A CONTRIBUTION TO THE THEORY OF THERMIONIC VACUUM TUBES

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Nicholas Anthony Begovich

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ABSTRACT

The elementary treatment of electron streams in vacuum tubes is based on the assumption that the velocity of the stream can be represented by a single valued function of the spatial coordinate. The Lagrangian method of treating the resulting electronic equations was introduced by Muller and extended by Llewellyn to general boundary conditions. In part III, this method is carried to the second order solution, which is important in the computation of distortion and detection properties of vacuum tubes at medium frequencies.

With the refinement of electronic techniques in the past decade, the useful radio frequency range for communication has been increased. In treating very high frequency tubes, the assumption of single valued velocity electron streams in close spaced vacuum tubes diverges sharply from physical fact. There arises a need for an electronic theory which includes the velocity spread of the electron stream. In part IV, the foundation of a multi-velocity theory is laid. Though in part IV treatment has been confined to one dimensional electron motion with only an electric field, the method can be readily extended to two or three dimensional flow with magnetic fields. The multi-velocity theory is based on a combination of Maxwell's equations and Liouville's Theorem of classical statistical mechanics. This fundamental approach in treating vacuum tubes, by focusing attention on the electron motion instead of boundary parameters, has been bypassed by prior investigators. The theory contains within its structure the explanation of all previously obtained results on one dimensional electron flow plus new answers to multi-velocity problems. In part V, some examples of stationary electron flow are treated; and in part VI, the time dependent solutions are formulated (though not carried out in complete detail).

In part VII, an interesting high frequency loading phenomena, observed by a number of investigators, is treated. From the results of the stationary flow in part V, this problem can be solved without the general theory developed in part VI. The solution obtained suggests a modification in the construction of very high frequency close spaced vacuum tubes.

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Part I

INTRODUCTION

Statement of Problem

Two infinite parallel planes are separated a distance d in a vacuum to form a generalized* diode (Fig. 1). Plane a is at a potential V_a; plane b is at a potential V_b. A beam of electrons is injected across the a plane into the region d. What is the relationship between the current flowing in the external circuit C and the potentials V_a and V_b?

Assumptions

The following assumptions will be made in the course of the solution.

1. The planes a and b are at uniform potentials, so that the electric field is everywhere perpendicular to the planes. Consequently, if these planes are grids, they must be made of very fine wire closely meshed, so that the electric field will be uniform over the entire grid.

2. Electron motion is always perpendicular to the planes a and b; i.e., the problem is one dimensional. The only spatial variable that will appear is the x coordinate which will be measured from the a plane.

3. The angular frequency of the time dependent part of the potentials is of such magnitude that propagation time for the electric

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*Neither plane is to be regarded as constituting a thermionic emitter or cathode.

field and potential from planes a to b is instantaneous.

4. The forces acting on the electrons will be produced by electric fields. No externally applied magnetic fields will exist. The magnetic field produced by electron convection current and displacement current can be neglected because of the small current density used.

5. The potentials V_a and V_b are of such magnitude that relativistic influences can be neglected.

Once the solution for a generalized diode is obtained the behavior of multi-element tubes (triodes, tetrodes, and etc.) is obtained by cascading diode* solutions.[1] In particular, the solution of the above problem is desired to explain the behavior of microwave disk-seal vacuum tubes. This class of microwave tubes, the best known members being called "lighthouse tubes", are of planar structure.[2] The only assumptions given above that would not be strongly satisfied by these tubes are 1 and 2. Some of the earlier model "lighthouse tubes" have very coarse grids so that the fields are far from uniform.[3] However, more recent tubes built at the M.I.T. Radiation Laboratory, in particular the Neher amplifier tubes,[4] satisfy the first two assumptions closely.

Classification of Solution

The solution of the general problem can be classified according to the velocity composition of the electron stream, namely, single valued velocity electron streams or multi-velocitied electron streams. For a single valued velocity stream all the electrons in a

*The first diode represents the cathode-grid space, the second diode the grid-screen grid space, and etc. to the last grid-anode region.

plane parallel to the a or b plane have the same momentary velocity.* A further solution specification would be the time behavior of the potentials. More detailed classification will become apparent in the sections that follow where these two types of electron streams are treated.

Maxwell's Equations and Units

The relationship between the fields and currents acting in the generalized diode is given by Maxwell's equations. The equations will be written in cgs - practical units. In this unit system, electrical quantities are measured in practical units, volts, amperes, coulombs, and ohms; while length, mass, and time are measured in centimeters, grams, and seconds.

$$\begin{aligned} \nabla \times \underline{H} &= \underline{Q} + \epsilon \frac{\partial \underline{E}}{\partial t} \\ \nabla \times \underline{E} &= -\mu \frac{\partial \underline{H}}{\partial t} \\ \nabla \cdot \epsilon \underline{E} &= \rho \\ \nabla \cdot \mu \underline{H} &= 0 \end{aligned} \tag{1.1}$$

In the above equations, $\underline{H}$ is the magnetic field strength (amperes per centimeter), $\underline{E}$ the electric field strength (volts per centimeter), $\underline{Q}$ the convection current density (amperes per square centimeter), ρ the charge density (coulombs per cubic centimeter), ϵ the permittivity of vacuum (farads per centimeter), and μ the permeability of vacuum (henrys per centimeter). Because of the fourth assumption, terms involving the magnetic field may be cancelled. Because the problem is one dimensional, all quantities will have only x direction components.

*The velocity is a single valued function of x and t.

Consequently, the vector equations reduce to scalar equations. Writing only the remaining terms in (1.1) there results*

$$-I(t) = Q + \epsilon \frac{\partial E}{\partial t}$$

and

$$\epsilon \frac{\partial E}{\partial x} = \rho \qquad (1.2)$$

where I is the total current density (amperes per square centimeter). It is to be noted that I is not a function of the spatial coordinate (x), but can only be constant or a function of time. This can be shown by taking the divergence of the left side of the first equation of (1.1). The convection (Q) and displacement current $(\epsilon \frac{\partial E}{\partial t})$ are, however, functions of x and t. The force law, in the above system of units, for the electric field acting on an electron is

$$F = m_e \frac{d^2 x}{dt^2} = -10^7 e E \qquad (1.3)$$

where F is the force in coulomb-volts per centimeters, m_e is the electron mass in grams, and e is the absolute value of the electron charge. A table of the above quantities and others to be introduced in subsequent pages is given in appendix 1.

*The total current density is written with a negative sign to conform with the usual custom in measuring currents in the opposite direction of negative electron flow. This is also useful in that the negative sign associated with the electron charge is cancelled in some equations.

Part II

SINGLE VALUED VELOCITY SOLUTION

Introduction

In the next section, the generalized diode will be treated under the condition that all electrons in a plane parallel to planes a or b have the same velocity. This condition restricts the electrons motion from the left to the right plane (Fig. 1) with no electrons passing each other. The criterion that the electron stream must satisfy to have a single valued velocity will be stated on page10.

A review of the papers on this problem prior to 1938 is given by Benham.[5] Most of the earlier papers assumed that the a plane was a thermionic emitter and electrons were liberated with zero velocity. However, for practical vacuum tubes the condition of single valued velocity and zero velocity emission is not satisfied for a number of reasons. They are:

1. Physical emitters eject electrons with a velocity distribution.
2. Tubes are generally operated space charge control so a negative field exists at the cathode with an accompanying potential minimum a short distance from the cathode (compare Part IV).
3. Because of the potential minimum, the major portion of the electrons emitted is returned to the cathode. Consequently, in the region between the cathode and the potential minimum, electrons are traveling away from and toward the emitter.

4. For very high frequency* tubes close element spacing must be used because of the effects of transit time. Consequently, the distance between the cathode and the potential minimum can be a major portion of, if not all, the distance between the cathode and the first grid. In addition, the velocity distribution will give different transit times for different electrons.

The above conditions practically void the single valued velocity solution when applied at very high frequencies to explain the behavior of the cathode-control grid region of tubes. An example of this divergence between experimental results for disk-seal micro-wave tubes and single valued velocity theory has been given in a recent paper.[(6)] The solution can be applied to medium frequency** tube analysis, however, since wider tube spacings are used and the potential minimum distance occupies then only a small portion of the cathode-control grid region.[(7)] In the output regions of tubes where the electron velocities are high (so that the emitter thermal velocity spread can be neglected), the single valued solution can be applied with some success even to microwave tubes. As was previously mentioned, if the diode is treated with general boundary conditions for the two planes, then the solution can be used for multi-element tubes by cascading. This possibility was first realized by Llewellyn,[(8)(9)] who extended Benham's[(10)] original diode solution first and then later Muller's[(11)] solution. The procedure in these solutions is to consider the potentials, currents, and other quantities as being composed of

*In the order of 10^9 cycles per second or higher.
**Less than 10^8 cps.

time independent (d.c.) plus time dependent (a.c.) components. The a.c. components are considered as small compared to the d.c. components, so that a.c. behavior is a perturbation on the d.c. solution. The resultant theory is called a "small signal theory".

Four different ways of solving the electronic equations have been developed. These methods are known by the form in which the differential equations are placed prior to solution or by the physical principle used in their formulation. They are known as the Eulerian,*[10] Lagrangian,*[11] electrostatic,[12] and the conservation of charge.[5] Llewellyn's most important paper[14] on small signal theory, which was later extended to a booklet,[15] is the generalization of the boundary conditions used by Muller.[11] He formulated the equations so that not only could the d.c. and small signal a.c. solution be obtained, but also higher order solutions. Unfortunately, Llewellyn has left out an important higher order term, so a closed form expression (circular functions) can not be obtained for a second order solution. In the section to follow, Muller's method of setting up the electronic equations will be used with Llewellyn's generalization of the boundary conditions. After the general theory has been formulated, Llewellyn's solution will be briefly given for the small signal case, namely, the zero and first order solution. Next, the as yet unpublished second order solution in closed form will be carried out in detail.** It must be re-emphasized that application of the single valued velocity theory

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*This designation comes from the analogous form of the equations to the hydrodynamic equations.[15]

**Benham[5] has carried out a second order solution by the conservation of charge method. However, his unorthodox use of complex notation reflects some apprehension on his results.

to practical vacuum tubes must always be done with utmost care, so that the basic assumptions of this theory are not violated.[6]

Single Valued Velocity Theory

The general theory of single valued velocity electron streams starts with (1.1) and (1.2). In (1.1), ρv is written for the convection current, where v is the velocity of the electrons in the x direction. In (1.2), -eE is written for the force F on a electron; and e, m_e, and 10^{-7} are written as the constant $\ell = 10^7 e/m_e$.*

$$-I(t) = \rho v + \epsilon \frac{\partial E}{\partial t} \tag{2.1}$$

$$-\ell E = \frac{d^2 x}{dt^2} = \frac{dv}{dt} = a \tag{2.2}$$

Using (1.2), in (2.1) can be written in terms of the electric field, then

$$-I(t) = \epsilon \left[\frac{\partial E}{\partial x} v + \frac{\partial E}{\partial t}\right] \tag{2.3}$$

The bracket term on the right side of the above equation is the total derivative of E, since $v = dx/dt$ for single valued velocity streams. If the stream was not single valued then for a given x, the velocity of different electrons would give different v's in the above expression; so identifying the right side of (2.3) as the total derivative of the electric field could not be done. Consequently for single valued streams

$$-I(t) = \epsilon \frac{dE}{dt} \tag{2.4}$$

Using (2.2), the electric field can be written in terms of the electron acceleration as

$$\frac{\ell}{\epsilon} I = \frac{da}{dt} \tag{2.5}$$

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*Wherever e appears it will mean the absolute magnitude of the electron charge. The negative sign of the charge will be written into the equations.

This is the fundamental equation that must be solved. The right side is the total rate of change of acceleration that an electron will experience as it moves from the a to the b plane. Note that the left side of this equation can only be a function of t or a constant. This suggests defining

$$\frac{\ell}{\epsilon} I = K + \phi'''(t) \qquad ,^* \tag{2.6}$$

where $K = \sum_{n=0}^{\infty} K_{2n}$ and $\phi'''(t) = \sum_{n=1}^{\infty} J_n(t)$

K is the sum of the even order independent currents multiplied by ℓ/ϵ; while $\phi'''(t)$ ** is the sum of the time dependent currents multiplied by ℓ/ϵ. The higher order contribution to K is to be noted since even harmonics of the fundamental frequency current result in a change in the d.c. Current. These higher order d.c. contribution to K are not included in Llewellyn's analysis. They obviously do not affect the zero or first order results, but will have pronounced influences on the higher order solutions. Writing (2.6) in (2.5) gives

$$\frac{da}{dt} = K + \phi'''(t) \tag{2.7}$$

Multiplying this equation by dt and integrating $t = t_a$ to $t = t$, where when $t = t_a$ the electron is at the a plane, result in

$$\int_{t_a}^{t} \frac{da}{dt} dt = a(t) - a(t_a) = \int_{t_a}^{t} [K + \phi'''(t)] dt = (t - t_a)K + \phi''(t) - \phi''(t_a) \tag{2.8}$$

The acceleration at the a plane, $a(t_a)$ can be written as the sum of a time independent plus a time dependent acceleration.

$$a(t_a) = a_a + \alpha(t_a)$$

Substituting this expression in (2.8) results in

---

* $\ell/\epsilon = 2.00 \cdot 10^{28}\, cm/sec^2$

** $\phi'''(t) = d^3\phi/dt^3$

$$a(t,t_a) = (t-t_a)K + \phi''(t) - \phi''(t_a) + a_a + \alpha(t_a) \tag{2.9}$$

Integrating again over the same limits gives

$$v(t,t_a) = \tfrac{1}{2}(t-t_a)^2K + \phi'(t) - \phi'(t_a) - (t-t_a)\phi''(t_a) + (t-t_a)a_a + (t-t_a)\alpha(t_a) + v_a + \nu(t_a) \tag{2.10}$$

where v_a is the velocity at $t = t_a$ and is not a function of t_a, while $\nu(t_a)$ is the time dependent velocity at the a plane. A third integration gives

$$x(t,t_a) = \tfrac{1}{6}(t-t_a)^3K + \phi(t) - \phi(t_a) - (t-t_a)\phi'(t_a) - \tfrac{1}{2}(t-t_a)^2\phi''(t_a) + \tfrac{1}{2}(t-t_a)^2a_a + \tfrac{1}{2}(t-t_a)^2\alpha(t_a) + (t-t_a)v_a + (t-t_a)\nu(t_a) \tag{2.11}$$

where when $t = t_a$, $x = 0$.

The single valued velocity criterion can be obtained from (2.11). The initial conditions at plane a must not be assigned so that electrons overtake each other anywhere between the two planes a and b. If electron passing occurs, then the velocity would not be a single valued function of x and the above derivation would be void. Muller(11) and recently Brillouin(16) have shown that the necessary and sufficient condition of single valued velocity is

$$\left(\frac{\partial x}{\partial t_a}\right)_t < 0 \tag{2.12}$$

Physically this condition means that electrons crossing the a plane at a given time cannot overtake electrons that have crossed the a plane at some previous time. A plot of x versus t for different t_a's readily shows the above requirement. If the time independent parts of v and a are large compared with the fluctuating components, then the above condition is usually satisfied. However, if there is any region in $0 \leq x \leq d$ where the electrons are being decelerated (retarding electric field), then the criterion must be carefully investigated.

Equations (2.9), (2.10), and (2.11) give the acceleration, velocity, and position at any time $t > t_a$ of the electrons that cross the a plane at $t = t_a$. However, since the electric field, the potential, and other quantities as a function of x and t are desired, the expressions (2.9) and (2.10) are not useful in their present form. In principle, (2.11) could be solved for t_a in terms of x and t and then substituted in (2.9) and (2.10). The formitable appearance of (2.11), with the additional complication that the time dependent functions for a steady state solution will be trigometric, precludes all possibilities of this direct approach. A method of perturbation on the time independent solution can be used.

If all the fluctuating components in (2.11) are zero and setting $t - t_a = T$, then

$$x = \frac{1}{6} T^3 K_0 + \frac{1}{2} T^2 a_a + T v_a \tag{2.13}$$

Solving this cubic equation for T and substituting in (2.9) and (2.10), the acceleration and velocity as a function of x and T would be obtained, if the fluctuating components are zero. When the fluctuating components are not zero, then let

$$t - t_a = T + \delta \tag{2.14}$$

where T is still defined by (2.13). δ is then the perturbation on the transit time T produced by the fluctuating quantities ϕ, α, ν; and as they approach zero, δ also approaches zero. Using (2.14), then any function of $(t-t_a)$ or t_a can be expanded in a Taylor series in terms of powers of δ. For a function of $t-t_a$,

$$f(t-t_a) = f(T+\delta) = f(T) + f'(T)\delta + \cdots = \sum_{n=0}^{\infty} \frac{\delta^n}{n!} f^{(n)}(T) \tag{2.15}$$

In a like manner, for a function of t_a

$$f(t_a) = f(t-T-\delta) = f(t-T) - f'(t-T)\delta + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta^n f^{(n)}(t-T) \quad (2.16)$$

Substituting (2.13) to (2.16) inclusive in (2.11) results in an expression involving t, T, δ, K, ϕ, and the acceleration and velocity at the a plane.

$$\begin{aligned} x = & \tfrac{1}{6} K\left[T^3 + 3T^2\delta + 3T\delta^2 + \delta^3\right] + \tfrac{1}{2} a_a\left[T^2 + 2T\delta + \delta^2\right] \\ & + v_a(\delta + T) + \phi(t) - \left[\phi(t-T) - \phi'(t-T)\delta + \tfrac{1}{2!}\phi''(t-T)\delta^2 - \right. \\ & \left. \cdots\right] - (T+\delta)\left[\phi'(t-T) - \phi''(t-T)\delta + \tfrac{1}{2!}\phi'''(t-T)\delta^2 - \right. \\ & \left. \cdots\right] - \tfrac{1}{2}(T^2 + 2T\delta + \delta^2)\left\{\left[\phi''(t-T) - \phi'''(t-T)\delta + \right.\right. \\ & \left. \tfrac{1}{2!}\phi''''(t-T)\delta^2 - \cdots\right] - \left[\alpha(t-T) - \alpha'(t-T)\delta + \tfrac{1}{2!}\alpha''(t-T)\delta^2 \right. \\ & \left.\left. \cdots\right]\right\} + (T+\delta)\left[\nu(t-T) - \nu'(t-T)\delta + \tfrac{1}{2!}\nu''(t-T)\delta^2 - \cdots\right] \end{aligned} \quad (2.17)$$

Though (2.17) can be arranged in a power series in δ, it can not be solved for δ as it stands. The important result of (2.17) is that δ, the perturbating transit time, has been removed from the fluctuating quantities ϕ, α, and ν. To solve (2.17), let δ, K, ϕ, α, and ν each be written as a series.*

$$\delta = \sum_{n=1}^{\infty} \delta_n \ , \quad K = K_0 + \sum_{n=1}^{\infty} K_{2n} \ , \quad \phi = \sum_{n=1}^{\infty} \phi_n \ , \quad \alpha = \sum_{n=1}^{\infty} \alpha_n \ , \quad \nu = \sum_{n=1}^{\infty} \nu_n \qquad (2.18)$$

After these quantities are substituted in (2.17), the resultant expression is resolved into groups such that the sum of the subscripts of each term of any particular group is the same.

---

*Compare (2.6).

$$
\begin{aligned}
0 = \Big\{ & \delta_1\left(\tfrac{1}{2}K_0 T^2 + a_a T + v_a\right) + \phi_1(t) - \phi_1(t-T) - T\phi_1'(t-T) \\
& - \tfrac{T}{2}\phi_1''(t-T) + \tfrac{1}{2}T^2\alpha_1(t-T) + T\nu_1(t-T)\Big\} + \Big\{\delta_2\left(\tfrac{1}{2}K_0 T^2\right. \\
& \left. + a_a T + v_a\right) + \delta_1^2\,\tfrac{1}{2}(K_0 T + a_a) + \tfrac{1}{6}K_2 T^3 + \delta_1\Big[\tfrac{1}{2}T^2\phi_1'''(t-T) \\
& - \tfrac{1}{2}T^2\alpha_1'(t-T) + T\alpha_1(t-T) - T\nu_1'(t-T) + \nu_1(t-T)\Big] + \phi_2(t) \\
& - \phi_2(t-T) - T\phi_2'(t-T) - \tfrac{1}{2}T^2\phi_2''(t-T) + \tfrac{1}{2}T^2\alpha_2(t-T) \\
& + T\nu_2(t-T)\Big\} + \Big\{\phi_3(t) - \phi_3(t-T) - T\phi_3'(t-T) \qquad (2.19) \\
& - \tfrac{1}{2}T^2\phi_3''(t-T) + \tfrac{1}{2}T^2\alpha_3(t-T) + T\nu_3(t-T) + \delta_3\left(\tfrac{1}{2}K_0 T^2 + \right. \\
& \left. T a_a + v_a\right) + \delta_2\Big[\tfrac{1}{2}T^2\phi_1'''(t-T) - \tfrac{1}{2}T^2\alpha_1'(t-T) + T\alpha_1(t-T) \\
& - T\nu_1'(t-T) + \nu_1(t-T)\Big] + \delta_1\Big[\tfrac{1}{2}K_2 T^2 + \tfrac{1}{2}T^2\phi_2'''(t-T) \\
& - \tfrac{1}{2}T^2\alpha_2'(t-T) + T\alpha_2(t-T) - T\nu_2'(t-T) + \nu_2(t-T)\Big] + \\
& \delta_1^2\Big[-\tfrac{1}{4}T^2\phi_1''''(t-T) + \tfrac{1}{2}T\phi_1'''(t-T) + \tfrac{1}{4}T^2\alpha_1''(t-T) - T\alpha_1'(t-T) \\
& + \tfrac{1}{2}\alpha_1(t-T) + \tfrac{1}{2}T\nu_1''(t-T) - \nu_1'(t-T)\Big] + \delta_1\delta_2\left(K_0 T + \right. \\
& \left. a_a\right) + \tfrac{1}{6}\delta_1^3 K_0\Big\} + \Big\{\cdots
\end{aligned}
$$

Since the left side of (2.19) is zero, then for the right side to be zero for all values of t, each group of terms on the right side of the same order must be zero. Equating the first group to zero and solving for δ_1 result in

$$
\delta_1 = -\frac{\left[\phi_1(t) - \phi_1(t-T) - T\phi_1'(t-T) - \tfrac{1}{2}T^2\phi_1''(t-T) + \tfrac{1}{2}T^2\alpha_1(t-T) + T\nu_1(t-T)\right]}{\tfrac{1}{2}K_0 T^2 + T a_a + v_a} \qquad (2.20)
$$

The second group gives

$$\delta_2 = - \frac{\begin{aligned} &\Big\{ \phi_2(t) - \phi_2(t-T) - T\phi_2'(t-T) - \tfrac{1}{2}T^2\phi_2''(t-T) + \tfrac{1}{2}T^2\alpha_2(t-T) \\ &+ T\nu_2(t-T) + \delta_1\big[\tfrac{1}{2}T^2\phi_1'''(t-T) - \tfrac{1}{2}T^2\alpha_1'(t-T) \\ &+ T\alpha_1(t-T) - T\nu_1'(t-T) + \nu_1(t-T)\big] + \tfrac{1}{2}\delta_1^2\big[K_0T + \\ &a_a\big] + \tfrac{1}{6}K_2T^3 \Big\} \end{aligned}}{\tfrac{1}{2}K_0T^2 + Ta_a + V_a} \tag{2.21}$$

The third group gives

$$\delta_3 = - \frac{\begin{aligned} &\Big\{ \phi_3(t) - \phi_3(t-T) - T\phi_3'(t-T) - \tfrac{1}{2}T^2\phi_3''(t-T) + \tfrac{1}{2}T^2\alpha_3(t-T) \\ &+ T\nu_3(t-T) + \delta_2\delta_1\big[K_0T + a_a\big] + \tfrac{1}{2}\delta_1K_2T^2 + \tfrac{1}{6}\delta_1^3K_0 \\ &+ \delta_1^2\big[-\tfrac{1}{4}T^2\phi_1''''(t-T) + \tfrac{1}{2}T\phi_1'''(t-T) + \tfrac{1}{4}T\alpha_1''(t-T) - T\alpha_1'(t-T) \\ &+ \tfrac{1}{2}\alpha_1(t-T) + \tfrac{1}{2}T\nu_1''(t-T) - \nu_1'(t-T)\big] + \delta_2\big[\tfrac{1}{2}T^2\phi_1'''(t-T) \\ &- \tfrac{1}{2}T^2\alpha_1'(t-T) + T\alpha_1(t-T) - T\nu_1'(t-T) + \nu_1(t-T)\big] \\ &+ \delta_1\big[\tfrac{1}{2}T^2\{\phi_2'''(t-T) - \alpha_2'(t-T)\} + T\{\alpha_2(t-T) - \nu_2'(t-T)\} + \nu_2(t-T)\big]\Big\} \end{aligned}}{\tfrac{1}{2}K_0T^2 + Ta_a + V_a} \tag{2.22}$$

