Analysis of a Multivibrator

Thesis by

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A theoretical analysis of the action of a fundamental multivibrator has been undertaken along the lines of Van der Pol's analysis.* A more exact approximation to the transfer characteristic has been used and an exact solution of the equations has been found in the case where the shunt capacities of the tube can be neglected. The analytical solution then allows one to localize the type of distortion in the "square wave" output and to suggest better circuits for the production of rectangular waves.

Experimentally several circuits are presented in which the ultimate limit to the "squareness" of the square waves is limited only by the shunt capacities of the tubes.

* Van der Pol, Phil. Mag. 2, pp. 978-992 (1926).

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I. Introduction

In electronics extensive use is being made of relaxation oscillations for the production of continuous "square waves." By "square wave" is meant almost any sort of voltage wave that has periodic abrupt discontinuities, and hence a high harmonic content. For example, the fact that it is possible to synchronize a harmonic of the square wave with a sine wave (subharmonic resonance) is made use of in the control of low frequencies with an accurately calibrated crystal oscillator. Also accurate television image formation and scanning is made possible by the use of relaxation oscillators.

Since the requirements of relaxation oscillators are becoming more exacting, a more accurate theory of the production of relaxation oscillations is required. There has been a large amount of work done on this subject and the allied subject of the non-linear theory of electric oscillations by Van der Pol(1) and collaborators in which they are concerned with the nature of relaxation oscillations and their synchronizations with sinusoidal oscillations. In the following work the nature of the relaxation oscillations will be discussed in more detail with what is hoped to be a more exact description of the phenomenon.

The multivibrator is a particular type of relaxation oscillator

(1) Non Linear Theory of Electric Oscillations, Balth Van der Pol, I.R.E. 22, pp. 1051-1086 (1934).

and is chosen because of its simplicity and fundamental nature. Its circuit diagram is as follows:

Basic Multivibrator Circuit:

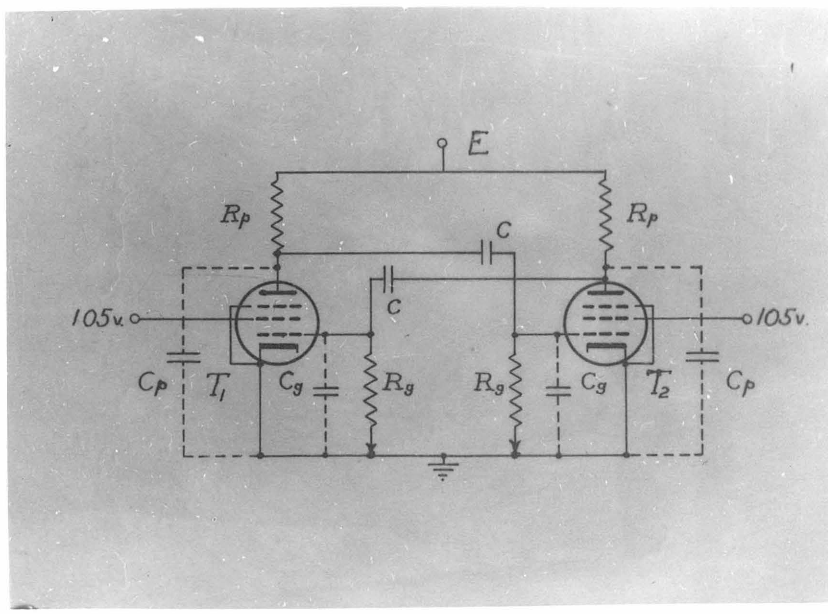

Fig 1.

The symmetrical arrangement detracts from the general solution of the problem, but does not detract from the general nature of the problem.

(a) Qualitative Operation of Multivibrator

Suppose initially that both tubes are conducting equally. Then the plate voltages and the grid voltages are equal. If now the grid voltage of tube T_1 is made slightly negative, the plate voltage of T_1

will go slightly positive because the tube is made less conducting and there is, therefore, less voltage drop in the plate resistor. This positive pulse will be transmitted through the coupling condenser to the grid of T_2 which makes T_2 more conducting and consequently makes the plate voltage of T_2 more negative. This negative pulse on the plate of T_2 is transmitted through the other coupling condenser to the grid of T_1. Thus a negative pulse on the grid of T_1 causes a negative pulse on the grid of T_1, or the action of the circuit is regenerative and the grid of T_1 goes as far negative as the ultimate conductivity of the tubes will allow (i.e., T_1 con-conducting, T_2 conducting). This equilibrium position is, however, only transient equilibrium for the grid voltages tend to approach zero voltage as their equilibrium value either by charging or discharging their respective coupling condensers. As the grid voltages approach zero, T_1 from the negative side and T_2 from the positive side, there will be a point reached where the tube T_1 begins to conduct and the tube T_2 becomes less conducting. But whenever the tubes are conducting the circuit is regenerative and the voltages go to their extremes. Thus if the grid voltage of T_1 is approaching zero from the negative side it will be thrown as far positive as possible. Likewise, the grid of T_2 will be thrown as far negative as possible because it is approaching zero from the positive side. As before a relaxation period will set in until the tube currents begin to change and another "flip-over" occurs. Summing up it may be said that the equilibrium values of the plate voltages are the extreme values while the equilibrium values of the grid voltages are zero. Since these two equilibrium values are mutually incompatible the plate voltages exchange their extreme values at the end of each relaxation cycle of the grid voltages. In this manner continuous relaxation oscillations of high harmonic content are generated.

The method of attack then consists of performing a circuit analysis using the complete set of voltage-current characteristics of the tubes. It is customary in circuit analysis to "linearize" the problem by restricting the amplitude to small quantities (i.e. using only a small portion of the tube characteristic). This leads to sinusoidal oscillations as the primary or ideal solution of the problem. In this case, however, the relaxation nature of the problem disappears if the amplitudes are restricted. Hence it is necessary to retain large amplitudes and to look for other primary or ideal solutions. That this can be done will be demonstrated, but the necessary generalization, which in the case of sinusoidal oscillations is the principle of superposition, is still lacking.

II. Circuit Analysis and the Relaxation Equation

(a) Circuit Analysis

There will be two equivalent circuits of the following type:

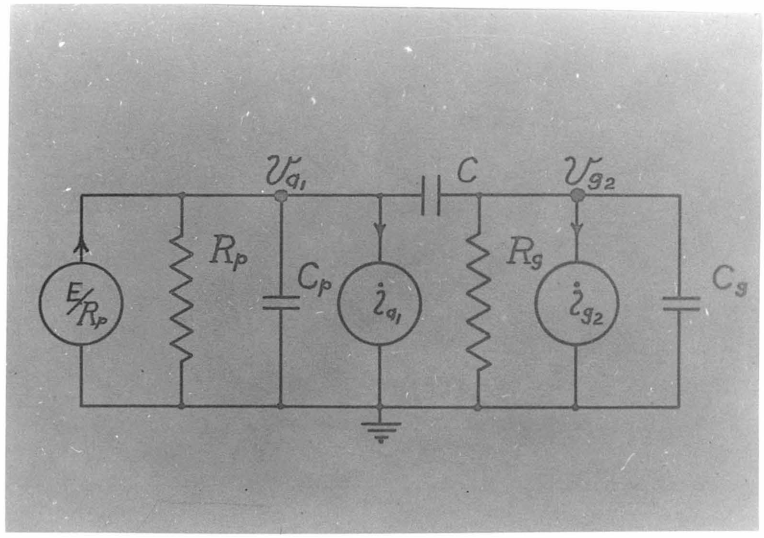

Fig. 2.

The other equivalent circuit will have the subscripts 1 and 2 interchanged. For the node voltages the circuit equations become:

Let $D = \frac{d}{dt}$:

$$\left[\frac{1}{R_p} + (C+C_p)D\right] v_{a_1} - C D v_{g_2} = \frac{E}{R_p} - i_{a_1} \quad \cdots\cdots (a)$$

$$-C D v_{a_1} + \left[\frac{1}{R_g} + (C+C_g)D\right] v_{g_2} = - i_{g_2} \quad \cdots\cdots (b)$$

$$\cdots\cdots (1)$$

$$\left[\frac{1}{R_p} + (C+C_p)D\right] v_{a_2} - C D v_{g_1} = \frac{E}{R_p} - i_{a_2} \quad \cdots\cdots (c)$$

$$-C D v_{a_2} + \left[\frac{1}{R_g} + (C+C_g)D\right] v_{g_1} = - i_{g_1} \quad \cdots\cdots (d)$$

Solving these for the grid voltages,

$$\begin{vmatrix} \frac{1}{R_p}+(C+C_p)D & -CD \\ -CD & \frac{1}{R_g}+(C+C_g)D \end{vmatrix} v_{g_1} = \begin{vmatrix} \frac{1}{R_p}+(C+C_p)D & -i_{a_2} \\ -CD & -i_{g_1} \end{vmatrix} \quad \ldots \ldots (2)$$

$$\begin{vmatrix} \frac{1}{R_p}+(C+C_p)D & -CD \\ -CD & \frac{1}{R_g}+(C+C_g)D \end{vmatrix} v_{g_2} = \begin{vmatrix} \frac{1}{R_p}+(C+C_p)D & -i_{a_1} \\ -CD & -i_{g_2} \end{vmatrix} \quad \ldots \ldots (3)$$

