APPLICATIONS OF THE TENSOR CALCULUS TO PROBLEMS ARISING IN DYNAMICS

Thesis by

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SUMMARY

The theory of the application of the tensor calculus to conservative dynamical systems is well known. However, the results of the actual application to specific systems are not. The thesis concerns itself with this aspect of the subject.

The gyroscope or spinning top is first considered. Necessary and sufficient conditions are developed in order that the Riemannian geometric space described by the system shall have constant Riemannian curvature, be an Einstein space, have a zero curvature invariant, etc. It is found that a simple relationship must exist between the moment of inertia coefficients. It is shown that the space described by the gyroscope is a special case of a general class of Riemannian spaces having the above properties.

A second type of conservative dynamical system is comsidered. It is shown that with a suitable choice of moment of inertias the above properties hold and that the geometric space obtained reduces to the gyroscope space.

An investigation is made into the form of the components ofthe fundamental metric tensor in order that general Riemannian spaces of three, four, and n-dimensions, suggested by the dynamical systems themselves, shall be Einstein spaces, possess constant Riemannian

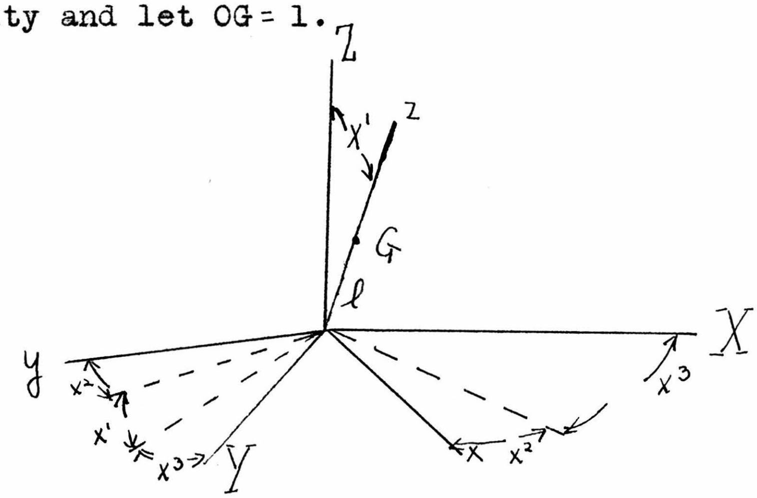
curvature and zero curvature invariant, and may be mapped conformally on a flat-space.

The author wishes to express his sincerest appreciation to Dr. A. D. Michal, under whose direction the thesis was written. Professor Michal's ever willingness to discuss and suggest ideas aided the author immeasurably.

A solid of revolution, having one point of its axis fixed, and whose initial rotation about its axis is given, is subject only to the external force of gravity.

We investigate the Riemannian geometry described by such a top or gyroscope.[1]

Taking Euler's angles x^1, x^2, x^3 for independent coordinates with the axis of spin being z, the position of a rigid body of this sort can be given at anytime through these angles. Let (fig. 1) G be the center of gravity and let OG = l.[2]



The Potential Function will be

1.1) $U = Mgl \cos x^1$

1. For the general theory of the application of the tensor calculus to dynamics see Brillouin (1), McConnell (1), pp.233-251, A. D. Michal (1) and (2).

2. For a more complete discussion see Whittaker, (1), pp. 163-164.

while the kinetic energy will be given by

1.2) $2T = A(\dot{x}^{1\,2} + \dot{x}^{3\,2}\sin^2 x^1) + B(\dot{x}^2 + \dot{x}^3\cos x^1)^2$

where A and B are the moments of inertia about the principal axes of the body.[3]

The three Lagrange's equations in x^1, x^2 and x^3 are

2.2) $A\dfrac{d\dot{x}^1}{dt} - (A - B)\sin x^1\cos x^1\,\dot{x}^{3\,2} + B\sin x^1\,\dot{x}^2\dot{x}^3 + \gamma\sin x^1 = 0$

3.2) $B\dfrac{d}{dt}(\dot{x}^2 + \cos x^1\,\dot{x}^3) = 0$

4.2) $\dfrac{d}{dt}((A\sin^2 x^1 + B\cos^2 x^1)\dot{x}^3 + B\cos x^1\,\dot{x}^2) = 0$

Using summation convention, the kinetic energy (1.2) of the system can be represented in the following manner.

5.2) $T = \frac{1}{2}a_{ij}\dot{x}^i\dot{x}^j$ where $i, j = 1, 2, 3$

and

6.2) $a_{11} = A,\quad a_{22} = B,\quad a_{33} = A\sin^2 x^1 + B\cos^2 x^1$

$a_{23} = a_{32} = B\cos x^1,\quad a_{12} = a_{21} = a_{13} = a_{31} = 0$

The quadratic form (5.2) is positive definite and hence we may define an element of arc-length, ds^2, for a Riemannian geometric space in the following manner:

7.2) $ds^2 = a_{ij}\,dx^i dx^j$[4] where the a_{ij}'s are defined as in (6.2).

We shall call this space the "Static Riemannian Space"

3. For the derivations of the equations for the potential function and the kinetic energy see Akimoff (1)pp135-136.

4. A complete explanation of the prinicple involved can be found in Brillouin (1), McConnell (1) and Michal (1).

of our dynamical system under investigation and proceed to develope the Christoffel symbols[5] for the Static Riemannian space.

All the Christoffel symbols of the first kind are zero with the exception of the following:

$$
\begin{aligned}
&[33,1] = (B-A)\ \sin x' \cos x' \\
&[12,3] = [21,3] = -\tfrac{1}{2}B\ \sin x' \\
1.3)\quad &[13,2] = [31,2] = -\tfrac{1}{2}B\ \sin x' \\
&[13,3] = [31,3] = (A-B)\ \sin x' \cos x' \\
&[23,1] = [32,1] = \tfrac{1}{2}B\ \sin x'
\end{aligned}
$$

In order to form the Christoffel symbols of the second kind we first compute the conjugate of a_{rp} where

$$a^{rp} = \frac{\text{cofactor of } a_{pr}}{|a|}$$

$$2.3)\quad |a| = \begin{vmatrix} A & 0 & 0 \\ 0 & B & B\cos x' \\ 0 & B\cos x' & A\sin x' + B\cos x' \end{vmatrix} = A^2 B \sin^2 x'$$

therefore

$$a^{11} = 1/A \qquad a^{22} = \frac{A \sin^2 x' + B \cos^2 x'}{AB \sin^2 x'} \qquad a^{33} = \frac{1}{A\sin^2 x'}$$

$$3.3)\quad a^{23} = a^{32} = \frac{-\cos x'}{A \sin^2 x'} \qquad a^{12} = a^{21} = a^{13} = a^{31} = 0$$

We calculate the Christoffel symbols of the second kind.

5. Symbol of the first kind: $[m,n,p] = \frac{1}{2}\left(\frac{d a_{np}}{d x^m} + \frac{d a_{pm}}{d x^n} - \frac{d a_{mn}}{d x^p}\right)$

Symbol of the second kind: $\left\{ {r \atop m\ n} \right\} = a^{rp}[m,n,p]$

$$\left\{ {1 \atop 33} \right\} = \frac{B-A}{A} \sin x' \cos x' \qquad \left\{ {1 \atop 23} \right\} = \left\{ {1 \atop 32} \right\} = \frac{B \sin x'}{2A}$$

$$\left\{ {2 \atop 12} \right\} = \left\{ {2 \atop 21} \right\} = \frac{B \cos x'}{2A \sin x'}$$

1.4) $$\left\{ {2 \atop 13} \right\} = \left\{ {2 \atop 31} \right\} = \frac{(B-A)\cos^2 x' - A}{2A \sin x'}$$

$$\left\{ {3 \atop 12} \right\} = \left\{ {3 \atop 21} \right\} = \frac{-B}{2A \sin x'}$$

$$\left\{ {3 \atop 13} \right\} = \left\{ {3 \atop 31} \right\} = \frac{(2A-B)\cos x'}{2A \sin x'}$$

We calculate the Riemann symbols of the first kind of the Static Riemannian space which are defined in the following manner: [6]

$$R_{hijk} = \frac{\partial}{\partial x^j}[ik,h] - \frac{\partial}{\partial x^k}[ij,h] + \left\{ {l \atop ij} \right\}[hk,l] - \left\{ {l \atop ik} \right\}[hj,l]$$

The six distinct components of the Riemann tensor are as follows:

2.4) $$R_{1212} = \frac{B^2}{4A}$$

$$R_{1313} = \frac{4A^2 + AB - B(A-B)\cos^2 x' - 4A(A\cos^2 x' + B\sin^2 x')}{4A}$$

6. For the development and properties of the components of the Riemann tensor see Eisenhart (1) pp. 20 - 47 (2) p. 101, 103, 120-122.

$$R_{2323} = \frac{B^2 \sin^2 x^1}{4A}$$

1.5) $R_{1231} = 0$

$R_{1323} = 0$

$$R_{1231} = \frac{-B^2 \cos x^1}{4A}$$

In computing the first covariant derivative[7] of the Riemann symbols we need concern ourselves only with the covariant derivative with respect to x^1.

$R_{1313,1} = B(A-B) \sin x^1 \cos x^1$

2.5) $R_{1231,1} = B(B-A) \sin x^1$

$R_{1212,1} = R_{2323,1} = R_{1232,1} = R_{1323,1} = 0.$

The components of the Ricci tensor[8] are defined in the following manner:

$$R_{ij} = \frac{\partial^2 \log\sqrt{a}}{\partial x^i \partial x^j} - \frac{\partial}{\partial x^k}\begin{Bmatrix} k \\ i\,j \end{Bmatrix} + \begin{Bmatrix} m \\ i\,k \end{Bmatrix}\begin{Bmatrix} k \\ m\,j \end{Bmatrix} - \begin{Bmatrix} m \\ i\,j \end{Bmatrix}\frac{\partial \log\sqrt{a}}{\partial x^m}$$

Since the following relationship is useful in calculations it is stated also.

$$\frac{\partial \log\sqrt{a}}{\partial x^m} = \begin{Bmatrix} i \\ i\,m \end{Bmatrix}$$

where i is a summation index.

3.5)
$$R_{11} = \frac{B-2A}{2A} \qquad R_{22} = \frac{-B^2}{2A^2} \qquad R_{33} = \frac{AB-2A^2+(2A^2-AB-B^2)\cos^2 x^1}{2A^2}$$

$$R_{23} = \frac{-B^2 \cos x^1}{2A^2}, \qquad R_{12} = R_{21} = R_{13} = R_{31} = 0$$

7. For the definition and concept of the first covariant derivative consult Eisenhart (1)pp27-31,(2)pp107-112, and McConnell (1) pp140-149.

8. For a complete discussion of the geometrical significance and derivation of the Ricci tensor see Eisenhart (1)pp22-47 (2)pp103-112.

As we use the curvature invariant, R,[9] in theorems which follow, it is computed here.

$$R = g^{ij} R_{ij}$$

1.6) $R = \dfrac{B - 4A}{2A^2}$

The conformal curvature tensor, R_{ijk}[10], is defined in the following manner:

$$R_{ijk} = R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)} (g_{ik} R_{,j} - g_{ij} R_{,k})$$

All components of this tensor for the Static Riemannian space are zero with the exception of the following:

2.6)
$$R_{113} = \frac{B(B-A) \sin x'}{2A^2} = -R_{231} = R_{312} = -R_{321}$$
$$R_{313} = \frac{B(A-B) \sin x' \cos x'}{A^2}$$

The Riemannian space in which the dynamical trajectories for a given energy constant, h, are geodesics, has an element of arc-length ds^2 given by

3.6) $ds^2 = 2(h-U)a_{ij}\, dx^i dx^j$

where U is the potential energy of the conservative dynamical system under consideration,[11] which in our case is $\gamma \cos x'$. Or the element of arc-length may be expressed

9. For the development and properties of the curvature invariant see Eisenhart (1) p.83.

10. The derivation and significance of the conformal curvature tensor is treated fully in Eisenhart (1) p.91.

11. For a complete discussion of the principle involved see McConnell (1) pp. 246-251.

in the following form:

1.7) $ds^2 = g_{ij}\, dx^i dx^j$

where

$$g_{11} = 2A(h - \gamma \cos x^1) \quad , \quad g_{22} = 2B(h - \gamma \cos x^1)$$

2.7) $g_{33} = 2h(A\sin^2 x^1 + B\cos^2 x^1) - 2\gamma(A\sin^2 x^1 \cos x^1 + B\cos^3 x^1)$

$$g_{23} = g_{32} = 2B \cos x^1(h - \gamma\cos x^1) \quad , \quad g_{12} = g_{21} = g_{13} = g_{31} = 0$$

The Riemannian space with the above line element we shall call the "Action Riemannian Space" and now proceed to develope its properties.

