independent of n and W_n is independent of x, then

64) $$\int_{-\infty}^{\infty} P_n(x)\, P_m(x)\, W(x)\, dx = \begin{cases} 0, & m \neq n \\ \dfrac{(1,n)}{W_n}, & m = n \end{cases}$$

so that $P_n(x)$ form a set of Tchebycheff polynomials, $\psi_n(x)$. And conversely, when the $P_n(x)$ are Tchebycheff polynomials, then $W_n(x)$ will certainly be of this form.

In the theory of the Tchebycheff polynomials there are several integrals and related functions which have significant meaning. We consider the extensions here:

The functions of the second kind $P_n^*(y)$ are defined by

65) $$\frac{1}{y-x} = \sum_m' \frac{P_m(x)\,P_m^*(y)}{(1,m)}$$

from this and the definition of the associate function $Q_n(y)$ we conclude that

66) $$P_m^*(y) = \frac{1}{y}\,Q_m\left(\frac{\Gamma}{y}\right)$$

where $\Gamma^m = m!$ (matrix notation). Also from 65) and 61)

67) $$\int_{-\infty}^{\infty} \frac{P_m(x)\,W_m(x)}{y-x}\,dx = \sum_m' \frac{P_m^*(y)}{(1,m)} \int_{-\infty}^{\infty} P_m(x)\,P_m(x)\,W_m(x)\,dx = P_m^*(y)$$

which would also follow symbolically from 66) and 62).

We may indicate a treatment of our expansion theory using the Cauchy complex variable theorems, in a manner customarily used for the Neumann, Legendre and Bessel function expansions. The question of the range of applicability of this to follow is another problem beyond our present purpose.

Let 65) be uniformly convergent with respect to y in some region R including the point z with y on the boundary C of R. Let

$$f(x) = \sum_m' \frac{a_m x^m}{(1,m)} = e^{ax}$$

be some function analytic inside and on C. Then, since 65) is uniformly convergent

68) $$f(z) = \frac{1}{2\pi i}\int_C \frac{f(y)}{y-z}\,dy = \sum_m \frac{P_m(z)}{(1,m)} \cdot \frac{1}{2\pi i}\int_C f(y)P_m^*(y)\,dy$$

But since the expansion in $P_n(x)$ is unique when the series converges, this implies

69) $$Q_m(a) = \frac{1}{2\pi i}\int_C e^{ay} P_m^*(y)\,dy$$

If, further, the weight function $W_n(x)$ does exist and the infinite integral 67) is uniformly convergent, then

70) $$\frac{1}{2\pi i}\int_C e^{ay} P_m^*(y)\,dy = \frac{1}{2\pi i}\int_C\int_{-\infty}^{\infty} \frac{e^{ay}P_m(x)W_m(x)}{y-x}\,dy\,dx$$
$$= \int_{-\infty}^{\infty} e^{ax} P_m(x)W_m(x)\,dx = Q_m(a)$$

in agreement with the inference above.

The two-variable theory outlined can also be treated in this manner. The method is obvious and since it contributes nothing of significance to our theory, we shall be content with the brief discussion of the one-variable case.

Chapter II - Application.

Functions of a Single Variable

We shall develop the application of this theory to the hypergeometric functions of one and two variables. All the classical orthogonal expansions are in this category and will appear as special cases in a logical sequence.

1. Hypergeometric Series

A single infinite series $\sum_{n=0}^{\infty} a_n$ is said to be hypergeometric if the ratio a_{n+1}/a_n is a rational function of n. When this is true, we may write:

$$\frac{a_{n+1}}{a_n} = \frac{P(n)}{Q(n)}$$

where P(n) and Q(n) are relatively prime polynomials in n. If they are of degrees p and q + 1 respectively, we may exhibit them in the factored form:

1) $$\frac{a_{n+1}}{a_n} = \frac{(n+\alpha_1)(n+\alpha_2)\cdots(n+\alpha_p)\lambda}{(n+\beta_1)(n+\beta_2)\cdots(n+\beta_q)(n+\beta_{q+1})}$$

λ being the ratio of the coefficients of the highest terms in P and Q.

Without loss of generality we set $\beta_{q+1} = 1$ according to established custom. In case any one of the β's is a negative integer, say $\beta_1 = -k$, then obviously a_k and all previous terms in the series must be zero. Hence, the series will begin with a_{k+1}, so a new set of β's is obtained with $\bar{\beta}_i = \beta_i + k + 1$. Thus, again, without any loss in generality we may assume that none of the β's are negative integers.

Applying 1) successively, we deduce

$$u_m = \frac{(\alpha_1, m)(\alpha_2, m) \cdots (\alpha_p, m)}{(\beta_1, m)(\beta_2, m) \cdots (\beta_q, m)} \cdot \frac{\lambda^m}{(1, m)} u_0$$

So that

$$\sum_{m=0}^{\infty} a_m = a_0 \sum_m \frac{(\alpha_1, m) \cdots (\alpha_p, m)}{(\beta_1, m) \cdots (\beta_q, m)} \cdot \frac{\lambda^m}{(1, m)}$$

For the series occurring in the second member of this equation, the standard notation is:

$${}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \cdots, \alpha_p \\ \beta_1, \beta_2, \cdots, \beta_q\end{matrix}; \lambda\right] = \sum_m \frac{(\alpha_1, m) \cdots (\alpha_p, m)}{(\beta_1, m) \cdots (\beta_q, m)} \cdot \frac{\lambda^m}{(1, m)}$$

In case any one of the α's is a negative integer, the terms become zero after a certain stage and the series is merely a polynomial in λ. In all other cases the series will be non-terminating and from the Weierstrass ratio test we infer the following convergence properties:

$p < q + 1$, - - - - - - - - - - -, a. c.

$p = q + 1$, $|\lambda| < 1$, - - - - - - - - -, a. c.

$p = q + 1$, $|\lambda| = 1$, $R(\sum \alpha - \sum \beta) < 0$, - -, a. c.

$p = q + 1$, $|\lambda| = 1$, $1 > R(\sum \alpha - \sum \beta) > 0$, $\lambda \neq 1$, c. c.

$p = q + 1$, $\lambda = 1$, $R(\sum \alpha - \sum \beta) \geq 0$, - -, d.

$p = q + 1$, $|\lambda| = 1$, $R(\sum \alpha - \sum \beta) \geq 1$, - -, d.

$p = q + 1$, $|\lambda| > 1$, - - - - - - - -, d.

$p > q + 1$, - - - - - - - - - - -, d.

a. c. - Absolutely convergent; c. c. - Conditionally convergent
d. - Divergent.

To avoid the complication of encounters with the divergent forms above, we shall consider them replaced when they do occur by the series for which the divergent forms represent an asymptotic expansion according to the treatment of Barnes,[1] the series being considered as functions of λ .

The formula, valid for $p > q+1$, is

$${}_pF_q\left[\begin{matrix}\alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}; \lambda\right]$$

$$= \sum_{r=1}^{p} \lambda^{r} \frac{\prod_{s=1}^{p}{}^{*} (-\alpha_r, \alpha_s)}{\prod_{s=1}^{q} (-\alpha_r, \beta_s)} {}_{q+1}F_{p-1}\left[\begin{matrix}\alpha_r, \alpha_r-\beta_1+1, \dots, \alpha_r-\beta_q+1 \\ \alpha_r-\alpha_1+1, \alpha_r-\alpha_2+1, \dots * \dots, \alpha_r-\alpha_p+1\end{matrix}; \frac{(-1)^{p+q}}{\lambda}\right]$$

where the asterisk indicates the omission of the term with equal subscripts.

When $p = q+1$, the same formula gives the analytic continuation of the function beyond the unit circle.

These formulae are particularly useful on occasion to replace certain polynomials by equivalent forms, the replacement being in these cases merely a reversion of the order of terms.

The various analytical properties of these functions have been considered in detail by a great many writers. Since they have no immediate connection with our formal expansion theory, we merely refer to these sources.[2]

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[1] E. W. Barnes, The Asymptotic Expansion of Integral Function defined by Generalized Hypergeometric Series, Proc. London Math. Soc., (2) V, 59-118.

[2] Appel et Kampe de Ferriet, Fonctions Hypergeometrique, (Paris, 1926); an extensive bibliography is geven here.
W. N. Bailey, Generalized Hypergeometric Series, (Cambridge, 1935).

2. Basic Sets of Associate Functions

All of the known expansions in hypergeometric functions of a single variable are deducible by our theory from two basic sets of associate functions. We are primarily concerned with such expansions and shall treat them in detail before proceeding to non-hypergeometric series.

The first basic pair arises as an extension of the well-known expansion in Hermite polynomials. The function $H_m^p(x)$ defined by the generating function

$$e^{zx - \frac{z^p}{p}} = \sum_m \frac{z^m}{(1,m)} H_m^p(x) = \sum_{r,s} (-)^s \frac{z^{r+ps}\, x^r}{p^s (1,r)(1,s)}$$

$$= \sum_{m,s} (-)^s \frac{z^m x^{m-ps}}{p^s (1,m-ps)(1,s)} = \sum_m \frac{z^m}{(1,m)} \sum_s (-)^{s(p+1)} \frac{(-m,sp)}{(1,s)} \frac{x^{m-ps}}{p^s}$$

has the explicit form:

$$H_m^p(x) = \sum_s (-)^{s(p+1)} \frac{(-m,sp)}{(1,s)} \frac{x^{m-ps}}{p^s}$$

$$= \sum_s (-)^{s(p+1)} \frac{p^{ps}\left(-\frac{m}{p},s\right)\left(-\frac{m-1}{p},s\right)\cdots\left(-\frac{m-p+1}{p},s\right)}{p^s (1,s)} x^{m-ps}$$

$$= x^m\, {}_pF_0\left[-\frac{m}{p}, -\frac{m-1}{p}, \ldots, -\frac{m-p+1}{p};\ \frac{(-p)^{p-1}}{x^p}\right]$$

The associate function, $N_m^p(y)$, is readily determined from the generating function, since:

$$e^{xy} = e^{xy - \frac{y^p}{p} + \frac{y^p}{p}} = \sum_m \frac{H_m^p(x)\, y^m e^{\frac{y^p}{p}}}{(1,m)}$$

Hence

$$N_m^p(y) = y^m e^{\frac{y^p}{p}} = \sum_r \frac{y^{m+pr}}{p^r (1,r)}$$

Expansions derived from these by application of our several theorems will be called Hermite type.

The well-known Neumann expansions in Bessel functions are special cases of the expansions obtained from the associate functions:

$$B_m^k(\alpha, x) = \sum_r \frac{x^{m+kr}}{(\alpha+\frac{2m}{k}+1, r)(1,r)}$$

$$L_m^k(\alpha, y) = \sum_s (-)^{ks}\frac{(-m, ks)\, y^{m-ks}}{(-\alpha-\frac{2m}{k}+1, s)(1,s)}$$

We show that these are associate by referring to the basic definition:

$$\sum_m \frac{B_m(x)L_m(y)}{(1,m)} = \sum_{m,r,s}(-)^{ks}\frac{(-m,ks)\, x^{m+kr}\, y^{m-ks}}{(\alpha+\frac{2m}{k}+1,r)(-\alpha-\frac{2m}{k}+1,s)(1,r)(1,s)(1,m)}$$

$$= \sum_p \frac{x^p y^p}{(1,p)}\sum_{r,s}(-)^{k(r+s)}\frac{(-p,kr)(-p+kr,ks)\, y^{-kr-ks}}{(-\alpha-\frac{2p}{k}+2r+1,s)(\alpha+\frac{2p}{k}-2r+1,r)(1,r)(1,s)}$$

$$= \sum_{p,q}\frac{x^p y^{p-kq}}{(1,p)(1,q)}\sum_r(-)^{kr}\frac{(-q,r)(-p,kr)(-p+kr,kq-kr)}{(-\alpha-\frac{2p}{k}+2r+1,q-r)(\alpha+\frac{2p}{k}-2r+1,r)(1,r)}$$

$$= \sum_{p,q}(-)^{kq}\frac{(-p,kq)\, x^p y^{p-kq}}{(-\alpha-\frac{2p}{k}+1,q)(1,p)(1,q)}\sum_r\frac{(-q,r)(-\alpha-\frac{2p}{k}+1,2r)(-\alpha-\frac{2p}{k},r)}{(-\alpha-\frac{2p}{k}+q+1,r)(-\alpha-\frac{2p}{k},2r)(1,r)}$$

$$= \sum_{p,q}(-)^{kq}\frac{(-p,kq)\, x^p y^{p-kq}}{(-\alpha-\frac{2p}{k}+1,q)(1,p)(1,q)}\; {}_4F_3\left[\begin{matrix}-\alpha-\frac{2p}{k},\ \frac{1}{2}(-\alpha-\frac{2p}{k}+1),\ \frac{1}{2}(-\alpha-\frac{2p}{k})+1,\ -q\\ \frac{1}{2}(-\alpha-\frac{2p}{k}),\ \frac{1}{2}(-\alpha-\frac{2p}{k}+1),\ -\alpha-\frac{2p}{k}+q+1\end{matrix}\right]$$

$$= \sum_{p,q}(-)^{kq}\frac{(-p,kq)\, x^p y^{p-kq}}{(-\alpha-\frac{2p}{k}+1,q)(1,p)(1,q)}\cdot\frac{(-\alpha-\frac{2p}{k}+1,q)(0,q)}{(\frac{-\alpha-\frac{2p}{k}+1}{2},q)(\frac{-\alpha-\frac{2p}{k}+1}{2},q)} \qquad *3$$

$$= \sum_p \frac{x^p y^p}{(1,p)} = e^{xy}$$

Thus B_n and L_n fulfill the definition of associate functions.

Expansions derived from these will be called Bessel type.

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[3] W. N. Bailey, loc. cit., p. 25.

In the discussion to follow we shall show how these two sets give rise to all the classical expansions by using Theorem 2.4. The two sets here given are essentially different in this respect, in that one cannot be derived **thus from** the other by application of Theorem 2.4. Further basic expansions of more complicated nature doubtless exist, but only provide expansions in functions of greater complexity than concern us.

3. Classical Expansions

The several well-known expansions in series of Bessel functions are included among both the Bessel and the Hermite type expansions above. The Neumann series of the first kind gives expansions in:

$$\left(\tfrac{1}{2}x\right)^{-\nu} J_{\nu+n}(x) = \sum_r (-)^r \frac{\left(\tfrac{1}{2}x\right)^{n+2r}}{\Gamma(\nu+n+r+1)\,(1,r)}$$

$$= \frac{1}{\Gamma(\nu+n+1)} \sum_r (-)^r \frac{\left(\tfrac{1}{2}x\right)^{n+2r}}{(\nu+n+1,r)(1,r)} = \frac{(-i)^n}{\Gamma(\nu+n+1)} B_n^2\left(\nu; \frac{ix}{2}\right)$$

Hence it is of the Bessel type and the associate function is:

$$\frac{\Gamma(\nu+n+1)}{(-i)^n} L_n^2(\nu; -2iy)$$

$$= \frac{\Gamma(\nu+n+1)}{(-i)^n} \sum_s \frac{(-n,2s)(-2i)^{n-2s}\, y^{n-2s}}{(-\nu-n+1,s)\,(1,s)} = \Gamma(\nu+n+1) \sum_s (-)^s \frac{(-n,2s)(2y)^{n-2s}}{(-\nu-n+1,s)\,(1,s)}$$

The Neumann expansion of the second kind gives expansions in:

$$\left(\tfrac{1}{2}x\right)^{-\frac{\rho+\nu}{2}} J_{\frac{\nu+n}{2}}(x)\, J_{\frac{\rho+n}{2}}(x) = \sum_{p,q} (-)^{p+q} \frac{\left(\tfrac{1}{2}x\right)^{n+2p+2q}}{\Gamma\left(\frac{\nu+n}{2}+p+1\right)\Gamma\left(\frac{\rho+n}{2}+q+1\right)(1,p)\,(1,q)}$$

$$= \sum_s (-)^s \frac{\left(\frac{\nu+\rho}{2}+n+1, 2s\right)\left(\tfrac{1}{2}x\right)^{n+2s}}{\Gamma\left(\frac{\nu+n}{2}+s+1\right)\Gamma\left(\frac{\rho+n}{2}+s+1\right)\left(\frac{\nu+\rho}{2}+n+1,s\right)(1,s)} = \frac{(-i)^n}{\Gamma\left(\frac{\nu+\rho}{2}+n+1\right)} B_n^2\left(\frac{\nu+\rho}{2}; \lambda x\right)$$

where $$\lambda_h = \frac{i^h \, \Gamma\left(\frac{\nu+\ell}{2} + h + 1\right)}{2^h \, \Gamma\left(\frac{\nu+h}{2} + 1\right) \Gamma\left(\frac{\ell+h}{2} + 1\right)}$$

again the Bessel type, so we obtain the associate function:

$$\frac{\Gamma\left(\frac{\nu+\ell}{2} + m + 1\right)}{(-i)^m} L_m^2\left(\frac{\nu+\ell}{2} ; \frac{y}{\lambda}\right) = (-i)^{-m} \Gamma\left(\frac{\nu+\ell}{2} + m + 1\right) \sum_s \frac{\Gamma\left(\frac{\nu+m-2s}{2}+1\right)\Gamma\left(\frac{\ell+m-2s}{2}+1\right)(-m, 2s)\,(-2iy)^{m-2s}}{\Gamma\left(\frac{\nu+\ell}{2} + m - 2s + 1\right)\left(-\frac{\nu+\ell}{2} - m + 1, s\right)(1, s)}$$

$$= \sum_s (-)^s \frac{(-m, 2s)\left(-\frac{\nu+\ell}{2} - m, 2s\right)\Gamma\left(\frac{\nu+m}{2} - s + 1\right)\Gamma\left(\frac{\ell+m}{2} - s + 1\right)}{\left(-\frac{\nu+\ell}{2} - m + 1, s\right)(1, s)} (2y)^{m-2s}$$

Related to these is the expansion of even functions in series of:

$$\left(\tfrac{1}{2}x\right)^{s-\nu} J_{\nu+s}(x) = \sum_r (-)^r \frac{\left(\frac{1}{2}x\right)^{2r+2s}}{\Gamma(\nu+s+r+1)(1,r)}$$

$$= (-)^s N_s'\left(\lambda \frac{x^2}{4}\right)$$

where $$\lambda_h = \frac{(-)^h}{\Gamma(\nu + h + 1)}$$

This is of the Hermite type, so the associate function is:

$$(-)^s H_s'\left(\frac{4y}{\lambda}\right) = (-)^s \sum_r \frac{(-s, r)}{(1, r)} \frac{(4y)^{s-r}}{\lambda_{s-r}} = \sum_r (-)^r \frac{(-s, r)\,\Gamma(\nu + s - r - 1)}{(1, r)} (4y)^{s-r}$$

In addition to these well-known expansions, we may construct from our basic set an almost unending variety of expansions in Bessel functions not discussed before. For example:

$$(-x)^{\frac{m}{2}} J_{\frac{m}{2}}(x) = \sum_r (-)^{\frac{m+2r}{2}} \frac{\left(\frac{1}{2}x\right)^{m+2r}}{\Gamma\left(\frac{m}{2} + r + 1\right)(1, r)}$$

$$= N_m^2\left(\lambda \frac{x}{2}\right), \qquad \lambda_h = \frac{(-2)^{\frac{h}{2}}}{\Gamma\left(\frac{h}{2} + 1\right)}$$

Thus we have a Hermite type with the associate function:

$$H_m^2\left(\frac{2y}{\lambda}\right) = \sum_s (-)^s \frac{(-m, 2s)\,(2y)^{m-2s}}{\lambda_{m-2s}\,(1, s)\,2^s} = (-2)^{-\frac{m}{2}} \sum_s \frac{\Gamma\left(\frac{m}{2} - s + 1\right)(-m, 2s)}{(1, s)} (2y)^{m-2s}$$

Again, the fact that

$$\mathcal{B}_m^k(\alpha, x) = (-1)^{\frac{m}{k}} \left(\sqrt{-x^k}\right)^{-\alpha} J_{\alpha + \frac{2m}{k}} \left(\sqrt{-4x^k}\right)$$

suggests a number of expansions.

The Jacobi polynomial may be similarly considered. We have the general form:

$$ {}_2F_1(-m, \alpha+m; \gamma; x) = \sum_r \frac{(-m,r)(\alpha+m,r)}{(\gamma,r)(1,r)} x^r$$

$$= \sum_r (-)^m \frac{(-m,r)(\alpha+m,m-r)}{(\gamma,m-r)(1,r)} x^{m-r} = (-)^m (\alpha+m,m) \sum_r (-)^r \frac{(-m,r) \quad x^{m-r}}{(-\alpha-2m+1,r)(\gamma,m-r)(1,r)}$$

$$= (-)^m (\alpha+m,m) \, L_m^1(\alpha, \lambda x), \qquad \lambda_h = \frac{1}{(\gamma,h)}$$

Hence expansions in Jacobi polynomials are of the Bessel type. The associate function is:

$$\frac{(-)^m}{(\alpha+m,m)} \mathcal{B}_m^1\left(\alpha; \frac{y}{\lambda}\right) = \frac{(-1)^m}{(\alpha+m,m)} \sum_s \frac{(\gamma, m+s) \, y^{m+s}}{(\alpha+2m+1,s)(1,s)}$$

Rather than discuss the Legendre polynomial, we consider the Gegenbauer function $C_m^\nu(x)$ which includes the former as the special case $\nu = \frac{1}{2}$. By definition

$$(1 - 2hx + h^2)^{-\nu} = \sum_m h^m C_m^\nu(x)$$

Hence
$$C_m^\nu(x) = \frac{(\nu,m)}{(1,m)} \sum_s \frac{(-m,2s) \quad (2x)^{m-2s}}{(-\nu-m+1,s)(1,s)}$$

$$= \frac{(\nu,m)}{(1,m)} L_m^2(\nu, 2x)$$

showing it to be of the Bessel type. The associate function here is:

$$\frac{(1,m)}{(\nu,m)} \mathcal{B}_m^2\left(\nu, \tfrac{1}{2}y\right) = \frac{(1,m)}{(\nu,m)} \sum_r \frac{\left(\frac{1}{2}y\right)^{m+2r}}{(\nu+m+1,r)(1,r)}$$

It may be observed here that thus, formally the expansions in Gegenbauer polynomials and the Neumann series of the first kind are, aside from trivial factors, equivalent in the light of our theory. The various expansions in generalized Legendre polynomials are of course related to the expansions in Jacobi polynomials mentioned above.