Assuming that the currents i_a and i_g depend only on the grid voltages (use of pentodes) then the equations (2) and (3), if they could be solved simultaneously, would represent a complete solution of the symmetrical multivibrator. It is possible to neglect the grid current i_g if a resistance is placed in series with the grid electrodes. This changes the high frequency response but for many frequencies the plate voltage wave form will be improved (more nearly square). The effect of grid current flow during the positive swing of the grid voltage will be discussed in an approximate manner later. Setting the grid currents equal to zero, the following equations result. Assume T_1 and T_2 are identical. Let:

$$p = \frac{1}{C\left[R_g\left(1+\frac{C_g}{C}\right)+R_p\left(1+\frac{C_p}{C}\right)\right]} \quad \ldots \ldots (4)$$

$$K = \frac{\left(1+\frac{C_p}{C}\right)\left(1+\frac{C_g}{C}\right)-1}{\left[\frac{1}{R_p}\left(1+\frac{C_g}{C}\right)+\frac{1}{R_g}\left(1+\frac{C_p}{C}\right)\right]\left[R_g\left(1+\frac{C_g}{C}\right)+R_p\left(1+\frac{C_p}{C}\right)\right]} \quad \ldots \ldots (5)$$

$$g_m = \left(\frac{d i_a}{d v_g}\right)_{v_g=0} \quad \ldots \ldots (6)$$

$$I_1(v_{g_1}) = \frac{1}{g_m} i_{a_1}(v_{g_1}) \quad \ldots\ldots\ldots\ldots \quad (7)$$

$$I_2(v_{g_2}) = \frac{1}{g_m} i_{a_2}(v_{g_2}) \quad \ldots\ldots\ldots\ldots \quad (7a)$$

$$b = \frac{g_m}{\frac{1}{R_p}\left(1+\frac{C_g}{C}\right) + \frac{1}{R_g}\left(1+\frac{C_p}{C}\right)} \quad \ldots\ldots\ldots \quad (8)$$

$$\tau = pt \quad \ldots\ldots\ldots\ldots \quad (9)$$

Then (2) becomes:

$$K\frac{d^2 v_{g_1}}{d\tau^2} + \frac{d v_{g_1}}{d\tau} + v_{g_1} = -b\, I'(v_{g_2}) \cdot \frac{d v_{g_2}}{d\tau} \quad \ldots \quad (10)$$

Likewise (3) becomes:

$$K\frac{d^2 v_{g_2}}{d\tau^2} + \frac{d v_{g_2}}{d\tau} + v_{g_2} = -b\, I'(v_{g_1}) \cdot \frac{d v_{g_1}}{d\tau} \quad \ldots \quad (11)$$

In order to take into account the actual plate current of the tubes, several grid transfer characteristics of pentodes were taken with a 5 megohm resistor placed in series with the control grid electrode. This insures that there will be no grid current flow for positive grid voltages. Since the tubes were pentodes, the anode current is a function only of the grid voltage for sufficiently high plate voltages. A typical curve is the following:

i_p (m.a.) vs. e_g (volts)

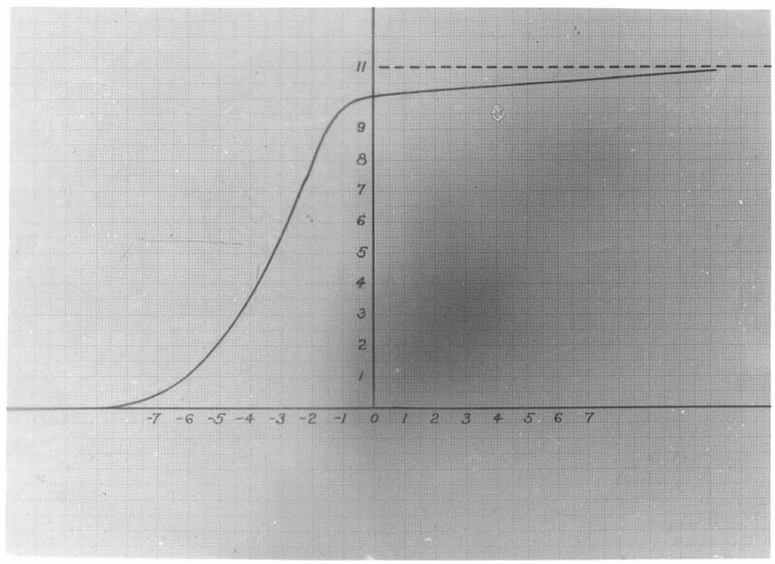

Fig 3

The zero of grid voltage can be selected by properly choosing the bias voltage applied to that electrode. The value of the saturation current i_o is constant for all plate voltages sufficiently high (greater than 100 v.) and depends only on the constant value of screen voltage.

As an analytical approximation to this transfer characteristic the following equation is chosen.

$$i_a = \frac{i_o}{\pi}\left(\frac{\pi}{2} + \tan^{-1}\frac{\pi g_m}{i_o} v_g\right) \quad \ldots\ldots \quad (12)$$

the graph of which is the following:

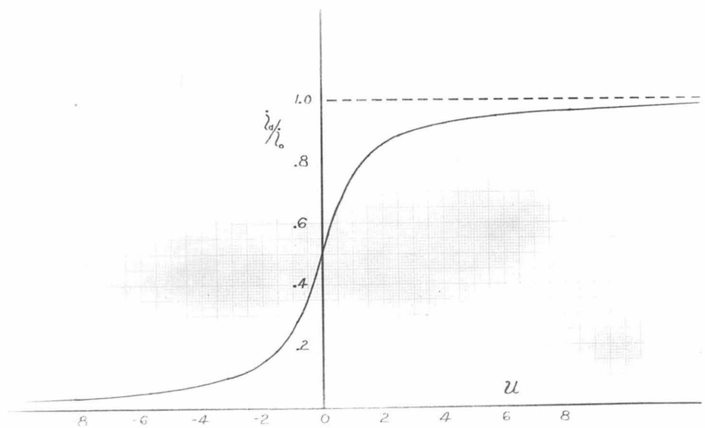

Fig 4.

This does not have the proper curvature in many regions of the actual characteristic but it preserves the general nature of the transfer characteristics. Furthermore it has the theoretical advantage of having an algebraic derivative. From eq. (7) and eq. (12) there results:

$$I'(v_g) = \frac{1}{1 + \left(\frac{\pi g_m}{l_0} v_g\right)^2} \quad \ldots\ldots \quad (12a)$$

Letting

$$a = \frac{\pi g_m}{l_0} \quad \ldots\ldots\ldots\ldots \quad (13)$$

$$u = a\, v_{g_1} \quad \ldots\ldots\ldots\ldots \quad (14)$$

$$v = a\, v_{g_2} \quad \ldots\ldots\ldots\ldots \quad (15)$$

Equations (10) and (11) become:

$$K\frac{d^2u}{d\tau^2} + \frac{du}{d\tau} + u + \frac{b}{1+v^2}\cdot\frac{dv}{d\tau} = 0 \quad \ldots\ldots \quad (16)$$

and

$$K\frac{d^2v}{d\tau^2} + \frac{dv}{d\tau} + v + \frac{b}{1+u^2}\cdot\frac{du}{d\tau} = 0 \quad \ldots\ldots \quad (17)$$

Now it can be shown (see appendix) that for general initial conditions, the oscillatory energy of the system will adjust itself either through addition to or dissipation of the initial energy so that in the steady state the grid voltages change in exactly opposite phase.

$$u = -v \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (18)$$

Thus the equation governing the steady state oscillations is given by

$$K\frac{d^2u}{d\tau^2} + \left(1 - \frac{b}{1+u^2}\right)\frac{du}{d\tau} + u = 0 \quad . \; . \; . \; . \; . \quad (19)$$

It should be noticed that only the steady state solution of this equation is desired since the initial transients are not governed by (19) but by the simultaneous set (16) and (17).

The essential feature of equation (19) is the coefficient of the dissipative term $1 - \frac{b}{1 + u^2}$. For small values of u it is negative which means that the damping of the oscillation is negative or energy is being added to the system thus tending to make the value of u larger. But for large values of u the coefficient is positive and the damping is positive or the amplitude u is decreasing. Thus oscillations take place in such a way that energy flows into the system when it gives signs of dying out (small amplitude) and energy flows out of the system when the amplitude gets too large. The maximum amplitude of oscillation adjusts itself so that the energy intake per cycle just balances the energy outflow per cycle. These oscillations are characteristic of relaxation oscillations and have been discussed by

Van der Pol[3] and others. A recent article by Levinson and Smith[4] gives an excellent description of the oscillations and gives the conditions necessary for a unique periodic solution of a generalized equation of relaxation oscillations.

$$\frac{d^2 u}{d\tau^2} + f\left(u, \frac{du}{d\tau}\right) \cdot \frac{du}{d\tau} + g(u) = 0 \quad . \quad . \quad . \quad . \quad . \quad (20)$$

(b) Particular Solutions of the Relaxation Equation

The pioneer work in the solution of equation (20) seems to have been undertaken by Van der Pol who used the method of isoclines as a graphical method for obtaining the solution. He approximates the grid transfer characteristic by a third degree parabola and hence gets for the coefficient of the damping term a second degree parabola.

