Applications of Combinatorial Analysis to the Calculation of the Partition Function of the Ising Model

Thesis by

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Abstract

The research work discussed in this thesis investigated the application of combinatorics and graph theory in the analysis of the partition function of the Ising Model.

Chapter 1 gives a general introduction to the partition function of the Ising Model and the Feynman Identity in the language of graph theory.

Chapter 2 describes and proves combinatorially the Feynman Identity in the special case when there is only one vertex and multiple loops.

Chapter 3 digresses into the number of cycles in a directed graph, along with its application in the special case to derive the analytical expression of the number of non-periodic cycles with positive and negative signs.

Chapter 4 comes back to the general case of the Feynman Identity. The Feynman Identity is applied to several special cases of the graph and a combinatorial identity is established for each case.

Chapter 5 concludes the thesis by summarizing the main ideas in each chapter.

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Chapter 1

Introduction

1.1 Review of the Ising Model and its partition function

Graph theory has a natural application in the calculation of the partition function of the Ising model. We will review the partition function of the two-dimensional planar Ising model and the key step in the calculation, the Feynman Identity.

The two-dimensional planar Ising model is defined on a square lattice Λ. Each site i of Λ contains

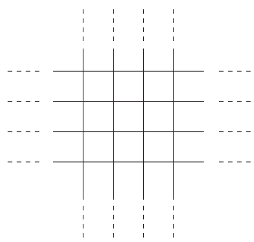

Figure 1.1: Ising Model

a 'particle' with two possible states. A generic example of this model is as follows: Consider a solid consisting of N identical atoms arranged in the regular lattice defined above. Each atom has a net electronic spin $\vec{S}$ and associated magnetic moment $\vec{\mu} = g\mu_0\vec{S}$. In the presence of an external applied magnetic field H_z along the z direction, the Hamiltonian $\mathcal{H}$ representing the interaction of the atoms with this field is then

$$\mathcal{H} = -g\mu_0 \sum_{j=1}^{N} \vec{S}_j \cdot H_z. \tag{1.1}$$

In addition, each atom is assumed to interact with neighboring atoms, but this is not just the magnetic dipole-dipole interaction. This interaction is in general too small to produce ferromagnetism. The dominant interaction is usually the exchange interaction, a consequence of the Pauli exclusion principle.

We will focus on the spin-1/2 situation and adapt the standard notation σ_i, instead of $\vec{S}_i$, to specify the spin state at site i. The interaction energy between two particles at the site i and j, each in the state σ_i and σ_j respectively, is described by

$$E_{ij} = \begin{cases} -J\sigma_i\sigma_j & \textit{if } i,\ j \textit{ are nearest neighbors(n.n.)} \\ 0 & \textit{otherwise} \end{cases} \tag{1.2}$$

where J is a parameter that describes the strength of the interaction and it tends to fall off rapidly with increasing separation between two particles. Therefore, we consider only nearest neighbor interaction. When J is greater than zero, the state of lower energy will be the one which favors parallel spin orientation, i.e., one tends to produce ferromagnetism.

Suppose Λ has N^2 sites, then there are 2^{N^2} distinct configurations of the spins. Call $S = \{\sigma\}$ the set of all configurations of the system. The energy E_σ of each configuration $\sigma \in S$ is given by

$$E_\sigma = -J \sum_{n.n.\in\sigma} \sigma_i\sigma_j \tag{1.3}$$

and the partition function is

$$Z = \sum_{\sigma\in S} e^{-\beta E_\sigma} \tag{1.4}$$

Since we will formulate everything in the language of graph theory, let's call the site *vertex* from here on. We need two definitions before we state the main theorem.

Definition 1: An even graph G is a graph whose vertex have even number of degrees. A even subgraph H of a graph K is a subgraph of K and H is an even graph itself.

Definition 2: Given a graph G, define the single-variable polynomial related to G as

$$I_G(u) = u^{|E(G)|} \tag{1.5}$$

where $|E(G)|$ is the total number of edges in G.

1.2 The partition function as a summation over graphs

For completeness, we will include the relevant part in [6] to show that the partition function can be written as a summation over even subgraphs.

Theorem 1.2.1 *Let $\mathcal{A}$ be the set of all even subgraphs of Λ which contains N^2 vertex, then*

$$Z = 2^{N^2}(1-u^2)^{-N(N-1)} \sum_{G\in\mathcal{A}} I_G(u) \tag{1.6}$$

where $u = tanh(K)$ and $K = J\beta$.

Proof. Let's rewrite the partition function (1.4) as

$$Z = \sum_{\sigma_1=\pm1} \cdots \sum_{\sigma_N=\pm1} \prod_{n.n.} e^{K\sigma_i\sigma_j} \tag{1.7}$$

by substitution of (1.3) into (1.4), where $K = J\beta$.

Since $\sigma_i\sigma_j = \pm1$, it follows that

$$e^{K\sigma_i\sigma_j} = e^{\pm K} = \cosh K \pm \sinh K; \tag{1.8}$$

therefore,

$$\prod_{n.n.} e^{K\sigma_i\sigma_j} = \prod_{n.n.} \cosh K \pm \sinh K$$

$$= \cosh^B K \prod_{n.n.}(1 \pm \tanh K) = (1-u^2)^{\frac{-B}{2}} \prod_{n.n.}(1 + u\sigma_i\sigma_j)$$

where $B = 2N(N-1)$ is the number of bonds in Λ. The factor $(1-u^2)^{\frac{-B}{2}}$ comes from $1-\tanh^2 K = \cosh^{-2} K$; therefore, $\cosh^B K = (1-\tanh^2 K)^{\frac{-B}{2}}$.

To each pair (i, j) of nearest neighbors of Λ, there corresponds a term $u\sigma_i\sigma_j$ and an edge. Since the number of pairs (i, j) of nearest neighbors coincides with the number of bonds B of Λ, the product on the right-hand side is a polynomial in u of degree B, that is ,

$$\prod_{n.n.}(1+u\sigma_i\sigma_j) = 1 + \sum_{p=1}^{B} u^p \sum_{n.n.}(\sigma_{i_1}\sigma_{i_2})\ldots(\sigma_{i_{2p-1}}\sigma_{i_{2p}}). \tag{1.9}$$

The second summation, i.e., the summation over n.n., is over all possible products of p pairs of nearest neighbors of Λ where a pair is not to occur twice in the same product. Each pair (σ_i, σ_j) is associated with an edge connecting vertices i and j; so each product of p pairs corresponds to a graph with p edges. We see that the second summation is over all graphs with p edges. The graph may have vertices of degree $0, 1, 2, 3$, or 4, in the situation of the planar lattice. The summations over the spins, i.e., σ_i's, eliminates graphs containing any odd-degree vertex, because $\sum \sigma_i = 0$ and $\sum \sigma_i^3 = 0$. The graphs left are those whose vertices all have even degrees. There is a factor of 2^{N^2} because each vertex of Λ contributes a factor of 2 from $\sum \sigma_i^{(\text{even degree})} = 2$, and the summations include all the σ_i's. Notice that in the proof, there is no need of planarity. □

We will get rid of the constant in front of the summation and call the summation as the partition function for simplicity.

The even graph we have here is not the physical diagram of the lattice itself. It is a diagrammatic description of the proof above. For example, a vertex with degree 4 doesn't mean that its corresponding site has a spin that is parallel to its four neighbors, but that it has appeared four times in the product.

1.3 The partition function as a product over cycles: The Feynman Identity

In order to facilitate the calculation of the partition function and therefore predict the thermal properties of the lattice, it is more convenient to write it in a product form. Feynman conjectured the following identity for a general graph embedded in a plane:

$$\sum_{G \in \mathcal{A}} I_G(u) = \prod_{[c] \in \mathbb{P}} (1 + J_c(u)) \tag{1.10}$$

where $\sigma(c)$ is the sign of the cycle $[c]$, defined in Chapter 2, $\mathbb{P}$ is the set of non-periodic reduced cycles, defined in Chapter 3, and $J_c(u) = \sigma(c) I_c(u)$.

There have been a few articles on the Feynman identity [1][7] and its proof. The special case when there is only one vertex first appeared in Sherman[2], but there is no rigorous proof of the identity in that paper. In the special case, one can interpret the loops as *generators* and the cycles as *word*, as will be shown in the next chapter. Therefore, we can assign combinatorial meanings to both sides of the Feynman identity, and by double counting, the proof is done in a clear and short way.

Chapter 2

A Combinatorial Proof of a Special Case of the Feynman Identity

2.1 Introduction

The identity stated as Theorem 2.1.1 below was given by S. Sherman [2] as a special case of the Feynman identity. This latter result has an involved geometric/topological proof. Sherman remarks in [2] that “a strictly algebraic and non-geometric proof of [this theorem] would be in order... Such an algebraic proof might shed light on the Ising problem in three dimensions.”

Let F_k be the free group on k generators $x_1, \ldots, x_k$. A *word* $w = a_1 a_2 \cdots a_n$ in F_k, where each *symbol* a_i is one of the generators $x_1, \ldots, x_k$ or their inverses $x_1^{-1}, \ldots, x_k^{-1}$, is *circularly-reduced*, or *c-reduced*, when $a_{i+1} \neq a_i^{-1}$ for $i = 1, 2, \ldots, n-1$ and also $a_1 \neq a_n^{-1}$. It will be convenient to exclude the identity in F_k as a c-reduced word. From here on, all words will be c-reduced, so we may skip the word when appropriate.

Let ρ be the permutation of the c-reduced words that takes $a_1 a_2 \cdots a_n$ to $a_n a_1 a_2 \cdots a_{n-1}$. The images of powers of ρ on a word w are the *rotations* of w. A *circular-word* $[w]$ is the set of all rotations of a given c-reduced word, i.e. an orbit of the cyclic group $\langle \rho \rangle$ generated by ρ on the set of c-reduced words in F_k. We remark that the circular-words are in one-to-one correspondence with the nontrivial conjugacy classes in F_k (every conjugacy class contains c-reduced words; two c-reduced word are conjugates if and only if one is a rotation of the other). As we switch to the formulation of a planar drawing of an oriented cotree in section 2.3, there is a natural correspondence between these two languages: each generator corresponds to a loop in the special case, and each cycle corresponds to a circular word, with two orientations considered as different.

Sherman defines a *sign* $\sigma(w) \in \{1, -1\}$ for every c-reduced word w. The definition will depend on an orientation of a planar drawing of a cotree (a graph with one vertex) with k loops labeled by the generators x_i. The definition of sign is geometric (involving paths and angles) and we must deal with that. For the momment, we note only that the signs of a word and any of its rotations are the same, so that σ is well-defined on circular-words.

The *period* of a word w (or circular-word $[w]$) is the least positive integer t so that $\rho^t(w) = w$. A word w or circular-word $[w]$ of length n is said to be *non-periodic* when the period of w is n.

Given $\epsilon = \pm 1$, and nonnegative integers $n_1, n_2, \ldots, n_k$, let $N(n_1, \ldots, n_k; \epsilon)$ denote the number of non-periodic c-reduced circular-words $[w]$ of length $n = n_1 + \cdots + n_k$ with $\sigma(w) = \epsilon$, and which contain a total of n_i of the symbols x_i and x_i^{-1} for each $i = 1, 2, \ldots, k$.

Theorem 2.1.1 *In the ring of formal power series in k (commuting) indeterminates $Z_1, Z_2, \ldots, Z_k$, we have*

$$\prod_{n_1,\cdots,n_k \geq 0} (1 + Z_1^{n_1} \cdots Z_k^{n_k})^{N(n_1,\cdots,n_k;+)}(1 - Z_1^{n_1} \cdots Z_k^{n_k})^{N(n_1,\cdots,n_k;-)} = \prod_{j=1}^{k} (1 + Z_j)^2 \tag{2.1}$$

where the product on the left is extended over all k-tuples $(n_1, \ldots, n_k)$ of nonnegative integers other than $(0, 0, \ldots, 0)$.

The proof of Theorem 2.1.1 is given in Section 2.2. In Section 2.3 we derive the algebraic properties of σ we use in our proof, and in Section 2.4 we observe a further property that allows us to compute the sign of a c-reduced word combinatorially from the word and the cotree without any reference to paths and angles as in the original definition.

The *Witt Identity* [3] concerns the numbers $M(n_1, \ldots, n_k)$ of non-periodic circular-words in the free semigroup generated by $x_1, \ldots, x_k$, with n_i occurrence of x_i for each $i = 1, \ldots, k$. Following the notation of Theorem 2.1.1,

$$\prod_{n_1,\cdots,n_k \geq 0} (1 - Z_1^{n_1} \cdots Z_k^{n_k})^{M(n_1,\cdots,n_k)} = 1 - Z_1 - Z_2 - \cdots - Z_k. \tag{2.2}$$

It is in view of (2.2) that (2.1) is called *an Analogue of the Witt Identity* in [2].