Among the functions expressible in terms of the confluent hypergeometric function:

$$W_{k,m}(x) = e^{-\frac{1}{2}z} z^k {}_2F_0\left(-m-k+\tfrac{1}{2},\ m-k+\tfrac{1}{2};\ -\tfrac{1}{z}\right)$$

we shall illustrate our theory for the following:

The Hermite polynomial:

$$H_m(x) = 2^{\frac{m}{2}+\frac{1}{4}} x^{-\frac{1}{2}} e^{-\frac{x^2}{4}} W_{\frac{m}{2}+\frac{1}{4},\frac{1}{4}}(x)$$

The parabolic cylinder function:

$$D_m(x) = e^{\frac{x^2}{4}} H_m(x) = 2^{\frac{m}{2}+\frac{1}{4}} x^{-\frac{1}{2}} W_{\frac{m}{2}+\frac{1}{4},\frac{1}{4}}(x)$$

The Sonine polynomial:

$$T_m^n(x) = \frac{1}{n!\,\Gamma(m+n+1)} x^{-\frac{1}{2}(m+1)} e^{\frac{1}{2}x} W_{n+\frac{1}{2}m+\frac{1}{2},\frac{1}{2}m}(x)$$

The Laguerre polynomial:

$$x^m L_m^{-\alpha-2m}\left(-\frac{1}{x}\right) = \frac{(-)^m}{(1,m)} e^{-\frac{1}{2x}} (-x)^{\frac{1-\alpha}{2}} W_{\frac{1-\alpha}{2},\frac{\alpha}{2}+m}\left(-\frac{1}{x}\right)$$

For the Hermite polynomial we have:

$$H_m(x) = \sum_s \frac{(-m,2s)}{(1,s)} \frac{x^{m-2s}}{2^s} = H_m^2(x)$$

Hence the associate function is:

$$N_m^2(y) = \sum_r{}' \frac{y^{m+2r}}{2^r (1,r)}.$$

The relation between the Hermite polynomial and the parabolic cylinder function points the way to obtain expansions in terms of the latter. If we wish to expand $f(x) = e^{ax}$ as a series $\sum' \frac{\alpha_m D_m(x)}{(1,m)}$, we consider the function

$$e^{ax - \frac{x^2}{4}} = \sum_m{}' \frac{x^m}{(1,m)} 2^{-\frac{m}{2}} H_m^2(a\sqrt{2}) = e^{bx}$$

where

$$b_m = 2^{-\frac{m}{2}} H_m^2(a\sqrt{2})$$

Then we have

$$e^{bx} = \sum_m{}' \frac{N_m^2(b) H_m(x)}{(1,m)}$$

so that

$$e^{ax} = \sum_m{}' \frac{D_m(x) N_m^2(b)}{(1,m)}$$

This device is one which may be profitably applied often to similar situations.

The Sonine polynomial:

$$T_m^m(x) = \frac{(-)^m}{(1,m)(1,m)} \sum_s{}' \frac{(-m,s)\, x^{m-s}}{(m+1,m-s)(1,s)}$$

$$= \frac{(-)^m}{(1,m)(1,m)} H_m'(\lambda x), \qquad \lambda_h = \frac{1}{(m+1,h)}$$

so the associate function is:

$$(-)^m(1,m)(1,m)\, N_m'\left(\frac{y}{\lambda}\right) = (-)^m(1,m)(1,m) \sum_r{}' \frac{(m+1,m+r)}{(1,r)}\, y^{m+r}$$

The Laguerre polynomial:

$$x^m L_m^{-\alpha-2m}\left(-\frac{1}{x}\right) = (-)^m \frac{(\alpha+m,m)}{(1,m)} x^m {}_1F_1\left(-m;-\alpha-2m+1;-\frac{1}{x}\right)$$

$$= \frac{(\alpha+m,m)}{(1,m)} \sum_s{}' (-)^{m+s} \frac{(-m,s)\,x^{m-s}}{(-\alpha-2m+1,s)(1,s)} = (-)^m \frac{(\alpha+m,m)}{(1,m)} L_m'(\alpha,x)$$

so the associate function is:

$$(-)^m \frac{(1,m)}{(\alpha+m,m)} B_m'(\alpha,y) = \frac{(-)^m(1,m)}{(\alpha+m,m)} \sum_r{}' \frac{y^{m+r}}{(\alpha+2m+1,r)(1,r)}$$

Just to mention one or two further expansions to illustrate the method, we mention first:

$$\sum_r{}' \frac{(\alpha_1,m+r)(\alpha_2,m+r)}{(\beta,m+r)\ (1,r)} x^{m+r} = \frac{(\alpha_1,m)(\alpha_2,m)}{(\beta,m)} x^m {}_2F_1(\alpha_1+m,\alpha_2+m;\beta+m;x)$$

$$= N_m'(\lambda x), \qquad \lambda_h = \frac{(\alpha_1,h)(\alpha_2,h)}{(\beta,h)}$$

which has the associate function:

$$H_m'\left(\frac{y}{\lambda}\right) = \sum_s{}' \frac{(-m,s)\,(\beta,m-s)}{(\alpha_1,m-s)(\alpha_2,m-s)} \frac{y^{m-s}}{(1,s)}$$

Again, the Bessel function of the third order introduced by P. Humbert,[4] illustrates the theory nicely. Consider expansion in terms of

$$\left(\tfrac{1}{3}x\right)^{-\alpha-\beta} J_{\alpha+\frac{2}{3}m,\ \beta+\frac{1}{3}m}(x)$$

analogous to the Neumann series of the first kind. By definition:

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[4] P. Humbert, C. R., CXC (1930) 59.

$$(\tfrac{1}{3}x)^{-\alpha-\beta} J_{\alpha+\frac{2}{3}m,\ \beta+\frac{1}{3}m}(x)$$

$$= \frac{(\frac{1}{3}x)^m}{\Gamma(\alpha+\frac{2}{3}m+1)\Gamma(\beta+\frac{1}{3}m+1)}\ {}_0F_2\left(\alpha+\tfrac{2}{3}m+1,\ \beta+\tfrac{1}{3}m+1;\ -\frac{x^3}{27}\right)$$

$$= \sum_r \frac{(-)^r\,(\frac{1}{3}x)^{m+3r}}{\Gamma(\alpha+\frac{2}{3}m+r+1)\Gamma(\beta+\frac{1}{3}m+r+1)\,(1,r)}$$

$$= \frac{1}{\Gamma(\alpha+\frac{2}{3}m+1)} \sum_r \frac{(-)^r\,(\frac{1}{3}x)^{m+3r}}{\Gamma(\beta+\frac{m+3r}{3}+1)(\alpha+\frac{2}{3}m+1,r)(1,r)}$$

$$= \frac{e^{-\frac{m\pi i}{3}}}{\Gamma(\alpha+\frac{2}{3}m+1)}\ B_m^3(\alpha,\lambda x), \quad \lambda_h = \frac{(\frac{1}{3}e^{\frac{\pi i}{3}})^h}{\Gamma(\beta+\frac{1}{3}h+1)}$$

Hence the associate function is:

$$e^{\frac{m\pi i}{3}}\Gamma(\alpha+\tfrac{2}{3}m+1)\, L_m^3\left(\alpha, \frac{y}{\lambda}\right)$$

$$= e^{\frac{m\pi i}{3}}\Gamma(\alpha+\tfrac{2}{3}m+1)\sum_s \frac{(-)^s(-m,3s)\,y^{m-3s}}{(-\alpha-\frac{2m}{3}+1,s)(1,s)} \cdot \frac{\Gamma(\beta+\frac{m-3s}{3}+1)}{(\frac{1}{3}e^{\frac{\pi i}{3}})^{m-3s}}$$

$$= \Gamma(\alpha+\tfrac{2}{3}m+1)\Gamma(\beta+\tfrac{1}{3}m+1)\sum_s \frac{(-)^s(-m,3s)\,(3y)^{m-3s}}{(-\alpha-\frac{2m}{3}+1,s)(-\beta-\frac{m}{3}+1,s)(1,s)}$$

These examples will suffice to illustrate the treatment of expansions in various hypergeometric functions. Further cases can be given at pleasure.

4. Non-hypergeometric Series

The Hermite type expansions considered above lead directly to one elegant application of our theory to functions of a more general nature than the hypergeometric series:

The ϕ polynomials of L. M. Milne-Thompson[5] are defined by the generating function:

$$f_\nu(y)\, e^{xy+g(y)} = \sum_m \frac{y^m}{(1,m)}\ \phi_m^{(\nu)}(x)$$

It can be shown that if $\phi_0^{(\nu)}(0) \neq 0$, $\phi_m^{(\nu)}(x)$ is a polynomial in x of degree n. The properties of the polynomials have been considered in detail. In addition to including as special cases

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[5] L. M. Milne-Thompson, Proc. London Math. Soc., (2) XXXV, (1933).

the generalized Hermite polynomials mentioned above, they also include the well-known Bernoulli and Euler polynomials.

The form of the generating function for $\phi_m^{(\nu)}(x)$ makes the application of our theory immediate. We have

$$e^{xy} = \sum_m \frac{\phi_m^{(\nu)}(x)}{(1,m)} \cdot y^m \frac{e^{-g(y)}}{f_\nu(y)}$$

Hence if we introduce the new function

$$\psi_m^{(\nu)}(y) = \frac{y^m e^{-g(y)}}{f_\nu(y)}$$

then $\phi_m^{(\nu)}(x)$ and $\psi_m^{(\nu)}(y)$ are associate and we can proceed to expand any arbitrary function in a series of either function.

Other general types of functions, such as powers of polynomials and any well-defined sequence, may be treated since we know from our previous theorems methods whereby the associate function may be constructed, although not always in such convenient form.

Chapter III - Applications

Functions of Two Variables

1. Hypergeometric Series

As in the case with single variable functions, the hypergeometric series are most elegantly suited to illustrate the theory.

A double infinite series $\sum_{m,n} a_{m,n}$ is said to be hypergeometric in case the two ratios: $a_{m,n+1}/a_{m,n}$ and $\frac{a_{m+1,n}}{a_{m,n}}$ are both rational functions of m and n. In such a case it can be shown, as in the one variable case, that ratios can be expressed:

$$\frac{a_{m+1,n}}{a_{m,n}} = \prod_{i=1}^{k} \frac{(\alpha_i + a_i m + b_i n,\ a_i)}{(\gamma_i + c_i m + e_i n,\ c_i)} \cdot \frac{x}{(m+1)}$$

$$\frac{a_{m,n+1}}{a_{m,n}} = \prod_{i=1}^{k} \frac{(\alpha_i + a_i m + b_i n,\ b_i)}{(\gamma_i + c_i m + e_i n,\ e_i)} \cdot \frac{y}{(n+1)}$$

where a_i, b_i, c_i, e_i are positive, negative integers, or zero. The general term of the series can then be written so we have finally the complete expression:

$$\sum_{m,n} \prod_{i=1}^{k} \frac{(\alpha_i,\ a_i m + b_i n)}{(\gamma_i,\ c_i m + e_i n)} \cdot \frac{x^m y^n}{m!\, n!} = F\left[\begin{matrix} \alpha_i ; & a_i, b_i ; & ; x, y \\ \gamma_i ; & c_i, e_i ; & \end{matrix}\right]$$

This latter notation we suggest as being completely descriptive, compact, and readily extensible to functions of many variables.

The convergence properties have been considered in detail by Horn.[1] We omit them as unessential here.

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[1] J. Horn, Über die Convergens der Hypergeometrisch Reihen zweier und dreier Veränderlichen, Math. Annalen, XXXIV, (1889), 544.

The analogue of the Barnes asymptotic expansion formula has not been given due to the uncertainty as to the exact meaning of asymptotic expansions of two variables. It is possible, however, to proceed by a method of double complex integration extending the Barnes integrals to obtain certain relations between series of an asymptotic appearance. These relations furthermore give immediately the formulae for reversion of series in case any of the series occurring are merely polynomials. As these are the only applications of the asymptotic formulae we need and since they may be obtained directly, we shall omit further discussion.

2. Expansions in Products

As with the one-variable functions, we refer our various expansions back to a few basic sets. Here, however, we shall be satisfied to illustrate with use of only combinations of the one-variable sets. Others of a more complicated nature not derivable from these doubtless can be exhibited.

The simplest two-variable expansions are those in products of one-variable functions: For example the Neumann Series of Bessel functions can be extended. Thus, if:

$$f(x,y) = \sum_{m,n}' a_{m,n} \frac{x^m}{(1,m)} \frac{y^n}{(1,n)}$$

then

$$f(x,y) = \sum_{m,n}' \alpha_{m,n} \frac{(\frac{1}{2}x)^{-\nu} (\frac{1}{2}y)^{-\mu}}{(1,m)\,(1,n)} J_{\nu+m}(x)\, J_{\mu+n}(y)$$

where:

$$\frac{\alpha_{m,n}}{\Gamma(\nu+m+1)\Gamma(\mu+n+1)} = \sum_{r,s}{}' (-)^{r+s} \frac{(-m,2r)(-n,2s)\,2^{m+n-2r-2s}}{(-\nu-m+1,r)(-\mu-n+1,s)} \frac{a_{m-2r,n-2s}}{(1,m)(1,n)}$$

where we have here made use of the associate functions for the Bessel functions previously obtained and applied Theorem 2.11.

Again, we can have expansions in mixed products. Thus for f(x,y) above, we have:

$$f(x,y) = \sum_{m,n}{}' \beta_{m,n} \frac{H_m(x)\,T_k^n(y)}{(1,m)(1,n)}$$

where:

$$\frac{(-)^m \beta_{m,n}}{(1,k)(1,n)} = \sum_{r,s}{}' \frac{(k+1,n+s)\,a_{m+2r,n+s}}{2^r\,(1,r)(1,s)}$$

giving the expansions in products of Hermite and Sonine polynomials. Again we have made use of the associate functions given before.

Such examples could be multiplied indefinitely, but these will suffice to illustrate the principle involved. Among the special functions which might be treated in this manner the following pseudo-addition formulae are very interesting:

$$(x\xi+y\eta)^{\frac{h-\nu}{2}} J_{\nu+h}\left(\sqrt{2(x\xi+y\eta)}\right)$$

$$= \sum_{m,n}{}' (-)^{m+n} \frac{A_{m,n}}{(1,m)(1,n)}\, x^{\frac{m-\rho}{2}}\, y^{\frac{n-\sigma}{2}}\, J_{m+\rho}(\sqrt{2x})\, J_{n+\sigma}(\sqrt{2y})$$

where:

$$A_{m,n} = \sum_{p,q}{}' \frac{(-m,p)(-n,q)(-p-q,h)\,\Gamma(\rho+p+1)\,\Gamma(\sigma+q+1)}{\Gamma(\nu+p+q+1)\,(1,p)(1,q)}\, \xi^p \eta^q$$

$$T_\gamma^{\nu}(x\xi + y\eta) = \sum_{m,n} (-1)^{m+n+\nu} \frac{A_{m,n}}{(1,\nu)} T_\alpha^{m}(x) T_\beta^{n}(y)$$

where: $$A_{m,n} = \sum_{r,s} \frac{(-r,m)(-s,n)\Gamma(\alpha+r+1)\Gamma(\beta+s+1)(-\nu,r+s)}{\Gamma(\gamma+r+s+1)\,(1,r)(1,s)} \xi^{r} \eta^{s}.$$

$$(x\xi + y\eta)^{\frac{h-\nu}{2}} J_{\nu+h}\left(\sqrt{2(x\xi+y\eta)}\right)$$

$$= \sum_{m,n} (-1)^{m+n} \frac{A_{m,n}}{(1,m)} x^{\frac{m-\mu}{2}} J_{m+\mu}(\sqrt{2x})\, T_\alpha^{n}(y)$$

where: $$A_{m,n} = \sum_{p,q} \frac{(-p-q,h)(-q,n)(-m,p)\Gamma(\mu+p+1)\Gamma(\alpha+q+1)}{\Gamma(\nu+p+q+1)\quad(1,p)(1,q)} \xi^{p} \eta^{q}$$

Again such examples could be extended at will, but these will suffice.

3. <u>Expansions in Two-Variable Hypergeometric Functions</u>

In addition to expansions in products, there can be derived expansions in distinctly two-variable polynomials and functions by application of Theorem 2.12.

P. Humbert[2] introduced the set of confluent hypergeometric functions of two variables, special cases of which can be variously interpreted as extensions of the Sonine and Jacobi polynomials as well as the Bessel functions. Several of these give particularly compact expansions of two-variable functions.

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[2] P. Humbert, Proc. Roy. Soc. Edin., XLI, (1921), 73.

The polynomial:

$$\mathcal{L}_2(-m,-n;\gamma;x,y)$$
$$=\sum_{r,s}\frac{(-m,r)(-n,s)}{(\gamma,r+s)(1,r)(1,s)}x^r y^s$$
$$=\sum_{r}\frac{(-m,r)}{(1,r)}x^r\lambda^r\sum_{s}\frac{(-n,s)}{(1,s)}y^s\mu^s$$
$$=(-)^{m+n}H_m'(\lambda x)\,H_n'(\mu y),\qquad \lambda^r\mu^s=\frac{1}{(\gamma,r+s)}$$

has the associate function:

$$(-)^{m+n}N_m'\left(\frac{\xi}{\lambda}\right)N_n'\left(\frac{\eta}{\mu}\right)$$
$$=\sum_{r,s}(-)^{m+n}\frac{\xi^{m+r}\,\eta^{n+s}}{\lambda^{m+r}\mu^{n+s}(1,r)(1,s)}$$
$$=\sum_{r,s}(-)^{m+n}\frac{(\gamma,r+s+m+n)}{(1,r)(1,s)}\xi^{m+r}\eta^{n+s}$$

So that

$$f(x,y)=\sum_{m,n}a_{m,n}\frac{x^m y^n}{(1,m)(1,n)}$$
$$=\sum_{m,n}(-)^{m+n}\frac{\mathcal{L}_2(-m,-n;\gamma;x,y)}{(1,m)(1,n)}\sum_{r,s}\frac{(\gamma,r+s+m+n)}{(1,r)(1,s)}a_{m+r,n+s}$$

The polynomial:

$$\mathcal{L}_1(-n,-m;\gamma;xy,y)$$
$$=\sum_{r,s}\frac{(-n,r+s)(-m,r)}{(\gamma,r+s)}\frac{x^r y^{r+s}}{(1,r)(1,s)}$$

$$= \sum_{r,s} \frac{(-n,s)\,(-m,r)}{(\gamma,s)(1,r-s)(1,r)}\, x^r y^s$$

$$= (-)^{m+n} H_m'(\lambda x)\, H_n'(\mu y), \qquad \lambda^r \mu^s = \frac{(1,s)}{(1,r-s)(\gamma,s)}$$

Hence the associate function is:

$$(-)^{m+n} N_m'\left(\frac{\xi}{\lambda}\right) N_n'\left(\frac{\eta}{\mu}\right)$$

$$= (-)^{m+n} \sum_{r,s} \frac{(1,r-s+m-n)(\gamma,s+n)}{(1,s+n)(1,r)\,(1,s)}\, \xi^{r+m} \eta^{s+n}$$

The polynomial:

$$\Xi_1(-m,-n,\ \beta+m;\ \gamma;\ x,y)$$

$$= \sum_{r,s} \frac{(-m,r)(-n,s)\,(\beta+m,r)}{(\gamma,r+s)\,(1,r)\,(1,s)}\, x^r y^s$$

$$= \sum_{r,s} (-)^{m+n} \frac{(-m,r)(-n,s)\,(\beta+m,m-r)}{(\gamma,m+n-r-s)\,(1,r)\,(1,s)}\, x^{m-r} y^{n-s}$$

$$= (-)^{m+n} (\beta+m,m) \sum_{r,s} (-)^r \frac{(-m,r)(-n,s)\, x^{m-r}\, y^{n-s}}{(-\beta-2m+1,r)(\gamma,m+n-r-s)(1,r)(1,s)}$$

$$= (-)^{m+n} (\beta+m,m)\, L_m'(\beta,\lambda x)\, H_n'(\mu y),$$

$$\lambda^r \mu^s = \frac{1}{(\gamma,r+s)}.$$

Hence the associate function is:

$$\frac{(-)^{m+n}}{(\beta+m,m)}\, B_m'\left(\beta,\frac{\xi}{\lambda}\right) N_m'\left(\frac{\eta}{\mu}\right)$$

$$= \frac{(-)^{m+n}}{(\beta+m,m)} \sum_{r,s} \frac{\lambda^{-(m+r)} \mu^{-(n+s)}\, \xi^{m+r} \eta^{n+s}}{(\beta+2m+1,r)\,(1,r)\,(1,s)}$$

$$= \frac{(-)^{m+n}}{(\beta+m,m)} \sum_{r,s} \frac{(\gamma, m+n+r+s)\ \xi^{m+r}\ \eta^{n+s}}{(\beta+2m+1, r)\ (1,r)\ (1,s)}$$