The equation he discusses is:

$$\frac{d^2 v}{d\tau^2} - \epsilon(1 - v^2)\frac{dv}{d\tau} + v = 0 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The transfer characteristic of the problem at hand does not permit the use of this equation although it is well suited to an analysis of smaller amplitudes. Using the method of successive approximations Appleton and Greaves[5] have given the solution of the

(3) Van der Pol, Phil. Mag. 2, pp. 978-992 (1926).
(4) Levinson and Smith, Relaxation Oscillations, Duke Math. Journ. 9, pp. 382-403 (June 1942).
(5) Appleton and Greaves, Phil. Mag. 45, pp. 401-414 (1923).

following equation:

$$\frac{d^2 v}{d\tau^2} + f(v)\frac{dv}{d\tau} + \omega^2 v = 0 \quad . \quad . \quad . \quad . \quad (22)$$

where f(v) is expanded in a power series. So far this seems to be the only successful method of getting exact analytical solutions, although recently Shohat[6] has given a new method of successive approximations applied to this equation giving similar results.

Refering back to equation (19) it will be seen that the term that is responsible for the relaxation nature of the oscillations is the first degree term.

If the circuit could be arranged to have the shunt capacity of the tubes C_p and C_g extremely small, the constant K would be essentially zero. If it is set equal to zero, the equation to be solved is then:

$$\left(1 - \frac{b}{1+u^2}\right)\frac{du}{d\tau} + u = 0 \quad . \quad . \quad . \quad . \quad . \quad (23)$$

This equation, however, has an exact solution which may be seen if it is put into the form:

$$\int_B^u \frac{du}{u} - \int_B^u \frac{b}{1+u^2}\cdot\frac{du}{u} = \int_0^\tau d\tau \quad . \quad . \quad . \quad . \quad (24)$$

By Pierce; Eq. 55

(6) J. Shohat, Journ. of App. Phys. 14, pp. 40-48 (1943)
(7) Pierce, A Short Table of Integrals, Ginn and Company.

$$\frac{u}{B}\left[\frac{B^2(1+u^2)}{u^2(1+B^2)}\right]^{\frac{b}{2}} = e^{-\tau} \quad . \ . \ . \ . \ . \ . \ (25)$$

The quantity B may be determined by observing that the magnitude of the charge on the condensers C cannot change during the "flip-over." Thus the voltage $v_{a_1} - v_{g_2}$ must remain the same for $\tau = 0 \pm \delta$, where $\delta \rightarrow 0$. But the value (u_o) of u, where the "flipover" occurs is determined by setting $\frac{du}{dr} = \infty$ in eq. (23). This gives

$$u_o = \pm\sqrt{b-1} \quad . \ . \ . \ . \ . \ . \ . \ (26)$$

Let us suppose that when $\tau = 0 + \delta$, $u = B$. Then at $\tau = 0 - \delta$, $u = -u_o$. To express $v_{a_1} - v_{g_2}$ in terms of v_{g_1} and v_{g_2}, set $C_p = C_g = 0$ in equations 1_a and 1_b; eliminate $\frac{dv_{a_1}}{dt}$, leaving:

$$v_{a_1} - v_{g_2} = E - R_p\, i_{a_1}(v_{g_1}) - \left(1 + \frac{R_p}{R_g}\right)v_{g_2} \quad . \ . \ (27)$$

Making use of the definitions of a and b:

$$v_{g_2} - v_{a_1} = \frac{R_p i_o}{\pi b}\left[b \tan^{-1} u + v + \frac{\pi b}{i_o} - \frac{\pi b E}{R_p i_o}\right] \quad . (28)$$

Since this is to remain the same before and after "flip-over" the constant terms and factors can be dropped. This gives, remembering that $u = -v$, the function $Z(u)$, which is to remain the same before and after "flip-over."

Error

Equation (31) should be:

$$\frac{B+u_0}{1-Bu_0} = \tan\left(\frac{B+u_0}{b}\right)$$

$$Z(u) = b \tan^{-1} u - u \quad \ldots\ldots\ldots \quad (29)$$

The condition thus reduces to:

$$Z(-u_0) = Z(B) \quad \ldots\ldots\ldots\ldots \quad (30)$$

or:

$$\frac{B + u_0}{1 + Bu_0} = \tan\left(\frac{B + u_0}{b}\right) \quad \ldots\ldots \quad (31)$$

This is the transcendental equation which must be solved for B. Remember that $B \geqslant 0$, and that equation (12) is restricted to $0 \leqslant i_a \leqslant i_o$. Hence the smallest positive solution of (31) is the required value of B. The graphical solution, remembering that B is a function of b only, is shown below:

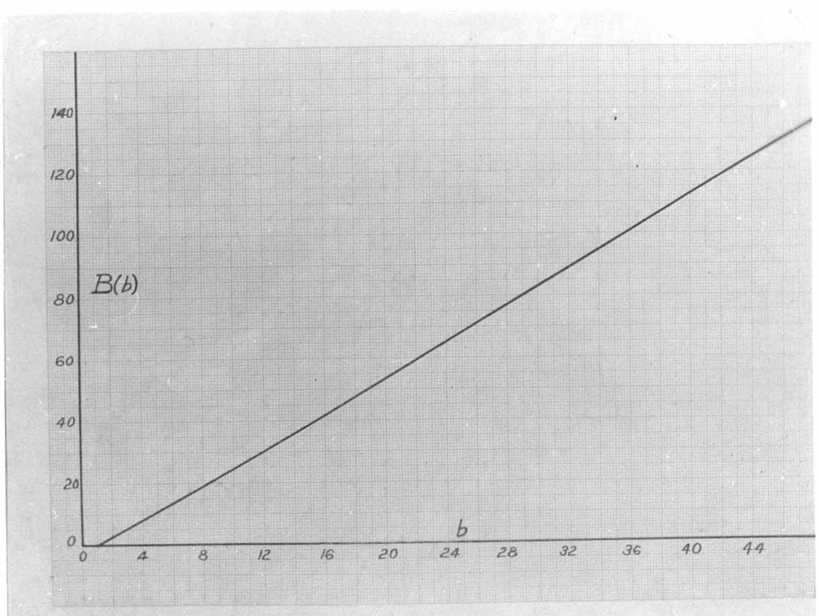

Fig 5.

The wave forms on the plate can then be found by means of equation (1_a) and (1_b).

$$v_{a_1} = E - R_p\, i_{a_1}(v_{g_1}) + \frac{R_p}{R_g}(v_{g_1}) \quad . \ . \ . \ . \quad (32)$$

(c) Wave Shapes, Theoretical and Experimental

The wave shapes on the grid and plate of the tubes are calculated from equations (25) and (32), and are compared with the observed oscillograph tracings for various experimental conditions. If the parameter K in equation (19) is set equal to zero, the wave shapes on the grids are seen to depend only on the parameter b which in this case becomes

$$b = \frac{g_m}{\frac{1}{R_p} + \frac{1}{R_g}}$$

The grid and plate waves are calculated for b = 25, and b = 5 in order to show the effect of this parameter b on the solution of equation (23). The plate and grid resistors were then chosen to give the same values of this parameter and the wave shapes were photographed. The timing sweep on the oscilloscope is rather definitely non-linear and hence the apparent increase in time between "flip-overs."