2.2 Proof of the identity

Properties of σ that we require for the proof are

$$\sigma(x_i) = \sigma(x_i^{-1}) = +1 \qquad \text{for every generator } x_i, \tag{2.3}$$

and

$$\sigma(u^d) = (-1)^{d-1}(\sigma(u))^d \qquad \text{for every c-reduced word } u. \tag{2.4}$$

For $\epsilon = \pm 1$ and nonegative integers $n_1, \ldots, n_k$, let $W(n_1, \ldots, n_k; \epsilon)$ denote the number of c-reduced words (not circular-words) that have sign ϵ and a total of n_i occurrences of x_i or x_i^{-1}, $i = 1, 2, \ldots, k$. We will need to know that given $n_1, n_2, \ldots, n_k$, at least *two* of which are positive, we have

$$W(n_1, n_2, \ldots, n_k; +) = W(n_1, n_2, \ldots, n_k; -). \tag{2.5}$$

Proofs of (2.3), (2.4), and (2.5) will be given in Section 2.3. Assuming these properties, the proof of Theorem 2.1.1 is purely combinatorial.

We first give a relation between the numbers $W(n_1, n_2, \ldots, n_k; \pm)$ and $N(n_1, n_2, \ldots, n_k; \pm)$.

Lemma 2.2.1 *Given $n_1, n_2, \ldots, n_k$, not all zero, let g be the greatest common divisor of $n_1, n_2, \ldots, n_k$ and write $n = n_1 + \cdots + n_k$. Then*

$$W(n_1, n_2, \ldots, n_k; +) = \sum_{d|g,\ d\ odd} \frac{n}{d} N(\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k}{d}; +), \tag{2.6}$$

and

$$\begin{aligned} W(n_1, n_2, \ldots, n_k; -) &= \sum_{d|g,\ d\ even} \frac{n}{d} N(\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k}{d}; +) \\ &+ \sum_{d|g} \frac{n}{d} N(\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k}{d}; -). \end{aligned} \tag{2.7}$$

Proof. The period of a c-reduced word with n_i occurrences of x_i and x_i^{-1} will be of the form n/d where d divides g. Given a divisor d of g, the c-reduced words with n_i occurrences of x_i and x_i^{-1}, and period n/d, may be obtained as follows: Start with a representative u of a non-periodic c-reduced circular-word $[u]$ of length n/d with n_i/d occurrences of x_i and x_i^{-1} and repeat it d times to get a c-reduced word $w = u^d$. Associate to $[u]$ the distinct rotations of w, namely the first n/d rotations $w, \rho(w), \ldots, \rho^{n/d-1}(w)$.

As d ranges over divisors of g, we obtain every c-reduced word w with n_i occurrences of x_i and x_i^{-1} exactly once in this way. By (2.4), the sign $\sigma(w)$ of $w = u^d$ is $(-1)^{d-1}(\sigma(u))^d$. If d is even, $\sigma(w) = -1$. If d is odd, $\sigma(w) = \sigma(u)$. Therefore, every word $[u]$ in $N(\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k}{d}; +)$ with d odd contribute n/d words in $W(n_1, n_2, \ldots, n_k; +)$, and the rest contribute to $W(n_1, n_2, \ldots, n_k; -)$. $\square$

Proof of Theorem 2.1.1. It will suffice to prove that the formal logarithms of both sides of (2.1) are equal. The logarithm of the right-hand side is

$$2\sum_{j=1}^{k}\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{Z_j^\ell}{\ell} \tag{2.8}$$

and the logarithm of the left-hand side is

$$\begin{aligned}\sum_{n_1,\cdots,n_k\geq 0} \quad & N(n_1,\cdots,n_k;+)\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{(Z_1^{n_1}\cdots Z_k^{n_k})^\ell}{\ell} \\ & - \sum_{n_1,\cdots,n_k\geq 0} N(n_1,\cdots,n_k;-)\sum_{\ell=1}^{\infty}\frac{(Z_1^{n_1}\cdots Z_k^{n_k})^\ell}{\ell}.\end{aligned} \tag{2.9}$$

Given $m_1, \ldots, m_k$, let g be the greatest common divisor of $m_1, \ldots, m_k$ and write $m = m_1 + \cdots + m_k$. Then the coefficient of a monomial $Z_1^{m_1} Z_2^{m_2} \cdots Z_k^{m_k}$ in (2.9) is

$$\sum_{d|g}(-1)^{d-1}\frac{1}{d}N(m_1/d, \ldots, m_k/d; +) \quad - \quad \sum_{d|g}\frac{1}{d}N(m_1/d, \ldots, m_k/d; -) \tag{2.10}$$

By Lemma 2.2.1, this is equal to

$$\frac{1}{m}\Big(W(m_1, m_2, \ldots, m_k; +) - W(m_1, m_2, \ldots, m_k; -)\Big). \tag{2.11}$$

By (2.5), this is 0 in the case that at least two of the m_i's are positive; that is, the coefficient of $Z_1^{m_1} \cdots Z_k^{m_k}$ in (2.9) and (2.8) are the same in this case.

Now consider the coefficient of $Z_j^{m_j}$ in (2.9), $m_j > 0$. There are only two c-reduced words with m_j occurences of x_j or x_j^{-1} (and all other $m_i = 0$), namely $x_j^{m_j}$ and $x_j^{-m_j}$. By (2.3) and (2.4), both of these have sign $(-1)^{m_j-1}$. Then the quantity in (2.11) is equal to $2(-1)^{m_j-1}/m_j$, which matches the coefficient of $Z_j^{m_j}$ in (2.8).

2.3 Definition and properties of the sign function

To define the sign function σ as in Sherman [2], we start with any planar drawing of an oriented cotree, a graph with one vertex v and k loops corresponding to the k pairs of generators and their inverse $x_1^{\pm 1}, \ldots, x_k^{\pm 1}$. See Figure 2.1 as an example of cotree with eight loops. To each generator x_i can be associated the closed path p_i that traverses the corresponding loop, starting and terminating at v, following the oriention of the loop; and x_i^{-1} can be associated with the reverse path p_i^{-1}. For a c-reduced word $w = x_{i_1}^{e_1} \cdots x_{i_\ell}^{e_\ell}$, the *sign* $\sigma(w)$ is defined as $(-1)^{r-1}$ where r is the number of revolutions made by the tangent vector to the associated path $p_{i_1}^{e_1} \cdots p_{i_\ell}^{e_\ell}$ as the path is traversed. Note that different drawings may give different sign functions.

For the definition of sign, it is convenient to assume the drawing of the graph is given so that

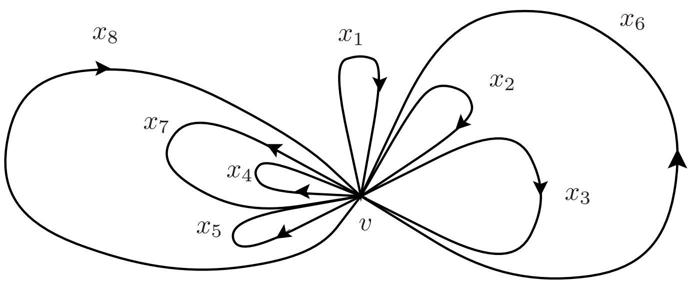

Figure 2.1: Cotree with eight loops

the loops are represented by smooth closed curves with tangent vectors that vary continuously over 'time' as the loop is traversed, except perhaps at v, but where there is a limiting value at the beginning and end of each loop. If t_1 (a unit vector in the plane) is the value of the tangent vector as the path nears the end of one loop and t_2 is the value of the tangent vector as the path begins another loop, we must imagine a short time in which the tangent vector rotates from t_1 to t_2 either clockwise or counter-clockwise, *whichever is the least motion*, i.e. whichever motion goes through an angle of less than 180°. We may assume that the graph is drawn so that $t_1 = -t_2$ never arises (which case would arise, e.g., if the word were not c-reduced).

It is clear that the sign of a rotation of a c-reduced word is the same as the sign of the original word. It is also clear that $\sigma(x_i) = \sigma(x_i^{-1}) = +1$ for any generator x_i. If the number of revolutions of the tangent vector made by traversing a path q associated with a c-reduced word u is r, then the number of revolutions of the tangent vector made by traversing q^d is dr, and so

$$\sigma(u^d) = (-1)^{dr-1} = (-1)^{d-1}(-1)^{d(r-1)} = (-1)^{d-1}(\sigma(u))^d.$$

This establishes (2.3) and (2.4).

Given a word w, let $L(w)$ be the induced planar drawing of the set of loops corresponding to the symbols in w. A loop in $L(w)$ is *extremal in* $L(w)$ when it either contains no other loops in $L(w)$, or contains all other loops in $L(w)$. A symbol a, which is one of the generators x_i or x_i^{-1}, is *extremal in* w when it occurs in w and its corresponding loop is extremal in $L(w)$. For example, if the drawing is as described in Figure 2.1 and $w = x_1^2 x_8^{-3} x_6 x_8 x_2$, then the extremal symbols in w are x_1, x_2, x_8, x_8^{-1}.

Lemma 2.3.1 *Let a be a symbol in a c-reduced word w, say $w = ua^k v$, where $k \neq 0$ and the symbols on either side of a in the cyclic order are different from a. (That is, u does not terminate with a, and v does not terminate with a when u is empty. Similarly, v does not begin with a and u does not begin with a if v is empty.) If a is an extremal symbol in w, then*

$$\sigma(ua^{-k}v) = -\sigma(ua^k v).$$

Proof. First consider the case when $k = 1$. The diagrams in Figure 2.2, where we have suppressed the vertex v and made the paths continuous, are suggestive. It seems clear the trace of the tangent vector around one of the paths has one more revolution than the other, so that the signs will differ. But we should be careful, for example in the case $c = b^{-1}$ below.

To be more precise, let b be the symbol that appears before a (at the end of u, if u is nonempty)

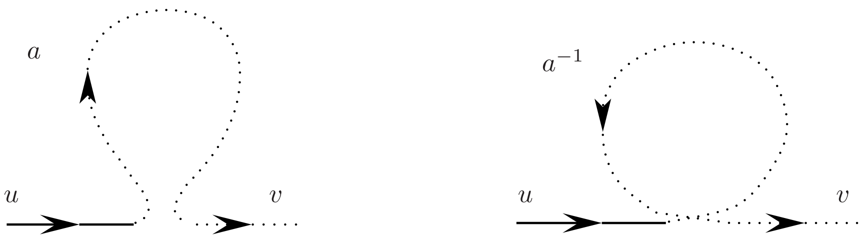

Figure 2.2: Illustrative graph for the change of revolution number

and c the symbol that appears just after a (at the beginning of v, if v is nonempty). It is important that the loops corresponding to b and c are both outside or both inside the loop corresponding to a, and this is ensured by our choice of a as extremal in w. We illustrate the case when they are both outside. Label the angles α, β, γ, and δ as shown in Figure 2.3, all taken clockwise as positive. We allow $c = b^{-1}$, in which case $\delta = 0$.

The difference of the number of revolutions between uav and $ua^{-1}v$ is

$$\{\beta + \gamma - (\pi - \alpha)\}/2\pi - \{-\alpha - \beta - \delta + (\pi - \gamma - \delta)\}2\pi = 1$$

Figure 2.3 illustrates the case when all angles are smaller than π, but the calculation holds when one of them is greater than π. For example, when δ is greater than π, it would be more intuitive to write the difference as

$$\{-(\delta - \pi)\}/2\pi - \{-2\pi - (\delta - \pi)\}/2\pi$$

but since $\alpha + \beta + \delta + \gamma = 2\pi$, the result is the same. So the signs are different when $k = 1$.

It is easy to see that $\sigma(ua^k v) = (-1)^{k-1}\sigma(uav)$ and $\sigma(ua^{-k}v) = (-1)^{k-1}\sigma(ua^{-1}v)$, for positive integers k. Thus the theorem holds for any k. □

Proof of (2.5). Fix $n_1, \ldots, n_k$ with at least *two* of the n_i's positive; let $n = n_1 + \cdots + n_k$. Let S be the set of c-reduced words w (not circular-words) in the free group generated by $x_1, \ldots, x_k$ with n_i occurrences of x_i or x_i^{-1}, $i = 1, 2, \ldots, k$. We will define a (fixed-point-free) involutory mapping

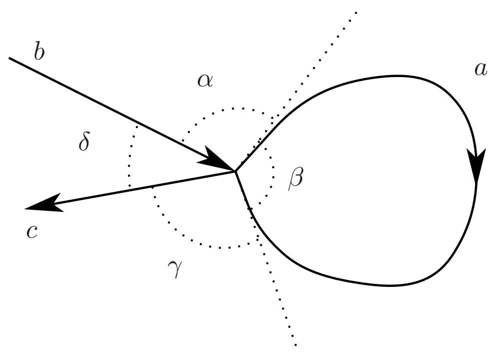

Figure 2.3: Illustrative graph for calculation of change of angles

$\phi : S \to S$, so that $\sigma(\phi(w)) = -\sigma(w)$ for all $w \in S$.