The semi-polynomial:

$$y^n\ \Psi_1(\alpha+m, -m\ ;\ \gamma, \gamma'+2n+1\ ;\ x, y)$$

$$= \sum_{r,s} \frac{(\alpha+n, r+s)(-m,r)}{(\gamma,r)(\gamma'+2n+1,s)} \frac{x^r y^{n+s}}{(1,r)(1,s)}$$

$$= \frac{1}{(\alpha,n)} \sum_{r,s} \frac{(\alpha, r+n+s)(-m,r)\ x^r\ y^{n+s}}{(\gamma,r)(\gamma'+2n+1,s)(1,r)(1,s)}$$

$$= \frac{1}{(\alpha,n)}\ H_m'(\lambda x)\ B_n'(\gamma', \mu y), \quad \lambda^r \mu^s = \frac{(\alpha, r+s)}{(\gamma,r)}$$

Hence the associate function is:

$$(\alpha,n)\ N_m'\left(\frac{\xi}{\lambda}\right)\ L_n'\left(\gamma', \frac{\eta}{\mu}\right)$$

$$= (\alpha,n) \sum_{r,s} (-)^s \frac{(-n,s)\quad \xi^{m+r}\ \eta^{n-s}}{(-\gamma'-2n+1,s)(1,r)(1,s)\ \lambda^{m+r} \mu^{n-s}}$$

$$= (\alpha,n) \sum_{r,s} (-)^s \frac{(-n,s)(\gamma, m+r)\ \xi^{m+r}\ \eta^{n-s}}{(-\gamma'-2n+1,s)(\alpha, m+n+r-s)(1,r)(1,s)}$$

The semi-polynomial:

$$x^m y^n\ F\left[\begin{matrix} -\frac{m}{2}, \frac{-m+1}{2}, \alpha+\frac{m+n}{2} \\ \gamma+n+1 \end{matrix} ; \begin{matrix} 1,0;\ 1,0;\ -1,1 \\ 0,1 \end{matrix} ; -\frac{2}{x^2}, y^2\right]$$

$$= \sum_{r,s} (-)^r \frac{2^r\left(-\frac{m}{2}, r\right)\left(\frac{-m+1}{2}, r\right)\left(\alpha+\frac{m+n}{2}, -r+s\right)}{(\gamma+n+1, s)\ (1,r)\ (1,s)}\ x^{m-2r}\ y^{n+2s}$$

$$= \frac{1}{(\alpha, \frac{m+n}{2})} \sum_{r,s} (-)^{r} \frac{(-m, 2r)(\alpha, \frac{m+n}{2} - r + s)}{2^{r}(\gamma+n+1, s)(1,r)(1,s)} x^{m-2r} y^{n+2s}$$

$$= \frac{1}{(\alpha, \frac{m+n}{2})} H_m^2(\lambda x) B_n^2(\gamma, \mu y), \qquad \lambda^r \mu^s = (\alpha, \tfrac{r+s}{2})$$

Hence the associate function is:

$$(\alpha, \tfrac{m+n}{2}) N_m^2 \left(\frac{\xi}{\lambda}\right) L_n^2 \left(\gamma; \frac{\eta}{\mu}\right)$$

$$= (\alpha, \tfrac{m+n}{2}) \sum_{r,s} (-)^{s} \frac{(-n, 2s)\, \xi^{m+2r} \eta^{n-2s}}{2^{r}(-\gamma-n+1, s)(1,r)(1,s)\lambda^{m+2r}\mu^{n-2s}}$$

$$= \sum_{r,s} (-)^{s} \frac{(-n, 2s)\, \xi^{m+2r} \eta^{n-2s}}{2^{r}(-\gamma-n+1, s)(\alpha+\frac{m+n}{2}, r-s)(1,r)(1,s)}$$

The Appel hypergeometric functions of two variables will give a similar extended list, as well as more complicated cases of the hypergeometric functions. As remarked before, further basic pairs of associate functions doubtlessly exist which could be used to give additional applications of the theory.

4. Non-hypergeometric Types

As in the case of one-variable functions, the theory is applicable to expansions in functions of a more general nature still than those considered above.

Let us consider the polynomials $\phi_{m,n}(x,y)$ defined by the generating function:

$$f(h,k)\, e^{hx+ky+g(h,k)} = \sum_{m,n} \frac{h^m k^n}{(1,m)(1,n)} \phi_{m,n}(x,y)$$

As in the one-variable case, the $\phi_{m,n}(x,y)$ can be shown in general to be polynomials in x and y of degree m,n.

The associate function is directly deducible since:

$$e^{x\xi+y\eta} = \sum_{m,n} \frac{\varphi_{m,n}(x,y)}{(1,m)(1,n)} \; \frac{\xi^m \eta^n e^{-g(\xi,\eta)}}{f(\xi,\eta)}$$

whence:

$$\psi_{m,n}(\xi,\eta) = \frac{\xi^m \eta^n e^{-g(\xi,\eta)}}{f(\xi,\eta)}$$

is the associate of $\varphi_{m,n}(x,y)$.

As an example of this, we mention the two-variable Hermite polynomials $H_{m,n}(x,y)$ defined by the expansion:

$$e^{h(ax+by)+k(bx+cy)-\frac{1}{2}(ah^2+2bhk+ck^2)}$$

$$= \sum_{m,n} \frac{h^m k^n}{(1,m)(1,n)} H_{m,n}(x,y)$$

If we write $\xi = ha+kb$, $\eta = hb+kc$ then

$$e^{x\xi+y\eta} = \sum_{m,n} \frac{H_{m,n}(x,y)}{(1,m)(1,n)} h^m k^n e^{\frac{1}{2}(ah^2+2bhk+ck^2)}$$

so that the function associate to $H_{m,n}(x,y)$ is

$$N(\xi,\eta) = \frac{(c\xi-b\eta)^m(-b\xi+a\eta)^n}{\Delta^{m+n}} e^{\frac{c\xi^2+2b\xi\eta+a\eta^2}{2\Delta}}$$

where $\Delta = ac-b^2 \neq 0$.

The several two-variable extensions of the Bernoulli and Euler polynomials can be similarly treated, using the generating function.

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[3] Ch. Hermite, Oeuvres, II, 293-308.

One could proceed to build up any number of examples in expansions of functions of three or more variables. The results so obtained would be in the large obvious generalizations of those given above. The many variable problems will introduce no complications in theory and only the manipulation of the algebra will be cause for any difficulties.

Part II

The Number of Representations Function for Binary Quadratic Forms

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Introduction

The problem of finding the number of representations of an arbitrary integer by a given binary quadratic form has yet to be solved in complete generality. In the two centuries that have followed the first general investigation by J. L. Lagrange[1)] of any part of the problem, the investigations have proceeded in two directions. A great number of specific forms have been considered individually for which more or less general solutions have been given. Again, certain general investigations have reduced the problem to more simple and direct questions. The early investigations of Dirichlet[2)] and more recently those of Pall[3)] are of this nature.

In the discussion to follow we offer as a contribution to the general problem, the general explicit expressions for the number of representations function for all forms whose discriminant is such that there is a single class in each genus. For the original suggestion of this problem and for continued advice and encouragement, the author wishes to express his deep appreciation to Professor E. T. Bell.

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1) J. L. Lagrange, Recherches d'Arithmetique, Oeuvres t III p 693 - 785.
2) G. L. Dirichlet, Zahlentheorie, ed. 4 (1894), p 229
3) G. Pall, Mathematische Zeitschrift, v 36, p 321-343 (1932)

In our terminology and notation we shall follow closely those of Dickson used in his ~~de~~Introduction to the Theory of Numbers. We are concerned with binary quadratic forms $ax^2+bxy+cy^2$, designated by $[a, b, c]$, of discriminant $-\Delta = b^2-4ac$, and shall examine the form of the number of representations function

$$N\,[m = ax^2+bxy+cy^2]$$

this being the number of solutions in integers, x and y, of

$$m = ax^2+bxy+cy^2$$

If a, b, and c have the common factor λ,

$$N[m = ax^2+bxy+cy^2] = 0 \quad or \quad N\left[m/\lambda = \frac{a}{\lambda}x^2 + \frac{b}{\lambda}xy + \frac{c}{\lambda}y^2\right]$$

according as m is or is not divisible by λ. If no such factor exists, the form is primitive. If $\Delta < 0$, $N[m = ax^2+bxy+cy^2]$ is infinite, and the form is called indefinite. When $\Delta > 0$, $ax^2+bxy+cy^2$ is either always greater than zero, or always less than zero, i.e., either positive definite or negative definite, there is no loss in generality if we restrict ourselves to positive definite, primitive forms. All forms hereinafter mentioned will be assumed as such.

Let

$$x = \alpha\xi + \beta\eta \quad , \quad y = \gamma\xi + \delta\eta$$

be a transformation whose determinant has unit absolute value, which replaces the form

$$ax^2+bxy+cy^2 \quad by \quad A\xi^2 + B\xi\eta + C\eta^2.$$

Two such forms are called equivalent and all equivalent forms are said to form a class. Clearly the discriminant is the same in both cases. Because of the restriction on the transformation, we have further

$$N[m = ax^2 + bxy + cy^2] = N[m = A\xi^2 + B\xi\eta + C\eta^2]$$

In view of this it is convenient to choose one form representation of the class, the reduced form. A form $[a, b, c]$ is reduced if $-a < b \leq a$, $c \geq a$ with $b \geq 0$ if $c = a$. It is unique.

We shall base our investigation on the following theorem of Dirichlet:

Theorem 1.1

Let m be positive and prime to Δ. The number of representations of m by all the reduced forms of discriminant $-\Delta$ is $\omega \sum_{q/m} (-\Delta/q)$, where $\omega = 2, \Delta > 4$, $\omega = 4$ $\Delta = 4$, $\omega = 6$ $\Delta = 3$ and $(-\Delta/q)$ is Kronecker's symbol.

If there existed invariants whereby we could determine whether a certain integer could be represented by a certain reduced form, we could proceed with the use of them and Theorem 1.1 to obtain the complete solution of our problem. The only known quantities associated with a particular form, invariant in that they are equal for all integers represented by said form, separate the forms given discriminant into genera which may or may not coincide with the several classes. These invariants, the so-called characters, are defined by the following:

Theorem 1.2

If $p_1, p_2, \cdots$ are the distinct odd prime factors of Δ, then (n/p_i) has the same value for all integers n prime to 2Δ, represented by a form $[a, b, c]$ of discriminant $-\Delta$. When Δ is even, $\Delta = -4D$, the same is true of

$$\delta = (-1)^{\frac{1}{2}(n-1)}, \quad \text{if } D \equiv 0 \text{ or } 3 \pmod 4$$

$$\epsilon = (-1)^{\frac{1}{8}(n^2-1)}, \quad \text{if } D \equiv 0 \text{ or } 2 \pmod 8$$

$$\delta\epsilon, \quad \text{if } D \equiv 0 \text{ or } 6 \pmod 8$$

We shall use the notation C_i for the set of characters for a certain form, excluding $\delta\epsilon$ if both δ and ϵ are characters. The notations ${}_mC_i$ and $C_i(m)$ are to represent the value of the characters C_i for m or for the form representing m.

All forms of a given discriminant whose characters have the same value are said to form a genus. Since equivalent forms represent the same numbers, all forms in the same class are in the same genus.

When there is a single class in each genus we may proceed using these characters to give the explicit form for the number of representations function for integers m prime to 2Δ. If $[a, b, c]$ represents the integer s, Theorem 1.2 states $C_i(m) = C_i(s)$ as a necessary condition that m be represented at all by $[a, b, c]$. Since we assume a single class in each genus, each reduced form has different values for the characters. Hence by Theorem 1.1 we obtain

Theorem 1.3.

Let $[a, b, c]$ be a form of discriminant Δ, $-\Delta < -4$ such that there is a single class of forms in each genus, with character C_i. If m is an integer prime to 2Δ

$$N[m = ax^2 + bxy + cy^2]$$
$$= \frac{1}{2^{k-1}} \prod_{i=1}^{k} \left\{ 1 + C_i(a) C_i(m) \right\} \sum_{y/m} (-\Delta/y)$$

Before we proceed to derive reduction formulae for the number of representations of integers not prime to double the discriminant so as to apply Theorem 1.3 to the complete solution of our problem, we shall make some general remarks on the set of discriminants of forms of which there is a single class in each genus. Since we are restricting ourselves to forms of such a discriminant, these results may prove useful.

Chapter I.

Discriminants with a Single Class to a Genus

It is known that the number of genera is equal to one half the number of characters, C_i , for forms of a particular discriminant. Further, it is well known that the number of classes is the same in each genus. Hence the question of the number of classes in the genus is closely related to the classical class-number problem. Following the analytic methods used in the study of this latter, S. Chowla has shown recently that the number of discriminats such that each genus has a specified number of classes is finite. Although this gives a certain qualitative satisfaction, yet it gives absolutely no light on the question as to just what particular discriminants have the specified number of classes in the genus.

Rather extensive specific lists of this nature have been given and all evidence points toward the conclusion that many are complete. In particular, it is an open question whether the known lists of discriminants with a single class in each genus are complete, although if there are others, they certainly must be very large.

In this connection we shall extend the known tests for there being a single class in each genus, and also give some conclusions regarding the elimination of discriminants satisfying certain congruences and also those whose factors are of a peculiar nature.

As a basis for our investigations we use the

Theorem 2.1

A necessary and sufficient condition that there be more than a single class of forms of a given discriminant in each genus is that there exist a reduced form of said discriminant, $[a, b, c]$, with $c > a > b > 0$

The sufficiency follows directly, since if $[a, b, c]$, $c > a > b > 0$ is a reduced form, $[a, -b, c]$ is also reduced, represents the same integers and is hence in the same genus. The necessity requires a detailed examination of several cases and as we do not require it now, we postpone its proof.

As it stands, this is scarcely convenient as a method to test certain discriminants, inasmuch as it requires a more or less exhaustive consideration of all reduced forms. It is certain in this respect, however, that having definitely listed all reduced forms, one can tell by immediate inspection whether there is more than a single class of forms in the genus. On this account it is essentially the final test that is always required. However, in order to make preliminary steps simpler, we derive from Theorem 2.1 several direct tests.

Using the sufficiency in Theorem 2.1, Dickson obtains tests for odd discriminants which we state for reference. Write $T_k = [\Delta + (2k+1)^2]/4$, then for positive integers, k, T_k must be a prime p, its square p^2, or, also, $T_1 = 3p, 3^3,$ or 3^4; $T_2 = 5p, 5^2,$ or 5^3, $T_3 = 7p$ or 7^2; $T_4 = 3p$ or 99, etc. Here as elsewhere we assume all primes to be positive.

The corresponding tests for even discriminants are similarly stated. Write $S_k = \frac{1}{4}\Delta + k^2$. We list below for successive integers, k, the S_k which are eliminated by the corresponding reduced forms satisfying the sufficient conditions of Theorem 2.1 or its equivalent interchanging a and c, and summarize the resulting necessary values for S_k. For S_3 we give the details of deducing these necessary values of S_k that there be a single class in each genus. The argument in the other cases is quite similar. We let (r,s,m) denote the greatest common divisor of r, s, and m

$S_1 = r\cdot s,\ r > s > 2,\ (r, s, 2) = 1,\quad [r, 2, s]$

$S_1 = 2^k,\ k > 5 \quad [8, 6, 2^{k-3}+1]$

$S_1 = 2^5, \quad [5, 4, 7]$

Hence S_1 must be a prime p, p^2, 2p, 2^3, or 2^4.

$S_2 = r\cdot s,\ r > s > 4,\ (r, s, 4) = 1,\quad [r, 4, s]$

$S_2 = 3\cdot m,\ m > 4, \quad [3, 2, m-1]$

$S_2 = 2^k,\ k > 8 \quad [32, 12, 2^{k-5}+1]$

$S_2 = 2^8 \quad [8, 4, 31]$

$S_2 = 2^7 \quad [3, 2, 41]$

Hence S_2 must be a prime p, p^2, 2p, 4p, 2^4, 2^5, or 2^6.

(1) $S_3 = r\cdot s,\ r > s > 6,\ (r, s, 6) = 1,\quad [r, 6, s]$

(2) $S_3 = 5m,\ m > 6, \quad [5, 4, m-1]$

(3) $S_3 = 4m,\ m \text{ odd} > 3, \quad [4, 2, m-2]$

(4) $S_3 = 3\cdot 2^k,\ k > 3, \quad [8, 2, 3\cdot 2^{k-3}-1]$

(5) $S_3 = 2^k,\ k > 4, \quad [8, 2, 2^{k-3}-1]$

(6) $S_3 = 2\cdot 3^k,\ k > 4, \quad [27, 12, 2\cdot 3^{k-3}+1]$

(7) $S_3 = 3^k, \ k > 5,$ $[27, 12, 3^{k-3}+1]$

(8) $S_3 = 2 \cdot 3^4,$ $[7, 2, 22]$

(9) $S_3 = 3^5,$ $[5, 2, 47]$

We consider exhaustively the possible forms which S_3 may take and refer to the numbered cases above to eliminate certain forms. Let $S_3 = p.n$ where p is the smallest prime factor of S_3, then $n \geq p$ or $n = 1$. If $p > 5$ or $p = 5$, S_3 is eliminated by (1) or (2) respectively unless $n = p$ or $n = 1$. If $p = 3$, let $n = q.m$ where q is the smallest prime factor of $n > 3$ if such exists. Then according as $q > 5$ or $q = 5$, we eliminate by (1) or (2) unless $m = 1$. If no such q exists then $S_3 = 2^k.3^j$, and when $k > 2$, $j > 1$ we use (1). For $k = 2$ consider (3); $k = 1$, (6) and (8); $k = 0$, (5); $j = 1$, (4); $j = 0$, (7) and (9). If $p = 2$ let $n = Q.m$ where Q is the smallest odd prime factor of n if such exists. Then according as $Q > 5$ or $Q = 5$ we eliminate by (1) or (2) unless $m = 1$, 2, or 3. If $m = 2$, we use (3). If $m = 3$ proceed as when $p = 3$. If no such Q exists, $S_3 = 2^k$ and we apply (5).

Hence S_3 must be a prime p, p^2, 2p, 3p, 6p, 2^4, 3^3, 3^4, $2^3.3$, or 2.3^3.

$S_4 = r \cdot s, \ r > s > 8, \ (r, s, 8) = 1,$ $[r, 8, s]$

$S_4 = 7 \cdot m, \ m > 8,$ $[7, 6, m-1]$

$S_4 = 5 \cdot m, \ m > 5, \ m \neq 8$ $[5, 2, m-3]$

$S_4 = 3m, \ m > 8$ $[3, 2, m-5]$

$S_4 = 2^k, \ k > 11$ $[128, 24, 2^{k-7}+1]$

$S_4 = 2^{11},$ $[47, 12, 44]$

$S_4 = 2^{10},$ $[11, 4, 92]$

$S_4 = 2^9,$ $[5, 4, 11]$

Hence S_4 must be a prime p, p^2, 2p, 4p, 8p, 2.3^2, 3.7, 2^5, 2^6, 2^7, or 2^8.

$S_5 = r \cdot s, \quad r > s > 10, \quad (r, s, 10) = 1, \qquad [r, 10, s]$

$S_5 = 7 \cdot n, \quad n > 7, \quad n \neq 10, \qquad [7, 4, n-3]$

$S_5 = 3n, \quad n > 11 \qquad [3, 2, n-8]$

$S_5 = 8n, \quad n \text{ odd } > 7, \qquad [8, 6, n-2]$

$S_5 = 4n, \quad n \text{ odd } > 7, \qquad [4, 2, n-6]$

$S_5 = 5 \cdot 2^k, \quad k > 3, \qquad [8, 2, 5 \cdot 2^{k-3} - 3]$

$S_5 = 2^k, \quad k > 5, \qquad [8, 2, 2^{k-3} - 3]$

$S_5 = 2 \cdot 5^k, \quad k > 3, \qquad [25, 20, 2 \cdot 5^{k-2} + 3]$

$S_5 = 5^k, \quad k > 3, \qquad [25, 20, 5^{k-2} + 3]$

$S_5 = 5^3, \qquad [8, 4, 13]$

Hence S_5 must be a prime p, p^2, 2p, 5p, 10p, 2^4, 2^5, 5^2, $2^3.5$, 2.5^3, 2.3^2, 3.7, $2^3.3$, 3^3, 3.11, or $2^2.7$.

To prove the necessity in Theorem 2.1 we must exhibit all the reduced forms occurring when there is a single class in each genus and show that when other reduced forms arise, giving more than a single class to a genus, they must be of the form stated. In addition to accomplishing this purpose, we will have available for later reference the explicit reduced forms for the disciminants we shall be considering.

The original definition of reduced forms allows, in addition to forms

$[a, b, c], \quad c > a > b > 0 \quad \text{or} \quad c > a > -b > 0$

only the three possibilities

$[a, 0, c], \quad c > a > 0; \quad [a, a, c], \quad c > a > 0; \quad [a, b, a], \quad a > b > 0.$

For these we have respectively

$\Delta = 4ac, \quad \Delta = a(4c - a), \quad \Delta = (2a - b)(2a + b)$

Restricting ourselves to these three types, we shall show that except in some cases where it is already known, there are more than a single class in each genus, and that all these possibilities give exactly the same number of reduced forms as there are genera. Hence when there is only a single class in each genus they are all the reduced forms and further when there are two or more reduced forms to a genus, among them must be at least one satisfying the conditions of Theorem 2.1.