Grid (Calc.) $b = 25$; $K = 0$

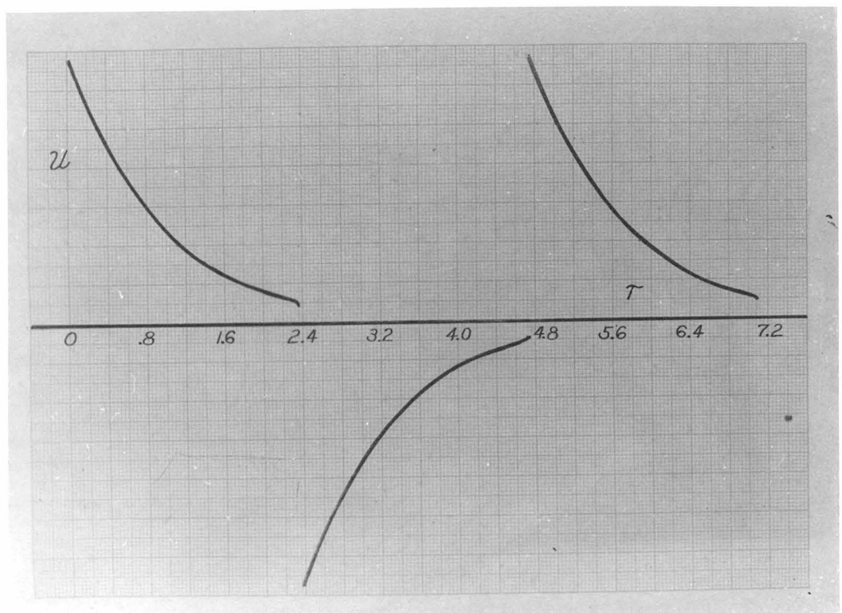

Fig 6

Grid (Obs.) $b = 25$; $K = 0$

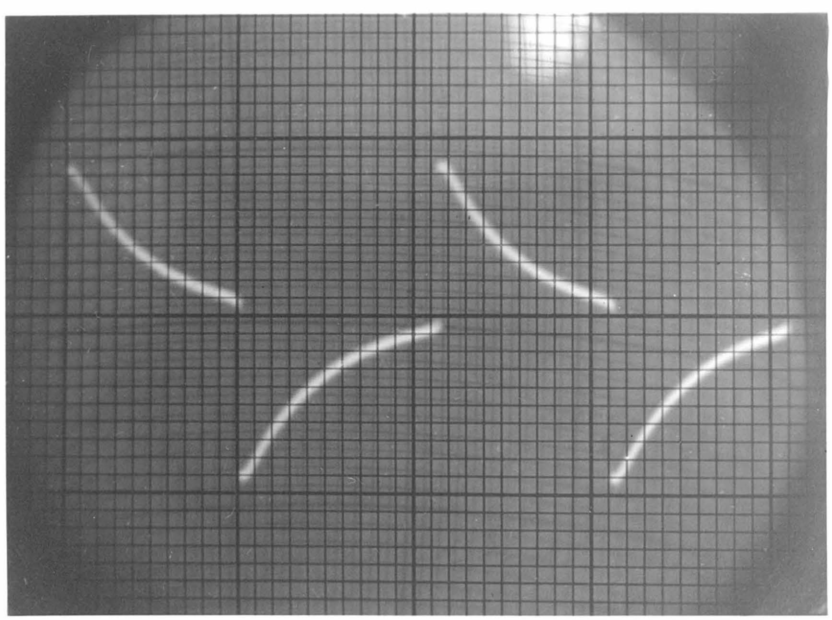

Fig 7.

Grid (Calc) $b = 5$; $K = 0$

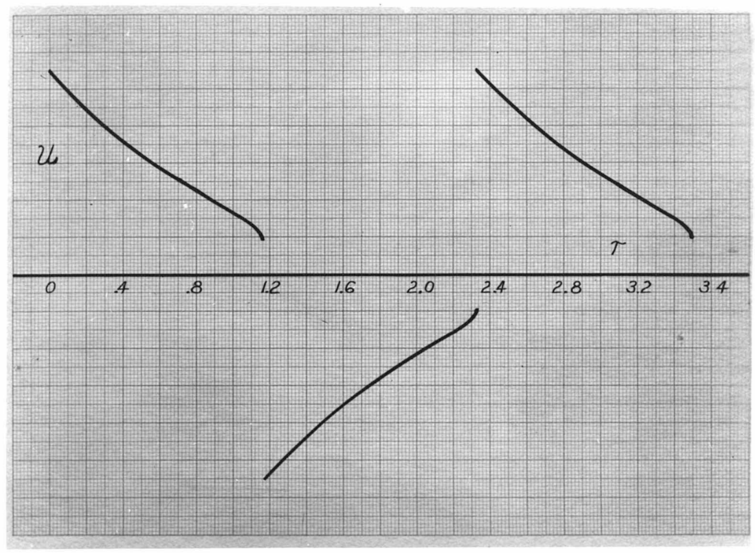

Fig 8.