Given a word c-reduced word $w \in S$, let $a = x_i^{\pm 1}$ be an extremal symbol in w whose subscript i is the least so that the corresponding loop in extremal in $L(w)$. Find the first occurrence of a term a^k in w and change it to a^{-k}, and also change the exponent of any term a^ℓ at the end of w in the case that w begins with a, to get $\phi(w)$. For example, suppose the loops corresponding to $x_1^{\pm 1}$ and $x_2^{\pm 1}$ are the only extremal loops in $L(w)$. Then

$$\phi(x_3^4\, x_2^{-2}\, x_1^{-7}\, x_4) = x_3^4\, x_2^{-2}\, x_1^7\, x_4 \quad \text{and} \quad \phi(x_1^4\, x_2^{-2}\, x_1^7\, x_4\, x_1^5) = x_1^{-4}\, x_2^{-2}\, x_1^7\, x_4\, x_1^{-5}.$$

It is clear that ϕ^2 is the identity and by Lemma 2.3.1, we have $\sigma(\phi(w)) = -\sigma(w)$. □

Remark. Even though the sign of a word depends on the particular drawing, both $N(n_1, \cdots, n_k; +)$ and $N(n_1, \cdots, n_k; -)$ are independent of the drawing. Of course $W(n_1, \cdots, n_k;) = W(n_1, \cdots, n_k; +) + W(n_1, \cdots, n_k; -)$ is independent of the drawing, and Lemma 2.3.1 shows that both $W(n_1, \cdots, n_k; +)$ and $W(n_1, \cdots, n_k; -)$ are independent of the drawing. Then by Lemma 2.2.1 and Möbius inversion,

$$nN(n_1, n_2, \ldots, n_k; +) = \sum_{d|g,\ d \text{ odd}} \mu(d)\, W(\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k}{d}; +), \tag{2.12}$$

$$nN(n_1, n_2, \ldots, n_k; -) = \sum_{d|g,\ d \text{ odd}} \mu(d)\, W(\frac{n_1}{d}, \frac{n_2}{d}, \ldots, \frac{n_k}{d}; -) \quad - \quad T, \tag{2.13}$$

where $T = 0$ when g is odd and $T = W(\frac{n_1}{2}, \frac{n_2}{2}, \ldots, \frac{n_k}{2}; -)$ when g is even. Here $\mu(d)$ is the number-theoretic Möbius function. Therefore, $N(n_1, \cdots, n_k; +)$ and $N(n_1, \cdots, n_k; -)$ are independent of the drawing.

2.4 More on the sign function

The following lemma provides a combinatorial method for computing the sign of a word, with respect to any drawing of the cotree. Let w_1 and w_2 be c-reduced words. For $i = 1, 2$, let out_i be a directed line segment starting at v and in the direction of the tangent to the path coresponding to w_i as it leaves v. For $i = 1, 2$, let in_i be a directed line segment terminating at v and in the direction of the tangent to the path corresponding to w_i as it returns to v. There will be a circular order on the four line segments at v. We say that w_1 and w_2 *interlace* when in_1 and in_2 separate out_1 and out_2 in this order. Figure 2.4 shows one type of interlacing.

Lemma 2.4.1 *If w_1 and w_2 are c-reduced words so that $w_1 w_2$ is also c-reduced, then, with the above notation,*

$$\sigma(w_1 w_2) = +\sigma(w_1)\sigma(w_2)$$

if w_1 and w_2 interlace, and

$$\sigma(w_1 w_2) = -\sigma(w_1)\sigma(w_2)$$

otherwise.

Proof. First suppose that w_1 and w_2 interlace and let α_1, α_2, β_1, and β_2 be the angles as indicated in Figure 2.4, as one of the possible situations. Others are similar and omitted. Let the number of revolutions of w_i be r_i, and we separate it into two parts: $2\pi r_i = 2\pi r_i^{'} + \alpha_i$. If r_{12} is the number of revolutions of $w_1 w_2$, start from out_1, then $2\pi r_{12} = 2\pi r_1^{'} + \beta_1 + 2\pi r_2^{'} + \beta_2$ Since $\alpha_1 + \alpha_2 - \beta_1 - \beta_2 = 2\pi$, we get $r_{12} = r_1 + r_2 - 1$, which will yield the correct sign rule. Had we taken clockwise as positive direction, then all the four angles will change sign, but the final result is not affected at all.

In the case when in_1 is at the "bottom" and in_2 is at the "top", then the calculation would be the same if α_1 is still defined as the angle from in_1 to out_1, β_1 the angle from in_1 to out_2, and similar for α_2 and β_2. When w_1 and w_2 do not interlace, the proof is similar.

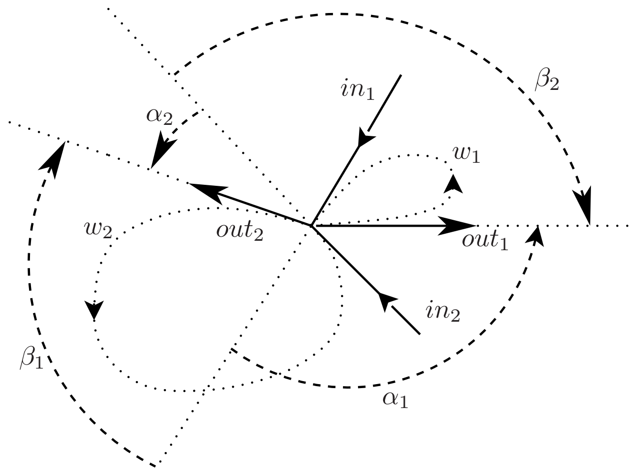

Figure 2.4: Interlacing graph

□

Here is an example of the computation of $\sigma(x_1x_2x_3x_4x_5x_6x_7x_8)$ when the cotree is as in Figure 2.1:

$$\begin{aligned}\sigma(x_1x_2x_3x_4x_5x_6x_7x_8) &= (-)\sigma(x_1x_2x_3x_4x_5x_6x_7)\sigma(x_8)\\ &= (-)(-)\sigma(x_1x_2x_3x_4x_5x_6)\sigma(x_7)\sigma(x_8)\\ &= \cdots = (-)(-)(+)(-)(+)(+)(-) = +.\end{aligned}$$

It is easy to see that every c-reduced word, involving at least two generators, can be written as the product of two c-reduced words, so Lemma 2.4.1, used successively, allows the computation of the sign of any word with respect to any drawing.

In case that the cotree is the "daisy", as the example shown in Figure 2.5 below, in the case of five loops, da Costa and Variane[5] have given an explicit rule to compute the sign of a c-reduced word, which we restate here as Theorem 2.4.2. We give a quick proof based on Lemma 2.4.1.

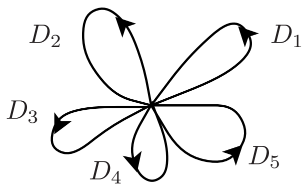

Figure 2.5: Daisy graph

Associated with each word $w = D_{i_1}^{e_{i_1}} D_{i_2}^{e_{i_2}} \cdots D_{i_l}^{e_{i_l}}$ a sequence $S_l = (i_1, i_2, \cdots, i_l)$, where D_{i_j}'s are symbols. Let s be the number of negative exponents e_{i_j}'s. The sequence is such that a loop i_j appears at least once in S, for all $j = 1, 2, \cdots, l$, and $i_k \neq i_{k+1}$ and $i_l \neq i_1$. Decompose S into T *subsequences* such that, each subsequence is an ordered set of numbers formed in the following way: (a) if p, q are two elements inside the subsequence and q comes after p, then $q > p$; (b) a new subsequence begins whenever this ordering is broken in the sequence by two adjacent elements not satisfying (a).

For instance, a sequence with three loops and $l = 14$ is decomposed as follows, with $T = 7$:

$$S_{14} = (12123213232312) = (12)(123)(2)(13)(23)(23)(12).$$

Theorem 2.4.2 (da Costa and Variane) *Let N be the total number of symbols in w, then the sign of w is $\sigma(w) = (-1)^{N+l+s+T+1}$.*

Proof. Recall that $\sigma(D_{i_1}^{e_{i_1}}) = (-1)^{e_{i_1}+1}$, where $D_j's$ are the counterclockwise loops as in Figure 2.5, we calculate the sign as we did in the previous example, and use induction on s and T. When $s = 0$ and $T = 1$, all the pairs in the calculation interlace, so

$$\sigma(D_{i_1}^{e_{i_1}} D_{i_2}^{e_{i_2}} \cdots D_{i_l}^{e_{i_l}}) = (+)^{l-1}\sigma(D_{i_1}^{e_{i_1}}) \cdots \sigma(D_{i_l}^{e_{i_l}}) = (-1)^{N+l}.$$

When $s = 0$ and there is one increment of T in the word, the sign will gain one extra factor of (-1) from the non-interlacing case of Lemma 2.4.1, which yields the same sign rule as [5]. For each increment of s, which implies one of the loops is traversed in the opposite direction, then the interlacing will change to non-interlacing, or vise versa. □

Chapter 3

The Number of Non-periodic Reduced Cycles in a Directed Graph

3.1 Outline

In this chapter, we will switch to cycles in a general directed graphs. By combining with the result in Chapter 2 and adapting the theorem in this chapter to the special case when there is only one vertex, a method for deriving the analytic formula of the number of non-periodic cycles with positive and negative signs is developed. If there are only two loops adjacent to the vertex, the formula first appeared in Costa [4]. We use our method and derive the same expression as well in Appendix B.

3.2 The number of non-periodic cycles

Given a directed graph D which may contain multiple edges and loops, let $E(D)$ be the set of all edges of D. The *origin* of an edge e is the starting vertex and the *end* of e is the ending vertex. The inverse $Inv(e)$ of e is the set of edges whose origin and end is the end and origin of e, respectively. If e is a loop, then $e \in Inv(e)$. The *successor* $E(e)$ of e is the set of edges whose origin is the end of e. If e is a loop, then $Inv(e) \subset E(e)$. The *proper successor* $P(e)$ of e is $E(e) \backslash Inv(e)$ when e is not a loop. When e is a loop, then $P(e)$ is the same as $E(e)$.

A *pointed cycle* p is a closed path in D with a specified starting edge, and a *cycle* $[c]$ is the equivalent class of cyclic permutations of c. A (pointed) cycle is called *reduced* if there is no immediate reversal as it traverses through an edge that is not a loop. Since we consider the inverse of a loop is the same as itself, it is ok that the reduced cycle traverses through a loop multiple times. Only reduced cycles are considered, so we may omit the word when appropriate.

Let ρ be the cyclic permutation of the pointed cycle $c = e_1 e_2 ... e_k$, i.e. $\rho(c) = e_k e_1 ... e_{k-1}$, then the *period* of a cycle $[c]$ is the least positive integer t so that $\rho^t(c) = c$. A cycle $[c]$ of length M is said to be *non-periodic* when the period of $[c]$ is M. A ℓ-cycle is a cycle of total length ℓ, i.e., the number of edges is ℓ.