The above expressions with (2.18), (2.16), (2.15), and (2.14) can now be used to express the acceleration (2.9) and the velocity (2.10) as a function of x and t, since T is a function only of x (2.13). Resolving the acceleration (2.9) in a series and combining terms in the manner used in (2.19) result in

$$a = a_0 + a_1 + a_2 + a_3 + \cdots \tag{2.23}$$

where*

$$a_0 = K_0 T + a_a \tag{2.24}$$

$$a_1 = K_0 \delta_1 + \phi_1''(t) - \phi_1''(t-T) + \alpha_1(t-T) \tag{2.25}$$

$$a_2 = K_0 \delta_2 + K_2 T + \phi_2''(t) - \phi_2''(t-T) + \delta_1 \phi_1'''(t-T) - \delta_1 \alpha_1'(t-T) + \alpha_2(t-T) \tag{2.26}$$

$$\begin{aligned} a_3 = {} & K_0 \delta_3 + \phi_3''(t) - \phi_3''(t-T) + \alpha_3(t-T) + \delta_2 \{ \phi_1'''(t-T) - \alpha_1'(t-T) \} \\ & + K_2 \delta_1 - \tfrac{1}{2!} \delta_1^2 \{ \phi_1''''(t-T) - \alpha_1''(t-T) \} + \delta_1 \{ \phi_2'''(t-T) - \alpha_2'(t-T) \} \end{aligned} \tag{2.27}$$

Using the same procedure for the velocity (2.10) results in

$$v = v_0 + v_1 + v_2 + v_3 + \cdots \tag{2.28}$$

where*

$$v_0 = \tfrac{1}{2} K_0 T^2 + T a_a + v_a \tag{2.29}$$

$$v_1 = \{ \delta_1 (K_0 T + a_a) - T\phi_1''(t-T) + \phi_1'(t) - \phi_1'(t-T) + T\alpha_1(t-T) + \nu_1(t-T) \} \tag{2.30}$$

$$\begin{aligned} v_2 = {} & \{ \delta_2 (K_0 T + a_a) - T\phi_2''(t-T) - \phi_2'(t-T) + \phi_2'(t) + T\alpha_2(t-T) + \nu_2(t-T) \\ & + \tfrac{1}{2} K_2 T^2 + \tfrac{1}{2} K_0 \delta_1^2 + \delta_1 [T\phi_1'''(t-T) + \alpha_1(t-T) - T\alpha_1'(t-T) - \nu'(t-T) \} \end{aligned} \tag{2.31}$$

$$\begin{aligned} v_3 = {} & \{ \delta_3 (K_0 T + a_a) - T\phi_3''(t-T) - \phi_3'(t-T) + \phi_3'(t) + T\alpha_3(t-T) + \nu_3(t-T) \\ & + \delta_2 [T\phi_1'''(t-T) - T\alpha_1'(t-T) + \alpha_1(t-T) - \nu_1'(t-T)] + K_0 \delta_1 \delta_2 \\ & + \delta_1^2 [\tfrac{1}{2}\phi_1'''(t-T) - \tfrac{1}{2} T\phi_1''''(t-T) - \alpha_1'(t-T) + \tfrac{1}{2}\nu_1''(t-T) + \tfrac{1}{2} T\alpha_1''(t-T)] \\ & + \delta_1 [K_2 T + T\phi_2'''(t-T) + \alpha_2(t-T) - T\alpha_2'(t-T) - \nu_2'(t-T)] \} \end{aligned} \tag{2.32}$$

After the δ's are substituted in the above expressions, the acceleration and velocity as a function of x and t and the initial conditions

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*In Llewellyn's[17] expressions for the above quantities, the K_2 terms are omitted.

at plane a will be obtained. Since there is only one spatial variable, the electric field can be written as the negative of the partial derivative of the potential,

$$E(x,t) = -\left.\frac{\partial V(x,t)}{\partial x}\right|_t . \tag{2.33}$$

Integrating from the a to b plane at a constant value of t,

$$-\int_a^b \left.\frac{\partial V}{\partial x}\right|_t dx = V_a - V_b = \int_a^b E(x,t)\,dx\Big|_t . \tag{2.34}$$

From (2.2),

$$E(x,t) = -\frac{1}{\ell}\, a(x,t)$$

Since (2.23) expresses the acceleration in terms of T and t, (2.13) can be used to change the integration variable in (2.34) and x to T. From (2.13) and (2.29)

$$dx = \left(\tfrac{1}{2}K_0 T^2 + T a_a + v_a\right) dT = v_0\, dT \tag{2.35}$$

Substituting in (2.34) and writing W for ℓV give

$$W_b - W_a = \int_0^T a\, v_0\, dT \tag{2.36}$$

Using (2.23), (2.36) can be written as

$$(W_b - W_a)_0 = \int_0^T a_0 v_0\, dT \tag{2.37}$$

$$(W_b - W_a)_1 = \int_0^T a_1 v_0\, dT \tag{2.38}$$

$$(W_b - W_a)_2 = \int_0^T a_2 v_0\, dT \tag{2.39}$$

$$(W_b - W_a)_3 = \int_0^T a_3 v_0\, dT \tag{2.40}$$

In evaluating these integrals, the proper δ's must be substituted in the accelerations before the integrations are performed.

Integrals (2.37) to (2.40) formally complete the single valued velocity solution up to the third order. Substituting particular functions for the current density (2.6) and the initial conditions at plane a, integrals (2.37) to (2.40) give the relation between the total current density and the potential difference between planes a and b.

Zero Order Solution

Many papers have been written on the time independent solution for a parallel plate diode.[18][19][20] Consequently, in this section only the expressions which will be of use for the higher order solutions will be given. Substituting (2.24) and (2.29) in (2.37) gives

$$(W_b - W_a)_0 = \frac{1}{8} K_0^2 T^4 + \frac{1}{2} K_0 a_0 T^3 + \frac{1}{2} a_a^2 T^2 + \frac{1}{2} v_a K_0 T^2 + a_a v_a T \qquad (2.41)$$

Note, the right side of this expression in terms of the velocity at plane b is by the use of (2.29)

$$\frac{1}{2}(v_b^2 - v_a) = (W_b - W_a), \qquad (2.42)$$

in terms of potentials V_a and V_b

$$e 10^7 (V_b - V_a) = \frac{1}{2} m_e (v_b^2 - v_a^2), \qquad (2.43)$$

which is the energy equation for d.c. electron flow. When V_a is zero (a_a and v_a then also zero), using (2.13) to solve for x in terms of T, substituting in (2.41), and using (2.6) give the familiar expression for Child's Law, i.e.,

$$I_0 = 2.33 \cdot 10^{-6} d^{-2} V_{b0}^{3/2} \quad \text{amps/cm}^2 \qquad (2.44)$$

where the numeric is $\frac{4}{9} \epsilon (2\ell)^{\frac{1}{2}}$

Another important equation is the value for the maximum current density that can be injected across the a plane for a given V_a and V_b. This expression may be called the generalized Child's Law. Using

(2.13) in (2.29) to eliminate a_a results in

$$V_b + V_a = \frac{1}{6} K_o T^2 + 2\frac{d}{T} \tag{2.45}$$

For a constant value of V_a and V_b, the left side is a constant; since from (2.43), the potential and velocity are related by

$$V_0 = \frac{1}{2}\frac{m_e}{e} 10^{-7} v_o^2 = \frac{1}{2\ell} v_o^2 \tag{2.46}$$

Differentiating the right side of (2.45) with d constant and solving for dK/dT result in

$$\frac{dK}{dT} = T^{-4} 12d - T^{-1} 2K_o \tag{2.47}$$

Setting the left side zero gives

$$K_{o_{max.}} = 6dT^{-3} = \frac{\ell}{\epsilon} I_{o_{max.}} \tag{2.48}$$

This is the value for the maximum injected current in terms of d and T. Substituting (2.48) in (2.45) gives

$$T_{I_{o_{max.}}} = 3d(V_a + V_b)^{-1} \tag{2.49}$$

When the convection current density is very small, space charge density (ρ) approaches zero, and a linear potential exists between the a and b plane, then

$$T_{linear} = 2d(V_a + V_b)^{-1} = \frac{2}{3} T_{I_{o_{max.}}} \tag{2.50}$$

Consequently, the transit time for complete space charge is increased by fifty percent over the transit time for zero space charge. Solving (2.48) for $T_{max.}$, substituting in (2.49), and using (2.46) give

$$I_{o_{max.}} = 2.33 \times 10^{-6} d^{-2} (V_{bo}^{1/2} + V_{ao}^{1/2})^3 \tag{2.51}$$

For this value of $I_{o_{max.}}$, there will exist a potential minimum between the a and b planes. Setting a_o equal to zero in (2.24), (since the field at the potential minimum is zero), gives the transit time to the potential minimum

$$T_{pot.\,min.} = -a_a (K_{o_{max}})^{-1} \tag{2.52}$$

Substitute (2.48) in (2.13) with x set equal to d, solving for a_a, and substituting T from (2.49) give

$$a_a = -2 v_a T^{-1} = -\frac{2}{3} v_a (v_a + v_b) d^{-1} \tag{2.53}$$

This expression shows that a retarding field exists at the a plane. Substituting (2.53) in (2.52) and using the resultant value of $T_{pot.min.}$ in (2.29) give the electron velocity at the potential minimum as

$$v_{pot.min.} = v_b \left(1 + \frac{v_b}{v_a}\right)^{-1}. \tag{2.54}$$

Using (2.46), this can be written in terms of the potentials

$$\left(V_{o\,pot.min.}/V_{ao}\right) = \left(V_{bo}/V_{ao}\right)\left[1 + \left(\frac{V_{bo}}{V_{ao}}\right)^{\frac{1}{2}}\right]^{-2}. \tag{2.55}$$

This expression then relates the potential minimum and the a and b plane voltages for the maximum current density given by (2.51). Substituting (2.52) in (2.13), using (2.48), (2.50), and (2.53), the distance to the potential minimum for the maximum current density (2.51) is

$$\frac{x_{pot.min}}{d} = \left(V_a/V_b\right)^3 \left(3 + V_a/V_b\right)\left[1 + \left(V_a/V_b\right)^3\right]^{-3} \tag{2.56}$$

or in terms of the potentials of the a and b planes

$$\frac{x_{pot.min.}}{d} = \left(V_a/V_b\right)\left(3 + \left\{V_a/V_b\right\}^{\frac{1}{2}}\right)\left(1 + \left\{V_a/V_b\right\}^{3/2}\right)^3. \tag{2.57}$$

In writing the first order solution, it is convenient to introduce a space charge factor

$$\zeta = 3\left(1 - \frac{T_{linear}}{T}\right) \tag{2.58}$$

Equation (2.50) shows that ζ equals one for complete space charge, maximum injected current density (2.51), and zero for zero injected current density. Using (2.50) in (2.58) to eliminate T_{linear} and substituting in (2.45) give

$$K = 2\zeta(V_a + V_b)T^{-2} \tag{2.59}$$

Dividing (2.59) by itself, where ζ is set equal to one, results in

$$K_o/K_{o_{max.}} = I_o/I_{o_{max.}} = \zeta(T_{max.}/T)^2 \tag{2.60}$$

Eliminating the transit time ratio by (2.58) gives

$$I_o/I_{o_{max.}} = \frac{9}{4}\,\zeta(1-\frac{\zeta}{3})^2 \quad * \tag{2.61}$$

For a given value of current density I, (2.61) gives the space charge factor; and using (2.58) and (2.50), the transit time is

$$T = 2(3-\zeta)^{-1}(6\epsilon d/eI_{o_{max}})^{\frac{1}{3}} = 2.88\times10^{-9}(3-\zeta)^{-1}(d/I_{o_{max}})^{\frac{1}{3}} \tag{2.62}$$

This equation is particularly useful in calculating the transit time.

First Order Solution

Llewellyn in his earlier papers(14)(15) and in a recent paper(1) has thoroughly investigated the first order solution. In this section, a brief description of the procedure and the final results will be given as the first order solution is required for the second order calculations.

Equation (2.37) shows that the first order potential and first order current density are linearly related;** and consequently, the principle of superposition holds for first order quantities of different frequencies. The linearity between current density and potential makes it possible to use the convenience of complex notation for time dependent quantities. To first order quantities only

*Equation (2.61) is plotted in appendix 2.

**Note, however, that the total potential and current density are not linearly related.

(2.6) gives

$$\frac{\ell}{\epsilon} I = K_o + \phi_1'''(t) \tag{2.63}$$

Let

$$\phi_1'''(t) = J_1 e^{pt} = \frac{\ell}{\epsilon} I_1 e^{pt} \quad * \tag{2.64}$$

then

$$\phi_1^{(n)}(t) = [J_1/p^{(3-n)}]e^{pt} \quad \text{and} \quad \phi_1^{(n)}(t-T) = [J_1/p^{(3-n)}]e^{p(t-T)} \tag{2.65}$$

Using the expression given by (2.65) in δ_1 (2.20), substituting in a_1 (2.25), multiplying by v_0 (2.29), substituting in (2.37), and integrating give the first order potential. The terms in the potential expression involve the zero and first order current densities, accelerations, and velocities at the a plane. A more convenient parameter than the time dependent acceleration α_1 at the a plane is the convection current density. Multiplying (2.1) by ℓ/ϵ, using (2.6) for the left side of the expression, expanding the convection current velocity by (2.28) and the charge density (ρ) in an analogous manner, and using (2.2) for the electric field in the displacement current density term give

$$-[K_o + J_1 + J_2 + \cdots] = \frac{\ell}{\epsilon}[(\rho_0+\rho_1+\cdots)(v_0+v_1+\cdots)] - \frac{\partial}{\partial t}[a_0+a_1+\cdots] \tag{2.66}$$

The first order terms of (2.66) are

$$-J_1 = \frac{\ell}{\epsilon} q_1 - \frac{\partial a_1}{\partial t} \tag{2.67}$$

where

$$q_1 = \rho_0 v_1 + \rho_1 v_0 \tag{2.67a}$$

All terms of (2.67) must have the same time dependence. Then since J's time dependence is specified by (2.64), (2.67) becomes (when e^{pt} is suppressed since it appears in all terms)

---

*$p = j\omega$, where $j = (-1)^{\frac{1}{2}}$ and ω is the angular frequency.

$$pa_1 = \frac{\ell}{\epsilon} q_1 + J_1 \tag{2.68}$$

At the **a** plane, $a_1 = \alpha_1$ so (2.68) becomes

$$\alpha_1 = \left(\frac{\ell}{\epsilon} q_a + J_1\right) \cdot p^{-1} \tag{2.69}$$

where q_a is the first order convection current at the **a** plane. Substituting (2.69) for α_1, in the potential integral mentioned above, and letting $\beta = pT$ give the first equation of (2.70). Substituting (2.69) in (2.25) and using (2.20) and (2.65) give the second equation in (2.70). Repeating the same operations on (2.30) gives the third equation in (2.70).

$$\begin{aligned} V_b - V_a &= \underline{A} I + \underline{B} q_a + \underline{C} v_a \\ q_b &= \underline{D} I + \underline{E} q_a + \underline{F} v_a \\ v_b &= \underline{G} I + \underline{H} q_a + \underline{I} v_a \end{aligned} \tag{2.70}$$

Equations (2.70) are known as the first order electronics equations.* The potential ($V_b - V_a$), current density (I), convection current density (q), and velocity (v) in (2.70) are all first order quantities of frequency ω .** The subscripts a and b are the respective first order values at the **a** and **b** planes. The bars under the coefficients in (2.70) indicate that the quantity may be complex. Because of the principle of superposition applies to first order quantities, (2.70) relates the various quantities for each different frequency. For example, if an injected current of one frequency and a

*The values of the coefficients are given in appendix 3.

**Since the time dependence is the same for all terms, the quantity e^{pt} is suppressed.

different frequency applied voltage ($V_a - V_b$) were placed across the a and b planes, then the resulting voltage produced by the injected current and the resulting current produced by the applied voltage would be independent as far as first order effects are concerned.

Part III

SECOND ORDER SINGLE VALUED VELOCITY DIODE SOLUTION

Introduction

In the sections to follow, the second order solution for a complete space charge diode, $V_a = 0$, will be obtained. The zero and first order potential, acceleration, and velocity at the a plane will be set equal to zero. The importance of the second order diode solution is in the computation of distortion and detection properties at medium frequencies.* In fact, transit time effects in diodes at moderately high frequencies were first observed in the behavior of rectification efficiency.[21]

Zero Transit Angle Second Order Solution

At very low frequencies where the transit angle of electrons is practically zero,** the relation between voltage and current density is given by (2.44), namely,

$$I_0 = c V_{b0}^{3/2} \quad amps/cm^2 \tag{3.00}$$

or

$$V_{b0} = (I_0/c)^{2/3} \quad volts \tag{3.00a}$$

where $c = 2.33 \times 10^{-6} d^{-2}$

If there is a small increment in the current density so that

$$I = I_0 + \Delta I \tag{3.01}$$

where $\Delta I / I_0 \ll 1$

---

*Note material that follows assumption 4 on page 2.

**The transit angle (θ) is ω times the transit time, $\theta = \omega T$. See appendix 3.

then the corresponding change in the voltage can be obtained by a Taylor series expansion for $(I_0+\Delta I)$ as

$$V = f(I_0+\Delta I) = f(I_0) + f'(I_0)\Delta I + \frac{1}{2!}f''(I_0)(\Delta I)^2 + \cdots \qquad (3.02)$$

The coefficients of this expansion are obtained by differentiating

(3.00a) $\quad f(I_0) = (I_0/c)^{\frac{2}{3}}, \quad f'(I_0) = \frac{2}{3}(c^2 I_0)^{-\frac{1}{3}}, \quad f''(I_0) = -\frac{2}{9}(cI^2)^{-\frac{2}{3}}, \cdots$

Substituting in (3.02), factoring out $(I_0/c)^{\frac{2}{3}}$ from the terms on the right side, and using (3.00a) result in

$$V = V_0\left[1 + \frac{2}{3}\left(\frac{\Delta I}{I_0}\right) - \frac{1}{9}\left(\frac{\Delta I}{I_0}\right)^2 + \frac{4}{81}\left(\frac{\Delta I}{I_0}\right)^3 + \cdots\right] \qquad (3.03)$$

Expanding the left side of (3.03) as

$$V = V_0 + V_1 + V_2 + V_3 + \cdots \qquad (3.04)$$

the following definitions can be made.

$$V_1 = \frac{2}{3}\left(\frac{\Delta I}{I_0}\right)V_0 = r_{0c}\Delta I, \quad V_2 = -\frac{1}{9}\left(\frac{\Delta I}{I_0}\right)^2 V_0, \quad V_3 = \frac{4}{81}\left(\frac{\Delta I}{I_0}\right)^3 V_0 \qquad (3.05)$$

The factor $\frac{2}{3}(V_0/I_0)$ in the V_1 expression is the inverse slope of the static characteristic of a diode operating with complete space charge, and is called the zero-frequency value of the diode resistance, and can be written r_{0c} .* Compare appendix 3 and (2.70). If ΔI in (3.01) is written as $I_1\cos\omega t$,** then (3.05) becomes

$$V_1 = r_{0c} I_1\cos\omega t, \quad V_2 = -\frac{1}{9}\left(\frac{I_1}{I_0}\right)^2 V_0\cos^2\omega t = -\frac{1}{18}\left(\frac{I_1}{I_0}\right)^2 V_0[1+\cos 2\omega t]$$

$$V_3 = \frac{4}{81}\left(\frac{I_1}{I_0}\right)^3 V_0\cos^3\omega t = \frac{1}{81}\left(\frac{I_1}{I_0}\right)^3[3\cos\omega t + \cos 3\omega t] \qquad (3.06)$$

Note that the second order voltage V_2 has a time independent component.

*The subscripts are for zero order (0), complete space charge diode (c), respectively.

**Note I_1 is the maximum value or $2^{\frac{1}{2}}$ times the rms value. In all equations to follow fluctuating components will also be written in terms of maximum values.

Equation (3.06) shows that to maintain a zero and a small* first order current density of a single frequency through the diode, voltage of a fundamental and higher harmonic frequency must be applied. In addition, if the d.c. value of the current density is to remain constant as I_1 increases in magnitude, a negative second order voltage also must be added. If the frequency is such that the transit angle is not zero, then the right side of each equation of (3.06) will be multiplied by a function of θ, the transit angle, or a function of β , the complex transit angle $(j\theta)$.

$$
\begin{aligned}
V_1 &= r_{oc}\, \underline{\Upsilon}_1(\beta)\, I_1 \cos\omega t \\
V_2 &= -\frac{1}{18}\left(\frac{I_1}{I_0}\right)^2 V_0 \left[\Upsilon_{20}(\theta) + \underline{\Upsilon}_{22}(\beta)\cos 2\omega t\right] \qquad (3.07) \\
V_3 &= \frac{1}{81}\left(\frac{I_1}{I_0}\right)^3 V_0 \left[3\,\underline{\Upsilon}_{31}(\beta)\cos\omega t + \underline{\Upsilon}_{33}(\beta)\cos 3\omega t\right] \; {**}
\end{aligned}
$$

The factor $r_{oc}\, \underline{\Upsilon}_1(\beta)$ is given in appendix 3. $\Upsilon_{20}(\theta)$ and $\underline{\Upsilon}_{22}(\beta)$ will be calculated in the sections that follow. The bar under $\underline{\Upsilon}_{22}(\beta)$ indicates that it may be complex; i.e., the second and first order voltages do not pass through zero simultaneously. The subscripts on the Υ_{nm} and Ψ_{nm} (next paragraph) functions are n for the order and m for the frequency dependence. For example, Υ_{20} indicates second order, zero frequency transit angle coefficient, while $\underline{\Upsilon}_{22}$ indicates second order, twice fundamental frequency coefficient.

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*If the first order current is not small compared to I_0, then more terms in the expansion (3.03) will be needed.

** $\underline{\Upsilon}_{nm}(\beta) \underset{\beta\to 0}{\to} 1$ and $\Upsilon_{2n,m}(\theta) \underset{\theta\to 0}{\to} 1$, for any n,m.

Using (3.00), an expansion for the case

$$V = V_0 + \Delta V \qquad \text{where} \qquad \Delta V / V_0 \ll 1 \tag{3.08}$$

can be performed in an analogous manner to that given above for an increment in the current density. The current density components that will result from the voltage (3.08) will be

$$I = I_0 \left[1 + \tfrac{3}{2}\left(\tfrac{\Delta V}{V_0}\right) + \tfrac{3}{8}\left(\tfrac{\Delta V}{V_0}\right)^2 - \tfrac{1}{48}\left(\tfrac{\Delta V}{V_0}\right)^3 + \cdots \right] \tag{3.09}$$

Expanding the left side of (3.09) as

$$I = I_0 + I_1 + I_2 + I_3 + \cdots \tag{3.10}$$

the following definitions can be made*

$$I_1 = \tfrac{3}{2}\left(\tfrac{\Delta V}{V_0}\right) I_0 \,, \quad I_2 = \tfrac{3}{8}\left(\tfrac{\Delta V}{V_0}\right)^2 I_0 \,, \quad I_3 = -\tfrac{1}{48}\left(\tfrac{\Delta V}{V_0}\right)^3 I_0 \tag{3.11}$$

I_2 can be written in terms of I, by eliminating the voltage ratio giving

$$I_2 / I_0 = \tfrac{1}{6}\left(I_1 / I_0 \right)^2 \tag{3.12}$$

If ΔV, in (3.08) is written as $V_1 \cos \omega t$, then (3.12) becomes

$$I_2 / I_0 = \tfrac{1}{6}\left(I_1 / I_0 \right)^2 \cos^2 \omega t = \tfrac{1}{12}\left(I_1 / I_0 \right)^2 \left[1 + \cos 2\omega t \right] \tag{3.13}$$

When the transit angle for the electrons can not be neglected then (3.13) can be written as

$$I_2 = \tfrac{1}{12}\left(I_1^2 / I_0 \right)\left[\Psi_{20}(\theta) + \underline{\Psi}_{22}(\theta) \cos 2\omega t \right] \tag{3.14}$$

where $\Psi_{20}(\theta)$ and $\underline{\Psi}_{22}$ ** equal one, when θ is zero. The values of Ψ will be calculated in the sections that follow.

Finite Transit Angle Second Order Diode Solution

Equation (2.39) gives the relationship between the second order voltage and current density. Since the second order solution

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*Van der Pol has carried out this expansion to terms of the tenth order.[22]

**Bar indicates $\underline{\Psi}_{22}$ is complex.

is to be carried out for a complete space charge diode with $V_a = 0$,* all quantities at the a plane must be set equal to zero. Consequently, the second order acceleration a_2 (2.26) and the δ's that are substituted in it will be those shown in (2.20) and (2.21), where α, ν, a_a, and v_a are set equal to zero. Multiplying the resultant a_2 by v_0 (2.29), where a_a and v_a in v_0 are set equal to zero and substituting in (2.39), the potential integral can be written as

$$W_{b2} = \int_0^T \Gamma_2 dT + \int_0^T \Gamma_1 dT + \frac{1}{3} K_0 K_2 \int_0^T T^3 dT \tag{3.15}$$

where

$$\int_0^T \Gamma_2 dT = K_0 \int_0^T [\phi_2(t-T) - \phi_2(t) + T\phi_2'(t-T) + \tfrac{1}{2} T^2 \phi_2''(t)] dT \tag{3.16}$$

and

$$\begin{aligned}\int_0^T \Gamma_1 dT = & -2\int_0^T \frac{1}{T^3}[\phi_1(t) - \phi_1(t-T)]^2 dT \\ & +4\int_0^T \frac{1}{T^2}\phi_1'(t-T)[\phi_1(t) - \phi_1'(t-T)] dT \\ & -2\int_0^T \frac{1}{T}[\{\phi_1'(t-T)\}^2 - \phi_1''(t-T)\{\phi_1(t) - \phi_1(t-T)\}] dT \\ & -\int_0^T \phi_1''(t-T)\{2\phi_1'(t-T) + \tfrac{1}{2} T\phi_1''(t-T)\} dT\end{aligned} \tag{3.17}$$

Note, the Γ_2 integral is composed of terms involving the second order current, while the Γ_1 integral has only the first order current.