All Christoffel symbols of the first kind of the Action space are zero with the exception of the following:

$$[11,1] = A\gamma \sin x^1$$

$$[22,1] = -B\gamma \sin x^1$$

$$[33,1] = 2h(B-A)\sin x^1 \cos x^1 + \gamma(2A-3B)\sin x^1 \cos^2 x^1 - A\gamma \sin^3 x^1$$

$$[12,2] = [21,2] = B\gamma \sin x^1$$

3.7) $[12,3] = [21,3] = -B(h-2\gamma\cos x^1)\sin x^1$

$$[13,2] = [31,2] = -B(h-2\gamma\cos x^1)\sin x^1$$

$$[13,3] = [31,3] = 2h(A-B)\sin x^1 \cos x^1 + \gamma(3B-2A)\sin x^1 \cos^2 x^1 + A\gamma \sin^3 x^1$$

$$[23,1] = [32,1] = B(h-2\gamma\cos x^1)\sin x^1$$

We calculate the conjugate ~~of~~ g^{rp} of g_{rp} where

$$g^{rp} = \frac{\text{cofactor of } g_{pr}}{|g|}$$

4.7) $|g| = 8A^2B \sin^2 x^1 (h - \gamma\cos x^1)^3$

therefore

5.7) $g^{11} = \dfrac{1}{2A(h - \gamma\cos x^1)} \quad , \quad g^{22} = \dfrac{A\sin^2 x^1 + B\cos^2 x^1}{2AB\sin^2 x^1 (h - \gamma\cos x^1)}$

1.8)
$$g^{33} = \frac{1}{2A\ \sin^2 x^1\,(h-\gamma\cos x^1)}, \qquad g^{12} = g^{21} = g^{13} = g^{31} = 0$$

$$g^{23} = g^{32} = \frac{-\cos x^1}{2A\ \sin^2 x^1\,(h-\gamma\cos x^1)}$$

All Christoffel symbols of the second kind or the Action space are zero with the exception of the following.

$$\left\{ {1 \atop 11} \right\} = \frac{\gamma \sin x^1}{2(h-\gamma\cos x^1)}, \qquad \left\{ {1 \atop 22} \right\} = \frac{-B\gamma\sin x^1}{2A(h-\gamma\cos x^1)}$$

$$\left\{ {1 \atop 33} \right\} = \frac{2h(B-A)\sin x^1\cos x^1 + \gamma(2A-3B)\sin x^1\cos^2 x^1 - A\gamma\sin^3 x^1}{2A(h-\gamma\cos x^1)}$$

$$\left\{ {1 \atop 23} \right\} = \left\{ {1 \atop 32} \right\} = \frac{B(h-2\gamma\cos x^1)\sin x^1}{2A(h-\gamma\cos x^1)}$$

$$\left\{ {2 \atop 12} \right\} = \left\{ {2 \atop 21} \right\} = \frac{\gamma(A\sin^2 x^1 - B\cos^2 x^1) + Bh\cos x^1}{2A\sin x^1(h-\gamma\cos x^1)}$$

2.8)
$$\left\{ {2 \atop 13} \right\} = \left\{ {2 \atop 31} \right\} = \frac{(B-A)\cos^2 x^1 - A}{2A\sin x^1}$$

$$\left\{ {3 \atop 12} \right\} = \left\{ {3 \atop 21} \right\} = \frac{-B}{2A\sin x^1}$$

$$\left\{ {3 \atop 13} \right\} = \left\{ {3 \atop 31} \right\} = \frac{(2A-B)(h-\gamma\cos x^1)\cos x^1 + A\gamma\sin^2 x^1}{2A\sin x^1(h-\gamma\cos x^1)}$$

The six distinct components of the Riemann symbols of the first kind for the Action space are as follows:

$$R_{1212} = \frac{2AB\gamma(\gamma - h\cos x') + B^2(h - \gamma\cos x')^2}{2A(h - \gamma\cos x')}$$

$$R_{1313} = -3AB\sin^2 x'(h^2 + \gamma^2\cos^2 x') + 4A^2 h\sin^2 x'(h - 3\gamma\cos x')$$
$$+ 2AB\gamma h\cos x'(4\sin^2 x' - \cos^2 x') + 2A^2\gamma^2\sin^2 x'(\sin^2 x' + 4\cos^2 x')$$
$$+ 2AB\gamma^2\cos^4 x' + B^2\cos^2 x'(h - \gamma\cos x')^2 \div 2A(h - \gamma\cos x')$$

1.9) $$R_{2323} = B\sin^2 x'(B(h - 2\gamma\cos x')^2 + \gamma\cos x'(B - A)$$
$$(2h - 3\gamma\cos x') - A\gamma^2) \div 2A(h - \gamma\cos x')$$

$$R_{1232} = 0$$

$$R_{1323} = 0$$

$$R_{1231} = AB\gamma h(3\cos^2 x' - 1) - AB\gamma^2(1 + \cos^2 x')\cos x'$$
$$-B^2(h - \gamma\cos x')^2\cos x' \div 2A(h - \gamma\cos x')$$

The non-zero first covariant derivatives of the components of the Riemannian tensor are

$$R_{1212,1} = 2A^2Bh^2\gamma + 2AB^2h^2\gamma - 6A^2B\gamma^3 + \cos x'(4A^2B\gamma^2 h + B^3h^3)$$
$$+\cos^2 x'(2A^2Bh^2\gamma - 2AB^2h^2\gamma - 2B^3\gamma h^2 + 6A^2B\gamma^3)$$
$$+\cos^3 x'(-4A^2B\gamma^2 h + B^3\gamma^2 h) \div 2A^2\sin x'(h - \gamma\cos x')^2$$

$$R_{1313,1} = 4A^2B\gamma h^2 - 6A^3\gamma^3 + (-10A^2B\gamma^2 h + 12A^3\gamma^2 h - 4AB^2h^3$$
$$+ 4A^2Bh^3)\cos x'$$

2.9) $$+\cos^2 x'(16A^3\gamma h^2 - 12A^2B\gamma h^2 - 10A^3\gamma^3 + 6AB^2h^2\gamma)$$
$$+\cos^3 x'(70A^2B\gamma^2 h - 72A^3\gamma^2 h - 2AB^2\gamma^2 h + 4AB^2h^3 - 4A^2Bh^3)$$
$$+\cos^4 x'(-40A^2B\gamma^3 + 12A^2Bh^2\gamma - 16A^3h^2\gamma + 70A^3\gamma^3 - 8AB^2\gamma^2 h - 4AB^2\gamma^3)$$
$$+\cos^5 x'(60A^3\gamma^2 h - 64A^2B\gamma^2 h + 4AB^2\gamma^2 h)$$
$$+\cos^6 x'(50A^2B\gamma^3 - 54A^3\gamma^3 + 4AB^2\gamma^3) \div 2A^2(h - \gamma\cos x')^2\sin x'$$

$$R_{1231,1} = (2AB^2h^3 - 2A^2Bh^3 - 2A^2hB\gamma^2) + \cos x'(2A^2B\gamma h^2 - 10AB^2h^2\gamma)$$
$$+\cos^2 x'(2A^2B\gamma^2 h - 4AB^2h^3 + 14AB^2\gamma^2 h + 2A^2Bh^3)$$
$$+\cos^3 x'(2A^2B\gamma h^2 + 16AB^2\gamma h^2 + 4A^2B\gamma^3 - 4AB^2\gamma^3)$$
$$+\cos^4 x'(-20AB^2\gamma^2 h - 8A^2B\gamma^2 h + B^3\gamma^2 h)$$
$$+\cos^5 x'(8AB^2\gamma^3 + 2A^2B\gamma^3) + 2A^2\sin x'(h - \gamma\cos x')^2$$

1.10)

$$R_{2323,1} = (3AB^2h^2\gamma + 2A^2Bh^2\gamma - A^2B\gamma^3)$$
$$+\cos x'(-6AB^2h\gamma^2 - 6A^2B\gamma^3 - 2A^2Bh\gamma^2)$$
$$+\cos^2 x'(3AB^2\gamma^3 + 12A^2B\gamma^3 - 10AB^2h^2\gamma - 4A^2Bh^2\gamma)$$
$$+\cos^3 x'(20AB^2h\gamma^2 + 10A^2Bh\gamma^2 + 6A^2B\gamma^3)$$
$$+\cos^4 x'(-10AB^2\gamma^3 - 15A^2B\gamma^3 + 7AB^2h^2\gamma + 2A^2Bh^2\gamma)$$
$$+\cos^5 x'(-14AB^2h\gamma^2 - 12A^2Bh\gamma^2)$$
$$+\cos^6 x'(7AB^2\gamma^3) + 2A^2(h - \gamma\cos x')^2\sin x'.$$

We develope the components of the Ricci tensor for the Action Space. All components are zero with the exception of the following.

$$R_{11} = (2ABh^2 - 4A^2\gamma^2 - 8A^2h^2) + \cos x'(14A^2\gamma h - 4AB\gamma h)$$
$$+\cos^2 x'(-2A^2\gamma^2 + 2AB\gamma^2 - 2ABh^2 + 4A^2h^2)$$
$$+\cos^3 x'(-14A^2\gamma h + 4AB\gamma h)$$
$$+\cos^4 x'(6A^2\gamma^2 - 2AB\gamma^2) + 4A^2\sin^2 x'(h - \gamma\cos x')^2$$

2.10)

$$R_{22} = (-AB\gamma^2 - 2B^2h^2) + \cos x'(4AB\gamma h + 4B^2\gamma h)$$
$$+\cos^2 x'(-2B^2\gamma^2 - 3AB\gamma^2) + 4A^2(h - \gamma\cos x')^2$$

$$R_{33} = 2AB\gamma h - 2B^2h^2\cos x' + \cos^2 x'(-2AB\gamma h + 4B^2\gamma h)$$
$$+\cos^3 x'(-4AB\gamma^2 - 2B^2\gamma^2) + 4A^2(h - \gamma\cos x')^2$$

$$R_{33} = 3ABh^2 - 4A^2h^2 - A^2\gamma^2 + \cos x'(-9AB\gamma h + 14A^2\gamma h)$$
$$+\cos^2 x'(-ABh^2 - 3B^2h^2 + 5AB\gamma^2 + 4A^2h^2 - 8A^2\gamma^2)$$
$$+\cos^3 x'(7AB\gamma h + 7B^2\gamma h - 14A^2\gamma h) \tag{1.11}$$
$$+\cos^4 x'(-5AB\gamma^2 - 4B^2\gamma^2 + 9A^2\gamma^2) \div 4A^2(h - \gamma\cos x')^2$$

The curvature invariant for the Action Riemannian space has the following form.

$$R = (-6A^2\gamma^2 - 12A^2h^2 + 3ABh^2) + \cos x'(-13AB\gamma h + 32A^2\gamma h)$$
$$+\cos^2 x'(-12A^2\gamma^2 + 8A^2h^2 + 4AB\gamma^2 - B^2h^2 - ABh^2)$$
$$+\cos^3 x'(15AB\gamma h + 3B^2\gamma h - 32A^2\gamma h) \tag{2.11}$$
$$+\cos^4 x'(18A^2\gamma^2 - 2B^2\gamma^2) \div 8A^3\sin^2 x'(h - \gamma\cos x')^3$$

All the components of the conformal curvature tensor are zero for the Action Riemannian space with the exception of the following.