Lemma 3.2.1 *Let T be a $|E(D)|$ by $|E(D)|$ square matrix whose entry $t(e_i, e_j)$ is 1 when e_j is in the proper successor $P(e_i)$ and zero otherwise. The diagonal entry $t(e_i, e_i)$ is 1 when e_i is a loop* *and* 0 *otherwise. Let* $N_D(\ell)$ *be the number of non-periodic reduced* ℓ*-cycles in* D*. Then*

$$\sum_{\ell|M} \ell N_D(\ell) = Tr(T^M). \tag{3.1}$$

Proof. To prove the lemma, we count the number of reduced *pointed* M-cycles in two different ways. First, given a divisor ℓ of M, a reduced pointed M-cycle w with period ℓ may be obtained as follows: Start with a representative c of a non-periodic reduced cycle $[c]$ of length ℓ and repeat it M/ℓ times to get a reduced pointed cycle $w = c^{M/\ell}$. Note that the first ℓ cyclic permutations $w, \rho(W), ..., \rho^{M/\ell-1}(w)$ are different pointed cycles. By the definition of $N_D(\ell)$, we conclude that the number of pointed M-cycles is $\sum_{\ell|M} \ell N_D(\ell)$. Here is the other way of counting. Let a and b be the edge of D. The $(a,b)^{th}$ entry of T^M is

$$T^M(a,b) = \sum_{e_{i_1}, e_{i_2}, ..., e_{i_{M-1}}} t(a, e_{i_1})t(e_{i_1}, e_{i_2})...t(e_{i_{M-1}}, b),$$

which counts the number of M-paths from the origin of edge a to the end of edge b. Taking the trace of T^M will ensure that only closed paths are counted and every starting edge is taken into account. Notice that a periodic cycle with period ℓ and total length M is also counted ℓ times in $\mathrm{Tr}(T^M)$ since , even though there are M entries, the number of different edges is ℓ. For example, if $M = 6$ and $\ell = 3$, then the 6-cycle, say $e_1e_2e_3e_1e_2e_3$, is counted once by each of $t^6(e_1, e_1)$, $t^6(e_2, e_2)$, and $t^6(e_3, e_3)$. □

Lemma 3.2.2 *Let* $\mu()$ *be the Mobius function. Then*

$$N_D(M) = \frac{1}{M} \sum_{\ell|M} \mu(\frac{M}{\ell}) Tr(T^\ell) \tag{3.2}$$

Proof. By the Mobius inversion formula, we can invert (3.1) to get

$$MN_D(M) = \sum_{\ell|M} \mu(\frac{M}{\ell}) Tr(T^\ell) \tag{3.3}$$

□

Theorem 3.2.3 *Define $\mathbb{P}$ as the set of non-periodic reduced cycles in a direct graph D and denote the length of its element $[c] \in \mathbb{P}$ by $|c|$. Then*

$$\prod_{[c]\in\mathbb{P}} (1 - u^{|c|}) = \det(I - uT) \tag{3.4}$$

Proof. It will suffice to prove that the formal logarithms of both sides of (3.4) are equal. The logarithm of the left-hand side is

$$\sum_{[c]\in\mathbb{P}} \sum_{\ell=1}^{\infty} \frac{u^{|c|\ell}}{\ell} = \sum_{M=1}^{\infty} \sum_{\ell|M} \frac{1}{\ell} N_D(\frac{M}{\ell}) u^M,$$

where $N_D(\frac{M}{\ell})$ has the same definition as above. The above equation comes from collecting terms that have the same power M.

The logarithm of the right-hand side is

$$\log\det(I - uT) = \mathrm{T}r \log(I - uT) = \mathrm{T}r \sum_{\ell=1}^{\infty} \frac{(uT)^\ell}{\ell} = \sum_{M=1}^{\infty} \frac{\mathrm{T}r(T^M)}{M} u^M.$$

Comparing the coefficient of u^M, we get the two sides of (3.1). □

3.3 The number of non-periodic cycles with specified occurrences of edges

Given a directed graph D which may contain multiple edges and loops, let $N_D(m_1, m_2, ..., m_{|E(D)|})$ denote the number of non-periodic reduced cycles of length $M = m_1 + m_2 + ... + m_{|E(D)|}$ which contain m_i of edge e_i for each $i = 1, 2, ..., |E(D)|$. In order to keep track of the edges, every edge e_i is assigned an indeterminate x_i. Define S as a square matrix whose entries $s(e_i, e_j)$ is $\sqrt{x_i x_j}$ when e_j is in the proper successor $P(e_i)$ of e_i and zero otherwise. For example, if D has only one vertex with two loops, then

$$S = \begin{pmatrix} x_1 & \sqrt{x_1 x_2} \\ \sqrt{x_1 x_2} & x_2 \end{pmatrix}.$$

Lemma 3.3.1 *Let $f(m_1, m_2, \ldots, m_k)$ be the coefficient of a monomial $x_1^{m_1} x_2^{m_2} \ldots x_k^{m_k}$ in $Tr(S^M)$, where $M = m_1 + m_2 + \cdots + m_k$, and g is the g.c.d. of $m_1, m_2, \ldots, m_k$, then*

$$\sum_{\ell | g} \frac{M}{\ell} N_D(\frac{m_1}{\ell}, \frac{m_2}{\ell}, \ldots, \frac{m_k}{\ell}) = f(m_1, m_2, \ldots, m_k). \tag{3.5}$$

Proof. We count the number of reduced pointed cycles with m_i occurrence of edge x_i in two ways. First given a divisor ℓ of g, a reduced pointed cycle w with period ℓ and m_i occurrence of x_i, for all i's, may be obtained as follows: Start with a representative c of a non-periodic reduced cycle $[c]$ of length M/ℓ and repeat it ℓ times to get a reduced pointed cycle $w = c^\ell$. By the definition of $N_D(\frac{m_1}{\ell}, \frac{m_2}{\ell}, \ldots, \frac{m_k}{\ell})$, we conclude that the number of pointed M-cycles with m_i occurrence of x_i is $\sum_{\ell | g} \frac{M}{\ell} N_D(\frac{m_1}{\ell}, \frac{m_2}{\ell}, \ldots, \frac{m_k}{\ell})$. Here is the other way of counting. In $Tr(S^M)$, the coefficient of a monomial $x_1^{m_1} x_2^{m_2} \ldots x_k^{m_k}$ is the number of pointed reduced cycles with m_i occurrence of x_i, for similar reasons as in Theorem 3.2.1. □

Theorem 3.3.2 *Let I_c be the product of x_i's that corresponds to the edges e_i traversed by the cycle $[c]$. Then*

$$\prod_{[c] \in \mathbb{P}} (1 - I_c) = \det(I - S). \tag{3.6}$$

Proof. This theorem is a generalization of Theorem 3.2.3 and the proof will follow a similar argument. The logarithm of the left-hand side is

$$\sum_{[c] \in \mathbb{P}} \sum_{\ell=1}^{\infty} \frac{I_p^\ell}{\ell} = \sum_{m_1, m_2, \ldots, m_k} \sum_{\ell | g} \frac{1}{\ell} N_D(\frac{m_1}{\ell}, \frac{m_2}{\ell}, \ldots, \frac{m_k}{\ell}) x_1^{m_1} x_2^{m_2} \ldots x_k^{m_k}$$

where g is the greatest common divisor of all m_i's. Multiplying by $M = m_1 + m_2 + \ldots + m_k$, we get the coefficient of $x_1^{m_1} x_2^{m_2} \ldots x_k^{m_k}$ equals the number of pointed M-cycles with $\frac{m_i}{\ell}$ occurrence of x_i for $i = 1, 2, \ldots, k$.

The logarithm of the right-hand side is

$$\log\det(I-S) = \mathrm{Tr}\log(I-S) = Tr\sum_{M=1}^{\infty}\frac{S^M}{M}.$$

After multiplying by M, the coefficient of $x_1^{m_1}x_2^{m_2}...x_k^{m_k}$ equals the number of pointed M-cycles. This is because taking the trace will ensure the path is closed and different starting positions are counted accordingly. □

3.4 Non-periodic cycles with positive and negative signs in the special case

The analytical expression of the number $N(m_1, m_2; \pm)$ of non-periodic cycles with signs in the special case of two loops is known[4], but there is no general expression when there are k loops, $k > 2$. Here we derive it as follows.

From Lemma 3.3.1, we can reverse (3.5) and get

$$N_D(m_1, m_2, ..., m_k) = \sum_{d|g}\frac{\mu(d)}{M}f(m_1/d, m_2/d, \ldots, m_k/d). \tag{3.7}$$

In order to apply this result to the special case as in Chapter 2, the directed graph now is a single vertex with loops embedded on a plane, and the inverse of the loop is always present and considered as different from itself due to the two orientations; therefore, it has to be excluded from the proper successor of itself. For example, the matrix S in the case of two loops, along with their inverse loops, is

$$S = \begin{pmatrix} x_1 & 0 & \sqrt{x_1x_2} & \sqrt{x_1x_2} \\ 0 & x_1 & \sqrt{x_1x_2} & \sqrt{x_1x_2} \\ \sqrt{x_1x_2} & \sqrt{x_1x_2} & x_2 & 0 \\ \sqrt{x_1x_2} & \sqrt{x_1x_2} & 0 & x_2 \end{pmatrix}. \tag{3.8}$$

We will use this to calculate $f(m_1/d, m_2/d)$ in Appendix A.

If we define $W_D(m_1, m_2, ..., m_k)$ as the number of pointed reduced M-cycles in D with m_i occurrence of e_i for $i = 1, 2, ..., k$ and $M = m_1 + m_2 + ...m_k$, then, by a similar argument as Lemma 2.2.1, we get

$$W_D(m_1, m_2, ..., m_k) = \sum_{\ell|g} \frac{M}{\ell} N_D(\frac{m_1}{\ell}, \frac{m_2}{\ell}, ..., \frac{m_k}{\ell}),$$

where $N_D(\frac{m_1}{\ell}, \frac{m_2}{\ell}, ..., \frac{m_k}{\ell})$ is calculated in (3.7).

The subscript D is to signify that $W_D(m_1, m_2, ..., m_k)$ applies to general directed graph. The function $W(m_1, m_2, ..., m_k; \pm)$, as defined in Chapter 2, applies to only the special case. Nonetheless, when we treat the special case as a directed graph and define

$$W(m_1, m_2, ..., m_k) = W(m_1, m_2, ..., m_k; +) + W(m_1, m_2, ..., m_k; -),$$

then $W(m_1, m_2, ..., m_k) = W_D(m_1, m_2, ..., m_k)$.

Put together,

$$\begin{aligned} W(m_1, m_2, ..., m_k) = & \quad W_D(m_1, m_2, ..., m_k) \\ = & \quad \sum_{\ell|g} \frac{M}{\ell} N_D(\frac{m_1}{\ell}, \frac{m_2}{\ell}, ..., \frac{m_k}{\ell}). \end{aligned}$$

Substitute (3.7) into the above expression to get

$$\sum_{\ell|g} \frac{M}{\ell} (\sum_{d|g'} \frac{\mu(d)}{M/\ell} f(\frac{m_1}{\ell d}, ..., \frac{m_k}{\ell d})) = \sum_{p|g} f(\frac{m_1}{p}, ..., \frac{m_k}{p}) \sum_{d'|p} \mu(d')$$

where g' is the greatest common divisor of m_i/ℓ. By using the property of the Mobiüs Function:

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \textit{if } n = 1 \\ 0 & \textit{otherwise} \end{cases}$$

we derive

$$W(m_1, m_2, ..., m_k) = f(m_1, ..., m_k). \tag{3.9}$$

The reason we introduce the function $W_D(m_1, m_2, ..., m_k)$ is to help us connect with the function $N(m_1, m_2, \ldots, m_k; \pm)$, which is defined in Chapter 2. Because

$$W(m_1, m_2, ..., m_k; +) = W(m_1, m_2, ..., m_k; -),$$

as proved in Chapter 2, we have

$$W(m_1, m_2, ..., m_k; +) = W(m_1, m_2, ..., m_k; -) = W(m_1, m_2, ..., m_k)/2. \tag{3.10}$$

If we replace $W(m_1, m_2, ..., m_k; \pm)$ in (2.12) and (2.13) by (3.10) and use $W(m_1, m_2, ..., m_k) = f(m_1, ..., m_k)$, we have

$$N(m_1, m_2, \ldots, m_k; +) = \sum_{d|g,\ d \text{ odd}} \frac{\mu(d)}{2M} f(m_1/d, m_2/d, \ldots, m_k/d), \tag{3.11}$$

and

$$N(m_1, m_2, \ldots, m_k; -) = \Big(\sum_{d|g,\ d \text{ odd}} \frac{\mu(d)}{2M} f(m_1/d, m_2/d, \ldots, m_k/d) \Big) \quad - \quad T \tag{3.12}$$

where $T = 0$ when g is odd and $T = W(\frac{m_1}{2}, \frac{m_2}{2}, \ldots, \frac{m_k}{2}; -)$ when g is even.

In principle, the function $f(m_1, ..., m_k)$ can be calculated from its definition; therefore, we can derive expressions for $N(m_1, m_2, ..., m_k; +)$ and $N(m_1, m_2, ..., m_k; -)$. We also show in Appendix A that (3.11) and (3.12) reduce to the expression in Costa [4] in the case of two loops.

Chapter 4

Theorems Derived from the Feynman Identity

4.1 Introduction

We will apply the Feynman Identity to certain graphs to derive combinatorial identities involving binomial numbers. A rigorous proof of each identity is provided separately.