As was shown in the zero transit angle second order solution, there are two distinct cases that can be treated in a second order solution. They are: A. Total current density composed of a small, single frequency, time dependent current (I_1) superimposed on a steady current (I_0). The problem is to determine the voltage that must be applied to the diode to satisfy this current

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*$W_a = 0$

density. B. Voltage impressed on diode is composed of a small, single frequency, time dependent voltage V_1, superimposed on a steady voltage V_0. The problem is to determine the currents that flow as a result of this impressed voltage.*

Problem A

The answer to problem A, when the transit angle for the electrons is zero, is given by (3.00a) and (3.06). When the transit angle is finite, (3.15) must be used, with Γ_2 and K_2 set equal to zero, since the current density is composed only of I_0 and I_1. Equation (3.15) will also contain the zero transit solution (3.06). To evaluate (3.17), let $\phi_1'''(t) = J_1 \cos wt$ so that**

$$
\begin{aligned}
\phi_1''(t) &= \frac{J_1}{w} \sin wt & \qquad \phi_1'''(t-T) &= J_1 \cos w(t-T) \\
\phi_1'(t) &= \frac{J_1}{w^2} \cos wt & \qquad \phi_1''(t-T) &= \frac{J_1}{w} \sin w(t-T) \\
\phi_1(t) &= \frac{J_1}{w^3} \sin wt & \qquad \phi_1'(t-T) &= -\frac{J_1}{w^2} \cos w(t-T) \\
& & \qquad \phi_1(t-T) &= -\frac{J_1}{w^3} \sin w(t-T)
\end{aligned}
\tag{3.18}
$$

Substituting the proper expressions of (3.18) in (3.17) and

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*Problem B seems to be of more experimental interest than Problem A. However, Dr. J.R. Pierce of the Bell Telephone Laboratories has informed the author that an experiment under the conditions of problem A has been performed. The divergence between the experimental results and Benham's published results on the frequency dependence of the d.c. second order voltage developed across the diode was the motivation for carrying out this general second order solution.

**The convenience of complex notation can not be readily used in the second order solution since the first order current terms appear as squares.

integrating give

$$\int_0^T \Gamma_1 dT = J_1^2 \Big\{ \frac{1}{\omega^6 T^2}\Big[1 - \tfrac{1}{2}\cos 2\omega t - \tfrac{1}{2}\cos 2\omega t \cos 2\omega T - \tfrac{1}{2}\sin 2\omega t \sin 2\omega T$$
$$+ \sin 2\omega t \sin \omega T + \cos 2\omega t \cos \omega T - \cos \omega T\Big]$$
$$+ \frac{1}{\omega^5 T}\Big[\sin 2\omega t \cos 2\omega T - \cos 2\omega t \sin 2\omega T$$
$$- \sin \omega T + \cos 2\omega t \sin \omega T - \sin 2\omega t \cos \omega T\Big] \qquad (3.19)$$
$$+ \frac{9}{16\omega^4}\Big[\sin 2\omega t \sin 2\omega T + \cos 2\omega t \cos 2\omega T\Big] +$$
$$\frac{T}{8\omega^3}\Big[\cos 2\omega t \sin 2\omega T - \sin 2\omega t \cos 2\omega T\Big] - \frac{1}{8}\frac{T^2}{\omega^2}\Big\}\Bigg|_0^T$$

The value of the integral at the upper limit is just the expression given on the right side of (3.19). However, in determining the value of the integral at the lower limit, T = 0 can not be directly substituted because of the inverse power of T. Before substitution is made, all terms involving T are expanded in a power series of T. Letting T in these expressions go to zero leaves only the terms

$$\frac{1}{2\omega^4}\Big[1 - \frac{1}{8}\cos 2\omega t\Big] \qquad (3.20)$$

Substituting in (3.15), writing $\theta = \omega T$ (transit angle), and dividing both sides by (2.41) (in which a_a and v_a have been set equal to zero), give

$$W_{b2}/W_{bo} = V_{b2}/V_{bo} = -\frac{1}{18}\Big(\frac{I_1}{I_o}\Big)^2\Big[\Upsilon_{20}(\theta) + X(\theta)\cos 2\omega t + Y(\theta)\sin 2\omega t\Big] \qquad (3.21)$$

where

$$\Upsilon_{20}(\theta) = -144\,\theta^{-6}\Big[1 - \cos\theta - \theta\sin\theta + \tfrac{1}{2}(\theta^2 - \tfrac{1}{4}\theta^4)\Big] \qquad (3.22)$$

$$X(\theta) = 72\,\theta^{-6}\Big[1 - 2\cos\theta + \cos 2\theta - 2\theta\sin\theta + 2\theta\sin 2\theta$$
$$- \tfrac{9}{8}\theta^2\cos 2\theta + \tfrac{1}{8}\theta^2 - \tfrac{1}{4}\theta^3\sin 2\theta\Big] \qquad (3.23)$$

$$Y(\theta) = -144\,\theta^{-6}\Big[\sin\theta - \tfrac{1}{2}\sin 2\theta - \theta\cos\theta + \theta\cos 2\theta$$
$$+ \tfrac{9}{16}\theta^2\sin 2\theta - \tfrac{1}{8}\theta^3\cos 2\theta\Big] \qquad (3.24)$$

For small transit angles, (3.22), (3.23), and (3.24) can be expanded in powers of θ giving *

$$\Upsilon_{20}(\theta) = \left[1 - \frac{1}{40}\theta^2 + \frac{1}{2800}\theta^4 - \frac{1}{302400}\theta^6 + \frac{1}{46569600}\theta^8 \cdots\right] \tag{3.25}$$

$$X(\theta) = \left[1 - \frac{31}{40}\theta^2 + \frac{351}{2800}\theta^4 - \frac{563}{60480}\theta^6 + \frac{18943}{46569600}\theta^8 \cdots\right] \tag{3.26}$$

$$Y(\theta) = \left[\frac{6}{5}\theta - \frac{37}{105}\theta^3 + \frac{341}{9240}\theta^5 - \frac{16}{17325}\theta^7 \cdots\right] \tag{3.27}$$

Using the complex transit angle $\beta = j\theta$, (3.26) and (3.27) can be combined so that (3.21) can be written as**

$$\frac{V_{b2}}{V_{bo}} = -\frac{1}{18}\left(\frac{I_1}{I_o}\right)^2\left[\Upsilon_{20}(\theta) + \Upsilon_{22}(\beta)\cos 2\omega t\right] \qquad *** \tag{3.28}$$

where

$$X(\theta) - jY(\theta) = \Upsilon_{22}(\beta) = 144\beta^{-3}\left[\frac{1}{2}\beta^{-3}(1 - 2e^{-\beta} + e^{-2\beta}) + \beta^{-2}(e^{-2\beta} - e^{-\beta}) + \frac{1}{16}\beta^{-1}(9e^{-2\beta} - 1) + \frac{1}{8}e^{-2\beta}\right] \tag{3.29}$$

When β is small,

$$\Upsilon_{22}(\beta) = \left[1 - \frac{6}{5}\beta + \frac{31}{40}\beta^2 - \frac{37}{105}\beta^3 + \frac{351}{2800}\beta^4 - \frac{341}{9240}\beta^5 + \frac{563}{60,480}\beta^6 - \frac{16}{17325}\beta^7 + \frac{18943}{46569600}\beta^8 \cdots\right] \tag{3.30}$$

With the writing of (3.21) to (3.29), problem A**** of the second order solution is completed. The procedure in using these answers for a particular problem is as follows. The given data would be the spacing d, current densities I_o and I_1, and the frequency ω of I_1.

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*Note that (3.21) gives (3.06) when $T \to 0$.

**$(a \cos 2\omega t + b \sin 2\omega t) = (a - jb)\cos 2\omega t$. Compare (401.2).[23]

***Compare (3.07).

****See page 29 .

V_{bo} for the particular I_o would be given by (3.00a). The transit angle required for the first and second order solution would be calculated by (2.62), which simplifies for complete space charge to

$$\theta = \omega T = 144 \cdot 10^{-9} \omega \left(\frac{d}{I_o}\right)^{\frac{1}{2}} \tag{2.62a}$$

The required $(V_b)_1$ would be given by the first order electronic equations (2.70);* $(V_b)_2$ would be obtained from (3.22) and (3.29), or for small transit angles** (3.25), (3.26), and (3.27). Figure 3 shows a plot of (3.22) and (3.29) for $0 \doteq \theta \doteq 13$.

Problem B

The answer to problem B*** when the transit angle for the electrons is zero is given by (3.00) and (3.13). When the transit angle is finite, (3.15) must be used with W_{b2} set equal to zero,

$$0 = \int_0^T \Gamma_2 \, dT + \int_0^T \Gamma_1 \, dT + \frac{1}{12} K_0 K_2 T^4 \tag{3.31}$$

or

$$\int_0^T \Gamma_2 \, dT + \frac{1}{12} K_0 K_2 T^4 = -\int_0^T \Gamma_1 \, dT \tag{3.31a}$$

For a first order current density $\phi_1'''(t) = J_1 \cos \omega t$, the right side of (3.31a) is given by (3.19) and (3.20), or in terms of $\Upsilon_{20}(\theta)$ and $\Upsilon_{22}(\beta)$,

$$\int_0^T \Gamma_1 \, dT = -\frac{1}{144} J_1^2 T^4 [\Upsilon_{20}(\theta) + \Upsilon_{22}(\beta) \cos 2\omega t]. \tag{3.32}$$

Using (3.16), the Γ_2 integral can be calculated after substituting $\phi_2'''(t) = J_2 \cos 2\omega t$. However, integration is unnecessary since the value of (3.16) has already been indirectly calculated. Note that the Γ_2

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*See appendix 2a.
** $\theta < 2.5$
***See page 29.

integral and the first order potential integral (2.37) with a_a and v_a set equal to zero are identical with the exception of the subscripts.* Consequently, the value of (3.16) can be written immediately from the coefficient $\underline{\Upsilon}_1$ ** with (2θ) substituted everywhere for θ

$$\int_0^T \Gamma_2 dT = \frac{1}{12} K_0 J_2 T^4 \underline{\Upsilon}(2\beta) \cos 2\omega t \qquad (3.33)$$

where $\underline{\Upsilon}_1(2\beta) = \frac{3}{2}\beta^{-4}[1 - \beta + \frac{2}{3}\beta^3 - c^{-2\beta}(\beta+1)]$

or $$\underline{\Upsilon}_1(2\beta) = \frac{3}{2}\theta^{-4}[(1 - \cos 2\theta - \theta \sin 2\theta) - j(\theta + \frac{2}{3}\theta^3 - \sin 2\theta + \theta \cos 2\theta)] \qquad (3.34)$$

Substituting (3.33) and (3.32) in (3.31a) and using (2.6) give

$$I_{20} = \frac{1}{12}\left(\frac{I_1^2}{I_0}\right)\Upsilon_{20}(\theta) \quad \text{and} \quad I_{22}(t) = \frac{1}{12}\left(\frac{I_1^2}{I_0}\right)\frac{\underline{\Upsilon}_{22}}{\underline{\Upsilon}_1}\cos 2\omega t \qquad (3.35)$$

or $$I_2 = I_{20} + I_{22}(t) = \frac{1}{12}\left(\frac{I_1^2}{I_0}\right)\left[\Upsilon_{20}(\theta) + \frac{\underline{\Upsilon}_{22}(\beta)}{\underline{\Upsilon}_1(2\beta)}\cos 2\omega t\right] \qquad (3.36)$$

Comparing with (3.14) gives

$$\Psi_{20}(\theta) = \Upsilon_{20}(\theta) \quad \text{and} \quad \underline{\Psi}_{22}(\beta) = \frac{\underline{\Upsilon}_{22}(\beta)}{\underline{\Upsilon}_1(2\beta)}$$

Equation (3.36) can be written in terms of the applied potentials** V_0 and V_1*** as

$$I_2 = \frac{3}{16}\left(\frac{V_1}{V_0}\right)^2 I_0 \left(|\underline{\Upsilon}(\beta)|^{-2}\right)\left[\Upsilon_{20}(\theta) + \frac{\underline{\Upsilon}_{22}(\beta)}{\underline{\Upsilon}_1(2\beta)}\cos 2\omega t\right] \qquad (3.37)$$

where $|\underline{\Upsilon}_1(\beta)|$ is the absolute value of $\underline{\Upsilon}_1(\beta)$. $\Upsilon_{20}(\theta)$ and $\underline{\Upsilon}_{22}(\beta)$ are given by (3.22) and (3.29) respectively. For small transit angles (3.25) and (3.30) can be used. $\Upsilon_{20}(\theta)$ and $\underline{\Upsilon}_{22}(\beta)$ are also plotted in figure 3. Using $\underline{\Upsilon}_1(\beta)$ (appendix 2a) and $\underline{\Upsilon}_{22}(\beta)$ (Fig. 3), the ratio $\underline{\Upsilon}_{22}(\beta)/\underline{\Upsilon}_1(2\beta)$ is plotted in figure 4.

With the writing of (3.36) or (3.37), problem B of the second order solution is completed. The procedure in using these

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* $W_{b1} = K_0\int_0^T [\phi_1(t-T) - \phi_1(t) + T\phi_1'(t-T) + \frac{1}{2}T^2\phi_1''(t-T)]dT$

**See appendix 2a.

***Compare (3.11) and appendix 2a.

answers for a particular problem is as follows. The given data would be the spacing d, the potentials V_0 and V_1, and the frequency ω of V_1. I_0 for a particular V_0 would be given by (3.00), and (2.62a)* would be used to calculate the transit angle. The first order current I_1 would be obtained from the first order electronics equations (2.70).**The second order current I_2 would be given by (3.36).

The above answers (3.28) and (3.37) for the second order solutions are in complete agreement with the results of the conservation of charge method used by Benham.[5] Though the conservation of charge method seems to involve slightly less labor, the Lagrangian method illustrated here seems to be more systematic once the fundamental expressions for the δ's are obtained. The agreement between the two results for the completely different procedures gives assurance as to the correctness of the final results.

For higher order solutions, a similar but longer procedure than that shown above for the second order solution would be followed. The contributions of the higher expressions to the current density and voltage for the zero transit case can be obtained by continuing the expansions (3.06) and (3.11).[22] For the third order solution, (2.40) would be used with the δ_1, δ_2, and δ_3 substitutions. The amount of labor required in performing these higher order calculations is of such magnitude that a strong motivation to explain some accurate experimental data would be required.

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*Page 32.
**See appendix 2a.

Part IV

MULTI-VELOCITY ELECTRON STREAMS

Introduction

As was stated previously,* the cathode-control grid region behavior of microwave tubes cannot be deduced in general from a single valued velocity electron stream theory. An exception to this statement would be the case when a tube is operated almost temperature limited** so that the potential minimum is small in magnitude and very close to the cathode compared to the cathode-grid spacing. Because of cathode surface irregularities, operation close to temperature emitted conditions are undesirable since an accelerating field acting on any part of the cathode surface seriously reduces cathode life. In addition, by this mode of operation the beneficial space charge reduction of the shot noise in the emitted electron stream would be lost. A theory to explain the behavior of microwave tubes must then take into account multi-velocitied electron streams since with the existence of a potential minimum electrons are traveling away and toward the cathode. Since microwave tubes must have very close element spacing to reduce the electron transit time, the distribution in electron velocities would in itself cause the electron stream to behave differently than a single valued theory would predict since different velocity groups

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*See introduction of Part I.

**Temperature limited operation would occur when all the electrons emitted are drawn away so that none return to the cathode.

of the electron stream would have appreciably different transit times.

In the sections to follow, a multi-velocitied electron stream theory will be formulated. The method of approach is similar to that Richardson used many years ago in his investigation of emission of electrons from hot metallic surfaces.[25][26] More recently, Dr. F. Gray, Bell Telephone Laboratories, in an unpublished memorandum has made a similar approach to the problem. Though the theory to be stated is not complete in all details, the method of approach is believed to be pregnant for further investigation of multi-velocity streams. The particular emphasis of the theory is in the mechanical properties, density, momentum, and kinetic energy of the electrons and their interaction with the electric field. In the single valued velocity theory, the approach has been to obtain relationships between boundary quantities, as applied voltage, total current density, conduction current density and velocities at the <u>a</u> and <u>b</u> planes. The interaction between the electrons and the electric field in the intervening space between the boundary planes is placed in the background in the steps to solution. The multi-velocitied stream theory to be formulated will be given in terms of quantities representing the electron stream and electric field interaction. This procedure brings to the foreground the fundamental physics of the problem and may make possible general conclusions in problems where exact solutions would be difficult.

To evaluate the mechanical properties of the electron stream, the procedure used in classical statistical mechanics[27] of introducing a distribution function seems to be the logical approach. Consider a function D(x,v,t,)* such that D(x,v,t,)dv gives the number of electrons per unit volume in the velocity range v to v + dv at the spatial position x and at the time t.** Following Gibb's[27] notation, D(x,v,t,) will be called the density-in-phase.*** Multiplying the density-in-phase by the electron mass and integrating over all electron velocities at spatial position x give the total mass density (N) of the electron stream.

$$N(x,t) = m_e \int D(x,v,t)\,dv \qquad \text{grams/cm}^3 \qquad (4.01)$$

The momentum of the electrons per unit volume, the momentum density, at space position x and time t is

$$P(x,t) = m_e \int D(x,v,t)\,v\,dv \qquad \text{grams/cm}^2\text{/sec} \qquad (4.02)$$

The kinetic energy of the electrons per unit volume, the kinetic energy density, at space position x and time t is

$$K(x,t) = \frac{1}{2} m_e \int D(x,v,t)\,v^2 dv \qquad \text{grams/cm/sec}^2 \qquad (4.03)$$

Using (4.01), the charge density of the electron stream is****

$$\rho = -\frac{e}{m_e} N(x,t) = -e \int D(x,v,t)\,dv \qquad (4.04)$$

Using (4.02), the convection current density (Q) at space position x and time t is

$$Q(x,t) = -\frac{e}{m_e} P(x,t) \qquad \text{amp/cm}^2 \qquad (4.05)$$

Equation (4.05) can be used to define an effective velocity (U) for the stream; that is, multiplying U by the charge density ρ, (4.04)

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* $D(x,v,t,) = [T]\cdot[L]^{-4}$

**If positive ions are present, analogous quantities, as that shown for electrons above, can be introduced.

***In what is to follow, the density-in-phase will, however, be used in the Maxwell and Boltzman sense. The density in phase will represent the distribution of the electrons of the stream in a two dimensional configuration and velocity phase space.

****See footnote on page 8 .

gives the same value of Q as (4.05).

$$Q(x,t) = \rho U = -\frac{e}{m_e} U N(x,t) \tag{4.06}$$

From (4.05) and (4.06)

$$U = \left[P(x,t) / N(x,t) \right] \tag{4.07}$$

Using (1.2), the total current density, convection plus displacement current density, is

$$-I(t) = Q + \epsilon \frac{\partial E}{\partial t} = -\frac{e}{m_e} P(x,t) + \epsilon \frac{\partial E}{\partial t} \qquad \text{amps/cm}^2 \tag{4.08}$$

Using (4.04), Gauss' Law (1.2) can be written as

$$\epsilon \frac{\partial E}{\partial x} = \rho = -\frac{e}{m_e} N(x,t) \tag{4.09}$$

The relations that exist between P, N, and K can be determined in the following way. Consider a unit area parallel to the a and b planes of the space contained between the space positions x and $x + \Delta x$ (Fig. 5). The total mass of electrons contained in this volume in $N\Delta x$, which can change by having electrons flow into or out of this region by crossing the two boundaries. The electron mass flow into the region across the left border for the velocity packet v to $v + dv$ is $(m_e D dv)v$. The total electron mass flow across the left border is

$$m_e \int D v \, dv = P \tag{4.10}$$

The total mass flow across the right boundary out of the region is

$$-\left\{ P(x,t) + \frac{\partial P(x,t)}{\partial x} \Delta x \right\} \tag{4.11}$$

Consequently,

$$\frac{\partial}{\partial t}(N \Delta x) = P(x,t) - \left\{ P(x,t) + \frac{\partial P(x,t)}{\partial x} \Delta x \right\} \tag{4.12}$$

Since Δx is constant, it can be removed out of the time derivative on the left side of (4.12) and cancelled from both sides giving

$$\frac{\partial N(x,t)}{\partial t} + \frac{\partial P(x,t)}{\partial x} = 0 \qquad (4.13)$$

Equation (4.13) is the conservation of charge for multi-velocitied streams. It is not a new relation between N and P since it is implicit in (4.08). By differentiating (4.08) with respect to x and using (4.09), (4.13) follows. The procedure used above to derive (4.13) is more desirable since the mode of thinking will be useful in obtaining further relations.

A new relation between N, P, and K not previously written into the equations can be supplied by applying the conservation of momentum to the Δx region (Fig. 6). The momentum of the electrons contained in the Δx region is $P\Delta x$. The momentum contained in Δx can change by momentum flowing into or out of the region or by the electric field acting on the electrons contained in Δx. The momentum flowing into Δx across the left boundary for the velocity packet v to v + dv is $(m_e D v dv)v$. The total momentum crossing the left boundary is

$$m_e \int D v^2 dv = 2K \qquad (4.14)$$

The momentum flowing out of the Δx region is

$$2\left\{K(x,t) + \frac{\partial K(x,t)}{\partial x}\Delta x\right\} \qquad (4.15)$$

The electric force acting on the electrons in the Δx region is*

$$F = 10^7 E\rho\Delta x = -10^7 \frac{e}{m_e} EN\Delta x = -\ell EN\Delta x \qquad (4.16)$$

Using (4.14), (4.15), and (4.16), the change of momentum in the x region is

$$\frac{\partial}{\partial t}(P\Delta x) = 2K - 2\left\{K + \frac{\partial K}{\partial x}\Delta x\right\} - \ell EN\Delta x$$

*See (1.3) for force equation in practical-cgs units.

Since Δx is time independent,

$$\frac{\partial P}{\partial t} + 2\frac{\partial K}{\partial x} + \ell E N = 0 \tag{4.17}$$

The mass density (N) in (4.17) can be eliminated by use of (4.09) so that (4.17) becomes

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}\left[2K - \frac{1}{2}10^7 \epsilon E^2\right] = 0 \tag{4.18}$$

The electric field energy density in the cgs-practical unit system is $W_E = \frac{\epsilon}{2} 10^7 E^2$ ergs/cm^3, so (4.18) can be written as

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}\left[2K - W_E\right] = 0 \tag{4.19}$$

Equations (4.19) or (4.18) in combination with (4.08) are the multi-velocitied stream equations. As the density-in-phase is completely general, these expressions hold for any kind of electron velocity distribution including the limiting case when the stream becomes single valued.* The two independent equations (4.19) and (4.08) have three unknown P, K, and E. Since P and K are integrals of the density-in-phase D, they are immediately determined when the form of D and the limits of the integrals for P and K are known. Thus, P and K can be considered as one unknown and (4.08) and (4.19) for a complete set. The density-in-phase D satisfies the fundamental equation of classical statistical mechanics, namely, Liouville's Theorem. For matter of completeness, Liouville's Theorem will be derived since it will point out some fundamental properties of the density-in-phase.**

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*For the single valued velocity stream, the density-in-phase becomes the impulse function (Dirac delta function).

**In the customary proof of Liouville's Theorem, the equations of motion in the canonical Hamiltonian form are used with a momentum-configuration phase space. However, for the one dimensional flow under investigation here, this generality is not necessary.

The total differential of the density-in-phase is

$$\frac{dD(x,v,t)}{dt} = \frac{\partial D}{\partial t} + \frac{\partial D}{\partial v}\frac{dv}{dt} + \frac{\partial D}{\partial x}\frac{dx}{dt} \tag{4.20}$$

Using (2.2) and $v = dx/dt$, (4.20) can be written as

$$\frac{dD}{dt} = \frac{\partial D}{\partial t} - \ell E\frac{\partial D}{\partial v} + v\frac{\partial D}{\partial x} \tag{4.21}$$

To calculate $\frac{\partial D}{\partial t}$, consider the two dimensional phase space shown in figure 7. The total number of electrons contained in the elemental phase space $\Delta v \Delta x$ situated at v and x at time t is $D(x,v,t,)\Delta v \Delta x$. If the boundaries of the elemental phase space are kept constant, the variation of the number of electrons contained within its boundaries as a function of time will determine $\frac{\partial D}{\partial t}$.