$$R_{212} = 18A^2B\gamma^3 + 8A^2B\gamma h^2 - 12AB^2\gamma h^2$$
$$+\cos x'(-12A^2B\gamma^2h + 38AB^2\gamma^2h + 2B^3h^3 - 4AB^2h^3)$$
$$+\cos^2 x'(-26A^2B\gamma^3 - 16A^2B\gamma h^2 - 8AB^2\gamma^3 + 32AB^2\gamma h^2 - 8B^3\gamma h^2)$$
$$+\cos^3 x'(39A^2B\gamma^2h - 108AB^2\gamma^2h + 8B^3\gamma^2h)$$
$$+\cos^4 x'(-6A^2B\gamma^3 + 8A^2B\gamma h^2 + 20AB^2\gamma^3 - 2B^2\gamma^3 - 20AB^2\gamma h^2)$$
$$+\cos^5 x'(-17A^2B\gamma^2h + 66AB^2\gamma^2h + 2B^3\gamma^3h)$$
$$+\cos^6 x'(-4AB^2\gamma^3 + 6A^2B\gamma^3 - 2B^3\gamma^3) \div 16A^3\sin^3 x'(h - \gamma\cos x')^3 \tag{3.11}$$

$$R_{221} = -R_{212}$$

$$R_{213} = -10AB^2h^3+28A^2B\gamma^2h+8A^2Bh^3$$
$$+\cos x'(-60A^2B\gamma h^2+32AB^2h^2\gamma+26A^2B\gamma^3)$$
$$+\cos^2x'(-76A^2B\gamma^2h-8AB^2\gamma^2h+12AB^2h^3+4B^3h^3-16A^2Bh^3)$$
$$+\cos^3x'(-58A^2B\gamma^3+4AB^2\gamma^3+112A^2B\gamma h^2-64AB^2h^2\gamma-16B^3h^2\gamma)$$
$$+\cos^4x'(99A^2B\gamma^2h-20AB^2\gamma^2h-6AB^2h^3+18B^3\gamma^2h-7B^3h^3+8A^2Bh^3)$$
$$+\cos^5x'(34A^2B\gamma^3+4AB^2\gamma^3-52A^2B\gamma h^2+32AB^2\gamma h^2-6B^3\gamma^3+8B^3\gamma h^2)$$
$$+\cos^6x'(-44A^2B\gamma^2h+24AB^2\gamma^2h-8B^3\gamma^2h)$$
$$+\cos^7x'(-10A^2B\gamma^3+2B^3\gamma^3)+16A^3\sin^3x'(h-\gamma\cos x')^3$$

$$R_{231} = -R_{213}$$

1.12)
$$R_{313} = -4A^2B\gamma h^2+18A^3\gamma^3+\cos x'(20A^3\gamma^2h-2A^2B\gamma^2h+28A^2Bh^3-26AB^2h^3)$$
$$+\cos^2x'(-108A^2B\gamma h^2+100AB^2\gamma h^2-68A^3\gamma^3+42A^2B\gamma^3)$$
$$+\cos^3x'(-45A^3\gamma^2h-86AB^2\gamma^2h+190A^2B\gamma^2h-44A^2Bh^3+58AB^2h^3+6B^3h^3)$$
$$+\cos^4x'(-126A^2B\gamma^3+22AB^2\gamma^3+92A^3\gamma^3+196A^2B\gamma h^2-168AB^2\gamma h^2-24B^3h^2\gamma)$$
$$+\cos^5x'(-35A^3\gamma^2h+112AB^2\gamma^2h-123A^2B\gamma^2h+36B^3\gamma^2h$$
$$-36AB^2h^3-4B^3h^3+16A^2Bh^3)$$
$$+\cos^6x'(98A^2B\gamma^3-12AB^2\gamma^3-96A^3\gamma^3-10B^3\gamma^3-84A^2B\gamma h^2+68AB^2\gamma h^2$$
$$+16B^3\gamma h^2)$$
$$+\cos^7x'(60A^3\gamma^2h-28AB^2\gamma^2h+42A^2B\gamma^2h-26B^3\gamma^2h)$$
$$+\cos^8x'(-22A^2B\gamma^3-2AB^2\gamma^3+18A^3\gamma^3+6B^3\gamma^3)$$
$$+16A^3\sin^3x'(h-\gamma\cos x')^3$$

In the theorems which follow, A and B stand for the moments of inertia of our spinning top computed with respect to the prinsipal axes.

<u>Theorem 1:</u> A necessary and sufficient condition in order that the Static Riemannian space be an Einstein space is that $A = B$.

Proof:

By an Einstein space is meant a space in which the Ricci tensor and the curvature invariant are related in the following manner: $R_{ij} = R\, a_{ij}/n$ where n is the dimension of the geometric space considered.[12]

From equations (3.5) and (1.6), to prove the sufficiency:

$$R_{11} = -\tfrac{1}{2}, \quad R_{22} = -\tfrac{1}{2}, \quad R_{33} = -\tfrac{1}{2}, \quad R_{23} = -\tfrac{1}{2}\cos x'$$

$$R = -3/2A$$

From equations (6.2)

$$\frac{a_{11}R}{3} = -\tfrac{1}{2} = R_{11} \qquad \frac{a_{33}R}{3} = -\tfrac{1}{2} = R_{33}$$

$$\frac{a_{22}R}{3} = -\tfrac{1}{2} = R_{22} \qquad \frac{a_{23}R}{3} = \frac{-\cos x'}{2} = R_{23}$$

This proves the sufficiency.

To prove the necessity:

Applying the necessity hypothesis to equations (3.5), (1.6) and (6.2)

$$R_{11} = \frac{B-2A}{2A} = \frac{A}{3}\left(\frac{B-4A}{2A}\right) \quad \text{or} \quad 2B-2A = 0 \quad \text{or} \quad A = B\,.$$

12. For a more complete discussion of Einstein spaces, see Eisenhart (1) pp.92-95, Schouten (1), and Weyl (1).

We state without proof a well known theorem on Riemannian curvature.

Lemma 1: A necessary and sufficient condition in order that the curvature of a Riemannian space be constant is that $R_{hijk} = b(a_{hj}\, a_{ik} - a_{hk}\, a_{ij})$ where b is constant and the a_{ij}'s are the components of the fundamental metric tensor.[13]

Theorem 2: A necessary and sufficient condition for the Static Riemannian space to have constant Riemannian curvature is that $A = B$.

Proof:

To prove the sufficiency.

Assuming $A = B$ using equations (2.4), (1.5) and (6.2), plus Lemma 1:

$$a_{11}\, a_{22} - a_{12}^2 = A^2 = 4A\, R_{1212}$$

$$a_{11}\, a_{33} - a_{13}^2 = A^2 = 4A\, R_{1313}$$

$$a_{22}\, a_{33} - a_{23}^2 = A^2 \sin^2 x^1 = 4A\, R_{2323}$$

$$a_{13}\, a_{21} - a_{11}\, a_{22} = -A^2 \cos x^1 = 4A\, R_{1231}$$

therefore,

$$b = 1/4A\ .$$

Applying the hypothesis of constant Riemannian curvature to equations (2.4), (1.5) and (6.2)

$$b = B/4A^2$$

$$b = \frac{4A^2 \sin^2 x^1 - 3AB \sin^2 x^1 + B^2 \cos^2 x^1}{A(A \sin^2 x^1 + B \cos^2 x^1)}$$

Solving these two equations simultaneously, we have

13. For the concept of curvature in a Riemannian space and the proof of this lemma see Eisenhart (1) pp.79-81.

$A = B$ and $b = 1/4A$.

Theorem 3: A necessary and sufficient condition for the Static Riemannian space to have the first covariant derivatives of the distinct components of the Riemann tensor equal to zero is that $A = B$.

Proof:

From equations (2.5) the proof is evident

In what follows a V_n geometric space stands for an n-dimensional Riemannian space. When the fundamental form is definite, V_n is a Euclidean space of n-dimensions. We denote by S_n a space for which the relationship of lemma 1 is satisfied with $b = 0$ and call it a flat space.

We state without proof the following well known result.

Lemma 2: A necessary and sufficient condition that a V_n can be mapped conformally on an S_n is that the conformal curvature tensor R_{ijk} be a zero tensor.[14]

Theorem 4: A necessary and sufficient condition in order that the Static Riemannian space can be mapped conformally on an S_3 is that $A = B$.

Proof:

Applying Lemma 2 to equations (2.6) the proof is evident.

14. For a complete discussion of the principles involved and the proof of this Lemma see Eisenhart (1)pp.90-92.

Combining theorems 1, 2, and 4 we have the following general result.

Theorem 5: A necessary and sufficient condition that the Static Riemannian space be an Einstein space with constant Riemannian curvature equal to 1/4A which can be mapped conformally upon an S_3 is that $A = B$.

It can be shown that the Static Riemannian space is a special case of a general class of Riemannian spaces which have the properties of the results of the preceding theorems. It is developed in the following manner.

Theorem 6: In a V_3 for which $g_{11} = g_{22} = g_{33} = A$, $g_{23} = Af(x)$, and $g_{12} = g_{21} = g_{13} = g_{31} = 0$, a necessary and sufficient condition in order that the V_3 have constant Riemannian curvature of value b, is that $g_{23} = A \sin 2\sqrt{bA}\,(x + C)$ where C is arbitrary and $x = x^1$, x^2, or x^3.

Proof:

For the Above V_3 the Riemann symbols of the first kind have the following form:

$$R_{1212} = \frac{A f'^2}{4(1, - f^2)} \qquad R_{1232} = 0$$

$$R_{1313} = \frac{A f'^2}{4(1 - f^2)} \qquad R_{1323} = 0$$

$$R_{2323} = \frac{A f'^2}{4} \qquad R_{1231} = \frac{A f''}{2} + \frac{A f f'^2}{4(1 - f^2)}$$

We shall go through the proof of R_{1212} only.

Applying the necessary and sufficient condition for constant curvature to R_{1212} :

$$\frac{f'^2}{4(1 - f^2)} = b\,A$$

or

$$\frac{f'}{\sqrt{(1 - f^2)}} = 2\sqrt{b\,A}$$

Solving this differential equation

$$\sin^{-1} f(x) = 2\sqrt{bA}(x + C)$$

or

$$f(x) = \sin 2\sqrt{bA}\,(x + C)$$

Corollary: The Static Riemannian space in which $A = B$ is a space of constant Riemannian curvature where $b = 1/4A$ and $C = \dfrac{\pi}{4\sqrt{bA}}$.

Theorem 7: In a V_3 for which $g_{11} = g_{22} = g_{33} = A$, $g_{23} = g_{32} = Af(x)$ and $g_{12} = g_{21} = g_{13} = g_{31} = 0$, a necessary and sufficient condition in order that the V_3 be an Einstein space is that $f(x) = \sin k(x + C)$ where k and C are arbitrary and $x = x^1$, x^2 or x^3 .

Proof:

The Ricci tensors for the above V_3 are as follows:

$$R_{11} = \frac{-f'^2 - 2f\,f'' - f^2 f'^2 + 2f^3\,f''}{2(1 - f^2)^2}$$

$$R_{22} = R_{33} = \frac{-f'^2}{2(1 - f^2)}$$

$$R_{23} = f''/2$$

The curvature invariant R is

$$R = \frac{-3f'^2 - 4f\,f'' + 4\,f^3\,f'' - f^2\,f'^2}{2A\,(1 - f^2)^2}$$

We shall go through the case of R_{11} only.

Applying the definition of an Einstein space, we obtain the following differential equation upon setting $f(x) \equiv y$.

$$\frac{d^2y}{dx^2}(1 - y^2) = -y\left(\frac{dy}{dx}\right)^2$$

Making the substitutions $\frac{dy}{dx} = p$ and $\frac{d^2y}{dx^2} = p\frac{dp}{dy}$ we have

$$p\frac{dp}{dy}(1 - y^2) = -y\,p^2$$

$$\frac{dp}{p} = \frac{-y\,dy}{1-y^2} \qquad \text{or} \qquad \ln p = \tfrac{1}{2}\ln k^2(1-y^2)$$

$$\frac{dy}{k\sqrt{1-y^2}} = dx$$

$$y = f(x) = \sin k(x + C)$$

Corollary: The Static Riemannian space in which $A = B$ is an Einstein space in which $k = 1$ and $\theta = \pi$.

Theorem 8: In a V_3 for which $g_{11} = g_{22} = g_{33} = A$, $g_{23} = g_{32} = Af(x)$ and $g_{12} = g_{21} = g_{13} = g_{31} = 0$, a necessary and sufficient condition that the first covariant derivative of the components of the Riemann tensor of the first kind be zero is that $f(x) = \sin k(x + C)$ where k and C are arbitrary and $x = x^1$, x^2, or x^3.

Proof:

We shall go through the case of R_{2323} only.

The first covariant derivative of R_{2323} is

$$R_{2323,1} = \frac{Af'f''}{2} + \frac{f'^3 f}{2(1-f^2)}$$

Setting this differential equation equal to zero, we obtain the following:

$$f'f'' - f^2 f' f'' + f f'^3 = 0$$

Setting $f = y$ and $\frac{dy}{dx} = p$, and $\frac{d^2y}{dx^2} = p\frac{dp}{dy}$

we have

$$\frac{dp}{dy}(1 - y^2) = -y\,p$$

Solving this equation

$$y = f(x) = \sin k(x + C)$$

Corollary: Setting $k = 1$ and $C = \pi$, in the Static Riemannian space in which $A = B$ the first covariant derivatives of the Riemann symbols is zero.

