Energy Correlators, Dispersive Sum Rules, and Modular Bootstrap in Conformal Field Theories

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Cyuan-Han Chang

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Cyuan-Han Chang
ORCID: 0000-0002-7133-3553

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ABSTRACT

Studying conformal field theories (CFTs) is an important topic in theoretical physics. CFTs describe critical phenomena, fixed points of RG flows, and UV-complete descriptions of quantum gravity. One powerful method to study CFTs is the bootstrap approach. The idea of bootstrap is to impose symmetries and basic consistency conditions and constrain the space of possible theories. In this thesis, we study constraints and observables in CFTs that come from putting the CFT in Lorentzian signature or on a torus.

In chapter 1, we give a brief overview of CFT and the idea of bootstrap. We also give a short summary of the rest of the thesis.

In chapter 2, we consider a product of two null-integrated operators on the same null plane in a CFT. We call the matrix elements of this product (two-point) energy correlators or event shapes. Energy correlators measure the distribution of energy flux at future null infinity, and they are important in real-world collider measurements. We show that the product admits an operator product expansion (OPE) in the direction transverse to the null plane. Such OPE is called the light-ray OPE. The light-ray OPE between two null-integrated stress-energy tensors contains light-ray operators at spin $J = 3, 5, 7, 9, \ldots$. The terms with spin $J = 3 + 2n$ have transverse spin $4 + 2n$, and are constructed using special conformally-invariant differential operators.

In chapter 3, we study three-point energy correlators (EEEC). We show that Lorentz symmetry implies that EEEC can be decomposed into "celestial blocks", which are functions completely fixed by symmetry. In the collinear limit where all three null-integrated operators approach each other in the transverse direction, the celestial block reduces to a four-point conformal block, with the position of the fourth point given by the momentum of the external state. Using the celestial block decomposition and leading order results for the EEEC in the collinear limit, we make all-order predictions for the EEEC in certain kinematic limits, in both weakly-coupled $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and QCD. We also derive a celestial inversion formula that computes the coefficients of the celestial block expansion. We then apply the formula to the EEEC in strongly-coupled $\mathcal{N} = 4$ SYM and obtain a complete celestial block decomposition.

In chapter 4, we use commutativity of null-integrated operators to generate a class of dispersive CFT sum rules for general spinning operators. Dispersive sum rules have

properties that their action on conformal blocks at the double-trace locations vanish, and they are useful for studying holographic CFTs. Our definition of the spinning dispersive sum rule is an integral of the four-point function over spacetime against a kernel. We show that the action on a conformal block with heavy dimensions can be computed by a saddle point approximation of this integral. Furthermore, the heavy action has a nice interpretation in terms of the flat space limit of the bulk scattering process. This allows us to build a dictionary between spinning dispersive CFT sum rules and flat space sum rules for the scattering of photons and gravitons.

In chapter 5, we use harmonic analysis to decompose the torus partition of 2d CFTs with $U(1)^c$ symmetry into a basis that is manifestly modular-invariant. From this decomposition, we can derive a crossing equation that acts only on the scalars. The crossing equation has interesting terms that are related to the zeros of the Riemann zeta function, and we can rephrase the Riemann hypothesis as a statement about the asymptotic density of scalar operators in $U(1)^c$ 2d CFTs. Finally, we use the scalar crossing equation to obtain bounds on the dimension of the lightest scalar operator.

PUBLISHED CONTENT AND CONTRIBUTIONS

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[4] Nathan Benjamin and Cyuan-Han Chang. "Scalar modular bootstrap and zeros of the Riemann zeta function". In: *JHEP* 11 (2022), p. 143. DOI: 10.1007/JHEP11(2022)143. arXiv: 2208.02259 [hep-th].
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Chapter 1

INTRODUCTION

Quantum field theory (QFT) is a crucial and foundational theoretical framework for modern physics. Since the early twentieth century, it has been used to describe a wide variety of physics phenomena. These include the interactions of elementary particles in the Standard Model, many-body systems in condensed matter physics, and also the study of gravitational theories.

One useful method for studying QFTs is perturbation theory, which can be applied to theories with weak coupling, such as the QED or QCD at high energy. However, there are also numerous QFTs that do not have a small parameter and then one has to study a strong-coupled system. It is therefore important to develop techniques that utilize the non-perturbative structures of the theory, such as symmetries, anomalies, and dualities. In this thesis, we will focus on a method, called the bootstrap, that aims to constrain the theories using symmetry and consistency conditions, such as unitarity and causality.

The bootstrap method is particularly powerful when the theory has enhanced symmetry. Typically, a physical system comes with a characteristic length scale, and the correlation functions of the theory decays exponentially as a function of the distance. However, there are systems that do not have any intrinsic length scales, and the correlations only decay as power laws. Such theories are scale-invariant, meaning they are invariant under rescaling (zooming in/out). Generically, a theory with scaling symmetry (and the usual Poincaré symmetry) has an even bigger symmetry group called the conformal group, which are angle-preserving transformations.[1] The conformal group consists of translations, rotations, rescaling, and special conformal transformations, which are compositions of inversions and translation. QFTs with conformal symmetry are called conformal field theories (CFTs).

There are many reasons why studying CFTs is important. First, they describe critical phenomena in phase transition. Crucially, critical points of different systems can be described by the same CFT. For example, the uniaxial magnet and the liquid-vapor transition of water both have the same power law behavior (with identical exponents)

[1] This statement can be proven when the spacetime dimension is 2. Note, however, that there are exceptions in higher dimensions for some free theories and non-unitary theories.

at their critical points, and they are both described by the 3d Ising CFT. One can then say that these two critical points are in the same universality class.

This type of behavior that multiple theories belonging to the same universality class also shows up in the study of QFTs. When changing the energy scale of a QFT, its behavior is described by the renormalization group (RG) flow. As one keeps lowering the energy, one possible scenario is that the theory flows to a nontrivial fixed point that is invariant under changing the energy scale, which corresponds to a CFT. Thus, CFTs can be thought of as important landmarks in the space of QFTs, and QFTs live on the RG flow between CFTs.

Finally, CFTs are also very important for understanding quantum gravity. This is due to the AdS/CFT correspondence, which says that a theory of quantum gravity in a $(d+1)$-dimensional Anti-de Sitter (AdS) space is equivalent to a d-dimensional CFT living on its boundary. Therefore, through AdS/CFT, CFTs can give non-perturbative definitions of quantum gravity. Additionally, in string theory, the worldsheet theory is given by a two-dimensional CFT.

How much does one need to know about a CFT to fully characterize it? First, each local operator carries two quantum numbers, spin and scaling dimension, describing how it transforms under conformal transformations. Furthermore, conformal symmetry allows one to show that each local operator in a CFT has a one-to-one correspondence to a state in its Hilbert space on a S^{d-1}. This is called the state-operator correspondence. As a result, local operators in a CFT are endowed with an operator product expansion (OPE) structure. That is, a product of two local operators can be written as a convergent sum of local operators. The coefficients in this sum are called the OPE coefficients. All correlation functions of local operators of the CFT are completely determined by these OPE coefficients and the quantum numbers of the operators, and they are called the "CFT data."

Let us now talk about how to study CFTs using bootstrap. The bootstrap approach is based on the philosophy that one should focus on the symmetries and consistency conditions of the CFT, and not care about its microscopic description. This idea of studying CFTs using symmetries and basic consistency conditions started in 1970s by Ferrara, Gatto, Grillo [1], and Polyakov [2]. It has later led to the solutions of a class of exactly solvable two-dimensional CFTs called the 2d minimal models [3]. In the two-dimensional case, the aforementioned conformal symmetry is enhanced even further to an infinite-dimensional symmetry group, called the Virasoro symmetry. Therefore, the bootstrap approach is even more powerful in 2d.

On the other hand, not much progress was made on solving $d \geq 3$ CFTs using bootstrap until the seminal work by Rattazzi, Rychkov, Tonni, and Vichi in 2008 [4]. Their main idea is to look at the crossing equation, which comes from imposing associativity of the OPE to a four-point correlation function. One important fact about this equation is that the coefficients in the sum are all positive for unitary CFTs. Instead of trying to completely solve the equation, they made rigorous numerical statements about the lowest scaling dimension that can appear in the OPE using the technique of linear programming. This idea of numerically solving the crossing equation was then further explored and then implemented in the semidefinite programming solver SDPB [5]. One of the most impressive results from this numerical bootstrap method is the world record determination of the critical exponents of the 3d Ising and $O(N)$ CFT [6].

Numerical conformal bootstrap mainly studies the crossing equation in Euclidean signature. More generally, CFTs can have different spacetime signature. Euclidean CFTs live on $\mathbb{R}^d$, and describe thermal critical points of statistical systems. Lorentzian CFTs live on $\mathbb{R}^{d-1,1}$ and describe quantum critical points. Formally, the Osterwalder-Schrader reconstruction implies that unitary Lorentzian CFTs are related to reflection-positive Euclidean CFTs by a Wick rotation. However, it turns out that many physics are deeply hidden in Euclidean signature, and Lorentzian signature can make more manifest many interesting observables and constraints. For example, consider the possible singularities of CFT correlation functions. In Euclidean signature, singularities can only appear in the OPE limit, where two local operators approach each other. On the other hand, in Lorentzian signature, one can have more singularities in different kinematic limits. One such example is the lightcone limit where two operators are lightlike separated.

Studying the crossing equation in the lightcone limit also gives interesting constraints. In [7, 8], it was shown that in the $\phi \times \phi$ OPE, there must exist a family of operators, called the double-twist operators, whose twist approaching $2\Delta_\phi$ at large spin, where Δ_ϕ is the scaling dimension of ϕ. Another important result derived in Lorentzian signature is the Lorentzian inversion formula. An inversion formula essentially inverts the OPE decomposition of the four-point function, and expresses the OPE data as an integral of the four-point against some kernel. In [9, 10], a Lorentzian inversion formula was derived by starting with an integral over the Euclidean regime and Wicking rotating to Lorentzian signature. Importantly, the expression of the integral is analytic in spin. This implies that the CFT data, which originally only make sense for integer spin, admit an analytic continuation in spin to all complex J, and the local operators of the CFT should organize themselves into continuous families called the Regge trajectory. It was then later understood that the operators with continuous spin are light-ray operators [11]. To better understand continuous-spin CFT dynamics encoded by these light-ray operators, it is important to study Lorentzian signature.

Besides considering CFTs in the Lorentzian signature, one can also put the theory on some other nontrivial manifold. This provides even more constraints that can be imposed when bootstrapping CFTs. In two dimensions, one concrete realization of this idea is to put the theory on a torus, and one new consistency condition is the modular invariance of the partition function.

In chapter 2, 3, 4, and 5 of this thesis, I present several different results related to $d > 2$ CFTs in Lorentzian signature and $d = 2$ CFTs on a torus. I first study an intrinsically Lorentzian observable called the energy correlator that is closely related to collider physics. Then, I show that by considering an integral of CFT four-point against a special kernel, one can obtain nontrivial constraints that are particularly useful for holographic CFTs. Finally, I consider the torus partition function of 2d CFTs, and apply harmonic analysis to prove new rigorous statements about the scalar spectrum of a special class of 2d CFTs. I discuss them in more detail below.

1.1 Conformal collider physics

We study energy correlators in CFTs in chapters 2 and 3. Energy correlators (or more generally event-shapes) are defined as the distribution of energy flux going through energy detectors placed at future null infinity. They are important observables in real-world colliders. In CFTs, they can be written as the matrix elements of light-ray operators. If there is only one energy detector, we have the one-point energy correlator. It is completely fixed by symmetry up to several OPE coefficients. It can also be shown that one-point energy correlators are always positive (known as the average null energy condition), and it leads to interesting bounds on the CFT data [12].

Energy correlators are perhaps one of the simplest intrinsically Lorentzian observables in CFT. By developing theoretical understanding of these observables and computing them in some CFTs, one might hope to measure them at a quantum critical point and confirm with theoretical predictions. This could pave the way for exploring nontrivial Lorentzian CFT dynamics through table-top experiments.

Another motivation for studying energy correlator is its connection to collider physics. They are observables that were measured in real-world colliders such as the LHC. Physicists have been developing many perturbative techniques to compute them in perturbative QCD. They often use the $\mathcal{N} = 4$ SYM at weak coupling, which is a close cousin of perturbative QCD, as a platform to test those methods. If we understand how to compute energy correlator in $\mathcal{N} = 4$ SYM using CFT techniques, it could be a nontrivial cross check between the two different methods. Moreover, phrasing the calculation in symmetry-based field theory language can also help understand better how to organize the complicated perturbative calculation.

In chapter 2, I study the two-point energy correlator. Although the energy detectors are nonlocal operators, they are point-like objects on the celestial sphere. We derive that when the two energy detectors get close to each other, there is still an OPE between them. We call this OPE the light-ray OPE. Interestingly, in order to produce high transverse spin[2] terms in the light-ray OPE, one needs a class of conformally-invariant differential operators which shift the transverse spin. The objects appearing in the OPE are light-ray operators with continuous spin $J = 3, 5, 7, \ldots$ (continuous in the sense that there are no local operators at those points).

In chapter 3, I study the three-point energy correlator (EEEC). The light-ray OPE of three energy detectors is not known yet. However, there are still some nontrivial statements one can make using symmetry. We show that Lorentz symmetry implies that EEEC can be decomposed into celestial blocks, which are special functions completely fixed by symmetry. In the collinear limit where all operators approach each other, the celestial block reduces to a conformal block, with the conformally-invariant cross ratios given by the ratio between the angles between the detectors. Using the celestial block expansion and known leading order data of EEEC, we can make all-order predictions in certain kinematic limits, even in QCD. We also derive a celestial inversion formula that computes the coefficients of the celestial block expansion from an integral of the EEEC over the celestial sphere.

1.2 Dispersive sum rules

Causality is an important consistency condition for any QFT. One condition that follows from causality is that operators commute when they are spacelike separated. In fact, the crossing equation can be understood as this statement by consider a four-point function with all spacelike separations. One can then try to get an even

[2]Transverse spins describes how operators transform under rotations on the celestial sphere.

stronger constraint by integrating over the positions of the operators, while making sure that the operators remain spacelike. This will give a "sum rule" that says an integral of the four-point function should vanish.[3] Such sum rules are powerful tools that allow us to rule out many seemingly sensible theories as being unphysical.

By carefully choosing the integration kernel, one can obtain a special class of sum rules called the dispersive sum rules. These sum rules have additional properties that their actions on conformal blocks at the double-twist locations $\Delta = 2\Delta_\phi + 2n$ vanish. Because of this, in a large-N CFT, dispersive sum rules can cleanly separate the contributions of single-trace and double-trace operators.

Now, let us consider applying this sum rule to a four-point function in a holographic CFT. We assume that the CFT has large N and a large gap Δ_{gap} in its single-trace higher-spin spectrum. By AdS/CFT, this means that the bulk dual should be described by a weakly-coupled gravitational EFT at low energy. On the CFT side, the action of the dispersive sum rules keep only contributions of single-trace operators below the gap. If we can find a sum rule whose contribution above the gap is always positive (which is a problem very similar to numerical bootstrap), then we can prove inequalities on the OPE coefficients of single-trace operators. On the bulk side, this means that one can prove bounds on the couplings of the bulk EFT. In particular, the bounds will be in terms of the gap Δ_{gap}, which is related to the energy cutoff of the EFT. One way to phrase the result is that dimensional analysis of gravitational EFTs in AdS follows from CFT axioms.

In chapter 4, we consider this problem for general spinning operators. Conceptually, there is no major difference between the spinning case and the scalar case (which was studied in [13]). However, the spinning case is technically much more challenging due to the large number of sum rules and tensor structures. We find that by using the position space language and writing the dispersive sum rule as an integral over a certain region in spacetime, it is straightforward to generalize the calculation to the spinning case. We then derive a list of CFT dispersive sum rules for spinning operators. The actions of the sum rules on heavy blocks have nice flat space interpretations. Thus, we can write down the dictionary between these sum rules and flat space sum rules for photons and gravitons when the operators are spin-1 currents and stress-energy tensors.

[3] It is called sum rule because one can turn this into a sum over the operator spectrum by apply OPE to the four-point function.

1.3 Scalar modular bootstrap

Modular bootstrap is a program that aims to derive bounds on 2d CFTs from modular invariance of the torus partition function. One can decompose the partition function into Virasoro characters, and the coefficients of this decomposition must be positive due to unitarity.[4] Modular invariance then implies a "modular crossing equation" for this decomposition. A method similar to the conformal bootstrap can then be applied to prove bounds on the spectrum of the CFT. One interesting question is how do these bounds scale in the limit of large central charge c, and whether they can be saturated by the BTZ black hole solution in AdS_3.

In chapter 5, we study the implication of modular invariance using a different approach. Using harmonic analysis, we can decompose the partition function into a basis that is manifestly modular-invariant. Therefore we do not have to impose modular invariance. However, unitarity in this decomposition becomes less clear. Eventually, we derive a new crossing equation that only acts on the scalar operators, and it has interesting terms related to the zeros of the Riemann zeta function. The crossing equation takes a simple form if the CFT has an enhanced $U(1)^c$ symmetry. One can use this crossing equation to turn the Riemann hypothesis into a statement about the asymptotic density of scalar operators in these theories. Moreover, using the scalar crossing equation, we also obtain bounds on the scaling dimension of the lightest scalar operators in 2d CFTs with $U(1)^c$ symmetry.

[4] In fact, the coefficients should be positive integers, since they count the number of operators. However, so far it has not been clearly understood how to impose integrality constraints into bootstrap.

Chapter 2

TRANSVERSE SPIN IN THE LIGHT-RAY OPE

This chapter is based on

[1] Cyuan-Han Chang, Murat Kologlu, Petr Kravchuk, David Simmons-Duffin, and Alexander Zhiboedov. "Transverse spin in the light-ray OPE". In: *JHEP* 05 (2022), p. 059. DOI: `10.1007/JHEP05(2022)059`. arXiv: `2010.04726 [hep-th]`.

2.1 Introduction

In Euclidean signature, the operator product expansion (OPE) gives a convergent expansion for correlation functions around coincident-point singularities. This expansion lets us formulate nonperturbative bootstrap conditions and perform myriad computations. Lorentzian correlators are in principle determined from Euclidean ones by analytic continuation. However, a given OPE may not commute with this continuation. Furthermore, Lorentzian signature allows for a much richer set of singularities than Euclidean signature [14]. It is important to develop nonperturbative tools for understanding these singularities and efficiently computing Lorentzian observables.

The work [15] introduced an intrinsically Lorentzian OPE for products of null-integrated operators on the same null plane.[1] This OPE can be applied to Lorentzian observables called "event shapes," which measure the distribution of energy (and other quantities) in a collider-like experiment [18, 19, 12]:

$$\langle\Psi|\mathcal{E}(\widehat{n}_1)\cdots\mathcal{E}(\widehat{n}_k)|\Psi\rangle. \tag{2.1}$$

Here $|\Psi\rangle$ is a state, for example created by sending particles from past null infinity and letting them scatter. Each operator $\mathcal{E}(\widehat{n}_i)$ is a stress-tensor integrated along retarded time at future null infinity, at a fixed position $\widehat{n}_i \in S^{d-2}$ on the celestial sphere. The $\mathcal{E}(\widehat{n}_i)$ act like calorimeters, measuring the energy flux at angle $\widehat{n}_i$. We refer to the $\mathcal{E}(\widehat{n}_i)$ as "detectors."

[1] See also [16] which derived the leading term in the expansion of a particular two-point event shape in $\mathcal{N}=4$ SYM theory, and [17] which derived leading terms in collinear limits of energy-energy correlators in the setting of perturbative gauge theories.

The OPE developed in [15] gives a nonperturbative expansion for event shapes in the separation between a pair of detectors $1-\widehat{n}_1\cdot\widehat{n}_2$. Specifically, [15] gave a precise description of the low "transverse spin" terms in this OPE. Here, "transverse spin" j refers to spin on the celestial sphere: it is conjugate to rotation of the points $\widehat{n}_1, \widehat{n}_2$ around each other, see figure 2.1. The spin J, a different quantum number, is related to selection rules for the light-ray OPE.

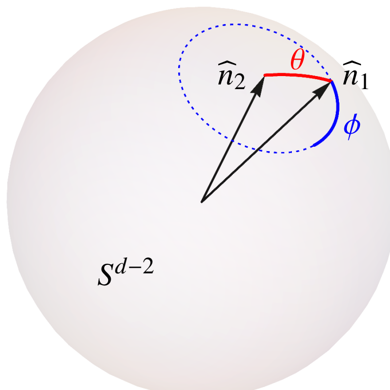

Figure 2.1: The celestial sphere S^{d-2} in a two-point event shape. The positions of the detectors are parametrized by $\widehat{n}_1, \widehat{n}_2 \in S^{d-2}$. The light-ray OPE is an expansion in the angle θ between detectors (solid red arc). Transverse spin j is conjugate to the angle ϕ of one detector around the other on the celestial sphere (solid blue arc).

The low transverse spin terms in the $\mathcal{E}\times\mathcal{E}$ OPE are given by spin $J=3$ light-ray operators [15], in accordance with an earlier analysis of the light-ray OPE by Hofman and Maldacena [12]. This is the complete OPE in 3d CFTs, where the transverse direction is 1-dimensional. Furthermore, low transverse spin terms are sufficient for studying two-point event shapes in rotationally-symmetric states in $d>3$ dimensions. This covers many of the cases studied in the literature, including simple energy two-point correlators in QCD [19, 20, 21, 22, 23] and $\mathcal{N}=4$ SYM [24, 25, 26].

In this work, we derive the remaining terms in the light-ray OPE, including arbitrary transverse spin. Higher transverse spin terms are important in $d>3$ dimensional theories when the initial state $|\Psi\rangle$ is not rotationally invariant or when other detectors are present. For example, in an energy three-point correlator $\langle\Psi|\mathcal{E}(\widehat{n}_1)\mathcal{E}(\widehat{n}_2)\mathcal{E}(\widehat{n}_3)|\Psi\rangle$, transverse spin in the $\mathcal{E}(\widehat{n}_1)\times\mathcal{E}(\widehat{n}_2)$ OPE encodes dependence of the event shape on the direction of the tangent vector from $\widehat{n}_1$ to $\widehat{n}_2$ relative to the third direction $\widehat{n}_3$.

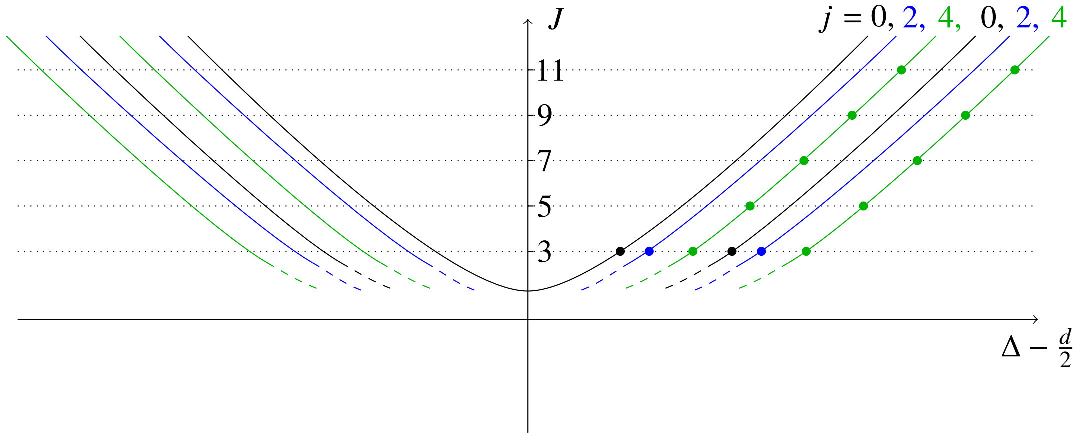

Figure 2.2: Chew-Frautschi plot of light-ray operators in the $\mathcal{E} \times \mathcal{E}$ OPE in a hypothetical CFT. We show Regge trajectories of even signature operators with transverse spins $j = 0$ (black curves), $j = 2$ (blue curves), and $j = 4$ (green curves). Light-ray operators that appear in the $\mathcal{E} \times \mathcal{E}$ OPE are marked with dots. They include operators with spin $J = 3$ and transverse spins $j = 0, 2, 4$. (These are the "low transverse spin" terms described in [15].) In addition, there are primary descendants of light-ray operators with transverse spin $j = 4$ and spin $J = 5, 7, 9, \ldots$.

In seeking the higher transverse spin terms in the $\mathcal{E} \times \mathcal{E}$ OPE, we initially encounter a puzzle: there are no spin $J = 3$ primary light-ray operators of the type defined in [11] that can do the job. The resolution is that higher transverse spin terms are "primary descendants." They are given by the action of special conformally-invariant differential operators $\mathcal{D}_{2n}$ on light-ray operators with higher spins $J = 3 + 2n$.[2] These conformally-invariant differential operators convert spin J into transverse spin j. They are well-defined only when acting on objects with special quantum numbers — precisely the quantum numbers that arise in the light-ray OPE. We find that the $\mathcal{E} \times \mathcal{E}$ OPE takes the schematic form

$$\mathcal{E} \times \mathcal{E} = \sum_i \left(\mathbb{O}^+_{i,J=3,j=0} + \mathbb{O}^+_{i,J=3,j=2} + \mathbb{O}^+_{i,J=3,j=4} \right) + \sum_{n,i} \mathcal{D}_{2n} \mathbb{O}^+_{i,J=3+2n,j=4}. \tag{2.2}$$

Here, $\mathbb{O}^+_{i,J,j}$ are light-ray operators on the i-th Regge trajectory with spin J and transverse spin j. The special differential operators $\mathcal{D}_{2n}$ act on $\mathbb{O}^+_{i,J=3+2n,j=4}$ to give higher transverse spin terms. A Chew-Frautschi plot of the light-ray operators in (2.2) is depicted in figure 2.2.

[2]In applications to event shapes such as (2.1), where the light-ray operators appearing in the OPE are inserted at spatial infinity (and extend along the future null infinity), the operators $\mathcal{D}_{2n}$ are actually polynomials in the special conformal generators K_μ. This happens because an inversion which sends a finite point to infinity maps the translation generators P_μ (which act by derivatives) to K_μ.

The higher transverse spin terms are new ingredients in the light-ray OPE. However, it is interesting that they do not require us to go outside the space of light-ray operators defined in [11]. Instead, they are hidden in an interesting way inside the usual space of light-ray operators — at higher values of J. Along the way to understanding higher transverse spin terms in the light-ray OPE, we will also find a much simpler derivation of the original light-ray OPE from [15].

We begin in section 2.2 with an introduction to the kinematics of transverse spin, focusing on the example of energy correlators in the process $e^+e^- \to$ hadrons. In section 2.3, we provide a more detailed introduction to the underlying representation theory and the special conformally-invariant differential operators that raise transverse spin. In section 2.4, we give a new derivation of the light-ray OPE, using null-integrated scalars as an example. In section 2.5, we generalize this discussion to the OPE of null integrals of arbitrary local operators. In section 2.6, we rederive the scalar light-ray OPE using the light-ray OPE formula for general operators, and give the light-ray operators that appear in the light-ray OPE of two charge detectors and the light-ray OPE of two energy detectors. In section 2.7, we check our formulas for an event shape in $\mathcal{N} = 4$ SYM with a non-rotationally-symmetric final state. We conclude in section 2.8 with discussion and future directions.

2.2 Event shapes, OPEs, and transverse spin

Event shapes describe patterns of excitations at future null infinity. Perhaps the most important examples are energy correlators

$$\langle\Psi|\mathcal{E}(\widehat{n}_1)\cdots\mathcal{E}(\widehat{n}_n)|\Psi\rangle, \tag{2.3}$$

which measure the distribution of energy at future null infinity in the state $|\Psi\rangle$. Here, $\mathcal{E}(\widehat{n}_i)$ are calorimeters inserted at future null infinity in the direction $\widehat{n}_i \in S^{d-2}$ on the celestial sphere. (We define them more precisely below.)

Suppose that $|\Psi\rangle$ is created by a linear combination of local operators $\mathcal{O}_i$ acting on the vacuum,

$$|\Psi\rangle = \int d^dx \sum_i f_i(x)\mathcal{O}_i(x)|\Omega\rangle. \tag{2.4}$$

The energy correlator is

$$\begin{aligned}\langle\Psi|\mathcal{E}(\widehat{n}_1)\cdots\mathcal{E}(\widehat{n}_n)|\Psi\rangle &= \int d^dx d^dx' \sum_{i,j} f_i^*(x) f_j(x')\langle\Omega|\mathcal{O}_i^\dagger(x)\mathcal{E}(\widehat{n}_1)\cdots\mathcal{E}(\widehat{n}_n)\mathcal{O}_j(x')|\Omega\rangle \\ &= \int \frac{d^dp}{(2\pi)^d}\sum_{i,j}\widetilde{f}_j^*(p)\widetilde{f}_i(p)\langle\mathcal{O}_i(p)|\mathcal{E}(\widehat{n}_1)\cdots\mathcal{E}(\widehat{n}_n)|\mathcal{O}_j(p)\rangle,\end{aligned} \tag{2.5}$$

where

$$|\mathcal{O}_i(p)\rangle = \int d^dx e^{ipx}\mathcal{O}_i(x)|\Omega\rangle, \tag{2.6}$$

and $\widetilde{f}_i(p)$ is the Fourier transform of $f_i(x)$. The matrix element on the last line of (2.5) is defined by stripping off a momentum-conserving delta function:

$$\langle\mathcal{O}_i(q)|\mathcal{E}(\widehat{n}_1)\cdots\mathcal{E}(\widehat{n}_n)|\mathcal{O}_j(p)\rangle = \langle\mathcal{O}_i(p)|\mathcal{E}(\widehat{n}_1)\cdots\mathcal{E}(\widehat{n}_n)|\mathcal{O}_j(p)\rangle(2\pi)^d\delta^d(p-q). \tag{2.7}$$

The delta-function $\delta^d(p-q)$ appears because the detectors $\mathcal{E}(\widehat{n}_i)$ are translation-invariant: translations do not alter the direction in which the excitations exit the system at future null infinity. Thus, we are naturally led to study expectation values of energy detectors in momentum eigenstates,

$$\langle\mathcal{O}_i(p)|\mathcal{E}(\widehat{n}_1)\cdots\mathcal{E}(\widehat{n}_n)|\mathcal{O}_j(p)\rangle. \tag{2.8}$$

2.2.1 Example: $e^+e^- \to$ hadrons

As a concrete example, consider the Standard Model process $e^+e^- \to \gamma^* \to$ hadrons, treated to leading order in the fine structure constant, but to all orders in the QCD coupling.[3] The relevant energy correlator is an expectation value in a state created by the electromagnetic current J_μ,

$$\epsilon^{*\nu}\langle J_\nu(p)|\mathcal{E}(\widehat{n}_1)\cdots\mathcal{E}(\widehat{n}_n)|J_\mu(p)\rangle\epsilon^\mu, \tag{2.9}$$

Here, ϵ_μ is a polarization vector for an off-shell photon that depends on the beam direction and helicities of the incoming e^+e^- pair. (We imagine that the collision occurs at high energies, so the electrons can be treated as massless.) Suppose particle 1 moves along the $\vec{e}_z$ direction and particle 2 moves along the $-\vec{e}_z$ direction,

[3] In practice, one considers the processes at high energies where hadronization corrections can be argued to be relatively small [27, 28], and computes the above event shape using perturbative QCD. Such calculations are one of the ways used to measure the strong coupling constant α_s, see [29] and references therein.

with total momentum $p_1 + p_2 = p = (E, 0, 0, 0)$. We henceforth set $E = 1$. If the incoming helicities are 1^+2^-, the corresponding polarization vector ϵ^μ is

$$\epsilon^\mu = \lambda_2 \sigma^\mu \overline{\lambda}_1 = (0, 1, i, 0). \tag{2.10}$$

More generally, for helicities $1^\pm 2^\mp$, we have

$$\epsilon^\mu = \epsilon_\pm^\mu \equiv (0, 1, \pm i, 0). \tag{2.11}$$

We can think of the ket and bra together in (2.9) as giving an (unnormalized) density matrix. For example, for helicities 1^+2^-, we have the pure state

$$\rho_{1^+2^-} = |J_\mu(p)\rangle\langle J_\nu(p)| \epsilon_+^\mu \epsilon_-^\nu, \tag{2.12}$$

where we used $\epsilon_+^{*\mu} = \epsilon_-^\mu$. More generally, we may wish to study a mixed state, for example by averaging over incoming helicities

$$\rho_{12}^{\text{av}} = |J_\mu(p)\rangle\langle J_\nu(p)| \cdot \frac{1}{2}(\epsilon_+^\mu \epsilon_-^\nu + \epsilon_-^\mu \epsilon_+^\nu). \tag{2.13}$$

It is common to additionally average over the beam direction, replacing $\epsilon_+^\mu \epsilon_-^\nu \to \frac{2}{3}(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2})$. However, this discards valuable information, as we explain below.

Symmetries and the density matrix

Let us understand how symmetries constrain energy correlators in the process $e^+e^- \to$ hadrons. Along the way, we will introduce the notion of light-ray operators with nonzero transverse spin and understand how they appear in event shapes. Instead of QCD, we will work in a general 4-dimensional CFT. However, much of our analysis will not depend on conformal symmetry. We point out when conformal symmetry is used below.

We concentrate on a two-point energy correlator in a density matrix where we average over helicities (but not the beam direction)

$$\text{Tr}(\rho_{12}^{\text{av}} \mathcal{E}(\widehat{n}_1)\mathcal{E}(\widehat{n}_2)) = \langle J_\nu(p)|\mathcal{E}(\widehat{n}_1)\mathcal{E}(\widehat{n}_2)|J_\mu(p)\rangle \cdot \frac{1}{2}(\epsilon_+^\mu \epsilon_-^\nu + \epsilon_-^\mu \epsilon_+^\nu). \tag{2.14}$$

Let us separate the tensor $\frac{1}{2}(\epsilon_+^\mu \epsilon_-^\nu + \epsilon_-^\mu \epsilon_+^\nu)$ into irreducible components under the rotation group $\text{SO}(3)$ that fixes p. Focusing on spatial components $i, j = 1, 2, 3$, we have the traceless-symmetric and trace parts

$$\frac{1}{2}(\epsilon_+^i \epsilon_-^j + \epsilon_-^i \epsilon_+^j) = \left(\frac{1}{2}(\epsilon_+^i \epsilon_-^j + \epsilon_-^i \epsilon_+^j) - \frac{2}{3}\delta^{ij}\right) + \frac{2}{3}\delta^{ij}. \tag{2.15}$$

The traceless-symmetric part can be written as a sum of products of null vectors $\vec{q} \in \mathbb{C}^3$:[4]

$$\frac{1}{2}(\epsilon_+^i \epsilon_-^j + \epsilon_-^i \epsilon_+^j) - \frac{2}{3}\delta^{ij} = \sum_{\vec{q}} q^i q^j. \tag{2.18}$$

Plugging this in, we have

$$\mathrm{Tr}(\rho_{12}^{\mathrm{av}} \mathcal{E}(\widehat{n}_1)\mathcal{E}(\widehat{n}_2)) = \sum_q \langle J_\nu(p)|\mathcal{E}(\widehat{n}_1)\mathcal{E}(\widehat{n}_2)|J_\mu(p)\rangle q^\mu q^\nu + \text{trace part}, \tag{2.19}$$

where $q = (0, \vec{q})$ and "trace part" refers to the contribution of the second term in (2.15). Note that each q appearing in the sum is null and orthogonal to p:

$$q^2 = q \cdot p = 0. \tag{2.20}$$

The form (2.19) makes it easy to analyze the constraints of symmetries, since now we have only a single vector q instead of a tensor.

Symmetries and detectors

Next we need a more precise definition of the detector $\mathcal{E}(\widehat{n})$. It can be expressed as an integral of the stress-tensor $T^{\mu\nu}$ over future null infinity. To state this more concretely, we can make a conformal transformation that maps future null infinity to the plane $x^- = 0$. We then have

$$\mathcal{E}(\widehat{n}) \to 2 \int dx^+ T_{++}(x^- = 0, x^+, \vec{x}), \tag{2.21}$$

where $\vec{x} \in \mathbb{R}^{d-2}$ is a function of $\widehat{n}$. As explained in detail in [11], such null-integrated stress tensors can be interpreted in terms of the "light-transform" of T, denoted by

$$\mathbf{L}[T](x, z), \tag{2.22}$$

[4] This is a general fact about traceless symmetric tensors. An example decomposition in this case is

$$\frac{1}{2}(\epsilon_+^i \epsilon_-^j + \epsilon_-^i \epsilon_+^j) - \frac{2}{3}\delta^{ij} = \sum_{\vec{q} \in Q} q^i q^j \tag{2.16}$$

where Q contains four null vectors,

$$Q = \left\{ \frac{1}{\sqrt{6}}(1, 0, \pm i), \frac{1}{\sqrt{6}}(0, 1, \pm i) \right\}. \tag{2.17}$$

where x is a space-time position marking the starting point of the null integral, and z is a future-directed null vector. The definition of $\mathbf{L}$ is given in (2.43) below. This description is useful because $\mathbf{L}[T](x,z)$ transforms like a primary operator at x. If we send x to past null infinity, $\mathbf{L}[T](x,z)$ becomes the null integral in (2.21). If we instead send x to spatial infinity, then $\mathbf{L}[T](x,z)$ becomes directly related to $\mathcal{E}(\widehat{n})$,

$$\mathcal{E}(\widehat{n}) = 2\mathbf{L}[T](\infty, z) \tag{2.23}$$

where $z = (1, \widehat{n})$.

In this work, we derive a nonperturbative OPE between light-ray operators that takes the schematic form

$$\mathbf{L}[T](x,z_1)\mathbf{L}[T](x,z_2) = \sum_{j=0,2,\ldots} \sum_i C_{\Delta_i-1,j}(z_1,z_2,\partial_z,\partial_w)\mathbb{D}_{\Delta_i,j}(x,z,w). \tag{2.24}$$

Here, $C_{\delta,j}(z_1,z_2,\partial_z,\partial_w)$ is a differential operator that is fixed by Lorentz symmetry, and $\mathbb{D}_{\Delta_i,j}(x,z,w)$ are light-ray operators that we characterize in more detail shortly. For now, the only information about $\mathbb{D}_{\Delta_i,j}(x,z,w)$ that we need are its Lorentz transformation properties. It is a homogeneous function of null vectors $z \in \mathbb{R}^{d-1,1}$, $w \in \mathbb{C}^{d-1,1}$ with homogeneities

$$\mathbb{D}_{\Delta_i,j}(x,\alpha z,\beta w) = \alpha^{1-\Delta_i}\beta^j \mathbb{D}_{\Delta_i,j}(x,z,w). \tag{2.25}$$

Furthermore, z, w are subject to the constraints

$$z \cdot z = w \cdot w = w \cdot z = 0, \tag{2.26}$$

and the gauge redundancy

$$w \sim w + \lambda z. \tag{2.27}$$

We can interpret z_1, z_2, and z as embedding space coordinates [30] on the celestial sphere S^{d-2}, and w as an embedding space polarization vector on the celestial sphere. The quantum number j labels spin on the celestial sphere, which we call "transverse spin." Further, $\Delta_i - 1$ is a dimension on the celestial sphere, and (2.24) takes the form of an OPE in a fictitious $d-2$-dimensional Euclidean CFT.

The dependence of $\mathcal{E}(\widehat{n})$, $\mathbb{D}_{\Delta,j}$, and $C_{\delta,j}$ on the coordinates z_1, z_2, z, w relies only on Lorentz symmetry — not full conformal symmetry.[5] We will use conformal

[5]Note that our use of Δ to denote one of the quantum numbers of $\mathbb{D}_{\Delta,j}$ might suggest that we are relying on conformal symmetry. We are not: here $1-\Delta$ is the Lorentz spin of $\mathbb{D}_{\Delta,j}$. The reason for this convention will become clear soon.

symmetry later to derive (2.24) — in particular to constrain which $\mathbb{D}_{\Delta_i,j}$ can appear. However, it is possible that a similar OPE exists in non-conformal theories, and our analysis of the contributions of light-ray operators $\mathbb{D}_{\Delta_i,j}$ in this section will also apply in that case.

Symmetries and matrix elements

Using the OPE in (2.19), it suffices to compute matrix elements

$$\langle J_\nu(p)|\mathbb{D}_{\Delta_i,j}(\infty,z,w)|J_\mu(p)\rangle q^\mu q^\nu. \tag{2.28}$$

In fact, let us analyze a more general matrix element where the density matrix has SO$(d-1)$ spin l:

$$\mathcal{M}(z,w;p,q) = \langle \mathcal{O}_{\mu_1\cdots\mu_k}(p)|\mathbb{D}_{\Delta_i,j}(\infty,z,w)|\mathcal{O}'_{\mu_{k+1}\cdots\mu_l}(p)\rangle q^{\mu_1}\cdots q^{\mu_l}. \tag{2.29}$$

The virtue of having classified the density matrix and light-ray operators into irreducible components is that $\mathcal{M}$ is fixed by symmetry. The argument is as follows. Because of the gauge redundancy (2.27), w can only appear in the gauge-invariant combination

$$[z,w]^{\mu\nu} = z^\mu w^\nu - w^\mu z^\nu. \tag{2.30}$$

Because of (2.26), the only antisymmetric tensor we can contract this with is $[p,q]$. Since w must appear with homogeneity j, we have

$$\mathcal{M}(z,w;p,q) = ([z,w]\cdot[p,q])^j \times \text{something}. \tag{2.31}$$

Finally, homogeneity in q and z fix the rest of the matrix element up to an overall coefficient:

$$\mathcal{M}(z,w;p,q) \propto ([z,w]\cdot[p,q])^j(-z\cdot q)^{l-j}(-p\cdot z)^{1-\Delta_i-l}. \tag{2.32}$$

A selection rule

The result (2.32) manifests a selection rule: Light-ray operators with transverse spin j only have nonzero expectation values in density matrices with SO$(d-1)$ spin l at least j. In other words, for $\mathcal{M}$ to be nonvanishing, we must have

$$j \leq l. \tag{2.33}$$

For example, if we average over the beam direction so that only density matrices with $l = 0$ appear, we discard information about light-ray operators with nonzero transverse spin.

Because of the selection rule (2.33), the low transverse spin terms in the $\mathcal{E}\times\mathcal{E}$ OPE (2.2), which have $j = 0, 2, 4$, are sufficient for computing two-point event shapes in density matrices with SO$(d-1)$ spin 4 or less. This includes the scalar density matrix studied in [21, 22, 25, 26]. Higher transverse spin terms are important for density matrices with higher spin and in multi-point event shapes.

The form of the light-ray OPE

The differential operator $\mathcal{C}_{\delta,j}$ appearing in (2.24) has an expansion in the angle θ between detectors, or equivalently in small $-2z_1\cdot z_2 \approx \theta^2$. To leading order in this expansion, $\mathcal{C}_{\delta,j}$ acts as

$$\mathcal{C}_{\delta,j}(z_1,z_2,\partial_z,\partial_w)([z,w]\cdot[p,q])^j f(z) = (-2z_1\cdot z_2)^{\frac{\delta-j-6}{2}}(-[z_1,z_2]\cdot[p,q])^j f(z_2)+\dots, \tag{2.34}$$

where $f(z)$ is any function of z with the correct homogeneity, and "..." indicates higher-order terms in θ. Applying this to (2.32), we find

$$\begin{aligned}&\langle\mathcal{O}_{\mu_1\cdots\mu_k}(p)|\mathbf{L}[T](\infty,z_1)\mathbf{L}[T](\infty,z_2)|\mathcal{O}'_{\mu_{k+1}\cdots\mu_l}(p)\rangle q^{\mu_1}\cdots q^{\mu_l}\\ &=\sum_{j,i}\lambda_{i,j}\left((-2z_1\cdot z_2)^{\frac{\Delta_i-j-7}{2}}(-[z_1,z_2]\cdot[p,q])^j(-z_2\cdot q)^{l-j}(-z_2\cdot p)^{1-\Delta_i-l}+\dots\right),\end{aligned} \tag{2.35}$$

where $\lambda_{i,j}$ are OPE coefficients that are not fixed by Lorentz symmetry. The "..." on the right-hand side are fixed by symmetry and re-sum into a celestial block [15].[6] We have written only the leading term of the celestial block for simplicity.

Let us specialize further to the kinematics of interest. Note that

$$[z_1,z_2]\cdot[p,q] = 2\vec{n}_{12}\cdot\vec{q}, \tag{2.36}$$

where $z_i = (1,\widehat{n}_i)$, $\vec{n}_{12} = \widehat{n}_1-\widehat{n}_2$, and $\vec{q}$ are the spatial components of q. From (2.35), we can compute the leading terms in the OPE for each possible value of l

[6]We give some example calculations of celestial blocks in section 2.7.2.

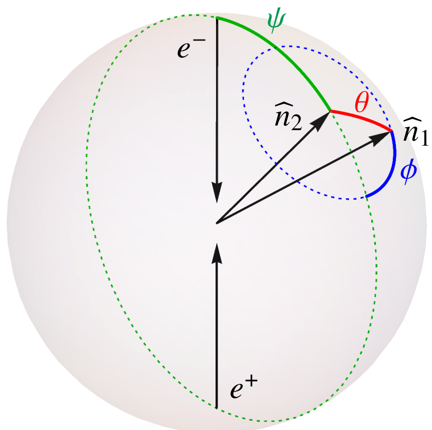

Figure 2.3: Kinematics of a two-point energy correlator in the process $e^+e^- \to$ hadrons. The picture shows the spatial geometry in the center of mass frame; time is suppressed. Particles e^+ and e^- propagate in along the $\pm\vec{e}_z$ directions, and we measure energy flux in the directions $\widehat{n}_1, \widehat{n}_2$. (The correlator is invariant under swapping e^+ and e^-.) ψ (solid green arc) is the angle between one of the detectors $\widehat{n}_2$ and the beam direction, θ (solid red arc) is the angle between detectors, and ϕ (solid blue arc) parametrizes the angle of $\widehat{n}_1$ around $\widehat{n}_2$ on the celestial sphere.

and j:

$$\begin{aligned}
&|\vec{n}_{12}|^{\Delta_i-9}\left((\vec{\epsilon}_+\cdot\vec{n}_{12})(\vec{\epsilon}_-\cdot\vec{n}_{12})-\frac{2}{3}\vec{n}_{12}^2\right)+\dots && (j=2, l=2),\\
&|\vec{n}_{12}|^{\Delta_i-7}\left((\vec{\epsilon}_+\cdot\widehat{n}_2)(\vec{\epsilon}_-\cdot\widehat{n}_2)-\frac{2}{3}\right)+\dots && (j=0, l=2),\\
&|\vec{n}_{12}|^{\Delta_i-7}+\dots && (j=0, l=0).
\end{aligned} \tag{2.37}$$

where we used (2.18) to replace the sum over q. In each case, the "$\dots$" are fixed by symmetry and resum into a celestial block. The fact that light-ray operators with $j = 0$ appear in two different ways reflects the fact that matrix elements of a given light-ray operator can admit multiple three-point structures, each of which comes with its own OPE coefficient. Light-ray operators with $j = 1$ do not appear in the $\mathcal{E} \times \mathcal{E}$ OPE due to permutation symmetry under $\widehat{n}_1 \leftrightarrow \widehat{n}_2$ (which follows from the fact that energy detectors commute).

To be completely explicit, let us parametrize the vectors as

$$\begin{aligned}
\widehat{n}_2 &= R_{zx}(\psi)(0,0,1)\\
\widehat{n}_1 &= R_{zx}(\psi)(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).
\end{aligned} \tag{2.38}$$

where $R_{zx}(\psi)$ is a rotation by ψ in the zx plane. Here, ψ is the angle between the (nearly coincident) detectors and the beam, θ is the angle between detectors, and ϕ

represents a rotation of the two detectors around each other on the celestial sphere, see figure 2.3. The expressions (2.37) become

$$\begin{aligned}
&\theta^{\Delta_i-7}\left(\frac{1}{3}-\cos^2\phi\sin^2\psi\right)+\dots && (j=2, l=2),\\
&\theta^{\Delta_i-7}\left(\sin^2\psi-\frac{2}{3}\right)+\dots && (j=0, l=2),\\
&\theta^{\Delta_i-7}+\dots && (j=0, l=0). \qquad (2.39)
\end{aligned}$$

where "..." are higher order terms in θ. As a check on the first line of (2.39), note that if we set $\psi = 0$ or π, so that the detectors are both approaching the beam direction, the dependence on ϕ goes away, reflecting the fact that our density matrix is invariant under rotations around the beam direction.

Typically both theoretical and experimental analysis of the $e^+e^- \longrightarrow$ hadrons process focus on observables averaged over the beam direction. As explained above, this amounts to throwing away the contribution of light-ray operators with non-zero transverse spin, which contain extra information and can provide further nontrivial tests of QCD. Event shapes that are not averaged over the beam direction, so-called *oriented event shapes*, were studied experimentally at LEP by the DELPHI [31] and OPAL [32] collaborations. For a recent discussion of oriented event shapes in QCD, see [33].

2.2.2 A transverse spin puzzle

So far, we have introduced the kinematics of the light-ray OPE inside a two-point event shape. The next question is: what are the operators $\mathbb{D}_{\Delta_i,j}$? In particular, what are the corresponding values of Δ_i and OPE coefficients? In [15], we derived the low transverse spin terms in the light-ray OPE. For simplicity, consider a product of light-transformed scalars $\mathbf{L}[\phi_1]\mathbf{L}[\phi_2]$. The result of [15] is[7]

$$\begin{aligned}
\mathbf{L}[\phi_1](x,z_1)\mathbf{L}[\phi_2](x,z_2) = \pi i \sum_i C_{\Delta_i-1,0}(z_1,z_2,\partial_{z_2})\mathbb{O}^+_{\Delta_i,J=-1}(x,z_2)\\
+\text{ higher transverse spin}. \qquad (2.40)
\end{aligned}$$

The operators $\mathbb{O}^+_{\Delta_i,J=-1}(x,z)$ are analytic continuations of null-integrated operators $\mathbf{L}[O_i]$ to spin $J=-1$, where O_i appears in the $\phi_1\times\phi_2$ OPE. The quantum number Δ_i becomes the analytic continuation of scaling dimensions of O_i. The operators

[7] As explained in [34], this product is only well-defined for a theory with a sufficiently low Regge intercept. Here we assume this is the case for the sake of illustration.

$\mathbb{O}^+_{\Delta_i,J=-1}(x,z)$ have transverse spin $j = 0$ — in particular they depend only on x and z and not on an additional polarization vector w. The reason is that only traceless symmetric tensors $\mathcal{O}_i$ appear in the OPE of scalar operators. The light-transform $\mathbf{L}[\mathcal{O}_i]$ of a traceless symmetric tensor has vanishing transverse spin, and thus so do its analytic continuations.

Note that in $d = 3$ all (bosonic) operators are traceless-symmetric tensors and correspondingly there is no transverse spin in $d - 2 = 1$ dimension. The higher transverse spin terms in (2.40) are absent in this case. However, in $d > 3$ the transverse spin j can be non-trivial and we expect infinitely many higher transverse spin terms to appear in (2.40), since there is no reason to expect the event shapes to be independent of the angle ϕ discussed above.[8]

It is natural to expect that higher transverse-spin terms in (2.40) should also be related to the $\phi_1 \times \phi_2$ OPE. However, this presents a puzzle: All the primary light-ray operators $\mathbb{O}_{\Delta_i,J}(x,z)$ built from $\phi_1 \times \phi_2$ using the construction of [11] have vanishing transverse spin. How can we build light-ray operators with nonzero transverse spin to play the role of $\mathbb{D}_{\Delta_i,j}$ with $j > 0$? It turns out that the $\mathbb{D}_{\Delta_i,j}$ are *primary descendants* of $\mathbb{O}_{\Delta_i,J}$ — i.e. carefully chosen derivatives of $\mathbb{O}_{\Delta_i,J}$ that nonetheless transform like conformal primaries. In the next section, we explain how such primary descendants arise.

2.3 Building transverse spin with differential operators

2.3.1 Local operators, light transforms, and shortening conditions

Consider a local operator $\mathcal{O}^{\mu_1\cdots\mu_J}(x)$ with dimension Δ and spin J. Throughout this work, we will use index-free notation, where we contract spin indices with an auxiliary null polarization vector $z \in \mathbb{R}^{d-1,1}$:

$$\mathcal{O}(x,z) = \mathcal{O}^{\mu_1\cdots\mu_J}(x) z_{\mu_1\cdots\mu_J}, \qquad (z^2 = 0). \tag{2.41}$$

By construction, $\mathcal{O}(x,z)$ is a homogeneous polynomial of degree J in z. Under a conformal transformation U, we have

$$U\mathcal{O}(x,z)U^{-1} = \Omega(x')^{\Delta}\mathcal{O}(x', R(x')z), \tag{2.42}$$

[8]The more accurate statement is that the first line of (2.40) already contains higher-transverse spin terms: they are generated by the operators $C_{\delta,0}$. However, since the operators $C_{\delta,0}$ are fixed by Lorentz symmetry, these contributions are determined in terms of $j = 0$ contributions and therefore the event shapes would still be over-constrained if there were no additional contributions to (2.40). For example, one would be able to write a differential equation in z_1, z_2 that the product $\mathbf{L}[\phi_1](x,z_1)\mathbf{L}[\phi_2](x,z_2)$ would have to satisfy. It can be checked that this differential equation is incompatible with the leading term of the $\mathbf{L}[\phi_2] \times \mathbf{L}[\phi_3]$ OPE in $\langle\Psi|\mathbf{L}[\phi_1]\mathbf{L}[\phi_2]\mathbf{L}[\phi_3]|\Psi\rangle$.

where $\Omega(x')$ and $R(x')$ are the local rescaling and rotation associated to U.

Index-free notation is more than a convenience. It allows us to describe a wider class of conformal representations than those associated to local operators. As an example, consider the light-transform

$$\mathbf{L}[O](x,z) = \int_{-\infty}^{\infty} d\alpha(-\alpha)^{-\Delta-J} O\left(x - \frac{z}{\alpha}, z\right), \tag{2.43}$$

which is an integral of O in the direction of its polarization vector. Using index-free notation, we can interpret the light-transform as a conformally-invariant transform, changing the quantum numbers (Δ, J) by

$$\mathbf{L} : (\Delta, J) \to (1-J, 1-\Delta). \tag{2.44}$$

In other words, $\mathbf{L}[O](x,z)$ satisfies the conformal transformation law (2.42), with Δ replaced by $1-J$ and J replaced by $1-\Delta$. (This is clearest from the definition of the light-transform in the embedding space [11].) In general, we define the spin of an object as its homogeneity in the polarization vector z. Because $\mathbf{L}[O](x,z)$ has non-integer spin $1-\Delta$, it cannot be written in terms of an underlying tensor with $1-\Delta$ indices.

Though index-free notation appears to treat the nonlocal operator $\mathbf{L}[O]$ in the same way as the local operator O, there is still something special about the representations associated to local operators. Specifically, a local operator is a polynomial in its polarization vector. This can be phrased as a kind of shortening condition. Morally speaking, $J+1$ derivatives of $O(x,z)$ with respect to z must vanish:

$$\text{``}\partial_z^{\mu_1} \cdots \partial_z^{\mu_{J+1}} O(x,z)\text{''} = 0. \tag{2.45}$$

We must take care to write this condition correctly because z is constrained, $z^2 = 0$. Let us parametrize z by

$$z = (1, \vec{y}^2, \vec{y}), \qquad \vec{y} \in \mathbb{R}^{d-2}, \tag{2.46}$$

where we use lightcone coordinates $z = (z^+, z^-, \vec{z})$ with metric $dz^2 = -dz^+ dz^- + d\vec{z} \cdot d\vec{z}$. A more proper shortening condition is

$$\partial_{\vec{y}}^{i_1} \cdots \partial_{\vec{y}}^{i_{J+1}} O(x,z) - \text{traces} = 0, \tag{2.47}$$

where we subtract traces using the metric on $\mathbb{R}^{d-2}$. The argument for (2.47) is as follows. $O(x,z)$ is a sum of products of J factors of $1, \vec{y}^2$ and the components of

$\vec{y}$. If $J+1$ $\vec{y}$-derivatives of some term is nonzero, at least two of those derivatives must act on the same $\vec{y}^2$ factor, resulting in a nonzero trace. By subtracting traces, we remove such terms.

The shortening condition (2.47) is naturally a traceless symmetric tensor with spin $J+1$ in the "transverse" space $\mathbb{R}^{d-2}$. We can write it more economically by introducing a null vector $\vec{s} \in \mathbb{C}^{d-2}$ such that $\vec{s}^2 = 0$:

$$(\vec{s} \cdot \partial_{\vec{y}})^{J+1} \mathcal{O}(x, z) = 0. \tag{2.48}$$

We can make (2.48) Lorentz-invariant by introducing a polarization vector $w = (0, 2\vec{y} \cdot \vec{s}, \vec{s}) \in \mathbb{C}^{d-1,1}$. By construction, w is null and transverse to z, i.e. $w \cdot z = w^2 = 0$. We would like w to encode only the $d-2$ degrees of freedom in $\vec{s}$, so we must additionally impose a gauge-redundancy $w \sim w + \lambda z$. The condition (2.48) finally becomes

$$(w \cdot \partial_z)^{J+1} \mathcal{O}(x, z) = 0. \tag{2.49}$$

One can check that the differential operator $(w \cdot \partial_z)^{J+1}$ preserves the ideal generated by $z^2 = w \cdot z = w^2 = 0$. More nontrivially, it is gauge invariant under $w \to w + \lambda z$ precisely when acting on functions with homogeneity J in z (we show this in (2.64) below).

The shortening condition (2.49) for $\mathcal{O}$ implies a related shortening condition for $\mathbf{L}[\mathcal{O}]$. For simplicity, suppose $\mathcal{O}$ is a scalar, i.e. $J = 0$. By integrating by parts inside the light-transform (2.43), we find

$$\mathbf{L}(w \cdot \partial_z) = \frac{1}{2-\Delta}\Big((z \cdot \partial_x)(w \cdot \partial_z) - (z \cdot \partial_z)(w \cdot \partial_x)\Big)\mathbf{L} \qquad \text{(acting on scalars with dimension } \Delta). \tag{2.50}$$

Consequently, $\mathbf{L}[\mathcal{O}]$ satisfies its own shortening condition

$$\Big((z \cdot \partial_x)(w \cdot \partial_z) - (z \cdot \partial_z)(w \cdot \partial_x)\Big)\mathbf{L}[\mathcal{O}](x, z) = 0 \qquad (\mathcal{O} \text{ scalar}), \tag{2.51}$$

inherited from the shortening condition (2.49) for $\mathcal{O}$. Just as $w \cdot \partial_z$ is Lorentz-invariant (also conformally-invariant) only when acting on scalar representations, $(z \cdot \partial_x)(w \cdot \partial_z) - (z \cdot \partial_z)(w \cdot \partial_x)$ is conformally-invariant only when acting on representations with the quantum numbers of $\mathbf{L}[\mathcal{O}]$, i.e. with dimension $1 - 0 = 1$.

2.3.2 Reducibility and primary descendants

The operator $(z\cdot\partial_x)(w\cdot\partial_z)-(z\cdot\partial_z)(w\cdot\partial_x)$ can be compared to other conformally-invariant differential operators that exist for special quantum numbers. A well-known example is the operator that takes a current to its divergence

$$\mathcal{J}^\mu \to \partial_\mu \mathcal{J}^\mu = (\partial_x\cdot\partial_z)\mathcal{J}(x,z). \tag{2.52}$$

On the right-hand side, we have written the divergence in index-free notation.

The operator $\partial_x\cdot\partial_z$ changes quantum numbers (Δ, J) by

$$\partial_x\cdot\partial_z : (d-1,1)\to(d,0). \tag{2.53}$$

It is conformally-invariant only when acting on operators with the correct dimension and spin. This has the following representation-theoretic interpretation. Let $V_{\Delta,J}$ be a long multiplet (i.e. a generalized Verma module, see e.g. [35]) of the conformal group with dimension Δ and spin J. $V_{\Delta,1}$ is irreducible for generic Δ. However it becomes reducible when $\Delta = d-1$:

$$V_{d-1,1}\supset V_{d,0}. \tag{2.54}$$

The quotient $\widetilde{V}_{d-1,1}\equiv V_{d-1,1}/V_{d,0}$ is the short representation associated to a conserved current. If $|\mathcal{J}(z)\rangle = z_\mu|\mathcal{J}^\mu\rangle$ is the highest-weight state of $V_{d-1,1}$, then the highest-weight state of $V_{d,0}\subset V_{d-1,1}$ is

$$P\cdot\partial_z|\mathcal{J}(z)\rangle \qquad \text{(primary descendant)}. \tag{2.55}$$

We say that (2.55) is a "primary descendant." More formally, there exists a homomorphism

$$\Phi : V_{d,0}\to V_{d-1,1}, \tag{2.56}$$

sending the highest-weight state of $V_{d,0}$ to (2.55) inside $V_{d-1,1}$. By replacing $P\to\partial_x$ inside the expression for the primary descendant, we find the conformally-invariant differential operator (2.53).

In summary, conformally-invariant differential operators are in correspondence with *reducible* generalized Verma modules — i.e. generalized Verma modules that contain primary descendants. The differential operators arising in this way have been studied in the mathematics literature, and also in the physics literature due to their relation to poles in conformal blocks [36, 37, 35, 38]. We refer the reader to [35] for a lucid discussion and classification. In order to compare our operator (2.51) to the classification in [35], we must understand the quantum numbers of the corresponding primary descendant.[9]

2.3.3 Index-free notation for general tensor representations

A general finite-dimensional tensor representation ρ of $\mathrm{SO}(d-1,1)$ has a Young diagram with rows of length $(m_1, m_2, \ldots, m_n)$, where $n = \lfloor \frac{d}{2} \rfloor$, see figure 2.4. We define spin $J = m_1$ as the length of the first row of the Young diagram of ρ. The remaining rows $\lambda = (m_2, \ldots, m_n)$ define a representation of $\mathrm{SO}(d-2)$. We define transverse spin $j = m_2$ as the length of the first row of the Young diagram of λ.

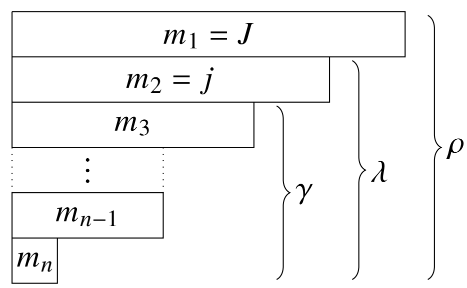

Figure 2.4: A Young diagram for an irreducible representation ρ of $\mathrm{SO}(d-1,1)$. The rows have length $m_1, m_2, \ldots, m_n$. We often write $m_1 = J$ (spin) and $m_2 = j$ (transverse spin). If we remove the first row of the Young diagram for ρ, the remaining rows $(m_2, \ldots, m_n)$ make a Young diagram for an irreducible representation λ of $\mathrm{SO}(d-2)$. If we remove another row, the remaining rows $(m_4, \ldots, m_n)$ make a Young diagram for an irreducible representation γ of $\mathrm{SO}(d-4)$.

An operator in the representation ρ has indices $\mathcal{O}^{\mu_1\cdots\mu_J;\nu_1\cdots\nu_{m_2};\cdots;\rho_1\cdots\rho_{m_n}}(x)$. Each index group delimited by semicolons is symmetric. Furthermore, $\mathcal{O}$ is traceless in all its indices, and satisfies some additional symmetry conditions that we describe below.[10] To use index-free notation, we introduce null polarization vectors

[9] Strictly speaking, we need to be more careful because the notion of generalized (aka parabolic) Verma module depends on a choice of a parabolic subalgebra of conformal algebra. This choice determines, in particular, which quantum numbers can be non-integer. In [35] the parabolic subalgebra is the maximal one for Euclidean conformal algebra, and it allows for non-integer Δ. Here, we are interested in both Δ and J being non-integer, which requires us to consider the maximal parabolic subalgebra of Lorentzian conformal algebra. Therefore, the classification of [35] is not, strictly speaking, applicable here. It would be interesting to have a full classification for Lorentzian conformal group. We take a more simplistic approach: our conformally-invariant differential operators can be identified with analytic continuations in J of the differential operators classified in [35], and this will be enough for our purposes.

[10] For simplicity, we only consider the bosonic representations, i.e. the representations of $\mathrm{SO}(d-1,1)$ and not of $\mathrm{Spin}(d-1,1)$. Furthermore, we ignore the possible self-duality conditions in even d.

$z, w_1, \ldots, w_{n-1}$ for each row:

$$\begin{aligned} O(x, z, \mathbf{w}) &= O(x, z, w_1, \ldots, w_{n-1}) \\ &= O^{\mu_1 \cdots \mu_J; \nu_1 \cdots \nu_{m_2}; \cdots ; \rho_1 \cdots \rho_{m_n}}(x) z_{\mu_1} \cdots z_{\mu_J} w_{1,\nu_1} \cdots w_{1,\nu_{m_2}} \cdots w_{n-1,\rho_1} \cdots w_{n-1,\rho_{m_n}}. \end{aligned} \tag{2.57}$$

Here we used the notation $\mathbf{w}$ to denote the collection of vectors w_i, $i = 1 \cdots n-1$. The polarization vectors satisfy the relations

$$\begin{aligned} 0 &= z^2 \\ 0 &= w_1^2 = w_1 \cdot z \\ 0 &= w_2^2 = w_2 \cdot w_1 = w_2 \cdot z \\ &\vdots \\ 0 &= w_{n-1}^2 = w_{n-1} \cdot w_{n-2} = \cdots = w_{n-1} \cdot z. \end{aligned} \tag{2.58}$$

The additional symmetry properties of O are equivalent to the statement that $O(x, z, \mathbf{w})$ is invariant under the gauge redundancies

$$\begin{aligned} w_1 &\sim w_1 + \# z \\ w_2 &\sim w_2 + \# w_1 + \# z \\ &\vdots \\ w_{n-1} &\sim w_{n-1} + \# w_{n-2} + \cdots + \# z. \end{aligned} \tag{2.59}$$

The object $O(x, z, \mathbf{w})$ is a homogeneous polynomial of the polarization vectors $z, w_1, \ldots, w_{n-1}$ with degrees $(J, m_2, \ldots, m_n)$. We often abbreviate $w_1 = w$.

The index-free formalism we have just developed is essentially the embedding space formalism for the Lorentz group.[11] In the embedding formalism, a traceless symmetric tensor operator in d-dimensions becomes a homogeneous function $O(X, Z)$ of variables $X, Z \in \mathbb{R}^{d+1,1}$ satisfying $X^2 = X \cdot Z = Z^2 = 0$ and a gauge redundancy $Z \sim Z + \lambda X$ [30]. The Lorentz group $\mathrm{SO}(d-1, 1)$ is of course the conformal group in $d-2$ dimensions. Thus the embedding formalism applies, with the $d+2$ dimensional X and Z replaced by the d-dimensional z and w. In particular, z can be interpreted as an embedding-space coordinate in $d-2$ dimensions.

2.3.4 Raising transverse spin

We can now recognize $w \cdot \partial_z$ (with $w_1 = w$) as a differential operator that raises transverse spin j, since it increases the degree in w by 1. Specifically, it changes the

[11] More generally, it comes from the Borel-Weil theorem, see [15] for details.

quantum numbers (Δ, J, j) by

$$w \cdot \partial_z : (\Delta, 0, 0) \to (\Delta, -1, 1). \tag{2.60}$$

The operator $(z \cdot \partial_x)(w \cdot \partial_z) - (z \cdot \partial_z)(w \cdot \partial_x)$, obtained by commuting $w \cdot \partial_z$ through the light-transform, changes (light-transformed) quantum numbers $(1 - J, 1 - \Delta, j)$ by

$$(z \cdot \partial_x)(w \cdot \partial_z) - (z \cdot \partial_z)(w \cdot \partial_x) : (1, 1 - \Delta, 0) \to (2, 1 - \Delta, 1). \tag{2.61}$$

More generally, let us define the operator

$$\mathcal{D}'_n = \frac{1}{n!}(w \cdot \partial_z)^n, \tag{2.62}$$

$$\mathcal{D}'_n : (\Delta, n + j - 1, j) \to (\Delta, j - 1, j + n). \tag{2.63}$$

The operator $\mathcal{D}'_n$ provides the shortening condition for the Lorentz representation of local operators with spin $J = n + j - 1$ and transverse spin j. Note that $\mathcal{D}'_n$ preserves the ideal generated by $z^2 = w \cdot z = w^2 = 0$. More nontrivially, it is gauge-invariant under $w \to w + \lambda z$. To see this, we act with the generator of a gauge transformation $z \cdot \partial_w$ on $\mathcal{D}'_n f(z, w)$, where $f(z, w)$ is gauge-invariant:

$$\begin{aligned} z \cdot \partial_w (w \cdot \partial_z)^n f(z, w) &= \sum_{j=0}^{n-1} (w \cdot \partial_z)^j (z \cdot \partial_z - w \cdot \partial_w)(w \cdot \partial_z)^{n-j-1} f(z, w) \\ &= \sum_{j=0}^{n-1} (2j - n + 1)(w \cdot \partial_z)^{n-1} f(z, w) \\ &= 0. \end{aligned} \tag{2.64}$$

In the first line, we commute $z \cdot \partial_w$ past other operators until it acts on $f(z, w)$, which it kills. In the second line, we use that $f(z, w)$ has homogeneity $n + j - 1$ in z and j in w. This computation shows that $\mathcal{D}'_n$ is Lorentz-invariant. Because local operators transform in irreducible representations of the Lorentz group, Schur's lemma implies that they must be killed by $\mathcal{D}'_n$.

By commuting $\mathcal{D}'_n$ through the light transform,

$$\mathbf{L}\mathcal{D}'_n = \mathcal{D}_n \mathbf{L}, \tag{2.65}$$

we find a new operator $\mathcal{D}_n$ given by

$$\begin{aligned} \mathcal{D}_n &= \frac{(-1)^n \Gamma(\Delta + j - 2)}{\Gamma(\Delta + j - 2 + n)\Gamma(n + 1)} (\partial_x \cdot \mathcal{D}^{0+}_{z,w})^n, \\ \mathcal{D}^{0+}_{z,w;\mu} &= w_\mu (w \cdot \partial_w - z \cdot \partial_z) + z_\mu w \cdot \partial_z, \end{aligned} \tag{2.66}$$

Here, $(\mathcal{D}^{0+}_{z,w})^\mu$ is a weight-shifting operator for the Lorentz group $\mathrm{SO}(d-1,1)$ in the vector representation [39]. It can be obtained from the vector weight-shifting operators for the conformal group in [39] by using the analogy of (z,w) with the embedding-space coordinates (X,Z). The operator $\mathcal{D}_n$ is conformally-invariant precisely when acting on the quantum numbers obtained by light-transforming (2.63):

$$\mathcal{D}_n : (2-n-j, 1-\Delta, j) \to (2-j, 1-\Delta, j+n). \tag{2.67}$$

The operator $\mathcal{D}_n$ falls into the classification of reducible Verma modules described in [35]. In the notation of [35], it has type $\mathrm{I}_{2,n}$.[12,9]

To summarize, by commuting $\mathcal{D}'_n$ through the light transform, we obtain a special conformally-invariant differential operator $\mathcal{D}_n$ that raises transverse spin. By construction, $\mathcal{D}_n$ vanishes when acting on the light transform of a local operator with the appropriate quantum numbers, since

$$\mathcal{D}_n \mathbf{L}[O] = \mathbf{L}[\mathcal{D}'_n O] = 0. \tag{2.68}$$

Shortly, we will encounter light-ray operators $\mathbb{O}^{\pm}_{i,J,j}(x,z)$ that are conformal primaries with quantum numbers $(1-J, 1-\Delta_i, j)$, but that are not light-transforms of local operators. When $J = n+j-1$ with $n \in \mathbb{Z}_{\geq 0}$, we can act on such operators with $\mathcal{D}_n$ to obtain new primary descendants with higher transverse spin

$$\mathcal{D}_n \mathbb{O}^{\pm}_{i,J=n+j-1,j}(x,z) : (2-j, 1-\Delta_i, j+n). \tag{2.69}$$

Primary descendants of this type will provide the higher transverse spin terms in the light-ray OPE.

2.4 The complete OPE of scalar detectors

In this section we will explicitly derive the OPE of the form (2.24) for two scalar detectors $\mathbf{L}[\phi_1]\mathbf{L}[\phi_2]$. As discussed in [34], products of scalar detectors are not in general well-defined non-perturbatively. Nevertheless, they make sense kinematically, and give us a nice playground to demonstrate the key concepts, which we will then generalize to arbitrary detectors in section 2.5. Furthermore, in section 2.7 we will see an example where $\mathbf{L}[\phi_1]\mathbf{L}[\phi_2]$ is well-defined and the formulas from this section can be verified.

---

[12] Similarly, we can think of $\mathcal{D}'$ as an invariant differential operator with respect to $\mathrm{SO}(d-1,1)$ (the Lorentz group), thought of as the conformal group in $d-2$ dimensions. In the notation of [35], it has type $\mathrm{I}_{1,n}$.

2.4.1 Kinematics of the OPE

Let us understand the general form that the OPE should take. First note that the product

$$\mathbb{W}(x, z_1, z_2) \equiv \mathbf{L}[\phi_1](x, z_1)\mathbf{L}[\phi_2](x, z_2), \tag{2.70}$$

is a conformal primary, if it is well-defined. This is because

$$\begin{aligned}[K_\mu, \mathbb{W}(0, z_1, z_2)] &= [K_\mu, \mathbf{L}[\phi_1](0, z_1)\mathbf{L}[\phi_2](0, z_2)] \\ &= [K_\mu, \mathbf{L}[\phi_1](0, z_1)]\mathbf{L}[\phi_2](0, z_2) + \mathbf{L}[\phi_1](0, z_1)[K_\mu, \mathbf{L}[\phi_2](0, z_2)] = 0.\end{aligned} \tag{2.71}$$

Similarly, the scaling dimension of $\mathbb{W}(x, z_1, z_2)$ is the sum of scaling dimensions of $\mathbf{L}[\phi_i]$,

$$\Delta_{\mathbb{W}} = (1 - J_1) + (1 - J_2) = 2. \tag{2.72}$$

However, the operator $\mathbb{W}(x, z_1, z_2)$ does not transform irreducibly under the Lorentz group — it depends on two polarization vectors, but they do not satisfy any of the relations described in section 2.3 among themselves. The OPE (2.24) decomposes $\mathbb{W}(x, z_1, z_2)$ into irreducible components, since the detectors $\mathbb{D}(x, z, w)$ are irreducible under the Lorentz group.

In fact, the problem of decomposing $\mathbb{W}(x, z_1, z_2)$ into irreducible representations of the Lorentz group is a familiar one. The Lorentz group $\mathrm{SO}(d-1, 1)$ is isomorphic to the Euclidean conformal group in $d-2$ dimensions. Under this isomorphism, z_i become embedding-space coordinates [40, 30] for a fictitious CFT_{d-2}. By the definition of spin we have $\mathbf{L}[\phi_i](x, \lambda z_i) = \lambda^{1-\Delta_i}\mathbf{L}[\phi_i](x, z_i)$, where Δ_i is the scaling dimension of ϕ_i.[13] At the same time, primary operators $\mathcal{P}_\delta$ in the embedding formalism for the fictitious CFT_{d-2} should satisfy $\mathcal{P}_\delta(\lambda z_1) = \lambda^{-\delta}\mathcal{P}_\delta(z_1)$, where δ is the $(d-2)$-dimensional scaling dimension. Thus, if we interpret the Lorentz group as a $(d-2)$-dimensional conformal group, the transformation properties of $\mathbb{W}(x, z_1, z_2)$ under Lorentz transformations can be described as[14]

$$\mathbb{W}(0, z_1, z_2) \sim \mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2), \tag{2.73}$$

where $\delta_i = \Delta_i - 1$. In particular, if we parameterize z_i as in (2.46),

$$z_i^+ = 1, \quad z_i^- = \vec{y}^2, \quad z_i^\mu = y^\mu \ (\mu = 2, \cdots, d-1), \tag{2.74}$$

[13]Recall that spin of $\mathbf{L}[\phi_i]$ is $1 - \Delta_i$.

[14]Provided that by Lorentz group we mean the group that fixes x. We take $x = 0$ and use the standard Lorentz group for concreteness.

for $\vec{y} \in \mathbb{R}^{d-2}$, then

$$\mathbb{W}(0, \vec{y}_1, \vec{y}_2) \sim \mathcal{P}_{\delta_1}(\vec{y}_1)\mathcal{P}_{\delta_2}(\vec{y}_2) \tag{2.75}$$

transforms exactly like a pair of scalar primaries with dimensions δ_1, δ_2 at coordinates $\vec{y}_1, \vec{y}_2$ in the fictitious CFT_{d-2}.

We are familiar with taking the usual OPE between two scalar primaries,

$$\mathcal{P}_{\delta_1}(\vec{y}_1)\mathcal{P}_{\delta_2}(\vec{y}_2) = \sum_{\delta,j} C_{\delta,j;\mu_1\cdots\mu_j}(\vec{y}_1, \vec{y}_2, \partial_{\vec{y}_2})\mathcal{P}^{\mu_1\cdots\mu_j}_{\delta,j}(\vec{y}_2), \tag{2.76}$$

where j is the $(d-2)$-dimensional traceless-symmetric spin.[15] The differential operator $C_{\delta,j;\mu_1\cdots\mu_j}(\vec{y}_1, \vec{y}_2, \partial_{\vec{y}_2})$ is fixed by $(d-2)$-dimensional conformal symmetry and is the usual operator in Euclidean CFT_{d-2}. It implicitly depends on δ_1, δ_2. For example, the leading term of this operator in the $\vec{y}_1 \to \vec{y}_2$ limit is given by

$$C^{\mu_1\cdots\mu_j}_{\delta,j}(\vec{y}_1, \vec{y}_2, \partial_{\vec{y}_2}) = |\vec{y}_{12}|^{\delta-\delta_1-\delta_2-j}(y_{12}^{\mu_1}\cdots y_{12}^{\mu_j} - \text{traces}) + \cdots . \tag{2.77}$$

It will be convenient to work with (2.76) written in embedding space coordinates. For this, note that traceless-symmetric primary operators $\mathcal{P}^{\mu_1\cdots\mu_j}_{\delta,j}(\vec{y})$ are described in the embedding formalism as functions of z, w satisfying $z^2 = w^2 = z\cdot w = 0$ subject to

$$\mathcal{P}_{\delta,j}(z, w) = \mathcal{P}_{\delta,j}(z, w + \alpha z), \tag{2.78}$$

$$\mathcal{P}_{\delta,j}(z, \lambda w) = \lambda^j \mathcal{P}_{\delta,j}(z, w). \tag{2.79}$$

In the notation of [30] we have $P_{\text{there}} = z_{\text{here}}$, $Z_{\text{there}} = w_{\text{here}}$. So we can rewrite (2.76) as

$$\mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2) = \sum_{\delta,j} C_{\delta,j}(z_1, z_2, \partial_{z_2}, \partial_{w_2})\mathcal{P}_{\delta,j}(z_2, w_2), \tag{2.80}$$

where $C_{\delta,j}(z_1, z_2, \partial_{z_2}, \partial_{w_2})$ is $C_{\delta,j;\mu_1\cdots\mu_j}(\vec{y}_1, \vec{y}_2, \partial_{\vec{y}_2})$, "lifted" to the embedding space. The embedding space expression for $C_{\delta,j}$ has been studied in a number of papers (e.g. [41, 42, 1, 43] to name a few), but we will not need its explicit form. All we need is that it is $\mathrm{SO}(d-1,1)$-invariant and upon restriction to the Poincaré section it becomes (2.76).

[15] Normally, we would have OPE coefficients in the right-hand side. Since here we are just discussing kinematics, we omit them. One can imagine that they have been absorbed into the definition of $\mathcal{P}_{\delta,j}$.

Translating this back to $\mathbb{W}(x, z_1, z_2)$, we see that the Lorentz symmetry requires the detector OPE to take the form

$$\mathbb{W}(x, \vec{y}_1, \vec{y}_2) = \sum_i C_{\delta_i, j_i;\mu_1\cdots\mu_{j_i}}(\vec{y}_1, \vec{y}_2, \partial_{\vec{y}_2})\mathbb{W}_i^{\mu_1\cdots\mu_{j_i}}(x, \vec{y}_2) \tag{2.81}$$

or

$$\mathbf{L}[\phi_1](x, z_1)\mathbf{L}[\phi_2](x, z_2) = \mathbb{W}(x, z_1, z_2) = \sum_i C_{\delta_i, j_i}(z_1, z_2, \partial_{z_2}, \partial_{w_2})\mathbb{W}_i(x, z_2, w_2), \tag{2.82}$$

for some primary operators $\mathbb{W}_i(x, z, w)$ with $(d-2)$-dimensional quantum numbers δ_i, j_i.[16] Note that w satisfies the same properties as the second-row polarization vector w in section 2.3, and so the operators $\mathbb{W}_i(x, z, w)$ transform in irreps of $\mathrm{SO}(d-1, 1)$ with two-row Young diagrams, with the first row of length $-\delta_i$ and the second row of length j_i.

2.4.2 Harmonic analysis on the celestial sphere

So far, (2.82) is an ansatz based on kinematics. In this section we will show under which conditions (2.82) holds, and provide a formula for $\mathbb{W}_i(x, z, w)$.

We will focus on a generic matrix element of $\mathbb{W}(x, z_1, z_2)$. Specifically, we will consider an expectation value of the form

$$W(z_1, z_2) \equiv \langle O_4(p)|\mathbb{W}(\infty, z_1, z_2)|O_3(p)\rangle, \tag{2.83}$$

for some local operators O_3, O_4. We allow O_3 and O_4 to carry arbitrary spin and do not require them to be primary.

In the language of the fictitious CFT_{d-2}, $W(z_1, z_2)$ is a function of two points that at each point transforms as a scalar primary of dimension δ_i. The main idea is then to decompose $W(z_1, z_2)$ into $(d-2)$-dimensional conformal partial waves [44, 45]. This is equivalent to constructing the $\mathrm{SO}(d-1, 1)$ Casimir operators from the Lorentz generators $M_{\mu\nu}$ defined by

$$M^{\mu\nu} = M_1^{\mu\nu} + M_2^{\mu\nu}, \tag{2.84}$$

$$M_i^{\mu\nu} = z_i^\nu \frac{\partial}{\partial z_{\mu,i}} - z_i^\mu \frac{\partial}{\partial z_{\nu,i}}, \tag{2.85}$$

[16] Above, we had set $x = 0$ but by translation invariance this is valid for any x.

and finding a basis of their common eigenfunctions.[17] In fact, note that the standard conformally-invariant three-point functions

$$\langle \mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z,w)\rangle = \frac{(-4)^j (w\cdot z_2 z_1\cdot z - w\cdot z_1 z_2\cdot z)^j}{(-2z_1\cdot z_2)^{\frac{\delta_1+\delta_2-\delta+j}{2}}(-2z_1\cdot z)^{\frac{\delta_1+\delta-\delta_2+j}{2}}(-2z_2\cdot z)^{\frac{\delta_2+\delta-\delta_1+j}{2}}} \tag{2.86}$$

are eigenfunctions of all the Casimirs for any z, w and δ, j.[18] This is because by conformal invariance, the action of the Casimirs on z_1, z_2 on (2.86) is equivalent to the action on z, w, and $\mathcal{P}_{\delta,j}$ is irreducible. Modulo technical details that we omit for simplicity (see [45, 15]), these eigenfunctions form a complete basis and we have

$$W(z_1,z_2) = \sum_{j=0}^{\infty}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{d\delta}{2\pi i}\int D^{d-2}z\,\langle \mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z,D_w)\rangle W'_{\widetilde{\delta},j}(z,w) \tag{2.87}$$

for some $W'_{\widetilde{\delta},j}(z,w)$, $\widetilde{\delta} = d-2-\delta$. Here the integration measure in z is the standard [46][19]

$$D^{d-2}z = 2\frac{d^d z\,\delta(z^2)\theta(z^0)}{\mathrm{vol}\,\mathrm{SO}(1,1)}, \tag{2.88}$$

and D_w is roughly equivalent to ∂_w [30] and serves to contract the indices encoded by w between W' and $\mathcal{P}_{\delta,j}$. This can be written in a form more similar to (2.82) by using the fact

$$\langle \mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z,w)\rangle = C_{\delta,j}(z_1,z_2,\partial_{z_2},\partial_{w_2})\langle \mathcal{P}_{\delta,j}(z_2,w_2)\mathcal{P}_{\delta,j}(z,w)\rangle, \tag{2.89}$$

for the normalization of $C_{\delta,j}$ as in (2.77) and for the standard two-point function given by[18]

$$\langle \mathcal{P}_{\delta,j}(z_1,w_1)\mathcal{P}_{\delta,j}(z_2,w_2)\rangle = \frac{(-2(w_1\cdot w_2)+2(w_1\cdot z_2)(w_2\cdot z_1)/(z_1\cdot z_2))^j}{(-2z_1\cdot z_2)^{\delta}}. \tag{2.90}$$

[17] These Casimir operators are only self-adjoint if $\delta_i \in \frac{d-2}{2}+i\mathbb{R}$. The forthcoming discussion is also only strictly rigorous under this condition and for square-integrable $W(z_1,z_2)$. In practice, however, usually one can analytically continue to general δ_i and work with non-square-integrable $W(z_1,z_2)$, provided certain care is taken in the process. See the discussions in [9, 10, 11] for details.

[18] Our conventions for tensor structures are summarized in appendix A.2.

[19] Upon restriction to the Poincaré section (2.74), this measure becomes simply $d^{d-2}\vec{y}$.

Using (2.89), we find[20]

$$W(z_1,z_2) = \sum_{j=0}^{\infty} \int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty} \frac{d\delta}{2\pi i} C_{\delta,j}(z_1,z_2,\partial_{z_2},\partial_{w_2}) W_{\delta,j}(z_2,w_2), \tag{2.91}$$

where

$$W_{\delta,j}(z_2,w_2) = \int D^{d-2}z \langle \mathcal{P}_{\delta,j}(z_2,w_2)\mathcal{P}_{\delta,j}(z,D_w)\rangle W'_{\widetilde{\delta},j}(z,w). \tag{2.92}$$

Using orthogonality of eigenfunctions, we can now find the inverse to (2.87) to compute W' and thus W. It is given by taking the inner product with the eigenfunctions,

$$W_{\delta,j}(z,w) \equiv \alpha_{\delta,j} \int D^{d-2}z_1 D^{d-2}z_2 \langle \widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z,w)\rangle W(z_1,z_2), \tag{2.93}$$

where the normalization coefficient has been computed, for example, in [15]

$$\alpha_{\delta,j} = \frac{(-1)^j \Gamma(j+\frac{d-2}{2})\Gamma(d-2+j-\delta)\Gamma(\delta-1)}{2\pi^{d-2}\Gamma(j+1)\Gamma(\delta-\frac{d-2}{2})\Gamma(\delta+j-1)} \frac{\Gamma(\frac{\delta+j+\delta_1-\delta_2}{2})\Gamma(\frac{\delta+j+\delta_2-\delta_1}{2})}{\Gamma(\frac{d-\delta+j+\delta_1-\delta_2}{2})\Gamma(\frac{d-\delta+j+\delta_2-\delta_1}{2})}, \tag{2.94}$$

and $\widetilde{\mathcal{P}}_\delta \equiv \mathcal{P}_{\widetilde{\delta}} = \mathcal{P}_{d-2-\delta}$.[21]

What is the relationship between the OPE ansatz (2.82) and the integral expression (2.91)? If $W_{\delta,j}(z,w)$ is meromorphic in the right half-plane of δ, we can deform the integration contour in (2.91).[22] In this way, we would obtain

$$\langle O_4(p)|\mathbb{W}_i(\infty,z,w)|O_3(p)\rangle = -\mathrm{res}_{\delta=\delta_i} W_{\delta,j_i}(z,w). \tag{2.95}$$

Since p, O_3, and O_4 were arbitrary, this completely determines $\mathbb{W}_i(\infty,z,w)$ as an operator.

[20] A subtlety is that (2.89) does not converge for all values of the coordinates, for similar reasons that the usual Euclidean OPE does not converge in all Euclidean configurations. The proper justification of this step is the same as in the case when one passes from conformal partial waves to conformal blocks in partial conformal wave expansion of a four-point function [47, 45].

[21] For $\delta_i \in \frac{d-2}{2} + i\mathbb{R}$ we have $\widetilde{\delta}_i = \delta_i^*$ and (2.93) just contains the complex conjugate of the wavefunction, as usual — recall footnote 17.

[22] In such deformations of the contour in the partial wave expansions one usually encounters "spurious" poles. Instead of representing physical contributions, they are purely kinematical and cancel among each other or against the possible discrete series contributions that we omitted in (2.87). See [9] for an example of cancellation of these poles in usual four-point functions, and [15] for an example of cancellation in event shapes (for low transverse spin). To simplify the discussion, in this paper we assume that the spurious poles always cancel.

2.4.3 Relation to light-ray operators

Review of light-ray operators

We now show that $W_{\delta,j}(z,w)$ is indeed expected to be meromorphic, with residues related to light-ray operators. Let us first review the definition and some basic properties of these operators.

Light-ray operators [11] are primary operators

$$\widehat{\mathbb{O}}^{\pm}_{i,J,\lambda}(x,z,\mathbf{w}), \tag{2.96}$$

parametrized by a spin J (the length of the first row of the $\mathrm{SO}(d-1,1)$ Young diagram), an $\mathrm{SO}(d-2)$ representation λ (the remaining rows of the $\mathrm{SO}(d-1,1)$ Young diagram), a Regge trajectory label i, and a signature $\pm$. Their defining property is that they are defined for generic complex spin J and are related to local operators for certain integer values of J. Specifically, we have

$$\widehat{\mathbb{O}}^{\pm}_{i,J,\lambda}(x,z,\mathbf{w}) = \mathbf{L}[\mathcal{O}](x,z,\mathbf{w}) \qquad (J \in \mathbb{Z}_{\geq 0} \text{ and } (-1)^J = \pm 1) \tag{2.97}$$

for some local operator $\mathcal{O}$ with $\mathrm{SO}(d-1,1)$ representation (J,λ) whenever J is a non-negative integer such that $(-1)^J = \pm 1$ (where $\pm$ indicates the signature). That is, the operators $\widehat{\mathbb{O}}^{+}_{i,J,\lambda}$ are related to even-spin local operators, while $\widehat{\mathbb{O}}^{-}_{i,J,\lambda}$ are related to odd-spin local operators. The light-ray operators $\widehat{\mathbb{O}}^{\pm}_{i,J,\lambda}$ thus organize the local operators into continuous families or "Regge trajectories," and we label different trajectories by the index i.[23]

In practice we construct light-ray operators from a product of two primaries $\mathcal{O}_1$ and $\mathcal{O}_2$.[24] In this section, we are interested in light-ray operators that appear in the ϕ_1,ϕ_2 OPE, so we use $\mathcal{O}_1=\phi_1, \mathcal{O}_2=\phi_2$ to construct them. These operators all have $\lambda=0$ since only traceless-symmetric operators appear in the OPE of two scalars, and analytic continuation in spin does not affect the second and higher rows of the

[23]The simple picture of operators organized into isolated Regge trajectories is not rigorously proven. We give a discussion of its correctness in appendix A.1. Even if it is not correct, the results of this paper are mostly unchanged, but become more awkward to phrase. For simplicity of presentation, we take it as an assumption in the main text, and delegate more nuanced discussion to appendix A.1.

[24]We expect that we get the same light-ray operators from any pair of local operators with the appropriate quantum numbers.

Young diagram. We define, following [11]

$$\mathbb{O}^{\pm}_{\Delta,J}(x,z) \equiv \pm \int'_{\substack{x_1\approx x^+\\ x_2\approx x}} d^dx_1 d^dx_2 K^t_{\Delta,J}(x_1,x_2;x,z)\phi_1(x_1)\phi_2(x_2)$$
$$+ \int'_{\substack{x_1\approx x\\ x_2\approx x^+}} d^dx_1 d^dx_2 K^u_{\Delta,J}(x_1,x_2;x,z)\phi_2(x_2)\phi_1(x_1), \tag{2.98}$$

where [11]

$$K^t_{\Delta,J}(x_1,x_2;x,z) = \beta_{\Delta,J}\langle 0|\widetilde{\phi}_1(x_1)\mathbf{L}[O]\widetilde{\phi}_2(x_2)|0\rangle,$$
$$K^u_{\Delta,J}(x_1,x_2;x,z) = \beta_{\Delta,J}\langle 0|\widetilde{\phi}_2(x_2)\mathbf{L}[O]\widetilde{\phi}_1(x_1)|0\rangle, \tag{2.99}$$

where $\langle 0|\cdots|0\rangle$ denote the standard tensor structures for the Wightman three-point functions as defined in appendix A.2.[25] The primes on the integrals indicate that we should restrict x_1 and x_2 to an arbitrary small neighborhood of the null cone of x. The dependence on this arbitrary choice will go away momentarily. We also used the notation $a \approx b$ to indicate that points a and b are spacelike, and x^+ denotes the image of x in the next Poincaré patch on the Lorentzian cylinder. The integrals are understood to be over the Lorentzian cylinder in order to preserve manifest conformal invariance. The coefficient $\beta_{\Delta,J}$ has an expression in terms of the Plancherel measure of $\mathrm{SO}(d+1,1)$, shadow coefficients, and Euclidean three-point pairings [11]. Here we will only need the explicit expression

$$\beta_{\Delta,J} = \frac{\Gamma(J+\frac{d}{2})\Gamma(d+J-\Delta)\Gamma(\Delta-1)}{2\pi^d\Gamma(J+1)\Gamma(\Delta-\frac{d}{2})\Gamma(\Delta+J-1)}\frac{\Gamma(\frac{\Delta+J+\Delta_{12}}{2})\Gamma(\frac{\Delta+J-\Delta_{12}}{2})}{\Gamma(\frac{d-\Delta+J+\Delta_{12}}{2})\Gamma(\frac{d-\Delta+J-\Delta_{12}}{2})}, \tag{2.100}$$

where $\Delta_{12} = \Delta_1 - \Delta_2$.

We then finally define

$$\mathbb{O}^{\pm}_{i,J}(x,z) = \mathrm{res}_{\Delta=\Delta_i(J)}\mathbb{O}^{\pm}_{\Delta,J}(x,z), \tag{2.101}$$

where $\Delta_i(J)$ are the scaling dimensions of the i-th Regge trajectory. To be more precise, from the Lorentzian inversion formula the matrix elements of $\mathbb{O}^{\pm}_{\Delta,J}(x,z)$ have poles at $\Delta = \Delta_i(J)$ for integer $J \geq 0$, $(-1)^J = \pm 1$, and these poles analytically continue to $J \in \mathbb{C}$ (modulo subtleties discussed in appendix A.1). The poles come from the integration region in (2.98) where x_1 and x_2 approach the null-cone of x, and thus the residues are independent of the precise choice of integration region for the primed integrals. The operators thus defined satisfy

$$\mathbb{O}^{\pm}_{i,J}(x,z) = f_{12O}\mathbf{L}[O](x,z), \qquad \text{when } J \in \mathbb{Z}_{\geq 0},\ (-1)^J = \pm 1, \tag{2.102}$$

[25] In particular, the structures appearing in K^t and K^u are not related to each other by a direct analytic continuation. Instead, they differ simply by the substitution $1 \leftrightarrow 2$.

for some local operator O, where f_{12O} is the OPE coefficient appearing in $\langle\phi_1\phi_2 O\rangle$. For general $J \in \mathbb{C}$ we have

$$\mathbb{O}^{\pm}_{i,J}(x,z) = f_{12i}(J)\widehat{\mathbb{O}}^{\pm}_{i,J}(x,z), \tag{2.103}$$

where $f_{12i}(J)$ is the analytic continuation of f_{12O} along the i-th Regge trajectory. Note that $\mathbb{O}^{\pm}_{i,J}(x,z)$ was constructed from a particular pair of local operators ϕ_1, ϕ_2. The above relation provides the connection between $\mathbb{O}^{\pm}_{i,J}(x,z)$ constructed from different pairs of local operators.

While the normalized operators $\widehat{\mathbb{O}}^{\pm}_{i,J}(x,z)$ are more fundamental, for our purposes it will be more convenient to work with the operators $\mathbb{O}^{\pm}_{i,J}(x,z)$ constructed from ϕ_1, ϕ_2.

The transverse spin puzzle

We now relate $\mathbb{W}_i(x,z,w)$ in (2.82) to the light-ray operators $\mathbb{O}^{\pm}_{i,J}(x,z)$. Consider the quantum numbers of $\mathbb{O}^{\pm}_{i,J}(x,z)$. By construction, they have scaling dimension $1-J$ and a one-row Lorentz representation with spin $1-\Delta_i(J)$. In particular, they have $j=0$. This is the analytic continuation of the fact that only traceless-symmetric operators appear in the OPE of two scalars.

On the other hand, the operators $\mathbb{W}_i(x,z,w)$ have scaling dimension 2 and two-row Lorentz representations with spin $-\delta_i$ and transverse spin j_i. If $j_i = 0$, we can match quantum numbers with light-ray operators by setting $J=-1$ and $\delta_i = \Delta_k(J=-1)-1$, leading us to expect

$$\mathbb{W}_i(x,z) \sim \mathbb{O}^{+}_{k,-1}(x,z) + \mathbb{O}^{-}_{k,-1}(x,z) \qquad (\text{for } \mathbb{W}_i \text{ with } j_i = 0). \tag{2.104}$$

Here "$\sim$" means that the left-hand side is a linear combination of objects on the right-hand side. Note that the expressions such as

$$\mathbb{W}_i(x,z) \sim \partial_x^2 \mathbb{O}^{+}_{k',1}(x,z), \qquad (\text{wrong}) \tag{2.105}$$

while allowed by dilatation and Lorentz symmetry, are forbidden by the full conformal symmetry, since $\mathbb{W}_i(x,z,w)$ must transform as primary operators.[26]

A relation of the form (2.104) was proven in [15], where $\mathbb{W}_i$ with $j_i = 0$ were called "low transverse-spin" terms. Having an expression for the low transverse-spin terms

[26] Note that ∂_x^2 can be conformally-invariant, i.e. produce a primary, but only if it acts on a scalar operator of dimension $\frac{d-2}{2}$, which is not the case here.

is sufficient for computing certain even shapes, as explored in [15], but does not provide a complete OPE expansion.

Finding an expression for $\mathbb{W}_i(x,z,w)$ with $j_i > 0$ in terms of $\mathbb{O}^{\pm}_{i,J}(x,z)$ might seem hopeless due to the mismatch of quantum numbers. Luckily, our discussion in section 2.3 provides us with a family of conformally-invariant differential operators $\mathcal{D}_n$ which have the property that

$$(\mathcal{D}_n \mathbb{O}^{\pm}_{i,J=n-1})(x,z,w) \tag{2.106}$$

transforms like a primary with scaling dimension 2, spin $1-\Delta_i$ and transverse spin n. We can thus conjecture that for all j_i

$$\mathbb{W}_i(x,z,w) \sim (\mathcal{D}_{j_i}\mathbb{O}^{+}_{k,J=j_i-1})(x,z,w) + (\mathcal{D}_{j_i}\mathbb{O}^{-}_{k,J=j_i-1})(x,z,w), \tag{2.107}$$

where $\delta_i = \Delta_k(J=j_i-1)-1$. (We use the convention $\mathcal{D}_0 = 1$.) This expression is similar in spirit to the wrong example (2.105) above, except that instead of ∂_x^2 we use a carefully-constructed differential operator $\mathcal{D}_n$ that makes the right-hand side transform as a primary. As discussed in section 2.3, such operators are rare and it is an intriguing mathematical conspiracy that the quantum numbers of $\mathbb{W}_i(x,z,w)$ are precisely such that $\mathcal{D}_n$ exist. In particular, one can verify that no other conformally-invariant differential operators exist in the classification of [35][9] that can be used in the ansatz for $\mathbb{W}_i(x,z,w)$.

Partial waves and light-ray operators

As we discussed in section 2.4.2, the matrix elements of the operators $\mathbb{W}_i(x,z,w)$ are residues of $W_{\delta,j}(x,z,w)$ defined by (2.93), which we can write as (restoring x-dependence)

$$W_{\delta,j}(x,z,w) = \alpha_{\delta,j}\int D^{d-2}z_1 D^{d-2}z_2 \langle \widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z,w)\rangle \langle \mathcal{O}_4|\mathbf{L}[\phi_1](x,z_1)\mathbf{L}[\phi_2](x,z_2)|\mathcal{O}_3\rangle. \tag{2.108}$$

By expanding the definition of $\mathbf{L}$, this can be understood as a particular integral of

$$\langle \mathcal{O}_4|\phi_1(x_1)\phi_2(x_2)|\mathcal{O}_3\rangle \tag{2.109}$$

over x_1, x_2.

Similarly, the matrix elements of $(\mathcal{D}_j\mathbb{O}^{\pm}_{i,J=j-1})(x,z,w)$ are residues of

$$\langle\mathcal{O}_4|(\mathcal{D}_j\mathbb{O}^{\pm}_{\Delta,J=j-1})(x,z,w)|\mathcal{O}_3\rangle, \tag{2.110}$$

which itself is a x_1, x_2-integral of (2.109). Therefore, we can prove the relation between $\mathbb{W}_i$ and $\mathcal{D}_j\mathbb{O}^{\pm}_{i,J=j-1}$ by simply showing that these integrals are the same. Concretely, we will show that

$$W_{\delta,j}(x,z,w) = A\langle\mathcal{O}_4|(\mathcal{D}_j\mathbb{O}^{+}_{\delta+1,J=j-1})(x,z,w) - (\mathcal{D}_j\mathbb{O}^{-}_{\delta+1,J=j-1})(x,z,w)|\mathcal{O}_3\rangle \tag{2.111}$$

for some constant A.

Let us start with rewriting (2.108) in terms of an integral of (2.109):

$$W_{\delta,j}(x,z,w) = \int d^dx_1d^dx_2\mathcal{L}_{\delta,j}(x_1,x_2;x,z,w)\langle\mathcal{O}_4|\phi_1(x_1)\phi_2(x_2)|\mathcal{O}_3\rangle. \tag{2.112}$$

By the definition of the light-transform, the kernel $\mathcal{L}_{\delta,j}$ is given by

$$\begin{aligned}\mathcal{L}_{\delta,j}(x_1,x_2;x,z,w) = \alpha_{\delta,j}\int d\alpha_1d\alpha_2D^{d-2}z_1D^{d-2}z_2\langle\widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z,w)\rangle\times\\ \times(-\alpha_1)^{-\delta_1-1}(-\alpha_2)^{-\delta_2-1}\delta^d(x-z_1/\alpha_1-x_1)\delta^d(x-z_2/\alpha_2-x_2).\end{aligned} \tag{2.113}$$

To evaluate the α_i, z_i-integrals, let us first consider the more general integral

$$\int d\alpha D^{d-2}z(-\alpha)^{-\delta-1}\delta^d(x+z/\alpha)f(z). \tag{2.114}$$

For the z-integration to make sense, we need $f(\lambda z) = \lambda^{-d+2+\delta}f(z)$. Otherwise, we keep f arbitrary. We restrict to x belonging to the first Poincaré patch, in which case α must be negative in order to satisfy $x + z/\alpha = 0$. We find

$$\begin{aligned}&\int_{\alpha<0} d\alpha D^{d-2}z(-\alpha)^{-\delta-1}\delta^d(x+z/\alpha)f(z)\\ &= \frac{2}{\mathrm{vol}\,\mathrm{SO}(1,1)}\int_{\alpha>0} d\alpha d^dz\delta(z^2)\theta(z>0)\alpha^{-\delta-1}\delta^d(x-z/\alpha)f(z)\\ &= \frac{2}{\mathrm{vol}\,\mathrm{SO}(1,1)}\int_{\alpha>0} d\alpha\delta(\alpha^2x^2)\theta(\alpha x>0)\alpha^{d-\delta-1}f(\alpha x)\\ &= \frac{2}{\mathrm{vol}\,\mathrm{SO}(1,1)}\int_{\alpha>0} d\alpha\delta(x^2)\theta(x>0)\alpha^{-1}f(x)\\ &= 2\delta(x^2)\theta(x>0)f(x),\end{aligned} \tag{2.115}$$

where we used that $\operatorname{vol} \mathrm{SO}(1,1) = \int_0^\infty d\alpha \alpha^{-1}$.[27] Using this result in (2.113), we find

$$\mathcal{L}_{\delta,j}(x_1,x_2;x_3,z,w) = 4\alpha_{\delta,j}\langle \widetilde{\mathcal{P}}_{\delta_1}(x_{13})\widetilde{\mathcal{P}}_{\delta_2}(x_{23})\mathcal{P}_{\delta,j}(z,w)\rangle \delta(x_{13}^2)\delta(x_{23}^2)\theta(x_{13}>0,x_{23}>0). \tag{2.116}$$

Let us now perform the same exercise for the right-hand side of (2.111). First, it is clear why we want the difference of $\mathbb{O}^+$ and $\mathbb{O}^-$ operators with equal coefficients: by comparing to (2.98), we see that this difference only involves the K^t kernel and the ordering $\phi_1\phi_2$, which is precisely the ordering we need. We find

$$\begin{aligned} &A\langle O_4|(\mathcal{D}_j\mathbb{O}^+_{\delta+1,J=j-1})(x,z,w) - (\mathcal{D}_j\mathbb{O}^-_{\delta+1,J=j-1})(x,z,w)|O_3\rangle \\ &= 2A\int' d^dx_1 d^dx_2 (\mathcal{D}_j K^t_{\delta+1,j-1}\theta(x_1\approx x^+, x_2\approx x))(x_1,x_2;x,z,w)\langle O_4|\phi_1(x_1)\phi_2(x_2)|O_3\rangle. \end{aligned} \tag{2.117}$$

Here we have taken into account the action of $\mathcal{D}_j$ on x in the integration limits in (2.98) by writing these limits as an explicit θ-function.

We would like to relate the kernel in (2.117) to the kernel $\mathcal{L}_{\delta,j}$ in (2.116). In doing so, there is an apparent problem: K^t is non-zero for generic configurations of points x_1, x_2, x, while the kernel $\mathcal{L}_{\delta,j}$ only contains delta-functions. This problem is resolved by two mechanisms. Firstly, when $j>0$, the differential operator $\mathcal{D}_j$ annihilates K^t for generic configurations of points. Hence, its action in (2.117) only has support on special loci — precisely the loci where $\mathcal{L}_{\delta,j}$ has support. Meanwhile, when $j=0$, we have $\mathcal{D}_0 = 1$ and this mechanism does not work. However, in this case the coefficient $\beta_{\delta+1,j-1}$ in (2.99) contains a 0 coming from $\Gamma(J+1)$ in the denominator of (2.100). The zero is only cancelled on special loci, giving rise to delta-functions again. In the next section, we describe in more detail how delta-functions of (2.116) emerge from (2.117).

2.4.4 Emergence of delta-functions from the light-ray kernel

The light transform entering K^t in (2.99) is given by [11]

$$\begin{aligned} &\langle 0|\widetilde{\phi}_1(x_1)\mathbf{L}[O](x_3,z)\widetilde{\phi}_2(x_2)|0\rangle \\ &= L(\widetilde{\phi}_1\widetilde{\phi}_2[O])\frac{(2z\cdot x_{23}x_{13}^2 - 2z\cdot x_{13}x_{23}^2)^{1-\Delta}}{(x_{12}^2)^{\frac{\widetilde{\Delta}_1+\widetilde{\Delta}_2-(1-J)+(1-\Delta)}{2}}(-x_{13}^2)^{\frac{\widetilde{\Delta}_{12}+(1-J)+(1-\Delta)}{2}}(x_{23}^2)^{\frac{-\widetilde{\Delta}_{12}+(1-J)+(1-\Delta)}{2}}} \\ &\qquad \text{(in the configuration } x_1\approx x_2, x_1>x_3, x_2\approx x_3), \end{aligned} \tag{2.118}$$

[27] In a more formal derivation, one would fix the SO(1,1) freedom by Faddeev-Popov procedure.

where

$$L(\widetilde{\phi}_1\widetilde{\phi}_2[\mathcal{O}]) = -2\pi i \frac{\Gamma(\Delta+J-1)}{\Gamma(\frac{\Delta+\Delta_{12}+J}{2})\Gamma(\frac{\Delta-\Delta_{12}+J}{2})}. \tag{2.119}$$

Acting with $\mathcal{D}_j$ given in (2.66) we find

$$\begin{aligned}&\langle 0|\widetilde{\phi}_1(x_1)(\mathcal{D}_j\mathbf{L}[\mathcal{O}])\widetilde{\phi}_2(x_2)|0\rangle\\ &= \frac{(-1)^j 2^{2j}(-J)_j L(\widetilde{\phi}_1\widetilde{\phi}_2[\mathcal{O}])(\Delta+j-2)}{(\Delta-2)\Gamma(j+1)}(w\cdot x_{13} z\cdot x_{23} - w\cdot x_{23} z\cdot x_{13})^j\\ &\quad\times \frac{(2z\cdot x_{23}x_{13}^2 - 2z\cdot x_{13}x_{23}^2)^{1-\Delta-j}}{(x_{12}^2)^{\frac{\widetilde{\Delta}_1+\widetilde{\Delta}_2-(1-J)+(1-\Delta)}{2}}(-x_{13}^2)^{\frac{\widetilde{\Delta}_{12}+(1-J)+(1-\Delta)}{2}}(x_{23}^2)^{\frac{-\widetilde{\Delta}_{12}+(1-J)+(1-\Delta)}{2}}}.\end{aligned} \tag{2.120}$$

If we set $J = j-1$ then for $j > 0$ this indeed vanishes due to the $(-J)_j$ factor, and thus $\mathcal{D}_j$ with $j > 0$ annihilates K^t, as promised above. This can also be immediately concluded from the fact that for $J = j-1 \geq 0$ we have above a light transform of an integer-spin three-point structure, and thus it must be annihilated by $\mathcal{D}_j$ based on general properties of $\mathcal{D}_j$ discussed in section 2.3.

However, this calculation of the action of $\mathcal{D}_j$ is not complete. This is because we have only computed the action of $\mathcal{D}_j$ on K^t in (2.117), but not on the θ-function. Derivatives hitting the theta-function will produce delta-function contributions that are needed to match (2.116). To derive the explicit form of these delta-functions, we will take a slightly more general approach.

Interlude: distributions and analytic continuation

To illustrate the approach, it is helpful to consider a toy example

$$\frac{\partial}{\partial x}\theta(x) = \delta(x). \tag{2.121}$$

While the right-hand side is obvious, let us re-derive it in a way that will be useful later.

We start with a new definition of the left-hand side. We can think of $\theta(x)$ as the case $a = 0$ of the 1-parameter family of distributions $p_a(x) = x^a\theta(x)$. The derivative $p'_a(x)$ is represented by a locally-integrable function when $\operatorname{Re} a > 0$:

$$p'_a(x) = ax^{a-1}\theta(x) + x^a\delta(x) = ax^{a-1}\theta(x) \qquad (\operatorname{Re} a > 0). \tag{2.122}$$

Specifically, when $\operatorname{Re} a > 0$, we can ignore $\partial/\partial x$ acting on $\theta(x)$. Let us define the left-hand side of (2.121) as the analytic continuation of $p'_a(x) = ax^{a-1}\theta(x)$ from $\operatorname{Re} a > 0$ to $a = 0$.

As we explain in detail in appendix A.7.1, the distribution $x^{a-1}\theta(x)$ has a pole at $a = 0$ with residue $\delta(x)$. It follows that $\lim_{a\to 0} ax^{a-1}\theta(x) = \delta(x)$. To see this, consider the integral against a test function $f(x)$ with a regular Taylor series expansion at $x = 0$,

$$\begin{aligned}\int_0^\infty dx\, ax^{a-1} f(x) &= a\int_0^\infty dx\, x^{a-1}\left(f(0) + (f(x) - f(0))\right)\\ &= a\left(\frac{f(0)}{a} + O(1)\right)\\ &= f(0) + O(a) \qquad\qquad (\mathrm{Re}\, a > 0).\end{aligned} \tag{2.123}$$

In the second line, we used the fact that $\int dx\, x^{a-1}(f(x) - f(0))$ is $O(1)$ as $a \to 0$, since $f(x) - f(0)$ vanishes at least linearly there. Finally, taking $a \to 0$, we obtain $f(0)$, as claimed.

Action of $\mathcal{D}_j$ on K^t

This strategy of starting in a region where the derivative of a kernel is simple (in particular where we can ignore derivatives acting on θ-functions), and analytically continuing away works well in the case at hand. First, let us define the expression

$$K^t_{\Delta,J}(x_1, x_2; x, z)\theta(x_1 \approx x^+, x_2 \approx x), \tag{2.124}$$

as a distribution. This is straightforward if we tune the parameters $\Delta_1, \Delta_2, \Delta, J$ to a region $U \in \mathbb{C}^4$ in which all the powers in (2.118) are such that (2.124) is locally bounded (in x, x_i, z). This is the analog of choosing $\mathrm{Re}\, a$ sufficiently large and positive in the toy example of section 2.4.4. For $(\Delta_1, \Delta_2, \Delta, J) \in U$ we automatically get a distribution in x, x_i, z that is analytic in $\Delta_1, \Delta_2, \Delta, J$. We can then define it for the values that we are interested in by analytic continuation. In the interior of U we can ignore the theta-function for the purposes of acting with $\mathcal{D}_j$: the theta function only jumps when some x_{ij}^2 in (2.118) vanishes, but inside U we have a product of all x_{ij}^2 appearing with positive powers, and so K^t vanishes on the jump of the theta-function. This is the analog of the computation (2.122) in our toy example. Therefore, in order to compute the action of $\mathcal{D}_j$ we can act in U just on K^t and then analytically continue to the desired values of $\Delta_1, \Delta_2, \Delta, J$.

Thus our goal is to analytically continue

$$\begin{aligned}&(\mathcal{D}_j K^t_{\Delta,J})(x_1, x_2; x_3, z, w)\theta(x_1 \approx x_3^+, x_2 \approx x_3)\\ &= \beta_{\Delta,J}\langle 0|\widetilde{\phi}_1(x_1)(\mathcal{D}_j \mathbf{L}[O])\widetilde{\phi}_2(x_2)|0\rangle\theta(x_1 \approx x_3^+)\theta(x_2 \approx x_3)\end{aligned} \tag{2.125}$$

with the Wightman function given in (2.120). We introduce the following coordinates,

$$s = 2z \cdot x_{23} x_{13}^2, \quad t = -2z \cdot x_{13} x_{23}^2. \tag{2.126}$$

Recall that the prime on the integral in (2.117) means that we should only include the configurations where x_2 is near the future null cone of x_3 and so we can assume $z \cdot x_{23} < 0$.[28] We can then rewrite (2.125), assuming for simplicity that all points are in the same Poincaré patch

$$\beta_{\Delta,J} \frac{(-1)^j 2^{2j} (-J)_j L(\widetilde{\phi}_1 \widetilde{\phi}_2 [O])(\Delta + j - 2)}{(\Delta - 2)\Gamma(j+1)} (w \cdot x_{13} z \cdot x_{23} - w \cdot x_{23} z \cdot x_{13})^j$$
$$\times \frac{(-2z \cdot x_{23})^{\frac{\widetilde{\Delta}_{12} + (1-J) + (1-\Delta)}{2}} (-2z \cdot x_{13})^{\frac{-\widetilde{\Delta}_{12} + (1-J) + (1-\Delta)}{2}}}{(x_{12}^2)^{\frac{\widetilde{\Delta}_1 + \widetilde{\Delta}_2 - (1-J) + (1-\Delta)}{2}}} \frac{(s+t)^{1-\Delta-j} \theta(s)\theta(t)}{s^{\frac{\widetilde{\Delta}_{12} + (1-J) + (1-\Delta)}{2}} t^{\frac{-\widetilde{\Delta}_{12} + (1-J) + (1-\Delta)}{2}}}. \tag{2.127}$$

The factors that are not written in terms of s, t will not be important in our analytic continuation: they are either analytic functions of the coordinates (the w-dependent factor), or can only become singular when x_1 becomes almost proportional to x_2, or when x_i becomes proportional to z, while at the moment we are interested in more generic configurations.[29]

To find the analytic continuation of the s, t-dependent factors, let us define

$$a = 1 - \Delta - j, \quad b = -\frac{\widetilde{\Delta}_{12} + (1-J) + (1-\Delta)}{2}, \quad c = -\frac{-\widetilde{\Delta}_{12} + (1-J) + (1-\Delta)}{2}. \tag{2.128}$$

Then the s, t-dependent part becomes

$$(s+t)^a s^b t^c \theta(s)\theta(t). \tag{2.129}$$

This is well-defined as a distribution analytic in a, b, c for $a, b, c > 0$. In appendix A.7.1 we show that near $a + b + c + 2 = 0$ its analytic continuation to general a, b, c has a pole of the form

$$(s+t)^a s^b t^c \theta(s)\theta(t) \sim \frac{1}{a+b+c+2} \frac{\Gamma(b+1)\Gamma(c+1)}{\Gamma(b+c+2)} \delta(s)\delta(t). \tag{2.130}$$

[28] This is not true when x_{23} is close to being proportional to z. However, here we focus on generic configurations.

[29] Following the logic similar to below, one can check that no new contributions appear in these limiting configurations.

With our definitions we have

$$a+b+c+2=J-j+1, \tag{2.131}$$

which we want to set to 0. The resulting divergence is canceled in (2.127) by $\beta_{\Delta,J}$ for $j=0$ or by the $(-J)_j$ factor for $j>0$. Thus, plugging (2.130) into (2.127) and setting $J=j-1$, we find

$$\begin{aligned}&(\mathcal{D}_j K^t_{\delta+1,j-1}\theta(x_1\approx x_3^+, x_2\approx x_3))(x_1,x_2;x_3,z,w)\\&=\frac{(-1)^j 2^{1+2j}}{i\pi}\alpha_{\delta,j}\frac{(w\cdot x_{13}z\cdot x_{23}-w\cdot x_{23}z\cdot x_{13})^j}{(-2x_{13}\cdot x_{23})^{\frac{\widetilde{\delta}_1+\widetilde{\delta}_2+j-\delta}{2}}(-2z\cdot x_{23})^{\frac{-\widetilde{\delta}_{12}+j+\delta}{2}}(-2z\cdot x_{13})^{\frac{\widetilde{\delta}_{12}+j+\delta}{2}}}\delta(x_{13}^2)\delta(x_{23}^2),\end{aligned} \tag{2.132}$$

Comparing this to (2.116) and to the expression (2.86) for the 3-point structure, which we reproduce here with the relevant quantum numbers,

$$\langle\widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z,w)\rangle=\frac{4^j(w\cdot z_1 z_2\cdot z_3-w\cdot z_2 z_1\cdot z_3)^j}{(-2z_1\cdot z_2)^{\frac{\widetilde{\delta}_1+\widetilde{\delta}_2-\delta+j}{2}}(-2z_1\cdot z_3)^{\frac{\widetilde{\delta}_1+\delta-\widetilde{\delta}_2+j}{2}}(-2z_2\cdot z_3)^{\frac{\widetilde{\delta}_2+\delta-\widetilde{\delta}_1+j}{2}}} \tag{2.133}$$

we find

$$(\mathcal{D}_j K^t_{\delta+1,j-1}\theta(x_1\approx x_3^+, x_2\approx x))(x_1,x_2;x_3,z,w)=\frac{(-1)^j}{2\pi i}\mathcal{L}_{\delta,j}(x_1,x_2;x_3,z,w). \tag{2.134}$$

Thus (2.111) indeed holds with

$$A=(-1)^j i\pi. \tag{2.135}$$

2.4.5 The final form of the scalar OPE

Combining equations (2.91), (2.111), and (2.135), we find the OPE formula

$$\begin{aligned}&\mathbf{L}[\phi_1](x,z_1)\mathbf{L}[\phi_2](x,z_2)\\&=\pi i\sum_{j=0}^{\infty}\int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty}\frac{d\delta}{2\pi i}C_{\delta,j}(z_1,z_2,\partial_{z_2},\partial_{w_2})(-1)^j\left((\mathcal{D}_j\mathbb{O}^+_{\delta+1,J=j-1})(x,z_2,w_2)-(\mathcal{D}_j\mathbb{O}^-_{\delta+1,J=j-1})(x,z_2,w_2)\right)\end{aligned} \tag{2.136}$$

Deforming the contour to the right and picking up the poles from light-ray operators,[22] we can write

$$\begin{aligned}&\mathbf{L}[\phi_1](x,z_1)\mathbf{L}[\phi_2](x,z_2)\\&=-\pi i\sum_{j=0}^{\infty}\sum_i C_{\Delta_i-1,j}(z_1,z_2,\partial_{z_2},\partial_{w_2})(-1)^j\left((\mathcal{D}_j\mathbb{O}^+_{i,J=j-1})(x,z_2,w_2)-(\mathcal{D}_j\mathbb{O}^-_{i,J=j-1})(x,z_2,w_2)\right).\end{aligned} \tag{2.137}$$

Here the differential operator $C_{\delta,j}$ is defined by (2.89). For $j = 0$, taking into account $\mathcal{D}_0 = 1$, the result (2.136) agrees with (3.96) in [15], which has been tested in a number of examples [15].[30] For $j > 0$ the result is new. We will explore an example in which the $j = 1$ term is important in section 2.7.

Note that in the above expressions, the terms involving $\mathbb{O}^+$ for odd j and terms involving $\mathbb{O}^-$ for even j are related to light-transforms of local operators and are thus annihilated by $\mathcal{D}_j$ for $j > 0$. Similarly, for $j = 0$ we have $\mathbb{O}^-_{\Delta,J=-1} = 0$ due to the superconvergence sum rule [34, 15], which holds whenever the leading Regge trajectory in $\phi_1 \times \phi_2$ OPE has intercept below $J_1 + J_2 - 1 = -1$. We indeed have to assume that this is the case, in order for the left-hand side of (2.136) to be well-defined [34]. We thus conclude that in fact the above expressions simplify to

$$\begin{aligned}&\mathbf{L}[\phi_1](x,z_1)\mathbf{L}[\phi_2](x,z_2)\\&= \pi i \sum_{j=0}^{\infty} \int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty} \frac{d\delta}{2\pi i} C_{\delta,j}(z_1,z_2,\partial_{z_2},\partial_{w_2})(\mathcal{D}_j \mathbb{O}^{(-1)^j}_{\delta+1,J=j-1})(x,z_2,w_2),\end{aligned} \tag{2.138}$$

and

$$\mathbf{L}[\phi_1](x,z_1)\mathbf{L}[\phi_2](x,z_2) = -\pi i \sum_{j=0}^{\infty} \sum_i C_{\Delta_i-1,j}(z_1,z_2,\partial_{z_2},\partial_{w_2})(\mathcal{D}_j \mathbb{O}^{(-1)^j}_{i,J=j-1})(x,z_2,w_2). \tag{2.139}$$

An interesting consequence of this simplification is vanishing of the commutator

$$[\mathbf{L}[\phi_1](x,z_1), \mathbf{L}[\phi_2](x,z_2)] = 0. \tag{2.140}$$

Indeed, according to the discussion in [15], this commutator has signature $(-1)^{J_1+J_2-1} = -1$. We claim that all the terms in (2.138) have signature [11, 15] +1 and thus there are no contributions to the commutator. To see why

$$\mathcal{D}_j \mathbb{O}^{(-1)^j}_{i,J=j-1} \tag{2.141}$$

has signature +1, we need to analyze two cases. Note that for $j = 0$ we have

$$\mathcal{D}_j \mathbb{O}^{(-1)^j}_{i,J=j-1} = \mathbb{O}^+_{i,J=-1}, \tag{2.142}$$

which manifestly has signature +1. For $j > 0$ the operator $\mathcal{D}_j$ is non-trivial, and we claim that it changes signature by $(-1)^j$, in which case we again see that

$$\mathcal{D}_j \mathbb{O}^{(-1)^j}_{i,J=j-1} \tag{2.143}$$

[30] The minus in front of $\mathbb{O}^-$ has to do with a more explicit treatment of the analytic continuation of $(-1)^J$ factors in this paper.

has signature +1. To understand how $\mathcal{D}_j$ changes signature, it is convenient to use CRT symmetry $\mathcal{J}_0$ under which the light-ray operators have the property

$$\left(\mathcal{J}_0 \mathbb{O}^{\pm}_{i,J}(0,z,\mathbf{w})\mathcal{J}_0^{-1}\right)^{\dagger} = \pm \mathbb{O}^{\pm}_{i,J}(0,z,\mathbf{w}) \tag{2.144}$$

for all $z, \mathbf{w}$.[31] Writing then

$$(\mathcal{D}_j \mathbb{O}^{\pm}_{i,J})(0,z,\mathbf{w}) \propto \mathcal{D}^{0+}_{z,w;\mu_1}\cdots\mathcal{D}^{0+}_{z,w;\mu_j}[P^{\mu_1},\cdots[P^{\mu_j},\mathbb{O}^{\pm}_{i,J}(0,z,\mathbf{w})]\cdots], \tag{2.145}$$

where $\mathcal{D}^{0+}_{z,w;\mu}$ is a differential operator in z, w defined in (2.66), and taking into account $\mathcal{J}_0 P^{\mu}\mathcal{J}_0^{-1} = -P^{\mu}$ and $P^{\dagger}_{\mu} = -P_{\mu}$, we find

$$\left(\mathcal{J}_0(\mathcal{D}_j \mathbb{O}^{\pm}_{i,J})(0,z,\mathbf{w})\mathcal{J}_0^{-1}\right)^{\dagger} = \pm(-1)^j(\mathcal{D}_j \mathbb{O}^{\pm}_{i,J})(0,z,\mathbf{w}), \tag{2.146}$$

and therefore $\mathcal{D}_j$ indeed changes the signature by $(-1)^j$. Finally, note that the differential operator $C_{\delta,j}$ does not affect signature because it acts in the transverse space (i.e. on $z, \mathbf{w}$) which is not affected by $\mathcal{J}_0$.

2.5 The complete OPE of general detectors

In the previous section, we derived the form of the light-ray OPE for a product of light-transformed scalar operators $\mathbf{L}[\phi_1]\mathbf{L}[\phi_2]$. We now derive a generalization for light-transforms of operators in arbitrary Lorentz representations, of which the scalar formula (2.136) is a special case. The generalized light-ray OPE formula is

$$\begin{aligned}
&\mathbf{L}[\mathcal{O}_1](x,z_1,\mathbf{w}_1)\mathbf{L}[\mathcal{O}_2](x,z_2,\mathbf{w}_2)\\
&= \pi i(-1)^{J_1+J_2}\sum_{\lambda\in\Lambda_{12}}\int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty}\frac{d\delta}{2\pi i}C^{(a)}_{\delta,\lambda}(z_1,\mathbf{w}_1,z_2,\mathbf{w}_2,\partial_{z_2},\partial_{\mathbf{w}_2})\mathbb{O}^{(-1)^{J_1+J_2}}_{\delta+1,J_1+J_2-1,\lambda,(a)}(x,z_2,\mathbf{w}_2)\\
&+\pi i(-1)^{J_1+J_2}\sum_{n=1}^{\infty}\sum_{\gamma\in\Gamma_{12}}\int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty}\frac{d\delta}{2\pi i}C^{(a)}_{\delta,\lambda_\gamma(+n)}(z_1,\mathbf{w}_1,z_2,\mathbf{w}_2,\partial_{z_2},\partial_{\mathbf{w}_2})(\mathcal{D}_n\mathbb{O}^{(-1)^{J_1+J_2+n}}_{\delta+1,J_1+J_2-1+n,\lambda_\gamma,(a)})(x,z_2,\mathbf{w}_2)
\end{aligned} \tag{2.147}$$

Owing to its generality, this expression is a bit unwieldy, so let us unpack it. (We also give several concrete examples in section 2.6.) The left-hand side is a product of light-transforms of operators $\mathcal{O}_1$ and $\mathcal{O}_2$, which have the quantum numbers $(\Delta_i, J_i, \lambda_i)$.[32] The right-hand side contains two sets of terms. The first set (on line

[31] The CRT transformation $\mathcal{J}_0$ can be obtained from the CRT transformation $\mathcal{J}_\Omega$ described around (2.172) below by conjugating with a conformal transformation which brings the point $(x^+ = -\infty, x^- = 0, x^i = 0)$ to $x = 0$ and the point $(x^+ = 0, x^- = -\infty, x^i = 0)$ to spatial infinity.

[32] For simplicity we focus on bosonic representations only.

2) contains what we refer to as the "low transverse spin" contributions, studied in detail in [15]. The second set (line 3) contains "higher transverse spin" contributions, which are new and require the use of the differential operators $\mathcal{D}_n$ from section 2.3.

The operators $C^{(a)}_{\delta,\lambda}$ are similar to the operators $C_{\delta,j}$ from section 2.4. They encode the kinematic structure of the OPE in a fictitious $(d-2)$-dimensional CFT on the celestial sphere. There are two important differences from section 2.4. Firstly, in the general case considered here, there can be several OPE coefficients that enter a given OPE, each with its own $C^{(a)}_{\delta,\lambda}$. The structure label (a) labels these different OPE coefficients. A sum over (a) is implicit in (2.147). As an example of the role of the structure label (a), recall that deforming the δ-contour in (2.147) to the right and picking up the poles, we obtain a discrete sum of terms

$$C^{(a)}_{\delta,\lambda}(z_1,\mathbf{w}_1,z_2,\mathbf{w}_2,\partial_{z_2},\partial_{\mathbf{w}_2})\mathbb{O}^{\pm}_{i,J_1+J_2-1,\lambda(a)}(x,z_2,\mathbf{w}_2). \tag{2.148}$$

Recall that $\mathbb{O}^{\pm}_{i,J_1+J_2-1,\lambda(a)}$ are built with reference to $\mathcal{O}_1,\mathcal{O}_2$ and are related to canonically-normalized light-ray operators as

$$\mathbb{O}^{\pm}_{i,J_1+J_2-1,\lambda(a)} = f_{12\mathcal{O}^\dagger,(a)}\widehat{\mathbb{O}}^{\pm}_{i,J_1+J_2-1,\lambda}, \tag{2.149}$$

where $f_{12\mathcal{O}^\dagger,(a)}$ are analytically-continued OPE coefficients in the $\mathcal{O}_1\times\mathcal{O}_2$ OPE. Equation (2.148) then becomes

$$f_{12\mathcal{O}^\dagger,(a)}C^{(a)}_{\delta,\lambda}(z_1,\mathbf{w}_1,z_2,\mathbf{w}_2,\partial_{z_2},\partial_{\mathbf{w}_2})\widehat{\mathbb{O}}^{\pm}_{i,J_1+J_2-1,\lambda}(x,z_2,\mathbf{w}_2), \tag{2.150}$$

Thus, the index (a) of $C^{(a)}_{\delta,\lambda}$ is naturally contracted with OPE coefficients. Similar statements hold for higher transverse spin terms as well.

The second distinction has to do with the normalization of $C^{(a)}_{\delta,\lambda}$. The operators $C_{\delta,j}$ were normalized by equation (2.89), which is formulated in terms of celestial three-point structures. On the other hand, the OPE coefficients $f_{12\mathcal{O}^\dagger,(a)}$ are defined in terms of d-dimensional tensor structures, and thus the normalization condition (2.89) is only correct due to our specific choice of conventions for both d-dimensional as well as celestial tensor structures. The main result of this section will be the proof of (2.147) together with the simple, convention-independent, equations (2.214) and (2.215) that determine the $C^{(a)}_{\delta,\lambda}$ in terms of d-dimensional data.[33] We refer to these formulas as "celestial map formulas" because they map

[33]Note that up to an action by an invertible matrix on the index (a), these operators are completely fixed by the $(d-2)$-dimensional conformal symmetry (equivalently, d-dimensional Lorentz symmetry). These equations thus simply determine a preferred basis of these operators.

the d-dimensional three-point tensor structures (which naturally pair with the OPE coefficients $f_{12\mathcal{O}^\dagger,(a)}$) to celestial sphere differential operators $\mathcal{C}^{(a)}_{\delta,\lambda}$.[34]

The remaining notation in (2.147) has to do with quantum numbers of exchanged operators and the respective selection rules. In the low transverse spin terms we are summing over transverse spins $\lambda \in \Lambda_{12}$ which simply means all the transverse spins that appear in the usual local OPE $\mathcal{O}_1 \times \mathcal{O}_2$ *and* the celestial OPE of operators with $(d-2)$-dimensional spins $\lambda_1 \times \lambda_2$. These are the transverse spins for which the operators $\mathbb{O}^{\pm}_{i,J,\lambda,(a)}$ can be constructed from $\mathcal{O}_1$ and $\mathcal{O}_2$ and for which the operators $\mathcal{C}^{(a)}_{\delta,\lambda}$ make sense.

As an example, when $\mathcal{O}_1, \mathcal{O}_2$ are scalars, Λ_{12} contains only the trivial representation of $\mathrm{SO}(d-2)$ because only traceless-symmetric operators appear in OPE of d-dimensional scalars. In this case, the constraint that $\lambda \in \Lambda_{12}$ should appear in the OPE of $(d-2)$ dimensional scalars is trivially satisfied. On the other hand, in the example of energy-energy OPE discussed in section 2.6 both constraints become non-trivial.

In the higher transverse spin terms, we sum over $\lambda_\gamma, \gamma \in \Gamma_{12}$. These are the transverse spins for which the action of the operators $\mathcal{D}_n$ is well-defined. Concretely, these are transverse spins with the first row of the $\mathrm{SO}(d-2)$ Young diagram (the second row of the $\mathrm{SO}(d-1,1)$ Young diagram) of length $J_1 + J_2$. We write $\lambda_\gamma = (J_1 + J_2, \gamma)$ with γ an $\mathrm{SO}(d-4)$ irrep. Since the action of $\mathcal{D}_n$ raises transverse spin, we use the OPE differential operator $\mathcal{C}^{(a)}_{\delta,\lambda_\gamma(+n)}$, where $\lambda_\gamma(+n) = (J_1 + J_2 + n, \gamma)$. The set Γ_{12} consists of γ for which transverse spin λ_γ appears in d-dimensional $\mathcal{O}_1 \times \mathcal{O}_2$ OPE and at the same time $\lambda_\gamma(+n)$ appears in the $(d-2)$-dimensional celestial OPE. (As explained in appendix A.5, these two conditions are in fact equivalent.)

Before proceeding with the derivation, let us comment again on the relation of this section to [15]. In [15], the celestial map formula for lower transverse spin was derived by a rather non-trivial procedure using the Lorentzian inversion formula. In this section, we will give a much simpler derivation of the celestial map formula for both lower and higher transverse spin. The drawback of this simpler derivation is that it is based on the assumption that the light-ray kernel $K^t_{\Delta,J,\lambda(a)}(x_1, x_2; x, z)$ satisfying (2.152) localizes on the null cone of x as we set J to certain values and possibly act with $\mathcal{D}_n$, analogously to the scalar case in section 2.4. This assumption is plausible in the sense that it gives a natural generalization of the pattern observed

[34] As we explain below, it will sometimes happen that $\mathcal{C}^{(a)}_{\delta,\lambda}$ vanishes for some values of a, i.e. not all of the OPE coefficients actually appear in (2.148).

in concrete examples, and is furthermore purely kinematical. It therefore appears to be a purely technical problem to prove it. (See sections 2.5.2 and 2.5.2 for the precise statement of our assumptions and the supporting evidence.)

For completeness, in appendix A.4, we derive (2.147) by generalizing the derivation in [15]. This derivation, although being more technical than the one in sections 2.5.2 and 2.5.2 (which is why it is relegated to an appendix), does not rely on the assumptions discussed above.

2.5.1 A formula for the light-ray operator kernel

Our starting point is the following intrinsically-Lorentzian description of light-ray operators. Recall that the light-ray operators are defined by the integral [11]

$$\mathbb{O}^{\pm}_{i,J,\lambda(a)}(x,z) = \operatorname{res}_{\Delta=\Delta_i} \pm \int_{\substack{x\approx 2\\ x\approx 1^-}} d^dx_1 d^dx_2 K^t_{\Delta,J,\lambda(a)}(x_1,x_2;x,z)\mathcal{O}_1\mathcal{O}_2 + \int_{\substack{x\approx 1\\ x\approx 2^-}} d^dx_1 d^dx_2 K^u_{\Delta,J,\lambda(a)}(x_2,x_1;x,z)\mathcal{O}_2\mathcal{O}_1. \quad (2.151)$$

We claim that the kernel K^t is determined by the equation

$$\int_{\substack{2>x'>1^-\\ x\approx 2,1^-}} \frac{d^dx_1 d^dx_2}{\operatorname{vol}(\mathrm{SO}(1,1))^2} K^t_{\Delta,J,\lambda(a)}(x_1,x_2;x,z)\langle 0|\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger](x',z')\mathcal{O}_1|0\rangle^{(b)}_{+} = \frac{1}{2\pi i}\langle \mathbf{L}[\mathcal{O}](x,z)\mathbf{L}[\mathcal{O}^\dagger](x',z')\rangle\delta^{(b)}_{(a)}, \quad (2.152)$$

together with the condition that it has the analyticity and conformal transformation properties of

$$\langle 0|\widetilde{\mathcal{O}}^\dagger_1(x_1)\mathcal{O}^L(x,z)\widetilde{\mathcal{O}}^\dagger_2(x_2)|0\rangle, \quad (2.153)$$

where $\mathcal{O}^L$ has the quantum numbers of $\mathbf{L}[\mathcal{O}]$. The kernel K^u is defined similarly and will be described below.[35] In (2.151) and in what follows we keep the transverse indices (encoded previously by polarization vectors $\mathbf{w}$) implicit in order to avoid excessive clutter in the notation. The way these indices are contracted should in all cases be clear from the context.

[35] An intuitive picture behind (2.152) is as follows. The Wightman three-point structure in (2.152), together with the condition $2 > x' > 1^-$, can be viewed as one of two parts of the light-transform of a time-ordered three-point structure [11]. The integration against K^t is then similar to producing $\mathbf{L}[\mathcal{O}]$ from $\mathcal{O}_1, \mathcal{O}_2$ inside of this time-ordered three-point function, which should be equal to the two-point function in the right-hand side of (2.152). It would be interesting if this intuitive reasoning could be made precise: our derivation of (2.152) is based on the generalized Lorentzian inversion formula of [11], where the latter is derived from Euclidean harmonic analysis. It would be instructive to bypass the Euclidean argumentation altogether.

Above, we use the notation that O_i is at point x_i. The kernel K^t carries Lorentz indices for points x_1, x_2 that are in the dual representations to O_1, O_2, and these indices are contracted with O_1, O_2 in (2.151) and (2.152). We prove (2.152) in appendix A.3. An advantage of (2.152) relative to the definition in [11] is that it makes reference only to Lorentzian objects. By contrast, the definition in [11] includes the Plancherel measure for the Euclidean conformal group and Euclidean shadow coefficients, and is thus more awkward to use in a purely Lorentzian setting.

In order to use (2.152) for non-integer J, it remains to explain the meaning of the objects

$$\langle \mathbf{L}[O](x,z)\mathbf{L}[O^\dagger](x',z')\rangle, \tag{2.154}$$

$$\langle 0|O_2\mathbf{L}[O^\dagger](x',z')O_1|0\rangle_+^{(b)} \tag{2.155}$$

for such J.

Analytic continuation in spin

We define (2.154) and (2.155) by extending the definition of the underlying structures

$$\langle O(x,z)O^\dagger(x',z')\rangle, \tag{2.156}$$

$$\langle 0|O_2 O^\dagger(x',z')O_1|0\rangle_+^{(b)} \tag{2.157}$$

to non-integer J and then taking the necessary light-transforms.

Note that fixing the normalization of local operators and setting the conventions for three-point structures of local operators involves specifying the expressions for

$$\langle O(x,z)O^\dagger(x',z')\rangle, \tag{2.158}$$

$$\langle O_1(x_1,z_1)O_2(x_2,z_2)O^\dagger(x,z)\rangle^{(b)}. \tag{2.159}$$

for integer spin J. We will now impose certain constraints on these choices which will simplify our general analysis with regard to factors of $(-1)^J$ which have ambiguous analytic continuation in J.