Grid (Obs.) $b = 5$; $K = 0$

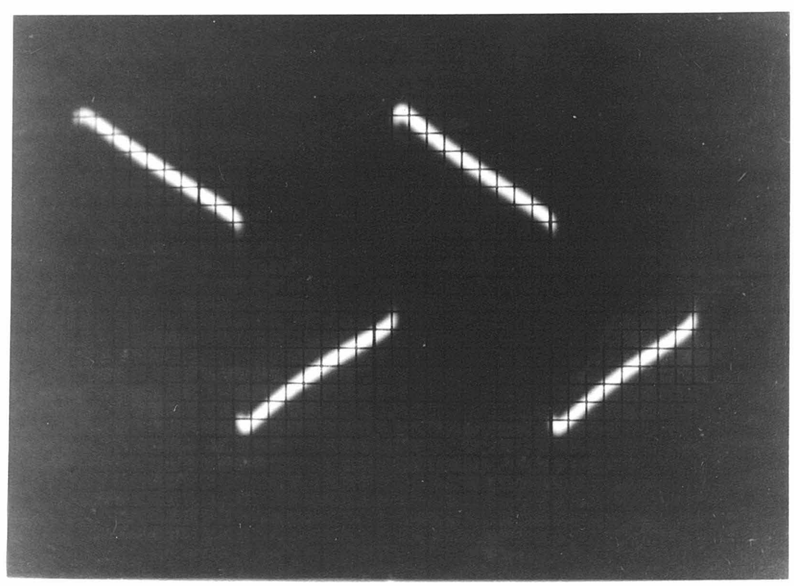

Fig. 9

Plate (Calc.) $b = 25$; $K = 0$

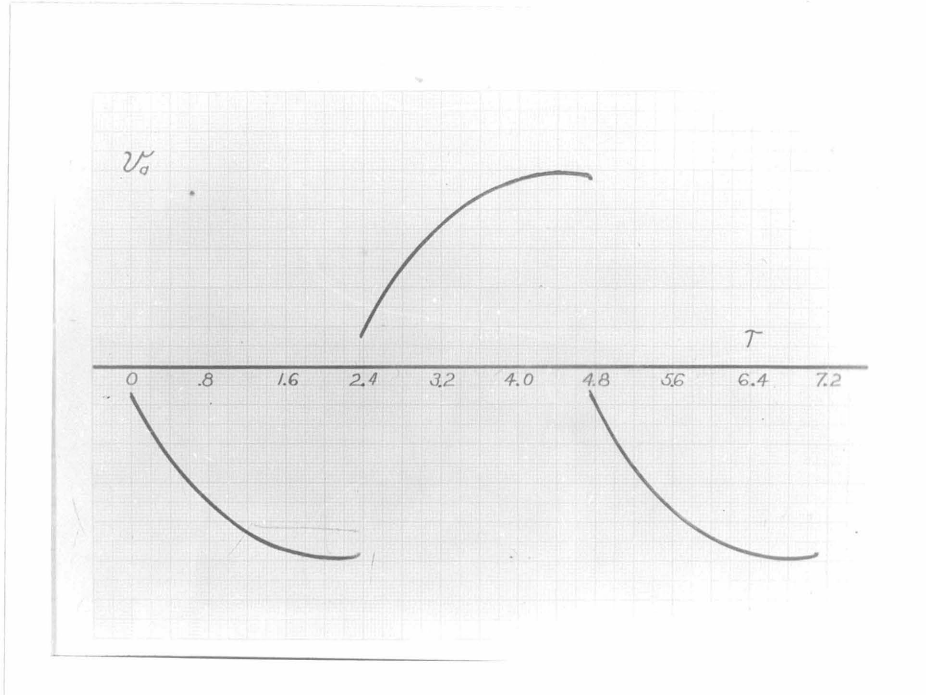

Fig 10

Plate (Obs.) $b = 25$; $K = 0$

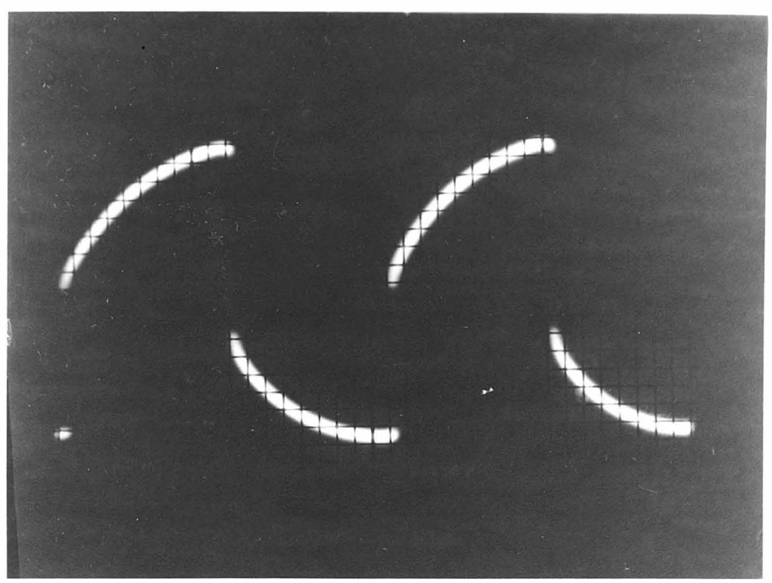

Fig. 11

Plate (Calc.) $b = 5$; $K = 0$

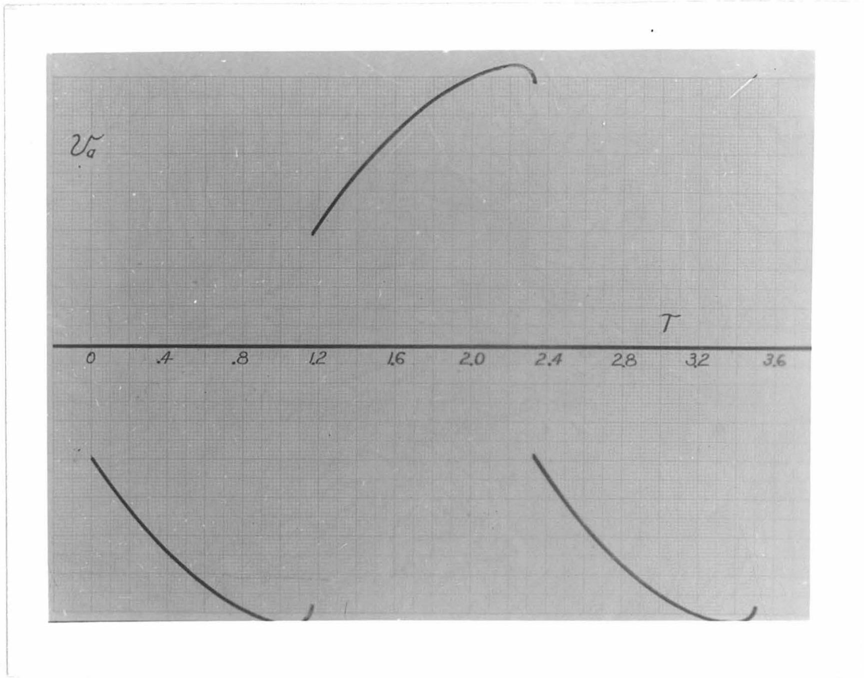

Fig 12

Plate (Obs.) $b = 5$; $K = 0$

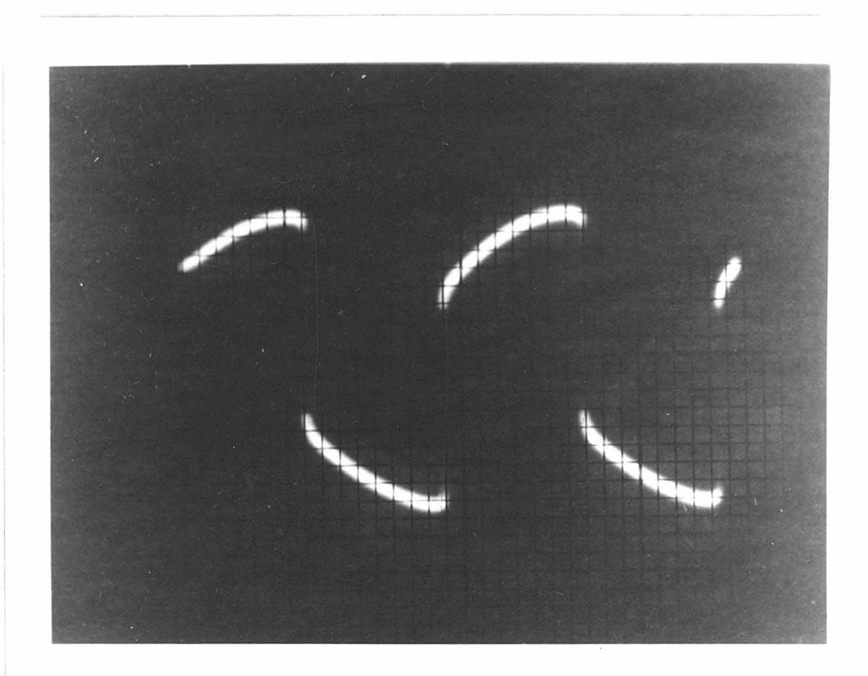

Fig 13

From these photographs it can be seen that the theory brings out the general relaxation nature of the multivibrator. Thus solutions of equation (23) represent the ideal relaxation oscillator in which the shunt capacities, represented by the parameter K, are neglected.

(d) Calculation of the Multivibrator Period.

Since the observed wave shapes agree with the theoretical, it is of considerable interest to calculate the period of the oscillations and check this experimentally. If T is the period then it can be found by noting that $\tau = \frac{pT}{2}$ when $u = u_0$. Hence equation (25) can be solved explicitly for the period. The result is:

$$T = \frac{2}{p}\left[\ln\frac{B}{\sqrt{b-1}} + \frac{b}{2}\ln\left(\frac{1+B^2}{B^2}\cdot\frac{b-1}{b}\right)\right] \quad . . (33)$$

For a given value of b the period in reduced time units (pT) can be found by first determining the value of B(b) from its graph and then solving equation (33) for pT. This has been plotted in Fig. 15

III. Experimental Check of Period

The following circuit arrangement was set up in order that the variation of the multivibrator period with circuit parameters could be observed.

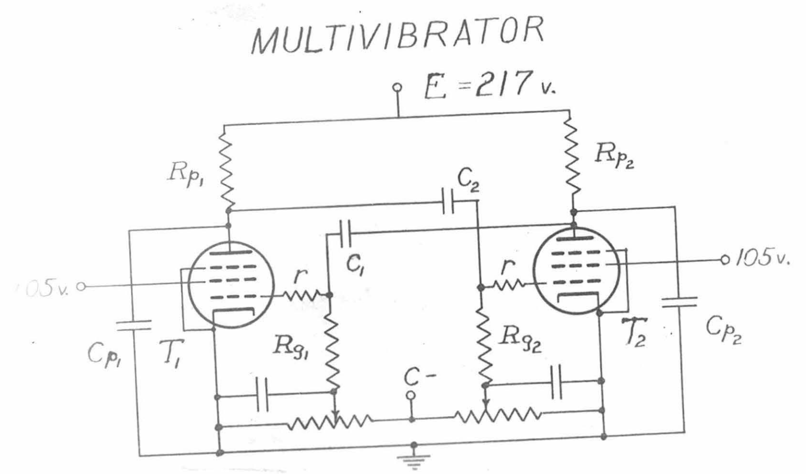

Fig 14.

(a) Experimental Considerations

The tubes, T_1 and T_2, were especially selected for their identical characteristics (same i_o and same g_m) insofar as it was possible. Since this was never exactly possible, whenever g_m appears as a measured quantity it is the average for the two tubes at the bias voltage chosen for each tube. The static characteristics of the tubes were measured within 2% and the slope of the grid transfer characteristics was then plotted as a function of the static grid voltage. Hence for a given setting of the grid bias on the tubes, the corresponding mutual conductance, g_m, can be found. This will be necessary in some later work.

Since it proves to be extremely difficult to analyse the action of the circuit when the series grid resistance r is included, the experimental conditions are chosen as an optimum between two opposing tendencies. First, for a given period (T) of the multivibrator the time constant rc_g, where c_g is the grid to cathode capacity, must be about 100 times smaller than the period T. This insures that there is no time lag between the actual voltage on the grid and the voltage applied to the grid. The other condition that must be satisfied is that r must be about 100 times larger than R_g, the grid resistor, in order to prevent grid current during the positive swing from short circuiting R_g. These two conditions require r to be both large and small at the same time, so actually there is only a limited range of multivibrator frequencies over which equation (33) can be checked within 2% or 3%.

Matched pairs of condensers were obtained by measuring a large number of Western Electric condensers to better than 1% on a decade bridge. The standard capacitance box was internally consistent to better than 1%.

The resistors R_p and R_g were made variable and could be measured to better than 1% on a Wheatstone bridge each time they were adjusted.

The frequency measurements were made by comparing on an oscilloscope the multivibrator wave with a sine wave from a Western Electric 13A oscillator. The dial of this oscillator was checked against the 60 cycle line voltage and found accurate to 1%. To correct for the drift, the 60 cycle point was checked frequently throughout the set of measurements.

It will be remembered also that, in order to make the plate current only a function of the grid voltage, the voltage on the plate must be greater than 100 volts. This meant that the plate resistors R_p must be made smaller than 10,000 ohms.

In the experimental test C_p and C_g were chosen as negligible and were placed equal to zero in the theory. Hence b and p become

$$p = \frac{1}{(R_g + R_p)C} \quad \ldots\ldots\ldots \quad (34)$$

$$b = \frac{g_m}{\frac{1}{R_p} + \frac{1}{R_g}} \quad \ldots\ldots\ldots \quad (35)$$

The observed frequencies were recorded with C = 0.0576 u.f., R_p varying from 1,500 ohms to 10,000 ohms, R_g varying from 600 to 9000 ohms. The series grid resistor r was set equal to 1 megohm, and the bias of the tubes was selected to give an average mutual conductance around 2200 umhos.