Theorem 9: In a V_3 for which $g_{11} = g_{22} = g_{33} = A$, $g_{23} = g_{32} = Af(x)$ and $g_{12} = g_{21} = g_{13} = g_{31} = 0$, a necessary and sufficient condition that the V_3 may be mapped conformally upon a S_3 is that $f(x) = \sin k(x + C)$ where k and C are arbitrary and $x = x^1$, x^2, ar x^3.

Proof:

As in the preceding theorems, by calculating R_{ijk} and setting the resulting differential equation equal to zero, the result is obtained.

In attempting to secure a corresponding set of theorems for the Action Riemannian space, insurmountable barriers are met. In fact, one may say that no relationship exists (with either physical or mathematical meaning) between

A, B, γ, and h which will enable one to obtain a set of theorems.

For example: Using equations (2.10), (1.11), and (2,11), the following set of equations must be satisfied in order for the Action Space to be an Einstein space.

1.20) $-6A^2\gamma^2 - 12A^2h^2 + 3ABh^2 = 0$

2.20) $20A^2\gamma h + AB\gamma h = 0$

3.20) $6A^2\gamma^2 + 2AB\gamma^2 - 5ABh^2 + 4A^2h^2 + B^2h^2 = 0$

4.20) $-20A^2\gamma h - 3AB\gamma h - 3B^2\gamma h = 0$

5.20) $-6AB\gamma^2 + 2B^2\gamma^2 = 0$

Equations (2.20) and (5.20) are clearly inconsistant unless $A = B = 0$.

Similarly in searching for possible conditions to obtain constant Riemannian curvature, we obtain equations which are inconsistant unless $h = \gamma = 0$.

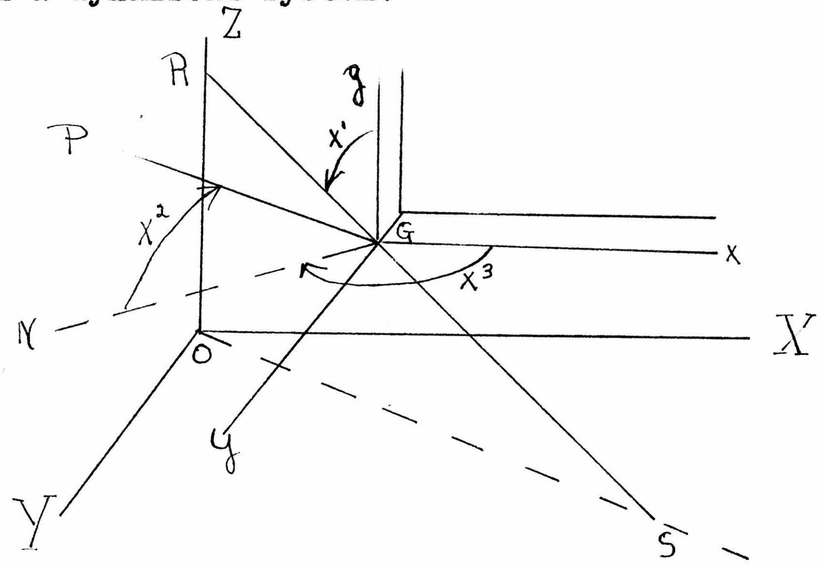
The same facts occur if one uses an approximation by neglecting all terms of higher order than the second. By neglecting all terms of higher order than the second, one can show that the first covariant derivative of R_{1212} will equal zero if $h = \gamma$ and $A = -\frac{B}{3}$. However, in the case of R_{2323} , and imaginary relationship must exist between A and B.

II

A dynamical system which resembles the preceding spinning top is the following.

A body of revolution is mounted on a straight wire R-S, coinciding with its axis, one end of which can slide along a vertical O-Z, while the lower end, S, can slide in any way on the horizontal plane x-O-y; no friction and no external forces except gravity.[15]

We investigate the Riemannian geometry described by such a dynamical system.



Once again using Euler's angles as independent coordinates, the kinetic and potential energies have the following form.[16]

15. For a complete discussion of this dynamical system see Akimoff (1) pp. 131-132.
16. For the derivation of these equations see Akimoff (1) pp. 132-135.

1.22) $2T = (A + Mr^2\cos^2 x^1 + M(1-r)^2 \sin^2 x^1)\dot{x}^{1^2} + C\dot{x}^{2^2}$

$+((A + Mr^2)\sin^2 x^1 + C\cos^2 x^1)\dot{x}^{3^2} + 2C\cos x^1 \;\dot{x}^2 \dot{x}^3$

2.22) $V = Mg(1-r)\cos x^1$

Lagranges equations in terms of x^1, x^2, and x^3 are as follows.

$\dot{x}^2 + \dot{x}^3 \cos x^1 = \text{const.} = a$

3.22) $\dot{x}^3 \sin^2 x^1 + m\cos x^1 = b$ where $m = \dfrac{a\,C}{A + Mr^2}$

$\dot{x}^{3^2} \sin^2 x^1 + (1 + n\sin^2 x^1)\dot{x}^{1^2} = p\cos x^1 + h$

where

$n = \dfrac{Ml(1-2r)}{A + Mr^2}$ and $p = \dfrac{2Mg(1-r)}{A + Mr^2}$

The kinetic energy (1.22) of the system can be represented in the following manner.

4.22) $T = \frac{1}{2} a_{ij}\, \dot{x}^i \dot{x}^j$ where i, $j = 1, 2, 3$ and

$a_{11} = A + Mr^2 + Ml(1-2r)\sin^2 x^1$

5.22) $a_{22} = C$, $a_{23} = C\cos x^1$

$a_{33} = (A + Mr^2)\sin^2 x^1 + C\cos^2 x^1 = C + (A + Mr^2 - C)\sin^2 x^1$

We define an element of arc-length ds^2 in the following manner.

6.22) $ds^2 = a_{ij}\, dx \;dx$ where the a_{ij} are defined as in (5.22)

We develope the Christoffel symbols for the Riemannian space with element of arc-length given by (6.22). We shall call this space the "Static Riemannian Space".

All Christoffel symbols of the first kind are zero with the exception of the following.

1.23)
$$
\begin{aligned}
&[11,1] = Ml(1-2r)\sin x'\cos x' \\
&[12,3] = [21,3] = -\tfrac{1}{2}C\sin x' \\
&[13,2] = [31,2] = -\tfrac{1}{2}C\sin x' \\
&[13,3] = [31,3] = (A+Mr^2-C)\sin x'\cos x' \\
&[23,1] = [32,1] = \tfrac{1}{2}C\sin x' \\
&[33,1] = (C-A-Mr^2)\sin x'\cos x' .
\end{aligned}
$$

We calculate $a^{rp} = \dfrac{\text{cofactor of } a_{pr}}{|a|}$

2.23) $|a| = C(A+Mr^2)(A+Mr^2+Ml(1-2r)\sin^2 x')\sin^2 x'$

therefore

3.23)
$$
a^{11} = \frac{1}{A+Mr^2+Ml(1-2r)\sin^2 x'}, \qquad a^{22} = \frac{C+(A+Mr^2-C)\sin^2 x'}{C(A+Mr^2)\sin^2 x'}
$$
$$
a^{33} = \frac{1}{(A+Mr^2)\sin^2 x'} \qquad a^{23} = \frac{-\cos x'}{(A+Mr^2)\sin^2 x'}
$$

All Christoffel symbols of the second kind are zero with the exception of the following.

4.23)
$$
\left\{ {1 \atop 1\;1} \right\} = \frac{Ml(l-2r)\sin x'\cos x'}{A+Mr^2+Ml(l-2r)\sin^2 x'}
$$
$$
\left\{ {1 \atop 3\;3} \right\} = \frac{(C-A-Mr^2)\sin x'\cos x'}{A+Mr^2+Ml(l-2r)\sin^2 x'}
$$

$$\left\{ {1 \atop 23} \right\} = \left\{ {1 \atop 32} \right\} = \frac{C \sin x'}{2(A+Mr^2+Ml(l-2r)\sin^2 x')}$$

$$\left\{ {2 \atop 21} \right\} = \left\{ {2 \atop 12} \right\} = \frac{C \cos x'}{2(A+Mr^2)\sin x'}$$

$$\left\{ {2 \atop 13} \right\} = \left\{ {2 \atop 31} \right\} = \frac{(C-A-Mr^2)\cos^2 x' - (A+Mr^2)}{2(A+Mr^2)\sin x'}$$

1.24)

$$\left\{ {3 \atop 12} \right\} = \left\{ {3 \atop 21} \right\} = \frac{-C}{2(A+Mr^2)\sin x'}$$

$$\left\{ {3 \atop 13} \right\} = \left\{ {3 \atop 31} \right\} = \frac{(2A+2Mr^2-C)\cos x'}{2(A+Mr^2)\sin x'}$$

The six distinct components of the Riemann tensor of the first kind have the following form.

$$R_{1212} = \frac{C^2}{4(A+Mr^2)}$$

$$R_{1313} = (A+Mr^2-C)(\sin^2 x' - \cos^2 x') + \frac{C(A+Mr^2)+(A+Mr^2-C)(4A+4Mr^2-C)\cos^2 x'}{4(A+Mr^2)} + \frac{Ml(l-2r)(A+Mr^2-C)\sin^2 x' \cos^2 x'}{A+Mr^2+Ml(l-2r)\sin^2 x'}$$

2.24)

$$R_{2323} = \frac{C^2 \sin^2 x'}{4(A+Mr^2+Ml(l-2r)\sin^2 x')}$$

$$R_{1232} = 0$$

$$R_{1323} = 0$$

$$R_{1231} = \frac{-C\cos x^1}{2} + \frac{C(2A+2Mr^2-C)\cos x^1}{4(A+Mr^2)} + \frac{CMl(1-2r)\sin^2 x^1 \cos x^1}{2(A+Mr^2+Ml(1-2r)\sin^2 x^1)} \tag{1.25}$$

We compute the non-zero components of the Ricci tensor for the Static Riemannian space.

$$R_{11} = \frac{(A+Mr^2)(C-2A-2Mr^2-2Ml(1-2r)) + CMl(1-2r)\sin^2 x^1}{2(A+Mr^2)(A+Mr^2+Ml(1-2r)\sin^2 x^1)}$$

$$R_{22} = \frac{-C^2}{2(A+Mr^2+Ml(1-2r)\sin^2 x^1)(A+Mr^2)}$$

$$\begin{aligned} R_{33} = {} & -C^2(A+Mr^2) + Ml(1-2r)(2(A+Mr^2)(C-A-Mr^2)-C^2)\sin^2 x^1 \\ & -CMl(1-2r)(A+Mr^2-C)\sin^4 x^1 \\ & + 2(A+Mr^2)(A+Mr^2+Ml(1-2r)\sin^2 x^1)^2 \end{aligned} \tag{2.25}$$

$$R_{23} = \frac{C(-C(A+Mr^2)+Ml(1-2r)(A+Mr^2-C))\cos x^1)}{2(A+Mr^2)(A+Mr^2+Ml(1-2r)\sin^2 x^1)^2}$$

The curvature invariant R is as follows.

$$\begin{aligned} R = {} & -2CMl(1-2r)(A+Mr^2-C) \\ & +\sin^2 x^1((A+Mr^2)^2(C-2A-2Mr^2-2Ml(1-2r)) + Ml(1-2r) \\ & \quad (2(A+Mr^2)(C-A-Mr^2) - C^2) - 2C^2(A+Mr^2) \\ & \quad + 2CMl(1-2r)(A+Mr^2-C) - C(A+Mr^2-C)(A+Mr^2) \\ & \quad - C^2 Ml(1-2r)) \\ & +\sin^4 x^1 (CMl(1-2r)(2C-A-Mr^2)) \\ & + 2(A+Mr^2)^2(A+Mr^2+Ml(1-2r)\sin^2 x^1)^2 \sin^2 x^1 . \end{aligned} \tag{3.25}$$

The Riemannian space in which the dynamical trajectories for a given energy constant, h, are geodesics, has an element of arc-length given by

1.26) $\quad ds^2 = 2(h-V)\, a_{ij}\, dx^i\, dx^j$

where V is the potential energy given by equation (2.22).