For the two-point function, note that for any fixed $n \in \mathbb{Z}$

$$\langle O(x_1,z_1)O^\dagger(x_2,z_2)\rangle(-2z_1\cdot I(x_{12})\cdot z_2)^{n-J} \tag{2.160}$$

is conformally-invariant with quantum numbers independent of J. Moreover, for sufficiently large n, the Lorentz weights with which it transforms are dominant[36] and

[36] I.e. the first row of Young diagram is at least as long as the second one.

we can take this structure to be equal to a fixed, J-independent, two-point function of local operators

$$\langle \mathcal{O}(x_1,z_1)\mathcal{O}^\dagger(x_2,z_2)\rangle(-2z_1\cdot I(x_{12})\cdot z_2)^{n-J} = f_0(x_1,z_1;x_2,z_2). \tag{2.161}$$

So we find that one can always choose

$$\langle \mathcal{O}(x_1,z_1)\mathcal{O}^\dagger(x_2,z_2)\rangle = f_0(x_1,z_1;x_2,z_2)(-z_1\cdot I(x_{12})\cdot z_2)^{J-n}. \tag{2.162}$$

We allow to modify this convention by exponential factors such as 2^J but not by $(-1)^J$.

Note that the expression (2.162) can be rewritten as

$$\begin{aligned} &f_0(x_1,z_1;x_2,z_2)(-z_1\cdot I(x_{12})\cdot z_2)^{J-n}\\ &= f_0(x_1,z_1;x_2,z_2)(2(z_1\cdot x_{12})(z_2\cdot x_{12})-(z_1\cdot z_2)x_{12}^2)^{J-n}(x_{12}^2)^{n-J}. \end{aligned} \tag{2.163}$$

Using the fact that $2(z_1\cdot x_{12})(z_2\cdot x_{12})-(z_1\cdot z_2)x_{12}^2>0$ for generic configurations,[37] we see that the usual time-ordered $i\epsilon$ prescription unambiguously defines the time-ordered two-point function (2.162) for generic configurations of x_i, z_i. This is sufficient to apply the light-transforms in (2.154).

A similar argument for $\langle \mathcal{O}_1(x_1,z_1)\mathcal{O}_2(x_2,z_2)\mathcal{O}^\dagger(x,z)\rangle^{(b)}$ shows that we can write, for sufficiently large integer J,

$$\langle \mathcal{O}_1(x_1,z_1)\mathcal{O}_2(x_2,z_2)\mathcal{O}^\dagger(x_3,z_3)\rangle^{(b)} = f_0^{(b)}(x_1,z_1;x_2,z_2;x_3,z_3)\frac{(2z\cdot x_{23}\,x_{13}^2-2z\cdot x_{13}\,x_{23}^2)^{J-n}}{x_{12}^{J-n}x_{23}^{J-n}x_{13}^{J-n}}, \tag{2.164}$$

where n is sufficiently large so that the Lorentz weights of f_0 are dominant, and the basis $f_0^{(b)}(x_1,z_1;x_2,z_2;x_3,z_3)$ can be chosen to be J-independent. There are two possible subtleties here. Firstly, the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ can be identical, in which case permutation invariance will typically constrain the structures for even and odd J differently. Secondly, some of $\mathcal{O}_1,\mathcal{O}_2$ can be conserved currents, in which case the conservation constraints will typically require some non-trivial polynomial dependence of $f_0^{(b)}$ on J. In both cases, our solution is to use the generic basis of structures, ignoring these constraints. The OPE coefficients computed in this basis may satisfy some linear equations, but this will not affect any of our arguments.

[37] To see this, note that one can always write $x_{12}=\alpha z_1+\beta z_2+x_\perp$ for $x_\perp\cdot z_i=0$. We have then $2(z_1\cdot x_{12})(z_2\cdot x_{12})-(z_1\cdot z_2)x_{12}^2=(-z_1\cdot z_2)x_\perp^2>0$, where we used that $x_\perp$ is spacelike due to being orthogonal to the timelike vector z_1+z_2.

The problem of defining three-point structures for non-integer J is complicated by the fact that $2z\cdot x_{23}\, x_{13}^2 - 2z\cdot x_{13}\, x_{23}^2$ is in general not sign-definite [11]. We will define the analytic continuation directly for $\langle 0|\mathcal{O}_2\mathcal{O}^\dagger(x_3,z_3)\mathcal{O}_1|0\rangle^{(b)}$ in the configuration $2>3, 1\approx 3, 1\approx 2$, where we have

$$2z\cdot x_{23}\, x_{13}^2 - 2z\cdot x_{13}\, x_{23}^2 < 0. \tag{2.165}$$

For integer J we have

$$\langle 0|\mathcal{O}_2\mathcal{O}^\dagger(x_3,z_3)\mathcal{O}_1|0\rangle^{(b)} = (-1)^J(-2z\cdot x_{23}\, x_{13}^2 + 2z\cdot x_{13}\, x_{23}^2)^J \times (\cdots), \tag{2.166}$$

where the dots represent the standard analytic continuation of all the other factors. For general $J\in\mathbb{C}$ we define

$$\langle 0|\mathcal{O}_2\mathcal{O}^\dagger(x_3,z_3)\mathcal{O}_1|0\rangle^{(b)}_{\pm} = \pm(-2z\cdot x_{23}\, x_{13}^2 + 2z\cdot x_{13}\, x_{23}^2)^J \times (\cdots). \tag{2.167}$$

In all other configurations these Wightman functions are determined by the usual analytic continuation, which is unambiguous for this ordering even for $J\in\mathbb{C}$ [11]

We will additionally use the "time-ordered" structures, which are defined to be equal to

$$\begin{aligned}\langle \mathcal{O}_1(x_1,z_1)\mathcal{O}_2(x_2,z_2)\mathcal{O}^\dagger(x_3,z_3)\rangle^{(b)}_{+} = f_0(x_1,z_1;x_2,z_2;x_3,z_3)\frac{(2z\cdot x_{23}\, x_{13}^2 - 2z\cdot x_{13}\, x_{23}^2)^{-n}}{x_{12}^{J-n}x_{23}^{J-n}x_{13}^{J-n}}\\ \times |2z\cdot x_{23}\, x_{13}^2 - 2z\cdot x_{13}\, x_{23}^2|^J\end{aligned} \tag{2.168}$$

for spacelike-separated points and are defined in other configurations by usual time-ordered $i\epsilon$-prescriptions applied to everything except $|\cdots|^J$. These structures are useful because

$$\langle \mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle^{(b)}_{+} = \langle 0|\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\mathcal{O}_1|0\rangle^{(b)}_{+}\theta(2>3>1^-) + (\cdots)\theta(1>3>2^-), \tag{2.169}$$

similarly to the relation between the light-transform of integer-spin time-ordered and Wightman correlators [11]. Here dots represent a structure that is a bit awkward to describe but which we will not need in what follows.

Our choice of conventions for traceless-symmetric operators described in appendix A.2 satisfies the above constraints.

Definition of K^u

The kernel K^u is defined by requiring that it has the analyticity and conformal transformation properties of

$$\langle 0|\widetilde{\mathcal{O}}_2^\dagger(x_2)\mathcal{O}^L(x,z)\widetilde{\mathcal{O}}_1^\dagger(x_1)|0\rangle, \tag{2.170}$$

and the requirement that when we set $x = -\infty z$ and $z^0 = z^1 = 1, z^i = 0$, the following equality holds

$$\int_{\substack{x\approx 1\\ x\approx 2^-}} d^d x_1 d^d x_2 K^u_{\Delta,J,\lambda(a)}(x_2,x_1;x,z)\mathcal{O}_2\mathcal{O}_1 = \overline{\int_{\substack{x\approx 2\\ x\approx 1^-}} d^d x_1 d^d x_2 K^t_{\Delta,J,\lambda(a)}(x_1,x_2;x,z)\mathcal{O}_1\mathcal{O}_2}. \tag{2.171}$$

Here, we have defined the linear operation

$$\overline{A} = (\mathcal{J}_\Omega A \mathcal{J}_\Omega^{-1})^\dagger. \tag{2.172}$$

Here $\mathcal{J}_\Omega$ is the anti-unitary operator implementing the CRT symmetry, where the reflection acts as $x^1 \to -x^1$. This relation should hold for all J. For integer J it is equivalent to

$$\begin{aligned}&\int_{\substack{1>x'>2^-\\ x\approx 1,2^-}} \frac{d^d x_1 d^d x_2}{\mathrm{vol}(\mathrm{SO}(1,1))^2} K^u_{\Delta,J,\lambda(a)}(x_1,x_2;x,z)\langle 0|\mathcal{O}_1\mathbf{L}[\mathcal{O}^\dagger](x',z')\mathcal{O}_2|0\rangle^{(b)}\\ &= \frac{1}{2\pi i}\langle \mathbf{L}[\mathcal{O}](x,z)\mathbf{L}[\mathcal{O}^\dagger](x',z')\rangle \delta^{(b)}_{(a)}.\end{aligned} \tag{2.173}$$

2.5.2 Derivation of the OPE formula

Harmonic analysis on celestial sphere

We have given a review of harmonic analysis on celestial sphere in section 2.4.2. In this section we need a generalization which we state here without proof, and refer the reader instead to [15].

Suppose we have a choice of two- and three-point structures on the celestial sphere

$$\langle \mathcal{P}_{\delta,\lambda}(z)\mathcal{P}^\dagger_{\delta,\lambda}(z')\rangle \tag{2.174}$$

$$\langle \mathcal{P}_{\delta_1,\lambda_1}(z_1)\mathcal{P}_{\delta_2,\lambda_2}(z_2)\mathcal{P}^\dagger_{\delta,\lambda}(z)\rangle^{(a)} \tag{2.175}$$

We assume that the three-point structures are linearly-independent and span the space of conformally-invariant tensor structures for the given quantum numbers. Given these structures, we can find OPE differential operators $\widehat{\mathcal{C}}^{(a)}_{\delta,\lambda}$ satisfying

$$\widehat{\mathcal{C}}^{(a)}_{\delta,\lambda}(z_1,z_2;\partial_{z_2})\langle \mathcal{P}_{\delta,\lambda}(z_2)\mathcal{P}^\dagger_{\delta,\lambda}(z')\rangle = \langle \mathcal{P}_{\delta_1,\lambda_1}(z_1)\mathcal{P}_{\delta_2,\lambda_2}(z_2)\mathcal{P}^\dagger_{\delta,\lambda}(z)\rangle^{(a)}. \tag{2.176}$$

Here and below, the $\mathcal{P}_{\delta,\lambda}$ carry SO$(d-2)$ indices for λ, which we suppress for brevity. These indices are implicitly contracted between $\widehat{C}^{(a)}_{\delta,\lambda}$ and $\mathcal{P}_{\delta,\lambda}(z_2)$.

Now suppose that conformally-invariant kernels $k_{\delta,\lambda,(a)}(z_1,z_2;z)$ solve the equation

$$\int \frac{D^{d-2}z_1 D^{d-2}z_2}{\mathrm{vol}(\mathrm{SO}(1,1))} k_{\delta,\lambda,(a)}(z_1,z_2;z)\langle \mathcal{P}_{\delta_1,\lambda_1}(z_1)\mathcal{P}_{\delta_2,\lambda_2}(z_2)\mathcal{P}^{\dagger}_{\delta,\lambda}(z')\rangle^{(b)} = \langle \mathcal{P}_{\delta,\lambda}(z)\mathcal{P}^{\dagger}_{\delta,\lambda}(z')\rangle \delta^{(b)}_{(a)}. \tag{2.177}$$

Here, $k_{\delta,\lambda,(a)}(z_1,z_2;z)$ carries SO$(d-2)$ indices dual to $\mathcal{P}_{\delta_1,\lambda_1}$ and $\mathcal{P}_{\delta_2,\lambda_2}$ and these indices are implicitly contracted in (2.177). If the above conditions hold for all λ's that can appear in the $(d-2)$-dimensional OPE $\mathcal{P}_{\delta_1,\lambda_1}\times\mathcal{P}_{\delta_2,\lambda_2}$, then we have

$$\mathbf{L}[\mathcal{O}_1](x,z_1)\mathbf{L}[\mathcal{O}_2](x,z_2) = \sum_{\lambda}\int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty}\frac{d\delta}{2\pi i}\widehat{C}^{(a)}_{\delta,\lambda}(z_1,z_2;\partial_{z_2})\mathbb{W}_{\delta,\lambda,(a)}(x,z_2), \tag{2.178}$$

where

$$\mathbb{W}_{\delta,\lambda,(a)}(x,z) \equiv \int D^{d-2}z_1 D^{d-2}z_2 k_{\delta,\lambda,(a)}(z_1,z_2;z)\mathbf{L}[\mathcal{O}_1](x,z_1)\mathbf{L}[\mathcal{O}_2](x,z_2). \tag{2.179}$$

For the validity of (2.178) the same caveats as in footnote 20 apply. Note that it is not important which basis of structures one chooses in (2.175) and (2.174) — the above statement is basis-independent. We utilize this freedom below, making convenient choices when appropriate.

Lower transverse spin

For lower transverse spin, we will assume that when we set $J \to J_1 + J_2 - 1$, the light-ray kernel $K^t_{\Delta,J,\lambda(a)}$ degenerates to

$$\begin{aligned}&\mathcal{N}^{(a)}_{(c),\delta,\lambda}\int_{\substack{x\approx 2\\ x\approx 1^-}} d^dx_1 d^dx_2 K^t_{\delta+1,J,\lambda,(a)}(x_1,x_2;x,z)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\\ &\to \int D^{d-2}z_1 D^{d-2}z_2 k_{\delta,\lambda,(c)}(z_1,z_2;z)\mathbf{L}[\mathcal{O}_1](x,z_1)\mathbf{L}[\mathcal{O}_2](x,z_2)\end{aligned} \tag{2.180}$$

where $k_{\delta,\lambda,(c)}$ is a basis of Lorentz-invariant kernels, and $\mathcal{N}^{(a)}_{(c),\delta,\lambda}$ is a rectangular matrix (implicit summation over (a) is understood). The matrix $\mathcal{N}$ is needed because not every $K^t_{\delta+1,J,\lambda,(a)}$ degenerates in this way. We assume that all Lorentz-invariant kernels $k_{\delta,\lambda,(c)}$ can be generated in this way i.e. $\mathcal{N}$ is full rank.

The evidence for this assumption comes from several lines of reasoning. First of all, it agrees with the scalar example in section 2.4. Moreover, we have additionally studied the example of $\mathbf{L}[\phi]\mathbf{L}[V]$ OPE, where V is an operator with spin 1, which has a non-trivial matrix $\mathcal{N}$ and verified this statement. Furthermore, from the counting of structures in appendix A.5 one can see that the dimensions of the spaces of structures match (specifically, the rank of $\mathcal{N}$ can be predicted from (2.186) below, and it matches the number of $k_{\delta,\lambda,(c)}$). Finally, (2.180) yields the same results as in [15] and in appendix A.4, where they have been obtained by an different, although more complicated, method. It would be interesting to find a direct derivation of (2.180).

Plugging (2.180) into (2.152), we obtain

$$\int D^{d-2}z_1 D^{d-2}z_2 k_{\delta,\lambda(c)}(z_1,z_2;z)\frac{\langle 0|\mathbf{L}^+[\mathcal{O}_2](x,z_2)\mathbf{L}[\mathcal{O}^\dagger](x',z')\mathbf{L}^-[\mathcal{O}_1](x,z_1)|0\rangle_+^{(a)}}{\mathrm{vol}(\mathrm{SO}(1,1))^2}$$
$$= \frac{1}{2\pi i}\langle \mathbf{L}[\mathcal{O}](x,z)\mathbf{L}[\mathcal{O}^\dagger](x',z')\rangle \mathcal{N}^{(a)}_{(c),\delta,\lambda}. \tag{2.181}$$

where $\mathbf{L}^+[\mathcal{O}_2]$ indicates that the light transform contour is restricted to $2 > x'$ and $\mathbf{L}^-[\mathcal{O}_1]$ indicates that the light transform contour is restricted to $x' > 1^-$. On the left-hand side, one factor of $\mathrm{vol}(\mathrm{SO}(1,1))$ cancels against a zero-mode in the integral over z_1, z_2, and the other factor cancels against a zero-mode in the triple light-transform of the three-point structure.

Let us define a matrix $\mathcal{M}^{(c)}_{(a),\delta,\lambda}$ with the property

$$\mathcal{M}^{(c)}_{(a),\delta,\lambda}\mathcal{N}^{(a)}_{(c'),\delta,\lambda} = \delta^{(c)}_{(c')}. \tag{2.182}$$

This is always possible to do since $\mathcal{N}$ is full rank. However, $\mathcal{M}$ defined in this way is in general ambiguous because we can replace

$$\mathcal{M}^{(c)}_{(a),\delta,\lambda} \to \mathcal{M}^{(c)}_{(a),\delta,\lambda} + m^{(c)}_{(a)} \tag{2.183}$$

for any $m^{(c)}_{(a)}$ such that $m^{(c)}_{(a)}\mathcal{N}^{(a)}_{(c'),\delta,\lambda} = 0$. We make a choice of $\mathcal{M}$ and rewrite (2.181) as

$$\int D^{d-2}z_1 D^{d-2}z_2 k_{\delta,\lambda(c)}(z_1,z_2;z)\frac{\langle 0|\mathbf{L}^+[\mathcal{O}_2](x,z_2)\mathbf{L}[\mathcal{O}^\dagger](x',z')\mathbf{L}^-[\mathcal{O}_1](x,z_1)|0\rangle_+^{(a)}}{\mathrm{vol}(\mathrm{SO}(1,1))^2}\mathcal{M}^{(c')}_{(a),\delta,\lambda}$$
$$= \frac{1}{2\pi i}\langle \mathbf{L}[\mathcal{O}](x,z)\mathbf{L}[\mathcal{O}^\dagger](x',z')\rangle \delta^{(c')}_{(c)}. \tag{2.184}$$

In the left-hand side we have a celestial "bubble integral" of $k_{\delta,\lambda(c)}(z_1,z_2;z)$ against

$$\frac{\langle 0|\mathbf{L}^+[\mathcal{O}_2](x,z_2)\mathbf{L}[\mathcal{O}^\dagger](x',z')\mathbf{L}^-[\mathcal{O}_1](x,z_1)|0\rangle_+^{(a)}}{\mathrm{vol}(\mathrm{SO}(1,1))}\mathcal{M}^{(c')}_{(a),\delta,\lambda}. \tag{2.185}$$

Since $k_{\delta,\lambda(c)}(z_1, z_2; z)$ form a basis, it follows that this bubble integral is non-degenerate [45], i.e. it vanishes for all (c) if and only if the above expression vanishes. Due to (2.183) it then follows that

$$\frac{\langle 0|\mathbf{L}^+[\mathcal{O}_2](x, z_2)\mathbf{L}[\mathcal{O}^\dagger](x', z')\mathbf{L}^-[\mathcal{O}_1](x, z_1)|0\rangle_+^{(a)}}{\mathrm{vol}(\mathrm{SO}(1,1))} m_{(a)}^{(c')} = 0 \tag{2.186}$$

for any $m_{(a)}^{(c)}$ such that $m_{(a)}^{(c)} \mathcal{N}_{(c'),\delta,\lambda}^{(a)} = 0$.

Equation (2.184) has almost the same structure as (2.177) if we identify, for fixed x, x',

$$\langle \mathcal{P}_{\delta_1,\lambda_1}(z_1)\mathcal{P}_{\delta_2,\lambda_2}(z_2)\mathcal{P}^\dagger_{\delta,\lambda}(z)\rangle^{(c)} \leftrightarrow \frac{\langle 0|\mathbf{L}^+[\mathcal{O}_2](x, z_2)\mathbf{L}[\mathcal{O}^\dagger](x', z')\mathbf{L}^-[\mathcal{O}_1](x, z_1)|0\rangle_+^{(a)}}{\mathrm{vol}(\mathrm{SO}(1,1))} \mathcal{M}_{(a),\delta,\lambda}^{(c)}, \tag{2.187}$$

$$\langle \mathcal{P}_{\delta,\lambda}(z)\mathcal{P}^\dagger_{\delta,\lambda}(z')\rangle \leftrightarrow \frac{1}{2\pi i}\langle \mathbf{L}[\mathcal{O}](x, z)\mathbf{L}[\mathcal{O}^\dagger](x', z')\rangle. \tag{2.188}$$

However, we cannot quite write equality signs in the relations above, because while z_1, z_2 and the transverse indices of $\mathbf{L}^+[\mathcal{O}_2], \mathbf{L}^-[\mathcal{O}_1]$ transform under the Lorentz group at x, the polarization z' and transverse indices of $\mathbf{L}[\mathcal{O}^\dagger]$ transform under the Lorentz group at x'. Fortunately, we can perform two steps to rectify this problem. First, there is a subgroup $L_{xx'} \simeq \mathrm{SO}(d-1, 1)$ of conformal transformations that fix both x and x', and act as Lorenz transformations locally at these points.[38] Second, we can "translate" the indices and polarizations of $\mathbf{L}[\mathcal{O}^\dagger]$ from x' to x, so that the action of $L_{xx'}$ is the same on all indices and polarizations. Specifically, we can write

$$\langle \mathcal{P}_{\delta_1,\lambda_1}(z_1)\mathcal{P}_{\delta_2,\lambda_2}(z_2)\mathcal{P}^\dagger_{\delta,\lambda}(z)\rangle^{(c)} = \frac{\langle 0|\mathbf{L}^+[\mathcal{O}_2](x, z_2)(I_{xx'}\mathbf{L}[\mathcal{O}^\dagger])(x', I_{xx'}z')\mathbf{L}^-[\mathcal{O}_1](x, z_1)|0\rangle_+^{(a)}}{\mathrm{vol}(\mathrm{SO}(1,1))} \mathcal{M}_{(a),\delta,\lambda}^{(c)}, \tag{2.189}$$

$$\langle \mathcal{P}_{\delta,\lambda}(z)\mathcal{P}^\dagger_{\delta,\lambda}(z')\rangle = \frac{1}{2\pi i}\langle \mathbf{L}[\mathcal{O}](x, z)(I_{xx'}\mathbf{L}[\mathcal{O}^\dagger])(x', I_{xx'}z')\rangle. \tag{2.190}$$

where

$$(I_{xx'})^\mu{}_\nu = \delta^\mu_\nu - 2x^\mu x_\nu / x^2, \tag{2.191}$$

and $I_{xx'}\mathbf{L}[\mathcal{O}^\dagger]$ indicates the action of I on the transverse spin indices. Importantly, the presence of $I_{xx'}$ means that while z' and other indices now transform in the

[38] For example, we can set $x = 0$ and $x' = \infty$, in which case the group in question is the standard Lorentz group.

same way under $L_{xx'}$ as the indices and polarizations at x, the representation that they form is reflected relative to that of $\mathbf{L}[O^\dagger]$. This reflection is crucial, because it is precisely the difference between the d-dimensional and $(d-2)$-dimensional $\dagger$ operation, and so the representations match in the above identifications.

Equation (2.184) implies that the set of three-point structures defined by (2.189) is complete, otherwise we wouldn't be able to get the Kronecker delta in the right-hand side of (2.184). We can therefore use the statement of section 2.5.2. Specifically, let us define operators $\widehat{C}^{(a)}_{\delta,\lambda}$ by (2.176), which now takes the form

$$\begin{aligned}&\frac{1}{2\pi i}\widehat{C}^{(c)}_{\delta,\lambda}(z_1,z_2,\partial_{z_2})\langle \mathbf{L}[O](x,z_2)(I_{xx'}\mathbf{L}[O^\dagger])(x',I_{xx'}z')\rangle\\ &=\frac{\langle 0|\mathbf{L}^+[O_2](x,z_2)(I_{xx'}\mathbf{L}[O^\dagger])(x',I_{xx'}z')\mathbf{L}^-[O_1](x,z_1)|0\rangle_+^{(a)}}{\mathrm{vol}(\mathrm{SO}(1,1))}\mathcal{M}^{(c)}_{(a),\delta,\lambda}. \qquad (2.192)\end{aligned}$$

In this equation, we can cancel the $I_{xx'}$ matrices on both sides

$$\frac{1}{2\pi i}\widehat{C}^{(c)}_{\delta,\lambda}(z_1,z_2,\partial_{z_2})\langle \mathbf{L}[O](x,z_2)\mathbf{L}[O^\dagger](x',z')\rangle=\frac{\langle 0|\mathbf{L}^+[O_2](x,z_2)\mathbf{L}[O^\dagger](x',z')\mathbf{L}^-[O_1](x,z_1)|0\rangle_+^{(a)}}{\mathrm{vol}(\mathrm{SO}(1,1))}\mathcal{M}^{(c)}_{(a)} \qquad (2.193)$$

Then we find

$$\mathbf{L}[O_1](x,z_1)\mathbf{L}[O_2](x,z_2)=\sum_\lambda\int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty}\frac{d\delta}{2\pi i}\widehat{C}^{(c)}_{\delta,\lambda}(z_1,z_2;\partial_{z_2})\mathbb{W}_{\delta,\lambda,(c)}(x,z_2), \qquad (2.194)$$

where

$$\mathbb{W}_{\delta,\lambda,(c)}(x,z)\equiv\int D^{d-2}z_1D^{d-2}z_2k_{\delta,\lambda,(c)}(z_1,z_2;z)\mathbf{L}[O_1](x,z_1)\mathbf{L}[O_2](x,z_2). \qquad (2.195)$$

In our case, the kernel $k_{\delta,\lambda,(c)}$ arose from K^t through (2.180). This means that we can alternatively rewrite $\mathbb{W}_{\delta,\lambda,(c)}(x,z)$ as

$$\mathbb{W}_{\delta,\lambda,(c)}(x,z)=\mathcal{N}^{(a)}_{(c),\delta,\lambda}\frac{1}{2}\left(\mathbb{O}^+_{\delta+1,J_1+J_2-1,\lambda,(a)}(x,z)-\mathbb{O}^-_{\delta+1,J_1+J_2-1,\lambda,(a)}(x,z)\right). \qquad (2.196)$$

Plugging this into (2.194) we conclude

$$\begin{aligned}&\mathbf{L}[O_1](x,z_1)\mathbf{L}[O_2](x,z_2)\\ &=\frac{1}{2}\sum_{\lambda\in\Lambda_{12}}\int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty}\frac{d\delta}{2\pi i}\mathcal{N}^{(a)}_{(c),\delta,\lambda}\widehat{C}^{(c)}_{\delta,\lambda}(z_1,z_2;\partial_{z_2})\left(\mathbb{O}^+_{\delta+1,J_1+J_2-1,\lambda,(a)}(x,z)-\mathbb{O}^-_{\delta+1,J_1+J_2-1,\lambda,(a)}(x,z)\right)+\cdots. \qquad (2.197)\end{aligned}$$

In the last step we fixed a subtlety that we glossed over before: the identification of three-point structures in (2.189) can only be performed for low transverse spin λ since the right-hand side simply does not exist for higher transverse spin. Therefore, in (2.197) we have only correctly identified the low transverse spin contributions, and $(\cdots)$ denotes the higher transverse spin contributions that we deal with in the next sections.

We can simplify equation (2.197) further, by defining

$$C^{(a)}_{\delta,\lambda}(z_1,z_2;\partial_{z_2}) = \frac{1}{2\pi i}\mathcal{N}^{(a)}_{(c),\delta,\lambda}\widehat{C}^{(c)}_{\delta,\lambda}(z_1,z_2;\partial_{z_2}). \tag{2.198}$$

Note that while $\widehat{C}^{(c)}_{\delta,\lambda}$ are linearly-independent, $C^{(a)}_{\delta,\lambda}$ are not. In terms of $C^{(a)}_{\delta,\lambda}$ we get

$$\begin{aligned}&\mathbf{L}[\mathcal{O}_1](x,z_1)\mathbf{L}[\mathcal{O}_2](x,z_2)\\ &= \pi i\sum_{\lambda\in\Lambda_{12}}\int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty}\frac{d\delta}{2\pi i}C^{(a)}_{\delta,\lambda}(z_1,z_2;\partial_{z_2})\left(\mathbb{O}^{+}_{\delta+1,J_1+J_2-1,\lambda,(a)}(x,z)-\mathbb{O}^{-}_{\delta+1,J_1+J_2-1,\lambda,(a)}(x,z)\right)+\cdots .\end{aligned} \tag{2.199}$$

Using (2.193) and (2.182) we can give an equivalent characterization of $C^{(a)}_{\delta,\lambda}$,

$$C^{(a)}_{\delta,\lambda}(z_1,z_2,\partial_{z_2})\langle\mathbf{L}[\mathcal{O}](x,z_2)\mathbf{L}[\mathcal{O}^\dagger](x',z')\rangle = \frac{\langle 0|\mathbf{L}^+[\mathcal{O}_2](x,z_2)\mathbf{L}[\mathcal{O}^\dagger](x',z')\mathbf{L}^-[\mathcal{O}_1](x,z_1)|0\rangle^{(a)}_+}{\mathrm{vol}(\mathrm{SO}(1,1))}. \tag{2.200}$$

Equations (2.200) and (2.199) reproduce equations (3.97) and (3.98) of [15], respectively.[39] We refer to (2.200) as the "celestial map" formula because it maps the d-dimensional tensor structures appearing on the right-hand side to the $(d-2)$-dimensional OPE differential operators $C^{(a)}_{\delta,\lambda}$. Note that (2.186) implies that $m_{(a)}$ such that $\mathcal{N}^{(a)}_{(c),\delta,\lambda}m_{(a)}=0$ satisfy

$$\frac{\langle 0|\mathbf{L}^+[\mathcal{O}_2](x,z_2)\mathbf{L}[\mathcal{O}^\dagger](x',z')\mathbf{L}^-[\mathcal{O}_1](x,z_1)|0\rangle^{(a)}_+}{\mathrm{vol}(\mathrm{SO}(1,1))}m_{(a)}=0, \tag{2.201}$$

i.e. the kernel of $\mathcal{N}^{(a)}_{(c),\delta,\lambda}$ is the same as the kernel of the celestial map.

Finally, to reproduce the low transverse spin terms in (2.147), we note, analogously to the discussion around (2.138), that in order for the left-hand side of (2.199) to be well-defined, we need the Regge intercept to satisfy $J_0 < J_1+J_2-1$, in

[39] There is a $(-)$ sign in (2.199) which is not in (3.98) of [15], which is due to the more explicit treatment of $(-1)^J$ signs in section 2.5.1 of this paper.

which case we can use a superconvergence sum rule [34, 15] which states that $\mathbb{O}^{(-1)^{J_1+J_2-1}}_{\delta+1,J_1+J_2-1,\lambda,(a)}(x,z)=0$. This allows us to rewrite (2.199) as

$$\begin{aligned}&\mathbf{L}[\mathcal{O}_1](x,z_1)\mathbf{L}[\mathcal{O}_2](x,z_2)\\&=\pi i(-1)^{J_1+J_2}\sum_{\lambda\in\Lambda_{12}}\int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty}\frac{d\delta}{2\pi i}C^{(a)}_{\delta,\lambda}(z_1,z_2;\partial_{z_2})\mathbb{O}^{(-1)^{J_1+J_2}}_{\delta+1,J_1+J_2-1,\lambda,(a)}(x,z)+\cdots,\end{aligned} \tag{2.202}$$

which reproduces the first sum in (2.147).

Higher transverse spin

Now we would like to understand the higher transverse spin terms in the OPE (2.147). The logic in this case is similar to the low transverse spin case, but with some extra complications. Similarly to (2.180), we assume that

$$\begin{aligned}&\mathcal{N}^{(a)}_{(c),\delta,\lambda_\gamma,n}\mathcal{D}_n\left(\int_{\substack{x\approx 2\\x\approx 1^-}}d^dx_1d^dx_2K^t_{\delta+1,J,\lambda_\gamma,(a)}(x_1,x_2;x,z)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\right)\\&\underset{J\to J_1+J_2-1+n}{\longrightarrow}\int D^{d-2}z_1D^{d-2}z_2k_{\delta,\lambda_\gamma(+n),(c)}(z_1,z_2;z)\mathbf{L}[\mathcal{O}_1](x,z_1)\mathbf{L}[\mathcal{O}_2](x,z_2).\end{aligned} \tag{2.203}$$

The evidence for this assumption is the same as for (2.180) and is discussed in section 2.5.2.

Using this statement in (2.152) requires some care. Indeed, plugging in this relation (again defining $\mathcal{M}$ to be the left inverse to $\mathcal{N}$), we find

$$\begin{aligned}&\int D^{d-2}z_1D^{d-2}z_2k_{\delta,\lambda_\gamma(+n),(c)}(z_1,z_2;z)\frac{\langle 0|\mathbf{L}^+[\mathcal{O}_2](x,z_2)\mathbf{L}[\mathcal{O}^\dagger](x',z')\mathbf{L}^-[\mathcal{O}_1](x,z_1)|0\rangle^{(a)}_+}{\mathrm{vol}(\mathrm{SO}(1,1))^2}\mathcal{M}^{(c')}_{(a),\delta,\lambda_\gamma,n}\\&=\frac{1}{2\pi i}\langle(\mathcal{D}_n\mathbf{L}[\mathcal{O}])(x,z)\mathbf{L}[\mathcal{O}^\dagger](x',z')\rangle\delta^{(c')}_{(c)}.\end{aligned} \tag{2.204}$$

The equality holds if $J=J_n\equiv J_1+J_2-1+n$. But for this value of J, the action of $\mathcal{D}_n$ is conformally-invariant, and the right-hand side of (2.204) contains a conformally-invariant two-point function for operators $\mathcal{D}_n\mathbf{L}[\mathcal{O}]$ and $\mathbf{L}[\mathcal{O}^\dagger]$. However, the quantum numbers of these operators are not Hermitian-conjugate to each other, and thus such two-point functions do not exist! This means that the right-hand side vanishes, and so the left-hand side must also vanish. Equation (2.204) at $J=J_n$ is thus trivially satisfied.

To obtain nontrivial information, we must shift away from $J = J_n$. For this, it helps to act with $\mathcal{D}_n$ on $\mathbf{L}[O^\dagger]$ on both sides of (2.204). We obtain on the right-hand side

$$\begin{aligned}
&\frac{1}{2\pi i}\langle(\mathcal{D}_n\mathbf{L}[O])(x,z)(\mathcal{D}_n\mathbf{L}[O^\dagger])(x',z')\rangle\delta_{(c)}^{(c')} \\
&= (J-J_n)\lim_{J\to J_n}\frac{1}{2\pi i}\frac{\langle(\mathcal{D}_n\mathbf{L}[O])(x,z)(\mathcal{D}_n\mathbf{L}[O^\dagger])(x',z')\rangle}{J-J_n}\delta_{(c)}^{(c')} + O((J-J_n)^2).
\end{aligned} \tag{2.205}$$

After acting with $\mathcal{D}_n$ on the second operator, the limit

$$\lim_{J\to J_n}\frac{1}{2\pi i}\frac{\langle(\mathcal{D}_n\mathbf{L}[O])(x,z)(\mathcal{D}_n\mathbf{L}[O^\dagger])(x',z')\rangle}{J-J_n} \tag{2.206}$$

is a conformally-invariant two-point function [38].

We now need to analyze the left-hand side of (2.152) away from $J = J_n$. One might worry that we would need to know the subleading term in (2.203) in order to determine the leading non-zero piece in the left-hand side of (2.204). Fortunately, this is not required. To see this, let us first write for general J, from (2.152) and the above,

$$\begin{aligned}
&\int \frac{d^dx_1 d^dx_2}{\mathrm{vol}(\mathrm{SO}(1,1))^2}\mathcal{D}_n\left(K^{t,\pm}_{\Delta,J,\lambda(a)}(x_1,x_2;x,z)\theta(x\approx 2,1^-)\right) \\
&\qquad\qquad \times \mathcal{D}_n\left(\langle 0|O_2\mathbf{L}[O^\dagger](x',z')O_1|0\rangle_+^{(b)}\theta(2>x'>1^-)\right) \\
&= (J-J_n)\lim_{J\to J_n}\frac{1}{2\pi i}\frac{\langle(\mathcal{D}_n\mathbf{L}[O])(x,z)(\mathcal{D}_n\mathbf{L}[O^\dagger])(x',z')\rangle}{J-J_n}\delta_{(a)}^{(b)} + O((J-J_n)^2)
\end{aligned} \tag{2.207}$$

Since the right-hand side is $O(J-J_n)$, this should be true for the left-hand side as well. We claim that in fact

$$\int \frac{d^dx_1 d^dx_2}{\mathrm{vol}(\mathrm{SO}(1,1))^2}F(x_1,x_2;x,z)\mathcal{D}_n\left(\langle 0|O_2\mathbf{L}[O^\dagger](x',z')O_1|0\rangle_+^{(b)}\theta(2>x'>1^-)\right) \in O(J-J_n) \tag{2.208}$$

for any conformally-invariant kernel F that transforms at x with the quantum numbers of $\mathcal{D}_n\mathbf{L}[O]$. This statement implies that we can use (2.204) directly at $J = J_n = J_1 + J_2 - 1 + n$ for the purposes of determining the $O(J-J_n)$ term.

We actually need the following refined version of (2.208),

$$\begin{aligned}
&\int \frac{d^dx_1 d^dx_2}{\mathrm{vol}(\mathrm{SO}(1,1))^2}F(x_1,x_2;x,z)\mathcal{D}_n\left(\langle 0|O_2\mathbf{L}[O^\dagger](x',z')O_1|0\rangle_+^{(b)}\theta(2>x'>1^-)\right) \\
&= (J-J_n)\int \frac{d^dx_1 d^dx_2}{\mathrm{vol}(\mathrm{SO}(1,1))}F(x_1,x_2;x,z)\mathcal{D}_n\left(\langle 0|O_2\mathbf{L}[O^\dagger](x',z')O_1|0\rangle_+^{(b)}\theta(2>x'>1^-)\right)\Big|_{J=J_n} \\
&\quad + O((J-J_n)^2).
\end{aligned} \tag{2.209}$$

Note that the expressions on the left-hand and right-hand sides differ by the power of the factor $\mathrm{vol}(\mathrm{SO}(1,1))$. The reason for this is that while at $J = J_n$ and $J \neq J_n$ both integrals have 2 reparameterization zero modes, in the integral on the right, one of the modes is similar to $x \to \alpha x$ in

$$\int dx\, \delta(x) = 1, \tag{2.210}$$

and does not require Faddeev-Popov fixing. A simple model for the above equation is provided by the integral

$$\int_0^\infty \frac{dx}{\mathrm{vol}(\mathrm{SO}(1,1))} 2 \cdot x^{-(J-J_n)} \partial_x (x^{J-J_n} \theta(x)) = 2(J - J_n). \tag{2.211}$$

Clearly, the coefficient 2 in front of $(J - J_n)$ can be computed by setting $J = J_n$ and removing the $\mathrm{vol}(\mathrm{SO}(1,1))$ factor. The result (2.209) can be shown by an explicit calculation in the case when $\mathcal{O}_1, \mathcal{O}_2$ are scalars,[40] and then noting that integrands in both sides have the same transformation properties under weight-shifting operators acting on $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}$ [39, 45].

Combining everything together, we find the following equation for the kernel $k_{\delta,\lambda_\gamma(+n),(a)}$,

$$\int \frac{D^{d-2}z_1 D^{d-2}z_2}{\mathrm{vol}(\mathrm{SO}(1,1))} k_{\delta,\lambda_\gamma(+n),(c)}(z_1,z_2;z) \mathcal{D}_n \left(\langle 0| \mathbf{L}^+[\mathcal{O}_2](x,z_2) \mathbf{L}[\mathcal{O}^\dagger](x',z') \mathbf{L}^-[\mathcal{O}_1](x,z_1) |0\rangle_+^{(a)} \right) \mathcal{M}^{(c')}_{(a),\delta,\lambda_\gamma}$$
$$= \lim_{J\to J_n} \frac{1}{2\pi i} \frac{\langle (\mathcal{D}_n \mathbf{L}[\mathcal{O}])(x,z) (\mathcal{D}_n \mathbf{L}[\mathcal{O}^\dagger])(x',z') \rangle}{J - J_n} \delta^{(c')}_{(c)}. \tag{2.212}$$

From this equation, following precisely the same steps as in the derivation of low transverse spin terms in the previous section, we find that if the operators $\mathcal{C}^{(a)}_{\delta,\lambda_\gamma(+n)}$ are defined by

$$\mathcal{C}^{(a)}_{\delta,\lambda_\gamma(+n)}(z_1,z_2,\partial_{z_2}) \lim_{J\to J_n} \frac{\langle (\mathcal{D}_n \mathbf{L}[\mathcal{O}])(x,z_2) (\mathcal{D}_n \mathbf{L}[\mathcal{O}^\dagger])(x',z') \rangle}{J - J_n}$$
$$= (-1)^n \mathcal{D}_n \left(\langle 0| \mathbf{L}^+[\mathcal{O}_2](x,z_2) \mathbf{L}[\mathcal{O}^\dagger](x',z') \mathbf{L}^-[\mathcal{O}_1](x,z_1) |0\rangle_+^{(a)} \right), \tag{2.213}$$

then the contribution of higher transverse spin terms is given by the second sum in (2.147).[41] In appendix A.5 we consider which representations $\lambda_\gamma(+n)$ can be generated in this way, and show that there are enough d-dimensional structures that survive this celestial map to account for all the celestial OPE structures.

[40] It is easy to convince oneself that it suffices to ensure that the result of this section agrees with the result of section 2.4.

[41] Here we introduced a sign $(-1)^n$ by hand in order to simplify (2.147).

2.5.3 Celestial map without light transforms

In the previous section we derived the celestial map formulas (2.200) and (2.213) that determine the OPE differential operators $C^{(a)}_{\delta,\lambda}$ appearing in (2.147). These formulas involve taking several light-transforms of tensor structures analytically continued to Wightman correlators, which in practice can be a difficult calculation. Fortunately, as first observed in [15] and proved for traceless-symmetric operators, there exists a simpler version of the celestial map formulas that contains only simple algebraic manipulations.

We will show that the following is equivalent to (2.200),

$$\begin{aligned}
&C^{(a)}_{\delta,\lambda}(z_1,\mathbf{w}_1,z_2,\mathbf{w}_2,\partial_{z_2},\partial_{\mathbf{w}_2})\left((-2H_{20})\langle\mathcal{O}(X_2,Z_2,\mathbf{W}_2)\mathcal{O}^\dagger(X_0,Z_0,\mathbf{W}_0)\rangle\right)\Big|_{\text{celestial}}\\
&\quad= X_{12}\left|-2V_{0,21}\right|\langle\mathcal{O}_1(X_1,Z_1,\mathbf{W}_1)\mathcal{O}^\dagger(X_0,Z_0,\mathbf{W}_0)\mathcal{O}_2(X_2,Z_2,\mathbf{W}_2)\rangle^{(a)}_+\Big|_{\text{celestial}},
\end{aligned} \tag{2.214}$$

and the following equivalent to (2.213),

$$\begin{aligned}
&C^{(a)}_{\delta,\lambda_\gamma(+n)}(z_1,\mathbf{w}_1,z_2,\mathbf{w}_2,\partial_{z_2},\partial_{\mathbf{w}_2})\left(\lim_{J\to J_n}(-2H_{20})\frac{\langle(\mathcal{D}'_n\mathcal{O})(X_2,Z_2,\mathbf{W}_2)(\mathcal{D}'_n\mathcal{O}^\dagger)(X_0,Z_0,\mathbf{W}_0)\rangle}{J-J_n}\right)\Bigg|_{\text{celestial}}\\
&= \frac{(-1)^n X_{12}}{\delta(-2V_{0,21})}\mathcal{D}'_n\left(\theta(V_{0,12})\langle\mathcal{O}_1(X_1,Z_1,\mathbf{W}_1)\mathcal{O}^\dagger(X_0,Z_0,\mathbf{W}_0)\mathcal{O}_2(X_2,Z_2,\mathbf{W}_2)\rangle^{(a)}_+\right)\Bigg|_{\text{celestial}}.
\end{aligned} \tag{2.215}$$

(Note that we have used the $\langle\cdots\rangle^{(a)}_+$ structures defined in section 2.5.1.) In the above equations, we have explicitly reintroduced polarization vectors $\mathbf{w}_i$ for the second and higher rows of Young diagrams of $\mathcal{O}_i$, as described in section 2.3. Furthermore, we have used embedding space notation [30] on the right-hand side, with the standard tensor structures

$$X_{ij} \equiv -2X_i\cdot X_j \tag{2.216}$$

$$V_{i,jk} \equiv \frac{Z_i\cdot X_j X_i\cdot X_k - Z_i\cdot X_k X_i\cdot X_j}{X_j\cdot X_k}, \tag{2.217}$$

$$H_{ij} \equiv -2(Z_i\cdot Z_j X_i\cdot X_j - Z_i\cdot X_j Z_j\cdot X_i). \tag{2.218}$$

Finally, the notation $(\cdots)|_{\text{celestial}}$ stands for substituting the following values for the embedding space coordinates,

$$\begin{aligned}
&Z_0 = -(1,0,0),\quad Z_1 = -(0,1,0),\quad Z_2 = -(0,1,0),\\
&X_i = (0,0,z_i),\quad W_{i,j} = (0,0,w_{i,j}),
\end{aligned} \tag{2.219}$$

where we specify coordinates in the order (Y^+, Y^-, Y^μ), $\mu = 0, \cdots, d-1$. Note that in (2.215) we divide by $\delta(-2V_{0,21})$. By this we mean that the result of the action of $\mathcal{D}'_n$ is proportional to $\delta(-2V_{0,21})$, and we simply read off the coefficient of this delta-function. Note that this coefficient is only well-defined in configurations where $V_{0,21} = 0$, which is indeed the case for the celestial locus (2.219).

Factoring out the light-transforms

We start by proving (2.215). The proof of (2.214) is only a simple modification that we comment on below.

The proof proceeds with evaluation of light-transforms in both sides of (2.213). We start with the right-hand side

$$\mathcal{D}_n \left(\langle 0|\mathbf{L}^+[\mathcal{O}_2](x, z_2, \mathbf{w}_2)\mathbf{L}[\mathcal{O}^\dagger](x_0, z_0, \mathbf{w}_0)\mathbf{L}^-[\mathcal{O}_1](x, z_1, \mathbf{w}_1)|0\rangle_+^{(a)} \right). \quad (2.220)$$

Here $\mathcal{D}_n$ acts on $(x_0, z_0, \mathbf{w}_0)$. We can rewrite this equivalently as

$$\begin{aligned} &= \mathcal{D}_n \mathbf{L}_1 \mathbf{L}_2 \left(\langle 0|\mathcal{O}_2(x, z_2, \mathbf{w}_2)\mathbf{L}[\mathcal{O}^\dagger](x_0, z_0, \mathbf{w}_0)\mathcal{O}_1(x, z_1, \mathbf{w}_1)|0\rangle_+^{(a)} \theta(2 > 0 > 1^-) \right) \\ &= \mathcal{D}_n \mathbf{L}_1 \mathbf{L}_2 \left(\langle \mathcal{O}_1(x, z_1, \mathbf{w}_1)\mathcal{O}_2(x, z_2, \mathbf{w}_2)\mathbf{L}[\mathcal{O}^\dagger](x_0, z_0, \mathbf{w}_0)\rangle_+^{(a)} \theta(V_{0,12}) \right), \quad (2.221) \end{aligned}$$

where $\mathbf{L}_i$ denotes light-transform acting on $(x_i, z_i, \mathbf{w}_i)$. In the last equality we used the following fact. First of all,

$$V_{0,12} = \frac{z_0 \cdot x_{10} x_{20}^2 - z_0 \cdot x_{20} x_{10}^2}{x_{12}^2} \quad (2.222)$$

is positive for $1 \approx 2$ and $2 > 0 > 1^-$, and is negative for $1 \approx 2$ and $1 > 0 > 2^-$. In (2.169) we have two terms, and multiplying by $\theta(V_{0,12})$ selects the first term, which is the one appearing in the first line of (2.221).

Now we can use (2.65) to rewrite this further as

$$= \mathbf{L}_1 \mathbf{L}_2 \mathbf{L}_0 \mathcal{D}'_n \left(\langle \mathcal{O}_1(x, z_1, \mathbf{w}_1)\mathcal{O}_2(x, z_2, \mathbf{w}_2)\mathcal{O}^\dagger(x_0, z_0, \mathbf{w}_0)\rangle_+^{(a)} \theta(V_{0,12}) \right), \quad (2.223)$$

where we took into account the easily verified fact that $V_{0,12}$ commutes with $\mathbf{L}_0$.

Appearance of $\delta(-2V_{0,12})$

Note that without the theta-function we would have

$$\mathcal{D}'_n \langle \mathcal{O}_1(x, z_1, \mathbf{w}_1)\mathcal{O}_2(x, z_2, \mathbf{w}_2)\mathcal{O}^\dagger(x_0, z_0, \mathbf{w}_0)\rangle^{(a)} = 0. \quad (2.224)$$

This is because for $J = J_n$, all possible three-point tensor structures above are polynomial in z_0, and hence killed by $\mathcal{D}'_n$. To see this, note that the way non-polynomial structures in z_0 appear is through the factors of the form

$$\langle \mathcal{O}_1(x, z_1, \mathbf{w}_1)\mathcal{O}_2(x, z_2, \mathbf{w}_2)\mathcal{O}^\dagger(x_0, z_0, \mathbf{w}_0)\rangle^{(a)} = (\cdots)V_{0,12}^{J-k} \tag{2.225}$$

when $J < k$, where $k > 0$ is the degree of z_0 in $(\cdots)$. Non-polynomiality cannot appear in any other way, because we require that all polarizations except z_0 enter polynomially, and $V_{0,12}$ is the only invariant that involves only z_0. We see that for sufficiently large integer J all structures are therefore polynomial, and the appearance of non-polynomial structures is indicated by the reduction in the number of polynomial structures. The number of polynomial structures can be computed using group-theoretic counting rules [48], and a simple calculation shows that the polynomial structures start disappearing for $J = J_1 + J_2 - 1$ (see appendix A.5). This means that for $J = J_1 + J_2 - 1$ the maximal non-polynomiality is $V_{0,12}^{-1}$, and for $J = J_n = J_1 + J_2 - 1 + n$ with $n > 0$ there are no non-polynomial structures. More generally, the smallest possible exponent of $V_{0,12}$ is

$$V_{0,12}^{J-J_1-J_2}. \tag{2.226}$$

Since all structures are polynomial for $J = J_n$, the properties of $\mathcal{D}'_n$ discussed in section 2.3 ensure (2.224).

However, we are interested in

$$\mathcal{D}'_n\left(\langle \mathcal{O}_1(x, z_1, \mathbf{w}_1)\mathcal{O}_2(x, z_2, \mathbf{w}_2)\mathcal{O}^\dagger(x_0, z_0, \mathbf{w}_0)\rangle_+^{(a)}\theta(V_{0,12})\right). \tag{2.227}$$

We can obtain derivatives of delta functions $\delta^{(m)}(-2V_{0,12})$ from $\mathcal{D}'_n$ hitting the theta-function. We claim that we get a result which is proportional to $\delta(-2V_{0,12})$. To see this, we regularize by analytic continuation in J. Note that in (2.227) the most negative power of $V_{0,12}$ is

$$V_{0,12}^{J-J_1-J_2-n} = V_{0,12}^{J-J_n-1}, \tag{2.228}$$

because $\mathcal{D}'_n$ has n derivatives which will therefore hit $V_{0,12}^{J-J_1-J_2}$, which is the most negative power of $V_{0,12}$ before acting with $\mathcal{D}'_n$, at most n times. In other words, we have for each (a)

$$\begin{aligned}&\mathcal{D}'_n\left(\langle \mathcal{O}_1(x, z_1, \mathbf{w}_1)\mathcal{O}_2(x, z_2, \mathbf{w}_2)\mathcal{O}^\dagger(x_0, z_0, \mathbf{w}_0)\rangle_+^{(a)}\theta(V_{0,12})\right)\\ &= (J - J_n)(\cdots)V_{0,12}^{J-J_n-1}\theta(V_{0,12}) + R,\end{aligned} \tag{2.229}$$

where R involves higher powers of $V_{0,12}$ or $(J-J_n)$. As we send $J \to J_n$, we then find $R \to 0$ and

$$(J-J_n)V_{0,12}^{J-J_n-1}\theta(V_{0,12}) \to \delta(V_{0,12}). \tag{2.230}$$

We therefore conclude that

$$\mathcal{D}'_n\left(\langle O_1(x,z_1,\mathbf{w}_1)O_2(x,z_2,\mathbf{w}_2)O^\dagger(x_0,z_0,\mathbf{w}_0)\rangle_+^{(a)}\theta(V_{0,12})\right) = \delta(-2V_{0,21}) \times \text{finite}. \tag{2.231}$$

Symmetries of the integrand

Defining

$$\begin{aligned} &f(X_1,Z_1,\mathbf{W}_1;X_2,Z_2,\mathbf{W}_2;X_0,Z_0,\mathbf{W}_0) \\ &\equiv \mathcal{D}'_n\left(\langle O_1(X_1,Z_1,\mathbf{W}_1)O_2(X_2,Z_2,\mathbf{W}_2)O^\dagger(X_0,Z_0,\mathbf{W}_0)\rangle_+^{(a)}\theta(V_{0,12})\right), \end{aligned} \tag{2.232}$$

and

$$f(\alpha_1,\alpha_2,\alpha_0) = f(Z_1-\alpha_1X_\infty,-X_\infty,\mathbf{W}_1;Z_2-\alpha_2X_\infty,-X_\infty,\mathbf{W}_2;Z_0-\alpha_0X_0,-X_0,\mathbf{W}_0), \tag{2.233}$$

where

$$X_\infty = (0,1,0), \quad X_0 = (1,0,0), \tag{2.234}$$

$$Z_i = (0,0,z_i), \quad W_{i,j} = (0,0,w_{i,j}), \tag{2.235}$$

we can rewrite (2.223) and thus (2.220) for $x=\infty, x'=0$ as

$$\mathcal{D}_n\langle 0|\mathbf{L}^+[O_2](\infty,z_2,\mathbf{w}_2)\mathbf{L}[O^\dagger](0,z_0,\mathbf{w}_0)\mathbf{L}^-[O_1](\infty,z_1,\mathbf{w}_1)|0\rangle_+^{(a)} = \int_{-\infty}^{+\infty} d\alpha_1 d\alpha_2 d\alpha_0 f(\alpha_1,\alpha_2,\alpha_0). \tag{2.236}$$

Our goal here will be to find an expression for $f(\alpha_1,\alpha_2,\alpha_0)$ based solely on symmetries. First, we check that the boost in the embedding space which sends

$$X_\infty \to \lambda X_\infty, \quad X_0 \to \lambda^{-1}X_0 \tag{2.237}$$

implies that for $\lambda > 0$

$$f(\alpha_1,\alpha_2,\alpha_0) = \lambda^{J_1+J_2-J'} f(\lambda\alpha_1,\lambda\alpha_2,\lambda^{-1}\alpha_0), \tag{2.238}$$

where $J' = J_n - n = J_1 + J_2 - 1$ is the spin we get after the action of $\mathcal{D}'_n$. That is, we have

$$f(\alpha_1, \alpha_2, \alpha_0) = \lambda f(\lambda\alpha_1, \lambda\alpha_2, \lambda^{-1}\alpha_0). \tag{2.239}$$

Second, the symmetry that in Minkowski space is represented by translation along z_0 acts on embedding space coordinates as

$$X_0 \to X_0 + \lambda Z_0, \quad Z_i \to Z_i + 2(z_i \cdot z_0)\lambda X_\infty \quad (i = 1, 2), \tag{2.240}$$

leaving all other coordinates invariant.[42] We can check that it implies the following equation for f,

$$f(\alpha_1, \alpha_2, \alpha_0) = (1 - \alpha_0\lambda)^{-\Delta - J'} f(\alpha_1 - 2(z_1 \cdot z_0)\lambda, \alpha_2 - 2(z_2, z_0)\lambda, (\alpha_0^{-1} - \lambda)^{-1}) \tag{2.242}$$

for $1 - \alpha_0\lambda > 0$.

Now recall from the previous discussion that $f(\alpha_1, \alpha_2, \alpha_0)$ is proportional to $\delta(-2V_{0,21})$, and so we can write

$$f(\alpha_1, \alpha_2, \alpha_0) = \delta\left(\frac{\alpha_1(z_0 \cdot z_2) - \alpha_2(z_0 \cdot z_2)}{(z_1 \cdot z_2)}\right) g(\alpha_2, \alpha_0), \tag{2.243}$$

for some g. Here we eliminated the α_1-dependence using the delta-function. In terms of g, the symmetries discussed above read

$$g(\alpha_2, \alpha_0) = g(\lambda\alpha_2, \lambda^{-1}\alpha_0), \tag{2.244}$$
$$g(\alpha_2, \alpha_0) = (1 - \alpha_0\lambda)^{-\Delta - J'} g(\alpha_2 - 2(z_2, z_0)\lambda, (\alpha_0^{-1} - \lambda)^{-1}), \tag{2.245}$$

for $\lambda > 0$ and $1 - \alpha_0\lambda > 0$ respectively. Let us consider g_0 defined by

$$g_0(\alpha_2, \alpha_0) = |\alpha_2\alpha_0 - 2(z_2 \cdot z_0)|^{-\Delta - J'}. \tag{2.246}$$

It is easy to check that g_0 satisfies the same symmetries as g, and thus g/g_0 is simply invariant under the above transformations of α_2, α_0. Note that we have two continuous families of transformations, and it is easy to verify that the 2 vector fields

[42] To be more precise, we have

$$W_{i,j} \to W_{i,j} + 2(w_{i,j} \cdot z_0)\lambda X_\infty \quad (i = 1, 2), \tag{2.241}$$

but since in (2.233) $W_{i,j}$ are inserted with first-row polarization $-X_\infty$, the shift by X_∞ has no effect on the value of f.

by which they are generated are generically linearly-independent. Since we have only two coordinates α_2, α_0, we find that g/g_0 should be locally constant and can only change value where these vector fields degenerate. This only happens when $\alpha_2\alpha_0 - 2(z_2 \cdot z_0) = 0$ or $\alpha_0 = 0$.

Note that $g(\alpha_2, \alpha_0)$ comes from a three-point structure, and the three-point structures are analytic away from $X_{ij} = 0$, which are given by

$$X_{12} = -2(z_1 \cdot z_2), \tag{2.247}$$

$$X_{10} = \alpha_1\alpha_0 - 2(z_1 \cdot z_0), \tag{2.248}$$

$$X_{20} = \alpha_2\alpha_0 - 2(z_2 \cdot z_0), \tag{2.249}$$

so g/g_0 cannot have a discontinuity at $\alpha_0 = 0$. We thus only need to determine how g/g_0 changes when crossing $X_{20} = 0$. Since g comes from a time-ordered three-point structure, a non-analyticity near $X_{20} = 0$ must have the form

$$(X_{20} + i\epsilon)^{\#}, \tag{2.250}$$

where we use the usual $i\epsilon$-prescription for time-ordered correlators. It follows that

$$g(\alpha_2, \alpha_0) = A(\alpha_2\alpha_0 - 2(z_2 \cdot z_0) + i\epsilon)^{-\Delta - J'} \tag{2.251}$$

for some A, for all real values of α_2, α_0. It is convenient to express A in terms of $g(0, 0)$,

$$g(\alpha_2, \alpha_0) = (-2z_2 \cdot z_0)^{\Delta + J'} g(0, 0)(\alpha_2\alpha_0 - 2(z_2 \cdot z_0) + i\epsilon)^{-\Delta - J'}. \tag{2.252}$$

Computing the light-transforms

We can now compute the integrals in (2.236),

$$\begin{aligned}
&\int d\alpha_1 d\alpha_2 d\alpha_0 f(\alpha_1, \alpha_2, \alpha_0) \\
&= \int d\alpha_1 d\alpha_2 \delta\left(\frac{\alpha_1(z_0 \cdot z_2) - \alpha_2(z_0 \cdot z_2)}{(z_1 \cdot z_2)}\right) \int d\alpha_0 (-2z_2 \cdot z_0)^{\Delta + J'} g(0, 0)(\alpha_2\alpha_0 - 2(z_2 \cdot z_0) + i\epsilon)^{-\Delta - J'} \\
&= g(0, 0)(-2z_2 \cdot z_0) \int d\alpha_1 d\alpha_2 \delta\left(\frac{\alpha_1(z_0 \cdot z_2) - \alpha_2(z_0 \cdot z_2)}{(z_1 \cdot z_2)}\right) \int d\alpha_0 (\alpha_2\alpha_0 + 1 + i\epsilon)^{-\Delta - J'} \\
&= \frac{-2\pi i}{\Delta + J' - 1} g(0, 0)(-2z_2 \cdot z_0) \int d\alpha_1 d\alpha_2 \delta\left(\frac{\alpha_1(z_0 \cdot z_2) - \alpha_2(z_0 \cdot z_2)}{(z_1 \cdot z_2)}\right) \delta(\alpha_2) \\
&= \frac{-2\pi i}{\Delta + J' - 1}(-2z_1 \cdot z_2) g(0, 0),
\end{aligned} \tag{2.253}$$

where we used the equation

$$\int_{-\infty}^{+\infty} \frac{dx}{(xy+1+i\epsilon)^a} = -\frac{2\pi i}{a-1}\delta(y) \tag{2.254}$$

which we prove in appendix A.7.2. Unwinding the definitions, we check that

$$(-2z_1 \cdot z_2) g(0,0)$$
$$= \frac{X_{12}}{\delta(-2V_{0,21})} \mathcal{D}'_n \left(\theta(V_{0,12})\langle \mathcal{O}_1(X_1, Z_1, \mathbf{W}_1)\mathcal{O}^\dagger(X_0, Z_0, \mathbf{W}_0)\mathcal{O}_2(X_2, Z_2, \mathbf{W}_2)\rangle_+^{(a)}\right)\Bigg|_{\text{celestial}}. \tag{2.255}$$

It only remains to compute the light-transforms in the left-hand side of (2.213). For this, note that

$$\lim_{J\to J_n} \frac{\langle (\mathcal{D}_n \mathbf{L}[\mathcal{O}])(x_2, z_2, \mathbf{w}_2)(\mathcal{D}_n \mathbf{L}[\mathcal{O}^\dagger])(x_0, z_0, \mathbf{w}_0)\rangle}{J-J_n}$$
$$= \mathbf{L}_2 \mathbf{L}_0 \lim_{J\to J_n} \frac{\langle (\mathcal{D}'_n \mathcal{O})(X_2, Z_2, \mathbf{W}_2)(\mathcal{D}'_n \mathcal{O}^\dagger)(X_0, Z_0, \mathbf{W}_0)\rangle}{J-J_n} \tag{2.256}$$

and defining

$$g'(X_2, Z_2, \mathbf{W}_2; X_0, Z_0, \mathbf{W}_0) \equiv \lim_{J\to J_n} \frac{\langle (\mathcal{D}'_n \mathcal{O})(X_2, Z_2, \mathbf{W}_2)(\mathcal{D}'_n \mathcal{O}^\dagger)(X_0, Z_0, \mathbf{W}_0)\rangle}{J-J_n}, \tag{2.257}$$

$$g'(\alpha_2, \alpha_0) \equiv g'(Z_2 - \alpha_2 X_\infty, -X_\infty, \mathbf{W}_2; Z_0 - \alpha_0 X_0, -X_0, \mathbf{W}_0), \tag{2.258}$$

we find that g' satisfies the same properties as g above, and the same arguments lead to

$$g'(\alpha_2, \alpha_0) = (-2z_2 \cdot z_0)^{\Delta+J'} g'(0,0)(\alpha_2\alpha_0 - 2(z_2\cdot z_0) + i\epsilon)^{-\Delta-J'}. \tag{2.259}$$

We can then similarly compute

$$\mathbf{L}_2 \mathbf{L}_0 \lim_{J\to J_n} \frac{\langle (\mathcal{D}'_n \mathcal{O})(X_2, Z_2, \mathbf{W}_2)(\mathcal{D}'_n \mathcal{O}^\dagger)(X_0, Z_0, \mathbf{W}_0)\rangle}{J-J_n} = \int d\alpha_2 d\alpha_0 g'(\alpha_2, \alpha_0)$$
$$= \frac{-2\pi i}{\Delta + J' - 1}(-2z_2 \cdot z_0) g'(0,0). \tag{2.260}$$

Since $(-2z_2 \cdot z_0) = -2H_{20}|_{\text{celestial}}$, we find

$$(-2z_2 \cdot z_0) g'(0,0) = \left(\lim_{J\to J_n} (-2H_{20}) \frac{\langle (\mathcal{D}'_n \mathcal{O})(X_2, Z_2, \mathbf{W}_2)(\mathcal{D}'_n \mathcal{O}^\dagger)(X_0, Z_0, \mathbf{W}_0)\rangle}{J-J_n}\right)\Bigg|_{\text{celestial}}. \tag{2.261}$$

Combining with (2.213), (2.236), (2.253), (2.256), (2.260), and (2.255), we arrive at (2.215).

Low transverse spin case

The only modification to the above proof required for the case of low transverse spin — (2.200) and (2.214) — concerns the appearance of the delta-function in the analogue of (2.227).

$$(\langle \mathcal{O}_1(x, z_1, \mathbf{w}_1)\mathcal{O}_2(x, z_2, \mathbf{w}_2)\mathcal{O}^\dagger(x_0, z_0, \mathbf{w}_0)\rangle_+^{(a)}\theta(V_{0,12})). \tag{2.262}$$

In principle, there is no delta-function here. Instead, from the discussion in section 2.5.3 we know that for $J = J_1 + J_2 - 1$ there are structures which contain the most negative power of $V_{0,21}$ which is

$$(\cdots)(-2V_{0,21})^{-1}\theta(V_{0,12}). \tag{2.263}$$

This inverse power of $V_{0,21}$ leads to the divergence in the triple light-transform in (2.200) which is canceled by the $\mathrm{SO}(1,1)$ factor. Similarly to the discussion around (2.209) we can remove the $\mathrm{vol}(\mathrm{SO}(1,1))$ factor and replace (in fact, this is exactly the same replacement as in (2.209))

$$(-2V_{0,21})^{-1}\theta(V_{0,12}) \to \delta(-2V_{0,21}) \tag{2.264}$$

and then the above proof can be applied to (2.214). The only remaining difference is that in the final formula we prefer to factor out $(-2V_{0,21})^{-1}$ before the above substitution instead of factoring out $\delta(-2V_{0,21})$ after the substitution.

2.6 Examples

2.6.1 Re-deriving the $\mathbf{L}[\phi_1] \times \mathbf{L}[\phi_2]$ OPE from general formulas

In this section we rederive the scalar detector OPE (2.138) from the general result (2.147).

The first step is to work out the low-transverse spin terms in (2.138). The set of transverse spins Λ_{12} is constrained to consist of traceless-symmetric representations λ of spin j because $\mathbf{L}[\phi_1]$ and $\mathbf{L}[\phi_2]$ both transform as scalars on the celestial sphere. In d-dimensional language, the transverse spin j is the length of the second row of the Young diagram of a $\mathrm{SO}(d-1,1)$ irrep, and is therefore constrained to be $j = 0$ since only traceless-symmetric light-ray operators appear in $\phi_1 \times \phi_2$ OPE. The low-transverse spin contributions are then given by

$$\mathbf{L}[\phi_1](x, z_1)\mathbf{L}[\phi_2](x, z_2) = \pi i \int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty} \frac{d\delta}{2\pi i} C_{\delta,j=0}(z_1, z_2, \partial_{z_2})\mathbb{O}^+_{\delta+1,J=-1}(x, z_2) + \cdots, \tag{2.265}$$

where we also substituted $J_1 = J_2 = 0$ and removed dependence on the transverse polarizations $\mathbf{w}_i$ since all operators are traceless-symmetric. In order for this to agree with the $j = 0$ term of (2.138), we need to verify that $C_{\delta,j=0}$ normalized as in (2.89) also satisfies the celestial map (2.214). We do this by computing the structures entering in (2.214) and comparing them to the structures in (2.89).

First of all, we need to determine the expression for the three-point structure entering (2.214). Our d-dimensional structures for integer J are defined in appendix A.2, and the analytic continuation should be performed following the conventions of section 2.5.1. Comparing (A.7) and (2.164) we see that $n = 0$ in (2.164) and f_0 is given by some product of distances x_{ij} which is positive for space-like separated points. Looking at (2.168) we see that in our case the analytically-continued three-point structures appearing in (2.214), when all points are spacelike-separated, are equal to the absolute value of (A.7).

We can therefore use (A.9) and substitute the celestial locus values (2.219) into it. We find for $J = -1$,

$$X_{12}|-2V_{3,12}|\langle\phi_1(X_1)\phi_2(X_2)O(X_3,Z_3)\rangle|_{\text{celestial}} = \left.\frac{1}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta-1}{2}}X_{13}^{\frac{\Delta_1+\Delta-\Delta_2-1}{2}}X_{23}^{\frac{\Delta_2+\Delta-\Delta_1-1}{2}}}\right|_{\text{celestial}}$$
$$= \frac{1}{(-2z_1\cdot z_2)^{\frac{\delta_1+\delta_2-\delta}{2}}(-2z_1\cdot z_3)^{\frac{\delta_1+\delta-\delta_2}{2}}(-2z_2\cdot z_3)^{\frac{\delta_2+\delta-\delta_1}{2}}}, \tag{2.266}$$

and since this is positive and all X_i are space-like separated in (2.219), it follows, according to the discussion above, that this is equal to the right-hand side of (2.214) after substitution $3 \to 0$.

The left-hand side of (2.214) is easily computed from (A.8) to be equal to

$$(-2H_{23})\langle O(X_2,Z_2)O(X_3,Z_3)\rangle|_{\text{celestial}} = (-2z_2\cdot z_3)^{-\delta}, \tag{2.267}$$

after substitution $3 \to 0$. Using these results in (2.214) we see immediately that it gives the same normalization of $C_{\delta,j=0}$ as (2.89).

A similar logic works for the higher transverse spin terms in (2.147). We have $\lambda_\gamma(+n) = (n,\gamma)$. The set Γ_{12} in (2.147) then consists of just the trivial representation, because only traceless-symmetric representations appear in the $\phi_1 \times \phi_2$ OPE. We thus find that the contribution of higher transverse spins is

$$\begin{aligned}&\mathbf{L}[O_1](x,z_1,\mathbf{w}_1)\mathbf{L}[O_2](x,z_2,\mathbf{w}_2)\\&=\pi i\sum_{n=1}^{\infty}\int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty}\frac{d\delta}{2\pi i}C_{\delta,j=n}(z_1,z_2,\partial_{z_2},\partial_{\mathbf{w}_2})(\mathcal{D}_n\mathbb{O}^{(-1)^n}_{\delta+1,n-1})(x,z_2,\mathbf{w}_2)+\cdots.\end{aligned} \tag{2.268}$$

Therefore, in order to verify that (2.147) reproduces (2.138), we need to check that the normalizations of $C_{\delta,j}$ defined by (2.89) and (2.215) are consistent.

Reasoning analogously to the lower transverse spin case, we find that we need to compute the action of $\mathcal{D}'_n$ defined by (2.62) on the absolute value of (A.7). Since in (2.215) we have $\theta(V_{0,12})$, we need to restrict to the region where $V_{0,12}$ is positive. In this region, we have

$$\begin{aligned}&\mathcal{D}'_n\langle\phi_1(x_1)\phi_2(x_2)O(x_3,z)\rangle_+=\mathcal{D}'_n\frac{(-2z\cdot x_{23}\,x_{13}^2+2z\cdot x_{13}\,x_{23}^2)^J}{x_{12}^{\Delta_1+\Delta_2-\Delta+J}x_{13}^{\Delta_1+\Delta-\Delta_2+J}x_{23}^{\Delta_2+\Delta-\Delta_1+J}}\\&=\frac{J(J-1)\cdots(J-n+1)}{n!}(-2w\cdot x_{23}\,x_{13}^2+2w\cdot x_{13}\,x_{23}^2)^n\frac{(-2z\cdot x_{23}\,x_{13}^2+2z\cdot x_{13}\,x_{23}^2)^{J-n}}{x_{12}^{\Delta_1+\Delta_2-\Delta+J}x_{13}^{\Delta_1+\Delta-\Delta_2+J}x_{23}^{\Delta_2+\Delta-\Delta_1+J}}.\end{aligned} \tag{2.269}$$

Multiplying by $\theta(V_{0,12})$ and taking limit $J\to n-1$, we find

$$\mathcal{D}'_n\left(\langle\phi_1(x_1)\phi_2(x_2)O(x_3,z)\rangle_+\theta(V_{0,12})\right)=\frac{1}{n}\frac{(-2w\cdot x_{23}\,x_{13}^2+2w\cdot x_{13}\,x_{23}^2)^n}{x_{12}^{\Delta_1+\Delta_2-\Delta+n+1}x_{13}^{\Delta_1+\Delta-\Delta_2+n-1}x_{23}^{\Delta_2+\Delta-\Delta_1+n-1}}\delta(-2V_{0,12}). \tag{2.270}$$

Lifting this to embedding space and evaluating at the celestial locus (2.219), we find that the right-hand side of (2.215) is

$$\begin{aligned}&\frac{(-1)^nX_{12}}{\delta(-2V_{0,21})}\mathcal{D}'_n\left(\theta(V_{0,12})\langle\phi_1(X_1)O(X_0,Z_0)\phi_2(X_2)\rangle_+^{(a)}\right)\Big|_{\text{celestial}}\\&=\frac{1}{n}\frac{(-4W\cdot X_2\,X_1\cdot X_3+4W\cdot X_1\,X_2\cdot X_3)^n}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta+n-1}{2}}X_{13}^{\frac{\Delta_1+\Delta-\Delta_2+n-1}{2}}X_{23}^{\frac{\Delta_2+\Delta-\Delta_1+n-1}{2}}}\Bigg|_{\text{celestial}}\\&=\frac{1}{n}\frac{(-4w\cdot z_2\,z_1\cdot z_3+4w\cdot z_1\,z_2\cdot z_3)^n}{(-2z_1\cdot z_2)^{\frac{\delta_1+\delta_2-\delta+n}{2}}(-2z_1\cdot z_3)^{\frac{\delta_1+\delta-\delta_2+n}{2}}(-2z_2\cdot z_3)^{\frac{\delta_2+\delta-\delta_1+n}{2}}}\end{aligned} \tag{2.271}$$

which up to a factor of $1/n$ agrees with the standard three-point structure (2.86) which appears in (2.89).

We now need to compute the left-hand side of (2.215). We have

$$
\begin{aligned}
&\langle(\mathcal{D}'_n\mathcal{O})(x_2,z_2)(\mathcal{D}'_n\mathcal{O})(x_0,z_0)\rangle = \mathcal{D}'_{n,2}\mathcal{D}'_{n,0}\frac{(-2z_2\cdot I(x_{20})\cdot z_0)^J}{x_{20}^{2\Delta}}\\
&= \frac{(J)_{(n)}}{n!}\mathcal{D}'_{n,2}(-2z_2\cdot I(x_{20})\cdot w_0)^n\frac{(-2z_2\cdot I(x_{20})\cdot z_0)^{J-n}}{x_{20}^{2\Delta}}\\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{(J)_{(n)}}{n!}\frac{(n)_{(k)}(J-n)_{(n-k)}}{n!}(-2w_2\cdot I(x_{20})\cdot w_0)^k(-2z_2\cdot I(x_{20})\cdot w_0)^{n-k}\\
&\qquad\times(-2w_2\cdot I(x_{20})\cdot z_0)^{n-k}\frac{(-2z_2\cdot I(x_{20})\cdot z_0)^{J-2n+k}}{x_{20}^{2\Delta}},
\end{aligned}\tag{2.272}
$$

where we have defined $(a)_{(b)}\equiv a(a-1)\cdots(a-b+1)$. We now send $J\to n-1$ to find

$$
\begin{aligned}
&\lim_{J\to n-1}\frac{1}{J-n+1}\langle(\mathcal{D}'_n\mathcal{O})(x_2,z_2)(\mathcal{D}'_n\mathcal{O})(x_0,z_0)\rangle\\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n}(-1)^{n-k}(-2w_2\cdot I(x_{20})\cdot w_0)^k(-2z_2\cdot I(x_{20})\cdot w_0)^{n-k}\\
&\qquad\times(-2w_2\cdot I(x_{20})\cdot z_0)^{n-k}\frac{(-2z_2\cdot I(x_{20})\cdot z_0)^{-1-n+k}}{x_{20}^{2\Delta}},\\
&= \frac{1}{n}\left((-2w_2\cdot I(x_{20})\cdot w_0)(-2z_2\cdot I(x_{20})\cdot z_0)-(-2w_2\cdot I(x_{20})\cdot z_0)(-2z_2\cdot I(x_{20})\cdot w_0)\right)^n\\
&\qquad\frac{(-2z_2\cdot I(x_{20})\cdot z_0)^{-1-n}}{x_{20}^{2\Delta}}\\
&= \frac{1}{n}\left((-2H_{20}^{WW})(-2H_{20})-(-2H_{20}^{WZ})(-2H_{20}^{ZW})\right)^n\frac{(-2H_{20})^{-1-n}}{X_{20}^{\Delta+n-1}}
\end{aligned}\tag{2.273}
$$

where H_{ij}^{AB} is defined by replacing Z_i by A_i and Z_j by B_j in H_{ij}. This can now be evaluated in the configuration (2.219) which yields, after multiplying by $-2H_{02}$, for the structure in the left-hand side of (2.215)

$$
\begin{aligned}
&\left.\left(\lim_{J\to n-1}(-2H_{20})\frac{\langle(\mathcal{D}'_n\mathcal{O})(X_2,Z_2,\mathbf{W}_2)(\mathcal{D}'_n\mathcal{O}^\dagger)(X_0,Z_0,\mathbf{W}_0)\rangle}{J-J_n}\right)\right|_{\text{celestial}}\\
&= \frac{1}{n}\frac{((4z_2\cdot z_0w_2\cdot w_0-4z_2\cdot w_0z_0\cdot w_2))^n}{(-2z_2\cdot z_0)^{\delta+n}}.
\end{aligned}\tag{2.274}
$$

This agrees up to a factor of $1/n$ with the standard two-point structure (2.90) which appears in (2.89). We thus find that both sides of (2.215) differ from (2.89) by a factor of $1/n$, and therefore the two equations are equivalent.

2.6.2 Selection rules in the $\mathbf{L}[\mathcal{J}] \times \mathbf{L}[\mathcal{J}]$ OPE

In this section we consider the light-ray operators that contribute to the light-ray OPE (2.147) of two identical charge detectors, i.e. to the two light-transforms of identical $U(1)$ currents $\mathbf{L}[\mathcal{J}]$. (The analysis for the non-abelian case is similar.[43]) For concreteness we focus on $d = 4$ and we do not assume parity symmetry. However, the results we find will be valid in any dimension $d \geq 4$.

In $d = 4$ the $\mathrm{SO}(d-1,1) = \mathrm{SO}(3,1)$ representations are parametrized by two-row Young diagrams, which can be supplemented with self- or anti-self duality constraints. However, since we are considering a non-chiral setup, it is convenient to use real tensor representations of $\mathrm{SO}(d-1,1)$ which do not have self-duality constraints. We thus parametrize these representations by pairs (J, j). Local operators always have $J \geq j$.

First, we consider the local OPE of a $U(1)$ current $\mathcal{J}$ with itself. Using the counting rules of [48] it is easy to see that for sufficiently large J we have operators in the $\mathcal{J} \times \mathcal{J}$ OPE in irreps $(J,0), (J,1)$ with even and odd J, and in irreps $(J,2)$ with even J. This generic-J behavior determines the light-ray operators that appear in the $\mathcal{J} \times \mathcal{J}$ OPE. To see this explicitly, recall that for even $J + j$ the number of structures is given by the dimension of [48]

$$\left(S^2 \square \otimes \mathrm{Res}^{\mathrm{SO}(3,1)}_{\mathrm{SO}(3)}(J,j)\right)^{\mathrm{SO}(3)}, \tag{2.275}$$

where $\square$ is the $\mathrm{SO}(3)$ vector irrep[44] and for odd $J + j$ we need instead

$$\left(\wedge^2 \square \otimes \mathrm{Res}^{\mathrm{SO}(3,1)}_{\mathrm{SO}(3)}(J,j)\right)^{\mathrm{SO}(3)}. \tag{2.276}$$

We have

$$S^2 \square = \square\square \oplus \bullet, \quad \wedge^2 \square = \square, \tag{2.277}$$

where $\bullet$ is the trivial representation, and the restriction of (J, j) to $\mathrm{SO}(3)$ is

$$\mathrm{Res}^{\mathrm{SO}(3,1)}_{\mathrm{SO}(3)}(J,j) = \sum_{l=j}^{J} (l) \tag{2.278}$$

where (l) is the spin-l irrep of $\mathrm{SO}(3)$. We get tensor structures by matching $\mathrm{SO}(3)$ irreps between (2.277) and (2.278). We immediately see that there are no structures

[43] Although note the discussion of contact terms in [15].

[44] Using an $\mathrm{SO}(3)$ irrep instead of an $\mathrm{SO}(4)$ irrep takes into account the conservation of $\mathcal{J}$.

with $j > 2$, and that for $j = 2$ the spin J must be even. For $j = 0, 1$ J can be of any parity.

The transverse spins that appear in the $\mathcal{J} \times \mathcal{J}$ OPE are thus $0, 1, 2$. All these spins are traceless-symmetric in $d - 2$ dimensions and thus are allowed to appear in the celestial OPE.[45] So, the set Λ_{12} in (2.147) is given by $\Lambda_{12} = \{0, 1, 2\}$. We then have the low transverse-spin contributions, schematically

$$\mathbf{L}[\mathcal{J}] \times \mathbf{L}[\mathcal{J}] = \sum_i \mathbb{O}^+_{i,J=1,j=0} + \mathbb{O}^+_{i,J=1,j=1} + \mathbb{O}^+_{i,J=1,j=2} + \cdots . \tag{2.279}$$

A subtlety here is that the celestial map (2.214) typically maps multiple three-point tensor structures to zero, see appendix A.5 for details. As we discuss there, the only structures that survive are those that contain $V^{-1}_{0,12}$, which are precisely those that are polynomial for $J > J_1 + J_2 - 1$ but stop being polynomial exactly at $J = J_1 + J_2 - 1$. In our case we are interested in even-spin structures which are polynomial for $J = 2$ but are not polynomial for $J = 1$, i.e. those which disappear from the counting above as we change $J = 2$ to $J = 1$.[46] We see that for $j = 2$ the number of structures changes from 1 to 0 and for $j = 0$ from 2 to 1. This happens because the SO(3) content of (J, j) changes: the spin-2 irrep disappears and cannot be paired with the spin-2 irrep in $S^2\square$. However, for $j = 1$ the number of structures does not change because the only structure comes from pairing with the spin-1 irrep in $\wedge^2\square$, and thus all $j = 1$ structures are annihilated by the celestial map.

The final form of the low transverse spin contributions is therefore

$$\mathbf{L}[\mathcal{J}] \times \mathbf{L}[\mathcal{J}] = \sum_i \mathbb{O}^+_{i,J=1,j=0} + \mathbb{O}^+_{i,J=1,j=2} + \cdots . \tag{2.280}$$

This is of course consistent with the fact that on the celestial sphere we have two identical scalars, and thus only even j should be allowed.[47]

For higher transverse spin we find that the label γ in (2.147) is trivial because $SO(d - 4)$ is trivial in our case. We have $\lambda(+n) = 2 + n$ and thus the higher transverse spin terms take the schematic form

$$\mathbf{L}[\mathcal{J}] \times \mathbf{L}[\mathcal{J}] = \sum_{n,i} \mathcal{D}_n \mathbb{O}^{(-1)^n}_{i,J=1+n,j=2} + \cdots . \tag{2.281}$$

[45] Note in $d > 5$ three-row Young diagrams would appear in the $\mathcal{J}\times\mathcal{J}$ OPE, but the corresponding transverse spins are not allowed to appear in the celestial OPE of two scalars.

[46] Here we need to detach the notion of signature (even-spin or odd-spin) from parity of J since we are analytically continuing in J. That is, for $J = 1$ we still use $S^2\square$ for $j = 0, 2$ and $\wedge^2\square$ for $j = 1$.

[47] Here we assume, as usual, that the product $\mathbf{L}[\mathcal{J}] \times \mathbf{L}[\mathcal{J}]$ is well-defined and thus the two light-transforms commute. This requires the Regge intercept J_0 to satisfy $J_0 < J_1 + J_2 - 1 = 1$ [34].

However, since $\mathbb{O}^-_{i,J,j=2}$ do not appear in the $\mathcal{J} \times \mathcal{J}$ OPE, we find

$$\mathbf{L}[\mathcal{J}] \times \mathbf{L}[\mathcal{J}] = \sum_{n,i} \mathcal{D}_{2n} \mathbb{O}^+_{i,J=1+2n,j=2} + \cdots . \tag{2.282}$$

Note that these contributions have even transverse spin $j = 2 + 2n$ and thus this expansion is again consistent with permutation symmetry on the celestial sphere.

Summarizing, we have the following schematic contributions to the OPE of two charge detectors in $d = 4$,

$$\mathbf{L}[\mathcal{J}] \times \mathbf{L}[\mathcal{J}] = \sum_{i} \left(\mathbb{O}^+_{i,J=1,j=0} + \mathbb{O}^+_{i,J=1,j=2} \right) + \sum_{n,i} \mathcal{D}_{2n} \mathbb{O}^+_{i,J=1+2n,j=2}. \tag{2.283}$$

2.6.3 Selection rules in the $\mathbf{L}[T] \times \mathbf{L}[T]$ OPE

We now discuss the case of the OPE of two energy detectors, i.e. two light-transforms of T. We use the same setup as in the previous section, i.e. we work in $d = 4$ and in terms of real tensor irreps of $\mathrm{SO}(3,1)$.

The transverse spins appearing in the $T \times T$ OPE are analogous to the $\mathcal{J} \times \mathcal{J}$ case. We have spins $j = 0, 1, 2, 3$ for both even and odd J and spin $j = 4$ for even J. This translates to the following analogue of (2.279)

$$\mathbf{L}[T] \times \mathbf{L}[T] = \sum_{i} \mathbb{O}^+_{i,J=3,j=0} + \mathbb{O}^+_{i,J=3,j=1} + \mathbb{O}^+_{i,J=3,j=2} + \mathbb{O}^+_{i,J=3,j=3} + \mathbb{O}^+_{i,J=3,j=4} + \cdots . \tag{2.284}$$

However, we again must take care of the fact that the celestial map (2.214) annihilates some tensor structures. In this case, using the same logic as before, we find that only $j = 0, 2, 4$ operators have tensor structures that are non-vanishing under the celestial map. Therefore, the low-transverse spin contribution is actually

$$\mathbf{L}[T] \times \mathbf{L}[T] = \sum_{i} \mathbb{O}^+_{i,J=3,j=0} + \mathbb{O}^+_{i,J=3,j=2} + \mathbb{O}^+_{i,J=3,j=4} + \cdots . \tag{2.285}$$

Again, this is consistent with the permutation symmetry on the celestial sphere that only allows even j.

The analysis of higher transverse spin contributions is also the same as in the $\mathcal{J} \times \mathcal{J}$ case. We have from (2.147)

$$\mathbf{L}[T] \times \mathbf{L}[T] = \sum_{n,i} \mathcal{D}_{n} \mathbb{O}^{(-1)^n}_{i,J=3+n,j=4} + \cdots . \tag{2.286}$$

Taking into account that only even-spin $j = 4$ operators appear in $T \times T$ and combining with the low transverse spin terms, we find

$$\mathbf{L}[T] \times \mathbf{L}[T] = \sum_i \left(\mathbb{O}^+_{i,J=3,j=0} + \mathbb{O}^+_{i,J=3,j=2} + \mathbb{O}^+_{i,J=3,j=4}\right) + \sum_{n,i} \mathcal{D}_{2n}\mathbb{O}^+_{i,J=3+2n,j=4}. \tag{2.287}$$

As mentioned above, although we have derived (2.287) in $d = 4$, the result is valid for any $d \geq 4$.

It is interesting to ask what are the leading operators appearing at various transverse spins in the above expansion in a weakly-coupled gauge theory. At $j = 0$ it is well-known that the leading twist is $\tau_0 = 2$ operators which can take the schematic form,

$$\overline{\phi}\partial_{\beta_1\dot{\alpha}_1}\cdots\partial_{\beta_J\dot{\alpha}_J}\phi, \quad \overline{\psi}_{\dot{\alpha}_1}\partial_{\beta_2\dot{\alpha}_2}\cdots\partial_{\beta_J\dot{\alpha}_J}\psi_{\beta_1}, \quad \overline{F}_{\dot{\alpha}_1\dot{\alpha}_2}\partial_{\beta_3\dot{\alpha}_3}\cdots\partial_{\beta_J\dot{\alpha}_J}F_{\beta_1\beta_2}, \tag{2.288}$$

where we assume that the gauge indices are implicitly contracted, and the dotted and undotted indices are implicitly symmetrized. At $j = 2$ the leading twist[48] is also $\tau_2 = 2$, for the operators

$$F_{\beta_1\beta_2}\partial_{\beta_5\dot{\alpha}_5}\cdots\partial_{\beta_{J+2}\dot{\alpha}_{J+2}}F_{\beta_3\beta_4}. \tag{2.289}$$

To see that this is the minimal possible twist, note that $\tau \geq 2$ is the unitarity bound in $d = 4$ for generic-J operators. To see that there are no other operators, note that the twist of a product of symbols is bounded from below by the sum of constituent twists, and the classical twist of all fundamental fields is 1, while the twist of a derivative is 0. Therefore only products of two fundamental fields and any number of derivatives can have twist ≈ 2, provided no indices are contracted. Fixing the value of transverse spin then leaves us with the above options.

From this argument it is clear that no $j = 4$ operators with twist $\tau = 2$ exist. To build the lowest-twist $j = 4$ operators we need to use more than 2 fundamental fields but as few as possible. Since F carries the most transverse spin among all fundamental fields, we find that the lowest-twist operators $j = 4$ have twist $\tau_4 = 4$ and are schematically given by,

$$F\partial^{J_1}F\partial^{J_2}F\partial^{J_3}F, \tag{2.290}$$

where we keep spinor indices uncontracted and symmetrized, with $J_1+J_2+J_3+4 = J$. Note that there are multiple ways in which the gauge indices can be contracted. The

[48]For all transverse spins we define twist as $\Delta - J$.

fact that $j = 4$ operators have higher twist than required by the unitarity bound is in general a consequence of the improved unitarity bounds of [49], which state

$$\tau \geq \max\{2, j\}. \tag{2.291}$$

In terms of the celestial quantum numbers, we get the following contributions. At $j = 0$ and $j = 2$ we have the leading contributions with dimension

$$\delta = \Delta - 1 \approx \tau_j + J - 1 = \tau_j + 2 = 4. \tag{2.292}$$

Note that this dimension corresponds to the singularity of the form $\theta^{\delta-\delta_1-\delta_2} = \theta^{-2}$. For $j = 4 + 2n \geq 4$ we get

$$\delta = \Delta - 1 \approx \tau_4 + J - 1 = \tau_4 + 2n + 2 = 6 + 2n, \tag{2.293}$$

which corresponds to the leading short-angle dependence of the form θ^{2n}. Since the leading classical twists τ_j are all at the (improved) unitarity bounds, the anomalous dimensions should be positive. Therefore, the leading short-angle asymptotics from these contributions in the interacting theory should be softer than the ones given above.

2.7 Example: event shape in $\mathcal{N} = 4$ SYM

In this section we consider an example of an event shape that includes the transverse spin structures discussed above. More precisely, we consider the following event shape in $\mathcal{N} = 4$ SYM:

$$\langle \mathcal{O}_{20'}(p)|\mathbf{L}[\mathcal{O}_{20'}](\infty, z_1)\mathbf{L}[\mathcal{O}_{20'}](\infty, z_2)|J(p, z_3)\rangle, \tag{2.294}$$

where J is the R-symmetry current, which is in the same multiplet as the half-BPS operator $\mathcal{O}_{20'}$. We will first compute (2.294) directly by performing the light transform of the relevant four-point function. Next, we compute it using the light-ray OPE formula. In both cases, we will derive a Ward identity that relates the event shape (2.294) to the energy-energy correlator calculated in [15]. The result is given by (2.323) and (2.357). Despite the simplicity of our result, the fact that the Ward identity can be derived in two independent ways still provides a nontrivial check of our formulas.

2.7.1 Direct computation

The computation takes a few steps, summarized as follows. We start with the expression for the correlator $\langle \mathcal{O}_{20'}\mathcal{O}_{20'}\mathcal{O}_{20'}J\rangle$ in terms of the scalar correlator

$\langle O_{20'} O_{20'} O_{20'} O_{20'} \rangle$. The two are related via the superconformal Ward identities [50]. Then, we go to the Mellin space representation of the correlator, and perform the light transforms there. Finally, we Fourier transform the separation of the in and out states to obtain the desired event shape.

The relevant four-point function is given by the following expression, see (3.15) in [50],

$$\begin{aligned}
&\langle O_{20'}(x_4) O_{20'}(x_1) O_{20'}(x_2) J_{\alpha\dot{\alpha},aa'}(x_3) \rangle \\
&= \frac{1}{4} (\partial_{x_3})^{\beta}{}_{\dot{\alpha}} (y_{12}^2 y_{14}^2 Y_{324} - v y_{12}^2 y_{24}^2 Y_{314} - u y_{24}^2 y_{14}^2 Y_{321})_{aa'} \\
&\quad \times \langle X_{324}, X_{314} \rangle_{(\alpha\beta)} \frac{\Phi(u,v)}{x_{12}^2 x_{24}^2 x_{14}^2},
\end{aligned} \tag{2.295}$$

where our spinor conventions can be found in appendix A.6. The function $\Phi(u,v)$ is the part of the four-point function of $O_{20'}$s that contains the nontrivial dynamical data, see e.g. section 7.2 in [15]. It satisfies $\Phi(u,v) = \Phi(v,u) = \frac{1}{v}\Phi(\frac{u}{v},\frac{1}{v})$. The rest of the ingredients are various kinematical factors that require some unpacking. The $\alpha, \dot{\alpha} = 1, 2$ are spinor indices for the Lorentz group $SU(2)_L \times SU(2)_R$. The y_i are auxiliary variables keeping track of the $SU(4)$ R-symmetry, see [50] for details. The structures X_{ijk} and Y_{ijk} are defined as follows:

$$(X_{ijk})_{\alpha\dot{\alpha}} = \frac{(x_{ij})_{\alpha\dot{\beta}}}{x_{ij}^2} (x_{jk})^{\dot{\beta}\gamma} \frac{(x_{ki})_{\gamma\dot{\alpha}}}{x_{ik}^2}, \quad (Y_{ijk})_{aa'} = (y_{ij})_{ab'} (y_{jk})^{b'b} (y_{ki})_{ba'}. \tag{2.296}$$

The commutator bracket $\langle \, , \, \rangle$ for spinor indices[49] is defined as

$$\langle a, b \rangle_{(\alpha\beta)} \equiv a_{\alpha\dot{\alpha}} b^{\dot{\alpha}\gamma} \epsilon_{\gamma\beta} - b_{\alpha\dot{\alpha}} a^{\dot{\alpha}\gamma} \epsilon_{\gamma\beta} \,. \tag{2.297}$$

Note that $\langle a, b \rangle_{(\alpha\beta)}$ is automatically symmetric under permutation of α and β.

To connect with the event shape, we set $x_4 = 0$, and x_3 will eventually be Fourier transformed with momentum p. For now, we can replace the derivative $(\partial_{x_3})^{\beta}{}_{\dot{\alpha}}$ with $\frac{i}{2} p^{\beta}{}_{\dot{\alpha}}$. We pass to index-free notation by introducing the polarization vector $\frac{1}{2} z_3^{\dot{\alpha}\alpha}$, such that $z_3^2 = 0$, and contracting $\frac{1}{2} z_3^{\dot{\alpha}\alpha} J_{\alpha\dot{\alpha}} = z_3^{\mu} J_{\mu}$. Contracting with the polarization vector produces a term

$$p^{\beta}{}_{\dot{\alpha}} z_3^{\dot{\alpha}\alpha} = \frac{1}{2} \langle z_3, p \rangle^{(\beta\alpha)} + (z_3 \cdot p) \epsilon^{\beta\alpha} \,. \tag{2.298}$$

[49] We use angular brackets to denote the commutator in spinor indices to avoid a clash with the commutator in vector indices denoted by the traditional square brackets.

In terms of the polarization vector, the correlator becomes

$$\langle O_{20'}(x_4)O_{20'}(x_1)O_{20'}(x_2)J_{aa'}(x_3,z_3)\rangle = \frac{i}{32}(y_{12}^2 y_{14}^2 Y_{324} - v y_{12}^2 y_{24}^2 Y_{314} - u y_{24}^2 y_{14}^2 Y_{321})_{aa'} \times \langle z_3, p\rangle^{(\alpha\beta)} \langle X_{324}, X_{314}\rangle_{(\alpha\beta)} \frac{\Phi(u,v)}{x_{12}^2 x_{24}^2 x_{14}^2}. \tag{2.299}$$

In this formula we only Fourier transformed the external derivative while keeping the rest in coordinate space. The contraction of the brackets can be performed by the identity

$$\langle a,b\rangle^{(\alpha\beta)}\langle c,d\rangle_{(\alpha\beta)} = 8(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma} + i\epsilon^{\mu\nu\rho\sigma})a_\mu b_\nu c_\rho d_\sigma\,. \tag{2.300}$$

To compute the event shape, we place the detectors at embedding space coordinates

$$X_i = X_\infty = (0,1,0), \quad Z_i = (0,0,z_i), \quad \text{for } i = 1,2, \tag{2.301}$$

and take the light-transforms

$$\mathbf{L}[O_{20'}](\infty, z_i) = \int_{-\infty}^{\infty} d\alpha_i\, O_{20'}(Z_i - \alpha_i X_\infty), \tag{2.302}$$

while the external states are placed at $X_3 = (1, x_3^2, x_3)$ and $X_4 = (1,0,0)$.

We find it convenient to work in the x Poincaré patch and approach the spatial infinity insertion as follows

$$\mathbf{L}[O_{20'}](\infty, z_i) = \int_{-\infty}^{\infty} d\alpha_i \lim_{r_i\to\infty} r_i^2 O_{20'}(r_i z_i + \alpha_i \bar{z}_i), \tag{2.303}$$

where $\bar{z}_i^2 = 0$, $(-z_i\cdot\bar{z}_i) = \frac{1}{2}$, and $\bar{z}_i$ is arbitrary otherwise. In the embedding space, (2.303) corresponds to choosing

$$Z_i = \lim_{r\to\infty} Z_i^r = \left(\frac{1}{r_i}, 0, z_i\right), \quad X_\infty = \lim_{r\to\infty} X_\infty^{r_i} = \left(0, 1, \frac{\bar{z}_i}{r_i}\right). \tag{2.304}$$

Note that $(Z_i^r)^2 = (X_\infty^{r_i})^2 = (Z_i^r \cdot X_\infty^{r_i}) = 0$. It is clear from the definition (2.302) that the final result does not depend on the particular choice of $\bar{z}_i$. For convenience, we also define the null coordinates

$$x_{i-} \equiv (-x_i\cdot z_i) = \alpha_i/2\,, \quad x_{3i-} \equiv (-x_3\cdot z_i) - x_{i-}\,, \tag{2.305}$$

for $i = 1, 2$.

Next we evaluate the integrand in the kinematics above. The cross ratios u and v take the form

$$u = \frac{x_3^2 z_1 \cdot z_2}{2x_{2-}x_{31-}} , \qquad v = \frac{x_{1-}x_{32-}}{x_{2-}x_{31-}} , \tag{2.306}$$

and the commutator becomes

$$\langle X_{324}, X_{314}\rangle_{(\alpha\beta)} = \frac{1}{2}\frac{1}{x_3^2 x_{32-}x_{31-}}\left(-x_{32-}\langle z_1, x_3\rangle_{(\alpha\beta)} + x_{31-}\langle z_2, x_3\rangle_{(\alpha\beta)} + \frac{1}{2}x_3^2\langle z_2, z_1\rangle_{(\alpha\beta)}\right). \tag{2.307}$$

With these expressions, the light transforms are evaluated by the integral

$$\begin{aligned}&\langle O_{20'}(0)|\mathbf{L}[O_{20'}](\infty, z_1)\mathbf{L}[O_{20'}](\infty, z_2)|J_{aa'}(x_3, z_3)\rangle \\ &\quad = \int_{-\infty}^{\infty} dx_{1-}dx_{2-}\frac{i}{32}(y_{12}^2 y_{14}^2 Y_{324} - v y_{12}^2 y_{24}^2 Y_{314} - u y_{24}^2 y_{14}^2 Y_{321})_{aa'} \\ &\qquad \times \langle z_3, p\rangle^{(\alpha\beta)}\langle X_{324}, X_{314}\rangle_{(\alpha\beta)}\frac{\Phi(u,v)}{(-2z_1\cdot z_2)x_{1-}x_{2-}}.\end{aligned} \tag{2.308}$$

To perform the light transform integrals above, it is very convenient to use the Mellin representation for $\Phi(u, v)$, see e.g. [51],

$$\begin{aligned}\Phi(u,v) &= \frac{v}{u}\int_{C_0}\frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2}\Gamma(\gamma_{12})^2\Gamma(\gamma_{14})^2\Gamma(2-\gamma_{12}-\gamma_{14})^2 M(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}} , \\ C_0 &: \quad \mathrm{R}e[\gamma_{12}] > -1, \quad \mathrm{R}e[\gamma_{14}], \mathrm{R}e[\gamma_{13}] > 1,\end{aligned} \tag{2.309}$$

where $\gamma_{12}+\gamma_{13}+\gamma_{14} = 2$. The condition $\mathrm{R}e[\gamma_{13}] > 1$ thus becomes $\mathrm{R}e[\gamma_{12}+\gamma_{14}] < 1$. The weak and strong coupling results take the form

$$\begin{aligned}M^{\text{weak}}(\gamma_{12},\gamma_{14}) &= -\frac{a}{4}\frac{\gamma_{12}^2}{(\gamma_{14}-1)^2(\gamma_{13}-1)^2}, \\ M^{\text{strong}}(\gamma_{12},\gamma_{14}) &= -\frac{1}{2}\frac{\gamma_{12}^2(1+\gamma_{12})}{(\gamma_{14}-1)(\gamma_{13}-1)} .\end{aligned} \tag{2.310}$$

We can plug the Mellin representation (2.309) above into (2.308), and perform the light-transform integrals using the formula

$$\int_{-\infty}^{\infty} dx_-\, (x_- + i\epsilon)^{-a}((-z\cdot x) - x_- + i\epsilon)^{-b} = -2\pi i(-z\cdot x)^{1-a-b}\frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)} . \tag{2.311}$$

The integral converges for $\text{Re}[a+b] > 1$.

At this point, we observe that the terms in the correlator proportional to y_{12}^2 produce a divergent result. This means that the event shapes for the corresponding R-symmetry structures are not well-defined. To obtain a well-defined event shape, we set $y_{12}^2 = 0$, so that only the $-uy_{24}^2 y_{14}^2 Y_{321}$ term survives. This term produces a finite result for the integral, which converges for $\text{Re}[\gamma_{12}] < 0$. Note that the condition $y_{12}^2 = 0$ keeps representations **84**, **105**, **175** in the OPE of scalars $\mathcal{O}_{20'}$ [51], whereas in the $J \times \mathcal{O}_{20'}$ OPE we have the representations

$$\mathbf{15} \times \mathbf{20'} = \mathbf{15} + \mathbf{20'} + \mathbf{45} + \overline{\mathbf{45}} + \mathbf{175}\,. \tag{2.312}$$

Therefore, the only representation that appears in the event shape determined by $y_{12}^2 = 0$ is **175**. From now on, we will focus on this finite event shape in the **175** R-symmetry channel:

$$\langle \mathcal{O}_{20'} | \mathbf{L}[\mathcal{O}_{20'}] \mathbf{L}[\mathcal{O}_{20'}] | J_{aa'} \rangle |_{y_{12}^2=0} = (y_{24}^2 y_{14}^2 Y_{321})_{aa'} \langle \mathcal{O}_{20'} | \mathbf{L}[\mathcal{O}_{20'}] \mathbf{L}[\mathcal{O}_{20'}] | J \rangle_{\mathbf{175}}. \tag{2.313}$$

Performing the light-transforms for the event shape in the **175** channel, we arrive at the expression

$$\begin{aligned}&\langle \mathcal{O}_{20'}(p) | \mathbf{L}[\mathcal{O}_{20'}](\infty, z_1) \mathbf{L}[\mathcal{O}_{20'}](\infty, z_2) | J(p, z_3) \rangle_{\mathbf{175}} \\ &\quad = -\frac{i}{32} \frac{1}{(z_1 \cdot z_2)^2} \int_{C_0} \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2} \frac{M(\gamma_{12}, \gamma_{14})}{\gamma_{12}} \frac{2\pi^4}{(\sin \pi\gamma_{12})^2} \mathcal{D}_p \int d^4x \frac{e^{ipx}}{x^4} \gamma^{\gamma_{12}-1},\end{aligned} \tag{2.314}$$

where $\gamma = 2\frac{(-x\cdot z_1)(-x\cdot z_2)}{x^2 (z_1 \cdot z_2)}$. The differential operator $\mathcal{D}_p$ is given by

$$\mathcal{D}_p \equiv \langle z_3, p \rangle^{(\alpha\beta)} \left((\gamma_{14} - 1) \langle z_1, \partial_p \rangle_{(\alpha\beta)} z_2 \cdot \partial_p - (\gamma_{13} - 1) \langle z_2, \partial_p \rangle_{(\alpha\beta)} z_1 \cdot \partial_p - \frac{1}{2} \gamma_{12} \langle z_2, z_1 \rangle_{(\alpha\beta)} \partial_p^2 \right). \tag{2.315}$$

Finally, the Fourier transform can be performed using the following master formula:

$$\int d^d x \frac{e^{ip\cdot x}}{(x^2 - i\epsilon x^0)^a} \gamma^b = \theta(p) \times 2^{1-2a+d} \pi^{1+\frac{d}{2}} (-p^2)^{a-\frac{d}{2}} \frac{\zeta^{-b}\, {}_2F_1(-b, -b, 1+a-b-\frac{d}{2}, \zeta)}{\Gamma(a+b)\Gamma(1+a-b-\frac{d}{2})}\,, \tag{2.316}$$

where we introduced $\theta(p) \equiv \theta(p^0)\theta(-p^2)$ and the cross ratio

$$\zeta = \frac{(-2z_1 \cdot z_2)(-p^2)}{(-2p \cdot z_1)(-2p \cdot z_2)}. \tag{2.317}$$

Setting $a = 2$, $b = \gamma_{12} - 1$ and $d = 4$ we obtain

$$\int d^4x \frac{e^{ipx}}{x^4} \gamma^{\gamma_{12}-1} = \frac{2\pi^2 \sin \pi\gamma_{12}}{\gamma_{12}} \int_0^{\zeta} dw w^{-\gamma_{12}} (1-w)^{\gamma_{12}-1}. \tag{2.318}$$

Acting with $\mathcal{D}_p$ on the result of the Fourier transform (2.318) and using crossing symmetry of the Mellin amplitude $M(\gamma_{12}, \gamma_{14}) = M(\gamma_{12}, \gamma_{13})$, we get

$$\begin{aligned} &\langle O_{20'}(p)|\mathbf{L}[O_{20'}](\infty, z_1)\mathbf{L}[O_{20'}](\infty, z_2)|J(p, z_3)\rangle_{\mathbf{175}} \\ &= i\frac{[z_3, p] \cdot [z_1, z_2]\theta(p)}{(-2z_1 \cdot z_2)^2} \int_{C_0} \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2} M(\gamma_{12}, \gamma_{14}) \frac{\pi^6}{\sin \pi\gamma_{12}} \frac{4}{p^2} \left(\frac{\zeta}{1-\zeta}\right)^{1-\gamma_{12}}, \end{aligned} \tag{2.319}$$

where the commutator $[a, b]^{\mu\nu}$ is defined in the same way as (2.30), and contracting a pair gives

$$[a, b] \cdot [c, d] = 2\left[(a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)\right]. \tag{2.320}$$

We can rewrite the result above as

$$\langle O_{20'}(p)|\mathbf{L}[O_{20'}](\infty, z_1)\mathbf{L}[O_{20'}](\infty, z_2)|J(p, z_3)\rangle_{\mathbf{175}} = 16i\pi^5 \frac{[z_3, p] \cdot [z_1, z_2](-p^2)\theta(p)}{(-2p \cdot z_1)^2(-2p \cdot z_2)^2} \mathcal{F}_{JO}(\zeta), \tag{2.321}$$

where

$$\mathcal{F}_{JO}(\zeta) = -\frac{1}{4\zeta^2} \int_{C_0} \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2} M(\gamma_{12}, \gamma_{14}) \frac{\pi}{\sin \pi\gamma_{12}} \left(\frac{\zeta}{1-\zeta}\right)^{1-\gamma_{12}}. \tag{2.322}$$

The result (2.322), see e.g. [24], immediately implies that

$$\mathcal{F}_{JO}(\zeta) = -\frac{\zeta}{2} \mathcal{F}_{\mathcal{E}}(\zeta), \tag{2.323}$$

where $\mathcal{F}_{\mathcal{E}}(\zeta)$ was defined in [15] and computes the energy-energy correlator.

Using (2.323) and the formulas [15], or directly computing the Mellin integral in (2.322) with the Mellin amplitudes (2.310), we get the results at weak and at strong coupling

$$\mathcal{F}_{JO}^{\text{weak}}(\zeta) = \frac{a}{8} \frac{\log(1-\zeta)}{\zeta(1-\zeta)}, \qquad \mathcal{F}_{JO}^{\text{strong}}(\zeta) = -\frac{\zeta}{4}. \tag{2.324}$$

2.7.2 Computation using the light-ray OPE

Now we compute the event shape (2.294) using the light-ray OPE formula. The complete light-ray OPE formula for two scalars including higher transverse spin terms is given by (2.138). Plugging the formula into (2.294), we have

$$\begin{aligned}
&\langle \mathcal{O}_{20'}(p)|\mathbf{L}[\mathcal{O}_{20'}](\infty,z_1)\mathbf{L}[\mathcal{O}_{20'}](\infty,z_2)|J(p,z_3)\rangle \\
&= \pi i \int_{2-i\infty}^{2+i\infty} \frac{d\Delta}{2\pi i}\Big(C_{\Delta-1,0}(z_1,z_2,\partial_z)\langle \mathcal{O}_{20'}(p)|\mathbb{O}^+_{\Delta,J=-1}(\infty,z)|J(p,z_3)\rangle \\
&\qquad\qquad + C_{\Delta-1,1}(z_1,z_2,\partial_z,\partial_w)\langle \mathcal{O}_{20'}(p)|(\mathcal{D}_1\mathbb{O}^-_{\Delta,J=0})(\infty,z,w)|J(p,z_3)\rangle\Big) \\
&= -\pi i \int_{2-i\infty}^{2+i\infty} \frac{d\Delta}{2\pi i}\Big(C^+_a(\Delta,-1) C_{\Delta-1,0}(z_1,z_2,\partial_z)\langle \mathcal{O}_{20'}(p)|\mathbf{L}[\mathcal{O}_{\Delta,-1}](\infty,z)|J(p,z_3)\rangle^{(a)}_+ \\
&\qquad\qquad + C^-_a(\Delta,0) C_{\Delta-1,1}(z_1,z_2,\partial_z,\partial_w)\langle \mathcal{O}_{20'}(p)|\mathcal{D}_1\mathbf{L}[\mathcal{O}_{\Delta,0}](\infty,z,w)|J(p,z_3)\rangle^{(a)}_-\Big),
\end{aligned} \tag{2.325}$$

where $C^\pm_a(\Delta,J)$ is a coefficient function that encodes the OPE data. It has poles of the form

$$C^\pm_a(\Delta,J) \sim -\frac{p^{(a)}_{i,J}}{\Delta-\Delta_{i,J}} \tag{2.326}$$

where $p^{(a)}_{i,J}$ and $\Delta_{i,J}$ are the product of OPE coefficients and the scaling dimension of an exchanged operator. Note that there's just one structure label in $C^\pm_a(\Delta,J)$ because the three-point function from the two $\mathcal{O}_{20'}$'s only has one tensor structure. In the second equality in (2.325), we use the relation between light-ray operators $\mathbb{O}^\pm_{\Delta,J}$ and $C^\pm_a(\Delta,J)$ [11]

$$\langle \mathcal{O}_{20'}\mathbb{O}^\pm_{\Delta,J}J\rangle_\Omega = -C^\pm_a(\Delta,J)\langle 0|\mathcal{O}_{20'}\mathbf{L}[\mathcal{O}_{\Delta,J}]J|0\rangle^{(a)}_\pm, \tag{2.327}$$

where $\langle 0|\mathcal{O}_{20'}\mathbf{L}[\mathcal{O}_{\Delta,J}]J|0\rangle^{(a)}_\pm$ is the analytic continuation of the continuous-spin Wightman function $\langle 0|\mathcal{O}_{20'}\mathbf{L}[\mathcal{O}_{\Delta,J}]J|0\rangle^{(a)}$ from either even or odd spin. It can be defined straightforwardly following section 2.5.1. We will give their explicit expressions later in section 2.7.2.

Furthermore, as discussed in section 2.7.1, we focus on the **175** R-symmetry channel event shape defined in (2.313). Since the **175** representation is antisymmetric under the exchange of $\mathcal{O}_{20'}$'s, the OPE should only contain operators with odd spin, and we have $C^+(\Delta,J)=0$ for all J. The **175** R-symmetry channel event shape is then given by

$$\langle \mathcal{O}_{20'}(p)|\mathbf{L}[\mathcal{O}_{20'}](\infty,z_1)\mathbf{L}[\mathcal{O}_{20'}](\infty,z_2)|J(p,z_3)\rangle_{\mathbf{175}}$$
$$= -\pi i \int_{2-i\infty}^{2+i\infty} \frac{d\Delta}{2\pi i} C_a'^{-}(\Delta,0) C_{\Delta-1,1}(z_1,z_2,\partial_z,\partial_w) \langle \mathcal{O}_{20'}(p)|\mathcal{D}_1 \mathbf{L}[\mathcal{O}_{\Delta,0}](\infty,z,w)|J(p,z_3)\rangle_{-}^{(a)}, \tag{2.328}$$

where $C_a'^{-}(\Delta,J)$ is simply given by the coefficient function $C_a^{-}(\Delta,J)$ without the R-symmetry factor $(y_{24}^2 y_{14}^2 Y_{321})$.

Celestial blocks

In order to compute (2.328), we first note that by Lorentz invariance and homogeneity,

$$\langle \mathcal{O}_{20'}(p)|\mathcal{D}_1 \mathbf{L}[\mathcal{O}_{\Delta,0}](\infty,z,w)|J(p,z_3)\rangle_{-}^{(a)} \propto (2w\cdot z_3 z\cdot p - 2w\cdot p z\cdot z_3)(-2p\cdot z)^{-\delta-1}. \tag{2.329}$$

Therefore, we need to solve for the $j=1$ celestial block defined by

$$C_{\delta,1}(z_1,z_2,\partial_z,\partial_w)\left((2w\cdot z_3 z\cdot p - 2w\cdot p z\cdot z_3)(-2p\cdot z)^{-\delta-1}\right). \tag{2.330}$$

This is the higher transverse spin version of the celestial block computed in [15]. Lorentz invariance and homogeneity imply that

$$C_{\delta,1}(z_1,z_2,\partial_z,\partial_w)\left([w,z]\cdot[z_3,p](-2p\cdot z)^{-\delta-1}\right)$$
$$= \frac{(-p^2)^{\frac{\delta_1+\delta_2-\delta+1}{2}}}{(-2p\cdot z_1)^{\delta_1+1}(-2p\cdot z_2)^{\delta_2+1}}$$
$$\times\left([z_3,p]\cdot[z_1,z_2]g(\zeta) + \{z_3,p\}\cdot\{z_1,z_2\}h(\zeta) - \frac{4p\cdot z_1 p\cdot z_2 p\cdot z_3}{p^2}h(\zeta)\right), \tag{2.331}$$

where once again $[a,b]^{\mu\nu}$ is defined in the same way as (2.30), and similarly

$$\{a,b\}^{\mu\nu} = a^\mu b^\nu + b^\mu a^\nu, \tag{2.332}$$

and the cross ratio ζ is given by (2.317). One way to obtain the functions $g(\zeta)$ and $h(\zeta)$ is using the fact that (2.331) is an eigenvector of the quadratic Casimir of the Lorentz group acting simultaneously on z_1, z_2 with eigenvalue $\delta(\delta-d+2)+d-3$. Proceeding this way, one gets two coupled second-order inhomogeneous differential equations of $g(\zeta)$ and $h(\zeta)$, and their boundary conditions are given by the OPE limit of (2.331). However, finding the solutions to these differential equations is a nontrivial task. Furthermore, the system of differential equations gets more and more complicated when one has even higher transverse spin. We would like a method that allows us to compute celestial blocks with general transverse spin. Fortunately, this can be achieved by using weight-shifting operators [39] and the $j = 0$ celestial block calculated in [15].

First, note that $[w, z] \cdot [z_3, p](-2p \cdot z)^{-\delta-1}$ can be written in terms of a "bubble diagram"

$$[w, z] \cdot [z_3, p](-2p \cdot z)^{-\delta-1} = \frac{\mathcal{D}^{0+}_{z,w} \cdot \mathcal{D}^{0-}_{z,w}}{(d-4)(\delta-1)(\delta-d+3)} \left([w, z] \cdot [z_3, p](-2p \cdot z)^{-\delta-1}\right), \tag{2.333}$$

where $\mathcal{D}^{0+\mu}_{z,w}$ and $\mathcal{D}^{0-\mu}_{z,w}$ are weight-shifting operators defined in [39], with the embedding space coordinates (X, Z) replaced with (z, w). The explicit expression of $\mathcal{D}^{0+\mu}_{z,w}$ is also given in (2.66). The operator $\mathcal{D}^{0+\mu}_{z,w}$ increases the transverse spin and $\mathcal{D}^{0-\mu}_{z,w}$ decreases the transverse spin, so acting $\mathcal{D}^{0+}_{z,w} \cdot \mathcal{D}^{0-}_{z,w}$ will basically give us the same expression with some overall factor. One can perform crossing on the weight-shifting operator $\mathcal{D}^{0-\mu}_{z,w}$ such that

$$\mathcal{D}^{0-\mu}_{z,w} \left([w, z] \cdot [z_3, p](-2p \cdot z)^{-\delta-1}\right) = \mathcal{D}^{\mu}_{p}(-2p \cdot z)^{-\delta}, \tag{2.334}$$

where $\mathcal{D}^{\mu}_{p}$ is a differential operator acting on p. By explicitly evaluating the left-hand side of (2.334), we find that $\mathcal{D}^{\mu}_{p}$ is given by

$$\mathcal{D}^{\mu}_{p} = -\frac{(d-4)}{\delta}\left((\delta-d+3)p^{\mu} z_3 \cdot \frac{\partial}{\partial p} + p^2 z_3 \cdot \frac{\partial}{\partial p}\frac{\partial}{\partial p_{\mu}} + (\delta+1) z_3 \cdot p \frac{\partial}{\partial p_{\mu}} + \delta(\delta-d+3) z_3^{\mu}\right). \tag{2.335}$$

Alternatively, $(-2p \cdot z)^{-\delta}$ and $[w, z] \cdot [z_3, p](-2p \cdot z)^{-\delta-1}$ can be viewed as spin 0 and spin 1 bulk-to-boundary propagators in AdS$_{d-1}$/CFT$_{d-2}$, and $\mathcal{D}^{\mu}_{p}$ is simply an AdS weight-shifting operator [52]. In particular, it is the AdS weight-shifting operator that increases spin by 1 multiplied by a bulk-to-boundary $6j$ symbol.

On the other hand, one can also perform crossing between $\mathcal{D}^{0+\mu}_{z,w}$ and the OPE differential operator $C_{\delta,1}$:

$$C^{(\delta_1,0,\delta_2,0)}_{\delta,1}\mathcal{D}^{0+\mu}_{z,w} = c_1\mathcal{D}^{0-\mu}_{z_1,w_1}C^{(\delta_1,1,\delta_2,0)}_{\delta,0} + c_2\mathcal{D}^{+0\mu}_{z_1,w_1}C^{(\delta_1-1,0,\delta_2,0)}_{\delta,0} + c_3\mathcal{D}^{-0\mu}_{z_1,w_1}C^{(\delta_1+1,0,\delta_2,0)}_{\delta,0}, \tag{2.336}$$

where $C^{(\delta_1,j_1,\delta_2,j_2)}_{\delta,j}$ is the OPE differential operator of $\mathcal{P}_{\delta,j}\in\mathcal{P}_{\delta_1,j_1}\times\mathcal{P}_{\delta_2,j_2}$. To obtain the coefficients c_1, c_2 and c_3, we apply (2.336) to a scalar two-point function $\langle\mathcal{P}_\delta\mathcal{P}_\delta\rangle$ and compare the two sides of the equation. The result is given by

$$\begin{aligned} c_1 &= \frac{\delta(\delta+\delta_1-\delta_2-1)}{(d-4)(\delta_1-1)(-\delta_1+d-3)} \\ c_2 &= \frac{\delta}{(\delta_1-1)(\delta_1-2)(-\delta_1+d-3)(2\delta_1-d+2)} \\ c_3 &= \frac{\delta(-\delta_1-\delta_2+\delta+d-3)(5-2d+\delta+3\delta_1-\delta_2)}{2(-\delta_1+d-3)(2\delta_1-d+2)}\,. \end{aligned} \tag{2.337}$$

Combining (2.333), (2.334) and (2.336), we have

$$\begin{aligned} &C_{\delta,1}(z_1,z_2,\partial_z,\partial_w)\left([w,z]\cdot[z_3,p](-2p\cdot z)^{-\delta-1}\right) \\ &= \frac{1}{(d-4)(\delta-1)(\delta-d+3)}\left(c_1\mathcal{D}_p\cdot\mathcal{D}^{0-}_{z_1,w_1}C^{(\delta_1,1,\delta_2,0)}_{\delta,0}\right. \\ &\quad \left.+c_2\mathcal{D}_p\cdot\mathcal{D}^{+0}_{z_1,w_1}C^{(\delta_1-1,0,\delta_2,0)}_{\delta,0}+c_3\mathcal{D}_p\cdot\mathcal{D}^{-0}_{z_1,w_1}C^{(\delta_1+1,0,\delta_2,0)}_{\delta,0}\right)(-2p\cdot z)^{-\delta}\,. \end{aligned} \tag{2.338}$$

Now the calculation is straightforward since $C_{\delta,0}(-2p\cdot z)^{-\delta}$ is simply the $j=0$ celestial block calculated in [15].[50] Finally, we obtain that the functions $g(\zeta)$ and $h(\zeta)$ in the $j=1$ celestial block (2.331) are given by

$$\begin{aligned} g(\zeta) &= \zeta^{\frac{\delta-\delta_1-\delta_2-1}{2}}\Big(\tfrac{1}{2}(1+\tfrac{\delta_2-\delta_1}{\delta-d+3})\,{}_2F_1(\tfrac{\delta+\delta_1-\delta_2-1}{2},\tfrac{\delta+\delta_2-\delta_1+1}{2},\delta+2-\tfrac{d}{2},\zeta) \\ &\qquad +\tfrac{1}{2}(1+\tfrac{\delta_1-\delta_2}{\delta-d+3})\,{}_2F_1(\tfrac{\delta+\delta_1-\delta_2+1}{2},\tfrac{\delta+\delta_2-\delta_1-1}{2},\delta+2-\tfrac{d}{2},\zeta)\Big) \\ h(\zeta) &= \frac{\zeta^{\frac{\delta-\delta_1-\delta_2+1}{2}}}{(1-\zeta)}\Big(\tfrac{1}{2}(1+\tfrac{\delta_2-\delta_1}{\delta-d+3})\,{}_2F_1(\tfrac{\delta+\delta_1-\delta_2-1}{2},\tfrac{\delta+\delta_2-\delta_1+1}{2},\delta+2-\tfrac{d}{2},\zeta) \\ &\qquad -\tfrac{1}{2}(1+\tfrac{\delta_1-\delta_2}{\delta-d+3})\,{}_2F_1(\tfrac{\delta+\delta_1-\delta_2+1}{2},\tfrac{\delta+\delta_2-\delta_1-1}{2},\delta+2-\tfrac{d}{2},\zeta)\Big)\,. \end{aligned} \tag{2.339}$$

One can check that (2.339) is indeed the solution to the Casimir differential equations of $g(\zeta)$ and $h(\zeta)$ obtained by applying the quadratic Casimir to (2.331). For $\delta_1=\delta_2=\delta_\phi$, we have $h(\zeta)=0$ and

$$\begin{aligned} g(\zeta) &= \zeta^{\frac{\delta-2\delta_\phi-1}{2}}\,{}_2F_1\left(\frac{\delta-1}{2},\frac{\delta+1}{2},\delta+2-\frac{d}{2},\zeta\right) \\ &= \zeta^{\frac{\Delta-2\Delta_\phi}{2}}\,{}_2F_1\left(\frac{\Delta-2}{2},\frac{\Delta}{2},\Delta+1-\frac{d}{2},\zeta\right). \end{aligned} \tag{2.340}$$

[50] The celestial block $C^{(\delta_1,1,\delta_2,0)}_{\delta,0}(-2p\cdot z)^{-\delta}$ was not calculated in [15], but it can be easily obtained using the Casimir equation method.

Having solved for the celestial blocks, we can now calculate $\langle \mathcal{O}_{20'}(p)|\mathcal{D}_1\mathbf{L}[\mathcal{O}_{\Delta,0}]|J(p,z_3)\rangle^{(a)}$ to obtain the proportionality constant in (2.329). For an operator $\mathcal{O}$ with weights (Δ, J), the three-point function $\langle 0|\mathcal{O}_{20'}\mathcal{O}J|0\rangle^{(a)}$ has two tensor structures. With our choice of conventions in appendix A.2, their expressions in the embedding space for integer J are given by

$$\langle 0|\mathcal{O}_{20'}(X_4)\mathcal{O}(X_2,Z_2)J(X_3,Z_3)|0\rangle^{(1)} = \frac{(-2V_{2,34})^J(-2V_{3,42})}{X_{24}^{\frac{\Delta+J-2}{2}}X_{23}^{\frac{\Delta+J+2}{2}}X_{34}^{\frac{6-\Delta-J}{2}}} \tag{2.341}$$

$$\langle 0|\mathcal{O}_{20'}(X_4)\mathcal{O}(X_2,Z_2)J(X_3,Z_3)|0\rangle^{(2)} = \frac{(-2V_{2,34})^{J-1}(-2H_{23})}{X_{24}^{\frac{\Delta+J-2}{2}}X_{23}^{\frac{\Delta+J+2}{2}}X_{34}^{\frac{6-\Delta-J}{2}}}, \tag{2.342}$$

where the structures $V_{i,jk}$ and H_{ij} are defined in (2.217) and (2.218). Note that after setting $J=0$, the first structure (2.341) is still a valid three-point function of local operators, and hence it should be annihilated by the shortening condition $\mathcal{D}'_1$ (or equivalently $\mathcal{D}_1\mathbf{L}$). So we can just consider the second structure (2.342). As explained in section 2.5.1, its analytic continuation for complex J should be given by

$$\langle 0|\mathcal{O}_{20'}(X_4)\mathcal{O}(X_2,Z_2)J(X_3,Z_3)|0\rangle^{(2)}_{\pm} = \mp\frac{(-2V_{2,43})^{J-1}(-2H_{23})}{X_{24}^{\frac{\Delta+J-2}{2}}X_{23}^{\frac{\Delta+J+2}{2}}X_{34}^{\frac{6-\Delta-J}{2}}}. \tag{2.343}$$

where $\mp$ is due to the $(-2V_{2,34})^{J-1}$ factor. Using the algorithm for computing light transform and Fourier transform of three-point functions described in [34] and applying the differential operator $\mathcal{D}_1$, we obtain

$$\begin{aligned}
&\langle \mathcal{O}_{20'}(p)|\mathcal{D}_1\mathbf{L}[\mathcal{O}_{\Delta,0}](\infty,z,w)|J(p,z_3)\rangle^{(2)}_{-} \\
&= 16\pi^4\frac{\Gamma(\Delta-2)}{\Gamma(\frac{\Delta}{2})^2\Gamma(\frac{\Delta-2}{2})\Gamma(\frac{4-\Delta}{2})} \\
&\qquad\times(-2p\cdot z)^{-\Delta}(2w\cdot z_3 z\cdot p-2w\cdot p z\cdot z_3)(-p^2)^{\frac{\Delta-2}{2}}\theta(p),
\end{aligned} \tag{2.344}$$

where $\theta(p)\equiv\theta(-p^2)\theta(p^0)$. Finally, combining (2.328), (2.331) and (2.344), we have

$$\langle \mathcal{O}_{20'}(p)|\mathbf{L}[\mathcal{O}_{20'}](\infty,z_1)\mathbf{L}[\mathcal{O}_{20'}](\infty,z_2)|J(p,z_3)\rangle_{\mathbf{175}} = 16i\pi^5\frac{[z_3,p]\cdot[z_1,z_2](-p^2)\theta(p)}{(-2p\cdot z_1)^2(-2p\cdot z_2)^2}\mathcal{F}_{J\mathcal{O}}(\zeta), \tag{2.345}$$

where the function $\mathcal{F}_{J\mathcal{O}}(\zeta)$ is

$$\mathcal{F}_{J\mathcal{O}}(\zeta) = -\int_{2-i\infty}^{2+i\infty}\frac{d\Delta}{2\pi i}C_2'^{-}(\Delta,0)\frac{\Gamma(\Delta-2)}{\Gamma(\frac{\Delta}{2})^2\Gamma(\frac{\Delta-2}{2})\Gamma(\frac{4-\Delta}{2})}g^{2,2}_{\Delta}(\zeta), \tag{2.346}$$

and $g_{\Delta}^{2,2}(\zeta)$ is the $j=1$ celestial block (2.340)

$$g_{\Delta}^{2,2}(\zeta)=\zeta^{\frac{\Delta-4}{2}}{}_2F_1\left(\frac{\Delta-2}{2},\frac{\Delta}{2},\Delta-1,\zeta\right). \tag{2.347}$$

Now the remaining task is to find the OPE data $C_2'^{-}(\Delta,J)$ at $J=0$ and calculate the event shape. In the next section, we will show that there's a superconformal Ward identity that relates $C_2'^{-}(\Delta,J)$ to the OPE data of the $\langle \mathcal{O}_{20'}\mathcal{O}_{20'}\mathcal{O}_{20'}\mathcal{O}_{20'}\rangle$ 4-point function. Using the identity, we can derive a simple relation between $\mathcal{F}_{J\mathcal{O}}(\zeta)$ and the energy-energy correlator calculated in [15].

Relation to energy-energy correlator

Deforming the contour of the Δ-integral in (2.346), we get

$$\mathcal{F}_{J\mathcal{O}}(\zeta)=-\sum_{\Delta}p_{\Delta,J=0}\frac{\Gamma(\Delta-2)}{\Gamma(\frac{\Delta}{2})^2\Gamma(\frac{\Delta-2}{2})\Gamma(\frac{4-\Delta}{2})}g_{\Delta}^{2,2}(\zeta), \tag{2.348}$$

where $p_{\Delta,J}$ is the three-point coupling of $\langle \mathcal{O}_{20'}\mathcal{O}_{20'}\mathcal{O}_{20'}J\rangle$ corresponding to the structure (2.342), analytically continued to $J=0$, and the sum is over Regge trajectories.

We can obtain $p_{\Delta,J}$ from the four-point function $\langle \mathcal{O}_{20'}\mathcal{O}_{20'}\mathcal{O}_{20'}J\rangle$, whose expression is given in (2.295). Note that the function $\Phi(u,v)$ can be written in terms of superconformal blocks as [53]

$$\Phi(u,v)=(2\pi)^4\frac{v}{u^3}\sum_{\Delta,J}a_{\Delta,J}g_{\Delta+4,J}(u,v), \tag{2.349}$$

where $g_{\Delta,J}$ is the usual 4d conformal block, and $a_{\Delta,J}$ is the product of the three-point couplings to a given superconformal primary. Plugging this into (2.295) and specializing to the configuration $x_1=0, x_3=1, x_4=\infty$, we find[51]

$$\begin{aligned}&\langle \mathcal{O}_{20'}(\infty)\mathcal{O}_{20'}(0)\mathcal{O}_{20'}(u,v)J(1)\rangle\\&=(2\pi)^4\sum_{\Delta,J}a_{\Delta,J}\left(-\frac{\Delta+1}{8}G_{\Delta+3,J+1}(u,v)+\frac{(J-1)(\Delta+1)}{8(J+1)}G_{\Delta+3,J-1}(u,v)\right.\\&\qquad+\frac{(\Delta+3)(\Delta+4)(\Delta+J+4)^2}{128(\Delta+2)(\Delta+J+3)(\Delta+J+5)}G_{\Delta+5,J+1}(u,v)\\&\qquad\left.-\frac{(J-1)(\Delta+3)(\Delta+4)(\Delta-J+2)^2}{128(J+1)(\Delta-J+3)(\Delta-J+1)(\Delta+2)}G_{\Delta+5,J-1}(u,v)\right),\end{aligned} \tag{2.350}$$

[51] To obtain this equation, we studied the small $z,\bar{z}$ limit on both sides and matched each term in the series expansion.

where $G_{\Delta,J}$ is the conformal block of one conserved current and three scalars with dimension 2, which can be calculated using e.g. [54]. The above expression should agree with the usual conformal block decomposition

$$\langle O_{20'}(\infty)O_{20'}(0)O_{20'}(u,v)J(1)\rangle = \sum_{\Delta,J} p_{\Delta,J}G_{\Delta,J}(u,v). \tag{2.351}$$

Therefore, there is a superconformal Ward identity that relates the three-point coupling coefficients $p_{\Delta,J}$ and $a_{\Delta,J}$:

$$\begin{aligned} p_{\Delta,J} = (2\pi)^4 \Bigg(&-\frac{\Delta-2}{8}a_{\Delta-3,J-1} + \frac{J(\Delta-2)}{8(J+2)}a_{\Delta-3,J+1} \\ &+ \frac{(\Delta-2)(\Delta-1)(\Delta+J-2)^2}{128(\Delta-3)(\Delta+J-3)(\Delta+J-1)}a_{\Delta-5,J-1} \\ &- \frac{J(\Delta-2)(\Delta-1)(\Delta-J-4)^2}{128(J+2)(\Delta-J-3)(\Delta-J-5)(\Delta-3)}a_{\Delta-5,J+1}\Bigg). \end{aligned} \tag{2.352}$$

Setting $J = 0$ in the above identity and plugging it into (2.348), we have

$$\begin{aligned} \mathcal{F}_{JO}(\zeta) &= -(2\pi)^4 \sum_{\Delta} \left(-\frac{\Delta-2}{8}a_{\Delta-3,-1} + \frac{(\Delta-2)^3}{128(\Delta-3)^2}a_{\Delta-5,-1} \right) \frac{\Gamma(\Delta-2)}{\Gamma(\frac{\Delta}{2})^2\Gamma(\frac{\Delta-2}{2})\Gamma(\frac{4-\Delta}{2})} g_{\Delta}^{2,2}(\zeta) \\ &= -(2\pi)^4 \sum_{\Delta} a_{\Delta-4,-1} \frac{\Gamma(\Delta-2)}{8\Gamma(\frac{\Delta-1}{2})^3\Gamma(\frac{3-\Delta}{2})} \left(g_{\Delta-1}^{2,2}(\zeta) + \frac{(\Delta-1)}{4(\Delta-2)}g_{\Delta+1}^{2,2}(\zeta) \right). \end{aligned} \tag{2.353}$$

The sum of $j = 1$ celestial blocks in the parentheses satisfies

$$g_{\Delta-1}^{2,2}(\zeta) + \frac{(\Delta-1)}{4(\Delta-2)}g_{\Delta+1}^{2,2}(\zeta) = \zeta f_{\Delta}^{4,4}(\zeta), \tag{2.354}$$

where $f_{\Delta}^{4,4}(\zeta)$ is the $j = 0$ celestial block:

$$f_{\Delta}^{4,4}(\zeta) = \zeta^{\frac{\Delta-7}{2}} {}_2F_1\left(\frac{\Delta-1}{2}, \frac{\Delta-1}{2}, \Delta-1, \zeta\right). \tag{2.355}$$

This gives

$$\mathcal{F}_{JO}(\zeta) = -\frac{\zeta}{2}\left(\sum_{\Delta} a_{\Delta-4,-1} \frac{4\pi^4\Gamma(\Delta-2)}{\Gamma(\frac{\Delta-1}{2})^3\Gamma(\frac{3-\Delta}{2})} f_{\Delta}^{4,4}(\zeta) \right). \tag{2.356}$$

Note that the term in the parentheses is simply the function $\mathcal{F}_{\mathcal{E}}(\zeta)$ related to the energy-energy correlator calculated in [15], see (7.11) and (7.16) there. Therefore, we have

$$\mathcal{F}_{JO}(\zeta) = -\frac{\zeta}{2}\mathcal{F}_{\mathcal{E}}(\zeta), \tag{2.357}$$

which agrees with (2.323) from direct computation.

2.8 Discussion and future directions

We have seen that a product of light-transformed local operators $\mathbf{L}[\mathcal{O}_1]\mathbf{L}[\mathcal{O}_2]$ is encoded in a nontrivial way inside the space of light-ray operators. Low transverse spin terms in the product are special linear combinations of light-ray operators with spin $J_1 + J_2 - 1$. Higher transverse spin terms are primary descendants, obtained by acting with the special conformally-invariant differential operators $\mathcal{D}_n$ on higher-J light-ray operators.

The differential operators $\mathcal{D}_n$ appear in the general classification of reducible generalized Verma modules described in [35]. Most of the operators in this classification act on multiplets with quantum numbers below the unitarity bound. Thus, when they were first identified, it was not obvious a priori what roles they could play in physical unitary CFTs. However, light-ray operators naturally have quantum numbers that violate the unitarity bound, and indeed we have identified a role for $\mathcal{D}_n$ acting on this space. It is interesting to ask whether there are similar roles in Lorentzian observables for other conformally-invariant differential operators. In addition, it would be interesting to further explore interrelationships between conformally-invariant differential operators, conformally-invariant pairings, conformally-invariant integral transforms, and weight-shifting operators.

One way to motivate the light-ray OPE is by thinking about null-integrated operators as primaries in a fictitious $d-2$-dimensional CFT. Now that we have a complete description of the terms in this OPE, can we push the analogy with CFT_{d-2} further? For example, what are the implications of the light-ray OPE for multi-point event-shapes, such as the three-point energy correlators studied in [55]? Associativity of the light-ray OPE should give rise to a nontrivial crossing equation satisfied by three-point event shapes, and it would be interesting to study this "celestial" crossing equation using bootstrap techniques. One of the first lessons of the analytic bootstrap is that the crossing equations imply the existence of "double-twist" operators with arbitrarily large spin, via the lightcone bootstrap [8, 7]. Similar arguments for the light-ray OPE could imply the existence of terms with arbitrarily-large transverse spin j. It would be interesting to understand the relationship between these operators and the usual large-spin operators in the lightcone bootstrap.

To fully develop a celestial bootstrap program, one would need to understand OPEs of more general light-ray operators, such as a product of a null-integrated local operator $\mathbf{L}[\mathcal{O}]$ with a general light-ray operator $\mathbb{O}_{\Delta,J}$, or even a product of two general light-ray operators. This is an important problem for the future. A key conceptual question is: do new types of operators appear beyond the ones constructed in [11]? For the OPE explored in this work, the answer turned out to be "no" for rather nontrivial reasons. For more general light-ray OPEs, the answer is less clear. It is natural to conjecture, however, that any light-ray operators allowed by symmetries will appear. For example, we expect that the leading light-ray operator in the three-fold OPE of average null energy operators is $\mathbf{L}[X_4]$, where X_4 is the lowest-twist spin-4 local operator. This claim and its implications should be testable using the results of [55].

Currently, the best available data about multi-point event shapes comes from Hofman and Maldacena's calculation of energy correlators in $\mathcal{N} = 4$ Super Yang Mills theory at large 't Hooft coupling [12]. They computed the first few terms in the large-λ expansion of a multi-point energy correlator up to order $1/\lambda^{3/2}$. It would be interesting to understand the structure of their result from the point of view of the light-ray OPE, and also the "t-channel" expansion of [34].