(b) Experimental Data

The experimental points (pT) were then plotted against b in Fig. 15 The value of b, however, was found using g_m = 5000 umhos instead of 2200 umhos as experimentally determined. It must be remembered that the analytical approximation used for the transfer characteristic does not have the proper curvature in many regions and in particular the region of "flip-over." Nevertheless if the value of g_m is increased by a constant factor, the theoretical transfer characteristic can be made to approximate very closely the actual transfer characteristic in the region of "flip-over." Thus there is a fairly good confirmation of the preceding theory. Had a better approximation to the transfer characteristic been used, it is conceivable that an exact check would be found.

The various conditions that were tried are indicated in the following table.

Table 1. Experimental Multivibrator Period

a. Series grid resistor = 1 megohm
b. Measured mutual conductance = 2400 umhos
c. Mutual conductance used in calculating b = 5000 umhos
d. Coupling condenser = 0.0576 u farads

R_p(ohms)	R_g(ohms)	f(cps)	pT	b
10000	9000	215	4.25	23.7
10000	5600	285	3.91	18.1
6250	5600	400	3.67	14.8
6250	3960	500	3.40	12.15
5300	3960	575	3.26	11.3
10000	1000	880	1.79	4.55
10000	1200	755	2.06	5.36

10000	1400	670	2.28	6.14
10000	1600	610	2.46	6.90
10000	1800	560	2.63	7.63
10000	2000	520	2.78	8.33
10000	2200	495	2.88	9.00
10000	2400	470	2.98	9.68
10000	2800	425	3.19	10.94
10000	3200	395	3.33	12.10
10000	3600	370	3.46	13.20
10000	4000	350	3.55	14.30
10000	5000	310	3.74	16.70
10000	7000	250	4.09	20.60
10000	9000	215	4.25	23.70
8000	9000	250	4.10	21.20
6000	9000	297	3.90	18.00
4000	9000	380	3.52	13.90
3000	9000	445	3.26	11.30
2000	9000	570	2.77	8.18
1500	9000	695	2.49	6.42
5000	1000	1620	1.79	4.16
5000	800	1975	1.52	3.45
5000	600	2460	1.26	2.67
5000	585	2470	1.26	2.62
11000	12000	171	4.42	28.8
11000	19000	125	4.64	34.8
11000	24000	106	4.68	37.6
11000	32000	85	4.76	41.0
11000	39000	73	4.77	42.7

5 megohm series grid resistor used on lower frequencies

11000	39000	71	4.97	42.7
11000	32000	84	4.82	41.0
11000	55000	54	4.88	45.9
11500	55000	49.6	5.28	47.6

Series grid resistor changed to 100,000 ohms to get higher frequencies. Coupling condenser reduced to 0.0110 u.f.

1200	9000	4365	2.04	5.3
1200	5000	7620	1.93	4.83
8000	5000	1940	3.61	15.4
4000	5000	3150	3.21	11.1

This data is plotted in Fig. 15 where it is compared with the theory.

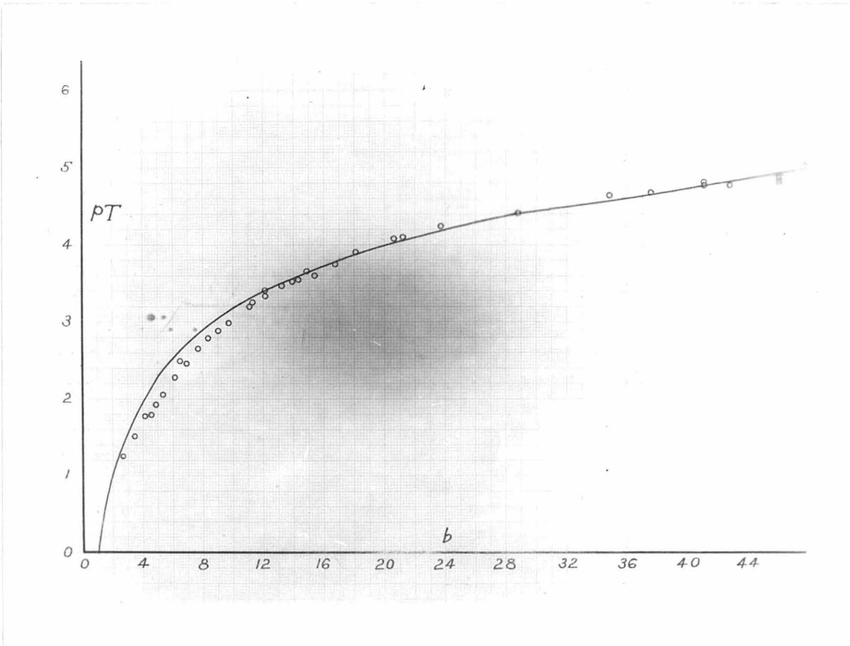

Fig. 15

IV. Effect of Shunt Capacity on Wave Forms.

(a) Wave Forms

In this section the effect of the second order term ($K \neq 0$) in equation (19) will be approximated. The equation is then:

$$K\frac{d^2u}{d\tau^2} + \left(1 - \frac{b}{1+u^2}\right)\cdot\frac{du}{d\tau} + u = 0 \quad \ldots\ldots \quad (36)$$

In order to facilitate the analysis, the effect of the first order term will be approximated as follows:

For large values of b the amplitude B of the grid wave can be approximated from equation (31). Thus:

$$B \cong \pi b \quad \text{for:} \quad \sqrt{b} \gg 1 \quad \ldots\ldots \quad (37)$$

Hence in equation (36), if u is not near the "flip over" region, for large values of b, its order of magnitude is b. Also for large values of b, and $u \neq u_0$, the damping term approaches 1. Thus equation (36) can be approximated, except in the region of "flip-over" by:

$$K\frac{d^2u}{d\tau^2} + \frac{du}{d\tau} + u = 0 \quad \ldots\ldots \quad (38)$$

This gives as a solution, $K \ll 1$.

$$u = Ae^{-\tau} + Be^{-\frac{\tau}{K}} \quad . \; . \; . \; . \; . \; . \quad (39)$$

When u is near zero the coefficient of the second term may be approximated by $-\underline{b}$, assuming $b \gg 1$. Thus near $u = 0$:

$$K\frac{d^2u}{d\tau^2} - b\frac{du}{d\tau} + u = 0 \quad . \; . \; . \; . \quad (40)$$

Again for $K \ll 1$, the solution is:

$$u = Ce^{\tau} + De^{\frac{b}{K}\tau} \quad . \; . \; . \; . \; . \; . \quad (41)$$

From equation (41) it can be seen that energy is injected into the oscillating circuit during the "flip-over" period, which in these reduced time units is equal to K/b. From equation (39) the dissipation of energy takes place as follows: The second term represents the form of the grid voltage while it is approaching its maximum amplitude. This approach to maximum requires a time K. Finally the first term of equation (39) represents the form of the grid voltage wave after it has reached its maximum amplitude. This continues on until the next "flip-over" is reached. The complete wave form is then somewhat as follows:

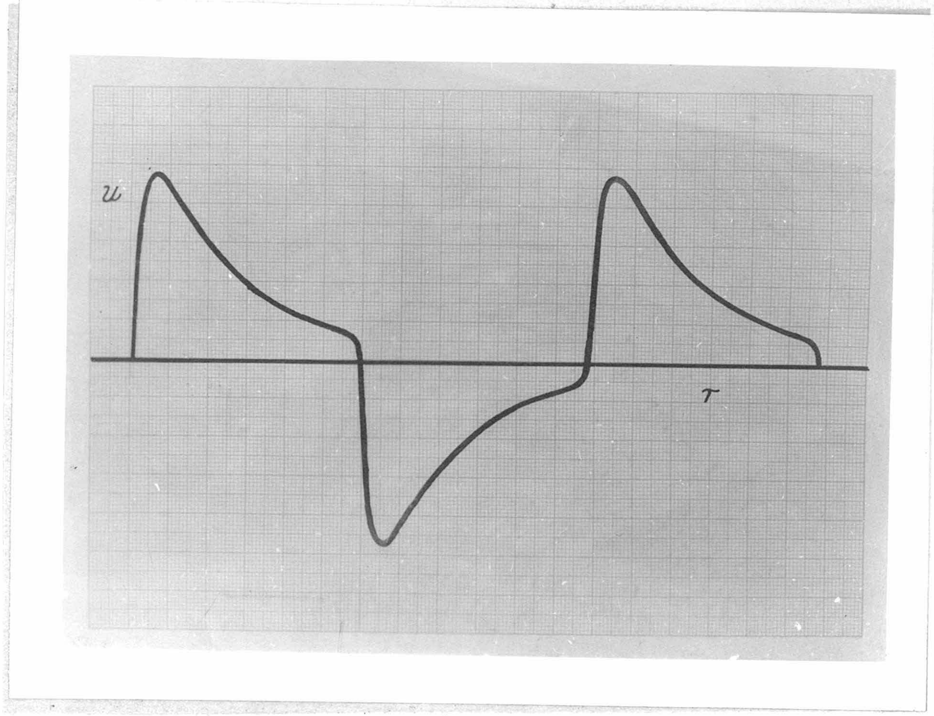

Fig 16

Under the various experimental conditions tabulated below, the following photographs were taken.