Or we may say

2.26) $\quad ds^2 = g_{ij}\, dx^i\, dx^j$

where

$$g_{11} = 2(h - Mg(l-r)\cos x^1)(A + Mr^2 + Ml(1-2r)\sin^2 x^1)$$

$$g_{22} = 2C(h - Mg(l-r)\cos x^1)$$

3.26) $$g_{33} = 2(h - Mg(l-r)\cos x^1)(C + (A + Mr^2 - C)\sin^2 x^1)$$

$$g_{23} = 2C(h - Mg(l-r)\cos x^1)\cos x^1$$

$$g_{12} = g_{13} = 0$$

The Riemannian space with element of arc-length (2.26) we shall call the "Action Riemannian space".

We calculate the Christoffel symbols of the Action space. All non-zero symbols of the first kind are

$$[11,1] = Mg(l-r)(A + Mr^2 - 2Ml(1-2r))\sin x^1 + 2Mlh(1-2r)\sin x^1 \cos x^1$$
$$+ 3M^2 lg(1-2r)(1-\overset{r}{4})\ \sin^3 x^1.$$

$$[22,1] = -CMg(l-r)\sin x^1$$

$$[33,1] = -Mg(l-r)(3C-2A-2Mr^2)\sin x^1 - 2h(A + Mr^2 - C)\sin x^1 \cos x^1$$
$$-3Mg(l-r)(A + Mr^2 - C)\ \sin^3 x^1$$

4.26)
$$[12,2] = [21,2] = CMg(l-r)\sin x^1$$

$$[12,3] = [21,3] = -Ch\sin x^1 + 2CMg(l-r)\sin x^1 \cos x^1$$

$$[13,2] = [31,2] = -Ch\sin x^1 + 2CMg(l-r)\sin x^1 \cos x^1$$

$$[13,3] = [31,3] = Mg(l-r)(3C-2A-2Mr^2)\sin x^1 + 2h(A + Mr - C)\sin x^1 \cos x^1$$
$$+ 3Mg(l-r)(A + Mr^2 - C)\ \sin^3 x^1$$

$$[23,1] = [32,1] = Ch\sin x^1 - 2CMg(l-r)\sin x^1 \cos x^1$$

We compute $g^{rp} = \dfrac{\text{cofactor of } g_{pr}}{|g|}$

$$1.27)\quad |g| = 8C(h-Mg(l-r)\cos x^1)^3 (A+Mr^2+Ml(l-2r)\sin^2 x^1)(A+Mr^2)\sin^2 x^1.$$

therefore

$$g^{11} = \frac{1}{2(h-Mg(l-r)\cos x^1)(A+Mr^2+Ml(l-2r)\sin^2 x^1)}$$

$$g^{22} = \frac{C+(A+Mr^2-C)\sin^2 x^1}{2C(A+Mr^2)(h-Mg(l-r)\cos x^1)\sin^2 x^1}$$

2.27)

$$g^{33} = \frac{1}{2(A+Mr^2)(h-Mg(l-r)\cos x^1)\sin^2 x^1}$$

$$g^{23} = \frac{-\cos x^1}{2(h-Mg(l-r)\cos x^1)(A+Mr^2)\sin^2 x^1}$$

$$g^{12} = g^{13} = 0$$

All Christoffel symbols of the second kind are zero with the exception of the following.

$$\left\{ {1 \atop 1\,1} \right\} = Mg(l-r)(A+Mr^2-2Ml(l-2r))\sin x^1 + 2Mlh(l-2r)\sin x^1\cos x^1 + 3M^2lg(l-2r)(l-r)\sin^3 x^1 \div 2(h-Mg(l-r)\cos x^1)(A+Mr^2+Ml(l-2r)\sin^2 x^1)$$

$$\left\{ {1 \atop 2\,2} \right\} = \frac{-CMg(l-r)\sin x^1}{2(h-Mg(l-r)\cos x^1)(A+Mr^2+Ml(l-2r)\sin^2 x^1)}$$

$$3.27)\quad \left\{ {1 \atop 3\,3} \right\} = -Mg(l-r)(3C-2A-2Mr^2)\sin x^1 - 2h(A+Mr^2-C)\sin x^1\cos x^1 - 3Mg(l-r)(A+Mr^2-C)\sin^3 x^1 \div 2(h-Mg(l-r)\cos x^1)(A+Mr^2+Ml(l-2r)\sin^2 x^1)$$

$$\left\{ {1 \atop 23} \right\} = \frac{Ch \sin x' - 2CMg(l-r)\sin x' \cos x'}{2(h - Mg(l-r)\cos x')(A + Mr^2 + Ml(l-2r)\sin^2 x')}$$

$$\left\{ {2 \atop 12} \right\} = \frac{-CMg(l-r) + Mg(l-r)(A + Mr^2 + C)\sin^2 x' + Ch\cos x'}{2(A + Mr^2)(h - Mg(l-r)\cos x')\sin x'}$$

$$\left\{ {2 \atop 13} \right\} = \frac{C - 2A - 2Mr^2 + (A + Mr^2 - C)\sin^2 x'}{2(A + Mr^2)\sin x'}$$

1.28)

$$\left\{ {3 \atop 12} \right\} = \frac{-C}{2(A + Mr^2)\sin x'}$$

$$\left\{ {3 \atop 13} \right\} = Mg(l-r)(C - 2A - 2Mr^2) + h\cos x'(2A + 2Mr^2 - C)$$
$$+ Mg(l-r)(3A + 3Mr^2 - C)\sin^2 x'$$
$$\div\, 2(A + Mr^2)(h - Mg(l-r)\cos x')\sin x'$$

We develope the six **distinct** components of the Riemann tensor of the first kind for the Action space.

$$R_{1212} = C^2(h^2 - M^2g^2(1-r)^2(A + Mr^2) + h(A + Mr^2))$$
$$+\cos x'(CMg(1-r)(-3A - 3Mr^2 - h - 3Ch - C(A + Mr^2))$$
$$+\cos^2 x'(CM^2g^2(1-r)^2(I + 2C)\cancel{cc}$$
$$+\cos x' \sin^2 x'(CM^2gl(1-2r)(1-r)(-I + Ch - 2C + 2h(A + Mr^2))$$
$$+\sin^2 x'(CM(Mg^2(1-r)^2(A + Mr^2 + C)(A + Mr^2) - CM^2lg^2(1-r)^2(1-2r)$$
$$+Clh(1-2r) + Mg^2(1-r)^2(A + Mr^2)(A + Mr^2 - 2Ml(1-2r)))$$
$$+\sin^4 x'(CM^3lg^2(1-r)(1-2r)(4A + 4Mr^2 + C)$$
$$\div\, 2(A + Mr^2)(h - Mg(1-r)\cos x')(A + Mr^2 + Ml(1-2r)\sin^2 x')$$

2.28)

$$R_{1313} = 4h^2(A + Mr^2)(A + Mr^2 - C) + Ch^2(A + Mr^2)^2 + M^2g^2(1-r)^2(A + Mr^2)^3$$
$$+\cos x'(Mgh(1-r)(A + Mr^2)^2(11C - 15A - 15Mr^2))$$
$$+\cos^2 x'(18Mgh(1-r)(A + Mr^2)^2(A + Mr^2 - C) + h^2(A + Mr^2)$$
$$(A + Mr^2 - C)(-4A - 4Mr^2 - C) + 2M^2g^2(1-r)^2(A + Mr^2)^2$$
$$(5A + 5Mr^2 - 4C))$$

$$+\cos^3 x'(9M^2g^2(1-r)^2(A+Mr^2)^2(C-A-Mr^2)+Mgh(1-r)(A+Mr^2)$$
$$(A+Mr^2-C)(-8A-8Mr^2+2C))$$
$$+\cos^4 x'(M^2g^2(1-r)^2(A+Mr^2)(C-A-Mr^2)(C-9A-9Mr^2))$$
$$+\sin^2 x'(Mh^2l(1-2r)(A+Mr^2)(4A+4Mr^2-C)$$
$$+M^3g^2l(1-r)^2(1-2r)(1+(A+Mr^2)^2)+M^2g^2(1-r)^2(A+Mr^2)^3)$$
$$+\sin^2 x'\cos x'(M^2glh(1-r)(1-2r)(A+Mr^2)(-11A-11Mr^2+10C)$$
$$+2Mgh(1-r)(A+Mr^2)(A+Mr^2-C))$$
$$+\sin^2 x'\cos^2 x'(18M^2glh(1-2r)(1-r)(A+Mr^2)(A+Mr^2-C)$$
$$-CMh^2l(1-2r)(A+Mr^2-C)+M^3lg^2(1-r)^2(1-2r)(A+Mr^2)$$
$$(4A+4Mr^2-5C)+3M^2g^2(1-r)^2(A+Mr^2)^2(C-A-Mr^2)$$
$$+\sin^2 x'\cos^3 x'(-18M^3g^2l(1-r)^2(1-2r)(A+Mr^2)(A+Mr^2-C)$$
$$+M^3glh(1-r)(1-2r)(A+Mr^2-C)(5C-12A-12Mr^2)$$
$$+M^2glh(1-r)(1-2r)(A+Mr^2-C)(20A+20Mr^2-C))$$
$$+\sin^2 x'\cos^4 x'(M^3g^2l(1-r)^2(1-2r)(C-A-Mr^2)(C-18A-18Mr^2))$$
$$\div\ 2(A+Mr^2)(h-Mg(1-r)\cos x')(A+Mr^2+Ml(1-2r)\sin^2 x')$$

1.29)

$$R_{2323} = \sin^2 x'(C^2h^2-CM^2g^2(1-r)^2(3C-2A-2Mr^2))$$
$$+\sin^2 x'\cos x'(2CMgh(1-r)(-C-A-Mr^2))$$
$$+\sin^2 x'\cos^2 x'\ (4C^2M^2g^2(1-r)^2)$$
$$+\sin^4 x'(3M^2g^2C(1-r)^2(C-A-Mr^2))$$
$$\div\ 2(h-Mg(1-r)\cos x')(A+Mr^2+Ml(1-2r)\sin^2 x')$$

$$R_{1231} = CMgh(1-r)(A+Mr^2)(C-A-Mr^2)-4CM^2lh(1-r)(1-2r)(A+Mr^2)$$
$$+4CM^3g^2l(1-r)^2(1-2r)(A+Mr^2)+CM^2lgh(1-2r)(1-r)(3A+3Mr^2+7C)$$
$$-C^2Mgh(1-r)(A+Mr^2)-C^2M^2ghl(1-r)(1-2r)$$
$$+\cos x'(-2C^2h^2(A+Mr^2)-2C^2h^2Ml(1-2r)-CM^2g^2(1-r)^2(A+Mr^2)^2$$
$$+2CM^2ghl(1-r)(1-2r)(A+Mr^2)+2CMlh^2(1-2r)(A+Mr^2)$$
$$-5CM^3g^2l(1-r)^2(1-2r)(A+Mr^2))$$

$$+\cos^2 x' \,(CMgh(1-r)(A+Mr^2)(3A+3Mr^2+2C)$$

$$+4CM^2hg(1-r)(1-2r)(A+M\ r^2) + 8CM^2lh(1-r)(1-2r)(A+Mr^2)$$

$$-12CM^3g^2l(1-r)^2(1-2r)(A+Mr^2)$$

$$+CM^2lgh(1-2r)(1-2)(-16A-16Mr^2+2C) + C^2h^2(A+Mr^2)$$

$$+C^2Mlh^2(1-2r))$$

$$+\cos^3 x' \,(2C^2h^2Ml(1-2r) - CM^2g^2(1-r)^2(A+Mr^2)(A+Mr^2+C)$$

$$-2M^2Cghl(1-r)(1-2r)(A+Mr^2) - 2CMlh^2(1-2r)(A+Mr^2)$$

$$+CM^3g^2l(1-r)^2(1-2r)(16A+16Mr^2-C))$$

$$+\cos^4 x' \,(-4CM^2hg(1-r)(1-2r)(A+Mr^2) - 4CM^2hl(1-r)(1-2r)(A+Mr^2)$$

$$+8CM^3g^2l(1-r)^2(1-2r)(A+Mr^2) + CM^2lgh(1-2r)(1-r)(13A+13Mr^2-2C)$$

$$-C^2Mlh^2(1-2r))$$

$$+\cos^5 x' \,(CM^3lg^2(1-2r)(1-r)^2(-11A-11Mr^2-C))$$

$$+\ 2(A+Mr^2)(h-Mg(1-r)\cos x')(A+Mr^2+Ml(1-2r)\sin^2 x')$$

1.30)

$$R_{1232} = 0$$

$$R_{1323} = 0$$

All components of the Ricci tensor R_{ij} for the Action space are zero with the exception of the following.