Correlators of average null energy operators do not capture complete information about the energy distribution of a state. In particular, they are blind to the value of retarded time when excitations reach future null infinity. To probe this more refined information, it is natural to weight integrals along null infinity by non-constant functions of retarded time. In [56], it was shown that meromorphic weighting functions with carefully-chosen poles can give rise to useful "dispersive sum rules" that constrain the data of a CFT. Alternative weighting functions may have other useful applications and are worth exploring.

Finally, it would be interesting to explore the structure of the short-angle expansion of energy-energy correlators and other event shapes in non-conformal theories, and in particular how the higher-transverse spin terms arise there [17]. When the conformal symmetry is present, the operators $\mathcal{D}_n$ acting on light-ray operators inserted at spatial infinity (in which case $\mathcal{D}_n$ become expressed in terms of special conformal generators) define some new discrete set of translationally-invariant detectors. As the conformal symmetry is broken, this relation between these detectors and the continuous Regge trajectories gets broken as well. We thus expect that in theories such as QCD some new discrete set of anomalous dimensions should appear in observables for which higher transverse spin is important (such as higher-point event shapes or oriented two-point event shapes).

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Chapter 3

THREE-POINT ENERGY CORRELATORS AND THE CELESTIAL BLOCK EXPANSION

This chapter is based on

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3.1 Introduction

Energy correlators [18, 19, 57] are natural Lorentzian observables with numerous applications in collider physics, conformal field theory, and string theory, see e.g. [58, 59, 60, 61, 62, 12, 63, 64, 24, 51, 25, 65, 66, 49, 67, 68, 21, 17, 22, 26, 55, 69, 23, 70, 71, 72, 73, 74, 75]. They are given by an expectation value of a product of energy flux operators $\mathcal{E}(\vec{n}_i)$ [76] that measure the flux of energy at locations $\vec{n}_i \in S^{d-2}$ on the celestial sphere:

$$\langle\Psi|\mathcal{E}(\vec{n}_1)\cdots\mathcal{E}(\vec{n}_k)|\Psi\rangle. \tag{3.1}$$

Energy correlators are examples of more general “event shapes,” which are expectation values of products of detectors at different locations on the celestial sphere.

The kinematics of energy correlators and event shapes exhibit many features of a (fictitious) Euclidean $d-2$-dimensional CFT on the celestial sphere. In particular, the Lorentz group $\mathrm{SO}(d-1,1)$ acts as the conformal group on the celestial sphere, so event shapes exhibit conformal symmetry.

However, other aspects of event shapes are different from $d-2$-dimensional CFT. Event shapes are not necessarily computed (in an obvious way) by a local path integral on the celestial sphere. Consequently, structures like radial quantization and a $d-2$-dimensional operator product expansion (OPE) cannot obviously be used to analyze them. Nevertheless, it was argued by Hofman and Maldacena [12] that a kind of OPE should exist between energy flux operators $\mathcal{E}(\vec{n}_1)\times\mathcal{E}(\vec{n}_2)$ in the limit $\vec{n}_1 \to \vec{n}_2$, i.e. as the corresponding points on the celestial sphere approach each other. In [15, 77], the OPE of two energy flux operators $\mathcal{E}(\vec{n}_1)\times\mathcal{E}(\vec{n}_2)$ was explicitly constructed in a general nonperturbative CFT_d, and it was shown that the objects appearing are the light-ray operators $\mathbb{O}_J(\vec{n})$ of [11]. This leads to a useful expansion for two-point energy correlators in special functions called "celestial blocks," which re-sum the contributions of light-ray operators and their descendants on the celestial sphere.

If it were possible to iterate the light-ray OPE, we would obtain a simple and beautiful procedure for evaluating higher-point energy correlators. However, the arguments of [15, 77] do not extend in a simple way to describe the OPE of an energy flux operator and a more general light-ray operator $\mathcal{E}(\vec{n}_1) \times \mathbb{O}_J(\vec{n}_2)$, or to describe an OPE of general light-ray operators $\mathbb{O}_{J_1}(\vec{n}_1) \times \mathbb{O}_{J_2}(\vec{n}_2)$. Perturbative studies of these more complicated OPEs were undertaken recently in [73]. Finding an appropriate nonperturbative generalization of the light-ray OPE is an important problem. However, we will not solve it in this work. Instead, we assume that a general light-ray OPE exists and study some of its consequences for higher-point energy correlators.

One consequence is that higher-point energy correlators should admit an expansion in a discrete sum of multi-point celestial blocks. Mathematically, harmonic analysis with respect to the Lorentz group [44] guarantees that energy correlators can be expanded in an integral of celestial "partial waves." However, going from a partial wave expansion to a celestial block expansion requires a dynamical assumption about poles in partial wave coefficients. We check this assumption by studying the celestial block expansion of three-point energy correlators (EEEC) at both weak coupling (in QCD and $\mathcal{N} = 4$ SYM) and strong coupling ($\mathcal{N} = 4$ SYM). In all cases, we find that a discrete celestial block expansion exists, and that the quantum numbers of objects appearing can be understood from symmetries. At weak coupling, we use the recent perturbative expressions for the EEEC in [55], and at strong coupling, we study Hofman and Maldacena's famous result for the EEEC [12].

A particularly interesting limit of the EEEC is the collinear limit [55, 73], where all three operators approach each other on the celestial sphere $|\vec{n}_{ij}| \to 0$ with $|\vec{n}_{ij}|/|\vec{n}_{kl}|$ fixed, where $\vec{n}_{ij} = \vec{n}_i - \vec{n}_j$. Physically, the collinear limit is obtained by simultaneously boosting all three detectors. By Lorentz-invariance, this is equivalent to boosting the state $|\Psi\rangle$ in the opposite direction, causing its momentum p to approach the null cone. However, a point on the null cone encodes a point on the celestial sphere $(1, \vec{n}_4) = p/p^0$, so the kinematics of the EEEC in the collinear limit are the same as for a conformal four-point function in CFT_{d-2}. In particular, celestial blocks have an expansion in the collinear limit, where the leading term is the usual four-point conformal block. This observation was made for the leading term in [73], and we will extend it to a systematic expansion around the collinear limit. Furthermore, event shapes inherit crossing symmetry from the d-dimensional bulk theory. This allows us to apply techniques from the analytic bootstrap for CFT four-point functions to the collinear EEEC, including the lightcone bootstrap [7, 8, 78, 79, 80, 81] and Lorentzian inversion formula [9, 10]. (Interestingly, the Lorentzian inversion formula requires analytically continuing to Lorentzian signature on the celestial sphere, which is $(d-2,2)$ signature from the point of view of the bulk theory.)

This paper is organized as follows. In section 3.2, we study implications of Lorentz symmetry for event shapes. We explain the form that the celestial block expansion should take for 2- and 3-point event shapes, and study the expansion of 3-point celestial blocks around the collinear limit. We furthermore explore general constraints of celestial crossing symmetry for the collinear EEEC using lightcone bootstrap methods. In section 3.3, we study recent leading-order weak-coupling results for the collinear EEEC in QCD and $\mathcal{N}=4$ SYM from the point of view of the celestial block expansion, using the Lorentzian inversion formula to extract celestial block coefficients. In section 3.4, we describe some predictions for higher orders in the weak coupling expansion that follow from a discrete celestial block expansion. In section 3.5, we discuss consequences of Ward identities, in particular using them to determine the leading nontrivial contact terms in the EEEC in weakly-coupled $\mathcal{N}=4$ SYM. In section 3.6, we study the EEEC in strongly-coupled $\mathcal{N}=4$ SYM for general configurations on the celestial sphere — not just the collinear limit. We explain how the corresponding celestial OPE data can be obtained from a three-point celestial inversion formula, and then apply the inversion formula to results from [12] to obtain simple analytic formulas for the full EEEC celestial OPE data at $O(1/\lambda)$. Finally, we conclude in section 3.7.

Note Added: This paper will appear simultaneously with a paper by Hao Chen, Ian Moult, Joshua Sandor, and Hua Xing Zhu, that also studies three-point correlators of light-ray operators from the perspective of the light-ray OPE. We thank these authors for coordinating submission.

3.2 Lorentz symmetry and event shapes

Because the Lorentz group $\mathrm{SO}(d-1,1)$ is also the conformal group on the celestial sphere S^{d-2}, event shapes can be decomposed into "celestial blocks," which are natural objects from the point of view of $d-2$ dimensional CFT. We will be particularly interested in three-point event shapes. In the "collinear" limit where the three detectors are close to each other, the kinematics of a three-point event shape become the same as a CFT four-point function, and celestial blocks become four-point conformal blocks. We will begin by reviewing event shapes in CFT. We then discuss celestial blocks for two-point event shapes, before introducing three-point celestial blocks and their collinear limit.

3.2.1 Review: event shapes and the light transform

An event shape can be thought of as a weighted cross section, or alternatively as the expectation value of an operator at future null infinity. For example, consider the three-point energy correlator (EEEC), conventionally defined by

$$\begin{aligned}&\mathrm{EEEC}(\zeta_{12},\zeta_{13},\zeta_{23})\\&=\sum_{i,j,k}\int d\sigma\frac{E_iE_jE_k}{Q^3}\delta\left(\zeta_{12}-\frac{1-\cos\theta_{ij}}{2}\right)\delta\left(\zeta_{13}-\frac{1-\cos\theta_{ik}}{2}\right)\delta\left(\zeta_{23}-\frac{1-\cos\theta_{jk}}{2}\right).\end{aligned} \tag{3.2}$$

Here, $d\sigma$ is the phase space measure multiplied by the squared amplitude for some state $|\mathcal{O}(p)\rangle$ to create outgoing particles, and the sum $\sum_{i,j,k}$ runs over triplets of outgoing particles. The definition (3.2) is convenient for perturbative calculations and deriving Ward identities (see section 3.5). However, it obscures some features like IR safety, and furthermore requires the existence of asymptotic states.

An alternative definition of the EEEC, that works in any nonperturbative QFT, is [24]

$$\begin{aligned}\mathrm{EEEC}(\zeta_{12},\zeta_{13},\zeta_{23})=\int d\Omega_{\vec{n}_1}d\Omega_{\vec{n}_2}d\Omega_{\vec{n}_3}\delta(\zeta_{12}-\tfrac{1-\vec{n}_1\cdot\vec{n}_2}{2})\delta(\zeta_{13}-\tfrac{1-\vec{n}_1\cdot\vec{n}_3}{2})\delta(\zeta_{23}-\tfrac{1-\vec{n}_2\cdot\vec{n}_3}{2})\\\times\frac{\langle\mathcal{O}(p)|\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{E}(\vec{n}_3)|\mathcal{O}(p)\rangle}{(-p^2)^{\frac{3}{2}}\langle\mathcal{O}(p)|\mathcal{O}(p)\rangle},\end{aligned} \tag{3.3}$$

where $\mathcal{E}(\vec{n})$ is an energy detector defined by

$$\mathcal{E}(\vec{n})=\lim_{r\to\infty}r^{d-2}\int_0^\infty dt\, n^iT^0{}_i(t,r\vec{n}). \tag{3.4}$$

In a CFT, $\mathcal{E}(\vec{n})$ is conformally equivalent to the average null energy operator ANEC — a null integral of the stress tensor [12]. Thus, we often refer to $\mathcal{E}(\vec{n})$ as ANEC operators. Here, and below, we use the shorthand notation where when a bra and ket have equal momenta, we implicitly strip off an overall momentum-conserving delta function. This is equivalent to Fourier-transforming only one of the operators:

$$\langle \mathcal{O}(p)|\cdots|\mathcal{O}(p)\rangle \equiv \int d^d x e^{ip\cdot x}\langle 0|\mathcal{O}(x)\cdots\mathcal{O}(0)|0\rangle. \tag{3.5}$$

The ANEC operator $\mathcal{E}(\vec{n})$ measures energy flux at a point on the celestial sphere $\vec{n}\in S^{d-2}$. In a CFT, it can be understood in terms of a conformally-invariant integral transform called the light transform [11]. To describe the light transform, we use index-free notation where we contract indices of an operator with an auxiliary null vector z: $\mathcal{O}(x,z) = \mathcal{O}^{\mu_1\cdots\mu_J}(x)z_{\mu_1}\cdots z_{\mu_J}$. The light transform of an operator $\mathcal{O}$ with scaling dimension Δ and spin J is

$$\mathbf{L}[\mathcal{O}](x,z) = \int_{-\infty}^{\infty} d\alpha(-\alpha)^{-\Delta-J}\mathcal{O}\left(x-\frac{z}{\alpha},z\right). \tag{3.6}$$

Under conformal transformations, $\mathbf{L}[\mathcal{O}](x,z)$ transforms like a primary operator at x with quantum numbers $(1-J, 1-\Delta)$. The ANEC operator defined in (3.4) can be written as the light transform of the stress-energy tensor placed at spatial infinity:

$$\mathcal{E}(\vec{n}) = 2\mathbf{L}[T](\infty, z=(1,\vec{n})). \tag{3.7}$$

In general, an (un-normalized) n-point event shape in CFT is the matrix element of a product of n light-transformed operators in a state $|\mathcal{O}(p)\rangle$:

$$\langle \mathcal{O}(p)|\mathbf{L}[\mathcal{O}_1](\infty,z_1)\cdots\mathbf{L}[\mathcal{O}_n](\infty,z_n)|\mathcal{O}(p)\rangle. \tag{3.8}$$

For the EEEC, we have $\mathcal{O}_1=\mathcal{O}_2=\mathcal{O}_3=T$.

3.2.2 Lorentz symmetry and celestial blocks

Two-point event shapes

Consider a two-point scalar event shape[1]

$$\langle \phi_4(p)|\mathbf{L}[\phi_1](\infty,z_1)\mathbf{L}[\phi_2](\infty,z_2)|\phi_3(p)\rangle. \tag{3.9}$$

For simplicity, we study event shapes built from scalars ϕ_i in this section, leaving spinning operators for later.

[1] This scalar event shape is only well-defined nonperturbatively if the theory has Regge intercept $J_0 < -1$ [34]. In this section, we are only interested in kinematics, so we assume this is the case.

Let us understand how the product $\mathbf{L}[\phi_1](\infty, z_1)\mathbf{L}[\phi_2](\infty, z_2)$ transforms under the Lorentz group $\mathrm{SO}(d-1,1)$. The Lorentz group is isomorphic to the Euclidean conformal group in $d-2$ dimensions. From this point of view, the polarization vector z can be thought of as an embedding-space coordinate for the celestial sphere S^{d-2}.

As convenient notation, let $\mathcal{P}_{\delta,j}(z,w)$ denote an operator with dimension δ and spin j in a fictitious CFT_{d-2} on the celestial sphere, in the embedding formalism. The embedding space coordinates are null vectors $z, w \in \mathbb{R}^{d-1,1}$ with a gauge redundancy $w \sim w + \alpha z$. $\mathcal{P}_{\delta,j}(z,w)$ is a homogeneous function of z and w with degrees $-\delta$ and j, respectively. See [15] for more details on this notation. We usually refer to the spin j on the celestial sphere as "transverse spin" to disambiguate it from the Lorentz spin J of a local operator in d-dimensions. When $j = 0$, we write simply $\mathcal{P}_\delta(z)$.

The light-transformed operator $\mathbf{L}[\phi_i](\infty, z)$ is homogeneous of degree $1 - \Delta_i$ in z. Thus, it transforms like a scalar on the celestial sphere with dimension $\delta_i = \Delta_i - 1$:

$$\mathbf{L}[\phi_i](\infty, z) \sim \mathcal{P}_{\delta_i}(z). \tag{3.10}$$

From this point of view, we can treat correlators of $\mathbf{L}[\phi_i](\infty, z)$ as if they were correlators of $\mathcal{P}_{\delta_i}(z)$ in a fictitious CFT_{d-2}. Note that we do not assert that there exists a *local* CFT on S^{d-2}. For our purposes, (3.10) is convenient notation for keeping track of symmetries.

Using this notation, a product $\mathbf{L}[\phi_1](\infty, z_1)\mathbf{L}[\phi_2](\infty, z_2)$ transforms like a product of scalars $\mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)$ in $d-2$ dimensions. It is natural to expand such a product in a $d-2$ dimensional OPE, where the objects that appear are spin-j traceless symmetric tensors:

$$\mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2) = \sum_{\delta,j} r_{\delta,j} C_{\delta,j}(z_1, z_2, \partial_{z_2}, \partial_{w_2})\mathcal{P}_{\delta,j}(z_2, w_2). \tag{3.11}$$

Here, the dimensions δ and "OPE coefficients" $r_{\delta,j}$ that appear depend on the theory. However, the differential operator $C_{\delta,j}$ is determined by symmetry and is defined by

$$C_{\delta,j}(z_1, z_2, \partial_{z_2}, \partial_{w_2})\langle \mathcal{P}_{\delta,j}(z_2, w_2)\mathcal{P}_{\delta,j}(z, w)\rangle = \langle \mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z, w)\rangle, \tag{3.12}$$

where $\langle \mathcal{P}_{\delta,j}(z_2, w_2)\mathcal{P}_{\delta,j}(z, w)\rangle$ and $\langle \mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z, w)\rangle$ are standard two- and three-point structures in the embedding space.

The light-ray OPE gives a concrete version of the expansion (3.11) where the objects on the right-hand side are light-ray operators. Taking $\mathbf{L}[\phi]\mathbf{L}[\phi]$ as an example, the operators appearing are [15, 77]

$$\mathbf{L}[\phi] \times \mathbf{L}[\phi] \sim \sum_i \mathbb{O}_{i,J=-1,j=0} + \sum_i \sum_{n=1}^{\infty} \mathcal{D}_{2n} \mathbb{O}_{i,J=-1+2n,j=0}. \tag{3.13}$$

Here, $\mathbb{O}_{i,J,j}$ denotes a light-ray operator on the i-th Regge trajectory with spin J and transverse spin j [11]. $\mathcal{D}_{2n}$ is a differential operator that decreases the spin J by $2n$ and increases the transverse spin j by $2n$. Light-ray operators are analytic continuations of light transformed operators $\mathbf{L}[O]$ in J, so they satisfy

$$\mathbb{O}_{i,J,j} = f_{\phi\phi O_{i,J,j}} \mathbf{L}[O_{i,J,j}], \qquad J \in \mathbb{Z}_{\geq 0},\ J \text{ even}. \tag{3.14}$$

The light-ray OPE thus establishes a relation between the scalar two-point event shape, defined as the matrix element of $\mathbf{L}[\phi]\mathbf{L}[\phi]$, and the OPE data of $\phi \times \phi$ analytically continued to $J = -1, 1, 3, \ldots$. For concrete calculations of two-point event shapes using the light-ray OPE, see [15, 77].

Let us apply the light-ray OPE to the event shape (3.9). As discussed in [77], there is a selection rule for the transverse spin j: in a state created by scalar operators $\langle\phi_4(p)| \cdot |\phi_3(p)\rangle$, only light-ray operators with $j = 0$ can have nonzero matrix elements. Thus, the sum in (3.11) collapses to just the $j = 0$ terms. We find

$$\langle\phi_4(p)|\mathbf{L}[\phi_1](\infty, z_1)\mathbf{L}[\phi_2](\infty, z_2)|\phi_3(p)\rangle = \sum_\delta r_{12\delta} C_\delta(z_1, z_2, \partial_{z_2})\langle\phi_4(p)|\mathbb{W}_\delta(z_2)|\phi_3(p)\rangle, \tag{3.15}$$

where $\mathbb{W}_\delta$ stands for transverse-spin zero light-ray operators $\mathbb{O}_{i,J=-1,j=0}$, and transforms as a scalar primary with dimension δ under $\mathrm{SO}(d-1, 1)$.

The form of the matrix element $\langle\phi_4(p)|\mathbb{W}_\delta(z_2)|\phi_3(p)\rangle$ is fixed by Lorentz symmetry, homogeneity in z_2, and dimensional analysis to be

$$\langle\phi_4(p)|\mathbb{W}_\delta(z_2)|\phi_3(p)\rangle = s_{34\delta}(-2p \cdot z_2)^{-\delta}(-p^2)^{\frac{\Delta_3+\Delta_4+\delta-2-d}{2}}. \tag{3.16}$$

Thus, the event shape can be written as

$$\langle\phi_4(p)|\mathbf{L}[\phi_1](\infty, z_1)\mathbf{L}[\phi_2](\infty, z_2)|\phi_3(p)\rangle = \sum_\delta r_{12\delta} s_{34\delta}(-p^2)^{\frac{\Delta_3+\Delta_4+\delta-2-d}{2}} C_\delta(z_1, z_2, \partial_{z_2})(-2p \cdot z_2)^{-\delta}. \tag{3.17}$$

The object $C_\delta(z_1, z_2, \partial_{z_2})(-2p \cdot z_2)^{-\delta}$ is called a "celestial block" and it is completely fixed by $\mathrm{SO}(d-1, 1)$ symmetry. It can be computed by solving a Casimir differential equation, similar to the method used by Dolan and Osborn to compute conventional conformal blocks [82]. The result is [15]

$$C_\delta(z_1, z_2, \partial_{z_2})(-2p\cdot z_2)^{-\delta} = \frac{(-p^2)^{\frac{\delta_1+\delta_2-\delta}{2}}}{(-2p\cdot z_1)^{\delta_1}(-2p\cdot z_2)^{\delta_2}} f_\delta^{\delta_1,\delta_2}(\zeta), \tag{3.18}$$

where ζ is a Lorentz-invariant cross ratio

$$\zeta = \frac{(-p^2)(-2z_1\cdot z_2)}{(-2p\cdot z_1)(-2p\cdot z_2)}, \tag{3.19}$$

and the function $f_\delta^{\delta_1,\delta_2}(\zeta)$ is given by

$$f_\delta^{\delta_1,\delta_2}(\zeta) = \zeta^{\frac{\delta-\delta_1-\delta_2}{2}}\,{}_2F_1\left(\tfrac{\delta+\delta_1-\delta_2}{2}, \tfrac{\delta+\delta_2-\delta_1}{2}, \delta+2-\tfrac{d}{2}, \zeta\right). \tag{3.20}$$

Combining everything, we obtain a celestial block expansion for the two-point event shape

$$\begin{aligned}&\langle\phi_4(p)|\mathbf{L}[\phi_1](\infty, z_1)\mathbf{L}[\phi_2](\infty, z_2)|\phi_3(p)\rangle\\ &= \frac{(-p^2)^{\frac{\delta_1+\delta_2+\delta_3+\delta_4-d}{2}}}{(-2p\cdot z_1)^{\delta_1}(-2p\cdot z_2)^{\delta_2}}\sum_\delta r_{12\delta}s_{34\delta}f_\delta^{\delta_1,\delta_2}(\zeta).\end{aligned} \tag{3.21}$$

Note that the form of the celestial block expansion (3.21) is completely dictated by symmetries. The light-ray OPE formula then predicts that the δ's appearing in the expansion (3.21) should be related to dimensions of light-ray operators in the CFT. Furthermore, it makes a prediction for the product of coefficients $r_{12\delta}s_{34\delta}$:

$$r_{12\delta}s_{34\delta} = \frac{2^{d+2-\delta_3-\delta_4}\pi^{\frac{d}{2}+3}e^{i\pi\frac{\delta_4-\delta_3}{2}}\Gamma(\delta-1)}{\Gamma(\frac{\delta+\delta_3-\delta_4}{2})\Gamma(\frac{\delta-\delta_3+\delta_4}{2})\Gamma(\frac{\delta_3+\delta_4-\delta}{2})\Gamma(\frac{\delta+\delta_3+\delta_4+2-d}{2})}\left(p^+_{\delta+1,J=-1}+p^-_{\delta+1,J=-1}\right), \tag{3.22}$$

where $p^+_{\Delta,J}(p^-_{\Delta,J})$ is the product of OPE coefficients of the four-point function $\langle\phi_4\phi_1\phi_2\phi_3\rangle$, analytically continued from even(odd) spin.

Even if we do not know the coefficients $r_{12\delta}$ and $s_{34\delta}$, we can still make some statements about the event shape. We will be particularly interested in the limit where all the detectors are close to each other. For the two-point case, this simply corresponds to $z_1 \to z_2$, or $\zeta \to 0$, and the event shape should behave as

$$\begin{aligned}&\langle\phi_4(p)|\mathbf{L}[\phi_1](\infty, z_1)\mathbf{L}[\phi_2](\infty, z_2)|\phi_3(p)\rangle\\ &= r_{12\delta_*}s_{34\delta_*}\frac{(-p^2)^{\frac{\delta_1+\delta_2+\delta_3+\delta_4-d}{2}}\zeta^{\frac{\delta_*-\delta_1-\delta_2}{2}}}{(-2p\cdot z_1)^{\delta_1}(-2p\cdot z_2)^{\delta_2}}+\cdots\\ &= r_{12\delta_*}s_{34\delta_*}(-p^2)^{\frac{\delta_*+\delta_3+\delta_4-d}{2}}\langle\mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta_*}(p)\rangle+\cdots,\end{aligned} \tag{3.23}$$

where δ_* is the smallest δ appearing in the $\sum_\delta$ sum.

Three-point event shapes

Next, consider a three-point event shape

$$\langle\phi_5(p)|\mathbf{L}[\phi_1](\infty,z_1)\mathbf{L}[\phi_2](\infty,z_2)\mathbf{L}[\phi_3](\infty,z_3)|\phi_4(p)\rangle. \tag{3.24}$$

We can use the light-ray OPE to decompose the product of a pair of detectors, say $\mathbf{L}[\phi_1]\mathbf{L}[\phi_2]$, as a sum of light-ray operators $\mathbb{O}_i$. However, we do not currently possess a more general light-ray OPE formula that lets us further decompose the product $\mathbb{O}_i\mathbf{L}[\phi_3]$. To make progress, let us use Lorentz symmetry to predict the form that a celestial block expansion for the three-point event shape should have.

Following the analysis in section 3.2.2, we treat $\mathbf{L}[\phi_1](\infty,z_1)\mathbf{L}[\phi_2](\infty,z_2)\mathbf{L}[\phi_3](\infty,z_3)$ as a product of three scalar primary operators $\mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta_3}(z_3)$ in a fictitious CFT_{d-2}. Formally taking consecutive OPEs, we have[2]

$$\begin{aligned}
&\mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta_3}(z_3)\\
&=\sum_{\delta,j} r_{12\mathcal{P}_{\delta,j}}C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})\mathcal{P}_{\delta,j}(z_2,w_2)\mathcal{P}_{\delta_3}(z_3)\\
&=\sum_{\delta',\lambda'}\sum_{\delta,j} r_{12\mathcal{P}_{\delta,j}}r'_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta',\lambda'}}C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta',\lambda'}}(z_2,z_3,\partial_{z_3},\partial_{\vec{w}_3})\mathcal{P}_{\delta',\lambda'}(z_3,\vec{w}_3).
\end{aligned} \tag{3.25}$$

Therefore, the three-point event shape should have the form

$$\begin{aligned}
&\langle\phi_5(p)|\mathbf{L}[\phi_1](\infty,z_1)\mathbf{L}[\phi_2](\infty,z_2)\mathbf{L}[\phi_3](\infty,z_3)|\phi_4(p)\rangle\\
&=\sum_{\delta'}\sum_{\delta,j} r_{12\mathcal{P}_{\delta,j}}r'_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}(z_2,z_3,\partial_{z_3})\langle\phi_5(p)|\mathbb{W}'_{\delta'}(z_3)|\phi_4(p)\rangle\\
&=\sum_{\delta'}\sum_{\delta,j} r_{12\mathcal{P}_{\delta,j}}r'_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}s'_{45\delta'}(-p^2)^{\frac{\delta_4+\delta_5+\delta'-1-d}{2}}\\
&\qquad\times C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}(z_2,z_3,\partial_{z_3})(-2z_3\cdot p)^{-\delta'},
\end{aligned} \tag{3.26}$$

where from the second to the third line, we again use the homogeneity of z_3 and p. In the expansion (3.26), we have three unknown coefficients $r_{12\mathcal{P}_{\delta,j}}$, $r'_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}$, and $s'_{45\delta'}$.

If an OPE expansion for a three-point event shape exists, symmetries ensure it must take the form (3.26). However, we do not know an argument guaranteeing the

[2] Here, we are following the notation of [15, 77], where $\vec{w}_3$ denotes a collection of polarization vectors for different rows of the Young diagram of an $\mathrm{SO}(d-2)$ representation. However, after taking the expectation value in a scalar density matrix $\langle\phi_5(p)|\cdot|\phi_4(p)\rangle$, only scalar representations λ' are allowed, so $\vec{w}_3$ immediately drops out and can be ignored.

existence of such an expansion. Mathematically, the only thing that is guaranteed is that (3.24) can be decomposed into a double-integral over complex δ and δ' of "celestial partial waves," defined below in equation (3.188). An expansion like (3.26) would arise if we can additionally close the contours to the right, picking up a set of discrete poles, as described in [47] for the conventional conformal block decomposition. In this work, we assume that such a contour maneuver is possible, at least to characterize the leading behavior of the event shape in the collinear limit. In the absence of nonperturbative arguments, it is also important to compare (3.26) to perturbative data, as we do in section 3.3.

The kinematic dependence of (3.26) is accounted for by the object

$$C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}(z_2,z_3,\partial_{z_3})(-2z_3\cdot p)^{-\delta'}, \tag{3.27}$$

which is completely fixed by Lorentz symmetry. We call (3.27) a three-point celestial block. Although we do not know a compact closed-form expression for it, we can still determine its expansion around the collinear limit, where z_1, z_2, z_3 are close to each other. The reason is that the collinear limit is equivalent (up to a Lorentz transformation) to a configuration where p is null. Writing p as z_0 in this null limit, the leading term of the three-point celestial block in the collinear limit becomes

$$\begin{aligned}&C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}(z_2,z_3,\partial_{z_3})(-2z_3\cdot p)^{-\delta'}\\ \to\ &C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}(z_2,z_3,\partial_{z_3})(-2z_3\cdot z_0)^{-\delta'} = g_{\delta,j}^{(\delta_1,\delta_2,\delta_3,\delta')}(z_1,z_2,z_3,z_0),\end{aligned} \tag{3.28}$$

where the second line is simply the definition of a four-point conformal block.

To characterize subleading terms in the expansion around the collinear limit (or equivalently the null p limit), it is helpful to introduce $|\mathcal{P}_{\delta'},p\rangle\rangle$, defined as the state in the conformal multiplet of $|\mathcal{P}_{\delta'}\rangle$ that is invariant under the little group $\mathrm{SO}(d-1)$ that fixes p. From the point of view of conformal symmetry, p is a point in the center of EAdS_{d-1}, so the overlap of $|\mathcal{P}_{\delta'},p\rangle\rangle$ with $|\mathcal{P}_{\delta'}(z_3)\rangle$ is a bulk-to-boundary propagator:

$$\langle\mathcal{P}_{\delta'}(z_3)|\mathcal{P}_{\delta'},p\rangle\rangle = (-2z_3\cdot p)^{-\delta'}. \tag{3.29}$$

The three-point celestial block can be written as

$$C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}(z_2,z_3,\partial_{z_3})\langle\mathcal{P}_{\delta'}(z_3)|\mathcal{P}_{\delta'},p\rangle\rangle. \tag{3.30}$$

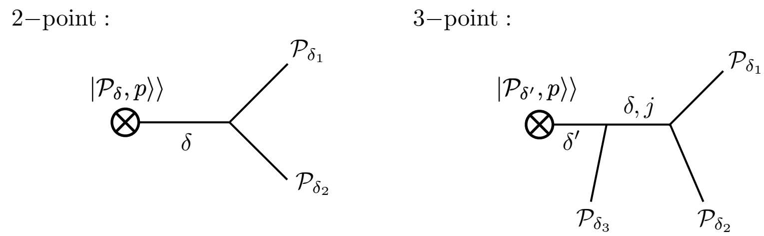

Figure 3.1: Diagrams representing the two-point and three-point celestial blocks. Each vertex should be understood as an OPE differential operator C_{ijk} in the CFT_{d-2}. The symbol $\otimes$ represents the factor (3.29).

This expression can be represented by the diagram in figure 3.1, where we also include the diagram for the two-point case. Theses diagrams can be thought of as the OPE decomposition of the usual three-point and four-point function in the fictitious CFT_{d-2}, but with one of the external operators replaced by $|\mathcal{P}_{\delta'}, p\rangle\rangle$.

Let us determine a more explicit expression for $|\mathcal{P}_{\delta'}, p\rangle\rangle$. Treating $\mathrm{SO}(d-1,1)$ as the $(d-2)$-dimensional conformal group, its generators are $\{D, M_{ab}, P_a, K_a\}$, where $a, b = 1, \cdots, d-2$, M_{ab} are $\mathrm{SO}(d-2)$ rotation generators, and P_a and K_a are the translation and special conformal generators in $\mathbb{R}^{d-2}$. For $p = (p^0, \vec{p}) = (1, \vec{0})$, the little group that fixes p is generated by M_{ab} and $P_a + K_a$. Therefore, the $|\mathcal{P}_{\delta'}, p\rangle\rangle$ must satisfy the conditions

$$\begin{aligned} M_{ab}|\mathcal{P}_{\delta'}, (1,\vec{0})\rangle\rangle &= 0, \\ (P_a + K_a)|\mathcal{P}_{\delta'}, (1,\vec{0})\rangle\rangle &= 0. \end{aligned} \tag{3.31}$$

The solution to these conditions was obtained in [83] with a somewhat different motivation (studying local probes of a dual AdS geometry). The result is

$$|\mathcal{P}_{\delta'}, (1,\vec{0})\rangle\rangle = \Gamma(\delta' + 2 - \tfrac{d}{2}) \left(\frac{\sqrt{P^2}}{2}\right)^{\frac{d-2}{2} - \delta'} J_{\delta' - \frac{d-2}{2}}\left(\sqrt{P^2}\right)|\mathcal{P}_{\delta'}\rangle, \tag{3.32}$$

where $|\mathcal{P}_{\delta'}\rangle$ is the CFT primary state killed by K_a, and $J_\nu(x)$ is a Bessel function of the first kind.

The action of momentum generators P_a on $|\mathcal{P}_{\delta'}\rangle$ can be written as derivatives of $|\mathcal{P}_{\delta'}(z)\rangle$ with respect to z. Equation (3.32) thus expresses $|\mathcal{P}_{\delta'}, (1,\vec{0})\rangle\rangle$ as an infinite-order differential operator acting on $|\mathcal{P}_{\delta'}(z)\rangle$. Plugging this into (3.30), we obtain the three-point celestial block as an infinite-order differential operator acting on a four-point conformal block:

$$\begin{aligned}
&C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}(z_2,z_3,\partial_{z_3})(-2z_3\cdot p)^{-\delta'} \\
&= \Gamma(\delta'+2-\tfrac{d}{2})\left(\frac{\sqrt{\partial_{\vec{y}}^2}}{2}\right)^{\frac{d-2}{2}-\delta'} J_{\delta'-\frac{d-2}{2}}\left(\sqrt{\partial_{\vec{y}}^2}\right) g_{\delta,j}^{(\delta_1,\delta_2,\delta_3,\delta')}(z_1,z_2,z_3,z_0), \qquad (3.33)
\end{aligned}$$

where $z_0 = (1, \vec{y}^2, \vec{y})$. Note that only even powers of $\sqrt{\partial_{\vec{y}}^2}$ appear in the expansion of the Bessel function, so the above differential operator is well-defined order-by-order in this expansion. Even though intermediate terms depend on the point z_0, the final result must be independent of z_0. In appendix B.1, we present an alternative derivation of this identity that leads to an expression where Lorentz invariance is more manifest.

Note that the state $|\mathcal{P}_{\delta'}, p\rangle\rangle$ breaks the Lorentz group $\mathrm{SO}(d-1,1)$ to the little group $\mathrm{SO}(d-1)$. This is the same pattern of symmetry breaking that occurs in the presence of a codimension-1 spherical boundary or defect [84]. Thus, celestial blocks are equivalent to boundary/defect conformal blocks [85]. Often, boundaries and defects are studied in Euclidean space, where the symmetry breaking pattern in CFT_{d-2} is $\mathrm{SO}(d) \to \mathrm{SO}(d-1)$. Since the signature of the corresponding orthogonal groups is different, our celestial blocks are related by analytic continuation to those blocks.

Expanding our expression to leading and subleading in the collinear limit $p^2 \to 0$, we find

$$\begin{aligned}
&C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}(z_2,z_3,\partial_{z_3})(-2z_3\cdot p)^{-\delta'} \\
&= T_{123\delta'}(z_1,z_2,z_3,p)\left(g_{\delta,j}^{(123\delta')}(u,v) + \frac{\zeta_{13}}{d-4-2\delta'}\mathcal{D}_{u,v}^{(1)} g_{\delta,j}^{(123\delta')}(u,v) + O((-p^2)^2)\right),
\end{aligned} \qquad (3.34)$$

where u and v are defined as

$$u = \frac{(-2z_1\cdot z_2)(-2z_3\cdot p)}{(-2z_1\cdot z_3)(-2z_2\cdot p)}, \quad v = \frac{(-2z_2\cdot z_3)(-2z_1\cdot p)}{(-2z_1\cdot z_3)(-2z_2\cdot p)}, \qquad (3.35)$$

and the overall factor $T_{123\delta'}(z_1,z_2,z_3,p)$ is

$$T_{123\delta'}(z_1,z_2,z_3,p) = \frac{\left(\frac{-2z_2\cdot p}{-2z_1\cdot p}\right)^{\frac{\delta_1-\delta_2}{2}}\left(\frac{-2z_1\cdot p}{-2z_1\cdot z_3}\right)^{\frac{\delta_3-\delta'}{2}}}{(-2z_1\cdot z_2)^{\frac{\delta_1+\delta_2}{2}}(-2z_3\cdot p)^{\frac{\delta_3+\delta'}{2}}}. \qquad (3.36)$$

The differential operator $\mathcal{D}^{(1)}_{u,v}$ generating the first subleading term is given by

$$\begin{aligned}\mathcal{D}^{(1)}_{u,v} =& \frac{1}{2}(\delta_1-\delta_2)(\delta_1-\delta_2-\delta_3+\delta')u + \frac{1}{2}(\delta_3+\delta')((\delta_1-\delta_2)v-\delta_1+\delta_2+\delta_3-\delta') \\ &+((2-2\delta_1+2\delta_2+\delta_3-\delta')u-(\delta_3+\delta')(v-1))v\partial_v \\ &+((2-\delta_1+\delta_2-\delta_3-\delta')v-(\delta_1-\delta_2-\delta_3+\delta')(u-1))u\partial_u \\ &+2uv(u\partial_u^2+v\partial_v^2+(u+v-1)\partial_u\partial_v). \end{aligned} \tag{3.37}$$

As an example, let us specialize to a three-point energy correlator in a 4d CFT. In this case, we have $\delta_1=\delta_2=\delta_3=3$, and (3.37) becomes

$$\begin{aligned}\mathcal{D}^{(1)\mathcal{E}\mathcal{E}\mathcal{E}}_{u,v} =& \frac{(3-\delta')(3+\delta')}{2} + ((5-\delta')u-(3+\delta')(v-1))v\partial_v - ((-3+\delta')(u-1)+(1+\delta')v)u\partial_u \\ &+2uv(u\partial_u^2+v\partial_v^2+(u+v-1)\partial_u\partial_v). \end{aligned} \tag{3.38}$$

Using (3.34), we can finally write down the expansion of the three-point event shape in the collinear limit:

$$\begin{aligned}&\langle\phi_5(p)|\mathbf{L}[\phi_1](\infty,z_1)\mathbf{L}[\phi_2](\infty,z_2)\mathbf{L}[\phi_3](\infty,z_3)|\phi_4(p)\rangle \\ &= \sum_{\delta'}\sum_{\delta,j} r_{12\mathcal{P}_{\delta,j}} r'_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}} s'_{45\delta'}(-p^2)^{\frac{\delta_4+\delta_5+\delta'-1-d}{2}} T_{123\delta'}(z_1,z_2,z_3,p) \\ &\qquad \times\left(g^{(123\delta')}_{\delta,j}(u,v)+\frac{\zeta_{13}}{d-4-2\delta'}\mathcal{D}^{(1)}_{u,v} g^{(123\delta')}_{\delta,j}(u,v)+O((-p^2)^2)\right). \end{aligned} \tag{3.39}$$

In sections 3.3 and 3.4 of this paper, we will mostly focus on the leading term, which is simply a four-point conformal block. In section 3.6, when we study $\mathcal{N}=4$ at strong coupling, we will derive results related to the full structure of the three-point celestial block.

We can also take the OPE of (3.24) in a different order. If we first take the 23 OPE, we obtain

$$\begin{aligned}&\langle\phi_5(p)|\mathbf{L}[\phi_1](\infty,z_1)\mathbf{L}[\phi_2](\infty,z_2)\mathbf{L}[\phi_3](\infty,z_3)|\phi_4(p)\rangle \\ &= \sum_{\delta'}\sum_{\delta,j} r_{23\mathcal{P}_{\delta,j}} r'_{\mathcal{P}_{\delta,j}1\mathcal{P}_{\delta'}} s'_{45\delta'}(-p^2)^{\frac{\delta_4+\delta_5+\delta'-1-d}{2}} T_{321\delta'}(z_3,z_2,z_1,p) \\ &\qquad \times\left(g^{(321\delta')}_{\delta,j}(v,u)+\frac{\zeta_{13}}{d-4-2\delta'}\mathcal{D}^{(1)}_{v,u} g^{(321\delta')}_{\delta,j}(v,u)+O((-p^2)^2)\right). \end{aligned} \tag{3.40}$$

Thus, we obtain a crossing equation

$$\begin{aligned}
&\sum_{\delta'}\sum_{\delta,j} r_{12\mathcal{P}_{\delta,j}} r'_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}} s'_{45\delta'} (-p^2)^{\frac{\delta_4+\delta_5+\delta'-1-d}{2}} T_{123\delta'}(z_1,z_2,z_3,p) \\
&\qquad\qquad \times \left(g^{(123\delta')}_{\delta,j}(u,v) + \frac{\zeta_{13}}{d-4-2\delta'} \mathcal{D}^{(1)}_{u,v} g^{(123\delta')}_{\delta,j}(u,v) + O((-p^2)^2) \right) \\
&= \sum_{\delta'}\sum_{\delta,j} r_{23\mathcal{P}_{\delta,j}} r'_{\mathcal{P}_{\delta,j}1\mathcal{P}_{\delta'}} s'_{45\delta'} (-p^2)^{\frac{\delta_4+\delta_5+\delta'-1-d}{2}} T_{321\delta'}(z_3,z_2,z_1,p) \\
&\qquad\qquad \times \left(g^{(321\delta')}_{\delta,j}(v,u) + \frac{\zeta_{13}}{d-4-2\delta'} \mathcal{D}^{(1)}_{v,u} g^{(321\delta')}_{\delta,j}(v,u) + O((-p^2)^2) \right).
\end{aligned} \tag{3.41}$$

The leading term of this equation looks like a usual four-point crossing equation in a $(d-2)$-dimensional CFT. We will study some of its implications in section 3.2.4 and appendix B.4. It would also be interesting to study (3.41) with subleading terms included.

3.2.3 Expansion of the EEEC in the collinear limit

We are now ready to study the expansion of the three-point energy correlator in the collinear limit. In what follows, we only keep the leading term (the conformal block). We also specialize to four dimensions, for simplicity. We can essentially follow section 3.2.2, replacing the scalar operators ϕ_1, ϕ_2, ϕ_3 with the stress-tensor T. Note that the ANEC operator $\mathcal{E}$ is still a scalar on the celestial sphere, so the only difference from our earlier analysis is the homogeneity in the momentum p. For sink/source states created by a scalar operator O, the result is[3]

$$\begin{aligned}
&\langle O(p)|\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{E}(\vec{n}_3)|O(p)\rangle \\
&= \sum_{\delta,j} r_{\mathcal{E}\mathcal{E}\mathcal{P}_{\delta,j}} r'_{\mathcal{P}_{\delta,j}\mathcal{E}\mathcal{P}_{\delta'_*}} s'_{OO\mathcal{P}_{\delta'_*}} \frac{(-p^2)^{\frac{2\Delta_O+\delta'_*-1}{2}} \left(\frac{-2z_2\cdot p}{-2z_1\cdot z_2}\right)^{\frac{3-\delta'_*}{2}}}{(-2z_2\cdot z_3)^3(-2z_1\cdot p)^{\frac{3+\delta'_*}{2}}} g^{(\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{P}_{\delta'_*})}_{\delta,j}(z,\bar{z}) + \dots,
\end{aligned} \tag{3.42}$$

where "…" denotes subleading terms in the collinear limit, and δ'_* is the smallest value of δ' that appears in the $\mathcal{P}_{\delta,j}\times\mathcal{E}$ OPE in the fictitious CFT$_2$. We assume for now that δ'_* is isolated.[4] We have also changed variables from u, v to $z, \bar{z}$, defined

[3] For later convenience, we relabel the points as $1\to 2, 2\to 3, 3\to 1$.

[4] We study a case where two operators have degenerate δ'_* at the lowest order in perturbation theory in section 3.4.2.

by

$$u = z\bar{z} = \frac{\zeta_{23}}{\zeta_{12}}, \quad v = (1-z)(1-\bar{z}) = \frac{\zeta_{13}}{\zeta_{12}}, \tag{3.43}$$

where $z, \bar{z}$ are complex conjugates of each other.[5]

Recall that the EEEC is defined by (3.3). Note that $\langle \mathcal{O}(p)|\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{E}(\vec{n}_3)|\mathcal{O}(p)\rangle$ depends only on angles between $\vec{n}_1, \vec{n}_2, \vec{n}_3$, which are localized by the delta functions in the first line of (3.3). The remaining Jacobian factor is

$$\begin{aligned} &\int d\Omega_{\vec{n}_1} d\Omega_{\vec{n}_2} d\Omega_{\vec{n}_3} \delta(\zeta_{12} - \tfrac{1-\vec{n}_1\cdot\vec{n}_2}{2})\delta(\zeta_{13} - \tfrac{1-\vec{n}_1\cdot\vec{n}_3}{2})\delta(\zeta_{23} - \tfrac{1-\vec{n}_2\cdot\vec{n}_3}{2}) \\ &\to \frac{64\pi^2}{|(\vec{n}_1 \times \vec{n}_2)\cdot\vec{n}_3|} \\ &= \frac{32\pi^2}{\sqrt{-\zeta_{12}^2 - \zeta_{13}^2 - \zeta_{23}^2 + 2\zeta_{12}\zeta_{13} + 2\zeta_{12}\zeta_{23} + 2\zeta_{13}\zeta_{23} - 4\zeta_{12}\zeta_{13}\zeta_{23}}}. \end{aligned} \tag{3.44}$$

Also, the total cross section is given by

$$\sigma^{\mathcal{O}}_{\text{total}} \equiv \int d^4x\, e^{ip\cdot x}\langle 0|\mathcal{O}^\dagger(x)\mathcal{O}(0)|0\rangle = \frac{2^{5-2\Delta_\mathcal{O}}\pi^3}{\Gamma(\Delta_\mathcal{O}-1)\Gamma(\Delta_\mathcal{O})}(-p^2)^{\frac{2\Delta_\mathcal{O}-4}{2}}\theta(p). \tag{3.45}$$

Combining these results with (3.42), the EEEC in the collinear limit is given by

$$\text{EEEC}(\zeta_{12},\zeta_{13},\zeta_{23}) = \frac{\zeta_{12}}{\zeta_{23}^3\sqrt{-\zeta_{12}^2 - \zeta_{13}^2 - \zeta_{23}^2 + 2\zeta_{12}\zeta_{13} + 2\zeta_{12}\zeta_{23} + 2\zeta_{13}\zeta_{23}}}\mathcal{G}(\zeta_{12}, z, \bar{z}) + \dots, \tag{3.46}$$

where

$$\begin{aligned} \mathcal{G}(\zeta_{12}, z, \bar{z}) &\equiv \zeta_{12}^{\frac{\delta'_*-5}{2}} \sum_{\delta,j} R_{\delta,j;\delta'_*} g_{\delta,j}^{(\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{P}_{\delta'_*})}(z,\bar{z}), \\ R_{\delta,j;\delta'_*} &\equiv \frac{2^{2\Delta_\mathcal{O}-9}\Gamma(\Delta_\mathcal{O}-1)\Gamma(\Delta_\mathcal{O})}{\pi} r_{\mathcal{E}\mathcal{E}\mathcal{P}_{\delta,j}} r'_{\mathcal{P}_{\delta,j}\mathcal{E}\mathcal{P}_{\delta'_*}} s'_{\mathcal{O}\mathcal{O}\mathcal{P}_{\delta'_*}}. \end{aligned} \tag{3.47}$$

Here, we have set $z_i = (1, \vec{n}_i)$ and $\zeta_{ij} = \frac{1-\vec{n}_i\cdot\vec{n}_j}{2}$.

Thus, the function $\mathcal{G}(\zeta_{12}, z, \bar{z})$ describing the leading behavior of the EEEC in the collinear limit can be written as a sum of conformal blocks, up to a power of ζ_{12}.

[5] Note that earlier we used z as a null polarization vector $z \in \mathbb{R}^{d-1,1}$, whereas here it is a complex number $z \in \mathbb{C}$. We hope that no confusion will arise from this overloaded notation.

The coefficients $R_{\delta,j;\delta'_*}$ appearing in the expansion are products of light-ray OPE coefficients $r_{\mathcal{E}\mathcal{E}\mathcal{P}_{\delta,j}}$ and $r'_{\mathcal{P}_{\delta,j}\mathcal{E}\mathcal{P}_{\delta'_*}}$ and 1-point functions $s'_{\mathcal{O}\mathcal{O}\mathcal{P}_{\delta'_*}}$. Since most of these quantities are unknown a-priori, we will mostly just work with the coefficients $R_{\delta,j;\delta'_*}$ in this paper. The detailed definition of $R_{\delta,j;\delta'_*}$ would become useful if one could derive a three-point light-ray OPE formula that relates $r_{\mathcal{E}\mathcal{E}\mathcal{P}_{\delta,j}} r'_{\mathcal{P}_{\delta,j}\mathcal{E}\mathcal{P}_{\delta'_*}} s'_{\mathcal{O}\mathcal{O}\mathcal{P}_{\delta'_*}}$ to the OPE data of the CFT (similar to (3.22)).

Although most of the coefficients in the expansion (3.47) are a-priori unknown, we *do know* a lot about the quantum numbers δ'_* and δ that appear. The dimension δ'_* is associated with the lowest dimension light-ray operator in the triple-$\mathcal{E}$ OPE. It is natural to guess that it is given by the lowest-twist spin-4 operator in the theory: $\delta'_* = \Delta_{\min}(J = 4)$, as we discuss in section 3.3.1. Furthermore, the dimensions δ that appear in the conformal block expansion (3.47) are controlled by the two-$\mathcal{E}$ light-ray OPE, which we understand much better — they are associated to light-ray operators with spin $J = 3, 5, \ldots$ [15, 77]. Below, we will confirm these expectations in examples.

3.2.4 Lightcone bootstrap constraints

The leading term of the crossing equation (3.41) can be written as[6]

$$\begin{aligned}\mathcal{G}(\zeta_{12}, z, \bar{z}) &= \zeta_{12}^{\frac{\delta'_*-5}{2}} \sum_{\delta,j} R_{\delta,j;\delta'_*} g_{\delta,j}^{(\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{P}_{\delta'_*})}(z,\bar{z}) \\ &= \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^3 \zeta_{12}^{\frac{\delta'_*-5}{2}} \sum_{\delta,j} R_{\delta,j;\delta'_*} g_{\delta,j}^{0,\frac{3-\delta'_*}{2}}(1-z, 1-\bar{z}). \end{aligned} \tag{3.48}$$

This looks like a four-point crossing equation in a $(d-2)$-dimensional CFT. It is thus interesting to ask what we can deduce about the original d-dimensional CFT from it. Unfortunately, the coefficients $R_{\delta,j;\delta'_*}$ do not satisfy any simple positivity conditions, so numerical bootstrap methods do not apply in an obvious way. In this section, we will instead study (3.48) from the point of view of the lightcone bootstrap [7, 8, 81], which does not require positivity conditions.

The usual lightcone bootstrap analysis begins by analytically continuing the four-point function into Lorentzian signature, and then considering the double lightcone limit $z \ll 1 - \bar{z} \ll 1$. In our setting, this would require analytically continuing

[6] Here, we use two different notations for conformal blocks, and we hope the meaning hereafter will be clear from context. The first notation is $g_{\delta,j}^{(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4)}$, where the block is labeled by the conformal multiplets of each individual external operator $\mathcal{O}_1, \cdots, \mathcal{O}_4$. The second notation is $g_{\delta,j}^{\delta_{12},\delta_{34}}$, where we use the fact that the block depends only on the differences of scaling dimensions $\delta_{12} = \delta_1 - \delta_2$ and $\delta_{34} = \delta_3 - \delta_4$.

celestial cross-ratios away from the Euclidean regime. However, our "correlator" $\mathcal{G}(\zeta_{12}, z, \bar{z})$ does not come from a local, reflection-positive Euclidean CFT$_{d-2}$, and thus it is not guaranteed that we can analytically continue it to $(d-2)$-dimensional Lorentzian signature (which would be $(2, d-2)$ signature from the point of view of the full d-dimensional theory).

However, we believe that the main conclusion of the analytic bootstrap, i.e. the existence of double-twist families at large spin, can still be obtained by staying in Euclidean signature. The idea is that by plugging the leading t-channel singularity into the Euclidean inversion formula, one can still deduce that the OPE coefficient density $C(\Delta, J)$ should behave as the lightcone bootstrap predicts at large spin J. Similarly, it should be possible to compute subleading corrections in J from subleading terms in the t-channel singularity. Hence, we expect that analytic continuation in $z, \bar{z}$ can be thought of as a proxy for a more complicated analysis using Euclidean partial waves.

Thus, let us proceed to studying implications of the lightcone bootstrap for the celestial crossing equation (3.48). We will actually consider a more general three-point event shape $\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]$, where $\mathcal{O}$ is an operator with spin J. The crossing equation reads

$$\sum_{\delta,j} R_{\delta,j,\delta'_*} g^{(0,s)}_{\delta,j}(z,\bar{z}) = \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\delta_{\mathcal{O}}} \sum_{\delta,j} R_{\delta,j,\delta'_*} g^{(0,s)}_{\delta,j}(1-z,1-\bar{z}), \quad (3.49)$$

where $s = \frac{\delta_{\mathcal{O}}-\delta'_*}{2}$. As we discuss later in section 3.3.1, we expect that $\delta'_* = \Delta'_* - 1$ is the celestial scaling dimension of the lowest-twist operator with spin $J' = 3J-2$ in the $\mathcal{O}\times\mathcal{O}\times\mathcal{O}$ OPE. Suppose the lowest celestial sphere twist $\tau_c = \delta - j$ appearing in the sum on the left-hand side is $\tau_c^* = 2h^*$. Then in the double lightcone limit $z \ll 1-\bar{z} \ll 1$, we have

$$R^*_{h^*} z^{h^*-2h_{\mathcal{O}}} k^{0,s}_{2\bar{h}^*}(\bar{z}) + \ldots = \sum_{h,\bar{h}} R_{h,\bar{h}} (1-\bar{z})^{h-2h_{\mathcal{O}}} k^{0,s}_{2\bar{h}}(1-z) + \ldots, \quad (3.50)$$

where we have introduced

$$h = \frac{\delta-j}{2} = \frac{\tau_c}{2}, \qquad \bar{h} = \frac{\delta+j}{2}. \quad (3.51)$$

Near the $\bar{z} \to 1$ limit, the $\mathrm{SL}(2,\mathbb{R})$ block $k^{0,s}_{2\bar{h}}(\bar{z})$ has the expansion [81]

$$\begin{aligned} k^{0,s}_{2\bar{h}}(\bar{z}) &= K^{0,s}_0(\bar{h}) + \ldots + K^{s,0}_0(\bar{h})(1-\bar{z})^{-s} + \ldots, \\ K^{r,s}_0(\bar{h}) &\equiv \frac{\Gamma(r-s)\Gamma(2\bar{h})}{\Gamma(\bar{h}+r)\Gamma(\bar{h}-s)}, \end{aligned} \quad (3.52)$$

where "..." represents terms like $(1-\bar{z})^k$ or $(1-\bar{z})^{k-s}$ where k is a positive integer. Therefore, (3.50) becomes

$$\begin{aligned} &R^*_{h^*} z^{h^*-2h_O}\left(K^{0,s}_0(\bar{h}) + \cdots + K^{s,0}_0(\bar{h})(1-\bar{z})^{-s} + \dots\right) + \dots \\ &= \sum_{h,\bar{h}} R_{h,\bar{h}}(1-\bar{z})^{h-2h_O} k^{0,s}_{2\bar{h}}(1-z) + \dots . \end{aligned} \tag{3.53}$$

Note that on the left-hand side, $z^{h^*-2h_O}$ is Casimir-singular in z (i.e. it can be made arbitrarily singular by repeatedly applying the quadratic Casimir), while on the right-hand side $k^{0,s}_{2\bar{h}}(1-z)$ is Casimir-regular in z. In order to reproduce the Casimir-singular term with the correct $\bar{z}$ behavior on the left hand side, we must have an infinite family of operators with $h \to 2h_O$ and $h \to 2h_O - s = h_O + h'_*$ as $\bar{h} \to \infty$. They can be thought of as the "celestial double-twist operators" $[\mathcal{P}_O\mathcal{P}_O]$ and $[\mathcal{P}_O\mathcal{P}_{O'_*}]$.

Since $[\mathcal{P}_O\mathcal{P}_O]$ has $h = 2h_O$ and large-j (which means that they must be higher transverse spin terms), the light-ray OPE [15, 77] predicts that they should come from operators with conventional twist $\tau = 2\delta_O + 2 = 2\Delta_O$. Thus, the existence of $[\mathcal{P}_O\mathcal{P}_O]$ predicts that for the maximally allowed transverse spin ($j_{max} = 2J$ in this case) in the $O \times O$ OPE, there must be a trajectory with $\tau = 2\tau_O + 2J$ at large spin. Similarly, the existence of $[\mathcal{P}_O\mathcal{P}_{O'_*}]$ predicts that there should be a trajectory with the maximally allowed transverse spin that has twist $\tau = \Delta_O + \Delta'_*$ at large spin.

The coefficients $R_{h,\bar{h}}$ at large $\bar{h}$ for the two types of celestial double-twist operators can also be determined. Using the formula [81]

$$\begin{aligned} &\sum_{\substack{\bar{h}=j+\bar{h}_0 \\ j=0,2,\ldots}} S^{r,s}_a(\bar{h}) k^{r,s}_{2\bar{h}}(1-z) = \frac{1}{2}\left(\frac{z}{1-z}\right)^a + \text{Casimir-regular}, \\ &S^{r,s}_a(\bar{h}) = \frac{1}{\Gamma(-a-r)\Gamma(-a-s)} \frac{\Gamma(\bar{h}-r)\Gamma(\bar{h}-s)}{\Gamma(2\bar{h}-1)} \frac{\Gamma(\bar{h}-a-1)}{\Gamma(\bar{h}+a+1)}, \end{aligned} \tag{3.54}$$

we find

$$\begin{aligned} R_{h=2h_O}(\bar{h}) &\sim 2R^*_{h^*} K^{0,s}_0(\bar{h}^*) S^{0,s}_{h^*-2h_O}(\bar{h}), \\ R_{h=h_O+h'_*}(\bar{h}) &\sim 2R^*_{h^*} K^{s,0}_0(\bar{h}^*) S^{0,s}_{h^*-2h_O}(\bar{h}), \end{aligned} \tag{3.55}$$

where $\sim$ means that both sides have the same leading behavior at large $\bar{h}$.

In (3.50), the left-hand side will have a $\log(1-\bar{z})$ term when $s = 0$. In the usual four-point lightcone bootstrap, the $\log(1-\bar{z})$ term determines the behavior of the anomalous dimensions of double-twist operators. However, unlike in the usual lightcone bootstrap, (3.50) does not possess an identity operator. As a result, the interpretation of $\log(1-\bar{z})$ as coming from anomalous dimensions does not work in this case. To see this, consider the case where $\mathcal{O}$ is a spin-1 conserved current $\mathcal{J}$. In this case, $\mathcal{O}'_*$ should be the lowest twist operator with spin $J'=1$ in the $\mathcal{J}\times\mathcal{J}\times\mathcal{J}$ OPE, which is just $\mathcal{J}$ itself. So, $\delta'_*=\delta_{\mathcal{J}}$ and $s=0$. For $s=0$, the leading term of left hand side of (3.50) becomes

$$-R^*_{h^*} z^{h^*-2h_{\mathcal{O}}} \frac{\Gamma(2\bar{h}^*)}{\Gamma(\bar{h}^*)^2}\left(2\psi(\bar{h}^*)-2\psi(1)+\log(1-\bar{z})\right). \tag{3.56}$$

To see how the above expression can be produced by the right hand side of (3.50), note that near $s=0$, the coefficients $R_{h=2h_{\mathcal{O}}}(\bar{h})$ and $R_{h=h_{\mathcal{O}}+h'_*}(\bar{h})$ in (3.55) are singular and take the form

$$\begin{aligned} R_{h=2h_{\mathcal{O}}}(\bar{h}) &= \frac{\widetilde{R}}{s}+\widetilde{R}^{(0)}+O(s), \\ R_{h=h_{\mathcal{O}}+h'_*}(\bar{h}) &= -\frac{\widetilde{R}}{s}+\widetilde{R}^{(0)}+O(s), \end{aligned} \tag{3.57}$$

where

$$\widetilde{R}=-2R^*_{h^*}\frac{\Gamma(2\bar{h}^*)}{\Gamma(\bar{h}^*)^2}S^{0,s}_{h^*-2h_{\mathcal{O}}}(\bar{h}), \qquad \widetilde{R}^{(0)}=(-\psi(1)+\psi(\bar{h}^*))\widetilde{R}. \tag{3.58}$$

Therefore, the contribution from the two celestial double-twist operators becomes (going back to the notation $g_{\delta,j}$ for conformal blocks)

$$\begin{aligned} &\lim_{s\to 0} R_{h=2h_{\mathcal{O}}}(\bar{h}) g^{0,s}_{4h_{\mathcal{O}}+j,j}+R_{h=h_{\mathcal{O}}+h'_*}(\bar{h}) g^{0,s}_{2h_{\mathcal{O}}-2s+j,j} \\ &=\lim_{s\to 0}\left(\frac{\widetilde{R}}{s}+\widetilde{R}^{(0)}\right)\left(g^{0,0}_{4h_{\mathcal{J}}+j,j}+s\partial_s g^{0,s}_{4h_{\mathcal{J}}+j,j}\right)+\left(-\frac{\widetilde{R}}{s}+\widetilde{R}^{(0)}\right)\left(g^{0,0}_{4h_{\mathcal{J}}+j,j}+s\partial_s g^{0,s}_{4h_{\mathcal{J}}-2s+j,j}\right) \\ &=2\widetilde{R}^{(0)} g^{0,0}_{4h_{\mathcal{J}}+j,j}+2\widetilde{R}\partial_\delta g^{0,0}_{\delta,j}|_{\delta\to 4h_{\mathcal{J}}+j}, \end{aligned} \tag{3.59}$$

which correctly reproduces (3.56) after performing the sum over $\bar{h}$. We see that the $\log(1-\bar{z})$ of (3.56) comes from a near-cancellation of coefficients between the two celestial double-twist families at the degenerate point $s=0$.

3.3 Extracting light-ray OPE data from the leading order collinear EEEC

In the previous section, we argued that the EEEC in the collinear limit can be decomposed into conformal blocks (up to a Jacobian factor). In this section, we study the decomposition of the leading-order collinear EEEC in $\mathcal{N}=4$ SYM and

QCD recently computed in [55]. In the case of $\mathcal{N} = 4$ SYM, the authors of [55] consider sink/source states created by the operator $\mathrm{Tr}F^2$ (with $\Delta_{\mathcal{O}} = 4$). For the QCD case, they consider both the gluon jet, created by $\mathrm{Tr}F^2$, and the quark jet, created by the quark contribution to the electromagnetic current J^μ. Specifically, they contract indices between the bra and the ket, so that the quark jet event shape is an expectation value in the density matrix $|J^\mu(p)\rangle\langle J_\mu(p)|$.

Note that [55] worked at low enough loop order that the β-function does not enter, so QCD can be thought of as conformal for the purposes of studying their results. At higher orders in perturbation theory, selection rules in J, such as those discussed in [12, 15] will be broken. However, the celestial block expansion should still be valid, since it relies on Lorentz invariance alone. We leave an investigation of these effects to the future.

The results of [55] for the leading-order collinear EEEC can be summarized by three functions of cross ratios, $G_{\mathcal{N}=4}(z,\bar{z})$ (equation (5.2) and (5.3) in [55]), $G^g_{\mathrm{QCD}}(z,\bar{z})$ (square bracket in equation (5.14) in [55]), and $G^q_{\mathrm{QCD}}(z,\bar{z})$ (square bracket in equation (5.16) in [55]). The relation between $G(z,\bar{z})$ and the EEEC is

$$\begin{aligned}
\mathrm{EEEC}(\zeta_{12},\zeta_{13},\zeta_{23}) &= \frac{g^4 N_c^2}{64\pi^5\zeta_{12}^2\sqrt{-\zeta_{12}^2-\zeta_{13}^2-\zeta_{23}^2+2\zeta_{12}\zeta_{13}+2\zeta_{12}\zeta_{23}+2\zeta_{13}\zeta_{23}}} \times G_{\mathcal{N}=4}(z,\bar{z}) + \ldots, \\
\mathrm{EEEC}(\zeta_{12},\zeta_{13},\zeta_{23}) &= \frac{g^4}{32\pi^5\zeta_{12}^2\sqrt{-\zeta_{12}^2-\zeta_{13}^2-\zeta_{23}^2+2\zeta_{12}\zeta_{13}+2\zeta_{12}\zeta_{23}+2\zeta_{13}\zeta_{23}}} \times G^{g/q}_{QCD}(z,\bar{z}) + \ldots.
\end{aligned} \tag{3.60}$$

In the weak coupling limit, we can expand (3.47) as

$$\begin{aligned}
\mathcal{G} &= a_0\left(\mathcal{G}^{(0)} + a\mathcal{G}^{(1)} + a^2\mathcal{G}^{(2)} + \ldots\right), \\
\delta_i &= \delta_i^{(0)} + a\gamma_i^{(1)} + a^2\gamma_i^{(2)} + \ldots, \\
\delta'_* &= 5 + a\gamma_*'^{(1)} + a^2\gamma_*'^{(2)} + \ldots, \\
R_{\delta,j;\delta'_*} &= a_0\left(R^{(0)}_{\delta,j;\delta'_*} + aR^{(1)}_{\delta,j;\delta'_*} + \ldots\right),
\end{aligned} \tag{3.61}$$

where for $\mathcal{N} = 4$ SYM we have $a_0 = \frac{g^2 N_c}{16\pi^3}$, $a = \frac{g^2 N_c}{4\pi^2}$, and for QCD $a_0 = \frac{g^2}{2\pi^3}$, $a = \frac{g^2}{16\pi^2}$.[7] As we explain later in section 3.3.1, we also set $\delta_*^{'(0)} = 5$ since it is the $J' = 4$ point on the twist-2 trajectory. Comparing (3.46), (3.60) and (3.61), we

[7] We choose a_0 such that there is no prefactor in (3.62).

immediately see that $\mathcal{G}^{(0)} = 0$ and[8]

$$\mathcal{G}^{(1)}(\zeta_{12}, z, \overline{z}) = (z\overline{z})^3 G(z, \overline{z}) \tag{3.62}$$

for both $\mathcal{N} = 4$ SYM and QCD. The expansion of (3.47) in the weak coupling limit is given by

$$\begin{aligned}
\mathcal{G}^{(0)}(\zeta_{12}, z, \overline{z}) &= \sum_{\delta^{(0)},j} \langle R^{(0)}_{\delta,j;\delta'_*} \rangle g^{(\mathcal{EEEP}_{\delta'_*=5})}_{\delta^{(0)},j}(z, \overline{z}) \\
\mathcal{G}^{(1)}(\zeta_{12}, z, \overline{z}) &= \sum_{\delta^{(0)},j} \left(\langle R^{(1)}_{\delta,j;\delta'_*} \rangle g^{(\mathcal{EEEP}_{\delta'_*=5})}_{\delta^{(0)},j}(z, \overline{z}) + \langle R^{(0)}_{\delta,j;\delta'_*} \gamma^{(1)}_{\delta,j} \rangle \partial_\delta g^{(\mathcal{EEEP}_{\delta'_*=5})}_{\delta^{(0)},j}(z, \overline{z}) \right) \\
&\quad + \gamma'^{(1)}_* \sum_{\delta^{(0)},j} \left(\langle R^{(0)}_{\delta,j;\delta'_*} \rangle \partial_{\delta'_*} g^{(\mathcal{EEEP}_{\delta'_*})}_{\delta^{(0)},j}(z, \overline{z}) + \frac{1}{2} \log(\zeta_{12}) \mathcal{G}^{(0)}(z, \overline{z}) \right),
\end{aligned} \tag{3.63}$$

where the notation $\langle \ldots \rangle$ represents a sum of the contributions from possibly degenerate operators, following e.g. [86].[9] Since $\mathcal{G}^{(0)} = 0$, we must have $\langle R^{(0)}_{\delta,j;\delta'_*} \rangle = 0$. The function $\mathcal{G}^{(1)}$ then becomes independent of ζ_{12} (which is consistent with (3.62)), and it can be written as

$$\mathcal{G}^{(1)}(z, \overline{z}) = \sum_{\delta^{(0)},j} \left(\langle R^{(1)}_{\delta,j;\delta'_*} \rangle g^{(\mathcal{EEEP}_{\delta'_*=5})}_{\delta^{(0)},j}(z, \overline{z}) + \langle R^{(0)}_{\delta,j;\delta'_*} \gamma^{(1)}_{\delta,j} \rangle \partial_\delta g^{(\mathcal{EEEP}_{\delta'_*=5})}_{\delta^{(0)},j}(z, \overline{z}) \right). \tag{3.64}$$

The values of $\delta^{(0)}$ and j appearing in the decomposition (3.64) can be related to the spectrum of the theory using the light-ray OPE formula. We describe this relation in section 3.3.1. We then obtain coefficients $\langle R^{(1)}_{\delta,j;\delta'_*} \rangle$ and $\langle R^{(0)}_{\delta,j;\delta'_*} \gamma^{(1)}_{\delta,j} \rangle$ using both a direct series expansion around the OPE limit (section 3.3.2) and the Lorentzian inversion formula (section 3.3.3). Our results are summarized in figure 3.2, where we plot the allowed values of δ, j in (3.64) and indicate points for which we obtained the coefficients $\langle R^{(1)}_{\delta,j;\delta'_*} \rangle$ and $\langle R^{(0)}_{\delta,j;\delta'_*} \gamma^{(1)}_{\delta,j} \rangle$ using either of the two methods.

[8]Note that (3.47) assumes that the sink/source states are created by scalar operators. Even though for the QCD quark jet case, the current J^μ is spin-1, we can still treat it as a scalar here since we contract indices between the bra and ket. This will produce an additional factor of 2 due to the fact that for a spin-1 operator V_μ, $\langle 0|V^\mu(x)V_\mu(0)|0\rangle = 2\langle 0|\phi(x)\phi(0)|0\rangle$ where ϕ is a scalar with dimension Δ_V, and we use the conventional two-point structures [30] for operators with spin 0 and 1 in CFT.

[9]The degeneracy can come from either operators with the same δ, j or operators with the same δ'_*.

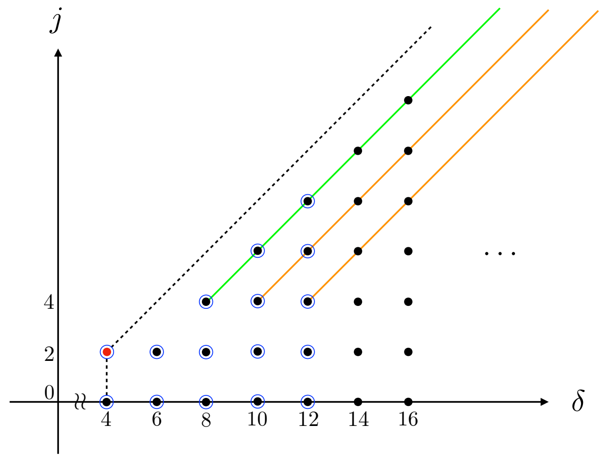

Figure 3.2: A plot showing the values of δ, j appearing in the conformal block decomposition (3.64). Black dashed line is the improved unitarity bound from (3.68). The red dot appears only in QCD, and the black dots appear in both QCD and $\mathcal{N}=4$ SYM. The coefficients $\langle R^{(1)}_{\delta,j;\delta'_*}\rangle$ and $\langle R^{(0)}_{\delta,j;\delta'_*}\gamma^{(1)}_{\delta,j}\rangle$ of dots with a blue circle can be obtained using the direct decomposition method. The $\langle R^{(1)}_{\delta,j;\delta'_*}\rangle$ coefficient of the green line and the $\langle R^{(0)}_{\delta,j;\delta'_*}\gamma^{(1)}_{\delta,j}\rangle$ coefficient of the orange line can be obtained using the Lorentzian inversion formula.

3.3.1 Predictions from the light-ray OPE

Schematically, the light-ray OPE for two stress tensors is given by [15, 77]

$$\mathbf{L}[T]\times\mathbf{L}[T]\sim\sum_i \mathbb{O}^+_{i,J=3,j=0}+\mathbb{O}^+_{i,J=3,j=2}+\mathbb{O}^+_{i,J=3,j=4}+\sum_{n,i}\mathcal{D}_{2n}\mathbb{O}^+_{i,J=3+2n,j=4}. \tag{3.65}$$

This formula allows us to predict which quantum numbers $\delta^{(0)}, j$ appear in the decomposition (3.64). First, we immediately see that only even values of j can appear. This is consistent with the fact that the OPE of two identical scalars on the celestial sphere only includes operators with even transverse spin.

To see the allowed values of δ, let us denote the conventional twist $\tau=\Delta-J$ of a trajectory with transverse spin j by τ_j.[10] Note that the $T\times T$ OPE only contains trajectories with transverse spin $j=0,2,4$. We also define a "celestial twist" $\tau_c\equiv\delta-j$, where δ, j are the quantum numbers appearing in (3.64).

Using (3.65), we can see that for $j=0,2$, the relation between τ_j and τ_c is given by

$$\tau_c(j=0,2)=\delta-j=\Delta(J=3)-1-j=\tau_j+2-j, \tag{3.66}$$

[10] We are working in perturbation theory, so the twist of each Regge trajectory is fixed and τ_j does not depend on J. In fact, we will simply use τ_j to label each Regge trajectory.

while for $j \geq 4$ we have

$$\tau_c(j) = \delta - j = \Delta(J = -1 + j) - 1 - j = \tau_{j=4} - 2. \tag{3.67}$$

In 4d, the conventional twist τ_j should satisfy the improved unitarity bound [49]

$$\tau_j \geq \max\{2, j\}. \tag{3.68}$$

Consequently, one might expect that the values of τ_j and $\tau_c(j)$ that can appear are

$$\begin{aligned} \tau_{j=0} &= 2, 4, 6, \ldots \quad \Rightarrow \quad \tau_c(j = 0) = 4, 6, 8, \ldots, \\ \tau_{j=2} &= 2, 4, 6, \ldots \quad \Rightarrow \quad \tau_c(j = 2) = 2, 4, 6, \ldots, \\ \tau_{j=4} &= 4, 6, 8, \ldots \quad \Rightarrow \quad \tau_c(j \geq 4) = 2, 4, 6, \ldots. \end{aligned} \tag{3.69}$$

However, as discussed in [87, 73], the contribution of the $j = 4$, $\tau_{j=4} = 4$ operator vanishes at the leading order for both $\mathcal{N} = 4$ SYM and QCD, and the contribution of the $j = 2$, $\tau_{j=2} = 2$ operator vanishes in $\mathcal{N} = 4$ SYM due to supersymmetry. So the actual values of τ_j and $\tau_c(j)$ appearing at the leading order should be

$$\begin{aligned} \tau_{j=0} &= 2, 4, 6, \ldots \quad \Rightarrow \quad \tau_c(j = 0) = 4, 6, 8, \ldots, \\ \tau_{j=2} &= \mathbf{2}, 4, 6, \ldots \quad \Rightarrow \quad \tau_c(j = 2) = \mathbf{2}, 4, 6, \ldots, \\ \tau_{j=4} &= 6, 8, 10, \ldots \quad \Rightarrow \quad \tau_c(j \geq 4) = 4, 6, 8, \ldots, \end{aligned} \tag{3.70}$$

where the bold $\tau_{j=2} = \mathbf{2}$ appears only in QCD.

What operators realize these quantum numbers? We will focus on operators with leading τ_c. We will identify light-ray operators by writing local operators on their Regge trajectories, with J as a free parameter. The actual light-ray operators are obtained by light-transforming and analytically continuing J to the appropriate value according to (3.65).

For $\mathcal{N} = 4$ SYM, the leading τ_c is 4, coming from operators with $j = 0, 2, 4$. For $j = 0$, $\tau_{j=0} = 2$, the operators can be schematically written as[11]

$$\overline{\phi}\partial^J \phi, \quad \overline{\psi}\partial^{J-1}\psi, \quad \overline{F}\partial^{J-2}F. \tag{3.71}$$

For $j = 2$, $\tau_{j=2} = 4$, we have

$$\psi\partial^{J_1}\psi\partial^{J_2}\psi\partial^{J-J_1-J_2-2}\psi, \quad F\partial^{J-2}\Box F, \quad \overline{F}\partial^{J_1}F\partial^{J_2}F\partial^{J-J_1-J_2-4}F, \tag{3.72}$$

[11] We use the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ spinor indices for the 4d Lorentz indices. The derivative is $\partial_{\dot{\alpha}\beta}$, and the field contents are $\overline{\phi}, \phi, \overline{\psi}_{\dot{\alpha}}, \psi_\beta, \overline{F}_{\dot{\alpha}_1\dot{\alpha}_2}, F_{\beta_1\beta_2}$. In (3.71)-(3.74), the Lorentz indices are implicitly symmetrized and the gauge indices are implicitly contracted.

and for $j = 4$, $\tau_{j=4} = 6$,

$$F\partial^{J_1}F\partial^{J_2}F\partial^{J-J_1-J_2-4}\Box F, \quad \overline{F}\partial^{J_1}F\partial^{J_2}F\partial^{J_3}F\partial^{J_4}F\partial^{J-J_1-J_2-J_3-J_4-6}F. \tag{3.73}$$

Note that there are degeneracies for all three values of j. (Supersymmetry relates some of these trajectories, but we do not study the consequences of supersymmetry here.) On the other hand, in QCD the leading τ_c is 2, and it is carried by one operator with $j = 2, \tau_{j=2} = 2$:

$$F\partial^{J-2}F. \tag{3.74}$$

Thus, the leading τ_c operator is non-degenerate in QCD.

Finally, let us discuss the value of δ'_* in (3.64). The value of δ'_* should be determined by a generalized light-ray OPE formula

$$\mathbf{L}[T] \times \mathbb{O}_{i,J=3,j} \sim \mathbb{O}'_{i,J=4,\lambda}, \tag{3.75}$$

where $\mathbb{O}'_{i,J=4,\lambda}$ is some unknown object that transforms like a primary operator with scaling dimension $\Delta_{\mathbb{O}'} = 1 - J = -3$. Though we do not have a rigorous definition of the object $\mathbb{O}'_{i,J=4,\lambda}$ for a general nonperturbative CFT, we expect that it should be related to light transforms of operators in the $T \times T \times T$ OPE with $J = 4$. Indeed, it has been shown in [87, 73] that for perturbative QCD, this object is just the light transformed operator $\mathbf{L}[O_{i,J=4,\lambda}]$ itself (at least at the leading order). Recently, there is also evidence from LHC data showing that the scaling behavior of the three-point energy correlator in the perturbative regime is governed by the twist-2 spin-4 anomalous dimension [74]. Therefore, in this paper we will assume that δ'_* is given by $\delta'_* = \Delta'_* - 1 = 5$ (at leading order in perturbation theory), since $\Delta'_* = 6$ is the scaling dimension of the leading twist-2, spin-4 operator in the $T \times T \times T$ OPE.[12]

3.3.2 Celestial block coefficients from direct decomposition

We now explain how to obtain the coefficients $\langle R^{(1)}_{\delta,j;\delta'_*}\rangle$ and $\langle R^{(0)}_{\delta,j;\delta'_*}\gamma^{(1)}_{\delta,j}\rangle$ in (3.64) using the known result for $\mathcal{G}^{(1)}(z,\bar{z})$ computed in [55]. Firstly, we can simply expand (3.64) in the OPE limit $z = re^{i\theta}, \bar{z} = re^{-i\theta}$ with $r \to 0$ and compare both sides order-by-order in r. Recall that the 2d block $g^{(\mathcal{EEEP}_{\delta'_*=5})}_{\delta,j}(z,\bar{z})$ in (3.64) is given by

$$\begin{aligned} g^{(\mathcal{EEEP}_{\delta'_*=5})}_{\delta,j}(z,\bar{z}) &= \frac{1}{1+\delta_{0,j}}(k^{0,-1}_{\delta+j}(z)k^{0,-1}_{\delta-j}(\bar{z}) + k^{0,-1}_{\delta-j}(z)k^{0,-1}_{\delta+j}(\bar{z})), \\ k^{r,s}_{\beta}(x) &= x^{\frac{\beta}{2}}{}_2F_1\left(\frac{\beta}{2}-r, \frac{\beta}{2}+s, \beta, x\right). \end{aligned} \tag{3.76}$$

[12]See also [75], where they show that in QCD, the leading correction to the scaling of collinear EEEC determined by $\delta'_* = 5$ can be used for top quark mass measurements.

The OPE limit of $\mathcal{G}^{(1)}$ corresponds to a "squeezed limit" on the celestial sphere, where two of the detectors are taken to be even closer after the collinear limit. The expansion of [55] in the squeezed limit has been studied in [87, 73] up to $O(r^{10})$.[13] Using these results, we find that the first few coefficients $\langle R^{(1)}_{\delta,j}\rangle$ and $\langle R^{(0)}_{\delta,j}\gamma^{(1)}_{\delta,j}\rangle$ in $\mathcal{N}=4$ SYM are given by

$$\begin{aligned}
&\langle R^{(1)}_{4,0}\rangle = 1, \quad \langle R^{(1)}_{6,0}\rangle = \frac{41}{12}-\frac{\pi^2}{4}, \quad \langle R^{(1)}_{6,2}\rangle = \frac{107}{60}-\frac{\pi^2}{6},\\
&\langle R^{(1)}_{8,0}\rangle = \frac{471}{600}-\frac{\pi^2}{15}, \quad \langle R^{(1)}_{8,2}\rangle = \frac{4883}{2800}-\frac{9\pi^2}{56}, \quad \langle R^{(1)}_{8,4}\rangle = \frac{2843}{5040}-\frac{\pi^2}{18},\\
&\langle R^{(0)}_{8,0}\gamma^{(1)}_{8,0}\rangle = \frac{2}{5}, \quad \langle R^{(0)}_{8,2}\gamma^{(1)}_{8,2}\rangle = -\frac{1}{20}.
\end{aligned} \tag{3.77}$$

In QCD, we use $R^{(1)g/q}_{\delta,j}$ to denote coefficients for the gluon/quark jet. For the QCD gluon jet, we find

$$\begin{aligned}
\langle R^{(1)g}_{4,0}\rangle &= \frac{98C_A^2+14C_An_fT_F+15C_Fn_fT_F}{1600},\\
R^{(1)g}_{4,2} &= \frac{C_A(C_A-2n_fT_F)}{2880},\\
\langle R^{(1)g}_{6,0}\rangle &= \frac{\left(636386-63000\pi^2\right)C_A^2+2\left(12600\pi^2-120899\right)C_An_fT_F-819C_Fn_fT_F}{403200},\\
\langle R^{(1)g}_{6,2}\rangle &= \frac{\left(834469-84000\pi^2\right)C_A^2+4\left(13125\pi^2-129587\right)C_An_fT_F+1260C_Fn_fT_F}{504000}.
\end{aligned} \tag{3.78}$$

Note that we do not use the bracket notation $\langle\cdots\rangle$ for the coefficient $R^{(1)}_{4,2}$, because from the discussion in section 3.3.1 it is non-degenerate. For the QCD quark jet we find

$$\begin{aligned}
\langle R^{(1)q}_{4,0}\rangle &= \frac{C_F(91C_A+240C_F+13n_fT_F)}{4800},\\
R^{(1)q}_{4,2} &= \frac{C_F(C_A-2n_fT_F)}{2880},\\
\langle R^{(1)q}_{6,0}\rangle &= \frac{C_F\left(\left(109200\pi^2-1077733\right)C_A-28\left(5100\pi^2-50929\right)C_F+3\left(111199-11200\pi^2\right)n_fT_F\right)}{403200},\\
\langle R^{(1)q}_{6,2}\rangle &= \frac{C_F\left(\left(157500\pi^2-1548703\right)C_A-210\left(1300\pi^2-12859\right)C_F+326n_fT_F\right)}{1008000}.
\end{aligned} \tag{3.79}$$

[13] We thank Hao Chen, Ian Moult, and Hua Xing Zhu for sending us a mathematica notebook containing the expansion.

We have further expanded the results of [55] up to $O(r^{12})$, which will be helpful when comparing to the Lorentzian inversion formula result in section 3.3.3. We record the coefficients up to $\delta = 12$ in appendix B.2.

3.3.3 The Lorentzian inversion formula on the celestial sphere

Direct decomposition yields OPE data at low dimensions δ, but becomes cumbersome as δ gets larger. Alternatively, we can use the Lorentzian inversion formula (LIF) [9] to extract OPE data from a four-point correlator. Like the lightcone bootstrap, the LIF requires us to analytically continue the correlator to Lorentzian signature — which in our case means complexifying the celestial sphere. Again, it is not clear whether this analytic continuation is admissible nonperturbatively. However, nothing prevents us from using the LIF as a tool in perturbation theory, as long as perturbative correlators are well-behaved. Indeed, we do not observe any pathologies when analytically continuing the results of [55] in $z, \overline{z}$. It would be interesting to study the analytic structure of the collinear EEEC as a function of $z, \overline{z}$ at higher orders in perturbation theory.

The LIF is only valid for $j > j_0$, where the "Regge intercept" j_0 controls the behavior of the correlator in the Regge limit. To reach the Regge limit, one should first take $\overline{z}$ around the branch point at 1, and then take both z and $\overline{z}$ to zero. From the point of view of celestial CFT, this is a strange kinematic regime, and we do not have rigorous bounds (much less physical intuition) for how the correlator should behave there. Note that the Regge limit on the celestial sphere has nothing to do with the Regge limit in d-dimensional Minkowski space (as far as we know), and that the celestial Regge intercept j_0 is not obviously related to the usual Regge intercept J_0. However, we can study this limit in perturbation theory.

In $\mathcal{N} = 4$ SYM, we find that the leading term of $\mathcal{G}^{(1)}(z, \overline{z})$ in the celestial Regge limit is given by

$$\begin{aligned}&\mathcal{G}^{(1)}_{\text{Regge}}(z, \overline{z})\\ &= \frac{i\pi r^3 (w^2+1)\left(2(w^2-1)\left((w^2-1)^2 \log(w+\frac{1}{w}) - 2w^2\right) - 2(w^6 - 3w^4 - 3w^2 + 1)\log w\right)}{w(w^2-1)^3},\end{aligned} \tag{3.80}$$

where we set $z = rw$ and $\overline{z} = r/w$. The scaling r^3 implies that the celestial Regge intercept is given by $j_0 = -2$ at this order in perturbation theory. The expressions for $\langle R^{(1)}_{\delta,j;\delta'_*}\rangle$ and $\langle R^{(0)}_{\delta,j;\delta'_*}\gamma^{(1)}_{\delta,j}\rangle$ for $\mathcal{N} = 4$ SYM obtained from the LIF should then be valid for all $j = 0, 2, 4, \ldots$.

For the QCD gluon jet, $\mathcal{G}^{(1)g}$ can be written as

$$\mathcal{G}^{(1)g} = C_F n_f T_F \mathcal{G}_1^{(1)g} + C_A n_f T_F \mathcal{G}_2^{(1)g} + C_A^2 \mathcal{G}_3^{(1)g}. \tag{3.81}$$

We find that the Regge intercept for $\mathcal{G}_1^{(1)g}$ is $j_0 = 0$, while for $\mathcal{G}_2^{(1)g}$ and $\mathcal{G}_3^{(1)g}$ the intercept is $j_0 = 2$. Therefore, for the coefficient proportional to $C_F n_f T_F$, the LIF result will agree with the result from direct decomposition only for $j > 0$. For the other two flavor structures ($C_A n_f T_F$ and C_A^2), we expect the result to agree only for $j > 2$. Similarly, for QCD quark jet, $\mathcal{G}^{(1)q}$ can be written as

$$\mathcal{G}^{(1)q} = C_F n_f T_F \mathcal{G}_1^{(1)q} + (C_A - 2C_F) C_F \mathcal{G}_2^{(1)q} + \mathcal{G}_3^{(1)q}, \tag{3.82}$$

where $\mathcal{G}_3^{(1)q}$ contains both C_F^2 and $C_F C_A$ factors. We find that the Regge intercept for $\mathcal{G}_2^{(1)q}$ is $j_0 = 0$, and the intercept for $\mathcal{G}_1^{(1)q}$ and $\mathcal{G}_3^{(1)q}$ is $j_0 = 2$.

Let us now briefly review the LIF. Consider a four-point function $g(z, \bar{z})$ in 2d with conformal block expansion

$$g(z, \bar{z}) = \sum_{\delta, j} p_{\delta, j} g_{\delta, j}^{\delta_i}(z, \bar{z}), \tag{3.83}$$

where only even j appear in the sum. The LIF in this case can be written as

$$C^+(\delta, j) = \frac{\kappa_{\delta+j}}{2} \int_0^1 \int_0^1 \frac{dz d\bar{z}}{(z\bar{z})^2} g_{j+1, \delta-1}^{\widetilde{\delta_i}}(z, \bar{z}) \mathrm{dDisc}_t[g(z, \bar{z})], \tag{3.84}$$

where $\widetilde{\delta_i} = 2 - \delta_i$, and

$$\kappa_\beta = \frac{\Gamma(\frac{\beta+\delta_1-\delta_2}{2})\Gamma(\frac{\beta-\delta_1+\delta_2}{2})\Gamma(\frac{\beta+\delta_3-\delta_4}{2})\Gamma(\frac{\beta-\delta_3+\delta_4}{2})}{2\pi^2 \Gamma(\beta-1)\Gamma(\beta)}. \tag{3.85}$$

The double discontinuity $\mathrm{dDisc}_t[g(z, \bar{z})]$ is

$$\begin{aligned} \mathrm{dDisc}_t[g(z, \bar{z})] &= \cos(\pi\phi) g(z, \bar{z}) - \frac{1}{2} e^{i\pi\phi} g^{\circlearrowleft}(z, \bar{z}) - \frac{1}{2} e^{-i\pi\phi} g^{\circlearrowright}(z, \bar{z}), \\ \phi &= \frac{\delta_2 - \delta_1 + \delta_3 - \delta_4}{2}, \end{aligned} \tag{3.86}$$

where $g^{\circlearrowleft}$ and $g^{\circlearrowright}$ indicate that we should take $\bar{z}$ around 1 in the direction shown, with z held fixed. Finally, the OPE coefficients $p_{\delta, j}$ are given by

$$p_{\delta^*, j} = -\mathrm{Res}_{\delta=\delta^*} C^+(\delta, j). \tag{3.87}$$

To compute $p_{\delta, j}$, it is convenient to define a generating functional

$$\begin{aligned} C(z, \delta, j) &= \kappa_{\delta+j} z^{\frac{\delta-j}{2}-1} \int_0^1 \frac{d\bar{z}}{\bar{z}^2} g_{j+1, \delta-1}^{\widetilde{\delta_i}}(z, \bar{z}) \mathrm{dDisc}_t[g(z, \bar{z})], \\ C^+(\delta, j) &= \int_0^1 \frac{dz}{2z} z^{-\frac{\delta-j}{2}} C(z, \delta, j). \end{aligned} \tag{3.88}$$

In the small z limit, $C(z,\delta,j)$ should have the expansion

$$C(z,\delta,j)=\sum_{\tau_c} C(\tau_c,\delta,j) z^{\frac{\tau_c}{2}}. \tag{3.89}$$

Using (3.84), we see that $C^+(\delta,j)$ has a pole when $\delta=j+\tau_c$, and therefore the OPE coefficient $p_{\delta,j}$ can be written as[14]

$$p_{j+\tau_c,j}=2C(\tau_c,j+\tau_c,j), \tag{3.90}$$

where the additional factor of 2 is due to $z\leftrightarrow\bar{z}$ symmetry.

Now, suppose we have a weak-coupling expansion

$$\begin{aligned} g(z,\bar{z}) &= a\left(g^{(0)}(z,\bar{z})+ag^{(1)}(z,\bar{z})+\cdots\right),\\ \delta &= \delta^{(0)}+a\gamma^{(1)}+\cdots,\\ p_{\delta,j} &= a\left(p^{(0)}_{\delta,j}+ap^{(1)}_{\delta,j}+\cdots\right), \end{aligned} \tag{3.91}$$

with coupling constant a, and we are interested in finding $p^{(1)}_{\delta,j}$ and $p^{(0)}_{\delta,j}\gamma^{(1)}_{\delta,j}$. Expanding (3.87) near $\delta^{(0)}$, we have

$$C^+(\delta,j)\sim a\left(-\frac{\langle p^{(0)}_{\delta,j}\rangle}{\delta-\delta^{(0)}}-a\frac{\langle p^{(1)}_{\delta,j}\rangle}{\delta-\delta^{(0)}}-a\frac{\langle p^{(0)}_{\delta,j}\gamma^{(1)}_{\delta,j}\rangle}{(\delta-\delta^{(0)})^2}+\cdots\right), \tag{3.92}$$

where $\langle\cdots\rangle$ indicates a sum over possible degenerate operators. For the generating functional, the expansion near $\delta^{(0)}$ is

$$\begin{aligned} C(z,\delta,j) &\sim a(C^{(0)}(\tau_c,\delta,j)+aC^{(1)}(\tau_c,\delta,j))z^{\frac{\delta^{(0)}+a\gamma^{(1)}-j}{2}}+\cdots\\ &\sim a\left(C^{(0)}(\tau_c,\delta,j)+aC^{(1)}(\tau_c,\delta,j)+a\frac{\gamma^{(1)}}{2}C^{(0)}(\tau_c,\delta,j)\log z\right)z^{\frac{\tau_c^{(0)}}{2}}, \end{aligned} \tag{3.93}$$

where $\tau_c^{(0)}=\delta^{(0)}-j$. Let us plug (3.93) into (3.88) and compare it with (3.92). We see that after integrating over z, the $C^{(1)}(\tau_c,\delta,j)$ term becomes a simple pole corresponding to $\langle p^{(1)}_{\delta,j}\rangle$. On the other hand, the $\gamma^{(1)}C^{(0)}(\tau_c,\delta,j)$ term has an additional $\log z$ and becomes a double pole, corresponding to $\langle p^{(0)}_{\delta,j}\gamma^{(1)}_{\delta,j}\rangle$. The precise formula is

$$\begin{aligned} \langle p^{(1)}_{j+\tau_c,j}\rangle &= 2C(z,\delta,j)\Big|_{z^{\frac{\tau_c}{2}}},\\ \langle p^{(0)}_{j+\tau_c,j}\gamma^{(1)}_{j+\tau_c,j}\rangle &= 4C(z,\delta,j)\Big|_{z^{\frac{\tau_c}{2}}\log z}. \end{aligned} \tag{3.94}$$

[14]In general, there will be an additional Jacobian factor coming from the dependence of τ_c on j. At the order we are working in, τ_c on each trajectory is a constant and this Jacobian factor is just 1.

$\mathcal{N} = 4$ **SYM**

Let us now apply the Lorentzian inversion formula to $\mathcal{G}^{(1)}(z, \bar{z})$ in $\mathcal{N} = 4$ SYM. After plugging in $g(z, \bar{z}) = \mathcal{G}^{(1)}_{\mathcal{N}=4}(z, \bar{z})$ and $\delta_1, \delta_2, \delta_3 = 3, \delta_4 = 5$ in (3.84), we find that $\langle R^{(1)}_{\delta,j;\delta'_*}\rangle$ is nonzero for even j and "celestial twists" $\tau_c = \delta - j = 4, 6, 8, \ldots$. Furthermore, $\langle R^{(0)}_{\delta,j;\delta'_*}\gamma^{(1)}_{\delta,j}\rangle$ is nonzero for even j and $\tau_c = 6, 8, 10, \ldots$ (except that $\langle R^{(0)}_{\delta=6,j=0;\delta'_*}\gamma^{(1)}_{\delta=6,j=0}\rangle = 0$). We can find analytical expressions for $\langle R^{(0)}_{j+\tau_c,j;\delta'_*}\gamma^{(1)}_{j+\tau_c,j}\rangle$ for general τ_c, and also $\langle R^{(1)}_{j+4,j;\delta'_*}\rangle$.[15] The results agree with those obtained from direct decomposition in section 3.3.2 and appendix B.2.

As an example, let us describe the detailed calculation for $\langle R^{(0)}_{\delta,j;\delta'_*}\gamma^{(1)}_{\delta,j}\rangle$ at celestial twist $\tau_c = 6$. For higher twists and $\langle R^{(1)}_{j+4,j;\delta'_*}\rangle$, we simply present the final result and leave details to appendix B.3. Using (3.94), we have

$$\begin{aligned}\langle R^{(0)}_{j+6,j;\delta'_*}\gamma^{(1)}_{j+6,j}\rangle =& 4\left(\kappa_{2j+6}z^2\int_0^1 \frac{d\bar{z}}{\bar{z}^2} g^{0,1}_{j+1,j+5}(z,\bar{z})\mathrm{dDisc}_t[\mathcal{G}(z,\bar{z})]\right)\Bigg|_{z^3\log z}\\ =& 4\kappa_{2j+6}\int_0^1 \frac{d\bar{z}}{\bar{z}^2} k^{0,1}_{2j+6}(\bar{z})\mathrm{dDisc}_t[\mathcal{G}(z,\bar{z})]|_{z^3\log z},\end{aligned} \tag{3.95}$$

where we have used the fact that the $\log z$ term of $\mathcal{G}(z, \bar{z})$ starts at z^3. We have also used the $\mathrm{SL}(2,\mathbb{R})$ expansion of the conformal block

$$\begin{aligned} g^{r,s}_{\delta,j}(z,\bar{z}) =& z^{\frac{\delta-j}{2}}k^{r,s}_{\delta+j}(\bar{z}) + O(z^{\frac{\delta-j}{2}+1}),\\ k^{r,s}_{\beta}(x) =& x^{\frac{\beta}{2}}{}_2F_1(\tfrac{\beta}{2}-r, \tfrac{\beta}{2}+s, \beta, x),\\ r = \frac{\delta_1-\delta_2}{2},& \qquad s = \frac{\delta_3-\delta_4}{2}. \end{aligned} \tag{3.96}$$

Next we should compute the double discontinuity $\mathrm{dDisc}_t[\mathcal{G}(z, \bar{z})]$. Note that the singularities of $\mathcal{G}(z, \bar{z})|_{z^3\log z}$ at $\bar{z} = 1$ only have integer powers or single logarithms of $1 - \bar{z}$. For example,

$$\mathcal{G}(z,\bar{z})|_{z^3\log z} = -\frac{1}{4(1-\bar{z})} - \frac{3}{2}\log(1-\bar{z}) + O((1-\bar{z})^0). \tag{3.97}$$

Naively, such terms have vanishing dDisc. However the correct interpretation is that their dDiscs are distributions localized at $\bar{z} = 1$. See [26] for examples of dealing with such distributions. To compute them, we insert a regulator ϵ so that the powers become non-integer, removing the regulator after taking the dDisc. For example,

[15] It would be interesting to calculate $\langle R^{(1)}_{j+\tau_c,j;\delta'_*}\rangle$ for general τ_c as well, but we leave that for future work.

inserting a regulator $1/(1-\bar{z})^{\epsilon}$, we have

$$\begin{aligned}
&\langle R^{(0)}_{j+6,j;\delta'_*}\gamma^{(1)}_{j+6,j}\rangle \\
&= \lim_{\epsilon\to 0} 4\kappa_\beta \int_0^1 \frac{d\bar{z}}{\bar{z}^2} k^{0,1}_\beta(\bar{z}) \mathrm{dDisc}_t \left[-\frac{1}{4(1-\bar{z})^{1+\epsilon}} - \frac{3}{2(1-\bar{z})^{\epsilon}} \log(1-\bar{z}) + O((1-\bar{z})^{-\epsilon})\right] \\
&= \lim_{\epsilon\to 0} \left(8\sin^2(\pi\epsilon)\kappa_\beta \int_0^1 \frac{d\bar{z}}{\bar{z}^2} k^{0,1}_\beta(\bar{z}) \left(\frac{1}{4(1-\bar{z})^{1+\epsilon}} + \frac{3}{2(1-\bar{z})^{\epsilon}} \log(1-\bar{z}) + O((1-\bar{z})^{-\epsilon})\right)\right. \\
&\qquad \left. - 8\pi \sin(2\pi\epsilon)\kappa_\beta \int_0^1 \frac{d\bar{z}}{\bar{z}^2} k^{0,1}_\beta(\bar{z}) \frac{3}{2(1-\bar{z})^{\epsilon}}\right),
\end{aligned} \tag{3.98}$$

where $\beta = 2j+6$ and we have used (B.30) for the double discontinuities. The $\bar{z}$ integrals will localize to $\bar{z}=1$ when taking the $\epsilon\to 0$ limit, so we can expand the SL_2 block $k^{0,1}_\beta(\bar{z})$ in this limit. For example, in the last line of (3.98), we have

$$\begin{aligned}
&- 8\pi\kappa_\beta \lim_{\epsilon\to 0} \sin(2\pi\epsilon) \int_0^1 \frac{d\bar{z}}{\bar{z}^2} k^{0,1}_\beta(\bar{z}) \frac{3}{2(1-\bar{z})^{\epsilon}} \\
=& - 8\pi\kappa_\beta \lim_{\epsilon\to 0} \sin(2\pi\epsilon) \int_0^1 \frac{d\bar{z}}{\bar{z}^2} \left(\frac{\Gamma(\beta)}{\Gamma(\frac{\beta}{2})\Gamma(\frac{\beta}{2}+1)} \frac{1}{1-\bar{z}} + \cdots\right) \frac{3}{2(1-\bar{z})^{\epsilon}} \\
=& - 8\pi\kappa_\beta \frac{3\Gamma(\beta)}{2\Gamma(\frac{\beta}{2})\Gamma(\frac{\beta}{2}+1)} \lim_{\epsilon\to 0} \sin(2\pi\epsilon) \int_0^1 \frac{d\bar{z}}{\bar{z}^2} \left(-\frac{1}{\epsilon}\delta(1-\bar{z}) + O(\epsilon^0)\right) \\
=& \frac{12\Gamma(\frac{\beta}{2}-1)\Gamma(\frac{\beta}{2})}{\Gamma(\beta-1)},
\end{aligned} \tag{3.99}$$

where we have used the expansion of $k^{0,1}_\beta(\bar{z})$ around $\bar{z}=1$ and the distributional identity $\frac{1}{x^{n+\epsilon}}\theta(x) = \frac{1}{\epsilon}\frac{(-1)^n}{(n-1)!}\delta^{(n-1)}(x) + O(\epsilon^0)$. Performing similar calculations for the other terms, we obtain

$$\langle R^{(0)}_{j+6,j;\delta'_*}\gamma^{(1)}_{j+6,j}\rangle = -\frac{j(j+5)\Gamma(j+2)^2}{2\Gamma(2j+4)}. \tag{3.100}$$

Similarly, we can use (3.94) to calculate $\langle R^{(0)}_{j+\tau_c,j;\delta'_*}\gamma^{(1)}_{j+\tau_c,j}\rangle$ with higher τ_c and $\langle R^{(1)}_{j+4,j}\rangle$. For example, for $\tau_c = 8$ we have (see (B.45) for the expression for general τ_c)

$$\langle R^{(0)}_{j+8,j}\gamma^{(1)}_{j+8,j}\rangle = \frac{\left(12(j+3)(j+4)S_1(j+3) - (19j^2+133j+192)\right)\Gamma(j+3)\Gamma(j+4)}{3\Gamma(2j+7)}, \tag{3.101}$$

where $S_1(n) = \sum_{k=1}^{n} \frac{1}{k}$ is the harmonic number. For $\langle R^{(1)}_{j+4,j} \rangle$, the result is

$$\langle R^{(1)}_{j+4,j} \rangle = \frac{\Gamma(j+2)^2}{6(j+1)\Gamma(2j+3)} \times \left(-9 + 2\pi^2 + j(j+3)(-6+\pi^2) + \frac{6(j+1)}{(j+3)^2(j+4)} {}_3F_2 \left(\begin{array}{ccc} 2, & 3, & j+3 \\ j+4, & j+5 & \end{array} ; 1 \right) \right). \tag{3.102}$$

QCD

We can perform a similar calculation for the QCD case by replacing $g(z,\bar{z})$ with $\mathcal{G}^{(1)g/q}$ in (3.84). For the gluon jet case, we again find that $\langle R^{(0)g}_{j+\tau_c,j} \gamma^{(1)}_{j+\tau_c,j} \rangle$ is nonzero for $\tau_c = 6, 8, \cdots$ and even j. The expressions for $\tau_c = 6$ and $\tau_c = 8$ are given by

$$\langle R^{(0)g}_{j+6,j} \gamma^{(1)}_{j+6,j} \rangle = -\frac{\Gamma(j+3)\Gamma(j+4)}{80\Gamma(2j+5)} C_A n_F T_F - \frac{(7j^2+35j-18)\Gamma(j+3)\Gamma(j+2)}{80\Gamma(2j+5)} C_A^2, \tag{3.103}$$

and

$$\begin{aligned} &\langle R^{(0)g}_{j+8,j} \gamma^{(1)}_{j+8,j} \rangle \\ &= -\frac{3\Gamma(j+4)\Gamma(j+5)}{40\Gamma(2j+7)} C_F n_F T_F - \frac{\left(28j^4+392j^3+1697j^2+2275j+204\right)\Gamma(j+3)\Gamma(j+4)}{336\Gamma(2j+7)} C_A n_F T_F \\ &\quad + \frac{\left(420\left(j^2+7j+12\right)S_1(j+3)+35j^4+490j^3+1052j^2-4641j-10056\right)\Gamma(j+3)\Gamma(j+4)}{840\Gamma(2j+7)} C_A^2. \end{aligned} \tag{3.104}$$

For the $\langle R^{(1)g}_{j+\tau_c,j} \rangle$ coefficient, although one can see from (3.78) that it has leading twist $\tau_c = 2$, the Lorentzian inversion formula will only give nonzero $\langle R^{(1)g}_{j+\tau_c,j} \rangle$ for $\tau_c = 4, 6, \ldots$. This is because the only nonzero $\langle R^{(1)g}_{j+\tau_c,j} \rangle$ with $\tau_c = 2$ has $j = 2$, and only contains flavor structures with Regge intercept $j_0 = 2$ (C_A^2 and $C_A n_f T_F$). Therefore we do not expect the Lorentzian inversion formula to give the correct result for $\tau_c = 2$. For $\tau_c = 4$, we obtain

$$\begin{aligned}
&\langle R^{(1)g}_{j+4,j}\rangle \\
&= \frac{\Gamma(j+2)\Gamma(j+3)}{80\Gamma(2j+3)} C_F n_F T_F \\
&+ \Bigg(-\frac{\left(31j^2+93j+17\right)\Gamma(j+1)\Gamma(j+2)}{7200\Gamma(2j+3)} \\
&\quad + \frac{\Gamma(j+1)\Gamma(j+2)^2\Gamma(j+3)}{4\Gamma(2j+3)\Gamma(j+4)\Gamma(j+5)}\Bigg(2\,{}_3F_2\left(\begin{matrix} 2, & 3, & j-2 \\ & j+4, & j+5 \end{matrix};1\right) \\
&\qquad -2\,{}_3F_2\left(\begin{matrix} 2, & 3, & j-1 \\ & j+4, & j+5 \end{matrix};1\right) + {}_3F_2\left(\begin{matrix} 2, & 3, & j \\ & j+4, & j+5 \end{matrix};1\right)\Bigg)\Bigg) C_A n_F T_F \\
&+ \Bigg(-\frac{\left(315+(j+1)(j+2)(-1117+150\pi^2)\right)\Gamma(j+1)\Gamma(j+2)}{7200\Gamma(2j+3)} \\
&\quad + \frac{\Gamma(j+1)\Gamma(j+2)^2}{8\Gamma(2j+3)\Gamma(j+4)} \sum_{k=0}^{4} c_k\, {}_3F_2\left(\begin{matrix} 1, & 2, & j-3+k \\ & j+3, & j+4 \end{matrix};1\right)\Bigg) C_A^2, \qquad (3.105)
\end{aligned}$$

where

$$c_0 = 1, \quad c_1 = -2, \quad c_2 = 3, \quad c_3 = -2, \quad c_4 = 1. \qquad (3.106)$$

As expected from the value of the Regge intercept, the results (3.103), (3.104) and (3.105) agree with (3.78), (B.24) and (B.25) for $j > 2$, but for the flavor structure $C_F n_f T_F$ they also agree at $j = 2$.

For the quark jet, the calculation is almost identical. We find that $\langle R^{(0)q}_{j+\tau_c,j}\gamma^{(1)}_{j+\tau_c,j}\rangle$ is nonzero for $\tau_c = 6, 8, \cdots$, and for $\tau_c = 6$ and $\tau_c = 8$ the expressions are

$$\langle R^{(0)q}_{j+6,j}\gamma^{(1)}_{j+6,j}\rangle = \frac{C_F(C_A(19j^2+95j-6) - 32C_F(2j^2+10j-3))\Gamma(j+2)\Gamma(j+3)}{480\Gamma(2j+5)}, \qquad (3.107)$$

and

$$\begin{aligned}
&\langle R^{(0)q}_{j+8,j}\gamma^{(1)}_{j+8,j}\rangle \\
&= -\frac{(10j^4+140j^3+637j^2+1029j+204)\Gamma(j+3)\Gamma(j+4)}{120\Gamma(2j+7)} C_F n_F T_F \\
&\quad - \frac{3(3j^2+21j+16)\Gamma(j+3)\Gamma(j+4)}{160\Gamma(2j+7)} C_F(C_A - 2C_F) - \frac{(23j^2+161j-24)\Gamma(j+3)\Gamma(j+4)}{60\Gamma(2j+6)} C_F^2 \\
&\quad + \frac{(10j^4+140j^3+401j^2-623j-3018+120(j+3)(j+4)S_1(j+3))\Gamma(j+3)\Gamma(j+4)}{240\Gamma(2j+7)} C_F C_A. \qquad (3.108)
\end{aligned}$$

For the $\langle R^{(1)q}_{j+\tau_c,j}\rangle$ coefficient, the leading twist $R^{(1)q}_{4,2}$ given in (3.79) also only contains flavor structure with Regge intercept $j_0 = 2$. So from the Lorentzian inversion formula the leading nonzero coefficient starts at $\tau_c = 4$, and it is given by

$$\begin{aligned}
&\langle R^{(1)q}_{j+4,j}\rangle \\
&= \Bigg(\frac{\left(1193 - 120\pi^2\right)\Gamma(j+2)\Gamma(j+3)}{5760\Gamma(2j+3)} \\
&\quad - \frac{\Gamma(j+1)\Gamma(j+2)^2\Gamma(j+3)}{160\Gamma(2j+3)\Gamma(j+4)\Gamma(j+5)} \Bigg(-(j^2+3j+26)\,{}_3F_2\left(\begin{matrix} 2, & 3, & j \\ j+4, & j+5 \end{matrix} ;1\right) \\
&\qquad + (j^2+3j+40)\,{}_3F_2\left(\begin{matrix} 2, & 3, & j+1 \\ j+4, & j+5 \end{matrix} ;1\right)\Bigg)\Bigg)(C_A - 2C_F)C_F \\
&+ \frac{C_F\left(C_A\left(167j^2+501j+220\right)+12C_F\left(13j^2+39j+58\right)\right)\Gamma(j+1)\Gamma(j+2)}{5760\Gamma(2j+3)} \\
&+ \frac{\Gamma(j+1)\Gamma(j+2)^2}{16\Gamma(2j+3)\Gamma(j+4)}\left({}_3F_2\left(\begin{matrix} 1, & 2, & j-2 \\ j+3, & j+4 \end{matrix} ;1\right) + {}_3F_2\left(\begin{matrix} 1, & 2, & j \\ j+3, & j+4 \end{matrix} ;1\right)\right) C_A C_F .
\end{aligned} \tag{3.109}$$

Similar to the gluon jet case, (3.107), (3.108) and (3.109) agree with (3.79), (B.26) and (B.27) for $j > 2$, but for the flavor structure $(C_A - 2C_F)C_F$ they also agree at $j = 2$.

3.4 Higher-order collinear EEEC

In this section, we use the celestial block decomposition (3.47) for the collinear EEEC and the leading order coefficients obtained in section 3.3 to make predictions for higher-order terms in the expansion of the EEEC in the coupling constant. Specifically, we study the $(n+1)$-st order expansion of $\mathcal{G}(\zeta_{12}, z, \bar{z})$ in a, which we denote by $\mathcal{G}^{(n+1)}(\zeta_{12}, z, \bar{z})$. Our key physical input is that contributions to (3.47) come from individual light-ray operators, whose contributions are fixed by symmetry in terms of their quantum numbers. In particular, this implies that anomalous dimensions should "exponentiate" to create the power laws predicted by symmetry.

Exponentiation is most powerful when the operators of interest are non-degenerate in perturbation theory. Thus, the $\tau_c = 2, j = 2$ non-degenerate operator (3.74) in QCD will play a crucial role in our arguments. Expanding (3.47) in the weak coupling limit, we find that $\mathcal{G}^{(n+1)}$ contains a term

$$\mathcal{G}^{(n+1)} \supset R^{(1)}_{4,2}(\gamma^{(1)}_{4,2})^n \partial^n_\delta g^{(\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{P}_{\delta'_*}=5)}_{\delta,2}(z,\bar{z})|_{\delta\to 4}. \tag{3.110}$$

Since $R^{(1)}_{4,2}$ is non-degenerate, and the anomalous dimension $\gamma^{(1)}_{4,2}$ is known [73], we can predict the coefficient of this term in $\mathcal{G}^{(n+1)}$ using available perturbative data. Moreover, it turns out that $\partial^n_\delta g^{(\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{P}_{\delta'_*=5})}_{\delta,j}(z,\bar{z})$ dominates in certain kinematics limits, and thus we have a prediction for the behavior of $\mathcal{G}^{(n+1)}$ in those limits. It is harder to apply the same argument for $\mathcal{G}^{(n+1)}$ in $\mathcal{N}=4$ SYM due to the fact that all the operators contributing to $\mathcal{G}^{(1)}$ in $\mathcal{N}=4$ SYM have tree-level degeneracies (see (3.71), (3.72) and (3.73) for the leading τ_c operators). This problem could be circumvented by using higher-order perturbative data to disentangle the degeneracies, or perhaps by organizing the EEEC in $\mathcal{N}=4$ into an appropriate super-celestial-block expansion. Regardless, we will focus on QCD in this section.

Before proceeding, let us comment on the nonzero β-function of QCD. Note that even in the presence of a nonzero β-function, the celestial block decomposition (3.47), which follows from Lorentz symmetry, should exist. However, some features will be different. Firstly, the spin selection rule $J=3$ for operators in the $\mathcal{E}\times\mathcal{E}$ OPE will be violated in the absence of conformal symmetry. We now expect light ray operators with spins $J=3+\delta J$ to appear, where contributions proportional to δJ come with additional factors of the β-function. In addition, without conformal symmetry, the quantum numbers of light-ray operators are no longer simply related to quantum numbers of local operators via the rule $(\Delta,J)\to(1-J,1-\Delta)$. Instead, light-ray operators at null infinity carry so-called "timelike" anomalous dimensions [88, 17].

While these issues are interesting to explore, here we will sidestep them by making predictions for "conformal QCD." Specifically, we work in dimensional regularization $d=4-\epsilon$, and tune the coupling constant a to a conformal fixed-point $\beta(\epsilon,a)=0$. The resulting predictions constrain the perturbative expansion of QCD away from the fixed point, up to terms proportional to the β-function. In fact, we expect that terms proportional to β do not affect the central predictions of this section. The reason is that the β-function corrections to the spin selection rule should only affect $R^{(n\geq 2)}_{\delta,j}$ and $\gamma^{(n\geq 2)}_{\delta,j}$, and the difference between spacelike and timelike anomalous dimensions should affect $\gamma^{(n\geq 2)}_{\delta,j}$. When deriving the results in this section, we only use known values of $R^{(1)}_{4,2}$ and $\gamma^{(1)}_{4,2}$. Therefore, they should still be true in the usual 4d QCD.

Let us now give concrete predictions for $\mathcal{G}^{(n+1)}(\zeta_{12},z,\bar{z})$ in conformal QCD. We define

$$\widetilde{\mathcal{G}}(z,\bar{z})=\sum_{\delta,j}R_{\delta,j;\delta'_*}g^{(\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{P}_{\delta'_*=5})}_{\delta,j}(z,\bar{z}), \tag{3.111}$$

so the collinear EEEC is

$$\mathcal{G}(\zeta_{12}, z, \bar{z}) = \zeta_{12}^{\frac{\delta'_*-5}{2}} \widetilde{\mathcal{G}}(z, \bar{z}). \tag{3.112}$$

We will first derive the leading behavior of $\widetilde{\mathcal{G}}^{(n+1)}(z, \bar{z})$ in three different kinematic limits, and then study the behavior of $\mathcal{G}^{(n+1)}(\zeta_{12}, z, \bar{z})$. Our main predictions for $\mathcal{G}^{(n+1)}(\zeta_{12}, z, \bar{z})$ are given by (3.143), (3.144), (3.147), and (3.148).

3.4.1 Predictions for $\widetilde{\mathcal{G}}^{(n+1)}(z, \bar{z})$

To make predictions for $\widetilde{\mathcal{G}}^{(n+1)}(z, \bar{z})$, we must take a kinematic limit where the $\tau_c = 2, j = 2$ non-degenerate operator gives the leading contribution. The first limit we consider is the OPE limit/squeezed limit, where $z, \bar{z} \ll 1$ with $z/\bar{z}$ fixed. We parametrize z and $\bar{z}$ as $z = re^{i\theta}, \bar{z} = re^{-i\theta}$. Using (3.111), we can fix the leading log term of the $(n+1)$-th order expansion of $\widetilde{\mathcal{G}}$. This is because $\log(r)$ can only come from a derivative of $g_{\delta,j}$ with respect to δ, and each derivative introduces an anomalous dimension factor $\gamma^{(1)}$. Hence, the leading log term, which has the most powers of $\log(r)$, should take the form $\langle R^{(1)}(\gamma^{(1)})^n\rangle \log^n(r)\partial_\delta^n g_{\delta,j}$. The quantum numbers with the lowest value of δ and nonzero $R^{(1)}$ are $(\delta, j) = (4, 0)$ and $(\delta, j) = (4, 2)$. Plugging these in, we obtain

$$\begin{aligned}
&\widetilde{\mathcal{G}}^{(n+1)}(r \to 0, \theta) \\
&= \frac{r^4 \log^n(r)}{n!}\left(\langle R^{(1)}_{4,0}(\gamma^{(1)}_{4,0})^n\rangle + 2R^{(1)}_{4,2}(\gamma^{(1)}_{4,2})^n \cos(2\theta)\right) + O(r^4 \log^{n-1}(r)).
\end{aligned} \tag{3.113}$$

As discussed in 3.3.1, the light-ray OPE formula implies that $\gamma^{(1)}_{4,2}$ should be the anomalous dimension of the operator (3.74) evaluated at $J = 3$. This has been calculated in e.g. [89, 73], and it is given by

$$\gamma^{(1)}_{F\partial^{J-2}F}(J) = 4C_A S_1(J) - \beta_0, \tag{3.114}$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}n_F T_F$. At $J = 3$, we then have

$$\gamma^{(1)}_{4,2} = \gamma^{(1)}_{F\partial^{J-2}F}(3) = \frac{22}{3}C_A - \left(\frac{11}{3}C_A - \frac{4}{3}n_F T_F\right) = \frac{11}{3}C_A + \frac{4}{3}n_F T_F. \tag{3.115}$$

Thus, the leading log term of $\widetilde{\mathcal{G}}^{(n+1)}$ in the OPE limit should be

$$\begin{aligned}
\widetilde{\mathcal{G}}^{(n+1)g/q}(r \to 0, \theta) = &\langle R^{(1)g/q}_{4,0}(\gamma^{(1)}_{4,0})^n\rangle \frac{r^4 \log^n(r)}{n!} \\
&+ 2R^{(1)g/q}_{4,2}\left(\frac{11}{3}C_A + \frac{4}{3}n_F T_F\right)^n \frac{r^4 \log^n(r)}{n!}\cos(2\theta) \\
&+ O(r^4 \log^{n-1}(r)),
\end{aligned} \tag{3.116}$$

where the superscript g/q denotes gluon jet or quark jet, and $R_{4,2}^{(1)g/q}$ are given in (3.78) and (3.79).

We see that the coefficient of the leading term $r^4 \log^n(r)\cos(2\theta)$ in the OPE limit is completely fixed. In fact, since the next quantum number with nonzero $R^{(1)}$ starts at $\delta = 6$, we can also predict the term proportional to $r^5 \log^n(r)\cos(3\theta)$ since it should come from the descendant of the $g_{\delta=4,j=2}$ block. The result is

$$\begin{aligned}
\widetilde{\mathcal{G}}^{(n+1)g/q}(r\to 0,\theta) = {} & \langle R_{4,0}^{(1)g/q}(\gamma_{4,0}^{(1)})^n\rangle \frac{r^4\log^n(r)}{n!} \\
& + 2R_{4,2}^{(1)g/q}\left(\frac{11}{3}C_A + \frac{4}{3}n_F T_F\right)^n \frac{r^4\log^n(r)}{n!}\cos(2\theta) \\
& + r^4(\cdots) \\
& + \langle R_{4,0}^{(1)g/q}(\gamma_{4,0}^{(1)})^n\rangle \frac{r^5\log^n(r)}{n!}\cos(\theta) \\
& + 2R_{4,2}^{(1)g/q}\left(\frac{11}{3}C_A + \frac{4}{3}n_F T_F\right)^n \frac{r^5\log^n(r)}{n!}\cos(3\theta) \\
& + O(r^5\log^{n-1}(r)).
\end{aligned} \tag{3.117}$$

To study other interesting limits where the non-degenerate operator gives the leading contribution, we can go to Lorentzian signature on the celestial sphere (where z and $\bar{z}$ are independent real variables) and consider the $z \ll 1$, fixed $\bar{z}$ limit. From the collider physics point of view, this limit might not be so useful since the kinematic region one can explore using collider experiments is intrinsically Euclidean (where z and $\bar{z}$ are complex conjugates of each other). Nevertheless, we still find a nontrivial constraint on the analytic expression of $\widetilde{\mathcal{G}}^{(n+1)}(z,\bar{z})$. In the limit $z \ll 1$, we can use the $\mathrm{SL}(2,\mathbb{R})$ expansion for the conformal block (3.96), and the leading term is from the operator with the leading celestial twist τ_c, which is exactly the non-degenerate operator with $\tau_c = 2$, $j = 2$. Using this, we find that the leading log of $\widetilde{\mathcal{G}}^{(n+1)}$ in the limit $z \ll 1$ is given by

$$\begin{aligned}
&\widetilde{\mathcal{G}}^{(n+1)g/q}(z \ll 1,\bar{z}) \\
&= R_{4,2}^{(1)g/q}(\gamma_{4,2}^{(1)})^n \frac{z\log^n(z)}{2^n n!} k_6^{0,-1}(\bar{z}) + O(z\log^{n-1} z) \\
&= 10R_{4,2}^{(1)g/q}\left(\frac{11}{3}C_A + \frac{4}{3}n_F T_F\right)^n \frac{z\log^n(z)}{2^n n!}\frac{\left(\bar{z}(12-12\bar{z}+\bar{z}^2) + 6(2-3\bar{z}+\bar{z}^2)\log(1-\bar{z})\right)}{\bar{z}^2} \\
&\quad + O(z\log^{n-1} z).
\end{aligned} \tag{3.118}$$

If we also write down the subleading $O(z\log^{n-1} z)$ order, we find

$$
\begin{aligned}
&\widetilde{\mathcal{G}}^{(n+1)g/q}(z \ll 1, \bar{z}) \\
&= R^{(1)g/q}_{4,2}(\gamma^{(1)}_{4,2})^n \frac{z\log^n(z)}{2^n n!} k^{0,-1}_6(\bar{z}) \\
&\quad + \left(\sum_{j=2,4,6,\cdots} \langle R^{(2)g/q}_{j+2,j}(\gamma^{(1)}_{j+2,j})^{n-1}\rangle k^{0,-1}_{2j+2}(\bar{z}) + R^{(1)g/q}_{4,2}(\gamma^{(1)}_{4,2})^n \partial_\delta k^{0,-1}_{\delta+2}(\bar{z})|_{\delta\to 4} \right. \\
&\qquad \left. + \langle R^{(1)g/q}_{j+2,2}\gamma'_*\rangle (\gamma^{(1)}_{j+2,2})^{n-1}\partial_{\delta'_*} k^{0,\frac{3-\delta'_*}{2}}_6(\bar{z})|_{\delta'_*\to 5} \right) \frac{z\log^{n-1}(z)}{2^{n-1}(n-1)!} + O(z\log^{n-2} z).
\end{aligned}
\tag{3.119}
$$

Note that for the subleading order coefficient $\langle R^{(2)}\rangle$, we should expect to get contribution from operators with $\tau_c = 2$ and all even j. The only term we know is $R^{(1)g/q}_{4,2}(\gamma^{(1)}_{4,2})^n \partial_\delta k^{0,-1}_{\delta+2}(\bar{z})|_{\delta\to 4}$. We can actually isolate this term by further taking the small $\bar{z}$ limit, in which we find $k^{0,-1}_{2j+2}(\bar{z}) = \bar{z}^{j+1} + O(\bar{z}^{j+2})$, $\partial_\delta k^{0,-1}_{\delta+2}(\bar{z})|_{\delta\to 4} = \frac{1}{2}\bar{z}^3\log\bar{z} + O(\bar{z}^4)$, and $\partial_{\delta'_*} k^{0,\frac{3-\delta'_*}{2}}_6(\bar{z})|_{\delta'_*\to 5} = O(\bar{z}^4)$. Therefore, in the $z \ll \bar{z} \ll 1$ limit, we can also predict the coefficient of the $z\log^{n-1} z\bar{z}^3\log\bar{z}$ term. It is given by

$$
\begin{aligned}
&\widetilde{\mathcal{G}}^{(n+1)g/q}(z \ll \bar{z} \ll 1) \\
&= 10R^{(1)g/q}_{4,2}\left(\frac{11}{3}C_A + \frac{4}{3}n_F T_F\right)^n \frac{z\log^n(z)}{2^n n!} \frac{(\bar{z}(12-12\bar{z}+\bar{z}^2) + 6(2-3\bar{z}+\bar{z}^2)\log(1-\bar{z}))}{\bar{z}^2} \\
&\quad + \frac{1}{2}R^{(1)g/q}_{4,2}\left(\frac{11}{3}C_A + \frac{4}{3}n_F T_F\right)^n \frac{z\log^{n-1}(z)}{2^{n-1}(n-1)!}\left(\bar{z}^3\log\bar{z} + O(\bar{z}^3)\right) \\
&\quad + O(z\log^{n-2} z).
\end{aligned}
\tag{3.120}
$$

It is also interesting to study the leading behavior of $\widetilde{\mathcal{G}}^{(n+1)}(z,\bar{z})$ in the double lightcone limit $z \ll 1-\bar{z} \ll 1$. From (3.118), we obtain

$$
\begin{aligned}
&\widetilde{\mathcal{G}}^{(n+1)g/q}(z \ll 1-\bar{z} \ll 1) \\
&= R^{(1)g/q}_{4,2}(\gamma^{(1)}_{4,2})^n \frac{z\log^n(z)}{2^n n!}(10 + O(1-\bar{z})) + O(z\log^{n-1} z),
\end{aligned}
\tag{3.121}
$$

and one can try to study how this term can be created using the crossing equation (3.48) and the lightcone bootstrap [81]. One will find that they come from the $R^{(n+1)}$ coefficients with $\tau_c = 6$ and large-j (which corresponds to double-trace operators with conventional twist $\tau = 8$ and $j = 4$). From the crossing equation, we can also predict the large-j behavior of $\langle R^{(n+1)}_{j+6,j}\rangle$. For example, we find that at large-j,

$$
\langle R^{(1)g/q}_{j+6,j}\rangle \sim \frac{5\sqrt{\pi}}{8} R^{(1)g/q}_{4,2} 4^{-j} j^{\frac{7}{2}}.
\tag{3.122}
$$

This result can be generalized to $\langle R^{(n+1)g/q}_{j+6,j}\rangle$ at any order. We describe the result and the details of the calculation in appendix B.4.

3.4.2 Degeneracies of $\mathcal{O}'_*$

So far in our analysis, we have assumed there is a unique isolated spin-4 operator $\mathcal{O}'_*$ that controls the collinear limit. However, it can happen that the leading-twist spin-4 operator is degenerate at tree-level, and thus we must take into account the contribution of multiple $\mathcal{O}'_*$'s in perturbation theory.

As discussed in section 3.3.1, $\Delta'_* = \delta'_* + 1$ should be the scaling dimension of the leading twist, spin-4 operator $\mathcal{O}'_*$ in the $T \times T \times T$ OPE. Also, since the sink/source states we consider are rotationally-invariant, $\mathcal{O}'_*$ should have zero transverse spin. There are only two such Regge trajectories in QCD:[16]

$$\begin{aligned}
\mathcal{O}_q(J) =&\frac{(-1)^J}{2^J}\sum_{k=1}^{J}(-1)^{k-1}\alpha^q_k(J)\overline{\psi}\gamma^{\{\mu_1}(i\overleftarrow{D}^{\mu_2})\cdots(i\overleftarrow{D}^{\mu_k})(iD^{\mu_{k+1}})\cdots(iD^{\mu_J\}})\psi - \text{traces},\\
\mathcal{O}_g(J) =&\frac{(-1)^J}{2^J}\sum_{k=1}^{J-1}(-1)^{k-1}\alpha^g_k(J)F_{a\nu}{}^{\{\mu_1}(i\overleftarrow{D}^{\mu_2})\cdots(i\overleftarrow{D}^{\mu_k})(iD^{\mu_{k+1}})\cdots(iD^{\mu_{J-1}})F_a^{\nu\mu_J\}} - \text{traces}.
\end{aligned} \tag{3.123}$$

where $\overleftarrow{D}^\mu$ (acting on the left) and D^μ (acting on the right) are covariant derivatives. The coefficients $\alpha^{q/g}_k(J)$ are chosen such that the entire expression of $\mathcal{O}_{q/g}(J)$ is a conformal primary, and they satisfy the normalization condition $\sum_{k=1}^J \alpha^q_k(J) = \sum_{k=1}^{J-1} \alpha^g_k(J) = 1$. For example, for $J = 2$ we have

$$\begin{aligned}
\mathcal{O}_q(2) =&\frac{1}{8}\left(\overline{\psi}\gamma^{\{\mu_1}iD^{\mu_2\}}\psi - \overline{\psi}\gamma^{\mu_1\}}i\overleftarrow{D}^{\{\mu_2}\psi\right) - \text{traces},\\
\mathcal{O}_g(2) =&\frac{1}{4}F_{a\nu}{}^{\mu_1}F^{a\nu\mu_2} - \text{traces}.
\end{aligned} \tag{3.124}$$

One can see that the sum $\mathcal{O}_q(2) + \mathcal{O}_g(2)$ is proportional to the stress-energy tensor.

For general J, there are many different methods to determine the coefficients $\alpha^{q/g}_k(J)$ that make $\mathcal{O}_{q/g}(J)$ a conformal primary [72, 90, 91, 92]. Here, we are only interested in the $J = 4$ case. So we take a simple approach: apply the generator K_μ of the special conformal transformation on $\mathcal{O}_{q/g}(4)$ and demand that its action vanishes.

[16] We are using mostly positive metric, and our convention for indices symmetrization is $T^{\{\mu_1\cdots\mu_n\}} = \frac{1}{n!}\sum_{P\in S_n} T^{\mu_{P(1)}\cdots\mu_{P(n)}}$

Using the basic commutation relation $[K_\mu, P_\nu] = 2\eta_{\mu\nu}D - 2M_{\mu\nu}$, we find

$$\mathcal{O}_q(4) = \frac{1}{224}\Big(\overline{\psi}\gamma^{\{\mu_1}(iD^{\mu_2})(iD^{\mu_3})(iD^{\mu_4\}})\psi - 6(iD^{\{\mu_2})\overline{\psi}\gamma^{\mu_1}(iD^{\mu_3})(iD^{\mu_4\}})\psi$$
$$+6(iD^{\{\mu_2})(iD^{\mu_3})\overline{\psi}\gamma^{\mu_1}(iD^{\mu_4\}})\psi - (iD^{\{\mu_2})(iD^{\mu_3})(iD^{\mu_4})\overline{\psi}\gamma^{\mu_1\}}\psi\Big) - \text{traces}, \tag{3.125}$$

and

$$\mathcal{O}_g(4) = \frac{3}{112}\left(F_{a\nu}{}^{\{\mu_1}(iD^{\mu_2})(iD^{\mu_3})F_a^{\nu\mu_4\}} - \frac{4}{3}(iD^{\{\mu_2})F_{a\nu}{}^{\mu_1}(iD^{\mu_3})F_a^{\nu\mu_4\}}\right) - \text{traces}. \tag{3.126}$$

The one-loop dilatation matrix for $\mathcal{O}_q(4)$ and $\mathcal{O}_g(4)$ is given by [93, 94, 95]

$$\widehat{\gamma} = \frac{\alpha_s}{4\pi}\begin{pmatrix} \frac{157C_F}{30} & -\frac{11n_FT_F}{15} \\ -\frac{11C_F}{15} & \frac{21C_A}{5} + \frac{4n_FT_F}{3} \end{pmatrix}. \tag{3.127}$$

Its eigenvalues are

$$\frac{21}{10}C_A + \frac{157}{60}C_F + \frac{2}{3}n_FT_F \mp \frac{1}{60}\sqrt{(126C_A - 157C_F)^2 + 32(315C_A - 332C_F)n_FT_F + 1600n_F^2T_F^2}, \tag{3.128}$$

and its left eigenvectors are

$$\begin{pmatrix}\alpha_\pm & 1\end{pmatrix}, \tag{3.129}$$

where

$$\alpha_\pm = \frac{126C_A - 157C_F + 40n_FT_F \pm \sqrt{(126C_A - 157C_F)^2 + 32(315C_A - 332C_F)n_FT_F + 1600n_F^2T_F^2}}{44n_FT_F}. \tag{3.130}$$

To resolve the degeneracy at 1-loop, we can define the following two operators:

$$\mathcal{O}'_{*1} = \alpha_+\mathcal{O}_q(4) + \mathcal{O}_g(4),$$
$$\mathcal{O}'_{*2} = \mathcal{O}_q(4) + \frac{1}{\alpha_-}\mathcal{O}_g(4). \tag{3.131}$$

In the $\mathcal{O}'_{*1}, \mathcal{O}'_{*2}$ basis, the anomalous dimension matrix is then diagonal. The first operator has anomalous dimension (suppressing the $\frac{\alpha_s}{4\pi}$ factor)

$$\gamma'_{*1} \equiv \frac{21}{10}C_A + \frac{157}{60}C_F + \frac{2}{3}n_FT_F - \frac{1}{60}\sqrt{(126C_A - 157C_F)^2 + 32(315C_A - 332C_F)n_FT_F + 1600n_F^2T_F^2}. \tag{3.132}$$

and γ'_{*2} is given by replacing $-\sqrt{\cdot} \to \sqrt{\cdot}$.

Taking into account the degeneracies of $\mathcal{O}'_*$, the decomposition (3.47) for $\mathcal{G}(\zeta_{12}, z, \bar{z})$ should become

$$\begin{aligned}\mathcal{G}(\zeta_{12}, z, \bar{z}) =& \frac{\pi^2}{16} \sum_{i=1,2} \frac{(-p^2)^{\frac{2\Delta_{\mathcal{O}}-4}{2}}}{\sigma^{\mathcal{O}}_{\text{total}}} s'^{(i)}_{\mathcal{O}\mathcal{O}\mathcal{P}_{\delta'_*}} \zeta_{12}^{\frac{\delta'_*-5}{2}} \sum_{\delta,j} r_{\mathcal{E}\mathcal{E}\mathcal{P}_{\delta,j}} r'^{(i)}_{\mathcal{P}_{\delta,j}\mathcal{E}\mathcal{P}_{\delta'_*}} g^{(\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{P}_{\delta'_*})}_{\delta,j}(z, \bar{z}) \\ =& \sum_{i=1,2} c^{\mathcal{O}}_i \zeta_{12}^{\frac{\delta'_*-5}{2}} \sum_{\delta,j} \widetilde{R}_{\delta,j;\delta'_*;i} g^{(\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{P}_{\delta'_*})}_{\delta,j}(z, \bar{z}), \end{aligned} \tag{3.133}$$

where in the first line we have used (3.45) to restore the total cross section in order to emphasize the dependence on the sink/source states, and in the second line we have defined

$$c^{\mathcal{O}}_i \equiv \frac{(-p^2)^{\frac{2\Delta_{\mathcal{O}}-4}{2}}}{\sigma^{\mathcal{O}}_{\text{total}}} s'^{(i)}_{\mathcal{O}\mathcal{O}\mathcal{P}_{\delta'_*}}, \qquad \widetilde{R}_{\delta,j;\delta'_*;i} \equiv \frac{\pi^2}{16} r_{\mathcal{E}\mathcal{E}\mathcal{P}_{\delta,j}} r'^{(i)}_{\mathcal{P}_{\delta,j}\mathcal{E}\mathcal{P}_{\delta'_*}}. \tag{3.134}$$

Note that unlike $R_{\delta,j;\delta'_*}$, the newly defined coefficient $\widetilde{R}_{\delta,j;\delta'_*;i}$ is independent of the operator $\mathcal{O}$, which creates the sink/source states. The only dependence on $\mathcal{O}$ is in the coefficient $c^{\mathcal{O}}_i$. Using (3.16), one can show that $c^{\mathcal{O}}_i$ can be determined using

$$\frac{\langle \mathcal{O}(p) | \mathbf{L}[\mathcal{O}'_{*;i}](\infty, z) | \mathcal{O}(p) \rangle}{\langle \mathcal{O}(p) | \mathcal{O}(p) \rangle} = c^{\mathcal{O}}_i (-2p \cdot z)^{-\delta'_*} (-p^2)^{\frac{\delta'_*+3}{2}}, \tag{3.135}$$

where we have replaced the object $\mathbb{W}_{\delta'}$ with $\mathbf{L}[\mathcal{O}'_{*;i}]$[17] based on the assumptions made in section 3.3.1. For us, the two degenerate operators can be chosen to be $\mathcal{O}'_{*1}$ and $\mathcal{O}'_{*2}$ defined in (3.131). The two different $\mathcal{O}$'s are $\text{Tr}F^2$ corresponding to the gluon jet and J^μ corresponding to the quark jet (averaged over polarization), so we will simply use $c^{g/q}_i$ for two cases.