The Feynman Identity can be presented as

$$\sum_{G\in\mathcal{A}} G(x_i) = \prod_{c\in\mathbb{P}}(1 + J_c(x_i)), \tag{4.1}$$

where x_i is the indeterminate corresponding to the edge e_i, $\mathcal{A}$ is the set of *even simple* subgraphs, and $G(x_i) = \prod_i x_i^{m_i}$ where m_i is the multiplicity of e_i in the subgraph G. Since G is a simple graph, m_i here equals either one or zero, but in general it is some non-negative number. For simplicity, G is used to denote both the subgraph in $\mathcal{A}$ and the corresponding monomial. On the right hand side, c is a class of non-periodic paths and $\mathbb{P}$ is the set of equivalent classes up to the "starting point" and orientation. For example, a C_4 is a path that has four starting point and after accounting for orientations, there are eight paths and they belong to the same class. $J_c(x_i) = \sigma(c)\prod_i x_i^{m_i}$ where m_i is the multiplicity of e_i in c and $\sigma(c)$ is the sign of c as defined in Chapter 2.

For future reference, we will call the left hand side of (4.1) *the graph side* and the right hand side *the path side*.

4.2 An alternative form of the Feynman Identity

The Feynman Identity can be rewritten by taking logarithm on both sides of (4.1) and comparing the coefficient of the monomial $\prod_i x_i^{m_i}$ where i runs over the non-zero m_i's. After taking logarithms, the coefficient on the path side is again

$$\frac{1}{M}(W(m_1, m_2, ..., m_k; +) - W(m_1, m_2, ..., m_k; -)),$$

as in the special case in Chapter 2. This is because the relation between $N(m_1, ..., m_k; \pm)$ and $W(m_1, ..., m_k, \pm)$ remains the same in the general case and the definition of the sign function is valid as long as the graph is embedded in a *two-dimensional orientable manifold*. The logarithm on the graph side, after Taylor expansion, is

$$\Big(\sum_{\mathcal{A}\backslash\emptyset} G(x_i)\Big) - \frac{1}{2}\Big(\sum_{\mathcal{A}\backslash\emptyset} G(x_i)\Big)^2 + \frac{1}{3}\Big(\sum_{\mathcal{A}\backslash\emptyset} G(x_i)\Big)^3 - ... = \sum_{B=1}^{\infty} \frac{(-1)^{B+1}}{B}\Big(\sum_{j} G_j(x_i)\Big)^B \tag{4.2}$$

where each of the summations on the left is over the set $\mathcal{A}\backslash\emptyset$ of non-empty even simple subgraphs.

So the identity can be rephrased as the following: Let $[f(x_1, x_2, ..., x_k)]_{x_1^{m_1} x_2^{m_2} ... x_k^{m_k}}$ denote the coefficient of the monomial term $x_1^{m_1} x_2^{m_2} ... x_k^{m_k}$ in the polynomial $f(x_1, x_2, ..., x_k)$. Then

$$\begin{aligned}&\Big[\Big(\sum_{\mathcal{A}\backslash\emptyset} G(x_i)\Big) - \frac{1}{2}\Big(\sum_{\mathcal{A}\backslash\emptyset} G(x_i)\Big)^2 + \frac{1}{3}\Big(\sum_{\mathcal{A}\backslash\emptyset} G(x_i)\Big)^3 - ...\Big]_{x_1^{m_1} x_2^{m_2} ... x_k^{m_k}} \\ =&\frac{1}{M}\Big(W(m_1, m_2, ..., m_k; +) - W(m_1, m_2, ..., m_k; -)\Big).\end{aligned} \tag{4.3}$$

4.3 The first identity

The first identity we will derive is in the following theorem.

Theorem 4.3.1 *Let $y_1, y_2, ..., y_L$ be positive intergers and X a positive number such that $X - 1 \geq \sum_{\ell=1}^{L} y_\ell$. Then*

$$\sum_{B=1}^{X} \frac{(-1)^{B+1}}{B} \sum_{j=B}^{max\{y_i\}} (-1)^{B+j} \binom{B}{j}\binom{j}{y_1}\binom{j}{y_2}...\binom{j}{y_L} = 0, \tag{4.4}$$

where the inner summation over j goes in a decreasing order. (The reason for the decreasing order will become clear in the proof.)

Proof using the Feynman Identity. Define the *degree-star* $\deg^*(v_i)$ of a vertex v_i as the number of edges incident with v_i and *counting multiple edges as* 1. For example, if there are totally three edges incident with v_i but two of them are mulple edges, then $deg^*(v_i) = 2$. We will look into the case when $deg^*(v_i) \leq 2$ for all v_i's.

Given a set of parameters $\{m_i\}$ which specifies the number of occurrence of each edge e_i, let $G_{\{m_i\}}$ be the graph with edge e_i of multiplicity m_i. When $G_{\{m_i\}}$ is connected and all m_i's are equal to, say, m, then the path side of (4.3) has nonzero coefficient. In this case, both sides equal $(-1)^{(m+1)}/m$. This is explained in Appendix B.

When $G_{\{m_i\}}$ is not connected, let the number of components be L. Since $deg^*(v_i) \leq 2$ for all vertices, each of the component C_ℓ of $G_{\{m_i\}}$ is actually a closed path repeated, say y_ℓ times, $\ell = 1, 2, ..., L$, where $y_\ell = m_i$ if $e_i \in C_l$. Without lose of generality, let $y_1 \geq y_2 \geq ... \geq y_L \geq 1$. The coefficient of $\prod_i x_i^{m_i}$ in the term $\left(\sum_{\mathcal{A}\backslash\emptyset} G(x_i)\right)^B$ is

$$\sum_{j=B}^{y_1} (-1)^{B+j} \binom{B}{j} \binom{j}{y_1} \binom{j}{y_2} ... \binom{j}{y_L}. \tag{4.5}$$

Notice that the summation over j goes in a decreasing order because we use inclusion-exclusion principle, which is explained in the next paragraph.

The first term $j = B$ comes from the following. First we choose y_ℓ out of B factors in the term

$$\left(\sum_{\mathcal{A}\backslash\emptyset} G(x_i)\right)^B$$

to compose the repeated component C_ℓ. Because each individual factor contains subgraphs as the direct composition of several C_i's, we can choose them "without replacement". But since the summation over $\mathcal{A}\backslash\emptyset$ does not contain the empty graph, which corresponds to 1, each factor has to contribute at least one component. We may use the inclusion-exclusion principle to get rid of those terms that contribute less than B factors.

The range of B for which the coefficient of $\prod_i x_i^{m_i}$ is nonzero goes from y_1, i.e., the maximum of all y_ℓ's, to $\sum_\ell y_\ell$. The lower value y_1 comes from the fact that the set $\mathcal{A}\backslash\emptyset$ contains only simple graphs without repeated edges, so we need at least y_1 factors to compose C_1 repeated y_1 times. The upper value comes from the fact that, if each factor contributes exactly one component, which is the most "spread-out" situation, then the total number of factors equals the total number of components, counting multiplicities, which is $\sum_\ell y_\ell$.

The above expression in (4.5) is the coefficient of $\prod_i x_i^{m_i}$ from the term $\left(\sum_{\mathcal{A}\setminus\emptyset} G(x_i)\right)^B$. If we sum over all B's, then the left hand side of (4.3) becomes

$$\sum_{B=1}^{X} \frac{(-1)^{B+1}}{B} \sum_{j=B}^{y_1} (-1)^{B+j} \binom{B}{j} \binom{j}{y_1} \binom{j}{y_2} \cdots \binom{j}{y_L}, \tag{4.6}$$

where X is some finite number such that all nonzero terms are taken into account. When B is larger than $\sum_\ell y_\ell$, because each factor of the term $\left(\sum_{\mathcal{A}\setminus\emptyset} G(x_i)\right)^B$ has nonzero contribution and totally we need only $\sum_\ell y_\ell$, the coefficient from the term $\left(\sum_{\mathcal{A}\setminus\emptyset} G(x_i)\right)^B$ has to be zero. Therefore the inner summation over j in (4.6) becomes zero when B is larger than $\sum_\ell y_\ell$. This is the expression we get from the graph side.

We can see easily that the path side of (4.3) is zero. Since $G_{\{m_i\}}$ is not connected, there is no way to traverse it in one stroke. So both $W(m_i;+)$ and $W(m_i;-)$ are zero. □

Alternative proof. Since for all $j < y_1$, the expression (4.5) is zero, we rewrite the summation index j as from 1 to B. After interchanging the order of summation, we have

$$\begin{aligned} &\sum_{B=1}^{X} \frac{1}{B} \sum_{j=1}^{B} (-1)^{j+1} \binom{B}{j} \binom{j}{y_1} \binom{j}{y_2} \cdots \binom{j}{y_L} \\ =&\sum_{j=1}^{X} \sum_{B=j}^{X} \frac{1}{B} \binom{B}{j} (-1)^{j+1} \binom{j}{y_1} \binom{j}{y_2} \cdots \binom{j}{y_L}. \end{aligned} \tag{4.7}$$

By using the identity (1.52) in [8], the inner summation on the right hand side of (4.7) can be written as

$$\sum_{B=j}^{X} \frac{1}{B} \binom{B}{j} = \sum_{B=j}^{X} j \binom{B-1}{j-1} = j \binom{X}{j} = X \binom{X-1}{j-1}.$$

Subsituting back into (4.7), we get

$$\begin{aligned} &\sum_{j=1}^{X} (-1)^{j+1} X \binom{X-1}{j-1} \binom{j}{y_1} \binom{j}{y_2} \cdots \binom{j}{y_L} \\ =&X \sum_{k=0}^{X-1} (-1)^k \binom{X-1}{k} \binom{k+1}{y_1} \binom{k+1}{y_2} \cdots \binom{k+1}{y_L}. \end{aligned} \tag{4.8}$$

Use the identity (x.16) in [8]

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a_1+b_1k}{m_1}\binom{a_2+b_2k}{m_2}...\binom{a_r+b_rk}{m_r}=0, \tag{4.9}$$

with the condition $0 \leq m_1+m_2+...+m_r \leq n$. Substitute with $n = X-1$, $r = L$, $m_i = y_i$, and $a_i = b_i = 1$ for all $i$in (4.9) to see that this is exactly the expression in (4.8). The condition on X is also satisfied since $X-1 \geq \sum y_i$, so the expression in (4.4) is zero. □

This complete the prove of the first theorem. We make some preliminary remarks before we state the second theorem.

4.4 A combinatorial lemma

Suppose there are A different objects, $A \geq 2$, and R of them are *special*. We want to partition them into B different baskets with the following rule: (1) None of the basket are empty and (2) the R special objects arein different baskets. Only when $B \leq A$ can the first rule be satisfied and only when $R \leq B$ can the second one be satisfied; so the number of partition is nonzero only when $R \leq B \leq A$.

A real life example may be, during Easter, we want to put R eggs into different baskets. None of the eggs are the same but we want to have at most one egg in each basket. The rest of the objects are candies and they can go to any basket. All the baskets must be nonempty.

We can partition the A objects in the following way. To partition the R special objects, or the *eggs* in the previous example, choose R baskets out of B, so there are $\binom{B}{R}R!$ ways of partition. Take J objects out of the rest of the $A-R$ objects, which are the *candies* in the previous example, and fill up the rest of the $B-R$ baskets. Since none of them is allowed to be empty, $J \geq B-R$. The rest of the $A-R-J$ objects then can go into any of the first R baskets. Let $F(N,k)$ denote the number of distributions of N different objects into k different nonempty basket. Then the number of the partitions of the candies is

$$\sum_{J=B-R}^{A-R} \binom{A-R}{J} F(J, B-R) R^{A-R-J}.$$

Combined together, the total number of distribution, denoted by $D(A, R; B)$, is

$$D(A, R, B) = \binom{B}{R} R! \sum_{J=B-R}^{A-R} \binom{A-R}{J} F(J, B-R) R^{A-R-J}. \tag{4.10}$$

Notice that each term in the summation is mutually exclusive due to the total number of candies in the no-egg baskets. If we denote the Stirling numbers of the second kind by $S_2(N, k)$, which is the number of distributions of N different objects into k identical baskets, then $F(J, B-R) = (B-R)! S_2(J, B-R)$.

Here is an interesting identify about $D(A, R; B)$.