The number of electrons contained in $\Delta v \Delta x$ can change by means of electrons crossing the four boundaries of the elemental phase space. The number of electrons crossing boundaries (1) and (3) is $(D\Delta v)\, dx/dt$ or $Dv\Delta v$, and $v(D + \frac{\partial D}{\partial x}\Delta x)\Delta v$ respectively. The net flow across these two boundaries is

$$\Delta v D v - v(D + \frac{\partial D}{\partial x}\Delta x)\Delta v = -v\frac{\partial D}{\partial x}\Delta x \Delta v \tag{4.22}$$

The net electron flow across boundaries (2) and (4) can be written with the aid of (2.2) as

$$D\frac{dv}{dt}\Delta x - (D + \frac{\partial D}{\partial v}\Delta v)\frac{dv}{dt} = -D\ell E\Delta x + \ell E(D + \frac{\partial D}{\partial v}\Delta v)\Delta x = \ell E\frac{\partial D}{\partial v}\Delta v \Delta x \tag{4.23}$$

The total flow across the border of the elemental phase space area $\Delta v \Delta x$ is

$$\frac{\partial}{\partial t}(D\Delta x \Delta v) = -v\frac{\partial D}{\partial x}\Delta x \Delta v + \ell E\frac{\partial D}{\partial v}\Delta v \Delta x \tag{4.24}$$

Since the boundaries are time constant, $\Delta x \Delta v$ can be removed outside the time differential giving

$$\frac{\partial D}{\partial t} = -v\frac{\partial D}{\partial x} + \ell E\frac{\partial D}{\partial v} \tag{4.25}$$

Substituting (4.25) in (4.21) gives

$$\frac{dD(x,v,t)}{dt} = 0 \qquad (4.26)$$

Equation (4.26) states that the total time rate of change of the density-in-phase vanishes. This expression is known as Liouville's Theorem and is of fundamental importance in treating multi-velocitied electron streams. Integrating (4.26) gives

$$D(x'',v'',t'') = D(x',v',t') \qquad (4.27)$$

The physical implication of (4.27) are as follows. Consider a group of electrons at time t', with a velocity range v' to v' + dv' and spatial position x' to x' + dx'. At some later time t">t' the electrons will have executed a motion such that they will occupy an elemental phase area at v" and x". Equation (4.27) states that the density of electrons at v" and x" will be the same as that at the prior position x' and v'.*

Another way of stating the physical implications of (4.26) has been shown by Gibbs,[28] and is called by him the "conservation of extension-in-phase". The number of electrons contained in a phase area is given by

$$\iint D(x,v,t)\,dx\,dv \qquad (4.28)$$

If the phase area is small, D can be regarded as a constant and moved outside the integral in (4.28).

$$D\iint dx\,dv \qquad (4.29)$$

As the electrons execute their motion, (4.29) remains constant since no electrons enter or leave the phase area, because the motion of the limits is identical with that of the electrons. However, (4.27) shows

*Because of this physical behavior, Gibb's refers to (4.27) as the "conservation of density-in-phase".

that D remains constant. Consequently, the integral

$$\iint dx\,dv \tag{4.30}$$

which Gibbs calls the "extension-in-phase", is also constant in time.*

Equations (4.08), (4.18), and (4.26) form the multi-velocitied stream equations. To summarize, they are repeated below.

$$-I(t) = -\frac{e}{m_e} P + \epsilon \frac{\partial E}{\partial t} \tag{4.08}$$

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}\left[2K - \frac{10^7}{2}\epsilon E^2\right] = 0 \tag{4.18}$$

$$\frac{dD}{dt} = 0 \tag{4.26}$$

where

$$P = m_e \int D v\,dv \tag{4.02}$$

and

$$K = \frac{m_e}{2}\int D v^2 dv \tag{4.03}$$

The similarity of the multi-velocity and the single valued velocity equations can be easily shown. Using (4.07), (4.08) can be written as

$$-I(t) = \rho U + \epsilon \frac{\partial E}{\partial t} \tag{4.31}$$

Using (4.07) again, the continuity equation (4.13) can be written as

$$-\frac{\partial}{\partial x}(UN) = \frac{\partial}{\partial t}\left(\frac{P}{U}\right) \tag{4.32}$$

Performing the indicated differentiations in (4.32) gives

$$\frac{\partial P}{\partial t} = \frac{P}{U}\frac{\partial U}{\partial t} - NU\frac{\partial U}{\partial x} - U^2\frac{\partial N}{\partial x} \tag{4.33}$$

Using (4.07) for the first term on the right side of the equal signs, adding, and subtracting $NU\frac{\partial U}{\partial x}$ give

$$\frac{\partial P}{\partial t} = N\frac{\partial U}{\partial t} + NU\frac{\partial U}{\partial x} - 2NU\frac{\partial U}{\partial x} - U^2\frac{\partial N}{\partial x} \tag{4.34}$$

The effective velocity U can be written as dx/dt reducing (4.34) to

*Gibb's definition is in terms of momentum and configuration coordinates. In the strict sense (4.30) should be multiplied by m_e before applying the designation of "extension-in-phase".

$$\frac{\partial P}{\partial t} = N\frac{dU}{dt} - \frac{\partial}{\partial x}(NU^2) \tag{4.35}$$

Substituting in (4.17) and dividing through by N give

$$\frac{dU}{dt} + \frac{2}{N}\frac{\partial}{\partial x}\left[K - \tfrac{1}{2}NU^2\right] = -\ell E \tag{4.36}$$

Equation (4.36) differs from the single-valued equation (2.2) only in the second term on the left side.* This term expresses the difference between the mean and the actual kinetic energy density of the stream. When the spread in velocity of the electrons in the stream becomes small, this term approaches zero and the behavior of the stream can be calculated neglecting the velocity spread.

Density-in-Phase Examples

Since the focal point of the multi-velocity theory is the density-in-phase, a number of examples of its behavior will be illustrated. For stationary fields,** the density-in-phase is only a function of the total energy (potential plus kinetic) of the electrons. This can be shown in a number of ways, but the most straight forward proof is obtained when the canonical Hamiltonian equations of motion are used. These equations for one dimension motion are

$$\frac{\partial H}{\partial(m_e v)} = \frac{dx}{dt} \quad \text{and} \quad \frac{\partial H}{\partial x} = -\frac{d}{dt}(m_e v) \tag{4.37}$$

where H(mv,x), the Hamiltonian, is the total energy of the electron. For a stationary field, the density-in-phase at any x and v is time independent so $\frac{\partial D}{\partial t} = 0$. Liouville's Theorem (4.26) with the aid of (4.37) is

$$-\frac{\partial D}{\partial v}\frac{\partial H}{\partial x} + \frac{\partial D}{\partial x}\frac{\partial H}{\partial(m_e v)} = 0 \tag{4.38}$$

*Equation (4.36) was first derived in a memorandum by Dr. F. Gray.
**Time dependent fields.

Equation (4.38) is a first order homogeneous linear partial differential equation with the general solution*

$$D = f(H) \tag{4.39}$$

Consequently, for the stationary fields the density-in-phase is only a function of the total energy. Since electron trajectories for stationary fields are along constant energy paths in phase space, (4.39) states that the density-in-phase is constant along every trajectory.

A simple example to illustrate the extension-in-phase (4.30) and (4.39) is given in figure 8. Electrons are injected across the x_0 plane into a region of retarding field. It is assumed that the electron space charge is small so that the field is linear. The magnitude of the retarding field (E) is 112.5 volts/cm.* The electron trajectories are parabolas and are drawn in steps of 0.1 electron volts. Consider the electrons enclosed in the extension-in-phase abcd at a time t, which are bound by the energies H_2 and H_1, and the velocities v_2 and v_1. At some later time $t_2 > t_1$, the extension-in-phase will have moved to a'b'c'd' bound by the velocities v_2' and v_1'. The extension-in-phase α at t_1 is

$$\alpha_{t=t_1} = (10^7 eE)^{-1}(H_2 - H_1)(v_2 - v_1) \tag{4.40}$$

and at t_2

$$\alpha_{t=t_2} = (10^7 eE)^{-1}(H_2 - H_1)(v_2' - v_1') \tag{4.41}$$

From the force equation (2.43) the primed and unprimed velocities are related by $m_e(v_2' - v_2) = -10^7 eE(t_2 - t_1)$ (4.42)

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*This can be readily verified by substituting in (4.38) and using (4.37). Compare [29].

**One volt applied across plate spacing of 8.89×10^{-3} cm. (3.5×10^{-3} in.).

and

$$m_e(v_1'-v_1) = -10^7 eE(t_2-t_1) \qquad (4.43)$$

Equating (4.42) and (4.43)

$$v_2'-v_1' = v_2-v_1 \qquad (4.44)$$

and substituting in (4.40) and (4.41) show that the extension-in-phase is constant as (4.30) demands. It is to be noted that though the extension-in-phase is constant, the shape varies along the phase paths. (Fig. 8).

If the plane x_0 is a thermionic cathode, the emitted electrons will have a Maxwell-Boltzmann distribution,*

$$D_{x_0} = D_0 \exp\left(-\frac{m_e v^2}{2kT}\right) \qquad (4.45)$$

where k is Boltzmann's constant and T is the absolute temperature of the emitter.** Using (2.43) and (4.39), the distribution for $x>x_0$ is

$$D(x,v) = D_0 \exp\left[-\frac{m_e v^2}{2kT} + \frac{10^7 e}{kT} V(x)\right] \qquad (4.46)$$

where $V(x)$ is the potential at x. Defining

$$V_T = \frac{kT10^{-7}}{e} = \frac{T}{11605} \quad \text{volts} \qquad (4.47)$$

(4.46) can be written as

$$D(x,v) = D_0 \exp\left[-\frac{v^2}{2\ell V_T} + \frac{V(x)}{V_T}\right] \qquad (4.48)$$

Equation (4.48) will be extensively used in Part V; further remarks will be included there.

The equation which is the basis of the conservation of charge method of treating single valued velocity electron streams in diodes[32] and velocity modulation tubes can be readily derived from the conservation of extension-in-phase. For a single valued velocity

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*Strictly speaking, the distribution is a Fermi-Dirac one; however, for the densities involved the two distributions are indistinguishable.
**See appendix 1. Compare [30],[31].

electron stream with a stationary accelerating field, the phase space plot will be similar to that shown in figure 9. The phase space electron flow is tube-like or solenoidal. The electrons entering the tube at the phase coordinates v_0 and x_0 at a time t_0 will be found at a later time t_1 at the phase coordinates v_1 and x_1. When the electric field is time dependent, the boundaries of the flow will pulsate with the fundamental and higher harmonics of the frequency of the electric field. For example, the fundamental frequency pulsating velocity at x_1 will be given by the third equation of (2.70) for the case of a very small time dependent field perturbing a much larger stationary field. Higher order pulsating velocities can be calculated from (2.31) and (2.32). The number of electrons crossing x_0 (Fig. 9) in the time dt_0 is

$$D(x_0, v_0, t_0)\, dv_0 \; v_0 dt_0 \tag{4.49}$$

or

$$D(x_0, v_0, t_0)\, dv_0 \, dx_0 \tag{4.50}$$

since $v_0 dt_0 = dx_0$

The number of electrons crossing x_1 in the time dt_1 is

$$D(x_1, v_1, t_1)\, dv_1 \; v_1 dt_1 \tag{4.51}$$

or

$$D(x_1, v_1, t_1)\, dv_1 dx_1 \tag{4.52}$$

since $v_1 dt_1 = dx_1$

From the conservation of extension-in-phase, (4.30) and (4.27),

$$D(x_0, v_0, t_0)\, dx_0 dv_0 = D(x_1, v_1, t_1)\, dx_1 dv_1 \;; \tag{4.53}$$

consequently, (4.49) and (4.51) are equal

$$D(x_0, v_0, t_0)\, v_0 dv_0 dt_0 = D(x_1, v_1, t_1)\, v_1 dv_1 \, dt_1 \,. \tag{4.54}$$

From (4.02) and (5.04) for a single valued velocity stream, (4.54) becomes $Q(x_0, t_0)dt_0 = Q(x_1, t_1)dt_1$

or

$$Q(x_1, t_1) = Q(x_0, t_0)\left.\frac{dt_0}{dt_1}\right|_{x = \text{constant}} \tag{4.55}$$

This is the conservation of charge equation which relates the convection current at x_1 and t_1 to that at a previous position x_0 and t_0. Though the above development is for the one dimensional electron flow between parallel planes, the same result can be obtained for other configurations (for example, coaxial circular cylinders), since the conservation of extension-in-phase applies for generalized coordinates and momenta.

PART V

STATIONARY ELECTRON STREAM SOLUTIONS

General Theory

For a stationary electric field,* the multi-velocity equations (4.08), (4.19), and (4.26) simplify to the following:

$$I = \frac{e}{m_e} P \tag{5.00}$$

$$\frac{\partial}{\partial x}[2K - W_E] = 0 \tag{5.01}$$

$$\frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial v}\frac{dv}{dt} = 0 \tag{5.02}$$

Since the current density (I) is constant for a stationary field, (5.00) shows that the momentum density is independent of the space position x (conservation of momentum density). The terms of (5.01) are only a function of x so that the partial derivative can be written as a total derivative. Integrating (5.01) gives

$$2[K(x_1) - K(x_0)] = W_E(x_1) - W_E(x_0) \tag{5.03}$$

Defining

$$K(x_1) - K(x_0) = \Delta K \tag{5.04}$$

and

$$W_E(x_1) - W_E(x_0) = \Delta W_E \tag{5.05}$$

(5.03) can be written as

$$\frac{\Delta W_E}{\Delta K} = 2 \tag{5.06}$$

Equations (5.06) and (5.00) are the fundamental equations for stationary field electron flow. They apply to any type of electron velocity distribution, multi-velocitied or the limiting case of single-valued

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*In the electronic literature, this would be referred to as a d.c. solution. This is an erroneous designation since the solution can be applied to any frequency provided the electron transit time is very small compared to the periodicity of the applied fields. Compare (2.70) and appendix 2 for the single valued velocity flow case.

velocity. The solution of (5.06) can be treated graphically step by step or analytically. Which method is chosen will depend on the accuracy desired and the form of ΔW_E and ΔK. Equation (5.06) gives a physical interpretation so readily that it will be called the theorem of stationary one dimensional electron flow.

Theorem

For a one dimensional electron flow acted on by a stationary electric field, the difference between the electric field energy density for two spatial positions is equal to twice the difference between the kinetic energy density of the electrons for the two spatial positions.*

Equation (5.06) with the use of (5.05) and the definition of W_E can be written as

$$\frac{10^7}{4}\epsilon\left[E^2(x) - E^2(x_0)\right] = K(x) - K(x_0) \qquad (5.07)$$

Writing the electric field in terms of the potential gives

$$-\int_{x_0}^{x} dx = (x_0 - x) = \int_{V_0}^{V}\left[E^2(x_0) + \frac{4\cdot 10^{-7}}{\epsilon}\{K - K(x_0)\}\right]^{-\frac{1}{2}} dV \qquad (5.08)$$

The relation between potential and distance in terms of the boundary value of electric field and kinetic energy density is given by (5.08). Using (5.00), the conservation momentum density, and (5.08), the stationary electron flow problem is solved for any electron velocity distribution, for example, Fermi-Dirac, Maxwellian, or the limiting case of single valued velocity.

*Fowler states the following theorem,(33) "The equilibrium state of the electron atmosphere is characterized by a minimum value of the ratio of the electrostatic energy to the kinetic energy of translation of the electrons." However, the electric and kinetic energy referred to are the total energies contained between the parallel plates and not the energy densities. Compare(34).

As was mentioned in part IV, for a single valued velocity stream, the velocity dependence of the density-in-phase becomes the impulse function. Using (5.00), (4.02) and (4.03) become

$$P = m_e \int D v dv = m_e n v = \frac{m_e}{e} I \tag{5.09}$$

and

$$K = \frac{m_e}{2} \int D v^2 dv = \frac{m_e}{2} n v^2 \tag{5.10}$$

where n is the electron number density (electrons/cm^3).

<u>Single Valued Velocity Electron Stream Example</u>

To illustrate (5.08) for a single valued velocity stream, consider the configuration shown in figure 1 where the a plane is at a potential V_a, the b plane at a potential V_b. Assume all the electrons injected across the a plane move to the b plane ($V_b > V_a$) and that the electric field at the a plane is zero. From (2.43) and (5.09), (5.10) becomes

$$K = \frac{m_e}{e} I \left(\frac{e}{2} V\right)^{\frac{1}{2}} \tag{5.11}$$

Substituting (5.11) in (5.08) and setting $E(x_0) = 0$ give

$$I = 2.66 \cdot 10^{-6} d^{-2} \left[\left(V_b^{\frac{1}{2}} + 2 V_a^{\frac{1}{2}} \right)^2 \left(V_b^{\frac{1}{2}} - V_a^{\frac{1}{2}} \right) \right] \quad \text{amps/cm}^2 \tag{5.12}$$

If $V_a = 0$, this reduces to Child's law (2.44). If the a plane has a field acting on it different from zero and a potential minimum exists between the a and b planes, (5.12) gives the relation between V_b and the potential minimum voltage V_a. Two expressions of the form of (5.12) can be combined (one relating the potential minimum voltage to the a plane voltage, and the other relating the potential minimum voltage to the b plane voltage) to give the relation between I, V_a, and V_b. For the maximum injected current, this expression reduces to (2.51). Any other potential combinations for the a and b planes can be solved by application of (5.08). The relations (5.12) and (2.51) have been obtained by a number of writers(18),(19) by the double integration of Poisson's equation. The method illustrated above is believed to be more straight forward and simple in application. Aside from the fact that (5.08) can be applied to any electron velocity distribution, its merit lies in the fact that attention is drawn to the mechanical properties of the electron stream. This is of particular importance in bringing forth the physics of the problem. For vacuum tube operation, in particular when time varying fields are considered, can only be properly understood in terms of field and electron interaction and not in terms of boundary properties as current and applied voltage.

Multi-Velocitied Electron Stream Examples

Consider figure 1 again, with the a plane a thermionic cathode ejecting electrons with a Maxwell-Boltzmann velocity distribution (4.45), and the b plane an anode at a negative voltage $-V_b$ with respect to the cathode such that there is a retarding field everywhere between the a and b planes.* For an electron just to reach the anode,** it must be emitted from the cathode with a velocity in terms of V_b given by***

$$v_\ell^2 = 2\ell / V_b / \qquad (5.13)$$

If an electron's emission velocity is greater than v_ℓ, it will reach the anode with a finite velocity; if it is less than v_ℓ, it will return and strike the cathode. The phase space picture of the above

*For large values of retarding voltage, the field between the a and b planes will be very close to linear and the phase space will be similar to that shown in figure 6.
**Zero velocity at the anode.
***See (2.43).

is given by figure 10. The velocity with which a returning electron strikes the cathode is of the same magnitude as its emission velocity but of opposite sign since it is moving in the -x direction. For any position x, there will be electrons returning to the cathode whose emission velocity lies in the velocity range $0<v<v_\ell$ (Fig. 10). The electrons which leave the cathode with a velocity v_ℓ^-,* and consequently return to the cathode, will determine the border between the density-in-phase given by (4.48) and D = 0. This border velocity (v') will determine the lower limit for the N, P, and K integrals, the upper limits being infinity.

The constant D_0, that appears in the density-in-phase (4.48), can be determined from the emission current density of the cathode. Since the off-cathode field is positive** (decelerating electron field), the emission current is only a function of the physical properties of the surface and its operating temperature, and is given by Dushman's equation(36)

$$I_e = AT^2 exp(-\frac{b}{T}) \qquad *** \qquad (5.14)$$

The relation between the emission current density**** and the emission momentum density is given by (5.00). Using (4.48) with V set equal to zero and (4.02) gives*****

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* $v_\ell^- = \lim_{\varepsilon \to 0} (v_\ell - \varepsilon)$

**This excludes the Schottky effect.(35) However, for thermionic tubes with negative off-cathode fields, the field is never of such magnitude that the Schottky effect must be considered.

***A and b are constants characteristic of the cathode surface; T is the absolute temperature of the surface.

****Note this is not the total current density at the cathode but only the current density produced by electrons leaving the surface. The total current density would have to take into consideration the returning electrons. See (5.18).

*****See appendix 3.

$$I_e = e\int_0^\infty D_0 \exp\left(-\frac{v^2}{2\ell V_T}\right) v\,dv \tag{5.15}$$

Integrating (5.15) and solving for D_0 result in

$$D_0 = I_e (e\ell V_T)^{-1} \tag{5.16}$$

The conservation of momentum density (5.00) states that the momentum density is constant (independent of x). Using (4.02) and (4.48) gives

$$P = m_e \int_{-v'}^{\infty} D_0 \exp\left(-\frac{v^2}{2\ell V_T} + \frac{V(x)}{V_T}\right) v\,dv \tag{5.17}$$

The border velocity v' equals v_ℓ^- at the cathode and approaches zero at the anode. Integrating (5.17) gives*

$$P = \frac{1}{2a^2} m_e D_0 \exp\left(-\frac{|V_b|}{V_T}\right) \tag{5.18}$$

where $\frac{1}{a^2} = 2\ell V_T$. Equation (5.18) verifies the constancy of the momentum density. Substituting (5.18) in (5.00) and using (5.16) result in

$$I = I_e \exp\left(-\frac{|V_b|}{V_T}\right) \tag{5.19}$$

Equation (5.19) is known as Boltzmann's equation,[37] and shows that the anode and emission current density are related by $\exp(-|V_b|/V_T)$.

For further calculations, it will be convenient to define a potential (V') measured relative to the anode. The relation between V' and V (the potential measured relative to the cathode) is

$$V' = V + |V_b| \tag{5.20}$$

Note $V' = V_b$ at the cathode where $V = 0$; and $V' = 0$ at the anode where $V = -|V_b|$. Comparing v' (the border velocity) and V', it is seen that they are related by $\qquad v'^2 = 2\ell V' \qquad$ (5.21)

Though the mass density N is not necessary in determining

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*See appendix 3.

the potential distribution, it will be calculated since it is of physical interest. Using (4.01) and (4.48) gives

$$N = m_e D_o \int_{-V'}^{\infty} exp\left(-\frac{v^2}{2\ell V_T} + \frac{V}{V_T}\right) dv \qquad (5.22)$$

Using the integrals given in appendix 3, (5.22) integrates to

$$N = \frac{\pi^{\frac{1}{2}}}{2a} m_e D_o \, exp\left[\frac{V'-|V_b|}{V_T}\right]\cdot\left[1 + erf\left(\frac{V'}{V_T}\right)^{\frac{1}{2}}\right] \qquad (5.23)$$

At the anode, $x = d$ and $V' = 0$, so

$$N(d) = \frac{\pi^{\frac{1}{2}}}{2a} m_e D_o \, exp\left(-\frac{|V_b|}{V_T}\right) \qquad (5.24)$$

Using (5.16) and (5.19), (5.24) can be written as

$$N(d) = 1.83 \cdot 10^{-14} I \, T^{-\frac{1}{2}} \qquad (5.25)$$

where the numeric is $\left(\pi m_e^3/2ke^2\right)^{\frac{1}{2}}$

Using (5.25) in (5.23),

$$N = N(d) \, exp\left(\frac{V'}{V_T}\right)\left[1 + erf\left(\frac{V'}{V_T}\right)^{\frac{1}{2}}\right] \qquad \text{grams/cm}^3 \qquad (5.26)$$

Multiplying (5.26) by $-\frac{e}{m_e}$ gives the charge density ρ in coulombs/cm^3 (compare (4.04). Using (4.07), (5.18), (5.25), and (5.26), the effective velocity (U) of the stream is

$$U = \left[a\pi^{\frac{1}{2}} \, exp\left(\frac{V'}{V_T}\right)\left[1 + erf\left(\frac{V'}{V_T}\right)^{\frac{1}{2}}\right]\right]^{-1} \qquad (5.27)$$

From (4.03) and (4.48), the kinetic energy density is

$$K = \frac{m_e}{2} D_o \int_{-V'}^{\infty} exp\left(-\frac{v^2}{2\ell V_T} + \frac{V}{V_T}\right) v^2 dv \qquad (5.28)$$

Resolving the above integral into two parts and using the formulae given in appendix 3, (5.28) integrates to

$$K = \frac{\pi^{\frac{1}{2}}}{8a^3} m_e D_o \, exp\left(\frac{V'-|V_b|}{V_T}\right)\left[1 + erf\left(\frac{V'}{V_T}\right)^{\frac{1}{2}} - 2\left(\frac{V'}{\pi V_T}\right)^{\frac{1}{2}} exp\left(-\frac{V'}{V_T}\right)\right] \qquad (5.29)$$

at $x = d$,

$$K(d) = \frac{\pi^{\frac{1}{2}}}{8a^3} m_e D_o \, exp\left(-\frac{|V_b|}{V_T}\right) = 1.39 \cdot 10^{-3} I \, T^{\frac{1}{2}} \qquad (5.30)$$

where the numeric is $\left(m_e \pi k/8e^2\right)^{\frac{1}{2}}$

In terms of N(d),

$$K(d) = 7.58 \cdot 10^{10} N(d)\, T \tag{5.31}$$

where $7.58 \times 10^{10} = \frac{k}{2m_e}$

Substituting (5.30) in (5.29), the kinetic energy density is

$$K = K(d)\, exp\left(\frac{V'}{V_T}\right)\left[1 + erf\left(\frac{V'}{V_T}\right)^{\frac{1}{2}} - 2\left(\frac{V'}{\pi V_T}\right)^{\frac{1}{2}} exp\left(-\frac{V'}{V_T}\right)\right] \tag{5.32}$$

Equation (5.32) gives the kinetic energy per unit volume. Dividing (5.32) by (5.26), the number of electrons per unit volume will give the mean kinetic energy per electron. The actual kinetic energy of any electron will differ from the mean kinetic energy and will depend on its emission velocity.