Let the discriminant be odd, then $\Delta \equiv -1 \pmod 4$. Write $\Delta = \lambda\mu$ with $\lambda \leq \mu$; from $\Delta \equiv -1 \pmod 4$ we have $\lambda + \mu \equiv 0 \pmod 4$. There can be no reduced forms $[a, 0, c]$ since Δ is odd. For $[a, a, c]$ we find the single solution of $\Delta = a(4c - a) = \lambda\mu$ satisfying $c > a > 0$, and $\lambda \leq \mu$ to give the form $[\lambda, \lambda, \frac{\lambda+\mu}{4}]$ with the restriction $\mu \geq 3\lambda$. Similarly, considering $[a, b, a]$, we solve $\Delta = (2a - b)(2a + b)$ under the conditions $a > b > 0$ to obtain the form $[\frac{\lambda+\mu}{4}, \frac{\mu-\lambda}{2}, \frac{\lambda+\mu}{4}]$ with $\mu \leq 3\lambda$. Now, if Δ is a prime or the power of a prime, the only primitive form among these is obtained when $\lambda = 1$, $\mu = \Delta$, i.e., the principal form

$$[1, \ 1, \ \frac{\Delta+1}{4}]$$

If Δ contains, say k, distinct prime factors, we obtain primitive reduced forms in addition to the principal form by separating Δ into all possible co-prime pairs λ and μ and using

$$[\lambda, \lambda, \frac{\lambda+\mu}{4}] \qquad \text{or} \qquad [\frac{\lambda+\mu}{4}, \frac{\mu-\lambda}{2}, \frac{\lambda+\mu}{4}]$$

according as $\mu \geq 3\lambda$ or $\mu \leq 3\lambda$.

This gives in all 2^{k-1} reduced forms, being the same as the number of genera.

The following, relating to odd discriminants stated by Dickson, is important for later consideration:

Theorem 2.2

When $\Delta \equiv -1 \pmod{8}$ there is more than a single class in each genus unless $\Delta = 7$ or 15.

Let $\Delta = 8k-1$, then $T_0 = 2k$ and Dickson's tests exclude all but k = 1 or 2, i.e., $\Delta = 7$ or 15

The known odd discriminants giving a single class in each genus are listed with their prime factors:

3	$99 = 3^2.11$	$435 = 3.5.29$
7	$115 = 5.23$	$483 = 3.7.23$
11	$123 = 3.41$	$555 = 3.5.37$
$15 = 5.3$	$147 = 3.7^2$	$595 = 5.7.17$
19	163	$627 = 3.11.19$
$27 = 3^3$	$187 = 11.17$	$715 = 5.11.13$
$35 = 5.7$	$195 = 3.5.13$	$795 = 3.5.53$
43	$235 = 5.47$	$1155 = 3.5.7.11$
$51 = 3.17$	$267 = 3.89$	$1435 = 5.7.41$
67	$315 = 3^2.5.7$	$1995 = 3.5.7.19$
$75 = 3.5^2$	$403 = 13.31$	$3003 = 3.7.11.13$
$91 = 7.13$	$427 = 7.61$	$3315 = 3.5.13.17$

Let the discriminants be even and $\Delta \equiv 4 \pmod{16}$, then $D \equiv 3 \pmod 4$, so δ is the single even character. Write $\Delta = 4\lambda\mu$ with $\lambda \leq \mu$. Reduced forms $[a, 0, c]$ will be $[\lambda, 0, \mu]$. For primitive $[a, a, c]$ we find the single ~~set~~ solution of $\Delta = a(4c-a) = 4\lambda\mu$ satisfying $c > a > 0$ and $\lambda \leq \mu$ to give the form $[2\lambda, 2\lambda, \frac{\lambda+\mu}{2}]$ with the restriction $\mu \geq 3\lambda$. Similarly considering $[a, b, a]$, we solve $\Delta = (2a-b)(2a+b) = 4\lambda\mu$ under the conditions $a > b > 0$ to obtain the form $\left[\frac{\lambda+\mu}{2}, \mu-\lambda, \frac{\lambda+\mu}{2}\right]$ with $\mu \leq 3\lambda$. If Δ is a prime or the power of a prime, the only primitive forms are obtained when $\lambda = 1$ and $\mu = \frac{\Delta}{4}$, give the principle form

$$\left[1, 0, \frac{\Delta}{4}\right], \quad \text{and} \quad \left[2, 2, \frac{\Delta/4 + 1}{2}\right]$$

When Δ contains k distinct prime factors, we obtain primitive reduced forms in addition to these two forms by separating $\Delta/4$ into all possible co-prime pairs λ and μ and using

$$[\lambda, 0, \mu]$$

and

$$\left[2\lambda, 2\lambda, \frac{\lambda+\mu}{2}\right] \quad \text{or} \quad \left[\frac{\lambda+\mu}{2}, \mu-\lambda, \frac{\lambda+\mu}{2}\right]$$

according as $\mu \geq 3\lambda$ or $\mu \leq 3\lambda$.

This gives in all 2^k reduced forms, being the same as the number of genera.

When $\Delta \equiv 8 \pmod{16}$, $D \equiv 2 \pmod 4$ so either ϵ or $\delta\epsilon$ is the single even character. We write $\Delta = 8\lambda\mu$ and observe that no reduced forms $[a, a, c]$ or $[a, b, a]$ exist by examining $8\lambda\mu = a(4c-a)$ and $8\lambda\mu = (2a-b)(2a+b)$ respectively. This leaves reduced forms $[\lambda, 0, 2\mu]$ and $[2\lambda, 0, \mu]$ or $[\mu, 0, 2\lambda]$ according as $2\lambda < \mu$ or $2\lambda > \mu$

obtained from $[a, 0, c]$ with $c > a > 0$. When $\Delta/8$ is prime or the power of a prime the two reduced forms are

$$[1, 0, \Delta/4] \quad \text{and} \quad [2, 0, \Delta/8]$$

When $\Delta/8$ contains k distinct prime factors, we obtain primitive reduced forms in addition to these two forms by separating $\Delta/8$ into all possible co-prime pairs λ and μ and using

$$[\lambda, 0, 2\mu]$$

and

$$[2\lambda, 0, \mu] \quad \text{or} \quad [\mu, 0, 2\lambda]$$

according as $2\lambda < \mu$ or $2\lambda > \mu$.

This gives in all 2^k reduced forms, being the same as the number of genera.

When $\Delta \equiv 12 \pmod{16}$, $D = 4k+3$ so that $\delta_1 = 4(k+1)$, there are then more than a single class of forms in each genus for all such discriminants except $\Delta = 12, 28, 60$. The reduced forms for these cases are given in the tables.

When $\Delta \equiv 0 \pmod{16}$, $D = 2^h(2k+1)$, $h \geq 2$. For $h \geq 4$, $\delta_4 = 16[2^{h-4}(2k+1)+1]$ so there are no discriminants with a single class in each genus except when $h = 4$ and $k = 0, 1, 3, 7$. For $h = 2$, $\delta_2 = 8(k+1)$, hence excluding all cases except $k = 0, 1, 3, 7$. The reduced forms for these cases are given in the tables.

The case $h = 3$, that is $\Delta \equiv 32 \pmod{64}$, requires separate treatment. Here $D \equiv 0 \pmod 8$ so that δ and ϵ are the characters. Write $\Delta = 32\lambda\mu$ with $\lambda \leq \mu$. Reduced forms $[a, 0, c]$ will be $[\lambda, 0, 8\mu]$ and $[\mu, 0, 8\lambda]$ or $[8\lambda, 0, \mu]$

according as $\mu < 8\lambda$ or $\mu > 8\lambda$.

To find the primitive reduced form $[a, a, c]$, we solve $32\lambda\mu = a(4c-a)$

to obtain the roots $a = 4\lambda,\ c = 2\mu + \lambda;\ a = 8\lambda,\ c = \mu + 2\lambda$ satisfying $c > a > 0$ and $\lambda \leq \mu$, whence the forms $[4\lambda, 4\lambda, 2\mu + \lambda]$ and $[8\lambda, 8\lambda, \mu + 2\lambda]$ with the restrictions $2\mu > 3\lambda$ and $\mu > 6\lambda$ respectively. Similarly, for $[a, b, a]$ the equation $32\lambda\mu = (2a - b)(2a + b)$ under the condition $a > b > 0$ has the roots $a = \lambda + 2\mu,\ b = 4\mu - 2\lambda;$ $a = 2\lambda + \mu,\ b = 2\mu - 4\lambda;\ a = 2\lambda + \mu,\ b = 4\lambda - 2\mu$ whence the reduced forms $[\lambda + 2\mu, 4\mu - 2\lambda, \lambda + 2\mu]$, $[2\lambda + \mu, 2\mu - 4\lambda, 2\lambda + \mu]$, $[2\lambda + \mu, 4\lambda - 2\mu, 2\lambda + \mu]$ under the respective conditions $2\mu < 3\lambda$, $2\lambda \leq \mu < 6\lambda$, $\mu < 2\lambda$.
If Δ is a prime or the power of a prime, the only primitive forms are obtained when $\lambda = 1, \mu = \Delta/32$ There are two or four according as $\mu = 1$ or $\mu > 1$:

$$[1, 0, 8\mu],\ \mu \geq 1$$

$$[4, 4, 2\mu + 1],\ \mu > 1$$

$$[8, 0, \mu],\ \mu > 8 \quad \text{or} \quad [\mu, 0, 8],\ 1 < \mu < 8$$

$$[8, 8, \mu + 2],\ \mu > 6 \quad \text{or} \quad [\mu + 2, 2\mu - 4, \mu + 2],\ 1 \leq \mu < 6$$

If $\Delta/32$ has k distinct prime factors, we obtain primitive reduced forms in addition to these four by separating $\Delta/32$ into all possible co-prime pairs λ and μ and using the four forms

$$[\lambda, 0, 8\mu]$$

$$[\mu, 0, 8\lambda],\ \mu < 8\lambda, \quad \text{or} \quad [8\lambda, 0, \mu],\ \mu > 8\lambda$$

$$[4\lambda, 4\lambda, 2\mu + \lambda],\ 2\mu > 3\lambda, \text{ or } [\lambda + 2\mu, 4\mu - 2\lambda, \lambda + 2\mu],\ 2\mu < 3\lambda$$

$$[8\lambda, 8\lambda, \mu + 2\lambda],\ \mu > 6\lambda, \text{ or } [2\lambda + \mu, 2\mu - 4\lambda, 2\lambda + \mu],\ 2\lambda \leq \mu < 6\lambda,$$

$$\text{or } [2\lambda + \mu, 4\lambda - 2\mu, 2\lambda + \mu],\ \mu < 2\lambda$$

This gives in all 2^{k+1} reduced forms, being the same as the number of genera.

Thus we have exhaustively considered all types of discriminants and have shown by actual exhibition sufficient reduced forms to account for each genus in every case where it is not shown definitely by the sufficiencies of Theorem 2.1 that there are more than a single class of forms to the genus, which reduced forms include all those not covered by the conditions of Theorem 2.1, thereby proving the necessity of Theorem 2.1.

For reference we list the known even discriminants with a single class in each genus according to the above division by congruences. When $\Delta \equiv 4 \pmod{16}$, Δ has the values:

4 = 4.1	148 = 4.37	660 = 4.3.5.11
20 = 4.5	180 = 4.3.3.5	708 = 4.3.59
36 = 4.3.3	228 = 4.3.19	1012 = 4.11.23
52 = 4.13	340 = 4.5.17	1092 = 4.3.7.13
84 = 4.3.7	372 = 4.3.31	1380 = 4.3.5.23
100 = 4.5.5	420 = 4.3.5.7	1428 = 4.3.7.17
132 = 4.3.11	532 = 4.7.19	1540 = 4.5.7.11
		5460 = 4.3.5.7.13

When $\Delta \equiv 8 \pmod{16}$ and $\Delta \equiv 32 \pmod{64}$, we may set $\Delta = 8m$ and $32m$ respectively, where m takes the values:

1	11	35 = 5.7	95 = 5.19
3	15 = 3.5	39 = 3.13	105 = 3.5.7
5	21 = 3.7	51 = 3.17	115 = 5.23
9 = 3.3	29	65 = 5.13	231 = 3.7.11

When $\Delta \equiv 0 \pmod{16}$, there are only the eight values:

$$\Delta = 16,\ 48,\ 112,\ 240;$$

$$\Delta = 64,\ 192,\ 448,\ 960$$

Finally, when $\Delta \equiv 12 \pmod{16}$, there remain the cases:

$$\Delta = 12,\ 28,\ 60$$

With these tables and the reduced forms given above in general form, we can write down explicitly the reduced forms for any known discriminant with a single class in each genus.

In addition to an explicit knowledge of the single reduced form in each genus and the factors of discriminant in our later discussion, it is necessary to treat separately the cases where the discriminant contains the square of any odd prime as a factor. In this connection we state:

Theorem 2.3

If the discriminant contains as a factor the square of any odd prime, there is more than a single class of forms in each genus except for the ten known cases

$$\Delta = 36,\ 72,\ 100,\ 180,\ 288;$$

$$\Delta = 27,\ 75,\ 99,\ 147,\ 315.$$

In proof, let Δ be even and contain as a factor the square of the positive odd prime q. Write $\Delta = 4q^2(r.s - 1)$. Then either $r.s = 2^h q^k$, or we may choose r as a positive odd prime distinct from q. In the latter case we consider the primitive form $[r, 2|cr - q|, sq^2 + c(cr - 2q)]$, where $c \geq 0$ is chosen so that $q = cr + b$, $0 < b < r/2$. The conditions of Theorem 2.1 are clearly satisfied when $r < sq^2 + c(cr - 2q)$. If $r > sq^2 + c(cr - 2q)$ then $r > q(sq - 2c)$, and $q < r$, if $sq - 2c > 0$. But $q > cr - r/2$ by choice of c. Hence $sq > scr - sr/2 = scr - 2c - sr/2 + 2c > 2c$, $c > 1$, since $scr - 2c - sr/2 > 0$ follows from $r \geq 3 > 4c/s(2c - 1)$; and $sq \geq 3 > 2 = 2c$, $c = 1$. If $q < r/2$, $c = 0$ and $2|cr - q| = 2q < sq^2 = sq^2 + c(cr - 2q)$. If $r/2 < q < r$, $c = 1$, $r < 2q < sq^2$,

$2|cr - q| = 2r - 2q < sq^2 + r - 2q = sq^2 + c\ (cr - 2q)$.

Hence the conditions of Theorem 2.1 are always satisfied.

If $\Delta = 4q^2(2^h q^k - 1)$, then $[q^2, 4q, 2^h q^k + 3]$ satisfies the required conditions when $q > 3$, $k > 1$; and $h > 1$, $k = 1$. If $q = 3$ then $S_3 = 2^h 3^{k+2}$, allowing only the cases $h = 1$, $k = 1$; $h = 0$, $k = 2$; $h = 0$, $k = 1$; and $h = 1$, $k = 0$; or $\Delta = 180$, 288, 72, and 36 respectively. When $h = 1$, $k = 1$ we use $[q + 4, 24, 2q^2 - 9q + 36]$, $q > 19$, and eliminate $q = 5, 7, 11, 13, 17, 19$ since $S_3 = 9.26$, $S_1 = 11.58$, $S_1 = 31.82$, $S_3 = 58.73$, $S_1 = 38.251$, $S_2 = 31.431$ respectively. Similarly for $h = 0$, $k = 1$ we use $[q + 3, 12, q^2 - 4q + 12]$, $q > 7$, eliminating $q = 5$ and 7 by $S_2 = 8.15$ and $S_1 = 5.59$ respectively. Finally, if $k = 0$, $S_1 = q^2(2^h - 1) + 1 = 4(2^{h-2}q^2 + (1 - q^2)/4)$, $h > 1$; and when $h = 1$, $S_1 = q^2 + 1$, $S_2 = q^2 + 4$, so that 5 divides S_2 for $q \equiv \pm 1 \pmod{10}$ and 5 divides S_1 for $q \equiv \pm 3 \pmod{10}$, thus excluding all cases except $q = 5$, namely, $\Delta = 100$.

Again, when Δ is odd and contains as a factor the square of the positive odd prime q, we may write $\Delta = q^2(4\ rs - 1)$, since $\Delta \equiv -1 \pmod 4$. Then either $r.s = q^k$ or we may choose r as a positive prime distinct from q. In the latter case we consider the primitive form $[r, |2\ cr - q|, sq^2 + c(cr - q)]$, where $c \geq 0$ is chosen so that $q = 2cr + b$, $0 < b < r$. The conditions of Theorem 2.1 are clearly satisfied when $r < sq^2 + c(cr - q)$. If $r > sq^2 + c(cr - q)$, then $r > q(sq - c)$, and $r > q$ if $sq - c > 0$. But $q > 2cr - r$ by choice of c. Hence $sq > 2scr - sr = 2scr - c - sr + c > c$, $c > 0$, since $2scr - c - sr > 0$ follows from

$r \geq 2 > c/s(2c - 1)$. When $r > q$, $c = 0$ so that $|2cr - q| = q < sq^2 = sq^2 + c(cr - q)$. Hence the conditions of Theorem 2.1 are again satisfied.

If $\Delta = q^2(4q^k - 1)$, then $[q^2, 3q, q^k + 2]$ satisfies the conditions of Theorem 2.1 when $q > 3$ and $k > 1$. If $q = 3$ then $T_1 = 3^{k+2}$, and hence by Dickson's test there are allowed only $k = 0, 1, 2$, namely, $\Delta = 27, 99, 315$. When $k = 1$ all remaining cases are eliminated by $[q + 2, q - 4, q^2 - 2q + 2]$.

Finally, when $k = 0$, namely, $\Delta = 3q^2$, we write $q = 6m + e$ where e is $+1$ or -1 as the case may be, so that $\Delta = 36m(3m + e) + 3$. Then $T_0 = 9m(3m + e) + 1 \equiv 0 \pmod 7$ when $m \equiv 4e$ or $5e \pmod 7$; $T_1 = 9m(3m + e) + 3 \equiv 0 \pmod 7$ when $m \equiv 3e$ or $6e \pmod 7$; $T_2 = 9m(3m + e) + 7 \equiv 0 \pmod 7$ when $m \equiv 0$ or $2e \pmod 7$; and $T_3 = 9m(3m + e) + 13 \equiv 0 \pmod{49}$ when $m \equiv e \pmod 7$. When $e = -1$ Dickson's tests eliminate all cases except $m = 1$, namely, $q = 5$, since $T_1 = 3.7$ and is allowed. When $e = +1$ all cases are eliminated except $m = 1$, namely, $q = 7$, since $T_3 = 49$ and is allowed. These give the final two allowable odd discriminants $\Delta = 75$, $\Delta = 147$.

Chapter II

Reduction Formulae

In stating Theorem 1.3 it was added that in order to apply it to the complete solution of the number of representations problem it would be necessary, among other things, to derive formulae whereby the number of representations for integers not prime to the discriminant could be expressed in terms of the number of representations function for integers prime to the discriminant. The results needed here have been given by Pall, who derived them and expressed them using the terminology and methods of the theory of composition of binary quadratic forms. In order that our treatment may present a unified completeness, we propose to deduce these theorems with some extensions by more elementary and direct methods. The essential part of our development will be based on certain directly derived properties of the quadratic characters defined in Theorem 1.2.

In most cases, if $(m, \Delta) = 1$ and g/Δ, then

$$N[gm = ax^2 + bxy + cy^2] = K\, N[m = a'x^2 + b'xy + c'y^2]$$

where K is some constant, $[a, b, c]$ has discriminant $-\Delta$, and $[a', b', c']$ has a discriminant derived from Δ by some prescribed method. Before giving theorems of this nature, we require the result on transformations of quadratic forms stated in

<u>Lemma 1</u>

<u>If the primitive quadratic form $ax^2 + bxy + cy^2$, where λ is a prime > 2, is such that $b^2 - 4ac \equiv 0 \pmod{\lambda^k}$, $k = 1$ or 2,</u>

then there exists a transformation

$$x = \alpha\xi + \beta\eta, \qquad y = \gamma\xi + \delta\eta, \qquad \alpha\delta - \beta\gamma = 1$$

such that

$$ax^2 + bxy + cy^2 = \bar{a}\xi^2 + \bar{b}\xi\eta + \bar{c}\eta^2$$

where $\bar{a} \equiv 0 \pmod{\lambda^k}$, $\quad \bar{b} \equiv 0 \pmod{\lambda}$

If all of a, b, c are $\equiv 0 \pmod{\lambda}$, the form is not primitive and so not considered. If $a \equiv 0 \pmod{\lambda}$ and $c \equiv 0 \pmod{\lambda}$, then since

$$b^2 - 4ac \equiv 0 \pmod{\lambda}, \qquad b \equiv 0 \pmod{\lambda}, \qquad \lambda \text{ prime}$$

reducing to the case mentioned.

Hence for all other cases we may, without loss of generality, choose $c \not\equiv 0 \pmod{\lambda}$.

Now if $a \equiv 0 \pmod{\lambda}$, then $b^2 \equiv 0 \pmod{\lambda}$ and $b \equiv 0 \pmod{\lambda}$, so $b^2 \equiv 0 \pmod{\lambda^2}$. Hence, since $b^2 - 4ac \equiv 0 \pmod{\lambda^k}$ and $c \not\equiv 0 \pmod{\lambda}$, $a \equiv 0 \pmod{\lambda^k}$. If $b \equiv 0 \pmod{\lambda}$, $b^2 \equiv 0 \pmod{\lambda^2}$ and $a \equiv 0 \pmod{\lambda^k}$. Therefore, if either a or $b \equiv 0 \pmod{\lambda}$, $a \equiv 0 \pmod{\lambda^k}$ and $b \equiv 0 \pmod{\lambda}$ in which case we use the identity transformation.