Lemma 4.4.1 *Based on the definition of $D(A, R; B)$ and with the condistion that $A > R$,*

$$\sum_{B=R}^{A} \frac{(-1)^{B+1}}{B} D(A, R; B) = 0. \tag{4.11}$$

Proof. Let the above summation be $G(A, R)$ and substitute the expression for $D(A, R; B)$ in (4.10) into (4.11) to obtain

$$G(A, R) = \sum_{B=R}^{A} \frac{(-1)^{B+1}}{B} \binom{B}{R} R! \sum_{J=B-R}^{A-R} \binom{A-R}{J} F(J, B-R) R^{A-R-J}.$$

Replace $F(J, B-R)$ by $(B-R)! S_2(J, B-R)$ and rearrange the terms to get

$$G(A, R) = R^{A-R} \sum_{B=R}^{A} \sum_{J=B-R}^{A-R} \frac{(-1)^{B+1}}{B} B! \binom{A-R}{J} S_2(J, B-R) R^{-J}.$$

Interchange the two summations along with the proper summation ranges to get

$$G(A,R) = R^{A-R} \sum_{J=0}^{A-R} \sum_{B=R}^{R+J} (-1)^{B+1} (B-1)! S_2(J, B-R) \binom{A-R}{J} R^{-J}.$$

Let $b = B - R$ and put those terms related to b together:

$$G(A,R) = R^{A-R} (-1)^{R+1} \sum_{J=0}^{A-R} \binom{A-R}{J} R^{-J} \sum_{b=0}^{J} (-1)^b (b+R-1)! S_2(J,b).$$

The second summation can be simplified using the generating function of S_2, which we will prove later:

$$\sum_{b=0}^{J} (-1)^b (b+R-1)! S_2(J,b) = (-R)^J (R-1)!. \tag{4.12}$$

Therefore,

$$G(A,R) = R^{A-R} (-1)^{R+1} \sum_{J=0}^{A-R} \binom{A-R}{J} R^{-J} (-R)^J (R-1)!.$$

Notice that $R^{-J}(-R)^J = (-1)^J$, so the part of summation over J becomes

$$\sum_{J=0}^{A-R} \binom{A-R}{J} (-1)^J = (1-1)^{A-R} = 0.$$

This completes the proof of (4.11). To finish the story we come back to (4.12) and use the generating function of the Stirling numbers of the second kind:

$$x^J = \sum_{b=0}^{J} S_2(J,b) x(x-1)...(x-b+1). \tag{4.13}$$

Let $x = -R$ in (4.13) and notice there are b terms following $S_2(J,b)$ on the right-hand side:

$$(-R)^J = \sum_{b=0}^{J} S_2(J,b) R(R+1)...(R+b-1)(-1)^b. \tag{4.14}$$

Complete the factorial by multiplying both sides by $(R-1)!$. Then (4.14) becomes (4.12):

$$(-R)^J(R-1)! = \sum_{b=0}^{J}(-1)^b(R+b-1)!S_2(J,b).$$

□

4.5 The second identity

Recall that $G_{\{m_i\}}$ is the graph with edge e_i of multiplicity m_i. A *repeated edge* is one that has multiplicity greater than one. We can decompose $G_{\{m_i\}}$ into a set of subpaths $S = \{p_j\}$ such that (1) the degrees of the vertices of p_j all equal two for all j and (2) the *disjoint union* of the edge set $E(p_j)$ is $E(G_{\{m_i\}})$; here disjoint means that the repeated edges in different edge set is not eliminated as we combine them together. Define A as the number of paths in S, i.e., $A = |S|$, and notice that $A \geq 2$ when there exists at least one vertex v_j with $deg^*(v_j) > 2$.

Use a *decomposition matrix* $\mathbf{S}_{A\times|E(G_{\{m_i\}})|}$ to denote the decomposition where $\mathbf{S}_{i,j} = 1$ when the sub-path p_i resulting from the decomposition S contains edge e_j and $\mathbf{S}_{i,j} = 0$ otherwise. Given a graph $G_{\{m_i\}}$, there may be different ways of decomposition that satisfy the above rule, and we will use a subcript, i.e., S_i to denote the set and $\mathbf{S}_i$ to denote the matrix when necessary.

After the decomposition, there are A subpaths, and it is desired to partition them into B parts, which we will call the *egg-candy partition procedure* (the name is inspired from the Easter example in the previous section): Partition the elements of the set S into B parts and put them into B different *baskets* such that no two subgraphs in the same basket have common edges. This way, the direct composition of the subpaths in the same basket corresponds to a unique even subgraph in the term $\sum_j G_j(x_i)$. Therefore, each choice of the above procedure contributes 1 to the coefficient of $\prod_i x_i^{m_i}$.

In view of the Easter example, those subpaths that have common edges are treated as eggs with common features, and they are not supposed to go to the same basket; eggs that have no features in common then can go to the same basket. Those subpaths without any repeated edges are candies and they can be in any basket.

For example, if $S = \{p_1, p_1, p_2, p_2\}$ where p_1 and p_2 *do not have any common edge*, so p_1 and p_2 are different kinds of eggs, but the two p_1's are considered the same kind of eggs. If $B = 2$, then the only elligible partition is to put p_1 and p_2 into the same basket, and repeat it again for the second basket. Therefore the term $(\sum_j G_j(x_i))^B$ contribute 1 to the coefficient of $\prod_i x_i^{m_i}$ when $B = 2$. This is because one can always find the unique subgraph, say G_j, which has exactly the same edge set as the disjoint union of $E(p_1)$ and $E(p_2)$ and the same subpaths cannot be put together in the same basket since the subgraphs are simple ones. If $B = 3$, There are 3! different ways to assign $\{\{p_1, p_2\}, \{p_1\}, \{p_2\}\}$ into three different basket, so the total contribution is 3!. If $B = 4$, then there are 4!/2!2! different partitions. If p_1 and p_2 *have any edge in common*, then the total is 4!/2!2! when $B = 4$ and zero otherwise.

Let $D(S; B)$ be the coefficient of $\prod_i x_i^{m_i}$ in the term $\left(\sum_j G_j(x_i)\right)^B$ contributed from the decomposition S. For example, take $S = \{p_1, p_1, p_2, p_2\}$ in the above paragraph where p_1 and p_2 don't have any common edge, then $D(S; 2) = 1, D(S; 3) = 3$, and $D(S; 4) = 6$.

Theorem 4.5.1 *Suppose the decomposition S is such that the matrix $\mathbf{S}$ has only one column with more than one nonzero entry, i.e., there is only one edge e_i with $m_i = R > 1$, and the other m_i's equal 1. Then*

$$\sum_{B=R}^{A} \frac{(-1)^{B+1}}{B} D(S; B) = 0, \tag{4.15}$$

where A is the number of rows of $\mathbf{S}$.

Proof using the Feynman Identity. Based on the discussion of $D(S; B)$, the coefficient of $\prod_i x_i^{m_i}$ on the graph side of (4.3) is

$$\sum_{B=1}^{\infty} \frac{(-1)^{B+1}}{B} D(S; B).$$

So the alternative form of the Feynman Identity becomes

$$\sum_{B=1}^{\infty} \frac{(-1)^{B+1}}{B} D(S; B) = \frac{1}{M} \big(W(m_1, m_2, ..., m_k; +) - W(m_1, m_2, ..., m_k; -) \big). \tag{4.16}$$

Since the nonzero terms start from $B = R$ to $B = A$, (4.16) actually has the same expression as

(4.15) on the left hand side.

We know that if there is any repeated edge, then the path side of the Feynman Identity (4.3) is zero. To be specific, because the paths are not allowed to go backward immediately, we can shrink the repeated edge e_R by connecting the vertices of e_R as a single vertex, and the situation is equivalent to the special case when there is a single vertex with several loops; therefore, we have a one-to-one mapping between the positive- and negative-sign paths. □

Alternative proof. In the case when the decomposition S produces $|A|$ subpaths and R of them have a common repeated edge, and there is no other repeated edge, $D(S;B)$ is the same as $D(A,R;B)$ in the previous section. Again, the nonzero term starts from $B=R$ and ends at $B=A$, as explained in the previous section. So the graph side of the Feynman Identity is same as the left hand side of (4.11) in Lemma 4.4.1. By Lemma 4.4.1, we know that this expression is zero.

□

4.6 Two more combinatorial lemmas

Lemma 4.6.1 *Let m be a positive integer and x is a positive number. Then*

$$\sum_{k=0}^{m} \frac{(-1)^k}{m+x+k} \binom{m+x+k}{k} \binom{m+x}{m-k} = 0 \tag{4.17}$$

Proof. Consider the first term when $k=0$. The left hand side of (4.17) can be written as

$$\Big(\sum_{k=1}^{m} \frac{(-1)^k}{m+x+k} \binom{m+x+k}{k} \binom{m+x}{m-k}\Big) + \frac{1}{m+x} \binom{m+x}{0} \binom{m+x}{m}.$$

So we will prove that

$$\sum_{k=1}^{m} \frac{(-1)^k}{m+x+k} \binom{m+x+k}{k} \binom{m+x}{m-k} = \frac{-1}{m+x} \binom{m+x}{m}. \tag{4.18}$$

Use

$$\binom{n}{t} = \frac{n}{t}\binom{n-1}{t-1}, \tag{4.19}$$

to get

$$\frac{1}{m+x+k}\binom{m+x+k}{k} = \frac{1}{k}\binom{m+x+k-1}{k-1}.$$

So the summation in (4.18) becomes

$$\sum_{k=1}^{m} \frac{(-1)^k}{k}\binom{m+x+k-1}{k-1}\binom{m+x}{m-k}. \tag{4.20}$$

We need to deal with $(-1)^k$. Using the definition

$$\binom{-z}{k} = \frac{1}{k}(-z)(-z-1)...(-z-k+1),$$

we get the following identity:

$$\binom{-z}{k} = (-1)^k\binom{z+k-1}{k}. \tag{4.21}$$

So

$$(-1)^{(k-1)}\binom{m+x+k+1}{k-1} = (-1)^{(k-1)}\binom{(m+x+1)+(k-1)+1}{k-1} = \binom{-(m+x+1)}{k-1}.$$

Substitute this expression back into the summation in (4.20), the expression (4.18) becomes

$$\sum_{k=1}^{m} \frac{(-1)^k}{k}\binom{m+x+k-1)}{k-1}\binom{m+x}{m-k} = \sum_{k=1}^{m} \frac{-1}{k}\binom{-(m+x-1)}{k-1}\binom{m+x}{m-k}.$$

Using (4.19) again with $n = -(m+x)$ and $t = k$, we get

$$\sum_{k=1}^{m} \frac{-1}{-(m+x)}\binom{-(m+x)}{k}\binom{m+x}{m-k} = \frac{1}{(m+x)}\sum_{k=1}^{m}\binom{-(m+x)}{k}\binom{m+x}{m-k}.$$

If we add and subtract the term $k = 0$, we get

$$\frac{1}{(m+x)}\Big(\sum_{k=0}^{m}\binom{-(m+x))}{k}\binom{m+x}{m-k} - \binom{-(m+x)}{0}\binom{m+x}{m}\Big).$$

Using

$$\sum_{k=0}^{m}\binom{a}{k}\binom{b}{m-k} = \binom{a+b}{m}, \tag{4.22}$$

which is true for general a and b, we see that the first summation becomes $\binom{0}{m}$, which equals zero. So put together, the left hand side of (4.18) becomes

$$\frac{1}{(m+x)}\Big(0 - \binom{-(m+x)}{0}\binom{m+x}{m}\Big) = \frac{-1}{m+x}\binom{m+x}{m},$$

which is the right hand side of (4.18) □

Lemma 4.6.2 *Let m be a positive integer and x a positive number. Then*

$$\sum_{k=0}^{m}(-1)^k\binom{m+x+k-1}{k}\binom{m+x}{m-k} = 0. \tag{4.23}$$

Proof. We will use (4.21) again to get

$$(-1)^k\binom{m+x+k-1}{k} = \binom{-(m+x)}{k}.$$

So the summation becomes

$$\sum_{k=0}^{m}(-1)^k\binom{m+x+k-1}{k}\binom{m+x}{m-k} = \sum_{k=0}^{m}\binom{-(m+x)}{k}\binom{m+x}{m-k},$$

which is zero by (4.22). □

We will use both lemmas in the proof of the following theorem.

4.7 The third identity

Suppose that after the decomposition, every subpath contains *exactly* one repeated edge. For example, if p_1, p_2, and p_3 have the common edge e_1, and p_4, p_5 have the common edge e_2, then the upper left part of the decomposition matrix $\mathbf{S}$ looks like the following:

$$\begin{pmatrix} 1 & 0 & \cdots \\ 1 & 0 & \cdots \\ 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

Let the multiplicity of edge e_i be R_i, $i = 1, 2, ..., n$, and let the coefficient in the Bth term $(\sum_j G_j(x_i))^B$ be $D(R_1, R_2, ..., R_n; B)$. We use $D(R_1, R_2, ..., R_n; B)$ instead of $D(S; B)$ since we will need the recursion relation on n. Without loss of generality, let $R_1 \geq R_2 \geq ... \geq R_n \geq 2$.