We are interested in $c^{(0)g/q}_i$ at the leading order in the coupling constant. To find $c^{(0)g/q}_i$, we first determine $\langle \mathcal{O}(x_1) \mathcal{O}_{g/q}(4)(x_2, z_2) \mathcal{O}(x_3) \rangle$ in position space using Wick contractions, where $\mathcal{O}$ is either $\text{Tr}F^2$ or J^μ. After doing the light transform and Fourier transform, we then obtain that at the leading order

$$\begin{aligned}\frac{\langle \text{Tr}F^2(p) | \mathbf{L}[\mathcal{O}_g(4)](\infty, z) | \text{Tr}F^2(p) \rangle}{\langle \text{Tr}F^2(p) | \text{Tr}F^2(p) \rangle} &= \frac{1}{64\pi} (-2p \cdot z)^{-5} (-p^2)^4 + O(\alpha_s), \\ \frac{\langle J_\mu(p) | \mathbf{L}[\mathcal{O}_q(4)](\infty, z) | J^\mu(p) \rangle}{\langle J_\mu(p) | J^\mu(p) \rangle} &= \frac{1}{64\pi} (-2p \cdot z)^{-5} (-p^2)^4 + O(\alpha_s), \end{aligned} \tag{3.136}$$

[17] The coefficient relating $\mathbb{W}_{\delta'}$ and $\mathbf{L}[\mathcal{O}'_{*;i}]$ can be absorbed into $r'^{(i)}_{\mathcal{P}_{\delta,j}\mathcal{E}\mathcal{P}_{\delta'_*}}$.

which implies that

$$c_1^{(0)g} = \frac{1}{64\pi}, \qquad c_2^{(0)g} = \frac{1}{\alpha_-}\frac{1}{64\pi}, \qquad c_1^{(0)q} = \alpha_+\frac{1}{64\pi}, \qquad c_2^{(0)q} = \frac{1}{64\pi}. \quad (3.137)$$

Therefore, the coefficients appearing in (3.64) can be rewritten as (we focus on $\langle R^{(1)}_{\delta,j;\delta'_*}\rangle$)

$$\begin{aligned} \langle R^{(1)g}_{\delta,j;\delta'_*}\rangle &= \frac{1}{64\pi}\left(\langle \widetilde{R}^{(1)}_{\delta,j;\delta'_*;1}\rangle + \frac{1}{\alpha_-}\langle \widetilde{R}^{(1)}_{\delta,j;\delta'_*;2}\rangle\right), \\ \langle R^{(1)q}_{\delta,j;\delta'_*}\rangle &= \frac{1}{64\pi}\left(\alpha_+\langle \widetilde{R}^{(1)}_{\delta,j;\delta'_*;1}\rangle + \langle \widetilde{R}^{(1)}_{\delta,j;\delta'_*;2}\rangle\right). \end{aligned} \quad (3.138)$$

Solving (3.138) for $\langle \widetilde{R}^{(1)}_{\delta,j;\delta'_*;1/2}\rangle$, we find

$$\begin{aligned} \langle \widetilde{R}^{(1)}_{\delta,j;\delta'_*;1}\rangle &= \frac{64\pi}{\alpha_+ - \alpha_-}\left(\langle R^{(1)q}_{\delta,j;\delta'_*}\rangle - \alpha_-\langle R^{(1)g}_{\delta,j;\delta'_*}\rangle\right), \\ \langle \widetilde{R}^{(1)}_{\delta,j;\delta'_*;2}\rangle &= \frac{64\pi\alpha_-}{\alpha_+ - \alpha_-}\left(\alpha_+\langle R^{(1)g}_{\delta,j;\delta'_*}\rangle - \langle R^{(1)q}_{\delta,j;\delta'_*}\rangle\right). \end{aligned} \quad (3.139)$$

3.4.3 Predictions for $\mathcal{G}^{(n+1)}(\zeta_{12}, z, \bar{z})$

We now explain how to use (3.133) and (3.139) to deal with the degeneracy of $\mathcal{O}'_*$ and make predictions for $\mathcal{G}^{(n+1)}(\zeta_{12}, z, \bar{z})$. If we expand (3.112) in small coupling assuming there are no degeneracies, we will get

$$\begin{aligned} &\mathcal{G}^{(n+1)}(\zeta_{12}, z, \bar{z}) \\ &= \sum_{k=0}^{n} \frac{\log^{n-k}\zeta_{12}}{2^{n-k}(n-k)!}\left(\sum_{p=0}^{k}\sum_{\substack{i_1+i_2+\cdots i_{p+1}=n-k \\ i_1+2i_2+\cdots+(p+1)i_{p+1}=n-k+p}} \frac{(n-k)!}{i_1!\cdots i_{p+1}!}\left(\gamma'^{(1)}_*\right)^{i_1}\cdots\left(\gamma'^{(p+1)}_*\right)^{i_{p+1}}\widetilde{\mathcal{G}}^{(k+1-p)}\right). \end{aligned}$$

$$(3.140)$$

More explicitly, the first few terms are given by

$$\begin{aligned} &\mathcal{G}^{(n+1)}(\zeta_{12}, z, \bar{z}) \\ &= \frac{\log^{n}\zeta_{12}}{2^n n!}\left(\gamma'^{(1)}_*\right)^n \widetilde{\mathcal{G}}^{(1)}(z,\bar{z}) \\ &+ \frac{\log^{n-1}\zeta_{12}}{2^{n-1}(n-1)!}\left(\left(\gamma'^{(1)}_*\right)^{n-1}\widetilde{\mathcal{G}}^{(2)}(z,\bar{z}) + (n-1)\left(\gamma'^{(1)}_*\right)^{n-2}\gamma'^{(2)}_*\widetilde{\mathcal{G}}^{(1)}(z,\bar{z})\right) \\ &+ \frac{\log^{n-2}\zeta_{12}}{2^{n-2}(n-2)!}\left(\left(\gamma'^{(1)}_*\right)^{n-2}\widetilde{\mathcal{G}}^{(3)}(z,\bar{z}) + (n-2)\left(\gamma'^{(1)}_*\right)^{n-3}\gamma'^{(2)}_*\widetilde{\mathcal{G}}^{(2)}(z,\bar{z})\right. \\ &\qquad\left. + \frac{(n-2)(n-3)}{2}\left(\gamma'^{(1)}_*\right)^{n-4}\left(\gamma'^{(2)}_*\right)^2\widetilde{\mathcal{G}}^{(1)}(z,\bar{z}) + (n-2)\left(\gamma'^{(1)}_*\right)^{n-3}\gamma'^{(3)}_*\widetilde{\mathcal{G}}^{(1)}(z,\bar{z})\right) \\ &+ \cdots. \end{aligned} \quad (3.141)$$

The above expression will be modified in the presence of degeneracies. In general, if we know the anomalous dimension $\gamma_*'^{(k)}$ to the k-th order, we will be able to rewrite the above expression up to the $\log^{n-k+1}\zeta_{12}$ term. In the previous section, we have only diagonalized the $\gamma_*'^{(1)}$ matrix, which then allows us to rewrite all the $(\gamma_*'^{(1)})^k$ terms.

First, let us focus on the leading logarithmic divergence $\log^n(\zeta_{12})$. Using (3.133), we find that in the presence of degeneracies, the leading log term of (3.140) should become

$$\begin{aligned}&\mathcal{G}^{(n+1)}(\zeta_{12},z,\bar{z})\\&=\frac{\log^n\zeta_{12}}{2^n n!}\left(c_1^{(0)\mathcal{O}}\left(\gamma'_{*1}\right)^n\sum_{\delta,j}\langle\widetilde{R}^{(1)}_{\delta,j;\delta'_*;1}\rangle g_{\delta,j}^{(\mathcal{EEEP}_{\delta'_*=5})}(z,\bar{z})+c_2^{(0)\mathcal{O}}\left(\gamma'_{*2}\right)^n\sum_{\delta,j}\langle\widetilde{R}^{(1)}_{\delta,j;\delta'_*;2}\rangle g_{\delta,j}^{(\mathcal{EEEP}_{\delta'_*=5})}(z,\bar{z})\right)\\&\quad+O(\log^{n-1}\zeta_{12}),\end{aligned}\tag{3.142}$$

where γ'_{*1} and γ'_{*2} are given by (3.132). Plugging in (3.139), we obtain for the gluon and quark jets

$$\begin{aligned}\mathcal{G}^{(n+1)g}(\zeta_{12},z,\bar{z})&=\frac{\log^n\zeta_{12}}{2^n n!}\left(\frac{\alpha_+\left(\gamma'_{*2}\right)^n-\alpha_-\left(\gamma'_{*1}\right)^n}{\alpha_+-\alpha_-}\mathcal{G}^{(1)g}+\frac{\left(\gamma'_{*1}\right)^n-\left(\gamma'_{*2}\right)^n}{\alpha_+-\alpha_-}\mathcal{G}^{(1)q}\right)\\&\quad+O(\log^{n-1}\zeta_{12}),\end{aligned}\tag{3.143}$$

$$\begin{aligned}\mathcal{G}^{(n+1)q}(\zeta_{12},z,\bar{z})&=\frac{\log^n\zeta_{12}}{2^n n!}\left(\frac{\alpha_+\left(\gamma'_{*1}\right)^n-\alpha_-\left(\gamma'_{*2}\right)^n}{\alpha_+-\alpha_-}\mathcal{G}^{(1)q}+\frac{\alpha_+\alpha_-\left(\left(\gamma'_{*2}\right)^n-\left(\gamma'_{*1}\right)^n\right)}{\alpha_+-\alpha_-}\mathcal{G}^{(1)g}\right)\\&\quad+O(\log^{n-1}\zeta_{12}),\end{aligned}\tag{3.144}$$

where $\mathcal{G}^{(1)g/q}$ are simply the leading order results for the gluon/quark jet, related to the known result of [55] by (3.62). Equations (3.143) and (3.144) show that the leading logarithmic divergence of the $(n+1)$-th order EEEC $\mathcal{G}^{(n+1)}(\zeta_{12},z,\bar{z})$ is $\log^n(\zeta_{12})$, and it is completely determined by the leading order result $\mathcal{G}^{(1)g}$ and $\mathcal{G}^{(1)q}$.

In fact, since we can rewrite all the $(\gamma_*'^{(1)})^k$ terms in (3.140), we can make further predictions for $\mathcal{G}^{(n+1)}(\zeta_{12},z,\bar{z})$ if we know how to separate out the contributions from $(\gamma_*'^{(1)})^k$. This can be done by taking the limits considered in section 3.4.1. For example, in the OPE limit we find

$$\begin{aligned}
&\mathcal{G}^{(n+1)g/q}(\zeta_{12}, r \to 0, \theta) \\
&= \sum_{k=0}^{n} \frac{\log^{n-k}\zeta_{12}}{2^{n-k}(n-k)!}\Big(c_1^{(0)g/q}(\gamma'_{*1})^{n-k}\widetilde{\mathcal{G}}_1^{(k+1)}(r\to 0,\theta) + c_2^{(0)g/q}(\gamma'_{*2})^{n-k}\widetilde{\mathcal{G}}_2^{(k+1)}(r\to 0,\theta) \\
&\qquad + O(r^4\log^{k-1} r)\Big). \qquad (3.145)
\end{aligned}$$

From (3.116) we know that $\widetilde{\mathcal{G}}^{(k+1)} \sim r^4 \log^k r$ in the OPE limit. Thus, at each $\log^{n-k}\zeta_{12}$ power, all the terms involving higher-loop anomalous dimensions ($\gamma_*'^{(p)}$ with $p > 1$) in (3.140) go as at most $r^4 \log^{k-1} r$, while the term involving $(\gamma_*'^{(1)})^k$ has the most divergent piece $r^4 \log^k r$. Therefore, we can determine the leading behavior in small r for each $\log^{n-k}\zeta_{12}$ power. The functions $\widetilde{\mathcal{G}}_1^{(k+1)}$ and $\widetilde{\mathcal{G}}_2^{(k+1)}$ only include the contribution from O'_{*1} and O'_{*2} respectively. They are defined as

$$\begin{aligned}
\widetilde{\mathcal{G}}_1^{(k+1)} &= \frac{64\pi}{\alpha_+ - \alpha_-}\left(\widetilde{\mathcal{G}}^{(k+1)q} - \alpha_-\widetilde{\mathcal{G}}^{(k+1)g}\right), \\
\widetilde{\mathcal{G}}_2^{(k+1)} &= \frac{64\pi\alpha_-}{\alpha_+ - \alpha_-}\left(\alpha_+\widetilde{\mathcal{G}}^{(k+1)g} - \widetilde{\mathcal{G}}^{(k+1)q}\right). \qquad (3.146)
\end{aligned}$$

Using (3.116), we then obtain, for example,

$$\begin{aligned}
&\mathcal{G}^{(n+1)g}(\zeta_{12}, r \to 0, \theta) \\
&= \sum_{k=0}^{n} \frac{\log^{n-k}\zeta_{12}}{2^{n-k-1}k!(n-k)!}\times \\
&\left(\left(\gamma_{4,2}^{(1)}\right)^k \left(\frac{\alpha_+\left(\gamma'_{*2}\right)^{n-k} - \alpha_-\left(\gamma'_{*1}\right)^{n-k}}{\alpha_+ - \alpha_-}R_{4,2}^{(1)g} + \frac{\left(\gamma'_{*1}\right)^{n-k} - \left(\gamma'_{*2}\right)^{n-k}}{\alpha_+ - \alpha_-}R_{4,2}^{(1)q}\right) r^4 \log^k r \cos(2\theta)\right. \\
&\left. + (\cdots)r^4\log^k r + O(r^4\log^{k-1} r)\right), \qquad (3.147)
\end{aligned}$$

where $\gamma_{4,2}^{(1)}$ and $R_{4,2}^{(1)g/q}$ are given in (3.115), (3.78) and (3.79) respectively. Thus, we have a prediction for the spin-2 part of the leading term of $\mathcal{G}^{(n+1)}$ in the OPE limit, at each logarithmic order in $\log\zeta_{12}$.

Similarly, $(\gamma_*'^{(1)})^k$ terms in (3.140) also give the dominant contribution in the $z \ll 1$ limit. Following the same calculation as that of the OPE limit, we obtain (using

(3.118))

$$
\begin{aligned}
&\mathcal{G}^{(n+1)g}(\zeta_{12}, z \ll 1, \bar{z}) \\
&= \sum_{k=0}^{n} \frac{\log^{n-k}\zeta_{12}}{2^n k!(n-k)!}\times \\
&\left(\left(\gamma_{4,2}^{(1)}\right)^k \left(\frac{\alpha_+ \left(\gamma'_{*2}\right)^{n-k} - \alpha_- \left(\gamma'_{*1}\right)^{n-k}}{\alpha_+ - \alpha_-} R_{4,2}^{(1)g} + \frac{\left(\gamma'_{*1}\right)^{n-k} - \left(\gamma'_{*2}\right)^{n-k}}{\alpha_+ - \alpha_-} R_{4,2}^{(1)q}\right) z \log^k(z) k_6^{0,-1}(\bar{z})\right. \\
&\left. + O(z \log^{k-1}(z))\right),
\end{aligned} \tag{3.148}
$$

and similarly for $\mathcal{G}^{(n+1)q}$. It is also straightforward to repeat the analysis in the $z \ll \bar{z} \ll 1$ limit and predict the $z \log^{k-1}(z)\bar{z}^3 \log(\bar{z})$ term at each $\log \zeta_{12}$ order using (3.120).

3.5 Contact terms and Ward identities

In this section, we study Ward identities satisfied by the EEEC, and also compute the EEEC at $O(g^0)$ and $O(g^2)$ order. Note that conventionally $O(g^4)$ is called the "leading order" (LO) since it is the lowest order at which the EEEC is nonzero for generic detector positions. However, contributions at $O(g^0)$ and $O(g^2)$ also exist: they are proportional to delta functions, which we call "contact terms", and so only become nonzero in special configurations. It was shown in [17, 15, 16] that Ward identities can be used to determine the contact terms in the two-point EEC. Here, we perform a similar analysis for the EEEC, and use perturbation theory and Ward identities to obtain the EEEC at $O(g^0)$ and $O(g^2)$ order.

3.5.1 EEEC′ and Ward identities

To study Ward identities, it is more convenient to write the EEEC as a function of the explicit positions on the celestial sphere $\vec{n}_i$, instead of the cross-ratios ζ_{ij}. Thus, we define

$$
\begin{aligned}
\text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3) &= \frac{\int d^d x\, e^{ip\cdot x} \langle 0|O^\dagger(x)\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{E}(\vec{n}_3)O(0)|0\rangle}{(-p^2)^{\frac{3}{2}} \int d^d x\, e^{ip\cdot x}\langle 0|O^\dagger(x)O(0)|0\rangle} \\
&= \sum_{i,j,k} \int d\sigma \frac{E_i E_j E_k}{Q^3} \delta\left(\vec{n}_1, \frac{\vec{p}_i}{E_i}\right) \delta\left(\vec{n}_2, \frac{\vec{p}_j}{E_j}\right) \delta\left(\vec{n}_3, \frac{\vec{p}_k}{E_k}\right),
\end{aligned} \tag{3.149}
$$

where the spherical delta function $\delta(\vec{n}_1, \vec{n}_2)$ is defined by

$$
\int d\Omega_{\vec{n}_2} \delta\left(\vec{n}_1, \vec{n}_2\right) f(\vec{n}_2) = f(\vec{n}_1). \tag{3.150}
$$

The first line in (3.149) is a nonperturbative definition, while the second line is suitable for perturbation theory. As before, $d\sigma$ represents an integration over phase space weighted by the scattering cross section, and $(E_i, \vec{p}_i)$ are energy and momentum of outgoing particles. The parametrization (3.149) of the EEEC is related to (3.2) by

$$\text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3) = \frac{\sin\theta_1 \sin\theta_2 |\sin\phi|}{64\pi^2} \text{EEEC}(\zeta_{12}, \zeta_{13}, \zeta_{23}), \tag{3.151}$$

where

$$\vec{n}_1 = (\sin\theta_1, 0, \cos\theta_1), \quad \vec{n}_2 = (\sin\theta_2\cos\phi, \sin\theta_2\sin\phi, \cos\theta_2), \quad \vec{n}_3 = (0, 0, 1). \tag{3.152}$$

To derive Ward identities, we simply integrate (3.149) over $\vec{n}_1, \vec{n}_2, \vec{n}_3$ and use energy and momentum conservation. For example,

$$\int d\Omega_{\vec{n}_1} d\Omega_{\vec{n}_2} d\Omega_{\vec{n}_3} \text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3) = \sum_{i,j,k} \int d\sigma \frac{E_i E_j E_k}{Q^3} = 1, \tag{3.153}$$

where we have used energy conservation $\sum_i E_i = Q$. Similarly, we also have

$$\int d\Omega_{\vec{n}_1} d\Omega_{\vec{n}_2} d\Omega_{\vec{n}_3} (1 - \vec{n}_1 \cdot \vec{n}_2) \text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3) = \sum_{i,j,k} \int d\sigma \frac{(E_i E_j - \vec{p}_i \cdot \vec{p}_j) E_k}{Q^3} = 1, \tag{3.154}$$

where we have used momentum conservation $\sum_i \vec{p}_i = 0$. In this way, we can derive the following Ward identities:

$$\begin{aligned}
&\int d\Omega_{\vec{n}_1} d\Omega_{\vec{n}_2} d\Omega_{\vec{n}_3} \text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3) = 1, \\
&\int d\Omega_{\vec{n}_1} d\Omega_{\vec{n}_2} d\Omega_{\vec{n}_3} (1 - \vec{n}_1 \cdot \vec{n}_3) \text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3) = 1, \\
&\int d\Omega_{\vec{n}_1} d\Omega_{\vec{n}_2} d\Omega_{\vec{n}_3} (1 - \vec{n}_2 \cdot \vec{n}_3) \text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3) = 1, \\
&\int d\Omega_{\vec{n}_1} d\Omega_{\vec{n}_2} d\Omega_{\vec{n}_3} (1 - \vec{n}_1 \cdot \vec{n}_2) \text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3) = 1, \\
&\int d\Omega_{\vec{n}_1} d\Omega_{\vec{n}_2} d\Omega_{\vec{n}_3} (1 - \vec{n}_1 \cdot \vec{n}_3)(1 - \vec{n}_2 \cdot \vec{n}_3) \text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3) = 1, \\
&\int d\Omega_{\vec{n}_1} d\Omega_{\vec{n}_2} d\Omega_{\vec{n}_3} (1 - \vec{n}_1 \cdot \vec{n}_3)(1 - \vec{n}_1 \cdot \vec{n}_2) \text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3) = 1, \\
&\int d\Omega_{\vec{n}_1} d\Omega_{\vec{n}_2} d\Omega_{\vec{n}_3} (1 - \vec{n}_2 \cdot \vec{n}_3)(1 - \vec{n}_1 \cdot \vec{n}_2) \text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3) = 1.
\end{aligned} \tag{3.155}$$

Using the Ward identities, we immediately see that the $O(g^0)$ EEEC′ must be given by

$$\text{EEEC}'(\vec{n}_1,\vec{n}_2,\vec{n}_3) = \frac{1}{16\pi}\Big(\delta(\vec{n}_1,\vec{n}_2)\delta(\vec{n}_1,\vec{n}_3) + \delta(\vec{n}_1,\vec{n}_2)\delta(\vec{n}_1,-\vec{n}_3) + \delta(\vec{n}_1,\vec{n}_3)\delta(\vec{n}_3,-\vec{n}_2) + \delta(\vec{n}_2,\vec{n}_3)\delta(\vec{n}_2,-\vec{n}_1)\Big). \quad (3.156)$$

The structure of (3.156) is easy to understand. The first term is supported when all three detectors are coincident. The other three terms appear when two of the detectors are coincident and the other is diametrically opposite on the celestial sphere. Physically, the $O(g^0)$ EEEC gets contributions only from two particle states. By energy and momentum conservation, these particles must fly in opposite directions, and thus can only be observed by detectors that are either coincident (observing the same particle) or diametrically opposite (observing the two different particles).

3.5.2 Tree-level ($O(g^2)$) EEEC′ in $\mathcal{N} = 4$ SYM

Let us now consider the EEEC at $O(g^2)$. We work in $\mathcal{N} = 4$ SYM and consider the case where the sink/source states are created by the operator $\text{Tr}F^2$. At this order, there are at most three particles in the outgoing state. Therefore, if the detectors are at different positions, the three vectors $\vec{n}_1, \vec{n}_2, \vec{n}_3$ must be coplanar in order for the total momentum to be zero. It follows that the $O(g^2)$ EEEC must take the form:

$$\begin{aligned}
&\text{EEEC}'(\vec{n}_1,\vec{n}_2,\vec{n}_3)|_{O(g^2)} \\
&= \mathcal{F}_0(\vec{n}_1,\vec{n}_2,\vec{n}_3)\delta\left((\vec{n}_1\times\vec{n}_2)\cdot\vec{n}_3\right) \\
&\quad + \mathcal{F}_1(\vec{n}_1,\vec{n}_2)\delta\left(\vec{n}_1,\vec{n}_3\right) + \mathcal{F}_1(\vec{n}_1,\vec{n}_3)\delta\left(\vec{n}_2,\vec{n}_3\right) + \mathcal{F}_1(\vec{n}_2,\vec{n}_3)\delta\left(\vec{n}_1,\vec{n}_2\right) \\
&\quad + 2\pi c_1\delta(\vec{n}_1,\vec{n}_2)\delta(\vec{n}_1,\vec{n}_3) \\
&\quad + 2\pi c_2[\delta(\vec{n}_1,\vec{n}_2)\delta(\vec{n}_1,-\vec{n}_3) + \delta(\vec{n}_1,\vec{n}_3)\delta(\vec{n}_3,-\vec{n}_2) + \delta(\vec{n}_2,\vec{n}_3)\delta(\vec{n}_2,-\vec{n}_1)].
\end{aligned} \quad (3.157)$$

On the right-hand side, the first line describes the configuration where the detectors are coplanar. The second line are the contact terms that appear when two of the detectors are at the same position and the third detector is at a generic position. The third and the fourth lines appear in the same configurations as (3.156), and can be thought of as the higher-order corrections.

Using perturbation theory, we can obtain the functions $\mathcal{F}_1(\vec{n}_1,\vec{n}_2)$ and $\mathcal{F}_0(\vec{n}_1,\vec{n}_2,\vec{n}_3)$.

We leave the details of the calculation in appendix B.5.1. The result for $\mathcal{F}_1$ is[18]

$$\mathcal{F}_1(\zeta) = -\frac{\zeta(-60+102\zeta-44\zeta^2+3\zeta^3)+(-60+132\zeta-90\zeta^2+19\zeta^3)\log(1-\zeta)}{256\pi^4(1-\zeta)\zeta^6}. \tag{3.158}$$

This expression is only valid for $0 < \zeta < 1$. The final expression should be a distribution and include contact terms at $\zeta = 0$ and $\zeta = 1$. To see the contact terms, we can further separate $\mathcal{F}_1$ into a singular part and a regular part:

$$\mathcal{F}_1(\zeta) = \frac{1}{512\pi^4\zeta} - \frac{1+\log(1-\zeta)}{256\pi^4(1-\zeta)} + \mathcal{F}_1^{\text{reg}}(\zeta). \tag{3.159}$$

The singular terms can be interpreted as

$$\mathcal{F}_1^{\text{sing}}(\zeta) = \frac{1}{512\pi^4}\left[\frac{1}{\zeta}\right]_0 - \frac{1}{256\pi^4}\left[\frac{1}{1-\zeta}\right]_1 - \frac{1}{256\pi^4}\left[\frac{\log(1-\zeta)}{1-\zeta}\right]_1 + a_1\delta(\zeta) + b_1\delta(1-\zeta), \tag{3.160}$$

where the distribution $\left[\frac{1}{\zeta}\right]_0$ is defined as the unique distribution that agrees with $\frac{1}{\zeta}$ for $\zeta > 0$ and satisfies

$$\int_0^1 d\zeta \left[\frac{1}{\zeta}\right]_0 = 0, \tag{3.161}$$

and $[\cdots]_1$ is defined in a similar way with $\zeta \to 1-\zeta$. The expressions (3.159) and (3.160) now specify $\mathcal{F}_1(\zeta)$ as a distribution. However, this distribution depends on unknown coefficients a_1, b_1. We will determine them using Ward identities.

Now consider the function $\mathcal{F}_0$. We find that it can be written as[19]

$$\mathcal{F}_0(\zeta_1,\zeta_2) = \sqrt{\zeta_1\zeta_2(1-\zeta_1)(1-\zeta_2)}\,\widetilde{\mathcal{F}}_0(\zeta_1,\zeta_2)\theta(\zeta_1+\zeta_2-1), \tag{3.162}$$

where $\theta(\cdots)$ is a step function (we explain its appearance in appendix B.5.1). The function $\widetilde{\mathcal{F}}_0$ is given by

$$\begin{aligned}\widetilde{\mathcal{F}}_0(\zeta_1,\zeta_2) = {} & \frac{1}{256\pi^4\zeta_1^2\zeta_2^2\sqrt{\zeta_1\zeta_2(1-\zeta_1)(1-\zeta_2)}\left(\sqrt{\zeta_1(1-\zeta_2)}+\sqrt{\zeta_2(1-\zeta_1)}\right)^4} \\ & \times\Big(-14\zeta_1^2\zeta_2^2 - 18\zeta_1\zeta_2 + 19\zeta_1\zeta_2(\zeta_1+\zeta_2) - 6(\zeta_1+\zeta_2)^2 + 12(\zeta_1+\zeta_2) - 6 \\ & + 2\sqrt{\zeta_1\zeta_2(1-\zeta_1)(1-\zeta_2)}\,(6-6(\zeta_1+\zeta_2)+7\zeta_1\zeta_2)\Big).\end{aligned} \tag{3.163}$$

[18] Note that $\mathcal{F}_1(\vec{n}_1,\vec{n}_2)$ is invariant under an overall rotation of $\vec{n}_1,\vec{n}_2$, so it can be written as a function of the cross ratio $\zeta = \frac{1-\vec{n}_1\cdot\vec{n}_2}{2}$.

[19] Again due to rotational invariance, we can write $\mathcal{F}_0$ as a function of two cross ratios ζ_1,ζ_2, where $\zeta_1 = \frac{1-\vec{n}_1\cdot\vec{n}_3}{2}$, $\zeta_2 = \frac{1-\vec{n}_2\cdot\vec{n}_3}{2}$.

It is convenient to study $\widetilde{\mathcal{F}}_0$ instead of $\mathcal{F}_0$. To interpret $\widetilde{\mathcal{F}}_0$ as a distribution, we again separate it into a singular part and regular part as $\widetilde{\mathcal{F}}_0 = \widetilde{\mathcal{F}}_0^{\text{sing}} + \widetilde{\mathcal{F}}_0^{\text{reg}}$. The singular part is given by

$$\widetilde{\mathcal{F}}_0^{\text{sing}}(\zeta_1, \zeta_2) = \frac{f(\theta_1)}{r_1^2} + \frac{f(\theta_2)}{r_2^2} + \frac{g(\theta_3)}{r_3^2}, \tag{3.164}$$

where

$$\begin{aligned} r_1 &= \sqrt{\zeta_1^2 + (1-\zeta_2)^2}, & \theta_1 &= \tan^{-1}\left(\frac{1-\zeta_2}{\zeta_1}\right), \\ r_2 &= \sqrt{\zeta_2^2 + (1-\zeta_1)^2}, & \theta_2 &= \tan^{-1}\left(\frac{1-\zeta_1}{\zeta_2}\right), \\ r_3 &= \sqrt{(1-\zeta_1)^2 + (1-\zeta_2)^2}, & \theta_3 &= \tan^{-1}\left(\frac{1-\zeta_2}{1-\zeta_1}\right). \end{aligned} \tag{3.165}$$

(r_i, θ_i) are the polar coordinates with respect to the points where $\widetilde{\mathcal{F}}_0$ becomes

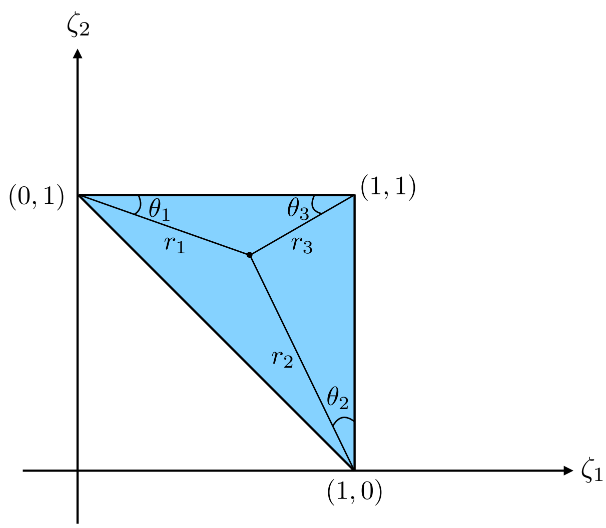

Figure 3.3: Definition of (r_i, θ_i). Due to the step function in (3.162), $\mathcal{F}_0(\zeta_1, \zeta_2)$ is nonzero only in the blue region. The three points $(1, 0), (0, 1), (1, 1)$ are where $\mathcal{F}_0(\zeta_1, \zeta_2)$ becomes singular.

singular. See figure 3.3. The two functions in (3.164) are given by

$$\begin{aligned} f(\theta) &= \frac{1}{256\pi^4 \sqrt{\sin(\theta)} \cos^{\frac{3}{2}}(\theta)}, \\ g(\theta) &= \frac{1}{256\pi^4 \sqrt{\sin(\theta)}\sqrt{\cos(\theta)} \left(\sqrt{\sin(\theta)} + \sqrt{\cos(\theta)}\right)^2}. \end{aligned} \tag{3.166}$$

One can then show that $\widetilde{\mathcal{F}}_0^{\text{sing}}$ as a distribution should be given by

$$\widetilde{\mathcal{F}}_0^{\text{sing}}(\zeta_1,\zeta_2) = a_0\left(\frac{\delta(r_1)}{r_1}+\frac{\delta(r_2)}{r_2}\right)+b_0\frac{\delta(r_3)}{r_3}+\frac{f_0(\theta_1)}{r_1}\left[\frac{1}{r_1}\right]_0+\frac{f_0(\theta_2)}{r_2}\left[\frac{1}{r_2}\right]_0+\frac{g_0(\theta_3)}{r_3}\left[\frac{1}{r_3}\right]_0. \tag{3.167}$$

Moreover, from (3.157) we see that $\mathcal{F}_0$ must be crossing symmetric. This condition implies[20]

$$b_0-\frac{1}{2}a_0 = \frac{\pi-\sqrt{2}\pi+2\log 2+\sqrt{2}\log(3-2\sqrt{2})}{256\pi^5}\approx 0.00003291. \tag{3.168}$$

Note that although we have six unknown coefficients $(c_1, c_2, a_1, b_1, a_0, b_0)$, there are actually only two types of contact terms in (3.157). By integrating against test functions, we find

$$\begin{aligned}
\frac{\delta(r_1)}{r_1}\sqrt{\zeta_1\zeta_2(1-\zeta_1)(1-\zeta_2)}\delta\left((\vec{n}_1\times\vec{n}_2)\cdot\vec{n}_3\right) &= \frac{\pi^2}{2}\delta\left(\vec{n}_1,\vec{n}_3\right)\delta\left(\vec{n}_2,-\vec{n}_3\right),\\
\frac{\delta(r_2)}{r_2}\sqrt{\zeta_1\zeta_2(1-\zeta_1)(1-\zeta_2)}\delta\left((\vec{n}_1\times\vec{n}_2)\cdot\vec{n}_3\right) &= \frac{\pi^2}{2}\delta\left(\vec{n}_1,-\vec{n}_3\right)\delta\left(\vec{n}_2,\vec{n}_3\right),\\
\frac{\delta(r_3)}{r_3}\sqrt{\zeta_1\zeta_2(1-\zeta_1)(1-\zeta_2)}\delta\left((\vec{n}_1\times\vec{n}_2)\cdot\vec{n}_3\right) &= \pi^2\delta\left(\vec{n}_1,-\vec{n}_3\right)\delta\left(\vec{n}_2,-\vec{n}_3\right),
\end{aligned} \tag{3.169}$$

and in addition

$$\begin{aligned}
\delta\left(\tfrac{1-\vec{n}_1\cdot\vec{n}_2}{2}\right)\delta\left(\vec{n}_2,\vec{n}_3\right) &= 4\pi\delta(\vec{n}_1,\vec{n}_2)\delta(\vec{n}_2,\vec{n}_3),\\
\delta\left(1-\tfrac{1-\vec{n}_1\cdot\vec{n}_2}{2}\right)\delta\left(\vec{n}_2,\vec{n}_3\right) &= 4\pi\delta(\vec{n}_1,-\vec{n}_2)\delta(\vec{n}_2,\vec{n}_3).
\end{aligned} \tag{3.170}$$

Collecting all the contact terms together, we find

$$\begin{aligned}
&2\pi\left(c_1+6a_1\right)\delta(\vec{n}_1,\vec{n}_2)\delta(\vec{n}_2,\vec{n}_3)\\
&+2\pi\left(c_2+2b_1+\frac{\pi}{4}a_0\right)\left(\delta(\vec{n}_1,\vec{n}_3)\delta(\vec{n}_2,-\vec{n}_3)+\delta(\vec{n}_1,-\vec{n}_3)\delta(\vec{n}_2,\vec{n}_3)\right)\\
&+2\pi\left(c_2+2b_1+\frac{\pi}{2}b_0\right)\delta\left(\vec{n}_1,-\vec{n}_3\right)\delta\left(\vec{n}_2,-\vec{n}_3\right).
\end{aligned} \tag{3.171}$$

Only the above linear combinations of coefficients are physically meaningful. Note that the coefficients of $\delta(\vec{n}_1,\vec{n}_3)\delta(\vec{n}_2,-\vec{n}_3)$ and $\delta\left(\vec{n}_1,-\vec{n}_3\right)\delta\left(\vec{n}_2,-\vec{n}_3\right)$ are different because of our choice of coordinates (3.165) to write the distributions $[\cdots]_0$. These coordinates are convenient for analyzing the singularities of $\mathcal{F}_0$, but they do not manifest crossing symmetry. When we perform a crossing transformation, we

[20] See appendix B.5 for a derivation.

rescale the arguments of the distributions $[\cdots]_0$, which can produce δ-functions via $[1/(\lambda x)]_0 = \lambda^{-1}[1/x]_0 - (\log\lambda/\lambda)\delta(x)$. Taking these δ-functions into account, one can show that as long as (3.168) is satisfied, the full expression is crossing symmetric. See appendix B.5.2 for details.

We can now use the Ward identities (3.155) to determine the coefficients of the contact terms. Plugging (3.157) into (3.155), we obtain

$$\begin{aligned}
&\int_0^1 d\zeta_1 \int_0^1 d\zeta_2\, \widetilde{\mathcal{F}}_0(\zeta_1,\zeta_2)\theta(\zeta_1+\zeta_2-1) + 6\int_0^1 d\zeta\, \mathcal{F}_1(\zeta) + c_1 + 3c_2 = 0,\\
&2\int_0^1 d\zeta_1 \int_0^1 d\zeta_2\, \zeta_1\widetilde{\mathcal{F}}_0(\zeta_1,\zeta_2)\theta(\zeta_1+\zeta_2-1) + 8\int_0^1 d\zeta\, \zeta\mathcal{F}_1(\zeta) + 4c_2 = 0,\\
&4\int_0^1 d\zeta_1 \int_0^1 d\zeta_2\, \zeta_1\zeta_2\widetilde{\mathcal{F}}_0(\zeta_1,\zeta_2)\theta(\zeta_1+\zeta_2-1) + 8\int_0^1 d\zeta\, \zeta^2\mathcal{F}_1(\zeta) + 4c_2 = 0,
\end{aligned} \tag{3.172}$$

where $\widetilde{\mathcal{F}}_0(\zeta_1,\zeta_2)$ is defined by (3.162). The integrals in (3.172) can be computed using (3.158), (3.160), (3.163), and (3.164). The integrals of the $\widetilde{\mathcal{F}}_0$ function are given by

$$\begin{aligned}
\int_0^1 d\zeta_1 \int_0^1 d\zeta_2\, \widetilde{\mathcal{F}}_0(\zeta_1,\zeta_2)\theta(\zeta_1+\zeta_2-1) &= \frac{\pi}{2}(a_0+b_0) + 0.00011205,\\
\int_0^1 d\zeta_1 \int_0^1 d\zeta_2\, \zeta_1\widetilde{\mathcal{F}}_0(\zeta_1,\zeta_2)\theta(\zeta_1+\zeta_2-1) &= \frac{\pi}{2}\left(\frac{a_0}{2}+b_0\right) + 0.0000815314,\\
\int_0^1 d\zeta_1 \int_0^1 d\zeta_2\, \zeta_1\zeta_2\widetilde{\mathcal{F}}_0(\zeta_1,\zeta_2)\theta(\zeta_1+\zeta_2-1) &= \frac{\pi}{2}b_0 + 0.0000884406,
\end{aligned} \tag{3.173}$$

and for the $\mathcal{F}_1$ integrals we get

$$\begin{aligned}
\int_0^1 d\zeta\mathcal{F}_1(\zeta) &= \frac{51+10\pi^2}{15360\pi^4} + a_1 + b_1,\\
\int_0^1 d\zeta\, \zeta\mathcal{F}_1(\zeta) &= \frac{3+2\pi^2}{3072\pi^4} + b_1,\\
\int_0^1 d\zeta\, \zeta^2\mathcal{F}_1(\zeta) &= \frac{-4+\pi^2}{1536\pi^4} + b_1.
\end{aligned} \tag{3.174}$$

Using these results, we find the solution to (3.172):

$$\begin{aligned}
b_0 - \frac{a_0}{2} &= 0.0000329062,\\
c_2 + 2b_1 + \frac{\pi}{4}a_0 &= -0.00021859,\\
c_1 + 6a_1 &= -0.000108274.
\end{aligned} \tag{3.175}$$

We see that the first line is consistent with (3.168), which comes from crossing symmetry of $\mathcal{F}_0(\zeta_1, \zeta_2)$. The second and the third line give the coefficients of the two types of contact terms that can appear in $\text{EEEC}'|_{O(g^2)}$.

3.5.3 Contact terms from the conformal block decomposition

It is interesting to ask how the δ-functions in $\text{EEEC}'|_{O(g^2)}$ are reproduced from the conformal block decomposition described in section 3.2. Since the conformal block decomposition describes the leading term in the collinear limit, we should study those δ-functions that survive when all $\vec{n}_i$'s are close to each other. More precisely, we have an expansion around the squeezed limit, where we first take $\vec{n}_2$ and $\vec{n}_3$ close together, and then $\vec{n}_1$.

Using (3.42), the collinear limit is

$$\text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3)|_{O(g^2)} = \frac{1}{32\pi^2} \frac{\zeta_{12}}{\zeta_{23}^3} \sum_{\delta,j} R_{\delta,j}|_{O(g^2)} g_{\delta,j}^{\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{P}_{\delta'_*}}(z, \bar{z}) + \dots , \quad (3.176)$$

where $z, \bar{z}$ are defined by

$$\frac{\zeta_{23}}{\zeta_{12}} = z\bar{z}, \qquad \frac{\zeta_{13}}{\zeta_{12}} = (1-z)(1-\bar{z}). \quad (3.177)$$

The small $z, \bar{z}$ limit corresponds to the limit where $\vec{n}_2$ and $\vec{n}_3$ become close. The contact term $\mathcal{F}_1(\vec{n}_1, \vec{n}_2)\delta(\vec{n}_2, \vec{n}_3)$ can then appear from the exchanged quantum numbers $\delta = 4$, $j = 0$. More precisely, we have

$$\text{EEEC}'(\vec{n}_1, \vec{n}_2, \vec{n}_3)|_{O(g^2)} \sim \frac{g^2 N_c}{512\pi^5} \frac{1}{\zeta_{12}^2} \lim_{\delta\to 4} \langle R_{\delta,0}^{(0)} \rangle (z\bar{z})^{\frac{\delta}{2}-3} = \frac{g^2 N_c}{64\pi^4} \left(\lim_{\delta\to 4} \frac{\langle R_{\delta,0}^{(0)} \rangle}{\delta - 4} \right) \frac{1}{\zeta_{12}} \delta(\vec{n}_2, \vec{n}_3). \quad (3.178)$$

This result should agree with the $\frac{1}{\zeta}$ term of $\mathcal{F}_1(\zeta)$ in (3.160). Matching the two expressions, we obtain

$$\lim_{\delta\to 4} \frac{\langle R_{\delta,0}^{(0)} \rangle}{\delta - 4} = \frac{1}{8}. \quad (3.179)$$

Thus, even though $\langle R_{4,0}^{(0)} \rangle$ vanishes, the zero of $\langle R_{\delta,0}^{(0)} \rangle$ at $\delta = 4$ is related to the EEEC at $O(g^2)$ order. It would be interesting to verify (3.179) using other methods.

3.6 EEEC at strong coupling

We now consider the EEEC at strong coupling. In [12], Hofman and Maldacena computed the EEEC at strong coupling in $\mathcal{N} = 4$ SYM up to $O(\lambda^{-3/2})$ order using

AdS/CFT for a sink/source state created by a massless closed string. In this paper, we focus on the leading order and $O(1/\lambda)$ correction. In terms of (3.149), the result of [12] is

$$\begin{aligned}&\mathrm{EEEC}'_{\text{strong}}(\vec{n}_1,\vec{n}_2,\vec{n}_3)\\&=\left(\frac{1}{4\pi}\right)^3\left(1+\frac{6\pi^2}{\lambda}\left((\vec{n}_1\cdot\vec{n}_2)^2+(\vec{n}_1\cdot\vec{n}_3)^2+(\vec{n}_2\cdot\vec{n}_3)^2-1\right)+O(\lambda^{-3/2})\right).\end{aligned} \tag{3.180}$$

Let us lift this expression into a more covariant form, using embedding space coordinates for the celestial sphere $z_i \in \mathbb{R}^{d-1,1}$ with $z_i^2 = 0$. We define

$$\mathcal{F}(z_1,z_2,z_3,p) \equiv \frac{8\int d^dx\, e^{ip\cdot x}\langle 0|\mathcal{O}^\dagger(x)\mathbf{L}[T](\infty,z_1)\mathbf{L}[T](\infty,z_2)\mathbf{L}[T](\infty,z_3)\mathcal{O}(0)|0\rangle}{(-p^2)^{\frac{3}{2}}\int d^dx\, e^{ip\cdot x}\langle 0|\mathcal{O}^\dagger(x)\mathcal{O}(0)|0\rangle}, \tag{3.181}$$

which is a homogeneous function of the z_i. The original EEEC can be recovered by specializing: $\mathcal{F}(z_i = (1,\vec{n}_i), p = (1,\vec{0})) = \mathrm{EEEC}'(\vec{n}_i)$. In embedding space coordinates, the expression (3.180) becomes

$$\begin{aligned}&\mathcal{F}_{\text{strong}}(z_1,z_2,z_3,p)\\&=\frac{8}{\pi^3}\frac{1}{(-2z_1\cdot p)^3(-2z_2\cdot p)^3(-2z_3\cdot p)^3}\left(1+\frac{6\pi^2}{\lambda}\left((1-2\zeta_{12})^2+(1-2\zeta_{23})^2+(1-2\zeta_{13})^2-1\right)\right)\\&\quad+O(\lambda^{-3/2}),\end{aligned} \tag{3.182}$$

where the cross ratios ζ_{ij} are defined in (3.19). For this section, we also assume that $-p^2 = 1$, since the $-p^2$ factors can be easily restored using dimensional analysis of p.

In the weak coupling limit, the EEEC at tree-level is only nonzero when the three detectors are coplanar, and the EEEC at 1-loop it is only known in the collinear limit. On the other hand, the strong-coupling EEEC given in (3.182) is valid for any detector positions on the celestial sphere.

Just as Mean Field Theory provides a simple example of a crossing-symmetric, conformally-invariant four-point function, the strong-coupling EEEC given in (3.182) gives a simple example of a crossing-symmetric, Lorentz invariant EEEC. It is therefore a perfect target for us to study its celestial block expansion and test the results of section 3.2. Because the strong-coupling EEEC is so simple, we will be able to compute its *complete* 3-point celestial block expansion — i.e. not just its conformal block expansion in the collinear limit.

The celestial block expansion can be written as[21]

$$\mathcal{F}_{\text{strong}}(z_1, z_2, z_3, p) = \sum_{\delta'} \sum_{\delta, j} P_{\delta,j;\delta'} G^c_{\delta,j;\delta'}(z_1, z_2, z_3, p), \tag{3.183}$$

where the celestial block is defined as

$$G^c_{\delta,j;\delta'}(z_1, z_2, z_3, p) = C_{12\mathcal{P}_{\delta,j}}(z_1, z_2, \partial_{z_2}, \partial_{w_2}) C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}(z_2, z_3, \partial_{z_3})(-2z_3 \cdot p)^{-\delta'}. \tag{3.184}$$

Let us first focus on the $j = 0$ sector of (3.183). The light-ray OPE formula gives a relation between the OPE data and the value of δ in the sum. In the strong coupling limit, the $T \times T$ OPE should contain double-trace operators with twists $\tau = 2\tau_T + 2n = 4 + 2n$. Therefore, the values of δ appearing in (3.183) should be

$$\delta = \Delta(J = 3) - 1 = 6 + 2n. \tag{3.185}$$

Similarly, the $T \times T \times T$ OPE should contain triple-trace operators with twists $\tau = 3\tau_T + 2k = 6 + 2k$. As argued in section 3.3.1, we expect that the values of δ' appearing in (3.183) are given by

$$\delta' = \Delta(J = 4) - 1 = 9 + 2k. \tag{3.186}$$

To study the celestial block expansion of the strong-coupling EEEC, we will use harmonic analysis for the Euclidean conformal group $\mathrm{SO}(d-1, 1)$ [44]. A modern review of harmonic analysis for Euclidean CFTs is given in [45], where they derive a Euclidean inversion formula that expresses OPE data as an integral of CFT four-point functions over Euclidean space. In this section, we use techniques from [45] to derive a "celestial inversion formula" that expresses the celestial block expansion data as an integral of the EEEC over the celestial sphere. We then consider the strong coupling limit and use this celestial inversion formula to obtain the celestial block expansion (3.183) for the strong-coupling EEEC (3.182).

3.6.1 Celestial inversion formula

We first derive the celestial inversion formula, following the derivation of the Euclidean inversion formula in section 2 of [45]. By harmonic analysis for $\mathrm{SO}(d-1, 1)$, the EEEC $\mathcal{F}(z_1, z_2, z_3, p)$ defined in (3.181) can be written as an integral of the form

$$\mathcal{F}(z_1, z_2, z_3, p) = \sum_j \int_{\frac{d-2}{2}}^{\frac{d-2}{2}+i\infty} \frac{d\delta}{2\pi i} \int_{\frac{d-2}{2}}^{\frac{d-2}{2}+i\infty} \frac{d\delta'}{2\pi i} I(\delta, j; \delta') \Psi^c_{\delta,j;\delta'}(z_1, z_2, z_3, p), \tag{3.187}$$

[21] The relation between $P_{\delta,j;\delta'}$ and the coefficient $R_{\delta,j;\delta'}$ defined in section 3.2 is $P_{\delta,j;\delta'} = \frac{16}{\pi^2} R_{\delta,j;\delta'}$.

where $\Psi^c_{\delta,j;\delta'}$ is a "celestial partial wave" defined by

$$\Psi^c_{\delta,j;\delta'}(z_1,z_2,z_3,p)$$
$$\equiv \int D^{d-2}z D^{d-2}z' \langle \mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z)\rangle\langle \widetilde{\mathcal{P}}_{\delta,j}(z)\mathcal{P}_{\delta_3}(z_3)\mathcal{P}_{\delta'}(z')\rangle \frac{1}{(-2z'\cdot p)^{\widetilde{\delta}'}}, \tag{3.188}$$

where $\widetilde{\mathcal{P}}_{\delta,j}$ is the shadow representation of $\mathcal{P}_{\delta,j}$, with scaling dimension $\widetilde{\delta}=d-2-\delta$. The measure $D^{d-2}z$ is defined by

$$D^{d-2}z = \frac{2d^d z\,\delta(z^2)\theta(z^0)}{\mathrm{vol}\,\mathbb{R}_+}, \tag{3.189}$$

where $\mathbb{R}_+$ acts by rescaling z. Note that the operators $\mathcal{P}_{\delta,j}(z)$ and $\widetilde{\mathcal{P}}_{\delta,j}(z)$ each carry j tangent-space indices on the celestial sphere. These indices are implicitly contracted in (3.188) and below.

By construction, Ψ^c is an eigenfunction of the Casimirs of $\mathrm{SO}(d-1,1)$, acting simultaneously on z_1, z_2 and simultaneously on z_1, z_2, z_3. So, we can study the behavior of Ψ^c in the OPE limit to determine its relation to the celestial block G^c. Following the logic in [45], the relation is

$$\begin{aligned}\Psi^c_{\delta,j;\delta'} =& S(\mathcal{P}_{\delta_3}\mathcal{P}_{\delta'}[\widetilde{\mathcal{P}}_{\delta,j}])I_{\delta'}(p)G^c_{\delta,j;\delta'} + S(\mathcal{P}_{\delta_3}\mathcal{P}_{\delta'}[\widetilde{\mathcal{P}}_{\delta,j}])S(\mathcal{P}_{\delta,j}\mathcal{P}_{\delta_3}[\mathcal{P}_{\delta'}])G^c_{\delta,j;\widetilde{\delta}'}\\ &+S(\mathcal{P}_{\delta_1}\mathcal{P}_{\delta_2}[\mathcal{P}_{\delta,j}])I_{\delta'}(p)G^c_{\widetilde{\delta},j;\delta'} + S(\mathcal{P}_{\delta_1}\mathcal{P}_{\delta_2}[\mathcal{P}_{\delta,j}])S(\widetilde{\mathcal{P}}_{\delta,j}\mathcal{P}_{\delta_3}[\mathcal{P}_{\delta'}])G^c_{\widetilde{\delta},j;\widetilde{\delta}'},\end{aligned} \tag{3.190}$$

where the coefficients $S(\cdots), I_{\delta'}(p)$ are defined by

$$\int D^{d-2}z\langle\widetilde{\mathcal{P}}(z')\widetilde{\mathcal{P}}(z)\rangle\langle\mathcal{P}_1\mathcal{P}_2\mathcal{P}(z)\rangle = S(\mathcal{P}_1\mathcal{P}_2[\mathcal{P}])\langle\mathcal{P}_1\mathcal{P}_2\widetilde{\mathcal{P}}(z')\rangle$$
$$\int D^{d-2}z'\langle\mathcal{P}_{\delta'}(z_3)\mathcal{P}_{\delta'}(z')\rangle\frac{1}{(-2z'\cdot p)^{\widetilde{\delta}'}} = I_{\delta'}(p)(-2z_3\cdot p)^{-\delta'}. \tag{3.191}$$

More explicitly, their expressions are

$$\begin{aligned}S(\mathcal{P}_1\mathcal{P}_2[\mathcal{P}_{\delta,j}]) =& \frac{\pi^{\frac{d-2}{2}}\Gamma(\delta-\frac{d-2}{2})\Gamma(\delta+j-1)\Gamma(\frac{\widetilde{\delta}+\delta_1-\delta_2+j}{2})\Gamma(\frac{\widetilde{\delta}-\delta_1+\delta_2+j}{2})}{\Gamma(\delta-1)\Gamma(d-2-\delta+j)\Gamma(\frac{\delta+\delta_1-\delta_2+j}{2})\Gamma(\frac{\delta-\delta_1+\delta_2+j}{2})},\\ S(\mathcal{P}_{\delta,j}\mathcal{P}_3[\mathcal{P}_{\delta'}]) =& \frac{\pi^{\frac{d-2}{2}}\Gamma(\delta'-\frac{d-2}{2})\Gamma(\frac{\widetilde{\delta}'+\delta_3-\delta+j}{2})\Gamma(\frac{\widetilde{\delta}'-\delta_3+\delta+j}{2})}{\Gamma(d-2-\delta')\Gamma(\frac{\delta'+\delta_3-\delta+j}{2})\Gamma(\frac{\delta'-\delta_3+\delta+j}{2})},\\ I_{\delta'}(p) =& \frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d-2}{2}-\delta')}{\Gamma(d-2-\delta')}(-p^2)^{\delta'-\frac{d-2}{2}}.\end{aligned} \tag{3.192}$$

Just like the four-point conformal partial wave, the celestial partial wave Ψ^c satisfies an orthogonality relation that can be derived using a "bubble" formula. Consider two celestial partial waves $\Psi^c_{\delta_5,j_5;\delta'_5}$ and $\Psi^{c(\widetilde{\delta}_i)}_{\widetilde{\delta}_6,j_6;\widetilde{\delta}'_6}$, where $\delta_{5,6} = \frac{d-2}{2} + is_{5,6}, \delta'_{5,6} = \frac{d-2}{2} + is'_{5,6}$ are on the principal series and the external dimensions of $\Psi^{c(\widetilde{\delta}_i)}$ are the shadows of Ψ^c. The orthogonality relation is (see Appendix B.6 for a derivation)

$$\begin{aligned}
&\int \frac{D^{d-2}z_1 D^{d-2}z_2 D^{d-2}z_3 d^{d-1}_{\mathrm{AdS}}p}{\mathrm{vol}(\mathrm{SO}(d-1,1))} \Psi^c_{\delta_5,j_5;\delta'_5}(z_1,z_2,z_3,p)\Psi^{c(\widetilde{\delta}_i)}_{\widetilde{\delta}_6,j_6;\widetilde{\delta}'_6}(z_1,z_2,z_3,p) \\
&= \frac{1}{2^{d-2}\,\mathrm{vol}(\mathrm{SO}(d-2))} B_{12\mathcal{P}_{\delta_5,j_5}} B_{\widetilde{\mathcal{P}}_{\delta_5,j_5}3\mathcal{P}_{\delta'_5}} \delta_{\mathcal{P}_5\mathcal{P}_6}\delta_{\mathcal{P}'_5\mathcal{P}'_6},
\end{aligned} \tag{3.193}$$

where the integral measure for p and the delta function $\delta_{\mathcal{P}_5\mathcal{P}_6}$ are defined by

$$\begin{aligned}
d^{d-1}_{AdS}p &= 2d^d p\,\delta(p^2+1)\theta(p^0), \\
\delta_{\mathcal{P}_5\mathcal{P}_6} &= 2\pi\delta(s_5-s_6)\delta_{j_5,j_6}.
\end{aligned} \tag{3.194}$$

The B-coefficients are "bubble" coefficients defined by

$$B_{12\mathcal{P}_{\delta,j}} = \frac{1}{\mu(\delta,j)}\left(\langle\mathcal{P}_1\mathcal{P}_2\mathcal{P}_{\delta,j}\rangle, \langle\widetilde{\mathcal{P}}_1\widetilde{\mathcal{P}}_2\widetilde{\mathcal{P}}_{\delta,j}\rangle\right), \tag{3.195}$$

where $\mu(\delta,j)$ is the Plancherel measure of $\mathrm{SO}(d-1,1)$, and $\left(\langle\mathcal{P}_1\mathcal{P}_2\mathcal{P}_{\delta,j}\rangle, \langle\widetilde{\mathcal{P}}_1\widetilde{\mathcal{P}}_2\widetilde{\mathcal{P}}_{\delta,j}\rangle\right)$ is a conformally-invariant three-point pairing. Their explicit expressions are given in [45].

Integrating both sides of celestial partial wave expansion (3.187) against a celestial partial wave $\Psi^{c(\widetilde{\delta}_i)}_{\widetilde{\delta},j;\widetilde{\delta}'}$, the orthogonality relation (3.193) gives

$$\begin{aligned}
&I(\delta,j;\delta') \\
&= \frac{2^{d-2}\,\mathrm{vol}(\mathrm{SO}(d-2))}{B_{12\mathcal{P}_{\delta,j}}B_{\widetilde{\mathcal{P}}_{\delta,j}3\mathcal{P}_{\delta'}}} \int \frac{D^{d-2}z_1 D^{d-2}z_2 D^{d-2}z_3 d^{d-1}_{\mathrm{AdS}}p}{\mathrm{vol}(\mathrm{SO}(d-1,1))} \mathcal{F}(z_1,z_2,z_3,p)\Psi^{c(\widetilde{\delta}_i)}_{\widetilde{\delta},j;\widetilde{\delta}'}(z_1,z_2,z_3,p).
\end{aligned} \tag{3.196}$$

This is the celestial inversion formula that expresses the celestial partial wave expansion data $I(\delta,j;\delta')$ as an integral of the EEEC $\mathcal{F}(z_1,z_2,z_3,p)$ over the celestial sphere. Finally, we must find the relation between the celestial partial wave expansion (3.187) and the celestial block expansion (3.183). Plugging (3.190) into

(3.187), we obtain

$$\mathcal{F}(z_1,z_2,z_3,p) = \sum_j \int_{\frac{d-2}{2}}^{\frac{d-2}{2}+i\infty} \frac{d\delta}{2\pi i} \int_{\frac{d-2}{2}}^{\frac{d-2}{2}+i\infty} \frac{d\delta'}{2\pi i} I(\delta,j;\delta')\times \\ \Big(S(\mathcal{P}_{\delta_3}\mathcal{P}_{\delta'}[\widetilde{\mathcal{P}}_{\delta,j}]) I_{\delta'}(p) G^c_{\delta,j;\delta'} + S(\mathcal{P}_{\delta_3}\mathcal{P}_{\delta'}[\widetilde{\mathcal{P}}_{\delta,j}]) S(\mathcal{P}_{\delta,j}\mathcal{P}_{\delta_3}[\mathcal{P}_{\delta'}]) G^c_{\delta,j;\widetilde{\delta}'} \\ + S(\mathcal{P}_{\delta_1}\mathcal{P}_{\delta_2}[\mathcal{P}_{\delta,j}]) I_{\delta'}(p) G^c_{\widetilde{\delta},j;\delta'} + S(\mathcal{P}_{\delta_1}\mathcal{P}_{\delta_2}[\mathcal{P}_{\delta,j}]) S(\widetilde{\mathcal{P}}_{\delta,j}\mathcal{P}_{\delta_3}[\mathcal{P}_{\delta'}]) G^c_{\widetilde{\delta},j;\widetilde{\delta}'} \Big). \tag{3.197}$$

Using the inversion formula (3.196), one can show that above expression remains the same when we keep only the first term $S(\mathcal{P}_{\delta_3}\mathcal{P}_{\delta'}[\widetilde{\mathcal{P}}_{\delta,j}]) I_{\delta'}(p) G^c_{\delta,j;\delta'}$ in the parentheses and extend the integration ranges to $\int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty}$. Therefore, we have

$$\mathcal{F}(z_1,z_2,z_3,p) = \sum_j \int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty} \frac{d\delta}{2\pi i} \int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty} \frac{d\delta'}{2\pi i} C(\delta,j;\delta') G^c_{\delta,j;\delta'}(z_1,z_2,z_3,p), \tag{3.198}$$

where

$$C(\delta,j;\delta') \equiv I(\delta,j;\delta') S(\mathcal{P}_{\delta_3}\mathcal{P}_{\delta'}[\widetilde{\mathcal{P}}_{\delta,j}]) I_{\delta'}(p). \tag{3.199}$$

Finally, we can close the contour of the δ and δ' integrals in (3.198) to the right and obtain the celestial block expansion.[22] In particular, for the strong-coupling EEEC (3.183), we have

$$\operatorname*{Res}_{\delta=\delta_i} \operatorname*{Res}_{\delta'=\delta'_i} C_{\text{strong}}(\delta,j;\delta') = P_{\delta_i,j;\delta'_i}. \tag{3.200}$$

3.6.2 Leading order

We first consider the leading order term of (3.182),

$$\mathcal{F}^{(0)}_{\text{strong}}(z_1,z_2,z_3,p) = \frac{8}{\pi^3(-2z_1\cdot p)^3(-2z_2\cdot p)^3(-2z_3\cdot p)^3}. \tag{3.201}$$

[22] As we will see later, closing the δ contour or the δ' contour first gives the same celestial block expansion, at least for the strong-coupling EEEC we consider in this paper. Moreover, in appendix B.6 we study the celestial block $G^c_{\delta,j;\delta'}$ at large δ and δ', and show that contributions at infinity of both contours vanish.

Plugging this into the celestial inversion formula (3.196), we find

$$\begin{aligned}
&I^{(0)}_{\text{strong}}(\delta, j; \delta') \\
&= \frac{8}{\pi^3}\frac{2^{d-2}\,\text{vol}(\text{SO}(d-2))}{B_{\mathcal{P}_{\mathcal{E}}\mathcal{P}_{\mathcal{E}}\mathcal{P}_{\delta,j}} B_{\widetilde{\mathcal{P}}_{\delta,j}\mathcal{P}_{\mathcal{E}}\mathcal{P}_{\delta'}}} \times \int \frac{D^{d-2}zD^{d-2}z'D^{d-2}z_1D^{d-2}z_2D^{d-2}z_3 d^{d-1}_{\text{AdS}}p}{\text{vol}(\text{SO}(d-1,1))} \\
&\quad \frac{1}{(-2z_1\cdot p)^3(-2z_2\cdot p)^3(-2z_3\cdot p)^3}\langle \mathcal{P}_{\widetilde{\mathcal{E}}}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\widetilde{\mathcal{P}}_{\delta,j}(z)\rangle\langle \mathcal{P}_{\delta,j}(z)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_3)\mathcal{P}_{\widetilde{\delta}'}(z')\rangle\frac{1}{(-2z'\cdot p)^{\delta'}}.
\end{aligned} \tag{3.202}$$

Let us study the z_1, z_2 integral,

$$\int D^{d-2}z_1D^{d-2}z_2\,\frac{1}{(-2z_1\cdot p)^3(-2z_2\cdot p)^3}\langle \mathcal{P}_{\widetilde{\mathcal{E}}}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\widetilde{\mathcal{P}}_{\delta,j}(z,w)\rangle. \tag{3.203}$$

After integration, the polarization vector w must appear in the combination $[z,w]^{\mu\nu} \equiv z^\mu w^\nu - z^\nu w^\mu$ due to Lorentz invariance. However, the only remaining vector that is left unintegrated is p, and $[z,w]$ cannot be contracted with anything. Therefore, this integral must vanish except when $j=0$. To compute the integral for $j=0$, we can use

$$\begin{aligned}
&\int D^{d-2}z_1D^{d-2}z_2\langle \widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_2}(z_2)\mathcal{P}_{\delta,j=0}(z)\rangle(-2p\cdot z_1)^{-\delta_1}(-2p\cdot z_2)^{-\delta_2} \\
&= C_{\delta_1,\delta_2;\delta}(-p^2)^{\frac{\delta-\delta_1-\delta_2}{2}}(-2p\cdot z)^{-\delta},
\end{aligned} \tag{3.204}$$

where

$$C_{\delta_1,\delta_2;\delta} = \frac{\pi^{d-2}\Gamma(\frac{\delta_1+\delta_2+\delta-d+2}{2})\Gamma(\frac{\delta_1+\delta_2-\delta}{2})\Gamma(\frac{d-2-\delta_2-\delta+\delta_1}{2})\Gamma(\frac{d-2-\delta_1-\delta+\delta_2}{2})}{\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(d-2-\delta)\Gamma(\frac{d-2}{2})}. \tag{3.205}$$

We give a derivation of (3.204) in appendix B.6. After integrating over z_1 and z_2, we obtain

$$\begin{aligned}
I^{(0)}_{\text{strong}}(\delta, 0; \delta') &= \frac{8}{\pi^3}\frac{2^{d-2}\,\text{vol}(\text{SO}(d-2))}{B_{\mathcal{P}_{\mathcal{E}}\mathcal{P}_{\mathcal{E}}\mathcal{P}_{\delta,j}} B_{\widetilde{\mathcal{P}}_{\delta,j}\mathcal{P}_{\mathcal{E}}\mathcal{P}_{\delta'}}} C_{3,3;\widetilde{\delta}} \times \int \frac{D^{d-2}zD^{d-2}z'D^{d-2}z_3 d^{d-1}_{\text{AdS}}p}{\text{vol}(\text{SO}(d-1,1))} \\
&\quad \frac{1}{(-2z_3\cdot p)^3(-2z\cdot p)^{\widetilde{\delta}}}\langle \mathcal{P}_{\delta,0}(z)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_3)\mathcal{P}_{\widetilde{\delta}'}(z')\rangle\frac{1}{(-2z'\cdot p)^{\delta'}}.
\end{aligned} \tag{3.206}$$

The remaining p-integral is just a three-point Witten diagram, and it is given by [96]

$$\begin{aligned}
&\int d^{d-1}_{AdS}p\,(-2p\cdot z_1)^{-\delta_1}(-2p\cdot z_2)^{-\delta_2}(-2p\cdot z_3)^{-\delta_3} \\
=&D_{\delta_1,\delta_2,\delta_3}\frac{1}{(-2z_1\cdot z_2)^{\frac{\delta_1+\delta_2-\delta_3}{2}}(-2z_1\cdot z_3)^{\frac{\delta_1+\delta_3-\delta_2}{2}}(-2z_2\cdot z_3)^{\frac{\delta_2+\delta_3-\delta_1}{2}}},
\end{aligned} \tag{3.207}$$

where

$$D_{\delta_1,\delta_2,\delta_3} = \frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{\delta_1+\delta_2+\delta_3-d+2}{2})}{2\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\delta_3)}\Gamma(\tfrac{\delta_1+\delta_2-\delta_3}{2})\Gamma(\tfrac{\delta_1+\delta_3-\delta_2}{2})\Gamma(\tfrac{\delta_2+\delta_3-\delta_1}{2}). \quad (3.208)$$

So, we have

$$\begin{aligned} I^{(0)}_{\text{strong}}(\delta,0;\delta') &= \frac{8}{\pi^3}\frac{2^{d-2}\,\text{vol}(\text{SO}(d-2))}{B_{\mathcal{P}_\mathcal{E}\mathcal{P}_\mathcal{E}\mathcal{P}_{\delta,0}}B_{\widetilde{\mathcal{P}}_{\delta,0}\mathcal{P}_\mathcal{E}\mathcal{P}_{\delta'}}}C_{3,3;\widetilde{\delta}}D_{\widetilde{\delta},3,\delta'}\times \\ &\int \frac{D^{d-2}zD^{d-2}z'D^{d-2}z_3}{\text{vol}(\text{SO}(d-1,1))}\langle\mathcal{P}_{\delta,0}(z)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_3)\mathcal{P}_{\widetilde{\delta}'}(z')\rangle\langle\mathcal{P}_\mathcal{E}(z_3)\mathcal{P}_{\widetilde{\delta}}(z)\mathcal{P}_{\delta'}(z')\rangle \\ &= \frac{8}{\pi^3}\frac{2^{d-2}\,\text{vol}(\text{SO}(d-2))}{B_{\mathcal{P}_\mathcal{E}\mathcal{P}_\mathcal{E}\mathcal{P}_{\delta,0}}B_{\widetilde{\mathcal{P}}_{\delta,0}\mathcal{P}_\mathcal{E}\mathcal{P}_{\delta'}}}C_{3,3;\widetilde{\delta}}D_{\widetilde{\delta},3,\delta'}\left(\langle\mathcal{P}_\delta\mathcal{P}_{\widetilde{\mathcal{E}}}\mathcal{P}_{\widetilde{\delta}'}\rangle,\langle\mathcal{P}_{\widetilde{\delta}}\mathcal{P}_\mathcal{E}\mathcal{P}_{\delta'}\rangle\right). \end{aligned} \quad (3.209)$$

Plugging in the explicit expressions using (3.195), (3.205) and (3.208), we finally obtain (after setting $d=4$)

$$\begin{aligned} &I^{(0)}_{\text{strong}}(\delta,0;\delta') \\ &= \frac{(1-\delta')\Gamma\left(3-\frac{\delta}{2}\right)\Gamma\left(\frac{\delta}{2}\right)^2\Gamma\left(\frac{\delta+4}{2}\right)\Gamma\left(\frac{-\delta-\delta'+5}{2}\right)\Gamma\left(\frac{-\delta+\delta'-1}{2}\right)\Gamma\left(\frac{-\delta+\delta'+3}{2}\right)\Gamma\left(\frac{\delta+\delta'+1}{2}\right)}{4\pi^5\Gamma(1-\delta)\Gamma(\delta-1)\Gamma(\delta'-1)}. \end{aligned} \quad (3.210)$$

Consequently, $C^{(0)}_{\text{strong}}(\delta,0;\delta')$ is given by

$$\begin{aligned} &C^{(0)}_{\text{strong}}(\delta,0;\delta') \\ &= \frac{\Gamma\left(3-\frac{\delta}{2}\right)\Gamma\left(\frac{\delta}{2}\right)^2\Gamma\left(\frac{\delta+4}{2}\right)\Gamma\left(\frac{\delta-\delta'+3}{2}\right)\Gamma\left(\frac{-\delta+\delta'+3}{2}\right)\Gamma\left(\frac{\delta+\delta'-3}{2}\right)\Gamma\left(\frac{\delta+\delta'+1}{2}\right)}{4\pi^3\Gamma(\delta-1)\Gamma(\delta)\Gamma(\delta'-1)}. \end{aligned} \quad (3.211)$$

To find the coefficients for the celestial block expansion, we have to close the δ and δ' contours in (3.198). It turns out that the resulting celestial block expansion does not depend on the order of contour deformations, so let us close the δ' contour first for simplicity. When δ is on the principal series, the only poles of δ' that are to the right of the principal series are at $\delta'=\delta+3+2k$, where k is a nonnegative integer. We then find that the residues

$$\operatorname*{Res}_{\delta'=\delta+3+2k} C^{(0)}_{\text{strong}}(\delta,0;\delta') \quad (3.212)$$

have poles at $\delta=6+2n$, where n is a nonnegative integer. Thus, the values of δ and δ' appearing in the celestial block expansion (3.183) should be

$$\delta=6+2n, \delta'=9+2n+2k. \quad (3.213)$$

This agrees with our previous predictions (3.185) and (3.186) from the light-ray OPE. The coefficients $P_{\delta,j;\delta'}$ are given by

$$\begin{aligned}&P^{(0)}_{6+2n,0;9+2n+2k}\\&=\frac{(k+1)(k+2)(-1)^{k+n}\Gamma(n+3)^2\Gamma(n+5)\Gamma(k+2n+6)\Gamma(k+2n+8)}{\pi^3\Gamma(n+1)\Gamma(2n+5)\Gamma(2n+6)\Gamma(2(k+n+4))}.\end{aligned}\tag{3.214}$$

As a consistency check, we can expand $\mathcal{F}^{(0)}_{\text{strong}}$ in the collinear limit and find the celestial block expansion order by order using the expansion of the celestial block G^c in the collinear limit given by (3.33) (or (B.18)). We have checked up to $O(\zeta_{13}^6)$ that the coefficients obtained in this way agree with (3.214).

3.6.3 $O(1/\lambda)$ correction

Let us now consider the $O(1/\lambda)$ term of (3.182),

$$\begin{aligned}&\mathcal{F}^{(1)}_{\text{strong}}(z_1,z_2,z_3,p)\\&=\frac{48}{\pi(-2p\cdot z_1)^3(-2p\cdot z_2)^3(-2p\cdot z_3)^3}\left(2-4\left(\zeta_{12}+\zeta_{13}+\zeta_{23}\right)+4\left(\zeta_{12}^2+\zeta_{13}^2+\zeta_{23}^2\right)\right).\end{aligned}\tag{3.215}$$

Plugging this into the celestial inversion formula, we get

$$\begin{aligned}&I^{(1)}_{\text{strong}}(\delta,j;\delta')\\&=\frac{48}{\pi}\frac{2^{d-2}\,\mathrm{vol}(\mathrm{SO}(d-2))}{B_{\mathcal{P}_\mathcal{E}\mathcal{P}_\mathcal{E}\mathcal{P}_{\delta,j}}B_{\widetilde{\mathcal{P}}_{\delta,j}\mathcal{P}_\mathcal{E}\mathcal{P}_{\delta'}}}\times\int\frac{D^{d-2}zD^{d-2}z'D^{d-2}z_1D^{d-2}z_2D^{d-2}z_3d^{d-1}_{\mathrm{AdS}}p}{\mathrm{vol}(\mathrm{SO}(d-1,1))}\\&\quad\frac{\left(2-4\left(\zeta_{12}+\zeta_{13}+\zeta_{23}\right)+4\left(\zeta_{12}^2+\zeta_{13}^2+\zeta_{23}^2\right)\right)}{(-2z_1\cdot p)^3(-2z_2\cdot p)^3(-2z_3\cdot p)^3}\langle\mathcal{P}_{\widetilde{\mathcal{E}}}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\widetilde{\mathcal{P}}_{\delta,j}(z)\rangle\langle\mathcal{P}_{\delta,j}(z)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_3)\mathcal{P}_{\widetilde{\delta}'}(z')\rangle\frac{1}{(-2z'\cdot p)}\end{aligned}\tag{3.216}$$

There are two types of z_1, z_2 integrals to consider,[23]

$$\begin{aligned}&\int D^{d-2}z_1D^{d-2}z_2\frac{\zeta_{12}^k}{(-2z_1\cdot p)^3(-2z_2\cdot p)^3}\langle\mathcal{P}_{\widetilde{\mathcal{E}}}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\widetilde{\mathcal{P}}_{\delta,j}(z,w)\rangle,\\&\int D^{d-2}z_1D^{d-2}z_2\frac{\zeta_{13}^k}{(-2z_1\cdot p)^3(-2z_2\cdot p)^3}\langle\mathcal{P}_{\widetilde{\mathcal{E}}}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\widetilde{\mathcal{P}}_{\delta,j}(z,w)\rangle,\end{aligned}\tag{3.217}$$

where $k=0,1,2$. Since the dependence of the integral on w should be $([z,w]\cdot[\cdots])^j$, the first line is only nonzero when $j=0$ and the second one is nonzero when

[23] Note that in (3.216) we implicitly contract the indices of $\widetilde{\mathcal{P}}_{\delta,j}(z)$ and $\mathcal{P}_{\delta,j}(z)$, and in (3.217) we contract the indices with a polarization vector w.

$j \leq k$. However, when $j = 1$, the three-point structure $\langle \mathcal{P}_{\widetilde{\mathcal{E}}}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\widetilde{\mathcal{P}}_{\delta,j}(z,w)\rangle$ is antisymmetric in $1 \leftrightarrow 2$, and the contribution from ζ_{13}^k and ζ_{23}^k will cancel. So, $I^{(1)}_{\text{strong}}(\delta, j; \delta')$ is nonzero for $j = 0$ and $j = 2$.

$j = 0$

We first consider the $j = 0$ case. The integral containing ζ_{12}^k can be computed by applying the differential operator $\partial_p \cdot \partial_p$ k times to (3.204). The result is

$$\int D^{d-2}z_1 D^{d-2}z_2 \frac{\zeta_{12}}{(-2z_1 \cdot p)^3(-2z_2 \cdot p)^3}\langle \mathcal{P}_{\widetilde{\mathcal{E}}}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\widetilde{\mathcal{P}}_{\delta}(z)\rangle = \frac{(\delta-6)(\widetilde{\delta}-6)}{36}C^{(0)}_{3,3;\widetilde{\delta}}(-2p\cdot z)^{-\widetilde{\delta}}, \tag{3.218}$$

and

$$\begin{aligned}&\int D^{d-2}z_1 D^{d-2}z_2 \frac{\zeta_{12}^2}{(-2z_1 \cdot p)^3(-2z_2 \cdot p)^3}\langle \mathcal{P}_{\widetilde{\mathcal{E}}}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\widetilde{\mathcal{P}}_{\delta}(z)\rangle \\ &= \frac{(\delta-6)(\delta-8)(\widetilde{\delta}-6)(\widetilde{\delta}-8)}{2304}C^{(0)}_{3,3;\widetilde{\delta}}(-2p\cdot z)^{-\widetilde{\delta}}.\end{aligned} \tag{3.219}$$

To compute the second type of integral in (3.217), which contains ζ_{13}^k or ζ_{23}^k, we can first study

$$z_1^\mu \langle \mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta}(z)\rangle. \tag{3.220}$$

The factor z_1^μ can be viewed as a weight-shifting operator [39] that decreases the scaling dimension of $\mathcal{P}_{\delta_1}(z_1)$ by 1. One can performing crossing on (3.220) to make the weight-shifting operators act on $\mathcal{P}_\delta(z)$. We find

$$\begin{aligned}&(z_3\cdot z_1)\langle \mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta}(z)\rangle \\ &= \frac{(\delta-\delta_1+\delta_2)(4-d+\delta-\delta_1+\delta_2)}{2(d-3-\delta)(d-2-2\delta)}(z_3\cdot z)\langle \mathcal{P}_{\delta_1-1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta+1}(z)\rangle \\ &+ \frac{1}{2-d+d\delta-2\delta^2}(z_3\cdot D_z)\langle \mathcal{P}_{\delta_1-1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta-1}(z)\rangle \\ &- \frac{\delta-\delta_1+\delta_2}{2(d-4)(d-3-\delta)(\delta-1)}(z_3\cdot \mathcal{D}^{0-}_{z,w})\langle \mathcal{P}_{\delta_1-1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta,1}(z,w)\rangle,\end{aligned} \tag{3.221}$$

where D_z is the Todorov operator and $\mathcal{D}^{0-}_{z,w}$ is the weight-shifting operator that decreases spin by 1 defined in [39]. As discussed below (3.217), the $\mathcal{D}^{0-}_{z,w}\langle \mathcal{P}_{\delta_1-1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta,1}(z,w)\rangle$ term will vanish after integrating over z_1, z_2.

After using (3.221) and a similar relation for ζ_{23}, we find

$$
\begin{aligned}
&\int D^{d-2}z_1 D^{d-2}z_2 \frac{\zeta_{13}+\zeta_{23}}{(-2z_1\cdot p)^3(-2z_2\cdot p)^3}\langle \mathcal{P}_{\widetilde{\mathcal{E}}}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\widetilde{\mathcal{P}}_{\delta}(z)\rangle \\
&= 2\int D^{d-2}z_1 D^{d-2}z_2 \frac{-2z_{3\mu}}{(-2z_1\cdot p)^4(-2z_2\cdot p)^3(-2z_3\cdot p)} \\
&\quad \left(\frac{\widetilde{\delta}^2}{4(\widetilde{\delta}-1)^2} z^{\mu}\langle \mathcal{P}_{\delta_{\widetilde{\mathcal{E}}}-1}(z_1)\mathcal{P}_{\mathcal{E}}(z_2)\mathcal{P}_{\widetilde{\delta}+1}(z)\rangle - \frac{1}{2(\widetilde{\delta}-1)^2} D_z^{\mu}\langle \mathcal{P}_{\delta_{\widetilde{\mathcal{E}}}-1}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\mathcal{P}_{\widetilde{\delta}-1}(z)\rangle\right) \\
&= \frac{2}{-2z_3\cdot p}\left(\frac{\widetilde{\delta}^2}{4(\widetilde{\delta}-1)^2}(-2z_3\cdot z)C^{(0)}_{4,3;\widetilde{\delta}+1}(-2p\cdot z)^{-\widetilde{\delta}-1} - \frac{1}{2(\widetilde{\delta}-1)^2}(-2z_3\cdot D_z)C^{(0)}_{4,3;\widetilde{\delta}-1}(-2p\cdot z)^{-\widetilde{\delta}+1}\right) \\
&= -\frac{1}{3}C^{(0)}_{3,3;\widetilde{\delta}}(-2p\cdot z)^{-\widetilde{\delta}}\left(\tfrac{\widetilde{\delta}-6}{2} - \widetilde{\delta}\zeta_{03}\right). \qquad (3.222)
\end{aligned}
$$

Similarly, for $\zeta_{13}^2+\zeta_{23}^2$ we have

$$
\begin{aligned}
&\int D^{d-2}z_1 D^{d-2}z_2 \frac{\zeta_{13}^2+\zeta_{23}^2}{(-2z_1\cdot p)^3(-2z_2\cdot p)^3}\langle \mathcal{P}_{\widetilde{\mathcal{E}}}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\widetilde{\mathcal{P}}_{\delta}(z)\rangle \\
&= \frac{1}{96(\widetilde{\delta}-3)}C^{(0)}_{3,3;\widetilde{\delta}}(-2p\cdot z)^{-\widetilde{\delta}}\left(\widetilde{\delta}\left(\widetilde{\delta}^2+6(\widetilde{\delta}-4)(\widetilde{\delta}+2)\zeta_{03}^2-2(\widetilde{\delta}-6)(3\widetilde{\delta}-4)\zeta_{03}-18\widetilde{\delta}+104\right)-192\right). \qquad (3.223)
\end{aligned}
$$

Combining (3.218), (3.219), (3.222), (3.223), we obtain

$$
\begin{aligned}
&\int D^{d-2}z_1 D^{d-2}z_2\, \mathcal{F}^{(1)}_{\text{strong}}(z_1,z_2,z_3,p)\langle \mathcal{P}_{\widetilde{\mathcal{E}}}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\widetilde{\mathcal{P}}_{\delta}(z)\rangle \\
&= \frac{48}{\pi}\frac{(\widetilde{\delta}-4)\widetilde{\delta}(\widetilde{\delta}+2)}{576(\widetilde{\delta}-3)}C^{(0)}_{3,3;\widetilde{\delta}}(-2p\cdot z)^{-\widetilde{\delta}}\left(\widetilde{\delta}^2-5\widetilde{\delta}+30+144(\zeta_{03}-1)\zeta_{03}\right). \qquad (3.224)
\end{aligned}
$$

Hence, the inversion formula (3.216) for $j=0$ is now given by

$$
\begin{aligned}
&I^{(1)}_{\text{strong}}(\delta,0;\delta') \\
&= \frac{48}{\pi}\frac{2^{d-2}\operatorname{vol}(\mathrm{SO}(d-2))}{B_{\mathcal{P}_{\mathcal{E}}\mathcal{P}_{\mathcal{E}}\mathcal{P}_{\delta,j}}B_{\widetilde{\mathcal{P}}_{\delta,j}\mathcal{P}_{\mathcal{E}}\mathcal{P}_{\delta'}}}\frac{(\widetilde{\delta}-4)\widetilde{\delta}(\widetilde{\delta}+2)}{576(\widetilde{\delta}-3)}C^{(0)}_{3,3;\widetilde{\delta}}\times\int\frac{D^{d-2}zD^{d-2}z'D^{d-2}z_3 d^{d-1}_{\text{AdS}}p}{\operatorname{vol}(\mathrm{SO}(d-1,1))} \\
&\quad \frac{1}{(-2p\cdot z)^{\widetilde{\delta}}(-2p\cdot z_3)^3}\left(\widetilde{\delta}^2-5\widetilde{\delta}+30+144(\zeta_{03}-1)\zeta_{03}\right)\langle \mathcal{P}_{\delta}(z)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_3)\mathcal{P}_{\widetilde{\delta}'}(z')\rangle\frac{1}{(-2z'\cdot p)^{\delta'}}. \qquad (3.225)
\end{aligned}
$$

The integral over p can be evaluated using (3.207). The result is

$$
\begin{aligned}
&I^{(1)}_{\text{strong}}(\delta,0;\delta') \\
&= 6\pi^2 A \frac{2^{d-2}\,\mathrm{vol}(\mathrm{SO}(d-2))}{B_{\mathcal{P}_\mathcal{E}\mathcal{P}_\mathcal{E}\mathcal{P}_{\delta,j}} B_{\widetilde{\mathcal{P}}_{\delta,j}\mathcal{P}_\mathcal{E}\mathcal{P}_{\delta'}}} \frac{(\widetilde{\delta}-4)\widetilde{\delta}(\widetilde{\delta}+2)}{576(\widetilde{\delta}-3)} C^{(0)}_{3,3;\widetilde{\delta}} D_{\widetilde{\delta},3,\delta'} \\
&\times \frac{\frac{3}{4}(\widetilde{\delta}-\delta'+3)(\widetilde{\delta}-\delta'+5)(\widetilde{\delta}+\delta'+1)(\widetilde{\delta}+\delta'+3)+(\widetilde{\delta}+1)((\widetilde{\delta}-18)\widetilde{\delta}(\widetilde{\delta}+1)+12(\delta'-3)(\delta'+1))}{\widetilde{\delta}(\widetilde{\delta}+1)} \\
&\times \left(\langle \mathcal{P}_\delta \mathcal{P}_{\widetilde{\mathcal{E}}} \mathcal{P}_{\widetilde{\delta}'}\rangle, \langle \mathcal{P}_{\widetilde{\delta}} \mathcal{P}_\mathcal{E} \mathcal{P}_{\delta'}\rangle\right), \qquad (3.226)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&C^{(1)}_{\text{strong}}(\delta,0;\delta') \\
&= \left(\delta^2\left(-6\delta'^2+12\delta'+10\right)+7\delta^4-28\delta^3+12\delta\left(\delta'^2-2\delta'+3\right)+3(\delta'-1)^2\left(\delta'^2-2\delta'-3\right)\right) \\
&\times (\delta-4)(\delta-1)(\delta+2)(\delta'-1)\frac{\Gamma\left(3-\frac{\delta}{2}\right)\Gamma\left(\frac{\delta}{2}\right)^2\Gamma\left(\frac{\delta+4}{2}\right)\Gamma\left(\frac{\delta-\delta'+3}{2}\right)\Gamma\left(\frac{-\delta+\delta'+3}{2}\right)\Gamma\left(\frac{\delta+\delta'-3}{2}\right)\Gamma\left(\frac{\delta+\delta'+1}{2}\right)}{1536\pi(\delta-3)(\delta+1)\Gamma(\delta)^2\Gamma(\delta')}. \qquad (3.227)
\end{aligned}
$$

We see that the Gamma functions are the same as the leading order result $C^{(0)}_{\text{strong}}$, and hence the locations of the poles are the same. In particular, we only get poles at

$$
\delta = 6+2n, \quad \delta' = 9+2n+2k, \qquad (3.228)
$$

where n, k are nonnegative integers. The coefficients $P^{(1)}_{\delta,0;\delta'}$ are given by the residues,

$$
\begin{aligned}
&P^{(1)}_{6+2n,0;9+2n+2k} \\
&= \left(3k^2\left(4n^2+38n+83\right)+12k^3(n+4)+3k^4+6k\left(6n^2+43n+76\right)+4n^4+40n^3+169n^2+381n+378\right) \\
&\times \frac{(-1)^{k+n}(n+1)(n+4)(2n+5)(k+n+4)\Gamma(k+3)\Gamma(n+3)^2\Gamma(n+5)\Gamma(k+2n+6)\Gamma(k+2n+8)}{3\pi(2n+3)(2n+7)\Gamma(k+1)\Gamma(n+1)\Gamma(2n+6)^2\Gamma(2k+2n+9)}. \qquad (3.229)
\end{aligned}
$$

We have checked that the coefficients obtained by expanding $\mathcal{F}^{(1)}_{\text{strong}}$ in the collinear limit up to $O(\zeta_{13}^6)$ agree with the above expression.

$j = 2$

Now let us consider the $j = 2$ case. From the discussion below (3.217), we must only consider the z_1, z_2-integral

$$
\int D^{d-2}z_1 D^{d-2}z_2 \frac{\zeta_{13}^2+\zeta_{23}^2}{(-2z_1\cdot p)^3(-2z_2\cdot p)^3}\langle \mathcal{P}_{\widetilde{\mathcal{E}}}(z_1)\mathcal{P}_{\widetilde{\mathcal{E}}}(z_2)\widetilde{\mathcal{P}}_{\delta,j=2}(z,w)\rangle. \qquad (3.230)
$$

We can again use crossing equations for weight-shifting operators. Only terms with a $j = 0$ three-point structure will be nonvanishing after we integrate over z_1 and z_2. There is only one such term for $(z_3 \cdot z_1)^2 \langle \mathcal{P}_{\delta_1} \mathcal{P}_{\delta_2} \mathcal{P}_{\delta,2}(z,w)\rangle$, and its $6j$ symbol is given by[24]

$$\begin{aligned}&(z_3 \cdot z_1)^2 \langle \mathcal{P}_{\delta_1} \mathcal{P}_{\delta_2} \mathcal{P}_{\delta,2}(z,w)\rangle \\ &= \frac{(\delta - \delta_1 + \delta_2 - d + 2)(\delta - \delta_1 + \delta_2 - d + 4)}{2(d-2)d\delta(\delta+1)(\delta - d + 1)(\delta - d + 2)} (z_3 \cdot \mathcal{D}^{0+}_{z,w}) \langle \mathcal{P}_{\delta_1 - 2}(z_1) \mathcal{P}_{\delta_2}(z_2) \mathcal{P}_{\delta}(z)\rangle + \dots .\end{aligned} \tag{3.231}$$

Our z_1, z_2 integral is thus given by

$$\begin{aligned}&\int D^{d-2}z_1 D^{d-2}z_2 \frac{\zeta_{13}^2 + \zeta_{23}^2}{(-2z_1 \cdot p)^3 (-2z_2 \cdot p)^3} \langle \mathcal{P}_{\widetilde{\mathcal{E}}}(z_1) \mathcal{P}_{\widetilde{\mathcal{E}}}(z_2) \widetilde{\mathcal{P}}_{\delta,2}(z,w)\rangle \\ &= \frac{\widetilde{\delta}}{2(\widetilde{\delta} - 3)} C^{(0)}_{5,3;\widetilde{\delta}} \frac{(2z \cdot p w \cdot z_3 - 2z \cdot z_3 w \cdot p)^2}{(-2z \cdot p)^{\widetilde{\delta}+2} (-2z_3 \cdot p)^2}.\end{aligned} \tag{3.232}$$

Plugging this into the inversion formula (3.216) for $j = 2$, we find

$$\begin{aligned}&I^{(1)}_{\text{strong}}(\delta, 2; \delta') \\ &= \frac{48}{\pi} \frac{2^{d-2} \operatorname{vol}(\mathrm{SO}(d-2))}{B_{\mathcal{P}_{\mathcal{E}} \mathcal{P}_{\mathcal{E}} \mathcal{P}_{\delta,2}} B_{\widetilde{\mathcal{P}}_{\delta,2} \mathcal{P}_{\mathcal{E}} \mathcal{P}_{\delta'}}} \frac{\widetilde{\delta}}{2(\widetilde{\delta} - 3)} C^{(0)}_{5,3;\widetilde{\delta}} \times \int \frac{D^{d-2}z D^{d-2}z' D^{d-2}z_3 d^{d-1}_{\text{AdS}} p}{\operatorname{vol}(\mathrm{SO}(d-1,1))} \\ &\quad \frac{(2z \cdot p)^2 z_{3\mu} z_{3\nu} - 2(2z \cdot p)(2z \cdot z_3) z_{3\mu} p_\nu + (2z \cdot z_3)^2 p_\mu p_\nu}{(-2z \cdot p)^{\widetilde{\delta}+2} (-2z_3 \cdot p)^5} \langle \mathcal{P}^{\mu\nu}_{\delta,2}(z) \mathcal{P}_{\widetilde{\mathcal{E}}}(z_3) \mathcal{P}_{\widetilde{\delta}'}(z')\rangle \frac{1}{(-2z' \cdot p)^{\delta'}}.\end{aligned} \tag{3.233}$$

To perform the p integral, we can view p_μ as a linear combination of the AdS weight-shifting operators [52] and then perform crossing. We find

$$p_\mu (-2p \cdot z)^{-\delta} = \frac{1}{(\delta - 1)(d - 2 - 2\delta)} D_{z\mu} (-2p \cdot z)^{-\delta+1} + \frac{2\delta}{2\delta - d + 2} z_\mu (-2p \cdot z)^{-\delta - 1}. \tag{3.234}$$

After using this relation, the integral over p is elementary. The result is

$$\begin{aligned}&I^{(1)}_{\text{strong}}(\delta, 2; \delta') \\ &= \frac{48}{\pi} \frac{2^{d-2} \operatorname{vol}(\mathrm{SO}(d-2))}{B_{\mathcal{P}_{\mathcal{E}} \mathcal{P}_{\mathcal{E}} \mathcal{P}_{\delta,2}} B_{\widetilde{\mathcal{P}}_{\delta,2} \mathcal{P}_{\mathcal{E}} \mathcal{P}_{\delta'}}} \frac{(\widetilde{\delta} + \delta' - 5)(\widetilde{\delta} + \delta' - 3)}{8(\widetilde{\delta} + 1)(\widetilde{\delta} - 3)} C^{(0)}_{5,3;\widetilde{\delta}} D_{\widetilde{\delta},5,\delta'} \left(\langle \mathcal{P}_{\mathcal{E}} \mathcal{P}_{\delta'} \widetilde{\mathcal{P}}_{\delta,2} \rangle, \langle \widetilde{\mathcal{P}}_{\mathcal{E}} \widetilde{\mathcal{P}}_{\delta'} \mathcal{P}_{\delta,2} \rangle \right).\end{aligned} \tag{3.235}$$

[24] Our convention here is $\langle \mathcal{P}_{\delta_1} \mathcal{P}_{\delta_2} \mathcal{P}_{\delta,j}(z,w)\rangle = \frac{(2z \cdot z_1 w \cdot z_2 - 2z \cdot z_2 w \cdot z_1)^j}{(-2z_1 \cdot z_2)^{\frac{\delta_1 + \delta_2 - \delta + j}{2}} (-2z_1 \cdot z)^{\frac{\delta_1 + \delta - \delta_2 + j}{2}} (-2z_2 \cdot z)^{\frac{\delta_2 + \delta - \delta_1 + j}{2}}}$.

Collecting together all the factors, we finally obtain

$$\begin{aligned} &C^{(1)}_{\text{strong}}(\delta, 2; \delta') \\ &= \frac{\Gamma\left(4-\frac{\delta}{2}\right)\Gamma\left(\frac{\delta}{2}\right)\Gamma\left(\frac{\delta+2}{2}\right)\Gamma\left(\frac{\delta+6}{2}\right)\Gamma\left(\frac{\delta-\delta'+5}{2}\right)\Gamma\left(\frac{-\delta+\delta'+5}{2}\right)\Gamma\left(\frac{\delta+\delta'-1}{2}\right)\Gamma\left(\frac{\delta+\delta'+3}{2}\right)}{24\pi\Gamma(\delta)\Gamma(\delta+2)\Gamma(\delta'-1)}. \end{aligned} \tag{3.236}$$

After contour deformations, the only poles that will contribute to the celestial block expansion are at

$$\delta = 8 + 2n, \qquad \delta' = \delta + 5 + 2k = 13 + 2n + 2k. \tag{3.237}$$

The corresponding coefficients are given by

$$\begin{aligned} &P^{(1)}_{8+2n,2;13+2n+2k} \\ &= \frac{(-1)^{k+n}\Gamma(k+5)\Gamma(n+5)\Gamma(n+7)\Gamma(n+4)\Gamma(k+2n+10)\Gamma(k+2n+12)}{6\pi\Gamma(k+1)\Gamma(n+1)\Gamma(2(n+5))\Gamma(2n+8)\Gamma(2(k+n+6))}. \end{aligned} \tag{3.238}$$

We have checked that this result agrees with the expansion of $\mathcal{F}^{(1)}_{\text{strong}}$ in the collinear limit up to $O(\zeta_{13}^6)$. The validity of these expressions for the celestial block expansion of the strong-coupling EEEC is a strong check on both our expressions for the collinear expansion of celestial blocks (3.33), and the celestial inversion formula (3.196).

3.7 Discussion and future directions

In this work, we studied aspects of the celestial block decomposition of the three-point energy correlator (EEEC). We found that, in both strong and weak coupling examples, the EEEC admits an expansion in a discrete sum of celestial blocks, corresponding to light-ray operators appearing in repeated OPEs of energy detectors. We derived useful formulas for 3-point celestial blocks in an expansion around the collinear limit. We then explored the celestial block expansion in the collinear limit using lightcone bootstrap techniques and the Lorentzian inversion formula, both in QCD and $\mathcal{N} = 4$ SYM. The symmetry structure of the celestial block expansion allowed us to make certain predictions for higher orders in perturbation theory. We also determined the leading and first subleading contact terms in the EEEC in $\mathcal{N} = 4$ SYM. Finally, using techniques from harmonic analysis, we studied the full celestial block decomposition of the strong-coupling EEEC in $\mathcal{N} = 4$ SYM, for generic configurations of detectors. Along the way, we encountered several puzzles and novel objects that we summarize below.

This celestial block expansion must be compatible with crossing symmetry, which leads to constraints similar to those studied in the traditional conformal bootstrap. It would be interesting to fully characterize the consistency conditions that multi-point energy correlators should satisfy, in order to set up a direct bootstrap program for them (and more general event shapes). It would also be interesting to incorporate "generalized detectors" [71, 72].

In the collinear EEEC, the density matrix $|\Psi\rangle\langle\Psi|$ in which we evaluate the event shape gets highly boosted, and essentially projected onto its component with boost eigenvalue δ'_* (the dimension of the lowest-twist spin-4 operator). These highly-boosted density matrices, which we might call "light-ray density matrices," are naturally dual to light-ray operators, via taking expectation values. It is interesting to ask whether they provide a useful basis for studying other physical observables. It may also be interesting to explore information-theoretic properties of light-ray density matrices.

Our lightcone bootstrap analysis of collinear event shapes reveals the existence of two important contributions to the light-ray OPE: double-twist operators built out of a pair of detectors $[\mathcal{P}_O\mathcal{P}_O]$, and double-twist objects $[\mathcal{P}_O\mathcal{P}_{O'_*}]$ that formally look like an OPE between a light-ray operator and the light-ray density matrix obtained from boosting $|\Psi\rangle\langle\Psi|$. This suggests that it may be possible more generally to make sense of an OPE between light-ray operators and light-ray density matrices.

In our analysis of the EEEC in QCD, we used the fact that QCD admits a conformal point at a particular value of the coupling in $d = 4 - \epsilon$ dimensions. The existence of this conformal point has implications for the structure of perturbation theory at each loop order, even away from the conformal point, allowing us to apply selection rules from conformal symmetry in our work. However, the existence of a celestial block expansion should require only Lorentz invariance. It would be interesting to characterize how the structure of the space of light-ray operators and light-ray OPEs changes in the presence of a nonzero β-function.

In applying the Lorentzian inversion formula to the collinear EEEC, we were led to continue the EEEC into a "doubly-Lorentzian" regime where the cross-ratios on the celestial sphere become independent real numbers. (This same analytic continuation to (a cover of) the celestial torus is frequently used in studying celestial amplitudes

[97].) Although this continuation is straightforward at the order in perturbation theory we studied, we have little understanding of whether it is admissible at higher orders or nonperturbatively. Another important question is how event shapes behave in the "celestial Regge limit" where the celestial cross ratios undergo the analytic continuation usually studied in the context of the Regge limit in CFT [98]. The two examples we studied (weakly-coupled $\mathcal{N} = 4$ SYM and QCD) exhibited very different behavior in this regime, and it would be interesting to understand what the general nonperturbative behavior can be.

Evidently, Lorentzian QFT observables still hold many mysteries. To better understand their structure, it will be important to study more examples and collect more data both in perturbation theory and beyond.

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Chapter 4

SPINNING DISPERSIVE CFT SUM RULES AND BULK SCATTERING

This chapter is based on

[1] Cyuan-Han Chang, Yakov Landau, and David Simmons-Duffin. "Spinning dispersive CFT sum rules and bulk scattering". In: (Nov. 2023). arXiv: `2311.04271 [hep-th]`.

4.1 Introduction

Causality and unitarity imply constraints on the space of low-energy effective field theories (EFTs) [99, 100]. In a $2 \to 2$ scattering process, causality and unitarity manifest as analyticity, Regge boundedness, and crossing symmetry of the S-matrix. Using these properties, one can derive dispersion relations that express the Wilson coefficients of a low-energy EFT in terms of data in the ultraviolet (UV), which have positive spectral density thanks to unitarity [101, 102]. By applying functionals (such as expanding around the forward limit) to dispersion relations, positivity of the UV spectral density can be used to obtain two-sided bounds on EFT Wilson coefficients, with the correct scaling in the EFT cutoff M expected by dimensional analysis. A systematic exploration of this approach was initiated recently in [103, 104, 105, 106, 107].

Including gravitational interactions introduces another layer of complexity, since graviton exchange produces divergences in the forward limit. This issue was overcome in [108] by considering functionals that measure the scattering amplitude at small impact parameter. This method has paved the way for deriving bounds on higher-derivative corrections to General Relativity in flat space [109, 110, 111, 112, 113].

Similar questions can be explored in Anti-de Sitter (AdS) space as well. Furthermore, via the AdS/CFT correspondence, these questions can be phrased — and potentially answered — in CFT language. For example, HPPS conjectured in [114] that a large-N CFT should have a local bulk EFT dual if its single trace spectrum has a large gap $\Delta_{\text{gap}} = MR_{\text{AdS}} \gg 1$ for $J > 2$, where R_{AdS} is the AdS radius and M is the cutoff scale of the bulk EFT. Significant progress towards establishing this conjecture using CFT techniques was made in e.g. [102, 115, 116, 117, 118, 9, 119, 120, 34]. Recently, part of the HPPS conjecture was established in [13] by building a dictionary between conformal bootstrap functionals and flat space dispersion relations. The authors of [13] considered a weakly-coupled EFT in AdS with a massless graviton and a scalar with mass $m_\phi = O(\frac{1}{R_{\text{AdS}}})$. Schematically, the low-energy effective action is

$$S = S_{\text{gravity}} + \int d^D x \sqrt{-g} \left(\frac{1}{2}\phi(\partial^2 - m_\phi^2)\phi + \sum_n g_n D^{2n} \phi^4 + \dots \right), \tag{4.1}$$

where S_{gravity} contains graviton interactions and will be given below. Using bootstrap methods, they derived bounds on the higher-derivative interactions g_n in (4.1) with the expected suppression in Δ_{gap}.

The main tool in this analysis is dispersive CFT sum rules [121, 122, 123, 124, 125, 56, 126, 127, 128], which have double zeros at the locations of most double-twist operators. Such sum rules allow one to separate out light double-trace contributions in holographic CFTs, and express the light contribution ($\Delta < \Delta_{\text{gap}}$) described by (4.1) in terms of a sum of heavy conformal blocks with positive coefficients. Remarkably, in a certain "flat space limit" that we review in section 4.2, dispersive CFT sum rules reduce to the flat space sum rules previously studied in [108]. As a result, flat space functionals can be uplifted to AdS and lead to bounds on the EFT couplings in (4.1).[1]

Given the bounds on graviton interactions in flat space obtained recently [109, 110, 111, 112, 113], we would like to derive similar bounds on graviton interactions in AdS, in particular establishing HPPS for purely gravitational theories. More precisely, let us consider an EFT with only a massless graviton with the effective action

$$S_{\text{gravity}} = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left(-2\Lambda + R + \alpha_2 R^2 + \alpha_3 R^3 + \dots \right), \tag{4.2}$$

where $\Lambda = -(D-1)(D-2)/(2R_{\text{AdS}}^2)$. We would like to obtain bounds like

$$|\alpha_2| \le G \times \frac{\#}{\Delta_{\text{gap}}^2}. \tag{4.3}$$

On the CFT side, this means we must construct dispersive sum rules for four-point functions of stress tensors whose flat space limit agrees with flat space sum rules for gravitons.

[1] The "uplifting" procedure only guarantees positivity of the functional in the "bulk point" regime discussed below. One must additionally check other regimes to derive a rigorous CFT bound.

Conceptually, this task is similar to the one undertaken in [13]. However, stress tensor four-point functions are technically more complicated due to the profusion of tensor structures, and we need to organize the calculation carefully. Our approach is to begin with "subtracted superconvergence" sum rules, which exploit the fact that the commutator of null-integrated operators on the same null plane should vanish [34]. In particular, we focus on the action of such sum rules on a heavy conformal block (with $\Delta > \Delta_{\text{gap}}$). This action is a spacetime integral that, in the flat space limit, localizes to a certain saddle configuration that we call the "scattering crystal." Thus, the computation of the heavy action turns into evaluating conformally-invariant structures at this saddle, which is straightforward for spinning operators. Our calculation gives a "spacetime interpretation" of the results of [13] for scalar operators.

The result of our saddle analysis has a simple interpretation in flat space. The inserted conformal blocks become flat space partial waves, and the subtracted superconvergence sum rule becomes a sum rule for flat space "shock" amplitudes [102, 34]. Thus, by comparing the action of CFT sum rules on heavy blocks to the action of flat-space sum rules on heavy states, we can deduce a concrete dictionary between dispersive CFT functionals and flat space sum rules.[2] Our sum rules for stress tensors become a subset of the known flat space sum rules for gravitons [110]. The full set of flat space sum rules for gravitons includes additional sum rules that cannot be expressed in terms of "shock amplitudes" (because they have different choices of external polarizations). We leave the problem of obtaining these additional sum rules using CFT techniques to future work. Our dictionary will be needed to convert the flat space functionals with positive action on partial waves into CFT functionals with positive action on heavy blocks.

This paper is organized as follows. In section 4.2, we briefly review superconvergence sum rules and the derivation of the scalar bounds in [13]. In section 4.3, we derive a simple formula for the flat space limit of superconvergence sum rules for scalars using a spacetime saddle-point analysis. We then generalize the formula to spinning operators. In section 4.4, we explain how this formula can be matched to flat space sum rules, and obtain the dictionary between spinning CFT sum rules and flat space sum rules for photons and gravitons. We conclude in section 4.5. We summarize our conventions in appendix C.1, and present an alternative derivation

[2] Once we know the dictionary, it follows that the contributions of light states must match between AdS and flat space as well, though we leave exploration of light states to future work.

of the flat space limit formula in appendix C.2. Technical details on matching the CFT sum rules to flat space are given in appendices C.3, C.4, and C.5.

4.2 Review: superconvergence sum rules

Let us start by reviewing the idea of superconvergence in CFT, and explain how it can be related to flat space dispersive sum rules. Along the way we will introduce notation that we use throughout this work.

4.2.1 The light transform and superconvergence

Subtracted superconvergence sum rules come from studying the commutator of two null-integrated operators on the same null plane [34]. For simplicity we first consider a four-point function of scalar operators. Let us choose lightcone coordinates $x = (u, v, \vec{y})$ with $x^2 = -uv + \vec{y}^2$. A subtracted superconvergence sum rule can be written explicitly as

$$\int_{-\infty}^{\infty} dv_1 \int_{-\infty}^{\infty} dv_3 f(v_1, v_3) \langle 0|\phi_4 [\phi_3(0, v_3, \vec{y}_3), \phi_1(0, v_1, \vec{y}_1)] \phi_2|0\rangle = 0, \tag{4.4}$$

where ϕ_1, ϕ_3 are placed on the same null plane $u = 0$. This integral vanishes for a simple reason: since $x_{13}^2 = \vec{y}_{13}^2 > 0$, the two operators ϕ_1, ϕ_3 are spacelike separated in the entire integration range, and therefore their commutator must vanish.

To obtain a sum rule, we would like to separate the two orderings $\phi_3\phi_1$ and $\phi_1\phi_3$ and perform the integral separately for each ordering. However, in order for this to be valid, we must check that each integral converges — in particular that there is no divergence from the endpoints of the integration contours. See [34] for a detailed analysis of this convergence condition. In the end, convergence can be achieved by a suitable choice of the function $f(v_1, v_3)$, which we call a "subtraction" factor. Expressions for subtraction factors will be given in section 4.3.

Our goal in this section is to rewrite (4.4) as a conformally-invariant spacetime integral, using tools from [11]. First, let us introduce index-free notation. For an ordinary integer-spin operator $\mathcal{O}$, we contract the indices with a null polarization vector z:

$$\mathcal{O}(x, z) = \mathcal{O}^{\mu_1 \cdots \mu_J} z_{\mu_1} \cdots z_{\mu_J}, \quad z^2 = 0. \tag{4.5}$$

More generally, for a representation ρ represented by a Young diagram with more than one row, we introduce polarizations $z, w, \widetilde{w}, \ldots$ for each row of the Young diagram. They should be null and mutually orthogonal. Due to antisymmetry between indices in different rows, we have gauge redundancies $w \sim w + \#z$, $\widetilde{w} \sim \widetilde{w} + \#w + \#z$, etc. The number of boxes in each row of the Young diagram encodes the homogeneity of the corresponding polarization vector. Generalizing to arbitrary homogeneity in the polarization vector z allows us to describe representations with continuous spin, which will be crucial in later calculations.[3]

Index-free notation can be viewed as the embedding space formalism [129, 30] for the Lorentz group $\mathrm{SO}(d-1,1)$. In the embedding space of the d-dimensional conformal group $\mathrm{SO}(d,2)$, one promotes the position x and polarization z to $X, Z \in \mathbb{R}^{d,2}$ satisfying $X^2 = X \cdot Z = Z^2 = 0$ and $Z \sim Z + \#X$. Then, conformal transformations act linearly on X, Z, and conformally-invariant structures are simply built from dot products of X and Z. We can recover the Minkowski space operator by the dictionary

$$\mathcal{O}(x,z) = \mathcal{O}(X = (1, x^2, x^\mu), Z = (0, 2x\cdot z, z^\mu)), \tag{4.6}$$

where we again use lightcone coordinates $X = (X^+, X^-, X^\mu)$, and $X^2 = -X^+X^- + X^\mu X_\mu$. We will often go between embedding space and Minkowski space when writing expressions for conformally-invariant structures. In (C.6), we review the embedding space description of operators with more complicated representations.

The embedding space vector X lives on a projective null cone in $\mathbb{R}^{d,2}$, which is topologically $S^1 \times S^{d-1}$. Correlation functions of Lorentzian CFTs live on the universal cover $\widetilde{\mathcal{M}}_d = \mathbb{R} \times S^{d-1}$, also called the Lorentzian cylinder. More precisely, CFT correlation functions on $\mathbb{R}^{d-1,1}$ can be analytically continued to their Lorentzian cylinder counterparts [130]. The Lorentzian cylinder is tiled by Poincaré patches, where each patch represents a Minkowski space $\mathbb{R}^{d-1,1}$. The conformal group acting on $\widetilde{\mathcal{M}}_d$ is the universal covering group $\widetilde{\mathrm{SO}}(d,2)$. There exists a symmetry $\mathcal{T}$ such that for each point $p \in \widetilde{\mathcal{M}}_d$, all light-rays emanating from p will converge at $\mathcal{T}p$ in the next Poincaré patch. The action of $\mathcal{T}$ on an operator is most easily described in the embedding space, where we have

$$\mathcal{T}\mathcal{O}(X, Z_1, Z_2, \ldots)\mathcal{T}^{-1} = \mathcal{O}(-X, -Z_1, -Z_2, \ldots). \tag{4.7}$$

We will also use the notation $p^+ \equiv \mathcal{T}p$, $p^- \equiv \mathcal{T}^{-1}p$.

The null-integrated operators in (4.4) are an example of a conformally-invariant integral transform called the light transform [11]. The light transform of a local

[3]For the Lorentzian conformal group $\mathrm{SO}(d,2)$, their unitary principal representations have two continuous parameters, Δ and J [11].

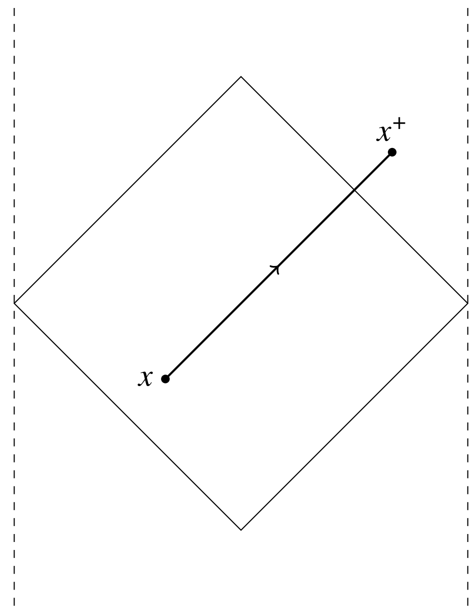


Figure 4.1: Contour of the light transform on the Lorentzian cylinder, where the two dashed lines should be identified. The contour starts at point x and ends at x^+ on the next Poincaré patch.

operator $\mathcal{O}$ with quantum numbers (Δ, J) is defined as

$$\mathbf{L}[\mathcal{O}](x,z) = \int_{-\infty}^{\infty} d\alpha (-\alpha)^{-\Delta-J} \mathcal{O}\left(x - \frac{z}{\alpha}, z\right). \tag{4.8}$$

Under conformal transformations, $\mathbf{L}[\mathcal{O}](x,z)$ transforms like a primary operator at x. The light transform contour is shown in figure 4.1. It starts at the point x, goes along the direction of the polarization z, and eventually ends at the point $\mathcal{T}x = x^+$. The previous description of the null-integrated operator in (4.4), where one puts the operator at $u = 0$ and integrates along v, can be obtained by putting x at past null infinity.

In the embedding space, the light transform (4.8) becomes

$$\mathbf{L}[\mathcal{O}](X,Z) = \int_{-\infty}^{\infty} d\alpha\, \mathcal{O}(Z - \alpha X, -X). \tag{4.9}$$

From this expression, it is not hard to see that the quantum numbers of $\mathbf{L}[\mathcal{O}]$ are $(1-J, 1-\Delta)$. The light transform of a local operator annihilates the vacuum [11]:

$$\mathbf{L}[\mathcal{O}](x,z)|\Omega\rangle = 0. \tag{4.10}$$

With all this in place, we can now write the subtracted superconvergence sum rule as a conformally-invariant spacetime integral. We define a functional $\Psi_{k,\nu}$ by[4]

$$\Psi_{k,\nu}[\mathcal{G}] \equiv \int \frac{d^d x_2 d^d x_4 d^d x_0 D^{d-2} z_1 D^{d-2} z_3}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))} K_{k,\nu}(x_2, x_4, x_0, z_1, z_3) \\ \times \langle \Omega | \phi_4(x_4) [\mathbf{L}[\phi_3](x_0, z_3), \mathbf{L}[\phi_1](x_0, z_1)] \phi_2(x_2) | \Omega \rangle, \qquad (4.11)$$

where $K_{k,\nu}$ is a kernel chosen such that the integral is conformally-invariant. Its explicit expression is given below in (4.33). The integral measure for the polarization $D^{d-2}z$ is defined as [46]

$$D^{d-2}z \equiv \frac{2 d^d z \delta(z^2) \theta(z^0)}{\mathrm{vol}\, \mathbb{R}_+}, \qquad (4.12)$$

where $\mathbb{R}_+$ acts by rescaling z. Then, the superconvergence condition (4.4) is equivalent to

$$\Psi_{k,\nu}[(\text{subtractions}) \times \mathcal{G}] = 0, \qquad (4.13)$$

where $\mathcal{G}$ is a four-point function and (subtraction) corresponds to the factor $f(v_1, v_3)$ in (4.4) (we discuss them in more detail below).

Using (4.10), we can rewrite (4.11) as

$$\Psi_{k,\nu}[\mathcal{G}] = \int \frac{d^d x_2 d^d x_4 d^d x_0 D^{d-2} z_1 D^{d-2} z_3}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))} K_{k,\nu}(x_2, x_4, x_0, z_1, z_3) \\ \times \langle \Omega | [\phi_4(x_4), \mathbf{L}[\phi_3](x_0, z_3)] [\mathbf{L}[\phi_1](x_0, z_1), \phi_2(x_2)] | \Omega \rangle - (1 \leftrightarrow 3). \qquad (4.14)$$

When we replace the four-point function $\mathcal{G}$ with an s-channel conformal block $G^s_{\Delta,J}$, the double commutator $\langle \Omega | [\phi_4, \mathbf{L}[\phi_3]] [\mathbf{L}[\phi_1], \phi_2] | \Omega \rangle$ will give a $\sin^2(\pi \frac{\Delta - J - 2\Delta_\phi}{2})$ factor. For double-twist operators $\Delta - J = 2\Delta_\phi + 2n$, the $\sin^2$ factor then becomes a double zero. Therefore, $\Psi_{k,\nu}$ is a dispersive functional, meaning that it gives double zeros for all but finitely many double-twist operators.[5]

4.2.2 The flat space limit of superconvergence sum rules

Let us now review the argument in [13] that leads to bounds on the scalar AdS EFT (4.1). As explained in the introduction, the idea is that in a certain "flat space limit," dispersive CFT sum rules reduce to flat space sum rules.

[4] This functional is equivalent to the $\widetilde{C}_{k,\nu}$ functional defined in the appendix of [13].

[5] When there is a subtraction factor, the $\sin^2$ factor gets modified and can become nonzero at finitely many double twist locations [56].

Review: flat space sum rules

Let us first briefly review flat space sum rules for identical real scalars. Flat space sum rules for photons and gravitons will be reviewed in section 4.4.

Let $\mathcal{M}(s,u)$ be a $2 \to 2$ scattering amplitude for identical scalars, which we assume satisfies analyticity, crossing symmetry, and Regge boundedness. This allows one to write down flat space dispersion relations for $\mathcal{M}(s,u)$ (see e.g., [104, 108] for more details),

$$-\oint_{\infty} \frac{ds}{2\pi i}\frac{1}{s}\frac{\mathcal{M}(s,u)}{(s(s+u))^{\frac{k}{2}}} = 0, \tag{4.15}$$

where s,u are the Mandelstam variables,

$$s = -(p_1+p_2)^2, \quad t = -(p_2+p_3)^2, \quad u = -(p_1+p_3)^2. \tag{4.16}$$

The contour in (4.15) can be deformed and separated into two parts: the low energy part, which can be computed using an EFT, and the high energy part, which can be decomposed into partial waves.

The partial wave decomposition of a scalar amplitude $\mathcal{M}(s,u)$ is given by

$$\mathcal{M}(s,u) = s^{\frac{4-D}{2}} \sum_J n_J^{(D)} a_J(s) \mathcal{P}_J\left(1+\frac{2u}{s}\right). \tag{4.17}$$

Here, $n_\rho^{(D)} = \frac{2^{d+1}(2\pi)^{d-1}\dim\rho}{\mathrm{vol}\, S^{d-1}}$, where in our case ρ is the spin-J traceless symmetric tensor representation of $\mathrm{SO}(d)$, and $d = D-1$. Meanwhile $\mathcal{P}_J$ is a Gegenbauer polynomial,

$$\mathcal{P}_J(x) = {}_2F_1(-J, J+d-2, \tfrac{d-1}{2}, \tfrac{1-x}{2}). \tag{4.18}$$

After the partial wave decomposition, the high energy part of the the dispersion relation (4.15) becomes

$$\left\langle \frac{2s+u}{s+u} \frac{\mathcal{P}_J(1+\frac{2u}{s})}{(s(s+u))^{\frac{k}{2}}} \right\rangle, \tag{4.19}$$

where $\langle\cdots\rangle$ is a heavy average, defined as a sum with positive coefficients over heavy states with mass m and spin J,

$$\langle\cdots\rangle = \frac{1}{\pi}\sum_J n_J^{(D)} \int_{M^2}^{\infty} \frac{dm^2}{m^2} m^{4-D}\, \mathrm{Im}\, a_J(s)\, (\cdots)\,. \tag{4.20}$$

Unitarity implies that the spectral density $\mathrm{Im}\, a_J(s)$ should be positive (or more generally a positive semidefinite matrix). Therefore, by applying a functional that has positive action on all partial wave contributions (4.19), one can derive inequalities for Wilson coefficients of the low energy EFT. Without gravity, the functional can simply include taking the forward limit $u \to 0$. For amplitudes with graviton exchange, one can integrate (4.19) over u against some kernel (see [108, 109, 110]).

The flat space limit

Consider a large-N CFT whose single-trace spectrum consists of a scalar operator ϕ, the stress tensor $T^{\mu\nu}$, and operators with twists $\tau = \Delta - J$ greater than $\Delta_{\text{gap}} \gg 1$. By the HPPS conjecture, this theory is expected to be dual to an EFT in AdS with the effective action (4.1). The s-channel conformal block decomposition of $\langle\phi\phi\phi\phi\rangle$ is given by

$$\langle\phi\phi\phi\phi\rangle = G_1^s + f_T^2 G_{T^{\mu\nu}}^s + \sum_{n,J} f_{[\phi\phi]_{n,J}}^2 G_{[\phi\phi]_{n,J}}^s + \sum_{\tau>\Delta_{\text{gap}}} f_{\Delta,J}^2 G_{\Delta,J}^s, \tag{4.21}$$

where f_O is the OPE coefficient, and $[\phi\phi]$ are double-trace operators built from ϕ. Consider now a dispersive functional ω whose action on the four-point function vanishes. Applying it to the conformal block decomposition gives us

$$-\,\omega|_{\text{light}} = \sum_{\tau>\Delta_{\text{gap}}} f_{\Delta,J}^2\, \omega[G_{\Delta,J}^s], \tag{4.22}$$

where

$$\omega|_{\text{light}} \equiv \omega[G_1^s] + f_T^2 \omega[G_{T^{\mu\nu}}^s] + \sum_{n,J} f_{[\phi\phi]_{n,J}}^2 \omega[G_{[\phi\phi]_{n,J}}^s]. \tag{4.23}$$

One can argue that $\omega|_{\text{light}}$ is determined by the Wilson coefficients of the low-energy EFT in AdS. Therefore, if we can find a functional ω such that

$$\omega[G_{\Delta,J}^s] \geq 0, \quad \tau > \Delta_{\text{gap}}, \tag{4.24}$$

then positivity of $f_{\Delta,J}^2$ implies that

$$-\,\omega|_{\text{light}} \geq 0, \tag{4.25}$$

which becomes an inequality on EFT Wilson coefficients.

To study the action of ω on heavy blocks, it is useful to consider two special limits.

- The "bulk point"/"flat space" limit is the regime of large Δ with $J/\Delta \ll 1$. As explained in [13], this corresponds to an AdS scattering process where the energy m and impact parameter β are given by

$$m^2 \approx \Delta^2, \quad \beta \approx \frac{2J}{\Delta} \ll 1. \tag{4.26}$$

  The term "flat space" comes from the fact that the impact parameter β is much smaller than the AdS radius (which is 1 in our conventions).

- The Regge limit is the regime of large Δ, J with fixed ratio Δ/J. This corresponds to an AdS scattering process with impact parameter comparable to the AdS radius.

A necessary condition for a functional ω to be positive is that it should be positive in the flat space regime. In this regime, one can relate the action of ω on conformal blocks to the action of simple flat-space dispersion relations. For example,

$$\Psi_{k,\nu;f_k}[\Delta, J] = \frac{2m^2 - \nu^2}{m^2 - \nu^2} \frac{\mathcal{P}_J(1 - \frac{2\nu^2}{m^2})}{(m^2(m^2 - \nu^2))^{\frac{k}{2}}} \left(1 + O\left(\frac{J^2}{m^2}\right)\right), \tag{4.27}$$

where $\mathcal{P}_J$ is a Gegenbauer polynomial given by (4.18). On the left-hand side $\omega[\Delta, J]$ denotes the action of ω on a conformal block, rescaled by some positive factors:

$$\omega[\Delta, J] \equiv \frac{1}{q_{\Delta,J}} \frac{\omega[G^s_{\Delta,J}]}{2\sin^2(\pi\frac{\Delta - J - 2\Delta_\phi}{2})}, \tag{4.28}$$

where

$$\begin{aligned} q_{\Delta,J} &= \frac{1}{\pi p^{\text{MFT}}_{\Delta,J}} \frac{2n_J}{m^{2d-4}} \frac{\Gamma(\Delta-1)(2\Delta-d)}{\Gamma(\Delta-d+2)}, \\ n_J &= \frac{2^{d+1}\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}(J+1)_{d-3}(2J+d-2). \end{aligned} \tag{4.29}$$

$p^{\text{MFT}}_{\Delta,J}$ is the OPE coefficient of the Mean Field Theory, whose expression can be found in [131].

The functional $\Psi_{k,\nu;f_k}$ denotes a subtracted version of $\Psi_{k,\nu}$, where we insert an additional factor f_k into the integrand. As discussed below (4.4) this factor is needed to modify the behavior near the endpoints of the integral. For example, for $k = 2$, an appropriate subtraction factor is given by

$$f_{k=2}(u', v') = \frac{v' - u'}{u'v'}, \tag{4.30}$$

where u', v' are conformally-invariant cross-ratios,

$$u' = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v' = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{4.31}$$

We give subtraction factors for general k in section 4.3.4.

The right hand side of (4.27) is exactly the contribution of a state with mass m and spin J to the flat space scalar sum rule (4.19) after identifying $s = m^2, u = -\nu^2$. This guarantees that the flat space functionals with positive action on (4.19) can be uplifted to AdS and will give us a CFT functional with positive action on all blocks in the bulk-point limit ($\Delta \gg 1$).

As explained in [13], to obtain bounds on the bulk EFT Wilson coefficients, one must also ensure that the functional ω is positive outside the bulk point limit, in particular in the Regge regime (and other regimes if necessary). We leave an exploration of the Regge regime for four-point functions of spinning operators to future work. The goal in this work is to derive the bulk-point limit result (4.27) and its generalization to spinning operators from a saddle analysis of the spacetime integral.

Before we proceed, let us comment on a subtlety regarding (4.27). In [13], (4.27) was derived (for a slightly different functional $C_{k,\nu}$ which is equivalent to $\Psi_{k,\nu}$ at large ν) for all $\nu \in [0, m)$. However, the derivation in this work will be in the limit where both ν and m are large (with fixed ratio). (Therefore, sometimes we will also refer to the bulk-point limit as the $\nu, m \gg 1$ limit.) In [13] it was shown that in the scalar case, the regimes of finite ν and large-ν are continuously connected at large m. In other words, we can safely compute at large ν, and then later consider all $\nu \in [0, m)$. We expect the same to be true in the spinning case as well.

4.3 Heavy action from a spacetime saddle-point analysis

In this section, we define our functional in position space, and study its action on a conformal block $G_{\Delta,J}$ with large scaling dimension Δ. We will show that in the bulk point limit $\nu, \Delta \gg 1$, the integral completely localizes to a saddle point, and the heavy action can be obtained by evaluating conformally-invariant structures at the saddle. This allows us to derive a simple formula for the functional in the bulk point limit. We will first use this formula to reproduce the results summarized in section 4.2, and then write down the generalized functional for spinning operators.

4.3.1 Functionals as spacetime integrals

As explained in section 4.2, our starting point for the position space functional is the fact that the commutator of light-transformed operators vanishes,

$$\langle\Omega|O_4(x_4)[\mathbf{L}[O_3](x_0,z_3),\mathbf{L}[O_1](x_0,z_1)]O_2(x_2)|\Omega\rangle = 0. \tag{4.32}$$

To make sure the light transform integrals $\mathbf{L}[O_1]\mathbf{L}[O_3]$ converge, we also need the condition $J_1 + J_3 - 1 > J_0$, where J_0 is the Regge intercept [34]. For the moment, let us assume this condition is satisfied, and we will come back to this issue later in section 4.3.4 and 4.3.5 by introducing subtraction factors.

We can integrate (4.32) against appropriate conformally-invariant structures to get different sum rules. We first consider the functional for scalar four-point functions. As we will see shortly, once we have derived this formula, the generalization to spinning operators will be obvious. For simplicity, we assume the external operators all have scaling dimension Δ_ϕ, but we will not impose that they are identical until section 4.3.4. Our functional is defined as

$$\Psi_{k,\nu}[\mathcal{G}] \equiv A_{k,\nu}\int_{\substack{2>4\\0\approx 2,4}}\frac{d^dx_2d^dx_4d^dx_0D^{d-2}z_0}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\langle 0|\phi_4(x_4^+)\mathbf{L}[O](x_0,z_0)\phi_2(x_2)|0\rangle^{-1}$$
$$\int D^{d-2}z_1D^{d-2}z_3\langle\widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_3}(z_3)\mathcal{P}_\delta(z_0)\rangle\langle\Omega|\phi_4(x_4^+)[\mathbf{L}[\phi_3](x_0,z_3),\mathbf{L}[\phi_1](x_0,z_1)]\phi_2(x_2)|\Omega\rangle, \tag{4.33}$$

where the coefficient $A_{k,\nu}$ is given by

$$A_{k,\nu} = 2^{-8+4\Delta_\phi}\pi^{-2-\frac{d}{2}}e^{\pi\nu}\nu^{2+d-2k-4\Delta_\phi}\Gamma\left(\tfrac{d-2}{2}\right)\Gamma(\Delta_\phi)^2\Gamma\left(\Delta_\phi-\tfrac{d-2}{2}\right)^2. \tag{4.34}$$

The coefficient is chosen such that the heavy action of $\Psi_{k,\nu}$ agrees with the flat space sum rule. We have also applied $\mathcal{T}_4$ to the integrand for later convenience. The causality relations between the points are shown in figure 4.2a. In (4.33), $2 > 4$ indicates that x_2 is in the future lightcone of 4, and $0 \approx 2,4$ indicates that x_0 is spacelike separated from x_2 and x_4.

Let us be explicit about the quantum numbers and structures in the definition (4.33). The operator O in the first line has scaling dimension $\Delta = \frac{d}{2}+i\nu$, and its spin is fixed by symmetry to be $J = -1$ [12, 15]. The structure $\langle 0|\phi_4(x_4^+)\mathbf{L}[O](x_0,z_0)\phi_2(x_2)|0\rangle^{-1}$ is a dual structure of a light transformed three-point function with respect to a

Lorentzian three-point pairing,

$$\Big(\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle, \langle \widetilde{\mathcal{O}}_1^\dagger \widetilde{\mathcal{O}}_2^\dagger \mathcal{O}^{S\dagger}\rangle\Big)_L \equiv \int_{\substack{2<1\\ x\approx 1,2}} \frac{d^dx_1 d^dx_2 d^dx D^{d-2}z}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}(x,z)\rangle \langle \widetilde{\mathcal{O}}_1^\dagger(x_1)\widetilde{\mathcal{O}}_2^\dagger(x_2)\mathcal{O}^{S\dagger}(x,z)\rangle. \tag{4.35}$$

The dual structure is defined by

$$\Big(\langle 0|\phi_4(x_4^+)\mathbf{L}[\mathcal{O}](x_0,z_0)\phi_2(x_2)|0\rangle^{-1}, \langle 0|\phi_4(x_4^+)\mathbf{L}[\mathcal{O}](x_0,z_0)\phi_2(x_2)|0\rangle\Big)_L = 1. \tag{4.36}$$

The explicit expression is [11][6]

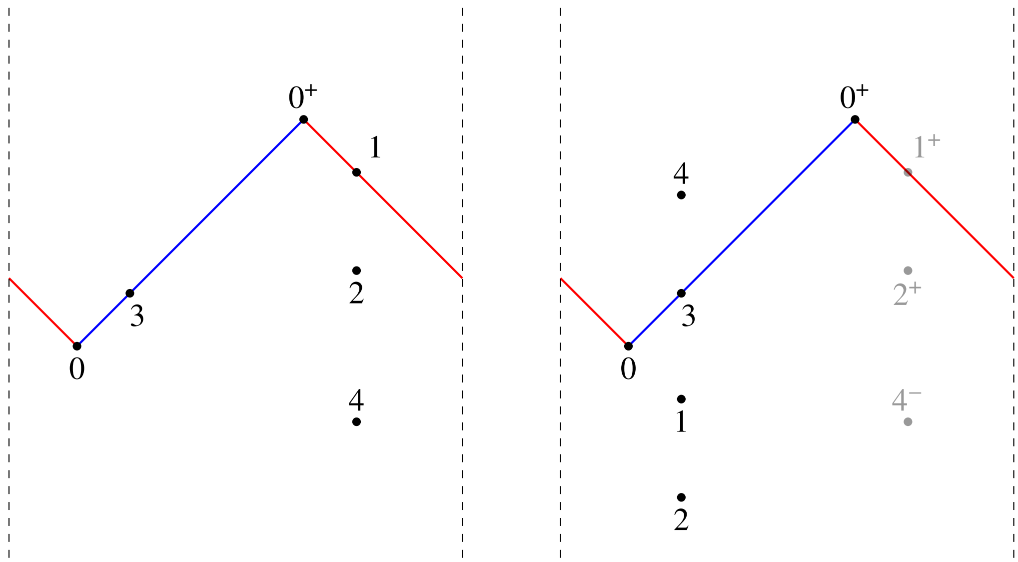
$$\langle 0|\phi_4(x_4^+)\mathbf{L}[\mathcal{O}_{\Delta,J}](x_0,z_0)\phi_2(x_2)|0\rangle^{-1} = \frac{2^{2d-2}\Gamma(\frac{\Delta+J}{2})^2\,\mathrm{vol}(\mathrm{SO}(d-2))}{2\pi i\Gamma(\Delta+J-2)} \frac{(2z_0\cdot x_{40}x_{20}^2 - 2z_0\cdot x_{20}x_{40}^2)^{\Delta-d+1}}{(-x_{24}^2)^{\frac{\widetilde{\Delta}_2+\widetilde{\Delta}_4+\Delta-J-2d+2}{2}}(x_{02}^2)^{\frac{\widetilde{\Delta}_2+\Delta+J-\widetilde{\Delta}_4}{2}}(x_{02}^2)^{\frac{\widetilde{\Delta}_4+\Delta+J-\widetilde{\Delta}_2}{2}}}. \tag{4.37}$$

In the second line of (4.33), $\langle \widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_3}(z_3)\mathcal{P}_\delta(z_0)\rangle$ is a three-point structure of a fictitious $(d-2)$-dimensional Euclidean CFT on the celestial sphere, whose conformal symmetry is the Lorentz symmetry $\mathrm{SO}(d-1,1)$ of the original CFT. The polarizations z_i should be viewed as embedding space coordinates of this CFT_{d-2}, and $\mathcal{P}_\delta(z)$ is a primary with scaling dimension δ on the celestial sphere. In order for the integral (4.33) to be conformally-invariant, we must have $\delta = \frac{d-2}{2} + i\nu$. We also use the notation $\widetilde{\mathcal{P}}_{\delta_i} = \mathcal{P}_{d-2-\delta_i}$, and $\delta_i = \Delta_i - 1$, while $\widetilde{\delta_i} = d-2-\delta_i$ is the shadow dimension with respect to the $(d-2)$-dimensional fictitious CFT. The celestial three-point structure is given by

$$\langle \widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_3}(z_3)\mathcal{P}_\delta(z_0)\rangle = \frac{1}{(-2z_1\cdot z_0)^{\frac{\delta+\widetilde{\delta}_1-\widetilde{\delta}_3}{2}}(-2z_3\cdot z_0)^{\frac{\delta+\widetilde{\delta}_3-\widetilde{\delta}_1}{2}}(-2z_1\cdot z_3)^{\frac{\widetilde{\delta}_1+\widetilde{\delta}_3-\delta}{2}}}. \tag{4.38}$$

Originally, the superconvergence sum rule (4.32) is parametrized by the positions and polarizations of the operators, subject to conformal invariance. By integrating it against the dual structure and celestial structure described above, we have

[6] Here, we choose the structures to be an analytic continuation from odd J, so the $(-1)^J$ factor in [11] becomes -1.



(a) Original configuration (b) Configuration after applying $\mathcal{T}$'s

Figure 4.2: Causality configuration of the functional. The left figure shows the configuration in the original definition (4.33). Points $1, 3$ both start from 0 and are integrated along two different null directions. The right figures shows the configuration after applying $\mathcal{T}_1^{-1}, \mathcal{T}_2^{-1}, \mathcal{T}_4$, which becomes $4 > 3 > 0 > 1 > 2$.

transformed the statement of superconvergence into ν-space, which parametrizes the quantum numbers of the structures. In fact, one can also show that the condition $\Psi_{k,\nu}[\mathcal{G}] = 0$ is equivalent to [15]

$$C^-(\tfrac{d}{2} + i\nu, J = -1) = 0, \quad J_0 < -1, \tag{4.39}$$

where $C^-(\Delta, J)$ is the coefficient function computed by the Lorentzian inversion formula [9, 10], and encodes the analytic continuation of the CFT data from odd spin.