If a, b, c are all $\not\equiv 0 \pmod{\lambda}$, then we must have

$$\bar{a} = a\alpha^2 + b\alpha\gamma + c\gamma^2 \equiv 0 \pmod{\lambda}, \qquad \text{and} \qquad \alpha\delta - \beta\gamma = 1$$

If $\quad a\alpha^2 + b\alpha\gamma + c\gamma^2 \equiv 0 \pmod{\lambda}$

$$4a^2\alpha^2 + 4ab\alpha\gamma + 4ac\gamma^2 = (2a\alpha + b\gamma)^2 + (4ac - b^2)\gamma^2$$

$$\equiv (2a\alpha + b\gamma)^2 \equiv 0 \pmod{\lambda}$$

since $4ac - b^2 \equiv 0 \pmod{\lambda}$, but since λ is prime

$$2a\alpha + b\gamma \equiv 0 \pmod{\lambda}$$

Conversely, if $2a\alpha + b\gamma \equiv 0 \pmod{\lambda}$,

$$4a(a\alpha^2 + b\alpha\gamma + c\gamma^2) \equiv 0 \pmod{\lambda}$$

and since λ is a prime > 2 and $a \not\equiv 0 \pmod{\lambda}$,

$$a\alpha^2 + b\alpha\gamma + c\gamma^2 \equiv 0 \pmod{\lambda}$$

so we wish a solution of

$$2a\alpha + b\gamma \equiv 0 \pmod{\lambda}, \quad \alpha\delta - \beta\gamma = 1$$

Take $\gamma = 1$; then since the 2α and λ are co-prime, α may be determined.

Then $\beta = \alpha\delta - 1$. So for all δ, β is fixed. Thus we have a solution and these values give a transformation which reduces the case to that where $a \equiv 0 \pmod{\lambda}$ considered above.

When λ is an odd prime dividing takes two essentially distinct forms according as λ^2 / Δ and $\lambda^2 \nmid \Delta$. These results are incorporated in the following:

<u>Theorem 3.1</u>

<u>If λ / Δ, λ an odd prime, and λ^2 / Δ, then</u>

$$\underline{N[\lambda^{2k} m = ax^2 + bxy + cy^2] = N[m = ax^2 + bxy + cy^2]}$$

$$\underline{N[\lambda^{2k+1} m = ax^2 + bxy + cy^2] = N[m = a'x^2 + b'xy + c'y^2]}$$

<u>where $4ac - b^2 = \Delta = \Delta' = 4a'c' - b'^2$</u>

By lemma 1

$$N[\lambda^2 m = ax^2 + bxy + cy^2]$$
$$= N[\lambda^2 m = \bar{a}\xi^2 + \bar{b}\xi\eta + \bar{c}\eta^2]$$

where $\bar{a} \equiv \bar{b} \equiv 0 \pmod{\lambda}$, $\bar{c} \not\equiv 0 \pmod{\lambda}$. Hence $\eta \equiv 0 \pmod{\lambda}$, but $\bar{a}\xi^2 \equiv 0 \pmod{\lambda^2}$ so $\xi \equiv 0 \pmod{\lambda}$. Therefore $\eta = \lambda y$, $\xi = \lambda x$ and

$$N[\lambda^2 m = ax^2 + bxy + cy^2]$$
$$= N[m = ax^2 + bxy + cy^2]$$

Also by lemma 1

$$N[\lambda m = ax^2 + bxy + cy^2]$$
$$= N[\lambda m = \bar{a}\xi^2 + \bar{b}\xi\eta + \bar{c}\eta^2]$$

where $\bar{a} \equiv \bar{b} \equiv 0 \pmod{\lambda}$, $\bar{c} \not\equiv 0 \pmod{\lambda}$. Hence $\eta \equiv 0 \pmod{\lambda}$. If we put $\bar{a} = \lambda a'$, $\bar{b} = \lambda b'$, $\eta = \lambda z$, then

$$N[\lambda m = \bar{a}\xi^2 + \bar{b}\xi\eta + c\eta^2] = N[m = a'\xi^2 + \lambda b'\xi z + \lambda\bar{c}z^2]$$

where $4a'\lambda\bar{c} - \lambda^2 b'^2 = 4\bar{a}\bar{c} - \bar{b}^2 = 4ac - b^2$. The theorem follows by successive application of these results:

<u>Theorem 3.2</u>

<u>If λ^2/Δ, λ an odd prime, then</u>

$$\underline{N[\lambda^2 m = ax^2 + bxy + cy^2]}$$
$$\underline{= N[m = a'x^2 + b'xy + c'y^2]}$$

<u>where $4a'c' - b'^2 = \Delta/\lambda^2$</u>

$$\underline{N[\lambda m = ax^2 + bxy + cy^2, (m, \lambda) = 1] = 0}$$

By lemma 1

$$N[\lambda^k m = ax^2 + bxy + cy^2]$$
$$= N[\lambda^k m = \bar{a}x^2 + \bar{b}xy + \bar{c}y^2], \quad k > 0$$

where $\bar{a} = \lambda^2 a'$, $\bar{b} = \lambda b'$, $\bar{c} \not\equiv 0 \pmod{\lambda}$, so

$$\lambda^k m = \lambda^2 a'x^2 + \lambda b'xy + \bar{c}y^2$$

which requires $x = \xi$, $y = \lambda\eta$. Hence

$$\lambda^k m = \lambda^2 a'\xi^2 + \lambda^2 b'\xi\eta + \lambda^2\bar{c}\eta^2$$

so if $k = 1$ and $m = m$ is prime to λ, there can be no representations and when $k = 2$

$$m = a'\xi^2 + b'\xi\eta + \bar{c}\eta^2$$

Also $\quad 4a'\bar{c} - b'^2 = 4\dfrac{\bar{a}}{\lambda^2}\bar{c} - \dfrac{\bar{b}^2}{\lambda^2} = \dfrac{\Delta}{\lambda^2}$

Hence the theorem follows.

As might be anticipated, the number of representations for $2n$ requires special treatment. Before proceeding, we recall that our development is being restricted to cases with a single class of forms to a genus. In theorem 2.2 we showed that if $\Delta \equiv -1 \pmod 8$, i.e., if the discriminants were odd and yet some of the coefficients of the form were even, there was more than a single class to a genus unless $\Delta = 7$ or 15. These cases we consider separately in

Theorem 3.3

If $[a,b,c]$ has discriminant $-\Delta = -7$

$$N[\,2^k m = ax^2 + bxy + cy^2,\ m \text{ odd}] = (k+1)\,N[\,m = ax^2 + bxy + cy^2\,]$$

If $[a,b,c]$ has discriminant $-\Delta = -15$

$$N[\,2^k m = ax^2 + bxy + cy^2,\ m \text{ odd}] = (k+1)\,N[\,m = a'x^2 + b'xy + c'y^2\,]$$

where $[a',b',c']$ is in the same class with $[a,b,c]$ or not according as k is even or odd.

When $\Delta = 7$ the single reduced form is $[1,1,2]$. Hence, if $2n = x^2 + xy + 2y^2$, then $x(x+y) \equiv 0 \pmod 2$ so that $x \equiv 0 \pmod 2$, $x + y \equiv 0 \pmod 2$, or both. If we make replacements according to the scheme: x even, $x \to 2x$, $y \to y$; $x + y$ even, $x \to -x + y$, $y \to x + y$; x and $x + y$ even, $x \to 2x$, $y \to 2y$, then we observe that

$$N[2n = x^2 + xy + 2y^2]$$

$$= N[n = x^2 + xy + 2y^2] \quad , \quad x \text{ even}$$

$$+ N[n = x^2 + xy + 2y^2] \quad , \quad x + y \text{ even}$$

$$- N[n = 2(x^2 + xy + 2y^2)] \quad , \quad x \text{ and } x + y \text{ even}$$

setting $n = 2^k m$, m odd and repeating the procedure, the result follows.

When $\Delta = 15$ the only reduced forms are $[1, 1, 4]$ and $[2, 1, 2]$. If

$$2n = x^2 + xy + 4y^2, \qquad x(x+y) \equiv 0 \pmod 2)$$

and x is even, or $x+y$ is even, or both are even. If we make replacements according to the scheme, x even, $x \to 2x$, $y \to y$; $x+y$ even, $x \to -2x-y$, $y \to y$; x and $x+y$ even, $x \to 2x$, $y \to 2y$, we observe:

$$N[2n = x^2 + xy + 4y^2]$$

$$= N[n = 2x^2 + xy + 2y^2] \quad , \quad x \text{ even}$$

$$+ N[n = 2x^2 + xy + 2y^2] \quad , \quad x + y \text{ even}$$

$$- N[n = 2(x^2 + xy + 4y^2)] \quad , \quad x \text{ and } x + y \text{ even}$$

If, however, $2n = 2x^2 + xy + 2y^2$, then $xy \equiv 0 \pmod 2$ and x is even, or y is even, or both x and y are even. So, if we make replacements: x even, $x \to 2y$, $y \to x$, y even, $x \to x$, $y \to 2y$; x and y even, $x \to 2x$, $y \to 2y$, we observe:

$$N[2n = 2x^2 + xy + 2y^2]$$

$$= N[n = x^2 + xy + 4y^2] \quad , \quad x \text{ even}$$

$$+ N[n = x^2 + xy + 4y^2] \quad , \quad y \text{ even}$$

$$- N[n = 2(2x^2 + xy + 2y^2)] \, , \quad x \text{ and } y \text{ even}$$

Combining these results with $n = 2^k m$, m odd, the theorem follows.

When $\Delta \equiv 3 \pmod 8$, that is for all other odd discriminants, all the coefficients must be odd, so the reduction theorem takes the form:

Theorem 3.4

If $\Delta \equiv 3 \pmod 8$

$$N[2^k m = ax^2 + bxy + cy^2,\ m \text{ odd}]$$
$$= N[m = ax^2 + bxy + cy^2],\quad k \text{ even}$$
$$= 0 \qquad\qquad, \quad k \text{ odd}$$

Since $\Delta \equiv 3 \pmod 8$, a, b, and c are odd; therefore

$$ax^2 + bxy + cy^2 \equiv x^2 + xy + y^2 \pmod 2$$

so that if x and y are such that

$$2m = ax^2 + bxy + cy^2$$

then $\qquad x^2 + xy + y^2 \equiv 0 \pmod 2$

or $\qquad (x+y)^2 - xy \equiv 0 \pmod 2$

So $x+y$ and xy are both odd or both even; but if xy is odd, x and y are both odd and $x+y$ is even. Hence xy and $x+y$ are even, so that x and y are both even $x = 2\xi$, $y = 2\eta$.

This requires

$$ax^2 + bxy + cy^2 \equiv 0 \pmod 4$$

and by a successive reduction the theorem follows.

For even discriminants it is convenient to make a separation into two types according as:

$$\Delta \equiv 4 \text{ or } 8 \pmod{16} \quad \text{and} \quad \Delta \equiv 0 \text{ or } 12 \pmod{16}$$

In which connection we state:

Lemma 2

A quadratic form $[a, 2b, c]$ has discriminant $-\Delta$, $\Delta \equiv 4$ or $8 \pmod{16}$ if, and only if, $ac \equiv 2$ or $ac \equiv 2b + 1 \pmod 4$; or equivalently if, and only if,

$$a \equiv 1 \pmod 2, \quad c \equiv 2 \pmod 4;$$

$$a \equiv 2 \pmod 4, \quad c \equiv 1 \pmod 2$$

or

$$a \equiv 1 \pmod 2, \quad c \equiv 1 \pmod 2$$

$$a(a + 2b + c) \equiv 2 \pmod 4$$

When $\Delta \equiv 4 \pmod{16}$, $b^2 - ac \equiv 3 \pmod 4$

$$\begin{aligned} ac &\equiv b^2 + 1 \\ &\equiv 1 \equiv 2b + 1, \quad b \text{ even} \\ &\equiv 2 \qquad\qquad\;\;, \quad b \text{ odd} \end{aligned}$$

When $\Delta \equiv 8 \pmod{16}$, $b^2 - ac \equiv 2 \pmod 4$

$$\begin{aligned} ac &\equiv b^2 + 2 \\ &\equiv 2 \qquad\qquad\;\;, \quad b \text{ even} \\ &\equiv 3 \equiv 2b + 1, \quad b \text{ odd} \end{aligned}$$

Conversely, when $ac \equiv 2 \pmod 4$

$$\begin{aligned} \Delta = 4ac - 4b^2 &\equiv 8 - 4b^2 \pmod{16} \\ &\equiv 8 \quad, \; b \text{ even} \\ &\equiv 4 \quad, \; b \text{ odd} \end{aligned}$$

When $ac \equiv 2b + 1 \pmod 4$

$$\begin{aligned} \Delta = 4ac - 4b^2 &\equiv -4b^2 + 8b + 4 \pmod{16} \\ &\equiv 4 \quad, \; b \text{ even} \\ &\equiv 8 \quad, \; b \text{ odd} \end{aligned}$$

Thus proving the first part of the lemma.

Then, if $$ac \equiv 2 \pmod 4$$

$$a \equiv 1 \pmod 2, \quad c \equiv 2 \pmod 4$$

or $$a \equiv 2 \pmod 4, \quad c \equiv 1 \pmod 2$$

and the converse is obvious.

Again, if $$ac \equiv 2b + 1 \pmod 4$$

$$a \equiv 1 \pmod 2, \quad c \equiv 1 \pmod 2$$

so that
$$a + c \equiv ac + 1 \pmod 4$$

hence
$$a + c \equiv 2b + c \pmod 4$$

or
$$a + 2b + c \equiv 2 \pmod 4$$

and finally
$$a(a + 2b + c) \equiv 2 \pmod 4$$

Conversely, if
$$a \equiv 1 \pmod 2, \quad c \equiv 1 \pmod 2$$

then
$$a(a + 2b + c) \equiv 2 \pmod 4$$

implies
$$a + 2b + c \equiv 2 \pmod 4$$

or
$$a + c \equiv 2b + 2 \pmod 4$$

but since a and c are odd, this is equivalent to

$$ac \equiv 2b + 1 \pmod 4$$

Theorem 3.5

Let $[a,2b,c]$ have even discriminant $-\Delta$. If $\Delta \equiv 4$ or $8 \pmod{16}$

$$N[2^{2k}m = ax^2 + 2bxy + cy^2] = N[m = ax^2 + 2bxy + cy^2]$$

<u>If $\Delta \equiv 0$ or $12 \pmod{16}$</u>

$$\underline{N[2^{2k} m = ax^2 + 2bxy + cy^2] = N[2^{2k-2} m = a'x^2 + b'xy + c'y^2]}$$

<u>where $[a', b', c']$ has discriminant $-\Delta/4$.</u>

We consider separately the two cases:

I. One of a and c odd, the other even

II. Both a and c odd

<u>Case I</u>

Without any loss of generality we choose $a \equiv 1 \pmod 2$, $c \equiv 0 \pmod 2$. Then

$$4m = ax^2 + 2bxy + cy^2$$
$$\equiv x^2 \pmod 2$$

so that $x = 2\xi$. Hence

$$4m = 4a\xi^2 + 4b\xi y + cy^2$$

Now if $\Delta \equiv 4$ or $8 \pmod{16}$, that is by lemma 2, if $c \equiv 2 \pmod 4$,

$$4m \equiv 2y^2 \pmod 4$$

so that $y = 2\eta$. Hence

$$m = a\xi^2 + 2b\xi\eta + c\eta^2$$

$$N[4m = ax^2 + 2bxy + cy^2] = N[m = ax^2 + 2bxy + cy^2]$$

However, if $\Delta \equiv 0$ or $12 \pmod{16}$, that is by lemma 2, if $c \equiv 0 \pmod 4$, then

$$m = a\xi^2 + b\xi y + \tfrac{1}{4}cy^2$$

and

$$N[4m = ax^2 + 2bxy + cy^2]$$
$$= N[m = a'x^2 + b'xy + c'y^2]$$

where $a' = a$, $b' = b$, $c' = \frac{1}{4}c$

$$4a'c' - b'^2 = ac - b^2 = \Delta/4$$

<u>Case II</u>.

The biunique transformation: $x = \xi + \eta$, $y = \eta$

replaces $ax^2 + 2bxy + cy^2$

by $\bar{a}\xi^2 + 2\bar{b}\xi\eta + \bar{c}\eta^2$

where $\bar{a} = a$, $\bar{b} = a + b$, $\bar{c} = a + 2b + c$

Since $a \equiv 1 \pmod 2$, $c \equiv 1 \pmod 2$,

$$\bar{c} = a + 2b + c \equiv 0 \pmod 2$$

Thus we reduce Case II to Case I.

When $\Delta \equiv 4$ or $8 \pmod{16}$, then by lemma 2 $\bar{c} = a + 2b + c \equiv 2 \pmod 4$;
and when $\Delta \equiv 0$ or $12 \pmod{16}$, $\bar{c} = a + 2b + c \equiv 0 \pmod 4$.

Thus in all cases:

When $\Delta \equiv 4$ or $8 \pmod{16}$

$$N[4m = ax^2 + 2bxy + cy^2] = N[m = ax^2 + 2bxy + cy^2]$$

When $\Delta \equiv 0$ or $12 \pmod{16}$

$$N[4m = ax^2 + 2bxy + cy^2] = N[m = a'x^2 + b'xy + c'y^2,\ \Delta' = \Delta/4]$$

The theorem follows then by mere repetition of the above process.

Theorem 3.6

Let $[a, 2b, c]$ have discriminant $-\Delta$, and let m be an odd integer; then:

If $\Delta \equiv 4 \text{ or } 8 \pmod{16}$

$$N[2m = ax^2 + 2bxy + cy^2]$$

$$= N[m = a'x^2 + 2b'xy + c'y^2]$$

where $ac - b^2 = a'c' - b'^2$

If $\Delta \equiv 0 \text{ or } 12 \pmod{16}$

$$N[2m = ax^2 + 2bxy + cy^2] = 0$$

Consider first the case where $a \cdot c$ is even. Without loss of generality we set a odd and c even. Then

$$2m = ax^2 + 2bxy + cy^2 \equiv x^2 \pmod{2}$$

so that $x = 2\xi$ and

$$m = a'\xi^2 + 2b\xi y + c'y^2$$

where $a' = 2a, \quad b' = b, \quad c' = \frac{1}{2}c$

Now when $\Delta \equiv 4 \text{ or } 8 \pmod{16}$ by lemma 2 $c \equiv 2 \pmod{4}$, $c' \equiv 1 \pmod{2}$

so that

$$N[2m = ax^2 + 2bxy + cy^2]$$

$$= N[m = a'x^2 + 2b'xy + c'y^2]$$

where $a'c' - b'^2 = 2a \cdot \frac{1}{2}c - b^2 = ac - b^2$

When $\Delta \equiv 0 \text{ or } 12 \pmod{16}$ by lemma 2 $c \equiv 0 \pmod{4}$, $c' \equiv 0 \pmod{2}$

so that

$$N[2m = ax^2 + 2bxy + cy^2]$$

$$= N[m = a'x^2 + 2b'xy + c'y^2] = 0$$

When a·c is odd, as in the previous theorem, the transformation $x = \xi + \eta,\ y = \eta$ reduces the situation to the case a·c even.

These theorems give the reduction formulae necessary for the forms we are to consider. In order to identify the several forms occurring in these formulae, we need some extensions on the properties of the characters as given by Dickson.

We generalize the statement of Dickson's Theorem 67 to read:

Theorem 3.7

$[a, b, c]$ has the discriminant $-\Delta$, $\Delta = 4ac - b^2$. If p_i are the distinct odd prime factors of Δ, then the characters (n/p_i) have the same value for all integers n represented by a form of discriminant $-\Delta$, excluding those characters, (n/p_j), where p_j/n. The same is true for the characters $\delta = (-1)^{\frac{1}{2}(n-1)}$, $\epsilon = (-1)^{\frac{1}{8}(n^2-1)}$, $\delta\epsilon$ for odd integers n.

In the proof of Dickson's Theorem defining the characters and giving their properties, it is only necessary to shift the emphasis that n be prime to Δ, hence allowing a consideration of all odd prime factors of Δ, to a restriction to consider only characters corresponding to those odd prime factors of Δ not dividing n. The details of proof remain unchanged.

In connection with the theorems to follow, we state:

Lemma 3

Let $[a, b, c]$ have the odd discriminant $-\Delta = -\lambda_1\lambda_2$. If λ_1 is square free and

$$\lambda_1 m = ax^2 + bxy + cy^2$$

there exist ξ and η such that

$$\lambda_2 m = a\xi^2 + b\xi\eta + c\eta^2$$

By hypothesis $\lambda_1 m = ax^2 + bxy + cy^2$

so that
$$4a\lambda_1 m = 4a^2x^2 + 4abxy + 4acy^2$$
$$= (2ax + by)^2 + \Delta y^2$$
$$= (2ax + by)^2 + \lambda_1\lambda_2 y^2$$

and similarly $4c\lambda_1 m = (2cy + bx)^2 + \lambda_1\lambda_2 x^2$

Since λ_1 is square free, this implies

$$bx + 2cy = \lambda_1\xi \quad , \quad 2ax + by = -\lambda_1\eta$$

to determine ξ and η, but from these

$$-\lambda_2 x = b\xi + 2c\eta \, , \quad \lambda_2 y = 2a\xi + b\eta$$

Hence
$$\lambda_1\lambda_2^2 m = a(b\xi + 2c\eta)^2 - b(b\xi + 2c\eta)(2a\xi + b\eta)$$
$$+ c(2a\xi + b\eta)^2$$
$$= (4ac \quad b^2)(a\xi^2 + b\xi\eta + c\eta^2)$$

or
$$\lambda_2 m = a\xi^2 + b\xi\eta + c\eta^2$$

Lemma 4

Let $[a, 2b, c]$ have discriminant $-\Delta = -4\lambda_1\lambda_2$. If λ_1 is square free and

$$\lambda_1 m = ax^2 + 2bxy + cy^2$$

there exist ξ and η such that

$$\lambda_2 m = a\xi^2 + 2b\xi\eta + c\eta^2$$

The proof parallels that of Lemma 3 starting with

$$a\lambda_1 m = (ax+by)^2 + \lambda_1\lambda_2 y^2$$

$$c\lambda_1 m = (cy+bx)^2 + \lambda_1\lambda_2 x^2$$

and using the determination of ξ and η:

$$\lambda_1\xi = bx + cy, \qquad -\lambda\eta = ax + by$$

It is unnecessary to repeat details.