In (5.26) and (5.32), the potentials V' and V_T always appear together and suggest the introduction of a new dimensionless variable φ, defined as

$$\varphi = \frac{V'}{V_T} = \frac{V + |V_b|}{V_T} \tag{5.33}$$

Using (5.33), the mass and kinetic energy density can be written as

$$N = N(d)\, e^{\varphi}\left[1 + erf\, \varphi^{\frac{1}{2}}\right] \tag{5.34}$$

$$K = K(d)\, e^{\varphi}\left[1 + erf\, \varphi^{\frac{1}{2}} - 2\left(\frac{\varphi}{\pi}\right)^{\frac{1}{2}} e^{-\varphi}\right] \tag{5.35}$$

Since K is written in terms of φ, it will be convenient to take x_0 and V_0 in (5.08) as the anode position and potential and to integrate (5.08) from the anode to the cathode. The final form of (5.08) can be simplified if the new variable ξ is introduced for x, where

$$\xi = \frac{(x - d)}{\beta} \tag{5.36}$$

and

$$\frac{1}{\beta} = \frac{1}{V_T}\left(\frac{4K(d)}{10^7 \epsilon}\right)^{\frac{1}{2}} = \left(\frac{2m_e \pi e^2 10^{14}}{\epsilon^2 k^3}\right)^{\frac{1}{4}} I^{\frac{1}{2}} T^{-\frac{3}{4}} = 9.18 \cdot 10^5 I^{\frac{1}{2}} T^{-\frac{3}{4}} \tag{5.37}$$

Introducing $F(\varphi) = \left[\frac{K - K(d)}{K(d)}\right]$

or in extenso $F(\varphi) = (1 + \operatorname{erf} \varphi^{\frac{1}{2}})e^{\varphi} - 2\left(\frac{\varphi}{\pi}\right)^{\frac{1}{2}} - 1$ (5.38)

(5.08) becomes*

$$-\int_0^{\xi_a} d\xi = \int_0^{\varphi_a} \left[\left(\frac{d\varphi}{d\xi}\right)^2_{\varphi=0} + F(\varphi)\right]^{-\frac{1}{2}} d\varphi \tag{5.39}$$

where

$$\left(\frac{d\varphi}{d\xi}\right)^2_{\varphi=0} = \left(\frac{\beta E(d)}{V_T}\right)^2 \tag{5.40}$$

and the upper limits

$$\xi_a = -\frac{d}{\beta} \qquad \text{and} \qquad \varphi_a = \frac{|V_b|}{V_T} \tag{5.41}$$

Integrating the left side of (5.39) and using (5.41), (5.37), and (5.19) give

$$-\int_0^{\xi_a} d\xi = 9.18 \times 10^5 d\, T^{-\frac{3}{4}} I_e^{\frac{1}{2}} e^{-\frac{\varphi_a}{2}} \tag{5.42}$$

Substituting (5.42) in (5.39) results in

$$9.18 \times 10^5 T^{-\frac{3}{4}} d\, I_e^{\frac{1}{2}} e^{-\frac{\varphi_a}{2}} = \int_0^{\varphi_a} \left[\left(\frac{d\varphi}{d\xi}\right)^2_{\varphi=0} + F(\varphi)\right]^{-\frac{1}{2}} d\varphi \tag{5.43}$$

The integrand on the right side of (5.43) is of such form that numerical integration must be used. To evaluate (5.43), values for T, d, and I_e would be assumed. The left side of (5.43) would then be tabulated in terms of φ. This can be done by plotting on logarithm graph paper. A value for the off-anode field is assumed which then determines $\left(\frac{d\varphi}{d\xi}\right)^2_{\varphi=0}$. Using (5.38), the right side of (5.43) is numerically integrated and its value as a function of φ is tabulated or drawn on the above mentioned graph. When the right and left side of (5.43) are numerically equal, the value of φ_a gives, by (5.41), the retarding potential $|V_b|$ for the particular assumed off-anode field E(d). Choosing a range of

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*An extensive tabulation has recently been given by Kleynen[38] for (5.39), when the off-anode field is zero $(d\varphi/d\xi)_{\varphi=0}$.

values for E(d), the potential distribution, retarding potential, and anode current can be determined.

Figure 11 shows the potential distributions for the following values of the quantities appearing in (5.43).*

$I_e = 0.1$ amp/cm^2, $T = 10^3$ deg., $d = 8.89 \times 10^{-3}$cm. $= 3.5 \times 10^{-3}$in.

Substituting the above values in (5.43) gives

$$14.5\, e^{-\frac{\varphi_a}{2}} = \int_0^{\varphi_a} \left[\left(\frac{d\varphi}{d\xi}\right)^2_{\varphi=0} + F(\varphi) \right]^{-\frac{1}{2}} d\varphi \tag{5.44}$$

where $\varphi_a = 11.605\,|V_b|$

and $\left(\frac{d\varphi}{d\xi}\right)_{\varphi=0} = -7.143 \cdot 10^{-3}\, e^{5.802\,|V_b|}\, E(d)$

For $\varphi \geq 5$, $F(\varphi)$ can be approximated by $2e^{\varphi}$. At $\varphi = 5$, this approximation gives a value of $F(\varphi)$ 0.82% too large; for $\varphi = 6$, 0.25% too large. When $F(\varphi) \gg \left(\frac{d\varphi}{d\xi}\right)^2_{\varphi=0}$, the right side of (5.44) can be written as

$$\int_0^{\varphi_a} [F(\varphi)]^{-\frac{1}{2}} \left[1 - \left(\frac{d\varphi}{d\xi}\right)^2_{\varphi=0} (2F(\varphi))^{-1} \right] d\varphi \tag{5.45}$$

or when $F(\varphi) \ll \left(\frac{d\varphi}{d\xi}\right)^2_{\varphi=0}$ as

$$\left(\frac{d\varphi}{d\xi}\right)^{-1}_{\varphi=0} \int_0^{\varphi_a} \left[1 - \tfrac{1}{2} F(\varphi) \left[\left(\frac{d\varphi}{d\xi}\right)_{\varphi=0} \right]^{-2} \right] d\varphi \tag{5.46}$$

After (5.44) is tabulated, the relation between V and x makes it possible to evaluate the number density (5.34), the kinetic energy density (5.35), and the electric field (5.07) as a function of x. In appendix 4, the retarding voltage, the off-anode field, and the off-cathode field are tabulated. In addition, the field at 0.254×10^{-3} cm. (0.1×10^{-3}in.) from the cathode** and the linear field that would exist if no electrons were present are given for comparison. The

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*In part VII, this example will be used for a calculation of high frequency loading.

**2.86% of cathode-anode spacing.

retarding voltage for a zero off-anode field is -0.3155 volts. As the retarding voltage is increased in magnitude, the electric field throughout the diode approaches linearity rapidly. Figure 12 shows a plot of the off-anode, off-cathode, 0.254×10^{-3} cm from the cathode, and linear field versus retarding voltage. The above mentioned rapid approach to linearity is clearly exhibited.

As the retarding potential is decreased in magnitude, below -0.3155, the zero field moves away from the cathode producing a potential minimum at a distance x_m from the cathode. The equations derived above for the retarding field diode can be used if (5.20) and (5.36) are re-defined as

$$V' = V + |V_M| \tag{5.47}$$

and

$$\xi = \frac{x - x_M}{\beta} \tag{5.48}$$

where $|V_M|$ is the potential minimum voltage and x_m is the distance of the potential minimum from the cathode. In addition, in the N and K expressions d is replaced by x_m. The anode current will be given by (5.19) with $|V_M|$ substituted for $|V_b|$. The definition (5.48) shows

$$\xi > 0 \qquad d \geq x \geq x_M$$

$$\xi < 0 \qquad x_M > x \geq 0$$

For convenience, the diode space where $\xi > 0$ will be called the A (accelerating field) space; where $\xi < 0$, the R (retarding field) space. The phase space given by figure 10 will hold now only for the R space. The total phase space for the present field configuration is given in figure 13.

The lower limit for the number density integral in the A space will now be v' instead of the previous -v'. Consequently, the second term in (5.34) for the A space will have an opposite sign.*

$$N = N(x_M)\, e^{\varphi}[1 - \operatorname{erf} \varphi^{\frac{1}{2}}] \tag{5.49}$$

By noting the change produced in the integrals in (5.28) when the sign of the lower limit is reversed, the kinetic energy in the A space will be given by

$$K = K(x_M)\, e^{\varphi}[1 - \operatorname{erf} \varphi^{\frac{1}{2}} + 2(\tfrac{\varphi}{\pi})^{\frac{1}{2}} e^{-\varphi}] \tag{5.50}$$

The N and K expressions for both A and R space can be written as

$$N = N(x_M)\, e^{\varphi}[1 \mp \operatorname{erf} \varphi^{\frac{1}{2}}] \tag{5.51}$$

and

$$K = K(x_M)\, e^{\varphi}[1 \mp \operatorname{erf} \varphi^{\frac{1}{2}} \pm 2(\tfrac{\varphi}{\pi})^{\frac{1}{2}} e^{-\varphi}] \tag{5.52}$$

where the upper sign is used in the A space and the lower sign in the R space.

The potential distribution integral (5.39) can be written for the A and R space as **

$$\pm\int_0^{\xi} d\xi = \int_0^{\varphi} [[1 \mp \operatorname{erf} \varphi^{\frac{1}{2}}]e^{\varphi} \pm 2(\tfrac{\varphi}{\pi})^{\frac{1}{2}} - 1]^{-\frac{1}{2}} d\varphi \tag{5.53}$$

Equation (5.53) is exactly the same expression that Professor P.S. Epstein[39] obtained for the above problem by double integration of Poisson's equation. Langmuir later obtained the same expression when he repeated the problem.[40] The method illustrated above using (5.08) is more straight forward and gives a better physical picture. Attention is focused on the electron behavior and the interaction between the electrons and the field is emphasized as it should be for

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*This is physically obvious since the electrons in the velocity range 0 to -v' added to the number density in the R space, while in the A space there are no electrons in the velocity range 0 to v'. The first term in (5.49) represents the density contribution for electrons in the velocity range zero to infinity.

** $(d\varphi/d\xi)_{\varphi=0} = 0$ since the field at the potential minimum is zero.

proper understanding of vacuum tube operation. Once the kinetic energy density is written, this step not being an unnecessary one since the quantity is of physical interest, the potential distribution (5.08) is given by one integration.

As was previously mentioned, Kleynen has recently published an extensive tabulation of (5.53).[38] In the calculation of electron transit time, an expression for the potential versus distance would be necessary. A series solution for this purpose can be obtained from (5.53) by a laborious calculation which will be briefly outlined. For convenience in writing, a new variable $\eta = \varphi^{\frac{1}{2}}$ is introduced in the intergrand on the right side of (5.53) which is then expanded in a power series in η. Next, the negative one half power of the series is taken, and the resultant series is integrated term by term. Resubstituting gives

$$\xi = a_0 + a_1 \varphi^{\frac{1}{2}} + a_2 \varphi + \cdot \quad \cdot \quad \cdot + a_n \varphi^{\frac{n}{2}} + \cdot \cdot \qquad (5.54)$$

The coefficients a_n are given in appendix 5. By inversion and squaring of (5.54), the series expansion for φ in terms of ξ is obtained.

$$\varphi = b_0 + b_1 \xi + b_2 \xi^2 + \cdot \quad \cdot \quad \cdot + b_n \xi^n + \cdot \cdot \qquad (5.55)$$

The coefficients b_n are given in appendix 5. In figure 14, the mass density ratio $N/N(x_m)$, the kinetic energy density ratio $K/K(x_m)$, and φ are plotted against ξ. Because φ and ξ are dimensionless, these curves can be applied to any value of the parameters I, T, V, and d. As figure 14 shows, the mass density and consequently the electron charge density decreases rapidly as x moves away from the cathode.

In the A space, for large values of φ * the mass density

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*Since $\varphi = V'/V_T$, φ can be large for small values of V'. For example, for $T = 10^3$, $V_T^{-1} = 11.605$.

ratio, with the use of the semiconvergent series for the error function,[23] can be written as

$$\frac{N}{N(x_M)} = \frac{1}{(\pi\varphi)^{\frac{1}{2}}}\left[1 - \frac{1}{2\varphi} + \frac{1\cdot 3}{(2\varphi)^2} - \cdot \cdot \cdot \cdot\right] \tag{5.56}$$

The error in using a given number of terms in this expansion is less than the last term used. For voltages V' greater than one or two volts

$$\frac{N}{N(x_M)} \cong (\pi\varphi)^{-\frac{1}{2}} \tag{5.57}$$

Using (4.07), (5.00), (5.57), and (5.25), the effective velocity in the A space for large values of φ becomes

$$U = \left(\frac{2k}{m_e}T\varphi\right)^{\frac{1}{2}} \tag{5.58}$$

With the use of (4.47), (5.33), and (5.47), (5.58) can be written as

$$U = \left(\frac{2e10^7}{m_e}[V + |V_M|]\right)^{\frac{1}{2}} \tag{5.59}$$

The average kinetic energy of an electron in the stream is

$$\frac{1}{2}m_e U^2 = e10^7(V + |V_M|) \tag{5.60}$$

An electron starting from the potential minimum with zero velocity would have a velocity U at the potential V given by (5.60). Consequently, for values of $V + |V_M|$ in the order of a few volts, the multi-velocitied stream behaves as if it were single valued. In analyzing vacuum tube behavior in the regions beyond the first grid, the electron stream can be usually treated as having a single valued velocity. The accuracy of this approximation for any case can be checked by the above equations. By a close examination of N, K, and U for different values of x; for example: at the cathode, potential minimum, and intervening positions, a complete picture of the electron stream behavior is revealed. Space limitation, however, prevents further discussion at this time.

For any other electron velocity distribution, the procedure in using (5.08) would be similar to that given above. Given the density-in-phase at the <u>a</u> plane, its value for any position throughout the diode is given by Liouville's Theorem. Calculating the kinetic energy density and substituting in (5.08), the potential distribution is obtained.

Part VI

TIME DEPENDENT ELECTRON STREAMS

The electron stream equations for a time dependent electric field are summarized on page 43. The first and second equations in the set are rewritten below.

$$-I(t) = -\frac{e}{m_e} P + \epsilon \frac{\partial E}{\partial t} \tag{6.00}$$

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}\left[2K - W_E\right] = 0 \tag{6.01}$$

Taking the time derivative of (6.00),* solving for $\frac{\partial P}{\partial t}$ and substituting in (6.01) give

$$\frac{m_e}{e}\frac{dI}{dt} + \frac{m_e \epsilon}{e}\frac{\partial^2 E}{\partial t^2} + \frac{\partial}{\partial x}\left[2K - W_E\right] = 0 \tag{6.02}$$

Integrating (6.02) from $x = a$ to $x = b$ at a time t gives

$$\frac{m_e}{e}(b-a)\frac{dI}{dt} + \frac{m_e \epsilon}{e}\int_a^b \frac{\partial^2 E}{\partial t^2}\bigg|_t \partial x + \left[2K - W_E\right]_{x=a}^{x=b}\bigg|_t = 0 \tag{6.03}$$

Steady State Small Signal Theory

For a steady state small signal theory, the quantities appearing in (6.03) can be written as

$$\begin{aligned} I &= I_0 + I_1(t) \\ E &= E_0(x) + E_1(x,t) \\ K(x,t) &= K_0(x) + K_1(x,t) \\ W_E(x,t) &= W_0(x) + W_{1E}(x,t) \end{aligned} \tag{6.04}$$

In (6.04), the subscripts (o) refer to the stationary part or zero order component, while the subscript (1) refers to the time dependent

*The partial time derivative of I can be written as a total derivative since I can only be a function of time.

or first order component. For every term, the first order part must be very small compared to the zero order term (time dependent quantities are a perturbation on their stationary counterparts). For a steady state solution, the first order terms can be written as the real parts of $\underline{I}_1 e^{j\omega t}$, $\underline{E}_1(x)e^{j\omega t}$, $\underline{K}_1(x)e^{j\omega t}$, and $\underline{W}_{1\epsilon}(x)e^{j\omega t}$. The bar under each term indicates it can be time complex. With the introduction of the exponential time function, a time derivative is replaced by multiplication by $j\omega$. Substituting (6.04) in (6.03) with the above time dependence for the first order solution gives

$$\frac{j\omega m_e}{e}(b-a)\underline{I}_1 + \frac{m_e\epsilon}{e}(j\omega)^2\int_a^b \underline{E}_1\,\partial x + \left[2\underline{K}_1 - \underline{W}_{1\epsilon}\right]_a^b = 0 \tag{6.05}$$

The bracket term contains only first order terms since the stationary part is equal to zero.* The electric field integral is

$$\int_a^b \underline{E}_1 \partial x = -\int_a^b \frac{\partial \underline{V}_1}{\partial x}\partial x = \underline{V}_1(a) - \underline{V}_1(b) \tag{6.06}$$

The complex admittance (Y) between the a and b plane can be defined as

$$\underline{Y} = \underline{I}_1\left[\underline{V}_1(b) - \underline{V}_1(a)\right]^{-1} \qquad \text{mhos/cm2} \tag{6.07}$$

Dividing (6.05) by (6.06), using (6.07), and solving for $\underline{Y}$ give

$$\underline{Y} = \frac{j\omega\epsilon}{(b-a)} + \frac{je}{m_e\omega}\frac{\left[2\underline{K}_1 - \underline{W}_{1\epsilon}\right]_a^b}{(b-a)\left[\underline{V}_1(b) - \underline{V}_1(a)\right]} \tag{6.08}$$

The first term on the **right** side of (6.08) is the susceptance of the condenser formed by the parallel a and b planes when no electrons are present in the intervening space. The second term of (6.08) is admittance introduced by the electron flow. An equivalent circuit

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*Compare (5.03).

for the diode formed by the a and b planes would be the diode capacity shunted by an admittance representing the electron flow. Using (6.08), the admittance can be written as the sum of two terms, Y_c plus Y_s, where Y_c is the capacitive susceptance, and Y_s is the stream admittance.

$$\underline{Y}_c = \frac{j\omega\epsilon}{(b-a)} = j\omega C \tag{6.09}$$

$$\underline{Y}_s = j\frac{e}{\omega m_e}\frac{[2\underline{K}_1 - \underline{W}_{1E}]_a^b}{(b-a)[\underline{V}_1(b) - \underline{V}_1(a)]} = \pm G_s - jB_s \tag{6.10}$$

The stream admittance (Y_s) clearly shows the electron stream and electric field interaction. Because of electron inertia, the kinetic energy and field energy densities will not be in time phase. Consequently, (6.10) will be composed of a real plus an imaginary term. The imaginary term will have a negative sign while the real term can have either a positive or negative sign. A positive real term would represent energy removed from the electric field, and consequently from the external circuit since steady state operation is being considered, and given to the electrons in the form of kinetic energy. An opposite sign for the stream conductance would represent energy given to the external circuit from the electrons.

A series impedance formulation analogous to (6.08) can be obtained by dividing (6.05), by $\underline{I}_1$ and using the inverse of (6.07) as the definition for impedance.

$$\underline{Z} = \frac{1}{\underline{Y}} = [\underline{V}_1(b) - \underline{V}_1(a)]\cdot \underline{I}^{-1} = \underline{Z}_c + \underline{Z}_s \tag{6.11}$$

where

$$\underline{Z}_c = -j\frac{(b-a)}{\epsilon\omega} \tag{6.12}$$

$$\underline{Z}_s = \frac{e}{m_e\epsilon\omega^2}[2\underline{K}_1 - \underline{W}_{1E}]_a^b\,\underline{I}_1^{-1} = \pm R_s + jX_s \tag{6.13}$$

This formulation gives a series equivalent circuit where the diode capacity is connected in series with the stream impedance. Physically, the admittance representation given by (6.08) is preferable.

To use (6.10) or (6.13) the first order kinetic energy density and electric field energy density would have to be determined. From (6.04)

$$W_E = \tfrac{1}{2}10^7\epsilon E^2 = \tfrac{1}{2}10^7\epsilon[E_0 + Rl(E_1 e^{j\omega t})] \tag{6.14}$$

or to the first order

$$W_{0E} = \tfrac{1}{2}10^7\epsilon E_0^2 \quad ; \quad \underline{W}_{1E} = 10^7\epsilon E_0 \underline{E}_1 e^{j\omega t} \tag{6.15}$$

The energy interchange term in (6.10) and (6.13) becomes

$$[2\underline{K}_1 - \underline{W}_{1E}]_a^b = 2[\underline{K}_1(b) - \underline{K}_1(a)] - 10^7\epsilon[E_0(b)\underline{E}_1(b) - E_0(a)\underline{E}_1(a)] \tag{6.16}$$

Using (4.03), the kinetic energy density can be evaluated by integrating the density-in-phase. The density-in-phase would be the particular solution of the general solution of (4.26) which would satisfy the given distribution at the $\underline{a}$ plane. If the $\underline{a}$ plane is a thermionic emitter, the Maxwellian distribution given by (4.45) would be used. Note, however, (4.39) can not be used since the energy for an electron at a particular space position can not be evaluated in terms of the potential at that position. The energy that an electron will have at any position will be determined by the interchange of energy with the electric field that the electron had formerly experienced.
For single valued velocity streams, this is clearly shown by the third equation of (2.70). For the analysis of shot noise produced by the cathode, the Maxwellian distribution would contain a function which would express the randomness of the number of electrons emitted for each velocity packet. The treatment of these above-mentioned problems will be reserved for future papers.

Part VII

HIGH FREQUENCY LOADING

Description of Experiment

An interesting high frequency loading phenomena occurring in very close spaced tubes is reported by a number of investigators.[(41),(42),(43)] The experimental set-up is as follows. The "Q" of the input gap between the cathode and the first grid of a triode is measured as a function of the retarding voltage on the grid (cathode anode region for a diode). (Fig. 1) The frequency used is 3×10^9 cps[(41),(43)] with the tube having a cathode grid spacing of 3.6×10^{-3} in. At very large retarding voltages, the shunt resistance across the input gap is the same whether the cathode is heated or not. With the cathode operating the Q is measured as the retarding voltage is decreased in magnitude. The Q sharply decreases and reaches a minimum value before any electrons pass through the first grid in the triode or are collected by the anode for the diode case.

Clearly, the loading of the input gap must be produced by the electrons that are returned to the cathode by the retarding field. On an average, the returning electrons must have an energy greater than their emission energy, this extra energy being supplied by the high frequency field which is converted into heat when the returning electrons strike the cathode. For the diode or a very fine grid triode, the stationary potential distribution would be that given by figure 11. For potentials greater than 1 or 2 volts, the field is very close to linear. A good approximation for the calculation of the electron motion would be to assume a linear field. This assumption is justified since the maximum loading occurs before any appreciable number of electrons strike the anode of the diode, so that the charge density of the electron is negligible beyond a very short distance from the cathode.