To further simplify the functional, let us focus on one term in the commutator $[\mathbf{L}[\phi_3], \mathbf{L}[\phi_1]]$ and consider

$$\begin{aligned}
\Psi_{k,\nu} &= \Psi^+_{k,\nu} - (1 \leftrightarrow 3), \\
\Psi^+_{k,\nu}[\mathcal{G}] &= A_{k,\nu} \int_{\substack{2>4 \\ 0\approx 2,4}} \frac{d^d x_2 d^d x_4 d^d x_0 D^{d-2} z_0}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))} \langle 0|\phi_4(x_4^+)\mathbf{L}[\mathcal{O}](x_0, z_0)\phi_2(x_2)|0\rangle^{-1} \\
&\quad \int D^{d-2} z_1 D^{d-2} z_3 \langle \widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_3}(z_3)\mathcal{P}_\delta(z_0)\rangle \langle \Omega|[\phi_4(x_4^+), \mathbf{L}[\phi_3](x_0, z_3)][\mathbf{L}[\phi_1](x_0, z_1), \phi_2(x_2)]|\Omega\rangle
\end{aligned} \tag{4.40}$$

Note that since light-transformed operators annihilate the vacuum, we can rewrite the four-point function as a double commutator. Moreover, in (4.40), we can rewrite the integral over the two light transform directions z_1, z_3 as an integral over the positions of ϕ_1, ϕ_3 with additional delta function constraints [77]. Explicitly, this gives

$$\Psi^{+}_{k,\nu}[\mathcal{G}] = 4A_{k,\nu}\int_{\substack{2>4\\0\approx 2,4}} \frac{d^dx_1 d^dx_2 d^dx_3 d^dx_4 d^dx_0 D^{d-2}z_0}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\langle 0|\phi_4(x_4^+)\mathbf{L}[\mathcal{O}](x_0,z_0)\phi_2(x_2)|0\rangle^{-1}$$
$$\langle\widetilde{\mathcal{P}}_{\delta_1}(x_{10})\widetilde{\mathcal{P}}_{\delta_3}(x_{30})\mathcal{P}_\delta(z_0)\rangle\delta(x_{10}^2)\delta(x_{30}^2)\theta(x_{10})\theta(x_{30})\langle\Omega|[\phi_4(x_4^+),\phi_3(x_3)][\phi_1(x_1),\phi_2(x_2)]|\Omega\rangle, \tag{4.41}$$

where the theta-functions $\theta(x_{10})\theta(x_{30})$ indicate that x_{10} and x_{30} must be timelike and future-pointing. Finally, let us make a change of variables $x_1 \to x_1^+, x_2 \to x_2^+, x_4 \to x_4^-$. This makes the causality configuration equivalent to $4 > 3 > 0 > 1 > 2$ (see figure 4.2b), and we arrive at the expression

$$\Psi^{+}_{k,\nu}[\mathcal{G}] = 4A_{k,\nu}\int_{4>3>0>1>2} \frac{d^dx_1 d^dx_2 d^dx_3 d^dx_4 d^dx_0 D^{d-2}z_0}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\langle 0|\phi_4(x_4)\mathbf{L}[\mathcal{O}](x_0,z_0)\phi_2(x_2^+)|0\rangle^{-1}$$
$$\langle\widetilde{\mathcal{P}}_{\delta_1}(x_{1^+0})\widetilde{\mathcal{P}}_{\delta_3}(x_{30})\mathcal{P}_\delta(z_0)\rangle\delta(x_{1^+0}^2)\delta(x_{30}^2)\theta(x_{1^+0})\theta(x_{30})\langle\Omega|[\phi_4(x_4),\phi_3(x_3)][\phi_1(x_1^+),\phi_2(x_2^+)]|\Omega\rangle. \tag{4.42}$$

4.3.2 Action on conformal blocks

Let us now apply the functional $\Psi^+_{k,\nu}$ to an s-channel conformal block $G^s_{\Delta,J}$ and study the action in the large Δ limit. In position space, the most useful expression for the block is the Lorentzian shadow representation [2], given by integrating a fifth point x_5 over a causal diamond $4 > 5 > 3$:

$$G^s_{\Delta,J} = \frac{1}{\beta_{\Delta,J}}\int_{4>5>3} d^dx_5 |\langle\phi_1\phi_2\mathcal{O}^{\mu_1\cdots\mu_J}(x_5)\rangle||\langle\widetilde{\mathcal{O}}^\dagger_{\mu_1\cdots\mu_J}(x_5)\phi_3\phi_4\rangle|. \tag{4.43}$$

The external points should satisfy the causality constraint $1 > 2, 4 > 3$ with all other pairs of points spacelike-separated. On the right-hand side, the operator $\mathcal{O}$ has quantum numbers (Δ, J), and the quantum numbers of $\widetilde{\mathcal{O}}^\dagger$ are related to those of $\mathcal{O}$ by a shadow transform and Hermitian conjugate:[7]

$$\begin{aligned}\widetilde{\mathcal{O}} &: (\Delta,\rho) \mapsto (d-\Delta,\rho^R),\\ \mathcal{O}^\dagger &: (\Delta,\rho)\mapsto(\Delta,\rho^\dagger = (\rho^R)^*).\end{aligned} \tag{4.44}$$

In (4.43), we contract the indices of the exchanged operator $\mathcal{O}_5$. This is more natural for our purpose because the spin of the inserted block corresponds to spin of the

[7] In the case where ρ is a traceless symmetric tensor representation with spin J, we have $\rho^R = \rho = (\rho^R)^*$.

exchanged massive particle in flat space, and hence we are only interested in the integer-spin case.[8] Moreover, later we will find a correspondence between blocks and flat space partial waves, which also come from contracting the indices of spin-J tensors. The shadow coefficient $\beta_{\Delta,J}$ in (4.43) in the scalar case is given by

$$\beta_{\Delta,J} = \frac{(-1)^J 2^{J-1}\pi^{\frac{d-2}{2}}\Gamma(\Delta+2-d)\Gamma(\frac{J+\Delta}{2})^2\Gamma(\frac{\Delta+2-d-J}{2})^2}{\Gamma(\Delta+\frac{2-d}{2})\Gamma(J+\Delta)\Gamma(\Delta+2-d-J)}. \tag{4.45}$$

For the three-point structures in (4.43), the notation $|\langle\ldots\rangle|$ means that all the x_{ij}'s in the denominator should come with absolute values (see appendix C.1 for definitions). In our case, the correct interpretation of the absolute-valued structures is in terms of a double commutator,

$$\begin{aligned}&\langle\Omega|[\phi_4(x_4),\phi_3(x_3)][\phi_1(x_1),\phi_2(x_2)]|\Omega\rangle^s_{\Delta,J}\\ &= \frac{-2(2\sin^2(\pi\frac{\Delta-J-2\Delta_\phi}{2}))}{\beta_{\Delta,J}}\int_{4>5>3} d^dx_5|\langle\phi_1\phi_2 O^{\mu_1\cdots\mu_J}(x_5)\rangle||\langle\widetilde{O}^\dagger_{\mu_1\cdots\mu_J}(x_5)\phi_3\phi_4\rangle|,\end{aligned} \tag{4.46}$$

where the left hand side denotes the contribution from the $G^s_{\Delta,J}$ block to the double commutator. Each commutator gives $2i\sin(\pi\frac{\Delta-J-2\Delta_\phi}{2})$ from the difference of two phase factors, so we have an additional $-2(2\sin^2(\pi\frac{\Delta-J-2\Delta_\phi}{2}))$ factor, which is accounted for on the right-hand side.

In our calculation, we will use absolute-valued structures as convenient notation in intermediate steps, and replace them with commutators at the end as appropriate. For example, for the structure $|\langle\phi_1(x_1)\phi_2(x_2)O(x_5)\rangle|$, which has causality relation $5\approx(1>2)$, we have the following identity:

$$\langle 0|O(x_5)[\phi_1(x_1),\phi_2(x_2)]|0\rangle = 2i\sin(\pi\frac{\Delta-J-2\Delta_\phi}{2})|\langle\phi_1(x_1)\phi_2(x_2)O(x_5)\rangle|, \quad 5\approx(1>2), \tag{4.47}$$

where the left-hand side is a commutator of the standard Wightman structure. For the other structure $|\langle\widetilde{O}^\dagger(x_5)\phi_3\phi_4\rangle|$, this identity is not well-defined due to the additional phase factors from x^2_{45}, x^2_{35}. However, in the final formula for the heavy action, we will be able to apply the above identity to both structures, enabling us to remove absolute values and write the final formula in terms of standard Wightman structures.

We can now plug the Lorentzian shadow representation into our functional. In the functional (4.42), the external points of the four-point function are $1^+, 2^+, 3, 4$ with

[8] This is in contrast to the continuous-spin version of the Lorentzian shadow representation given in [11] that involves integrating over the polarization vector of the exchanged operator.

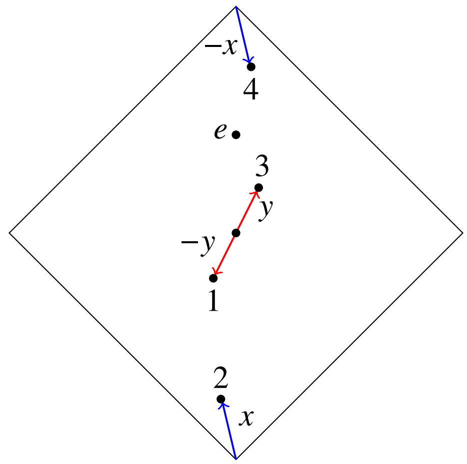


Figure 4.3: Our gauge fixing condition is $x_3 = -x_1 = y, x_4 = -x_2 = \frac{x}{-x^2}, x_5 = e$. The causality constraint is $4 > 5 > 3 > 0 > 1 > 2$.

the condition $4 > 3 > 0 > 1 > 2$, so the causality configuration agrees with the Lorentzian shadow representation (4.43). See also figure 4.2b. Therefore, using (4.43), the action of $\Psi^+_{k,\nu}$ on a conformal block can be written

$$\begin{aligned}\Psi^+_{k,\nu}[G^s_{\Delta,J}] =&\frac{-8(2\sin^2(\pi\frac{\Delta-J-2\Delta_\phi}{2}))A_{k,\nu}}{\beta_{\Delta,J}}\int_{4>5>3>0>1>2}\frac{d^dx_1d^dx_2d^dx_3d^dx_4d^dx_5d^dx_0D^{d-2}z_0}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\\&\times\langle 0|\phi_4\mathbf{L}[O](x_0,z_0)\phi_{2^+}|0\rangle^{-1}\langle\widetilde{\mathcal{P}}_{\delta_1}(x_{1^+0})\widetilde{\mathcal{P}}_{\delta_3}(x_{30})\mathcal{P}_\delta(z_0)\rangle\\&\times\delta(x_{1^+0}^2)\delta(x_{30}^2)\theta(x_{1^+0})\theta(x_{30})|\langle\phi_{1^+}\phi_{2^+}O^{\mu_1\cdots\mu_J}(x_5)\rangle||\langle\widetilde{O}^\dagger_{\mu_1\cdots\mu_J}(x_5)\phi_3\phi_4\rangle|.\end{aligned} \tag{4.48}$$

For brevity, we have used the short-hand notation $\phi_i = \phi_i(x_i), \phi_{i^+} = \phi_i(x_i^+)$.

Now we use conformal symmetry to gauge fix the integral. We will choose the gauge fixing in [13] and fix $x_1 + x_3$ to the origin, $x_{2^+} + x_{4^-}$ to spatial infinity, and x_5 to the unit time vector e. More precisely, we choose (see figure 4.3)

$$x_3 = -x_1 = y, \quad x_4 = -x_2 = \frac{x}{-x^2}, \quad x_5 = e, \quad x, y > 0. \tag{4.49}$$

For this gauge fixing, the stabilizer group is $\mathrm{SO}(d-1)$, and the Faddeev-Popov determinant is [13]

$$2^d(1 + 2x\cdot y + x^2y^2)(1 - 2x\cdot y + x^2y^2)(1 - x^2y^2)^{d-2}. \tag{4.50}$$

After the gauge fixing, we obtain

$$\begin{aligned}\Psi^+_{k,\nu}[G^s_{\Delta,J}] =& \frac{-8(2\sin^2(\pi\frac{\Delta-J-2\Delta_\phi}{2}))A_{k,\nu}}{\beta_{\Delta,J}\,\mathrm{vol}(\mathrm{SO}(d-1))}\int \frac{d^dx d^dy d^dx_0 D^{d-2}z_0}{(-x^2)^{2d}}\\ &\times 2^d(1+2x\cdot y+x^2y^2)(1-2x\cdot y+x^2y^2)(1-x^2y^2)^{d-2}\\ &\times \langle 0|\phi_4\mathbf{L}[O](x_0,z_0)\phi_{2^+}|0\rangle^{-1}\langle\widetilde{\mathcal{P}}_{\delta_1}(x_{1^+0})\widetilde{\mathcal{P}}_{\delta_3}(x_{30})\mathcal{P}_\delta(z_0)\rangle\\ &\times \delta(x^2_{1^+0})\delta(x^2_{30})\theta(x_{1^+0})\theta(x_{30})|\langle\phi_{1^+}\phi_{2^+}O^{\mu_1\cdots\mu_J}(x_5)\rangle||\langle\widetilde{O}^\dagger_{\mu_1\cdots\mu_J}(x_5)\phi_3\phi_4\rangle|\Big|_{\text{gauge-fixed}},\end{aligned} \tag{4.51}$$

where the last two lines should be evaluated in the gauge-fixed configuration (4.49).

The integral over z_0 can also be fixed using $\mathrm{SO}(d-1)$ invariance. After integrating over x, y, x_0, the remaining vectors in the integrand of (4.51) are z_0 and e. Homogeneity of z_0 then implies the z_0-integral must be of the form

$$\int D^{d-2}z_0\,(-2z_0\cdot e)^{2-d} = \frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d-2}{2})}{\Gamma(d-2)}. \tag{4.52}$$

Thus, we can eliminate the z_0-integral by setting z_0 to be a fixed null vector z_0^*, and then introduce a factor

$$(-2z_0^*\cdot e)^{d-2}\frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d-2}{2})}{\Gamma(d-2)}. \tag{4.53}$$

Let us introduce lightcone coordinate $x = (u, v, \vec{x}_\perp)$, where $x^2 = -uv + \vec{x}_\perp^2$, and choose $z_0^* = (1, 0, \vec{0})$. Then (4.51) becomes

$$\begin{aligned}\Psi^+_{k,\nu}[G^s_{\Delta,J}] =& \frac{-8(2\sin^2(\pi\frac{\Delta-J-2\Delta_\phi}{2}))A_{k,\nu}}{\beta_{\Delta,J}\,\mathrm{vol}(\mathrm{SO}(d-1))}\frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d-2}{2})}{\Gamma(d-2)}\int \frac{d^dx d^dy d^dx_0}{(-x^2)^{2d}}\\ &\times 2^d(1+2x\cdot y+x^2y^2)(1-2x\cdot y+x^2y^2)(1-x^2y^2)^{d-2}\\ &\times \langle 0|\phi_4\mathbf{L}[O](x_0,z_0^*)\phi_{2^+})|0\rangle^{-1}\langle\widetilde{\mathcal{P}}_{\delta_1}(x_{1^+0})\widetilde{\mathcal{P}}_{\delta_3}(x_{30})\mathcal{P}_\delta(z_0^*)\rangle\\ &\times \delta(x^2_{1^+0})\delta(x^2_{30})\theta(x_{1^+0})\theta(x_{30})|\langle\phi_{1^+}\phi_{2^+}O^{\mu_1\cdots\mu_J}(x_5)\rangle||\langle\widetilde{O}^\dagger_{\mu_1\cdots\mu_J}(x_5)\phi_3\phi_4\rangle|\Big|_{\text{gauge-fixed}}.\end{aligned} \tag{4.54}$$

4.3.3 Saddle point analysis

We now study the action $\Psi^+_{k,\nu}[G^s_{\Delta,J}]$ in the bulk-point limit, where we take both ν and Δ to be large (and $\Delta \sim m$). To study the large ν, large Δ limit of (4.54), we will consider the factors that depend exponentially on ν and Δ and look for a saddle

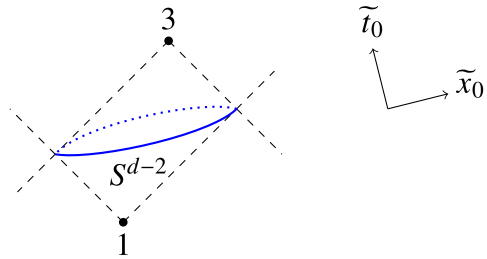


Figure 4.4: The two delta functions $\delta(x_{10}^2)\delta(x_{30}^2)$ force x_0 to be on an S^{d-2} (the blue curves). The directions of the variables $\widetilde{t}_0, \widetilde{x}_0$ defined in (4.59) are also indicated.

point. The important factors in (4.54) are given by

$$\left(\frac{(2z_0\cdot x_{40}x_{20}^2 - 2z_0\cdot x_{20}x_{40}^2)^2(2x_{10}\cdot x_{30})}{(-x_{24}^2)(-x_{40}^2)(-x_{20}^2)(2z_0\cdot x_{10})(-2z_0\cdot x_{30})}\right)^{\frac{i\nu}{2}}\left(\frac{(-x_{35}^2)(-x_{45}^2)(-x_{12}^2)}{(-x_{15}^2)(-x_{25}^2)(-x_{34}^2)}\right)^{\frac{m}{2}}\Bigg|_{\text{gauge-fixed},z_0=z_0^*}$$
$$= e^{f_{\nu,m}(x,y,x_0)}, \tag{4.55}$$

where the function $f_{\nu,m}(x,y,x_0)$ is given by

$$f_{\nu,m}(x,y,x_0) = \frac{i\nu}{2}\log\left(\frac{4(z_0^*\cdot x(1+x^2x_0^2) - 2z_0^*\cdot x_0x^2x\cdot x_0)^2(-y^2)}{(-x^2)(1-2x\cdot x_0+x^2x_0^2)(1+2x\cdot x_0+x^2x_0^2)(z_0^*\cdot(y-x_0))(z_0^*\cdot(y+x_0))}\right)$$
$$+\frac{m}{2}\log\left(\frac{(e-x)^2(e-y)^2}{(e+x)^2(e+y)^2}\right). \tag{4.56}$$

The integral we have to consider in the bulk-point limit then takes the form

$$\int d^dxd^dyd^dx_0\delta((y+x_0)^2)\delta((y-x_0)^2)\theta(y-x_0)\theta(y+x_0)e^{f_{\nu,m}(x,y,x_0)}g(x,y,x_0), \tag{4.57}$$

where $g(x,y,x_0)$ are all the factors that do not depend exponentially on ν and m in the integrand of (4.54).

To find the location of the saddle, it is convenient to consider lightcone coordinates

$$x = (u_x, v_x, \vec{x}_\perp), \quad y = (u_y, v_y, \vec{y}_\perp), \quad x_0 = (u, v, \vec{y}_0). \tag{4.58}$$

Let us also define

$$\widetilde{t}_0 = \frac{1}{2}\left(\frac{u}{u_y}+\frac{v}{v_y}-2\frac{\vec{y}_0\cdot\vec{y}_\perp}{u_yv_y}\right), \quad \widetilde{x}_0 = \frac{1}{2}\left(\frac{u}{u_y}-\frac{v}{v_y}\right). \tag{4.59}$$

In our functional, there are two delta functions $\delta(x_{1^+0}^2)\delta(x_{30}^2)$ in the integrand due to the light transforms. These delta functions restrict x_0 to lie on the intersection of the past lightcone of x_3 and the future lightcone of x_1. So, the x_0-integral is forced to be on an S^{d-2}, as shown by figure 4.4. In terms of the variables defined above, the delta functions will localize $\widetilde{t}_0$ and $|\vec{y}_0|$ to be

$$\widetilde{t}_0 = 0, \quad |\vec{y}_0|^2 = \frac{u_y v_y (u_y v_y (1 - \widetilde{x}_0^2) - \vec{y}_\perp^2)}{u_y v_y - (\vec{n}_0 \cdot \vec{y}_\perp)^2}, \tag{4.60}$$

where $\vec{n}_0 = \vec{y}_0 / |\vec{y}_0|$ is the unit vector in the $\vec{y}_0$ direction. The remaining integral over S^{d-2} then becomes an integral over $\widetilde{x}_0 \in [-(1 - \frac{\vec{y}_\perp^2}{u_y v_y})^{\frac{1}{2}}, (1 - \frac{\vec{y}_\perp^2}{u_y v_y})^{\frac{1}{2}}]$ and $\vec{n}_0 \in S^{d-3}$.

After going to lightcone coordinates and removing the delta functions, (4.57) becomes

$$\frac{(u_y v_y)^{\frac{d-2}{2}}}{32} \int d\widetilde{x}_0 du_x dv_x du_y dv_y d\Omega_{\vec{n}_0} d^{d-2}\vec{x}_\perp d^{d-2}\vec{y}_\perp (u_y v_y - (\vec{n}_0 \cdot y_\perp)^2)^{\frac{2-d}{2}} (u_y v_y (1 - \widetilde{x}_0^2) - \vec{y}_\perp^2)^{\frac{d-4}{2}}$$
$$\times e^{f_{\nu,m}(x,y,x_0)} g(x, y, x_0). \tag{4.61}$$

The large ν and large m limit localizes all the variables in the integral, except for $\vec{n}_0$. In fact, the $f_{\nu,m}$ function leads to sixteen different saddles at

$$x_0 = (0, 0, \vec{y}_0), \quad x = \left(-i\tfrac{m \pm \sqrt{m^2-\nu^2}}{\nu}, i\tfrac{m \pm \sqrt{m^2-\nu^2}}{\nu}, \vec{0}\right), \; y = \left(i\tfrac{m \pm \sqrt{m^2-\nu^2}}{\nu}, -i\tfrac{m \pm \sqrt{m^2-\nu^2}}{\nu}, \vec{0}\right), \tag{4.62}$$

where $|\vec{y}_0|^2 = -y^2$. Each saddle corresponds to a locus S^{d-3} parametrized by $\vec{n}_0$. To find the correct saddle approximation for the integral (4.61), we need to deform the integration contour into steepest descendant flows and see which saddle locus the contour goes through (see [132] for a pedagogical introduction). The analysis is essentially identical to [13], which also studies saddle point of x, y in the bulk-point limit. The result is that the dominant saddle depends on the relative size of ν and m. The case we will be particularly interested in is the region related to the flat space functional, where one considers $\nu \in [0, \Delta_{\text{gap}}]$ and $m > \Delta_{\text{gap}}$. In this case ($\nu < m$), the saddle locus that dominates the integral is given by

$$x_0 = (0, 0, \vec{y}_0), \quad x = \left(-i\tfrac{m - \sqrt{m^2-\nu^2}}{\nu}, i\tfrac{m - \sqrt{m^2-\nu^2}}{\nu}, \vec{0}\right), \; y = \left(i\tfrac{m - \sqrt{m^2-\nu^2}}{\nu}, -i\tfrac{m - \sqrt{m^2-\nu^2}}{\nu}, \vec{0}\right). \tag{4.63}$$

Expanding the function $f_{\nu,m}$ around the dominant saddle (4.63), we find

$$f_{\nu,m} = -\pi\nu + i\nu\log 2 - \frac{1}{2}\begin{pmatrix}\widetilde{x}_0 & u_x & v_x & u_y & v_y & \vec{x}_\perp & \vec{y}_\perp\end{pmatrix} M_{\nu,m,\vec{y}_0}\begin{pmatrix}\widetilde{x}_0\\ u_x\\ v_x\\ u_y\\ v_y\\ \vec{x}_\perp\\ \vec{y}_\perp\end{pmatrix} + \ldots, \tag{4.64}$$

where ... are higher-order terms in the expansion. The Hessian $M_{\nu,m,\vec{y}_0}$ has determinant

$$\mathrm{Det}M_{\nu,m,\vec{y}_0} = -\frac{i\nu^{6d+1}(m^2-\nu^2)^2(m-\sqrt{m^2-\nu^2})^{-4d}}{16m^4}. \tag{4.65}$$

Therefore, the large ν, large m limit of the integral (4.57) can be written as

$$\begin{aligned}&\lim_{\substack{\nu,m\gg 1\\ \nu<m}}\int d^dxd^dyd^dx_0\delta((y+x_0)^2)\delta((y-x_0)^2)\theta(y-x_0)\theta(y+x_0)e^{f_{\nu,m}(x,y,x_0)}g(x,y,x_0)\\ =&\frac{\nu^{4-d}(m-\sqrt{m^2-\nu^2})^{d-4}}{32}\sqrt{\frac{(2\pi)^{2d+1}}{\mathrm{Det}M_{\nu,m,\vec{y}_0}}}e^{-\pi\nu}2^{i\nu}\int d\Omega_{\vec{n}_0}\, g(x,y,x_0)|_{\text{saddle}}\,,\end{aligned} \tag{4.66}$$

where saddle stands for the configuration (4.63). Furthermore, the function $g(x,y,x_0)$ evaluated at the saddle does not include any $(d-2)$-dimensional vectors other than $\vec{n}_0$, and hence the remaining integral over the S^{d-3} locus must be trivial. The final expression of the integral is then given by

$$\begin{aligned}&\lim_{\substack{\nu,m\gg 1\\ \nu<m}}\int d^dxd^dyd^dx_0\delta((y+x_0)^2)\delta((y-x_0)^2)\theta(y-x_0)\theta(y+x_0)e^{f_{\nu,m}(x,y,x_0)}g(x,y,x_0)\\ =&\frac{e^{i\frac{\pi}{4}}(2\pi)^{d+\frac{1}{2}}m^2\nu^{-4d+\frac{7}{2}}(m-\sqrt{m^2-\nu^2})^{3d-4}}{8(m^2-\nu^2)}\,\mathrm{vol}(S^{d-3})\; e^{f_{\nu,m}(x,y,x_0)}g(x,y,x_0)\Big|_{\text{saddle}},\end{aligned} \tag{4.67}$$

where we have plugged in the Hessian determinant and used the fact that $e^{-\pi\nu}2^{i\nu}$ comes from evaluating $e^{f_{\nu,m}(x,y,x_0)}$ at the saddle.

Now, we can compare (4.67) and the functional (4.54) and find the expression for $\Psi^{+}_{k,\nu}[G^{s}_{\Delta,J}]$ in the bulk-point limit. From the above saddle analysis, we see that the integral in the bulk-point limit can be obtained by simply evaluating the integrand at the saddle and multiplying by the additional factors coming from the Jacobian and

Hessian. This implies that in the bulk-point limit, the calculation of the heavy action of our functional can be simplified to evaluating conformally-invariant structures in the saddle configuration (4.63), which leads to

$$\begin{aligned}
&\lim_{\substack{\nu,m\gg 1\\ \nu<m}} \frac{\Psi^{+}_{k,\nu}[G^{s}_{\Delta,J}]}{-2(2\sin^2(\pi\frac{\Delta-J-2\Delta_\phi}{2}))}\\
&= A_{k,\nu}\frac{2^{\frac{5}{2}+3d}\pi^{\frac{3d-1}{2}}e^{i\frac{\pi}{4}}m^{d+2}\nu^{\frac{7}{2}-4d}(m^2-\nu^2)^{\frac{d-2}{2}}(m-\sqrt{m^2-\nu^2})^{d-4}}{\beta_{\Delta,J}\Gamma\left(\frac{d-2}{2}\right)\mathrm{vol}(\mathrm{SO}(d-2))}\\
&\times\langle 0|\phi_4\mathbf{L}[\mathcal{O}](x_0,z_0^*)\phi_{2^+}|0\rangle^{-1}\langle\widetilde{\mathcal{P}}_{\delta_1}(x_{1^+0})\widetilde{\mathcal{P}}_{\delta_3}(x_{30})\mathcal{P}_{\delta}(z_0^*)\rangle\\
&\times|\langle\phi_{1^+}\phi_{2^+}\mathcal{O}^{\mu_1\cdots\mu_J}(x_5)\rangle||\langle\widetilde{\mathcal{O}}^{\dagger}_{\mu_1\cdots\mu_J}(x_5)\phi_3\phi_4\rangle|\Big|_{\text{saddle}}. \qquad (4.68)
\end{aligned}$$

The saddle calculation done above involves computing the determinant of a $(2d+1)\times(2d+1)$ Hessian matrix. An alternative derivation of the above formula that can avoid this technically involved calculation is by first studying the large ν limit of the functional, and then use the result of the large m saddle analysis that was already done in [13]. We describe this calculation in appendix C.2.

The shadow coefficient

Finally, the remaining task is to study the shadow coefficient $\beta_{\Delta,J}$ in the large Δ limit. For the scalar case, its explicit expression is known and given by (4.45). For the spinning case, the shadow transform will involve mixing of different tensor structures and therefore the shadow coefficients become a matrix. In principle it can be computed using e.g., weight-shifting operators [39]. However, as we discuss below, it turns out that the shadow transform at large Δ also gets localized to a saddle. Hence, the shadow transform at large Δ becomes algebraic and the coefficients can again be computed by evaluating structures at the saddle configuration.

Let us demonstrate this idea by studying the shadow transform in the scalar case. We will derive a formula for $\beta_{\Delta,J}$ that reproduces the known answer (4.45). However, the advantage of this formula is that its generalization to the spinning case is almost trivial.

Considering the OPE limit of (4.43), we see that the shadow coefficient $\beta_{\Delta,J}$ should satisfy

$$\begin{aligned}
&\frac{1}{\beta_{\Delta,J}}\int_{4>5>3} d^d x_5\langle\mathcal{O}^{\dagger}_{\nu_1\cdots\nu_J}(x_6)\mathcal{O}^{\mu_1\cdots\mu_J}(x_5)\rangle|\langle\widetilde{\mathcal{O}}^{\dagger}_{\mu_1\cdots\mu_J}(x_5)\phi_3\phi_4\rangle|\\
&=|\langle\mathcal{O}^{\dagger}_{\nu_1\cdots\nu_J}(x_6)\phi_3\phi_4\rangle|. \qquad (4.69)
\end{aligned}$$

Let us choose the external points x_3, x_4, x_6 to be

$$x_3 = y|_{\text{saddle}}, \quad x_4 = \frac{x}{-x^2}\bigg|_{\text{saddle}}, \quad x_6 = (-e)^+, \tag{4.70}$$

where x, y are evaluated at the saddle configuration (4.63). With this choice of external points, we find that at large Δ, the integrand in (4.69) has a saddle point at exactly $x_5 = e$, which agrees with the saddle configuration of the original integral of the functional $\Psi^+_{k,\nu}$. After computing the determinant factor coming from the Gaussian integral, we obtain that in the large Δ limit, (4.69) becomes

$$\begin{aligned} &\lim_{\Delta\gg1} \frac{2^d \pi^{\frac{d}{2}} \Delta^{-\frac{d}{2}}}{\beta_{\Delta,J}} \langle O^\dagger_{\nu_1\cdots\nu_J}(x_6) O^{\mu_1\cdots\mu_J}(x_5)\rangle |\langle \widetilde{O}^\dagger_{\mu_1\cdots\mu_J}(x_5)\phi_3\phi_4\rangle|\Big|_{\substack{\text{saddle}\\ x_6=(-e)^+}} \\ &= |\langle O^\dagger_{\nu_1\cdots\nu_J}(x_6)\phi_3\phi_4\rangle|\Big|_{\substack{\text{saddle}\\ x_6=(-e)^+}}. \end{aligned} \tag{4.71}$$

To recap, we see that at large Δ, the shadow transform gets localized to a saddle point. Furthermore, with a good choice of external points $x_6 = (-e)^+$, we can make the saddle point location agree with the saddle configuration for the functional (4.63). We have also checked that (4.71) is consistent with the known scalar result (4.45).

To simplify the formula of the functional in the bulk-point limit given by (4.68), we need the combination $\frac{1}{\beta_{\Delta,J}}|\langle\phi_{1^+}\phi_{2^+}O^{\mu_1\cdots\mu_J}(x_5)\rangle||\langle\widetilde{O}^\dagger_{\mu_1\cdots\mu_J}(x_5)\phi_3\phi_4\rangle|$, which is different from the structure appearing in (4.71). To get the correct structure, let us first note that the two-point structure at the saddle $\langle O^{\dagger\nu_1\cdots\nu_J}((-e)^+)O^{\mu_1\cdots\mu_J}(e)\rangle$ can be viewed as an invertible matrix for the spin-J representation of the Lorentz group.[9] Therefore, we can define an inverse "r-tensor" that satisfies

$$r_{\Delta,J;\rho_1\cdots\rho_J}{}^{\nu_1\cdots\nu_J}\langle O^\dagger_{\nu_1\cdots\nu_J}((-e)^+)O^{\mu_1\cdots\mu_J}(e)\rangle = \delta_{\{\rho_1}{}^{\{\mu_1}\cdots\delta_{\rho_J\}}{}^{\mu_J\}} - \text{traces}. \tag{4.72}$$

Here, $\{\mu_1\cdots\mu_J\}$ means we symmetrize the indices, and "$-$traces" means we subtract terms proportional to $\delta^{\mu_i\mu_j}$ and $\delta_{\rho_i\rho_j}$ to make the result traceless. Concretely, in this case $r_{\Delta,J}$ is given by a reflection in the time direction,

$$r_{\Delta,J;\rho_1\cdots\rho_J}{}^{\nu_1\cdots\nu_J} = (-1)^J 2^{-J+2\Delta}\left(\delta_{\{\rho_1}{}^{\{\nu_1} + 2e_{\{\rho_1}e^{\{\nu_1}\right)\cdots\left(\eta_{\rho_J\}}{}^{\nu_J\}} + 2e_{\rho_J\}}e^{\nu_J\}}\right) - \text{traces}. \tag{4.73}$$

[9]More generally, if O has representation ρ, it should be a map from ρ^* to $\rho^\dagger = (\rho^*)^R$.

By contracting both sides of (4.71) with $|\langle\phi_{1^+}\phi_{2^+}O^{\rho_1\cdots\rho_J}(x_5)\rangle| r_{\Delta,J;\rho_1\cdots\rho_J}{}^{\nu_1\cdots\nu_J}$, we obtain

$$\lim_{\Delta\gg 1}\frac{1}{\beta_{\Delta,J}}\ |\langle\phi_{1^+}\phi_{2^+}O^{\mu_1\cdots\mu_J}(x_5)\rangle||\langle\widetilde{O}^\dagger_{\mu_1\cdots\mu_J}(x_5)\phi_3\phi_4\rangle|\Big|_{\substack{\text{saddle}\\ x_6=(-e)^+}}$$
$$= 2^{-d}\pi^{-\frac{d}{2}}\Delta^{\frac{d}{2}}\ |\langle\phi_{1^+}\phi_{2^+}O^{\mu_1\cdots\mu_J}(x_5)\rangle| r_{\Delta,J;\mu_1\cdots\mu_J}{}^{\nu_1\cdots\nu_J}|\langle O^\dagger_{\nu_1\cdots\nu_J}(x_6)\phi_3\phi_4\rangle|\Big|_{\substack{\text{saddle}\\ x_6=(-e)^+}}. \tag{4.74}$$

The left hand side of the above equation now agrees with the structure in the formula (4.68). We can then plug this equation into (4.68) and find a formula without the shadow coefficient. We write down this final formula in the next subsection.

Finally, let us discuss a natural formula for the $r_{\Delta,\rho}$-tensor that will make it easier for us to generalize to the spinning case. It also makes it clear that the tensor depends on a choice of two-point structure convention. For a general representation ρ, the corresponding r-tensor is defined as

$$r_{\Delta,\rho;a'}{}^{\overline{b}}\langle O^\dagger_{\overline{b}}((-e)^+)O^a(e)\rangle = \delta^a{}_{a'}, \tag{4.75}$$

where the operator O has quantum numbers (Δ,ρ). Here, a, a' are the indices of the representations ρ,ρ^*, and $\overline{b}$ labels the indices of the reflected representation ρ^R and its dual $\rho^\dagger$. On the right-hand side, $\delta^a{}_{a'}$ is the identity matrix of the ρ representation (i.e., $\delta^a{}_{a'}T^{a'} = T^a$ for any tensor T with representation ρ).

Motivated by symmetry, we can write down an ansatz for the $r_{\Delta,\rho}$-tensor using the two-point function of the shadow operator $\widetilde{O}$,

$$r_{\Delta,\rho;a'}{}^{\overline{b}} = C_{\Delta,\rho}\langle\widetilde{O}^\dagger_{a'}(e)\widetilde{O}^{\overline{b}}((-e)^+)\rangle. \tag{4.76}$$

Plugging this ansatz into the definition of $r_{\Delta,\rho}$ and taking the trace, we obtain

$$C_{\Delta,\rho}\langle\widetilde{O}^\dagger_a(e)\widetilde{O}^{\overline{b}}((-e)^+)\rangle\langle O^\dagger_{\overline{b}}((-e)^+)O^a(e)\rangle = \delta^a{}_a = \dim(\rho). \tag{4.77}$$

We see that the unknown coefficient $C_{\Delta,\rho}$ in the ansatz can be expressed in terms of the dimension of the representation and a pairing of the two-point functions.

We define a natural Euclidean two-point pairing [45],[10]

$$\begin{aligned}\frac{\left(\langle\widetilde{O}^\dagger\widetilde{O}\rangle,\langle O^\dagger O\rangle\right)}{\mathrm{vol}(\mathrm{SO}(1,1))} &= \int\frac{d^dx d^dy}{\mathrm{vol}(\mathrm{SO}(d+1,1))}\langle\widetilde{O}^\dagger_a(x)\widetilde{O}^{\overline{b}}(y)\rangle\langle O^\dagger_{\overline{b}}(y)O^a(x)\rangle,\\ &= \frac{1}{2^d\,\mathrm{vol}(\mathrm{SO}(d))\,\mathrm{vol}(\mathrm{SO}(1,1))}\langle\widetilde{O}^\dagger_a(0)\widetilde{O}^{\overline{b}}(\infty)\rangle\langle O^\dagger_{\overline{b}}(\infty)O^a(0)\rangle.\end{aligned} \tag{4.78}$$

[10] Compared to the definition in [45], we have absorbed an infinite vol(SO(1, 1)) factor to make the pairing finite.

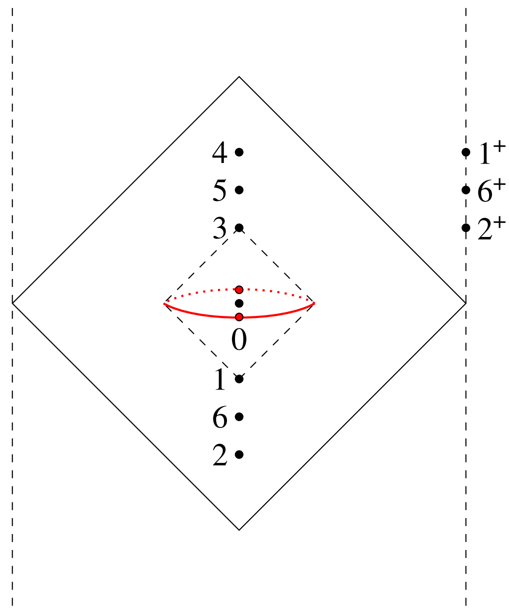


Figure 4.5: The bulk-point limit localizes the spacetime integral to a saddle locus that we call the scattering-crystal. The entire saddle satisfies the causality constraint $4 > 5 > 3 > 0 > 1 > 6 > 2$. The two delta functions restrict x_0 to be on an S^{d-2} shown by the red circle in the figure. The saddle locus further restricts x_0 to an S^{d-3}, which is represented by the two red dots (0-sphere). Points 5 and 6 are localized at $\pm e$. Points $1, 2, 3, 4$ are restricted to be timelike vectors, although in practice they have purely imaginary spatial components (see (4.81)).

Originally, in (4.77) we want to compute the pairing between Lorentzian two-point structures. Since the two points $e, (-e)^+$ are spacelike separated, the structures can be thought of as Euclidean two-point structures with distance $\sqrt{-(2e)^2} = 2$, and we can replace the pairing with a Euclidean two-point pairing (4.78). Using the two-point pairing, one can then show that (4.77) gives

$$C_{\Delta,\rho} = \frac{2^d \dim(\rho)}{\mathrm{vol}(\mathrm{SO}(d)) \left(\langle \widetilde{O}^\dagger \widetilde{O} \rangle, \langle O^\dagger O \rangle \right)}, \tag{4.79}$$

and the r-tensor can be written as

$$r_{\Delta,\rho;a'}{}^{\overline{b}} = \frac{2^d \dim(\rho)}{\mathrm{vol}(\mathrm{SO}(d)) \left(\langle \widetilde{O}^\dagger \widetilde{O} \rangle, \langle O^\dagger O \rangle \right)} \langle \widetilde{O}^\dagger_{a'}(e) \widetilde{O}^{\overline{b}}((-e)^+) \rangle. \tag{4.80}$$

4.3.4 Scalar sum rules

Let us briefly summarize what we have done and write down the final formula for the scalar functional in the bulk-point limit. We started with the commutator of two light-transformed operators and wrote down a functional $\Psi_{k,\nu}$ ((4.33)) whose action on any physical four-point function vanishes. The functional integrates the four-point function against a specific kernel over the external points $x_{1,2,3,4}$ and internal variables x_0, z_0. Using the Lorentzian shadow representation of conformal blocks, we wrote down the action of the functional on blocks, with an additional internal point x_5 from the shadow representation. We then found that in the bulk-point limit ($\nu, m \gg 1$ with $\nu < m$), the integral gets completely localized to a saddle configuration (4.63). Furthermore we removed the shadow coefficient by introducing a final internal point x_6.

In the end, we obtain a formula for the action of the functional on a conformal block in the bulk-point limit. The formula is given by a known coefficient and a product of conformally-invariant structures evaluated at a configuration that fixes all the positions and polarizations, $x_{0,1,2,3,4,5,6}, z_0$. In lightcone coordinates, the configuration is given by (see figure 4.5)

$$\begin{aligned}
&x_3 = -x_1 = y, \quad x_4 = -x_2 = \frac{x}{-x^2}, \quad x_5 = e = (1,1,\vec{0}), \quad x_6 = -e, \\
&x = \left(-i\tfrac{m-\sqrt{m^2-\nu^2}}{\nu}, i\tfrac{m-\sqrt{m^2-\nu^2}}{\nu}, \vec{0}\right), \quad y = \left(i\tfrac{m-\sqrt{m^2-\nu^2}}{\nu}, -i\tfrac{m-\sqrt{m^2-\nu^2}}{\nu}, \vec{0}\right), \\
&z_0 = (1,0,\vec{0}), \quad x_0 = (0,0,\vec{y}_0), \quad |\vec{y}_0|^2 = \frac{(m-\sqrt{m^2-\nu^2})^2}{\nu^2}.
\end{aligned} \tag{4.81}$$

As we will show later, evaluating the conformal structures in this configuration reproduces the heavy action of a flat space sum rule and allows us to study the bulk scattering process. This is a generalization of the saddle point that leads to the *spacelike scattering* phenomenon in [13]. We will call the configuration (4.81) the "scattering-crystal."

By plugging (4.74) in (4.68) to remove the shadow coefficient, we find

$$\begin{aligned}
&\lim_{\substack{\nu,m\gg 1\\ \nu<m}} \frac{\Psi^+_{k,\nu}[G^s_{\Delta,J}]}{-2(2\sin^2(\pi\frac{\Delta-J-2\Delta_\phi}{2}))} \\
&= A_{k,\nu}\frac{2^{\frac{5}{2}+2d}\pi^{d-\frac{1}{2}}e^{i\frac{\pi}{4}}m^{\frac{3d}{2}+2}\nu^{\frac{7}{2}-4d}(m^2-\nu^2)^{\frac{d-2}{2}}(m-\sqrt{m^2-\nu^2})^{d-4}}{\Gamma\left(\frac{d-2}{2}\right)\mathrm{vol}(\mathrm{SO}(d-2))} \\
&\times \langle 0|\phi_4\mathbf{L}[\mathcal{O}](x_0,z_0)\phi_{2^+}|0\rangle^{-1}\langle\widetilde{\mathcal{P}}_{\delta_1}(x_{1^+0})\widetilde{\mathcal{P}}_{\delta_3}(x_{30})\mathcal{P}_\delta(z_0)\rangle \\
&\times |\langle\phi_{1^+}\phi_{2^+}\mathcal{O}^{\mu_1\cdots\mu_J}(x_5)\rangle| r_{\Delta,J;\mu_1\cdots\mu_J}{}^{\nu_1\cdots\nu_J}|\langle\mathcal{O}^\dagger_{\nu_1\cdots\nu_J}(x_6^+)\phi_3\phi_4\rangle|\Big|_{\text{scattering-crystal}}.
\end{aligned} \tag{4.82}$$

A nice feature of the scattering-crystal is that both structures with absolute values satisfy the identity (4.47) in this configuration. Therefore, we can replace those structures with commutators of standard Wightman structures, and remove the

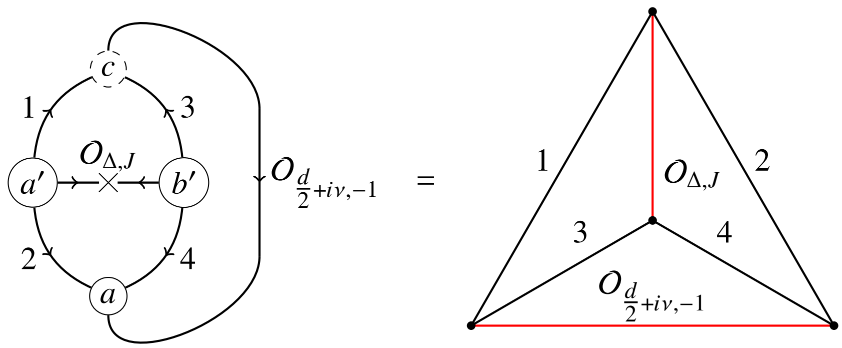


Figure 4.6: Our functional is a special $6j$ symbol (or a tetrahedron) where one of the three-point structures is a celestial structure. The cross in the left figure stands for the contraction of the $\langle \widetilde{\mathcal{O}}^\dagger \widetilde{\mathcal{O}} \rangle$ tensor. The flat space limit is the limit where the two red legs have large quantum numbers. (More precisely, the two legs have large scaling dimensions.)

$-2((2\sin^2(\pi\frac{\Delta-J-2\Delta_\phi}{2}))$ factor on the left-hand side. Moreover, by applying the natural formula for the r-tensor given by (4.80), we arrive at the final formula for the action of the scalar functional in the bulk-point limit:[11]

$$
\begin{aligned}
&\lim_{\substack{\nu,m\gg 1\\ \nu<m}} \Psi^+_{k,\nu}[G^s_{\Delta,J}] \\
&= A_{k,\nu}\frac{2^{\frac{5}{2}+2d}\pi^{d-\frac{1}{2}}e^{i\frac{\pi}{4}}m^{\frac{3d}{2}+2}\nu^{\frac{7}{2}-4d}(m^2-\nu^2)^{\frac{d-2}{2}}(m-\sqrt{m^2-\nu^2})^{d-4}}{\Gamma\left(\frac{d-2}{2}\right)\mathrm{vol}(\mathrm{SO}(d-2))}\frac{2^d\dim(\rho)}{\mathrm{vol}(\mathrm{SO}(d))} \\
&\times \langle 0|\phi_4\mathbf{L}[\mathcal{O}](x_0,z_0)\phi_{2^+}|0\rangle^{-1}\langle \widetilde{\mathcal{P}}_{\delta_1}(x_{1^+0})\widetilde{\mathcal{P}}_{\delta_3}(x_{30})\mathcal{P}_\delta(z_0)\rangle \\
&\times \left.\frac{\langle 0|[\phi_{1^+},\phi_{2^+}]\mathcal{O}^{\mu_1\cdots\mu_J}(x_5)|0\rangle\langle \widetilde{\mathcal{O}}^\dagger_{\mu_1\cdots\mu_J}(x_5)\widetilde{\mathcal{O}}^{\nu_1\cdots\nu_J}(x_6^+)\rangle\langle 0|\mathcal{O}^\dagger_{\nu_1\cdots\nu_J}(x_6^+)[\phi_4,\phi_3]|0\rangle}{\left(\langle \widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle,\langle \mathcal{O}^\dagger\mathcal{O}\rangle\right)}\right|_{\text{scattering-crystal}}.
\end{aligned}
\tag{4.83}
$$

This formula can be viewed as a kind of $6j$ symbol, or equivalently a tetrahedron formed by gluing conformal three-point structures (see figure 4.6). Compared to the usual $6j$ symbol [133], in our case one of the three-point structures becomes a celestial structure, which originally comes from setting one of the legs to have $J = -1$. Furthermore, the bulk-point limit, which reproduces the flat space functional, corresponds to setting two of the legs to have large quantum numbers. In this limit, one of them has large and positive dimension $\Delta \gg 1$ and the other one has dimension

[11] To keep track of $(-1)^J$ factors, we should clarify our convention for $\langle 0|\mathcal{O}^\dagger(x_6)[\phi_4,\phi_3]|0\rangle$. Our convention is that it should be the analytic continuation of (C.2) with $1 \to 3, 2 \to 4, 3 \to 6$.

$\frac{d}{2} + i\nu$ with $\nu \gg 1$. For usual $6j$ symbols in the classical limit, where all legs have large quantum number, they can be computed as the volume of a tetrahedron [134]. It would be interesting to see if there is a similar argument for the bulk-point limit of our functional.

Before using this formula, we still need to address two more issues. First, note that the original functional $\Psi_{k,\nu}$ in (4.33) is given by

$$\Psi_{k,\nu} = \Psi^{+}_{k,\nu} - (1 \leftrightarrow 3). \tag{4.84}$$

Therefore, when we apply $\Psi_{k,\nu}$ to a four-point function of identical operators, which is manifestly symmetric under $(1 \leftrightarrow 3)$, we will get a trivial sum rule since its action on each conformal block is simply zero. The second issue is about the convergence in the Regge limit discussed below (4.32). For external scalars, in order for the light transforms to converge, we need the Regge intercept to satisfy $J_0 < -1$, while for non-perturbative CFTs we only know that $J_0 \leq 1$ [115, 9, 34].

It turns out that both issues can be resolved by introducing a "subtraction factor" to the functional, that is, we consider a more general functional

$$\Psi_{k,\nu,f}[\mathcal{G}] = \Psi^{+}_{k,\nu}[f(u',v')\mathcal{G}] - (1 \leftrightarrow 3), \tag{4.85}$$

where $f(u',v')$ is a meromorphic function of the conformally-invariant cross-ratios u',v' defined in (4.31). Note that the kernel of $\Psi_{k,\nu,f}$ is again manifestly antisymmetric under $1 \leftrightarrow 3$. On the other hand, points 1 and 3 are always spacelike in the range of integration of the functional, which comes from the light transforms. Therefore, causality/crossing guarantees that any physical correlator $\mathcal{G}$ should be symmetric under $1 \leftrightarrow 3$ in the integral, and the action of the general functional $\Psi_{k,\nu,f}$ should vanish on physical four-point functions. Additionally, by choosing subtraction factors that are antisymmetric under $1 \leftrightarrow 3$, which swaps u' and v', we can get nontrivial sum rules for identical external scalars, meaning its action on each conformal block is nonzero. Furthermore, by restricting to subtraction factors with nice behaviors in the Regge limit, we can make sure the integral of the functional is convergent in the Regge limit.

For the scalar case, a nice choice of subtraction factors with the above properties is[12]

$$f_k^{\text{scalar}}(u',v') = \begin{cases} \frac{v'-u'}{(u'v')^{\frac{k+2}{4}}} & k \equiv 2 \bmod 4 \\ \frac{(v'-u')^3-(v'-u')}{4(u'v')^{\frac{k+4}{4}}} & k \equiv 0 \bmod 4 \end{cases}, \qquad k = 2,4,6,\dots. \tag{4.87}$$

At the scattering-crystal locus, these subtraction factors become

$$f_k^{\text{scalar}}(u',v')\big|_{\text{scattering-crystal}} = -\nu^{2k+2}\frac{(2m^2-\nu^2)}{m^2(m^2-\nu^2)}\frac{1}{(m^2(m^2-\nu^2))^{\frac{k}{2}}}. \tag{4.88}$$

Note that this satisfies $f_k^{\text{scalar}}(u',v') = -f_k^{\text{scalar}}(v',u')$. In the Regge limit, the subtraction factor increases the effective Regge spin of the kernel by $k+1$. (One can see this from the large-m scaling of (4.88).) Therefore, the condition from convergence in the Regge limit becomes $J_0 < k$, which is indeed true for any $k \geq 2$.

Using the subtraction factors (4.87), we can then write down nontrivial sum rules for identical scalars,

$$\Psi_{k,\nu,f_k^{\text{scalar}}}[\mathcal{G}_{\phi\phi\phi\phi}] = 0. \tag{4.89}$$

Its action on a single s-channel block is given by

$$\Psi_{k,\nu,f_k^{\text{scalar}}}[G^s_{\Delta,J}] = 2\Psi^+_{k,\nu}[f_k^{\text{scalar}}(u',v')G^s_{\Delta,J}]. \tag{4.90}$$

In the bulk-point limit, since the subtraction factors do not have any ν and m dependence, the formula (4.83) gets modified to

$$\begin{aligned} &\lim_{\substack{\nu,m\gg 1\\ \nu<m}} \Psi_{k,\nu,f_k^{\text{scalar}}}[G^s_{\Delta,J}] \\ &= A_{k,\nu}\frac{2^{\frac{7}{2}+2d}\pi^{d-\frac{1}{2}}e^{i\frac{\pi}{4}}m^{\frac{3d}{2}+2}\nu^{\frac{7}{2}-4d}(m^2-\nu^2)^{\frac{d-2}{2}}(m-\sqrt{m^2-\nu^2})^{d-4}}{\Gamma\left(\frac{d-2}{2}\right)\text{vol}(\text{SO}(d-2))}\frac{2^d\dim(\rho)}{\text{vol}(\text{SO}(d))} \\ &\times \langle 0|\phi_4\mathbf{L}[\mathcal{O}](x_0,z_0)\phi_{2^+}|0\rangle^{-1}\langle\widetilde{\mathcal{P}}_{\delta_1}(x_{1^+0})\widetilde{\mathcal{P}}_{\delta_3}(x_{30})\mathcal{P}_\delta(z_0)\rangle f_k^{\text{scalar}}(u',v') \\ &\times \left.\frac{\langle 0|[\phi_{1^+},\phi_{2^+}]\mathcal{O}^{\mu_1\cdots\mu_J}(x_5)0|\rangle\langle\widetilde{\mathcal{O}}^\dagger_{\mu_1\cdots\mu_J}(x_5)\widetilde{\mathcal{O}}^{\nu_1\cdots\nu_J}(x_6^+)\rangle\langle 0|\mathcal{O}^\dagger_{\nu_1\cdots\nu_J}(x_6^+)[\phi_4,\phi_3]|0\rangle}{\left(\langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle,\langle\mathcal{O}^\dagger\mathcal{O}\rangle\right)}\right|_{\text{scattering-crystal}}. \end{aligned} \tag{4.91}$$

[12]The $k \equiv 2 \bmod 4$ case follows straightforwardly from the discussion in appendix D of [13]. The other case can be motivated by taking the large-ν limit of (D.12) in the same paper, and using the identity

$$\left((v'-u')^3+\frac{3}{d-8}(v'-u')\right)G_{d-1,\frac{2-d}{2}+i\nu} \sim 8G_{d-5,\frac{2-d}{2}+i\nu}, \quad \nu\to\infty. \tag{4.86}$$

We can then evaluate the structures and and the subtraction factor at the scattering-crystal locus to find an explicit expression. Moreover, for symmetric traceless tensors, the pairing $\frac{\mathrm{vol}(\mathrm{SO}(d))}{2^d \dim(\rho)}\left(\langle \widetilde{O}^\dagger \widetilde{O}\rangle, \langle O^\dagger O\rangle\right)$ becomes 4^{-d} [45]. After combining all the factors and writing the result in terms of the "heavy action" defined in (4.28), we obtain

$$\lim_{\substack{\nu,m\gg 1\\ \nu<m}} \Psi_{k,\nu,f_k^{\text{scalar}}}[\Delta, J] = \frac{2m^2-\nu^2}{m^2-\nu^2}\frac{\mathcal{P}_J(1-\frac{2\nu^2}{m^2})}{(m^2(m^2-\nu^2))^{\frac{k}{2}}}. \tag{4.92}$$

As reviewed in section 4.2, this is precisely the heavy state contribution of the flat space sum rule for scalars.

4.3.5 Spinning sum rules

We are now ready to write down the sum rules for spinning operators. The above analysis for the bulk-point limit can be generalized straightforwardly. A spinning functional can be defined by

$$\Psi^{(a),(c)}_{k,\nu,\lambda'}[\mathcal{G}] \equiv \int_{\substack{2>4\\ 0\approx 2,4}} \frac{d^dx_2 d^dx_4 d^dx_0 D^{d-2}z_0}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\left(\langle 0|O_4(x_4^+)\mathbf{L}[O](x_0,z_0)O_2(x_2)|0\rangle^{(a)}\right)^{-1}$$
$$\int D^{d-2}z_1 D^{d-2}z_3 \langle \widetilde{\mathcal{P}}^\dagger_{\delta_1,\lambda_1}(z_1)\widetilde{\mathcal{P}}^\dagger_{\delta_3,\lambda_3}(z_3)\mathcal{P}_{\delta,\lambda'}(z_0)\rangle^{(c)}\langle\Omega|O_4(x_4^+)[\mathbf{L}[O_3](x_0,z_3),\mathbf{L}[O_1](x_0,z_1)]O_2(x_2)|\Omega\rangle. \tag{4.93}$$

Compared to the scalar case (4.33), the main difference is that the operators O and $\mathcal{P}_{\delta,\lambda'}$ can have more complicated representations, and there are multiple allowed dual structures and celestial structures.

Moreover, in the spinning case we do not include any coefficient in front of the integral. In the scalar case, there is a natural choice for this coefficient $A_{k,\nu}$, which is the factor needed to exactly reproduce the action of flat space sum rule (4.92). However, as we will see in the next section, for spinning operators, the CFT sum rules and flat space sum rules are related by a matrix. Since there is no natural choice for the overall factor, we do not include a factor in our definition of the spinning functional.

Let us introduce some notation to describe spinning operators. A spinning operator with $\mathrm{SO}(d-1,1)$ representation ρ can be described by a Young diagram with rows of length $(m_1, m_2, \ldots, m_n)$, where $n = \lfloor \frac{d}{2} \rfloor$. We will also use the notation $\rho = (J, \lambda)$, where $J = m_1$ is the spin, and $\lambda = (m_2, \ldots, m_n)$ specifies an $\mathrm{SO}(d-2)$ representation (which we sometimes call the "transverse" representation). We also define $j = m_2$ to be the transverse spin of the operator.

In (4.93), the external operators have quantum numbers $(\Delta_i, J_i, \lambda_i)$, and the operator O in the first line has quantum numbers $(\frac{d}{2} + i\nu, J_1 + J_3 - 1, \lambda')$. In the second line, the celestial operators, which are primary operators that live on the celestial sphere, also carry $\mathrm{SO}(d-2)$ representations. $\widetilde{\mathcal{P}}^\dagger_{\delta_1,\lambda_1}$ and $\widetilde{\mathcal{P}}^\dagger_{\delta_3,\lambda_3}$ have the same $\mathrm{SO}(d-2)$ representations as the transverse representations of the external operators O_1, O_3, and $\mathcal{P}_{\delta,\lambda'}$ have the same transverse representation as O. Note that the $\mathrm{SO}(d-1,1)$ indices of the ρ_2, ρ_4 representations, and the $\mathrm{SO}(d-2)$ indices of $\lambda_1, \lambda_3, \lambda'$ are implicitly contracted in (4.93).

In addition to the ν variable, the spinning functional also depends on a choice of $\mathrm{SO}(d-2)$ representation λ'. Moreover, the functional has two tensor structure labels $(a), (c)$. The first label (a) corresponds to the tensor structure of the spinning dual structure, defined as

$$\left(\left(\langle 0|O_4(x_4^+)\mathbf{L}[O](x_0,z_0)O_2(x_2)|0\rangle^{(a)}\right)^{-1}, \langle 0|O_4(x_4^+)\mathbf{L}[O](x_0,z_0)O_2(x_2)|0\rangle_{(b)}\right)_L = \delta^a_b. \tag{4.94}$$

The allowed number of labels (a) is the number of continuous-spin structures of $\langle O_2 O_4 O\rangle$, which is given by [48, 15]

$$\left(\lambda' \otimes \left(\mathrm{Res}^{\mathrm{SO}(d-1,1)}_{\mathrm{SO}(d-2)} \rho_2 \otimes \rho_4\right)\right)^{\mathrm{SO}(d-2)}, \tag{4.95}$$

where Res^G_H denotes the restriction of a representation of G to its subgroup H, and $(\dots)^H$ denotes the number of H-singlets. The second tensor structure label (c) corresponds to the structures of a celestial three-point function with $\mathrm{SO}(d-2)$ representation $\lambda_1, \lambda_3, \lambda'$. The number of those structures is [48]

$$\left(\mathrm{Res}^{\mathrm{SO}(d-2)}_{\mathrm{SO}(d-3)} \lambda_1 \otimes \lambda_3 \otimes \lambda'\right)^{\mathrm{SO}(d-3)}. \tag{4.96}$$

Combining (4.95) and (4.96), we see that given the representations of the external operators, the allowed choice of the transverse representation λ' is

$$\left\{\lambda' | \lambda' \in \mathrm{Res}^{\mathrm{SO}(d-1,1)}_{\mathrm{SO}(d-2)} \rho_2 \otimes \rho_4, \left(\mathrm{Res}^{\mathrm{SO}(d-2)}_{\mathrm{SO}(d-3)} \lambda_1 \otimes \lambda_3 \otimes \lambda'\right)^{\mathrm{SO}(d-3)} \neq 0\right\}. \tag{4.97}$$

In section 4.4, we will give the set of allowed λ''s for some concrete examples.

Similar to the scalar case, commutativity of the light-transformed operators implies

$$\Psi^{(a),(c)}_{k,\nu,\lambda'}[\mathcal{G}] = 0, \quad J_1 + J_3 - 1 > J_0. \tag{4.98}$$

Again, this condition can be equivalently written in terms of the $C^{\pm}_{ab}(\Delta,\rho)$ coefficient function,

$$C^{(-1)^{J_1+J_3-1}}_{ab}(\tfrac{d}{2}+i\nu, J_1+J_3-1,\lambda') = 0, \quad J_1+J_3-1 > J_0, \tag{4.99}$$

where a is the same tensor structure label of the functional (4.93), which corresponds to the structures of the $\mathcal{O}_2\times\mathcal{O}_4$ OPE. On the other hand, b is a subset of the structures of the $\mathcal{O}_1\times\mathcal{O}_3$ OPE that have the same counting rule as the celestial structures (4.96) (see [77] for more details).

The spinning heavy action

We now derive a formula for the heavy action of the spinning functional in the bulk-point limit, similar to the scalar case (4.83). For simplicity, in what follows we assume the external operators are symmetric traceless tensors with spin J_e, and they all have scaling dimensions Δ_e. By rewriting the light transforms in (4.93), the spinning functional can be written as

$$\Psi^{(a),(c)}_{k,\nu,\lambda'} = \Psi^{+(a),(c)}_{k,\nu,\lambda'} - (1\leftrightarrow 3), \tag{4.100}$$

and

$$\begin{aligned}
&\Psi^{+(a),(c)}_{k,\nu,\lambda'}[\mathcal{G}] \\
&= 4\int_{4>3>0>1>2} \frac{d^dx_1d^dx_2d^dx_3d^dx_4d^dx_0D^{d-2}z_0}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\delta(x^2_{1^+0})\delta(x^2_{30})\theta(x_{1^+0})\theta(x_{30}) \\
&\times\Big(\langle 0|\mathcal{O}_4(x_4)\mathbf{L}[\mathcal{O}](x_0,z_0)\mathcal{O}_2(x_2^+)|0\rangle^{(a)}\Big)^{-1}\langle\widetilde{\mathcal{P}}^\dagger_{\delta_1,\lambda_1}(x_{1^+0})\widetilde{\mathcal{P}}^\dagger_{\delta_3,\lambda_3}(x_{30})\mathcal{P}_{\delta,\lambda'}(z_0)\rangle^{(c)} \\
&\times\langle\Omega|[\mathcal{O}_4(x_4),\mathcal{O}_3(x_3,x_{30})][\mathcal{O}_1(x_1^+,x_{1^+0}),\mathcal{O}_2(x_2^+)]|\Omega\rangle.
\end{aligned} \tag{4.101}$$

Note that the polarizations of $\mathcal{O}_1,\mathcal{O}_3$ are fixed to be x_{30}, x_{1^+0} (which are null vectors due to the delta functions in the integrand). This is because of the light transforms $\mathbf{L}[\mathcal{O}_3]\mathbf{L}[\mathcal{O}_1]$ in (4.93).

To study the action of $\Psi^{+(a),(c)}_{k,\nu,\lambda'}$ on a conformal block, we again use the Lorentzian shadow representation of the block and consider the bulk-point limit, in which $\Delta\sim m$ and $\nu, m\gg 1, \nu<m$. To perform the saddle point analysis, we separate the integrand into a quickly-varying part and slowly-varying part. Crucially, in the bulk-point limit, the quickly-varying part of the integrand is the same as the scalar case! To see why, recall that the factor in the scalar case is given by (4.55), and it comes from powers of x^2_{ij}, $z_0\cdot x_{10}$, $z_0\cdot x_{30}$, $x_{10}\cdot x_{30}$, and $V_{0,24}$ in the conformally-invariant

structures in the integrand. For spinning structures, the powers of these factors will only differ by integer values that depend on the representations and choice of tensor structures, and hence they do not matter in the $\nu, m \gg 1$ limit. Since the quickly-varying part in the scalar and spinning cases are identical, the saddle point analysis for the spinning case is identical to what we did in section 4.3.3. As a result, the formula in the scalar case immediately generalizes to the spinning case, and by rewriting (4.83) we obtain that the formula in the spinning case is given by

$$
\begin{aligned}
&\lim_{\substack{\nu,m\gg 1\\ \nu<m}} \Psi^{+(a),(c)}_{k,\nu,\lambda'}[G^s_{\Delta,\rho,(a'b')}] \\
&= \frac{2^{\frac{5}{2}+2d}\pi^{d-\frac{1}{2}}e^{i\frac{\pi}{4}}m^{\frac{3d}{2}+2}\nu^{\frac{7}{2}-4d}(m^2-\nu^2)^{\frac{d-2}{2}}(m-\sqrt{m^2-\nu^2})^{d-4}}{\Gamma\left(\frac{d-2}{2}\right)\mathrm{vol}(\mathrm{SO}(d-2))}\frac{2^d\dim(\rho)}{\mathrm{vol}(\mathrm{SO}(d))}\times \\
&\left(\langle 0|\mathcal{O}_4(x_4)\mathbf{L}[\mathcal{O}](x_0,z_0)\mathcal{O}_2(x_2^+)|0\rangle^{(a)}\right)^{-1}\langle\widetilde{\mathcal{P}}^\dagger_{\delta_1,\lambda_1}(x_{1^+0})\widetilde{\mathcal{P}}^\dagger_{\delta_3,\lambda_3}(x_{30})\mathcal{P}_{\delta,\lambda'}(z_0)\rangle^{(c)}\times \\
&\left.\frac{\langle 0|[\mathcal{O}_1(x_{1^+},x_{1^+0}),\mathcal{O}_2(x_{2^+})]\mathcal{O}(x_5)|0\rangle_{(a')}\langle\widetilde{\mathcal{O}}^\dagger(x_5)\widetilde{\mathcal{O}}(x_6^+)\rangle\langle 0|\mathcal{O}^\dagger(x_6^+)[\mathcal{O}_4(x_4),\mathcal{O}_3(x_3,x_{30})]|0\rangle_{(b')}}{\left(\langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle,\langle\mathcal{O}^\dagger\mathcal{O}\rangle\right)}\right|_{\substack{\text{scattering-}\\ \text{crystal}}},
\end{aligned}
\tag{4.102}
$$

where (a'), (b') are the three-point tensor structure labels of the block. The formula should be evaluated at the same "scattering-crystal" as the scalar case, given by (4.81), and we have implicitly contracted all the $\mathrm{SO}(d-1,1)$ indices of ρ_2, ρ_4, ρ and the $\mathrm{SO}(d-2)$ indices of $\lambda_1, \lambda_3, \lambda'$.

Subtraction factors

To finish the discussion of spinning sum rules, let us give the subtraction factors that should be used in the spinning case. We consider a more general functional

$$
\Psi^{(a),(c)}_{k,\nu,\lambda',f}[\mathcal{G}] = \Psi^{+(a),(c)}_{k,\nu,\lambda'}[f(u',v')\mathcal{G}] - (1 \leftrightarrow 3). \tag{4.103}
$$

As discussed in section 4.3.4, to have nontrivial sum rules for identical external operators, the kernel of the $\Psi^{+(a),(c)}_{k,\nu,\lambda'}$ functional times the subtraction factor $f(u',v')$ should be antisymmetric under $(1 \leftrightarrow 3)$. For the spinning functionals, the signature under swapping 1 and 3 depends on the transverse representation λ'. We will focus on the case where the external operators are symmetric traceless tensors, so the allowed λ' satisfying the selection rule (4.97) should be a symmetric traceless tensor itself (as an $\mathrm{SO}(d-2)$ representation). Therefore, the spinning functionals in this case are labeled by an integer j', which is the transverse spin. Due to the celestial structure $\langle\widetilde{\mathcal{P}}_1\widetilde{\mathcal{P}}_3\mathcal{P}_{\delta,j'}\rangle$, the functional $\Psi^{+(a),(c)}_{k,\nu,j'}$ has signature $(-1)^{j'}$ under $(1\leftrightarrow 3)$. Therefore, the nontrivial spinning sum rules are given by

$$\Psi^{+(a),(c)}_{k,\nu,j'}[f^{(-1)^{j'}}(u',v')\mathcal{G}]=0, \tag{4.104}$$

where the subtraction factors should satisfy

$$f^{\pm}(v',u')=\mp f^{\pm}(u',v'). \tag{4.105}$$

We find that for external operators with spin J_e, the subtraction factors with even j' can be chosen to be

$$f^{+}_{k;J_e}(u',v')=\begin{cases}\frac{v'-u'}{(u'v')^{\frac{k-2J_e+2}{4}}} & k\equiv 2J_e+2 \bmod 4\\ \frac{(v'-u')^3-(v'-u')}{4(u'v')^{\frac{k-2J_e+4}{4}}} & k\equiv 2J_e \bmod 4\end{cases},\qquad k=2,4,6,\dots, \tag{4.106}$$

which reduces to the scalar case (4.87) when $J_e=0$. On the other hand, for odd j' we have

$$f^{-}_{k;J_e}(u',v')=\begin{cases}-\frac{1}{(u'v')^{\frac{k-2J_e+1}{4}}} & k\equiv 2J_e+3 \bmod 4\\ -\frac{(u'-v')^2-1}{4(u'v')^{\frac{k-2J_e+3}{4}}} & k\equiv 2J_e+1 \bmod 4\end{cases},\qquad k=3,5,7,\dots. \tag{4.107}$$

When evaluating these subtraction factors at the scattering crystal (4.81), we find

$$\begin{aligned}f^{+}_{k;J_e}(u',v')\Big|_{\text{scattering-crystal}}&=-\nu^{2(k-2J_e+1)}\frac{(2m^2-\nu^2)}{m^2(m^2-\nu^2)}\frac{1}{(m^2(m^2-\nu^2))^{\frac{k-2J_e}{2}}},\quad k=2,4,6,\dots,\\ f^{-}_{k;J_e}(u',v')\Big|_{\text{scattering-crystal}}&=-\nu^{2(k-2J_e)}\frac{\nu^2}{m^2(m^2-\nu^2)}\frac{1}{(m^2(m^2-\nu^2))^{\frac{k-2J_e-1}{2}}},\quad k=3,5,7,\dots.\end{aligned} \tag{4.108}$$

From the flat space point of view, the expressions for the subtraction factors at the saddle are precisely what we need. We expect that functionals with even j' are related to the $s\leftrightarrow t$ symmetric part of the flat space amplitude. Therefore, the corresponding flat space sum rule should have a factor

$$\left(\frac{1}{s}+\frac{1}{s+u}\right)\frac{1}{(s(s+u))^{\frac{k}{2}}}=\frac{2s+u}{s(s+u)}\frac{1}{(s(s+u))^{\frac{k}{2}}}. \tag{4.109}$$

On the other hand, functionals with odd j' should be related to the $s\leftrightarrow t$ antisymmetric part, and the corresponding flat space sum rule should have the factor

$$\left(\frac{1}{s}-\frac{1}{s+u}\right)\frac{1}{(s(s+u))^{\frac{k}{2}}}=\frac{u}{s(s+u)}\frac{1}{(s(s+u))^{\frac{k}{2}}}. \tag{4.110}$$

Under the identification $s = m^2, u = -\nu^2$, we see that these expressions agree with $f^{\pm}_{k;J_e}(u', v')$ at the saddle. We make this connection more precise in section 4.4.

Finally, let us explain the allowed range of k in (4.106) and (4.107). In the Regge limit, the subtraction factor f^+ changes the Regge spin by $k - 2J_e + 1$, and f^- changes the Regge spin by $k - 2J_e$. Therefore, for the integral of the functional to be convergent, we must have $k > J_0$ for even j' and $k - 1 > J_0$ for odd j'. For non-perturbative CFTs, we have $J_0 \leq 1$. So, we should choose $k \geq 2$ for even j' and $k \geq 3$ for odd j'. As we will show in the next section, k also agrees with the Regge spin of the corresponding flat space sum rule.

Normally, introducing subtraction factors comes with a price that the action of the functional becomes non-vanishing on finitely many light double-traces [56]. However, if J_e is large enough, sometimes we can have subtraction factors that make the Regge behavior of the kernel worse. For these unsubtracted sum rules, the action on all double-traces will remain zero. On the flat space side, the EFT series of the corresponding flat space sum rule should automatically truncate, since there is no denominator in the integrand. As an example, for $J_e = 2$ (which corresponds to flat space gravitons), we have two such subtraction factors from $k = 2, k = 3$. Their expressions at the saddle are given by

$$\begin{aligned} f^{+}_{k=2;J_e=2}(u', v')\Big|_{\text{scattering-crystal}} &= -\nu^{-2}(2m^2 - \nu^2), \\ f^{-}_{k=3;J_e=2}(u', v')\Big|_{\text{scattering-crystal}} &= 1. \end{aligned} \tag{4.111}$$

These unsubtracted sum rules for gravitons will be given in section 4.4.

4.4 Flat space interpretation

We now discuss the flat space interpretation of the CFT sum rules derived in the previous section. On the flat space side, we are mainly interested in sum rules of photons and gravitons, which are dual to conserved operators on the CFT side. Our goal is to relate the CFT heavy action formula (4.102) to the heavy state contribution of the flat space sum rule. The heavy action formula consists of a block part (fourth line of (4.102)) from the block insertion, and a kernel part (third line of (4.102)) from the kernel of the functional. We will be able to understand both of them in flat space. Throughout the discussion, we will study the photon case in detail and explain how to match the CFT result and the flat space result. For the graviton case, we simply give the final relation between CFT sum rules and flat space sum rules.

Hereafter, we use D to denote the spacetime dimension of flat space, and $d = D - 1$ is the spacetime dimension of the CFT.

4.4.1 Conservation at large Δ

The flat space limit of a dispersive functional corresponds to an AdS scattering process where the incoming particles are boosted to high energies. Our main interest is in massless particles like photons and gravitons, which are dual to conserved CFT operators. In this highly boosted limit, conservation will substantially simplify the dictionary that we find. We expect that for external massive particles, it is most natural to use the goldstone equivalence theorem to break them into massless excitations, and use our dictionary for each excitation.

We first note that the conservation condition for CFT three-point structures simplifies when one of the operator dimensions Δ becomes large. For a three point function $\langle O_1(x_1, z_1) O_2(x_2, z_2) O_3(x_3, z_3)\rangle$, with O_1 conserved we can write its conservation condition as

$$0 = \partial_{\mu_1} \langle O_1^{\mu_1 \cdots \mu_J}(x_1) O_2(x_2, z_2) O_3(x_3, z_3)\rangle. \tag{4.112}$$

We claim that at large $\Delta = \Delta_3$ this simplifies to

$$v_{1,23}^{\mu_1} \langle O_{1;\mu_1 \cdots \mu_J} O_2 O\rangle = 0, \tag{4.113}$$

where

$$v_{1,23}^{\mu} = \frac{1}{2} \partial_1^{\mu} \log \frac{x_{12}^2}{x_{13}^2} = \frac{x_{12}^{\mu} x_{13}^2 - x_{13}^{\mu} x_{23}^2}{x_{12}^2 x_{13}^2}. \tag{4.114}$$

To see why, we can isolate the Δ dependence in the three-point structure:

$$\langle O_1^{\mu_1 \cdots \mu_J}(x_1) O_2(x_2, z_2) O_3(x_3, z_3)\rangle = \left(\frac{x_{12}^2}{x_{13}^2 x_{23}^2}\right)^{\Delta/2} \times (\cdots), \tag{4.115}$$

where $(\cdots)$ does not depend exponentially on Δ. Applying the conservation condition (4.112), the leading contribution in the large-Δ limit comes from the prefactor in (4.115), which gives (4.113).

Let us check this claim in the explicit example $\langle JJO\rangle$, where J is a spin-1 conserved operator, and O is a symmetric traceless tensor. Before imposing conservation, the general three-point function $\langle JJO\rangle$ in the embedding space (using the conventions in (C.9)) is given by

$$\langle J(X_1,Z_1)J(X_2,Z_2)O(X_3,Z_3)\rangle = \frac{c_1(-2V_3)^{J-2}H_{23}H_{13} + c_2H_{12}(-2V_3)^J + c_3V_2H_{13}(-2V_3)^{J-1} + c_4V_1H_{23}(-2V_3)^{J-1} + c_5V_1V_2(-2V_3)^J}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta+2-J}{2}} X_{13}^{\frac{\Delta_1-\Delta_2+\Delta+J}{2}} X_{23}^{\frac{-\Delta_1+\Delta_2+\Delta+J}{2}}}. \tag{4.116}$$

Let us now impose the conservation condition (4.112). Starting with $\langle JJO\rangle$, we can see that $\partial_{X_1}\cdot\mathcal{D}_{Z_1}\langle JJO\rangle$ transforms like $\langle\phi JO\rangle$. It therefore has two independent structures; one proportional to V_2 and one proportional to H_{23}. These must independently vanish. Furthermore, we get two more equations from imposing that J_2 be conserved. If however we impose $\Delta_1 = d-1 = \Delta_2$ as required for a conserved spin 1 operator, we see that only 3 of these equations are independent. Imposing them and denoting $\Delta_3 = \Delta$ we get

$$\begin{aligned}
(4-2d-J+\Delta)c_1 + 4Jc_2 + 2c_3 + 2(d-1+\Delta)c_4 &= 0,\\
(2J+2\Delta)c_2 + (2(d-1)+J-\Delta)c_3 + 2(d-1+\Delta)c_5 &= 0,\\
(4-2d-J+\Delta)c_1 + 4Jc_2 + 2(d-1+\Delta)c_3 + 2c_4 &= 0,\\
(2J+2\Delta)c_2 - (2(d-1)+J-\Delta)c_4 + 2(d-1+\Delta)c_5 &= 0.
\end{aligned} \tag{4.117}$$

As mentioned above, of these four equations, only 3 are linearly independent. We can then take the large Δ limit to get

$$c_1 - 2c_4 = 0 \quad 2c_2 + c_4 + 2c_5 = 0 \quad c_3 + c_4 = 0. \tag{4.118}$$

One can then verify that this is the same as (4.113), which is equivalent to demanding that in the embedding space

$$\begin{aligned}
V_1\cdot D_{Z_1}\langle O_1(X_1,Z_1)O_2(X_2,Z_2)O(X,Z)\rangle &= 0.\\
V_2\cdot D_{Z_2}\langle O_1(X_1,Z_1)O_2(X_2,Z_2)O(X,Z)\rangle &= 0,
\end{aligned} \tag{4.119}$$

where V_i^A is the usual embedding space structure V_i defined in (C.4) with Z_i^A stripped off, and $D_{Z_i}^A = \left(\frac{d}{2}-1+Z_i\cdot\partial_{Z_i}\right)\partial_{Z_i}^A - \frac{1}{2}Z_i^A\partial_{Z_i}^2$ is the Todorov/Thomas operator [30].

Importantly, the simplified conservation condition (4.113) is algebraic. It allows us to describe the conserved operator using a boundary polarization vector with one fewer degree of freedom, which will be necessary for writing a simple correspondence between boundary and bulk polarizations.

4.4.2 CFT 3-point structures and flat space 3-point amplitudes

Our goal now is to interpret the last line of (4.102) in flat space language. The conformal three-point structures will become three-point amplitudes, and their contractions will become a partial wave. Recall that the partial wave decomposition of a spinning flat space amplitude is given by [110]

$$\mathcal{M}(s,u) = s^{\frac{4-D}{2}} \sum_{\rho} n_{\rho}^{(D)} \sum_{ab} (a_{\rho}(s))_{ab} \pi_{\rho,(ab)}, \tag{4.120}$$

where ρ is an irrep of $\mathrm{SO}(d)$, a, b represent the three-point structures (see below), and $n_{\rho}^{(D)} = \frac{2^{d+1}(2\pi)^{d-1}\dim\rho}{\mathrm{vol}\,S^{d-1}}$ is a normalization factor. This is a generalization of the scalar partial wave decomposition (4.17), where instead of the Gegenbauer polynomial $\mathcal{P}_J$ we now have a more general partial wave $\pi_{\rho,(ab)}$.

The partial wave $\pi_{\rho,(ab)}$ transforms in the representation ρ of the $\mathrm{SO}(d)$ group that stabilizes $P^{\mu} = p_1^{\mu} + p_2^{\mu}$. It can be obtained by gluing two vertices,

$$\pi_{\rho,(ab)} \equiv \left(\overline{v}_b(n', e_3, e_4), v_a(n, e_1, e_2)\right), \tag{4.121}$$

where the pairing $(\ldots,\ldots)$ represents the contraction of $\mathrm{SO}(d)$ indices, and $\overline{f}(x) = f(x^*)^*$ is Schwarz reflection. The vertex $v_a(n, e_1, e_2)$ is a three-point amplitude of two massless particles and a massive particle. The massive particle has momentum P^{μ} and transforms in the representation ρ under the little group $\mathrm{SO}(d)$. We use a to label different independent vertices, and we define n^{μ} and e_i^{μ} as

$$n^{\mu} = \frac{p_1^{\mu} - p_2^{\mu}}{\sqrt{(p_1 - p_2)^2}}, \qquad e_i^{\mu} = \epsilon_i^{\mu} - p_i^{\mu} \frac{\epsilon_i \cdot P}{p_i \cdot P}. \tag{4.122}$$

They satisfy $n \cdot e_i = e_i^2 = 0, n^2 = 1$. The other vertex can be defined in the same way with momenta p_3^{μ}, p_4^{μ}.

Let us demonstrate how to obtain vertices $v_a(n, e_1, e_2)$ from CFT structures by studying the example of photon scattering. As we saw in the previous section, for a CFT three-point function between two conserved spin one currents and an operator with large Δ, conservation places a simple constraint on the allowed three point structures. For example, when $\mathcal{O}$ is a traceless symmetric tensor, we have

$$\begin{aligned}&\langle J_1(X_1,Z_1) J_2(X_2,Z_2) \mathcal{O}(X_5,Z_5)\rangle \\ &= \frac{c_1(-2V_5)^{J-2} H_{25} H_{15} + c_2 H_{12} (-2V_5)^J + c_3 V_2 H_{15} (-2V_5)^{J-1} + c_4 V_1 H_{25} (-2V_5)^{J-1} + c_5 V_1 V_2 (-2V_5)^J}{X_{12}^{\frac{2d-\Delta-J}{2}} X_{15}^{\frac{\Delta+J}{2}} X_{25}^{\frac{\Delta+J}{2}}}.\end{aligned} \tag{4.123}$$

At large Δ, conservation demands that we have no terms proportional to V_1 or V_2. Moreover, at the saddle configuration of (4.102) we find

$$-v_{1,25}|_{\text{saddle}} = v_{2,51}|_{\text{saddle}} = v_{3,46}|_{\text{saddle}} = -v_{4,63}|_{\text{saddle}} = e, \tag{4.124}$$

where e is the unit vector in the time direction. Therefore, if we choose the polarization vectors z_1, z_2 for the conserved currents to have no time component, the conserved structures of $\langle JJO\rangle$ at large Δ will satisfy

$$\begin{aligned}
\langle J_1 J_2 O\rangle^{(1)}_{\text{saddle},z_i=(0,z_i^x,\vec{z}_{i\perp})} &= \left.\frac{(-2V_5)^{J-2}H_{15}H_{25}}{X_{12}^{\frac{2d-\Delta-J}{2}}X_{15}^{\frac{\Delta+J}{2}}X_{25}^{\frac{\Delta+J}{2}}}\right|_{\text{saddle},z_i=(0,z_i^x,\vec{z}_{i\perp})}, \\
\langle J_1 J_2 O\rangle^{(2)}_{\text{saddle},z_i=(0,z_i^x,\vec{z}_{i\perp})} &= \left.\frac{(-2V_5)^{J}H_{12}}{X_{12}^{\frac{2d-\Delta-J}{2}}X_{15}^{\frac{\Delta+J}{2}}X_{25}^{\frac{\Delta+J}{2}}}\right|_{\text{saddle},z_i=(0,z_i^x,\vec{z}_{i\perp})}.
\end{aligned} \tag{4.125}$$

We would like to interpret these expressions in such a way that they give rise to flat-space three-point vertices. We can describe the possible vertices using the formalism of [135, 110]. The two massless spin-1 particles have momenta p_1 and p_2, as well as polarizations e_1 and e_2. We additionally have the momentum of the third particle $p_3 := P = p_1 + p_2$ and another Lorentz invariant quantity $n \propto p_1 - p_2$. We have the freedom to normalize n, and to shift the polarizations e_i by p_i. As the third particle is massive, we can go to its rest frame and parameterize its momentum as $P = (m, 0, \cdots, 0)$. We can then use the symmetry of our saddle point to find the flat space kinematics.

Since the vertices are tensors with $\mathrm{SO}(d)$ indices, we can introduce index free notation and contract the indices with polarization vectors $w_1, w_2, \ldots \in \mathbb{C}^d$ for each row of the Young diagram. For example, if ρ is a symmetric traceless tensor, we have a single polarization vector w_1, and from these quantities we can build two three-point amplitudes:

$$\begin{aligned}
v^{(1)}_{\gamma\gamma J}(n, e_1, e_2) &= (n \cdot w_1)^{J-2}(e_1 \cdot w_1)(e_2 \cdot w_1), \\
v^{(2)}_{\gamma\gamma J}(n, e_1, e_2) &= (n \cdot w_1)^{J}(e_1 \cdot e_2).
\end{aligned} \tag{4.126}$$

These two amplitudes should correspond to (4.125) through the appropriate dictionary.

Our saddle point (4.81) defines an $\mathrm{SO}(d-1) \subset \mathrm{SO}(d,2)$ subgroup that fixes the locations of the operators J_1, J_2, O. The vector $v^{\mu}_{5,12}$ points along the direction preserved by this $\mathrm{SO}(d-1)$ at the location of $\mathcal{O}$. In the bulk, there is a corresponding $\mathrm{SO}(d-1)$ that preserves n. Thus it is natural to impose

$$(z_5 \cdot v_{5,12})^J \leftrightarrow (w_1 \cdot n)^J. \tag{4.127}$$

The left-hand side is the projection of the polarization of $\mathcal{O}$ along the $\mathrm{SO}(d-1)$-invariant direction, while the right-hand side is the projection of the polarization of the massive particle along the $\mathrm{SO}(d-1)$-invariant direction.

Therefore we should impose $p_1^\mu - p_2^\mu \propto (0, v_{5,12}^\mu)$. Evaluating this on the saddle (4.81) we find $n^\mu = (0, \frac{\sqrt{m^2-\nu^2}}{m}, \frac{\nu}{m}, 0, \cdots, 0)$, while we can choose the momenta of the scattering process to be:

$$\begin{aligned} p_1 =& \frac{1}{2}(m, \sqrt{m^2-\nu^2}, \nu, 0, \ldots, 0), \qquad p_2 = \frac{1}{2}(m, -\sqrt{m^2-\nu^2}, -\nu, 0, \ldots, 0), \\ p_3 =& \frac{1}{2}(m, \sqrt{m^2-\nu^2}, -\nu, 0, \ldots, 0), \quad p_4 = \frac{1}{2}(m, -\sqrt{m^2-\nu^2}, \nu, 0, \ldots, 0). \end{aligned} \tag{4.128}$$

Here, p_1, p_2 are incoming and p_3, p_4 are outgoing.

It now remains to parameterize the bulk polarization vectors e_1, e_2, w_1 in terms of CFT polarization vectors z_1, z_2, z_5 to get (4.125) to agree with (4.126). Our CFT polarizations can be expressed as

$$z_1 = (0, z_1^x, \vec{z}_{1\perp}), \quad z_2 = (0, z_2^x, \vec{z}_{2\perp}), \quad z_5 = (z_5^t, z_5^x, \vec{z}_{5\perp}), \tag{4.129}$$

where we have explicitly set the time component of z_1 and z_2 to be zero, in accordance with (4.124). Meanwhile, we can express the bulk polarizations as

$$e_1^\mu = e_1^n n_\perp^\mu + (0,0,0,\vec{e}_{1\perp}), \quad e_2^\mu = e_2^n n_\perp^\mu + (0,0,0,\vec{e}_{2\perp}), \quad w_1^\mu = (0, w_1^t, w_1^x, \vec{w}_{1\perp}), \tag{4.130}$$

where $n_\perp^\mu$ is a vector perpendicular to n^μ, given by

$$n_\perp^\mu = (0, -\tfrac{\nu}{m}, \tfrac{\sqrt{m^2-\nu^2}}{m}, 0, \ldots, 0). \tag{4.131}$$

As we argued above, the $\mathrm{SO}(d-1)$ singlets should match between CFT and flat space. In particular, this imposes

$$\begin{aligned} &\left.\frac{H_{15}}{X_{15}}\right|_{\text{saddle}} \leftrightarrow e_1 \cdot w_1, \qquad \left.\frac{H_{25}}{X_{25}}\right|_{\text{saddle}} \leftrightarrow e_2 \cdot w_1, \\ &\left.\frac{H_{12}}{X_{12}}\right|_{\text{saddle}} \leftrightarrow e_1 \cdot e_2, \qquad V_{5,12}\big|_{\text{saddle}} \leftrightarrow -in \cdot w_1, \end{aligned} \tag{4.132}$$

which we can achieve by setting

$$z_1^x = e_1^n, \quad \vec{z}_{1\perp} = \vec{e}_{1\perp}, \quad z_2^x = -e_2^n, \quad \vec{z}_{2\perp} = \vec{e}_{2\perp}, \quad z_5^t = -iw_1^t, \quad z_5^x = w_1^x, \quad \vec{z}_5 = \vec{w}_{1\perp}. \tag{4.133}$$

Let us emphasize that even though we use a t superscript for the w_1^t component, it is actually a spatial direction in the bulk, while z_5^t is the time direction in the CFT. This is why we need an i factor in their relation. Note also that the relation maps the condition $z_i^2 = 0$ to $e_i^2 = w_1^2 = 0$. If the Young diagram of the representation ρ has more than one row, then we need to introduce more polarization vectors for both the flat space vertex and the CFT three-point structure. The map between these polarizations is the same as the relation between z_5 and w_1 in (4.133).

The polarization map (4.133) turns each CFT three-point structure building block into an amplitude building block, and thus the CFT three-point structures naturally become flat space vertices.

Note that in the CFT sum rule (4.102), the three-point structures appear as commutators of Wightman functions. So we will use the commutator to define vertices, and divide by the $\sin(\ldots)$ factor from the commutator by hand. For the cases we consider in this paper (photons and gravitons), the Young diagram of the exchanged representation ρ has at most three rows. We will often write $\rho = (J, j, \widetilde{j})$ to denote the length of each row. For external operators $\mathcal{O}_1, \mathcal{O}_2$ with quantum numbers (Δ_i, J_i) and exchanged representation ρ, we define the vertices as[13]

$$\begin{aligned} &v_a(n, e_1, e_2, w_i) \\ &\equiv (-x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}} \left(\frac{x_{15}^2}{x_{25}^2}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left.\frac{\langle 0|[\mathcal{O}_1(x_{1^+}, z_1), \mathcal{O}_2(x_{2^+}, z_2)]\mathcal{O}(x_5, z_5, \vec{w}_5)]|0\rangle_{(a)}}{2i(\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_1-J_1-\Delta_2-J_2}{2}))}\right|_{\text{saddle},(4.133)}, \end{aligned} \tag{4.134}$$

where w_i on the left-hand side are the flat space polarizations, and $z_5, \vec{w}_5$ on the right-hand side are the CFT polarizations. The $2i(\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_1-J_1-\Delta_2-J_2}{2}))$ factor comes from taking the commutator, and $\widetilde{\tau}_\rho = \Delta - J + j + \widetilde{j}$ for $\rho = (J, j, \widetilde{j})$.[14] We introduce the factor $(-x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}} \left(\frac{x_{15}^2}{x_{25}^2}\right)^{\frac{\Delta_1-\Delta_2}{2}}$ to remove factors of ν.

[13]Reference [136] presents a similar relation between flat space 3-point amplitudes and CFT 3-point structures, but with a different configuration of CFT positions and polarizations X_i, Z_i. We expect that our relation is equivalent up to a choice of conformal frame.

[14]We find this $\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_1-J_1-\Delta_2-J_2}{2})$ factor by studying the commutator of (C.9). It would be interesting to determine what $\widetilde{\tau}_\rho$ should be for general ρ.

As an example, we can apply the polarization map (4.133) to the $\langle JJO\rangle$ CFT three-point structures (4.125). We get

$$(-x_{12}^2)^{d-1}\left\{\frac{\langle 0|[J(x_{1^+},z_1),J(x_{2^+},z_2)]O(x_5,z_5)|0\rangle_{(1,2)}}{2i\left(\sin(\pi\frac{\Delta-J-2d}{2})\right)}\right\}\Bigg|_{\text{saddle},(4.133)}$$
$$= -(-i)^J 2^J\left\{2^{-2}(n\cdot w_1)^{J-2}(e_1\cdot w_1)(e_2\cdot w_1),(n\cdot w_1)^J(e_1\cdot e_2)\right\}. \qquad (4.135)$$

As expected, the result correctly reproduces the three-point amplitudes (4.126).

The other vertex $\overline{v}(n',e_3,e_4)$ can be defined using the $\langle O_3O_4O_6\rangle$ CFT three-point structure in a similar way. The polarization map is given by

$$z_3^x = e_3^{n'},\quad \vec{z}_{3\perp} = \vec{e}_{3\perp},\quad z_4^x = -e_4^{n'},\quad z_{4\perp} = \vec{e}_{4\perp},\quad z_6^t = -iw_1^t,\quad z_6^x = w_1^x,\quad \vec{z}_6 = \vec{w}_{1\perp}. \qquad (4.136)$$

We again set the time component of z_3, z_4 to be zero, and

$$e_3^\mu = e_3^{n'}n_\perp'^\mu + (0,0,0,\vec{e}_{3\perp}),\quad e_4^\mu = e_4^{n'}n_\perp'^\mu + (0,0,0,\vec{e}_{4\perp}),\quad w_1^\mu = (0,w_1^t,w_1^x,\vec{w}_{1\perp}), \qquad (4.137)$$

where

$$n_\perp'^\mu = (0,\tfrac{\nu}{m},\tfrac{\sqrt{m^2-\nu^2}}{m},0,\dots,0), \qquad (4.138)$$

which is perpendicular to $n'^\mu = (0,\frac{\sqrt{m^2-\nu^2}}{m},-\frac{\nu}{m},0,\dots,0) \propto p_3^\mu - p_4^\mu$.

The definition of the $\overline{v}(n',e_3,e_4)$ is given by

$$\overline{v}_b(-n',e_3,e_4,w_i)$$
$$\equiv (-x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}\left(\frac{x_{36}^2}{x_{46}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}}\mathcal{I}_e^\rho\,\frac{\langle 0|O^\dagger(x_{6^+},z_6,\vec{w}_6)[O_4(x_4,z_4),O_3(x_3,z_3)]|0\rangle_{(b)}}{2i(\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_3-J_3-\Delta_4-J_4}{2}))}\Bigg|_{\text{saddle},(4.136)}. \qquad (4.139)$$

On the right hand side, $\mathcal{I}_e^\rho$ is a tensor that reflects all the indices in the time direction. More precisely, for any tensor T with representation ρ, in the index-free notation we have

$$(\mathcal{I}_e^\rho T)(z,\vec{w}) = T(I_e\cdot z, I_e\cdot\vec{w}), \qquad (4.140)$$

where $I_{e\ \nu}^{\mu} = \delta^\mu{}_\nu + 2e^\mu e_\nu$ is the usual reflection in time direction.

Note that (4.139) gives a definition for $\overline{v}(-n', e_3, e_4)$ instead of $\overline{v}(n', e_3, e_4)$, which appears in the actual partial wave. This is to ensure that the two definitions (4.134) and (4.139) are consistent. In particular, one can show that the partial waves are always positive in the forward limit. Furthermore, the definitions for $v(n, e_1, e_2)$ and $\overline{v}(-n', e_3, e_4)$ are related by CRT symmetry on the CFT side. We give more details in appendix C.3, in which we also write down the relation between the partial waves coming from CFT and the Young tableaux basis of [135, 110].

The appearance of $\mathcal{I}_e^\rho$ in (4.139) is due to the two-point structure $\langle \widetilde{O}^\dagger(e)\widetilde{O}((-e)^+)\rangle$ in the block part of the CFT sum rule (4.102). The effect of this two-point structure is a reflection of all the indices of O in the time direction with some overall factor. For a representation $\rho = (J, j, \widetilde{j})$, we find

$$\frac{2^d \dim(\rho)}{\mathrm{vol}(\mathrm{SO}(d))} \frac{\langle \widetilde{O}_a^\dagger(x_5)\widetilde{O}^{\overline{b}}(x_6^+)\rangle}{\left(\langle \widetilde{O}^\dagger\widetilde{O}\rangle, \langle O^\dagger O\rangle\right)}\Bigg|_{\text{saddle}} = 2^{-J+2\Delta}(-1)^{J-j+\widetilde{j}} R_\rho \left(\mathcal{I}_e^\rho\right)_a{}^{\overline{b}}, \qquad (4.141)$$

where the lower index a and upper index $\overline{b}$ are indices in the dual ρ^* and reflected ρ^R representations respectively. As one can see from the left-hand side, the coefficient R_ρ depends on our convention of the two-point structure. For the two-point convention (C.5), we have

$$R_{\rho=(J,j,\widetilde{j})} = \frac{(j-\widetilde{j}+1)(J-j+1)(J-\widetilde{j}+2)}{(j+1)(J+1)(J+2)}. \qquad (4.142)$$

This result can be obtained by using weight-shifting operators [39] to derive a recursion relation of R_ρ. We give an example in appendix C.3.

Using the momentum configuration and maps of polarizations discussed above, we can now write down the precise flat space interpretation of the block part of the CFT sum rule. Let us define

$$\mathcal{B}_{\Delta,\rho,(a'b')} = \frac{2^d \dim(\rho)}{\mathrm{vol}(\mathrm{SO}(d))}\times$$
$$\frac{\langle 0|[O_1(x_{1^+}, z_1), O_2(x_{2^+}, z_2)]O(x_5)|0\rangle_{(a')}\langle \widetilde{O}^\dagger(x_5)\widetilde{O}(x_6^+)\rangle\langle 0|O^\dagger(x_6^+)[O_4(x_4, z_4), O_3(x_3, z_3)]|0\rangle_{(b')}}{\left(\langle \widetilde{O}^\dagger\widetilde{O}\rangle, \langle O^\dagger O\rangle\right)}\Bigg|_{\substack{\text{saddle}\\ (4.133),(4.}}$$

(4.143)

From the definitions (4.134), (4.139), and (4.141), we find

$$\frac{\mathcal{B}_{\Delta,\rho,(a'b')}}{-2(2\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_1-J_1-\Delta_2-J_2}{2})\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_3-J_3-\Delta_4-J_4}{2}))}$$
$$= 2^{-J+2\Delta-\Delta_1-\Delta_2-\Delta_3-\Delta_4} m^{-\Delta_1-\Delta_2-\Delta_3-\Delta_4} \nu^{2(\Delta_1+\Delta_3)} \left(m-\sqrt{m^2-\nu^2}\right)^{-\Delta_1+\Delta_2-\Delta_3+\Delta_4} R_\rho \pi_{\rho,(a'b')}, \tag{4.144}$$

where $\pi_{\rho,(a'b')}$ is the flat space partial wave (4.121) in the CFT three-point structure basis, and R_ρ is given by (4.142). Note that when turning the vertex $\overline{v}(-n', e_3, e_4)$ in (4.139) into $\overline{v}(n', e_3, e_4)$ to get the partial wave, we get a $(-1)^{J-j+\widetilde{j}}$ factor, which exactly cancels with the same factor in (4.141).

Recall that in the scalar case, where the exchanged operators are symmetric traceless tensors, we define a heavy action $\Psi[\Delta, J]$ in (4.28) by rescaling the action of the functional on the block by a positive factor. Relation (4.144) in fact suggests a way to generalize this definition to more general representation ρ. Since the other parts of the sum rule should not know about the exchanged quantum numbers of the inserted block, the heavy action should remove the m- and ρ-dependent factors that we do not expect to appear in the flat space answer, such as the R_ρ coefficient. For external operators with quantum numbers (Δ_i, J_i) and a general exchanged representation ρ, we define the large $\Delta \sim m$ heavy action $\Psi[\Delta, \rho]$ to be[15]

$$\begin{aligned}\Psi[\Delta,\rho]_{(a'b')} \equiv & \frac{2^{J-2\Delta-\Delta_1-\Delta_2-\Delta_3-\Delta_4} m^{\Delta_1+\Delta_2+\Delta_3+\Delta_4-2-\frac{d}{2}}}{R_\rho} \\ & \times \frac{\Psi[G^s_{\Delta,\rho,(a'b')}]}{2\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_1-J_1-\Delta_2-J_2}{2})\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_3-J_3-\Delta_4-J_4}{2})}.\end{aligned} \tag{4.145}$$

4.4.3 Kernels and the shock frame

We now consider the kernel part of our spinning sum rules. It is given by evaluating a celestial structure $\langle\widetilde{\mathcal{P}}_1\widetilde{\mathcal{P}}_3\mathcal{P}_{\delta,j'}\rangle$ and a dual structure $\langle 0|O_4\mathbf{L}[O]O_2|0\rangle^{-1}$ at the scattering-crystal. In the heavy action formula, we first evaluate these structures at the scattering crystal (4.81), and then we can use the maps of CFT and flat space polarizations (4.133), (4.136) to get the corresponding flat space answer. We claim that the resulting expression can be seen to correspond to a shockwave amplitude in flat space. The shockwave amplitude is defined as the amplitude of an elastic

[15] In the scalar case (4.28), the factor $q_{\Delta,J}$ in the heavy action definition comes from the OPE coefficient of Mean Field Theory. So, to find the analogous factor for more general external operators, one can consider the corresponding OPE coefficients in the Mean Field Theory. However, we do not do this in this paper, and the definition (4.145) is good enough for our purpose.

scattering of a probe particle X and two shockwave gravitons g^* [34],

$$g^*(p_1)X(p_2) \to g^*(p_3)X(p_4). \tag{4.146}$$

In our case, we will take the probe particle X to be a graviton. The two shockwave gravitons are dual to the two light transformed operators $\mathbf{L}[O_1], \mathbf{L}[O_3]$ in the CFT functional. A convenient choice of momenta for shockwave amplitudes is given by (in all incoming convention)

$$\begin{aligned} p_1^\mu &= p_{\bar{s}}\bar{s}^\mu + q_1^\mu, \quad & p_2^\mu &= p_s s^\mu - q_3^\mu \\ p_3^\mu &= -p_{\bar{s}}\bar{s}^\mu + q_3^\mu, \quad & p_4^\mu &= -p_s s^\mu - q_1^\mu, \end{aligned} \tag{4.147}$$

where s^μ and $\bar{s}^\mu$ are two null directions, and the polarization of the shock gravitons should be in the $\bar{s}^\mu$ direction. The transverse momenta q_1^μ, q_3^μ are perpendicular to the null directions, and in order to make the external momenta on-shell, we must have $q_1^2 = q_3^2 = 0$. We will call the momentum configuration (4.147) the shock frame. More explicitly, let $D = d + 1$ be the spacetime dimension of the bulk, in the shock frame we have two null vectors $s^\mu, \bar{s}^\mu$ and $D - 2$ transverse unit vectors $v_{1;\perp}^\mu, v_{2;\perp}^\mu, \ldots, v_{D-2;\perp}^\mu$ satisfying

$$s^2 = \bar{s}^2 = s \cdot v_{i;\perp} = \bar{s} \cdot v_{i;\perp} = 0, \quad s \cdot \bar{s} = -2, \quad v_{i;\perp} \cdot v_{j;\perp} = \delta_{ij}. \tag{4.148}$$

In the previous subsection, we have shown that in order to match the blocks to flat space partial waves, we should choose the flat space momentum to be (4.128). The shock frame momenta (4.147) should also agree with these center of mass frame momenta (4.128). By relating the two frames, we obtain that in terms of the bulk Minkowski coordinates used in (4.128), the vectors in the shock frame can be written as

$$\begin{aligned} \bar{s}^\mu &= \left(\tfrac{\sqrt{m^2-\nu^2}}{m}, 1, 0, \tfrac{i\nu}{m}, 0, \ldots, 0\right) \\ s^\mu &= \left(\tfrac{\sqrt{m^2-\nu^2}}{m}, -1, 0, \tfrac{i\nu}{m}, 0, \ldots, 0\right) \\ v_{1;\perp}^\mu &= \left(\tfrac{i\nu}{m}, 0, 0, \tfrac{\sqrt{m^2-\nu^2}}{m}, 0, \ldots\right) \\ v_{2;\perp}^\mu &= (0, 0, 1, 0, 0, \ldots, 0) \\ v_{3;\perp}^\mu &= (0, 0, 0, 0, 1, 0, \ldots, 0) \\ &\vdots \\ v_{D-2;\perp}^\mu &= (0, 0, 0, 0, 0, 0, \ldots, 1), \end{aligned} \tag{4.149}$$

and

$$p_{\bar{s}} = p_s = \frac{\sqrt{m^2 - \nu^2}}{2}, \quad q_1^\mu = -\frac{i\nu}{2} v_{1;\perp}^\mu + \frac{\nu}{2} v_{2;\perp}^\mu, \quad q_3^\mu = \frac{i\nu}{2} v_{1;\perp}^\mu + \frac{\nu}{2} v_{2;\perp}^\mu. \tag{4.150}$$

One motivation for relating the CFT kernel to the shock frame is that we set the CFT polarization vectors z_1, z_3 to be x_{1^+0}, x_{30} in the CFT sum rule (4.102). Using the map of CFT and flat space polarizations (4.133),(4.136) derived in the previous subsection, we find that in flat space the two polarizations become (up to $e_i \sim e_i + \alpha p_i$ gauge redundancy)

$$x_{1^+0}|_{(4.133)} \sim i\frac{m(m - \sqrt{m^2 - \nu^2})}{\nu^2}\bar{s}^\mu, \qquad x_{30}|_{(4.136)} \sim i\frac{m(m - \sqrt{m^2 - \nu^2})}{\nu^2}\bar{s}^\mu, \tag{4.151}$$

where $\bar{s}^\mu$ is given by (4.149). We see that the polarizations of particles 1 and 3 should be set to the same null direction $\bar{s}^\mu$, which agrees with the fact that the two shock gravitons in the shockwave amplitude should have the same polarizations along the direction of the shock.

We can now discuss the CFT kernel in the shock frame. Explicitly, evaluating the celestial three-point structure at the saddle configuration gives

$$\begin{aligned} &\langle \widetilde{\mathcal{P}}^\dagger_{\delta_1}(x_{1^+0}) \widetilde{\mathcal{P}}^\dagger_{\delta_3}(x_{30}) \mathcal{P}_{\delta,j'}(z_0, w_0)\rangle\Big|_{\text{saddle}} \\ &= (-1)^{j'}(-i)^{-\delta} 2^{j'+\delta-\widetilde{\delta}_1-\widetilde{\delta}_3} \nu^{\widetilde{\delta}_1+\widetilde{\delta}_3} (m - \sqrt{m^2 - \nu^2})^{-\widetilde{\delta}_1-\widetilde{\delta}_3} (\vec{n}_0 \cdot \vec{w}_{0\perp})^{j'}, \end{aligned} \tag{4.152}$$

where we have introduced a polarization $w_0 = (0, 0, \vec{w}_{0\perp})$ for the transverse indices of $\mathcal{P}_{\delta,j'}$, and $\vec{n}_0 = \vec{y}_0/|\vec{y}_0|$ is the unit vector in the transverse $\vec{y}_0$ direction in the scattering crystal (4.81).

For each transverse spin j', the celestial three-point function has a unique structure which gives (4.152) at the saddle. On the other hand, the dual structure $\langle 0|\mathcal{O}_4 \mathbf{L}[\mathcal{O}]\mathcal{O}_2|0\rangle^{-1}$ can have multiple tensor structures. Moreover, they are subject to constraints from imposing conservation on $\mathcal{O}_2$ and $\mathcal{O}_4$. Interestingly, the space of allowed structures becomes easier to study when we just focus on the saddle configuration. This fact can be seen from the identity

$$- V_{2,40} V_{0,24}\big|_{\text{saddle}, z_i = (0, z_i^x, \vec{z}_{i\perp})} = H_{20}\big|_{\text{saddle}, z_i = (0, z_i^x, \vec{z}_{i\perp})}, \tag{4.153}$$

and a similar identity for $V_{4,02}$ and H_{40}. Note that we also set the time component of the polarizations to zero, which is due to the large-Δ conservation analysis in section

4.4.2. This identity allows us to just remove all $V_{2,40}, V_{4,02}$ structures when writing down the basis for the spinning dual structures. Additionally, it can be shown that the basis constructed without $V_{2,40}, V_{4,02}$ gives the correct number of independent tensor structures of continuous-spin three-point function of two conserved operators, which should be given by [48, 15]

$$\left(\lambda' \otimes \mathrm{Res}^{\mathrm{SO}(d-1)}_{\mathrm{SO}(d-2)} \rho_2 \otimes \rho_4\right)^{\mathrm{SO}(d-2)}. \tag{4.154}$$

We give the argument for this statement in appendix C.4, and also explain how to compute these spinning dual structures.

As an example, let us consider $j' = 0$. Focusing just on the spinning part of $\mathcal{O}_2, \mathcal{O}_4$, we see that the allowed structures should be

$$H_{20}^{J_2-n} H_{40}^{J_4-n} H_{24}^{n}, \tag{4.155}$$

where $n = 0, 1, \ldots, \min(J_2, J_4)$. The number of structures is $\min(J_2, J_4) + 1$, which agrees with (4.154).

We now study the four-photon example in detail, where all external operators should be conserved spin-1 currents. The selection rule (4.97) implies that the allowed values of transverse spin should be $j' = 0, 1, 2$. Using the algorithm explained in appendix C.4, we obtain that the dual structures for $j' = 0$ in the large ν limit are given by (for $4 > 0 > 2^-$)

$$\begin{aligned}
&\left\{\left(\langle 0|J(X_4,Z_4)\mathbf{L}[\mathcal{O}](X_0,Z_0)J(X_2,Z_2)|0\rangle^{(1)}\right)^{-1}, \left(\langle 0|J(X_4,Z_4)\mathbf{L}[\mathcal{O}](X_0,Z_0)J(X_2,Z_2)|0\rangle^{(2)}\right)^{-1}\right\} \\
&= 2^{\frac{3d-5}{2}-i\nu} e^{-\frac{i\pi}{4}} \sqrt{\frac{\nu}{\pi}}\, \mathrm{vol}\,\mathrm{SO}(d-2) \frac{1}{X_{24}^{\frac{4-\Delta_F-J_F}{2}} X_{02}^{\frac{\Delta_F+J_F}{2}} (-X_{04})^{\frac{\Delta_F+J_F}{2}}} \\
&\times \left\{ \frac{16(d-1)}{d-2}(-2V_{0,42})^{J_F-2} H_{20}H_{40} + \frac{4}{d-2}(-2V_{0,42})^{J_F} H_{24}, \right. \\
&\qquad \left. \frac{4}{d-2}(-2V_{0,42})^{J_F-2} H_{20}H_{40} + \frac{1}{d-2}(-2V_{0,42})^{J_F} H_{24} \right\} + \ldots,
\end{aligned} \tag{4.156}$$

where $\Delta_F = d, J_F = \frac{2-d}{2} + i\nu$, and $\ldots$ are subleading terms at large ν. The first structure is dual to the light-transformed structure $\langle 0|J\mathbf{L}[\mathcal{O}]J|0\rangle$ with $H_{20}H_{40}$ (see (C.67)), and the second structure is dual to the one with H_{24}.

Then, the next step is to evaluate (4.156) at the saddle (after applying a $\mathcal{T}_2$ to (4.156) to make the causality configuration agree with the saddle). This will give us an expression that depends on the external polarizations z_2, z_4 (note that z_0 is fixed in (4.81)). For example, for the H_{24} structure, we have

$$\frac{H_{24}}{X_{24}}\bigg|_{\text{saddle},z_i=(0,z_i^x,\vec{z}_{i\perp})} = -z_2^x z_4^x + \vec{z}_{2\perp}\cdot\vec{z}_{4\perp}. \tag{4.157}$$

Using (4.133) and (4.136) to map this to flat space, we find

$$-z_2^x z_4^x + \vec{z}_{2\perp}\cdot\vec{z}_{4\perp} \to -e_2^n e_4^{n'} + \vec{e}_{2\perp}\cdot\vec{e}_{4\perp} = e_{2\mu_2}e_{4\mu_4}\left(-n_\perp^{\mu_2}n_\perp'^{\mu_4} + \delta_\perp^{\mu_2\mu_4}\right). \tag{4.158}$$

Our goal is to rewrite this expression in the shock frame (4.149). Because of the gauge redundancy, we can shift everything that is contracted with e_2 by p_2. Furthermore, thanks to the momentum configuration we choose in section 4.4.2, we have $e_2\cdot p_1 = 0$, so we can also shift by p_1. Similarly, we can shift things that are contracted with e_4 by p_3, p_4. Using this, we can make a gauge choice to remove all the $s^\mu, \bar{s}^\mu$, and we find that the H_{24} structure mapped to the shock frame can be written as

$$\frac{H_{24}}{X_{24}}\bigg|_{\text{saddle},(4.133),(4.136)} = e_{2\mu_2}e_{4\mu_4}\left(\delta_{\perp;D-4}^{\mu_2\mu_4} - \frac{1}{\nu^2(m^2-\nu^2)^2}(\mathcal{V}_2^{\mu_2}\mathcal{V}_4^{\mu_4} + \mathcal{W}_2^{\mu_2}\mathcal{W}_4^{\mu_4})\right), \tag{4.159}$$

where we define

$$\begin{aligned}
\mathcal{V}_2^{\mu_2} &= m^2 q_1^{\mu_2} + (m^2-\nu^2)q_3^{\mu_2},\\
\mathcal{V}_4^{\mu_4} &= (m^2-\nu^2)q_1^{\mu_4} + m^2 q_3^{\mu_4},\\
\mathcal{W}_2^{\mu_2} &= m^2 q_1^{\mu_2} - (m^2-\nu^2)q_3^{\mu_2},\\
\mathcal{W}_4^{\mu_4} &= (m^2-\nu^2)q_1^{\mu_4} - m^2 q_3^{\mu_4}.
\end{aligned} \tag{4.160}$$

The momenta q_1^μ, q_3^μ are given by (4.149), (4.150), and $\delta_{\perp;D-4}^{\mu_2\mu_4}$ is the metric in the $(D-4)$-dimensional transverse space spanned by $v_{3;\perp}^\mu, \dots, v_{D-2;\perp}^\mu$. Similarly, the structure with $H_{20}H_{40}$ becomes

$$\frac{H_{20}H_{40}}{X_{20}X_{40}}\bigg|_{\text{saddle},(4.133),(4.136)} = \frac{1}{4\nu^2(m^2-\nu^2)}e_{2\mu_2}e_{4\mu_4}\mathcal{V}_2^{\mu_2}\mathcal{V}_4^{\mu_4}. \tag{4.161}$$

Using (4.156), (4.159), and (4.161), we can then obtain the flat space kernels for the two $j'=0$ dual structures.

For nonzero j', we also need to contract the transverse indices carried by $\mathcal{O}$ in the dual structure and $\mathcal{P}_{\delta,j'}$ in the celestial structure. Let us define

$$\mathcal{K}_{\nu,j'}^{4\gamma,(a)} = \left(\langle 0|J(x_4,z_4)\mathbf{L}[\mathcal{O}_{j'}](x_0,z_0)J(x_{2^+},z_2)|0\rangle^{(a)}\right)^{-1}\langle\widetilde{\mathcal{P}}_{\delta_J}^\dagger(x_{1^+0})\widetilde{\mathcal{P}}_{\delta_J}^\dagger(x_{30})\mathcal{P}_{\delta,j'}(z_0)\rangle\bigg|_{\substack{\text{saddle},(4.133),(4.136)\\ \nu\to\infty}}, \tag{4.162}$$

where $\nu \to \infty$ indicates that we take the leading term at large ν. The results for the four-photon kernels are given by

$$\begin{aligned}
\mathcal{K}^{4\gamma,(1)}_{\nu,j'=0} &= \frac{K^{4\gamma}_\nu}{2} e_{2\mu_2} e_{4\mu_4} \left((\mathcal{V}^{\mu_2}_2 \mathcal{V}^{\mu_4}_4 + \mathcal{W}^{\mu_2}_2 \mathcal{W}^{\mu_4}_4 - \nu^2 (m^2-\nu^2)^2 \delta^{\mu_2\mu_4}_{\perp;D-4}) - (D-2)\mathcal{V}^{\mu_2}_2 \mathcal{V}^{\mu_4}_4 \right), \\
\mathcal{K}^{4\gamma,(2)}_{\nu,j'=0} &= \frac{K^{4\gamma}_\nu}{8} e_{2\mu_2} e_{4\mu_4} \left(\mathcal{W}^{\mu_2}_2 \mathcal{W}^{\mu_4}_4 - \nu^2 (m^2-\nu^2)^2 \delta^{\mu_2\mu_4}_{\perp;D-4} \right), \\
\mathcal{K}^{4\gamma,(1)}_{\nu,j'=1} &= K^{4\gamma}_\nu e_{2\mu_2} e_{4\mu_4} \mathcal{W}^{\mu_2}_2 \mathcal{V}^{\mu_4}_4, \\
\mathcal{K}^{4\gamma}_{\nu,j'=2} &= -\frac{4K^{4\gamma}_\nu}{(D-1)} e_{2\mu_2} e_{4\mu_4} \left(\mathcal{W}^{\mu_2}_2 \mathcal{W}^{\mu_4}_4 + \frac{\nu^2(m^2-\nu^2)^2}{D-4} \delta^{\mu_2\mu_4}_{\perp;D-4} \right),
\end{aligned} \tag{4.163}$$

where

$$K^{4\gamma}_\nu = \frac{2^{-\frac{d}{2}+i\nu} e^{-\pi\nu} m^{-d} \nu^{-4+3d} (m^2-\nu^2)^{-2-\frac{d}{2}} (m - \sqrt{m^2-\nu^2})^{2-d}}{d-2}. \tag{4.164}$$

Note that for $j' = 1$ there are two tensor structures (see (C.9)). However, they are just related by $2 \leftrightarrow 4$, and as we will see later, kernels that are related by $2 \leftrightarrow 4$ will give the same sum rule, so we just include one of the structures here.

Finally, let us comment on the symmetry properties of the kernel. On the CFT side, the celestial three-point structure has a stabilizer group $\mathrm{SO}(d-3)$, which is also the stabilizer group of the scattering crystal (4.81). In the shock frame, this group becomes the $\mathrm{SO}(D-4)$ that fixes the shockwave amplitude momenta (4.147), and the group only rotates in the $v^\mu_{3;\perp}, \ldots, v^\mu_{D-2;\perp}$ directions. On the other hand, for the continuous-spin dual structure we can choose a conformal frame to fix all three points to be on the same line, and fix the continuous-spin polarization vector in a null direction. The configuration should be fixed by an $\mathrm{SO}(d-2)$, and the transverse spin j' is a representation of this group. In flat space, this should correspond to an $\mathrm{SO}(D-3)$. From the expressions of the kernels (4.163), it is not clear what this $\mathrm{SO}(D-3)$ should be. It turns out that (at least in the Regge limit) the group can be understood as the rotation group that fixes the $s^\mu, \bar{s}^\mu, v^\mu_{2;\perp}$ shock frame vectors. To see this we will first need to study how to apply the kernels (4.163) to four-photon amplitudes and write down the full sum rules for photons. We will therefore explain this $\mathrm{SO}(D-3)$ group and the meaning of transverse spin in the bulk in section 4.4.5.

4.4.4 Review: flat space sum rules for photons

After understanding how to map each part of the CFT bulk-point limit formula (4.102) to flat space, we can now derive the dictionary between CFT and flat space sum rules for photons and gravitons. Let us first review flat space sum rules for photons [137, 110]. The graviton sum rules will be considered in section 4.4.6.

The four-photon amplitude $\mathcal{M}_{4\gamma}(p_i, \epsilon_i)$ is a function of photon momenta p_i and polarizations ϵ_i for $i = 1,2,3,4$. They should satisfy $p_i^2 = p_i \cdot \epsilon_i = \epsilon_i^2 = 0$ and $\epsilon_i \sim \epsilon_i + \# p_i$. In general, the amplitude can be written as a sum of Lorentz-invariant polynomials of p_i, ϵ_i times functions of Mandelstam variables defined in (4.16). Depending on the choice of the polynomials, sometimes the functions of Mandelstam variables can develop spurious poles. It turns out that there exists a "local module" such that the amplitude will not have any spurious poles [138]. Namely, an amplitude that is a polynomial can always be written as generators of the local module times polynomials of Mandelstam variables.

To describe the generators of the local module, let us introduce some basic building blocks,[16]

$$\begin{aligned} H_{ij} &= F^{\mu}_{i\nu} F^{\nu}_{j\mu}, & H_{ijk} &= F^{\mu}_{i\nu} F^{\nu}_{j\sigma} F^{\sigma}_{k\mu}, \\ H_{ijkl} &= F^{\mu}_{i\nu} F^{\nu}_{j\sigma} F^{\sigma}_{k\rho} F^{\rho}_{l\mu}, & V_{i,jk} &= p_{k\mu} F^{\mu}_{i\nu} p^{\nu}_j, \end{aligned} \tag{4.165}$$

where $F^{\mu}_{i\nu} = p^{\mu}_i \epsilon_{i\nu} - \epsilon^{\mu}_i p_{i\nu}$. We also define

$$\begin{aligned} X_{ijkl} &= H_{ijkl} - \frac{1}{4} H_{ij} H_{kl} - \frac{1}{4} H_{ik} H_{jl} - \frac{1}{4} H_{il} H_{jk}, \\ S &= V_{1,24} H_{234} + V_{2,31} H_{341} + V_{3,42} H_{412} + V_{4,13} H_{123}. \end{aligned} \tag{4.166}$$

Then, the four-photon amplitude can be written as

$$\mathcal{M}_{4\gamma}(s,u) = \Big(H_{14} H_{23} \mathcal{M}^{(1)}_{4\gamma}(s,u) + X_{1243} \mathcal{M}^{(2)}_{4\gamma}(s,u) + \text{permutations} \Big) + S \mathcal{M}^{(3)}_{4\gamma}(s,t), \tag{4.167}$$

where $\mathcal{M}^{(1)}_{4\gamma}, \mathcal{M}^{(2)}_{4\gamma}$ are symmetric in their two arguments, and $\mathcal{M}^{(3)}_{4\gamma}$ is symmetric under all permutations of s, t, u.

To write down dispersion relations for $\mathcal{M}_{4\gamma}(s,u)$, we need to study how the polarization structures behave in the fixed-u Regge limit, where we take $s \to \infty$. This will tell us how many subtractions we should have in the dispersion relations. Moreover, for fixed-u sum rules it is useful to separately consider structures that are symmetric and antisymmetric under $s \leftrightarrow t$. The polarization structures in the Regge limit scale as [110]

$$H_{12}, H_{14} \sim s, \quad H_{13} \sim s^0, \quad X_{1234}, X_{1243} - X_{1324} \sim s, \quad X_{1243} + X_{1324} \sim s^2, \quad S \sim s^2, \tag{4.168}$$

[16] We are using V and H for the building blocks of both CFT tensor structures and flat space amplitudes. We hope the meaning of the notations will be clear from the context and this will not cause too much confusion.

where other structures can be obtained by permutations.

The amplitude itself $\mathcal{M}_{4\gamma}$ satisfies boundedness and analyticity properties due to unitarity and causality [108, 13, 109, 111]. As shown by [111], for a suitably smeared amplitude,

$$\mathcal{M}_{\Psi}(s) \equiv \int_0^M dp \Psi(p) \mathcal{M}(s, -p^2), \tag{4.169}$$

along any complex direction of s it should satisfy

$$|\mathcal{M}_{\Psi}(s)|_{s\to\infty} \leq s \times \text{constant}. \tag{4.170}$$

The Regge behaviors (4.168), (4.170) then combine to give boundedness conditions on each coefficient function $\mathcal{M}^{(i)}$. For example, since $X_{1243} - X_{1324} \sim s$ in the Regge limit, $\mathcal{M}^{(2)}(s, u) - \mathcal{M}^{(2)}(t, u)$ cannot grow faster than s^0 in the Regge limit. Performing this analysis for all the structures, we can then determine the subtractions needed for the dispersion relation of each coefficient $\mathcal{M}^{(i)}$. The complete list of flat space fixed-u sum rules for photons is given by

$$\begin{aligned}
\left\{C^{+(1,2,3,4,5)}_{4\gamma;k,u}\right\} &= -\oint_\infty \frac{ds}{4\pi i} \frac{1}{(-st)^{\frac{k}{2}}} \Big\{(s-t)\mathcal{M}^{(1,2)+}_{4\gamma}(s,u), (s-t)\left(\tfrac{-u}{2}\right)\mathcal{M}^{(3)}_{4\gamma}(s,t), \\
&\qquad \mathcal{M}^{(2)-}_{4\gamma}(s,u), \tfrac{(s-t)}{(-st)}\mathcal{M}^{(1)}_{4\gamma}(s,t)\Big\}, \quad k = 2, 4, \ldots, \\
\left\{C^{-(1,2)}_{4\gamma;k,u}\right\} &= -\oint_\infty \frac{ds}{4\pi i} \frac{1}{(-st)^{\frac{k-1}{2}}} \Big\{\mathcal{M}^{(1)-}_{4\gamma}(s,u), \tfrac{(s-t)}{(-st)}\mathcal{M}^{(2)}_{4\gamma}(s,t)\Big\}, \quad k = 3, 5, \ldots,
\end{aligned} \tag{4.171}$$

where we have used the notation $\mathcal{M}^{(i)\pm}(s,u) = \mathcal{M}^{(i)}(s,u) \pm \mathcal{M}^{(i)}(t,u)$. For later convenience, we introduce an additional $\frac{-u}{2}$ factor for $\mathcal{M}^{(3)}_{4\gamma}$. This would also make all the sum rules have the same power counting since the S structure has one more $p_i \cdot p_j$ compared to the other structures. The parameter k is the Regge spin of the sum rule. A sum rule with Regge spin k would be convergent if the amplitude $\mathcal{M}$ satisfies $\mathcal{M}/s^k \to 0$ in the Regge limit.

At low energy, the amplitude can be computed by an EFT with cutoff M. To derive bounds from the dispersion relations, one can deform the contour in (4.171) and separate the contributions from high energy $s \geq M^2$ and low energy $s < M^2$. Our main focus is the high energy part, where one decomposes the amplitude into SO(d) partial waves and imposes unitarity.

When applying the dispersion relations (4.171) to the partial wave expansion (4.120), for each sum rule C we can write its heavy contribution as

$$\left\langle C[m^2,\rho]_{(ab)}\right\rangle, \tag{4.172}$$

where $\langle\cdots\rangle$ is a heavy average defined as

$$\langle\cdots\rangle = \frac{1}{\pi}\sum_{\rho} n_\rho^{(D)} \sum_{ab}\int_{M^2}^{\infty}\frac{dm^2}{m^2} m^{4-D}\,\mathrm{Im}(a_\rho(s))_{ab}\,(\cdots)\,. \tag{4.173}$$

For example, for the $C^{+(1)}_{4\gamma;k,u}$ sum rule in (4.171), we have

$$C^{+(1)}_{4\gamma;k,u}[m^2,\rho]_{(ab)} = \frac{(2m^2+u)}{[m^2(m^2+u)]^{\frac{k}{2}}}\pi^{(1)}_{\rho,(ab)}(m^2,u), \tag{4.174}$$

where we have decomposed the partial wave into different polarization structures similar to (4.167), so $\pi^{(1)}_{\rho,(ab)}$ is the $H_{14}H_{23}$ component of the partial wave.

This dictionary we obtain below in sections 4.4.5 and 4.4.6 will be a relation between the action of CFT sum rules on conformal blocks in the bulk-point limit (4.102) and the action of flat space sum rules on partial waves (4.172). This will allow us to find CFT sum rules with positive action on conformal blocks with large Δ straightforwardly from the flat space result.

4.4.5 From CFT to flat space for photons

Now, we combine the discussion about the block/partial wave in section 4.4.2 and the discussion about the kernel/shockwave amplitude in section 4.4.3 to write down the relation between four-photon flat space sum rules and the bulk-point limit of our CFT sum rules.

When combining the two parts, we first have to understand how to contract the indices of the external operators. Recall that in the CFT functional, the polarizations z_1, z_3 are set to be $z_1 = x_{1^+0}, z_3 = x_{30}$. As discussed in 4.4.3, this implies that in the bulk, we should set $e_1^\mu = e_3^\mu = i\frac{m(m-\sqrt{m^2-\nu^2})}{\nu^2}\bar{s}^\mu$. For 2 and 4, we have to contract the indices of the kernel and the block. Note that we have chosen the CFT polarization vectors to have no time component. In the bulk, this means that e_2 has no time and n^μ components (see (4.130)), and similarly e_4^μ has no time and n'^μ components. Let us consider the following index contraction in the CFT sum rule:

$$(z_2^x K^x + \vec{z}_{2\perp}\cdot\vec{K}_\perp)\odot(z_2^x G^x + \vec{z}_{2\perp}\cdot\vec{G}_\perp), \tag{4.175}$$

where $K^x, \vec{K}_\perp$ are from the kernel and $G^x, \vec{G}_\perp$ are from the block, and the symbol $\odot$ represents the index contraction. For spin-1, it is simply stripping off the z_2's and contracting both sides, and we see that after the index contraction, we should get $K^x G^x + \vec{K}_\perp \cdot \vec{G}_\perp$. On the other hand, mapping (4.175) to the bulk, we have

$$\begin{aligned}
&(-e_2^n K^x + \vec{e}_{2\perp} \cdot \vec{K}_\perp) \odot (-e_2^n G^x + \vec{e}_{2\perp} \cdot \vec{G}_\perp) \\
=&e_{2\mu_2}(-n_\perp^{\mu_2} K^x + \delta_\perp^{\mu_2 i} \vec{K}_{\perp i}) \odot e_{2\rho_2}(-n_\perp^{\rho_2} G^x + \delta_\perp^{\rho_2 i} \vec{G}_{\perp i}) \\
=&e_{2\mu_2} K^{\mu_2} \odot e_{2\rho_2} G^{\rho_2},
\end{aligned} \tag{4.176}$$

where K^{μ_2} is $-n_\perp^{\mu_2} K^x + \delta_\perp^{\mu_2 i} \vec{K}_{\perp i}$ up to a shift by $p_1^{\mu_2}, p_2^{\mu_2}$ and similarly for G^{ρ_2}. For example, in the $j' = 2$ kernel given by (4.163) we have $K^{\mu_2} \propto m^2 q_1^{\mu_2} - (m^2 - \nu^2) q_3^{\mu_2}$. To correctly reproduce $K^x G^x + \vec{K}_\perp \cdot \vec{G}_\perp$, we should perform a polarization sum in e_2 using[17]

$$e_{2\mu_2} e_{2\rho_2} \longrightarrow \eta_{\mu_2\rho_2} - \frac{p_{1\mu_2} p_{2\rho_2} + p_{1\rho_2} p_{2\mu_2}}{p_1 \cdot p_2}. \tag{4.177}$$

Then, we obtain

$$K^{\mu_2} G^{\rho_2} \left(\eta_{\mu_2\rho_2} - \frac{p_{1\mu_2} p_{2\rho_2} + p_{1\rho_2} p_{2\mu_2}}{p_1 \cdot p_2} \right) = K^x G^x + \vec{K}_\perp \cdot \vec{G}_\perp \tag{4.178}$$

In summary, for the flat space sum rules we should set the polarizations e_1, e_3 to be proportional to $\bar{s}$, and perform polarization sums for e_2, e_4. More precisely,

$$\begin{aligned}
e_{1\mu} = e_{3\mu} &= i \frac{m(m - \sqrt{m^2 - \nu^2})}{\nu^2} \bar{s}_\mu, \\
e_{2\mu_2} e_{2\rho_2} \longrightarrow \eta_{\mu_2\rho_2} - \frac{p_{1\mu_2} p_{2\rho_2} + p_{1\rho_2} p_{2\mu_2}}{p_1 \cdot p_2}&, \quad e_{4\mu_4} e_{4\rho_4} \longrightarrow \eta_{\mu_4\rho_4} - \frac{p_{3\mu_4} p_{4\rho_4} + p_{3\rho_4} p_{4\mu_4}}{p_3 \cdot p_4}.
\end{aligned} \tag{4.179}$$

To study how the spinning CFT sum rules are mapped to flat space, we first start with the heavy action formula (4.102). Then, we further decompose the conformal block $G^s_{\Delta,\rho}$ into independent four-point tensor structures. Namely, we want to consider

$$\Psi^{+(a)}_{k,\nu,j'}[G^s_{\Delta,\rho,(a'b'),I} Q^I], \tag{4.180}$$

where Q^I is a basis of CFT four-point structures. We will choose the CFT four-point structures such that under the polarization maps (4.133),(4.136), they get mapped

[17] For a general polarization ϵ_2, p_1 can be replaced with any null vector q. However, here we have chosen the polarization to satisfy the condition $e_2 \cdot p_1 = e_2 \cdot p_2 = 0$, so the correct choice here is $q = p_1$.

to the generators of the local module of the four-photon amplitude. Namely, the four-point structures satisfy

$$Q^I\Big|_{\text{saddle},(4.133),(4.136)} = \frac{16}{\nu^4}\left\{H_{14}H_{23}, H_{13}H_{24}, H_{12}H_{34}, X_{1243}, X_{1234}, X_{1324}, \frac{2}{\nu^2}S\right\}, \tag{4.181}$$

where the structures on the right hand side is the basis of polarization structures of the four-photon amplitude introduced in section 4.4.4. We give the explicit expressions of Q^I's in appendix C.5.

We can now apply the polarization map (4.133),(4.136) to the right hand side of the heavy action formula (4.102). From the previous discussions, we know that the kernel part should become $\mathcal{K}^{(a)}_{\nu,j'}$ defined in (4.162) and it acts on the four-point tensor structures Q^I's through (4.179). For the block part, by (4.143), (4.144), it should turn into four-photon partial waves which are further decomposed in four-point structures. In summary, we have

$$\begin{aligned}
&\lim_{\substack{\nu,m\gg 1\\ \nu<m}} \frac{\Psi^{+(a)}_{k,\nu,j'}[G^s_{\Delta,\rho,(a'b')}]}{-2(2\sin^2(\pi\frac{\widetilde{\tau}_\rho-2d}{2}))}\\
&= \frac{2^{\frac{5}{2}+2d}\pi^{d-\frac{1}{2}}e^{i\frac{\pi}{4}}m^{\frac{3d}{2}+2}\nu^{\frac{7}{2}-4d}(m^2-\nu^2)^{\frac{d-2}{2}}(m-\sqrt{m^2-\nu^2})^{d-4}}{\Gamma\left(\frac{d-2}{2}\right)\text{vol}(\text{SO}(d-2))}\\
&\times \mathcal{K}^{(a)}_{\nu,j'}[Q^I]\\
&\times 2^{-J+2m-4d+4}m^{-4d+4}\nu^{4d-4}R_\rho \pi^I_{\rho,(a'b')}.
\end{aligned} \tag{4.182}$$

Our remaining task is to evaluate the above expression explicitly. We perform the polarization sum of $\mathcal{K}^{(a)}_{\nu,j'}[Q^I]$ using (4.163), (4.179), and also insert the subtraction factors $f^{(-1)^{j'}}_{k;J_e}$ given by (4.106), (4.107) with $J_e = 1$. Finally, by comparing the result with the heavy state contribution of the four-photon flat space sum rules (4.171), we obtain the dictionary between the four-photon CFT and flat space sum rules.

$$\begin{aligned}
&\lim_{\substack{\Delta,\nu\gg 1\\ \nu<m}} \begin{pmatrix}\Psi^{(1)}_{4\gamma;k,\nu,j'=0;f^+_{k;J_e=1}}[\Delta,\rho]\\ \Psi^{(2)}_{4\gamma;k,\nu,j'=0;f^+_{k;J_e=1}}[\Delta,\rho]\\ \Psi_{4\gamma;k,\nu,j'=2;f^+_{k;J_e=1}}[\Delta,\rho]\end{pmatrix} = \frac{1}{A^{4\gamma}_{k,\nu}}M^+_{4\gamma}\begin{pmatrix}C^{+(1)}_{4\gamma;k,-\nu^2}[m^2,\rho]\\ C^{+(2)}_{4\gamma;k,-\nu^2}[m^2,\rho]\\ C^{+(3)}_{4\gamma;k,-\nu^2}[m^2,\rho]\end{pmatrix}, \quad k=2,4,=\ldots,\\
&\lim_{\substack{\Delta,\nu\gg 1\\ \nu<m}} \left(\Psi^{(1)}_{4\gamma;k,\nu,j'=1;f^-_{k;J_e=1}}[\Delta,\rho]\right) = \frac{1}{A^{4\gamma}_{k,\nu}}M^-_{4\gamma}\left(C^{-(1)}_{4\gamma;k,-\nu^2}[m^2,\rho]\right), \quad k=3,5,\ldots,
\end{aligned} \tag{4.183}$$

where $A^{4\gamma}_{k,\nu}$ is

$$A^{4\gamma}_{k,\nu} = 2^{5d-3-2j'} e^{\pi\nu} \pi^{1-d} \nu^{6-3d-2k} \Gamma(\tfrac{d}{2}). \tag{4.184}$$

The two matrices $M^{+}_{4\gamma}, M^{-}_{4\gamma}$ are our main results. They are given by

$$M^{+}_{4\gamma} = \begin{pmatrix} -4(D-2) & -2(D-4) & 0 \\ -1 & -\frac{D-4}{2} & (D-3) \\ \frac{2}{(D-1)} & -\frac{1}{(D-1)} & 0 \end{pmatrix},$$
$$M^{-}_{4\gamma} = \begin{pmatrix} 2 \end{pmatrix}. \tag{4.185}$$

The left-hand side of the dictionary is written as the heavy action defined by (4.145), and the right hand side is the heavy state contribution given by e.g., (4.174). The dictionary should be true for all tensor structures of the block/partial wave, so we suppress their labels $(a'), (b')$ in (4.183). Moreover, since $(-1)^{j'}$ gives the signature of the integrand under $1 \leftrightarrow 3$, or equivalently $s \leftrightarrow t$, we can separately consider the CFT sum rules with even/odd j' and the flat space sum rules that are symmetric/antisymmetric under $s \leftrightarrow t$. On the CFT side, k is a parameter of the functional and the subtraction factor, which controls the behavior of the integrand in the Regge limit. On the flat space side, the same k gives the Regge spin of the sum rule.

We also notice that the all the CFT sum rules correspond to the "lowest-subtracted" flat space sum rules, meaning that they have the minimal number of subtractions (for a given Regge spin) among all the sum rules in (4.171). (In other words, their corresponding polarization structures grow the fastest in the Regge limit.) Interestingly, upon setting $e_1 = e_3$ to be along the $\bar{s}^{\mu}$ direction following (4.179), we find that the only four-photon amplitude structures that are still non-vanishing are the ones that appear in the lowest subtracted sum rules. It would be nice to better understand why this happens.

Transverse spin of sum rules

By construction, our CFT sum rules are labeled by a transverse spin j'. As discussed below the dictionary (4.183), the signature $(-1)^{j'}$ tells us if the flat space sum rule is symmetric or antisymmetric under $s \leftrightarrow t$. However, j' is a representation of $\mathrm{SO}(d-2)$, so it should encode more information than just a signature. On the CFT side, the transverse spin appears in the dual structure

$\left(\langle 0|\mathcal{O}_4(x_4)\mathbf{L}[\mathcal{O}](x_0,z_0)\mathcal{O}_2(x_2^+)\rangle^{(a)}\right)^{-1}$, and j' can be realized as a representation of the SO$(d-2)$ group that stabilizes the saddle configuration of x_{2^+}, x_4, x_0, z_0. On the flat space side, this should correspond to an SO$(D-3)$. What is this SO$(D-3)$ in flat space? Moreover, in the dictionary (4.183), (4.185), each CFT sum rule with a fixed j' gets mapped to a linear combination of the coefficient functions $\mathcal{M}^{(i)}_{4\gamma}$. How can we interpret this linear combination as having a transverse spin j' directly in flat space?

We find that the simplest way to understand transverse spin in flat space is by considering the fixed-u Regge limit in the shock frame (4.149). First, note that in the Regge limit, the $\mathcal{V}, \mathcal{W}$ vectors defined in (4.160) become $\mathcal{V}^\mu \sim q_1^\mu + q_3^\mu = \nu v_{2;\perp}^\mu, \mathcal{W}^\mu \sim q_1^\mu - q_3^\mu = -i\nu v_{1;\perp}^\mu$. Therefore, the four kernels in (4.163) become

$$\begin{aligned}
\mathcal{K}^{4\gamma,(1)}_{j'=0}\Big|_{\text{Regge}} &\propto e_{2\mu_2}e_{4\mu_4}\left((D-3)v_{2;\perp}^{\mu_2}v_{2;\perp}^{\mu_4} + v_{1;\perp}^{\mu_2}v_{1;\perp}^{\mu_4} + \delta_{\perp;D-4}^{\mu_2\mu_4}\right),\\
\mathcal{K}^{4\gamma,(2)}_{j'=0}\Big|_{\text{Regge}} &\propto e_{2\mu_2}e_{4\mu_4}\left(v_{1;\perp}^{\mu_2}v_{1;\perp}^{\mu_4} + \delta_{\perp;D-4}^{\mu_2\mu_4}\right),\\
\mathcal{K}^{4\gamma,(1)}_{j'=1}\Big|_{\text{Regge}} &\propto e_{2\mu_2}e_{4\mu_4}v_{1;\perp}^{\mu_2}v_{2;\perp}^{\mu_4},\\
\mathcal{K}^{4\gamma}_{j'=2}\Big|_{\text{Regge}} &\propto e_{2\mu_2}e_{4\mu_4}\left(v_{1;\perp}^{\mu_2}v_{1;\perp}^{\mu_4} - \frac{v_{1;\perp}^{\mu_2}v_{1;\perp}^{\mu_4} + \delta_{\perp;D-4}^{\mu_2\mu_4}}{D-3}\right).
\end{aligned} \tag{4.186}$$

From the expression of the kernels in the Regge limit, we see that the SO$(D-3)$ that realizes the transverse spin in flat space should be the stabilizer group of $s, \bar{s}, v_{2;\perp}$, and $v_{1;\perp}^{\mu_2}v_{1;\perp}^{\mu_4} + \delta_{\perp;D-4}^{\mu_2\mu_4}$ is the metric of the corresponding $(D-3)$-dimensional space. In fact, we can see this more clearly by studying the four-photon amplitude in the Regge limit. To make the action of the SO$(D-3)$ more manifest, we can parametrize the polarizations as

$$\begin{aligned}
e_2^\mu &= e_{2s}s^\mu + e_{2\bar{s}}\bar{s}^\mu + e_{2+}v_{2;\perp}^\mu + z_2^\mu,\\
e_4^\mu &= e_{4s}s^\mu + e_{4\bar{s}}\bar{s}^\mu + e_{4+}v_{2;\perp}^\mu + z_4^\mu,
\end{aligned} \tag{4.187}$$

where z_2^μ, z_4^μ are $(D-3)$-dimensional vectors that transform under the SO$(D-3)$ group. Using transversality $p_2 \cdot e_2 = 0$ and the gauge redundancy $e_2 \sim e_2 + \# p_2$, we can express $e_{2s}, e_{2\bar{s}}$ in terms of e_{2+}, z_2 and the shock frame momenta (and similarly for e_4). Therefore, we can parametrize the polarizations using e_{2+}, e_{4+} and z_2, z_4.