Grid $b = 20.6$; $K = 0.072$

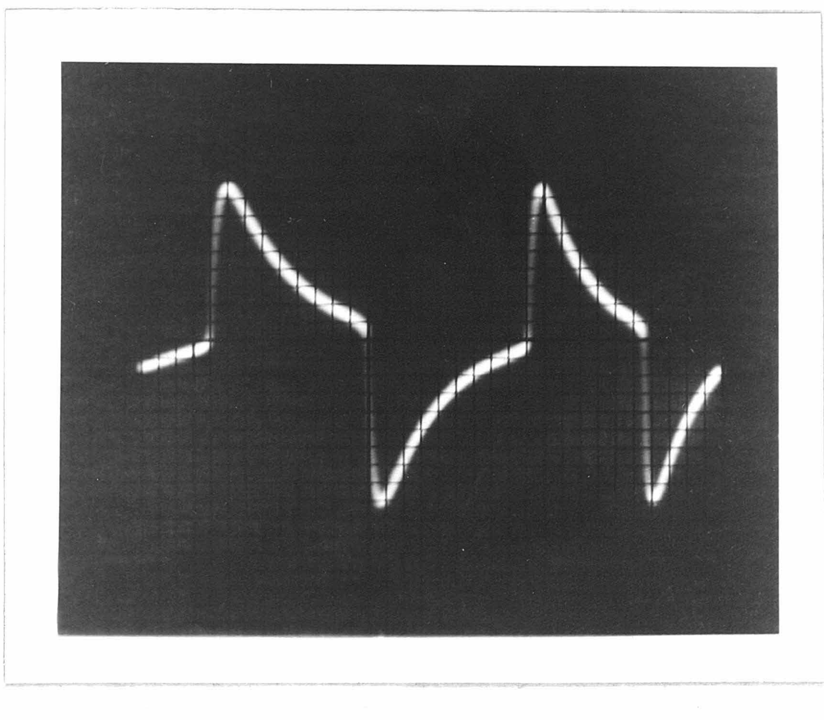

Fig 17

Grid $b = 24.0$; $K = 0.0204$

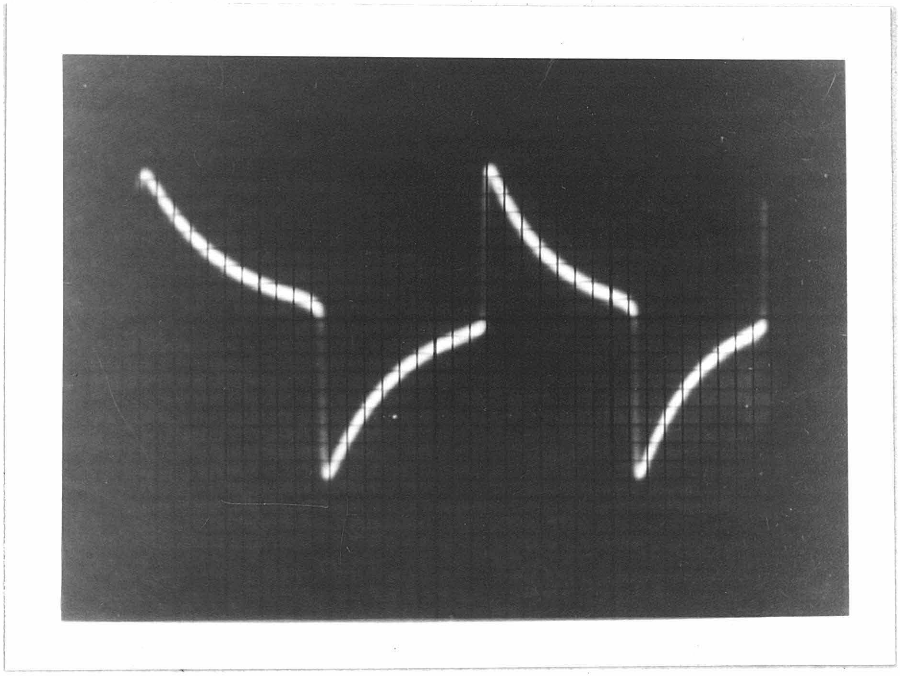

Fig 18

Grid $b = 24.7$; $K = 0.00495$

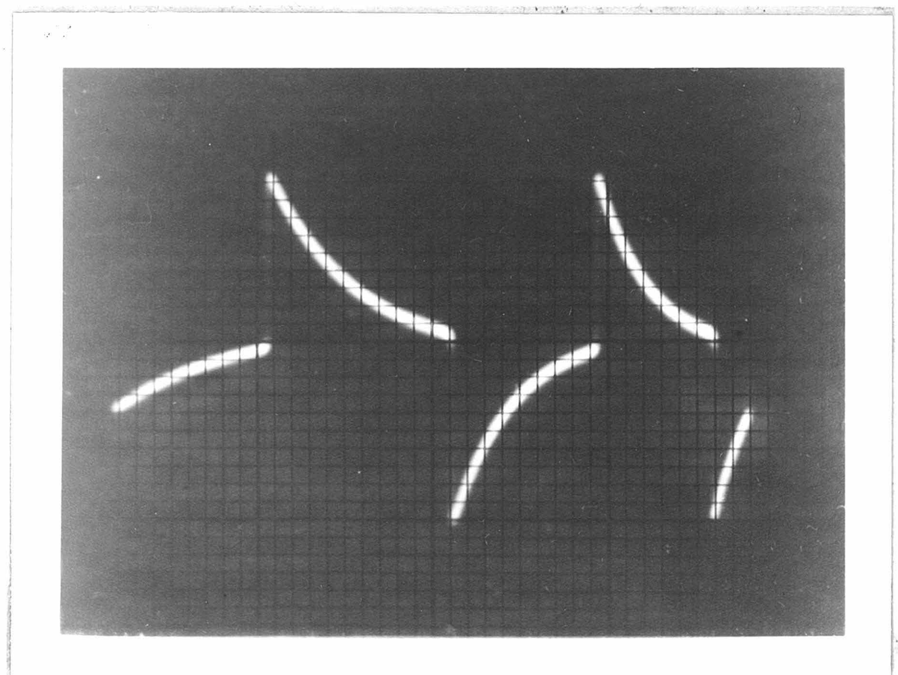

Fig 19

Grid $b = 5.83$; $K = 0.333$

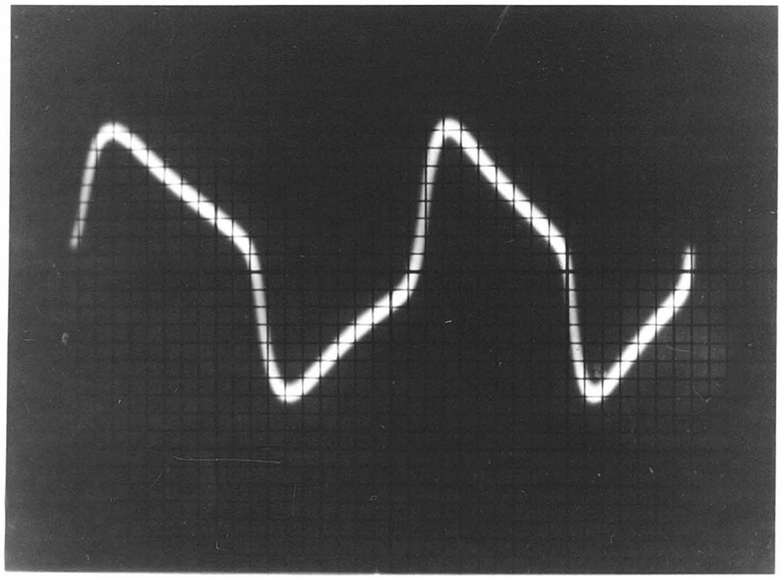

Fig 20

Grid $b = 5.63$; $K = 0.045$

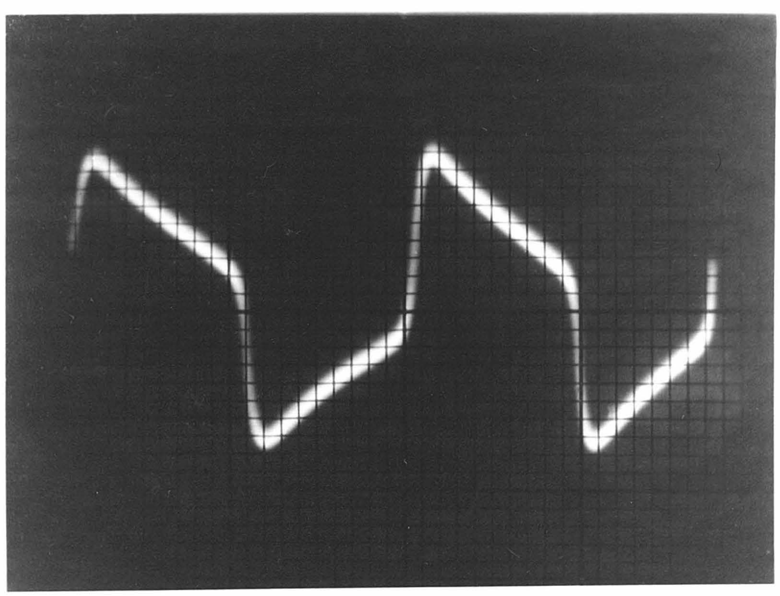

Fig. 21

Grid $f = 3.71$; $K = 0.00495$

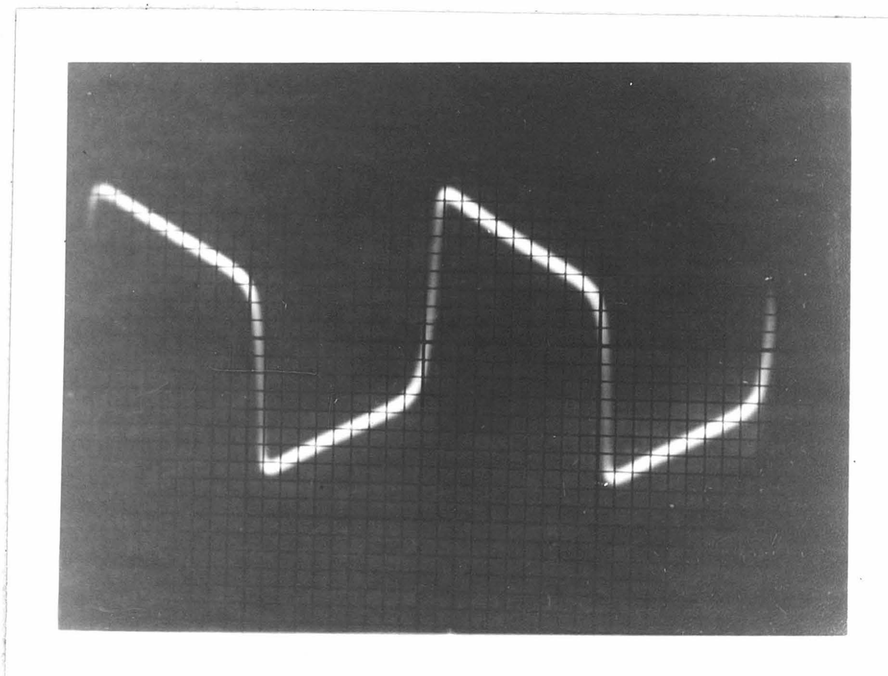

Fig 22

(b) Experimental Period when Shunt Capacity is Included.

It is conceivable that the major correction to a multivibrator period when the shunt capacities C_p and C_g are included would be in a redefinition of the quantities p and b. Experimentally, it is desirable to keep C_g, the grid to cathode capacity, as small as possible in order that r, the series grid resistor, may be as large as possible and still have a small time constant rC_g. Hence the effect of the shunt capacity was determined solely from C_p, the plate to ground capacity. Returning to the original definitions of p and b, if $C_g \ll C$

$$p = \frac{1}{C[R_g + R_p(1+\frac{C_p}{C})]} \quad \ldots\ldots (42)$$

$$b = \frac{g_m}{\frac{1}{R_g} + \frac{1}{R_p}(1+\frac{C_p}{C})} \quad \ldots\ldots (43)$$

If the experimental date (pT) is compared with the same theory (Eq. 33) using as before g_m = 5000 umhos it is seen to check fairly well.

Table 2. Experimental Data Including Shunt Capacity

(a) Series grid resistor = 100,000 ohms
(b) Mutual conductance used in calculating b = 5000 umhos.

R_p(ohms)	R_g(ohms)	C(uf)	C_p(uf)	f(cps)	b	(Obs) pT	(Calc) pT
10000	7000	.0576	.00945	248	18.7	3.75	3.87
10000	7000	.1233	.00945	121	19.7	3.78	3.96
10000	10000	.1233	.00945	98.9	24.1	3.95	4.18
Changed series grid resistor to 5 megohm							
10000	10000	0.1233	.00945	92.4	24.1	4.23	4.18
Changed series grid resistor to 1 megohm							
10000	10000	0.1233	.00945	94	24.1	4.16	4.18