Theorem 4.7.1 *Let $D(R_1, R_2, ..., R_n; B)$ be defined as in the last paragraph. Then*

$$\sum_{B=R_1}^{R_1+R_2+...+R_n} \frac{(-1)^{B+1}}{B} D(R_1, R_2, ..., R_n; B) = 0. \tag{4.24}$$

Proof using the Feynman Identity. Since the expression for the coefficient of $\prod_i x_i^{m_i}$ is $D(R_1, R_2, ..., R_n; B)$, the left hand side of (4.24) is the same as the graph side of (4.3). Again, the nonzero terms start from $B = R_1$ to $B = A$. Also we know that the whenever there is repeated edge, the path side of the Feynman Identity yields zero. Therefore, the summation result is zero. □

Alternative proof. We will prove the identity using induction, and the base case is when $n = 2$. For simplicity of notation, we will use (4.25), an expression for $D(R_1, R_2; B)$, to prove (4.24) when $n = 3$, instead of general n, but the procedure is similar.

Here is the expression for $D(R_1, R_2; B)$:

$$D(R_1, R_2; B) = \binom{B}{R_1} R_1! \binom{R_2}{B - R_1} (B - R_1)! \binom{R_1}{R_2 - (B - R_1)} (R_2 - (B - R_1))!. \tag{4.25}$$

This is because to put $R_1 + R_2$ objects into B non-empty different baskets, where only for $B = R_1, R_1 + 1, ..., R_1 + R_2$ is the expression nonzero, such that none of the first kind of R_1 objects are in the same basket, and neither are the second kind of R_2 objects, we choose R_1 out of B baskets and fill them with the first R_1 objects. Since the baskets are different, we need the factor of $R_1!$. Then we choose $B - R_1$ out of R_2 second kind of objects and put into the rest of the baskets. For the rest $R_2 - (B - R_1)$ second kind of objects, they need to go into the baskets filled with the first kind of objects.

We can prove (4.24) in the case when $n = 2$ by substituting (4.25) into the summation in (4.24). Let $k = B - R_1$. Then (4.24) becomes

$$\begin{aligned}
&\sum_{B=R_1}^{R_1+R_2} \frac{(-1)^{B+1}}{B} D(R_1, R_2; B) \\
=&\sum_{B=R_1}^{R_1+R_2} \frac{(-1)^{B+1}}{B} \binom{B}{R_1} R_1! \binom{R_2}{B - R_1} (B - R_1)! \binom{R_1}{R_2 - (B - R_1)} (R_2 - (B - R_1))! \\
=&\sum_{k=0}^{R_2} \frac{(-1)^{k+R_1+1}}{R_1 + k} \binom{R_1 + k}{R_1} R_1! \binom{R_2}{k} k! \binom{R_1}{R_2 - k} (R_2 - k)!.
\end{aligned} \tag{4.26}$$

In view of

$$\binom{R_2}{k} k!(R_2 - k)! = R_2!,$$

the expression (4.26) becomes

$$(-1)^{R_1+1} R_1! R_2! \sum_{k=0}^{R_2} \frac{(-1)^k}{R_1 + k} \binom{R_1 + k}{k} \binom{R_1}{R_2 - k}. \tag{4.27}$$

Using (4.17) again from Lemma 4.6.1 and substituting with $m = R_2$ and $x = R_1 - R_2$, we see that the expression (4.27) is zero.

This complete the proof of the base case. To prove (4.24) for general n, we need the recursion relation

$$D(R_1,...,R_n;B) = R_n! \sum_{i=B-R_n}^{B} \binom{B}{i}\binom{i}{R_n-(B-i)} D(R_1,...,R_{n-1};B). \tag{4.28}$$

We can easily understand (4.35) from the recursion relation between $D(R_1, R_2, R_3; B)$ and $D(R_1, R_2; B)$:

$$D(R_1,R_2,R_3;B) = R_3! \sum_{i=B-R_3}^{B} \binom{B}{i}\binom{i}{R_3-(B-i)} D(R_1,R_2;i). \tag{4.29}$$

Choose i out of B baskets to put the first and second kind of objects, which is accounted by $D(R_1, R_2; i)$, and the remaining $B - i$ baskets have to be filled up with the third kind of objects. Now there are $R_3 - (B - i)$ third kind of objects left. Choose $R_3 - (B - i)$ out of i baskets which already contain the first and/or the second kinds of objects. Since the baskets and the objects are different, there is a factor of $R_3!$.

Now we will prove

$$\sum_{B=R_1}^{R_1+R_2+R_3} \frac{(-1)^{B+1}}{B} D(R_1,R_2,R_3;B) = 0 \tag{4.30}$$

and the proof for general n is similar.

Substitute (4.29) into the left hand side of (4.30) and interchange the two summations:

$$\begin{aligned}
&\sum_{B=R_1}^{R_1+R_2+R_3} \frac{(-1)^{B+1}}{B} D(R_1,R_2,R_3;B) \\
=&\sum_{B=R_1}^{R_1+R_2+R_3} \frac{(-1)^{B+1}}{B} R_3! \sum_{i=B-R_3}^{B} \binom{B}{i}\binom{i}{R_3-(B-i)} D(R_1,R_2;i) \\
=&\Bigg(\sum_{i=R_1-R_3}^{R_1-1} \sum_{B=R_1}^{i+R_3} + \sum_{i=R_1}^{R_1+R_2} \sum_{B=i}^{i+R_3} + \sum_{i=R_1+R_2+1}^{R_1+R_2+R_3} \sum_{B=i}^{R_1+R_2+R_3} \Bigg) \\
&\frac{(-1)^{B+1}}{B} \binom{B}{i}\binom{i}{R_3-(B-i)} D(R_1,R_2;i)
\end{aligned}$$

Since $D(R_1, R_2; i) = 0$ unless $i = R_1, R_1 + 1, ...R_1 + R_2$, only the second summation is nonzero.

Focus on the inner summation:

$$\begin{aligned}&\sum_{B=i}^{i+R_3}\frac{(-1)^{B+1}}{B}\binom{B}{i}\binom{i}{R_3-(B-i)}\\=&\frac{1}{i}\sum_{B=i}^{i+R_3}(-1)^{B+1}\binom{B-1}{i-1}\binom{i}{R_3-(B-i)}\end{aligned}$$

Set $k = B - i$; then we have

$$\begin{aligned}&\frac{(-1)^{i+1}}{i}\sum_{k=0}^{R_3}(-1)^k\binom{k+i-1}{i-1}\binom{i}{R_3-k}\\=&\frac{(-1)^{i+1}}{i}\sum_{k=0}^{R_3}(-1)^k\binom{k+i-1}{k}\binom{i}{R_3-k}.\end{aligned}$$

Using (4.23) in Lemma 4.6.2 with $m = R_3$ and $x = i - R_3$, we see that the left hand side of (4.30) is zero. □

4.8 The fourth identity

Theorem 4.8.1 *Let L be a positive integer greater than* 1. *Then*

$$\sum_{k=1}^{L-1}\sum_{j=1}^{k}(-1)^j\binom{k}{j}j^{L-1} = (-1)^{L-1}. \tag{4.31}$$

Proof using the Feynman Identity. The special situation of the decomposition that will lead to this combinatorial identity is the following. Every subpath has only one repeated edge, except one of them, which we call *the bridge.* The repeated edges all have multiplicity two, and one of them is contained in the bridge and the other one is in some other subpath. Use the decomposition matrix and put the bridge in the first row, the left upper part of the decomposition matrix **S** looks like the following:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ . & . & . & \cdots \end{pmatrix}$$

Let the number of subpaths, including the bridge, be L, so L is greater than one. In order that the coefficient of $\prod_i x_i^{m_i}$ be nonzero, we need to take at least two factors, or two *baskets* in the egg-candy partition procedure, which means B has to be greater or equal to 2 in the summation of the right hand side of (4.2). The partition in this case is that, the bridge goes into one basket and the remaining $(L-1)$ subpaths go to the other basket, so total number of ways is 2. In the case of three baskets, the bridge has three choices to begin; after the bridge chooses one basket, and the remaining subpaths have two choices of baskets. In order to make sure every basket is non-empty, we need to subtract the two cases when both of the subpaths choose the same basket. Therefore, the total number is $3(2^{L-1}-2)$. In general for $k+1$ basket, $k=1,2,...,L-2$, we use the inclusion-exclusion principle to ensure that there is no empty basket, so

$$\begin{aligned}&(k+1)\Big(k^{L-1} - \binom{k}{k-1}(k-1)^{L-1} + \binom{k}{k-2}(k-2)^{L-1} + \cdots + (-1)^{k+1}\binom{k}{1}1^{L-1}\Big) \\ =&(k+1)\sum_{j=k}^{1}(-1)^{k+j}\binom{k}{j}j^{L-1}. \end{aligned} \tag{4.32}$$

When there are L baskets, then each basket has to be fulfilled, therefore the number of choices is $L!$. If we substitute with $k=L-1$ in (4.32), then we have

$$L\sum_{j=L-1}^{1}(-1)^{L-1+j}\binom{L-1}{j}j^{L-1} = (-1)^{L-1}L\sum_{j=L-1}^{1}(-1)^{j}\binom{L-1}{j}j^{L-1}.$$

Using the follwing identity which we will prove later:

$$\sum_{j=0}^{L-1}(-1)^{j}\binom{L-1}{j}j^{L-1} = (-1)^{L-1}(L-1)!, \tag{4.33}$$

we recognize that the expression in (4.32) when $k = L - 1$ is $L!$. So the expression (4.32) applies when $k = L - 1$ too.

Summing over the index k for the number of baskets to find the coefficient on the graph side of (4.3):

$$-\frac{1}{2}2 + \frac{1}{3}3(2^{L-1} - 2) + \cdots + \frac{(-1)^{k+2}}{k+1}(k+1)\sum_{j=1}^{k}(-1)^{k+j}\binom{k}{j}j^{L-1} + \ldots + \frac{(-1)^{L+1}}{L}L!$$
$$= \sum_{k=1}^{L-1}\sum_{j=1}^{k}(-1)^{j}\binom{k}{j}j^{L-1}$$

The path side of (4.3) has the coefficient of $\prod_i x_i^{m_i}$ as $(-1)^{L-1}$, as explained in the following. Since there is only one possibility to compose a single-stroke path, which is to start from some vertex of the bridge, and visit every other subpath using the repeated edge as a "bridge", the number of crossings equals $L - 1$.

To finish the story, we come back to (4.33). Use the identity (1.6) in [8]:

$$\sum_{j=0}^{k}(-1)^{j}\binom{x}{j}j^{n-1} = \sum_{j=0}^{n-1}(-1)^{j}\binom{x}{j}\binom{k-x}{k-j}j!S_2(n-1,k), \tag{4.34}$$

where $S_2(n-1,k)$ is the Stirling numbers of the second kind. Set $k = x = n - 1 = L - 1$ in (4.34):

$$\sum_{j=0}^{k}(-1)^{j}\binom{L-1}{j}j^{L-1} = \sum_{j=0}^{L-1}(-1)^{j}\binom{L-1}{j}\binom{0}{L-1-j}j!S_2(L-1,j).$$

Noticing that $\binom{0}{L-1-j} = 0$ when $L-1 \neq j$ and $\binom{0}{L-1-j} = 1$ when $L-1 = j$ and $S_2(L-1, L-1) = 1$, we get $(-1)^{L-1}(L-1)!$ on the right hand side. This complete the proof using the Feynman Identity. □

Alternative proof. Use the identity (1.6) in [8] again:

$$\sum_{j=0}^{k}(-1)^{j}\binom{x}{j}j^{n-1} = \sum_{j=0}^{n-1}(-1)^{j}\binom{x}{j}\binom{k-x}{k-j}j!S_2(n-1,k).$$

Set $x = k$ and $n = L$ in the above expression; we get

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^{L-1} = \sum_{j=0}^{L-1}(-1)^j\binom{k}{j}\binom{0}{k-j}j!S_2(L-1,k).$$

Notice that $\binom{0}{k-j}$ equals 0 when $k \neq j$ and 1 when $k = j$. Therefore, the only nonzero term on the right hand side is when $j = k$. So

$$\sum_{j=1}^{k}(-1)^j\binom{k}{j}j^{L-1} = (-1)^k k!S_2(L-1,k)$$

Putting this expression into the inner summation on the left hand side of (4.31) , we get

$$\sum_{k=1}^{L-1}(-1)^k k!S_2(L-1,k) = (-1)^{L-1}$$

This identity can be proved using the generating function of the Stirling numbers of the second kind:

$$x^n = \sum_{k=1}^{n} S_2(n,k)x(x-1)(x-2)...(x-k+1)$$

Set $x = -1$, $n = L - 1$, and the proof is complete. □

4.9 The fifth recursion relation

When the decomposition $S = \{p_j | j = 1, 2, ..., A\}$ of $G_{\{m_i\}}$ contains a subpath p_k, called *the special path*, whose edges are not repeated, i.e., for all $e_i \in E(p_k)$, $m_i = 1$. We say S is a *special decomposition* if it contains a special subpath.