We now take the four-point amplitude and set $e_1 = e_3 = i\frac{m(m-\sqrt{m^2-\nu^2})}{\nu^2}\bar{s}$ as required by our functional. As mentioned above, after this step only the polarization structures that appear in the lowest-subtracted sum rules are non-vanishing. We then take the

Regge limit and consider the leading term. With the parametrization (4.187), we find

$$\begin{aligned}\mathcal{M}_{4\gamma}^{\text{Regge}} &= \frac{m^2\nu^2(D-1)}{8}\left(\tfrac{2}{D-1}\mathcal{M}^{(1)+}(s,u)-\tfrac{1}{D-1}\mathcal{M}^{(2)+}(s,u)\right)E^{4\gamma}_{24,j'=2}\\ &+\frac{im^2\nu^2}{8}\left(2\mathcal{M}^{(1)-}(s,u)\right)E^{4\gamma}_{24,j'=1}\\ &-\frac{m^2\nu^2}{4(D-3)}\left(-\mathcal{M}^{(1)+}(s,u)-\tfrac{D-4}{2}\mathcal{M}^{(2)+}(s,u)+(D-3)\tfrac{\nu^2}{2}\mathcal{M}^{(3)}\right)E^{4\gamma,(1)}_{24,j'=0}\\ &-\frac{m^2\nu^2}{16(D-3)}\left(-4(D-2)\mathcal{M}^{(1)+}(s,u)-2(D-4)\mathcal{M}^{(2)+}(s,u)\right)E^{4\gamma,(2)}_{24,j'=0},\end{aligned}\tag{4.188}$$

where each $E^{4\gamma}_{24,j'}$ is a tensor built from e_2, e_4 that transforms as a spin-j' representation under the $\mathrm{SO}(D-3)$ group. They are defined as

$$\begin{aligned}E^{4\gamma}_{24,j'=2} &= z_2\cdot v_{1;\perp}z_4\cdot v_{1;\perp}-\frac{z_2\cdot z_4}{D-3}, \qquad & E^{4\gamma}_{24,j'=1} &= e_{4+}z_2\cdot v_{1;\perp}-e_{2+}z_4\cdot v_{1;\perp},\\ E^{4\gamma,(1)}_{24,j'=0} &= z_2\cdot z_4-e_{2+}e_{4+}, & E^{4\gamma,(2)}_{24,j'=0} &= e_{2+}e_{4+}.\end{aligned}\tag{4.189}$$

We see that by decomposing $\mathcal{M}_{4\gamma}^{\text{Regge}}$ into different representations under $\mathrm{SO}(D-3)$, the linear combinations of the coefficient functions exactly agree with the ones in the dictionary (4.185). In other words, we can write $\mathcal{M}_{4\gamma}^{\text{Regge}}$ as

$$\begin{aligned}\mathcal{M}_{4\gamma}^{\text{Regge}} &= \begin{pmatrix}-\frac{m^2\nu^2}{4(D-3)}E^{4\gamma,(1)}_{24,j'=0} & -\frac{m^2\nu^2}{16(D-3)}E^{4\gamma,(2)}_{24,j'=0} & \frac{m^2\nu^2(D-1)}{8}E^{4\gamma}_{24,j'=2}\end{pmatrix}M^{+}_{4\gamma}\begin{pmatrix}\mathcal{M}^{(1)+}(s,u)\\ \mathcal{M}^{(2)+}(s,u)\\ \frac{\nu^2}{2}\mathcal{M}^{(3)}(s,t)\end{pmatrix}\\ &+\left(\tfrac{im^2\nu^2}{8}E^{4\gamma}_{24,j'=1}\right)M^{-}_{4\gamma}\left(\mathcal{M}^{(1)-}(s,u)\right),\end{aligned}\tag{4.190}$$

where $M^{+}_{4\gamma}, M^{-}_{4\gamma}$ are the matrices in the dictionary (4.185).

Another way to see why we get the same matrices that appear in the dictionary is that under the polarization sum, the pairing between the Regge limit kernels (4.186) and the spin-j' tensors (4.189) form a diagonal matrix.

For the $j'=0$ case, there are two different tensors structures. By comparing $E^{(1)}_{j'=0}, E^{(2)}_{j'=0}$ and the CFT structures that give the two kernels for $j'=0$, we can even write down a dictionary between the embedding space CFT structures and spin-j' tensors:

$$H_{20}H_{40}\leftrightarrow e_{2+}e_{4+}, \quad H_{24}\leftrightarrow z_2\cdot z_4-e_{2+}e_{4+}. \tag{4.191}$$

A similar dictionary between CFT structures and polarization structures in a conformal frame is also given in [48].

4.4.6 Sum rules for gravitons

In this section we give the dictionary between the CFT sum rules and flat space sum rules for gravitons. We first review the flat space graviton sum rules.

The local module of the graviton amplitude has 29 generators, and 28 of them can be constructed from taking products of the photon generators defined in section 4.4.4. The remaining generator $\mathcal{G}$ can be written as the Gram determinant of all dot products between $(p_1, p_2, p_3, e_1, e_2, e_3, e_4)$. The general four-graviton amplitude in generic spacetime dimension ($D \geq 8$) takes the form [138]

$$\begin{aligned}\mathcal{M}_{4g}(s,u) =& \mathcal{G}\mathcal{M}^{(1)}_{4g}(s,u) + S^2\mathcal{M}^{(10)}_{4g}(s,u)\\ &+ \Big(H_{14}^2 H_{23}^2 \mathcal{M}^{(2)}_{4g}(s,u) + H_{12}H_{13}H_{24}H_{34}\mathcal{M}^{(3)}_{4g}(s,u)\\ &\quad + H_{14}H_{23}(X_{1243} - X_{1234} - X_{1324})\mathcal{M}^{(4)}_{4g}(s,u) + X_{1243}^2\mathcal{M}^{(6)}_{4g}(s,u)\\ &\quad + X_{1234}X_{1324}\mathcal{M}^{(7)}_{4g}(s,u) + H_{14}H_{23}S\mathcal{M}^{(8)}_{4g}(s,u)\\ &\quad + X_{1243}S\mathcal{M}^{(9)}_{4g}(s,u) + \text{triplet permutations}\Big)\\ &+ H_{12}H_{34}X_{1243}\mathcal{M}^{(5)}_{4g}(s,u) + \text{sextuplet permutations}.\end{aligned} \tag{4.192}$$

The functions that multiply the polarization structures $\mathcal{G}$ and S^2 are symmetric under all permutations of s, t, u, and the ones with "triplet permutations" are symmetric in their two arguments. For the $\mathcal{M}^{(5)}$ function one should include all six permutations.

By repeating the analysis of the Regge limit behavior reviewed in section 4.4.4, one can obtain the dispersion relations for the graviton amplitude. There are 19 independent fixed-u sum rules with even Regge spin k and are symmetric under $s \leftrightarrow t$ [110]. Similar to the photon case, the number of subtractions for a given Regge spin of these sum rules can be different. In the dictionary for photons, we see that our CFT sum rules all correspond to the lowest-subtracted flat space sum rules. This turns out to be true for gravitons as well. For graviton sum rules with even Regge spin, 7 of them are the lowest-subtracted. They are given by

$$\begin{aligned}\left\{C^{+(1-7)}_{4g;k,u}\right\} = -\oint_\infty \frac{ds}{4\pi i}\frac{(s-t)}{(-st)^{\frac{k-2}{2}}}\Big\{&\mathcal{M}^{(3)}_{4g}(s,t), \left(\tfrac{-u}{2}\right)^2\mathcal{M}^{(10)}_{4g}(s,t), \mathcal{M}^{(2,5)+}_{4g}(s,u),\\ &\left(\tfrac{-u}{2}\right)\mathcal{M}^{(8,9)+}_{4g}(s,u), \mathcal{M}^{(6)+}_{4g}(s,u) + \mathcal{M}^{(7)}_{4g}(s,t)\Big\}, \quad k = 2, 4, \ldots.\end{aligned} \tag{4.193}$$

For the remaining 12 sum rules that have higher number of subtractions, see appendix A of [110]. We again use the notation $\mathcal{M}^{(i)\pm}(s,u) = \mathcal{M}^{(i)}(s,u) \pm \mathcal{M}^{(i)}(t,u)$, and rescale some of the sum rules by $\left(\frac{-u}{2}\right)$ to make all of them have the same power counting.

For graviton sum rules with odd Regge spin, there are 10 independent sum rules, and 3 of them are the lowest-subtracted,

$$\left\{C^{-(1-3)}_{4g;k,u}\right\} = -\oint_{\infty} \frac{ds}{4\pi i} \frac{1}{(-st)^{\frac{k-3}{2}}} \left\{\mathcal{M}^{(2,5)-}_{4g}(s,u), \left(\tfrac{-u}{2}\right) \mathcal{M}^{(8)-}_{4g}(s,u)\right\}, \quad k = 3, 5, \ldots. \tag{4.194}$$

When performing the CFT analysis for gravitons, all the discussions in section 4.4.2 and 4.4.3 can be generalized straightforwardly. In particular, we can also obtain a simple relation similar to (4.144) that relates the block part of the four-stress-tensor functional to the four-graviton partial waves, which include 20 different cases in total [135, 110]. Moreover, we can obtain the graviton kernels that are similar to (4.163). The number of independent kernels are 3 for $j' = 0$, 2 for $j' = 1$, 3 for $j' = 2$, 1 for $j' = 3$, 1 for $j' = 4$. Therefore, we have 7 even j' and 3 odd j' sum rules in total, and they indeed agree with the number of "lowest-subtracted" flat space graviton sum rules that are symmetric or antisymmetric under $s \leftrightarrow t$ given in (4.193) and (4.194).

Finally, the only remaining difference in the graviton case is that instead of (4.179), the rules for polarization sum of e_2, e_4 become

$$e_i^\mu e_i^\nu e_i^\alpha e_i^\beta \to \frac{1}{2}\left(P^{\mu\alpha}_{e_i} P^{\nu\beta}_{e_i} + P^{\nu\alpha}_{e_i} P^{\mu\beta}_{e_i}\right) - \frac{1}{D-2} P^{\mu\nu}_{e_i} P^{\alpha\beta}_{e_i}, \quad i = 2, 4, \tag{4.195}$$

where

$$\begin{aligned} P^{\mu\nu}_{e_2} &= \eta^{\mu\nu} - \frac{p_1^\mu p_2^\nu + p_1^\nu p_2^\mu}{p_1 \cdot p_2}, \\ P^{\mu\nu}_{e_4} &= \eta^{\mu\nu} - \frac{p_3^\mu p_4^\nu + p_3^\nu p_4^\mu}{p_3 \cdot p_4}. \end{aligned} \tag{4.196}$$

Our final dictionary for gravitons is given by

$$\lim_{\substack{\Delta,\nu\gg 1\\ \nu<m}} \begin{pmatrix} \Psi^{(1)}_{4g;k,\nu,j'=0;f^{+}_{k;Je=2}}[\Delta,\rho] \\ \Psi^{(2)}_{4g;k,\nu,j'=0;f^{+}_{k;Je=2}}[\Delta,\rho] \\ \Psi^{(3)}_{4g;k,\nu,j'=0;f^{+}_{k;Je=2}}[\Delta,\rho] \\ \Psi^{(1)}_{4g;k,\nu,j'=2;f^{+}_{k;Je=2}}[\Delta,\rho] \\ \Psi^{(2)}_{4g;k,\nu,j'=2;f^{+}_{k;Je=2}}[\Delta,\rho] \\ \Psi^{(3)}_{4g;k,\nu,j'=2;f^{+}_{k;Je=2}}[\Delta,\rho] \\ \Psi_{4g;k,\nu,j'=4;f^{+}_{k;Je=2}}[\Delta,\rho] \end{pmatrix} = \frac{1}{A^{4g}_{k,\nu}} M^{+}_{4g} \begin{pmatrix} C^{+(1)}_{4g;k,-\nu^2}[m^2,\rho] \\ C^{+(2)}_{4g;k,-\nu^2}[m^2,\rho] \\ C^{+(3)}_{4g;k,-\nu^2}[m^2,\rho] \\ C^{+(4)}_{4g;k,-\nu^2}[m^2,\rho] \\ C^{+(5)}_{4g;k,-\nu^2}[m^2,\rho] \\ C^{+(6)}_{4g;k,-\nu^2}[m^2,\rho] \\ C^{+(7)}_{4g;k,-\nu^2}[m^2,\rho] \end{pmatrix}, \quad k=2,4,=\ldots,$$

$$\lim_{\substack{\Delta,\nu\gg 1\\ \nu<m}} \begin{pmatrix} \Psi^{(1)}_{4g;k,\nu,j'=1;f^{-}_{k;Je=2}}[\Delta,\rho] \\ \Psi^{(2)}_{4g;k,\nu,j'=1;f^{-}_{k;Je=2}}[\Delta,\rho] \\ \Psi^{(1)}_{4g;k,\nu,j'=3;f^{-}_{k;Je=2}}[\Delta,\rho] \end{pmatrix} = \frac{1}{A^{4g}_{k,\nu}} M^{-}_{4g} \begin{pmatrix} C^{-(1)}_{4g;k,-\nu^2}[m^2,\rho] \\ C^{-(2)}_{4g;k,-\nu^2}[m^2,\rho] \\ C^{-(3)}_{4g;k,-\nu^2}[m^2,\rho] \end{pmatrix}, \quad k=3,5,=\ldots, \tag{4.197}$$

where the coefficient $A^{4g}_{k,\nu}$ is

$$A^{4g}_{k,\nu} = 2^{5d+3-2j'} e^{\pi\nu} \pi^{1-d} \nu^{2-3d-2k} \Gamma\left(\tfrac{d+2}{2}\right), \tag{4.198}$$

and the two matrices are given by

$$M^{+}_{4g} = \begin{pmatrix} 16(D-4)(D-2) & 0 & 16D(D+2) & 8(D-4)D & 0 & 0 & 4(D-4)(D-2) \\ 16 & 0 & 16D & 2\left(D^2-3D-2\right) & -4(D-2)(D-1) & -2(D-4)(D-1) & 2\left(D^2-6D+7\right) \\ 2 & (D-3)(D-1) & 2 & \frac{D-3}{2} & 1-D & -\frac{1}{2}(D-4)(D-1) & \frac{1}{4}\left(D^2-6D+7\right) \\ -\frac{8D}{D+1} & 0 & \frac{8(D+2)}{D+1} & -\frac{4}{D+1} & 0 & 0 & \frac{2}{D+1} \\ -\frac{32}{D+1} & 0 & -\frac{32(D+2)}{D+1} & \frac{16}{D+1} & 0 & 0 & \frac{4(D-1)}{D+1} \\ -\frac{8}{D+1} & 0 & -\frac{8}{D+1} & -\frac{D-3}{D+1} & 2 & -1 & \frac{D-1}{D+1} \\ \frac{24}{(D+1)(D+3)} & 0 & \frac{24}{(D+1)(D+3)} & -\frac{12}{(D+1)(D+3)} & 0 & 0 & \frac{6}{(D+1)(D+3)} \end{pmatrix},$$

$$M^{-}_{4g} = \begin{pmatrix} -16(D+2) & 4(D-4) & 0 \\ -8 & (D-3) & 2(D-1) \\ \frac{24}{D+1} & \frac{6}{D+1} & 0 \end{pmatrix}. \tag{4.199}$$

Finally, the transverse spin analysis done in section 4.4.5 can also be performed in the graviton case. In this case we have transverse spin up to $j'=4$. As an example, let us consider the spin-4 tensor which is defined as

$$E^{4g}_{24,j'=4} = (z_2\cdot v_{1;\perp})^2 (z_4\cdot v_{1;\perp})^2 - \text{traces}. \tag{4.200}$$

When we isolate the spin-4 part of the four-graviton amplitude in the Regge limit (and after setting e_1, e_3 to be in the $\overline{s}$ direction), we obtain

$$\mathcal{M}_{4g}^{\text{Regge}} = \frac{m^4\nu^4}{64}\Big(4\mathcal{M}^{(3)}(s,t) + 4\mathcal{M}^{(2)+}(s,u) - 2\mathcal{M}^{(5)+}(s,u) \\ +\mathcal{M}^{(6)+}(s,u) + \mathcal{M}^{(7)}(s,t)\Big)\, E^{4g}_{24,j'=4} + \dots, \tag{4.201}$$

where ... are other terms with $j' < 4$. Indeed, we see that the linear combination of the coefficient functions agrees with the dictionary (4.199) (the last row of M^+_{4g}).

4.5 Discussion

In this work, we studied a basis of CFT dispersive sum rules for spinning operators. The basis was constructed using the fact that the commutator of two null-integrated operators on the same null plane vanishes, also known as superconvergence. Using the Lorentzian shadow representation of conformal blocks, we expressed the action of our sum rule on a conformal block as an integral over spacetime. We showed that in the bulk point limit, where ν (parameter of the sum rule) and Δ (dimension of the block) both become large, the spacetime integral gets completely localized to a "scattering-crystal." This enabled us to derive a simple formula for the block action of our sum rule in the bulk point limit, generalizing the results in [13] to the spinning case.

The main result of this paper is a dictionary between the block action of CFT sum rules in the bulk point limit and the heavy state contribution of flat space sum rules. We fixed the dictionary by exploring flat space interpretations for the inserted block and the kernel in the simple formula of the CFT sum rule. We showed that CFT three-point structures at the saddle form a natural basis for the flat space three-point amplitudes, allowing the inserted block to be directly related to flat space partial waves in this basis. On the other hand, the kernels of the sum rule can be interpreted as actions on shockwave amplitudes in flat space. Combining the two results, we derived the dictionary between our CFT sum rules and the flat space sum rules for photons and gravitons.

By construction, the CFT sum rules are labeled by a transverse spin j'. Through the dictionary, this gives a basis of flat space sum rules each labeled by a transverse spin (an $\mathrm{SO}(D-3)$ representation). This $\mathrm{SO}(D-3)$ stabilizes the shock directions and the momentum transfer $p_1 + p_3$ of the shockwave amplitude, and the transverse spin of each sum rule can be realized by studying the amplitude in the Regge limit.

However, from the flat space perspective, it remains unclear how this "transverse spin basis" is useful in practice. It would be interesting to explore this further.

Our construction of spinning CFT sum rules is an important step toward lifting the flat space gravitational bounds in [110] to AdS/CFT. It provides functionals with positive action on conformal blocks with large Δ and fixed J. To complete the bootstrap argument, one also needs to check positivity in the Regge limit, where Δ and J are both large. Moreover, one has to derive a similar dictionary that relates the low energy EFT contribution of the CFT and flat space sum rules. In the scalar case [13], the light contribution analysis also involves a saddle point. Therefore we expect we should be able to generalize it to the spinning case by arguments similar to section 4.3. Once these calculations are done, one can then derive bounds on the higher derivative corrections of gravitational AdS EFTs in terms of the large CFT gap. These bounds would give a sharp form of the HPPS conjecture [114] at the level of four-point functions of stress tensors.

It would be interesting to also test the HPPS conjecture beyond four-point functions using superconvergence. For instance, one could consider commutators of light-transformed operators in a multi-point correlation function. In flat space, we expect this should correspond to commutators of multiple shockwaves. Another interesting direction is studying superconvergence in a thermal state by using a folded light transform contour proposed in [139].

So far, we have only found a dictionary that relates CFT sum rules to some of the known flat space sum rules. In particular, we only get sum rules that have the fewest subtractions. As discussed in 4.4.5, to get other sum rules we have to leave the configuration where the polarizations of the shock gravitons are along the shock direction. In CFT this condition comes from setting the polarizations of the light-transformed operators $\mathcal{O}_1, \mathcal{O}_3$ to be the null vectors x_{10}, x_{30}. Therefore, it is conceivable that new sum rules can be obtained by studying null-integrated operators that are not integrated along their polarizations. Equivalently, we can consider taking derivatives of the light transformed operators. Schematically, we would like to find a differential operator $\mathcal{D}$ such that

$$\mathcal{D}_{x,z_1,z_3}\mathbf{L}[\mathcal{O}_1](x,z_1)\mathbf{L}[\mathcal{O}_3](x,z_3) \tag{4.202}$$

is a conformal primary at x. These conformally-invariant differential operators have interesting connections to reducible representations of the conformal group [35]. We hope to address this problem in future.

In [13], it was shown that superconvergence sum rules also exist at $J = -2, -4, \ldots$. In these sum rules, one integrates a commutator against derivatives of delta functions, which again constrains the two operators in the commutator to be spacelike separated. It would be interesting to find spinning versions of these sum rules.

In flat space, unitarity additionally imposes an upper bound on the spectral density. This condition has been used in the context of EFT amplitudes through primal bootstrap methods [140, 141]. However, it is not clear what the corresponding CFT statement is. To understand how this upper bound is realized in CFT, our dictionary will be an important ingredient.

Crossing-symmetric dispersion relations in flat space [107, 142] have also proven valuable in studying dispersive bounds on EFTs. They have the advantage of making low-energy crossing almost trivial to impose, and are more well-behaved when including loop EFT corrections [143]. Analogous CFT dispersion relations have been established in Mellin space [144], and crossing-symmetric dispersion relations in position space were recently studied in [145]. We expect that position space methods will be more straightforward to generalize to spinning case.

Including loop EFT corrections often makes the forward limit in flat space divergent [103, 146]. Consequently, one has to consider a limited set of sum rules that give weaker positivity bounds. On the other hand, loop corrections in AdS are automatically regularized by R_{AdS}. By computing the loop contributions from summing over double-trace exchanged operators [147, 148, 149], one should be able to obtain finite loop corrections to the tree-level AdS bounds.

Acknowledgements

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Chapter 5

SCALAR MODULAR BOOTSTRAP AND ZEROS OF THE RIEMANN ZETA FUNCTION

This chapter is based on

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5.1 Introduction

The conformal bootstrap is a powerful program used to highly constrain quantum field theories starting from basic consistency conditions. In two dimensional conformal field theory (CFT), one avatar of this program is the so-called modular bootstrap which uses modular invariance of the genus one partition function to constrain possible allowed spectra of 2d CFTs. This program started with the work of [150] and has led to many interesting results (see e.g. [151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162] for a non-exhaustive list). This has several applications, including constraining theories of quantum gravity in AdS_3.

In many (but not all) cases, the *spinless* bootstrap equations are studied, in which one throws away information about the spin of the original operators and only looks at their energies. This is done by grading the partition function only by the energies of the operators, and using S-invariance, rather than the full $SL(2,\mathbb{Z})$-invariance of the partition function. In particular, we have

$$Z(y) := \sum_{\mathcal{O}} e^{-2\pi y(\Delta_{\mathcal{O}}-\frac{c}{12})} = Z(y^{-1}), \tag{5.1}$$

where the sum over $\mathcal{O}$ is a sum over all local operators in the theory, and $\Delta_{\mathcal{O}}$ is the scaling dimension of operator $\mathcal{O}$. Any bound derived from (5.1) will by definition be insensitive to the spins of the operators $\mathcal{O}$. For example, the current strongest bound on the lightest nontrivial Virasoro primary operator at large central charge c is in [156], which showed at large c,

$$\Delta_{\text{gap}}^{\text{Virasoro}} \lesssim \frac{c}{9.1}. \tag{5.2}$$

However it makes no claim on what the spin of that operator is, or what the lightest spin j operator is. A similar result using the spinless bootstrap was found for a simpler class of theories, those with a $U(1)^c$ chiral algebra, in [162]

$$\Delta_{\text{gap}}^{U(1)^c} \lesssim \frac{c}{9.869}. \tag{5.3}$$

In this paper we derive a novel one-dimensional crossing equation using the technology of harmonic analysis. In the case of CFTs with $U(1)^c$ symmetry, this crossing equation acts *only* on the scalar primary operators of the theory (with respect to the $U(1)^c$ chiral algebra). This allows us to place new bounds on the scalar gap of all $U(1)^c$ conformal field theories for any integer c. This is more refined information than the bound in e.g. (5.3) since it provides explicit information about the spin of the operator. Indeed the scalar gap is a natural object to consider. Scalar operators can be added to the Lagrangian while still preserving Lorentz invariance. The scalar gap is then related to questions about, for instance, if the CFT has a relevant operator or not. Another application is in the study of boundary conformal field theory. There, the bulk scalars show up in some crossing equations rather than all bulk operators, which can lead to interesting bounds that are conditional on the scalar gap [163].

Remarkably, our crossing equation has an intimate relation with the nontrivial zeros of the Riemann zeta function. In a sense which we will explain, hidden inside the scalar operators of any 2d CFT with $U(1)^c$ symmetry are the nontrivial zeros of the zeta function. As a result, we can rephrase the Riemann hypothesis as a statement about the behavior of scalar operators of any $U(1)^c$ CFT.

We also discuss a generalization to Virasoro CFTs. We derive a more complicated one-dimensional crossing equation that involves operators of all spins. The nontrivial zeros of the zeta function again play an important role. This leads to the Riemann hypothesis being equivalent to a more complicated statement about the asymptotic density of a signed count of all operators (of any spin) in any CFT. Unfortunately we run into some technical obstacles in bounding physical quantities such as the scalar gap for Virasoro CFTs.

This paper is organized as follows. In Section 5.2 we review harmonic analysis on the fundamental domain of $SL(2,\mathbb{Z})$, which will play an important role in deriving our scalar crossing equation. In Section 5.3 we apply this to the study of $U(1)^c$ CFTs and derive the scalar crossing equation. We present the numerical results for the scalar gap of $U(1)^c$ theories for various values of c. In Section 5.4 we discuss generalizations to theories with only Virasoro symmetry. In Section 5.5 we study more explicitly the connections between 2d CFTs and the Riemann hypothesis. We discuss various potentially interesting future directions in Section 5.6. Some detailed calculations and derivations are banished to the appendices.

5.2 Review of Harmonic Analysis

In this section we will review harmonic analysis on the space $\mathbb{H}/SL(2,\mathbb{Z})$, where $\mathbb{H}$ is the upper half plane. For much of this discussion, we refer to [164]. We will use the notation of [165] in this section.

The main idea is to decompose square-integrable modular invariant functions into eigenfunctions of the Laplacian on the space $\mathbb{H}/SL(2,\mathbb{Z})$. If $\tau \in \mathbb{H}$, with real and imaginary parts x, y respectively, then there is a natural metric on $\mathbb{H}$ given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{5.4}$$

The Laplacian on this space is given by

$$\Delta = -y^2(\partial_x^2 + \partial_y^2). \tag{5.5}$$

Square-integrable modular-invariant functions $f(\tau)$ are those with finite L^2 norm under the measure (5.4), meaning

$$\int_{-1/2}^{1/2} dx \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} |f(\tau)|^2 < \infty. \tag{5.6}$$

If $f(\tau)$ is a square-integrable, modular-invariant function, it has a unique decomposition into eigenfunctions of the Laplacian (5.5). These eigenfunctions have been classified and they come in three types:

- The constant function 1, with eigenvalue 0.
- An infinite, continuous family of eigenfunctions known as real analytic Eisenstein series, $E_s(\tau)$, with $s = \frac{1}{2} + it$, t real, with eigenvalue $\frac{1}{4} + t^2$. Any real t is permissible.
- An infinite, discrete family of eigenfunctions known as Maass cusp forms, denoted $\nu_n^{\pm}(\tau)$, $n = 1, 2, \cdots$. These have sporadic eigenvalues, which we denote $\frac{1}{4} + (R_n^{\pm})^2$, for $R_n^{\pm}$ a positive real number. Both ν_n^+ and ν_n^- are ordered in increasing eigenvalue, i.e. $R_1^+ < R_2^+ < \cdots$, and likewise for R_n^-. The superscript $\pm$ refers to whether the cusp form is even or odd under parity.

The decomposition of $f(\tau)$ is then given by:

$$f(\tau) = \frac{(f,1)}{(1,1)} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds E_s(\tau)(f, E_s) + \sum_{n=1}^{\infty}\sum_{\epsilon=\pm} \nu_n^\epsilon(\tau)\frac{(f,\nu_n^\epsilon)}{(\nu_n^\epsilon,\nu_n^\epsilon)}, \qquad (5.7)$$

where the overlap function is given by the Petersson inner product:

$$(f,g) \coloneqq \int_{-1/2}^{1/2} dx \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} f(\tau)\overline{g(\tau)}. \qquad (5.8)$$

The decomposition (5.7) is known as the Roelcke-Selberg decomposition.

Let us be more explicit about the eigenfunctions of the Laplacian. The real analytic Eisenstein series $E_s(\tau)$, $s \in \mathbb{C}$ are defined as a modular sum of y^s:

$$E_s(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} y^s|_\gamma, \qquad (5.9)$$

where Γ_∞ is the subgroup of $SL(2,\mathbb{Z})$ generated by $\tau \to \tau+1$. The sum (5.9) converges if $\mathrm{Re}(s) > 1$. However, it admits an analytic continuation everywhere in the s plane:

$$E_s(\tau) = y^s + \frac{\Lambda(1-s)}{\Lambda(s)} y^{1-s} + \sum_{j=1}^{\infty} \frac{4\sigma_{2s-1}(j)\sqrt{y}K_{s-\frac{1}{2}}(2\pi j y)}{\Lambda(s) j^{s-\frac{1}{2}}}\cos(2\pi j x), \qquad (5.10)$$

where $\sigma_{2s-1}(j)$ is the divisor sigma function, K is the modified Bessel function of second kind, and Λ is defined as

$$\Lambda(s) \coloneqq \pi^{-s}\zeta(2s)\Gamma(s). \qquad (5.11)$$

The function $\Lambda(s)$ obeys a useful identity:

$$\Lambda(s) = \Lambda(\tfrac{1}{2} - s). \qquad (5.12)$$

From (5.10) we also see that the real analytic Eisenstein series obey a useful identity:

$$\Lambda(s)E_s(\tau) = \Lambda(1-s)E_{1-s}(\tau). \qquad (5.13)$$

The remaining eigenfunctions, the Maass cusp forms, are more mysterious. They take the following functional form:

$$\begin{aligned} \nu_n^+(\tau) &= \sum_{j=1}^{\infty} a_j^{(n,+)}\sqrt{y}K_{iR_n^+}(2\pi j y)\cos(2\pi j x) \\ \nu_n^-(\tau) &= \sum_{j=1}^{\infty} a_j^{(n,-)}\sqrt{y}K_{iR_n^-}(2\pi j y)\sin(2\pi j x), \end{aligned} \qquad (5.14)$$

where $R_n^\pm$ and $a_j^{(n,\pm)}$ are a set of sporadic real numbers. For example, we have the following first few values of $R_n^\pm$:

$$\begin{aligned} R_1^+ &\approx 13.77975, \quad R_1^- \approx 9.53370 \\ R_2^+ &\approx 17.73856, \quad R_2^- \approx 12.17301 \\ R_3^+ &\approx 19.42348, \quad R_3^- \approx 14.35851. \end{aligned} \tag{5.15}$$

For more numerical data on the Maass cusp forms, see the online database [LMFDB]. One key feature the Maass cusp forms have is, unlike the real analytic Eisenstein series, they all lack a scalar piece:

$$\int_{-1/2}^{1/2} dx \nu_n^\pm(\tau) = 0. \tag{5.16}$$

5.3 $U(1)^c$ CFTs

We begin with studying a family of particularly simple conformal field theories, with an extended current algebra of $U(1)^c$. Examples of such CFTs include Narain's family of c free bosons compactified on a c-dimensional lattice, parameterized by the moduli space $O(c,c,\mathbb{Z})\backslash O(c,c)/O(c)\times O(c)$. It is believed that this family of CFTs fully classifies all theories with $U(1)^c$ current algebra. However, this has not been proven. Our results in this section will apply to all theories with $U(1)^c$ symmetry; we do not need to assume the theory is a Narain CFT.

5.3.1 Harmonic decomposition

In [165], the harmonic decomposition of $U(1)^c$ CFT partition functions were calculated, which we review here. The characters of the $U(1)^c$ chiral algebra are given by

$$\chi^h(\tau) = \frac{q^h}{\eta(\tau)^c}, \tag{5.17}$$

where $\eta(\tau)$ is the Dedekind eta function. Instead of decomposing the full partition function $Z(\tau)$, we instead consider the primary-counting partition function

$$\begin{aligned} \widehat{Z}^c(\tau,\mu) &:= y^{c/2}|\eta(\tau)|^{2c}Z(\tau) \\ &= y^{c/2}\sum_{h,\overline{h}} q^h\overline{q}^{\overline{h}}, \end{aligned} \tag{5.18}$$

where in (5.18) the sum over $h,\overline{h}$ goes over the $U(1)^c$ primary operators. In (5.18), we write $\widehat{Z}^c(\tau,\mu)$ to emphasize that the (reduced) partition function depends not only on the worldsheet modulus τ, but also on an abstract target space coordinate μ.[1]

The function (5.18) is not yet square-integrable, but once we subtract out the Eisenstein series $E_{c/2}(\tau)$ (defined in (5.10)), this yields a square-integrable function that admits a unique spectral decomposition[2]. In [168, 169, 165] the spectral decomposition was given as follows[3]:

$$\widehat{Z}^c(\tau,\mu) = E_{c/2}(\tau) + 3\pi^{-\frac{c}{2}}\Gamma\left(\frac{c}{2}-1\right)\mathcal{E}^c_{\frac{c}{2}-1}(\mu) + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds\pi^{s-\frac{c}{2}}\Gamma\left(\frac{c}{2}-s\right)\mathcal{E}^c_{\frac{c}{2}-s}(\mu)E_s(\tau)$$
$$+\sum_{n=1}^{\infty}\sum_{\epsilon=\pm}\frac{(\widehat{Z}^c,\nu_n^\epsilon)(\mu)}{(\nu_n^\epsilon,\nu_n^\epsilon)}\nu_n^\epsilon(\tau). \tag{5.19}$$

The coefficients $\mathcal{E}^c_s(\mu)$ were called constrained Epstein zeta series in [169], and are defined as:

$$\mathcal{E}^c_s(\mu) \coloneqq \sum_{\Delta\in\mathcal{S}}(2\Delta)^{-s}, \tag{5.20}$$

where we define the set $\mathcal{S}$ to be the dimensions of all non-vacuum scalar primary operators under the $U(1)^c$ chiral algebra (with multiplicity). This sum converges for $\text{Re}(s) > c-1$, but like for the $SL(2,\mathbb{Z})$ Eisenstein series (5.9), they admit an analytic continuation everywhere in the complex s plane. They also obey a functional equation:

$$\mathcal{E}^c_{\frac{c}{2}-s}(\mu) = \frac{\Gamma(s)\Gamma(s+\frac{c}{2}-1)\zeta(2s)}{\pi^{2s-\frac{1}{2}}\Gamma(\frac{c}{2}-s)\Gamma(s-\frac{1}{2})\zeta(2s-1)}\mathcal{E}^c_{\frac{c}{2}+s-1}(\mu). \tag{5.21}$$

This equation is inherited from the functional equation that the Eisenstein series obey (5.13), combined with the definition of $\mathcal{E}^c_s(\mu)$ as an overlap of $\widehat{Z}^c(\tau,\mu)$ with the Eisenstein series:

$$(\widehat{Z}^c - E_{\frac{c}{2}}, E_s) = \pi^{s-\frac{c}{2}}\Gamma\left(\frac{c}{2}-s\right)\mathcal{E}^c_{\frac{c}{2}-s}(\mu). \tag{5.22}$$

---

[1]For Narain theories, we can view μ as a parameter $\mu \in O(c,c;\mathbb{Z})\backslash O(c,c)/O(c)\times O(c)$. The target space of Narain theories is parametrized by a symmetric metric G_{ab} and an antisymmetric B-field B_{ab}, where a,b indices run from $1,2,\cdots,c$. Here, however, we can just view μ as some abstract coordinate.

[2]For Narain CFTs, $E_{c/2}(\tau)$ has the interpretation of the averaged partition function [166, 167].

[3]Note that due to the pole structure of $\Lambda(s)$ and the real analytic Eisenstein series $E_s(\tau)$, the decompositions of $c=1$ and $c=2$ are slightly different than other c, so we will assume $c\neq 1,2$ for the rest of this section. We revisit $c=1$ and $c=2$ in Appendix D.3.

For Narain CFTs, (5.20) can be rewritten as

$$\mathcal{E}_s^c(\mu) = \sum_{\vec{n},\vec{m}\in\mathbb{Z}^c}' \frac{\delta_{\vec{n}\cdot\vec{w},0}}{M_{\vec{n},\vec{w}}(\mu)^{2s}}, \tag{5.23}$$

with

$$M_{\vec{n},\vec{w}}(\mu)^2 \coloneqq G^{ab}(n_a + B_{ac}w^c)(n_b + B_{bd}w^d) + G_{cd}w^c w^d, \tag{5.24}$$

and the prime over the summation indicating we should not sum over the vacuum state (with $\vec{n} = \vec{w} = \vec{0}$).

5.3.2 Crossing equation

Since the Maass cusp forms have no scalar piece (i.e. (5.16)), the scalar part of (5.19) is particularly simple:

$$\begin{aligned}\int_{-1/2}^{1/2} dx \widehat{Z}^c(\tau,\mu) = y^{\frac{c}{2}} + \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} y^{1-\frac{c}{2}} + 3\pi^{-\frac{c}{2}}\Gamma\left(\frac{c}{2}-1\right)\mathcal{E}^c_{\frac{c}{2}-1}(\mu) \\ + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \pi^{s-\frac{c}{2}}\Gamma\left(\frac{c}{2}-s\right)\mathcal{E}^c_{\frac{c}{2}-s}(\mu)\left(y^s + \frac{\Lambda(1-s)}{\Lambda(s)}y^{1-s}\right),\end{aligned} \tag{5.25}$$

where as usual $\tau = x + iy$, and $\Lambda(s)$ is defined as in (5.11).

As a reminder, the set $\mathcal{S}$ is the set of conformal weights of all non-vacuum scalar primaries under the $U(1)^c$ chiral algebra (with multiplicity). We can rewrite the LHS of (5.25) as

$$\int_{-1/2}^{1/2} dx \widehat{Z}^c(\tau,\mu) = y^{\frac{c}{2}}\left(1 + \sum_{\Delta\in\mathcal{S}} e^{-2\pi\Delta y}\right). \tag{5.26}$$

This gives

$$\sum_{\Delta\in\mathcal{S}} e^{-2\pi\Delta y} = \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} y^{1-c} + \varepsilon_c(\mu) y^{-\frac{c}{2}} + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \pi^{s-\frac{c}{2}}\Gamma\left(\frac{c}{2}-s\right)\mathcal{E}^c_{\frac{c}{2}-s}(\mu) y^{s-\frac{c}{2}}, \tag{5.27}$$

where we have defined $\varepsilon_c(\mu) \coloneqq 3\pi^{-\frac{c}{2}}\Gamma\left(\frac{c}{2}-1\right)\mathcal{E}^c_{\frac{c}{2}-1}(\mu)$, and used the symmetry between $s \leftrightarrow 1-s$ in the integral over s.

The remaining task is to do the integral in (5.27). We will do the integral over s by moving the contour to the right of $s = \frac{c}{2}$. It turns out the only poles we enclose after moving the contour are at $s = \frac{c}{2}, \frac{1+z_n}{2}, \frac{1+z_n^*}{2}$, where z_n are the nontrivial zeros of the Riemann zeta function with positive imaginary part (i.e. $z_1 \approx \frac{1}{2} + 14.135i$, $z_2 \approx \frac{1}{2} + 21.022i$, etc.). See Fig. 5.1 for a picture of the pole structure (shown for $c = 3$). We derive the pole structure in Appendix D.1. After moving the contour, (5.27) becomes

$$1 + \sum_{\Delta\in\mathcal{S}} e^{-2\pi\Delta y} = \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} y^{1-c} + \varepsilon_c(\mu) y^{-\frac{c}{2}} + \sum_{k=1}^{\infty} \mathrm{Re}\left(\delta_{k,c}(\mu) y^{-\frac{c}{2}+1-\frac{z_k}{2}}\right) + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds \pi^{s-\frac{c}{2}} \Gamma\left(\frac{c}{2}-s\right) \mathcal{E}^c_{\frac{c}{2}-s}(\mu) y^{s-\frac{c}{2}}, \tag{5.28}$$

where $\gamma > \frac{c}{2}$. The terms $\varepsilon_c(\mu)$ and $\delta_{k,c}(\mu)$ are moduli-dependent constants, which have an explicit formula as

$$\begin{aligned} \varepsilon_c(\mu) &= \frac{3}{\pi}\int_{\mathcal{F}} \frac{dxdy}{y^2}(\widehat{Z}^c(\tau,\mu) - E_{c/2}(\tau)) \\ \delta_{k,c}(\mu) &= \int_{\mathcal{F}} \frac{dxdy}{y^2}(\widehat{Z}^c(\tau,\mu) - E_{c/2}(\tau)) \mathrm{Res}_{s=z_k/2} E_s(\tau), \end{aligned} \tag{5.29}$$

where

$$\mathrm{Res}_{s=z_k/2} E_s(\tau) = \frac{\sqrt{\pi}\zeta(z_k-1)\Gamma(\frac{z_k-1}{2})}{2\zeta'(z_k)\Gamma(\frac{z_k}{2})} y^{1-\frac{z_k}{2}} + \sum_{j=1}^{\infty} \frac{2\pi^{\frac{z_k}{2}} \cos(2\pi jx)\sigma_{z_k-1}(j)\sqrt{y}K_{\frac{z_k-1}{2}}(2\pi jy)}{j^{\frac{z_k-1}{2}}\zeta'(z_k)\Gamma(\frac{z_k}{2})}. \tag{5.30}$$

Now let us consider the integral in (5.28). We first rewrite the integral using the functional identity (5.21):

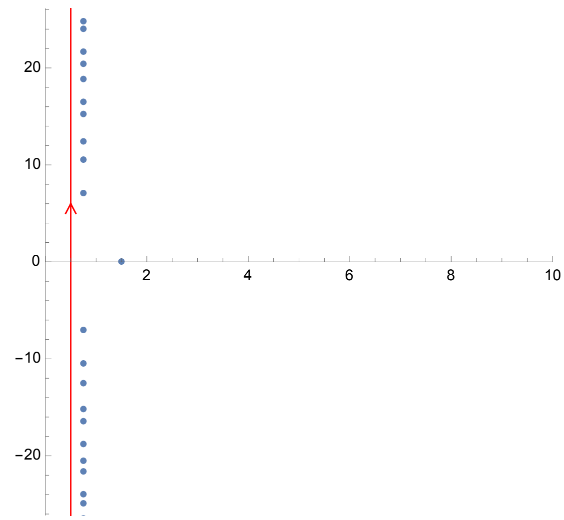
$$\int_{\gamma-i\infty}^{\gamma+i\infty} ds \pi^{s-\frac{c}{2}} \Gamma\left(\frac{c}{2}-s\right) \mathcal{E}^c_{\frac{c}{2}-s}(\mu) y^{s-\frac{c}{2}} = \int_{\gamma-i\infty}^{\gamma+i\infty} ds \frac{\Gamma(s)\Gamma(s+\frac{c}{2}-1)\zeta(2s)}{\pi^{s+\frac{c-1}{2}}\Gamma(s-\frac{1}{2})\zeta(2s-1)} \mathcal{E}^c_{\frac{c}{2}+s-1}(\mu) y^{s-\frac{c}{2}}. \tag{5.31}$$

Because we take $\gamma > \frac{c}{2}$, this means that $\mathrm{Re}(\frac{c}{2}+s-1) > c-1$, which means we can write this as the following convergent sum:

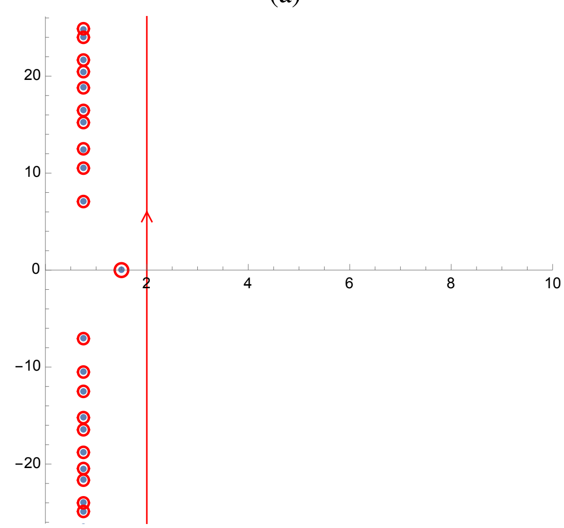
$$\mathcal{E}^c_{\frac{c}{2}+s-1}(\mu) = \sum_{\Delta\in\mathcal{S}} (2\Delta)^{-\frac{c}{2}-s+1}. \tag{5.32}$$

Moreover we will expand the ratio of zeta functions

$$\frac{\zeta(2s)}{\zeta(2s-1)} = \sum_{n=1}^{\infty} b(n) n^{-2s}, \tag{5.33}$$



(a)



(b)

Figure 5.1: (a) Pole structure of the integral in (5.27) in the complex s plane. The poles are located at $s = \frac{c}{2}, \frac{1+z_n}{2}, \frac{1+z_n^*}{2}$ (shown here for $c = 3$), where z_n are the nontrivial zeros of the Riemann zeta function with positive imaginary part. If the Riemann hypothesis is true, the tower of poles in the figure all occur at real part $\frac{3}{4}$, except for the pole at $s = \frac{c}{2}$. (b) Contour deformation of the integral to $\mathrm{Re}(s) > \frac{c}{2}$.

where $b(n)$ is a number-theoretic function defined as

$$b(n) \coloneqq \sum_{k|n} k\mu(k). \tag{5.34}$$

where $\mu(n)$ is the Mobius function:

$$\mu(n) \coloneqq \begin{cases} (-1)^{\text{number of prime factors of } n} & n \text{ is square-free} \\ 0 & n \text{ is divisible by a prime squared.} \end{cases} \tag{5.35}$$

We can then rewrite (5.31) as

$$\begin{aligned} &\int_{\gamma-i\infty}^{\gamma+i\infty} ds \pi^{s-\frac{c}{2}} \Gamma\left(\frac{c}{2}-s\right) \mathcal{E}^{c}_{\frac{c}{2}-s}(\mu) y^{s-\frac{c}{2}} \\ &\qquad = \sum_{\Delta\in\mathcal{S}} \sum_{n=1}^{\infty} b(n) \int_{\gamma-i\infty}^{\gamma+i\infty} ds \frac{\Gamma(s)\Gamma(s+\frac{c}{2}-1)}{\pi^{s+\frac{c-1}{2}}\Gamma(s-\frac{1}{2})} (2\Delta)^{-\frac{c}{2}-s+1} y^{s-\frac{c}{2}} n^{-2s}. \end{aligned} \tag{5.36}$$

The integral in (5.36) is related to a confluent hypergeometric function of the second kind (see 13.4.18 of [170]), which we denote as U (and is given by `HypergeometricU` in Mathematica):

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} ds \frac{\Gamma(s)\Gamma(s+\frac{c}{2}-1)}{\pi^{s+\frac{c-1}{2}}\Gamma(s-\frac{1}{2})} (2\Delta)^{-\frac{c}{2}-s+1} y^{s-\frac{c}{2}} n^{-2s} = \frac{y^{1-c}}{\sqrt{\pi}} n^{c-2} U\left(-\frac{1}{2}, \frac{c}{2}, \frac{2\pi n^2\Delta}{y}\right) e^{-\frac{2\pi n^2\Delta}{y}}. \tag{5.37}$$

Thus we get a final crossing equation of:

$$\boxed{\begin{aligned} 1 + \sum_{\Delta\in\mathcal{S}} e^{-2\pi\Delta y} &= \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} y^{1-c} + \varepsilon_c(\mu) y^{-\frac{c}{2}} + \sum_{k=1}^{\infty} \mathrm{Re}\left(\delta_{k,c}(\mu) y^{-\frac{c}{2}+1-\frac{z_k}{2}}\right) \\ &\quad + \frac{y^{1-c}}{\sqrt{\pi}} \sum_{\Delta\in\mathcal{S}} \sum_{n=1}^{\infty} b(n) n^{c-2} U\left(-\frac{1}{2}, \frac{c}{2}, \frac{2\pi n^2\Delta}{y}\right) e^{-\frac{2\pi n^2\Delta}{y}}. \end{aligned}} \tag{5.38}$$

In addition to a rigorous derivation we have also numerically checked (5.38) for various values of c, y to a precision of 1 part in 10^{70}.

Another consistency check of (5.38) one can perform analytically is to consider the large y limit. In this limit, the LHS is dominated by 1 from the identity, but each term on the RHS is perturbatively small at large y. Similar to the lightcone bootstrap of four-point functions [7, 8], it turns out that the leading term on the LHS is reproduced by the infinite sum over Δ in the RHS. More precisely, one can show that

$$\frac{y^{1-c}}{\sqrt{\pi}}\sum_{n=0}^{\infty} b(n)n^{c-2}\int_0^{\infty} d\Delta \frac{2\pi^c\zeta(c-1)\Delta^{c-2}}{\zeta(c)\Gamma(\frac{c}{2})^2} U\left(-\frac{1}{2},\frac{c}{2},\frac{2\pi n^2\Delta}{y}\right) e^{-\frac{2\pi n^2\Delta}{y}} = 1 \tag{5.39}$$

where $\frac{2\pi^c\zeta(c-1)\Delta^{c-2}}{\zeta(c)\Gamma(\frac{c}{2})^2}$ is the leading large Δ behavior of the spectral density (and which is the average spectral density for Narain theories; see [166, 167]). It might also be interesting to understand how the perturbatively small terms at large y on the RHS of (5.38) cancel among each other to give the non-perturbatively small corrections on the LHS.

5.3.3 Functionals

We would now like to apply linear functionals to (5.38) to obtain sum rules that can constrain the possible sets $\mathcal{S}$. In particular we would like to put a bound on the scalar gap, meaning the lightest operator present in $\mathcal{S}$. One immediate problem is that not every term in (5.38) is sign-definite. The term $\varepsilon_c(\mu)$ is not sign-definite, and the infinite terms $\delta_{k,c}(\mu)$ are also not sign-definite for any k. To remove the $\varepsilon_c(\mu)$ term is straightforward. Let us start by rewriting (5.38) as:

$$\sum_{\Delta\in\mathcal{S}}\left[y^{\frac{c}{2}}e^{-2\pi\Delta y} - \frac{y^{1-\frac{c}{2}}}{\sqrt{\pi}}\sum_{n=1}^{\infty} b(n)n^{c-2}U\left(-\frac{1}{2},\frac{c}{2},\frac{2\pi n^2\Delta}{y}\right)e^{-\frac{2\pi n^2\Delta}{y}}\right]$$
$$= -y^{\frac{c}{2}} + \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)}y^{1-\frac{c}{2}} + \varepsilon_c(\mu) + \sum_{k=1}^{\infty}\mathrm{Re}\left(\delta_{k,c}(\mu)y^{1-\frac{z_k}{2}}\right). \tag{5.40}$$

Taking a derivative with respect to y removes the $\varepsilon_c(\mu)$ term. If we then redefine $t^2 := y^{-1}$ we get:

$$\sum_{\Delta\in\mathcal{S}}\Bigg[t^{-c}(4\pi\Delta - ct^2)e^{-\frac{2\pi\Delta}{t^2}} - \frac{t^c}{\sqrt{\pi}}\sum_{n=1}^{\infty} b(n)n^{c-2}e^{-2\pi\Delta n^2t^2}\times$$
$$\left((c-2-4\pi n^2t^2\Delta)U\left(-\frac{1}{2},\frac{c}{2},2\pi n^2t^2\Delta\right) + 2\pi n^2\Delta t^2 U\left(\frac{1}{2},\frac{c}{2}+1,2\pi n^2t^2\Delta\right)\right)\Bigg]$$
$$= ct^{2-c} + \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)}(c-2)t^c + \sum_{k=1}^{\infty}\mathrm{Re}\left(\delta_{k,c}(\mu)(z_k-2)t^{z_k}\right). \tag{5.41}$$

Now we need a functional acting on (5.41) to remove terms of the form t^{z_k} where z_k is a nontrivial zero of the Riemann zeta function. To accomplish this we use the following family of functionals[4].

Consider an even function $\varphi(t)$ that satisfies the following properties:

- $\varphi(t)$ and $\widehat{\varphi}(t)$ both decay rapidly (faster than any polynomial) at infinity
- $\varphi(t)$ and $\widehat{\varphi}(t)$ have no singularities at finite t
- $\varphi(0) = \widehat{\varphi}(0) = 0$
- $\int_0^\infty \frac{dt}{t}\varphi(t)t^s$ admits an analytic continuation to all $s \in \mathbb{C}$ (which we will call $M_\varphi(s)$),

where $\widehat{\varphi}$ is the Fourier transform of φ:

$$\widehat{\varphi}(p) \coloneqq \int_{-\infty}^{\infty} dx\, e^{-2\pi i p x}\varphi(x). \tag{5.42}$$

We define

$$\Phi(t) \coloneqq \sum_{n=1}^{\infty} \varphi(nt). \tag{5.43}$$

The function $\Phi(t)$ can also be rewritten via the Poisson resummation formula as

$$\begin{aligned}\Phi(t) &= -\frac{1}{2}\varphi(0) + \frac{1}{2t}\widehat{\varphi}(0) + \frac{1}{t}\sum_{n=1}^{\infty}\widehat{\varphi}\left(\frac{n}{t}\right)\\ &= \frac{1}{t}\sum_{n=1}^{\infty}\widehat{\varphi}\left(\frac{n}{t}\right).\end{aligned} \tag{5.44}$$

Combining (5.43) and (5.44) and the properties listed above, we see that $\Phi(t)$ decays faster than any polynomial at both small t and large t.

Now, we define a functional $\mathcal{F}^\varphi[h(t)]$ by

$$\mathcal{F}^\varphi[h(t)] \coloneqq \int_0^\infty \frac{dt}{t} h(t)\Phi(t). \tag{5.45}$$

Let us first consider the action of the functional on a power of t:

$$\mathcal{F}^\varphi[t^s] = \int_0^\infty \frac{dt}{t} t^s \Phi(t). \tag{5.46}$$

[4] We are extremely grateful to Danylo Radchenko for explaining this strategy to us. See [171] for further generalizations of this. The construction of the functionals in [171] seems to be reminiscent of the analytic functionals in [121]. It might be interesting to explore the connection further.

Because of the properties of $\Phi(t)$ discussed above, $\mathcal{F}^{\varphi}[t^s]$ is an analytic function on the entire complex s plane. Moreover, for $\mathrm{Re}(s) > 1$, we can exchange the integration and the summation, which gives

$$\begin{aligned}\mathcal{F}^{\varphi}[t^s] &= \int_0^\infty dt\, t^{s-1}\Phi(t)\\ &= \int_0^\infty dt\, t^{s-1}\sum_{n=1}^\infty \varphi(nt)\\ &= \sum_{n=1}^\infty n^{-s}\int_0^\infty dt\, t^{s-1}\varphi(t)\\ &= \zeta(s)\int_0^\infty dt\, t^{s-1}\varphi(t), \qquad \mathrm{Re}(s) > 1. \end{aligned} \tag{5.47}$$

Properties of analytic continuation then imply that for all $s \in \mathbb{C}$,

$$\mathcal{F}^{\varphi}[t^s] = \zeta(s)M_{\varphi}(s). \tag{5.48}$$

From (5.48) we see that the functional $\mathcal{F}^{\varphi}$ will remove the final sign-indefinite terms $\delta_{k,c}t^{z_k}$ in our crossing equation (5.41). We can then apply the functional $\mathcal{F}^{\varphi}$ to (5.41) to get a positive sum rule the scalar operators must satisfy. Let us consider the situation where $\varphi(t)$ is a (finite) linear combination of Gaussians, for which $M_{\varphi}(s)$ is a sum of Gamma functions. In particular we consider the following family of $\varphi(t)$ defining the functionals:

$$\varphi(t) = \sum_{i=1}^N \alpha_i e^{-\pi k_i t^2}, \tag{5.49}$$

where k_i, α_i are an arbitrary set of N real numbers. In order for $\varphi(t)$ to satisfy $\varphi(0) = \widehat{\varphi}(0) = 0$, we choose k_i, α_i subject to the constraints

$$\begin{aligned}\sum_{i=1}^N \alpha_i &= 0,\\ \sum_{i=1}^N \alpha_i k_i^{-1/2} &= 0.\end{aligned} \tag{5.50}$$

With this definition of φ, we can define Φ and the action of the functional $\mathcal{F}$ by using (5.43) and (5.45). If we then apply this functional to our crossing equation (5.41), we get a positive sum rule for the operators Δ. In Appendix D.2, we write down explicit formulas for the action of this functional on (5.41) with a single Gaussian $\varphi(t) = e^{-\pi k t^2}$ as a function of Δ and k.

Although in principle we could choose any functional via (5.49) obeying (5.50), for numerical calculations it will be more convenient to use functionals consisting of derivatives with respect to k, evaluated at $k = 1$ instead. To be more explicit, the sum rule we get after applying the functional from (5.49) is given by

$$\sum_{i=1}^{N} \alpha_i \mathrm{vac}(k_i) + \sum_{i=1}^{N} \sum_{\Delta} \alpha_i f(k_i, \Delta) = 0, \tag{5.51}$$

subject to the constraints (5.50). $f(k, \Delta)$ and $-\mathrm{vac}(k)$ are the actions of the functional on the LHS and RHS respectively of (5.41) (with explicit formulas given in Appendix D.2, see e.g. (D.7)). Let us consider the action of a single Gaussian of width k (i.e not yet obeying the constraints above):

$$\mathrm{vac}(k) + \sum_{\Delta} f(k, \Delta). \tag{5.52}$$

The expression (5.52) is not equal to 0 because we have not obeyed the constraints (5.50). However, the only functions of k that it can be equal to are a constant term and a term proportional to $k^{-1/2}$. Any other term would allow some combination of functionals obeying (5.50) to not vanish, and thus contradict (5.51). Therefore we have

$$\mathrm{vac}(k) + \sum_{\Delta} f(k, \Delta) = c_0 + c_1 k^{-1/2}, \tag{5.53}$$

where c_0, c_1 are k-independent constants (they could be theory-dependent however). From an explicit calculation of $\mathrm{vac}(k)$ and $f(k, \Delta)$ in Appendix D.2, we see that

$$\mathrm{vac}'(1) = \partial_k f(k, \Delta)|_{k=1} = 0, \tag{5.54}$$

which implies $c_1 = 0$.[5] Thus we have

$$\mathrm{vac}^{(n)}(1) + \sum_{\Delta} (\partial_k)^n f(k, \Delta)|_{k=1} = 0, \quad n \geq 2 \tag{5.55}$$

which will be the basis for our functionals. (Only even values of n will provide independent equations, however.)

Notice that

$$\mathrm{vac}^{(n)}(1) = \lim_{\Delta \to 0} (\partial_k)^n f(k, \Delta)|_{k=1}, \quad n \geq 2 \tag{5.56}$$

so indeed the vac term in (5.55) is precisely the contribution of the vacuum ($\Delta = 0$) to the sum rule (and the same is true in (5.51)).

[5] In fact it turns out that c_0 is related to $\varepsilon_c(\mu)$ (defined in (5.29)) via $c_0 = \frac{\pi \varepsilon_c(\mu)}{6}$. This in principle leads to a stronger crossing equation but we find that numerically it gives very similar bounds on the scalar gap, so we will not explore it further in this paper.

5.3.4 Numerical results

In this section, we present the numerical results for bounds on the scalar gap of $U(1)^c$ CFTs for various values of c obtained from using the basis of functionals (5.55). Note that the hypergeometric function in (5.38) for odd values of c reduces to an elementary function, which greatly simplifies the technical calculations. We therefore focus on odd values of c (although there is nothing in principle stopping the following from working for even c). We first consider the functional obtained from taking 2 and 4 derivatives of (5.55), and obtain a bound on $\Delta_{\text{scalar gap}}$ from these two sum rules, following the approach in [150]. Since we take at most 4 derivatives, we denote this bound as $\Delta^{(4)}_{\text{scalar gap}}$ (and more generally define a bound from at most n derivatives as $\Delta^{(n)}_{\text{scalar gap}}$). Note that $\Delta^{(n)}_{\text{scalar gap}}$ is obtained from $\frac{n}{2}$ functionals.

We have computed $\Delta^{(4)}_{\text{scalar gap}}$ for odd central charge up to 251.[6] The results are plotted in Fig. 5.2. The bound at large c numerically appears to grow quadratically with c. Fitting it to a quadratic function gives

$$\Delta^{(4)}_{\text{scalar gap}}(c) \sim 0.0253303c^2 + 0.13506c + 0.400. \tag{5.57}$$

The coefficient of the leading term is very close to $\frac{1}{4\pi^2} \approx 0.0253302959$. It may be possible to analytically prove that $\Delta^{(4)}_{\text{scalar gap}}(c) \sim \frac{c^2}{4\pi^2}$ at large c. Note that in this analysis we only considered 4 derivatives of (5.55), but it may be the case that if we take $c \to \infty$ with fixed number of derivatives, the leading asymptotics for the bound is independent of the number of derivatives. This is indeed what happens in the spinless modular bootstrap, where the large c bound at any fixed number of derivatives scales as $\frac{c}{6}$ [152].

It would be better to do the analysis with the opposite order of limits, where we take the number of derivatives to large before taking c large (as in [154]) and then extrapolate in c. We can obtain bounds from including a larger number of derivatives in (5.55) by using the semidefinite program solver SDPB [5, 172]. More precisely, we consider the sum rule

$$\sum_{n=2,4,\ldots,n_{\max}} \alpha_n \text{vac}^{(n)}(1) + \sum_{n=2,4,\ldots,n_{\max}} \alpha_n \sum_{\Delta} (\partial_k)^n f(k,\Delta)|_{k=1} = 0. \tag{5.58}$$

Unfortunately, the function $(\partial_k)^n f(k,\Delta)|_{k=1}$ in (5.55) does not have a good approximation as a product of a positive function of Δ and a polynomial in Δ. Therefore,

[6] At $c = 1$ the crossing equation we use is slightly different due to a divergence of the zeta function at 1; see Appendix D.3.1 for discussion.

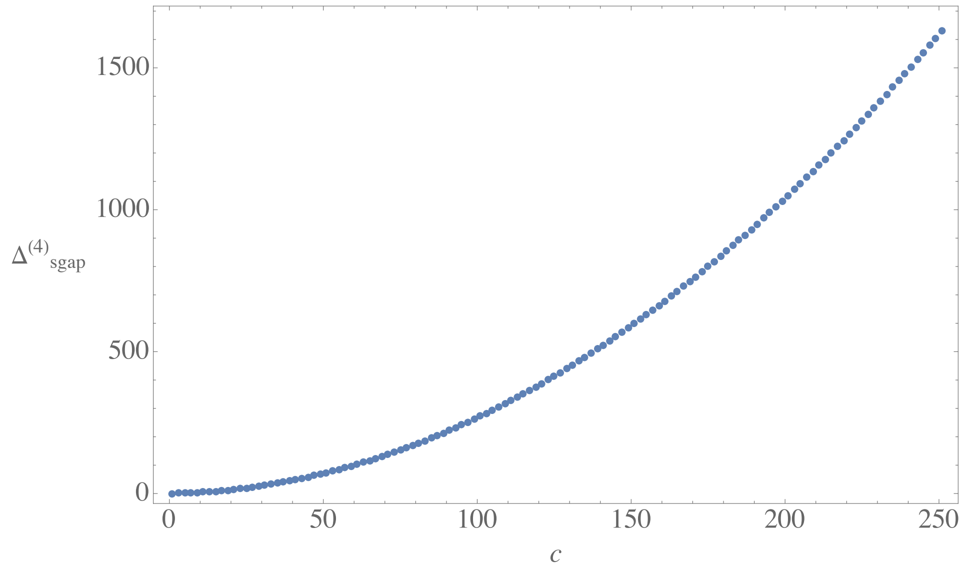


Figure 5.2: Plot of a bound on the scalar gap for $U(1)^c$ CFTs with 4 derivatives, up to central charge $c = 251$. The numerical data seems to be well-approximated by a quadratic function with leading coefficient $\frac{1}{4\pi^2}$ (see (5.57)).

we discretize in Δ-space and sample the function $(\partial_k)^n f(k,\Delta)|_{k=1}$ at various points $\Delta_1, \Delta_2, \ldots, \Delta_M$ above the scalar gap assumption, and use SDPB as a linear programming solver to look for a functional that satisfies

$$\begin{aligned} \sum_{n=2,4,\ldots,n_{\max}} \alpha_n \text{vac}^{(n)}(1) &= 1, \\ \sum_{n=2,4,\ldots,n_{\max}} \alpha_n (\partial_k)^n f(k,\Delta)|_{k=1} &\geq 0, \qquad \Delta = \Delta_1, \ldots, \Delta_M. \end{aligned} \tag{5.59}$$

Finally, we check the positivity of the obtained functional for all $\Delta \geq \Delta^{(n_{\max})}_{\text{scalar gap}}$ by hand. If there is a negative region, we sample more points there and rerun SDPB, and repeat this procedure until the functional is positive or SDPB gives a primal feasible solution[7].

Using the method described above, we have computed $\Delta^{(n)}_{\text{scalar gap}}$ for $n = 10, 20, \cdots, 60$ for central charge odd $c \leq 27$. Our bounds are summarized in Table 5.1 and plotted in Fig. 5.3.[8] We were not able to go to high enough central charge to do a reliable extrapolation to large c. There are two obstacles in going to large central charge. The

[7]We are extremely grateful to David Simmons-Duffin for explaining this approach to us.

[8]In Table 1 of [166], a bound on the gap (of any spin) was computed using the spinless modular bootstrap. Our results in Table 5.1 are specifically for scalars, and so in general are orthogonal. However, for $c = 3$, the bound in [166] is less than 1 and so must be a scalar, and is stronger than the bound we found at $c = 3$.

c	$\Delta^{(10)}_{\text{scalar gap}}$	$\Delta^{(20)}_{\text{scalar gap}}$	$\Delta^{(30)}_{\text{scalar gap}}$	$\Delta^{(40)}_{\text{scalar gap}}$	$\Delta^{(50)}_{\text{scalar gap}}$	$\Delta^{(60)}_{\text{scalar gap}}$	$\Delta_{\text{avg sgap}}$
1	0.507	$\frac{1}{2}+7\times10^{-5}$	$\frac{1}{2}+2\times10^{-6}$	$\approx\frac{1}{2}$	$\approx\frac{1}{2}$	$\approx\frac{1}{2}$	ill-defined
3	0.910	0.864	0.863	0.863	0.863	0.863	0.136
5	1.444	1.310	1.304	1.303	1.302	1.302	0.324
7	2.129	1.843	1.820	1.814	1.813	1.813	0.471
9	2.972	2.476	2.419	2.400	2.397	2.396	0.606
11	3.980	3.219	3.110	3.063	3.055	3.051	0.736
13	5.155	4.078	3.897	3.808	3.789	3.779	0.863
15	6.500	5.058	4.788	4.638	4.602	4.581	0.989
17	8.018	6.614	5.786	5.558	5.497	5.458	1.113
19	9.709	7.399	6.895	6.570	6.477	6.412	1.237
21	11.576	8.765	8.118	7.680	7.545	7.445	1.360
23	13.619	10.266	9.460	8.890	8.705	8.561	1.482
25	15.839	11.903	10.922	10.202	9.959	9.762	1.604
27	18.238	13.679	12.506	11.620	11.310	11.049	1.725

Table 5.1: Upper bounds on the scalar gap from $U(1)^c$ CFTs with odd $c \leq 27$ after taking up to $10, 20, \cdots, 60$ derivatives of our crossing equation (i.e., the maximum value of n in (5.55)) computed to three decimal places. We also compare it to the average Narain scalar gap, defined in (5.60) (though note that the optimal bound is different from the average). See Fig. 5.3 for a plot.

first is that the convergence of the bound as the derivative order $n \to \infty$ becomes slower for larger c. The second obstacle is that the number of terms in the sum rule (5.55) grows as c^4 (see the sum in (D.12)), which makes evaluating derivatives with respect to k very slow. It would be good if there were a more efficient way to compute the derivatives.

It is interesting to compare the bounds on the $U(1)^c$ scalar gap we get to the average Narain scalar gap. In [166] an expression for the average scalar gap of Narain theories was computed, by first calculating the average density of states for all Narain theories (under the Zamolodchikov measure), and determining when the integral of the average density of states is 1. By looking at the average density of

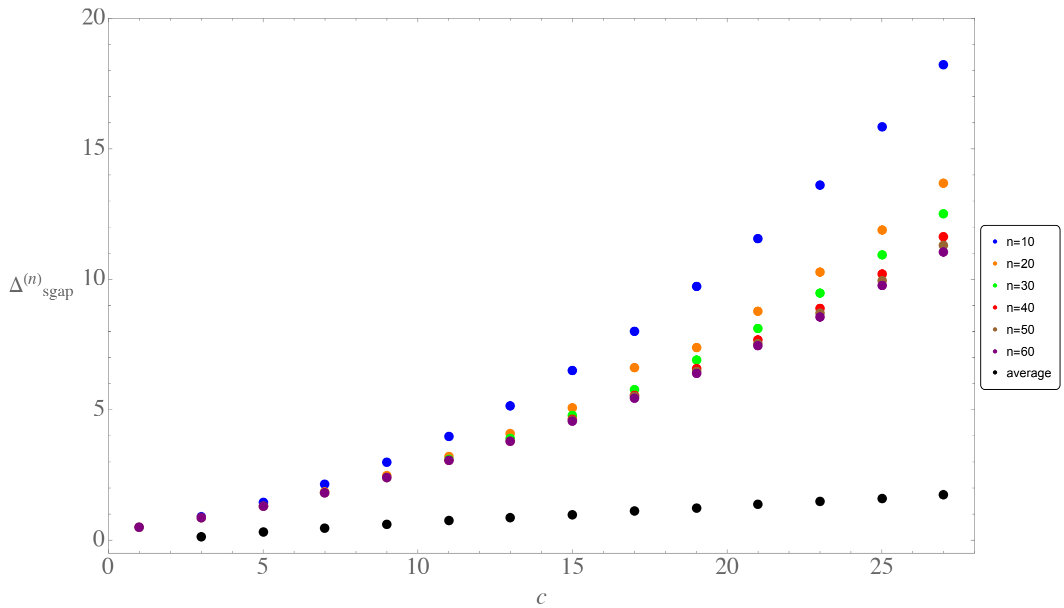


Figure 5.3: Plot of a bound on the scalar gap for $U(1)^c$ CFTs at odd $c \leq 27$. The colors blue, orange, green, red, brown, and purple represent the bound we get at $10, 20, \cdots, 60$ derivatives, respectively. The color black represents the average Narain scalar gap, for comparison. (However, there is no a priori reason the average Narain scalar gap and the optimal $U(1)^c$ scalar gap should be similar.) See Table 5.1 for the numerical data.

scalars, [166] got an average scalar gap of [9]

$$\begin{aligned}\Delta_{\text{avg sgap}} &= \left(\frac{\zeta(c)\Gamma\left(\frac{c}{2}\right)^2(c-1)}{\zeta(c-1)2\pi^c}\right)^{\frac{1}{c-1}} \\ &= \frac{c}{2\pi e} + \frac{\log c}{2\pi e} + O(1). \end{aligned} \tag{5.60}$$

Our numerical bounds at large c (including our bounds with four derivatives extrapolated to large c) appear to be very far from both the average Narain scalar gap and the bound on the gap of the lightest operator of any spin (see (5.3)). It would be interesting to explore further if our bounds on the scalar gap can be substantially improved by considering other crossing equations. Of course, it is possible that the optimal scalar gap behaves differently from both the average Narain scalar gap and the optimal gap at large c.

[9] Note that choosing the integrated average to be 1, as opposed to any other $O(1)$ number less than 1, is somewhat of a convention. However, if we choose another cutoff, the result (5.60) changes very little.

5.4 General 2d CFTs

So far our discussion has been restricted to a very special class of CFTs, namely those with $U(1)^c$ chiral algebra. In this section we generalize to generic 2d CFTs, which only have Virasoro symmetry and no extended chiral algebra (though we pause to note that we do not have any explicit examples of such theories, even numerically [154]).

The main obstacle to repeating our analysis to general 2d CFTs is that the partition function is not square-integrable, due to the Casimir energy of the theory on a cylinder. For theories with $U(1)^c$ chiral algebra, when we factored out the characters of the theory and considered the primary counting partition function $\widehat{Z}$, the resulting function grew only polynomially ($\sim y^{c/2}$) at the cusp (see (5.18)). For theories with only Virasoro symmetry, however, the (Virasoro) primary counting partition function will grow as $\sim e^{2\pi\frac{c-1}{12}y}$ at large y. Although there are various ways we can get around this (see Sec. 4 of [165] for some discussions of other approaches), in this section we will simply take the partition function multiply by the same cusp form as we did for theories with $U(1)^c$ symmetry, and bound the resulting function we get. This will not give us a crossing equation acting only on the Virasoro scalar operators, but instead will give us an equation acting on a more complicated combination of operators of all spin.

To be more precise, let us consider any compact 2d CFT with $c > 1$ and only Virasoro symmetry as its maximal chiral algebra (although generalizations to other chiral algebras are simple). Suppose the partition function of this theory is $Z(\tau)$. We define the "fake scalars" of this theory as

$$Z^{\text{fake scalars}}(y) = \int_{-1/2}^{1/2} dx|\eta(\tau)|^{2c}Z(\tau). \tag{5.61}$$

Note that the central charge c is not necessarily an integer in this analysis. We call this function "fake scalars" because if this theory were to have a $U(1)^c$ chiral algebra, then (5.61) would be a count of the scalars (under the $U(1)^c$ algebra). However, since the theory only has Virasoro symmetry, then $Z^{\text{fake scalars}}(y)$ does not in general have a positive q-expansion.

Even without the full $U(1)^c$ chiral algebra, the logic in deriving the crossing equation (5.38) in Sec. 5.3 will apply to $Z^{\text{fake scalars}}(y)$. We can still apply harmonic analysis to $y^{c/2}|\eta(\tau)|^{2c}Z(\tau) - E_{c/2}(\tau)$ and derive an analogous crossing equation for $Z^{\text{fake scalars}}(y)$. To be precise, the equation we derive is the following.

Let $a_c(n)$ be defined as[10]

$$\sum_{n=0}^{\infty} a_c(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^{c-1}. \tag{5.62}$$

Then we have the following crossing equation in terms of the Virasoro primary operators of any $c > 1$ compact CFT:

$$\begin{aligned}
&\sum_{n=0}^{\infty} e^{-4\pi y n} a_c(n)^2 + \sum_{\Delta, j \in \mathcal{S} \cup \mathcal{S}^{\text{null}}} \sum_{n=0}^{\infty} e^{-2\pi y(\Delta+j+2n)} a_c(n) a_c(n+j) = \\
&\frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} y^{1-c} + \varepsilon y^{-\frac{c}{2}} + \sum_{n=1}^{\infty} \mathrm{Re}\left[\delta_n y^{-\frac{c}{2}+1-\frac{z_n}{2}}\right] \\
&\quad + \frac{y^{1-c}}{\sqrt{\pi}} \sum_{\Delta, j \in \mathcal{S} \cup \mathcal{S}^{\text{null}}} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} b(k) k^{c-2} U\left(-\frac{1}{2}, \frac{c}{2}, \frac{2\pi k^2(\Delta+j+2n)}{y}\right) e^{-\frac{2\pi k^2(\Delta+j+2n)}{y}} a_c(n) a_c(n+j) \\
&\quad + \frac{y^{1-c}}{\sqrt{\pi}} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} b(k) k^{c-2} U\left(-\frac{1}{2}, \frac{c}{2}, \frac{4\pi k^2 n}{y}\right) e^{-\frac{4\pi k^2 n}{y}} a_c(n)^2.
\end{aligned} \tag{5.63}$$

In (5.63), $\mathcal{S}$ is the set of all non-vacuum Virasoro primary operators, labeled by their dimension $\Delta = h + \overline{h}$ and their spin $j = |h - \overline{h}|$. Moreover we define $\mathcal{S}^{\text{null}}$ formally as a set containing -2 operators of weight 1, spin 1 and 1 operator of weight 2, spin 0. This is simply to take into account the level 1 null state in the Virasoro vacuum block (i.e. that L_{-1} and $\overline{L}_{-1}$ annihilate the vacuum). The LHS of (5.63) is precisely what we called $Z^{\text{fake scalars}}(y)$ above, written in terms of the Virasoro primary operators of the theory, which we denoted by the set $\mathcal{S}$. For convenience we have assumed the theory has no additional conserved currents, but it is simple to generalize (5.63) to allow for them.

We have tested (5.63) numerically on the pure gravity partition function of [173, 174], which we will denote as $Z^{\text{MWK}}(\tau)$, at various values of the central charge. For simplicity we have ignored the null state at level 1 (even though this leads to a inconsistent chiral algebra due to the lack of charged twist zero states [175], the resulting function is still modular invariant with a gap to the first primary operator, so it will obey (5.63), without including the contribution from $\mathcal{S}^{\text{null}}$). Strictly speaking we glossed over a subtlety in deriving (5.63). When we derived (5.38) we used the fact that $\mathcal{E}_s^c = \sum_{\Delta \in \mathcal{S}} \Delta^{-2s}$ for $\mathrm{Re}(s) > c - 1$ because the sum converges for those values of s. However if we define $\mathcal{E}_s^c$ analogously for the "fake scalars," it

[10] At central charge 25, $a_{c=25}(n)$ is the Ramanujan tau function (up to a shift of the argument by 1).

could potentially be the case that there is no s such that the sum converges, due to the Cardy growth of the Virasoro primary operators. Nonetheless, we numerically find that (5.63) is still satisfied. It might be interesting to present a more rigorous derivation of this step.

We can then apply the same functionals on (5.63) as discussed in Sec. 5.3.3 to remove the sign-indefinite terms related to the nontrivial zeros of the zeta function. This gives sum rules the CFT must satisfy, where now all operators (instead of just scalars) participate. For example, at $c = 3$, and taking two derivatives in (5.55), we get

$$\sum_{\Delta, j \in \mathcal{S} \cup \mathcal{S}^{\text{null}}} \sum_{n=0}^{\infty} a_{c=3}(n) a_{c=3}(n+j) f(\Delta + j + 2n) + \sum_{n=1}^{\infty} a_{c=3}(n)^2 f(2n) = \frac{\pi}{4}, \quad (5.64)$$

where

$$f(\Delta) := \frac{\pi \left(-3 + 8\pi^2 \Delta + (3 + 4\pi^2 \Delta) \cosh(2\sqrt{2}\pi\sqrt{\Delta}) - 6\sqrt{2}\pi\sqrt{\Delta} \sinh(2\sqrt{2}\pi\sqrt{\Delta})\right)}{8 \sinh^4\left(\sqrt{2}\pi\sqrt{\Delta}\right)}. \quad (5.65)$$

(This comes from evaluating $\partial_k^2|_{k=1}$ on (D.9).) We can apply the same family of functionals discussed in Sec. 5.3.3 to (5.63) more generally and try to derive bounds on the various quantities (e.g. scalar gap, gap, etc.) from this crossing equation. Unfortunately we run into two distinct issues that stop us from bounding generic theories.

First, we see that at large Δ, (5.65) falls off as $\sim e^{-2\pi\sqrt{2\Delta}}$. In fact from Appendix D.2 we see that regardless of the central charge or derivative order, the functionals used in Sec. 5.3.3 fall off with the same leading asymptotics. However, the asymptotic growth of operators in $\mathcal{S}$ comes from the Cardy formula [176] and is $\sim e^{2\pi\sqrt{\frac{\Delta(c-1)}{3}}}$. We thus see that if $c \geq 7$, the sum rule does not obviously converge. Note that for $U(1)^c$ CFTs this was not an issue because there, the asymptotic density of primary operators grew polynomially in Δ ($\sim \Delta^{c-2}$). As a check we have verified (5.64) for $Z^{\text{MWK}}(\tau)$ at $c = 3$, but the analogous computation at $c = 9$ diverges (even though both obey (5.63)). We have also verified (5.64) for various rational CFTs with $c < 7$ where we only decompose into Virasoro characters.

It is unfortunate that we only get a falloff in $e^{-\#\sqrt{\Delta}}$ in our sum rules. This only happened after we integrated against the function $\Phi(t)$ in (5.45). Before this integral (e.g. in (5.38) and (5.63)), we had a falloff as $e^{-\#\Delta}$, which will always overwhelm the Cardy growth at any central charge. It would be interesting to see if there were another choice of functional that would both remove the sign-indefinite terms related to nontrivial zeros of the zeta function, but still preserve the faster falloff in dimension.

Second, the asymptotically large Δ behavior of

$$\sum_{n=0}^{\infty} a_c(n) a_c(n+j) f(\Delta + j + 2n) \tag{5.66}$$

does not have fixed sign: for some spins the asymptotic Δ value is positive and for some spins it is negative. (This is true when one takes any number of derivatives of the crossing equation, not just two.) The root of this problem is that $a_c(n)a_c(n+j)$ does not have a definite sign. Thus there is no obvious way to construct functionals that have fixed sign for all spin and all dimensions larger than some cutoff.

We note that we have chosen to multiply the partition function by the cusp form $y^{c/2}|\eta(\tau)|^{2c}$ to render the partition function square-integrable. However any cusp form with a gap to the first excited state and that falls off at least as fast as $(q\overline{q})^{c/24}$ would be sufficient and give a similar crossing equation as (5.63). It might be useful to explore constraints one gets from other cusp forms.

Finally we end this section with an interesting observation. Our crossing equation (5.63) for Virasoro theories involves operators of all spins, since multiplying by $y^{c/2}|\eta(\tau)|^{2c}$ does not have an obvious physical interpretation for theories without a $U(1)^c$ extended current algebra. It would be better to have a crossing equation or sum rule that only involved scalar Virasoro primary operators. Surprisingly, we find strong hints that such a sum rule exists.

In order to get a sum rule acting only on scalar Virasoro primary operators, the natural thing to do is to multiply the partition function by $y^{1/2}|\eta(\tau)|^2$. This is the same object that we multiply for $U(1)^c$ theories for $c = 1$. Recall that there, we derived the following sum rule (see (D.15) and App. D.3.1):

$$\log k + \sum_{\Delta \in \mathcal{S}} \left[h(k, \Delta) - h(k^{-1}, \Delta) \right] = 0, \tag{5.67}$$

where

$$h(k,\Delta) := \sqrt{2}\pi\sqrt{k\Delta}(1 - \coth(\sqrt{2}\pi\sqrt{k\Delta})) + 2\pi^2 k\Delta \operatorname{csch}^2(\sqrt{2}\pi\sqrt{k\Delta}) + \log(1 - e^{-2\sqrt{2}\pi\sqrt{k\Delta}}). \tag{5.68}$$

(The expression (5.68) is just (D.14) at $c = 1$, where we multiplied through by a factor of -4 for convenience.)

Remarkably, we numerically find that (5.67) also holds for general Virasoro CFTs, where $\mathcal{S}$ is now the set of conformal dimensions of scalar Virasoro primary operators (minus $\frac{c-1}{12}$) subject the following constraints. First of all, due to the null state structure of the Virasoro vacuum character, we introduce an additional term in $\mathcal{S}$ of $\Delta - \frac{c-1}{12} = -\frac{c-25}{12}$ (assuming no spin 1 currents). Second of all, we do not include the $\log k$ term in the sum rule (since there is not necessarily a state with $\Delta - \frac{c-1}{12} = 0$ in the spectrum). Finally, and most importantly, the sum rule does not converge for sufficiently large c. At large Δ, we have

$$h(k,\Delta) - h(k^{-1},\Delta) \sim e^{-2\sqrt{2}\pi\sqrt{\Delta\times\min(k,k^{-1})}}, \tag{5.69}$$

whereas

$$\rho^{\text{scalar primaries}}(\Delta) \sim e^{2\pi\sqrt{\frac{\Delta(c-1)}{3}}}, \tag{5.70}$$

so our sum rule only converges if

$$c < 1 + 6\min(k,k^{-1}), \tag{5.71}$$

which implies $c < 7$.[11]

For various theories obeying (5.71), we very surprisingly find that the sum rule

$$\int_{-\frac{c-1}{12}}^{\infty} d\Delta \rho^{\text{scalars}}(\Delta)\left[h(k,\Delta) - h(k^{-1},\Delta)\right] = 0 \tag{5.72}$$

is obeyed to arbitrarily high precision. For $c < 7$ we can use this to bound the Virasoro scalar gap. However, our bounds from this so far seem to be substantially weaker than those found in [154]. It would be extremely interesting if there were a way to analytically continue the sum in (5.72) to arbitrary central charge (and also to prove, or more honestly derive, (5.72)). If so, this could be a way to derive a Virasoro scalar gap for all central charge[12].

[11]Note also $h(k,\Delta) - h(k^{-1},\Delta)$ has poles at $\Delta = -\frac{n^2}{2k}$ and $\Delta = -\frac{n^2 k}{2}, n \in \mathbb{N}$, which may be problematic for convergence. For example, if $c = 1 + 6kn^2$ or $c = 1 + 6k^{-1}n^2$, with $n \in \mathbb{N}$, then the vacuum term contributes as a pole.

[12]In [154], it was shown that no bound on the Virasoro scalar gap could be derived for $c \geq 25$ using the traditional modular bootstrap. This was due to the existence of a "spurious solution" to crossing of $\frac{J(\tau)+\overline{J}(\overline{\tau})}{\sqrt{\tau_2}|\eta(\tau)|^2}$, which lacks scalar primary operators (see discussion around Eqn (3.2) of [154]). However, if there exists a convergent sum rule like (5.72) for all c that only acts on scalar primary operators, then by definition it would vanish on the spurious solution found in [154], and one may be able to find a bound for $c \geq 25$.

5.5 2d CFTs and the Riemann Hypothesis

One interesting feature of our crossing equation (5.38) is that in the small y (high temperature) limit, the asymptotics are controlled by the real parts of the nontrivial zeros of the Riemann zeta function. Let us rewrite (5.38), defining the temperature $T := y^{-1}$, as

$$\begin{aligned} 1 + \sum_{\Delta \in \mathcal{S}} e^{-\frac{2\pi\Delta}{T}} &= \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} T^{c-1} + \varepsilon_c T^{\frac{c}{2}} \\ &+ \sum_{k=1}^{\infty} T^{\frac{c}{2}-1+\frac{\mathrm{Re}(z_k)}{2}} \left[\mathrm{Re}(\delta_{k,c}) \cos(\mathrm{Im}(z_k)\log T) - \mathrm{Im}(\delta_{k,c}) \sin(\mathrm{Im}(z_k)\log T)\right] \\ &+ O\left(e^{-2\pi\Delta_{\mathrm{gap}} T}\right). \end{aligned} \tag{5.73}$$

At high temperature, second line of (5.73) behaves as a highly oscillatory function with an overall envelope controlled by $\mathrm{Re}(z_k)$. The Riemann hypothesis says that for all k,

$$\mathrm{Re}(z_k) = 1/2, \tag{5.74}$$

which would fix the envelope to be $T^{\frac{c}{2}-\frac{3}{4}}$. In other words, if the Riemann hypothesis is true, (5.73) can be written as

$$\begin{aligned} 1 + \sum_{\Delta \in \mathcal{S}} e^{-\frac{2\pi\Delta}{T}} &= \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} T^{c-1} + \varepsilon_c T^{\frac{c}{2}} \\ &+ \sum_{k=1}^{\infty} T^{\frac{c}{2}-\frac{3}{4}} \left[\mathrm{Re}(\delta_{k,c}) \cos(\mathrm{Im}(z_k)\log T) - \mathrm{Im}(\delta_{k,c}) \sin(\mathrm{Im}(z_k)\log T)\right] \\ &+ O\left(e^{-2\pi\Delta_{\mathrm{gap}} T}\right). \end{aligned} \tag{5.75}$$

However, if the Riemann hypothesis is false, then there is at least one z_k with real part greater than $1/2$,[13] which changes the large temperature scaling in the second line of (5.75).[14] Since the leading term of (5.73) is essentially the Cardy formula, then in some sense, the Riemann hypothesis makes a claim about the overall size of the "subsubleading" corrections to the Cardy formula.

[13] By the functional equation (5.12) and meromorphicity, the Riemann hypothesis being false implies a pair of zeros of the zeta function with identical imaginary part: one with real part greater than $1/2$, one with real part less than $1/2$.

[14] Note that there is a possibility that the residue at that zero vanishes, meaning $\delta_{k,c}$ vanishes in (5.75). However, this will only happen in a real codimension 2 subspace of the moduli space. Thus for a generic theory the scaling will change at large temperature. We thank Per Kraus for raising this question to us.

We can illustrate this with an explicit example. Let us consider the $SU(3)_1$ WZW model, and decompose the theory under the $U(1)^2$ chiral algebra (note that this is not the maximal chiral algebra). The scalar partition function is given by

$$
\begin{aligned}
Z^{\text{scalars}}_{SU(3)_1}(T) &:= 1 + \sum_{\Delta \in \mathcal{S}_{SU(3)_1}} e^{-\frac{2\pi\Delta}{T}} \\
&= 1 + \sum_{n=1}^{\infty} 48 \left(\sum_{k|n} (-1)^k \sin(\tfrac{k\pi}{3}) \right)^2 e^{-\frac{4\pi n}{T}} + 24 \left(\sum_{k|3n-2} (-1)^k \sin(\tfrac{k\pi}{3}) \right)^2 e^{-\frac{4\pi(n-\frac{2}{3})}{T}} \\
&= 1 + 18 e^{-\frac{4\pi}{3T}} + 36 e^{-\frac{4\pi}{T}} + 18 e^{-\frac{16\pi}{3T}} + 72 e^{-\frac{28\pi}{3T}} + \cdots . \qquad (5.76)
\end{aligned}
$$

For $c = 2$ the crossing equation (5.75) is slightly modified due to a pole at $\Lambda(1/2)$. As derived in (D.37), the crossing equation we get for a $c = 2$ Narain theory is

$$
\begin{aligned}
1 + \sum_{\Delta \in \mathcal{S}} e^{-\frac{2\pi\Delta}{T}} &= \frac{3}{\pi} T \log T + \left[\widehat{E_1}(\rho) + \widehat{E_1}(\sigma) + \frac{3}{\pi} \left(\gamma_E + \log(4\pi) + 24\zeta'(-1) - 2 \right) \right] T \\
&+ \sum_{k=1}^{\infty} \text{Re} \left(\frac{4\pi^{\frac{z_k}{2}} \Lambda(\frac{1+z_k}{2})^2 E_{\frac{1+z_k}{2}}(\rho) E_{\frac{1+z_k}{2}}(\sigma)}{2\Gamma\left(\frac{z_k}{2}\right) \zeta'(z_k)} T^{\frac{z_k}{2}} \right) \\
&+ \frac{T}{\sqrt{\pi}} \sum_{\Delta \in \mathcal{S}} \sum_{n=1}^{\infty} b(n) U\left(-\frac{1}{2}, 1, 2\pi\Delta n^2 T\right) e^{-2\pi\Delta n^2 T}, \qquad (5.77)
\end{aligned}
$$

where $\widehat{E_1}$ is defined in (D.34). From the explicit form of the sum over k in (5.77), we see that the coefficient in front of $T^{\frac{z_k}{2}}$ falls off exponentially in k, so the sum converges rapidly.

For the case of the $SU(3)_1$ WZW model, we have $\rho = \sigma = e^{2\pi i/3}$. At large temperature, the last line of (5.77) becomes non-perturbatively small. Therefore if we subtract the first two terms on the RHS of (5.77) and go to large temperature, we should be able to probe the real part of the nontrivial zeros of the Riemann zeta function. Indeed, by evaluating (5.76) up to $T = 300$, we numerically are able to recover the first few nontrivial zeros of the Riemann zeta function. We plot this in Fig. 5.4. Of course for any 2d CFT we could make a similar plot using (5.63); here we picked this particular theory for concreteness.

We pause to note that we can only numerically go up to certain fixed temperature (e.g. $T = 300$) because we only computed a finite number of terms in (5.76). Since the residue falls off exponentially in $\text{Im}(z_k)$, this means we only numerically test the Riemann hypothesis up to a fixed imaginary part. Since the Riemann hypothesis has already been checked up to imaginary part 3×10^{12} [177], we emphasize that our numerics are *not* an independent check of the Riemann hypothesis.

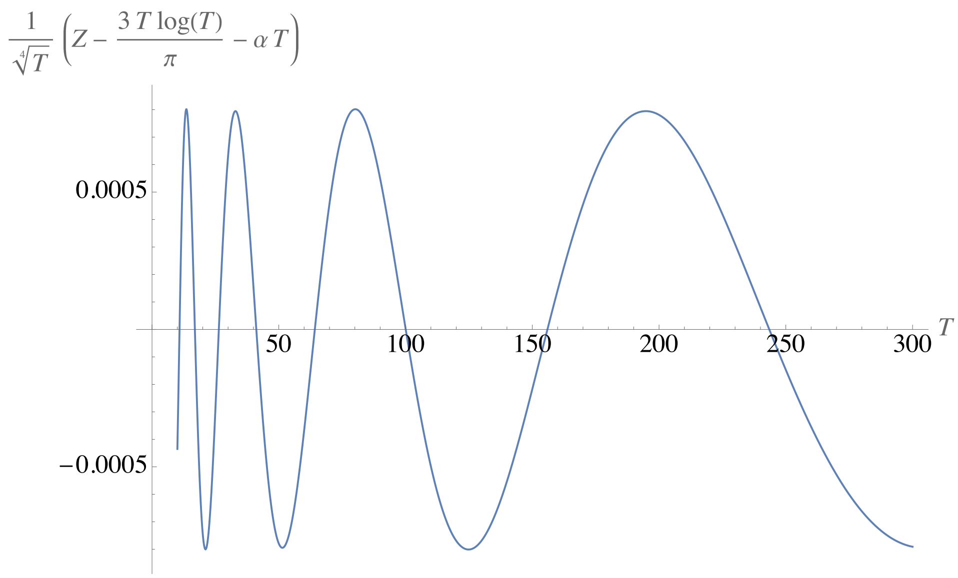


Figure 5.4: Scalar part of the $SU(3)_1$ WZW model with first two leading terms subtracted, rescaled by $T^{1/4}$, plotted up to $T = 300$. If the Riemann hypothesis is true, then at large temperature, this function will remain bounded. However, if the Riemann hypothesis is false, at large temperatures the oscillations will grow in size and become unbounded (modulo the subtlety explained in footnote 14). In this plot, $\alpha := 2\widehat{E_1}(e^{2\pi i/3}) + \frac{3}{\pi}(\gamma_E + \log(4\pi) + 24\zeta'(-1) - 2) \approx 0.975$ (see (5.77)). By fitting this plot with oscillating functions in $\log(T)$, we can numerically recover the first few nontrivial zeros of the zeta function. (A similar plot can be made for any $c > 1$ CFT.)

However, it would be extremely interesting if there were a physical reason why the scalar partition function, with the first two leading terms subtracted off, had to scale as $T^{\frac{c}{2}-\frac{3}{4}}$. This would give a "physics explanation" of the Riemann hypothesis. We leave this problem as an exercise to the reader.

5.6 Future directions

In this paper we have derived a crossing equation acting only on the scalar operators of certain 2d CFTs. Rather curiously the crossing equation is intimately related to the nontrivial zeros of the Riemann zeta function. This allows us to rephrase the Riemann hypothesis purely in terms of the growth of states of scalar operators of $U(1)^c$ CFTs. By applying clever choices of linear functionals, we are able to derive positive sum rules that the scalar operators must satisfy, which lead to nontrivial bounds on the lightest non-vacuum scalar operator in $U(1)^c$ CFTs. We discuss generalizations to theories with only Virasoro symmetry. There are various future directions that may be interesting to pursue.

Virasoro scalar crossing equation?

In Sec. 5.4 we derived a crossing equation acting on all operators for theories with Virasoro symmetry. In order to make the partition function square-integrable, we multiplied by a cusp form, namely $y^{c/2}|\eta(\tau)|^{2c}$, which led to the inclusion of all spins to the crossing equation. It would be nice if there exists a crossing equation that does not rely on this, and acts only on the scalar Virasoro primary operators. In order to derive such an equation (if it exists), it might be necessary to consider some generalization of harmonic analysis to allow for exponential divergences as $y \to \infty$.

In the end of Sec. 5.4, we guessed such a sum rule for Virasoro CFTs with $c < 7$. It would be interesting to derive it more rigorously and somehow analytically continue the sum rule so it makes sense for arbitrary central charge.

Crossing equation for spin j?

In this paper we considered crossing equations acting on scalar operators of $U(1)^c$ CFTs (or "fake scalars" for the case of Virasoro CFTs). This was largely to avoid the Maass cusp forms in the spectral decomposition (which lack scalars – see (5.16)). It would be interesting if there were a generalization of our crossing equation to any fixed spin partition function.

In fact, the techniques we studied almost immediately generalize to any spin $j \neq 0$ crossing equation. Let us denote $\mathcal{J}$ as the set of spin j primary operators of a $U(1)^c$ CFT. The spin j partition function is given by

$$\begin{aligned}\sum_{\Delta\in\mathcal{J}} e^{-2\pi\Delta y} = &\frac{2\sigma_{c-1}(j) y^{\frac{1-c}{2}} K_{\frac{c-1}{2}}(2\pi j y)}{\Lambda\left(\frac{c}{2}\right) j^{\frac{c-1}{2}}} \\ &+ \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \pi^{s-\frac{c}{2}} \Gamma\left(\frac{c}{2}-s\right) \mathcal{E}^{c}_{\frac{c}{2}-s} \frac{\sigma_{2s-1}(j) K_{s-\frac{1}{2}}(2\pi j y) y^{\frac{1-c}{2}}}{\Lambda(s) j^{s-\frac{1}{2}}} \\ &+ \sum_{n=1}^{\infty}\sum_{\epsilon=\pm} \frac{a_j^{(n,\pm)}(\widehat{Z}, \nu_n^{\pm})}{2(\nu_n^{\pm}, \nu_n^{\pm})} y^{\frac{1-c}{2}} K_{iR_n^{\pm}}(2\pi j y). \qquad (5.78)\end{aligned}$$

Unfortunately the last line in (5.78) seems very difficult to deal with analytically due to the sporadic nature of the Maass cusp form eigenvalues, but we can in fact do the integral in the second line using the same techniques as in Sec. 5.3.2. We move the contour of integration to the right, past $\mathrm{Re}(s) = \frac{c}{2}$, so that we can expand the function $\mathcal{E}^{c}_{\frac{c}{2}+s-1}$ in terms of the scalar primary operators and then change the order of the sum and integral. From the discussion in Appendix D.1, we know the only poles in $\mathcal{E}^c_{\frac{c}{2}-s}$ to the right of the contour occur at $s = \frac{c}{2}, \frac{1+z_n}{2}, \frac{1+z_n^*}{2}$ (see Fig. 5.1). The additional terms do not introduce any additional poles to the right of the contour. We thus get a crossing equation in terms of the spin j operators on the LHS and the scalars on the RHS (as well as the cusp forms). It may be interesting to analyze this equation further.

Better bounds on $U(1)^c$ theories?

In Table 5.1 our numerical bounds on the scalar operators of $U(1)^c$ theories are quite far from the average Narain scalar gap. For instance our numerical bounds seem to grow quadratically with c instead of linearly. This may be an indication that our crossing equation is not strong enough to pinpoint the CFT with the largest scalar gap. It would be interesting if we could modify the set of crossing equations we consider to get better bounds. For instance we could include both our crossing and the "traditional" modular invariance (or four-point function) crossing equations to see if we can get better bounds. Other avenues to explore may be to consider different functionals from the ones used in Sec. 5.3.3 (for example not just considering $\varphi(t)$ in (5.43) to be Gaussians) or somehow incorporate the residues at the nontrivial zeros of the zeta function into the crossing equation. It would also be nice to get numerical results for even c.

Four-point functions?

There is a well-known relation between crossing symmetry acting on a four-point functions of four scalar operators and modular covariance. For four identical operators, under an appropriate coordinate transformation, the four-point function should be modular invariant. (For different operators, it will transform as some vector-valued modular function.) It would be interesting if one could derive a crossing equation on certain correlation functions where only a one-dimensional slice of operators are exchanged (e.g. only scalar operators are exchanged instead of operators of all spin). It would be especially interesting if this could generalize to higher dimensions.

Applications to $\mathcal{N} = 4$ SYM?

Besides in 2d CFT, another natural place that modular invariance shows up in string theory is in S-duality of $\mathcal{N} = 4$ super Yang-Mills theory. In [178] (see also [179]), harmonic analysis was used extensively to study various integrated correlators as a function of the complexified Yang-Mills coupling. It would be interesting if there were some sort of crossing equation acting only on the zero-instanton sector (but note that the pole structure of the overlap with Eisensteins is different because there is no notion of a “scalar gap”; see [178] for discussions on this).

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Appendix A

APPENDICES TO CHAPTER 2

A.1 Comments on existence of light-ray operators

In section 2.4.3 we described a picture in which light-ray operators exist in any CFT and provide analytic continuation of local operators in the sense of equation (2.97). This picture is supported, for example, by perturbative examples, where explicit expressions can be given for $\widehat{\mathbb{O}}^{\pm}_{i,J,\lambda}(x, z, \mathbf{w})$ in terms of fundamental fields and (2.97) can be verified. However, this picture is not rigorously known to be valid in non-perturbative CFTs as we now review.

In [11] a construction was given for light-ray operators in general CFTs, which we reviewed in 2.4.3. The following statements need to be established before it can be claimed that the story of section 2.4.3 is correct.

First of all, it needs to be shown that the functions

$$\langle \Psi | \mathbb{O}^{\pm}_{\Delta,J}(x, z) | \Phi \rangle \tag{A.1}$$

are meromorphic in Δ for general J. Furthermore, we need to prove that the positions of the poles are independent of the choice of the states Ψ, Φ, as well as of the operators ϕ_1, ϕ_2 used to define $\mathbb{O}^{\pm}_{\Delta,J}$. Finally, we have to argue that the residues of these poles depend on ϕ_1, ϕ_2 only through an OPE coefficient $f_{12O^\dagger}$.

All these statements are known to be true for non-negative integer J (with $(-1)^J = \pm$) [11] but, to the best of our knowledge, no proof is known for other values. In the main text we assume that these are true, since this simplifies the statement of our results. However, even if none of the above statements hold, the results of this paper, and in particular (2.147), continue to hold in the following sense. One needs to take matrix element of (2.147) between the states of interest, which for concreteness we take to be created by single insertions of primaries O_3, O_4. The right-hand side is then given by δ integrals of

$$\langle O_3 \mathbb{O}^{\pm}_{\delta+1,J,\lambda,(a)} O_4 \rangle, \tag{A.2}$$

which can be expressed in terms of the function $C_{ab}(\Delta, J, \lambda)$ which is computed by the Lorentzian inversion integral [11] for the four-point function

$$\langle O_1 O_2 O_3 O_4 \rangle. \tag{A.3}$$

The Lorentzian inversion integral and thus $C_{ab}(\Delta, J, \lambda)$ is well-defined on the δ integration contour of (2.147), and so we get a rigorous interpretation of (2.147) without assuming any of the above facts about the light-ray operators. Moreover, the entire derivation of (2.147) can be carried out in this language, which is what is essentially done in [15] and appendix A.4.

In this language, where the matrix elements

$$\langle O_3 \mathbf{L}[O_1]\mathbf{L}[O_2]O_4\rangle \tag{A.4}$$

are expressed as an integral of $C_{ab}(\Delta, J, \lambda)$ along principal series in Δ, the analysis of the small angle limit between the detectors can be carried out in the usual way, by analytically continuing $C_{ab}(\Delta, J, \lambda)$ away from the principal series and deforming the integration contour to the right, picking up the singularities that one encounters on the way.

A.2 Conventions for two- and three-point structures

We follow the same conventions for two- and three-point structures of traceless-symmetric operators as in [11]. In particular, we define

$$\langle O(x_1, z_1) O(x_2, z_2)\rangle = \frac{(-2z_1 \cdot I(x_{12}) \cdot z_2)^J}{x_{12}^{2\Delta}} \tag{A.5}$$

where

$$I^\mu{}_\nu(x) = \delta^\mu{}_\nu - 2\frac{x^\mu x_\nu}{x^2}. \tag{A.6}$$

For three-point structures we define

$$\langle \phi_1(x_1)\phi_2(x_2) O(x_3, z)\rangle = \frac{(2z\cdot x_{23}\, x_{13}^2 - 2z\cdot x_{13}\, x_{23}^2)^J}{x_{12}^{\Delta_1+\Delta_2-\Delta+J} x_{13}^{\Delta_1+\Delta-\Delta_2+J} x_{23}^{\Delta_2+\Delta-\Delta_1+J}}. \tag{A.7}$$

In terms of embedding-space formalism these become

$$\langle O(X_1, Z_1) O(X_2, Z_2)\rangle = \frac{(-2H_{12})^J}{X_{12}^{\Delta+J}}, \tag{A.8}$$

$$\langle \phi_1(X_1)\phi_2(X_2) O(X_3, Z_3)\rangle = \frac{(-2V_{3,12})^J}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta-J}{2}} X_{13}^{\frac{\Delta_1+\Delta-\Delta_2+J}{2}} X_{23}^{\frac{\Delta_2+\Delta-\Delta_1+J}{2}}}, \tag{A.9}$$

where as usual

$$X_{ij} \equiv -2X_i \cdot X_j \tag{A.10}$$

$$V_{i,jk} \equiv \frac{Z_i \cdot X_j X_i \cdot X_k - Z_i \cdot X_k X_i \cdot X_j}{X_j \cdot X_k}, \tag{A.11}$$

$$H_{ij} \equiv -2(Z_i \cdot Z_j X_i \cdot X_j - Z_i \cdot X_j Z_j \cdot X_i). \tag{A.12}$$

Recall that we project from embedding space using

$$(X^+, X^-, X^\mu) = (1, x^2, x^\mu), \quad (Z^+, Z^-, Z^\mu) = (0, 2x\cdot z, z^\mu), \tag{A.13}$$

and the embedding space metric is

$$X^2 = -X^+X^- + X^\mu X_\mu. \tag{A.14}$$

We use these conventions both for d-dimensional structures as well as for $(d-2)$-dimensional celestial structures. For example, replacing $X \to z, Z \to w, \Delta \to \delta$ and $J \to j$ we find

$$\begin{aligned}\langle \phi_1(z_1)\phi_2(z_2)O(z,w)\rangle &= \frac{(-2\frac{w\cdot z_1 z\cdot z_2 - w\cdot z_2 z\cdot z_1}{z_1\cdot z_2})^j}{(-2z_1\cdot z_2)^{\frac{\delta_1+\delta_2-\delta-j}{2}}(-2z_1\cdot z)^{\frac{\delta_1+\delta-\delta_2+j}{2}}(-2z_2\cdot z)^{\frac{\delta_2+\delta-\delta_1+j}{2}}}\\ &= \frac{(4w\cdot z_1 z\cdot z_2 - 4w\cdot z_2 z\cdot z_1)^j}{(-2z_1\cdot z_2)^{\frac{\delta_1+\delta_2-\delta+j}{2}}(-2z_1\cdot z)^{\frac{\delta_1+\delta-\delta_2+j}{2}}(-2z_2\cdot z)^{\frac{\delta_2+\delta-\delta_1+j}{2}}}\\ &= \frac{(-4)^j(w\cdot z_2 z\cdot z_1 - w\cdot z_1 z\cdot z_2)^j}{(-2z_1\cdot z_2)^{\frac{\delta_1+\delta_2-\delta+j}{2}}(-2z_1\cdot z)^{\frac{\delta_1+\delta-\delta_2+j}{2}}(-2z_2\cdot z)^{\frac{\delta_2+\delta-\delta_1+j}{2}}},\end{aligned} \tag{A.15}$$

in agreement with (2.86).

Sometimes we need the standard tensor structures for continuous spin. We define them for the Wightman functions as

$$\langle 0|\phi_1(x_1)O(x_3,z)\phi_2(x_2)|0\rangle = \frac{(2z\cdot x_{23}\,x_{13}^2 - 2z\cdot x_{13}\,x_{23}^2)^J}{x_{12}^{\Delta_1+\Delta_2-\Delta+J}x_{13}^{\Delta_1+\Delta-\Delta_2+J}x_{23}^{\Delta_2+\Delta-\Delta_1+J}}. \tag{A.16}$$

For non-integer J, this is defined to be positive for $x_{ij}^2 > 0$ and $z\cdot x_{23}\,x_{13}^2 - z\cdot x_{13}\,x_{23}^2 > 0$. The values in other configurations are obtained by analytic continuation assuming standard analyticity properties of Wightman functions.

A.3 A Lorentzian formula for the light-ray kernel

In this appendix, we prove the Lorentzian formula for the light-ray kernel (2.152). We follow the notation of [11]. Our starting point is the generalized Lorentzian inversion formula

$$C^t_{ab}(\Delta,J,\lambda) = \frac{-1}{2\pi i}\int_{\substack{3>4\\1>2}}\frac{d^dx_1\cdots d^dx_4}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\mathcal{T}_2\mathcal{T}_4\langle\Omega|[O_4,O_1][O_2,O_3]|\Omega\rangle G_O, \tag{A.17}$$

$$G_O = \frac{\left(\mathcal{T}_2\langle 0|O_2\mathbf{L}[O^\dagger]O_1|0\rangle^{(a)}\right)^{-1}\left(\mathcal{T}_4\langle 0|O_4\mathbf{L}[O]O_3|0\rangle^{(b)}\right)^{-1}}{\langle\mathbf{L}[O]\mathbf{L}[O^\dagger]\rangle^{-1}}. \tag{A.18}$$

We have written the formula in slightly different conventions relative to [11]. Firstly, we made the change of variables $x_2 \to \mathcal{T}_2 x_2$ and $x_4 \to \mathcal{T}_4 x_4$, so that the causal relationships become $3 > 4$ and $1 > 2$ with all other pairs spacelike separated. This choice makes it simpler to apply the Lorentzian two and three-point pairings defined in [11]. In addition, we only wrote the t-channel term in the inversion formula. The treatment of the u-channel term is analogous. In our notation, $\mathcal{O}$ is a representation with quantum numbers (Δ, J, λ).

The object $G_{\mathcal{O}}$ is a conformal block with internal quantum numbers $(J+d-1, \Delta-d+1, \lambda)$. In (A.18), we have written it in schematic notation, where the three-point structures in the numerator should be glued using the two-point structure in the denominator. In more precise notation, $G_{\mathcal{O}}$ is defined by

$$G_{\mathcal{O}} = \int_{1>x>2} d^d x D^{d-2} z A_{(a)}(x_1, x_2, x, z) \left(\mathcal{T}_4 \langle 0 | \mathcal{O}_4 \mathbf{L}[\mathcal{O}] \mathcal{O}_3 | 0 \rangle^{(b)} \right)^{-1} \tag{A.19}$$

where the kernel $A_{(a)}$ satisfies

$$\int_{1>x>2} d^d x D^{d-2} z A_{(a)}(x_1, x_2, x, z) \langle \mathbf{L}[\mathcal{O}](x,z) \mathbf{L}[\mathcal{O}^\dagger](x', z') \rangle^{-1} = \left(\mathcal{T}_2 \langle 0 | \mathcal{O}_2 \mathbf{L}[\mathcal{O}^\dagger](x', z') \mathcal{O}_1 | 0 \rangle^{(a)} \right)^{-1} \tag{A.20}$$

Recall that dual structures $\langle \cdots \rangle^{-1}$ are defined using the Lorentzian two and three-point pairings defined in [11]. In [11], the block $G_{\mathcal{O}}$ was defined by specifying its behavior in the OPE limit $x_1 \to x_2$. The above definition in terms of the integral kernel $A_{(a)}$ is equivalent and more convenient for our purposes.

The light-ray kernel should satisfy

$$\int_{1>x>2^-} d^d x_1 d^d x_2 K^t_{\Delta, J.\lambda(a)}(x_1, x_2; x, z) \langle \Omega | \mathcal{O}_4 \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 | \Omega \rangle = -C^t_{ab}(\Delta, J, \lambda) \langle 0 | \mathcal{O}_4 \mathbf{L}[\mathcal{O}] \mathcal{O}_3 | 0 \rangle^{(b)}. \tag{A.21}$$

In the expression (A.21), we can replace $\langle \Omega | \mathcal{O}_4 \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 | \Omega \rangle$ with a double commutator. The reason is that the kernel $K^t_{\Delta, J, \lambda(a)}$ factors through null integrals of x_1 and x_2, which annihilate the past and future vacuum, see [11] for details. Furthermore, let us make the change of variables $x_2 \to \mathcal{T}_2 x_2$ and $x_4 \to \mathcal{T}_4 x_4$, so that we have

$$\int_{1>x>2} d^d x_1 d^d x_2 (\mathcal{T}_2 K^t_{\Delta, J.\lambda(a)}) \mathcal{T}_2 \mathcal{T}_4 \langle \Omega | [\mathcal{O}_4, \mathcal{O}_1][\mathcal{O}_2, \mathcal{O}_3] | \Omega \rangle = -C^t_{ab}(\Delta, J, \lambda) \mathcal{T}_4 \langle 0 | \mathcal{O}_4 \mathbf{L}[\mathcal{O}] \mathcal{O}_3 | 0 \rangle^{(b)}. \tag{A.22}$$

For brevity, here and in the following, we assume the arguments of $\mathcal{T}_2 K^t_{\Delta, J.\lambda(a)}$ are $(x_1, x_2; x, z)$. The arguments of $\mathbf{L}[\mathcal{O}]$ will always be (x, z), and we use the notation $\mathbf{L}[\mathcal{O}'^\dagger]$ to indicate $\mathbf{L}[\mathcal{O}^\dagger](x', z')$.

Pairing both sides with the dual of the right-hand structure, we obtain

$$-C^t_{ab}(\Delta,J,\lambda) = \int_{\substack{1>x>2\\3>4}} \frac{d^dx_1 d^dx_2 d^dx_3 d^dx_4 d^dx D^{d-2}z}{\mathrm{vol}\,\widetilde{\mathrm{SO}}(d,2)} (\mathcal{T}_2 K^t_{\Delta,J.\lambda(a)}) \left(\mathcal{T}_4\langle 0|\mathcal{O}_4 \mathbf{L}[\mathcal{O}]\mathcal{O}_3|0\rangle^{(b)}\right)^{-1}$$
$$\times \mathcal{T}_2\mathcal{T}_4\langle\Omega|[\mathcal{O}_4,\mathcal{O}_1][\mathcal{O}_2,\mathcal{O}_3]|\Omega\rangle. \tag{A.23}$$

Comparing with (A.17), (A.19), and (A.20), we conclude

$$\int_{1>x>2} d^dx D^{d-2}z(\mathcal{T}_2 K^t_{\Delta,J.\lambda(a)})\langle \mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}'^\dagger]\rangle^{-1} = \frac{1}{2\pi i}\left(\mathcal{T}_2\langle 0|\mathcal{O}_2\mathbf{L}[\mathcal{O}'^\dagger]\mathcal{O}_1|0\rangle^{(a)}\right)^{-1}. \tag{A.24}$$

This is essentially the desired result, written in terms of dual structures. To put it in a more conventional form, we must apply the two- and three-point pairings that appear in the definition of the dual structures. Pairing both sides with $\mathcal{T}_2\langle 0|\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger](x',z')\mathcal{O}_1|0\rangle^{(b)}$, we find

$$\int_{\substack{1>x>2\\x'\approx 1,2}} \frac{d^dx_1 d^dx_2 d^dx D^{d-2}z d^dx' D^{d-2}z'}{\mathrm{vol}\,\widetilde{\mathrm{SO}}(d,2)} (\mathcal{T}_2 K^t_{\Delta,J.\lambda(a)}) \left(\mathcal{T}_2\langle 0|\mathcal{O}_2\mathbf{L}[\mathcal{O}'^\dagger]\mathcal{O}_1|0\rangle^{(b)}\right)\langle \mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}'^\dagger]\rangle^{-1}$$
$$= \frac{1}{2\pi i}\delta^{(b)}_{(a)} \tag{A.25}$$

Comparing with the definition of the Lorentzian two-point pairing in [11], finally gives

$$\int_{1>x>2} \frac{d^dx_1 d^dx_2}{(\mathrm{vol}\,\mathrm{SO}(1,1))^2}(\mathcal{T}_2 K^t_{\Delta,J.\lambda(a)})\mathcal{T}_2\langle 0|\mathcal{O}_2\mathbf{L}[\mathcal{O}'^\dagger]\mathcal{O}_1|0\rangle^{(b)} = \frac{1}{2\pi i}\delta^{(b)}_{(a)}\langle \mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}'^\dagger]\rangle, \tag{A.26}$$

where the equality is valid if $x' \approx 1,2$. Finally, after changing variables $x_2 \to \mathcal{T}_2^{-1}x_2$, we obtain (2.152).

A.4 An alternative derivation for the light-ray OPE formula

In this appendix, we give another derivation for the light-ray OPE formula (2.147). We will first review the derivation for the low transverse spin terms given in [15]. Then we will derive the "higher transverse spin" terms by generalizing the derivation in [15].

A.4.1 Review: derivation of [15]

We first briefly review the proof given in [15]. We are interested in an expansion for

$$\mathbf{L}[\mathcal{O}_1](x,z_1)\mathbf{L}[\mathcal{O}_2](x,z_2). \tag{A.27}$$

For simplicity, we assume O_1, O_2 are traceless symmetric tensors. Generalization to arbitrary representations will become straightforward after finishing the proof. We will study the matrix element

$$W(z_1, z_2) = \langle\Omega|O_4\mathbf{L}[O_1](x, z_1)\mathbf{L}[O_2](x, z_2)O_3|\Omega\rangle, \tag{A.28}$$

where O_3, O_4 are some local primary operators. Then we can apply harmonic analysis on the celestial sphere to expand $W(z_1, z_2)$ into partial waves. The result is given by (2.91) and (2.93). The object $W_{\delta,j}(z_1, z_2)$ can be further written as

$$\begin{aligned} W_{\delta,j}(x, z) &= \alpha_{\delta,j} \int D^{d-2}z_1 D^{d-2}z_2 \langle\widetilde{\mathcal{P}}^\dagger_{\delta_1}(z_1)\widetilde{\mathcal{P}}^\dagger_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z)\rangle\langle\Omega|O_4\mathbf{L}[O_1](x, z_1)\mathbf{L}[O_2](x, z_2)O_3|\Omega\rangle \\ &= \int d^dx_1 d^dx_2 D^{d-2}z_1 D^{d-2}z_2\ \mathcal{L}_{\delta,j}(x_1, z_1, x_2, z_2; x, z)\langle\Omega|O_4O_1(x_1, z_1)O_2(x_2, z_2)O_3|\Omega\rangle, \end{aligned} \tag{A.29}$$

and the kernel $\mathcal{L}_{\delta,j}$ is

$$\begin{aligned} &\mathcal{L}_{\delta,j}(x_1, z_1, x_2, z_2; x, z) \\ &= \alpha_{\delta,j}\langle\widetilde{\mathcal{P}}^\dagger_{\delta_1}(z_1)\widetilde{\mathcal{P}}^\dagger_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z)\rangle \\ &\quad\times \int_{-\infty}^{\infty} d\alpha_1 d\alpha_2(-\alpha_1)^{-\delta_1-J_1-1}(-\alpha_2)^{-\delta_2-J_2-1}\delta^{(d)}\left(x - z_1/\alpha_1 - x_1\right)\delta^{(d)}\left(x - z_2/\alpha_2 - x_2\right). \end{aligned} \tag{A.30}$$

Note that we have suppressed the Lorentz indices carried by $W_{\delta,j}$ and $\mathcal{L}_{\delta,j}$. They are contracted with the indices carried by $C_{\delta,j}$ in (2.91).

Using (A.29) and the fact that $\mathbf{L}[O_i]$ annihilates the vacuum, we have

$$\begin{aligned} W_{\delta,j}(x, z) = \int d^dx_1 d^dx_2 D^{d-2}z_1 D^{d-2}z_2 \mathcal{L}_{\delta,j}(x_1, z_1, x_2, z_2; x, z)\theta(4 > 1)\theta(2 > 3) \\ \times\langle\Omega|[O_4, O_1(x_1, z_1)][O_2(x_2, z_2), O_3]|\Omega\rangle. \end{aligned} \tag{A.31}$$

By conformal invariance, we also have

$$W_{\delta,j}(x, z) = A_b(\delta, j)\langle 0|O_4\mathbf{L}[O](x, z)O_3|0\rangle_+^{(b)}. \tag{A.32}$$

where O has quantum numbers $(\Delta, J, \lambda) = (\delta + 1, J_1 + J_2 - 1, j)$, (b) is the tensor structure index, and $\langle 0|O_4\mathbf{L}[O](x, z)O_3|0\rangle_+^{(b)}$ is the continuous-spin structure analytically continued from even spin.[1] In order to proceed, we need to introduce

[1]The analytic continuation of $\langle 0|O_4\mathbf{L}[O](x, z)O_3|0\rangle^{(b)}$ to complex spin has to be done separately for even and odd spin due to our convention of the three-point function, so we have to make a choice in the right-hand side of (A.32). However, this choice does not affect our final result. It will only change how we relate $C^\pm(\Delta, J)$ and $\mathbb{O}^\pm(\Delta, J)$ in the final step of our derivation. With our choice in (A.32), we need to identify $C^+(\Delta, J) \to \mathbb{O}^+_{\Delta,J}$ and $C^-(\Delta, J) \to -\mathbb{O}^-_{\Delta,J}$.

conformally-invariant pairings for two-point and three-point continuous spin structures in the Lorentzian signature. The pairings are described in detail in appendix E of [11]. For a two-point structure of $\mathcal{O}$ in representation (Δ, J, λ), it can be paired with a two-point structure of $\mathcal{O}^S$ in representation $(d-\Delta, 2-d-J, \lambda)$. The two-point pairing is defined by

$$\begin{aligned}
&\frac{\left(\langle\mathcal{O}\mathcal{O}^\dagger\rangle, \langle\mathcal{O}^S\mathcal{O}^{S\dagger}\rangle\right)_L}{\mathrm{vol}(\mathrm{SO}(1,1))^2} \\
&\equiv \int_{x_1\approx x_2} \frac{d^dx_1 d^dx_2 D^{d-2}z_1 D^{d-2}z_2}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\langle\mathcal{O}^a(x_1,z_1)\mathcal{O}^{b\dagger}(x_2,z_2)\rangle\langle\mathcal{O}^S_b(x_2,z_2)\mathcal{O}^{S\dagger}_a(x_1,z_1)\rangle \\
&= \frac{\langle\mathcal{O}^a(0,z_1)\mathcal{O}^{b\dagger}(\infty,z_2)\rangle\langle\mathcal{O}^S_b(\infty,z_2)\mathcal{O}^{S\dagger}_a(0,z_1)\rangle}{2^{2d-2}\,\mathrm{vol}(\mathrm{SO}(d-2))}\frac{1}{(-2z_1\cdot z_2)^{2-d}},
\end{aligned} \tag{A.33}$$

where in the last line we use $\widetilde{\mathrm{SO}}(d,2)$ transformations to gauge-fix to $x_1 = 0, x_2 = \infty$, and a, b are the indices carried by the representation λ. The three-point pairing is defined by

$$\begin{aligned}
&\left(\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle, \langle\widetilde{\mathcal{O}}_1^\dagger\widetilde{\mathcal{O}}_2^\dagger\mathcal{O}^{S\dagger}\rangle\right)_L \\
&\equiv \int_{\substack{2<1\\ x\approx 1,2}} \frac{d^dx_1 d^dx_2 d^dx D^{d-2}z}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}(x,z)\rangle\langle\widetilde{\mathcal{O}}_1^\dagger(x_1)\widetilde{\mathcal{O}}_2^\dagger(x_2)\mathcal{O}^{S\dagger}(x,z)\rangle \\
&= \frac{1}{2^{2d-2}\,\mathrm{vol}(\mathrm{SO}(d-2))}\frac{\langle\mathcal{O}_1(e^0)\mathcal{O}_2(0)\mathcal{O}(\infty,z)\rangle\langle\widetilde{\mathcal{O}}_1^\dagger(e^0)\widetilde{\mathcal{O}}_2^\dagger(0)\mathcal{O}^{S\dagger}(\infty,z)\rangle}{(-2z\cdot e^0)^{2-d}}.
\end{aligned} \tag{A.34}$$

Similarly, in the last line, we gauge-fixed $x_1 = e^0, x_2 = 0, x = \infty$.

We can then obtain $A_b(\delta, j)$ by taking a Lorentzian three-point pairing of both sides with a dual structure

$$\begin{aligned}
A_b(\delta,j) &= \left(\left(\mathcal{T}_4\langle 0|\mathcal{O}_4\mathbf{L}[\mathcal{O}](x,z)\mathcal{O}_3|0\rangle_+^{(b)}\right)^{-1}, \mathcal{T}_4 W_{\delta,j}(x,z)\right)_L \\
&= \int_{\substack{4>1\\2>3}} \frac{d^dx_1 d^dx_2 d^dx_3 d^dx_4 D^{d-2}z_1 D^{d-2}z_2}{\mathrm{vol}\,\widetilde{\mathrm{SO}}(d,2)}\langle\Omega|\left[\mathcal{O}_4,\mathcal{O}_1(x_1,z_1)\right]\left[\mathcal{O}_2(x_2,z_2),\mathcal{O}_3\right]|\Omega\rangle \\
&\quad\times\mathcal{T}_2^{-1}\mathcal{T}_4^{-1}\Bigg[\int d^dx D^{d-2}z\left(\mathcal{T}_4\langle 0|\mathcal{O}_4\mathbf{L}[\mathcal{O}](x,z)\mathcal{O}_3|0\rangle_+^{(b)}\right)^{-1} \\
&\qquad\times(\mathcal{T}_2\mathcal{L}_{\delta,j})(x_1,z_1,x_2,z_2;x,z)\theta(4^+>1)\theta(2^+>3)\Bigg],
\end{aligned} \tag{A.35}$$

where $\mathcal{T}_2, \mathcal{T}_4$ translate the points x_2, x_4 to the next Poincaré patch on the Lorentzian cylinder. They are introduced so that the causality configuration in the three-point

Lorentzian pairing is satisfied. The structure $\left(\mathcal{T}_4\langle 0|O_4\mathbf{L}[O](x,z)O_3|0\rangle_+^{(b)}\right)^{-1}$ is defined by

$$\left(\left(\mathcal{T}_4\langle 0|O_4\mathbf{L}[O](x,z)O_3|0\rangle_+^{(b)}\right)^{-1}, \mathcal{T}_4\langle 0|O_4\mathbf{L}[O](x,z)O_3|0\rangle_+^{(d)}\right)_L = \delta_{(b)}^{(d)}. \quad \text{(A.36)}$$

The derivation up to this point is true for all transverse spin j. In the rest of this section we first finish the derivation assuming $j \leq j_{\text{max}}$, and explain what goes wrong when $j > j_{\text{max}}$, where j_{max} is the maximal allowed transverse spin in the $O_1 \times O_2$ OPE. We then give the derivation for $j > j_{\text{max}}$ in section A.4.3. Note that since we assume O_1, O_2 are traceless symmetric tensors, in what follows we will simply use $j_{\text{max}} = J_1 + J_2$.

In (A.35), the term in the bracket is a conformally-invariant four-point structure, and is an eigenfunction of the quadratic Casimir acting on 1,2 (or 3,4). Therefore it is a linear combination of conformal blocks, and we can study it by taking the OPE limit. In the OPE limit $x_3, x_4 \to x'$, the 34 three-point structure should be given by a linear operator B_{34O} acting on a two-point function:[2]

$$\begin{aligned}&\left(\mathcal{T}_4\langle 0|O_4\mathbf{L}[O](x,z)O_3|0\rangle_+^{(b)}\right)^{-1}\\ &= B_{34O}(x_3, x_4, \partial_{x'}, \partial_{z'})\langle O^F(x', z')O^{F\dagger}(x,z)\rangle,\end{aligned} \quad \text{(A.37)}$$

where O^F has quantum numbers $(\Delta_F, J_F, j_F) = (J + d - 1, \Delta - d + 1, j)$, where $J = J_1 + J_2 - 1$ and $\Delta = \delta + 1$. Plugging this into (A.35), one can then show that the bracketed term is the conformal block that appears in the Lorentzian inversion formula, and therefore we can relate $A_b(\delta, j)$ to $C^{\pm}(\delta + 1, J_1 + J_2 - 1, j)$. Using (A.32) and the relation between light-ray operators $\mathbb{O}^{\pm}_{\Delta,J}$ and $C^{\pm}(\Delta, J)$, we obtain that

$$\begin{aligned}&\mathbf{L}[O_1](x,z_1)\mathbf{L}[O_2](x,z_2)\\ &= \sum_{j \leq J_1+J_2} \int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty} \frac{d\delta}{2\pi i} C^{(a)}_{\delta,j}(z_1, z_2, \partial_z)\left(\mathbb{O}^+_{\delta+1,J_1+J_2-1,j(a)}(x,z) - \mathbb{O}^-_{\delta+1,J_1+J_2-1,j(a)}(x,z)\right)\\ &\quad + \text{higher transverse spin},\end{aligned} \quad \text{(A.38)}$$

where the differential operator $C^{(a)}_{\delta,j}(z_1, z_2, \partial_z)$ can be obtained from a celestial map formula

$$C^{(a)}_{\delta,j}(z_1, z_2, \partial_z)\langle\mathbf{L}[O](\infty, z_2)\mathbf{L}[O^\dagger](0, z_1)\rangle = \frac{\langle 0|\mathbf{L}^+[O_2](\infty, z_2)\mathbf{L}[O^\dagger](0,z)\mathbf{L}^-[O_1](\infty, z_1)|0\rangle_+^{(a)}}{\text{vol}\,\text{SO}(1,1)}. \quad \text{(A.39)}$$

[2]For continuous spin, the operator B_{34O} should be an integral operator. See appendix H of [11].

This result agrees with the first sum in (2.147).

When $j > J_1 + J_2$, the invalid step in this derivation is (A.37). The reason is that the linear operator $B_{34\mathcal{O}}$ becomes divergent when $j > J_1 + J_2$. To see this, consider a conformal block

$$C_{34\mathcal{O}}(x_3, x_4, \partial_{x'})\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}(x')\rangle. \tag{A.40}$$

It has been shown that this conformal block has simple poles in Δ, the scaling dimension of $\mathcal{O}$, due to null descendant states [35, 38].[3] Furthermore, the poles come from the differential operator $C_{34\mathcal{O}}$ and have the form

$$C_{34\mathcal{O}} \sim \frac{\mathcal{N}}{\Delta - \Delta_*} C_{34\mathcal{O}'}\mathcal{D}, \tag{A.41}$$

where $\mathcal{N}$ is some coefficient, $\mathcal{O}'$ is a primary descendant of $\mathcal{O}$, $\mathcal{D}$ is a differential operator such that $\mathcal{O}' = \mathcal{D}\mathcal{O}$ (at $\Delta = \Delta_*$), and $C_{34\mathcal{O}'}$ is the OPE operator for $\mathcal{O}' \in \mathcal{O}_3 \times \mathcal{O}_4$. The possible pole positions Δ_* are classified in [35] using representation theory of the conformal group. For us, the relevant cases are what are called type I and type II poles in [35]. If the exchanged operator has representation $(l_1, l_2, \ldots, l_N)$, where l_k is the number of boxes of the k-th row of the Young diagram, then the positions of the type I and type II poles are

$$\begin{aligned} \Delta^*_{\mathrm{I}_k,n} &= k - l_k - n && (n = 1, 2, \ldots, l_{k-1} - l_k) \\ \Delta^*_{\mathrm{II}_k,n} &= d + l_k - k - n && (n = 1, 2, \ldots, l_k - l_{k+1}), \end{aligned} \tag{A.42}$$

and we call $\mathcal{D}$ for the type I and type II case $\mathcal{D}_{\mathrm{I}_k,n}$ and $\mathcal{D}_{\mathrm{II}_k,n}$ respectively.[4] In (A.37), the operator $\mathcal{O}^F$ has scaling dimension $\Delta_F = J + d - 1 = d + J_1 + J_2 - 2$. Therefore, if $j = J_1 + J_2 + n$ for some $n \geq 1$, then $\Delta^*_{\mathrm{II}_{k=2},n} = d + j - 2 - n = \Delta_F$. This implies that the linear operator $B_{34\mathcal{O}}$ is divergent for all $j > J_1 + J_2$. To fix (A.37), we can separate the pole part and the finite part of $B_{34\mathcal{O}}$ near $\Delta_F = \Delta^*_{\mathrm{II}_2,n}$:

$$B_{34\mathcal{O}} = \frac{1}{\Delta_F - \Delta^*_{\mathrm{II}_2,n}} C_{34\mathcal{O}'}\mathcal{D}_{\mathrm{II}_2,n} + B^{\text{finite}}_{34\mathcal{O}}, \tag{A.43}$$

where $n = j - J_1 - J_2$, and $C_{34\mathcal{O}'}$ is a new linear operator proportional to the OPE operator of $\mathcal{D}_{\mathrm{II}_2,n}\mathcal{O}^F \in \mathcal{O}_3 \times \mathcal{O}_4$. Plugging the above expression for $B_{34\mathcal{O}}$ into (A.37),

[3] In even d, some of the poles can become double poles when more than one simple poles are at the same position. However, one can explicitly check that for the case we are discussing (type II poles with $k = 2$), the poles do not overlap with other poles and hence remain simple poles in even $d \geq 4$.

[4] $\mathcal{D}_{\mathrm{I}_2,n}$ is the differential operator $\mathcal{D}_n$ we use extensively in the main text.

we obtain

$$\begin{aligned}&\left(\mathcal{T}_4\langle 0|O_4\mathbf{L}[O](x,z)O_3|0\rangle_+^{(b)}\right)^{-1}\\&=B_{34O}^{\text{finite}}(x_3,x_4,\partial_{x'},\partial_{z'})\langle O^F(x',z')O^{F\dagger}(x,z)\rangle\\&\quad+C_{34O'}(x_3,x_4,\partial_{x'},\partial_{z'})\langle \mathcal{D}_{\mathrm{II}_2,n}O^F(x',z')O^{F\dagger}(x,z)\rangle_{\text{lim}},\end{aligned}\tag{A.44}$$

where

$$\langle \mathcal{D}_{\mathrm{II}_2,n}O^F(x',z')O^{F\dagger}(x,z)\rangle_{\text{lim}}=\lim_{\Delta_F\to\Delta^*_{\mathrm{II}_2,n}}\frac{\langle \mathcal{D}_{\mathrm{II}_2,n}O^F(x',z')O^{F\dagger}(x,z)\rangle}{\Delta_F-\Delta^*_{\mathrm{II}_2,n}}.\tag{A.45}$$

Note that $\langle \mathcal{D}_{\mathrm{II}_2,n}O^F(x',z')O^{F\dagger}(x,z)\rangle\sim O(\Delta_F-\Delta^*_{\mathrm{II}_2,n})$ because when $\Delta_F=\Delta^*_{\mathrm{II}_2,n}$ it is a two-point function between two primaries with different scaling dimensions and therefore is zero. So the above limit should be finite. This structure is not conformally-invariant, since it does not vanish under the special conformal transformation K_μ. However, the result we get by integrating it against $\mathcal{T}_2\mathcal{L}_{\delta,j}$ is still a conformally-invariant three-point structure. To see this, note that $[K_\mu,\mathcal{D}_{\mathrm{II}_2,n}]=O(\Delta_2-\Delta^*_{\mathrm{II}_2,n})$ since $\mathcal{D}_{\mathrm{II}_2,n}$ is conformally-invariant at $\Delta^*_{\mathrm{II}_2,n}$. So, we can define

$$\mathcal{D}_\mu\equiv\lim_{\Delta_F\to\Delta^*_{\mathrm{II}_2,n}}\frac{[K_\mu,\mathcal{D}_{\mathrm{II}_2,n}]}{\Delta_F-\Delta^*_{\mathrm{II}_2,n}}.\tag{A.46}$$

Then, we have

$$\begin{aligned}&K_\mu\int d^dxD^{d-2}z\langle \mathcal{D}_{\mathrm{II}_2,n}O^F(x',z')O^{F\dagger}(x,z)\rangle_{\text{lim}}\mathcal{T}_2\mathcal{L}_{\delta,j}(x_1,z_1,x_2,z_2,x,z)\\&=\int d^dxD^{d-2}z\langle \mathcal{D}_\mu O^F(x',z')O^{F\dagger}(x,z)\rangle\mathcal{T}_2\mathcal{L}_{\delta,j}(x_1,z_1,x_2,z_2,x,z).\end{aligned}\tag{A.47}$$

This is simply a derivative of the Lorentzian shadow transform of $\mathcal{T}_2\mathcal{L}_{\delta,j}$. In [15] it has been shown that the Lorentzian shadow transform of $\mathcal{T}_2\mathcal{L}_{\delta,j}$ vanishes for $j>J_1+J_2$, and therefore (A.47) should vanish as well.

The operator $\mathcal{D}_{\mathrm{II}_2,n}O^F$ has quantum numbers $(J+d-1+n,\Delta-d+1,j-n)$. Thus, the appearance of $C_{34O'}$ in (A.44) suggests that for $j=J_1+J_2+n$, $A_b(\delta,j)$ should come from exchanged operators with quantum numbers $(\delta+1,J_1+J_2-1+n,J_1+J_2)$. As we will see briefly, this is indeed the case.

A.4.2 Relation between $\mathcal{D}_{\mathrm{I}_2,n}$ and $\mathcal{D}_{\mathrm{II}_2,n}$

We now describe an interesting relation between the differential operators $\mathcal{D}_{\mathrm{I}_2,n}$ and $\mathcal{D}_{\mathrm{II}_2,n}$. This relation will be used later in the derivation for the $j>J_1+J_2$ case.