The characters of forms representing certain products of integers occurring in certain of the above theorems are related according to the following four theorems:

Theorem 3.8

Let $-\Delta$ be a discriminant such that $\Delta = \lambda\mu$, λ an odd prime and $(\lambda,\mu) = 1$. Then, if λm, m odd, is represented by a form of discriminant $-\Delta$ whose characters are ${}_{\lambda m}C_i$, n is represented by a form of discriminant $-\Delta$ whose characters are ${}_{m}C_i$, where

$$ {}_{m}C_i \; {}_{\lambda}C_i = {}_{\lambda m}C_i $$

${}_{\lambda}C_i$ being the value of the characters for the form representing λ.

Consider first odd discriminant, $-\Delta$, let

$$\lambda m = au^2 + buv + cv^2$$

$$m = ar^2 + brs + cs^2$$

so that
$$4\lambda m m = x^2 + \Delta y^2 \qquad (1)$$

where

$$x = 2aur + bus + brv + 2cvs$$

$$y = us - rv$$

If $\Delta = \lambda\mu$, by lemma 3 there exist u' and v' such that

$$\mu m = au'^2 + bu'v' + cv'^2$$

Hence

$$4\mu m m = x'^2 + \Delta y'^2 \qquad (2)$$

But (1) and (2) state

$$4\lambda m m \equiv x^2 (\text{mod } p), \qquad 4\mu m m \equiv x'^2 (\text{mod } p)$$

for all p prime factors of Δ.

Therefore

$$(\lambda/p)(m/p)(m/p) = 1, \qquad p \text{ not in } \lambda$$

$$(\mu/p)(m/p)(m/p) = 1, \qquad p \text{ in } \lambda$$

But by lemma 3 λ and μ are represented by the same form, hence (λ/p) or (μ/p) include all ${}_\lambda C_\lambda$ and the theorem is proved for odd disciminants.

For even discriminants the proof begins similarly. Let

$$\lambda m = au^2 + 2buv + cv^2$$

$$m = ar^2 + 2brs + cs^2$$

so that

$$\lambda m m = x^2 - D y^2 \qquad (3)$$

where

$$x = aur + bus + brv + cvs$$

$$y = us - rv$$

If $-D = \lambda\mu$, by lemma 4 there exist u' and v', such that

$$\mu m = au'^2 + 2bu'v' + cv'^2$$

Hence

$$\mu m m = x'^2 - D y'^2 \qquad (4)$$

But (3) and (4) state

$$\lambda m m \equiv x^2 (\text{mod } p), \qquad \mu m m \equiv x'^2 (\text{mod } p)$$

for all p, odd prime factors of D.

Therefore

$$(\lambda / p)(n/p)(m/p) = 1, \qquad p \neq \lambda$$

$$(\mu / p)(n/p)(m/p) = 1, \qquad p = \lambda$$

But by lemma 4 λ and μ are represented by the same form, hence (λ/p) or (μ/p) include all ${}_{\lambda}C_{\lambda}$ of this type.

Let $D \equiv 3 \pmod 4$. Then $\lambda n m \equiv x^2 + y^2 \pmod 4$ but λ, n, and m are odd, so that $\lambda n m \equiv 1 \pmod 4$ whence $\lambda + n - 1 \equiv m \pmod 4$ so

$$(-)^{\frac{1}{2}(m-1)} = (-)^{\frac{1}{2}[(\lambda-1)+(n-1)]} \quad \text{hence} \quad \delta(m) = \delta(\lambda)\,\delta(n)$$

If $D \equiv 2 \pmod 8$, then x is odd in (3). According as y is even or odd, $\lambda n m \equiv +1$ or -1, $\lambda n \equiv \pm m \pmod 8$. Hence $\lambda^2 n^2 \equiv m^2 \pmod{16}$, but since λ and n are odd

$$\lambda^2 n^2 \equiv \lambda^2 + n^2 - 1 \equiv m^2 \pmod{16}$$

Hence $(-1)^{\frac{1}{8}(m^2-1)} = (-1)^{\frac{1}{8}[(\lambda^2-1)+(n^2-1)]}$

hence

$$\epsilon(m) = \epsilon(\lambda)\,\epsilon(n)$$

When $D \equiv 6 \pmod 8$, then $\lambda n m \equiv 1$ or 3, $\lambda n \equiv m$ or $3m \pmod 8$, whence $\lambda^2 n^2 \equiv m^2$ or $9m^2 \pmod{16}$. For the first alternative

$$\delta(m) = \delta(\lambda)\,\delta(n) \quad \text{and} \quad \epsilon(m) = \epsilon(\lambda)\,\epsilon(n)$$

For the second

$$(-)^m\,\delta(m) = \delta(\lambda)\,\delta(n), \qquad (-)^{m^2}\,\epsilon(m) = \epsilon(\lambda)\,\epsilon(n)$$

so that for both $\delta(\lambda)\,\epsilon(\lambda)\,\delta(n)\,\epsilon(n) = \delta(m)\,\epsilon(m)$

If $D \equiv 0 \pmod 4$, $\lambda n m \equiv x^2 \equiv 1 \pmod 4$, and $\delta(\lambda)\delta(m) = \delta(m)$

If $D \equiv 0 \pmod 8$, $\lambda n m \equiv x^2 \equiv 1 \pmod 8$, and $\epsilon(\lambda)\epsilon(m) = \epsilon(m)$

<u>Theorem 3.9</u>

<u>Let $-\Delta$ be an even disciminant. If 2m, m odd, is represented by a form of discriminant, $-\Delta$, whose characters are ${}_{2m}C_i$, m is represented by a form with characters ${}_{m}C_i$ where</u>

$$\underline{{}_2C_i \, {}_mC_i = {}_{2m}C_i}$$

<u>${}_2C_i$ being the value of the characters of the form representing 2.</u>

The proof follows the second part of theorem 3.8 for the characters (m/p_i). (That 2 is even is immaterial dealing with odd characters).

Since m is assumed odd, we require $\Delta \equiv 4 \text{ or } 8 \pmod{16}$. (Theorem 2)

If $\Delta \equiv 8 \pmod{16}$, $\frac{\Delta}{8} m$ is represented by the same form as 2m by Lemma 4 and may be used in the manner of Theorem 3.8 to complete the proof for even characters.

When $\Delta \equiv 4 \pmod{16}$, $D \equiv 3 \pmod 4$ and as before $2mn = x^2 - Dy^2$, since m, n, and D are odd, x and y must be odd. Hence $x^2 \equiv y^2 \equiv 1 \pmod 8$

so $$2mn \equiv 1 - D \pmod 8$$

But $$2mn \equiv 2m + 2n - 2 \equiv 2m - 2n + 2 \pmod 8$$

so that $$1 - D \equiv 2m - 2n + 2 \pmod 8$$

and $$2n \equiv 2m + 1 + D \pmod 8$$

Hence $$(-1)^{\frac{1+D}{4}} (-1)^{\frac{m-1}{2}} = (-1)^{\frac{n-1}{2}}$$

But $\frac{1-D}{2}$ is represented by the same form that represents 2.

Therefore $$\delta(2)\,\delta(m) = \delta(2m)$$

δ is the only even character here.

Theorem 3.10

Let 4m be represented by a form of discriminant, $-\Delta$, with characters ${}_{4m}C_i$. m will be represented by a form of discriminant $-\Delta$ or $-\Delta/4$ whose characters

$${}_{m}C_i = {}_{4m}C_i$$

for all i for which both exist.

For the odd characters, (m/p_i), the proof parallels that of Theorem 3.8 replacing λ everywhere by 4 and omitting mention of μ as unnecessary. If Δ is odd or $\Delta \equiv 12 \pmod{16}$ there are no even characters. When $\Delta \equiv 4$ or $8 \pmod{16}$, by Theorem 3.5 4m and m are represented by the same form, hence all the characters must be identical.

When $\Delta \equiv \pmod{32}$, δ, ϵ, and $\delta\epsilon$ are all characters, but when we go to $\Delta/4$ two of these characters will no longer occur, leaving one as follows:

$$\delta, \quad \Delta \equiv 0 \pmod{64}$$
$$\epsilon, \quad \Delta \equiv 96 \pmod{128}$$
$$\delta\epsilon, \quad \Delta \equiv 32 \pmod{128}$$

We assume

$$4m = au^2 + 2buv + cv^2$$
$$m = ar^2 + 2brs + cs^2, \quad m, m \text{ odd}$$

and as before $4mm = x^2 - Dy^2$

Now, if $\Delta \equiv 0 \pmod{64}$, $D \equiv 0 \pmod{16}$, and

$$4mm \equiv x^2 \equiv 4 \pmod{16}, \text{ since } m \text{ and } m \text{ are odd}$$

so $\quad mm \equiv 1 \pmod 4, \quad m \equiv m \pmod 4)$

and $\quad \delta(m) = \delta(m)$

If $\Delta \equiv 96 \pmod{128}$, $D \equiv 8 \pmod{32}$

$$4mn \equiv x^2 \pmod{32}, \quad y \text{ even}$$
$$\equiv x^2 - 8 \pmod{32}, \quad y \text{ odd}$$

so that $mn \equiv \pm 1 \pmod 8$, y even or odd

$$m^2 \equiv n^2 \pmod{16}$$

and $\epsilon(m) = \epsilon(n)$

If $\Delta \equiv 32 \pmod{128}$, $D \equiv 24 \pmod{32}$

$$4mn \equiv x^2 \pmod{32} \quad y \text{ even}$$
$$\equiv x^2 - 24 \pmod{32} \quad y \text{ odd}$$

so that $mn \equiv 1 \text{ or } 3 \pmod 8$

and $\delta(m)\,\epsilon(m) = \delta(n)\,\epsilon(n)$

<u>Theorem 3.11</u>

<u>Let $\Delta = \lambda^2 u$, λ an odd prime, If $\lambda^2 m$, m odd is represented by a form of discriminant $-\Delta$ with characters ${}_{\lambda^2 m}C_i$, m is represented by a form of discriminant $-\Delta/\lambda^2$ whose characters</u>

$$\underline{{}_m C_i = {}_{\lambda^2 m} C_i}$$

<u>for all i for which both exist.</u>

Let $\lambda^2 m = ax^2 + bxy + cy^2$

where $4ac - b^2 = \Delta = \lambda^2 u$

By lemma 1 there exists a primitive form in the same class of the form $[\lambda^2 a', \lambda b', c']$. We may equally well consider this. But

$$\lambda^2 m = \lambda^2 a' x^2 + \lambda b' xy + c' y^2$$

implies $y = \lambda \eta$, hence $\lambda^2 m = \lambda^2 a' x^2 + \lambda^2 b' x \eta + \lambda^2 c' \eta^2$

and $m = a'x^2 + b'x\eta + c'\eta^2$

Now the form $[a', b', c']$ representing m has discriminant $-\Delta/4$ and certainly represents all integers represented by $[a, b, c]$ and its equivalent $[\lambda^2 a', \lambda b', c']$. An integer prime to 2Δ, and hence also to $2\Delta/\lambda^2$, is represented by both. Hence all common characters must be equal as they are identical.

Chapter III

The Number of Representations Function

The explicit formulae for the number of representations function for all integers in the cases summarized above where there is a single class of forms to each genus follow now in logical summary. These are given in the theorems below. We let s uniformly stand for some odd integer prime to the discriminant, $-\Delta$, represented by the form $[a, b, c]$. Unless otherwise specified, k will stand for the number of characters in any particular case as determined according to Theorem 1.2. The connection with this k and the number of prime factors of the discriminant is used.

<u>Theorem 4.1</u>

<u>If $\Delta \equiv p_1 p_2 \cdots p_k \not\equiv 7$ or $15 \pmod{16}$ and $\Delta > 3$ where the p_i are distinct odd primes, and $[a, b, c]$ has discriminant $-\Delta$, then:</u>

$$\underline{N[2^\lambda p_1^{\alpha_1} \cdots p_k^{\alpha_k} m = ax^2 + bxy + cy^2;\ (m, 2\Delta) = 1]}$$

$$\underline{= \frac{1}{2^k}[1 + (-1)^\lambda] \prod_{i=1}^{k} \Big\{ 1 + \prod_{j=1}^{k} [C_i(p_j)]^{\alpha_j} C_i(s)\, C_i(m) \Big\} \sum_{\mu / m} (-\Delta / \mu)}$$

By Theorem 3.4

$$N[2^\lambda p_1^{\alpha_1} \cdots p_k^{\alpha_k} m = ax^2 + bxy + cy^2]$$

$$= \frac{1}{2}[1 + (-1)^\lambda]\, N[p_1^{\alpha_1} \cdots p_k^{\alpha_k} m = ax^2 + bxy + cy^2]$$

By repeated application of Theorem 3.1

$$N[p_1^{\alpha_1} \cdots p_k^{\alpha_k} m = ax^2 + bxy + cy^2]$$

$$= N[m = a'x^2 + b'xy + c'y^2]$$

where $4a'c' - b'^2 = 4ac - b^2$, and by Theorem 3.8 if q is an odd integer prime to Δ represented by $[a', b', c']$

$$C_i(q) = \prod_{j=1}^{k} [C_i(p_j)]^{\alpha_j} C_i(s)$$

Hence by Theorem 1.3 the result follows.

Theorem 4.2

If $\Delta = p_1^{\theta} p_2 \cdots p_k \equiv 0 \text{ or } 12 \pmod{16}$, where $p_1 = 2$, $\theta = 2 \text{ or } 3$ and the remaining p_i are distinct odd primes, and $[a, b, c]$ has discriminant $-\Delta$, then:

$$N[p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} m = ax^2 + bxy + cy^2;\ (m, \Delta) = 1]$$

$$= \frac{1}{2^{k-1}} \prod_{i=1}^{k} \left\{ 1 + \prod_{j=1}^{k} [C_i(p_j)]^{\alpha_j} C_i(s) C_i(m) \right\} \sum_{\mu | m}{}' (-\Delta | \mu)$$

By repeated application of Theorems 3.1 and 3.6,

$$N[p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} m = ax^2 + bxy + cy^2]$$
$$= N[m = a'x^2 + b'xy + c'y^2]$$

where $4a'c' - b'^2 = 4ac - b^2$ and by repeated application of Theorems 3.8, 3.9, and 3.10, if q is an integer prime to Δ represented by $[a', b', c']$,

$$C_i(q) = \prod_{j=1}^{k} [C_i(p_j)]^{\alpha_j} C_i(s)$$

Hence the result follows from Theorem 1.3.

Theorem 4.3

If $\Delta = p_1^{5} p_2 \cdots p_{k-1}$ where $p_1 = 2$ and the p_i are distinct odd primes, and $[a, b, c]$ has discriminant $-\Delta$, then:

$$N[p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{k-1}^{\alpha_{k-1}} m = ax^2 + bxy + cy^2; (m, \Delta) = 1]$$

$$= \frac{1}{2^{k-1}} \prod_{i=1}^{k} \left\{ 1 + \prod_{j=1}^{k-1} [C_i(p_j)]^{\alpha_j} C_i(s) C_i(m) \right\} \sum_{\mu/m}{}' (-\Delta/\mu), \quad \alpha_1 = 0$$

$$= 0, \qquad \alpha_1 = 1$$

$$= \frac{1}{2^{k-2}} \prod_{i=1}^{k-1} \left\{ 1 + \prod_{j=1}^{k-1} [C_i(p_j)]^{\alpha_j} C_i(s) C_i(m) \right\} \sum_{\mu/m}{}' (-\Delta/\mu), \quad \alpha_1 \geq 2$$

where C_k is the character for Δ , not a character for $\Delta/4$.

By repeated application of Theorem 3.1

$$N[p_1^{\alpha_1} \cdots p_{k-1}^{\alpha_{k-1}} m = ax^2 + bxy + cy^2]$$

$$= N[p_1^{\alpha_1} m = a'x^2 + b'xy + c'y^2]$$

where $4a'c' - b'^2 = 4ac - b^2$, and if $\alpha_1 = 0$, q being an odd integer represented by $[a', b', c']$

$$C_i(q) = \prod_{j=1}^{k-1} [C_i(p_j)]^{\alpha_j} C_i(s)$$

and by Theorem 1.3, the first part follows.

If $\alpha_1 = 1$, Theorem 3.6 indicates

$$N[p_1 m = a'x^2 + b'xy + c'y^2; (m, \Delta) = 1] = 0$$

If $\alpha_1 \geq 2$, Theorem 3.5 indicates

$$N[p_1^{\alpha_1} \cdots p_k^{\alpha_k} m = ax^2 + bxy + cy^2]$$

$$= N[p_1^{\alpha_1 - 2} p_2^{\alpha_2} \cdots p_k^{\alpha_k} m = a''x^2 + b''xy + c''y^2]$$

where $4(4a''c'' - b''^2) = 4ac - b^2$

Hence this falls under Theorem 4.2 and by Theorem 3.10 our result follows, noting that $(-\Delta/\mu) = (-\frac{\Delta}{4}/\mu)$.

Theorem 4.4

By Theorem 2.3 the odd discriminants containing a square factor and such that there are a single class of forms in each genus, are all of the form $\Delta = p_1^2 p_2 p_3$ where the p_i are all odd primes on one. If $[a, b, c]$ has discriminant $-\Delta$, then

$$N[2^{\lambda} p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} m = ax^2 + bxy + cy^2, \ (m, 2\Delta) =]$$

$$= \frac{1}{8}[1 + (-1)^{\lambda}] \prod_{i=1}^{3} \left\{ 1 + \prod_{j=1}^{3} [C_i(p_j)]^{\alpha_j} C_i(s)\, C_i(m) \right\} \sum_{\mu / m}{}' (-\Delta / \mu), \quad \alpha_1 = 0$$

$$= 0 \qquad , \quad \alpha_1 = 1$$

$$= \frac{1}{4}[1 + (-1)^{\lambda}] \prod_{i=2}^{3} \left\{ 1 + \prod_{j=2}^{3} [C_i(p_j)]^{\alpha_j} C_i(s)\, C_i(p_1^{\alpha_1 - 2} m) \right\} \sum_{\mu / p_1^{\alpha_1 - 2} m}{}' \left(-\frac{\Delta}{p_1^2} / \mu\right), \quad \alpha_1 \geq 2$$

where C_i is the character determined from p_i .

By Theorem 3.4

$$N[2^{\lambda} p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} m = ax^2 + bxy + cy^2]$$

$$= \frac{1}{2}[1 + (-1)^{\lambda}]\ N[p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} m = ax^2 + bxy + cy^2]$$

By Theorem 3.1

$$N[p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} m = ax^2 + bxy + cy^2]$$

$$= N[p_1^{\alpha_1} m = a'x^2 + b'xy + c'y^2]$$

where $4a'c' - b'^2 = 4ac - b^2$ and if q is an odd integer prime to Δ represented by $[a', b', c']$, by Theorem 3.8.

$$C_i(q) = \prod_{j=2}^{3} [C_i(p_j)]^{\alpha_j} C_i(s)$$

If $\alpha_1 = 0$ or 1 the result follows from Theorem 1.3 and 3.2 respectively.

If $\alpha_1 > 1$, by Theorem 3.2

$$N[p_1^{\alpha_1} m = a'x^2 + b'xy + c'y^2]$$

$$= N[p_1^{\alpha_1 - 2} m = a''x^2 + b''xy + c''y^2]$$

where $p_1^2(4a''c'' - b''^2) = 4a'c' - b'^2$ and if q' is an odd integer prime to represented by $[a'', b'', c'']$, by Theorem 3.11.

$$C_i(q') = C_i(q)$$

Hence Theorem 1.3 gives the final result.