Theorem 4.9.1 *Let S be a special decomposition of $G_{\{m_i\}}$, and p_k is the special subpath of S. Then*

$$D(S;B) = B\Big(D(S\backslash\{p_k\};B) + D(S\backslash\{p_k\};B-1)\Big), \tag{4.35}$$

where $S\backslash\{p_k\} = \{p_j | j = 1, ..., A$ and $j \neq k\}$.

Proof. The above is true because we can put the last object p_k in any of the B baskets once we have done the allocation of the $A-1$ objects into B baskets, none of them are empty. We may as well allocation the $A-1$ objects into $B-1$ non-empty baskets, and put the last object p_k into the last basket. Since the baskets are distinct, again we have B choices. $\square$

Chapter 5

Conclusion

5.1 Conclusion

In Chapter 2, the special case of the Feynman Identity was proved combinatorially. By taking logarithms and interpreting both sides of the Feynman Identity in a combinatorial way, we saw that the identity arises by counting the difference of the numbers of c-reduced words that have positive and negative signs. The definition of the sign function is explained by the planar drawing of an oriented cotree, but its properties (2.3) and (2.4), as well as Lemma 2.3.1 and 2.4.1 are independent of the particular drawings. Based on the latter Lemma, we were able to give a quick prove of Theorem 2.4.2.

Chapter 3 digresses into the number of non-periodic reduced cycles in a directed graph. We apply the result to the special case of a single vertex and develope a method of calculating $N(m_1, m_2, ..., m_k; +)$ and $N(m_1, m_2, ..., m_k; -)$. When the number k of loops is 2, these numbers were calculated by Costa[4]. In Appendix A, we applied the methods of Chapter 3 to derive the same expressions for these numbers.

Chapter 4 presents the Feynman Identity (4.1) and its alternative form (4.3) after taking logarithms. Four theorems involving combinatorial identities are derived from the alternative form as we look into several special conditions on the graph. We introduced the function $D(A, R; B)$ to denote the number of distributions of A objects into B non-empty different baskets, with the condition that, among the A objects, R of them have to be in different baskets. This facilitates our discussion of the alternative form of the Feynman Identity and we are able to endow the graph side with a combinatorial interpretation.

Appendix A

The Number of Pointed Cycles in the Case of Two Loops

A.1 Relation with Costa's equation

In Costa[4], an explicit expression for $N(m_1, m_2; +)$ is presented in eq (3.36). To simplify, let $m_1 = a$, $m_2 = b$, and $a + b = M$. If we rewrite that expression in terms of the notation here, we have

$$N(a,b;+) = \sum_{d|g,\ d \text{ odd}} \frac{\mu(d)}{d} \sum_{\ell=1}^{min[a,b]/d} \frac{2^{2\ell-1}}{\ell} \binom{a/d-1}{\ell-1} \binom{b/d-1}{\ell-1}. \tag{A.1}$$

If we reduce (3.11) to the case of two loops and compare with the corresponding equation (A.1), we find the expression for the function $f_C(a/d, b/d)$:

$$f_C(a/d,b/d) = \sum_{\ell=1}^{min[a,b]/d} \frac{2M}{d} \frac{2^{2\ell}}{2\ell} \binom{a/d-1}{\ell-1} \binom{b/d-1}{\ell-1}. \tag{A.2}$$

Or,

$$f_C(a,b) = \sum_{\ell=1}^{min[a,b]} (a+b) \frac{2^{2\ell}}{\ell} \binom{a-1}{\ell-1} \binom{b-1}{\ell-1}. \tag{A.3}$$

The subscript C is to signify that this function is the one from Costa. We will derive (A.3) by applying the result from Chapter 3.

Notice that in the notation of Chapter 3, we distinguish a path from its reverse-direction one, but in Costa[4], page 1016, they are considered as in the same class. So the final result here should be twice as in Costa. Therefore, we will prove that

$$f(a,b) = 2f_C(a,b)$$

in the following section.

A.2 Derivation of the function for the number of pointed cycles $f(a,b)$

We will derive this formula from the matrix S in (3.8). Replace x_1 in S by x and y_1 by y for simplicity. Diagonalize S to get

$$S' = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & \frac{1}{2}(x+y+\sqrt{x^2+14xy+y^2}) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(x+y-\sqrt{x^2+14xy+y^2}) \end{pmatrix}$$

Define $A = \frac{1}{2}(x+y+\sqrt{x^2+14xy+y^2})$ and $B = \frac{1}{2}(x+y-\sqrt{x^2+14xy+y^2})$. According to Lemma 3.3.1 when $a > 0$ and $b > 0$,

$$f(a,b) = [Tr(S'^{a+b})]_{x^a y^b} = [x^{a+b} + y^{a+b} + A^{a+b} + B^{a+b}]_{x^a y^b}.$$

So only $A^{a+b} + B^{a+b}$ is relevant. Use $A^t + B^t = (A+B)(A^{t-1} + B^{t-1}) - AB(A^{t-2} + B^{t-2})$, $A + B = x + y$, and $AB = -3xy$, we get the recursion relation

$$f(a,b) = f(a-1,b) + f(a,b-1) + 3f(a-1,b-1) \tag{A.4}$$

for $a \geq 1$ and $b \geq 1$.The initial conditions are $f(1,0) = f(0,1) = 2$, $f(0,0) = 4$. The last one comes from the fact that the trace of a 4×4 identity matrix is 4.

Define

$$f(x,y) = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} f(a,b)x^a y^b,$$

then from the recursion relation and the initial conditions,

$$f(x,y) = 2\frac{2-(x+y)}{1-(x+y+3xy)}.$$

Let $h(a,b) = [(1-(x+y+3xy))^{-1}]_{x^a y^b}$, i.e., the coefficient of $x^a y^b$ in $h(x,y) = (1-(x+y+3xy))^{-1}$, then

$$h(a,b) = \sum_{\ell=0}^{min[a,b]} \frac{(a+b-\ell)!}{(a-\ell)!(b-\ell)!\ell!} 3^\ell,$$

and

$$f(a,b) = 2(2h(a,b) - h(a-1,b) - h(a,b-1)).$$

Put together, we get

$$f(a,b) = 2(a+b) \sum_{\ell=0}^{min[a,b]} \frac{(a+b-\ell-1)!}{(a-\ell)!(b-\ell)!\ell!} 3^\ell. \tag{A.5}$$

This completes the derivation of the function $f(a,b)$ from Chapter 3.

A.3 Comparison of the two functions

In order to show that $f(a,b)$ and $2f_C(a,b)$ are equal, we need the following lemma.

Lemma A.3.1

$$\sum_{\ell=0}^{min[a,b]} \frac{(a+b-\ell-1)! Z^\ell}{(a-\ell)!(b-\ell)!\ell!} = \sum_{\ell=1}^{min[a,b]} \frac{(Z+1)^\ell}{\ell} \binom{a-1}{\ell-1} \binom{b-1}{\ell-1} \tag{A.6}$$

for any $Z > 0$. The expressions we have for $f(a,b)$ and $f_C(a,b)$ apply when $Z = 3$.

Proof. Let $m = min[a,b]$ and the coefficient of Z^k be $[Z^k]$. Treat both sides as polynomials in the variable Z, then on the left-hand side $[Z^k]$ is equal to

$$\frac{(a+b-k-1)!}{(a-k)!(b-k)!k!} = \frac{1}{a+b-k} \binom{a}{k} \binom{a+b-k}{a},$$

and on the right-hand side $[Z^k]$ is equal to

$$\sum_{\ell=k}^{m} \frac{1}{\ell} \binom{a-1}{\ell-1} \binom{b-1}{\ell-1} \binom{\ell}{k} = \sum_{\ell=k}^{m} \frac{1}{a} \binom{a}{k} \binom{a-k}{\ell-k} \binom{b-1}{\ell-1}.$$

For the last step, we use

$$\binom{a-1}{\ell-1} = \frac{\ell}{a}\binom{a}{\ell}$$

and

$$\binom{a}{\ell}\binom{\ell}{k} = \binom{a}{k}\binom{a-k}{\ell-k}.$$

After eliminating $\binom{a}{k}$, we get the equivalent statement

$$\frac{a}{a+b-k}\binom{a+b-k}{a} = \sum_{\ell=k}^{m}\binom{a-k}{\ell-k}\binom{b-1}{\ell-1}.$$

Or, by using

$$\frac{a}{a+b-k}\binom{a+b-k}{a} = \binom{a+b-k-1}{a-1}$$

and rewriting the dummy variable $\ell = \ell' + k$, we get

$$\binom{a+b-k-1}{a-1} = \sum_{\ell'=0}^{m-k}\binom{a-k}{\ell'}\binom{b-1}{\ell'+k-1}. \tag{A.7}$$

To prove (A.7), the right-hand side can be written as

$$\sum_{\ell'=0}^{m-k}\binom{a-k}{\ell'}\binom{b-1}{(b-k)-\ell'},$$

and when $a \geq b$, we use Vandermonde Convolution [8]:

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}$$

where $x = a-k$, $y = b-1$, and $n = b-k$ in this case. When $a < b$, we use [8]

$$\sum_{k=0}^{n}\binom{n}{k}\binom{x+y-n}{x-k} = \binom{x+y}{x}$$

where $x = b - k$, $y = a - 1$, and $n = a - k$ in this case. So the two expressions are the same. $\square$This complete the verification that our general method for deriving $N(m_1, m_2, ..., m_k; +)$ is combatible with Costa's analytical expression when $k = 2$.

A.4 Combinatorial interpretation of $f_C(a, b)$

We provide a combinatorial meaning of Costa's formula (A.3) as follows. The number of pointed cycles with a total of a occurrences of the first loop and b occurrences of the second loop can be counted by forming a word that has ℓ "blocks". Each Block contains at least one first loop and at least one second loop. The possible number of blocks goes from 1 to $min[a, b]$.

Now we need to divide a loops into ℓ blocks. We know that $\binom{a-1}{\ell-1}$ is the number of positive solutions of the equation $x_1 + x_2 + ... + x_\ell = a$. Divide a identical objects into ℓ blocks and the same for another b identical objects. Since each block, consisted of a and b, can be reversed, so there are $2^{2\ell}$ possibilities. This accounts for the part

$$2^{2\ell}\binom{a-1}{\ell-1}\binom{b-1}{\ell-1}.$$

For a pair of the solutions, say $\{(x_1^*, ..., x_\ell^*), (y_1^*, ..., y_\ell^*)\}$ of the two equations $x_1+x_2+\cdots+x_\ell = a$ and $y_1+y_2+\cdots+y_\ell = b$, respectively, its $\ell-1$ rotations: $\{(x_2^*, ..., x_{\ell-1}^*), (y_2^*, ..., y_{\ell-1}^*)\}, ..., \{(x_\ell^*, ..., x_1^*), (y_\ell^*, ..., y_1^*)\}$ lead to the same pointed cycles. So after dividing it by ℓ, we can get the total number. Since we are looking for pointed cycles, there are $a + b$ possible starting points. Therefore, there is a correction factor of $\frac{a+b}{\ell}$.

Appendix B

Explicit Expression of the Feynman Identity When the Graph Is Connected

B.1 A special case of (4.3) when the graph is connected

We will discuss the special case of (4.3) when $G_{\{m_i\}}$ is a connected graph, where m_i is the power of the indeterminate x_i corresponding to the edge e_i, and for every vertex v_i, $deg^*(v_i) \leq 2$.

Notice that each of the summations on the graph side of (4.3) is over even *simple* subgraphs. So only when m_i's are equal, say m, that the coefficient of $\prod_i x_i^{m_i}$ on the graph side of (4.3) can possibly be nonzero. Under this situation, the only nonzero term is $\frac{(-1)^{m+1}}{m}(\sum_j G_j(x_i))^m$; therefore, the coefficient is $(-1)^{m+1}/m$.

The expression from the path side reduces to $(-1)^{m+1}/m$ under this situation too. Again only when all the m_i's are equal can the expression possibly be non-zero . This is because the path is not allowed to go backward immediately; if the vertex v_i has two distinct edges e_j and e_k, i.e., $deg^*(v_i) = 2$, each with multiplicity m_j and m_k respectively, then since every path that goes through e_j has to continue to e_k, $m_j = m_k$. When all the m_i's are equal, say m, the graph is actually a non-self-crossing path, say p, repeated m times. The path p^m has sign equals $(-1)^{m+1}$. Because $W(m_i; \pm)$ are the numbers of *pointed* paths, when m is odd, $W(m_i; +) = |E(p)|$ and $W(m_i; -) = 0$; when m is even, $W(m_i; +) = 0$ and $W(m_i; -) = |E(p)|$. Therefore, the expression on the path side equals $(-1)^{m+1}|E(p)|/M = (-1)^{m+1}/m$.

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