First, note that $\mathcal{D}_{\mathrm{I}_2,n}$ and $\mathcal{D}_{\mathrm{II}_2,n}$ change the quantum numbers in the following way:

$$\begin{aligned}
\mathcal{D}_{\mathrm{I}_2,n} &: (2-j, l_1, j-n) \to (2-j+n, l_1, j) \\
\mathcal{D}_{\mathrm{II}_2,n} &: (d+j-n-2, 2-d-l_1, j) \to (d+j-2, 2-d-l_1, j-n),
\end{aligned} \tag{A.48}$$

and their explicit expressions can be chosen to be [5]

$$\begin{aligned}
\mathcal{D}_{\mathrm{I}_2,n} &= \frac{(-1)^n\Gamma(j-n-1-l_1)}{\Gamma(j-1-l_1)\Gamma(n+1)}\left(\partial_x \cdot \mathcal{D}^{0+}_{z,w}\right)^n \\
\mathcal{D}_{\mathrm{II}_2,n} &= \frac{(-1)^n\Gamma(j-n-1-l_1)}{\Gamma(j-1-l_1)\Gamma(n+1)}\left(\partial_x \cdot \mathcal{D}^{0-}_{z,w}\right)^n,
\end{aligned} \tag{A.49}$$

where $\mathcal{D}^{0+}_{z,w}$ and $\mathcal{D}^{0-}_{z,w}$ are weight-shifting operators that increase and decrease the transverse spin, respectively [39]. As shown in [45], in a pairing between operators one can always integrate weight-shifting operators by parts. In particular, we have

$$(\mathcal{D}^{0-}_{z,w}|_{J,j})^* = -2j(h-2+j)\mathcal{D}^{0+}_{z,w}|_{2-d-J,j-1}, \tag{A.50}$$

where $h = \frac{d-2}{2}$, and $\mathcal{D}|_{J,j}$ indicates that $\mathcal{D}$ acts on a multiplet with usual spin J and transverse spin j. Using this relation, one can show that

$$(\mathcal{D}_{\mathrm{II}_2,n}|_{d-\Delta-n,2-d-l_1,j})^* = N_{n,j}\mathcal{D}_{\mathrm{I}_2,n}|_{\Delta,l_1,j-n}, \tag{A.51}$$

where

$$N_{n,j} = 2^n \frac{\Gamma(j+1)\Gamma(h-1+j)}{\Gamma(j-n+1)\Gamma(h-1+j-n)}. \tag{A.52}$$

Note that this relation holds for general Δ, but $\mathcal{D}_{\mathrm{I}_2,n}$ and $\mathcal{D}_{\mathrm{II}_2,n}$ are conformally-invariant only when $\Delta = 2-j$. More explicitly, for an operator $\mathcal{O}_1$ with quantum numbers $(\Delta, l_1, j-n)$ and $\mathcal{O}_2$ with quantum numbers $(d-\Delta-n, 2-d-l_1, j)$, we have

$$\int d^dxD^{d-2}z\, \mathcal{O}_1(x,z)(\mathcal{D}_{\mathrm{II}_2,n}\mathcal{O}_2)(x,z) = N_{n,j}\int d^dxD^{d-2}z\, (\mathcal{D}_{\mathrm{I}_2,n}\mathcal{O}_1)(x,z)\mathcal{O}_2(x,z). \tag{A.53}$$

In other words, $\mathcal{D}_{\mathrm{II}_2,n}$ and $N_{n,j}\mathcal{D}_{\mathrm{I}_2,n}$ are adjoint to each other.

We end this section by deriving two relations that will be used later in the derivation for the higher-transverse spin terms. First, consider operators $\mathcal{O}_1$ with quantum

[5] After replacing $j-n \to j$ and $l_1 \to 1-\Delta$, the definition of $\mathcal{D}_{\mathrm{I}_2,n}$ agrees with (2.66).

numbers $(\Delta_1, l_1, j-n)$ and $\mathcal{O}_2$ with quantum numbers $(\Delta_2, 2-d-l_1, j)$, where $\Delta_1+\Delta_2 = d$. Then by (A.51), we have for general Δ_1, Δ_2

$$\begin{aligned}&\int_{x\approx x'} d^dx d^dx' D^{d-2}z D^{d-2}z' \langle \mathcal{D}_{\mathrm{II}_2,n}\mathcal{O}_2 \mathcal{D}_{\mathrm{II}_2,n}\mathcal{O}_2'^{\dagger}\rangle \langle \mathcal{O}_1\mathcal{O}_1'^{\dagger}\rangle \\ &= N_{n,j}^2 \int_{x\approx x'} d^dx d^dx' D^{d-2}z D^{d-2}z' \langle \mathcal{O}_2\mathcal{O}_2'^{\dagger}\rangle \langle \mathcal{D}_{\mathrm{I}_2,n}\mathcal{O}_1 \mathcal{D}_{\mathrm{I}_2,n}\mathcal{O}_1'^{\dagger}\rangle, \end{aligned} \tag{A.54}$$

where we use the short-hand notation that $\mathcal{O}_i$ is at point (x,z) and $\mathcal{O}_i'$ is at point (x',z'). Setting $\Delta_1 \to \Delta^*_{\mathrm{I}_2,n}$ and $\Delta_2 \to \Delta^*_{\mathrm{II}_2,n}$, this equation can be rewritten as

$$\begin{aligned}&(\Delta_2 - \Delta^*_{\mathrm{II}_2,n}) \int_{x\approx x'} d^dx d^dx' D^{d-2}z D^{d-2}z' \lim_{\Delta_2\to\Delta^*_{\mathrm{II}_2,n}} \frac{\langle \mathcal{D}_{\mathrm{II}_2,n}\mathcal{O}_2 \mathcal{D}_{\mathrm{II}_2,n}\mathcal{O}_2'^{\dagger}\rangle}{\Delta_2 - \Delta^*_{\mathrm{II}_2,n}} \langle \mathcal{O}_1\mathcal{O}_1'^{\dagger}\rangle \\ &= N_{n,j}^2 (\Delta_1 - \Delta^*_{\mathrm{I}_2,n}) \int_{x\approx x'} d^dx d^dx' D^{d-2}z D^{d-2}z' \langle \mathcal{O}_2\mathcal{O}_2'^{\dagger}\rangle \lim_{\Delta_1\to\Delta^*_{\mathrm{I}_2,n}} \frac{\langle \mathcal{D}_{\mathrm{I}_2,n}\mathcal{O}_1 \mathcal{D}_{\mathrm{I}_2,n}\mathcal{O}_1'^{\dagger}\rangle}{\Delta_1 - \Delta^*_{\mathrm{I}_2,n}}. \end{aligned} \tag{A.55}$$

Finally, using $(\Delta_2 - \Delta^*_{\mathrm{II}_2,n}) = -(\Delta_1 - \Delta^*_{\mathrm{I}_2,n})$, we can conclude that

$$\left(\lim_{\Delta_2\to\Delta^*_{\mathrm{II}_2,n}} \frac{\langle \mathcal{D}_{\mathrm{II}_2,n}\mathcal{O}_2 \mathcal{D}_{\mathrm{II}_2,n}\mathcal{O}_2^{\dagger}\rangle}{\Delta_2 - \Delta^*_{\mathrm{II}_2,n}}, \langle \mathcal{O}_1\mathcal{O}_1^{\dagger}\rangle \right)_L = -N_{n,j}^2 \left(\langle \mathcal{O}_2\mathcal{O}_2^{\dagger}\rangle, \lim_{\Delta_1\to\Delta^*_{\mathrm{I}_2,n}} \frac{\langle \mathcal{D}_{\mathrm{I}_2,n}\mathcal{O}_1 \mathcal{D}_{\mathrm{I}_2,n}\mathcal{O}_1^{\dagger}\rangle}{\Delta_1 - \Delta^*_{\mathrm{I}_2,n}} \right)_L. \tag{A.56}$$

The second relation is about the integral

$$\int d^dx_1 d^dx_2 d^dx' D^{d-2}z' \langle \mathcal{D}_{\mathrm{II}_2,n}\mathcal{O}^F(x',z')\mathcal{O}^{F\dagger}(x,z)\rangle F(x_1,x_2,x,z) \mathcal{T}_2\langle 0|\mathcal{O}_2 \mathbf{L}[\mathcal{O}^\dagger](x',z')\mathcal{O}_1|0\rangle_{+}\theta((1>2) \approx \tag{A.57}$$

where $\mathcal{O}^F$ has quantum numbers (Δ_F, J_F, j), $\mathcal{O}$ has $(\Delta, J, j-n)$, and $F(x_1,x_2,x,z)$ is a conformally-invariant kernel that transforms as $\langle \widetilde{\mathcal{O}}_1^\dagger \widetilde{\mathcal{O}}_2^\dagger \mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[\mathcal{O}]\rangle$. Note that $\Delta_F + n + (1-J) = d$ in order for the integral to be conformally-invariant. By applying (A.51), we have for general Δ_F

$$\begin{aligned}&\int d^dx_1 d^dx_2 d^dx' D^{d-2}z' \langle \mathcal{D}_{\mathrm{II}_2,n}\mathcal{O}^F(x',z')\mathcal{O}^{F\dagger}(x,z)\rangle F(x_1,x_2,x,z) \mathcal{T}_2\langle 0|\mathcal{O}_2 \mathbf{L}[\mathcal{O}^\dagger](x',z')\mathcal{O}_1|0\rangle_{+}\theta((1>2) \approx \\ &= N_{n,j} \int d^dx_1 d^dx_2 d^dx' D^{d-2}z' \langle \mathcal{O}^F(x',z')\mathcal{O}^{F\dagger}(x,z)\rangle F(x_1,x_2,x,z) \\ &\qquad \times \mathcal{D}_{\mathrm{I}_2,n}\left(\mathcal{T}_2\langle 0|\mathcal{O}_2 \mathbf{L}[\mathcal{O}^\dagger](x',z')\mathcal{O}_1|0\rangle_{+}\theta((1>2)\approx x') \right). \end{aligned} \tag{A.58}$$

Now we set $\Delta_F \longrightarrow \Delta^*_{\mathrm{II}_2,n}$ and $J \longrightarrow J^*_I = \Delta^*_{\mathrm{II}_2,n} - d + 1 + n$, then $\mathcal{D}_{\mathrm{I}_2,n}$ and $\mathcal{D}_{\mathrm{II}_2,n}$ are conformally-invariant, and the above equation becomes

$$
\begin{aligned}
&(\Delta_F - \Delta^*_{\mathrm{II}_2,n}) \int d^dx_1 d^dx_2 d^dx' D^{d-2}z' \lim_{\Delta_F \to \Delta^*_{\mathrm{II}_2,n}} \frac{\langle \mathcal{D}_{\mathrm{II}_2,n} O^F(x',z') O^{F\dagger}(x,z)\rangle}{\Delta_F - \Delta^*_{\mathrm{II}_2,n}} F(x_1,x_2,x,z) \\
&\qquad\qquad \times \mathcal{T}_2 \langle 0|O_2 \mathbf{L}[O^\dagger](x',z') O_1|0\rangle_+ \theta((1>2)\approx x') \\
&= N_{n,j}(J - J^*_I) \int d^dx_1 d^dx_2 d^dx' D^{d-2}z' \langle O^F(x',z') O^{F\dagger}(x,z)\rangle F(x_1,x_2,x,z) \\
&\qquad\qquad \times \lim_{J\to J^*_I} \frac{1}{J - J^*_I} \mathcal{D}_{\mathrm{I}_2,n} \Big(\mathcal{T}_2 \langle 0|O_2 \mathbf{L}[O^\dagger](x',z') O_1|0\rangle_+ \theta((1>2)\approx x') \Big).
\end{aligned}
\tag{A.59}
$$

Moreover, in the right-hand side we can apply (2.209) and finally obtain

$$
\begin{aligned}
&\int d^dx_1 d^dx_2 d^dx' D^{d-2}z' \lim_{\Delta_F \to \Delta^*_{\mathrm{II}_2,n}} \frac{\langle \mathcal{D}_{\mathrm{II}_2,n} O^F(x',z') O^{F\dagger}(x,z)\rangle}{\Delta_F - \Delta^*_{\mathrm{II}_2,n}} F(x_1,x_2,x,z) \\
&\qquad\qquad \times \mathcal{T}_2 \langle 0|O_2 \mathbf{L}[O^\dagger](x',z') O_1|0\rangle_+ \theta((1>2)\approx x') \\
&= \mathrm{vol}(\mathrm{SO}(1,1)) N_{n,j} \int d^dx_1 d^dx_2 d^dx' D^{d-2}z' \langle O^F(x',z') O^{F\dagger}(x,z)\rangle F(x_1,x_2,x,z) \\
&\qquad\qquad \times \mathcal{D}_{\mathrm{I}_2,n} \Big(\mathcal{T}_2 \langle 0|O_2 \mathbf{L}[O^\dagger](x',z') O_1|0\rangle_+ \theta((1>2)\approx x') \Big),
\end{aligned}
\tag{A.60}
$$

where we also use the fact that $(\Delta_F - \Delta^*_{\mathrm{II}_2,n}) = (J - J^*_I)$.

A.4.3 Derivation for the higher transverse spin case

Now we give the derivation for the higher-transverse spin terms, where $j > J_1 + J_2$. We can follow the same steps for the $j \le J_1 + J_2$ until (A.31), but we should change (A.32) to [6]

$$
W_{\delta,j}(x,z) = A_b(\delta,j) \langle 0|O_4 \mathcal{D}_{\mathrm{I}_2,n} \mathbf{L}[O](x,z) O_3|0\rangle^{(b)}_+, \tag{A.61}
$$

where O has quantum numbers $(\Delta, J, j) = (\delta+1, J_1+J_2-1+n, J_1+J_2)$ and $n = j - J_1 - J_2$ in order for $\mathcal{D}_{\mathrm{I}_2,n}$ to be conformally-invariant. We expect that $A_b(\delta,j)$ is related to the OPE data of $O_1 \times O_2$ OPE. In particular, we want to show that $A_b(\delta,j)$ is proportional to $C^+(\delta+1, J_1+J_2-1+n, J_1+J_2) + C^-(\delta+1, J_1+J_2-1+n, J_1+J_2)$.

[6]Note that this is just a rewriting of (A.32), but in (A.32) the operator O has quantum numbers $(\delta+1, J_1+J_2-1, j)$. Here we use the relation $\mathbf{L}[O_{\delta+1,J_1+J_2-1,j}] \propto \mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[O_{\delta+1,J_1+J_2-1+n,J_1+J_2}]$ and call the operator $O_{\delta+1,J_1+J_2-1+n,J_1+J_2}$ as O. We hope that this does not cause confusion.

We follow the old derivation and take the Lorentz pairing with a dual structure:

$$\begin{aligned}
A_b(\delta,j) &= \left(\left(\mathcal{T}_4\langle 0|O_4\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[O](x,z)O_3|0\rangle_+^{(b)}\right)^{-1}, \mathcal{T}_4 W_{\delta,j}(x,z)\right)_L \\
&= \int_{\substack{4>1\\2>3}} \frac{d^dx_1 d^dx_2 d^dx_3 d^dx_4 D^{d-2}z_1 D^{d-2}z_2}{\mathrm{vol}\,\widetilde{\mathrm{SO}}(d,2)} \langle\Omega|\left[O_4, O_1(x_1,z_1)\right]\left[O_2(x_2,z_2), O_3\right]|\Omega\rangle \\
&\quad \times \mathcal{T}_2^{-1}\mathcal{T}_4^{-1}\left[\int d^dx D^{d-2}z \left(\mathcal{T}_4\langle 0|O_4\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[O](x,z)O_3|0\rangle_+^{(b)}\right)^{-1}\right. \\
&\qquad \left. \times (\mathcal{T}_2\mathcal{L}_{\delta,j})(x_1,z_1,x_2,z_2;x,z)\theta(4^+>1)\theta(2^+>3)\right].
\end{aligned} \tag{A.62}$$

In the OPE limit where $x_3, x_4 \to x'$, the step functions $\theta(4^+ > 1)\theta(2^+ > 3)$ should become $\theta((1 > 2) \approx x')$. Also, from the discussion in section A.4.1, we know that the three-point function $\left(\mathcal{T}_4\langle 0|O_4\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[O](x,z)O_3|0\rangle^{(b)}\right)^{-1}$ in the OPE limit is given by

$$\begin{aligned}
&\left(\mathcal{T}_4\langle 0|O_4\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[O](x,z)O_3|0\rangle_+^{(b)}\right)^{-1} \\
&= B_{34O}^{\mathrm{finite}}(x_3,x_4,\partial_{x'},\partial_{z'})\langle O^F(x',z')O^{F\dagger}(x,z)\rangle \\
&\quad + C_{34O'}(x_3,x_4,\partial_{x'},\partial_{z'})\langle\mathcal{D}_{\mathrm{II}_2,n}O^F(x',z')O^{F\dagger}(x,z)\rangle_{\mathrm{lim}}.
\end{aligned} \tag{A.63}$$

As shown in [15], one gets zero after integrating the finite part $B_{34O}^{\mathrm{finite}}\langle O^F O^{F\dagger}\rangle$ against $\mathcal{T}_2\mathcal{L}_{\delta,j}$. Therefore, we have

$$\begin{aligned}
&\int d^dx D^{d-2}z \left(\mathcal{T}_4\langle 0|O_4\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[O]O_3|0\rangle_+^{(b)}\right)^{-1}(\mathcal{T}_2\mathcal{L}_{\delta,j}(x_1,z_1,x_2,z_2;x,z))\theta(4^+>1)\theta(2^+>3) \\
&= C_{34O'}(x_3,z_3,x_4,z_4,\partial_{x'},\partial_{z'}) \lim_{\Delta_F\to\Delta^*_{\mathrm{II}_2,n}} \frac{1}{\Delta_F-\Delta^*_{\mathrm{II}_2,n}} \\
&\quad \times \int_{x\approx x'} d^dx D^{d-2}z\langle\mathcal{D}_{\mathrm{II}_2,n}O^F(x',z')O^{F\dagger}(x,z)\rangle\mathcal{T}_2\mathcal{L}_{\delta,j}(x_1,z_1,x_2,z_2;x,z)\theta((1>2)\approx x') \\
&= C_{34O'}(x_3,z_3,x_4,z_4,\partial_{x'},\partial_{z'})(\mathcal{D}_{\mathrm{II}_2,n}\mathbf{S}[\mathcal{T}_2\mathcal{L}_{\delta,j}])_{\mathrm{lim}}(x_1,z_1,x_2,z_2;x',z')\theta((1>2)\approx x'),
\end{aligned} \tag{A.64}$$

where $\mathbf{S}$ represents the Lorentzian shadow transform, and we have defined

$$\begin{aligned}
&(\mathcal{D}_{\mathrm{II}_2,n}\mathbf{S}[\mathcal{T}_2\mathcal{L}_{\delta,j}])_{\mathrm{lim}}(x_1,z_1,x_2,z_2;x',z') \\
&= \lim_{\Delta_F\to\Delta^*_{\mathrm{II}_2,n}} \frac{1}{\Delta_F-\Delta^*_{\mathrm{II}_2,n}} \int_{x\approx x'} d^dx D^{d-2}z\langle\mathcal{D}_{\mathrm{II}_2,n}O^F(x',z')O^{F\dagger}(x,z)\rangle\mathcal{T}_2\mathcal{L}_{\delta,j}(x_1,z_1,x_2,z_2;x,z).
\end{aligned} \tag{A.65}$$

This is a conformally-invariant three-point function that transforms like $\langle\widetilde{O}_1^\dagger\widetilde{O}_2^\dagger\mathcal{D}_{\mathrm{II}_2,n}O^F\rangle$. The expression in (A.64) should then give a conformal block. To compute it, let us first act $\mathcal{D}_{\mathrm{II}_2,n}$ on both sides of (A.63):

$$\begin{aligned}&\mathcal{D}_{\mathrm{II}_2,n}\left(\mathcal{T}_4\langle 0|O_4\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[O](x,z)O_3|0\rangle_+^{(b)}\right)^{-1}\\&=C_{34O'}(x_3,x_4,\partial_{x'},\partial_{z'})\lim_{\Delta_F\to\Delta^*_{\mathrm{II}_2,n}}\frac{\langle\mathcal{D}_{\mathrm{II}_2,n}O^F(x',z')\mathcal{D}_{\mathrm{II}_2,n}O^{F\dagger}(x,z)\rangle}{\Delta_F-\Delta^*_{\mathrm{II}_2,n}}.\end{aligned}\tag{A.66}$$

Using (A.51) and the definition of the three-point dual structure $\left(\mathcal{T}_4\langle 0|O_4\mathbf{L}[O](x,z)O_3|0\rangle^{(b)}\right)^{-1}$, one can find

$$\mathcal{D}_{\mathrm{II}_2,n}\left(\mathcal{T}_4\langle 0|O_4\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[O](x,z)O_3|0\rangle_+^{(b)}\right)^{-1}=N_{n,j}\left(\mathcal{T}_4\langle 0|O_4\mathbf{L}[O](x,z)O_3|0\rangle_+^{(b)}\right)^{-1}.\tag{A.67}$$

Combining (A.64), (A.66), and (A.67), we obtain

$$\begin{aligned}&\int d^dxD^{d-2}z\left(\mathcal{T}_4\langle 0|O_4\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[O]O_3|0\rangle_+^{(b)}\right)^{-1}(\mathcal{T}_2\mathcal{L}_{\delta,j}(x_1,z_1,x_2,z_2;x,z))\\&=N_{n,j}\frac{((\mathcal{D}_{\mathrm{II}_2,n}\mathbf{S}[\mathcal{T}_2\mathcal{L}_{\delta,j}])_{\mathrm{lim}}\theta((1>2)\approx x'))\left(\mathcal{T}_4\langle 0|O_4\mathbf{L}[O]O_3|0\rangle_+^{(b)}\right)^{-1}}{\lim_{\Delta_F\to\Delta^*_{\mathrm{II}_2,n}}\frac{\langle\mathcal{D}_{\mathrm{II}_2,n}O^F\mathcal{D}_{\mathrm{II}_2,n}O^{F\dagger}\rangle}{\Delta_F-\Delta^*_{\mathrm{II}_2,n}}}.\end{aligned}\tag{A.68}$$

This is a conformal block whose exchanged operator has quantum numbers $(J_1+J_2+d-2+n,\delta-d+2,J_1+J_2)$. Plugging this into (A.62) and comparing with the Lorentzian inversion formula, we find

$$\begin{aligned}A_b(\delta,j)&=-2\pi i\times\frac{1}{2}(C^+_{ab}(\delta+1,J_1+J_2-1+n,J_1+J_2)+C^-_{ab}(\delta+1,J_1+J_2-1+n,J_1+J_2))\\&\quad\times N_{n,j}\frac{\langle\mathbf{L}[O]\mathbf{L}[O^\dagger]\rangle^{-1}}{\lim_{\Delta_F\to\Delta^*_{\mathrm{II}_2,n}}\frac{\langle\mathcal{D}_{\mathrm{II}_2,n}O^F\mathcal{D}_{\mathrm{II}_2,n}O^{F\dagger}\rangle}{\Delta_F-\Delta^*_{\mathrm{II}_2,n}}}\\&\quad\times\left((\mathcal{D}_{\mathrm{II}_2,n}\mathbf{S}[\mathcal{T}_2\mathcal{L}_{\delta,j}])_{\mathrm{lim}}\theta((1>2)\approx x'),\mathcal{T}_2\langle 0|O_2\mathbf{L}[O^\dagger]O_1|0\rangle_+^{(a)}\right)_L.\end{aligned}\tag{A.69}$$

This result can be simplified by integrating $\mathcal{D}_{\mathrm{II}_2,n}$ by parts. For the second line, using (A.56) we have

$$N_{n,j}\frac{\langle\mathbf{L}[O]\mathbf{L}[O^\dagger]\rangle^{-1}}{\lim_{\Delta_F\to\Delta^*_{\mathrm{II}_2,n}}\frac{\langle\mathcal{D}_{\mathrm{II}_2,n}O^F\mathcal{D}_{\mathrm{II}_2,n}O^{F\dagger}\rangle}{\Delta_F-\Delta^*_{\mathrm{II}_2,n}}}=N_{n,j}^{-1}\frac{\left(\lim_{J\to J^*_{\mathrm{I}}}\frac{\langle\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[O]\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[O^\dagger]\rangle}{J-J^*_{\mathrm{I}}}\right)^{-1}}{\langle O^FO^{F\dagger}\rangle},\tag{A.70}$$

where $J_{\rm I}^* = J_1 + J_2 - 1 + n$. For the third line, using (A.60) we have

$$\begin{aligned}&\Big((\mathcal{D}_{\mathrm{II}_2,n}\mathbf{S}[\mathcal{T}_2\mathcal{L}_{\delta,j}])_{\lim}\theta((1>2)\approx x'),\mathcal{T}_2\langle 0|\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\mathcal{O}_1|0\rangle_+^{(a)}\Big)_L\\ &=\mathrm{vol}(\mathrm{SO}(1,1))N_{n,j}\int_{x\approx x'}\frac{d^dx_1d^dx_2d^dx'd^dxD^{d-2}z_1D^{d-2}z_2D^{d-2}z'D^{d-2}z}{\mathrm{vol}\,\widetilde{\mathrm{SO}}(d,2)}\langle\mathcal{O}^F(x',z')\mathcal{O}^{F\dagger}(x,z)\rangle\\ &\quad\times\mathcal{T}_2\mathcal{L}_{\delta,j}(x_1,z_1,x_2,z_2;x,z)\mathcal{D}_{\mathrm{I}_2,n}\Big(\mathcal{T}_2\langle 0|\mathcal{O}_2(x_2,z_2)\mathbf{L}[\mathcal{O}^\dagger](x',z')\mathcal{O}_1(x_1,z_1)|0\rangle_+^{(a)}\theta((1>2)\approx x')\Big).\end{aligned}\tag{A.71}$$

Plugging in the definition of $\mathcal{L}_{\delta,j}$ in (A.30), we find that $A_b(\delta,j)$ is given by

$$\begin{aligned}A_b(\delta,j)&=-\pi i(C^+_{ab}(\delta+1,J_1+J_2-1+n,J_1+J_2)+C^-_{ab}(\delta+1,J_1+J_2-1+n,J_1+J_2))\\ &\quad\times\frac{\left(\left(\lim_{J\to J_{\rm I}^*}\frac{\langle\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[\mathcal{O}]\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[\mathcal{O}^\dagger]\rangle}{J-J_{\rm I}^*}\right)^{-1},Q^{(a)}_{\delta,j}\right)_L}{\mathrm{vol}\,\mathrm{SO}(1,1)},\end{aligned}\tag{A.72}$$

where

$$\begin{aligned}Q^{(a)}_{\delta,j}(x,z,x',z')&=\alpha_{\delta,j}\int D^{d-2}z_1D^{d-2}z_2\,\langle\widetilde{\mathcal{P}}^\dagger_{\delta_1}(z_1)\widetilde{\mathcal{P}}^\dagger_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z)\rangle\\ &\quad\times\langle 0|\mathbf{L}^+[\mathcal{O}_2](x,z_2)\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[\mathcal{O}^\dagger](x',z')\mathbf{L}^-[\mathcal{O}_1(x,z_1)]|0\rangle_+^{(a)},\end{aligned}\tag{A.73}$$

where $\mathbf{L}^+[\mathcal{O}_2]$ indicates that the light transform contour is restricted to $2>x'$, and $\mathbf{L}^-[\mathcal{O}_1]$ is restricted to $1\approx x'$. Finally, comparing the expression of $A_b(\delta,j)$ with the old derivation for the lower transverse spin case, we find that the for higher transverse spin $j>J_1+J_2$, we have

$$\begin{aligned}&\mathbf{L}[\mathcal{O}_1](x,z_1)\mathbf{L}[\mathcal{O}_2](x,z_2)\\ &=\sum_{n=1}^{\infty}(-1)^n\int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty}\frac{d\delta}{2\pi i}C^{(a)}_{\delta,j}(z_1,z_2,\partial_z)\\ &\quad\times\Big(\mathcal{D}_{\mathrm{I}_2,n}\mathbb{O}^+_{\delta+1,J_1+J_2-1+n,J_1+J_2(a)}(x,z)-\mathcal{D}_{\mathrm{I}_2,n}\mathbb{O}^-_{\delta+1,J_1+J_2-1+n,J_1+J_2(a)}(x,z)\Big)\\ &\quad+\text{lower transverse spin},\end{aligned}\tag{A.74}$$

where $n=j-J_1-J_2$. The celestial map formula for $C^{(a)}_{\delta,j}$ is given by

$$\begin{aligned}&C^{(a)}_{\delta,j}(z_1,\mathbf{w}_1,z_2,\mathbf{w}_2,\partial_{z_2},\partial_{\mathbf{w}_2})\left(\lim_{J\to J_{\rm I}^*}\frac{\langle\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[\mathcal{O}](\infty,z_2,\mathbf{w}_2)\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[\mathcal{O}^\dagger](0,z_0,\mathbf{w}_0)\rangle}{J-J_{\mathrm{I}^*}}\right)\\ &=(-1)^n\langle 0|\mathbf{L}^+[\mathcal{O}_2](\infty,z_2,\mathbf{w}_2)\mathcal{D}_{\mathrm{I}_2,n}\mathbf{L}[\mathcal{O}^\dagger](0,z_0,\mathbf{w}_0)\mathbf{L}^-[\mathcal{O}_1](\infty,z_1,\mathbf{w}_1)|0\rangle_+^{(a)}.\end{aligned}\tag{A.75}$$

This result agrees with the second sum in (2.147).

A.5 Kernel of the celestial map

In the main text we have given the celestial map formulas (2.214) and (2.215) which map d-dimensional three-point tensor structures to OPE differential operators in $(d-2)$-dimensional space. The latter are in turn in one-to-one correspondence with three-point tensor structures in $(d-2)$-dimensional space. In general the space T_d of three-point structures in d and the space T_{d-2} of three-point structures in $d-2$ dimensions have different dimensionality. Therefore, the celestial map $T_d \to T_{d-2}$ in general has non-trivial kernel or cokernel. In this appendix we identify a part $K_0 \subseteq K$ of the kernel $K \subseteq T_d$ of this map and conjecture that it is in fact the entire kernel ($K = K_0$) and that the cokernel is trivial, i.e. that the celestial map is surjective. We give support to this conjecture by matching the dimension of T_d/K_0 with the dimension of T_{d-2}. Finally, we consider the $\mathrm{SO}(d-2)$ representations λ that can be generated by the celestial map and show that they cover all the representations appearing in the $(d-2)$-dimensional OPE.

We begin by identifying K_0. Let us start with the low transverse spin case (2.200). We are instructed to evaluate the structure from T_d in configuration (2.219) after multiplying by $V_{0,12}$. Since in this configuration $V_{0,12}$ vanishes, the only structures that survive are those which contain $V_{0,12}^{-1}$. Structures with more negative powers of $V_{0,12}$ will be singular and the structures with non-negative powers will vanish. We claim that no structures have more singular power of $V_{0,12}$ and that there is a simple rule for counting those with $V_{0,12}^{-1}$.

To see this, let us label the $\mathrm{SO}(d-1,1)$ irreps of the operators as $\rho_1 = (J_1, \lambda_1)$, $\rho_2 = (J_2, \lambda_2)$ and $\rho = (J, \lambda)$. For large J the number of three-point tensor structures is J-independent. This is because the number of structures is given [11] by the dimension of

$$(\rho_1 \otimes \rho_2 \otimes \rho)^{\mathrm{SO}(d-1)}. \tag{A.76}$$

The dependence on J can be exhibited by first computing the $\mathrm{SO}(d-1)$ content of $\rho_1 \otimes \rho_2$ and matching $\mathrm{SO}(d-1)$ irreps there to dual irreps in $\mathrm{SO}(d-1)$ content of ρ. As J is increased, all that happens is that new irreps appear in $\mathrm{SO}(d-1)$ decomposition of ρ, but any given irrep appears at most once.[7] Since $\rho_1 \otimes \rho_2$ contains finitely many $\mathrm{SO}(d-1)$ irreps, at some point the number of matching dual pairs stabilizes.

[7] In what follows we use facts about dimensional reduction of $\mathrm{SO}(N)$ irreps. See, e.g., [180] for a review.

As J is decreased to sufficiently low values, some structures disappear from this counting. In terms of explicit expressions this happens because the structures at large J depend on $V_{0,12}^{J-n}$ for various n, and as J becomes less than n such structures cease to be polynomial in z and have to disappear from the above counting. We are interested in the maximal n among all structures, which is therefore the same as the value of J at which the number of structures stabilizes. As J is increased by 1, the $\mathrm{SO}(d-1)$ content of ρ is appended by representations with the first row of length J. Since the maximal length of the first row of $\mathrm{SO}(d-1)$ representations in $\rho_1 \otimes \rho_2$ is $J_1 + J_2$,[8] it follows that the stable number of representations is achieved starting from at most $J = J_1 + J_2$. This implies that $n \leq J_1 + J_2$. This means that the power of $V_{0,12}$ is no smaller than

$$V_{0,12}^{J-J_1-J_2}. \tag{A.77}$$

This finishes the proof of the claim that structures analytically continued to $J = J_1 + J_2 - 1$ have at most $V_{0,12}^{-1}$ singularity.

We thus find that the map (2.214) is well-defined and the structures with $V_{0,12}$ to non-negative powers get mapped to 0. These structures constitute the set K_0. We conjecture that (2.214) is non-degenerate on the remaining structures in T_d, i.e. those which contain $V_{0,12}^{-1}$. To support this conjecture, let us count these structures and match their number to the number of structures in T_{d-2}.

To do that, note that the above discussion implies that the number of structures with $V_{0,12}^{-1}$ is precisely the difference between the number of polynomial structures for $J = J_1 + J_2$ and the number of polynomial structures for $J = J_1 + J_2 - 1$, i.e. the dimension of[9]

$$\begin{aligned}(\rho_1 \otimes \rho_2 \otimes (J_1 + J_2, \lambda))^{\mathrm{SO}(d-1)} &\ominus (\rho_1 \otimes \rho_2 \otimes (J_1 + J_2 - 1, \lambda))^{\mathrm{SO}(d-1)} \\ &= (\rho_1 \otimes \rho_2 \otimes \mathrm{Res}^{\mathrm{SO}(d-1,1)}_{\mathrm{SO}(d-1)}((J_1 + J_2, \lambda) \ominus (J_1 + J_2 - 1, \lambda)))^{\mathrm{SO}(d-1)}.\end{aligned} \tag{A.78}$$

We can simplify this by noting that

$$\mathrm{Res}^{\mathrm{SO}(d-1,1)}_{\mathrm{SO}(d-1)}((J_1 + J_2, \lambda) \ominus (J_1 + J_2 - 1, \lambda)) = \bigoplus_{\tau \in \mathrm{Res}^{\mathrm{SO}(d-2)}_{\mathrm{SO}(d-3)} \lambda} (J_1 + J_2, \tau). \tag{A.79}$$

[8] To see this, note that the length of the first row of the Young diagram gives the maximal eigenvalue under any given boost. Since it is J_i for ρ_i, it must be $J_1 + J_2$ for $\rho_1 \otimes \rho_2$. Choosing a boost in (a complexification of) $\mathrm{SO}(d-1)$ yields the desired result.

[9] Here we use the formal difference $\ominus$. To make this precise one can interpret all identities involving it as character identities.

In particular, this only involves $\mathrm{SO}(d-1)$ irreps with the first row of length J_1+J_2. As mentioned above, this is the maximal length of the first row in $\mathrm{SO}(d-1)$ irreps in $\rho_1\otimes\rho_2$, and this part of the tensor product $\rho_1\otimes\rho_2$ simplifies[10]

$$\mathrm{Res}^{\mathrm{SO}(d-1,1)}_{\mathrm{SO}(d-1)}\rho_1\otimes\rho_2=\bigoplus_{\tau\in\mathrm{Res}^{\mathrm{SO}(d-2)}_{\mathrm{SO}(d-3)}\lambda_1\otimes\lambda_2}(J_1+J_2,\tau)\oplus\cdots,\tag{A.80}$$

where the dots represent irreps with shorter first row. By comparing the last two equations we see that

$$\begin{aligned}&(\rho_1\otimes\rho_2\otimes(J_1+J_2,\lambda))^{\mathrm{SO}(d-1)}\ominus(\rho_1\otimes\rho_2\otimes(J_1+J_2-1,\lambda))^{\mathrm{SO}(d-1)}\\&=(\lambda_1\otimes\lambda_2\otimes\lambda)^{\mathrm{SO}(d-3)}\end{aligned}\tag{A.81}$$

which is the same rule as for counting the structures in T_{d-2}.

We now turn to the higher transverse spin case. In this case the celestial map is applied to three-point structures with representations (J_i,λ_i) and (J,λ) where $J=J_1+J_2-1+n$ and $\lambda=(J_1+J_2,\gamma)$. As discussed in section 2.5.3, the structures that contain $V_{0,12}$ to powers higher than the minimal possible $J-J_1-J_2$ are mapped to zero by the celestial map (2.215). Similarly to the above, we can determine the number of structures which contain $V_{0,12}^{J-J_1-J_2}$ by taking the difference between the number of polynomial structures at $J=J_1+J_2$ and $J=J_1+J_2-1$. However, since $\lambda=(J_1+J_2,\gamma)$, there are no polynomial structures for $J=J_1+J_2-1$. Therefore, all tensor structures with such λ contain $V_{0,12}^{J-J_1-J_2}$. In this case K_0 is trivial and we simply would like to match the dimensions of T_d and T_{d-2}.

Since $\lambda=(J_1+J_2,\gamma)$, the minimal length of the first row in $\mathrm{SO}(d-1)$ irreps contained in (J,λ) with sufficiently large J[11] is J_1+J_2, in particular

$$\mathrm{Res}^{\mathrm{SO}(d-1,1)}_{\mathrm{SO}(d-1)}(J,\lambda)=\bigoplus_{\tau\in\mathrm{Res}^{\mathrm{SO}(d-2)}_{\mathrm{SO}(d-3)}\lambda}(J_1+J_2,\tau)+\cdots,\tag{A.82}$$

where the dots represent irreps with longer first row. Taking into account (A.80), we find

$$(\rho_1\otimes\rho_2\otimes(J,\lambda))^{\mathrm{SO}(d-1)}=(\lambda_1\otimes\lambda_2\otimes\lambda)^{\mathrm{SO}(d-3)}.\tag{A.83}$$

[10] One can see this by treating $\mathrm{SO}(d-1,1)$ irreps as shortened parabolic Verma modules of $\mathrm{SO}(d-1,1)$, in which case it is analogous to the statement that all primaries of dimension $\Delta_1+\Delta_2$ one can build out of primaries $\mathcal{O}_1^a,\mathcal{O}_2^b$ of dimensions Δ_1,Δ_2 are those given by decomposing $\mathcal{O}_1^a\mathcal{O}_2^b$ into irreducible Lorentz irreps.

[11] We need the number of analytically-continued tensor structures, which is the same as for very large J.

Since λ_i has first row that is no larger than J_i, it follows that $\lambda_1 \otimes \lambda_2$ only contains $\mathrm{SO}(d-3)$ irreps with first row at most of length $J_1 + J_2$. Since the first row of λ is already $J_1 + J_2$, it makes no difference to increase it by n,

$$(\rho_1 \otimes \rho_2 \otimes (J, \lambda))^{\mathrm{SO}(d-1)} = (\lambda_1 \otimes \lambda_2 \otimes \lambda(+n))^{\mathrm{SO}(d-3)}. \tag{A.84}$$

This establishes the equality of dimensions of T_d and T_{d-2}.

Finally, let us discuss which representations λ can be generated through celestial map. The low transverse spin terms contain λ's which are parts of the $\rho = (J, \lambda)$ representations which appear in $\mathcal{O}_1 \times \mathcal{O}_2$ OPE. We claim that they cover all λ's that can appear in $(d-2)$-dimensional OPE and have first row length at most $J_1 + J_2$. Indeed, using any such λ in (A.81) we find that

$$\dim(\rho_1 \otimes \rho_2 \otimes (J, \lambda))^{\mathrm{SO}(d-1)} \geq \dim(\lambda_1 \otimes \lambda_2 \otimes \lambda)^{\mathrm{SO}(d-3)} \tag{A.85}$$

for $J = J_1 + J_2$ and thus for any larger J. Since the right-hand side is non-zero whenever λ appears in $(d-2)$-dimensional OPE, and the left-hand side being non-zero implies that (J, λ) appears in d-dimensional OPE, we obtain the desired result.

It remains to establish that the higher-transverse spin terms cover all λ's in $(d-2)$-dimensional OPE with first row of length more than $J_1 + J_2$. Any such λ' can be represented as $\lambda(+n)$ where λ has first row $J_1 + J_2$. This can be then used in (A.84) to conclude that (J, λ) appears in $\mathcal{O}_1 \times \mathcal{O}_2$ OPE for generic J.

A.6 Spinor Conventions

We use conventions from [50] but switch to mostly plus signature, see appendix A in that paper. A four-dimensional vector $z^\mu = (z^0, \vec{z})$ is represented by 2×2 matrix

$$\begin{aligned} z_{\alpha\dot{\alpha}} &= z_\mu (\overline{\sigma}^\mu)_{\alpha\dot{\alpha}} = z^0 \sigma^0 + \vec{z} \cdot \vec{\sigma}, \\ z^{\dot{\alpha}\alpha} &= z_\mu (\sigma^\mu)^{\dot{\alpha}\alpha} = -z^0 \sigma^0 + \vec{z} \cdot \vec{\sigma} \end{aligned} \tag{A.86}$$

where $\sigma^\mu = (1, \vec{\sigma})$, $\overline{\sigma}^\mu = (-1, \vec{\sigma})$ and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$.

Convention for lowering and raising of indices is

$$\begin{aligned} x^{\dot{\beta}\beta} &= \epsilon^{\beta\alpha} x_{\alpha\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\beta}}, \\ a^{\alpha\beta} &= \epsilon^{\alpha\gamma} a_{\gamma\delta} \epsilon^{\delta\beta} . \end{aligned} \tag{A.87}$$

The Levi-Civita tensors are normalized as follows

$$\epsilon^{12} = \epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = \epsilon^{\dot{1}\dot{2}} = 1 . \tag{A.88}$$

The X matrices are defined as follows

$$(X_{ijk})_{\alpha\dot{\alpha}} = \frac{(x_{ij})_{\alpha\dot{\beta}}}{x_{ij}^2}(x_{jk})^{\dot{\beta}\gamma}\frac{(x_{ki})_{\gamma\dot{\alpha}}}{x_{ik}^2}\,. \tag{A.89}$$

Using sigma matrix identities we can simplify

$$X_{ijk} = -\frac{x_{ij}\cdot x_{jk}x_{ki\mu} + x_{jk}\cdot x_{ki}x_{ij\mu} - x_{ij}\cdot x_{ki}x_{jk\mu}}{x_{ij}^2 x_{ki}^2}\overline{\sigma}^{\mu}\,. \tag{A.90}$$

A.7 Distributional formulas

In this appendix we formally derive some of the expressions involving distributions that were used in the main text.

A.7.1 Analytic continuation of distributions

In this section we prove (2.130), i.e. we study the analytic continuation of the distribution

$$(s+t)^a s^b t^c \theta(s)\theta(t) \tag{A.91}$$

from the region $a, b, c > 0$, where it is represented by a locally-integrable function, to general complex a, b, c.

First, let us clarify the idea of analytic continuation of a distribution. Let us restrict to one-variable case with one parameter, i.e. we consider a distribution $g_a(x)$ defined for values of parameter $a \in U \subseteq \mathbb{C}$. Assume that this distribution depends on a holomorphically. That is, for any test function $f(x)$ the pairing

$$\langle g_a, f\rangle \equiv \int dx g_a(x) f(x) \tag{A.92}$$

is a holomorphic function of a. We say that a distribution $h_a(x)$ defined and holomorphic for $a \in V \subseteq \mathbb{C}$ is an analytic continuation of $g_a(x)$ if $U \subseteq V$ and for any test function $f(x)$ we have

$$\langle h_a, f\rangle = \langle g_a, f\rangle \tag{A.93}$$

for all $a \in U$. Similarly, we say that $g_a(x)$ is meromorphic for $a \in U$ if $\langle g_a, f\rangle$ is meromorphic for any test function f and the set of poles is independent of f (of course, some poles may disappear for a specially chosen f). We say that $h(x)$ is the residue of $g_a(x)$ at a_* if $\langle h, f\rangle = \mathrm{res}_{a=a_*}\langle g_a, f\rangle$ for any f, etc. All these notions generalize straightforwardly to the case of several variables and several parameters.

Before studying (A.91), let us consider a simpler example,

$$g_a(x) = x^a\theta(x). \tag{A.94}$$

For $\operatorname{Re} a > -1$ this is an integrable function of x, and is holomorphic in a as a distribution. We claim that it admits analytic continuation to $\mathbb{C} \setminus \mathbb{Z}_{<0}$ that is meromorphic in $\mathbb{C}$ with simple poles at negative integer a. As a simple example, consider a test function $f(x)$ that is equal to e^{-x} for $x \geq 0$ and for $x < 0$ is completed in some smooth way so that it decays quickly at $x \to -\infty$.[12] We have then

$$\langle g_a, f\rangle = \int_0^\infty e^{-x}x^a dx = \Gamma(a+1), \tag{A.95}$$

which is indeed meromorphic and has simple poles at negative integer a. To see that this statement holds for more general test functions, recall that

$$(x \pm i\epsilon)^a \tag{A.96}$$

is a distribution that is an entire function of a.[13] For $\operatorname{Re} a > -1$ we can write the equality of distributions

$$x^a\theta(x) = \frac{(x-i\epsilon)^a e^{i\pi a} - (x+i\epsilon)^a e^{-i\pi a}}{2i\sin\pi a}, \tag{A.97}$$

where the right-hand side is in fact analytic for all $a \in \mathbb{C} \setminus \mathbb{Z}$. We can compute the residue at $a = -n$ as

$$\begin{aligned}\operatorname{res}_{a=-n}\frac{(x-i\epsilon)^a e^{i\pi a} - (x+i\epsilon)^a e^{-i\pi a}}{2i\sin\pi a} &= \frac{(x-i\epsilon)^{-n} - (x+i\epsilon)^{-n}}{2i\pi}\\ &= \frac{(-1)^{n-1}}{(n-1)!}\delta^{(n-1)}(x).\end{aligned} \tag{A.98}$$

In particular, this vanishes for $n \leq 0$, consistently with $g_a(x)$ being analytic for $\operatorname{Re} a > -1$.

We conclude that $g_a(x) = x^a\theta(x)$ can be analytically continued so that it has simple poles at $a = -n$ with residues

$$\operatorname{res}_{a=-n} x^a\theta(x) = \frac{(-1)^{n-1}}{(n-1)!}\delta^{(n-1)}(x). \tag{A.99}$$

[12] Everywhere in this section we can work with tempered distributions, so that e^{-x} is an appropriate test function for $x > 0$ (i.e. it is Schwartz).

[13] The notation $(x \pm i\epsilon)^a$ means the boundary value of x^a on real line, approached either from above or below. Since x^a has at most power-law singularity for any a, Vladimirov's theorem [181] ensures that its boundary values are well-defined tempered distributions in x, analytic in the parameter a. See, e.g. [182] for a review of these facts.

Let us perform a simple check with the $f(x)$ defined above. We have

$$\mathrm{res}_{a=-n}\langle g_a, f\rangle = \mathrm{res}_{a=-n}\Gamma(a+1) = \frac{(-1)^{n-1}}{(n-1)!} \tag{A.100}$$

and this is indeed equal to

$$\langle \mathrm{res}_{a=-n} g_a, f\rangle = \frac{(-1)^{n-1}}{(n-1)!}\langle \delta^{(n-1)}, f\rangle = \frac{(-1)^{n-1}}{(n-1)!}. \tag{A.101}$$

Note that the non-trivial part of this analytic continuation is taking care of the singularity at $x = 0$, since for $x > x_0 > 0$ the function $x^a\theta(x)$ is locally-integrable for any $a \in \mathbb{C}$.

This simple result for $x^a\theta(x)$ can be extended a more general setup: consider a finite set of k smooth functions $q_i(x)$, $x \in \mathbb{R}^n$ and consider the function

$$g_a(x) = \prod_{i=1}^{k} \theta(q_i(x))q_i(x)^{a_i}. \tag{A.102}$$

For Re $\alpha_i > 0$ this defines a locally-integrable function. Provided that the functions q_i are in general position (clarified below), we claim that the distribution $g_a(x)$ can be analytically continued to a distribution meromorphic for $a \in \mathbb{C}^k$. To see this, suppose we want to define the analytic continuation in a neighborhood of some point x_0 where r functions $q_{i_1}(x), \cdots, q_{i_r}(x)$ vanish. Provided that the matrix of derivatives

$$M_{jl} = \partial_l q_{i_j}(x_0) \tag{A.103}$$

has rank r (in particular, $r \leq n$), we can use $y_j \equiv q_{i_j}(x)$ as the first r coordinates in a neighborhood U of x_0. This condition is what we mean by "general position" above. With this choice of coordinates, we simply have

$$g_a(y) = \widetilde{g}_a(y)\prod_{j=1}^{r} y_j^{a_{i_j}}\theta(y_j), \tag{A.104}$$

where $\widetilde{g}_a(y)$ is a smooth function in U. Each of $y_j^{a_{i_j}}\theta(y_j)$ can be analytically continued as above, and we can take their product because they are distributions in different variables y_j. We can then finally multiply the resulting distribution by the smooth function $\widetilde{g}_a(y)$.

This more general result still does not apply to (A.91) because in (A.91) we have functions

$$q_1(s,t) = s, \quad q_2(s,t) = t, \quad q_3(s,t) = s+t \tag{A.105}$$

which are not in general position near $s, t = 0$. The conclusion about analytic continuation, however, still holds. To establish it, one needs to use a general result about resolution of singularities. We do not reproduce this argument, and instead refer to [183, 184]. Here we simply work out the required resolution in the concrete example of (A.91). Our goal is to define the integral

$$\int dsdt(s+t)^a s^b t^c \theta(s)\theta(t) f(s,t) \tag{A.106}$$

for test functions f. To do so, we define new coordinates u, v by

$$s = uv, \quad t = u(1-v). \tag{A.107}$$

The region $s, t > 0$ is mapped one-to-one onto the region $u > 0, 0 < v < 1$, and the integral (A.106) can be written as

$$\int dudvu^{a+b+c+1} v^b (1-v)^c \theta(u)\theta(v)\theta(1-v)\widetilde{f}(u,v) \tag{A.108}$$

where

$$\widetilde{f}(u,v) \equiv f(uv, u(1-v)). \tag{A.109}$$

Importantly, $\widetilde{f}(u,v)$ is a test function in u, v. Therefore, if we manage to define

$$u^{a+b+c+1} v^b (1-v)^c \theta(u)\theta(v)\theta(1-v) \tag{A.110}$$

as a distribution meromorphic in a, b, c, we are done. In this case, setting

$$q_1(u,v) = u, \quad q_2(u,v) = v, \quad q_3(u,v) = 1-v, \tag{A.111}$$

we find that q_i are in general position at all points x_0 where at least one function vanishes. (The coordinates u, v "resolve the singularity" that we had at $s, t = 0$.) In particular, we find poles at

$$a + b + c + 1 = -n \tag{A.112}$$

with residues proportional to $\delta^{(n)}(u)$, poles at

$$b = -n \tag{A.113}$$

with residues proportional to $\delta^{(n)}(v)$, and poles at

$$c = -n \tag{A.114}$$

with residues proportional to $\delta^{(n)}(1-v)$. As explained in the main text, we are interested in the pole near

$$a+b+c+1=-1, \tag{A.115}$$

in which case from the analysis above we get

$$u^{a+b+c+1}v^b(1-v)^c\theta(u)\theta(v)\theta(1-v) \sim \frac{1}{a+b+c+2}\delta(u)v^b(1-v)^c\theta(v)\theta(1-v). \tag{A.116}$$

Now we only need to pull this back to s,t coordinates, i.e. evaluate (A.108) with $\widetilde{f}$ given by (A.109). We find

$$\begin{aligned}
&\int dsdt(s+t)^a s^b t^c\theta(s)\theta(t)f(s,t)\\
&=\int dudvu^{a+b+c+1}v^b(1-v)^c\theta(u)\theta(v)\theta(1-v)\widetilde{f}(u,v)\\
&\sim\frac{1}{a+b+c+2}\int dudv\delta(u)v^b(1-v)^c\theta(v)\theta(1-v)\widetilde{f}(u,v)\\
&=\frac{1}{a+b+c+2}\int dvv^b(1-v)^c\theta(v)\theta(1-v)\widetilde{f}(0,v)\\
&=\frac{f(0,0)}{a+b+c+2}\int dvv^b(1-v)^c\theta(v)\theta(1-v)\\
&=\frac{f(0,0)}{a+b+c+2}\frac{\Gamma(b+1)\Gamma(c+1)}{\Gamma(b+c+2)}.
\end{aligned} \tag{A.117}$$

This implies that

$$(s+t)^a s^b t^c\theta(s)\theta(t) \sim \frac{1}{a+b+c+2}\frac{\Gamma(b+1)\Gamma(c+1)}{\Gamma(b+c+2)}\delta(s)\delta(t), \tag{A.118}$$

as stated in (2.130).

A.7.2 An identity

In this section, we show

$$\int_{-\infty}^{+\infty}\frac{dx}{(xy+1+i\epsilon)^a}=-\frac{2\pi i}{a-1}\delta(y) \tag{A.119}$$

We define the integral by analytic continuation from the region $\operatorname{Re} a>1$. Thus, let us evaluate it assuming $\operatorname{Re} a>1$. Suppose first that y is nonzero. If $y>0$, the integrand is holomorphic in the upper-half plane for x. Furthermore, because $\operatorname{Re} a>1$, it decays sufficiently quickly at infinity that the integration contour can be deformed into the upper half-plane, giving zero. If $y<0$, a similar argument shows that the integral can be deformed into the lower half-plane, giving zero. It follows that the distribution (A.119) is supported at $y = 0$.

Now consider the integral against a test function $f(y)$. Because (A.119) is supported at $y = 0$, we can restrict the y integral to the range $[-1, 1]$ (or any finite-size interval containing the origin). We furthermore substitute the Taylor expansion of $f(y)$ and integrate term by term:

$$\int_{-1}^{1} dy \int_{-\infty}^{+\infty} \frac{dx}{(xy + 1 + i\epsilon)^a} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} y^n \tag{A.120}$$

Let us evaluate the term proportional to $f(0)$. Swapping the order of integration, we have

$$f(0) \int_{-\infty}^{+\infty} dx \int_{-1}^{1} dy \frac{1}{(xy + 1 + i\epsilon)^a} = f(0) \int_{-\infty}^{+\infty} dx \frac{(1 + i\epsilon + x)^{1-a} - (1 + i\epsilon - x)^{1-a}}{(1 - a)x} \tag{A.121}$$

Because the integrand on the right-hand side is holomorphic at $x = 0$, we can deform the contour so that it moves slightly above the origin (staying below the singularity at $x = 1 + i\epsilon$). We denote this by $\int_{-\infty}^{\infty} \to \int_{\frown}$. After this deformation, we split the integrand into two terms

$$\frac{f(0)}{1 - a} \int_{\frown} dx \left(\frac{(1 + i\epsilon + x)^{1-a}}{x} - \frac{(1 + i\epsilon - x)^{1-a}}{x} \right) \tag{A.122}$$

The first term is holomorphic in the positive imaginary direction for x, so we can deform the contour that direction and obtain zero. The second term is holomorphic in the negative imaginary direction for x, except for the pole at $x = 0$. Thus, we can deform the contour that direction and pick up only the residue at $x = 0$. We obtain

$$\frac{2\pi i}{1 - a} f(0). \tag{A.123}$$

Finally, consider the terms in (A.120) proportional to $f^{(n)}(0)$. For these terms, note that

$$\frac{y^n}{(xy + 1 + i\epsilon)^a} \propto \partial_x^n \frac{1}{(xy + 1 + i\epsilon)^{a-n}} \tag{A.124}$$

This is a total derivative in x, and hence integrates to zero. Together with (A.123), this establishes (A.119).

Appendix B

APPENDICES TO CHAPTER 3

B.1 Alternative derivation of the three-point celestial block

In this section we give an alternative derivation of an expression for the three-point celestial block

$$C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}(z_2,w_2,z_3,\partial_{z_3})(-2p\cdot z_3)^{-\delta'}. \tag{B.1}$$

Lorentz invariance and homogeneity imply that[1]

$$C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}(z_2,z_3,\partial_{z_3})(-2p\cdot z_3)^{-\delta'} = \frac{(-[z_2,w_2]\cdot[z_3,p])^j(-p^2)^{\frac{\delta+\delta_3+j-\delta'}{2}}}{(-2p\cdot z_2)^{\delta+j}(-2p\cdot z_3)^{\delta_3+j}}f'_{\delta'}(\zeta_{23}). \tag{B.2}$$

From the definition of $C_{\mathcal{P}_{\delta,j}3\mathcal{P}_{\delta'}}$, the leading term of $f'_{\delta'}(\zeta_{23})$ should be given by

$$f'_{\delta'}(\zeta_{23}) = \zeta_{23}^{\frac{\delta'-\delta-j-\delta_3}{2}}\left(1+O(\zeta_{23})\right). \tag{B.3}$$

Solving the Casimir equation with this boundary condition, we find

$$f'_{\delta'}(\zeta_{23}) = \zeta_{23}^{\frac{\delta'-\delta-j-\delta_3}{2}}\,{}_2F_1\left(\tfrac{\delta'-\delta_3+\delta+j}{2},\tfrac{\delta'+\delta_3-\delta+j}{2},\delta'+2-\tfrac{d}{2},\zeta_{23}\right). \tag{B.4}$$

This is the two-point celestial block where one of the external operators has nonzero j, generalizing the result derived in [15].

Now, we must compute

$$C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})\left(\frac{(-[z_2,w_2]\cdot[z_3,p])^j(-p^2)^{\frac{\delta+\delta_3+j-\delta'}{2}}}{(-2p\cdot z_2)^{\delta+j}(-2p\cdot z_3)^{\delta_3+j}}f'_{\delta'}(\zeta_{23})\right). \tag{B.5}$$

Expanding the two-point celestial block $f'_{\delta'}(\zeta_{23})$ given in (B.4), we can rewrite (B.5) as

$$\sum_{n=0}^{\infty}\frac{\left(\frac{\delta'-\delta_3+\delta+j}{2}\right)_n\left(\frac{\delta'+\delta_3-\delta+j}{2}\right)_n}{\left(\delta'+2-\frac{d}{2}\right)_n n!}(-p^2)^n C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})\left(\langle\mathcal{P}_{\delta,j}(z_2,w_2)\mathcal{P}_{\delta_3}(z_3)\mathcal{P}_{\delta'+2n}(p)\rangle\right), \tag{B.6}$$

[1] We define $[a,b]\cdot[c,d] = 2[(a\cdot c)(b\cdot d)-(a\cdot d)(b\cdot c)]$

where

$$\langle \mathcal{P}_{\delta,j}(z_2,w_2)\mathcal{P}_{\delta_3}(z_3)\mathcal{P}_{\delta'+2n}(p)\rangle = \frac{(-[z_2,w_2]\cdot[z_3,p])^j}{(-2p\cdot z_2)^{\frac{\delta'+\delta+j-\delta_3}{2}+n}(-2p\cdot z_3)^{\frac{\delta'+\delta_3+j-\delta}{2}+n}(-2z_2\cdot z_3)^{\frac{\delta+j+\delta_3-\delta'}{2}-n}}. \tag{B.7}$$

Note that the function $\langle \mathcal{P}_{\delta,j}(z_2,w_2)\mathcal{P}_{\delta_3}(z_3)\mathcal{P}_{\delta'+2n}(p)\rangle$ is not a conformally invariant three-point function since p is not null, and therefore $C_{12\mathcal{P}_{\delta,j}}$ acting on it does not give a conformal block. To compute the action of $C_{12\mathcal{P}_{\delta,j}}$ on $\langle \mathcal{P}_{\delta,j}(z_2,w_2)\mathcal{P}_{\delta_3}(z_3)\mathcal{P}_{\delta'+2n}(p)\rangle$, we would like to express it as an expansion in conformal three-point functions in the limit that p becomes null. To do so, we express p as a linear combination of two null vectors $p = z_0 + v$. In the limit $v \to 0$, p approaches the point z_0. We define the null vectors by

$$z_0^\mu = p^\mu - \frac{p^2}{2p\cdot z_1}z_1^\mu, \quad v^\mu = \frac{p^2}{2p\cdot z_1}z_1^\mu, \tag{B.8}$$

essentially using z_1 as a reference direction. Consequently, subsequent expressions will not be manifestly symmetric with respect to $1 \leftrightarrow 2$, but the symmetry will be restored in the final answer. Note that z_0 and v satisfy

$$z_0^2 = v^2 = 0,\ p = z_0 + v,\ -p^2 = -2z_0\cdot v. \tag{B.9}$$

The cross ratios can then be written

$$\zeta_{12} = \frac{-2z_2\cdot v}{-2z_2\cdot(z_0+v)}, \quad \zeta_{13} = \frac{-2z_3\cdot v}{-2z_3\cdot(z_0+v)}, \quad \zeta_{23} = \frac{(-2z_2\cdot z_3)(-2z_0\cdot v)}{(-2z_2\cdot(z_0+v))(-2z_3\cdot(z_0+v))}. \tag{B.10}$$

In these variables, the expansion around the collinear limit is an expansion in small v. We can write the three-point celestial block as

$$\sum_{n=0}^{\infty}\frac{\left(\frac{\delta'-\delta_3+\delta+j}{2}\right)_n\left(\frac{\delta'+\delta_3-\delta+j}{2}\right)_n}{\left(\delta'+2-\frac{d}{2}\right)_n n!}(-2z_0\cdot v)^n C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})\left(\langle \mathcal{P}_{\delta,j}(z_2,w_2)\mathcal{P}_{\delta_3}(z_3)\mathcal{P}_{\delta'+2n}(z_0+v)\rangle\right). \tag{B.11}$$

We find that the function $\langle \mathcal{P}_{\delta,j}(z_2,w_2)\mathcal{P}_{\delta_3}(z_3)\mathcal{P}_{\delta'+2n}(z_0+v)\rangle$ has the following expansion:

$$\begin{aligned}&\langle \mathcal{P}_{\delta,j}(z_2,w_2)\mathcal{P}_{\delta_3}(z_3)\mathcal{P}_{\delta'+2n}(z_0+v)\rangle\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{m} c_{m,k}(-2v\cdot z_0)^k(v\cdot D_{z_0})^{m-k}\langle \mathcal{P}_{\delta,j}(z_2,w_2)\mathcal{P}_{\delta_3}(z_3)\mathcal{P}_{\delta'+2n+2k}(z_0)\rangle,\end{aligned} \tag{B.12}$$

where D_{z_0} is the Todorov operator acting on z_0. The coefficients $c_{m,k}$ can be determined by expanding both sides of the above equation order by order in small v:

$$c_{m,k} = \frac{\left(\frac{j+2n+\delta-\delta_3+\delta'}{2}\right)_k \left(\frac{j+2n-\delta+\delta_3+\delta'}{2}\right)_k}{(m-k)!k!} \frac{\left(\frac{d}{2}-(2k+1+2n+\delta')\right)\Gamma\left(\frac{d}{2}-(m+k+1+2n+\delta')\right)}{\Gamma\left(\frac{d}{2}-(k+2n+\delta')\right)}. \tag{B.13}$$

Since the Todorov operator D_{z_0} commutes with $C_{12\mathcal{P}_{\delta,j}}$, the three-point celestial block is therefore given by

$$\begin{aligned}
&C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})\left(\frac{(-[z_2,w_2]\cdot[z_3,p])^j(-p^2)^{\frac{\delta+\delta_3+j-\delta'}{2}}}{(-2p\cdot z_2)^{\delta+j}(-2p\cdot z_3)^{\delta_3+j}} f'_{\delta'}(\zeta_{23})\right)\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{\left(\frac{\delta'-\delta_3+\delta+j}{2}\right)_n\left(\frac{\delta'+\delta_3-\delta+j}{2}\right)_n}{\left(\delta'+2-\frac{d}{2}\right)_n n!}c_{m,k}(-2v\cdot z_0)^{n+k}(v\cdot D_{z_0})^{m-k}g_{\delta,j}^{(\delta_1,\delta_2,\delta_3,\delta'+2n+2k)}(z_1,z_2,z_3,z_0)\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}B_{n,m,k}(-2v\cdot z_0)^{n+k}(v\cdot D_{z_0})^{m-k}g_{\delta,j}^{(\delta_1,\delta_2,\delta_3,\delta'+2n+2k)}(z_1,z_2,z_3,z_0),
\end{aligned} \tag{B.14}$$

where $g_{\delta,j}^{(\delta_1,\delta_2,\delta_3,\delta'+2n+2k)}(z_1,z_2,z_3,z_0)$ is a usual four-point conformal block, and

$$B_{n,m,k} = \frac{\left(\frac{j+\delta-\delta_3+\delta'}{2}\right)_{n+k}\left(\frac{j-\delta+\delta_3+\delta'}{2}\right)_{n+k}}{\left(\delta'+2-\frac{d}{2}\right)_n n!(m-k)!k!} \frac{\left(\frac{d}{2}-(2k+1+2n+\delta')\right)\Gamma\left(\frac{d}{2}-(m+k+1+2n+\delta')\right)}{\Gamma\left(\frac{d}{2}-(k+2n+\delta')\right)}. \tag{B.15}$$

The expression (B.14) can be further simplified. To do so, we redefine $n' \equiv n+k$, $N \equiv n+m$, which gives

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}B_{n,m,k}(-2v\cdot z_0)^{n+k}(v\cdot D_{z_0})^{m-k}g_{\delta,j}^{(\delta_1,\delta_2,\delta_3,\delta'+2n+2k)}(z_1,z_2,z_3,z_0)\\
&=\sum_{N=0}^{\infty}\sum_{n'=0}^{N}\sum_{k=0}^{n'}B_{n'-k,N-n'+k,k}(-2v\cdot z_0)^{n'}(v\cdot D_{z_0})^{N-n'}g_{\delta,j}^{(\delta_1,\delta_2,\delta_3,\delta'+2n')}(z_1,z_2,z_3,z_0).
\end{aligned} \tag{B.16}$$

It turns out that the coefficients $B_{n,m,k}$ given in (B.15) satisfy

$$\sum_{k=0}^{n'}B_{n'-k,N-n'+k,k} = B_{0,N,0}\delta_{n',0}. \tag{B.17}$$

Therefore, two of the sums in (B.14) collapse, and the three-point celestial block becomes

$$
\begin{aligned}
&C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})\left(\frac{(-[z_2,w_2]\cdot[z_3,p])^j(-p^2)^{\frac{\delta+\delta_3+j-\delta'}{2}}}{(-2p\cdot z_2)^{\delta+j}(-2p\cdot z_3)^{\delta_3+j}}f'_{\delta'}(\zeta_{23})\right)\\
&=\sum_{N=0}^{\infty}\frac{\Gamma(\frac{d}{2}-1-\delta'-N)}{\Gamma(\frac{d}{2}-1-\delta')N!}(v\cdot D_{z_0})^N g_{\delta,j}^{(\delta_1,\delta_2,\delta_3,\delta')}(z_1,z_2,z_3,z_0)\\
&=\Gamma(\delta'+2-\tfrac{d}{2})(\sqrt{v\cdot D_{z_0}})^{\frac{d}{2}-\delta'-1}J_{\delta'+1-\frac{d}{2}}\left(2\sqrt{v\cdot D_{z_0}}\right)g_{\delta,j}^{(\delta_1,\delta_2,\delta_3,\delta')}(z_1,z_2,z_3,z_0).
\end{aligned}
\tag{B.18}
$$

This expression is more manifestly Lorentz-invariant than (3.33), at the cost of singling out z_1 as special.

It is convenient to write the result in terms of conformally-invariant cross ratios. Before doing so, we relabel the vector $v\to v'$ to avoid confusion with the cross-ratio v. Factoring out the homogeneity factor in the conformal block, we find (up to linear order in v', for simplicity)

$$
\begin{aligned}
&C_{12\mathcal{P}_{\delta,j}}(z_1,z_2,\partial_{z_2},\partial_{w_2})\left(\frac{(-[z_2,w_2]\cdot[z_3,p])^j(-p^2)^{\frac{\delta+\delta_3+j-\delta'}{2}}}{(-2p\cdot z_2)^{\delta+j}(-2p\cdot z_3)^{\delta_3+j}}f'_{\delta'}(\zeta_{23})\right)\\
&=\left(1+\frac{1}{\frac{d}{2}-2-\delta'}(v'\cdot D_{z_0})+\cdots\right)T_{123\delta'}(z_1,z_2,z_3,z_0)g_{\delta,j}^{(123\delta')}(u,v)\,,
\end{aligned}
\tag{B.19}
$$

where u, v and $T_{123\delta'}$ are defined in (3.35) and (3.36) (with p replaced by z_0). Finally, we can replace $z_0\to p-\frac{p^2}{2p\cdot z_1}z_1$, $v'\to\frac{p^2}{2p\cdot z_1}z_1$. After expanding in $(-p^2)$, we obtain (3.34) in the main text.

We can also derive the same result from (3.33). To compute the celestial block using (3.32), we start with the conformal block

$$
T_{123\delta'}(z_1,z_2,z_3,z_0)g_{\delta,j}^{(123\delta')}\left(\frac{(-2z_1\cdot z_2)(-2z_3\cdot z_0)}{(-2z_1\cdot z_3)(-2z_2\cdot z_0)},\frac{(-2z_2\cdot z_3)(-2z_1\cdot z_0)}{(-2z_1\cdot z_3)(-2z_2\cdot z_0)}\right),
\tag{B.20}
$$

and go to the frame where z_0 is near the origin and z_1,z_2,z_3 are near infinity. Explicitly, let us set

$$
z_0=(1,\vec{y}^2,\vec{y}),\quad z_i=(\lambda^2\vec{y}_i^2,1,\lambda\vec{y}_i),
\tag{B.21}
$$

where $i = 1, 2, 3$, and λ is an expansion parameter. The first two terms, for example, from expanding out the Bessel function in (3.33) then become

$$\left(1 - \frac{\partial^2_{\vec{y}}}{4(\delta' + 2 - \frac{d}{2})}\right) \left(T_{123\delta'}(z_1, z_2, z_3, z_0) g^{(123\delta')}_{\delta,j} \left(\frac{(-2z_1 \cdot z_2)(-2z_3 \cdot z_0)}{(-2z_1 \cdot z_3)(-2z_2 \cdot z_0)}, \frac{(-2z_2 \cdot z_3)(-2z_1 \cdot z_0)}{(-2z_1 \cdot z_3)(-2z_2 \cdot z_0)}\right)\right)\Bigg|_{\vec{y}_0 \to 0}$$
$$= T_{123\delta'}(z_1, z_2, z_3, p)\left(g^{(123\delta')}_{\delta,j}(u, v) + \frac{\zeta_{13}}{d - 4 - 2\delta'} \mathcal{D}^{(1)}_{u,v} g^{(123\delta')}_{\delta,j}(u, v)\right)\Bigg|_{\substack{p \to (1,1,0) \\ z_i = (\lambda^2 \vec{y}_i^2, 1, \lambda \vec{y}_i)}} + O(\lambda^4), \tag{B.22}$$

which agrees with (3.34). More generally, when acting on the appropriate class of functions, $v' \cdot D_{z_0} = \partial^2_{\vec{y}}/4$, so (3.33) and (B.18) are equivalent.

B.2 $\langle R^{(1)}_{\delta,j} \rangle$ and $\langle R^{(0)}_{\delta,j} \gamma^{(1)}_{\delta,j} \rangle$ from direct decomposition

In this appendix, we give the coefficients $\langle R^{(1)}_{\delta,j} \rangle$ and $\langle R^{(0)}_{\delta,j} \gamma^{(1)}_{\delta,j} \rangle$ up to $\delta = 12$ for $\mathcal{N} = 4$ SYM, the QCD gluon jet, and the QCD quark jet obtained by expanding the collinear EEEC order-by-order in small r, as in section 3.3.2.

B.2.1 $\mathcal{N} = 4$ SYM

$$\begin{aligned}
&\langle R^{(1)}_{10,0} \rangle = \tfrac{141301}{352800} - \tfrac{2\pi^2}{49}, \quad \langle R^{(1)}_{10,2} \rangle = \tfrac{107129}{529200} - \tfrac{19\pi^2}{1260}, \\
&\langle R^{(1)}_{10,4} \rangle = \tfrac{33394601}{85377600} - \tfrac{5\pi^2}{132}, \quad \langle R^{(1)}_{10,6} \rangle = \tfrac{189283}{1801800} - \tfrac{3\pi^2}{286}, \\
&\langle R^{(1)}_{12,0} \rangle = \tfrac{84401}{3175200} - \tfrac{\pi^2}{378}, \quad \langle R^{(1)}_{12,2} \rangle = \tfrac{547098707}{5378788800} - \tfrac{9\pi^2}{1232}, \\
&\langle R^{(1)}_{12,4} \rangle = \tfrac{2674437767}{81162081000} - \tfrac{16\pi^2}{6435}, \quad \langle R^{(1)}_{12,6} \rangle = \tfrac{1220098669}{19675656000} - \tfrac{7\pi^2}{1144}, \\
&\langle R^{(1)}_{12,8} \rangle = \tfrac{1030567}{68612544} - \tfrac{\pi^2}{663}, \\
&\langle R^{(0)}_{10,0} \gamma^{(1)}_{10,0} \rangle = \tfrac{89}{210}, \quad \langle R^{(0)}_{10,2} \gamma^{(1)}_{10,2} \rangle = \tfrac{8}{105}, \quad \langle R^{(0)}_{10,4} \gamma^{(1)}_{10,4} \rangle = -\tfrac{1}{154}, \\
&\langle R^{(0)}_{12,0} \gamma^{(1)}_{12,0} \rangle = \tfrac{253}{630}, \quad \langle R^{(0)}_{12,2} \gamma^{(1)}_{12,2} \rangle = \tfrac{28247}{194040}, \quad \langle R^{(0)}_{12,4} \gamma^{(1)}_{12,4} \rangle = \tfrac{893}{90090}, \\
&\langle R^{(0)}_{12,6} \gamma^{(1)}_{12,6} \rangle = -\tfrac{1}{1560}.
\end{aligned} \tag{B.23}$$

B.2.2 QCD

For the gluon jet, we find

$$
\begin{aligned}
\langle R^{(1)g}_{8,0}\rangle &= \frac{-2\left(56889000\pi^2-563610307\right)C_A^2+2\left(39690000\pi^2-399089339\right)C_A n_f T_F-3108375C_F n_f T_F}{317520000},\\
\langle R^{(1)g}_{8,2}\rangle &= \frac{C_A\left(\left(241456351-24418800\pi^2\right)C_A+2\left(11207700\pi^2-110698537\right)n_f T_F\right)}{50803200},\\
\langle R^{(1)g}_{8,4}\rangle &= \frac{\left(1713863-173600\pi^2\right)C_A^2+2\left(107800\pi^2-1063963\right)C_A n_f T_F+120C_F n_f T_F}{403200},\\
\langle R^{(1)g}_{10,0}\rangle &= \frac{\left(4851797956-487317600\pi^2\right)C_A^2+2\left(251143200\pi^2-2512589293\right)C_A n_f T_F+7121499C_F n_f T_F}{1303948800},\\
\langle R^{(1)g}_{10,2}\rangle &= \frac{\left(104601961181-10560904200\pi^2\right)C_A^2}{23051952000}+\frac{\left(13728260700\pi^2-136102341667\right)C_A n_f T_F}{23051952000}\\
&\quad-\frac{389C_F n_f T_F}{1411200},\\
\langle R^{(1)g}_{10,4}\rangle &= \frac{C_A\left(\left(31701145719-3211362000\pi^2\right)C_A+14\left(336659400\pi^2-3323123011\right)n_f T_F\right)}{4098124800},\\
\langle R^{(1)g}_{10,6}\rangle &= \frac{\left(459012085-46506600\pi^2\right)C_A^2+\left(76381200\pi^2-753853073\right)C_A n_f T_F+2520C_F n_f T_F}{86486400},\\
\langle R^{(1)g}_{12,0}\rangle &= \frac{\left(4613713731326-363812248800\pi^2\right)C_A^2}{1558311955200}+\frac{\left(254307853800\pi^2-2498045171789\right)C_A n_f T_F}{779155977600}\\
&\quad+\frac{788981C_F n_f T_F}{113836800},\\
\langle R^{(1)g}_{12,2}\rangle &= -\frac{\left(885896411400\pi^2-9950914029409\right)C_A^2}{2727045921600}+\frac{\left(139361777100\pi^2-1376830794827\right)C_A n_f T_F}{283329446400}\\
&\quad+\frac{9197C_F n_f T_F}{19514880},\\
\langle R^{(1)g}_{12,4}\rangle &= -\frac{\left(56981204280\pi^2-577018051339\right)C_A^2}{129859329600}+\frac{\left(655611356400\pi^2-6475204191751\right)C_A n_f T_F}{865728864000}\\
&\quad-\frac{86819C_F n_f T_F}{4122518400},\\
\langle R^{(1)g}_{12,6}\rangle &= \frac{C_A\left(3\left(372815843400\pi^2-3679625658343\right)n_f T_F-7\left(91148557500\pi^2-899626801693\right)C_A\right)}{865728864000},\\
\langle R^{(1)g}_{12,8}\rangle &= \frac{\left(3609756605-365743840\pi^2\right)C_A^2+16\left(41815620\pi^2-412703677\right)C_A n_f T_F+2352C_F n_f T_F}{914833920},
\end{aligned}
\tag{B.24}
$$

and

$$
\begin{aligned}
\langle R^{(0)g}_{8,0}\gamma^{(1)}_{8,0}\rangle &= \frac{-62C_A^2-110C_A n_f T_F-135C_F n_f T_F}{5040}, & \langle R^{(0)g}_{8,2}\gamma^{(1)}_{8,2}\rangle &= -\frac{C_A(73C_A+16n_f T_F)}{20160},\\
\langle R^{(0)g}_{10,0}\gamma^{(1)}_{10,0}\rangle &= \frac{1986C_A^2-6740C_A n_f T_F-1407C_F n_f T_F}{47040}, & &\\
\langle R^{(0)g}_{10,2}\gamma^{(1)}_{10,2}\rangle &= \frac{5743C_A^2-11981C_A n_f T_F-594C_F n_f T_F}{332640}, & \langle R^{(0)g}_{10,4}\gamma^{(1)}_{10,4}\rangle &= -\frac{C_A(39C_A+7n_f T_F)}{73920},\\
\langle R^{(0)g}_{12,0}\gamma^{(1)}_{12,0}\rangle &= \frac{491846C_A^2-1164457C_A n_f T_F-102141C_F n_f T_F}{6486480}, & &\\
\langle R^{(0)g}_{12,2}\gamma^{(1)}_{12,2}\rangle &= \frac{999298C_A^2-1763153C_A n_f T_F-34398C_F n_f T_F}{24216192}, & &\\
\langle R^{(0)g}_{12,4}\gamma^{(1)}_{12,4}\rangle &= \frac{3197C_A^2-6135C_A n_f T_F-126C_F n_f T_F}{720720}, & \langle R^{(0)g}_{12,6}\gamma^{(1)}_{12,6}\rangle &= -\frac{C_A(37C_A+6n_f T_F)}{686400}.
\end{aligned}
\tag{B.25}
$$

For the quark jet, we find

$$
\begin{aligned}
\langle R_{8,0}^{(1)q}\rangle &= \frac{\left(27930000\pi^2-275120246\right)C_AC_F}{35280000}-\frac{\left(206976\pi^2-2041751\right)C_F^2}{282240}+\frac{\left(201264317-20580000\pi^2\right)C_Fn_fT_F}{35280000},\\
\langle R_{8,2}^{(1)q}\rangle &= \frac{\left(10224900\pi^2-100838071\right)C_AC_F}{16934400}-\frac{\left(1774500\pi^2-17530127\right)C_F^2}{2822400}+\frac{\left(39243247-3981600\pi^2\right)C_Fn_fT_F}{8467200},\\
\langle R_{8,4}^{(1)q}\rangle &= \frac{C_F\left(7\left(15000\pi^2-148003\right)C_A+\left(1685981-170800\pi^2\right)C_F\right)}{403200},\\
\langle R_{10,0}^{(1)q}\rangle &= \frac{\left(1709457750\pi^2-16812815347\right)C_AC_F}{1792929600}-\frac{\left(21218400\pi^2-209515081\right)C_F^2}{39513600}\\
&\quad+\frac{\left(168438023821-17142602400\pi^2\right)C_Fn_fT_F}{14343436800},\\
\langle R_{10,2}^{(1)q}\rangle &= \frac{\left(13877671500\pi^2-136733593943\right)C_AC_F}{15367968000}-\frac{\left(64499400\pi^2-636790073\right)C_F^2}{127008000}\\
&\quad+\frac{\left(2801569019-284592000\pi^2\right)C_Fn_fT_F}{256132800},\\
\langle R_{10,4}^{(1)q}\rangle &= \frac{\left(6747741000\pi^2-66583998913\right)C_AC_F}{12294374400}+\frac{\left(46333633219-4693827600\pi^2\right)C_F^2}{12294374400}\\
&\quad-\frac{\left(999600\pi^2-9863251\right)C_Fn_fT_F}{1330560},\\
\langle R_{10,6}^{(1)q}\rangle &= \frac{C_F\left(4\left(9538200\pi^2-94135219\right)C_A+\left(540714493-54784800\pi^2\right)C_F\right)}{172972800},\\
\langle R_{12,0}^{(1)q}\rangle &= \frac{\left(83057703300\pi^2-830250288473\right)C_AC_F}{111307996800}-\frac{\left(702332400\pi^2-7516807777\right)C_F^2}{3073593600}\\
&\quad+\frac{\left(2504233505507-261779918400\pi^2\right)C_Fn_fT_F}{222615993600},\\
\langle R_{12,2}^{(1)q}\rangle &= \frac{\left(112390544366100\pi^2-1118604851318477\right)C_AC_F}{159986694067200}-\frac{\left(18487161000\pi^2-193326296189\right)C_F^2}{86060620800}\\
&\quad+\frac{\left(20817079707113-2158310154000\pi^2\right)C_Fn_fT_F}{1904603500800},\\
\langle R_{12,4}^{(1)q}\rangle &= \frac{\left(3216440026800\pi^2-31809587301651\right)C_AC_F}{5194373184000}-\frac{\left(518049932400\pi^2-5194727659819\right)C_F^2}{2597186592000}\\
&\quad+\frac{\left(488517802327-49855085280\pi^2\right)C_Fn_fT_F}{51943731840},\\
\langle R_{12,6}^{(1)q}\rangle &= -\frac{\left(31815610577491-3223780560000\pi^2\right)C_AC_F}{10388746368000}-\frac{\left(177924146400\pi^2-1756123265947\right)C_F^2}{1484106624000}\\
&\quad-\frac{\left(9737280\pi^2-96097693\right)C_Fn_fT_F}{17297280},\\
\langle R_{12,8}^{(1)q}\rangle &= \frac{C_F\left(\left(4699648800\pi^2-46383418021\right)C_A+14\left(4303513889-436035600\pi^2\right)C_F\right)}{41167526400},
\end{aligned}
\tag{B.26}
$$

and

$$\langle R^{(0)q}_{8,0}\gamma^{(1)}_{8,0}\rangle = -\tfrac{C_F(24C_A+4C_F+67n_fT_F)}{1680}, \qquad \langle R^{(0)q}_{8,2}\gamma^{(1)}_{8,2}\rangle = \tfrac{C_F(13C_A-40C_F)}{6720},$$
$$\langle R^{(0)q}_{10,0}\gamma^{(1)}_{10,0}\rangle = -\tfrac{C_F(-23799C_A+3652C_F+86219n_fT_F)}{517440},$$
$$\langle R^{(0)q}_{10,2}\gamma^{(1)}_{10,2}\rangle = -\tfrac{C_F(-4825C_A+682C_F+9012n_fT_F)}{221760}, \qquad \langle R^{(0)q}_{10,4}\gamma^{(1)}_{10,4}\rangle = \tfrac{C_F(113C_A-368C_F)}{443520},$$
$$\langle R^{(0)q}_{12,0}\gamma^{(1)}_{12,0}\rangle = -\tfrac{C_F(-343735C_A+5382C_F+865310n_fT_F)}{4324320},$$
$$\langle R^{(0)q}_{12,2}\gamma^{(1)}_{12,2}\rangle = -\tfrac{C_F(-38990713C_A+463606C_F+69682564n_fT_F)}{887927040},$$
$$\langle R^{(0)q}_{12,4}\gamma^{(1)}_{12,4}\rangle = -\tfrac{C_F(-7296C_A+655C_F+13016n_fT_F)}{1441440}, \qquad \langle R^{(0)q}_{12,6}\gamma^{(1)}_{12,6}\rangle = \tfrac{C_F(13C_A-43C_F)}{514800}. \tag{B.27}$$

B.3 More details on the Lorentzian inversion formula calculation

In this section, we compute in full detail the coefficients $\langle R^{(1)}_{j+4,j}\rangle$ and $\langle R^{(0)}_{j+\tau_c,j;\delta'_*}\gamma^{(1)}_{j+\tau_c,j}\rangle$ for $\mathcal{N}=4$ SYM using the Lorentzian inversion formula. We compute $\langle R^{(1)}_{j+4,j}\rangle$ in the first subsection, and then we compute $\langle R^{(0)}_{j+\tau_c,j;\delta'_*}\gamma^{(1)}_{j+\tau_c,j}\rangle$ in the next subsection.

B.3.1 $\langle R^{(1)}_{j+4,j}\rangle$

From (3.94), we have

$$\langle R^{(1)}_{j+4,j}\rangle = 2\kappa_{2j+6}\int_0^1 \frac{d\bar{z}}{\bar{z}^2} k^{0,1}_{2j+6}(\bar{z})\mathrm{dDisc}_t[\mathcal{G}^{(1)}(z,\bar{z})]|_{z^2}. \tag{B.28}$$

Therefore, we first need $\mathcal{G}^{(1)}(z,\bar{z})|_{z^2}$ in order to compute $\langle R^{(1)}_{j+4,j}\rangle$. We find

$$\mathcal{G}^{(1)}(z,\bar{z})|_{z^2} = \frac{1}{4(1-\bar{z})^2}\Big(2\bar{z}^4(\bar{z}-2)\zeta_2 - \bar{z}(1-\bar{z})\log(1-\bar{z})(4+2\bar{z}(\bar{z}-2)+\bar{z}(1-\bar{z})^2\log(1-\bar{z})) \\ +2\bar{z}^3(3-3\bar{z}+\bar{z}^2)\mathrm{Li}_2(\bar{z})\Big). \tag{B.29}$$

Using the double discontinuities

$$\begin{aligned}
\mathrm{dDisc}_t[(1-\bar{z})^\alpha] &= -2\sin^2(\pi\alpha)(1-\bar{z})^\alpha,\\
\mathrm{dDisc}_t[(1-\bar{z})^\alpha\log(1-\bar{z})] &= -2\sin^2(\pi\alpha)(1-\bar{z})^\alpha\log(1-\bar{z}) - 2\pi\sin(2\pi\alpha)(1-\bar{z})^\alpha,\\
\mathrm{dDisc}_t[(1-\bar{z})^\alpha\mathrm{Li}_2(\bar{z})] &= -2\sin^2(\pi\alpha)(1-\bar{z})^\alpha\mathrm{Li}_2(\bar{z}) + 2\pi\sin(2\pi\alpha)(1-\bar{z})^\alpha\log(\bar{z}),\\
\mathrm{dDisc}_t[(1-\bar{z})^\alpha\log^2(1-\bar{z})] &= -2\sin^2(\pi\alpha)(1-\bar{z})^\alpha\log^2(1-\bar{z}) - 4\pi\sin(2\pi\alpha)(1-\bar{z})^\alpha\log(1-\bar{z})\\
&\quad - 4\pi^2\cos(2\pi\alpha)(1-\bar{z})^\alpha,
\end{aligned} \tag{B.30}$$

we obtain (after introducing a small regulator ϵ)

$$
\begin{aligned}
&\mathrm{dDisc}_t[\mathcal{G}^{(1)}(z,\bar{z})]|_{z^2} \\
&= -2\sin^2(\pi\epsilon)\mathcal{G}^{(1)}(z,\bar{z})|_{z^2} + 2\pi\sin(2\pi\epsilon)\times\left(-\frac{\bar{z}^{1+\epsilon}}{4(1-\bar{z})^{1+\epsilon}}\right)(4+2\bar{z}^2-4\bar{z}) \\
&\quad - 2\pi\sin(2\pi\epsilon)\times\frac{\bar{z}^{3+\epsilon}}{2(1-\bar{z})^{2+\epsilon}}(3-3\bar{z}+\bar{z}^2)\log(\bar{z}) \\
&\quad + 4\pi\sin(2\pi\epsilon)\times\left(-\frac{\bar{z}^{2+\epsilon}}{4(1-\bar{z})^{-1+\epsilon}}\right)\log(1-\bar{z}) \\
&\quad - 4\pi^2\cos(2\pi\epsilon)\times\left(-\frac{\bar{z}^{2+\epsilon}}{(1-\bar{z})^{-1+\epsilon}}\right).
\end{aligned}
\tag{B.31}
$$

We can then plug (B.31) into (B.28) and integrate each term over $\bar{z}$. After taking the $\epsilon \to 0$ limit, we obtain (3.102).

B.3.2 $\langle R^{(0)}_{j+\tau_c,j;\delta'_*}\gamma^{(1)}_{j+\tau_c,j}\rangle$

To do the calculation for general twist operators, we will need to compute dDisc of more complicated functions than in the leading twist case. It is convenient to first define the linear functional

$$
I_\beta[f] \equiv \lim_{\epsilon\to 0}\kappa_\beta\int_0^1 \frac{d\bar{z}}{\bar{z}^2}k^{0,1}_\beta(\bar{z})\mathrm{dDisc}_t\left[\frac{\bar{z}^\epsilon}{(1-\bar{z})^\epsilon}f(\bar{z})\right], \tag{B.32}
$$

where κ_β is defined in (3.85), and

$$
k^{0,1}_\beta(\bar{z}) = \bar{z}^{\frac{\beta}{2}}{}_2F_1\left(\tfrac{\beta}{2},\tfrac{\beta}{2}+1,\beta;\bar{z}\right). \tag{B.33}
$$

To compute anomalous dimensions, we will need to apply this functional to functions with power-law divergences or logarithmic divergences near $\bar{z}=1$. Using (B.30), we can obtain

$$
\begin{aligned}
I_\beta\left[\frac{\bar{z}^b}{(1-\bar{z})^a}\right] =& \frac{\Gamma(\frac{\beta}{2})^2\Gamma(\frac{\beta}{2}+1)\Gamma(-1+b+\frac{\beta}{2})}{\Gamma(a)\Gamma(a+1)\Gamma(\beta-1)\Gamma(1-a+\frac{\beta}{2})\Gamma(b-a+\frac{\beta}{2})} \\
&\times {}_3F_2\left(\begin{matrix}1-a, & b-a, & 1+\frac{\beta}{2} \\ 1-a+\frac{\beta}{2}, & b-a+\frac{\beta}{2} & \end{matrix};1\right),
\end{aligned}
\tag{B.34}
$$

where a should be a positive integer (for zero or negative a the right hand side is zero). Also, for logarithmic divergences we can take the derivative of (B.34) with respect to a. For $a=0$, we have

$$
I_\beta\left[\bar{z}^b\log(1-\bar{z})\right] = -\frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{\beta}{2}-1)}{\Gamma(\beta-1)}, \tag{B.35}
$$

and for $a = 1$

$$I_\beta\left[\frac{\bar{z}^b}{(1-\bar{z})}\log(1-\bar{z})\right]$$
$$= \frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{\beta}{2}+1)}{\Gamma(\beta-1)}\left(1 - S_1(\tfrac{\beta}{2}-1) - S_1(\tfrac{\beta}{2}+b-2) + {}_3F_2^{(0,0,1),(0,0)}\left(\begin{array}{ccc} 1-b, & 1+\frac{\beta}{2}, & 0 \\ -1+b+\frac{\beta}{2}, & \frac{\beta}{2} & \end{array};1\right)\right), \tag{B.36}$$

where S_1 is the harmonic number, and

$${}_3F_2^{(0,0,1),(0,0)}\left(\begin{array}{ccc} a_1, & a_2, & a_3 \\ b_1, & b_2 & \end{array};1\right) = \partial_{a_3}\left({}_3F_2\left(\begin{array}{ccc} a_1, & a_2, & a_3 \\ b_1, & b_2 & \end{array};1\right)\right). \tag{B.37}$$

(B.34), (B.35), and (B.36) are the main results we need for the functional I_β in order to do the Lorentzian inversion formula calculation for general twists. In particular, for (B.36), it suffices to consider the $b = 1$ case, which is given by

$$I_\beta\left[\frac{\bar{z}}{(1-\bar{z})}\log(1-\bar{z})\right] = \frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{\beta}{2}+1)\left(1-2S_1(\frac{\beta}{2}-1)\right)}{\Gamma(\beta-1)}. \tag{B.38}$$

By (3.94), the anomalous dimension coefficient $\langle R^{(0)}_{j+\tau_c,j}\gamma^{(1)}_{j+\tau_c,j}\rangle$ with celestial twist τ_c should be given by

$$\langle R^{(0)}_{j+\tau_c,j}\gamma^{(1)}_{j+\tau_c,j}\rangle = 4\kappa_{2j+\tau_c}\left(z^{\frac{\tau_c}{2}-1}\int_0^1\frac{d\bar{z}}{\bar{z}^2}g^{\widetilde{\delta}_i}_{j+1,j+\tau_c-1}(z,\bar{z})\mathrm{dDisc}_t[\mathcal{G}(z,\bar{z})]\right)\bigg|_{z^{\frac{\tau_c}{2}}\log z}. \tag{B.39}$$

The 2d conformal block $g^{\widetilde{\delta}_i}_{j+1,j+\tau_c-1}(z,\bar{z})$ is

$$g^{\widetilde{\delta}_i}_{j+1,j+\tau_c-1}(z,\bar{z}) = k^{0,1}_{2-\tau_c}(z)k^{0,1}_{2j+\tau_c}(\bar{z}) + (z\leftrightarrow\bar{z}). \tag{B.40}$$

Expanding the above expression in small z, we find (the $(z \leftrightarrow \bar{z})$ term does not contribute)

$$\langle R^{(0)}_{j+\tau_c,j}\gamma^{(1)}_{j+\tau_c,j}\rangle = 4\sum_{m=0}^{\frac{\tau_c}{2}-3}\frac{(1-\frac{\tau_c}{2})_m(2-\frac{\tau_c}{2})_m}{(2-\tau_c)_m m!}I_{2j+\tau_c}\left[\mathcal{G}(z,\bar{z})\right]|_{z^{\frac{\tau_c}{2}-m}\log z}, \tag{B.41}$$

where we have used the fact that $\mathcal{G}(z,\bar{z})$ starts at z^3 to restrict the range of the sum over m. Therefore, our task now is to find $I_\beta\left[\mathcal{G}(z,\bar{z})\right]|_{z^n\log z}$ for general n. Focusing on the singular terms near $\bar{z} = 1$, we find

$$\begin{aligned}
&\mathcal{G}(z,\bar{z})|_{\log z} \\
&= z^3 \times \Bigg(-\frac{1+z+2z^2}{2z(1-z)^2}\frac{1}{1-\bar{z}} - \frac{1}{2(1-z)^2}\frac{\bar{z}^3}{(1-\bar{z})^2}\left(1+\frac{z\bar{z}}{2} - 2z\bar{z}\sum_{k=1}^{\infty}\frac{z^k\bar{z}^k}{k(k+2)}\right) \\
&\qquad - \frac{(1+z)\log(1-z)}{2z(1-z)}\frac{\bar{z}^2}{(1-\bar{z})^2} - \frac{(1-z+2z^3)\log(1-z)}{2z^2(1-z)^2}\frac{\bar{z}^2}{(1-\bar{z})} \\
&\qquad - \frac{\bar{z}^2(z\bar{z}-(1-z)(1-\bar{z}))(z\bar{z}(1-z)(1-\bar{z}))}{2z(1-z)(1-\bar{z})}\sum_{k=1}^{\infty}\frac{z^k}{k}\left(1-\frac{1}{(1-\bar{z})^k}\right) \\
&\qquad - \frac{z}{2(1-z)}\frac{\bar{z}\log(1-\bar{z})}{1-\bar{z}} - \frac{3(1+2z^2-z^3)}{2(1-z)^3}\bar{z}\log(1-\bar{z})\Bigg) + O((1-\bar{z})^0).
\end{aligned} \tag{B.42}$$

We can then apply the functional I_β to the above expression and expand in small z. After several lines of algebra, we obtain that for $n \geq 1$

$$\begin{aligned}
&I_\beta\left[\mathcal{G}(z,\bar{z})\right]\Big|_{z^{3+n}\log z} \\
&= \left(-n-2+\frac{3-7n-4n^2+2n(n+2)S_1(n)+(n+2)S_1(n+1)}{2(n+2)}\right) I_\beta\left[\frac{1}{1-\bar{z}}\right] \\
&\quad + \frac{1}{2}\left(S_1(n+1)+S_1(n)\right) I_\beta\left[\frac{\bar{z}^2}{(1-\bar{z})^2}\right] \\
&\quad - \frac{1}{2}\left((n+1)I_\beta\left[\frac{\bar{z}^3}{(1-\bar{z})^2}\right] + \frac{n}{2}I_\beta\left[\frac{\bar{z}^4}{(1-\bar{z})^2}\right] - 2\sum_{k=1}^{n-1}\frac{n-k}{k(k+2)}I_\beta\left[\frac{\bar{z}^{k+4}}{(1-\bar{z})^2}\right]\right) \\
&\quad - \frac{1}{2}\left(\frac{I_\beta\left[\frac{\bar{z}^2}{(1-\bar{z})^n}\right]}{n+1} - \frac{I_\beta\left[\frac{\bar{z}^2}{(1-\bar{z})^{n-1}}\right]}{n} + S_1(n-1)I_\beta\left[\frac{\bar{z}^4}{(1-\bar{z})}\right] - \sum_{k=1}^{n-1}\frac{1}{k}I_\beta\left[\frac{\bar{z}^4}{(1-\bar{z})^{1+k}}\right]\right) \\
&\quad - \frac{1}{2}I_\beta\left[\frac{\bar{z}}{(1-\bar{z})}\log(1-\bar{z})\right] - \frac{3}{2}n(n+2)I_\beta\left[\log(1-\bar{z})\right].
\end{aligned} \tag{B.43}$$

For the leading order term $n = 0$, we simply have

$$\begin{aligned}
I_\beta\left[\mathcal{G}(z,\bar{z})\right]\Big|_{z^3\log z} &= -\frac{1}{4}I_\beta\left[\frac{1}{1-\bar{z}}\right] - \frac{3}{2}I_\beta\left[\log(1-\bar{z})\right] \\
&= -\frac{(\beta-6)(\beta+4)\Gamma(\frac{\beta}{2})\Gamma(\frac{\beta}{2}-1)}{16\Gamma(\beta-1)}.
\end{aligned} \tag{B.44}$$

Combining (B.34), (B.35), (B.36), (B.41), (B.43), and (B.44), we can then determine

$\langle R^{(0)}_{j+\tau_c,j}\gamma^{(1)}_{j+\tau_c,j}\rangle$ for any twist $\tau_c = 6, 8, 10, \ldots$. For general τ_c, we obtain

$$\begin{aligned}
&\langle R^{(0)}_{j+\tau_c,j}\gamma^{(1)}_{j+\tau_c,j}\rangle \\
&= \frac{(-1)^{\frac{\tau_c}{2}}(\tau_c^2 - 2\tau_c + 8(-1)^{\frac{\tau_c}{2}} - 16)\Gamma(\frac{\tau_c}{2})\Gamma(\frac{\tau_c}{2}+1)\Gamma(j+\frac{\tau_c}{2})\Gamma(j+\frac{\tau_c}{2}+1)S_1(j+\frac{\tau_c}{2}-1)}{2\Gamma(\tau_c-1)\Gamma(2j+\tau_c-1)} \\
&\quad + \frac{\Gamma(j+\frac{\tau_c}{2})\Gamma(j+\frac{\tau_c}{2}-1)}{\Gamma(2j+\tau_c-1)}P_{\tau_c}(j),
\end{aligned} \tag{B.45}$$

where $P_{\tau_c}(j)$ is a polynomial in j. It is given by

$$P_{\tau_c}(j) = \frac{(-1)^{\frac{\tau_c}{2}}(\frac{\tau_c}{2}-2)\Gamma(\frac{\tau_c}{2})\Gamma(\frac{\tau_c}{2}+2)(j+\frac{\tau_c}{2}-3)(j+\frac{\tau_c}{2}+2)}{2\Gamma(\tau_c-1)} + \sum_{m=0}^{\frac{\tau_c}{2}-4}\frac{(1-\frac{\tau_c}{2})_m(2-\frac{\tau_c}{2})_m}{m!(2-\tau_c)_m}Q_{m,\tau_c}(j), \tag{B.46}$$

and

$$\begin{aligned}
&Q_{m,\tau_c}(j) \\
&= -2\tau_c\left(j^2 - 2j(m+2) + 4m + 6\right) + j^2(4m+6) + \tau_c^2(-2j+m+4) + \frac{4(j+m+2)(j+m+3)}{-2m+\tau_c-6} \\
&\quad + \frac{4(j+m+1)(j+m+2)}{-2m+\tau_c-4} + \frac{4(j+m)(j+m+1)}{-2m+\tau_c-2} + j(6-4m) + 6(m+1)(m+4) - \frac{\tau_c^3}{2} \\
&\quad + 2\Gamma(j+\tfrac{\tau_c}{2}+1)^2\left(\frac{{}_3F_2\left(\begin{matrix}5+m-\frac{\tau_c}{2}, & 5+m-\frac{\tau_c}{2}, & j+\frac{\tau_c}{2} \\ j+4+m, & j+6+m\end{matrix};1\right)}{\Gamma(j+4+m)\Gamma(j+6+m)\Gamma(\frac{\tau_c}{2}-m-2)\Gamma(\frac{\tau_c}{2}-m-4)}\right. \\
&\quad \left. - \frac{{}_3F_2\left(\begin{matrix}4+m-\frac{\tau_c}{2}, & 4+m-\frac{\tau_c}{2}, & j+\frac{\tau_c}{2} \\ j+3+m, & j+5+m\end{matrix};1\right)}{\Gamma(j+3+m)\Gamma(j+5+m)\Gamma(\frac{\tau_c}{2}-m-1)\Gamma(\frac{\tau_c}{2}-m-3)}\right) \\
&\quad + 2\sum_{k=1}^{\frac{\tau_c}{2}-4-m}\frac{\Gamma(j+\frac{\tau_c}{2})\Gamma(j+\frac{\tau_c}{2}+1)\Gamma(j+\frac{\tau_c}{2}+3)\,{}_3F_2\left(\begin{matrix}3-k, & -k, & j+\frac{\tau_c}{2}+1 \\ j+\frac{\tau_c}{2}-k, & j+\frac{\tau_c}{2}+3-k\end{matrix};1\right)}{k\Gamma(k+1)\Gamma(k+2)\Gamma(j+\frac{\tau_c}{2}-k)\Gamma(j+\frac{\tau_c}{2}+3-k)\Gamma(j+\frac{\tau_c}{2}-1)}.
\end{aligned} \tag{B.47}$$

B.4 Lightcone bootstrap and large-j behavior of $\langle R^{(n+1)g/q}_{j+6,j}\rangle$

In this appendix, we describe how to obtain the large-j behavior of the coefficients $\langle R^{(n+1)g/q}_{j+6,j}\rangle$ using the double lightcone limit result (3.121) and the crossing equation (3.48).

B.4.1 Infinite sums of $\mathrm{SL}(2,\mathbb{R})$ blocks

We will use lightcone bootstrap techniques from [81] to study the large-j behavior of the coefficients. An important ingredient for the lightcone bootstrap are identities for infinite sums of $\mathrm{SL}(2,\mathbb{R})$ blocks. One identity derived in [81] is given in (3.54). For the calculation in this appendix, we can write it as (assuming a is negative)

$$\sum_{\substack{h=j+h_0\\ j=0,2,\ldots}} S_a^{r,s}(h) k_{2h}^{r,s}(1-z) = \frac{1}{2} z^a + O(z^{a+1}),$$
$$S_a^{r,s}(h) = \frac{1}{\Gamma(-a-r)\Gamma(-a-s)} \frac{\Gamma(\bar{h}-r)\Gamma(\bar{h}-s)}{\Gamma(2\bar{h}-1)} \frac{\Gamma(\bar{h}-a-1)}{\Gamma(\bar{h}+a+1)}. \tag{B.48}$$

For our calculation, we will also need to compute an infinite sum of derivatives of the $\mathrm{SL}(2,\mathbb{R})$ blocks, such as $\partial_h k_{2h}^{r,s}(1-z)$ or $\partial_s k_{2h}^{r,s}(1-z)$. In particular, we want to compute

$$\sum_{j=0,2,\cdots} S_a^{0,-1}(j+3) \partial_\delta^n k_{\delta+j}^{0,-1}(1-z)|_{\delta\to j+6}, \tag{B.49}$$
$$\sum_{j=0,2,\cdots} S_a^{0,-1}(j+3) \partial_{\delta_*'}^n k_{2j+6}^{0,\frac{3-\delta_*'}{2}}(1-z)|_{\delta_*'\to 5}. \tag{B.50}$$

Let us first consider (B.49) with $n=1$. One way of computing it is to follow the original argument for lightcone bootstrap given in [7, 8]. In the limit $h\to\infty$, $\partial_h k_{2h}(z) \sim \frac{(4\rho)^h}{\sqrt{1-\rho^2}} \log(4\rho)$, and therefore the dominant contribution of the sum in h is at $h \sim \frac{1}{1-\rho} \sim \frac{1}{\sqrt{1-z}}$. Thus, we should study the large h limit of $k_{2h}^{0,-1}(z)$ with $h\sqrt{1-z}$ fixed. Using the Euler integral representation of $k_{2h}(z)$, we find that

$$k_{2h}^{0,-1}(z) \sim 2^{2h} \sqrt{\frac{y}{\pi h}} K_1(2\sqrt{y}), \quad y = h^2(1-z), \tag{B.51}$$

in the fixed y, $h\to\infty$ limit. Now, we can replace $1-z\to z$ in (B.51) and plug it into (B.49) with $n=1$. Focusing on the leading term in the small z limit, we find

$$\sum_{h=0,2,\ldots} S_a^{0,-1}(h+3) \frac{1}{2} \partial_h k_{2h+6}^{0,-1}(1-z)$$
$$\approx \frac{1}{4} \int_0^\infty dh \frac{2^{-4-2h} h^{-2a-\frac{1}{2}} \sqrt{\pi}}{\Gamma(1-a)\Gamma(-a)} \partial_h \left(2^{2h+6} \sqrt{\frac{h^2(1-\bar{z})}{\pi h}} K_1(2h\sqrt{z}) \right)$$
$$= \frac{1}{2} z^a \left(\log 2 + \sqrt{z} \frac{(1+4a)\Gamma(-\frac{1}{2}-a)\Gamma(\frac{1}{2}-a)}{4\Gamma(1-a)\Gamma(-a)} \right) + O(z^{a+1}), \tag{B.52}$$

where $\approx$ means that the Casimir-singular terms are the same. We have checked numerically that this formula is correct in the $z \to 0$ limit.

Interestingly, we find that if we simply move the derivative ∂_h of $\sum_{h=0,2,\ldots} S_a^{0,-1}(h+3)\frac{1}{2}\partial_h k_{2h+6}^{0,-1}(1-z)$ to the function $S_a^{0,-1}(h+3)$, we will also get the same result in the small z limit. One way of understanding this is by following the discussion in [81]. We can write (B.49) as a contour integral

$$\int_{-\epsilon-i\infty}^{\epsilon+i\infty} dh \frac{\pi(1+e^{i\pi h})}{2\tan(\pi h)} S_a^{0,-1}(h+3)\frac{1}{2^n}\partial_h^n k_{2h+6}^{0,-1}(1-z), \tag{B.53}$$

and then integrate by parts to move all the derivatives ∂_h. The leading term at small z should come from $\partial_h^n S_a^{0,-1}(h+3)$. In conclusion, the infinite sum (B.49) in the small z limit is given by

$$\sum_{j=0,2,\cdots} S_a^{0,-1}(j+3)\partial_\delta^n k_{\delta+j}^{0,-1}(1-z)|_{\delta\to j+6} = \left(\frac{\log 2}{2}\right)^n z^a + O(z^{a+\frac{1}{2}}). \tag{B.54}$$

We now consider (B.50). We claim that it is given by

$$\sum_{j=0,2,\cdots} S_a^{0,-1}(j+3)\partial_{\delta'_*}^n k_{2j+6}^{0,\frac{3-\delta'_*}{2}}(1-z)|_{\delta'_*\to 5} = c_n z^a \log^n z + O(z^a \log^{n-1} z). \tag{B.55}$$

The coefficient c_n can be determined by applying $\partial_{\delta'_*}$ to (B.48) n times. Since (B.48) is independent of δ'_*, we should get zero. Hence, we have

$$\begin{aligned}
&\sum_{j=0,2,\cdots} S_a^{0,-1}(j+3)\partial_{\delta'_*}^n k_{2j+6}^{0,\frac{3-\delta'_*}{2}}(1-z)|_{\delta'_*\to 5} \\
&= -\sum_{m=1}^{n}\sum_{j=0,2,\cdots} \frac{n!}{m!(n-m)!}\left(\partial_{\delta'_*}^m S_a^{0,\frac{3-\delta'_*}{2}}(j+3)|_{\delta'_*\to 5}\right)\left(\partial_{\delta'_*}^{n-m} k_{2j+6}^{0,\frac{3-\delta'_*}{2}}(1-z)|_{\delta'_*\to 5}\right) \\
&= -\sum_{m=1}^{n}\sum_{j=0,2,\cdots} \frac{n!}{m!(n-m)!}\frac{(-1)^m}{4^m}\partial_a^m S_a^{0,-1}(j+3)\left(\partial_{\delta'_*}^{n-m} k_{2j+6}^{0,\frac{3-\delta'_*}{2}}(1-z)|_{\delta'_*\to 5}\right) + O(z^a \log^{n-1} z) \\
&= -\sum_{m=1}^{n} \frac{n!}{m!(n-m)!}\frac{(-1)^m}{4^m} c_{n-m} z^a \log^n z + O(z^a \log^{n-1} z).
\end{aligned} \tag{B.56}$$

This leads to the recursion relation

$$c_n = -\sum_{m=1}^{n} \frac{n!}{m!(n-m)!}\frac{(-1)^m}{4^m} c_{n-m}. \tag{B.57}$$

With the initial condition $c_0 = \frac{1}{2}$, one can then determine that $c_n = 2^{-1-2n}$. Thus,

$$\sum_{j=0,2,\cdots} S_a^{0,-1}(j+3)\partial_{\delta'_*}^n k_{2j+6}^{0,\frac{3-\delta'_*}{2}}(1-z)|_{\delta'_*\to 5} = \frac{1}{2^{1+2n}} z^a \log^n z + O(z^a \log^{n-1} z). \tag{B.58}$$

B.4.2 $\langle R^{(n+1)}_{j+6,j}\rangle$ in the large-j limit

We are now ready to compute the large-j behavior of $\langle R^{(n+1)g/q}_{j+6,j}\rangle$. As a warmup, we first consider the leading order ($n = 0$) case of the double lightcone limit (3.121). Since we know that $R^{(1)}_{j+2,j}$ is nonzero only when $j = 2$, the $z(1-\bar{z})^0$ term of $\widetilde{\mathcal{G}}^{(1)}(z \ll 1-\bar{z} \ll 1)$ should be produced by the first subleading term in the $\mathrm{SL}(2,\mathbb{R})$ expansion of the $\tau_c = 4$ operator or the leading $\mathrm{SL}(2,\mathbb{R})$ expansion of the $\tau_c = 6$ operator. More explicitly, the crossing equation (3.48) at leading order in the double lightcone limit is given by (suppressing the g/q superscript for brevity)

$$\begin{aligned}
&10R^{(1)}_{4,2}z(1-\bar{z})^0+\cdots\\
&=\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^3\left[\sum_{j=0,2,\cdots}\langle R^{(1)}_{j+4,j}\rangle\left((1-\bar{z})^2k^{0,-1}_{2j+4}(1-z)+(1-\bar{z})^3\sum_{n=-1}^{1}C_nk^{0,-1}_{2j+4+2n}(1-z)+\cdots\right)\right.\\
&\qquad\left.+\sum_{j=0,2,\cdots}\langle R^{(1)}_{j+6,j}\rangle(1-\bar{z})^3k^{0,-1}_{2j+6}(1-z)+\cdots\right], \qquad \text{(B.59)}
\end{aligned}$$

where $\cdots$ are all subleading terms in the double lightcone limit. The coefficients C_n are from the $\mathrm{SL}(2,\mathbb{R})$ block expansion of the conformal block, but their expressions will not be important for our discussion. The reason is that the $\langle R^{(1)}_{j+4,j}\rangle$ coefficient actually has no contribution to the leading term in the left hand side. There are two ways of seeing this. First, we can simply look at the expression for $\langle R^{(1)}_{j+4,j}\rangle$ in (3.105) and (3.109) we obtained from the Lorentzian inversion formula and take the large-j limit.[2] For both the gluon jet and quark jet, we find

$$\langle R^{(1)}_{j+4,j}\rangle \sim 4^{-j}j^{\frac{3}{2}} \sim S^{0,-1}_{a=-1}(j), \qquad \text{(B.61)}$$

and therefore

$$z^3\sum_{j=0,2,4,\cdots}\langle R^{(1)}_{j+4,j}\rangle k^{0,-1}_{2j+4+2n}(1-z) \sim z^2, \qquad \text{(B.62)}$$

[2] When taking the large-j limit of the ${}_3F_2$ functions in $\langle R^{(1)}_{j+4,j}\rangle$, it is more convenient to first use the Euler representation of the ${}_3F_2$ function to write it as an integral of the ${}_2F_1$ function, and then study the large-j limit of the integral numerically. For example,

$${}_3F_2\left(\begin{matrix}2, & 3, & j+1\\ & j+4, & j+5\end{matrix};1\right)=\frac{\Gamma(j+5)}{\Gamma(j+1)\Gamma(4)}\int_0^1 dz\, z^j(1-z)^3{}_2F_1\left(2,3,j+4,z\right)\sim 1+\frac{6}{j}-\frac{18}{j^2} \qquad \text{(B.60)}$$

Sometimes the ${}_2F_1$ is even explicitly known, and one can study the large-j limit after evaluating the integral.

which is indeed subleading in (B.59). The other argument is to consider the term

$$\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^3 (1-\bar{z})^2 \sum_j \langle R^{(1)}_{j+4,j}\rangle k^{0,-1}_{2j+4}(1-z). \tag{B.63}$$

If $\langle R^{(1)}_{j+4,j}\rangle$ grows like $S^{0,-1}_{a=-2}(j)$ or faster at large-j, we should expect to see $z(1-\bar{z})^{-1}$ on the left hand side of (B.59). However, such terms do not exist, so $z^3\sum_j \langle R^{(1)}_{j+4,j}\rangle k^{0,-1}_{2j+4+2n}(1-z)$ must give subleading contribution to z.

Based on the above discussion, the crossing equation (B.59) becomes

$$10R^{(1)}_{4,2} z(1-\bar{z})^0 + \cdots = \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^3 \sum_{j=0,2,\cdots} \langle R^{(1)}_{j+6,j}\rangle (1-\bar{z})^3 k^{0,-1}_{2j+6}(1-z) + \cdots . \tag{B.64}$$

Using (B.48), we obtain

$$\langle R^{(1)}_{j+6,j}\rangle \sim 20R^{(1)}_{4,2} S^{0,-1}_{a=-2}(j+3) \sim \frac{5\sqrt{\pi}}{8} R^{(1)}_{4,2} 4^{-j} j^{\frac{7}{2}}. \tag{B.65}$$

which is (3.122) in the main text.

Let us now consider $n=1$. In this case, we should allow nonzero $\langle R^{(2)}_{j+\tau_c,j}\rangle$ for general τ_c. At this order, the crossing equation in the double lightcone limit becomes

$$\begin{aligned}
&10R^{(1)}_{4,2}\gamma^{(1)}_{4,2} z\log z(1-\bar{z})^0 + \cdots \\
&= \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^3 \Bigg[\sum_{\tau_c<6}\sum_j \langle R^{(2)}_{j+\tau_c,j}\rangle \sum_{n=0}^{\infty}(1-\bar{z})^{\frac{\tau_c}{2}+n} \sum_{m=-n}^{n} C^{\tau_c}_{n,m} k^{0,-1}_{2j+\tau_c+2m}(1-z) \\
&\quad + \sum_{\tau_c<6}\sum_j \langle R^{(1)}_{j+\tau_c,j}\gamma^{(1)}_{j+\tau_c,j}\rangle \Bigg(\sum_{n=0}^{\infty}\frac{1}{2}(1-\bar{z})^{\frac{\tau_c}{2}+n}\log(1-\bar{z}) \sum_{m=-n}^{n} C^{\tau_c}_{n,m} k^{0,-1}_{2j+\tau_c+2m}(1-z) \\
&\qquad + \sum_{n=0}^{\infty}(1-\bar{z})^{\frac{\tau_c}{2}+n} \sum_{m=-n}^{n} C^{\tau_c}_{n,m}\partial_\delta k^{0,-1}_{\delta+j+2m}(1-z)|_{\delta\to j+\tau_c}\Bigg) \\
&\quad + \sum_{\tau_c<6}\sum_j \langle R^{(1)}_{j+\tau_c,j}\gamma'^{(1)}_{*}\rangle \sum_{n=0}^{\infty}(1-\bar{z})^{\frac{\tau_c}{2}+n} \sum_{m=-n}^{n} C^{\tau_c}_{n,m}\partial_{\delta'_*} k^{0,\frac{3-\delta'_*}{2}}_{2j+\tau_c+2m}(1-z)|_{\delta'_*\to 5} \\
&\quad + \sum_j \langle R^{(2)}_{j+6,j}\rangle (1-\bar{z})^3 k^{0,-1}_{2j+6}(1-z) \\
&\quad + \sum_j \langle R^{(1)}_{j+6,j}\gamma^{(1)}_{j+6,j}\rangle \left(\frac{1}{2}(1-\bar{z})^3\log(1-\bar{z}) k^{0,-1}_{2j+6}(1-z) + (1-\bar{z})^3\partial_\delta k^{0,-1}_{\delta+j}(1-z)|_{\delta\to}\right. \\
&\quad + \sum_j \langle R^{(1)}_{j+6,j}\gamma'^{(1)}_{*}\rangle (1-\bar{z})^3 \partial_{\delta'_*} k^{0,\frac{3-\delta'_*}{2}}_{2j+6}(1-z)|_{\delta'_*\to 5} + \cdots \Bigg].
\end{aligned} \tag{B.66}$$

We can use (B.48) and (B.54) to argue that for $\tau_c < 6$, $\langle R^{(2)}_{j+\tau_c,j}\rangle$ and $\langle R^{(1)}_{j+\tau_c,j}\gamma^{(1)}_{j+\tau_c,j}\rangle$ cannot grow as fast as $\partial_a S^{0,-1}_a|_{a\to -2}$ at large j, otherwise the $z\log z$ term in the left hand side will have the wrong leading behavior for $1-\bar{z} \ll 1$. Similarly, $\langle R^{(1)}_{j+6,j}\gamma^{(1)}_{j+6,j}\rangle$ should not give any leading contribution, or we will get $z\log z\log(1-\bar{z})$. Thus, we have

$$\begin{aligned}&10R^{(1)}_{4,2}\gamma^{(1)}_{4,2}z\log z+\cdots\\&=z^3\left(\sum_j\langle R^{(2)}_{j+6,j}\rangle k^{0,-1}_{2j+6}(1-z)+\sum_j\langle R^{(1)}_{j+6,j}\gamma'^{(1)}_*\rangle\partial_{\delta'_*}k^{0,\frac{3-\delta'_*}{2}}_{2j+6}(1-z)|_{\delta'_*\to 5}+\cdots\right).\end{aligned} \tag{B.67}$$

As discussed in section 3.4, we also need to deal with the degeneracies coming from γ'_*. To do so, let us make an ansatz for the large-j behavior of $R^{(n+1)}_{j+6,j}$. We will assume

$$\begin{aligned}R^{(n+1)g}_{j+6,j}&=\left(c^{(0)g}_1A^{(n+1)}_1+c^{(0)g}_2A^{(n+1)}_2\right)\partial^n_aS^{0,-1}_a(j+3)|_{a\to-2}+O(4^{-j}j^{\frac{7}{2}}\log^{n-1}j),\\R^{(n+1)q}_{j+6,j}&=\left(c^{(0)q}_1A^{(n+1)}_1+c^{(0)q}_2A^{(n+1)}_2\right)\partial^n_aS^{0,-1}_a(j+3)|_{a\to-2}+O(4^{-j}j^{\frac{7}{2}}\log^{n-1}j),\end{aligned} \tag{B.68}$$

where the coefficients $c^{(0)g/q}_i$ are given in (3.137), and our goal is to determine $A^{(n+1)}_1$ and $A^{(n+1)}_2$. In general, the coefficient $c^{\mathcal{O}}$ at subleading order can also contribute, but we expect them to appear as $c^{(k)\mathcal{O}}_iA^{(n+1-k)}_i\partial^{n-k}_aS_a|_{a\to-2}$. Since $\partial^{n-k}_aS_a|_{a\to-2}$ is subleading at large-j, we can just consider $c^{(0)\mathcal{O}}_i$ for the leading large-j behavior. Note that the ansatz (B.68) implies that for any nonnegative integers n and k,

$$\begin{aligned}\langle R^{(n+1)g/q}_{j+6,j}(\gamma'_*)^k\rangle=&\left(c^{(0)g/q}_1A^{(n+1)}_1(\gamma'_{*1})^k+c^{(0)g/q}_2A^{(n+1)}_2(\gamma'_{*2})^k\right)\partial^n_aS^{0,-1}_a(j+3)|_{a\to-2}\\&+O(4^{-j}j^{\frac{7}{2}}\log^{n-1}j),\end{aligned} \tag{B.69}$$

where γ'_{*1} and γ'_{*2} are given by (3.132). Comparing (B.68) to the $n=0$ result (B.65), we find

$$\begin{aligned}A^{(1)}_1=&\frac{1280\pi}{\alpha_+-\alpha_-}(R^{(1)q}_{4,2}-\alpha_-R^{(1)g}_{4,2}),\\A^{(1)}_2=&\frac{1280\pi}{\alpha_+-\alpha_-}\alpha_-(\alpha_+R^{(1)g}_{4,2}-R^{(1)q}_{4,2}).\end{aligned} \tag{B.70}$$

For the $n=1$ crossing equation (B.67), we can also plug in the ansatz (B.68), and use (B.48) and (B.58) to compute the infinite sums. After taking care of the degeneracies, we obtain

$$\begin{aligned} A_1^{(2)} =& \frac{320\pi}{\alpha_+ - \alpha_-}(R_{4,2}^{(1)q} - \alpha_- R_{4,2}^{(1)g})(4\gamma_{4,2}^{(1)} - \gamma'_{*1}), \\ A_2^{(2)} =& \frac{320\pi}{\alpha_+ - \alpha_-}\alpha_-(\alpha_+ R_{4,2}^{(1)g} - R_{4,2}^{(1)q})(4\gamma_{4,2}^{(1)} - \gamma'_{*2}). \end{aligned} \tag{B.71}$$

For $n > 1$, one can repeat the above argument and determine $A_1^{(n+1)}, A_2^{(n+1)}$. In particular, terms involving $R_{j+\tau_c,j}^{(n+1-p)}$ with $\tau_c < 6$ should not contribute to the $z\log^n z(1-\bar{z})^0$ term. Moreover, only $\langle R_{j+6,j}^{(n+1)}\rangle$ and $\langle R_{j+6,j}^{(n+1-m)}(\gamma_*'^{(1)})^m\rangle$ can produce $z\log^n z$. So, we have

$$\begin{aligned} &10R_{4,2}^{(1)}\left(\gamma_{4,2}^{(1)}\right)^n z\log^n z + \cdots \\ &= z^3\left(\sum_j \langle R_{j+6,j}^{(n+1)}\rangle k_{2j+6}^{0,-1}(1-z) + \sum_{m=1}^n\sum_j \langle R_{j+6,j}^{(n+1-m)}(\gamma_*'^{(1)})^m\rangle \partial_{\delta'_*}^m k_{2j+6}^{0,\frac{3-\delta'_*}{2}}(1-z)|_{\delta'_*\to 5} + \cdots\right). \end{aligned} \tag{B.72}$$

After using the infinite sum formula (B.48), (B.58) and the relation (B.69), we find that the solution takes a suprisingly simple form for $n \geq 1$:

$$\begin{aligned} A_1^{(n+1)} =& \frac{320\pi(R_{4,2}^{(1)q} - \alpha_- R_{4,2}^{(1)g})(\gamma_{4,2}^{(1)})^{n-1}(4\gamma_{4,2}^{(1)} - \gamma'_{*1})}{\alpha_+ - \alpha_-}, \\ A_2^{(n+1)} =& \frac{320\pi\alpha_-(\alpha_+ R_{4,2}^{(1)g} - R_{4,2}^{(1)q})(\gamma_{4,2}^{(1)})^{n-1}(4\gamma_{4,2}^{(1)} - \gamma'_{*2})}{\alpha_+ - \alpha_-}. \end{aligned} \tag{B.73}$$

It would be interesting to use some other methods to verify our results for the large-j behavior of $\langle R_{j+6,j}^{(n+1)}\rangle$. For example, $\langle R_{j+6,j}^{(1)}\rangle$ can be obtained by performing a higher-twist version of the Lorentzian inversion formula calculation described in section 3.3.3. If the collinear EEEC at subleading order is known, one should also be able to find $\langle R_{j+6,j}^{(n+1)}\rangle$ at higher values of n.

B.5 Tree-level EEEC

In this appendix, we give the details of the calculation of tree-level EEEC in section 3.5. In the first section, we compute the functions $\mathcal{F}_0$ and $\mathcal{F}_1$ in (3.157). In the second section, we derive the relation (3.168) using crossing symmetry.

B.5.1 Computing $\mathcal{F}_0$ and $\mathcal{F}_1$

We first calculate the squared amplitude for the initial state created by $\mathrm{Tr}F^2$. We focus on processes with three out-going particles since $\mathcal{F}_0$ and $\mathcal{F}_1$ only get contributions from those processes. At tree-level, there are three possible processes with three outgoing particles. The first one includes three gluons, and its amplitude squared is given by (we use mostly positive metric)

$$\begin{aligned}|\mathcal{M}_{g+g+g}|^2 =& -64g^2N_c(N_c^2-1)\times \\ &\Bigg(6(p_1\cdot p_2+p_1\cdot p_3+p_2\cdot p_3) \\ &+2\left(\frac{(p_1\cdot p_2)^2+(p_2\cdot p_3)^2}{p_1\cdot p_3}+\frac{(p_1\cdot p_3)^2+(p_2\cdot p_3)^2}{p_1\cdot p_2}+\frac{(p_1\cdot p_2)^2+(p_1\cdot p_3)^2}{p_2\cdot p_3}\right) \\ &+\frac{(p_1\cdot p_2)^3}{p_1\cdot p_3 p_2\cdot p_3}+\frac{(p_1\cdot p_3)^3}{p_1\cdot p_2 p_2\cdot p_3}+\frac{(p_2\cdot p_3)^3}{p_1\cdot p_2 p_1\cdot p_3} \\ &+3\left(\frac{p_1\cdot p_3 p_2\cdot p_3}{p_1\cdot p_2}+\frac{p_1\cdot p_2 p_2\cdot p_3}{p_1\cdot p_3}+\frac{p_1\cdot p_2 p_1\cdot p_3}{p_2\cdot p_3}\right)\Bigg). \end{aligned} \tag{B.74}$$

The second process has one gluon and two Weyl spinors

$$|\mathcal{M}_{g(p_1)+\lambda(p_2)+\bar{\lambda}(p_3)}|^2 = -16g^2N_c(N_c^2-1)\frac{(p_1\cdot p_2)^2+(p_1\cdot p_3)^2}{p_2\cdot p_3}. \tag{B.75}$$

Finally, the third process has one gluon and two scalars

$$|\mathcal{M}_{g(p_1)+\phi(p_2)+\phi(p_3)}|^2 = -32g^2N_c(N_c^2-1)\frac{p_1\cdot p_2 p_1\cdot p_3}{p_2\cdot p_3}. \tag{B.76}$$

We can then define the total amplitude squared $|\mathcal{M}|^2$ as

$$|\mathcal{M}|^2 = \frac{1}{3!}|\mathcal{M}_{g+g+g}|^2 + 4|\mathcal{M}_{g(p_1)+\lambda(p_2)+\bar{\lambda}(p_3)}|^2 + \frac{6}{2!}|\mathcal{M}_{g(p_1)+\phi(p_2)+\phi(p_3)}|^2. \tag{B.77}$$

Note that due to the three identical gluons and two identical scalars in the final states, we should include symmetry factors $\frac{1}{3!}$ and $\frac{1}{2!}$ in the phase space measure for the corresponding final state. But here we choose to include those factors in $|\mathcal{M}|^2$, so that we can just use the same phase space measure for all the final states.

Comparing (3.149) and (3.157), one can show that the function $\mathcal{F}_1$ is defined as

$$\begin{aligned}&\mathcal{F}_1(\vec{n}_1,\vec{n}_2) \\ &=\sum_{i,j}\int d\sigma\frac{E_i^2E_j}{Q^3}\delta\left(\vec{n}_1,\frac{\vec{p}_i}{E_i}\right)\delta\left(\vec{n}_2,\frac{\vec{p}_j}{E_j}\right) \\ &=\frac{1}{\sigma_{\text{tot}}}\sum_{\{i,j\}\subset\{a,b,c\}}\int\frac{d^3\vec{p}_a}{(2\pi)^3}\frac{d^3\vec{p}_b}{(2\pi)^3}\frac{d^3\vec{p}_c}{(2\pi)^3}\frac{1}{2E_a}\frac{1}{2E_b}\frac{1}{2E_c}|\mathcal{M}|^2(2\pi)^4\delta(Q-E_a-E_b-E_c)\delta^{(3)}\left(\vec{p}_a+\vec{p}_b+\vec{p}_c\right) \\ &\qquad\times\frac{E_i^2E_j}{Q^3}\delta\left(\vec{n}_1,\frac{\vec{p}_i}{E_i}\right)\delta\left(\vec{n}_2,\frac{\vec{p}_j}{E_j}\right),\end{aligned} \tag{B.78}$$

and $\mathcal{F}_0$ is defined as

$$\mathcal{F}_0(\vec{n}_1,\vec{n}_2,\vec{n}_3)\delta((\vec{n}_1\times\vec{n}_2)\cdot\vec{n}_3)$$
$$= \frac{1}{\sigma_{\text{tot}}}\sum_{\{i,j,k\}=\{a,b,c\}}\int \frac{d^3\vec{p}_a}{(2\pi)^3}\frac{d^3\vec{p}_b}{(2\pi)^3}\frac{d^3\vec{p}_c}{(2\pi)^3}\frac{1}{2E_a}\frac{1}{2E_b}\frac{1}{2E_c}|\mathcal{M}|^2(2\pi)^4\delta(Q-E_a-E_b-E_c)\delta^{(3)}(\vec{p}_a+\vec{p}_b+\vec{p}_c)$$
$$\times\frac{E_iE_jE_k}{Q^3}\delta\left(\vec{n}_1,\frac{\vec{p}_i}{E_i}\right)\delta\left(\vec{n}_2,\frac{\vec{p}_j}{E_j}\right)\delta\left(\vec{n}_3,\frac{\vec{p}_k}{E_k}\right). \tag{B.79}$$

We can then plug in (B.77) for the squared amplitude $|\mathcal{M}|^2$. Also, the total cross section is

$$\sigma_{\text{tot}} = \frac{N_c^2-1}{2\pi}Q^4. \tag{B.80}$$

Performing the integral in (B.78), we then obtain that $\mathcal{F}_1$ is given by (3.158). For (B.79), the delta function $\delta((\vec{n}_1\times\vec{n}_2)\cdot\vec{n}_3)$ on the left-hand side will be canceled by one of the delta functions in $\delta^{(3)}(\vec{p}_a+\vec{p}_b+\vec{p}_c)$ on the right-hand side. After performing the calculation, we then find that $\mathcal{F}_0$ is given by (3.162) and (3.163). The step function $\theta(\zeta_1+\zeta_2-1)$ in (3.162) comes from the condition $\vec{p}_a+\vec{p}_b+\vec{p}_c=0$. If $\zeta_1+\zeta_2<1$, one can easily draw a line such that all three momenta $\vec{p}_{a,b,c}$ lie on the same side of the line, and therefore there are no solutions to $\vec{p}_a+\vec{p}_b+\vec{p}_c=0$.

B.5.2 Crossing symmetry of $\mathcal{F}_0$

We now consider crossing symmetry of $\mathcal{F}_0(\vec{n}_1,\vec{n}_2,\vec{n}_3)$. If $\vec{n}_1,\vec{n}_2,\vec{n}_3$ are all different from each other, it is not too hard to check that the function $\mathcal{F}_0(\vec{n}_1,\vec{n}_2,\vec{n}_3)$ given by (3.162) and (3.163) is crossing symmetric. So we want to focus on the delta functions and show that they are also crossing symmetric. In particular, we will consider $2\leftrightarrow 3$. Note that (3.162) can also be written as

$$\mathcal{F}_0(\vec{n}_1,\vec{n}_2,\vec{n}_3) = \frac{1}{16}\sqrt{(\vec{n}_1-\vec{n}_3)^2(\vec{n}_1+\vec{n}_3)^2(\vec{n}_2-\vec{n}_3)^2(\vec{n}_2+\vec{n}_3)^2}\widetilde{\mathcal{F}}_0(\vec{n}_1,\vec{n}_2,\vec{n}_3), \tag{B.81}$$

and the singular part of $\widetilde{\mathcal{F}}_0$ is given by (3.167).

We now preform the crossing $2\leftrightarrow 3$ and first look at the contact term at $r_2=0$. Naively, we expect that the contact term looks like

$$\frac{\delta(r_2')}{r_2'}\frac{1}{16}\sqrt{(\vec{n}_1-\vec{n}_2)^2(\vec{n}_1+\vec{n}_2)^2(\vec{n}_2-\vec{n}_3)^2(\vec{n}_2+\vec{n}_3)^2}\delta\left((\vec{n}_1\times\vec{n}_2)\cdot\vec{n}_3\right), \tag{B.82}$$

where the new variable r_2' is given by

$$r_2' = \frac{1}{4}\sqrt{(\vec{n}_1+\vec{n}_2)^4+(\vec{n}_2-\vec{n}_3)^4} = h_2(\theta_2)r_2+O(r_2^2), \tag{B.83}$$

and the function $h_2(\theta_2)$ is

$$h_2(\theta_2) = \sqrt{\cos^2\theta_2 + (\sqrt{\cos\theta_2} - \sqrt{\sin\theta_2})^4}. \tag{B.84}$$

If we integrate (B.82) against a test function $F(\vec{n}_1, \vec{n}_2, \vec{n}_3)$, we find

$$\begin{aligned}
&\int d\Omega_{\vec{n}_1} d\Omega_{\vec{n}_2} d\Omega_{\vec{n}_3} F(\vec{n}_1, \vec{n}_2, \vec{n}_3) \frac{\delta(r_2')}{r_2'} \frac{1}{16} \sqrt{(\vec{n}_1 - \vec{n}_2)^2 (\vec{n}_1 + \vec{n}_2)^2 (\vec{n}_2 - \vec{n}_3)^2 (\vec{n}_2 + \vec{n}_3)^2} \delta\left((\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_3\right) \\
&= 8\pi^2 \int_0^{\frac{\pi}{4}} d\theta_2 \frac{1}{h_2(\theta_2)^2} \frac{\sqrt{\cos\theta_2} - \sqrt{\sin\theta_2}}{\sqrt{\sin\theta_2}} \int dr_2 \delta(r_2) F(\vec{n}_1, \vec{n}_2, \vec{n}_3) \\
&= 2\pi^3 F(-\vec{n}_3, \vec{n}_3, \vec{n}_3),
\end{aligned} \tag{B.85}$$

which agrees with the original contact term before crossing. However, there is actually another delta function coming from the $[\cdots]_0$ distribution. This is due to the relation

$$\left[\frac{1}{ax}\right]_0 = \frac{1}{a}\left[\frac{1}{x}\right]_0 + \frac{\log a}{a}\delta(x). \tag{B.86}$$

Therefore, from the $[\cdots]_0$ distribution we have

$$\frac{f_0(\theta_2')}{r_2'}\left[\frac{1}{r_2'}\right]_0 \to \frac{f_0(\theta_2')}{h_2(\theta_2) r_2} \frac{\log h_2(\theta_2)}{h_2(\theta_2)} \delta(r_2), \tag{B.87}$$

where

$$\theta_2' = \tan^{-1}\left(\frac{(\vec{n}_1 + \vec{n}_2)^2}{(\vec{n}_2 - \vec{n}_3)^2}\right) = \tan^{-1}\left((1 - \sqrt{\tan\theta_2})^2\right) + O(r_2). \tag{B.88}$$

Therefore, we should also consider

$$8\pi^2 \int_0^{\frac{\pi}{4}} d\theta_2 \frac{f_0(\theta_2') \log h_2(\theta_2)}{h_2(\theta_2)^2} \frac{\sqrt{\cos\theta_2} - \sqrt{\sin\theta_2}}{\sqrt{\sin\theta_2}} \int dr_2 \delta(r_2) F(\vec{n}_1, \vec{n}_2, \vec{n}_3). \tag{B.89}$$

It turns out that the θ_2 integral actually vanishes. So, for the $r_2 = 0$ contact term, the $[\cdots]_0$ distribution does not produce new delta function after crossing $2 \leftrightarrow 3$.

Now we consider the other two contact terms. After $2 \leftrightarrow 3$, r_1 and r_3 become

$$\begin{aligned}
r_1' &= \frac{1}{4}\sqrt{(\vec{n}_1 - \vec{n}_2)^4 + (\vec{n}_2 + \vec{n}_3)^4} = h_{1\to 3}(\theta_3) r_3 + O(r_3^2), \\
r_3' &= \frac{1}{4}\sqrt{(\vec{n}_1 + \vec{n}_2)^4 + (\vec{n}_2 + \vec{n}_3)^4} = h_{3\to 1}(\theta_1) r_1 + O(r_1^2),
\end{aligned} \tag{B.90}$$

where

$$h_{1\to3}(\theta_3) = \sqrt{\sin^2\theta_3 + \left(\sqrt{\cos\theta_3} + \sqrt{\sin\theta_3}\right)^4},$$
$$h_{3\to1}(\theta_1) = \sqrt{\sin^2\theta_1 + \left(\sqrt{\cos\theta_1} - \sqrt{\sin\theta_1}\right)^4}. \tag{B.91}$$

Also, θ_1 and θ_3 become

$$\theta_1' = \tan^{-1}\left(\frac{(\vec{n}_2+\vec{n}_3)^2}{(\vec{n}_1-\vec{n}_2)^2}\right) = \tan^{-1}\left(\frac{\sin\theta_3}{(\sqrt{\cos\theta_3}+\sqrt{\sin\theta_3})^2}\right) + O(r_3),$$
$$\theta_3' = \tan^{-1}\left(\frac{(\vec{n}_2+\vec{n}_3)^2}{(\vec{n}_1+\vec{n}_2)^2}\right) = \tan^{-1}\left(\frac{\sin\theta_1}{(\sqrt{\cos\theta_1}-\sqrt{\sin\theta_1})^2}\right) + O(r_1). \tag{B.92}$$

Integrating the $\delta(r_1')$ term, we get

$$\int d\Omega_{\vec{n}_1} d\Omega_{\vec{n}_2} d\Omega_{\vec{n}_3} F(\vec{n}_1,\vec{n}_2,\vec{n}_3)\frac{\delta(r_1')}{r_1'}\frac{1}{16}\sqrt{(\vec{n}_1-\vec{n}_2)^2(\vec{n}_1+\vec{n}_2)^2(\vec{n}_2-\vec{n}_3)^2(\vec{n}_2+\vec{n}_3)^2}\delta\left((\vec{n}_1\times\vec{n}_2)\cdot\vec{n}_3\right)$$
$$= 8\pi^2\int_0^{\frac{\pi}{2}} d\theta_3 \frac{1}{h_{1\to3}(\theta_3)^2}\frac{\sqrt{\cos\theta_3}+\sqrt{\sin\theta_3}}{\sqrt{\cos\theta_3}}\int dr_3\delta(r_3)F(\vec{n}_1,\vec{n}_2,\vec{n}_3)$$
$$= 2\pi^3 F(-\vec{n}_3,-\vec{n}_3,\vec{n}_3). \tag{B.93}$$

We should also include the contribution from the $\left[\frac{1}{r_1'}\right]_0$ distribution. This term will give

$$8\pi^2\int_0^{\frac{\pi}{2}} d\theta_3 \frac{f_0(\theta_1')\log h_{1\to3}(\theta_3)}{h_{1\to3}(\theta_3)^2}\frac{\sqrt{\cos\theta_3}+\sqrt{\sin\theta_3}}{\sqrt{\cos\theta_3}}\int dr_3\delta(r_3)F(\vec{n}_1,\vec{n}_2,\vec{n}_3)$$
$$= 8\pi^2\times\frac{\pi-\sqrt{2}\pi+2\log 2+\sqrt{2}\log(3-2\sqrt{2})}{512\pi^4}F(-\vec{n}_3,-\vec{n}_3,\vec{n}_3). \tag{B.94}$$

Comparing this result with the contact terms before crossing, we find that for the delta functions to be crossing-symmetric, we must have (3.168). Also, if we consider the contact term at $r_1 = 0$, we will get the same condition.