<u>Theorem 4.5</u>

<u>If $\Delta = 2^\theta p$; $\theta = 0, 2, 4, 6$; $p = 7, 15$; $p = p_2 \cdot p_3$, $p_1 = 2$, and $[a, b, c]$ has discriminant $-\Delta$, then:</u>

$$\underline{N[p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} m = ax^2 + bxy + cy^2; (m, 2\Delta) = 1]}$$

$$\underline{= \frac{(\alpha_1 - \theta + 1)}{2^k} \prod_{i=1}^{k} \left\{1 + \prod_{j=1}^{3} [C_i(p_j)]^{\alpha_j} C_i(s) C_i(m)\right\} \sum_{\mu / m} \left(-\frac{\Delta}{2^\theta} \Big| \mu\right), \quad \alpha_1 \geq \theta}$$

$$\underline{= \frac{1}{2^k}[1 + (-1)^{\alpha_1}] \prod_{i=1}^{k} \left\{1 + \prod_{j=1}^{3} [C_i(p_j)]^{\alpha_j} C_i(s) C_i(m)\right\} \sum_{\mu / m} \left(-\frac{\Delta}{2^{\alpha_1}} \Big| \mu\right), \quad \alpha_1 \leq \theta}$$

By Theorems 3.6 and 3.5, if $\alpha_1 \leq \theta$

$$N[p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} m = ax^2 + bxy + cy^2]$$

$$= \frac{1}{2}[1 + (-1)^{\alpha_1}]\, N[p_2^{\alpha_2} p_3^{\alpha_3} m = a'x^2 + b'xy + c'y^2]$$

$[a', b', c']$ has discriminant $-\Delta/2^{\alpha_1}$. If q, an integer prime to Δ, is represented by $[a', b', c']$, then by Theorem 3.10

$$C_i(s) = C_i(q)$$

If $\alpha_1 \geq \theta$, by Theorems 3.6 and 3.3

$$N\cdot[p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} m = ax^2 + bxy + cy^2]$$

$$= (\alpha_1 - \theta + 1)\, N[p_2^{\alpha_2} p_3^{\alpha_3} m = a''x^2 + b''xy + c''y^2]$$

where $[a'', b'', c'']$ has discriminant $-\Delta/2^\theta$, and if q', an integer prime to Δ, is represented by $[a'', b'', c'']$

$$C_i(q') = [C_i(p_1)]^{\alpha_1} C_i(s)$$

By Theorem 3.1, if $[\alpha, \beta, \gamma]$ has discriminant $-\Delta/2^h$, then:

$$N[p_2^{\alpha_2} p_3^{\alpha_3} m = \alpha x^2 + \beta xy + \gamma y^2]$$

$$= N[m = \alpha' x^2 + \beta' xy + \gamma' y^2]$$

Where $[\alpha', \beta', \gamma']$ has the same discriminant as $[\alpha, \beta, \gamma]$, and if ρ and ρ' are integers prime to Δ and represented respectively by $[\alpha, \beta, \gamma]$ and $[\alpha', \beta', \gamma']$,

Theorem 3.8 indicates

$$C_i(\rho') = [C_i(p_2)]^{\alpha_2} [C_i(p_3)]^{\alpha_3} C_i(\rho)$$

Combining these results with the above and applying Theorem 1.3, the statement of the theorems follows.

<u>Theorem 4.6</u>

<u>If $\Delta = 2^\theta \cdot 3$; $\theta = 0, 2, 4, 6$, and $[a, b, c]$ has discriminant $-\Delta$, then:</u>

$$\underline{N[2^\alpha 3^\beta m = ax^2 + bxy + cy^2,\ (m, 6) = 1]}$$

$$= \frac{3}{2^k}[1 + (-1)^\alpha] \prod_{i=1}^{k} \{1 + C_i(3)^\beta C_i(s) C_i(m)\} \sum_{\mu/m} (-3/\mu), \quad \alpha \geq \theta$$

$$= \frac{1}{2^k}[1 + (-1)^\alpha] \prod_{i=1}^{k} \{1 + C_i(3)^\beta C_i(s) C_i(m)\} \sum_{\mu/m} (-2^{\theta-\alpha} \cdot 3/\mu), \quad \alpha < \theta$$

By Theorems 3.4, 3.5, 3.6

$$N[2^{\alpha} m = ax^2 + bxy + cy^2]$$

$$= \tfrac{1}{2}[1 + (-1)^{\alpha}]\; N[m = a'x^2 + b'xy + c'y^2]$$

where $[a', b', c']$ has discriminant $-\Delta = -3$, if $\alpha \geq 0$ and $\Delta = 2^{\theta - \alpha} 3$ if $\alpha < \theta$; and if q is an odd integer prime to 2Δ represented by $[a', b', c']$, by Theorem 3.10

$$C_i(q) = C_i(s)$$

By Theorem 3.1

$$N[3^{\beta} m = ax^2 + bxy + cy^2]$$

$$= N[m = a''x^2 + b''xy + c''y^2]$$

where if q' is an odd integer prime to 2Δ represented by $[a'', b'', c'']$ by Theorem 3.8

$$C_i(q') = C_i(3)^{\beta} C_i(s)$$

Hence by Theorems 1.1, 1.3, and these results, the Theorem follows.

Similarly, the number of representations formulae for the remaining discriminants to be considered: those even ones according to Theorem 2.3 containing an odd square factor $\Delta = 36, 72, 100,$ 180, 280; and the even ones $\Delta = 4, 16, 64$. We shall omit the derivations here as evident from Theorems 3.1 - 3.11, 1.1 and 1.3.

$$\Delta = 288 = 2^5 \cdot 3^2$$

$$N[2^\alpha 3^\beta m = ax^2 + bxy + cy^2,\ (m,\Delta) = 1]$$

$$= \frac{1}{2^{k-1}} \prod_{i=1}^{k} [1 + C_i(8) C_i(m)] \sum_{\mu/m}{}' (-\Delta/\mu), \quad \alpha = \beta = 0$$

$$= 0, \quad \alpha = 1 \text{ or } \beta = 1$$

$$= \frac{1}{2^{k-1}} \prod_{i=1}^{k} [1 + C_i(2)^\alpha C_i(8) C_i(3^{\beta-2} m)] \sum_{\mu/3^{\beta-2} m}{}' \left(-\frac{\Delta}{3^2}/\mu\right) \quad \begin{matrix} \alpha \geq 2 \\ \beta \geq 2 \end{matrix}$$

$$\Delta = 180 = 2^2 \cdot 3^2 \cdot 5$$

$$N[2^\alpha 3^\beta 5^\gamma m = ax^2 + bxy + cy^2,\ (m,\Delta) = 1]$$

$$= \frac{1}{2^{k-1}} \prod_{i=1}^{k} [1 + C_i(2)^\alpha C_i(5)^\gamma C_i(8) C_i(m)] \sum_{\mu/m}{}' (-\Delta/\mu), \quad \beta = 0$$

$$= 0, \quad \beta = 1$$

$$= \frac{1}{2^{k-1}} \prod_{i=1}^{k} [1 + C_i(2)^\alpha C_i(5)^\gamma C_i(8) C_i(3^{\beta-2} m)] \sum_{\mu/3^{\beta-2} m}{}' \left(-\frac{\Delta}{3^2}/\mu\right), \quad \beta > 1$$

$$\Delta = 100 = 2^2 \cdot 5^2$$

$$N[2^\alpha 5^\beta m = ax^2 + bxy + cy^2,\ (m,\Delta) = 1]$$

$$= \frac{1}{2^{k-1}} \prod_{i=1}^{k} [1 + C_i(2)^\alpha C_i(8) C_i(m)] \sum_{\mu/m}{}' (-\Delta/\mu), \quad \beta = 0$$

$$= 0, \quad \beta = 1$$

$$= 4 \sum_{\mu/5^{\beta-2} m}{}' (-1/\mu), \quad \beta \geq 2 .$$

$$\Delta = 72 = 2^3 \cdot 3^2$$

$$N[2^\alpha \cdot 3^\beta m = ax^2 + bxy + cy^2,\ (m, \Delta) = 1]$$

$$= \frac{1}{2^{k-1}} \prod_{i=1}^{k} [1 + C_i(2)^\alpha C_i(3) C_i(m)] \sum_{u/m}{}' (-\Delta/u),\quad \beta = 0$$

$$= 0,\quad \beta = 1$$

$$= \frac{1}{2^{k-1}} \prod_{i=1}^{k} [1 + C_i(2)^\alpha C_i(3) C_i(m)] \sum_{u/3^{\beta-2}m}{}' \left(-\frac{\Delta}{3^2}/u\right),\quad \beta \geq 2$$

$$\Delta = 36 = 2^2 \cdot 3^2$$

$$N[2^\alpha 3^\beta m = ax^2 + bxy + cy^2,\ (m, \Delta) = 1]$$

$$= \frac{1}{2^{k-1}} \prod_{i=1}^{k} [1 + C_i(2)^\alpha C_i(3) C_i(m)] \sum_{u/m}{}' (-\Delta/u),\quad \beta = 0$$

$$= 0,\quad \beta = 1$$

$$= 4 \sum_{u/3^{\beta-2}m}{}' (-1/u),\quad \beta \geq 2$$

$$\Delta = 4 = 2^2$$

$$N[2^\alpha m = ax^2 + bxy + cy^2,\ (m, \Delta) = 1]$$

$$= 4 \sum_{u/m}{}' (-1/u)$$

$$\Delta = 16 = 2^4$$

$$N[2^\alpha m = ax^2 + bxy + cy^2,\ (m, \Delta) = 1]$$

$$= 2 \sum_{u/m}{}' (-1/u), \quad \alpha = 0$$

$$= 0, \quad \alpha = 1$$

$$= 4 \sum_{u/m}{}' (-1/u), \quad \alpha \geq 2$$

$$\Delta = 64 = 2^6$$

$$N[2^\alpha m = ax^2 + bxy + cy^2,\ (m, \Delta) = 1]$$

$$= \tfrac{1}{2}[1 + \delta(s)\delta(m)][1 + \epsilon(s)\epsilon(m)] \sum_{u/m}{}' (-1/u), \quad \alpha = 0$$

$$= 0, \quad \alpha = 1, 3$$

$$= 2 \sum_{u/m}{}' (-1/u), \quad \alpha = 2$$

$$= 4 \sum_{u/m}{}' (-1/u), \quad \alpha \geq 4$$

In these cases [a,b,c] is to be any form of discriminant indicated.

Appendix A

Tables of Reduced Forms and Characters

Chapter II exhibits in detail the appearance of the several reduced forms when there is a single class of forms in each genus. In order to provide complete reference, however, we append here a complete table of all reduced forms for all known discriminants having a single class to a genus. Further, in order to facilitate the calculation of the number of representations function, we indicate the several characters for each discriminant and indicate the values these characters assume for numbers represented by each of the reduced forms. The reduced forms which represent the several odd prime factors of the discriminant are likewise indicated.

An explanation in detail for one case will suffice to show the organization of the table. Consider the selection:

$280 = 2^3.5.7$	ϵ	(n/5)	(n/7)
[1,0,70]	1	1	1
[2,0,35],2	−1	−1	1
[5,0,14],5	−1	1	−1
[7,0,10],7	1	−1	−1

The negative discriminant, $-280 = -2^3.5.7$, has the reduced forms: [1,0,70], [2,0,35], [5,0,14], [7,0,10]. The prime factors of the discriminant, 2,5, and 7, are represented by the last three of these respectively. There are three characters, ϵ , (n/5), (n/7), which take on the values listed for numbers represented by the form opposite.

3		(n/3)	
	[1,1,1],3	1	
$4 = 2^2$		δ	
	[1,0,1],2	1	
7		(n/7)	
	[1,1,2],7	1	
$8 = 2^3$		$\delta\ \epsilon$	
	[1,0,2],2	1	
11		(n/11)	
	[1,1,3],11	1	
$12 = 2^2.3$		(n/3)	
	[1,0,3],3	1	
$15 = 3.5$		(n/3)	(n/5)
	[1,1,4]	1	1
	[2,1,2],3,5	−1	−1
$16 = 2^4$		$\delta\ \epsilon$	
	[1,0,4],4	1	
19		(n/19)	
	[1,1,5],19	1	
$20 = 2^2.5$		δ	(n/5)
	[1,0,5],5	1	1
	[2,2,3],2	−1	−1
$24 = 2^3.3$		ϵ	(n/3)
	[1,0,6]	1	1
	[2,0,3],2,3	−1	−1
$27 = 3^3$		(n/3)	
	[1,1,7],9	1	
$28 = 2^2.7$		(n/7)	
	[1,0,7],7	1	
$32 = 2^5$		δ	ϵ
	[1,0,8],8	1	1
	[3,2,3]	−1	−1
$35 = 5.7$		(n/5)	(n/7)
	[1,1,9]	1	1
	[3,1,3],5,7	−1	−1

$36 = 2^2 . 3^2$		δ	(n/3)	
	[1,0,9],9	1	1	
	[2,2,5],2	1	-1	
$40 = 2^3 . 5$		$\delta\epsilon$	(n/5)	
	[1,0,10]	1	1	
	[2,0,5],2,5	-1	-1	
43		(n/43)		
	[1,1,11],43	1		
$48 = 2^4 . 3$		δ	(n/3)	
	[1,0,12]	1	1	
	[3,0,4],4,3	-1	1	
$51 = 3.17$		(n/3)	(n/17)	
	[1,1,13]	1	1	
	[3,3,5],3,17	-1	-1	
$52 = 2^2 . 13$		δ	(n/13)	
	[1,0,13],13	1	1	
	[2,2,7],2	-1	-1	
$60 = 2^2 . 3 . 5$		(n/3)	(n/5)	
	[1,0,15]	1	1	
	[3,0,5],8,3,5	-1	-1	
$64 = 2^6$		δ	ϵ	
	[1,0,16]	1	1	
	[4,4,5],4	1	-1	
67		(n/67)		
	[1,1,17],67	1		
$72 = 2^3 . 3^2$		$\delta\epsilon$	(n/3)	
	[1,0,18]	1	1	
	[2,0,9],2,9	1	-1	
$75 = 3 . 5^2$		(n/3)	(n/5)	
	[1,1,19],25	1	1	
	[3,3,7],3	1	-1	
$84 = 2^2 . 3 . 7$		δ	(n/3)	(n/7)
	[1,0,21]	1	1	1
	[2,2,11],2	-1	-1	1
	[3,0,7],3,7	-1	1	-1
	[5,4,5]	1	-1	-1
$88 = 2^3 . 11$		ϵ	(n/11)	
	[1,0,22]	1	1	
	[2,0,11],2,11	-1	-1	

91 = 7.13		(n/7)	(n/13)	
	[1,1,23]	1	1	
	[5,3,5],7,13	-1	-1	

96 = 2^5.3		δ	ϵ	(n/3)
	[1,0,24]	1	1	1
	[3,0,8],3	-1	-1	-1
	[4,4,7],4	-1	1	1
	[5,2,5]	1	-1	-1

99 = 3^2.11		(n/3)	(n/11)	
	[1,1,25]	1	1	
	[5,1,5],9,11	-1	-1	

100 = $2^2.5^2$		δ	(n/5)	
	[1,0,25],25	1	1	
	[2,2,12],2	1	-1	

112 = 2^4.7		δ	(n/7)	
	[1,0,28]	1	1	
	[4,0,7],4,7	-1	1	

115 = 5.23		(n/5)	(n/23)	
	[1,1,29]	1	1	
	[5,5,7],5,23	-1	-1	

120 = 2^3.3.5		ϵ	(n/3)	(n/5)
	[1,0,30]	1	1	1
	[2,0,15],2	1	-1	-1
	[3,0,10],3	-1	1	-1
	[5,0,6],5	-1	-1	1

123 = 3.41		(n/3)	(n/41)	
	[1,1,31]	1	1	
	[3,3,11],3,41	-1	-1	

132 = 2^2.3.11		δ	(n/3)	(n/11)
	[1,0,33]	1	1	1
	[2,2,17],2	1	-1	-1
	[3,0,11],3,11	-1	-1	1
	[6,6,7]	-1	1	-1

147 = 3.7^2		(n/3)	(n/7)	
	[1,1,37],49	1	1	
	[3,3,13],3	1	-1	

148 = 2^2.37		δ	(n/37)	
	[1,0,37],37	1	1	
	[2,2,19],2	-1	-1	

$160 = 2^5.5$		δ	ϵ	(n/5)
	[1,0,40]	1	1	1
	[4,4,11],4	−1	−1	1
	[5,0,8],5	1	−1	−1
	[7,6,7]	−1	1	−1

163		(n/163)
	[1,1,41],163	1

$168 = 2^3.3.7$		$\delta\epsilon$	(n/3)	(n/7)
	[1,0,42]	1	1	1
	[2,0,21],2	−1	−1	1
	[3,0,14],3	1	−1	−1
	[6,0,7],7	−1	1	−1

$180 = 2^2.3^2.5$		δ	(n/3)	(n/5)
	[1,0,45]	1	1	1
	[2,2,23],2	−1	−1	−1
	[5,0,9],5,9	1	−1	1
	[7,4,7]	−1	1	−1

$187 = 11.17$		(n/11)	(n/17)
	[1,1,47]	1	1
	[7,3,7],11,17	−1	−1

$192 = 2^6.3$		δ	ϵ	(n/3)
	[1,0,48]	1	1	1
	[3,0,16],3	−1	−1	1
	[4,4,13],4	1	−1	1
	[7,2,7]	−1	1	1

$195 = 3.5.13$		(n/3)	(n/5)	(n/13)
	[1,1,49]	1	1	1
	[3,3,17],3	−1	−1	1
	[5,5,11],5	−1	1	−1
	[7,1,7],13	1	−1	−1

$228 = 2^2.3.19$		δ	(n/3)	(n/19)
	[1,0,57]	1	1	1
	[2,2,29],2	1	−1	−1
	[3,0,19],3,19	−1	1	−1
	[6,6,11]	−1	−1	1

$232 = 2^3.29$		$\delta\epsilon$	(n/29)
	[1,0,58]	1	1
	[2,0,29],2,29	−1	−1

$235 = 5.47$		(n/5)	(n/47)
	[1,1,59]	1	1
	[5,5,13],5,47	−1	−1

$240 = 2^4.3.5$

	δ	(n/3)	(n/5)
[1,0,60]	1	1	1
[3,0,20],3	-1	-1	-1
[4,0,15],4	-1	1	1
[5,0,12],5	1	-1	-1

$267 = 3.89$

	(n/3)	(n/89)
[1,1,67]	1	1
[3,3,23],3,89	-1	-1

$280 = 2^3.5.7$

	ϵ	(n/5)	(n/7)
[1,0,70]	1	1	1
[2,0,35],2	-1	-1	1
[5,0,14],5	-1	1	-1
[7,0,10],7	1	-1	-1

$288 = 2^5.3^2$

	δ	ϵ	(n/3)
[1,0,72]	1	1	1
[4,4,19],4	-1	-1	1
[8,0,9],9	1	1	-1
[8,8,11]	-1	-1	-1

$312 = 2^3.3.13$

	ϵ	(n/3)	(n/13)
[1,0,78]	1	1	1
[2,0,39],2	1	-1	-1
[3,0,26],3	-1	-1	1
[6,0,13],13	-1	1	-1

$315 = 3^2.5.7$

	(n/3)	(n/5)	(n/7)
[1,1,79]	1	1	1
[5,5,17],5	-1	-1	-1
[7,7,13],7	1	-1	-1
[9,9,11],9	-1	1	1

$340 = 2^2.5.17$

	δ	(n/5)	(n/17)
[1,0,85]	1	1	1
[2,2,43],2	-1	-1	1
[5,0,17],5,17	1	-1	-1
[10,10,11]	-1	1	-1

$352 = 2^5.11$

	δ	ϵ	(n/11)
[1,0,88]	1	1	1
[4,4,23],4	-1	1	1
[8,0,11],11	-1	-1	-1
[8,8,13]	1	-1	-1

$372 = 2^2.3.31$

	δ	(n/3)	(n/31)
[1,0,93]	1	1	1
[2,2,47],2	-1	-1	1
[3,0,31],3,31	-1	1	-1
[6,6,17]	1	-1	-1

$403 = 13 \cdot 31$		(n/13)	(n/31)		
	[1,1,101]	1	1		
	[11,9,11],13,31	−1	−1		

$408 = 2^3 \cdot 3 \cdot 17$		ϵ	(n/3)	(n/17)	
	[1,0,102]	1	1	1	
	[2,0,51],2	−1	−1	1	
	[3,0,36],3	−1	1	−1	
	[17,0,24],17	1	−1	−1	

$420 = 2^2 \cdot 3 \cdot 5 \cdot 7$		δ	(n/3)	(n/5)	(n/7)
	[1,0,105]	1	1	1	1
	[2,2,53],2	1	−1	−1	1
	[3,0,35],3	−1	−1	−1	−1
	[5,0,21],5	1	−1	1	−1
	[6,6,19]	−1	1	1	−1
	[7,0,15],7	−1	1	−1	1
	[10,10,13]	1	1	−1	−1
	[11,8,11]	−1	−1	1	1

$427 = 7 \cdot 61$		(n/7)	(n/61)		
	[1,1,107]	1	1		
	[7,7,17],7,61	−1	−1		

$435 = 3 \cdot 5 \cdot 29$		(n/3)	(n/5)	(n/29)	
	[1,1,109]	1	1	1	
	[3,3,37],3	1	−1	−1	
	[5,5,23],5	−1	−1	1	
	[11,7,11],29	−1	1	−1	

$448 = 2^6 \cdot 7$		δ	ϵ	(n/7)	
	[1,0,112]	1	1	1	
	[4,4,29],4	1	−1	1	
	[7,0,16],7	−1	1	1	
	[11,6,11]	−1	−1	1	

$480 = 2^5 \cdot 3 \cdot 5$		δ	ϵ	(n/3)	(n/5)
	[1,0,120]	1	1	1	1
	[3,0,40],3	−1	−1	1	−1
	[4,4,31],4	−1	1	1	1
	[5,0,24],5	1	−1	−1	1
	[8,0,15]	−1	1	−1	−1
	[8,8,17]	1	1	−1	−1
	[11,2,11]	−1	−1	−1	1
	[12,12,13]	1	−1	1	−1

$483 = 3 \cdot 7 \cdot 23$		(n/3)	(n/7)	(n/23)	
	[1,1,121]	1	1	1	
	[3,3,41],3	−1	−1	1	
	[7,7,19],7	1	−1	−1	
	[11,1,11],23	−1	1	−1	

$520 = 2^3.5.13$

	ϵ	(n/5)	(n/13)
[1,0,130]	1	1	1
[2,0,65],2	1	−1	−1
[5,0,26],5	−1	1	−1
[10,0,13],13	−1	−1	1

$532 = 2^2.7.19$

	δ	(n/7)	(n/19)
[1,0,133]	1	1	1
[2,2,67],2	−1	1	−1
[7,0,19],7,19	−1	−1	1
[13,12,13]	1	−1	−1

$555 = 3.5.37$

	(n/3)	(n/5)	(n/37)
[1,1,139]	1	1	1
[3,3,47],3	−1	−1	1
[5,5,29],5	−1	1	−1
[13,11,13],37	1	−1	−1