$$R_{11} = (A+Mr^2)^3(-4h^2M^2g^2(1-r)^2) + (A+Mr^2)\ (-4Mlh^2+2C-M^3g^2l(1-r)^2(1-2r))$$

$$+(A+Mr^2)(C^2M^2g^2(1-r)^2 - 2CM^2g^2(1-r)^2 + 2CMl(1-2r) + 2Ch^2)$$

$$+(A+Mr^2+C)(M^2g^2(1-r)^2(A+Mr^2) + M^3g^2l(1-r)^2(1-2r))$$

2.30)

$$+CM^3g^2l(1-r)^2(1-2r)(C-2)$$

$$+\cos x' \,(18Mgh(1-r)(A+Mr^2)^3 - 4M^2g^2h(1-r)^2(A+Mr^2)^3$$

$$+14M^2ghl(1-r)(1-2r)(A+M\ r^2)^2 - 4M^3g^2lh(1-r)^2(1-2r)(A+Mr^2)^2$$

$$+2CMgh(1-r)(A+Mr^2)^2 - 2C^2M^2ghl(1-r)(1-2r)$$

$$+2CM^2ghl(1-r)(1-2r)(A+Mr^2+C) - 4ChMg(1-r)(A+Mr^2)$$

$$+2Mgh(1-r)(A+Mr^2)(2A+2Mr^2-C)+2M^2ghl(1-r)(1-2r)(2A+2Mr^2-C))$$

$$+\cos^2 x'(4h^2(A+Mr^2)^3-18M^2g^2(1-r)^2(A+Mr^2)^3+4M^3g^3(1-r)^3(A+Mr^2)^3$$

$$-4h^2Mg(1-r)(A+Mr^2)^3+8Mlh^2(A+Mr^2)^2-13M^3g^2l(1-r)^2(1-2r)(A+Mr^2)^2$$

$$+4M^4g^3l(1-r)^3(1-2r)(A+Mr^2)^2-4h^2M^2gl(1-r)(1-2r)(A+Mr^2)$$

$$-2M^2g^2(1-r)^2(A+Mr^2)^2(A+Mr^2+C)+C^2H^2(A+Mr^2)$$

$$+2CM^3g^2l(1-r)^2(1-2r)-3M^3g^2l(1-r)^2(1-2r)(A+Mr^2+C)$$

$$+2CM^3g^2l(1-r)^2(1-2r)(2A+2Mr^2+C)+C^2Mlh^2(1-2r)$$

$$+2C(A+Mr^2)(C-A-Mr^2)-2CMl(1-2r)(A+Mr^2)+2CMl(1-2r)(C-A-Mr^2)$$

$$+2Ch^2(C-A-Mr^2)+2CM^2g^2(1-r)^2(A+Mr^2)-2M^2g^2(1-r)^2(A+Mr^2)^2(3A+3Mr^2-C)$$

$$+h^2(A+Mr^2)(2A+2Mr^2-C)^2-4Mlh^2(1-2r)(A+Mr^2)^2)+Mh^2l(l-2r)(2A+2Mr^2-C)^2$$
$$-2M^3g^2l(l-r)^2(l-2r)(A+Mr^2)(3A+3Mr^2-C)$$

$$+\cos^3 x'(-18Mgh(1-r)(A+Mr^2)^3+16M^2g^2h(1-r)^2(A+Mr^2)^3$$

$$-36M^2ghl(1-r)(1-2r)(A+Mr^2)^2+8M^3g^2lh(1-r)^2(1-2r)(A+Mr^2)^2$$

$$+12M^3g^2lh(1-r)^2(1-2r)-2CMgh(1-r)(\cancel{xxxxx}(A+Mr^2+C)(A+Mr^2)$$

$$+2C^2M^2ghl(1-r)(1-2r)-4CM^2ghl(1-r)(1-2r)(A+Mr^2+C)$$

$$-4ChMg(1-r)(C-A-Mr^2)-2Mgh(1-r)(A+Mr^2)(2A+2Mr^2-C)$$

$$(3A+3Mr^2-C)-2M^2ghl(1-r)(1-2r)(2A+2Mr^2-C)$$

$$-2M^2ghl(1-r)(1-2r)(2A+2Mr^2-C)(3A+3Mr^2-C))$$

1.31)

$$+\cos^4 x'(20M^2g^2(1-r)^2(A+Mr^2)^3-12M^3g^3(1-r)^3(A+Mr^2)^3$$

$$+4Mgh^2(1-r)(A+Mr^2)^3-4Mlh^2(A+Mr^2)^2+50M^3g^2l(1-r)^2(1-2r)$$

$$(A+Mr^2)^2-16M^4g^2l(1-r)^3(1-2r)(A+Mr^2)^2$$

$$+8M^3gh^2l(1-r)(1-2r)(A+Mr^2)-C^2MLh^2(1-2r)$$

$$+4M^3g^2l(1-r)^2(1-2r)(A+Mr^2)-2CMl(1-2r)(C-A-Mr^2)$$

$$+2CM^2g^2(1-r)^2(C-A-Mr^2)+4M^2g^2(1-r)^2(A+Mr^2)^2$$

$$-Mh^2l(2A+2Mr^2-C)(1-2r)+4Mlh^2(1-2r)(A+Mr^2)^2$$

$$+\cos^5 x'(-12M^2g^2h(1-r)^2(A+Mr^2)^3+22M^2ghl(1-r)(1-2r)(A+Mr^2)^2$$

$$-4M^3g^2l(1-r)^2(1-2r)(A+Mr^2)^2-24M^3g^2lh(1-r)^2(1-2r)$$

$$+2CM^2ghl(1-r)(1-2r)(A+Mr^2+C)$$
$$+2M^2ghl(1-r)(1-2r)(2A+2Mr^2-C)(3A+3Mr^2-C))$$
$$+\cos^6 x'(8M^3g^3(1-r)^3(A+Mr^2)^3-28M^3g^2l(1-r)^2(1-2r)(A+Mr^2)^2$$
$$+20M^4g^3l(1-r)^3(1-2r)(A+Mr^2)^2-4M^2gh^2l(1-r)(1-2r)(A+Mr^2)^2$$
$$-4M^3g^2l(1-r)^2(1-2r)(A+Mr^2))$$
$$+\cos^7 x'(12M^3g^2hl(1-r)^2(1-2r))$$
$$+\cos^8 x'(-8M^4g^3l(1-r)^3(1-2r)(A+Mr^2)^2)$$
$$+\;4(A+Mr^2)^2\sin^2 x'(h-Mg(1-r)\cos x')^2(A+Mr^2+Ml(1-2r)\sin^2 x')$$

$$R_{22}=2h(A+Mr^2)^2+2Mlh(1-2r)(A+Mr^2)-2CMg(1-r)(A+Mr^2)^2$$
$$-2CM^2gl(1-r)(1-2r)(A+Mr^2)\;-2C^2h^2(A+Mr^2)-2C^2Mh^2l(1-2r)$$
$$+CM^2g^2(1-r)^2(A+Mr^2)^2+CM^3g^2l(1-r)^2(1-2r)(A+Mr^2)$$
$$+\cos x'(-2Mg(1-r)(A+Mr^2)^2-2M^2gl(1-r)(1-2r)(A+Mr^2)$$
$$+CMgh(1-r)(A+Mr^2)(4C+2A+2Mr^2)$$
$$+4CM^2ghl(1-r)(1-2r)(A+Mr^2))$$
$$+\cos^2 x'(-2Mlh(1-2r)(A+Mr^2)+2CMg(1-r)(A+Mr^2)^2$$
1.32)
$$+4CM^2gl(1-r)(1-2r)(A+Mr^2)+2C^2Mh^2l(1-2r)$$
$$-CM^2g^2(1-r)^2(A+Mr^2)(2C+3A+3Mr^2)$$
$$-2CM^3g^2l(1-r)^2(1-2r)(3A+3Mr^2+C))$$
$$+\cos^3 x'(2M^2gl(1-r)(1-2r)(A+Mr^2)-4CM^2ghl(1-r)(1-2r)(A+Mr^2+C))$$
$$+\cos^4 x'(-2CM^2gl(1-r)(1-2r)(A+Mr^2)$$
$$+CM^3g^2l(1-r)^2(1-2r)(2C+5A+5Mr^2))$$
$$+\;4(A+Mr^2)(h-Mg(1-r)\cos x')(A+Mr^2+Ml(1-2r)\sin^2 x')^2$$

$$R_{23}=-CMgh(1-r)(A+Mr^2)^2-CM^2glh(1-2r)(1-r)(A+Mr^2)$$
$$+\cos x'(-2C^2h^2(A+Mr^2)^2-2CMlh^2(1-2r)(A+Mr^2-C))$$
$$+\cos^2 x'(4C^2M^2glh(1-r)(1-2r)+CMgh(1-r)(A+Mr^2)(5A+5Mr^2+4C))$$

$$+\cos^3 x' \,(-2CM^2g^2(1-r)^2(A+Mr^2)(2A+2Mr^2+C)$$
$$-2C^2M^3g^2l(1-r)^2(1-2r)\;-2CMlh^2(1-2r)(A+Mr^2-C))$$
$$+\cos^4 x'\,(CM^2glh\,(1-r)(1-2r)(A+Mr^2-4C))$$
$$+\cos^5 x'\,(2C^2M^3g^2l(1-r)^2(1-2r))$$
$$+\;4(A+Mr^2)(h-Mg(1-r)\cos x')^2(A+Mr^2+Ml(1-2r)\sin^2 x')^2$$

1.33)

$$R_{33}=-2h^2(A+Mr^2)^2(2A+2Mr^2-C)-2Mlh^2(1-2r)(A+Mr^2)(2A+2Mr^2-6)$$
$$-2M^2g^2(1-r)^2(A+Mr^2)^3\;-2M^3g^2l(1-r)^2(1-2r)(A+Mr^2)^2$$
$$+CM^2g^2(1-r)^2(A+Mr^2)(2A+2Mr^2-6)$$
$$+M^2g^2(1-r)^2(A+Mr^2)(A+Mr^2+C)(3C-2A-2Mr^2)$$
$$-M^3g^2l(1-r)^2(1-2r)(A+Mr^2)(5A+5Mr^2-6C)$$
$$+\cos x'\,(Mgh(1-r)(A+Mr^2)(16A+16Mr^2-14C)$$
$$+14M^2ghl(1-r)(1-2r)(A+Mr^2)(A+Mr^2-C)$$
$$+CMgh(1-r)(A+Mr^2)(8C-A-Mr^2)+CM^2glh(1-r)(1-2r)(8C-A-Mr^2)$$
$$+Mgh(1-r)(A+Mr^2)(A+Mr^2-C)(3C-2A-2Mr^2)$$
$$+M^2ghl(1-r)(1-2r)(C-2A-2Mr^2)(2C-A-Mr^2)$$
$$-4M^2ghl(1-r)(1-2r)(A+Mr^2-C)^2$$
$$+\cos^2 x'\,(2h^2(A+Mr^2)(2A+2Mr^2+C)(A+Mr^2-C)$$
$$+M^2g^2(1-r)^2(A+Mr^2)^2(9C-10A-10Mr^2)$$
$$+2Mlh^2(1-2r)(A+Mr^2)(4A+4Mr^2-3C)$$
$$+M^3g^2l(1-r)^2(1-2r)(A+Mr^2)(9C-7A-7Mr^2)$$
$$+CM^2g^2(1-r)^2(A+Mr^2)(3C-7A-7Mr^2)$$
$$+3CM^3g^2l(1-r)^2(1-2r)(3C-4A-4Mr^2)$$
$$+M^2g^2(1-r)^2(A+Mr^2)(A+Mr^2-C)(9C+14A+14Mr^2)$$
$$+M^3g^2l(1-r)^2(1-2r)(C-2A-2Mr^2)(4A+4Mr^2-3C)$$
$$-2Mlh^2(1-2r)(A+Mr^2-C)(2A+2Mr^2-C)$$

$$+M^3g^2l(1-r)^2(1-2r)(3A+3Mr^2-C)(5A+5Mr^2-3C)$$
$$-M^2g^2(1-r)^2(A+Mr^2)(A+Mr^2+C)(3C-2A-2Mr^2)$$
$$+M^3g^2l(1-r)^2(1-2r)(A+Mr^2+C)(13A+13Mr^2-15C))$$
$$+\cos^3x'(6Mgh(1-r)(A+Mr^2)(A+Mr^2-C)(A+Mr^2-2C)$$
$$+28M^2ghl(1-r)(1-2r)(A+Mr^2)(C-A-Mr^2)$$
$$+CM^2glh(1-r)(1-2r)(10A+10Mr^2-17C)$$
1.34)
$$+M^2ghl(1-r)(1-2r)(C-2A-2Mr^2)^2+8M^2glh(1-r)(1-2r)(A+Mr^2-C)^2)$$
$$+\cos^4x'(2M^2g^2(1-r)^2(A+Mr^2-C)(3A+3Mr^2-C)$$
$$+M^3g^2l(1-r)^2(1-2r)(3C-2A-2Mr^2)(2C-3A-3Mr^2)$$
$$+CM^3g^2l(1-r)^2(1-2r)(3A+3Mr^2+C)$$
$$-3M^3g^2l(1-r)^2(1-2r)(A+Mr^2-C)(5A+5Mr^2+C)-2CMlh^2(1-2r)(A+Mr^2-C))$$
$$+\cos^5x'(2M^2glh(1-r)(1-2r)(A+Mr^2-C)(2A+2Mr^2-C))$$
$$+\cos^6x'(M^3g^2l(1-r)^2(1-2r)(A+Mr^2-C)(4C-15A-15Mr^2))$$
$$+\quad 4(A+Mr\ )(h-Mg(1-r)\cos\ x')^2(A+Mr^2+Ml(1-2r)\sin^2x')^2$$