To calculate the energy of the returning electrons, (1.3)* must be integrated to determine the transit time.

$$m_e \frac{d^2x}{dt^2} = -10^7 eE = -10^7 e\left[\frac{V_{bo}}{d} - \frac{V_{b1}}{d}\cos(\omega t + \varphi)\right] \tag{7.00}$$

The stationary voltage at the **b** plane is written as V_{bo}, the time dependent voltage as V_{b1}. At the cathode $(x = 0)$, the initial conditions are $v = v_o$ at time $t = 0$. Integrating (7.00) once and using the initial conditions cited, the velocity at time t is

$$\frac{dx}{dt} - v_o = -\frac{\ell}{d}\left[V_{bo}t + \frac{V_{b1}}{\omega}\{\sin\varphi - \sin(\omega t + \varphi)\}\right] \tag{7.01}$$

The position at time t is

$$x = v_o t - \frac{\ell}{d}\left[\frac{1}{2}V_{bo}t^2 + \frac{V_{b1}}{\omega}t\sin\varphi + \frac{V_{b1}}{\omega^2}\{\cos(\omega t + \varphi) - \cos\varphi\}\right] \tag{7.02}$$

The stationary transit time** will be determined by (7.02) when $x = 0$.

$$0 = v_o t_o - \frac{\ell}{2d} V_{bo} t_o^2 \tag{7.03}$$

Equation (7.03) has two solutions, a trivail one being $t_o = 0$, and the other being the to and fro time for the electron emitted from the cathode with a velocity v_o. The subscript (o) on t is to indicate it is the transit time when V_{bo} acts alone. Solving (7.03) gives

$$t_o = 2\frac{v_o d}{\ell V_{bo}} \tag{7.04}$$

*This method of calculating the transit time has been used by a number of investigators.[44],[45]

**Transit time when $V_{b1} = 0$.

With V_{b1} different from zero, the transit time T can be written as

$$T = t_o + \delta_1 p + \delta_2 p^2 + \cdots \tag{7.05}$$

where $\quad p = \frac{V_{b1}}{V_{bo}}$

Clearly, p is the proper parameter for the transit time expansion since the ratio of V_{b1} to V_{bo} will have a direct influence on the perturbation of the transit time t_o. Using the parameter p, (7.02) can be written as

$$x = v_o t - 2\frac{v_o}{t_o}\left[\frac{1}{2}t^2 + \frac{p}{\omega}\left\{\cos(\omega t + \varphi) - \cos\varphi + \omega t \sin\varphi\right\}\right] \tag{7.06}$$

Writing (7.05) for t in (7.06) with x set equal to zero and grouping the resultant expression in terms of powers of p give, to the second order in p,

$$\begin{aligned} 0 &= p\left[v_o\delta_1 + \frac{2}{\omega}v_o \sin\varphi - \frac{2}{\omega^2 t_o}v_o\left\{\cos\varphi - \cos(\omega t_o + \varphi)\right\}\right] \\ &+ p^2\left[v_o\delta_2 + \frac{v_o}{t_o}\delta_1^2 + \frac{2v_o\delta_1}{\omega t_o}\left\{\sin\varphi - \sin(\omega t_o + \varphi)\right\}\right] \end{aligned} \tag{7.07}$$

Since the left side of (7.07) is to be equal to zero for any value of p, the bracket term for each power of p must be separately equal to zero. Setting the bracket quantities equal to zero and solving for δ_1 and δ_2 give

$$\delta_1 = -\frac{2}{\omega^2 t_o}\left[(\omega t_o - \sin\omega t_o)\sin\varphi + (\cos\omega t_o - 1)\cos\varphi\right] \tag{7.08}$$

$$\delta_2 = -\frac{\delta_1^2}{t_o} - \frac{2\delta_1}{\omega t_o}\left[(1 - \cos\omega t_o)\sin\varphi - \sin\omega t_o \cos\varphi\right] \tag{7.09}$$

The velocity of the returning electrons emitted with a velocity v_o will be given by (7.01), where (7.05) is substituted for t. Squaring (7.01) and calling the velocity of the returning electrons at the cathode v_1 give

$$v_1^2 - v_0^2 = 4p\left(\frac{v_0}{t_0}\right)^2\Big\{\Big[\delta_1 t_0 + \frac{t_0}{\omega}\{\sin\varphi - \sin(\omega t_0 + \varphi)\}\Big] + p\Big[t_0\delta_2 + \delta_1^2$$
$$- \delta_1 t_0 \cos(\omega t_0 + \varphi) + \frac{1}{\omega^2}\{\sin\varphi - \sin(\omega t_0 + \varphi)\}^2 \tag{7.10}$$
$$+ \frac{2\delta_1}{\omega}\{\sin\varphi - \sin(\omega t_0 + \varphi)\}\Big]\Big\}$$

Using (7.08) and (7.09), after some laborious algebra the following results

$$v_1^2 - v_0^2 = 4p\left(\frac{v_0}{\omega t_0}\right)^2\Big\{4\left(\sin\frac{\omega t_0}{2} - \frac{\omega t_0}{2}\cos\frac{\omega t_0}{2}\right)\Big\{\sin\left(\frac{\omega t_0}{2} + \varphi\right)$$
$$+ p\sin\frac{\omega t_0}{2}\Big\} + p(\omega t_0 - \sin\omega t_0)(\sin[2\varphi + \omega t_0])\Big\} \tag{7.11}$$

Defining $\frac{\omega t_0}{2} = \theta$ (where θ is the one way transit angle)* and $\Psi(\theta) = (\sin\theta - \theta\cos\theta)$, (7.11) can be written as

$$v_1^2 - v_0^2 = 4p\left(\frac{v_0}{\theta}\right)^2\Big\{\Psi(\theta)\sin(\theta + \varphi) + \frac{p}{4}(2\theta - \sin 2\theta)\sin 2(\theta + \varphi)$$
$$+ p\,\Psi(\theta)\sin\theta\Big\} \tag{7.12}$$

Multiplying (7.12) by $m_e/2$ will give the difference in the emitted and returning kinetic energy for an electron leaving the cathode with a velocity v_0 and phase angle φ for the potential V_{b1} at the time of exit. The average energy interchange per electron for electrons leaving the cathode with a velocity v_0 will be the integration of (7.12) over a complete cycle of φ.

$$\frac{1}{2} m_e \overline{(v_1^2 - v_0^2)} = \frac{1}{2\pi}\int_{\varphi}^{\varphi + 2\pi} \frac{1}{2} m_e (v_1^2 - v_0^2)\, d\varphi \tag{7.13}$$

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*Transit angle from cathode to position in field where electron velocity is zero.

Substituting (7.12) in (7.13) and integrating, the only remaining term will be the last one given in (7.12) since it is not a function of φ .

$$\frac{1}{2} m_e \overline{(v_1^2 - v_0^2)} = 2\left(\frac{v_0}{\theta}\rho\right)^2 m_e \, \Psi(\theta) \sin\theta \qquad (7.14)$$

The border emission velocity, the velocity of emission which separates the electrons reaching the anode from those returning to the cathode, is

$$v_\ell^2 = 2\ell\,|V_{bo}| \quad * \qquad (7.15)$$

When V_{b1} is different from zero, (7.15) should be modified. However, as will be seen shortly, this will be an unimportant correction.** The number of electrons per sec per cm^2 ($\Delta n/\Delta t$) leaving the cathode in the velocity range v_0 to $v_0 + dv_0$ is given by (5.15) and (5.16) as

$$\frac{\Delta n}{\Delta t} = \frac{dI_e}{e} = \frac{I_e}{\ell e V_T} \quad exp\left(-\frac{v_0^2}{2\ell V_T}\right) v_0 dv_0 \qquad (7.16)$$

The energy interchange per sec per cm^2 ($\Delta K/\Delta t$) between the cathode and the emitted and returning electrons in the velocity range v_0 to $v_0 + dv_0$ is (7.14) multiplied by (7.16).

$$\frac{\Delta K}{\Delta t} = \frac{1}{2} m_e \overline{(v_1^2 - v_0^2)} \;\frac{\Delta n}{\Delta t} \qquad \text{ergs/sec/cm}^2 \qquad (7.17)$$

Dividing (7.17) by 10^7 and substituting (7.16), the energy interchanged per sec per cm^2 for the velocity packet dv_0 is given as

$$\Delta P = \frac{2I_e}{V_T}\left(\frac{v_0 \rho}{\ell\theta}\right)^2 \Psi(\theta)\sin\theta \;\; exp\left(-\frac{v_0^2}{2\ell V_T}\right) v_0 dv_0 \qquad \text{watts/cm}^2 \quad (7.18)$$

Since the transit angle θ is a function of v_0, it will be more convenient to write (7.18) completely in terms of θ instead of v_0. Introducing the following parameters

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*Compare (5.13).
**Compare (7.30).

$$h = \ell V_{bo}^2 (2\omega^2 d^2 V_T)^{-1} = 1.021 \cdot 10^9 \left(\frac{V_{bo}}{\omega d}\right)^2 \frac{1}{T} \tag{7.19}$$

and

$$\alpha = \frac{\omega d}{\ell V_{bo}} \tag{7.19a}$$

(7.18) can be written as

$$\Delta P = 4 I_e h \frac{\rho^2}{\ell \alpha^2} \Psi(\theta) \sin\theta \, e^{-h\theta^2} \theta d\theta \tag{7.20}$$

The upper limit of integration for (7.20) is

$$\theta_\ell = \alpha v_\ell = 2.115 \cdot 10^{-7} fd (V_{bo})^{-\frac{1}{2}} \tag{7.21}$$

where f is the frequency of V_{bl}. Integrating (7.20) and using (7.19) and (7.19a) give

$$P = 7.188 \cdot 10^{34} \left(\frac{V_{bl} V_{bo}}{\omega^2 d^2}\right)^2 \frac{I_e}{T} \int_0^{\theta_\ell} \Psi(\theta) \sin\theta \, e^{-h\theta^2} \theta d\theta \tag{7.22}$$

The power can be written in terms of a stream conductance G_s and V_{bl}^2 as*

$$P = \frac{1}{2} V_{bl}^2 G_s \tag{7.23}$$

Using (7.23) for the right side of (7.22) and solving for G_s give

$$G = 1.438 \cdot 10^{35} \frac{I_e}{T} \left(\frac{V_{bo}}{\omega^2 d^2}\right)^2 \int_0^{\theta_\ell} \Psi(\theta) \sin\theta \, e^{-h\theta^2} \theta d\theta \tag{7.24}$$

By repeated integration by parts, the integral in (7.24) can be written as

$$\int_0^{\theta_\ell} \Psi(\theta) \sin\theta \, e^{-h\theta^2} \theta d\theta = \frac{1}{4h^2} e^{-h\theta_\ell^2} \left[(h+1)\cos 2\theta_\ell + h(\theta_\ell \sin 2\theta_\ell - 1)\right] - \frac{1}{4h^2} + \frac{(h+2)}{4h^2} \int_0^{\theta_\ell} \sin 2\theta \, e^{-h\theta^2} d\theta \tag{7.25}$$

Because of the rapid decay of the exponential in the sin2θ integral

*The half appears because V_{bl} is the maximum value while P is in r.m.s. watts/cm².

in (7.25), the upper limit can be replaced by infinity. The maximum possible error introduced by this change of the upper limit can be evaluated by setting sin2θ equal to one and integrating from θ to infinity. The value of this integral is given in terms of the error function as

$$\int_{\theta_\ell}^{\infty} e^{-h\theta^2} d\theta = \frac{1}{2}\left(\frac{\pi}{h}\right)^{\frac{1}{2}}\left[1 - erf(\theta_\ell h^{\frac{1}{2}})\right] = \frac{1}{2}\left(\frac{\pi}{h}\right)^{\frac{1}{2}} erfc(\theta_\ell h^{\frac{1}{2}}) \qquad (7.26)$$

The sin2θ integral in (7.25) with infinity as the upper limit is evaluated in appendix 6 and can be written as

$$\int_0^{\infty} \sin 2\theta \, e^{-h\theta^2} d\theta = h^{-\frac{1}{2}} e^{-\frac{1}{h}} \int_0^{\frac{1}{h^{1/2}}} e^{t^2} dt \qquad (7.27)$$

or

$$= h^{-1} e^{-\frac{1}{h}} {}_1F_1(\tfrac{1}{2};\tfrac{3}{2};\tfrac{1}{h}) \qquad (7.28)$$

The exponential integral in (7.27) is tabulated in Jahnke and Emde,[46] page 32. The ${}_1F_1$ function appearing in (7.28) is the confluent hypergeometric function.[47] A curve is given on page 279 for ${}_1F_1(1/2;3/2;1/h)$ in Jahnke and Emde.*

Substituting (7.25) and (7.27) in (7.24), the stream conductance is

$$G_s = 3.44\times10^{-4} I_e T V_{bo}^{-2}\Big\{e^{-h\theta_\ell^2}\left[(h+1)\cos 2\theta_\ell + h(\theta_\ell \sin 2\theta_\ell - 1)\right] - 1 + \frac{1}{h}(h+2)\, e^{-\frac{1}{h}} {}_1F_1(\tfrac{1}{2};\tfrac{3}{2};\tfrac{1}{h})\Big\} \qquad (7.29)$$

where θ_ℓ is given by (7.21) and h by (7.19).

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*In Jahnke and Emde's notation ${}_1F_1(1/2;3/2;1/h)$ is written as $M(1/2;3/2;1/h)$.

<u>Examples</u>

1. To illustrate (7.29), the constants given by Kuper[43] for the frequency and spacing will be used.

$f = 3 \times 10^9$ cps; $d = 8.89 \times 10^{-3}$ cm; $T = 10^3$ deg.*; $I_e = 0.1$ amp/cm.*
The value of the parameters h and θ' are $0.3625 V_{bo}^2$ and $5.66\,(|V_b|)^{\frac{1}{2}}$ respectively.** In figure 15, the electron stream resistance R_s (ohms/cm^2) (the reciprocal of G_s), is plotted against the retarding potential V_{bo}. The behavior of the curve and order of magnitude of R_s is the same as that given by Kuper[43] and Smyth.[41] It is to be noted that figure 15 is a plot of the stream resistance per square centimeter (ohms/cm^2), while the curve given by Kuper is the gap and stream resistance for an area of 0.08 cm^2.

2. Another interesting example is the variation of loading with frequency for a constant value of retarding voltage. The range of frequency used in this calculation is from 3×10^8 cps, to 1.05×10^{10} cps, or in terms of wavelength from 100 cms to 2.86 cm. The constant value of retarding voltage was -3 volts. Figure 16 shows G_s(mho/cm^2) plotted against the frequency. Note that the loading increases practically linearly from 1.5×10^9 to 4.5×10^9 cps and has a maximum value at 6.4×10^9 cps. Kuper states that the loading should decrease as the frequency increases though experimental results seem to indicate the opposite. As figure 16 shows, this

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*These values are assumed as being representative values for T and I_e.
**For these values of h and θ', the term involving $\exp(-h\theta^2)$ is completely negligible so the loading is given by

$$G_s = 3.44 \cdot 10^{-4} I_e T V_{bo}^{-2} \left[\frac{1}{h}(h+2)\, e^{-\frac{1}{h}}\, {}_1F_1\left(\tfrac{1}{2}; \tfrac{3}{2}; \tfrac{1}{h}\right) - 1 \right] \qquad (7.30)$$

calculation is in agreement with the experiments cited.

3. Another interesting calculation is on the distance the electrons that produce the maximum loading penetrate in the retarding field. This can be obtained by differentiating (7.20) with respect to θ and setting the resultant equal to zero. Performing this operation, the following transcendental equation is obtained for the transit angle producing maximum loading.

$$\cos 2\theta\left[1+2\theta^2(1-h)\right] = 1 + 2h\theta^2\left[\theta \sin 2\theta - 1\right] \qquad (7.31)$$

The constants used were those given in example 1. Results of this toilsome calculation are shown in figure 17. On the phase space diagram (Fig.11), the arrows on the curves for retarding potential -1 and -2 volts indicate the distance the maximum loading electrons travel. It is to be noted that these maximum loading distances are sufficiently far out from the cathode that the assumption of linear field is very good. Thus, the loading plotted in figure 15 for one volt retarding potential (57.1 ohms/cm^2) should be in agreement with experimental results.*

The large magnitude of loading calculated above indicates that, for close spaced high frequency tubes, the cathode-first grid region should not be in the radio frequency circuit. By using the first and second grids as the input gap** with the potential minimum residing in the cathode-first grid region, shot noise reduction can be obtained[24] without undesirable loading.

*Compare[41].
**Triode operation of tetrode.

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Appendix 1

PRINCIPAL SYMBOLS

Symbol	Meaning and Dimension	First Appearance (equation no.)
a	x directed electron acceleration, cm/sec^2	(2.1)
a^2	Constant $(2\ell V_T)^{-1}$, sec^2/cm^2	(5.18)
$(\ )_a$	Quantity ( ) evaluated at <u>a</u> plane	(2.8)
$\alpha(t_a)$	Time dependent electron acceleration at <u>a</u> plane, cm/sec^2	(2.9)
α	Extension in phase used only in part IV, cm^2/sec	(4.40)
α	Constant used in part VII, $(\omega d/\ell V_{bo})$	(7.19a)
$(\ )_b$	Quantity ( ) evaluated at <u>b</u> plane	(2.34)
B_s	Electron stream susceptance	(6.10)
β	Complex transit angle $(j\omega T)$	(2.70)
$1/\beta$	Parameter, cm^{-1}	(5.37)
c	Numeric, $2.33 \times 10^{-6} d^{-2}$ mhos/cm^2/volts$^{\frac{1}{2}}$	(3.00)
Γ_1, Γ_2	Integrand in second order potential	(3.15)
d	Spacing between <u>a</u> and <u>b</u> planes, cm	(2.44)
D	Density-in-phase, sec/cm^4	(4.01)
δ	Perturbation on electron stationary transit time	(2.14)
e	Absolute value of electronic charge, 1.59×10^{-19} coulombs	(1.3)
E	Electric field strength, volts/cm	(1.1)
ϵ	Permittivity of vacuum, 8.85×10^{-14} farods/cm	(1.1)
f	Frequency, sec^{-1}	
${}_1F_1$	Confluent hypergeometric function	(7.28)

ζ	Space charge factor	(2.58)
G_s	Electron stream conductance per cm^2, $mhos/cm^2$	(6.10)
$\underline{H}$	Magnetic field strength, amps/cm	(1.1)
H	Hamiltonian, ergs	(4.37)
h	Constant	(7.19)
θ	Transit angle	(2.70)
θ_ℓ	Transit angle corresponding to v_ℓ	(7.21)
I	Total current density, $amps/cm^2$	(1.2)
I_e	Emission current density, $amps/cm^2$	(5.14)
j	$(-1)^{\frac{1}{2}}$	(2.64)
k	Boltzmann constant, 1.380×10^{-16} ergs/deg	(4.45)
K	Kinetic energy density of electrons, $grams/cm/sec^2$	(4.03)
K	Time independent part of current density multiplied by ℓ/e, cm/sec^3	(2.6)
$\Delta K/\Delta t$	Electron and field energy interchange per sec per cm^2	(7.17)
ℓ	Constant, $10^7 e/m_e$ 1.76×10^{15} coul/grams	(2.2)
m_e	Electronic mass, 9.03×10^{-28} grams	(1.3)
μ	Permeability of vacuum, 1.257×10^{-8} henrys/cm	(1.1)
$(\;)_n$	"n"th order of ( )	
$(\Delta n/\Delta t)$	Number of electrons per second leaving cathode per cm^2	(7.16)
N	Mass density of electrons, $grams/cm^3$	(4.01)
$\nu(t_a)$	Time dependent electron velocity at $\underline{a}$ plane, cm/sec	(2.10)
ξ	Dimensionless parameter $(x-d)/\rho$	(5.36)
p	complex angular frequency $(j\omega)$, sec^{-1}	(2.64)
p	Dimensionless parameter, V_{b1}/V_{bo} (part VII)	(7.05)

P	Momentum density of electrons, grams/cm^2/sec	(4.02)
P	Power in watts/cm^2	(7.18)
Q	Convection current density, amps/cm^2	(1.1)
r_{oc}	Complete space charge, zero frequency diode resistance	(3.06)
R_s	Resistance of electron stream ohms/cm^2	(6.11)
ρ	Charge density, coulombs/cm^3	(1.1)
t_o	Stationary electron transit time	(7.03)
T	Stationary electron transit time(parts I, II, III)	(2.13)
T	Absolute temperature of cathode, degrees (parts IV to VII)	(4.45)
U	Effective velocity of electron stream	(4.07)
Υ_{nm}	nth order, mth harmonic of fundamental frequency transit angle coefficient. m = 0 indicates zero frequency	(3.07)
v	x directed electron velocity, cm/sec	(2.1)
v_ℓ	Phase space border emission velocity at cathode	(5.13)
v'	Phase space border emission velocity at x > 0	(5.17)
V	Potential, volts	(2.34)
V'	Voltage measured from potential minimum or anode	(5.20)
V_T	Thermal voltage $kT10^{-7}/e$ T/11605, volts	(4.47)
V_M	Voltage of potential minimum referred to cathode	(5.47)
ϕ'''	Time dependent part of current density multiplied by ℓ/e	(2.6)
φ	Dimensionless parameter, V'/V_T	(5.33)
W	ℓV, volt-cm^2/sec^2	(2.36)
W_E	Electric field energy density, ergs/cm^3 (parts IV to VI)	(4.19)
x_m	Value of x at potential minimum	(5.48)

$X(\theta)$	Real part of $\Upsilon_{22}(\beta)$	(3.29)
$Y(\theta)$	Imaginary part of $\Upsilon_{22}(\beta)$	(3.29)
$\underline{Y}$	Diode admittance per cm^2, mhos/cm^2	(6.07)
$\underline{Y}_c$	Diode admittance per cm^2 produced by capacity of <u>a</u> and <u>b</u> planes	(6.09)
$\underline{Y}_s$	Diode admittance per cm^2 produced by electron stream	(6.10)
$\Psi(\theta)$	$\sin\theta - \theta\cos\theta$	(7.12)
Z	Diode impedance per cm^2, ohms/cm^2	(6.09)
Z_c	Diode impedance per cm^2 produced by capacity of <u>a</u> and <u>b</u> planes	(6.10)
Z_s	Diode impedance per cm^2 produced by electron stream	(6.11)
ω	Angular frequency	(2.64)

Appendix 2

FIRST ORDER ELECTRONIC COEFFICIENTS

$$\underline{A} = \frac{1}{\epsilon}(v_{oa}+v_{ob})\frac{T}{2}\frac{1}{\beta}\left[1-\frac{\zeta}{3}\left(1-\frac{12S}{\beta^3}\right)\right]$$

$$\underline{B} = \frac{1}{\epsilon}\frac{T^2}{\beta^3}\left[v_{oa}(P-\beta Q)-v_{oa}P+\zeta(v_{oa}+v_{ob})P\right]$$

$$\underline{C} = -\frac{2}{\ell}\zeta(v_{oa}+v_{ob})\frac{P}{\beta^2}$$

$$\underline{D} = 2\zeta\left(1+\frac{v_{oa}}{v_{ob}}\right)\frac{P}{\beta^2}$$

$$\underline{E} = \left[1-\zeta\left(1+\frac{v_{oa}}{v_{ob}}\right)\right]e^{-\beta}$$

$$\underline{F} = \frac{\epsilon}{\ell}\frac{2\zeta}{T^2}\left[1+\frac{v_{oa}}{v_{ob}}\right]\beta e^{-\beta}$$

$$\underline{G} = -\frac{\ell}{\epsilon}\frac{T^2}{\beta^3}\left[(P-\beta Q) - \frac{v_{oa}}{v_{ob}}P + \zeta\left(1+\frac{v_{oa}}{v_{ob}}\right)P\right]$$

$$\underline{H} = -\frac{\ell}{\epsilon}\frac{T^2}{2}\left(1+\frac{v_{oa}}{v_{ob}}\right)(1-\zeta)\frac{e^{-\beta}}{\beta}$$

$$\underline{I} = \left[\frac{v_{oa}}{v_{ob}} + \zeta\left(1+\frac{v_{oa}}{v_{ob}}\right)\right]e^{-\beta}$$

where $\beta = j\theta$, $P = 1 - e^{-\beta} - \beta e^{-\beta}$, $Q = 1 - e^{-\beta}$ and $S = 2P - \beta Q$

The space charge factor ζ is defined in (2.58). v_{oa} and v_{ob} are the zero order velocities at the a and b planes, respectively.

For small values of β

$$P = \frac{1}{2}\beta^2 - \frac{1}{3}\beta^3 + \frac{1}{8}\beta^4 + \cdots$$

$$Q = \beta - \frac{1}{2}\beta^2 + \frac{1}{6}\beta^3 - \frac{1}{24}\beta^4 \cdots$$

$$S = -\frac{1}{6}\beta^3 + \frac{1}{12}\beta^4 - \frac{1}{40}\beta^5 + \frac{1}{180}\beta^6 \cdots$$

Appendix 2a

COMPLETE SPACE CHARGE IMPEDANCE COEFFICIENT

For complete space charge $(\zeta = 1)$ with $V_{ao} = 0$,

$$A = \frac{2}{3}\frac{V_{bo}}{I_o}\left[\frac{2}{\beta} + \frac{12S}{\beta^4}\right] = r_{oc}\,\underline{\Upsilon}_1(\beta)$$

where

$$\underline{\Upsilon}_1(\beta) = \left[\frac{2}{\beta} + \frac{12S}{\beta^4}\right] = \frac{12}{\theta^4}\left[(2 - 2\cos\theta - \theta\sin\theta) + j(2\sin\theta - \theta - \theta\cos\theta - \frac{1}{6}\theta^3)\right]$$

and

$$r_{oc} = \frac{2}{3}\frac{V_{bo}}{I_o}$$

Appendix 3

TABLE OF INTEGRALS FOR PART V

The following integrals were used in the evaluation of equations in part V. The integrals are listed in approximately the order they are used in the text.

$$\int_0^\infty v e^{-a^2v^2} dv = \frac{1}{2a^2}$$

$$\int_0^{v'} v e^{-a^2v^2} dv = \frac{1}{2a^2}\left(1 - e^{-a^2v'^2}\right)$$

$$\int_0^\infty e^{-a^2v^2} dv = \frac{\pi^{\frac{1}{2}}}{2a}$$

$$\int_0^{v'} e^{-a^2v^2} dv = \frac{\pi^{\frac{1}{2}}}{2a} \operatorname{erf}(av')$$

$$\int_0^\infty v^2 e^{-a^2v^2} dv = \frac{\pi^{\frac{1}{2}}}{4a^3}$$

$$\int_0^{v'} v^2 e^{-a^2v^2} dv = \frac{\pi^{\frac{1}{2}}}{4a^3} \operatorname{erf}(av') - \frac{v'}{2a^2} e^{-a^2v'^2}$$

where $\quad \operatorname{erf} x = \frac{2}{\pi^{\frac{1}{2}}}\int_0^x e^{-t^2} dt$

The error integral (erf) is tabulated on pages 23 to 32 in Jahnke and Emde.(46)

Appendix 4

RETARDING FIELD DIODE TABLE

$\left(\frac{d\varphi}{d\xi}\right)_{\varphi=0}$	φ_a	$-\lvert V_b\rvert$	E^*_{anode}	$E^*_{cathode}$	E^{**}	E^*_{linear}
0	3.664	0.3155	0	0.494	0.3745	0.0902
4.55	5.84	0.503	0.0854	0.508	0.381	0.1438
240	11.59	1.0	0.262	0.571	0.448	0.2855
1.75×10^5	23.23	2.0	0.563	0.754	0.634	0.571
		3.5	1.0	1.12		1.0
		5.0	1.43	1.52		1.43
		7.5	2.425	2.22		2.425
		10.0	2.86	2.905		2.86

*volts/10^{-3}in.
**Field 10^{-4} in. from cathode in volts/10^{-3}in.