$595 = 5.7.17$

	(n/5)	(n/7)	(n/17)
[1,1,149]	1	1	1
[5,5,31],5	1	−1	−1
[7,7,23],7	−1	1	−1
[13,9,13],17	−1	−1	1

$627 = 3.11.19$

	(n/3)	(n/11)	(n/19)
[1,1,157]	1	1	1
[3,3,53],3	−1	1	−1
[11,11,17],11	−1	−1	1
[13,7,13],19	1	−1	−1

$660 = 2^2.3.5.11$

	δ	(n/3)	(n/5)	(n/11)
[1,0,165]	1	1	1	1
[2,2,83],2	−1	−1	−1	−1
[3,0,55],3	−1	1	−1	1
[5,0,33],5	1	−1	−1	1
[6,6,29]	1	−1	1	−1
[10,10,19]	−1	1	1	−1
[11,0,15],11	−1	−1	1	1
[13,4,13]	1	1	−1	−1

$672 = 2^5.3.7$

	δ	ϵ	(n/3)	(n/7)
[1,0,168]	1	1	1	1
[3,0,56],3	−1	−1	−1	−1
[4,4,43],4	−1	−1	1	1
[7,0,24],7	−1	1	1	−1
[8,0,21]	1	−1	−1	1
[8,8,23]	−1	1	−1	1
[12,12,17]	1	1	−1	−1
[13,2,13]	1	−1	1	−1

$708 = 2^2.3.59$

	δ	(n/3)	(n/59)
[1,0,177]	1	1	1
[2,2,89],2	1	-1	-1
[3,0,59],3,59	-1	-1	1
[6,6,31]	-1	1	-1

$715 = 5.11.13$

	(n/5)	(n/11)	(n/13)
[1,1,179]	1	1	1
[5,5,37],5	-1	1	-1
[11,11,19],11	1	-1	-1
[13,13,17],13	-1	-1	1

$760 = 2^3.5.19$

	ϵ	(n/5)	(n/19)
[1,0,190]	1	1	1
[2,0,95],2	1	-1	-1
[5,0,38],5	-1	-1	1
[10,0,19],19	-1	1	-1

$795 = 3.5.53$

	(n/3)	(n/5)	(n/53)
[1,1,199]	1	1	1
[3,3,67],3	1	-1	-1
[5,5,41],5	-1	1	-1
[15,15,17],53	-1	-1	1

$840 = 2^3.3.5.7$

	$\delta\epsilon$	(n/3)	(n/5)	(n/7)
[1,0,210]	1	1	1	1
[2,0,105],2	1	-1	-1	1
[3,0,70],3	1	1	-1	-1
[5,0,42],5	-1	-1	-1	-1
[6,0,35]	1	-1	1	-1
[7,0,30],7	-1	1	-1	1
[10,0,21]	-1	1	1	-1
[14,0,15]	-1	-1	1	1

$928 = 2^5.29$

	δ	ϵ	(n/29)
[1,0,232]	1	1	1
[4,4,59],4	-1	-1	1
[8,0,29],29	1	-1	-1
[8,8,31]	-1	1	-1

$960 = 2^6.3.5$

	δ	ϵ	(n/3)	(n/5)
[1,0,240]	1	1	1	1
[3,0,80]	-1	-1	-1	-1
[4,4,61],4	1	-1	1	1
[5,0,48],5	1	-1	-1	-1
[12,12,23]	-1	1	-1	-1
[15,0,16]	-1	1	1	1
[16,16,19]	-1	-1	1	1
[17,14,17]	1	1	-1	-1

$1012 = 2^2.11.23$

	δ	(n/11)	(n/23)
[1,0,253]	1	1	1
[2,2,127],2	−1	−1	1
[11,0,23],11,23	−1	1	−1
[17,12,17]	1	−1	−1

$1082 = 2^2.3.7.13$

	δ	(n/3)	(n/7)	(n/13)
[1,0,273]	1	1	1	1
[2,2,137],2	1	−1	1	−1
[3,0,91],3	−1	1	−1	1
[6,6,47]	−1	−1	−1	−1
[7,0,39],7	−1	1	1	−1
[13,0,21],13	1	1	−1	−1
[14,14,23]	−1	−1	1	1
[17,8,17]	1	−1	−1	1

$1120 = 2^5.5.7$

	δ	ϵ	(n/5)	(n/7)
[1,0,280]	1	1	1	1
[4,4,71],4	−1	1	1	1
[5,0,56],5	1	−1	1	−1
[7,0,40],7	−1	1	−1	−1
[8,0,35]	−1	−1	−1	1
[8,8,37]	1	−1	−1	1
[17,6,17]	1	1	−1	−1
[19,18,19]	−1	−1	1	−1

$1155 = 3.5.7.11$

	(n/3)	(n/5)	(n/7)	(n/11)
[1,1,289]	1	1	1	1
[3,3,97],3	1	−1	−1	1
[5,5,59],5	−1	1	−1	1
[7,7,43],7	1	−1	1	−1
[11,11,28],11	−1	1	1	−1
[15,15,23]	−1	−1	1	1
[17,1,17]	−1	−1	−1	−1
[19,17,19]	1	1	−1	−1

$1248 = 2^5.3.13$

	δ	ϵ	(n/3)	(n/13)
[1,0,312]	1	1	1	1
[3,0,104],3	−1	−1	−1	1
[4,4,79],4	−1	1	1	1
[8,0,39]	−1	1	−1	−1
[8,8,41]	1	1	−1	−1
[12,12,29]	1	−1	−1	1
[13,0,24],13	1	−1	1	−1
[19,14,19]	−1	−1	1	−1

$1320 = 2^3.3.5.11$

	δε	(n/3)	(n/5)	(n/11)
[1,0,330]	1	1	1	1
[2,0,165],2	−1	−1	−1	−1
[3,0,110],3	1	−1	−1	1
[5,0,66],5	−1	−1	1	1
[6,0,55]	−1	1	1	−1
[10,0,33]	1	1	−1	−1
[11,0,30],11	1	−1	1	−1
[15,0,22]	−1	1	−1	1

$1380 = 2^2.3.5.23$

	δ	(n/3)	(n/5)	(n/23)
[1,0,345]	1	1	1	1
[2,2,173],2	1	−1	−1	1
[3,0,115],3	−1	1	−1	1
[5,0,69],5	1	−1	1	−1
[6,6,59]	−1	−1	1	1
[10,10,37]	1	1	−1	−1
[15,0,23],23	−1	−1	−1	−1
[19,8,19]	−1	1	1	−1

$1428 = 2^2.3.7.17$

	δ	(n/3)	(n/7)	(n/17)
[1,0,357]	1	1	1	1
[2,2,179],2	−1	−1	1	1
[3,0,119],3	−1	−1	−1	−1
[6,6,61]	1	1	−1	−1
[7,0,51],7	−1	1	1	−1
[14,14,29]	1	−1	1	−1
[17,0,21],17	1	−1	−1	1
[19,4,19]	−1	1	−1	1

$1435 = 5.7.41$

	(n/5)	(n/7)	(n/41)
[1,1,359]	1	1	1
[5,5,73],5	−1	−1	1
[7,7,53],7	−1	1	−1
[19,3,19],41	1	−1	−1

$1540 = 2^2.5.7.11$

	δ	(n/5)	(n/7)	(n/11)
[1,0,385]	1	1	1	1
[2,2,193],2	1	−1	1	−1
[5,0,77],5	1	−1	−1	1
[7,0,55],7	−1	−1	−1	−1
[10,10,41]	1	1	−1	−1
[11,0,35],11	−1	1	1	−1
[14,14,31]	−1	1	−1	1
[22,22,23]	−1	−1	1	1

$1632 = 2^5.3.17$

	δ	ϵ	(n/3)	(n/17)
[1,0,403]	1	1	1	1
[4,4,103],4	−1	1	1	−1
[3,0,136],3	−1	−1	1	−1
[8,0,51]	−1	−1	−1	1
[8,8,53]	1	−1	−1	1
[12,12,37]	1	−1	1	−1
[17,0,24],17	1	1	−1	−1
[23,22,23]	−1	1	−1	−1

$1844 = 2^3.3.7.11$

	ϵ	(n/3)	(n/7)	(n/11)
[1,0,462]	1	1	1	1
[2,0,231],2	1	−1	1	−1
[3,0,154],3	−1	1	−1	1
[6,0,77]	−1	−1	−1	−1
[7,0,66],7	1	1	−1	−1
[11,0,42],11	−1	−1	1	1
[14,0,33]	1	−1	−1	1
[21,0,22]	−1	1	1	−1

$1995 = 3.5.7.19$

	(n/3)	(n/5)	(n/7)	(n/19)
[1,1,499]	1	1	1	1
[3,3,167],3	−1	−1	−1	−1
[5,5,101],5	−1	1	−1	1
[7,7,73],7	1	−1	−1	1
[15,15,37]	1	−1	1	−1
[19,19,31],19	1	1	−1	−1
[21,21,29]	−1	1	1	−1
[23,11,23]	−1	−1	1	1

$2080 = 2^5.5.13$

	δ	ϵ	(n/5)	(n/13)
[1,0,520]	1	1	1	1
[4,4,131],4	−1	−1	1	1
[5,0,104],5	1	−1	1	−1
[8,0,65]	1	1	−1	−1
[8,8,67]	−1	−1	−1	−1
[13,0,40],13	1	−1	−1	1
[20,20,31]	−1	1	1	−1
[23,6,23]	−1	1	−1	1

$3003 = 3.7.11.13$

	(n/3)	(n/7)	(n/11)	(n/13)
[1,1,751]	1	1	1	1
[3,3,251],3	−1	−1	1	1
[7,7,109],7	1	1	−1	−1
[11,11,71],11	−1	1	1	−1
[13,13,61],13	1	−1	−1	1
[21,21,41]	−1	−1	−1	−1
[29,19,29]	−1	1	−1	1
[31,29,31]	1	−1	1	−1

$3315 = 3.5.13.17$

	(n/3)	(n/5)	(n/13)	(n/17)
[1,1,829]	1	1	1	1
[3,3,277],3	1	-1	1	-1
[5,5,167],5	-1	-1	-1	-1
[13,13,67],13	1	-1	-1	1
[15,15,59]	-1	1	-1	1
[17,17,53],17	-1	-1	1	1
[29,7,29]	-1	1	1	-1
[31,23,31]	1	1	-1	-1

$3360 = 2^5.3.5.7$

	δ	ϵ	(n/3)	(n/5)	(n/7)
[1,0,840]	1	1	1	1	1
[3,0,280],3	-1	-1	1	-1	-1
[4,4,211],4	-1	-1	1	1	1
[5,0,168],5	1	-1	-1	-1	-1
[7,0,120],7	-1	1	1	-1	1
[8,0,105]	1	1	-1	-1	1
[8,8,107]	-1	-1	-1	-1	1
[12,12,73]	1	1	1	-1	-1
[15,0,56]	-1	1	-1	1	1
[20,20,47]	-1	1	-1	-1	-1
[21,0,40]	1	-1	1	1	-1
[24,0,35]	1	1	-1	1	-1
[28,28,37]	1	-1	1	-1	1
[29,2,29]	1	-1	-1	1	1
[31,22,31]	-1	1	1	1	-1

$5280 = 2^5.3.5.11$

	δ	ϵ	(n/3)	(n/5)	(n/11)
[1,0,1320]	1	1	1	1	1
[3,0,440],3	-1	-1	-1	-1	1
[4,4,331],4	-1	-1	1	1	1
[5,0,264],5	1	-1	-1	1	1
[8,0,165]	1	-1	-1	-1	-1
[8,8,167]	-1	1	-1	-1	-1
[11,0,120],11	-1	-1	-1	1	-1
[12,12,113]	1	1	-1	-1	1
[15,0,88]	-1	1	1	-1	1
[20,20,71]	-1	1	-1	1	1
[24,0,55]	-1	1	1	1	-1
[24,24,61]	1	-1	1	1	-1
[33,0,40]	1	1	1	-1	-1
[37,14,37]	1	-1	1	-1	1
[40,40,43]	-1	-1	1	-1	-1
[41,38,41]	1	1	-1	1	-1

$5460 = 2^2 . 3 . 5 . 7 . 13$

	δ	(n/3)	(n/5)	(n/7)	(n/13)
[1,0,1365]	1	1	1	1	1
[2,2,683],2	-1	-1	-1	1	-1
[3,0,455],3	-1	-1	-1	-1	1
[5,0,273],5	1	-1	-1	-1	-1
[6,6,229]	1	1	1	-1	-1
[7,0,195],7	-1	1	-1	-1	-1
[10,10,139]	-1	1	1	-1	1
[13,0,105],13	1	1	-1	-1	1
[14,14,101]	1	-1	1	-1	1
[15,0,91]	-1	1	1	1	-1
[21,0,65]	1	-1	1	1	-1
[26,26,59]	-1	-1	1	-1	-1
[30,30,53]	1	-1	-1	1	1
[35,0,39]	-1	-1	1	1	1
[37,4,37]	1	1	-1	1	-1
[42,42,43]	-1	1	-1	1	1

$7392 = 2^5 . 3 . 7 . 11$

	δ	ϵ	(n/3)	(n/7)	(n/11)
[1,0,1848]	1	1	1	1	1
[3,0,616],3	-1	-1	1	-1	1
[4,4,463],4	-1	1	1	1	1
[7,0,264],7	-1	1	1	-1	-1
[8,0,231]	-1	1	-1	1	-1
[8,8,233]	1	1	-1	1	-1
[11,0,168],11	-1	-1	-1	1	1
[12,12,157]	1	-1	1	-1	1
[21,0,88]	1	-1	1	1	-1
[24,0,77]	1	-1	-1	-1	-1
[24,24,83]	-1	-1	-1	-1	-1
[28,28,73]	1	1	1	-1	-1
[33,0,56]	1	1	-1	-1	1
[43,2,43]	-1	-1	1	1	-1
[44,44,53]	1	-1	-1	1	1
[47,38,47]	-1	1	-1	-1	1

Appendix B

Numerical Examples

In order to make explicit the application of the results obtained, we proceed to calculate in detail the number of representations function for the two forms:

$$2x^2 + 35y^2 \quad , \quad 7x^2 + xy + 7y^2.$$

These should illustrate sufficiently the methods required.

The form $2x^2 + 35y^2$ has the discriminant $-280 = -2^3.5.7$ and characters, $\epsilon = (-1)^{\frac{1}{8}(n^2-1)}$, $(n/5)$, $(n/7)$. The number of representations function is given for this case by Theorem 4.2. Referring to the table of Appendix A to obtain the values of the characters for the form, we obtain explicitly:

$$N[2^{\alpha_1}5^{\alpha_2}7^{\alpha_3}m = 2x^2 + 35y^2, (m,70) = 1]$$

$$= \frac{1}{4}[1-(-1)^{\alpha_1+\alpha_2}\epsilon(m)][1-(-1)^{\alpha_1+\alpha_3}(m/5)][1+(-1)^{\alpha_2+\alpha_3}(m/7)]\sum_{\mu/m}(-280/\mu)$$

We have further,

$$\begin{aligned} \epsilon(m) &= 1, & m &\equiv 1 \text{ or } 7 \pmod 8 \\ &= -1, & m &\equiv 3 \text{ or } 5 \quad (\quad '' \quad) \\ (m/5) &= 1, & m &\equiv 1 \text{ or } 4 \pmod 5 \\ &= -1, & m &\equiv 2 \text{ or } 3 \quad (\quad '' \quad) \\ (m/7) &= 1, & m &\equiv 1, 2, \text{ or } 4 \pmod 7 \\ &= -1, & m &\equiv 3, 5, \text{ or } 6 \quad (\quad '' \quad) \end{aligned}$$

Hence we may seperate integers, m, $(m,70) = 1$, into residue classes, modulo 280, with the triplet $\epsilon(m)$, $(m/5)$, $(m/7)$, identical for all integers in the class:

	$\epsilon(m)$	(m/5)	(m/7)	
1)	+	+	+	1, 9, 39, 71, 79, 81, 121, 151, 169, 191, 239, 249.
2)	−	+	+	11, 29, 51, 99, 109, 141, 149, 179, 211, 219, 261, 221.

	$\epsilon(m)$	(m/5)	(m/7)	
3)	+	−	+	23,57, 113, 127, 177, 183, 193, 207, 233, 247, 263, 137.
4)	−	−	+	37, 53, 67, 93, 107, 123, 163, 197, 253, 267, 277, 43.
5)	+	+	−	31, 41, 111, 89, 129, 159, 199, 201, 209, 241, 271, 279.
6)	−	+	−	19, 59, 61, 69, 101, 131, 139, 171, 181, 229, 251, 269.
7)	+	−	−	17, 33, 47, 73, 87, 97, 103, 153, 223, 257, 143, 167.
8)	−	−	−	3, 13, 27, 83, 117, 157, 173, 187, 213, 227, 237, 243.

According to the parity of α_1, α_2, α_3 we have the four cases:

	$\alpha_1+\alpha_2$	$\alpha_1+\alpha_3$	$\alpha_2+\alpha_3$	α_1	α_2	α_3
a)	even	even	even	even	even	even
				odd	odd	odd
b)	odd	odd	even	odd	even	even
				even	odd	odd
c)	odd	even	odd	even	odd	even
				odd	even	odd
d)	even	odd	odd	odd	odd	even
				even	even	odd

Applying these results to the number of representations function as given above, we see that it is possible to state further:

$$N[2^{\alpha_1}\cdot 5^{\alpha_2}\cdot 7^{\alpha_3} m = 2x^2+35y^2,\ (m,70)=1]$$

$$= 2\sum_{\mu/m}{}' (-280/\mu)$$

when α_1, α_2, α_3 and m are paired, according to the divisions above, a),4); b),1); c),7); d),6); otherwise the number of representations is zero.

The law of quadratic reciprocity shows that:

$$(-280/\mu) = (-)^{\frac{1}{8}(\mu^2-1)}(\mu/5)(\mu/7) = \epsilon(\mu)(\mu/5)(\mu/7)$$

Hence $\sum_{\mu/m}(-280/\mu)$ is equal to the excess of the divisors of m in classes 1), 4), 6), 7) over those in classes 2), 3), 5), 8).

We illustrate further by verifying the formula for the number of representations of 23,902. If

$$23,902 = 2x^2 + 35y^2,$$

we obtain empirically the solutions : $x = \pm 11$, $y = \pm 26$; $x = \pm 59$, $y = \pm 22$; $x = \pm 101$, $y = \pm 10$; $x = \pm 109$, $y = \pm 2$; with all choices of sign permissable. Hence the number of representations is 16. To check this with our formula, we observe that 23,902 = 2.17.19.37, hence

$$N[23,902 = 2x^2 + 35y^2] = N[2\cdot 17\cdot 19\cdot 37 = 2x^2 + 35y^2]$$

$$= 2\sum_{\mu/11951}{}^{1} \epsilon(\mu)(\mu/5)(\mu/7)$$

Since, referring to the previous notation, $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 0$, 17.19.37 = 11951 ≡ 191 (mod 280), we have case b),1). Furthermore, for the prime factors of 11951, $\epsilon(\mu)$, $(\mu/5)\cdot(\mu/7)$ is seen by reference to 4), 6), and 7) to be + 1. Hence the same is true for all factors. In all, 11951 has eight factors: 1; 17; 19; 37; 17.19; 17.37; 19.37; 17.19.37. Thus, finally

$$N[23902 = 2x^2 + 35y^2] = 2.8 = 16,$$

to agree with the empirical result.

The form $7x^2 + xy + 7y^2$ has the discriminant −195 = −3.5.13, hence characters, (n/3), (n/5), (n/13). The number of representations function is given by Theorem 4.1. Referring to Appendix A to obtain the values of characters for this form, we obtain

$$N[2^{\lambda}\cdot 3^{\alpha_1}\cdot 5^{\alpha_2}\cdot 13^{\alpha_3} m = 7x^2 + xy + 7y^2, (m, 390) = 1]$$

$$= \frac{1}{8}\left[1 + (-1)^{\lambda}\right]\left[1 + (-1)^{\alpha_1+\alpha_2}(m/3)\right]\left[1 - (-1)^{\alpha_1+\alpha_3}(m/5)\right]$$

$$\left[1 - (-1)^{\alpha_2+\alpha_3}(m/13)\right]\sum_{\mu/m}{}' (-195/\mu)$$

Using the same table as above relating the parity of $\alpha_1, \alpha_2, \alpha_3$ to that of $\alpha_1+\alpha_2, \alpha_1+\alpha_3, \alpha_2+\alpha_3$, we see that when λ is even

$$N[2^{\lambda}\cdot 3^{\alpha_1}\cdot 5^{\alpha_2}\, 13^{\alpha_3} m = 7x^2 + xy + 7y^2, (m, 390) = 1]$$

$$= 2\sum_{\mu/m}{}' (-195/\mu)$$

in the cases,

	(m/3)	(m/5)	(m/13)
a)	1	1	−1
b)	−1	−1	−1
c)	−1	1	1
d)	1	−1	1

In all other cases, including λ odd, the number of representations is zero.

By the law of quadratic reciprocity, we have

$$(-195/\mu) = (-1)^{\frac{1}{2}(\mu-1)}(\mu/3)(\mu/5)(\mu/13)$$

In a similar manner as in our first example we may separate numbers, n, into classes modulo 4.195 according to the values taken by the set $(-1)^{\frac{1}{2}(n-1)}$, (n/3), (n/5), (n/13), so as to first test by the above whether the number may be represented at all. Then by combining into two classes according as (−195/n) is + 1 or − 1, $\sum_{\mu/m}{}' (-195/\mu)$ is obtained as the excess of the divisors of m in the first classes over those in the second. Further details are routine and are omitted.