The curvature invariant of the Action space has the following form.

$$R=-4CMlh^2(A+Mr^2)^2+2C^2(A+Mr^2)^2+2C^2Ml(1-2r)(A+Mr^2)$$
$$+2C^2h^2(A+Mr^2)+C^3M^3g^2l(1-2r)(1-r)^2+2h(A+Mr^2)^3$$
$$+2Mlh(1-2r)(A+Mr^2)^2\ -2CMg(1-r)(A+Mr^2)^3\ -2CM^2gl$$
$$(1-r)(1-2r)(A+Mr^2)^2-2CM^2g^2(1-r)^2(A+Mr^2)^3$$
$$+CM^2g^2(1-r)^2(A+Mr^2)(A+Mr^2+C)(3C-2A-2Mr^2)-8Ch^2(A+Mr^2)^3$$
$$+2C^2M^2g^2(1-r)^2(A+Mr^2)(A+Mr^2-4C)$$
$$+CM^2g^2(1-r)^2(A+Mr^2)(A+Mr^2-C)$$
$$+CM^3g^2l(1-r)^2(1-2r)(A+Mr^2-C)-4CMlh^2(1-2r)(A+Mr^2)^2$$
$$-CM^3g^2l(1-r)^2(1-2r)(A+Mr^2)(7A+7Mr^2-6C)$$

$$+\cos x'(-4CM^2g^2h(1-r)^2(A+Mr^2)^3-4CM^3g^2lh(1-r)^2(1-2r)(A+Mr^2)^2$$

$$+2CM^2ghl(1-r)(1-2r)(2A+2Mr^2-C)-2Mg(1-r)(A+Mr^2)^3$$

$$-2M^2gl(1-r)(1-2r)(A+Mr^2)^2+CMgh(1-r)(A+Mr^2)(A+Mr^2-C)(3C-2A-2Mr^2)$$

$$+CM^2ghl(1-r)(1-2r)(C-2A-2Mr^2)(2C-A-Mr^2)$$

$$-4CM^2ghl(1-r)(1-2r)(A+Mr^2-C)$$

$$+C^2Mgh(1-r)(A+Mr^2)(3A+3Mr^2+8C)$$

$$+4CMgh(1-r)(A+Mr^2)^2(5A+5Mr^2+C)+16CM^2ghl(1-r)(1-2r)$$

$$(A+Mr^2)(2A+2Mr^2+C)\ +20CMgh(1-r)(A+Mr^2)(A+Mr^2-C)$$

$$+C^2M^2ghl(1-r)(1-2r)(8C+A+Mr^2))$$

1.35)

$$+\cos^2x'(4Ch^2(A+Mr^2)^3+4CM^3g^3(1-r)^3(A+Mr^2)^3-4CMgh^2(1-r)(A+Mr^2)^3$$

$$+18CMlh^2(A+Mr^2)+4CM^4g^3l(1-r)^3(1-2r)(A+Mr^2)^2$$

$$-4Ch^2M^2gl(1-r)(1-2r)(A+Mr^2)+C^3h^2(A+Mr^2)$$

$$+2C^2(A+Mr^2)(C-A-Mr^2)+2C^2h^2(C-A-Mr^2)+2C^2M^2g^2(1-r)^2(A+Mr^2)$$

$$+2h(A+Mr^2)^2(C-A-Mr^2)+CM^3g^2l(1-r)^2(1-2r)((C-2A-2Mr^2)$$

$$(4A+4Mr^2-3C)+(3A+3Mr^2-C)(5A+5Mr^2-3C)$$

$$+(A+Mr^2+C)(13A+13Mr^2-15C))\ -2C^3h^2(A+Mr^2)^2$$

$$+8CM^2g^2(1-r)^2(A+Mr^2)^2(C-5A-5Mr^2)+3CM^3g^2l(1-r)^2(1-2r)$$

$$(A+Mr^2)(4C-9A-9Mr^2)-3CM^3g^2l(1-r)^2(1-2r)(3A+3Mr^2-C)$$

$$+C^2M^3g^2l(1-r)^2\ (1-2r)(11C-8A-8Mr^2)+C^3Mlh^2(1-2r)$$

$$+2C^2Ml(1-2r)(C-2A-2Mr^2)+Ch^2(A+Mr^2)(8(A+Mr^2)^2-4C(A+Mr^2)-3C^2)$$

$$+CMlh^2(1-2r)(2A+2Mr^2-C)(4A+4Mr^2-3C)$$

$$+4CMlh^2(1-2r)(A+Mr^2)(A+Mr^2-C)$$

$$+2Mlh(1-2r)(A+Mr^2)(C-2A-2Mr^2)+2CMg(1-r)(A+Mr^2)^2(2A+2Mr^2-C)$$

$$+2CM^2gl(1-r)(1-2r)(A+Mr^2)(3A+3Mr^2-C)$$

$$+CM^2g^2(1-r)^2(A+Mr^2)(-13AC-13CMr^2+16A^2+32AMr^2+14M^2r^4-9C^2))$$

$$+\cos^3x'(16CM^2g^2h(1-r)^2(A+Mr^2)^3+8CM^3g^2lh(1-r)^2(1-2r)(A+Mr^2)^2$$

$$+12CM^3g^2lh(1-r)^2(1-2r)-2CM^2ghl(1-r)(1-2r)(2A+2Mr^2-C)$$

$$
\begin{aligned}
&-2Mg(1-r)(A+Mr^2)^2(C-A-Mr^2)-2CMgh(1-r)(A+Mr^2)^2(13A+13Mr^2+9C)\\
&+2M^2gl(1-r)(1-2r)(A+Mr^2)(2A+2Mr^2-C)\\
&+4CM^2ghl(1-r)(1-2r)(-18(A+Mr^2)^2+6C(A+Mr^2)-5C^2))\\
&+\cos^4x'(-12CM^3g^3(1-r)^3(A+Mr^2)^3+4CMgh^2(1-r)(A+Mr^2)^3\\
&-4CMlh^2(A+Mr^2)^2-16CM^4g^3l(1-r)^3(1-2r)(A+Mr^2)^2\\
&+8CM^3gh^2l(1-r)(1-2r)(A+Mr^2)^2+4CM^3g^2l(1-r)^2(1-2r)(A+Mr^2)\\
&-2C^2Ml(1-2r)(C-A-Mr^2)-2Mlh(1-2r)(A+Mr^2)(C-A-Mr^2)\\
&+2CMg(1-r)(A+Mr^2)^2(C-A-Mr^2)\\
&+CM^2g^2(1-r)^2(A+Mr^2)(23(A+Mr^2)^2+7C(A+Mr^2)+2C^2)\\
&+CM^3g^2l(1-r)^2(1-2r)(A+Mr^2)(55A+55Mr^2+2C)-C^3Mlh^2(1-2r)\\
&+2CM^2g^2(1-r)^2(2C^2-5C(A+Mr^2)+5(A+Mr^2)^2)\\
&+2C^2glh(1-2r)(A+Mr^2)(2C-3A-3Mr^2)\\
&+C^2Mlh^2(1-2r)(4A+4Mr^2-C)\\
&+CM^3g^2l(1-r)^2(1-2r)(-3(A+Mr^2)^2-2C(A+Mr^2)+12C^2))\\
&+\cos^5x'(-12CM^2g^2h(1-r)^2(A+Mr^2)^3-4CM^3g^2l(1-r)^2(1-2r)(A+Mr^2)^2\\
&-24CM^3g^2hl(1-r)^2(1-2r)+2M^2gl(1-r)(1-2r)(A+Mr^2)(C-A-Mr^2)\\
&+2CM^2ghl(1-r)(1-2r)(21(A+Mr^2)^2-8(A+Mr^2)+5C^2))\\
&+\cos^6x'(8CM^3g^3(1-r)^3(A+Mr^2)^3+20CM^4g^3l(1-r)^3(1-2r)(A+Mr^2)^2\\
&-4CM^2gh^2l(1-r)(1-2r)(A+Mr^2)^2-4CM^3g^2l(1-r)^2(1-2r)(A+Mr^2)\\
&-2CM^2gl(1-r)(1-2r)(A+Mr^2)(C-A-Mr^2)\\
&+2CM^3g^2l(1-r)^2(1-2r)(-24(A+Mr^2)^2+11C(A+Mr^2)-2C^2))\\
&+\cos^7x'(12CM^3g^2hl(1-r)^2(1-2r))\\
&+\cos^8x'(-8CM^4g^3l(1-r)^3(1-2r)(A+Mr^2)^2)\\
&+8C(A+Mr^2)^2(h-Mg(1-r)\cos x')^3(A+Mr^2+Ml(1-2r)\sin^2x')\sin^2x'.
\end{aligned}
\tag{1.36}
$$

In the following theorems C refers to the moment of inertia of the body about R-S, and A is the moment of inertia about a principal axis in the plane perpendicular to R-S. M is the whole mass considered concentrated at the center of gravity G, while r is the distance RG and l is the distance RS.

<u>Theorem10</u>: If $C = A + Mr^2$ and $l = 2r$, then the Static Riemannian space is a Riemannian space of constant Riemannian curvature in which $b = 1/4C$, where b is the value of the curvature.

Proof:

From equations (2.24) and (1.25), applying the necessary and sufficient condition for constant Riemannian curvature (Lemma 1), we have

$$\frac{C}{A + Mr^2} = b(A + Mr^2 + Ml(l-2r)\sin^2 x')$$

if

$l = 2r$, then $b = \dfrac{C}{4(A + Mr^2)^2}$

From equation (2.24) for R_{2323}

$$\frac{C^2 \sin^2 x'}{4(A + Mr^2 + Ml(l-2r)\sin^2 x')} = bC(A + Mr^2)\sin^2 x'$$

if $l = 2r$

$$b = \frac{C}{4(A + Mr^2)^2}$$

Similarly from equation (2.24) for R_{1231}

$$\frac{C^2 \cos x'}{4(A + Mr^2)} = bC(A + Mr^2)\cos x'$$

or

$$b = \frac{C}{4(A + Mr^2)^2}$$

From equation (1.25) for R_{1313}

$$C^2 + \sin^2 x'(A + Mr^2 - C)(4A + 4Mr^2 + C)$$
$$= 4bC(A + Mr^2) + 4b(A + Mr^2)^2(A + Mr^2 - C)\sin^2 x'.$$

As far as the constant term is concerned

$$b = \frac{C}{4(A + Mr^2)^2}$$

But for the $\sin^2 x'$ term, we must further impose that

$C = A + Mr^2$, and hence $b = 1/4C$

We note that if x' is very small and we may neglect $\sin^2 x'$, which is the same as letting $l = 2r$ in the case of R_{1212}, R_{1231}, and R_{2323}, we then would have a space of constant Riemannian curvature where $b = \dfrac{C}{4(A + Mr^2)^2}$

Under the hypothesis of $C = A + Mr^2$ and $l = 2r$, our space becomes identical with the Static Riemannian space of our spinning top or gyroscope discussed in I. Therefore we may write the following theorems.

<u>Theorem 11:</u> If $l = 2r$, a necessary and sufficient condition in order that the Static Riemannian space be an Einstein space with constant Riemannian curvature equal to $1/4C$ which can be mapped conformally upon an S_3 is that $C = A + Mr^2$.

<u>Theorem 12:</u> If $l = 2r$, a necessary and sufficient condition in order that the Static Riemannian space have the first covariant derivative of the six components of the Riemann tensor of the first kind equal to zero is that $C = A + Mr^2$.