Appendix 5

SERIES EXPANSION COEFFICIENTS FOR ξ AND φ FOR DIODE SOLUTION

$$\xi = a_0 + a_1\varphi^{\frac{1}{2}} + a_2\varphi + a_3\varphi^{\frac{3}{2}} + \cdot \quad \cdot \quad \cdot + a_n\varphi^{\frac{n}{2}} + \cdot \; \cdot \; \cdot \qquad (5.54)$$

where

$a_0 = 0$

$a_1 = 2$

$a_2 = \pm\frac{2}{3\pi^{\frac{1}{2}}}$

$a_3 = \frac{4}{9\pi} - \frac{1}{6}$

$a_4 = \pm\frac{5\cdot2}{3^3\pi^{\frac{3}{2}}} \mp \frac{7}{5\cdot3\cdot2^2\pi^{\frac{1}{2}}}$

$a_5 = \frac{7\cdot2^2}{3^4\pi^2} - \frac{3}{5^2\pi} + \frac{1}{5\cdot3\cdot2^4}$

$$a_6 = \pm\frac{7\cdot 2^2}{3^4\pi^{5/2}} \mp \frac{11}{3^4\pi^{3/2}} \pm \frac{37}{7\cdot 5\cdot 3^2\cdot 2^4\pi^{1/2}}$$

$$a_7 = \frac{11\cdot 2^3}{3^5\pi^3} - \frac{13}{3^4\pi^2} + \frac{1103}{7^2\cdot 5^2\cdot 3^2\cdot 2^3\pi} + \frac{1}{7\cdot 3\cdot 2^6}$$

$$a_8 = \pm\frac{13\cdot 11\cdot 2}{3^6\pi^{7/2}} \mp \frac{7}{3^2\cdot 2^2\pi^{5/2}} \pm \frac{1819}{7\cdot 5\cdot 3^4\cdot 2^5\pi^{3/2}} \pm \frac{151}{7\cdot 5\cdot 3^3\cdot 2^8\pi^{1/2}}$$

$$a_9 = \frac{13\cdot 11\cdot 5\cdot 2^2}{3^8\pi^4} - \frac{17\cdot 11\cdot 7\cdot 2}{5\cdot 3^7\cdot \pi^3} - \frac{53\cdot 17}{5\cdot 3^6\cdot 2^3\pi^2} - \frac{577}{7\cdot 5^2\cdot 3^4\cdot 2^5\pi} - \frac{1}{5\cdot 3^2\cdot 2^{10}}$$

Coefficients a_0 to a_7 inclusive were obtained by two different methods of expansion; a_8 and a_9 were not rechecked.

$$\varphi = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + \cdot\ \cdot\ \cdot \tag{5.55}$$

where

$$b_0 = b_1 = 0$$

$$b_2 = \frac{1}{4}$$

$$b_3 = \mp\frac{1}{12\pi^{1/2}}$$

$$b_4 = \frac{1}{3^2\cdot 2^4\pi} + \frac{1}{3\cdot 2^5}$$

$$b_5 = \mp\frac{13}{5\cdot 3\cdot 2^7\pi^{1/2}}$$

$$b_6 = \pm\frac{1}{3^3\cdot 2\pi^{5/2}} \mp \frac{19}{5^2\cdot 3^3\cdot 2\pi^{3/2}} \pm \frac{417}{7\cdot 5\cdot 3\cdot 2^{11}\pi^{1/2}}$$

Appendix 6

EVALUATION OF (7.27) AND (7.28)

Let
$$u = \int_0^\infty e^{-hx^2}\sin 2px\,dx \tag{a}$$

Differentiating with respect to p gives
$$\frac{du}{dp} = 2\int_0^\infty x e^{-hx^2}\cos 2px\,dx \tag{b}$$

The integrand can be written as

$$2xe^{-hx^2}\cos 2px\,dx = d\left[-\frac{1}{h}e^{-hx^2}\cos 2px\right] - \frac{2p}{h}e^{-hx^2}\sin 2px\,dx \tag{c}$$

Substituting (c) in (b) gives

$$\frac{du}{dp} = \left[-\frac{1}{h}\, e^{-hx^2} \cos 2px\right]_0^\infty - \frac{2p}{h}\int_0^\infty e^{-hx^2} \sin 2px\, dx$$

or $$\frac{du}{dp} + \frac{2p}{h}\, u = \frac{1}{h} \tag{d}$$

Equation (d) is a linear differential equation of the first order whose solution is

$$u = e^{-\frac{p^2}{h}}\left[\frac{1}{h}\int e^{\frac{p^2}{h}} dp + C\right]$$

The constant of integration C is zero since from (a) $u = 0$, when $p = 0$. Changing the integration variable

$$u = \frac{1}{h^{\frac{1}{2}}}\, e^{-\frac{p^2}{h}} \int_0^{\frac{p}{h^{1/2}}} e^{t^2} dt \tag{e}$$

For $p = 1$

$$\left(\frac{1}{h^{\frac{1}{2}}}\right) e^{-\frac{1}{h}} \int_0^{\frac{1}{h^{1/2}}} e^{t^2} dt = \int_0^\infty e^{-hx^2} \sin 2x\, dx \tag{7.27}$$

Multiplying (7.27) by h and using (850.4) in Dwight[23] result in

$$h^{\frac{1}{2}} \int_0^{\frac{1}{h^{1/2}}} e^{t^2} dt = \sum_{m=0}^{\infty} \frac{1}{m!\,(2m+1)\, h^m} = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{h}\right)^m \Gamma(2m+1)}{\Gamma(2m+2)\,\Gamma(m+1)} \tag{f}$$

By use of the Legendre's duplication formula given on page 225 of Copson[47]

$$\frac{\Gamma(2m+2)}{\Gamma(2m+1)} = 2\,\frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+\frac{1}{2})} \tag{g}$$

Substituting (g) in (f) gives

$$h^{\frac{1}{2}} \int_0^{\frac{1}{h^{\frac{1}{2}}}} e^{t^2} dt = \frac{1}{2}\sum_{m=0}^{\infty} \frac{\left(\frac{1}{h}\right)^m \Gamma(m+\frac{1}{2})}{\Gamma(m+1)\,\Gamma(m+\frac{3}{2})} \tag{h}$$

The confluent hypergeometric function is defined on page 247 and 260 of Copson[47] as

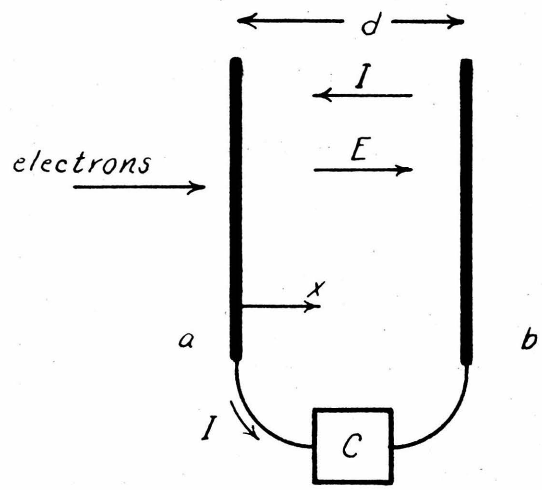
$$_1F_1(\alpha;\beta;z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{z^m\, \Gamma(\alpha+m)}{\Gamma(\beta+m)\,\Gamma(m+1)} \tag{i}$$

Comparing (i) and (h) and using (850.6) and (850.7) in Dwight[23] give

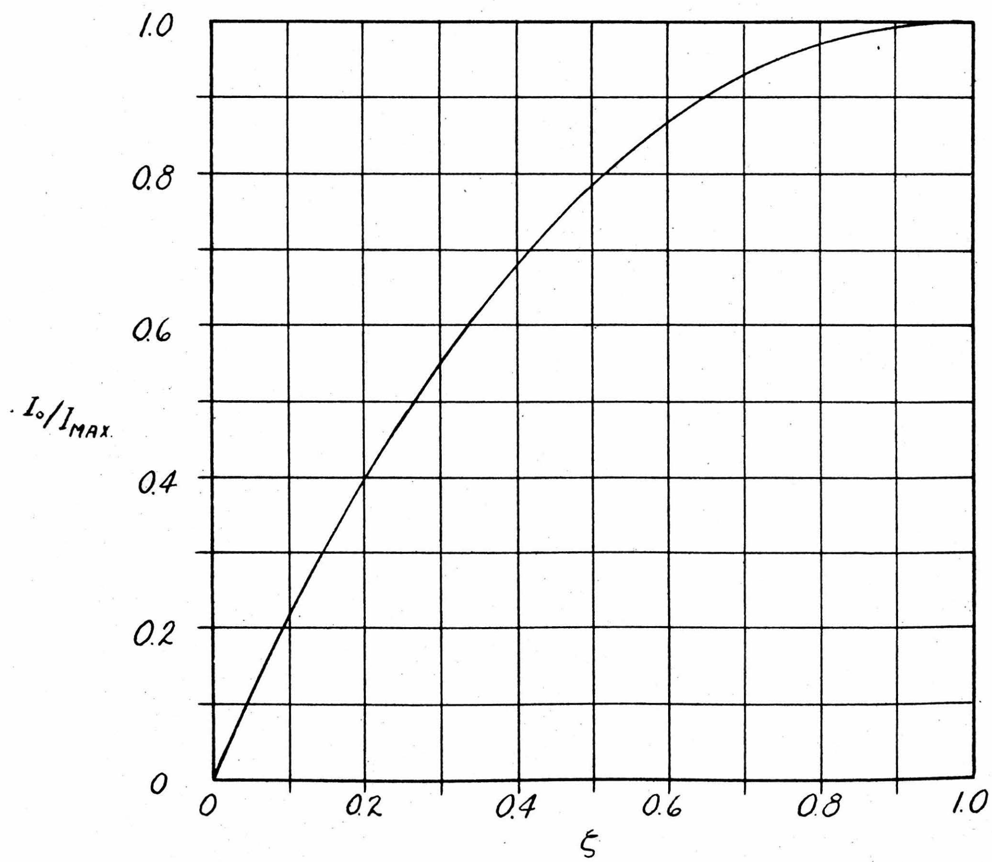
$$h^{\frac{1}{2}}\int_0^{\frac{1}{h^{\frac{1}{2}}}} e^{t^2}dt = {}_1F_1\left(\tfrac{1}{2};\tfrac{3}{2};\tfrac{1}{h}\right) \qquad \text{(j)}$$

Substituting (j) in (7.27) gives

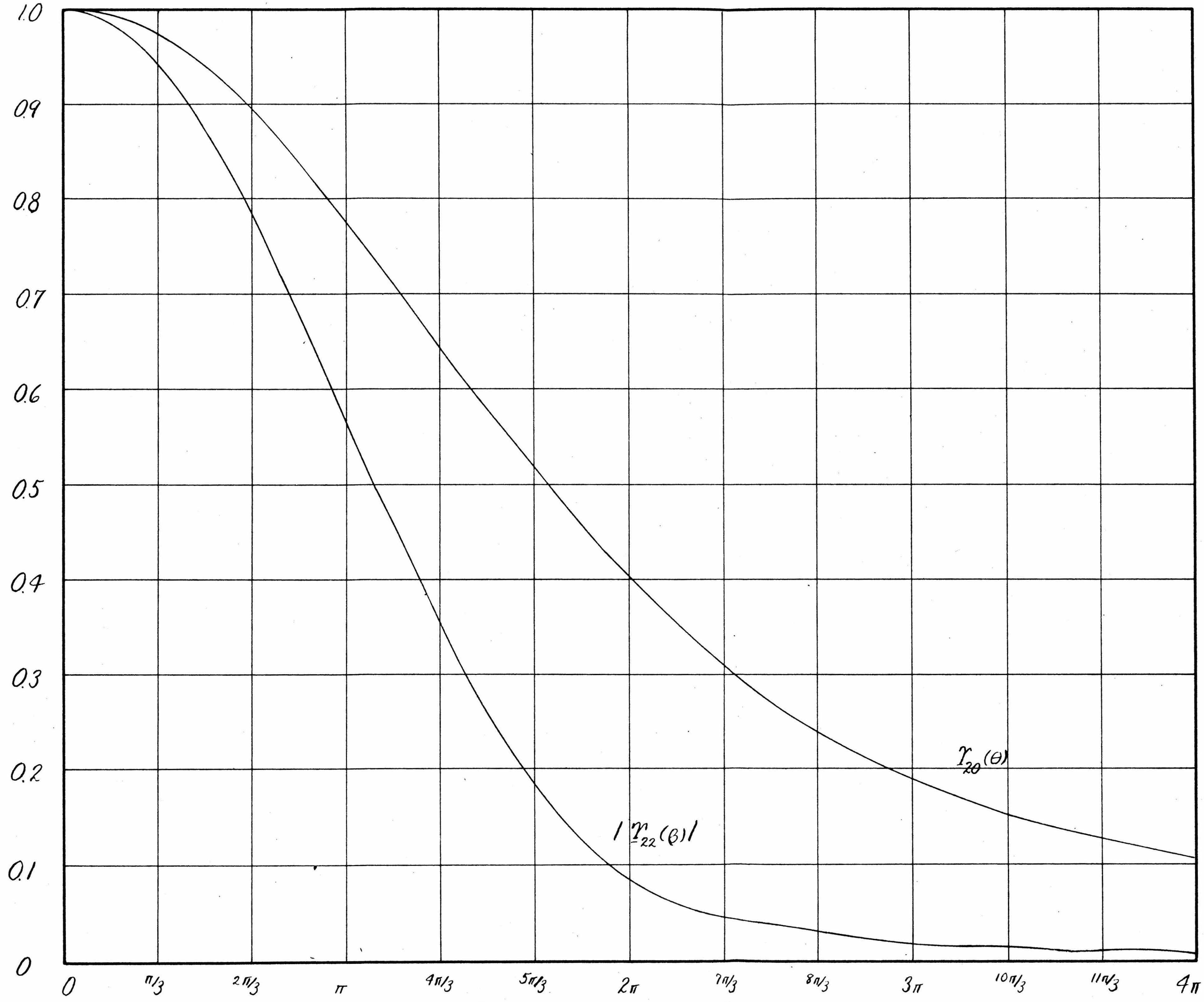
$$\int_0^\infty e^{-hx^2}\sin 2x\,dx = \frac{1}{h}e^{-\frac{1}{h}}\,{}_1F_1\left(\tfrac{1}{2};\tfrac{3}{2};\tfrac{1}{h}\right) \qquad (7.28)$$



Basic Picture for Electronic Analysis
Figure 1.

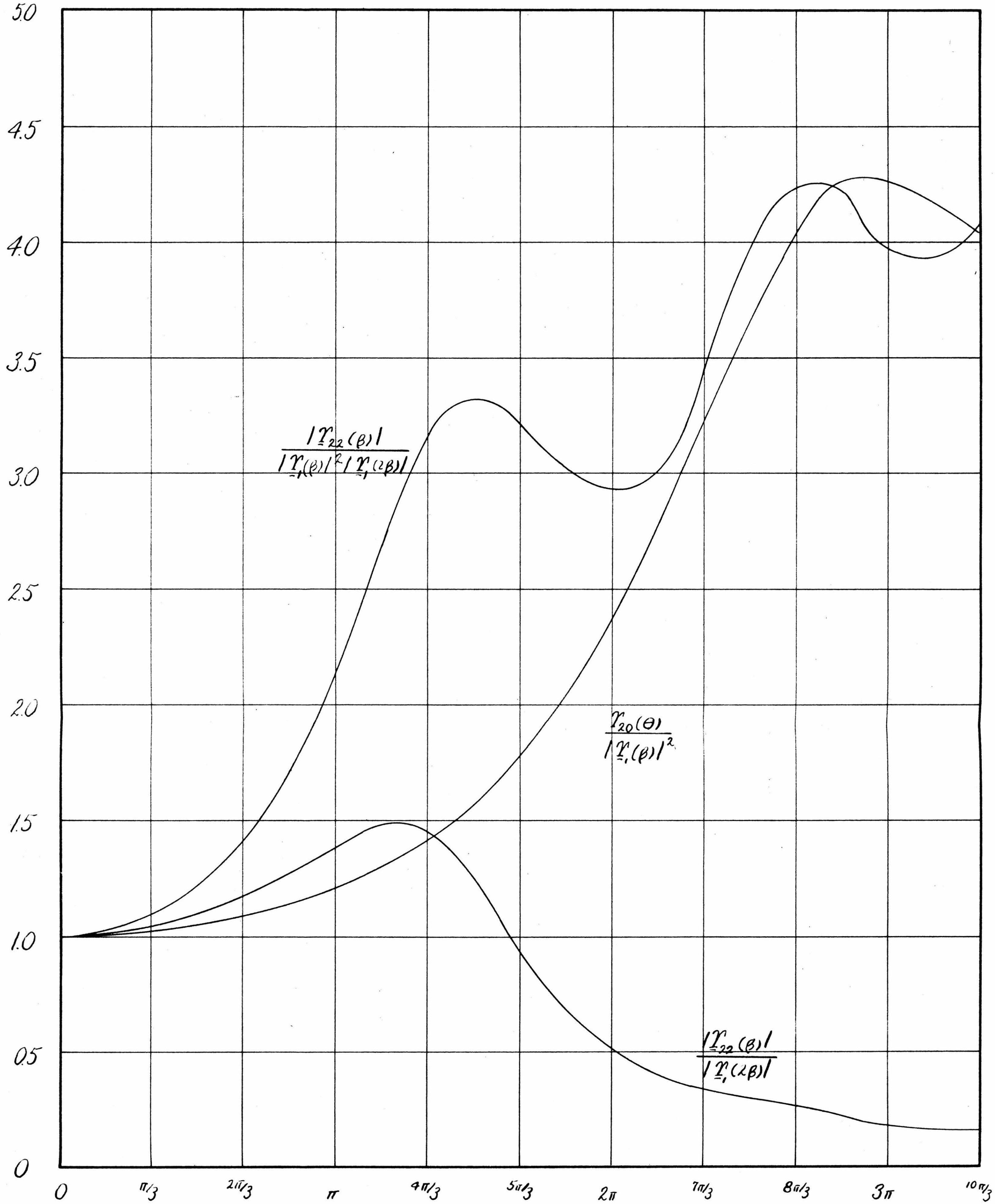


Plot of Equation (2.61)
Figure 2.



Second Order Transit Time Coefficients

Figure 3.



Transit Angle (Θ), Radians

Second Order Transit Time Coefficients

Figure 4.

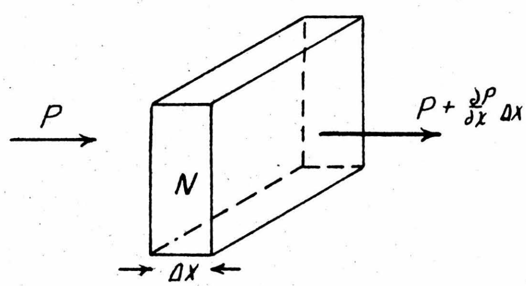


Figure for Equation (4.13)
Figure 5.

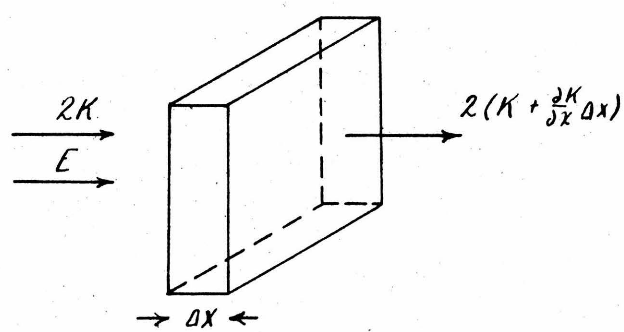
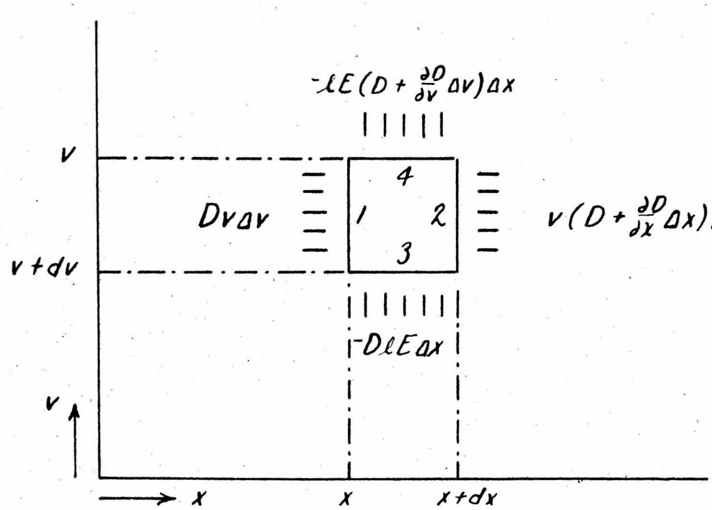


Figure for Equation (4.17)
Figure 6.



Phase Space
Figure 7.