B.6 More details on the celestial inversion formula

In this appendix, we give the derivation for the orthogonality relation of celestial partial waves (3.193) and the integral identity (3.204). We also show that the contributions at infinity of the contour deformations of (3.198) vanish.

B.6.1 Orthogonality of celestial partial waves

To derive the orthogonality relation, let us consider a natural pairing

$$\int D^{d-2}z_1 D^{d-2}z_2 D^{d-2}z_3 d^{d-1}_{\text{AdS}}p\ \Psi^c_{\delta_5,j_5;\delta_5'}(z_1,z_2,z_3,p)\Psi^{c(\widetilde{\delta}_i)}_{\widetilde{\delta}_6,j_6;\widetilde{\delta}_6'}(z_1,z_2,z_3,p), \tag{B.95}$$

where $d^{d-1}_{AdS}p = 2d^d p\delta(p^2+1)\theta(p^0)$ is an integral over the AdS space defined by $p^2 = -1$. Plugging in the definition of the celestial partial wave, we get

$$\begin{aligned}
&\int D^{d-2}z_1 D^{d-2}z_2 D^{d-2}z_3 d^{d-1}_{\text{AdS}}p D^{d-2}z D^{d-2}z' D^{d-2}z'' D^{d-2}z''' \\
&\langle \mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta_5,j_5}(z)\rangle\langle\widetilde{\mathcal{P}}_{\delta_5,j_5}(z)\mathcal{P}_{\delta_3}(z_3)\mathcal{P}_{\delta'_5}(z')\rangle\frac{1}{(-2z'\cdot p)^{\widetilde{\delta}'_5}} \\
&\times\langle\widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_2}(z_2)\widetilde{\mathcal{P}}_{\delta_6,j_6}(z'')\rangle\langle\mathcal{P}_{\delta_6,j_6}(z'')\widetilde{\mathcal{P}}_{\delta_3}(z_3)\widetilde{\mathcal{P}}_{\delta'_6}(z''')\rangle\frac{1}{(-2z'''\cdot p)^{\delta'_6}} \\
&= B_{12\mathcal{P}_{\delta_5,j_5}}\delta_{\mathcal{P}_5\mathcal{P}_6}\int D^{d-2}z_3 d^{d-1}_{\text{AdS}}p D^{d-2}z D^{d-2}z' D^{d-2}z''' \\
&\langle\widetilde{\mathcal{P}}_{\delta_5,j_5}(z)\mathcal{P}_{\delta_3}(z_3)\mathcal{P}_{\delta'_5}(z')\rangle\frac{1}{(-2z'\cdot p)^{\widetilde{\delta}'_5}}\langle\mathcal{P}_{\delta_5,j_5}(z)\widetilde{\mathcal{P}}_{\delta_3}(z_3)\widetilde{\mathcal{P}}_{\delta'_6}(z''')\rangle\frac{1}{(-2z'''\cdot p)^{\delta'_6}} \\
&= B_{12\mathcal{P}_{\delta_5,j_5}}B_{\widetilde{\mathcal{P}}_{\delta_5,j_5}3\mathcal{P}_{\delta'_5}}\delta_{\mathcal{P}_5\mathcal{P}_6}\delta_{\mathcal{P}'_5\mathcal{P}'_6} \\
&\times\int d^{d-1}_{\text{AdS}}p D^{d-2}z'\frac{1}{(-2z'\cdot p)^{\widetilde{\delta}'_5}}\frac{1}{(-2z'\cdot p)^{\delta'_5}}.
\end{aligned} \tag{B.96}$$

In the first and the second equality above, we have used the bubble formula (eq. (2.32) in [45]), and $B_{12\mathcal{P}_{\delta_5,j_5}}$, $B_{\widetilde{\mathcal{P}}_{\delta_5,j_5}3\mathcal{P}_{\delta'_5}}$ are bubble coefficients given by (3.195). Moreover, the integral in the last line of (B.96) is given by

$$\int\frac{D^{d-2}z d^{d-1}_{\text{AdS}}p}{\text{vol}(\text{SO}(d-1,1))}(-2p\cdot z)^{-\delta}(-2p\cdot z)^{-\widetilde{\delta}} = \frac{1}{2^{d-2}\,\text{vol}(\text{SO}(d-2))}, \tag{B.97}$$

where we use the conformal group to gauge fix $p = (1,0,0,\ldots,0)$ and $z = (1,1,0,\ldots,0)$. The stabilizer group after the gauge-fixing is $\text{SO}(d-2)$, and the Fadeev-Popov determinant for the gauge-fixing is 1.

Therefore, the orthogonality relation for the celestial partial wave is

$$\begin{aligned}
&\int\frac{D^{d-2}z_1 D^{d-2}z_2 D^{d-2}z_3 d^{d-1}_{\text{AdS}}p}{\text{vol}(\text{SO}(d-1,1))}\,\Psi^c_{\delta_5,j_5;\delta'_5}(z_1,z_2,z_3,p)\Psi^{c(\widetilde{\delta}_i)}_{\widetilde{\delta}_6,j_6;\widetilde{\delta}'_6}(z_1,z_2,z_3,p) \\
&= \frac{1}{2^{d-2}\,\text{vol}(\text{SO}(d-2))}B_{12\mathcal{P}_{\delta_5,j_5}}B_{\widetilde{\mathcal{P}}_{\delta_5,j_5}3\mathcal{P}_{\delta'_5}}\delta_{\mathcal{P}_5\mathcal{P}_6}\delta_{\mathcal{P}'_5\mathcal{P}'_6},
\end{aligned} \tag{B.98}$$

where

$$\delta_{\mathcal{P}_5\mathcal{P}_6} = 2\pi\delta(s_5-s_6)\delta_{j_5,j_6}, \tag{B.99}$$

for $\delta_5 = \frac{d-2}{2}+is_5$, $\delta_6 = \frac{d-2}{2}+is_6$ with $s_5, s_6 > 0$.

B.6.2 Derivation of (3.204)

We now consider the identity

$$\int D^{d-2}z_1 D^{d-2}z_2 \langle \widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_2}(z_2)\mathcal{P}_{\delta,j=0}(z)\rangle(-2p\cdot z_1)^{-\delta_1}(-2p\cdot z_2)^{-\delta_2}$$
$$= C_{\delta_1,\delta_2;\delta}(-p^2)^{\frac{\delta-\delta_1-\delta_2}{2}}(-2p\cdot z)^{-\delta}. \tag{B.100}$$

By Lorentz symmetry and homogeneity of p and z, the right-hand side must be proportional to $(-p^2)^{\frac{\delta-\delta_1-\delta_2}{2}}(-2p\cdot z)^{-\delta}$. Thus, our goal here is showing that the coefficient $C_{\delta_1,\delta_2;\delta}$ is given by (3.205). Our strategy is to first fix $p^2=-1$, and integrate both sides of (3.204) against $(-2p\cdot z)^{-\widetilde{\delta}}$ over z and p. More precisely, for the right-hand side, we have

$$\int \frac{D^{d-2}z\, d^{d-1}_{\mathrm{AdS}}p}{\mathrm{vol}(\mathrm{SO}(d-1,1))}C_{\delta_1,\delta_2;\delta}(-2p\cdot z)^{-\delta}(-2p\cdot z)^{-\widetilde{\delta}} = C_{\delta_1,\delta_2;\delta}\frac{1}{2^{d-2}\,\mathrm{vol}(\mathrm{SO}(d-2))}, \tag{B.101}$$

which follows from (B.97). For the left-hand side, we want to compute

$$\int \frac{D^{d-2}z_1 D^{d-2}z_2 D^{d-2}z\, d^{d-1}_{\mathrm{AdS}}p}{\mathrm{vol}(\mathrm{SO}(d-1,1))}\langle \widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_2}(z_2)\mathcal{P}_{\delta,0}(z)\rangle(-2p\cdot z_1)^{-\delta_1}(-2p\cdot z_2)^{-\delta_2}(-2p\cdot z)^{-\widetilde{\delta}}. \tag{B.102}$$

One can immediately recognize that the p integral is a three-point Witten diagram, and can be evaluated using (3.207). Furthermore, the remaining integral over z_1, z_2, z is a conformally-invariant three-point pairing. Therefore, the left-hand side is given by

$$\int \frac{D^{d-2}z_1 D^{d-2}z_2 D^{d-2}z\, d^{d-1}_{\mathrm{AdS}}p}{\mathrm{vol}(\mathrm{SO}(d-1,1))}\langle \widetilde{\mathcal{P}}_{\delta_1}(z_1)\widetilde{\mathcal{P}}_{\delta_2}(z_2)\mathcal{P}_{\delta,0}(z)\rangle(-2p\cdot z_1)^{-\delta_1}(-2p\cdot z_2)^{-\delta_2}(-2p\cdot z)^{-\widetilde{\delta}}$$
$$= D_{\delta_1,\delta_2,\widetilde{\delta}}\left(\langle \widetilde{\mathcal{P}}_{\delta_1}\widetilde{\mathcal{P}}_{\delta_2}\mathcal{P}_{\delta}\rangle, \langle \mathcal{P}_{\delta_1}\mathcal{P}_{\delta_2}\widetilde{\mathcal{P}}_{\delta}\rangle\right)$$
$$= \frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{\delta_1+\delta_2+\widetilde{\delta}-d+2}{2})\Gamma(\frac{\delta_1+\delta_2-\widetilde{\delta}}{2})\Gamma(\frac{\delta_1+\widetilde{\delta}-\delta_2}{2})\Gamma(\frac{\delta_2+\widetilde{\delta}-\delta_1}{2})}{2\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\widetilde{\delta})}\frac{1}{2^{d-2}\,\mathrm{vol}(\mathrm{SO}(d-3))}. \tag{B.103}$$

Finally, comparing (B.101) and (B.103), we obtain

$$C_{\delta_1,\delta_2;\delta} = \frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{\delta_1+\delta_2+\widetilde{\delta}-d+2}{2})\Gamma(\frac{\delta_1+\delta_2-\widetilde{\delta}}{2})\Gamma(\frac{\delta_1+\widetilde{\delta}-\delta_2}{2})\Gamma(\frac{\delta_2+\widetilde{\delta}-\delta_1}{2})}{2\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\widetilde{\delta})}\,\mathrm{vol}(S^{d-3}), \tag{B.104}$$

which agrees with (3.205).

B.6.3 Celestial block at large δ and δ'

In this section, we study (3.198),

$$\mathcal{F}(z_1, z_2, z_3, p) = \sum_j \int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty} \frac{d\delta}{2\pi i} \int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty} \frac{d\delta'}{2\pi i} C(\delta, j; \delta') G^c_{\delta,j;\delta'}(z_1, z_2, z_3, p), \tag{B.105}$$

and make sure that the contributions at infinity vanish when doing the contour deformations. For concreteness, we consider the leading order strong-coupling EEEC, so $C(\delta, j; \delta')$ is given by (3.211). When we first close the δ' contour to the right, the locations of the poles are at $\delta' = \delta + 3 + 2k$. Furthermore, (3.211) in the large δ' limit behaves like

$$C^{(0)}_{\text{strong}}(\delta, j; \delta' \to \infty) \sim 2^{-\delta'} (\cdots) , \tag{B.106}$$

where $(\cdots)$ grows sub-exponentially at large δ'. Therefore, a sufficient condition for the contribution at infinity of the δ' contour to vanish is

$$\lim_{\text{Re}(\delta')\to\infty} 2^{-\delta'} (\text{sub-exponential}) \, G^c_{\delta,j;\delta'}(z_1, z_2, z_3, p) = 0. \tag{B.107}$$

After closing the δ' contour, (3.198) becomes

$$\mathcal{F}(z_1, z_2, z_3, p) = \sum_j \sum_{k=0}^{\infty} \int_{\frac{d-2}{2}-i\infty}^{\frac{d-2}{2}+i\infty} \frac{d\delta}{2\pi i} \left(\text{Res}_{\delta'=\delta+3+2k} C^{(0)}_{\text{strong}}(\delta, j; \delta')\right) G^c_{\delta,j;\delta'=\delta+3+2k}(z_1, z_2, z_3, p). \tag{B.108}$$

On the δ-plane, $\text{Res}_{\delta'=\delta+3+2k} C^{(0)}_{\text{strong}}$ has poles at $\delta = 6 + 2n$. The contributions from these poles will reproduce the celestial block coefficients (3.214). Note that at large δ,

$$\text{Res}_{\delta'=\delta+3+2k} C^{(0)}_{\text{strong}} \sim 2^{-\delta} (\cdots) , \tag{B.109}$$

where $(\cdots)$ grows sub-exponentially at large δ. Thus, a sufficient condition for the contribution at infinity of the δ contour to vanish is

$$\lim_{\text{Re}(\delta)\to\infty} 2^{-\delta} (\text{sub-exponential}) \, G^c_{\delta,j;\delta+3+2k}(z_1, z_2, z_3, p) = 0. \tag{B.110}$$

If the two conditions (B.107) and (B.110) are true, the celestial block expansion (3.183) can be obtained from (3.198) by contour deformation.

To show that (B.107) and (B.110) are true, we will need to understand the behavior of the celestial block G^c at large δ or δ'. For four-point conformal blocks, one can determine their large Δ behavior by studying the Casimir equation in the limit $\Delta \to \infty$ [36, 37, 35, 180, 38]. For the celestial block, the analogous Casimir equations are

$$\begin{aligned}&\left(-\frac{1}{2}L^{(12)}_{\mu\nu}L^{(12)\mu\nu}-\delta(\delta-d+2)-j(j+d-4)\right)G^c_{\delta,j;\delta'}=0,\\&\left(-\frac{1}{2}L^{(123)}_{\mu\nu}L^{(123)\mu\nu}-\delta'(\delta'-d+2)\right)G^c_{\delta,j;\delta'}=0,\end{aligned} \tag{B.111}$$

where

$$\begin{aligned}L^{(12)}_{\mu\nu}&=\sum_{i=1}^{2}z_{i\mu}\frac{\partial}{\partial z_i^\nu}-z_{i\nu}\frac{\partial}{\partial z_i^\mu},\\L^{(123)}_{\mu\nu}&=\sum_{i=1}^{3}z_{i\mu}\frac{\partial}{\partial z_i^\nu}-z_{i\nu}\frac{\partial}{\partial z_i^\mu}.\end{aligned} \tag{B.112}$$

If we take the limit given by (B.107), the second line of (B.111) will give a differential equation that the leading behavior of G^c in this limit must satisfy. Similarly, for the limit given by (B.110), (B.111) will give two differential equations. However, since G^c can depend nontrivially on three cross ratios, $\zeta_{12}, \zeta_{13}, \zeta_{23}$, these differential equations are not as simple as as the conformal block case, and it is difficult to solve them directly.

Thus, we are led to consider an alternative method for studying G^c at large δ or δ'. We find that this can be achieved by writing down an integral representation with finite integration range for the celestial block. When we consider the limits given by (B.107) and (B.110), the integral will be dominated by a saddle point and the behavior of G^c can be determined.

Warmup: conformal blocks at large Δ revisited

The method also applies to the conformal block, so let us first consider this simpler case. They key idea is to use the *Lorentzian* shadow representation of the block [2]. Here, we follow the notation of [11], where the block can be written

$$G_{\Delta,J}(x_i)\sim\int_{1>x_0>2}d^dx_0D^{d-2}z|T_{d-\Delta,2-d-J}(x_1,x_2,x_0,z)|T_{\Delta,J}(x_3,x_4,x_0,z), \tag{B.113}$$

where the causality configuration is $1>2, 3>4$, and all other points are spacelike. Here, $\sim$ means that the two sides can differ by a factor independent of the positions

x_i, and

$$T_{\Delta,J}(x_1,x_2,x_0,z) = \frac{(2z\cdot x_{20}x_{10}^2 - 2z\cdot x_{10}x_{20}^2)^J}{(-x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta+J}{2}}(x_{10}^2)^{\frac{\Delta_1+\Delta-\Delta_2+J}{2}}(x_{20}^2)^{\frac{\Delta_2+\Delta-\Delta_1+J}{2}}} \tag{B.114}$$

is a conformal three-point structure. Since we are interested in the large Δ limit, we focus on the Δ-dependence of the integrand,

$$\left(\frac{(-x_{10}^2)(-x_{20}^2)(-x_{34}^2)}{(-x_{12}^2)(-x_{30}^2)(-x_{40}^2)}\right)^{\frac{\Delta}{2}}. \tag{B.115}$$

Let us choose lightcone coordinates $x = (u, v, x_\perp)$, where $x^2 = uv + x_\perp^2$, and set $x_1 = (u_1, v_1, 0), x_2 = (0,0,0), x_3 = (1,1,0), x_4 = \infty$. The conditions $1 > 2$ and $1 \approx 3$ become $v_1 < 0, 0 < u_1 < 1$. Our integral becomes

$$(-u_1v_1)^{-\frac{\Delta}{2}} \int dudvd^{d-2}x_\perp \Big((-uv-r^2)(-(u-u_1)(v-v_1)-r^2)\Big)^{\frac{\Delta}{2}} \Big((u-1)(v-1)+r^2\Big)^{-\frac{\Delta}{2}} (\cdots)\,, \tag{B.116}$$

where $(\cdots)$ are independent of Δ. In the large Δ limit, a saddle point appears at $u = 1-\sqrt{1-u_1}, v = 1-\sqrt{1-v_1}, r = 0$. Therefore, the leading large Δ behavior is given by

$$\mathcal{G}_{\Delta,J} \sim (-u_1v_1)^{-\frac{\Delta}{2}} \Big((1-\sqrt{1-u_1})(\sqrt{1-v_1}-1)\Big)^{\Delta} (\cdots)\,. \tag{B.117}$$

Going back to the more familiar cross ratios $\rho, \overline{\rho}$ [185], we find

$$\frac{-u_1v_1}{(1-\sqrt{1-u_1})^2(\sqrt{1-v_1}-1)^2} = \frac{1}{\rho\overline{\rho}}. \tag{B.118}$$

Therefore, in the large Δ limit, $G_{\Delta,J}$ is proportional to $(\rho\overline{\rho})^{\frac{\Delta}{2}}(\cdots)$. One can fix the position-independent factor by considering the OPE limit of $G_{\Delta,J}$. Eventually, we obtain

$$G_{\Delta,J} \sim 4^{\Delta}(\rho\overline{\rho})^{\frac{\Delta}{2}}(\cdots)\,, \tag{B.119}$$

which agrees with the known result. To match the full result in [36, 37, 35, 180, 38], we could additionally include the 1-loop determinant around the saddle point. However, that such subleading terms will not be important in our analysis.

Lorentzian integrals for the celestial block

For the three-point celestial block G^c, we can write down a similar integral representation with finite integration range by continuing to "double Lorentzian" signature, as in section 3.2.4. That is, we must analytically continue the celestial sphere to a Lorentzian signature space, so that the full spacetime has signature $(2, d-2)$. We then have

$$G^c_{\delta,j;\delta'}(z_i,p) \sim \int_{\substack{1>0>2\\0>0'>3}} D^{d-2}z_0 D^{d-2}z'_0 \frac{1}{(-2p\cdot z'_0)^{\delta'}} \langle \widetilde{\mathcal{P}}_{\delta'}(z'_0)\mathcal{P}_3\mathcal{P}_{\delta,j}(z_0)\rangle\langle\widetilde{\mathcal{P}}_{\delta,j}(z_0)\mathcal{P}_1\mathcal{P}_2\rangle \tag{B.120}$$

where z_i, p are in $(2, d-2)$ signature. We can view z_i as the embedding space coordinates of $\mathbb{R}^{1,d-3}$, and the causality constraints $1 > 0 > 2$ and $0 > 0' > 3$ should be understood in this space. Since the right-hand side is a solution of the Casimir equations (B.111) by construction, one can show that it is proportional to G^c (up to a factor independent of z_i and p) by considering its various OPE limits. To study the integral more explicitly, we will pick the frame $z_i = (1, \vec{y}_i^{\,2}, \vec{y}_i)$, $p = (p_+, 1, \vec{y}_p)$, where

$$\vec{y}_1 = (1,-1,0), \quad \vec{y}_2 = (0,0,0), \quad \vec{y}_3 = (-1,1,0), \quad \vec{y}_p = (y_p^+, y_p^-, 0), \tag{B.121}$$

where the coordinate for $\vec{y}$ is $(y^+, y^-, \vec{y}_\perp)$, and $\vec{y}^{\,2} = y^+y^- + \vec{y}_\perp^{\,2}$. This satisfies the causality constraint $1 > 2 > 3$. Also, the relation between p_+, y_p^+, y_p^- and $\zeta_{12}, \zeta_{13}, \zeta_{23}$ can be obtained straightforwardly using the definition of the cross ratios.

Let us now consider the limit $\delta' \to \infty$ corresponding to (B.107). For this limit, it is more convenient to study

$$\int_{1>2>0'>3} D^{d-2}z'_0 \frac{1}{(-2p\cdot z'_0)^{\delta'}} g^{(\delta_1,\delta_2,\delta_3,\widetilde{\delta}')}_{\delta,j}(z_1,z_2,z_3,z'_0). \tag{B.122}$$

In the limit $\delta' \to \infty$, the conformal block $g^{(\delta_1,\delta_2,\delta_3,\widetilde{\delta}')}_{\delta,j}(z_1,z_2,z_3,z'_0)$ behaves like [186]

$$g^{(\delta_1,\delta_2,\delta_3,\widetilde{\delta}')}_{\delta,j}(z_1,z_2,z_3,z'_0) \sim \left(\frac{(-2z_2\cdot z_3)}{(-2z_2\cdot z'_0)(-2z_3\cdot z'_0)}\right)^{-\frac{\delta'}{2}} (\cdots), \tag{B.123}$$

where $(\cdots)$ grows sub-exponentially. Therefore, we must consider the integral

$$\int_{1>2>0'>3} D^{d-2}z'_0 \frac{1}{(-2p\cdot z'_0)^{\delta'}} \left(\frac{(-2z_2\cdot z_3)}{(-2z_2\cdot z'_0)(-2z_3\cdot z'_0)}\right)^{-\frac{\delta'}{2}}. \tag{B.124}$$

After considering the integral in the frame $z_i = (1, \vec{y}_i^2, \vec{y}_i)$, $p = (p_+, 1, \vec{y}_p)$ given by (B.121) and solving for its saddle point in the large δ' limit, we can find the behavior of the integral at large δ'. We can further determine the position-independent factor in (B.120) by matching the integral with the collinear limit of G^c given by (3.34).[3,4] We find that at large δ',

$$\begin{aligned} G^c_{\delta,j;\delta'\to\infty} \sim\, & T_{123\delta'}(z_1, z_2, z_3, p)\left(\frac{2(1-\sqrt{1-\zeta_{13}})}{\zeta_{13}}\right)^{\delta'} \text{(sub-exponential)} \\ & + T_{123\delta'}(z_1, z_2, z_3, p)\left(\frac{2(1-\sqrt{1-\zeta_{23}})}{\sqrt{\zeta_{13}\zeta_{23}}}\right)^{\delta'} \text{(sub-exponential)}, \end{aligned} \tag{B.125}$$

where $T_{123\delta'}$ is the homogeneity factor defined in (3.36). We have checked that this result indeed solves the Casimir equations (B.111) in the δ' limit. Furthermore, if z_i's are on the celestial sphere, all the cross ratios should satisfy $\zeta_{ij} \in (0, 1)$. Using (B.125), we find that for $\zeta_{ij} \in (0, 1)$, $2^{-\delta'} G^c_{\delta,j;\delta'}$ is always decaying exponentially at large δ', and thus the condition (B.107) is true.

Finally, we consider the limit corresponding to (B.110), where we set $\delta' = \delta + 3 + 2k$ and take $\delta \to \infty$. For this limit, we must consider the integral

$$\begin{aligned} \int_{\substack{1>0>2\\0>0'>3}} D^{d-2}z_0 D^{d-2}z'_0 & \frac{1}{(-2p\cdot z'_0)^{\delta}} \frac{1}{(-2z'_0\cdot z_3)^{-\delta}(-2z_0\cdot z_3)^{\delta}} \\ & \times \frac{1}{(-2z_0\cdot z_1)^{-\frac{\delta}{2}}(-2z_0\cdot z_2)^{-\frac{\delta}{2}}(-2z_1\cdot z_2)^{\frac{\delta}{2}}}. \end{aligned} \tag{B.126}$$

In the large δ limit, it turns out that the dominant contribution of the z'_0 integral comes from the top of the diamond $0 > 0' > 3$. Hence, we should set $z'_0 = z_0$ and solve for the saddle point of the z_0 integral. After comparing the saddle point result

[3] When comparing the saddle point result to the collinear limit of the celestial block, one should note that in the main text we always set $-p^2 = 1$. However, in the frame (B.121), $-p^2$ depends on the cross ratios and is not equal to 1. Therefore, we should first factor out the homogeneity factors on both sides of (B.120) and just compare the remaining functions that depend on the cross ratios, which are independent of the choice of conformal frame.

[4] From (3.34), one can show that if we first take the collinear limit and then the $\delta' \to \infty$ limit, the leading behavior of G^c at large δ' is $(\max(\zeta_{13}, \zeta_{23}))^{\frac{\delta'}{2}} \times$ (sub-exponential).

to the collinear limit of the celestial block,[5] we obtain

$$G^c_{\delta\to\infty,j;\delta'=\delta+3+2k\to\infty} \sim T_{123\delta'}(z_1,z_2,z_3,p)\left(\frac{2(1-\sqrt{1-\zeta_{12}})}{\sqrt{\zeta_{13}\zeta_{12}}}\right)^{\delta} \text{(sub-exponential)} \tag{B.127}$$

The result is indeed a solution to the Casimir equations (B.111) in the corresponding limit. Moreover, (B.127) implies that the condition (B.110) holds for $\zeta_{ij} \in (0,1)$.

[5] If we first take the collinear limit and then the limit corresponding to (B.110), the leading behavior of G^c is $\zeta_{12}^{\frac{\delta}{2}} \times$ (sub-exponential)

Appendix C

APPENDICES TO CHAPTER 4

C.1 Conventions for two-point and three-point structures

In this appendix, we summarize the conventions for the conformally-invariant structures we use in the main text.

For standard two-point and three-point structures, we use

$$\begin{aligned} \langle \mathcal{O}(x_1, z_1)\mathcal{O}(x_2, z_2)\rangle &= \frac{(2z_1 \cdot I(x_{12}) \cdot z_2)^J}{x_{12}^{2\Delta}}, \\ I^{\mu}{}_{\nu}(x) &= \delta^{\mu}{}_{\nu} - \frac{2x^{\mu}x_{\nu}}{x^2}, \end{aligned} \tag{C.1}$$

and

$$\langle \phi_1(x_1)\phi_2(x_2)\mathcal{O}(x_3, z_3)\rangle = \frac{(2z_3 \cdot x_{23}x_{13}^2 - 2z_3 \cdot x_{13}x_{23}^2)^J}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta+J}{2}}(x_{13}^2)^{\frac{\Delta_1+\Delta+J-\Delta_2}{2}}(x_{23}^2)^{\frac{\Delta_2+\Delta+J-\Delta_1}{2}}}. \tag{C.2}$$

In the embedding space, the standard structures can be written as

$$\begin{aligned} \langle \mathcal{O}(X_1, Z_1)\mathcal{O}(X_2, Z_2)\rangle &= \frac{(2H_{12})^J}{X_{12}^{\Delta+J}}, \\ \langle \phi_1(X_1)\phi_2(X_2)\mathcal{O}(X_3, X_3)\rangle &= \frac{(-2V_{3,12})^J}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta-J}{2}}X_{13}^{\frac{\Delta_1+\Delta+J-\Delta_2}{2}}X_{23}^{\frac{\Delta_2+\Delta+J-\Delta_1}{2}}}, \end{aligned} \tag{C.3}$$

where

$$\begin{aligned} X_{ij} &= -2X_i \cdot X_j, \\ V_{i,jk} &= \frac{Z_i \cdot X_j X_i \cdot X_k - Z_i \cdot X_k X_i \cdot X_j}{X_j \cdot X_k}, \\ H_{ij} &= -2(Z_i \cdot Z_j X_i \cdot X_j - Z_i \cdot X_j Z_j \cdot X_i). \end{aligned} \tag{C.4}$$

We also need the structures with more complicated representations. In the main text, the most complicated case will be when $\mathcal{O}_1, \mathcal{O}_2$ are symmetric traceless tensors with spins J_1, J_2, and $\mathcal{O}$ has representation ρ whose Young diagram has three rows of length $(J, j, \widetilde{j})$. Our convention for the two-point structure is

$$\langle \mathcal{O}(X_1, Z_1, W_1, \widetilde{W}_1)\mathcal{O}(X_2, Z_2, W_2, \widetilde{W}_2)\rangle = \frac{(2H_{12})^{J-j}(2Y_{12})^{j-\widetilde{j}}(2\widetilde{Y}_{12})^{\widetilde{j}}}{X_{12}^{\Delta+J}}. \tag{C.5}$$

For the additional two rows of the Young diagram, we introduce two null polarization vectors $W_i, \widetilde{W}_i$ with the conditions

$$\begin{aligned} X_i \cdot W_i = Z_i \cdot W_i = X_i \cdot \widetilde{W}_i = Z_i \cdot \widetilde{W}_i = W_i \cdot \widetilde{W}_i = 0, \\ W_i \sim W_i + \# X_i + \# Z_i, \\ \widetilde{W}_i \sim \widetilde{W}_i + \# X_i + \# Z_i + \# W_i. \end{aligned} \tag{C.6}$$

The structures $Y_{12}, \widetilde{Y}_{12}$ are index contractions of the antisymmetrization of the embedding space vectors $X_i, Z_i, W_i, \widetilde{W}_i$. Let us introduce the notation

$$[V_1, V_2, \dots, V_n] \cdot [W_1, W_2, \dots, W_n] \equiv \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) V_{1\alpha_1} V_{2\alpha_2} \dots V_{n\alpha_n} W_1^{\alpha_{\sigma(1)}} W_2^{\alpha_{\sigma(2)}} \dots W_n^{\alpha_{\sigma(n)}}. \tag{C.7}$$

Then, the structures $Y_{12}, \widetilde{Y}_{12}$ are defined as

$$\begin{aligned} Y_{ij} &= -2[X_i, Z_i, W_i] \cdot [X_j, Z_j, W_j], \\ \widetilde{Y}_{ij} &= -2[X_i, Z_i, W_i, \widetilde{W}_i] \cdot [X_j, Z_j, W_j, \widetilde{W}_j]. \end{aligned} \tag{C.8}$$

One can check that subject to the conditions (C.6), (C.5) is the only conformally-invariant structure with the correct homogeneity of the embedding space vectors. We have chosen the two-point convention such that (C.5) itself is Rindler positive.

For three-point structures, we have

$$\begin{aligned} &\langle O_1(X_1, Z_1) O_2(X_2, Z_2) O(X_3, Z_3, W_3, \widetilde{W}_3) \rangle^{(a)} \\ &= \frac{(-2V_{3,12})^{m_3} V_{1,23}^{m_1} V_{2,31}^{m_2} H_{12}^{n_{12}} H_{13}^{n_{13}} H_{23}^{n_{23}} (-2U_{3,12})^{k_{31}} (-2U_{3,21})^{k_{32}} (-2\widetilde{U}_{3,12})^{\widetilde{j}}}{X_{12}^{\frac{\Delta_1+J_1+\Delta_2+J_2-\Delta-J-j-\widetilde{j}}{2}} X_{13}^{\frac{\Delta_1+J_1+\Delta+J+j+\widetilde{j}-\Delta_2-J_2}{2}} X_{23}^{\frac{\Delta_2+J_2+\Delta+J+j+\widetilde{j}-\Delta_1-J_1}{2}}}, \end{aligned} \tag{C.9}$$

where the tensor structure is labeled by the nonnegative integers m_i, n_{ij}, k_{ij}, subject to the constraints

$$\begin{aligned} m_1 + n_{12} + n_{13} + k_{31} &= J_1 - \widetilde{j}, \\ m_2 + n_{12} + n_{23} + k_{32} &= J_2 - \widetilde{j}, \\ m_3 + n_{13} + n_{23} &= J - j, \\ k_{31} + k_{32} &= j - \widetilde{j}. \end{aligned} \tag{C.10}$$

The structures $U_{i,jk}, \widetilde{U}_{i,jk}$ are defined as

$$\begin{aligned} U_{i,jk} &= \frac{X_{ij}}{X_{jk}} [X_i, Z_i, W_i] \cdot [X_j, Z_j, X_k], \\ \widetilde{U}_{i,jk} &= \frac{X_{ij} X_{ik}}{X_{jk}} [X_i, Z_i, W_i, \widetilde{W}_i] \cdot [X_j, Z_j, X_k, Z_k]. \end{aligned} \tag{C.11}$$

When writing the Lorentzian shadow representation of the block (e.g., (4.43)), we need to use the three-point structures with absolute values. Also, note that the two structures in (4.43) have different causality configurations. Their explicit expressions are given by

$$\begin{aligned}
|\langle\phi_1(x_1)\phi_2(x_2)O(x_5,z_5)\rangle| &= \frac{(2z_5\cdot x_{25}x_{15}^2-2z_5\cdot x_{15}x_{25}^2)^J}{(-x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta+J}{2}}(x_{15}^2)^{\frac{\Delta+J+\Delta_1-\Delta_2}{2}}(x_{25}^2)^{\frac{\Delta+J+\Delta_2-\Delta_1}{2}}}, \quad 1>2,5\approx 1,2 \\
|\langle\phi_3(x_3)\phi_4(x_4)O(x_5,z_5)\rangle| &= \frac{(2z_5\cdot x_{45}x_{35}^2-2z_5\cdot x_{35}x_{45}^2)^J}{(-x_{43}^2)^{\frac{\Delta_4+\Delta_3-\Delta+J}{2}}(-x_{45}^2)^{\frac{\Delta+J+\Delta_4-\Delta_3}{2}}(-x_{35}^2)^{\frac{\Delta+J+\Delta_3-\Delta_4}{2}}}, \quad 4>5>3.
\end{aligned} \tag{C.12}$$

From the expression of $|\langle\phi_1(x_1)\phi_2(x_2)O(x_5,z_5)\rangle|$, one can explicitly verify the identity (4.47).

In the definition of the functional (4.93), we introduce a celestial three-point structure. This structure is a standard three-point structure in a Euclidean $(d-2)$-dimensional CFT, where z_i are viewed as the embedding space coordinates. Therefore, we can get this structure by taking the d-dimensional three-point structure in (C.3) and make the replacement $X_i \to z_i, Z_i \to w_i$. This gives

$$\langle\mathcal{P}_{\delta_1}(z_1)\mathcal{P}_{\delta_2}(z_2)\mathcal{P}_{\delta,j}(z,w)\rangle = \frac{(4w\cdot z_1 z\cdot z_2-4w\cdot z_2 z\cdot z_1)^j}{(-2z_1\cdot z)^{\frac{\delta_1+\delta+j-\delta_2}{2}}(-2z_2\cdot z)^{\frac{\delta_2+\delta+j-\delta_1}{2}}(-2z_1\cdot z_2)^{\frac{\delta_1+\delta_2-\delta+j}{2}}}. \tag{C.13}$$

Finally, in the functional we also use a dual structure $\langle 0|O_4\mathbf{L}[O]O_{2^+}|0\rangle^{-1}$, which should be a continuous-spin structure. In particular, we are interested in the structure with the causality constraint $4>0>2$. We define the standard continuous-spin structure with this configuration as

$$\begin{aligned}
&\langle 0|\phi_4(x_4)O(x_0,z_0)\phi_2(x_2^+)|0\rangle \\
&= \frac{(2z_0\cdot x_{40}x_{20}^2-2z_0\cdot x_{20}x_{40}^2)^J}{(-x_{24}^2)^{\frac{\Delta_2+\Delta_4-\Delta+J}{2}}(-x_{02}^2)^{\frac{\Delta+J+\Delta_2-\Delta_4}{2}}(-x_{04}^2)^{\frac{\Delta+J+\Delta_4-\Delta_2}{2}}}, \quad 4>0>2.
\end{aligned} \tag{C.14}$$

In the embedding space, it is given by

$$\begin{aligned}
&\langle 0|\phi_4(X_4)O(X_0,Z_0)\phi_2(X_2^+)|0\rangle \\
&= \frac{(-2V_{0,42}(-X_{24}))^J}{(-X_{24})^{\frac{\Delta_2+\Delta_4-\Delta+J}{2}}(-X_{02})^{\frac{\Delta+J+\Delta_2-\Delta_4}{2}}(-X_{04})^{\frac{\Delta+J+\Delta_4-\Delta_2}{2}}}, \quad 4>0>2.
\end{aligned} \tag{C.15}$$

Note that the combination $-2V_{0,42}(-X_{24})$ is always positive due to the causality constraint $4 > 0 > 2$. The actual dual structure will be the above structure with the replacement $\Delta \to J + d - 1, J \to \Delta - d + 1, \Delta_i \to \widetilde{\Delta}_i$, multiplied by a computable prefactor. In the scalar case, it is given by (4.37). We discuss the spinning case in more detail in appendix C.4.

C.2 Heavy action formula from the large ν limit

In the main text of this paper we have taken the large ν limit of the kernel and large Δ limit of the block at the same time. However, in this appendix we compute the large ν limit independently. Taking the large ν limit of the kernel alone will slightly differ from taking Δ and ν limits simultaneously. Recall that when we perform the gauge fixing, we also fix a fifth point x_5 coming from the conformal block we want to act the functional on. If we just study the action of the functional on a general four-point function, we can again use conformal symmetry to fix $x_1 + x_3$ to the origin and $x_{2^+} + x_{4^-}$ to spatial infinity, and since we do not have a fifth point, the stabilizer group is generated by Lorentz transformations and dilatations. We can then use Lorentz transformation to fix all four points to be on the same plane. After performing the large ν analysis (which will be done below), one can then obtain a kernel that only depends on the cross ratios (see (C.34)). Therefore, we can undo the gauge fixing and go back the the bulk-point gauge fixing again, and then the rest of the calculation is the same as [13].

We now want to take the large ν limit of

$$\Psi^{+}_{k,\nu}[\mathcal{G}] = 4A_{k,\nu} \int_{4>3>0>1>2} \frac{d^d x_1 d^d x_2 d^d x_3 d^d x_4 d^d x_0 D^{d-2} z_0}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))} \langle 0|\phi_4(x_4)\mathbf{L}[O](x_0,z_0)\phi_2(x_2^+)|0\rangle^{-1}$$
$$\langle \widetilde{\mathcal{P}}_{\delta_1}(x_{1^+0})\widetilde{\mathcal{P}}_{\delta_3}(x_{30})\mathcal{P}_{\delta}(z_0)\rangle \delta(x_{1^+0}^2)\delta(x_{30}^2)\theta(x_{1^+0})\theta(x_{30})\langle \Omega|[\phi_4(x_4),\phi_3(x_3)][\phi_1(x_1^+),\phi_2(x_2^+)]|\Omega\rangle. \tag{C.16}$$

The only ν dependence is in dual and celestial structures, and so we need only focus on

$$\int d^d x_0 D^{d-2} z \langle 0|\phi_4(x_4)\mathbf{L}[O](x_0,z_0)\phi_2(x_2^+)|0\rangle^{-1}\delta(x_{1^+0}^2)\delta(x_{30}^2)$$
$$\times \langle \mathcal{P}_{\delta}(z)\mathcal{P}_{\widetilde{\delta}_1}(x_{1^+0})\mathcal{P}_{\widetilde{\delta}_3}(x_{30})\rangle (-x_{13}^2)^{\widetilde{\Delta}_\phi}(-x_{24}^2)^{\widetilde{\Delta}_\phi}, \tag{C.17}$$

which we can rewrite using the explicit expression for the dual structure (4.37) which gives us

$$\int d^dx_0 D^{d-2}z \times \langle \mathcal{P}_\delta(z)\mathcal{P}_{\widetilde{\delta}_1}(x_{1^+0})\mathcal{P}_{\widetilde{\delta}_3}(x_{30})\rangle(-x_{13}^2)^{\widetilde{\Delta}_\phi}(-x_{24}^2)^{\widetilde{\Delta}_\phi}\delta(x_{1^+0}^2)\delta(x_{30}^2) \frac{(2z_0\cdot x_{40}x_{20}^2 - 2z_0\cdot x_{20}x_{40}^2)^{\Delta-d+1}}{(-x_{24}^2)^{\frac{\widetilde{\Delta}_2+\widetilde{\Delta}_4+\Delta-J-2d+2}{2}}(x_{02}^2)^{\frac{\widetilde{\Delta}_2+\Delta+J-\widetilde{\Delta}_4}{2}}(x_{02}^2)^{\frac{\widetilde{\Delta}_4+\Delta+J-\widetilde{\Delta}_2}{2}}}, \tag{C.18}$$

where we have dropped the overall coefficient in (4.37). It will be useful to notice that the dual structure has the form $\langle\widetilde{\phi}(x_4)O^F(x_0,z_0)\widetilde{\phi}(x_2)\rangle$, which is a three-point function of two scalars with shadow dimensions $\widetilde{\Delta}_\phi = d-\Delta_\phi$, and a third operator with $(\Delta_F, J_F) = (J+d-1, \Delta-d+1)$, where J and Δ are the dimension and spin of the light transformed operator. In particular we will focus on the limit $J_F = 2-d+\delta = \frac{2-d}{2}+i\nu$, $\nu\to\infty$.

From the analysis in the cross-ratio space [13], we know that in the bulk-point limit x and y will get fixed to saddle points of the form

$$x = (u_x, v_x, \vec{0}), \qquad y = (u_y, v_y, \vec{0}), \tag{C.19}$$

and therefore we can just study the large ν limit for this particular x and y using the gauge fixing procedure described above.

Let us choose the coordinates of x_0 and z to be $x_0 = (u, v, \vec{y}_0)$, $z = (1, \vec{y}_z^2, \vec{y}_z)$. In the integrand, the J_F dependent factors are (we replace δ with J_F+d-2 as well as Δ with J_F+d-1)

$$\begin{aligned}
&\langle 0|\widetilde{\phi}_4(x_4)O^F(x_0,z)\widetilde{\phi}_2(x_2^+)|0\rangle\langle\mathcal{P}_\delta(z)\mathcal{P}_{\widetilde{\delta}_1}(x_{1^+0})\mathcal{P}_{\widetilde{\delta}_3}(x_{30})\rangle \\
=&\frac{(-2V_{0,24})^{J_F}}{(x_{2^+4}^2)^{\frac{2\widetilde{\Delta}_\phi-d+2-J_F}{2}}(-x_{40}^2)^{\frac{d-2+J_F}{2}}(x_{2^+0}^2)^{\frac{d-2+J_F}{2}}}\times \\
&\frac{1}{(-2z\cdot x_{1^+0})^{\frac{d-2+J_F}{2}}(-2z\cdot x_{30})^{\frac{d-2+J_F}{2}}(-2x_{1^+0}\cdot x_{30})^{\frac{2\widetilde{\delta}_\phi-d+2-J_F}{2}}}.
\end{aligned} \tag{C.20}$$

Note that we have the delta functions $\delta(x_{1^+0}^2)\delta(x_{30}^2)$, and therefore $-2x_{1^+0}\cdot x_{30} = (x_{1^+0}-x_{30})^2 = x_{1^+3}^2$.

The two delta functions can be written as

$$\delta(x_{1^+0}^2)\delta(x_{30}^2) = \delta((u+u_y)(v+v_y)-\vec{y}_0^2)\delta(-(u-u_y)(v-v_y)+\vec{y}_0^2). \tag{C.21}$$

In terms of $\widetilde{t}_0, \widetilde{x}_0$ defined in (4.59) (in this case we have $\vec{y}_\perp = 0$), the delta functions fix $\widetilde{t}_0 = 0$ and $r_0 \equiv |\vec{y}_0| = (u_y v_y(1-\widetilde{x}_0^2))^{\frac{1}{2}}$. The Jacobian relating $\widetilde{t}_0, \widetilde{x}_0$ and u, v is given by $dudv = 2u_y v_y d\widetilde{t}_0 d\widetilde{x}_0$. This gives (for any function $f(u,v,\vec{y}_0)$)

$$\frac{1}{2}\int dudvd^{d-2}\vec{y}_0\ \delta(-(u-u_y)(v-v_y)+\vec{y}_0^2)\delta((u+u_y)(v+v_y)-\vec{y}_0^2)f(u,v,\vec{y}_0)$$
$$=\int d\widetilde{x}_0 d\Omega_{\vec{n}_0}\frac{(u_y v_y(1-\widetilde{x}_0^2))^{\frac{d-4}{2}}}{8}f(\widetilde{t}_0=0,\widetilde{x}_0,r_0=(u_y v_y(1-\widetilde{x}_0^2))^{\frac{1}{2}},\Omega_{\vec{n}_0}), \quad \text{(C.22)}$$

where $\vec{n}_0$ is the unit vector in the $\vec{y}_0$ direction.

After removing the delta functions, (C.17) becomes

$$\int d\widetilde{x}_0 d\Omega_{\vec{n}_0} d^{d-2}\vec{y}_z \frac{(u_y v_y(1-\widetilde{x}_0^2))^{\frac{d-4}{2}}}{8}\frac{2^{-J_F}}{(-x^2_{2^+4^-})^{\frac{2\widetilde{\Delta}_\phi-d+2-J_F}{2}}(x^2_{4^-0})^{\frac{d-2}{2}}(x^2_{2^+0})^{\frac{d-2}{2}}}\times$$
$$\frac{1}{(-2z\cdot x_{1^+0})^{\frac{d-2}{2}}(-2z\cdot x_{30})^{\frac{d-2}{2}}(-2x_{1^+0}\cdot x_{30})^{\frac{2\widetilde{\delta}_\phi-d+2-J_F}{2}}}e^{J_F\times h(\widetilde{x}_0,\vec{y}_z,\Omega_{\vec{n}_0})}(4u_y v_y)^{(d-\Delta_\phi)}(\tfrac{4}{u_x v_x})^{(d-\Delta_\phi)},$$
$$\text{(C.23)}$$

where the function $h(\widetilde{x}_0,\vec{y}_z,\Omega_{\vec{n}_0})$ is given by

$$h(\widetilde{x}_0,\vec{y}_z,\Omega_{\vec{n}_0}) = \log\left(\frac{1}{u_x}+\frac{\vec{y}_z^2}{v_x}-u_y v_y(1-\widetilde{x}_0^2)(v_x+u_x\vec{y}_z^2)-u_x v_y^2\widetilde{x}_0^2-u_y^2 v_x\widetilde{x}_0^2\vec{y}_z^2+2(u_y v_x-v_y u_x)\widetilde{x}_0\vec{y}_0\cdot\vec{y}_z\right)$$
$$-\frac{1}{2}\log\left((v_y(1-\widetilde{x}_0)-2\vec{y}_0\cdot\vec{y}_z+u_y(1+\widetilde{x}_0)\vec{y}_z^2)(v_y(1+\widetilde{x}_0)+2\vec{y}_0\cdot\vec{y}_z u_y(1-\widetilde{x}_0)\vec{y}_z^2)\right)$$
$$-\frac{1}{2}\log\left(\frac{(1-u_y v_y u_x v_x-(u_y v_x-v_y u_x)\widetilde{x}_0)(1-u_y v_y u_x v_x+(u_y v_x-u_x v_y)(1-\widetilde{x}_0))}{u_x^2 v_x^2}\right).$$
$$\text{(C.24)}$$

We find that at large J_F, the integral has two saddle loci: one at $\widetilde{x}_0=0, \vec{y}_z=0$ and one at $\widetilde{x}_0=0, \vec{y}_z=\infty$. Namely,

$$\frac{\partial}{\partial\widetilde{x}_0}h(\widetilde{x}_0,\vec{y}_z,\Omega_{\vec{n}_0})\Big|_{\widetilde{x}_0\to 0,\vec{y}_z\to 0} = \frac{\partial}{\partial\vec{y}_z}h(\widetilde{x}_0,\vec{y}_z,\Omega_{\vec{n}_0})\Big|_{\widetilde{x}_0\to 0,\vec{y}_z\to 0}=0$$
$$\frac{\partial}{\partial\widetilde{x}_0}h(\widetilde{x}_0,\vec{y}_z,\Omega_{\vec{n}_0})\Big|_{\widetilde{x}_0\to 0,\vec{y}_z\to \infty} = \frac{\partial}{\partial\vec{y}_z}h(\widetilde{x}_0,\vec{y}_z,\Omega_{\vec{n}_0})\Big|_{\widetilde{x}_0\to 0,\vec{y}_z\to \infty}=0. \quad \text{(C.25)}$$

Let us first consider the locus at $\widetilde{x}_0=\vec{y}_z=0$. We will see what the other saddle should give shortly. Expanding $h(\widetilde{x}_0,\vec{y}_z,\Omega_{\vec{n}_0})$ around $\widetilde{x}_0=\vec{y}_z=0$, we get

$$h(\widetilde{x}_0,\vec{y}_z,\Omega_{\vec{n}_0}) \approx -\log\left(\frac{v_y}{v_x}\right)+\frac{(1+u_y^2v_x^2-2u_x u_y v_x v_y)(1-u_x^2v_y^2)}{2(1-u_y v_y u_x v_x)^2}\widetilde{x}_0^2$$
$$+\frac{2(1-u_x^2v_y^2)}{v_y(1-u_y v_y u_x v_x)}\widetilde{x}_0\vec{y}_0\cdot\vec{y}_z+\frac{2v_x(\vec{y}_0\cdot\vec{y}_z)^2-v_y(u_y v_x-v_y u_x)\vec{y}_z^2}{v_x v_y^2}+\dots$$
$$=-\log\left(\frac{v_y}{v_x}\right)+\frac{1}{2}\begin{pmatrix}\widetilde{x}_0 & \vec{y}_z\end{pmatrix}M_{\vec{y}_0}\begin{pmatrix}\widetilde{x}_0\\ \vec{y}_z\end{pmatrix}. \quad \text{(C.26)}$$

Alternatively, one can also choose to not integrate $\widetilde{t}_0, |\vec{y}_0|$ and keep the delta functions. Then the function h will also depend on $\widetilde{t}_0$ and $|\vec{y}_0|$, and one can show that there are saddle loci at $\widetilde{t}_0 = \widetilde{x}_0 = 0, \vec{y}_z = 0, \infty$. We can then expand h around the saddle locus and use the delta functions to fix $\widetilde{t}_0$ and $|\vec{y}_0|$. This calculation should give the same result.

Now let us compute the saddle integral in (C.23). By rotational invariance, the determinant of the Hessian $M_{\vec{y}_0}$ in (C.26) does not depend on the direction of $\vec{y}_0$, so we can pick a direction that makes computing the determinant simple. For example, $\vec{y}_0 = |\vec{y}_0|(1, 0, \ldots, 0)$. We then find

$$\mathrm{Det}M_{\vec{y}_0} = \frac{2^{d-2}(1-u_y^2v_x^2)(1-v_y^2u_x^2)(v_yu_x-u_yv_x)^{d-2}}{(v_xv_y)^{d-2}(1-u_yv_yu_xv_x)^2}. \tag{C.27}$$

One also has to check the sign of each eigenvalue of $\mathrm{Det}M_{\vec{y}_0}$ in order to get the correct phase in the saddle integral. We find that for $v_yu_x - u_yv_x > 0$, all eigenvalues are positive, and for $v_yu_x - u_yv_x < 0$, one eigenvalue is positive and the other $d-2$ eigenvalues are negative. Both cases turn out to give the same result, so let us assume $v_yu_x - u_yv_x > 0$ here. The saddle integral in (C.23) is then given by

$$\begin{aligned}&\int d\widetilde{x}_0 d^{d-2}\vec{y}_z\,(\cdots)\,e^{J_F\times h(\widetilde{x}_0,\vec{y}_z,\Omega_{\vec{n}_0})}\\ &\approx (\cdots)\,|_{\widetilde{x}_0\to 0,\vec{y}_z\to 0} \times \left(\frac{v_x}{v_y}\right)^{J_F} e^{\frac{i\pi}{2}(d-1)} J_F^{-\frac{d-1}{2}} \sqrt{\frac{(2\pi)^{d-1}}{\mathrm{Det}M_{\vec{y}_0}}} + \ldots,\end{aligned} \tag{C.28}$$

where $(\cdots)$ are the other factors in the integrand of (C.23), and $\ldots$ are subleading terms at large J_F. Evaluating the remaining factors on the saddle locus $\widetilde{x}_0 = 0, \vec{y}_z = 0$, we obtain

$$\begin{aligned}&(4u_yv_y)^{(d-\Delta_\phi)}\left(\tfrac{4}{u_xv_x}\right)^{(d-\Delta_\phi)} \int d\Omega_{\vec{n}_0} \frac{(u_yv_y)^{\frac{d-4}{2}}}{8} \left(\frac{v_x}{v_y}\right)^{J_F} e^{\frac{i\pi}{2}(d-1)} J_F^{-\frac{d-1}{2}} \sqrt{\frac{(2\pi)^{d-1}}{\mathrm{Det}M_{\vec{y}_0}}}\times\\ &\left.\frac{2^{-J_F}}{(-x_{2^+4^-}^2)^{\frac{2\widetilde{\Delta}_\phi-d+2-J_F}{2}}(x_{4^-0}^2)^{\frac{d-2}{2}}(x_{2^+0}^2)^{\frac{d-2}{2}}} \frac{1}{(-2z\cdot x_{1^+0})^{\frac{d-2}{2}}(-2z\cdot x_{30})^{\frac{d-2}{2}}(-2x_{1^+0}\cdot x_{30})^{\frac{2\widetilde{\delta}_\phi-d+2-J_F}{2}}}\right|_{\widetilde{x}_0\to 0,\vec{y}_z\to 0}\\ &= 2^{-\frac{9}{2}+2d+J_F}\pi^{\frac{d-1}{2}}\,\mathrm{vol}(S^{d-3})e^{\frac{i\pi}{2}(d-1)}J_F^{-\frac{d-1}{2}}\times\\ &\quad\frac{(v_yu_x)^{\frac{d-2-J_F}{2}}(u_yv_x)^{\frac{J_F}{2}+d-2}(v_yu_x-u_yv_x)^{\frac{2-d}{2}}(1-u_yv_yu_xv_x)^{3-d}}{\sqrt{(1-u_y^2v_x^2)(1-v_y^2u_x^2)}}.\end{aligned} \tag{C.29}$$

We can write the result in terms of the conformally-invariant cross-ratios r and η defined as

$$u' = \frac{(1-2r\eta+r^2)^2}{16r^2}, \quad v' = \frac{(1+2r\eta+r^2)^2}{16r^2}, \tag{C.30}$$

where u', v' are defined in (4.31). In terms of r, η, the u-channel Regge limit corresponds to $r \to 0$ with fixed η. In our gauge fixing, they are given by

$$r = |x||y| = (u_y v_y u_x v_x)^{\frac{1}{2}}, \qquad \eta = -\frac{x \cdot y}{|x||y|} = \frac{v_y u_x + u_y v_x}{2(u_y v_y u_x v_x)^{\frac{1}{2}}}. \tag{C.31}$$

Then, in terms of r, η, (C.29) becomes

$$2^{-\frac{9}{2}+2d+J_F}\pi^{\frac{d-1}{2}}\,\mathrm{vol}(S^{d-3})e^{\frac{i\pi}{4}(d-1)}\nu^{-\frac{d-1}{2}}\frac{(1-r^2)^{3-d}r^{d-2}w^{-i\nu}(w-\frac{1}{w})^{\frac{2-d}{2}}}{((1+r^2)^2-4r^2\eta^2)^{\frac{1}{2}}}, \tag{C.32}$$

where we have used $J_F = \frac{2-d}{2} + i\nu$ and $\eta = \frac{w+1/w}{2}$. One can then notice that the leading large ν behavior is similar to

$$\mathcal{P}_{\frac{2-d}{2}+i\nu}\left(\frac{w+1/w}{2}\right) \sim \frac{\Gamma(d-2)}{\Gamma(\frac{d-2}{2})}\frac{\nu^{\frac{2-d}{2}}}{(w-1/w)^{\frac{d-2}{2}}}\left(e^{-\frac{i\pi}{4}(d-2)}w^{i\nu} + e^{\frac{i\pi}{4}(d-2)}w^{-i\nu}\right), \quad \nu \to \infty, |w| > 1. \tag{C.33}$$

In particular, the second term agrees with (C.32), and the first term comes from the other saddle locus. Therefore, after including both saddle loci, the integral (C.23) becomes

$$2^{-\frac{9}{2}+2d+J_F}\pi^{\frac{d-1}{2}}e^{\frac{i\pi}{4}}\,\mathrm{vol}(S^{d-3})\nu^{-\frac{1}{2}}\frac{\Gamma(\frac{d-2}{2})}{\Gamma(d-2)}\frac{(1-r^2)^{3-d}r^{d-2}}{((1+r^2)^2-4r^2\eta^2)^{\frac{1}{2}}}\mathcal{P}_{\frac{2-d}{2}+i\nu}(\eta). \tag{C.34}$$

The expression we start with, (C.17), is essentially the shadow representation of the conformal block $G^{\widetilde{\Delta}_\phi}_{\frac{d}{2}+i\nu,J=-1}$, where one of the three-point structures becomes a celestial structure and delta functions due to setting $J=-1$ [77]. What we have shown here is that its large ν limit is given by (C.34). Although the same result can be obtained in the cross-ratio space using the Casimir equation [11], we believe that the above calculation from saddle point can be more straightforwardly generalized to spinning operators. Since $G^{\widetilde{\Delta}_\phi}_{\frac{d}{2}+i\nu,J=-1}$ is exactly the kernel appearing in the Lorentzian inversion formula for $C(\Delta = \frac{d}{2}+i\nu, J=-1)$, the calculation here could also be helpful for understanding the OPE data at large Δ [187].

Finally, to reproduce the heavy action formula, we can take the conformally-invariant expression (C.34) and plug it back in (C.16). Similar to the main text, we can then study its action on conformal block and choose the gauge fixing (4.49), which introduces the Faddeev-Popov factor (4.50). This gives (4.51) with the kernel part replaced with (C.34). Combining with the subtraction factors (4.87), we find that we get an integral of the form

$$\int d^dxd^dy\frac{r^{k+d-1}(1-r^4)\eta}{((1+r^2)^2-4r^2\eta^2)^{\frac{k+1}{2}}}\times \\ \frac{1}{(-x^2)^{2d}}\frac{1}{(-x_{12}^2)^{\widetilde{\Delta}_\phi}(-x_{34}^2)^{\widetilde{\Delta}_\phi}}\ |\langle\phi_{1^+}\phi_{2^+}O^{\mu_1\cdots\mu_J}(x_5)\rangle||\langle\widetilde{O}^\dagger_{\mu_1\cdots\mu_J}(x_5)\phi_3\phi_4\rangle|\Big|_{\text{gauge-fixed}}. \tag{C.35}$$

The only difference between this integral and the one considered in [13] (see (3.25) and (3.37)) is a factor of η, which becomes 1 in the bulk-point limit. One can further check that the overall factors also agree. Hence, the calculation in [13] implies that we recover the same heavy action formula (4.92).

C.3 Details on matching partial waves

In this appendix, we give some more details on the partial wave discussion in section 4.4.2.

C.3.1 Partial waves in monomial basis

In the main text, we define the vertices $v(n, e_1, e_2)$ directly from evaluating the CFT three-point structures at the saddle and applying the polarization map. From the flat space perspective, it is more convenient to define the vertices as monomials, which can be easily expressed using Young tableaux, as given in [110]. Furthermore, the bootstrap calculation done in [110] also uses the monomial basis (up to a Gram-Schmidt procedure to convert it into a orthonormal basis). The dictionary found in the main text tells us that we can take the flat space functional found in [110] and construct a positive CFT functional with positive action on heavy blocks. If we want to further study the OPE coefficients in the bootstrap calculation, we will need to know the relation between the two different bases.

One can find the relation between the two bases by explicitly evaluating the CFT three-point structures at the saddle for all the exchanged representations ρ in the photon and graviton case (see [110] for a complete list). It turns out that there is a clear map between the building blocks of CFT three-point tensor structures and the building blocks of the monomial basis, given by columns of the Young tableaux. We find that the map between the structures of $\langle O_1O_2O_5\rangle$ and $v(n, e_1, e_2)$ is given by

$$\begin{array}{|c|}\hline n\\\hline\end{array} \leftrightarrow [X_5, Z_5]\cdot[X_1, X_2], \qquad e_1\cdot e_2 \leftrightarrow [X_1, Z_1]\cdot[X_2, Z_2],$$
$$\begin{array}{|c|}\hline e_1\\\hline n\\\hline\end{array} \leftrightarrow [X_5, Z_5, W_5]\cdot[X_1, Z_1, X_2], \qquad \begin{array}{|c|}\hline e_2\\\hline n\\\hline\end{array} \leftrightarrow [X_5, Z_5, W_5]\cdot[X_2, Z_2, X_1],$$
$$\begin{array}{|c|}\hline e_1\\\hline e_2\\\hline n\\\hline\end{array} \leftrightarrow [X_5, Z_5, W_5, \widetilde{W}_5]\cdot[X_1, Z_1, X_2, Z_2], \tag{C.36}$$

where we use the notation (C.7) for the CFT structures. The Young tableaux columns represent the antisymmetrization of the vectors in the boxes (see [110] for the precise definition). For example,

$$\begin{array}{|c|}\hline e_1\\\hline n\\\hline\end{array} = e_1\cdot w_1 n\cdot w_2 - n\cdot w_1 e_1\cdot w_2, \tag{C.37}$$

where w_1, w_2 are the polarizations of the vertex. Both sides of the map agree up to an overall factor after one evaluates the CFT structure at the scattering-crystal configuration (4.81) and applies the polarization map (4.133).

The map given in (C.36) allows us to unambiguously relate the structure labels of the two bases, and we find that the partial waves computed in the two bases are related by

$$\pi^{\text{CFT}}_{\rho,(a'b')} = 2^{2J-2j-n^{a'}_{15}-n^{a'}_{25}-n^{b'}_{36}-n^{b'}_{46}}(-1)^{n^{a'}_{12}+n^{a'}_{25}+n^{b'}_{34}+n^{b'}_{46}}\pi^{\text{tableaux}}_{\rho,(a'b')}, \tag{C.38}$$

where $n^{a'}_{15}$ counts the number of H_{15} of the (a') tensor structure, and other $n^{a'}_{ij}, n^{b'}_{ij}$ are defined similarly (see (C.9)). Note that the above relation is true for each a', b'. Alternatively, one can think of the factor relating the two partial waves as two diagonal matrices that rescale the vertices.

We also see that this change of basis preserves the positivity of the partial wave actions, so we can reuse the positive functional in the monomial basis. In the forward limit, where $n = n', e_3 = e_1^*, e_4 = e_2^*$, the partial wave in the monomial basis is positive (or positive semi-definite for multiple tensor structures) by construction. In the CFT basis, (C.38) implies that the partial wave is also positive. (Note that if we instead define $\overline{v}(n', e_3, e_4)$ in (4.139), then the partial wave wouldn't be positive in the forward limit.)

C.3.2 Polarization map and CRT symmetry

We now explain why the two definitions of vertices (4.134), (4.139) are consistent with each other. In particular, we will show that by using the definition of $v(n, e_1, e_2)$ and the CRT symmetry, we can recover the definition of $\overline{v}(-n', e_3, e_4)$.

For any CFT, there is an anti-unitary CRT symmetry J satisfying $J^2 = 1$, and it acts on local operators as (assuming the operator is bosonic)

$$J\mathcal{O}(x,z)J^{-1} = \mathcal{O}^\dagger(\overline{x}, \overline{z}^*), \tag{C.39}$$

where $\overline{x} = (-x^0, -x^1, x^2, \ldots)$ (and similarly for $\overline{z}^*$) is a Rindler reflection. Since J is anti-unitary, any CFT correlation function should satisfy

$$\langle \mathcal{O}_1(x_1, z_1)\cdots\mathcal{O}_n(x_n, z_n)\rangle = \langle \mathcal{O}_1^\dagger(\overline{x}_1, \overline{z}_1^*)\cdots\mathcal{O}_n^\dagger(\overline{x}_n, \overline{z}_n^*)\rangle^*, \tag{C.40}$$

where we have applied a CRT to each operator.

More generally, as explained in [34], for any spacelike points A and B, one can define two Rindler wedges $B > x > A^-$ and $A > x > B^-$. Then, there exists a Rindler conjugation J_{AB} that exchanges the two wedges. Explicitly,

$$J_{AB}(X) = X - 2\frac{X\cdot X_A}{X_A\cdot X_B}X_B - 2\frac{X\cdot X_B}{X_A\cdot X_B}X_A. \tag{C.41}$$

It turns out that if we choose A to be the future infinity and B^- to be the origin, under the corresponding Rindler conjugation J_{AB} we have $3 \to 1^+, 4 \to 2^+, 6^+ \to 5$ in the scattering crystal configuration (4.81). Therefore, using this J_{AB}, one might expect

$$\langle 0|\mathcal{O}^\dagger(x_{6^+}, z_6)\mathcal{O}_4(x_4, z_4)\mathcal{O}_3(x_3, z_3)|0\rangle\Big|_{\text{saddle}} \stackrel{?}{=} \langle 0|\mathcal{O}(x_5, \overline{z}_6^*)\mathcal{O}_4^\dagger(x_{2^+}, \overline{z}_4^*)\mathcal{O}_3^\dagger(x_{1^+}, \overline{z}_3^*)|0\rangle^*\Big|_{\text{saddle}}. \tag{C.42}$$

This is however too fast. A funny feature of our saddle configuration is that although the points $x_{1,2,3,4}$ are timelike, they all have purely imaginary spatial components. The correct way to think about this should be we first consider e.g. $x_3 = (u_y, v_y, 0)$ for real u_y, v_y, and at the end analytically continue to $u_y \to -i\frac{m-\sqrt{m^2-\nu^2}}{\nu}, v_y \to i\frac{m-\sqrt{m^2-\nu^2}}{\nu}$. To take into account the fact that there are imaginary spatial components, we should apply an additional reflection to them, which can then be shifted to act on the polarizations. In summary, the correct CRT relation becomes

$$\begin{aligned}&\langle 0|\mathcal{O}^\dagger(x_{6^+}, z_6)\mathcal{O}_4(x_4, z_4)\mathcal{O}_3(x_3, z_3)|0\rangle\Big|_{\text{saddle}}\\ &= \langle 0|\mathcal{O}(x_5, I_e\cdot z_6^*)\mathcal{O}_4^\dagger(x_{2^+}, I_e\cdot z_4^*)\mathcal{O}_3^\dagger(x_{1^+}, I_e\cdot z_3^*)|0\rangle^*\Big|_{\text{saddle}},\end{aligned} \tag{C.43}$$

where the polarizations are reflected only in the time direction. Since (C.43) is simply a statement about the CFT structures at the saddle configuration, we can explicitly verify that it is true for different representations ρ.

Using the CRT relation (C.43), we can write

$$(-x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}} \left(\frac{x_{36}^2}{x_{46}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}} \mathcal{I}_e^\rho \left. \frac{\langle 0|O^\dagger(x_{6^+}, z_6)[O_4(x_4, z_4), O_3(x_3, z_3)]|0\rangle_{(b)}}{2i(\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_3-J_3-\Delta_4-J_4}{2}))}\right|_{\text{saddle}}$$
$$= (-x_{12}^2)^{\frac{\Delta_3+\Delta_4}{2}} \left(\frac{x_{15}^2}{x_{25}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}} \left. \frac{\langle 0|[O_3(x_{1^+}, z_3^*), O_4(x_{2^+}, z_4^*)]O(x_5, z_6^*)|0\rangle_{(b)}^*}{-2i(\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_3-J_3-\Delta_4-J_4}{2}))}\right|_{\text{saddle}}. \tag{C.44}$$

We assume there is only one polarization z_6 for simplicity, but the argument for more polarizations $\vec{w}_6$ is the same. We have also imposed conservation to set the time components of z_3, z_4 to zero. So, $I_e \cdot z_3 = z_3, I_e \cdot z_4 = z_4$, and the reflection tensor $\mathcal{I}_e^\rho$ cancels with the I_e acting on z_6. The additional minus sign in the denominator comes from changing the operator order in the commutator.

Since the positions are now at x_{1^+}, x_{2^+}, x_5, we can use the definition (4.134) for $v(n, e_1, e_2)$ (although now it is a vertex for external particles with spin J_3, J_4). We can then write (C.44) as

$$(-x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}} \left(\frac{x_{36}^2}{x_{46}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}} \mathcal{I}_e^\rho \left. \frac{\langle 0|O^\dagger(x_{6^+}, z_6)[O_4(x_4, z_4), O_3(x_3, z_3)]|0\rangle_{(b)}}{2i(\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_3-J_3-\Delta_4-J_4}{2}))}\right|_{\text{saddle}}$$
$$= v(n, z_3^{x*} n_\perp + \vec{z}_{3\perp}^*, -z_4^{x*} n_\perp + \vec{z}_{4\perp}^*, w_1 = (0, iz_6^{t*}, z_6^{x*}, \vec{z}_{6\perp}^*))^*. \tag{C.45}$$

When contracting the indices of the vertex $\overline{v}$, we should contract the polarizations after taking the complex conjugate. So, using $(v \cdot z)^* = v^* \cdot z^*$, we can write the above equation as a Schwarz reflection,

$$(-x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}} \left(\frac{x_{36}^2}{x_{46}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}} \mathcal{I}_e^\rho \left. \frac{\langle 0|O^\dagger(x_{6^+}, z_6)[O_4(x_4, z_4), O_3(x_3, z_3)]|0\rangle_{(b)}}{2i(\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_3-J_3-\Delta_4-J_4}{2}))}\right|_{\text{saddle}}$$
$$= \overline{v}(n, z_3^x n_\perp + \vec{z}_{3\perp}, -z_4^x n_\perp + \vec{z}_{4\perp}, w_1 = (0, -iz_6^t, z_6^x, \vec{z}_{6\perp})). \tag{C.46}$$

Rotational invariance demands that the vertex should just contain dot products of the polarizations and n^μ. Thus we are free to apply a reflection to all the vectors since it leaves the dot products invariant. In the bulk Minkowski coordinates $(t^{\text{bulk}}, x_1^{\text{bulk}}, x_2^{\text{bulk}}, \ldots)$, we will apply a reflection in the x_1^{bulk} direction. This will send $n^\mu \to -n'^\mu, n_\perp^\mu \to n_\perp'^\mu$. (See section 4.4.2 for their expressions in Minkowski coordinates.) After this reflection, we finally arrive at

$$\begin{aligned}&(-x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}\left(\frac{x_{36}^2}{x_{46}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}} \mathcal{I}_e^\rho \left.\frac{\langle 0|\mathcal{O}^\dagger(x_{6^+},z_6)[\mathcal{O}_4(x_4,z_4),\mathcal{O}_3(x_3,z_3)]|0\rangle_{(b)}}{2i(\sin(\pi\frac{\widetilde{\tau}_\rho-\Delta_3-J_3-\Delta_4-J_4}{2}))}\right|_{\text{saddle}}\\ &= \overline{v}(-n', z_3^x n'_\perp + \vec{z}_{3\perp}, -z_4^x n'_\perp + \vec{z}_{4\perp}, w_1 = (0, iz_6^t, z_6^x, \vec{z}_{6\perp})),\end{aligned} \tag{C.47}$$

which agrees with the $\overline{v}(-n', e_3, e_4)$ definition (4.139) and the polarization map (4.136).

C.3.3 Computing R_ρ

In the main text, we claim that

$$\frac{2^d \dim(\rho)}{\text{vol}(\text{SO}(d))} \left.\frac{\langle \widetilde{\mathcal{O}}_a^\dagger(x_5)\widetilde{\mathcal{O}}^{\overline{b}}(x_6^+)\rangle}{\left(\langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle, \langle\mathcal{O}^\dagger\mathcal{O}\rangle\right)}\right|_{\text{saddle}} = 2^{-J+2\Delta}(-1)^{J-j+\widetilde{j}} R_\rho \left(\mathcal{I}_e^\rho\right)_a{}^{\overline{b}}, \tag{C.48}$$

where the R_ρ coefficient is given by (4.142). Here, we give a derivation for (4.142) in the case where $\rho = (J, j)$.

By the two-point pairing definition (4.78), we can rewrite the left-hand side of the above equation as

$$\dim(\rho) \frac{\langle \widetilde{\mathcal{O}}_a^\dagger(e)\widetilde{\mathcal{O}}^{\overline{b}}((-e)^+)\rangle}{\langle \widetilde{\mathcal{O}}_{a'}^\dagger(e)\widetilde{\mathcal{O}}^{\overline{b}'}((-e)^+)\rangle\langle \mathcal{O}^{a'}(e)\mathcal{O}_{\overline{b}'}^\dagger((-e)^+)\rangle}. \tag{C.49}$$

It is not hard to see that both the shadow two-point $\langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle$ and the two-point structure $\langle\mathcal{O}^\dagger\mathcal{O}\rangle$ are proportional to the reflection tensor $\mathcal{I}_e^\rho$. However, the coefficient R_ρ should only depend on the convention of the two-point structure $\langle\mathcal{O}^\dagger\mathcal{O}\rangle$. From (C.5), we get

$$\begin{aligned}&\langle \mathcal{O}^\dagger(e, z_5, w_5, \widetilde{w}_5)\mathcal{O}((-e)^+, z_6, w_6, \widetilde{w}_6)\rangle\\ &= 2^{J-2\Delta}(-1)^{J-j+\widetilde{j}}\widehat{I}_e^\rho(z_5, w_5, \widetilde{w}_5; z_6, w_6, \widetilde{w}_6),\end{aligned} \tag{C.50}$$

where

$$\begin{aligned}&\widehat{I}_e^\rho(z_5, w_5, \widetilde{w}_5; z_6, w_6, \widetilde{w}_6)\\ &= (z_6\cdot I_e\cdot z_5)^{J-j}\left(\sum_{\sigma\in S_2} z_{6\alpha_1}w_{6\alpha_2}I_e^{\alpha_1}{}_{\beta_1}I_e^{\alpha_2}{}_{\beta_2}z_5^{\beta_{\sigma(1)}}w_5^{\beta_{\sigma(2)}}\right)^{j-\widetilde{j}}\\ &\times\left(\sum_{\sigma\in S_3} z_{6\alpha_1}w_{6\alpha_2}\widetilde{w}_{6\alpha_3}I_e^{\alpha_1}{}_{\beta_1}I_e^{\alpha_2}{}_{\beta_2}I_e^{\alpha_3}{}_{\beta_3}z_5^{\beta_{\sigma(1)}}w_5^{\beta_{\sigma(2)}}\widetilde{w}_5^{\beta_{\sigma(3)}}\right)^{\widetilde{j}},\end{aligned} \tag{C.51}$$

where $I^{\mu}_{e\,\nu} = \delta^{\mu}{}_{\nu} + 2e^{\mu}e_{\nu}$, and we have introduced polarization vectors $z, w, \widetilde{w}$ for the three rows of the Young diagram of ρ. By comparing (C.48), (C.49), (C.50), and using the fact that $\left(\mathcal{I}^{\rho}_{e}\right)_{a}{}^{\overline{b}}\left(\mathcal{I}^{\rho}_{e}\right)^{a}{}_{\overline{b}} = \dim\rho$, we obtain

$$\mathcal{I}^{\rho}_{e} = R_{\rho}\widehat{I}^{\rho}_{e}. \tag{C.52}$$

This implies

$$(\widehat{I}^{\rho}_{e})_{a}{}^{\overline{b}}(\widehat{I}^{\rho}_{e})^{a}{}_{\overline{b}} = R_{\rho}^{-2}\dim\rho. \tag{C.53}$$

Now, let us specialize to $\rho = (J, j)$ and compute $(\widehat{I}^{\rho}_{e})_{a}{}^{\overline{b}}(\widehat{I}^{\rho}_{e})^{a}{}_{\overline{b}}$. The main idea is that we can use weight-shifting operators to derive a recursion relation [39, 45]. In particular, we will use

$$\begin{aligned}
\mathcal{D}^{0+\mu}_{z,w}\Big|_{J,j} &= (j-J)w^{\mu} + z^{\mu}z\cdot\frac{\partial}{\partial w},\\
\mathcal{D}^{0-\mu}_{z,w}\Big|_{J,j} &= \left((-J-d+4-j)\delta^{\mu}{}_{\nu} + z^{\mu}\frac{\partial}{\partial z^{\nu}}\right)\left((d-6+2j)\frac{\partial}{\partial w_{\nu}} - w^{\nu}\partial_{w}^{2}\right),
\end{aligned} \tag{C.54}$$

where $\mathcal{D}^{0+}_{z,w}$ increases the transverse spin j by 1 and $\mathcal{D}^{0-}_{z,w}$ decreases j by 1. These operators are the weight-shifting operators of the SO$(d-1,1)$ group in the vector representation [39]. Using them, we can build a "bubble diagram,"

$$\mathcal{D}^{0+}_{z_5,w_5}\cdot\mathcal{D}^{0-}_{z_5,w_5}\widehat{I}^{\rho}_{e}(z_5,w_5;z_6,w_6) = j(d-6+2j)(J+j+d-4)(J-j+2)\widehat{I}^{\rho}_{e}(z_5,w_5;z_6,w_6). \tag{C.55}$$

Furthermore, one can perform crossing on a weight-shift operator and move it to the other leg of $\widehat{I}_{e}$,

$$\mathcal{D}^{0-\mu}_{z_5,w_5}\widehat{I}^{\rho}_{e}(z_5,w_5;z_6,w_6) = \frac{j(d-6+2j)(J+j+d-4)}{J-j+1}\mathcal{D}^{0+\mu}_{z_6,w_6}\widehat{I}^{\rho'}_{e}(z_5,w_5;z_6,w_6), \tag{C.56}$$

where $\rho' = (J, j-1)$. Combining the above two equations, we get

$$\widehat{I}^{\rho}_{e}(z_5,w_5;z_6,w_6) = \frac{1}{(J-j+1)(J-j+2)}\mathcal{D}^{0+}_{z_5,w_5}\cdot\mathcal{D}^{0+}_{z_6,w_6}\widehat{I}^{\rho'}_{e}(z_5,w_5;z_6,w_6). \tag{C.57}$$

Our goal is to compute the index contraction of two $\widehat{I}^{\rho}_{e}$ tensors. Using (C.57) we can rewrite it as the index contraction of $\mathcal{D}^{0+}_{z_5,w_5}\cdot\mathcal{D}^{0+}_{z_6,w_6}\widehat{I}^{\rho'}_{e}$ and $\widehat{I}^{\rho}_{e}$. Then, we can "integrate by parts" to move the weight-shifting operators to act on $\widehat{I}^{\rho}_{e}$. More precisely, let us denote the index contraction by a pairing $(\cdots, \cdots)$. Then the integration by parts relation is

$$\left(\mathcal{D}^{0+\mu}_{z,w}\widehat{I}^{\rho'}_e, \widehat{I}^{\rho}_e\right) = \frac{J-j+2}{j(d-6+2j)(J+j+d-4)}\left(I^{\rho'}_e, \mathcal{D}^{0-\mu}_{z,w} I^{\rho}_e\right). \tag{C.58}$$

Note that this relation is different from the one given in [45]. This is simply because they are adjoint relations with respect to different pairings. To obtain the adjoint with respect to index contraction, a simple way is to use the identity of the $\mathcal{D}^{(h)}$ operator given in [110]. (Note that the $\mathcal{D}^{0-}$ operator defined here is just a special case of $\mathcal{D}^{(h)}$ with $h = 2$.)

The adjoint relation enables us to turn the computation into the index contraction of $\widehat{I}^{\rho'}_e$ and $\mathcal{D}^{0-}_{z_5,w_5}\cdot\mathcal{D}^{0-}_{z_6,w_6}\widehat{I}^{\rho}_e$, which then becomes the contraction of two $\widehat{I}^{\rho'}_e$'s. Together with (C.53), this gives the recursion relation

$$R^{-2}_{\rho}\dim\rho = \frac{(j+d-5)(2j+d-4)(J-j+2)(J+j+d-3)}{j(2j+d-6)(J-j+1)(J+j+d-4)}R^{-2}_{\rho'}\dim\rho'. \tag{C.59}$$

The relation between $\dim\rho$ and $\dim\rho'$ can be obtained from standard formula [180, 45], or from the recursion relation for the Plancherel measure of the $\mathrm{SO}(d-1,1)$ Lorentz group [45]. Eventually, we get

$$\frac{R^2_{\rho'=(J,j-1)}}{R^2_{\rho=(J,j)}} = \frac{(J-j+2)^2}{(J-j+1)^2}. \tag{C.60}$$

With the initial condition $R_{\rho=(J,0)} = 1$, this leads to (4.142) in the $\widetilde{j} = 0$ case.

C.4 Dual structures at large ν

In this appendix, we discuss how to compute the dual structure $\left(\langle 0|\mathcal{O}_4\mathbf{L}[\mathcal{O}]\mathcal{O}_2|0\rangle^{(a)}\right)^{-1}$ that appears in the kernel of the functional. In particular, we are interested in the large ν limit, where ν parametrizes the scaling dimension of $\mathcal{O}$ as $\Delta = \frac{d}{2} + i\nu$.

C.4.1 Light transform at large dimension

To understand the dual structure, we should first study the light transform of the three-point function $\langle 0|\mathcal{O}_4\mathcal{O}\mathcal{O}_2|0\rangle$, and then consider its Lorentzian three-point pairing. Hence, let us first consider the light transform $\mathbf{L}[\mathcal{O}]$ in the limit where $\mathcal{O}$ has large scaling dimension. Recall that the light transformed three-point function is given by

$$\langle 0|\mathcal{O}_4\mathbf{L}[\mathcal{O}_{\Delta,J,\lambda'}](X_0,Z_0)\mathcal{O}_2|0\rangle = \int d\alpha\,\langle 0|\mathcal{O}_4\mathcal{O}_{\Delta,J,\lambda'}(Z_0-\alpha X_0,-X_0)\mathcal{O}_2|0\rangle \tag{C.61}$$

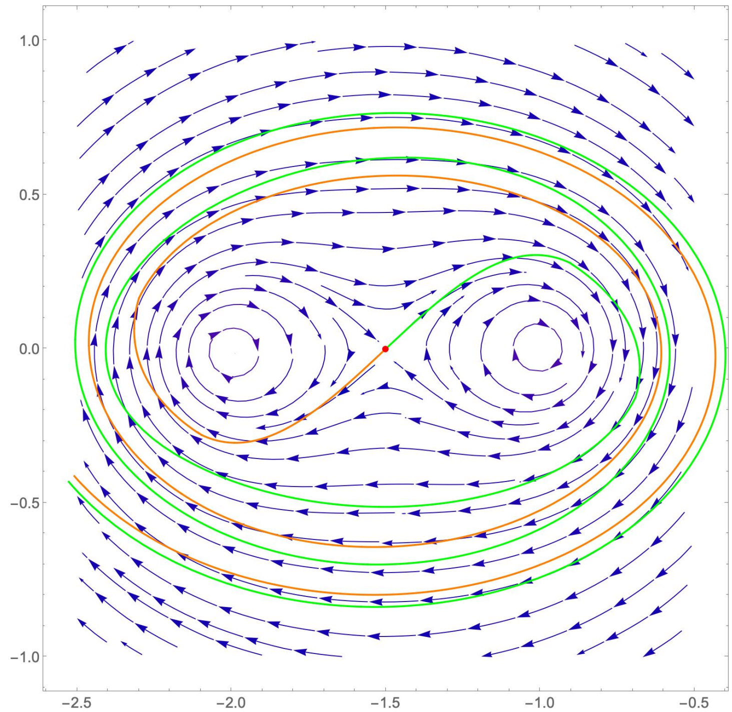

Figure C.1: The steepest descent flow of the saddle integral (C.64). The red point is the saddle point. We show the deformed contour passing through the saddle in green and orange. The green part starts at infinity, goes in the opposite direction of the flow, and gradually spirals in toward the saddle. Along the green contour, the integrand keeps increasing and reaches maximum at the saddle point. It passes through the saddle point along a direction that is rotated by $e^{-\frac{3\pi i}{4}}$ relative to the real axis. After passing through the saddle point, the contour (in orange) goes in the same direction as the flow and gradually spirals out to infinity. The integrand at infinity goes as $\alpha^{-i\nu}$, so by making the contour spiral many times clockwise at infinity, we ensure that the contribution $\alpha^{-i\nu} \propto e^{2\pi\nu \arg(\alpha)}$ can be made arbitrarily small.

We will focus on the case $4 > 0 > 2^-$ and consider the limit $\Delta = \frac{d}{2} + i\nu$, $\nu \to \infty$. In this limit, we see that the quickly varying part of the integrand is

$$e^{-\frac{i\nu}{2}(\log(-2(Z_0-\alpha X_0)\cdot X_2)+\log(-2(Z_0-\alpha X_0)\cdot X_4))}, \tag{C.62}$$

which implies that the integral has a saddle point at

$$\alpha_* = \frac{2X_0 \cdot X_2 Z_0 \cdot X_4 + 2X_0 \cdot X_4 Z_0 \cdot X_2}{X_{02}X_{04}}. \tag{C.63}$$

As one can check, this point is spacelike from both 2 and 4. Therefore, the light transform gets localized to the point in the middle of the region that is spacelike from 2 and 4.

Expanding the quickly varying part around α_*, we find that the saddle integral we have to do is

$$\int d\alpha\, e^{\frac{i\nu}{2}\left(\frac{X_{02}X_{04}}{V_{0,24}X_{24}}\right)^2(\alpha-\alpha_*)^2} = e^{-\frac{3i\pi}{4}}\sqrt{\frac{2\pi}{\nu}\frac{(-V_{0,42})X_{24}}{X_{02}(-X_{04})}}. \tag{C.64}$$

The phase factor $e^{-\frac{3i\pi}{4}}$ comes from deforming the contour to a steepest descent contour. We show the steepest descent flow and the deformed contour in figure C.1. The contour passes through the saddle point with angle $-\frac{3\pi}{4}$, and hence we have the $e^{-\frac{3i\pi}{4}}$ factor in (C.64). Also, note that when taking $\left(\frac{X_{02}X_{04}}{V_{0,42}X_{24}}\right)^2$ out of the square root, we have to make sure all factors are positive. In summary, the above analysis implies that the large Δ limit of $\langle 0|\mathcal{O}_4\mathbf{L}[\mathcal{O}_{\Delta,J}](X_0,Z_0)\mathcal{O}_2|0\rangle$ should be given by

$$\lim_{\substack{\Delta=\frac{d}{2}+i\nu\\ \nu\gg 1}}\langle 0|\mathcal{O}_4\mathbf{L}[\mathcal{O}_{\Delta,J,\lambda'}](X_0,Z_0)\mathcal{O}_2|0\rangle = e^{-\frac{3i\pi}{4}}\sqrt{\frac{2\pi}{\nu}\frac{(-V_{0,42})X_{24}}{X_{02}(-X_{04})}}\langle 0|\mathcal{O}_4\mathcal{O}_{\Delta,J,\lambda'}(Z_0-\alpha_*X_0,-X_0)\mathcal{O}_2|0\rangle. \tag{C.65}$$

Furthermore, we can study how each building block of the three-point structure transforms under $X_0\to Z_0-\alpha_*X_0$, $Z_0\to -X_0$. We will focus on the structures that appear when λ' has a single row of length j'. We find

$$\begin{aligned}
X_{02} &\to \frac{(-V_{0,42})X_{24}}{-X_{04}}, & V_{2,04} &\to -\left(V_{2,04}+\frac{H_{20}}{V_{0,42}}\right), & U_{0,24} &\to \frac{(-V_{0,42})X_{24}}{(-X_{04})X_{02}}U_{0,24},\\
X_{04} &\to \frac{(-V_{0,42})X_{24}}{X_{02}}, & V_{4,20} &\to -\left(V_{4,20}+\frac{H_{40}}{V_{0,42}}\right), & U_{0,42} &\to \frac{(-V_{0,42})X_{24}}{X_{04}X_{02}}U_{0,42},
\end{aligned} \tag{C.66}$$

and all other structures are invariant (assuming $\mathcal{O}_2$ and $\mathcal{O}_4$ have no transverse spin). Therefore, starting with the three-point structure (C.9), its light transform at large Δ is given by

$$\lim_{\substack{\Delta=\frac{d}{2}+i\nu\\ \nu\gg 1}}\langle 0|\mathcal{O}_4\mathbf{L}[\mathcal{O}_{\Delta,J,j'}](X_0,Z_0,W_0)\mathcal{O}_2|0\rangle = 2^{\Delta+J-1}e^{-\frac{3i\pi}{4}}\sqrt{\frac{2\pi}{\nu}}$$
$$\times\frac{(-2V_{0,42})^{1-\Delta-J+m_0}\left(-V_{2,04}-\frac{H_{20}}{V_{0,42}}\right)^{m_2}\left(-V_{4,20}-\frac{H_{40}}{V_{0,42}}\right)^{m_4}H_{24}^{n_{24}}H_{02}^{n_{02}}H_{04}^{n_{04}}(-2U_{0,24})^{k_{02}}(2U_{0,42})^{k_{04}}}{X_{24}^{\frac{\Delta_2+J_2+\Delta_4+J_4-2+\Delta+J-j'}{2}}(-X_{04})^{\frac{\Delta_4+J_4+2-\Delta-J+j'-\Delta_2-J_2}{2}}X_{02}^{\frac{\Delta_2+J_2+2-\Delta-J+j'+\Delta_2+J_2}{2}}}. \tag{C.67}$$

The result can also be written as

$$2^{\Delta+J-1}e^{-\frac{3i\pi}{4}}\sqrt{\frac{2\pi}{\nu}}\langle 0|\mathcal{O}_4\mathcal{O}^L(X_0,Z_0,W_0)\mathcal{O}_2|0\rangle\Bigg|_{V_{2,04}\to-\left(V_{2,04}+\frac{H_{20}}{V_{0,42}}\right),V_{4,20}\to-\left(V_{4,20}+\frac{H_{40}}{V_{0,42}}\right),U_{0,42}\to-U_{0,42}}, \tag{C.68}$$

where the operator O^L has quantum numbers $(1-J, 1-\Delta, j')$.

Computation of light transform of general spinning three-point functions has been discussed in [34] for general Δ (see eq. (5.69)). One can check that the above result agrees with the large Δ_1 limit of the result in [34], which contains an Appell F_2. The Appell F_2 function in this limit will simplify and lead to the replacement rule $V_{2,04} \to -\left(V_{2,04} + \frac{H_{20}}{V_{0,42}}\right), V_{4,20} \to -\left(V_{4,20} + \frac{H_{40}}{V_{0,42}}\right)$ given above.

We also need to understand how to impose conservation condition. In section 4.4.1, we study the conservation condition when the exchanged operator O has large dimension. Here, after the light transform, the dimension and spin are swapped, and therefore we have to instead consider conservation at large spin. However, the strategy in both cases are the same. We should identify the quickly-varying part of the three-point structure and take its derivative. By taking a x_4-derivative of the Δ-dependent factor of (C.67), we find that the conservation condition $\partial_{x_4}\langle O_4 O O_2\rangle = 0$ after the light transform becomes equivalent to

$$\left(V_{4,20} + \frac{H_{40}}{V_{0,42}}\right)_A D^A_{Z_4}\langle 0|O_4 \mathbf{L}[O] O_2|0\rangle = 0, \tag{C.69}$$

where A is an embedding space index, and its meaning as a subscript is that we should take the structures $V_{4,20}, H_{40}$ and strip off the Z_4. $D^A_{Z_4}$ is the Todorov/Thomas operator [30]. This condition is perfectly consistent with the original large Δ conservation (4.113) and the replacement rule from the light transform given by (C.66).

C.4.2 Dual structures

To get the dual structures, we can start with a standard basis of continuous-spin three-point structures $\langle 0|\widetilde{O}_4^\dagger O^F \widetilde{O}_2^\dagger|0\rangle^{(a)}$ by adding spinning structures to the scalar convention (C.15) following (C.9). The quantum numbers of $\widetilde{O}_2^\dagger, \widetilde{O}_4^\dagger$ are $(d-\Delta_2, J_2)$ and $(d-\Delta_4, J_4)$. For O^F, we should have $(\Delta_F, J_F) = (J+d-1, \Delta-d+1)$, where $\Delta = \frac{d}{2} + i\nu$ and $J = J_1 + J_3 - 1$ depends on the spin of the external operators. The actual dual structures should be given by

$$\left(\langle 0|O_4 \mathbf{L}[O] O_2|0\rangle^{(a)}\right)^{-1} = \alpha^{(a)}{}_{(b)}\langle 0|\widetilde{O}_4^\dagger O^F \widetilde{O}_2^\dagger|0\rangle^{(b)}, \tag{C.70}$$

and our goal is to find the coefficients $\alpha^{(a)}{}_{(b)}$.

Following the definition of dual structures (4.94), we have

$$\alpha^{(a)}{}_{(b)}\left(\langle 0|\widetilde{O}_4^\dagger O^F \widetilde{O}_2^\dagger|0\rangle^{(b)}, \langle 0|O_4 \mathbf{L}[O] O_2|0\rangle_{(c)}\right)_L = \delta^a_c, \tag{C.71}$$

which gives

$$\alpha^{(a)}{}_{(b)} = \left(\langle 0|\widetilde{O}_4^\dagger O^F \widetilde{O}_2^\dagger|0\rangle^{(b)}, \langle 0|O_4 \mathbf{L}[O] O_2|0\rangle_{(a)}\right)_L^{-1}$$
$$=2^{2d-2}\,\mathrm{vol}(\mathrm{SO}(d-2))\left(\langle 0|\widetilde{O}_4^\dagger(0^+) O^F(\infty, z_0^*)\widetilde{O}_2^\dagger(e)|0\rangle^{(b)}\langle 0|O_4(0^+)\mathbf{L}[O](\infty, z_0^*)O_2(e)|0\rangle_{(a)}\right)^{-1}, \tag{C.72}$$

where the in the second line we have used the conformal group to fixed all the points to the configuration $x_4 = 0^+, x_2 = e, x_0 = \infty$ and $z_0 = (1, 1, \vec{0})$ (in lightcone coordinates). The prefactor $2^{2d-2}\,\mathrm{vol}(\mathrm{SO}(d-2))$ comes from the Faddeev-Popov determinant and volume of the stabilizer group of the gauge-fixed configuration [11].

Then, the calculation of dual structures is now reduced to evaluating the continuous-spin basis and the light transformed structures in this standard configuration. Since the continuous-spin basis can be obtained from (C.15), (C.9), and the light transformed structures at large ν are given by (C.67), the dual structures at large ν can be computed straightforwardly using (C.72).

Lastly, when we impose conservation on the external operators O_2, O_4, the dual structures get a gauge redundancy due to the conservation equations. At large ν, the statement becomes

$$\langle 0|O_4\mathbf{L}[O]O_2|0\rangle^{-1} \sim \langle 0|O_4\mathbf{L}[O]O_2|0\rangle^{-1} + \left(V_{4,20} + \frac{H_{40}}{V_{0,42}}\right)(\ldots) + \left(V_{2,04} + \frac{H_{20}}{V_{0,42}}\right)(\ldots). \tag{C.73}$$

The idea is that whenever we have a $\left(V_{4,20} + \frac{H_{40}}{V_{0,42}}\right)$ factor, we can always integrate it by parts in the pairing [45] and get the large-ν conservation equation (C.69). The argument for the other factor is similar. Fortunately, thanks to the identity (4.153), when we evaluate the structures at the saddle and set the time component of the external polarizations to zero, the result is independent of this gauge redundancy.

In summary, to compute the dual structures for conserved external operators, we can use the gauge redundancy to remove all the V_i structures for the external operators and simply consider a basis with just the H_{i0}, H_{ij} structures, as stated in section 4.4.3. We can then use (C.72) to compute the dual structure coefficients, where the label (b) only includes continuous-spin structures without the V_i's, and (a) are the conserved structures. These two structures both have the same counting given by (4.154).

C.5 CFT four-point structures

In this appendix, we give the expressions of the CFT four-point structures that we use in the main text. In particular, they should satisfy (4.181). From the group-theoretic counting argument [48], we already know that the number of CFT four-point structures should agree with the number of flat space polarization structures. To write down the CFT structures, let us first define

$$\begin{aligned} H^{CFT}_{ijk} &= [X_i, Z_i]^{A_1A_2}[X_j, Z_j]^{A_2A_3}[X_k, Z_k]^{A_3A_1}, \\ H^{CFT}_{ijkl} &= [X_i, Z_i]^{A_1A_2}[X_j, Z_j]^{A_2A_3}[X_k, Z_k]^{A_3A_4}[X_l, Z_l]^{A_4A_1}, \\ X^{CFT}_{ijkl} &= H^{CFT}_{ijkl} - \frac{1}{4}\left(H^{CFT}_{14}H^{CFT}_{23} + H^{CFT}_{13}H^{CFT}_{24} + H^{CFT}_{12}H^{CFT}_{34}\right), \\ S^{CFT} &= V^{CFT}_{1,24}H^{CFT}_{234} + V^{CFT}_{2,31}H^{CFT}_{341} + V^{CFT}_{3,42}H^{CFT}_{412} + V^{CFT}_{4,13}H^{CFT}_{123}, \end{aligned} \tag{C.74}$$

where A_i are embedding space indices, and $[X, Z]^{AB} = X^A Z^B - X^B Z^A$. The superscript CFT is to distinguish the CFT structure and the flat space structure, and V^{CFT}, H^{CFT}_{ij} are given in (C.4). Then, we find

$$\begin{aligned} &\{H^{CFT}_{14}H^{CFT}_{23}, H^{CFT}_{13}H^{CFT}_{24}, H^{CFT}_{12}H^{CFT}_{34}, X^{CFT}_{1243}, X^{CFT}_{1234}, X^{CFT}_{1324}, S^{CFT}\}\big|_{\text{saddle},(4.133),(4.136)} \\ =&\frac{16}{\nu^4}\left\{H_{14}H_{23}, H_{13}H_{24}, H_{12}H_{34}, X_{1243}, X_{1234}, X_{1324}, \frac{2}{\nu^2}S\right\}. \end{aligned} \tag{C.75}$$

Thus, we choose the left hand side of the above equation to be our four-point structure basis Q^I in the main text.

The CFT four-point structures we choose all have homogeneity 1 for all $X_{1,2,3,4}, Z_{1,2,3,4}$. Under the map of polarizations, they reproduce the expected amplitude structures, with the correct Regge behavior. The additional factors of ν should not be a big issue since we can absorb them into the prefactors. The only different case is that we have to introduce a $\frac{2}{\nu^2}$ factor for the S structure. This is reasonable from dimensional analysis since the S structure has an additional $p_i \cdot p_j$. It also seems to suggest that from the CFT point of view, $\frac{2}{\nu^2}S$ is a more natural choice for the structure.

For the four-graviton case, the construction given above can be easily generalized. The only four-graviton flat space structure that cannot be written in terms of H_{ij}, X_{ijkl}, S is the Gram determinant of all dot products between $(p_1, p_2, p_3, e_1, e_2, e_3, e_4)$, which we write as $\mathcal{G}_{p_1,p_2,p_3,e_1,e_2,e_3,e_4}$. Even though this structure does not appear in the graviton sum rule dictionary given in section 4.4.6 as it grows more slowly in the Regge limit, let us still give the corresponding CFT structure for completeness.

The corresponding CFT structure can be written as

$$X_{24}\left(\frac{X_{14}X_{34}}{X_{12}X_{23}}\right)^{\frac{1}{2}}\mathcal{G}_{X_1,X_2,X_3,Z_1,Z_2,Z_3,Z_4}-(\ldots)\mathcal{G}_{X_1,X_2,X_3,Z_1,Z_2,Z_3,X_4}. \tag{C.76}$$

This structure is manifestly invariant under $Z_i \to Z_i + \#X_i$ thanks to properties of determinant. The only exception is the $Z_4 \to Z_4 + \#X_4$ gauge redundancy. This is why we introduce the second term, and $(\ldots)$ is a factor that fixes homogeneity. The structure is constructed such that its homogeneity is the same as the other structures (has homogeneity 1 for all X_i, Z_i). When evaluating the above expression at the saddle and applying the polarization map, we find that the second term vanishes, and we have

$$X_{24}\left(\frac{X_{14}X_{34}}{X_{12}X_{23}}\right)^{\frac{1}{2}}\mathcal{G}_{X_1,X_2,X_3,Z_1,Z_2,Z_3,Z_4}\bigg|_{\text{saddle},(4.133),(4.136)}=\frac{256}{\nu^6}\mathcal{G}_{p_1,p_2,p_3,e_1,e_2,e_3,e_4}. \tag{C.77}$$

So, (C.76) should be the correct CFT four-point structure that gives $\mathcal{G}_{p_1,p_2,p_3,e_1,e_2,e_3,e_4}$.

Appendix D

APPENDICES TO CHAPTER 5

D.1 Pole structure of scalar crossing equation

In this appendix we will carefully derive the pole structure of the constrained Epstein zeta series $\mathcal{E}^c_s(\mu)$. Much of this analysis is in Sec. 3.2 of [165]. Let us look at the scalar sector of (5.19):

$$\begin{aligned} y^{\frac{c}{2}}\left(1+\sum_{\Delta\in\mathcal{S}} e^{-2\pi\Delta y}\right) &:= \int_{-1/2}^{1/2} dx \widehat{Z}^c(\tau,\mu) \\ &= y^{\frac{c}{2}} + \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} y^{1-\frac{c}{2}} + 3\pi^{-\frac{c}{2}}\Gamma\left(\frac{c}{2}-1\right)\mathcal{E}^c_{\frac{c}{2}-1}(\mu) \\ &\quad + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \pi^{s-\frac{c}{2}}\Gamma\left(\frac{c}{2}-s\right)\mathcal{E}^c_{\frac{c}{2}-s}(\mu)\left(y^s + \frac{\Lambda(s-\frac{1}{2})}{\Lambda(s)}y^{1-s}\right) \\ &= y^{\frac{c}{2}} + \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} y^{1-\frac{c}{2}} + 3\pi^{-\frac{c}{2}}\Gamma\left(\frac{c}{2}-1\right)\mathcal{E}^c_{\frac{c}{2}-1}(\mu) \\ &\quad + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \pi^{s-\frac{c}{2}}\Gamma\left(\frac{c}{2}-s\right)\mathcal{E}^c_{\frac{c}{2}-s}(\mu) y^s. \end{aligned} \tag{D.1}$$

In the last line of (D.1), we used the functional equation that $\mathcal{E}^c_{\frac{c}{2}-s}(\mu)$ obeys, (5.21). We would like to move the contour in (D.1) to the right, so we again need to classify all simple poles of the integrand with $\text{Re}(s) > \frac{1}{2}$. As was argued in [165], there can only be poles we cross at $s = \frac{c}{2}$ and $s = \frac{1+z_n}{2}, \frac{1+z_n^*}{2}$. Let us review the argument.

The idea is to take the inverse Laplace transform of (D.1) to get the scalar density of states. We then integrate from 0 to some number Δ (not including the vacuum), and demand that this vanishes for sufficiently small Δ. This is due to the fact that the spectrum for a compact CFT is discrete, so in general there is a gap between the vacuum and first excited scalar state. A simple calculation shows the number of scalar operators (excluding the vacuum) below Δ is

$$\begin{aligned}
N_0(\Delta) &= \frac{2\pi^c\zeta(c-1)\Delta^{c-1}}{(c-1)\Gamma(\frac{c}{2})^2\zeta(c)} + 12\frac{2^{\frac{c}{2}}\Delta^{\frac{c}{2}}\mathcal{E}^c_{\frac{c}{2}-1}(\mu)}{c(c-2)} - \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \frac{2^{\frac{c}{2}-s}\Delta^{\frac{c}{2}-s}\mathcal{E}^c_{\frac{c}{2}-s}(\mu)}{s-\frac{c}{2}} \\
&= \frac{2\pi^c\zeta(c-1)\Delta^{c-1}}{(c-1)\Gamma(\frac{c}{2})^2\zeta(c)} + 12\frac{2^{\frac{c}{2}}\Delta^{\frac{c}{2}}\mathcal{E}^c_{\frac{c}{2}-1}(\mu)}{c(c-2)} \\
&\quad + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \frac{2^{\frac{c}{2}-s}\Delta^{\frac{c}{2}-s}\Gamma(s)\Gamma(s+\frac{c}{2}-1)\zeta(2s)\mathcal{E}^c_{\frac{c}{2}+s-1}(\mu)}{\pi^{2s-\frac{1}{2}}\Gamma(\frac{c}{2}+1-s)\Gamma(s-\frac{1}{2})\zeta(2s-1)}
\end{aligned} \tag{D.2}$$

Let us look at the last line of (D.2). In the limit of small Δ, we must get 0 for the integrated density of states, which means the integral must cancel the two power laws in Δ coming from the first two terms. In the integral in the last line of (D.2), we must close the contour to the left in the s-plane since Δ is small. This will tell us about the pole structure of $\mathcal{E}^c_{\frac{c}{2}+s-1}(\mu)$ for $\mathrm{Re}(s) < \frac{1}{2}$ (if we wanted to know the pole structure for $\mathrm{Re}(s) > \frac{1}{2}$ we would look at the first line of (D.2) and again close the contour to the left). In order to cancel the term that goes as Δ^{c-1}, we need a pole at $s = 1 - \frac{c}{2}$. This comes from the term $\Gamma(s + \frac{c}{2} - 1)$ in the numerator, with the others being finite. (Although the other gamma and zeta functions naively contribute poles and zeros for integer c, their combination is always finite.) Moreover in order for the residue to match, this fixes

$$\mathcal{E}^c_0(\mu) = -1. \tag{D.3}$$

We also need to cancel the second polynomial in (D.2). This comes from a pole at $s = 0$, coming from the $\Gamma(s)$ term. We see the residue already matches the coefficient in (D.2) so we cannot constrain the value of $\mathcal{E}^c_{\frac{c}{2}-1}(\mu)$. Finally there can be no other poles with $\mathrm{Re}(s) < \frac{1}{2}$. Naively this tells us that $\mathcal{E}^c_{\frac{c}{2}+s-1}(\mu)$ cannot have any poles for $\mathrm{Re}(s) < \frac{1}{2}$, but this is too fast – if the prefactor vanishes then $\mathcal{E}^c_{\frac{c}{2}+s-1}(\mu)$ can have a pole. The only zeros with $\mathrm{Re}(s) < \frac{1}{2}$ in the prefactor of the integrand are when $s = \frac{z_n}{2}, \frac{z_n^*}{2}$, coming from the $\zeta(2s)$ term. Thus $\mathcal{E}^c_{\frac{c}{2}+s-1}(\mu)$ can have a pole at $s = \frac{z_n}{2}, \frac{z_n^*}{2}$. We also know that $\mathcal{E}^c_{\frac{c}{2}+s-1}(\mu)$ must have zeros at $s = -\frac{c}{2}, -\frac{c}{2} - 1, \cdots$ to cancel the poles from $\Gamma(s + \frac{c}{2} - 1)$.

Thus, looking at the integrand in (D.1), we see the only poles to the right of the contour of integration are at $s = \frac{c}{2}$ and $s = 1 - \frac{z_n}{2}, 1 - \frac{z_n^*}{2}$. (Using the functional equation for the zeta function, we can rewrite the last term as $s = \frac{1+z_n}{2}, \frac{1+z_n^*}{2}$.) The residue of the pole at $s = \frac{c}{2}$ is given in (D.3) and the residue at $s = \frac{1+z_n}{2}, \frac{1+z_n^*}{2}$ is just given by reading off the pole from integrating the partition function against an Eisenstein series at $s = \frac{z_n}{2}$ (see (5.29)).

This fully reproduces the pole structure which we used to derive (5.38).

D.2 Functional action on crossing equation

Let us consider the functional

$$\mathcal{F}_k[h(t)] := \int_0^\infty \frac{dt}{t} h(t) \sum_{m=1}^\infty e^{-\pi k t^2 m^2}. \tag{D.4}$$

We would like to apply this functional to each of the terms in (5.41). To do so let us first compute:

$$\begin{aligned}\mathcal{F}_k[t^s e^{-\frac{A}{t^2}}] &:= \sum_{m=1}^\infty f_m^k(s,A)\\ \mathcal{F}_k[t^s e^{-Bt^2} U(\alpha,\beta,Bt^2)] &:= \sum_{m=1}^\infty g_m^{\alpha,\beta,k}(s,B),\end{aligned} \tag{D.5}$$

with $f_m^k(s,A)$ and $g_m^{\alpha,\beta,k}(s,B)$ defined as

$$\begin{aligned}f_m^k(s,A) &= \int_0^\infty dt\; t^{s-1} e^{-\frac{A}{t^2}-\pi k t^2 m^2}\\ &= A^{s/4} k^{-s/4} m^{-s/2} \pi^{-s/4} K_{s/2}(2m\sqrt{k\pi A})\\ g_m^{\alpha,\beta,k}(s,B) &= \int_0^\infty dt\; t^{s-1} e^{-Bt^2-\pi k t^2 m^2} U(\alpha,\beta,Bt^2)\\ &= \frac{1}{2}\left(B+k\pi m^2\right)^{-s/2}\left(\frac{\Gamma(\frac{s}{2})\Gamma(1-\beta)\,{}_2F_1(\alpha,\frac{s}{2},\beta;\frac{B}{B+k\pi m^2})}{\Gamma(1+\alpha-\beta)}\right.\\ &\quad\left.+\frac{\Gamma(\beta-1)(B+k\pi m^2)^{\beta-1}\Gamma(1-\beta+\frac{s}{2})\,{}_2F_1(1+\alpha-\beta,1-\beta+\frac{s}{2},2-\beta;\frac{B}{B+k\pi m^2})}{\Gamma(\alpha)B^{\beta-1}}\right).\end{aligned} \tag{D.6}$$

Applying this term by term to (5.41) we then get:

$$\begin{aligned}&\sum_{\Delta\in\mathcal{S}}\sum_k \alpha_k \sum_{m=1}^\infty \Bigg[4\pi\Delta f_m^k(-c,2\pi\Delta) - c f_m^k(2-c,2\pi\Delta)\\ &\qquad - \sum_{n=1}^\infty \frac{b(n)n^{c-2}}{\sqrt{\pi}}\Big((c-2)g_m^{-\frac{1}{2},\frac{c}{2},k}(c,2\pi n^2\Delta) - 4\pi n^2\Delta g_m^{-\frac{1}{2},\frac{c}{2},k}(c+2,2\pi n^2\Delta)\\ &\qquad\qquad + 2\pi n^2 \Delta g_m^{\frac{1}{2},\frac{c+2}{2},k}(c+2,2\pi n^2\Delta)\Big)\Bigg]\\ &\quad = \zeta(c-1)\Gamma\left(\frac{c-1}{2}\right)\pi^{\frac{1-c}{2}}\sum_k \alpha_k\left(\frac{c}{2}k^{\frac{c}{2}-1}+\left(\frac{c}{2}-1\right)k^{-\frac{c}{2}}\right).\end{aligned} \tag{D.7}$$

The above equation is summed over an arbitrary choice of k's and α_k's, subject to the constraints in (5.50).

Remarkably, for odd $c \geq 3$, we can get closed form expressions for the sums over m in (D.7). For $c = 3$, (D.7) reduces to

$$\sum_{\Delta\in\mathcal{S}}\sum_k \alpha_k \left[\frac{\sqrt{2} - e^{2\sqrt{2}\pi\sqrt{k\Delta}}(\sqrt{2} - 2\pi\sqrt{k\Delta})}{2(-1 + e^{2\sqrt{2}\pi\sqrt{k\Delta}})^2\sqrt{\Delta}} + \sum_{n=1}^{\infty} \frac{b(n)n\pi\cosh(\sqrt{2}n\pi\sqrt{\frac{\Delta}{k}})}{4k^{\frac{3}{2}}\sinh^3(\sqrt{2}n\pi\sqrt{\frac{\Delta}{k}})}\right]$$
$$= \frac{\pi}{6}\sum_k \alpha_k \left(\frac{3}{2}k^{\frac{1}{2}} + \frac{1}{2}k^{-\frac{3}{2}}\right). \tag{D.8}$$

The sum over n can be simplified to give

$$\sum_{\Delta\in\mathcal{S}}\sum_k \alpha_k \left[\frac{\sqrt{2} - e^{2\sqrt{2}\pi\sqrt{k\Delta}}(\sqrt{2} - 2\pi\sqrt{k\Delta})}{2(-1 + e^{2\sqrt{2}\pi\sqrt{k\Delta}})^2\sqrt{\Delta}} + \frac{\pi e^{2\sqrt{2}\pi\sqrt{\frac{\Delta}{k}}}}{(-1 + e^{2\sqrt{2}\pi\sqrt{\frac{\Delta}{k}}})^2 k^{3/2}}\right]$$
$$= \frac{\pi}{6}\sum_k \alpha_k \left(\frac{3}{2}k^{\frac{1}{2}} + \frac{1}{2}k^{-\frac{3}{2}}\right). \tag{D.9}$$

To simplify (D.7) for c odd, $c \geq 5$, we first define the auxiliary functions:

$$\begin{aligned}
v_1(c,0,m) &:= \frac{(-1)^{\frac{c+1}{2}+m}\Gamma(c)}{(c-2)\Gamma(m+1)\Gamma(\frac{c+1}{2}-m)},\\
v_1(c,n,m) &:= (-1)^{n+\frac{c-1}{2}+m}2^{2n}(n+1)(c-\frac{(n+1)(n+2)}{2})\Gamma(c-n-2)\\
&\quad\times\sum_{i=0}^{m-1}\sum_{j=0}^{i+1}\frac{(-1)^{i+j}(i-j+1)^n}{\Gamma(j+1)\Gamma(n-j+2)\Gamma(m-i)\Gamma(\frac{c+3}{2}-n-m+i)}, \qquad n\neq 0\\
v_2(c,n,m) &:= 2^{-2m}(c-3-2m)(c-1-2m)(c+1-2m)(c+3-2m)\Gamma(\frac{c-3}{2}+m)\\
&\quad\times\sum_{i=0}^{\frac{c-1}{2}-m}\sum_{j=0}^{i+1}\frac{(-1)^{m+n+i+j}(i-j+1)^{\frac{c+1}{2}-m}}{\Gamma(j+1)\Gamma(\frac{c+5}{2}-m-j)\Gamma(n-i+1)\Gamma(m-n+i+1)}\\
v_3(c,n,m) &:= (-1)^{n+\frac{c-1}{2}+m}2^{2n}(n+1)(n+2)(n+3)\Gamma(c-n-4)\\
&\quad\times\sum_{i=0}^{m}\sum_{j=0}^{i+1}\frac{(-1)^{i+j}(i-j+1)^{n+2}}{\Gamma(j+1)\Gamma(n-j+4)\Gamma(m-i+1)\Gamma(\frac{c-3}{2}-n-m+i)}.
\end{aligned}\tag{D.10}$$

Then (D.7) becomes:

$$\sum_{\Delta\in\mathcal{S}}\sum_{k}\alpha_k\left[\frac{1}{2^{\frac{3c}{2}-3}\pi^{\frac{c-3}{2}}\Delta^{\frac{c-2}{2}}(-1+e^{2\sqrt{2}\pi\sqrt{k\Delta}})^{\frac{c+1}{2}}}\sum_{i=0}^{\frac{c-1}{2}}\sum_{j=0}^{\frac{c-1}{2}}\nu_1(c,i,j)e^{2\sqrt{2}\pi\sqrt{k\Delta}j}(\sqrt{2k\Delta}\pi)^i\right.$$
$$\left.+\sum_{n=1}^{\infty}\frac{b(n)e^{2\sqrt{2}\pi n\sqrt{\frac{\Delta}{k}}}n^{\frac{c-1}{2}}\pi}{2^{\frac{c+13}{4}}k^{\frac{c+3}{4}}\Delta^{\frac{c-3}{4}}(-1+e^{2\sqrt{2}\pi n\sqrt{\frac{\Delta}{k}}})^{\frac{c+3}{2}}}\sum_{i=0}^{\frac{c-1}{2}}\sum_{j=0}^{\frac{c-5}{2}}\frac{\nu_2(c,i,j)k^{\frac{j}{2}}e^{2i\sqrt{2}n\pi\sqrt{\frac{\Delta}{k}}}}{2^{\frac{j}{2}}n^j\pi^j\Delta^{\frac{j}{2}}}\right]$$
$$=\zeta(c-1)\Gamma\left(\frac{c-1}{2}\right)\pi^{\frac{1-c}{2}}\sum_k\alpha_k\left(\frac{c}{2}k^{\frac{c}{2}-1}+\left(\frac{c}{2}-1\right)k^{-\frac{c}{2}}\right),\qquad c\text{ odd},\ c\geq 5. \tag{D.11}$$

The sum over n in (D.11) can be done exactly, which gives:

$$\sum_{\Delta\in\mathcal{S}}\sum_{k}\alpha_k\left[\frac{1}{2^{\frac{3c}{2}-3}\pi^{\frac{c-3}{2}}\Delta^{\frac{c-2}{2}}(-1+e^{2\sqrt{2}\pi\sqrt{k\Delta}})^{\frac{c+1}{2}}}\sum_{n=0}^{\frac{c-1}{2}}\sum_{m=0}^{\frac{c-1}{2}}\nu_1(c,n,m)e^{2\sqrt{2}\pi\sqrt{k\Delta}m}(\sqrt{2k\Delta}\pi)^n\right.$$
$$\left.+\frac{e^{2\sqrt{2}\pi\sqrt{\frac{\Delta}{k}}}}{2^{\frac{3c}{2}-7}\pi^{\frac{c-7}{2}}k^2\Delta^{\frac{c-4}{2}}(-1+e^{2\sqrt{2}\pi\sqrt{\frac{\Delta}{k}}})^{\frac{c+1}{2}}}\sum_{n=0}^{\frac{c-5}{2}}\sum_{m=0}^{\frac{c-3}{2}}\nu_3(c,n,m)e^{2\sqrt{2}\pi\sqrt{\frac{\Delta}{k}}m}(\sqrt{\frac{2\Delta}{k}}\pi)^n\right]$$
$$=\zeta(c-1)\Gamma\left(\frac{c-1}{2}\right)\pi^{\frac{1-c}{2}}\sum_k\alpha_k\left(\frac{c}{2}k^{\frac{c}{2}-1}+\left(\frac{c}{2}-1\right)k^{-\frac{c}{2}}\right),\qquad c\text{ odd},\ c\geq 5. \tag{D.12}$$

In the notation of (5.53),

$$\mathrm{vac}(k)=-\zeta(c-1)\Gamma\left(\frac{c-1}{2}\right)\pi^{\frac{1-c}{2}}\left(\frac{c}{2}k^{\frac{c}{2}-1}+\left(\frac{c}{2}-1\right)k^{-\frac{c}{2}}\right), \tag{D.13}$$

and $f(k,\Delta)$ is the term in the brackets of (D.12). Using these definitions, an explicit calculation verifies the claim in (5.54).

By examining the crossing equation (D.12), we notice something interesting. Acting on the crossing equation with $k^{3/2}\partial_k$ gives us an expression that is antisymmetric under $k\leftrightarrow k^{-1}$. This gives us another way to rewrite the crossing equation that will turn out to work for all c (not just odd c). Let us define the following function, using

(D.6)

$$h(c,k,\Delta) := \sum_{m=1}^{\infty} k^{3/2}\partial_k \left(4\pi\Delta f_m^k(-c,2\pi\Delta) - c f_m^k(2-c,2\pi\Delta)\right)$$
$$= \sum_{m=1}^{\infty}\Bigg[2^{-\frac{c-4}{4}} k^{\frac{c+2}{4}} m^{\frac{c}{2}} \Delta^{-\frac{c-4}{4}} \pi c K_{\frac{c}{2}}(2\sqrt{2}m\pi\sqrt{k\Delta})$$
$$- 2^{-\frac{c+2}{4}} k^{\frac{c}{4}} m^{\frac{c-2}{2}} \Delta^{-\frac{c-2}{4}} (c(c-2)+8\pi^2\Delta k m^2) K_{\frac{c-2}{2}}(2\sqrt{2}m\pi\sqrt{k\Delta})\Bigg]. \tag{D.14}$$

The sum can be evaluated exactly in closed form for odd c, but exists and converges for any c. An equivalent formulation of our scalar crossing equation is:

$$k^{3/2}\text{vac}'(k) + \sum_{\Delta\in\mathcal{S}} h(c,k,\Delta) - h(c,k^{-1},\Delta) = 0. \tag{D.15}$$

The sum rules used in (5.55) are just the odd derivatives of k (evaluated at $k = 1$) of (D.15). Finally, note that the term $k^{3/2}\text{vac}'(k)$ is simply the contribution of the vacuum state:

$$k^{3/2}\text{vac}'(k) = \frac{\Lambda\left(\frac{c-1}{2}\right)c(c-2)}{4}\left(k^{\frac{1-c}{2}} - k^{\frac{c-1}{2}}\right)$$
$$= \lim_{\Delta\to 0}\left(h(c,k,\Delta) - h(c,k^{-1},\Delta)\right). \tag{D.16}$$

D.3 $c = 1$ and $c = 2$ revisited

In this appendix, we reconsider $U(1)^c$ theories at $c = 1$ and $c = 2$. Due to the pole structure of the function $\Lambda(s) := \pi^{-s}\Gamma(s)\zeta(2s)$, the spectral decomposition and scalar crossing equation for these theories are slightly different than for $c > 2$. This is related to the fact that the average genus 1 partition function for $c = 1$ and $c = 2$ Narain CFTs diverges [166, 167]. For both $c = 1$ and $c = 2$ we will first consider Narain CFTs, and then the potentially more general $U(1)^c$ theories. We will use the notation

$$\tilde{E}_s(\tau) := \Lambda(s)E_s(\tau)$$
$$= \Lambda(s)y^s + \Lambda(1-s)y^{1-s} + \sum_{j=1}^{\infty}\frac{4\sigma_{2s-1}(j)\sqrt{y}K_{s-\frac{1}{2}}(2\pi j y)}{j^{s-\frac{1}{2}}}\cos(2\pi j x) \tag{D.17}$$

which we can see obeys $\tilde{E}_s(\tau) = \tilde{E}_{1-s}(\tau)$. This will make $s \leftrightarrow 1-s$ crossing manifestly invariant.

D.3.1 $c = 1$ reconsidered

The $c = 1$ free boson is labeled by a radius r. In our convention, we will take the self-dual point (i.e. the $SU(2)_1$ WZW model) to be $r = 1$ so that T-duality acts as $r \leftrightarrow r^{-1}$. The spectral decomposition of the reduced $c = 1$ partition function is:

$$\widehat{Z}^{c=1}(\tau, r) = r + r^{-1} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds 2\tilde{E}_s(\tau)(r^{2s-1} + r^{1-2s}). \tag{D.18}$$

(See e.g. Sec. 3.1.1 of [165] for derivation.) Notice that there are no Maass cusp forms in (D.18).

At $c = 1$, our scalar crossing equation (5.38) reduces to

$$1 + \sum_{\Delta\in\mathcal{S}} e^{-2\pi\Delta y} = -1 + \varepsilon_{c=1}(\mu)y^{-\frac{1}{2}} + \sum_{k=1}^{\infty}\text{Re}\left(\delta_{k,c=1}(\mu)y^{\frac{z_k}{2}}\right) + \sum_{\Delta\in\mathcal{S}}\sum_{n=1}^{\infty} b(n)\sqrt{\frac{2\Delta}{y}}e^{-\frac{2\pi n^2\Delta}{y}}, \tag{D.19}$$

where as usual μ is some abstract coordinate that we include to emphasize which terms are theory-dependent.

Let us verify (D.19) for a free boson at radius r. From the explicit spectral decomposition (D.18), we know that the free boson at radius r has $\varepsilon_{c=1}(\mu) = r + r^{-1}$ and $\delta_{k,c=1}(\mu) = 0$. Moreover, the set of scalar operators $\mathcal{S}$ are simply operators with either zero momentum or zero winding number (recall at $c = 1$, the spin of an operator is just the product of its momentum and winding number). Thus the set $\mathcal{S}$ is simply operators of dimension $\frac{m^2}{2r^2}$ and $\frac{m^2r^2}{2}$ for $m \in \mathbb{Z}_{>0}$, each with degeneracy 2. Thus (D.19) reduces to

$$2y^{\frac{1}{2}} + 2\sum_{m=1}^{\infty}(e^{-\pi m^2r^2y} + e^{-\pi m^2r^{-2}y})y^{\frac{1}{2}} = r + r^{-1} + 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b(n)m(re^{-\frac{\pi n^2m^2r^2}{y}} + r^{-1}e^{-\frac{\pi n^2m^2}{yr^2}}). \tag{D.20}$$

We can rewrite the RHS with new variables $m' = nm, n' = n$ (and dropping primes)

$$2y^{\frac{1}{2}} + 2\sum_{m=1}^{\infty}(e^{-\pi m^2r^2y} + e^{-\pi m^2r^{-2}y})y^{\frac{1}{2}} = r + r^{-1} + 2\sum_{m=1}^{\infty}\sum_{n|m} b(n)\frac{m}{n}(re^{-\frac{\pi m^2r^2}{y}} + r^{-1}e^{-\frac{\pi m^2}{yr^2}}). \tag{D.21}$$

It can be shown from properties of the M'obius μ function that

$$\sum_{n|m} b(n)\frac{m}{n} = 1 \tag{D.22}$$

for all m. Our crossing equation is then equivalent to

$$2y^{\frac{1}{2}}+2\sum_{m=1}^{\infty}(e^{-\pi m^2r^2y}+e^{-\pi m^2r^{-2}y})y^{\frac{1}{2}}=r+r^{-1}+2\sum_{m=1}^{\infty}(re^{-\frac{\pi m^2r^2}{y}}+r^{-1}e^{-\frac{\pi m^2}{yr^2}}). \tag{D.23}$$

This simply follows from the modular transformation properties of the Jacobi theta functions.

We would now like to derive a more general bound for $U(1)^c$ CFTs at $c=1$, without assuming the theory is a free boson compactified on a circle. This means we cannot assume that the $\delta_{k,c=1}$ terms in (D.19) necessarily vanish, so we need to apply the same functionals that we considered in Sec. 5.3.3. We first take a derivative with respect to y to remove the $\varepsilon_{c=1}(\mu)$ term. This gives the analog of (5.41):

$$\begin{aligned}\sum_{\Delta\in\mathcal{S}}&\left[\left(\frac{4\pi\Delta}{t}-t\right)e^{-\frac{2\pi\Delta}{t^2}}+\sum_{n=1}^{\infty}b(n)4\pi\sqrt{2}n^2t^4\Delta^{\frac{3}{2}}e^{-2\pi\Delta n^2t^2}\right]\\&=2t+\sum_{k=1}^{\infty}\operatorname{Re}\left(\delta_{k,c=1}(\mu)(z_k-2)t^{z_k}\right).\end{aligned} \tag{D.24}$$

We next would like to apply the functional (5.45) to (D.24), but there a slight subtlety. Recall that (5.45) was designed so that

$$\mathcal{F}[t^s]\propto\zeta(s). \tag{D.25}$$

The last line of (D.24) has a term $2t$, which will naively give something proportional to $\zeta(1)$ which diverges. However, it can be shown the integral (5.45) converges. The reason is that $M_\varphi(s)$ in (5.48) vanishes at $s=1$ which cancels the divergence of the zeta function. A careful analysis shows that if we choose $\varphi(t)=\sum_{i=1}^N\alpha_i e^{-\pi k_i t^2}$ (subject to the constraints (5.50)), then

$$\mathcal{F}^\varphi[2t]=\sum_{i=1}^N\alpha_i\left(-\frac{\log k_i}{2\sqrt{k_i}}\right). \tag{D.26}$$

We then apply the same functional $\mathcal{F}^\varphi$ to the LHS of (D.24). This gives

$$\begin{aligned}\sum_k\alpha_k\Bigg(\frac{\log k}{2\sqrt{k}}+\sum_{\Delta\in\mathcal{S}}\Bigg[&\frac{\pi\sqrt{\Delta}(\coth(\sqrt{2}\pi\sqrt{k\Delta})-1)}{\sqrt{2}}+\frac{\log(1-e^{-2\sqrt{2}\pi\sqrt{k\Delta}})}{2\sqrt{k}}\\&+\sum_{n=1}^{\infty}b(n)\frac{\sqrt{k}\coth\left(\sqrt{2}n\pi\sqrt{\frac{\Delta}{k}}\right)+\sqrt{2}n\pi\sqrt{\Delta}\operatorname{csch}^2\left(\sqrt{2}n\pi\sqrt{\frac{\Delta}{k}}\right)}{4kn}\Bigg]\Bigg)=0.\end{aligned} \tag{D.27}$$

The sum over n in (D.27) formally diverges but we can replace $\coth\left(\sqrt{2}n\pi\sqrt{\frac{\Delta}{k}}\right)$ with $\coth\left(\sqrt{2}n\pi\sqrt{\frac{\Delta}{k}}\right)-1$ since the term we add is multiplied by 0 from (5.50). This gives the following convergent sum rule:

$$\sum_k \alpha_k\Bigg(\frac{\log k}{2\sqrt{k}}+\sum_{\Delta\in\mathcal{S}}\Bigg[\frac{\pi\sqrt{\Delta}(\coth(\sqrt{2}\pi\sqrt{k\Delta})-1)}{\sqrt{2}}+\frac{\log(1-e^{-2\sqrt{2}\pi\sqrt{k\Delta}})}{2\sqrt{k}} \\ +\sum_{n=1}^{\infty}b(n)\frac{\sqrt{k}\left(\coth\left(\sqrt{2}n\pi\sqrt{\frac{\Delta}{k}}\right)-1\right)+\sqrt{2}n\pi\sqrt{\Delta}\,\mathrm{csch}^2\left(\sqrt{2}n\pi\sqrt{\frac{\Delta}{k}}\right)}{4kn}\Bigg]\Bigg)=0. \tag{D.28}$$

The sum over n can be done exactly to give:

$$\sum_k \alpha_k\Bigg(\frac{\log k}{2\sqrt{k}}+\sum_{\Delta\in\mathcal{S}}\Bigg[\frac{\pi\sqrt{\Delta}(\coth(\sqrt{2}\pi\sqrt{k\Delta})-1)}{\sqrt{2}}+\frac{\log(1-e^{-2\sqrt{2}\pi\sqrt{k\Delta}})}{2\sqrt{k}} \\ +\frac{\pi\sqrt{\Delta}(\coth(\sqrt{2}\pi\sqrt{\frac{\Delta}{k}})-1)}{\sqrt{2}k}-\frac{\log(1-e^{-2\sqrt{2}\pi\sqrt{\frac{\Delta}{k}}})}{2\sqrt{k}}\Bigg]\Bigg)=0. \tag{D.29}$$

Again from the same arguments as used to derive (5.53) we know that the term in parenthesis in (D.29) must be $c_0+c_1k^{-1/2}$ for some (theory-dependent) constants c_0, c_1. Moreover we see after evaluating $\partial_k|_{k=1}$ on each term, that $c_1=-1$. Therefore we can write our crossing equation as

$$\mathrm{vac}^{(n)}(1)+\sum_{\Delta\in\mathcal{S}}(\partial_k)^n f(k,\Delta)|_{k=1}=0,\quad n\geq 2,\quad n\text{ even}, \tag{D.30}$$

with

$$\mathrm{vac}(k)=\frac{2+\log k}{2\sqrt{k}}$$
$$f(k,\Delta)=\frac{\pi\sqrt{\Delta}(\coth(\sqrt{2}\pi\sqrt{k\Delta})-1)}{\sqrt{2}}+\frac{\log(1-e^{-2\sqrt{2}\pi\sqrt{k\Delta}})}{2\sqrt{k}} \\ +\frac{\pi\sqrt{\Delta}(\coth(\sqrt{2}\pi\sqrt{\frac{\Delta}{k}})-1)}{\sqrt{2}k}-\frac{\log(1-e^{-2\sqrt{2}\pi\sqrt{\frac{\Delta}{k}}})}{2\sqrt{k}}. \tag{D.31}$$

Note that the equations (D.30) are indeed equivalent to derivatives (with respect to k, evaluated at $k=1$) of (D.15) at $c=1$.

D.3.2 $c = 2$ reconsidered

The $c = 2$ free boson is labeled by a metric and B field, which gives four real moduli in total. These can be repackaged into two elements of the upper half plane as [188]:

$$\rho = B + i\sqrt{\det G}, \quad \sigma = \frac{G_{12}}{G_{11}} + i\frac{\sqrt{\det G}}{G_{11}}. \tag{D.32}$$

T-duality acts as two independent elements of $SL(2,\mathbb{Z})$ acting on ρ and σ in the usual way. In terms of these coordinates, the spectral decomposition of the reduced $c = 2$ partition function is:

$$\begin{aligned}\widehat{Z}^{c=2}(\tau,\rho,\sigma) = \widehat{E_1}(\tau) + \widehat{E_1}(\rho) + \widehat{E_1}(\sigma) - \frac{3}{\pi}\left(4 - \gamma_E - 3\log(4\pi) - 48\zeta'(-1)\right)\\ + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \frac{\tilde{E}_s(\tau)\tilde{E}_s(\rho)\tilde{E}_s(\sigma)}{\Lambda(s)\Lambda(1-s)}\\ + 8\sum_{n=1}^{\infty}\frac{\nu_n^+(\tau)\nu_n^+(\rho)\nu_n^+(\sigma)}{(\nu_n^+,\nu_n^+)} - 8i\sum_{n=1}^{\infty}\frac{\nu_n^-(\tau)\nu_n^-(\rho)\nu_n^-(\sigma)}{(\nu_n^-,\nu_n^-)}.\end{aligned} \tag{D.33}$$

(See e.g. Sec. 3.1.2 of [165] for derivation.) In (D.33), the function $\widehat{E_1}$ is defined as

$$\begin{aligned}\widehat{E_1}(\tau) &\coloneqq \lim_{s\to 1} E_s(\tau) - \frac{3/\pi}{s-1}\\ &= y - \frac{3}{\pi}\log y + \frac{6}{\pi}\left(1 - 12\zeta'(-1) - \log 4\pi\right) + \sum_{j=1}^{\infty}\frac{12\sigma_1(j)e^{-2\pi j y}\cos(2\pi j x)}{j}.\end{aligned} \tag{D.34}$$

Let us derive the scalar crossing equation at $c = 2$. We first assume the theory is a Narain CFT. As usual let us denote the set of scalar operators under the $U(1)^2$ chiral algebra excluding the vacuum, as $\mathcal{S}$. (Of course, $\mathcal{S}$ depends on the moduli of the theory, which for $c = 2$ we denote by ρ, σ, but we will suppress that.) The partition function of these scalars is given by

$$\begin{aligned}y\left(1 + \sum_{\Delta\in\mathcal{S}} e^{-2\pi\Delta y}\right) &\coloneqq \int_{-1/2}^{1/2} dx \widehat{Z}^{c=2}(\tau,\rho,\sigma)\\ &= y - \frac{3}{\pi}\log y + \widehat{E_1}(\rho) + \widehat{E_1}(\sigma) + \frac{3}{\pi}\left(-2 + 24\zeta'(-1) + \gamma_E + \log 4\pi\right)\\ &\quad + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \frac{(\Lambda(s)y^s + \Lambda(1-s)y^{1-s})\tilde{E}_s(\rho)\tilde{E}_s(\sigma)}{\Lambda(s)\Lambda(1-s)}\\ &= y - \frac{3}{\pi}\log y + \widehat{E_1}(\rho) + \widehat{E_1}(\sigma) + \frac{3}{\pi}\left(-2 + 24\zeta'(-1) + \gamma_E + \log 4\pi\right)\\ &\quad + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \frac{y^s\tilde{E}_s(\rho)\tilde{E}_s(\sigma)}{\Lambda(1-s)}.\end{aligned} \tag{D.35}$$

Let us move the contour in s to the right past all the poles. The function $\tilde{E}_s$ has simple poles at $s = 0, 1$ (which can be seen from (D.17)). Moreover, $\Lambda(1-s) = \pi^{\frac{1}{2}-s}\Gamma(s-\frac{1}{2})\zeta(2s-1)$ has zeros whenever $2s-1$ is a nontrivial zero of the Riemann zeta function. Thus the integrand has simple poles that we cross at $s = 1, \frac{1+z_n}{2}, \frac{1+z_n^*}{2}$, where z_n is a nontrivial zero of the Riemann zeta function (with positive imaginary part).[1] A picture of the pole structure is given in Fig. 5.1 (where we move the pole at $s = \frac{c}{2}$ to $s = 1$).

We then get the equation:

$$y\left(1+\sum_{\Delta\in\mathcal{S}} e^{-2\pi\Delta y}\right) = -\frac{3}{\pi}\log y + \frac{3}{\pi}\left(-2+24\zeta'(-1)+\gamma_E+\log 4\pi\right) + \widehat{E_1}(\rho) + \widehat{E_1}(\sigma)$$
$$+\sum_{k=1}^{\infty}\mathrm{Re}\left(\frac{4\pi^{\frac{z_k}{2}}\Lambda(\frac{1+z_k}{2})^2 E_{\frac{1+z_k}{2}}(\rho)E_{\frac{1+z_k}{2}}(\sigma)}{2\Gamma\left(\frac{z_k}{2}\right)\zeta'(z_k)}y^{\frac{1+z_k}{2}}\right)$$
$$+\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\frac{y^s\tilde{E}_s(\rho)\tilde{E}_s(\sigma)}{\Lambda(1-s)}, \tag{D.36}$$

where $\gamma > \frac{c}{2} = 1$. This integral is a special case of the one studied in (5.37), which can be done exactly to give us:

$$1+\sum_{\Delta\in\mathcal{S}} e^{-2\pi\Delta y} = -\frac{3}{\pi}\frac{\log y}{y} + \frac{\frac{3}{\pi}\left(-2+24\zeta'(-1)+\gamma_E+\log 4\pi\right)+\widehat{E_1}(\rho)+\widehat{E_1}(\sigma)}{y}$$
$$+\sum_{k=1}^{\infty}\mathrm{Re}\left(\frac{4\pi^{\frac{z_k}{2}}\Lambda(\frac{1+z_k}{2})^2 E_{\frac{1+z_k}{2}}(\rho)E_{\frac{1+z_k}{2}}(\sigma)}{2\Gamma\left(\frac{z_k}{2}\right)\zeta'(z_k)}y^{\frac{-1+z_k}{2}}\right)$$
$$+\frac{1}{y\sqrt{\pi}}\sum_{\Delta\in\mathcal{S}}\sum_{n=1}^{\infty} b(n)U\left(-\frac{1}{2},1,\frac{2\pi\Delta n^2}{y}\right)e^{-\frac{2\pi\Delta n^2}{y}}. \tag{D.37}$$

The sum over k in (D.37) falls off exponentially in k so the sum is indeed convergent.

The generalization to any $U(1)^2$ CFT at $c = 2$ is straightforward. We again need to subtract $\widehat{E_1}(\tau)$ to render the reduced partition function square-integrable, and a gap to the first excited state constrains the poles we cross in s to only be at $s = 1, \frac{1+z_n}{2}, \frac{1+z_n^*}{2}$ (see Appendix D.1). Finally, the same arguments as in Sec 5.3.2 let us compute the

[1] The double pole at $s = 1$ in the numerator of the integrand becomes a simple pole when canceled by the simple pole at $s = 1$ in the denominator. There is also a pole at $s = 0$, but since we move the contour to the right we can ignore it.

non-perturbative corrections at high temperature to get:

$$1 + \sum_{\Delta \in \mathcal{S}} e^{-2\pi\Delta y} = -\frac{3}{\pi}\frac{\log y}{y} + \frac{\varepsilon_{c=2}(\mu)}{y} + \sum_{k=1}^{\infty} \mathrm{Re}\left(\delta_{k,c=2}\, y^{\frac{-1+z_k}{2}}\right) + \frac{1}{y\sqrt{\pi}} \sum_{\Delta \in \mathcal{S}} \sum_{n=1}^{\infty} b(n) U\left(-\frac{1}{2}, 1, \frac{2\pi\Delta n^2}{y}\right) e^{-\frac{2\pi\Delta n^2}{y}}. \tag{D.38}$$