Theorem 13: If $C = A + Mr^2$, a necessary and sufficient condition in order that the Static Riemannian space have constant Riemannian curvature is that $l = 2r$.

Proof:

Under the hypothesis of $C = A + Mr^2$

$$R_{1212} = C/4 \quad , \quad R_{1313} = C/4 \quad , \quad R_{2323} = \frac{C^2 \sin^2 x'}{4(C + Ml(1-2r)\sin^2 x')}$$

$$R_{1231} = \frac{-C \cos x' (C - Ml(1-2r)\sin^2 x')}{4(C + Ml(1-2r)\sin^2 x')}$$

By direct calculation, one can show that the condition for constant Riemannian curvature is satisfied if and only if $l = 2r$.

Theorem 14: If $C = A + Mr^2$, a necessary and sufficient condition in order that the Static Riemannian space be an Einstein space is that $l = 2r$.

Proof:

Under the hypothesis of $C = A + Mr^2$

$$R_{11} = \frac{-C - 2Ml\,(1-2r) + CMl(1-2r)\sin^2 x'}{2(C + Ml(1-2r)\sin^2 x')}$$

$$R_{22} = \frac{-C}{2(C + Ml(1-2r)\sin^2 x')} \quad , \quad R_{23} = \frac{-C}{2(C + Ml(1-2r)\sin^2 x')^2}$$

$$R_{33} = \frac{-C - 2CMl(1-2r)\sin^2 x'}{2(C + Ml(1-2r)\sin^2 x')^2} \quad , \quad R = \frac{-3C - 4Ml(1-2r) + Ml(1-2r)\sin^2 x'}{2(C + Ml(1-2r)\sin^2 x')}$$

By direct calculation, one can show that the definition of an Einstein space is satisfied if and only if $l = 2r$.

In attempting to secure a corresponding set of theorems for the Action Riemannian space, we know that a hypothesis of $l = 2r$ and $C = A + Mr^2$ will give no results, since under this hypothesis our space becomes the spinning top Action space.

Concerning an investigation of a general nature, the following occurs.

Applying the condition for constant Riemannian curvature to R_{1212} , we find that either $l = r$ or $l = 2r$. Under the hypothesis of $l = 2r$, the following equation must be satisfied:

$$8C(A+Mr^2)^3 \quad M^3g^3r^3 = 0$$

None of the solutions to this equation have physical significance. Concerning the solution $C = 0$ or $A = -Mr^2$, our problem is reduced to an absurdity.

If M is extremely small, and we approximate by $M = 0$

$$R_{1212} = \frac{C^2(h+A)}{2\ A^2} \quad , \text{ giving us } b = \frac{C(h+A)}{8A^3\ h^2}$$

$$R_{1313} = \frac{4Ah-4hC+CAh+\cos^2 x'(-4A^2h+3ACh+C^2h)}{2\ A}$$

giving us two conditions on b

$$b = \frac{4A - 4C + AC}{8A^3h} \quad , \quad b = \frac{-4A^2 + 3AC + C^2}{8A^2h\ (A - C)}$$

Solving these simultaneously,

$$C = \frac{2\ A(A - 2)}{2 - A}$$

which gives us

$$b = \frac{A - 5}{8Ah(2-A)}$$

Solving this simultaneously with the b of R_{1212} we see that h, the energy constant, no longer remains arbitrary.

Similarly in all other investigations, we find that a necessary condition is that either 1 must equal r or 2r. Imposing this condition, h nolonger remains arbitrary which makes the necessity hypothesis untenable.

III.

The investigations of the preceding dynamical systems suggested the following problem.

We investigate the Riemannian geometry of a V_3 in which the components of the fundamental metric have the form $g_{ij} = \delta^i_j f(x)$ where $\delta^i_j = 0$ if $i \neq j$ and $\delta^i_j = 1$ if $i = j$, and $x = x^1$, x^2, or x^3.

With element of arc-length $ds^2 = g_{ij}\, dx^i dx^j$, the non-zero Christoffel symbols have the following form.

1.42)
$$
\begin{aligned}
[11,1] &= \frac{1}{2}\frac{df}{dx} & \qquad \left\{ {1 \atop 11} \right\} &= g^{11}[11,1] \\
[12,2] &= \frac{1}{2}\frac{df}{dx} & \left\{ {1 \atop 22} \right\} &= -\left\{ {1 \atop 11} \right\} \\
[13,3] &= \frac{1}{2}\frac{df}{dx} & \left\{ {1 \atop 33} \right\} &= -\left\{ {1 \atop 11} \right\} \\
[22,1] &= -\frac{1}{2}\frac{df}{dx} & \left\{ {2 \atop 12} \right\} &= \left\{ {1 \atop 11} \right\} \\
[33,1] &= -\frac{1}{2}\frac{df}{dx} & \left\{ {3 \atop 13} \right\} &= \left\{ {1 \atop 11} \right\}
\end{aligned}
$$

The three non-zero components of the Riemann tensor of the first kind are as follows.

2.42)
$$
\begin{aligned}
R_{1212} &= -\frac{1}{2}\frac{d^2 f}{dx^2} + \frac{1}{2f}\left(\frac{df}{dx}\right)^2 \\
R_{1313} &= -\frac{1}{2}\frac{d^2 f}{dx^2} + \frac{1}{2f}\left(\frac{df}{dx}\right)^2 \\
R_{2323} &= -\frac{1}{4f}\left(\frac{df}{dx}\right)^2
\end{aligned}
$$

All components of the Ricci tensor are zero with the exception of the following.

$$1.43)\quad R_{11} = 2 \quad \frac{d\left\{ {1 \atop 11} \right\}}{dx}$$

$$R_{22} = \frac{d}{dx}\left\{ {1 \atop 11} \right\} + \left\{ {1 \atop 11} \right\}^2$$

$$R_{33} = R_{22}$$

The curvature invariant may be expressed as

$$2.43)\quad R = \frac{d}{dx}\left\{ {1 \atop 11} \right\} + \frac{1}{2}\left\{ {1 \atop 11} \right\}^2$$

All components of the conformal curvature tensor are zero with the exception of the following.

$$3.43)\quad R_{212} = R_{313} = \frac{d}{dx}\left\{ {1 \atop 11} \right\} - \left\{ {1 \atop 11} \right\}^2$$

For a Riemannian space with the above structure, one may prove the following theorems.

Theorem 15: In a V_3 Riemannian space in which $g_{ij} = \delta^i_j f(x)$, a necessary and sufficient condition in order that the space have constant Riemannian curvature of value b is that $f(x) = \dfrac{1}{k(x+C)^2}$ where $b = -k$ and C is an arbitrary constant.

Proof:

By direct calculation the sufficiency of the condition may be shown. Assuming constant Riemannian curvature we find the two following differential equations must be satisfied:

From R_{1212} and R_{1313} (2.42)

$$-\frac{1}{2}\frac{d^2 f}{dx^2}+\frac{1}{2f}\left(\frac{df}{dx}\right)^2=bf^2$$

From R_{2323} (2.42)

$$-\frac{1}{4f}\left(\frac{df}{dx}\right)^2=bf^2$$

Solving we have

$$f(x)=\frac{1}{k(x+C)^2} \qquad \text{where } k=-b.$$

Theorem 16: In a V_3 Riemannian space in which $g_{ij}=\delta^i_j f(x)$, a necessary and sufficient condition in order that the V_3 be an Einstein space is that $f(x)=\dfrac{1}{k(x+C)^2}$ where k and C are arbitrary constants.

Proof:

The sufficiency can be shown by direct calculation.

Assuming our necessity hypothesis we find that R_{ii} will equal $g_{ii}R/3$ if

$$\frac{d}{dx}\left\{ {1 \atop 1\,1} \right\}=\left\{ {1 \atop 1\,1} \right\}^2$$

or

$$-\left\{ {1 \atop 1\,1} \right\}=\frac{1}{x+C}$$

or

$$\frac{-2}{x+C}=\frac{1}{f}\frac{df}{dx}$$

Solving this differential equation

$$f(x)=\frac{1}{k(x+C)^2}$$

Theorem 17: In a V_3 Riemannian space in which $g_{ij} = \delta^i_j f(x)$, a necessary and sufficient condition in order that the space can be mapped conformally upon an S_3 is that $f(x) = \frac{1}{k(x+C)^2}$ where k and C are arbitrary constants.

Proof:

$$R_{212} = R_{313} = -R_{22,1} + \tfrac{1}{4} g_{11} \; R_{,1} = 0$$

which gives us the differential equation

$$\frac{\partial}{\partial x}\left\{ {1 \atop 1\ 1} \right\} = \left\{ {1 \atop 1\ 1} \right\}^2$$

or $$f(x) = \frac{1}{k(x+C)^2}$$

Theorem 18: In a V_3 Riemannian space in which $g_{ij} = \delta^i_j f(x)$ a necessary and sufficient condition in order that the curvature invariant $R = 0$ is that $f(x) = k(x+C)^4$ where k and C are arbitrary constants.

Proof:

Setting equation(2.43) equal to zero we have

$$\frac{\partial}{\partial x}\left\{ {1 \atop 1\ 1} \right\} = -\frac{1}{2}\left\{ {1 \atop 1\ 1} \right\}^2$$

or $$\left\{ {1 \atop 1\ 1} \right\} = \frac{2}{x+C}$$

or $$\frac{1}{2\, f(x)} \frac{df}{dx} = \frac{2}{x+C}$$

Solving, we get

$$f(x) = k(x+C)^4$$

We may extend the above theorems to a V_4 with a similarly defined arc-length.

<u>Theorem 19</u>: In a V_4 Riemannian space in which $g_{ij} = \delta^i_j f(x)$, a necessary and sufficient condition in order that it be a space of constant Riemannian curvature of value b, is that $f(x) = \frac{1}{k(x+C)^2}$ where $k = -b$ and C is arbitrary.

Proof:

The Riemann symbols of the first kind have the following form:

$$R_{1212} = R_{1313} = R_{1414} = -\frac{d}{dx}[11,1] + 2\left\{ {1 \atop 1\,1} \right\}[11,1] \tag{1.46}$$

$$R_{2323} = R_{2424} = R_{3434} = -\left\{ {1 \atop 1\,1} \right\}[11,1]$$

Proceeding as in Theorem 15, the proof is completed.

The results of the theorem may be extended to a Riemannian space of n-dimensions.

<u>Theorem 20</u>: In a V_n Riemannian space in which $g_{ij} = \delta^i_j f(x)$, a necessary and sufficient condition in order that the space have constant Riemannian curvature of value b is that $f(x) = \frac{1}{k(x+C)^2}$ where $k = -b$ and C is arbitrary.

Proof:

The proof is the obvious extension of Theorems 15 and 19.

<u>Theorem 21</u>: In a V_4 Riemannian space in which $g_{ij} = \delta^i_j f(x)$, a necessary and sufficient condition in order that the space be an Einstein space is that $f(x) = \dfrac{1}{k(x+C)^2}$ where k and C are arbitrary.

Proof:

The Ricci tensor has the following form:

$$1.47) \quad \begin{aligned} R_{11} &= 3\frac{d}{dx}\left\{ {1 \atop 1\;1} \right\} \\ R_{22} = R_{33} = R_{44} &= \frac{d}{dx}\left\{ {1 \atop 1\;1} \right\} + 2\left\{ {1 \atop 1\;1} \right\}^2 \end{aligned}$$

Proceeding as in Theorem 16, the proof is completed.

<u>Theorem 22:</u> In a V_n Riemannian space in which $g_{ij} = \delta^i_j f(x)$, a necessary and sufficient condition in order that the space be an Einstein space is that $f(x) = \dfrac{1}{k(x+C)^2}$ where k and C are arbitrary.

Proof:

The proof is obtained by the obvious extension of the proofs in Theorems 16 and 21.

In a similar manner we may obtain the following.

<u>Theorem 23:</u> In a V_n Riemannian space in which $g_{ij} = \delta^i_j f(x)$, a necessary and sufficient condition in order that it may be mapped on an S_n is that $f(x) = \dfrac{1}{k(x+C)^2}$

Theorems 20, 22 and 23 may be combined into the following general theorem.

<u>Theorem 24</u>: In a V_n Riemannian space in which $g_{ij} = \delta^i_j f(x)$, a necessary and sufficient condition in order that the space be of constant Riemannian curvature of value b, an Einstein space, and may be mapped conformally on an S_n is that $f(x) = \dfrac{1}{k(x+C)^2}$ where $k = -b$ and C is an arbitrary constant.

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