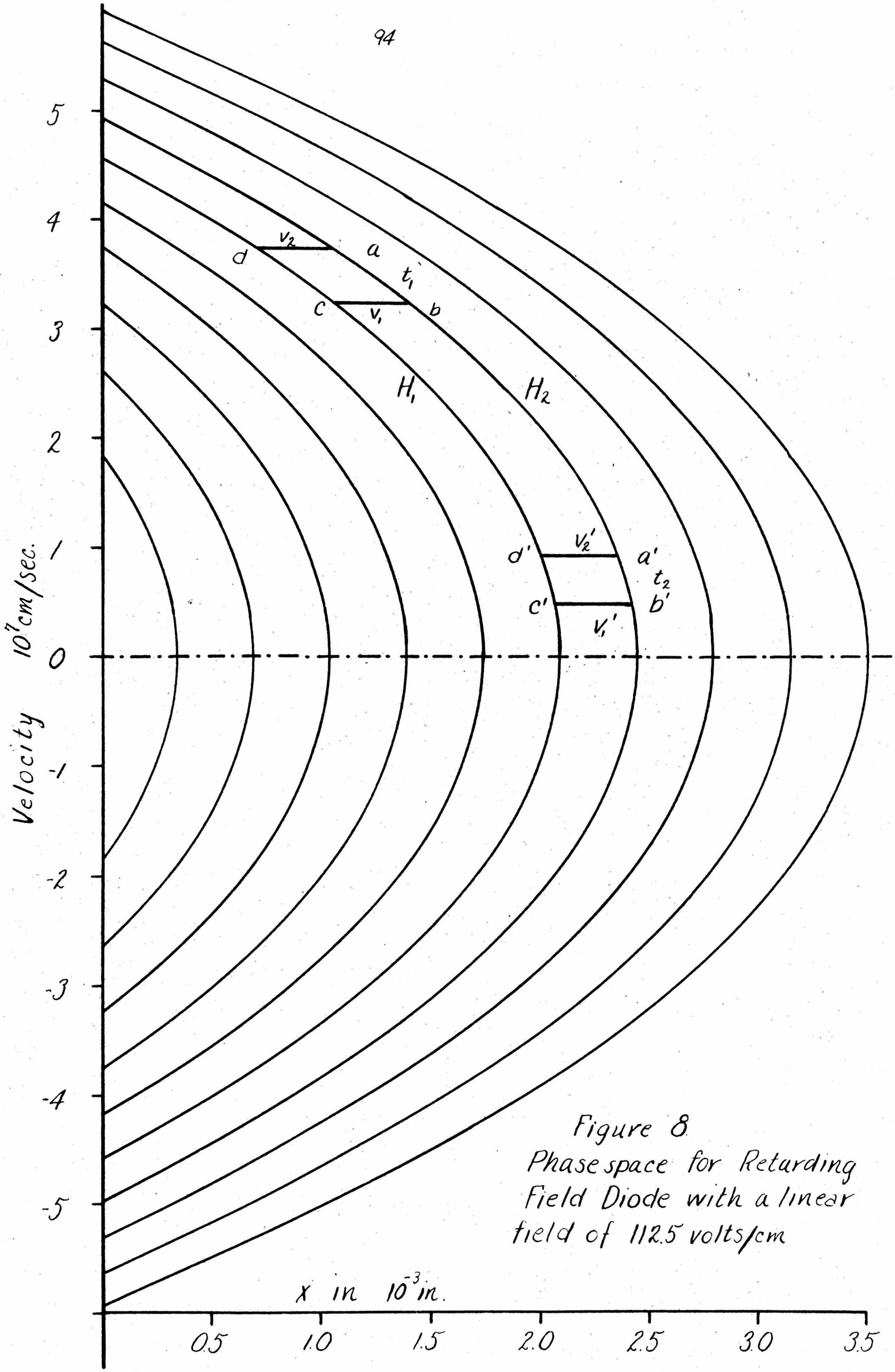
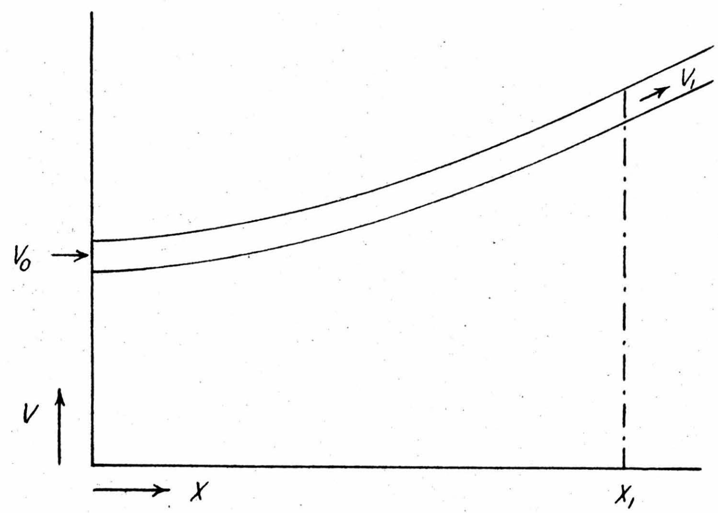
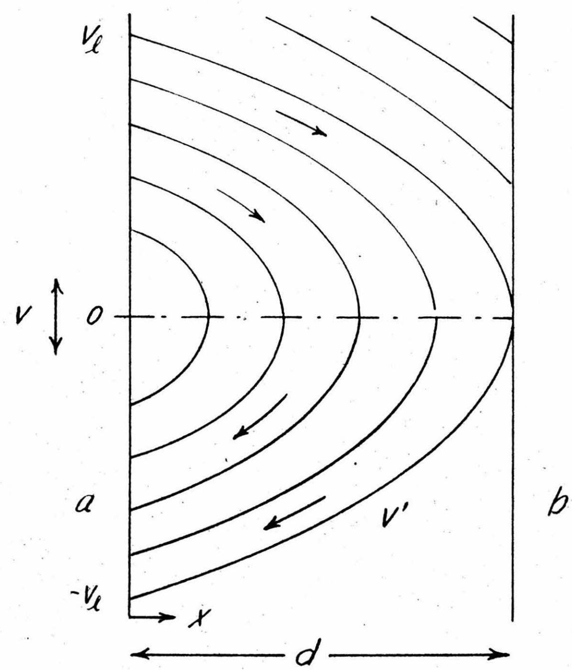


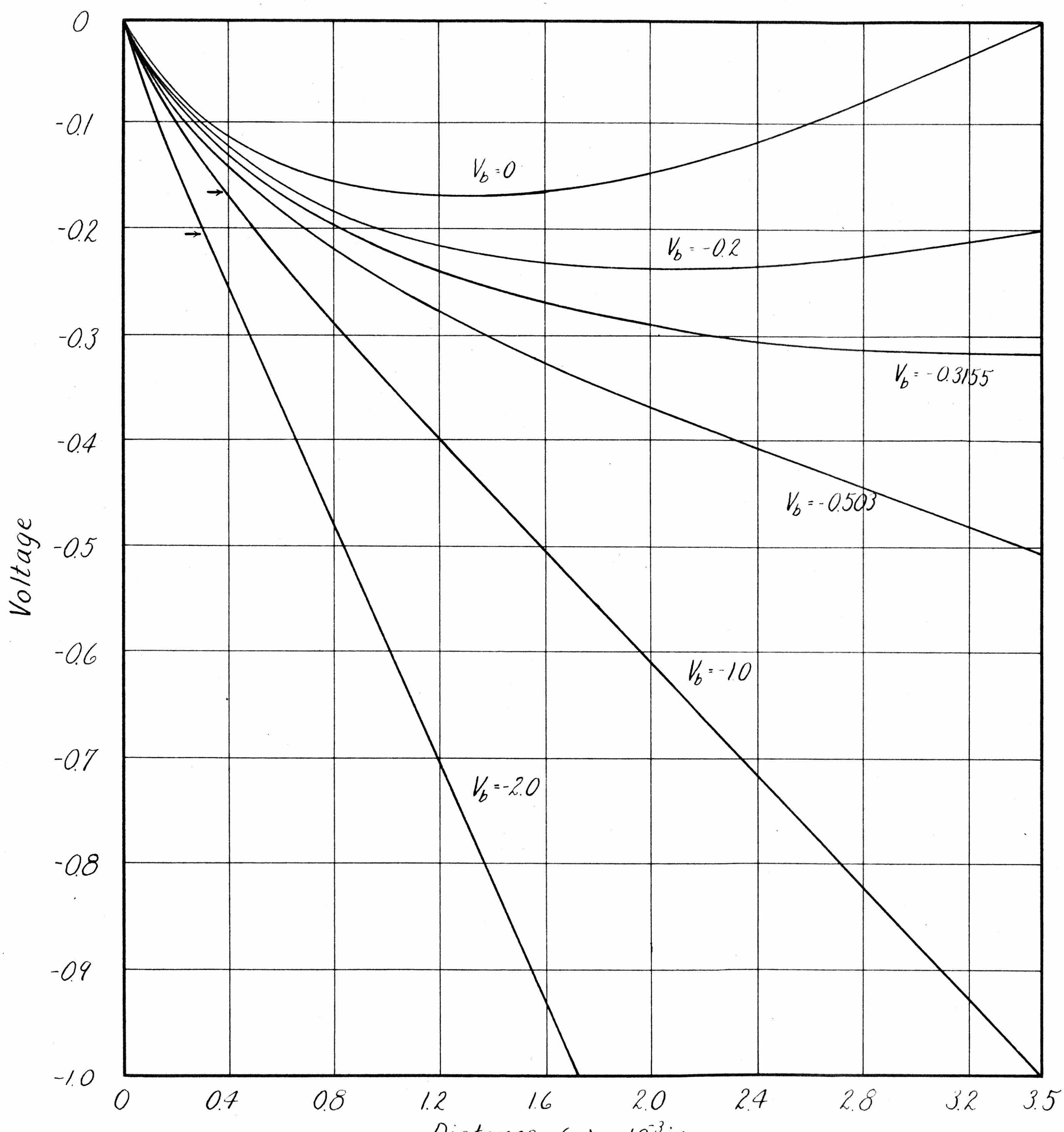
Figure 8
Phase space for Retarding Field Diode with a linear field of 112.5 volts/cm



Phase Space for Single Valued Velocity Stream
Figure 9.

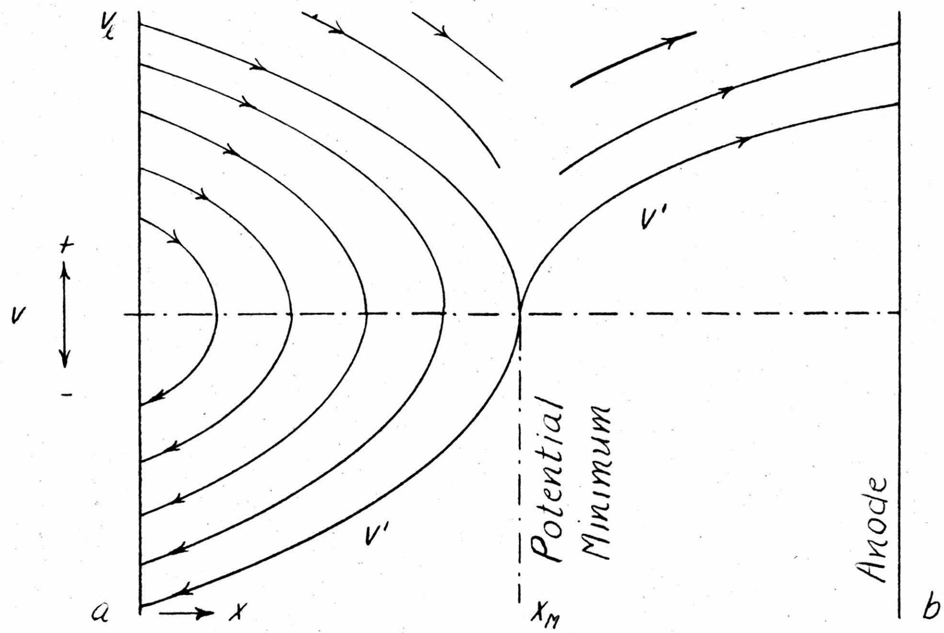


Phase Space for Retarding Field Diode
Figure 10.

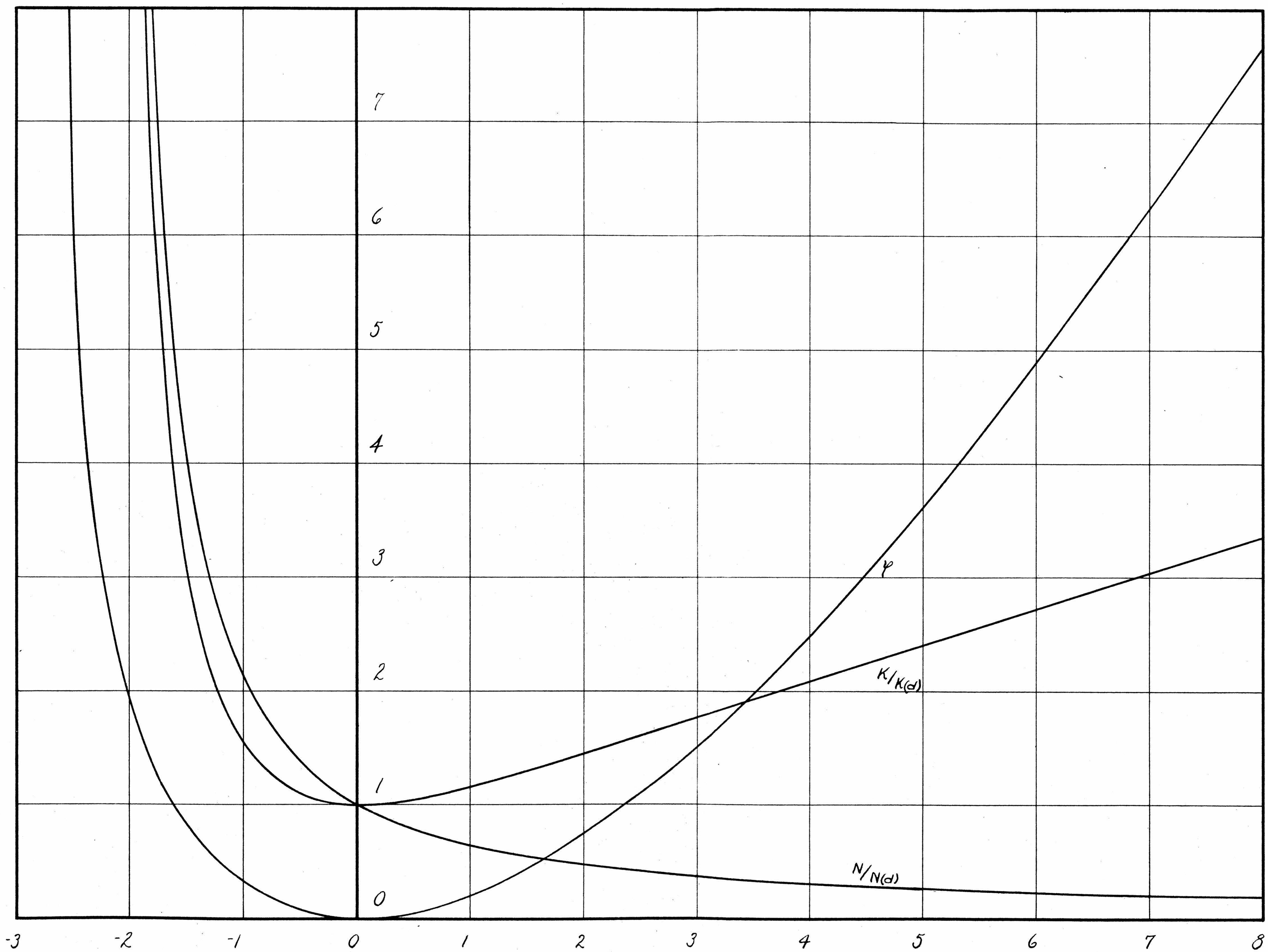


Potential Distribution in Diode

Figure 11.

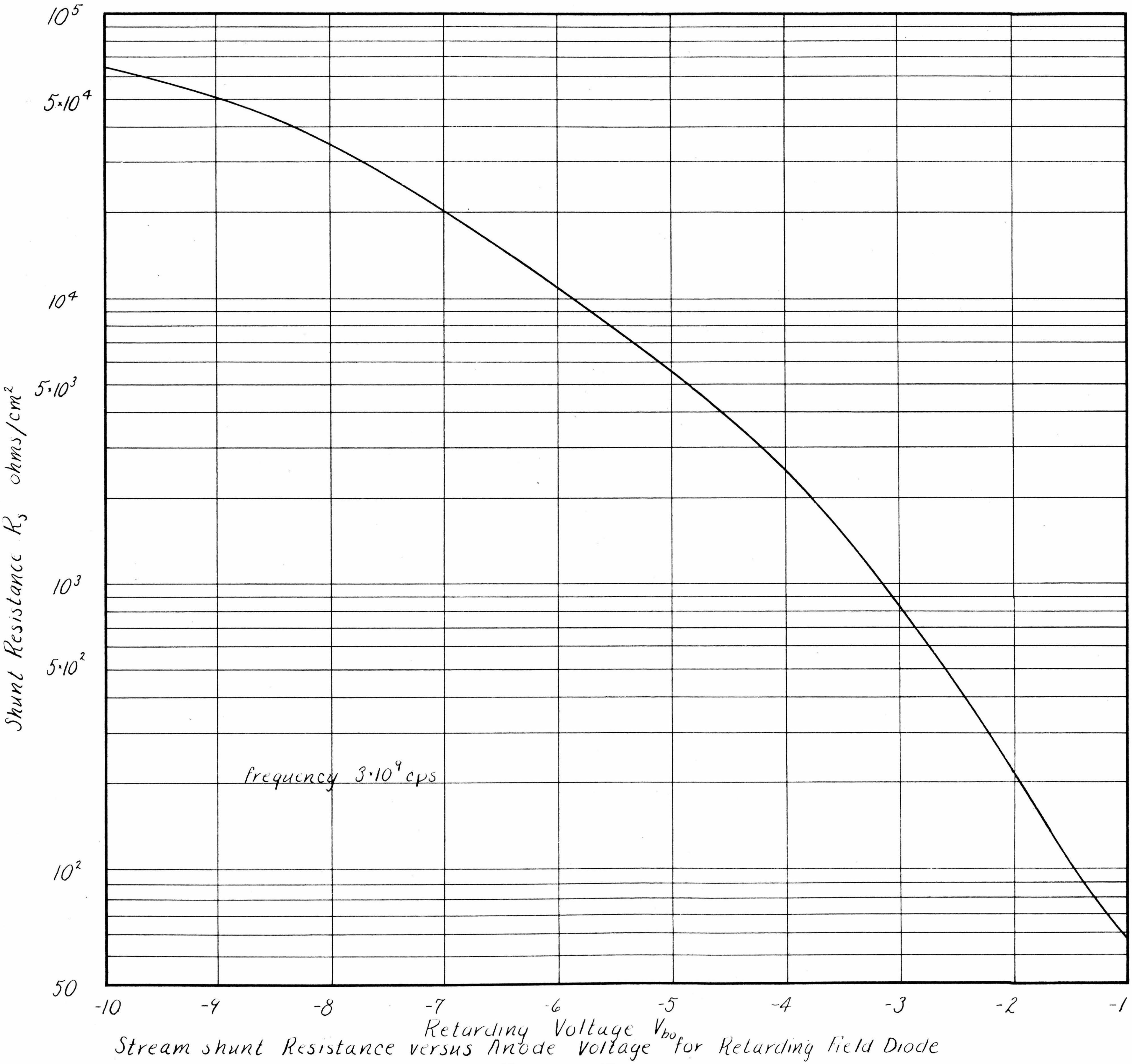


Phase Space for Diode with
Potential Minimum
Figure 13



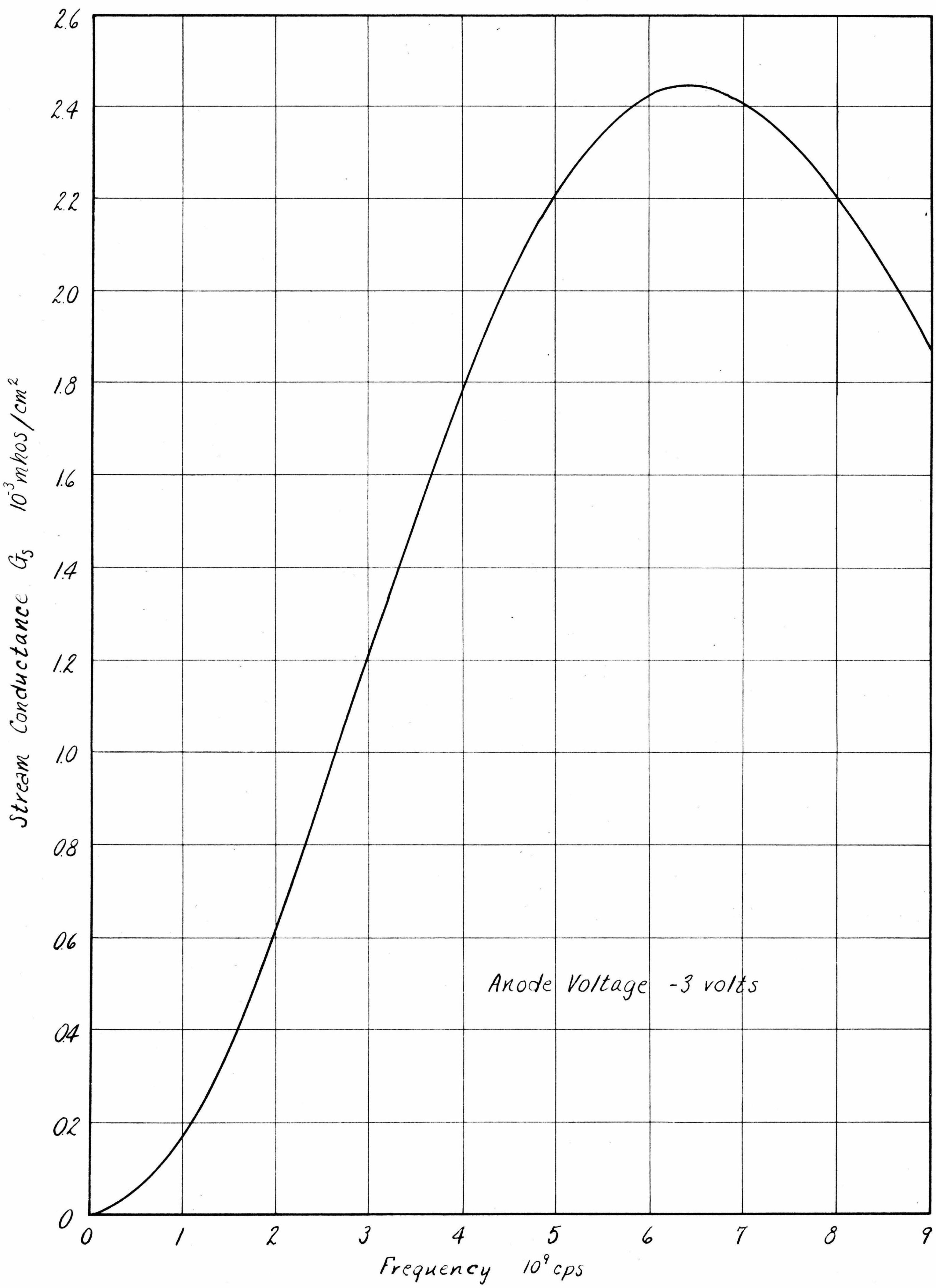
Kinetic Energy Density ($K/K(d)$), Mass Density ($N/N(d)$), and φ versus ξ for a Complete Space Charge Diode

Figure 14

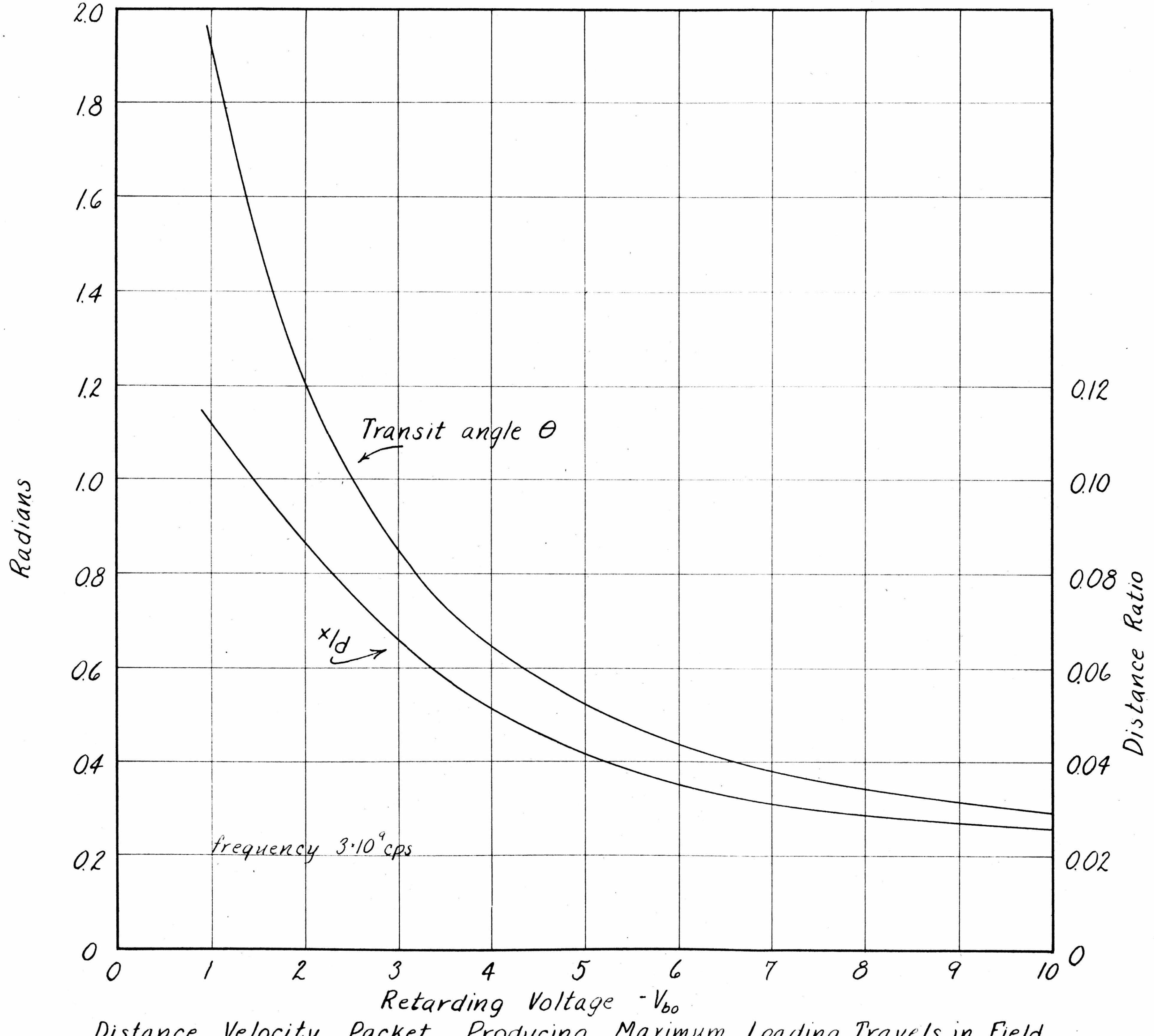


Stream shunt Resistance versus Anode Voltage for Retarding Field Diode

Figure 15.



Stream Conductance versus Frequency for Retarding Field Diode
Figure 16.



Distance Velocity Packet Producing Maximum Loading Travels in Field
Figure 17.