10000	10000	.0576	.00945	190	23.1	4.24	4.14
10000	7000	.0576	.00945	236	18.7	3.94	3.90
10000	7000	.1233	.00945	115	19.7	3.98	3.96
10000	5000	.1233	.00945	140	15.9	3.67	3.70

V. Linearized Solution and First Order Approximation

(a) Wave Forms

By referring to the graph of B(b) it is seen that the amplitude of the grid voltage approaches zero as b approaches 1. However, if shunt capacity is included the second order term requires that sine waves be produced of vanishing amplitude. That this proves to be true will be obvious after considering the first order approximation worked out by Appleton and Greaves[8] using the method

(8) Appleton and Greaves, Phil. Mag., 45, pp. 401-414 (1923).

of successive approximations. The equation discussed in this paper, where f(v) is an arbitrary power series is:

$$\frac{d^2 v}{d\tau^2} + f(v)\frac{dv}{d\tau} + \omega^2 v = 0 \quad . \quad . \quad . \quad . \quad (44)$$

In equation (36) let:

$$u = \sqrt{b-1}\, v = \sqrt{\epsilon}\, v \quad . \quad . \quad . \quad . \quad (45)$$

$$\omega^2 = \frac{1}{K} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$f(v) = -\frac{\epsilon}{K}\frac{(1-v^2)}{(1+\epsilon v^2)} \quad . \quad . \quad . \quad . \quad (47)$$

Expanding f(v) there results:

$$f(v) = -\frac{\epsilon}{K} + \frac{\epsilon}{K}(1+\epsilon)v^2 - \frac{\epsilon^2}{K}(1+\epsilon)v^4 \quad . \quad . \quad (48)$$

Appleton and Greaves give as the solution:

$$u = \sqrt{\frac{b-1}{b}}\left[2\sin\omega_0\tau + \frac{b-1}{8K\omega_0}\left(\cos\omega_0\tau - \cos 3\omega_0\tau\right)\right] \quad . \quad . \quad (49)$$

where

$$\omega_0^2 = \frac{1}{K}\left(1 - \frac{\epsilon^2}{8K}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (49a)$$

A plot of this function is as follows:

$b = 1.3$; $K = 0.045$

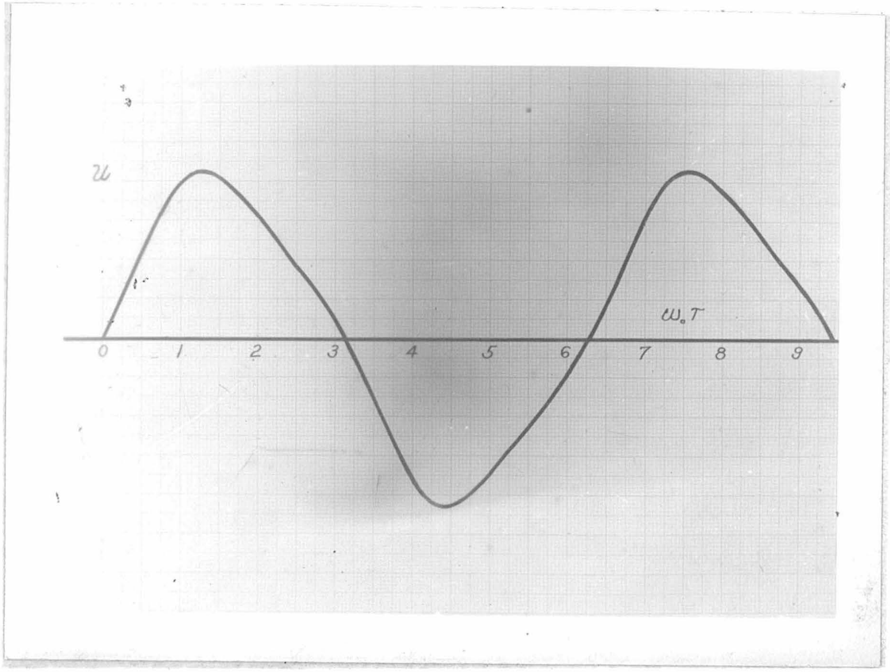

Fig 23

Experimentally the wave shape on the grid is as follows:

$b = 1.3$; $K = 0.045$

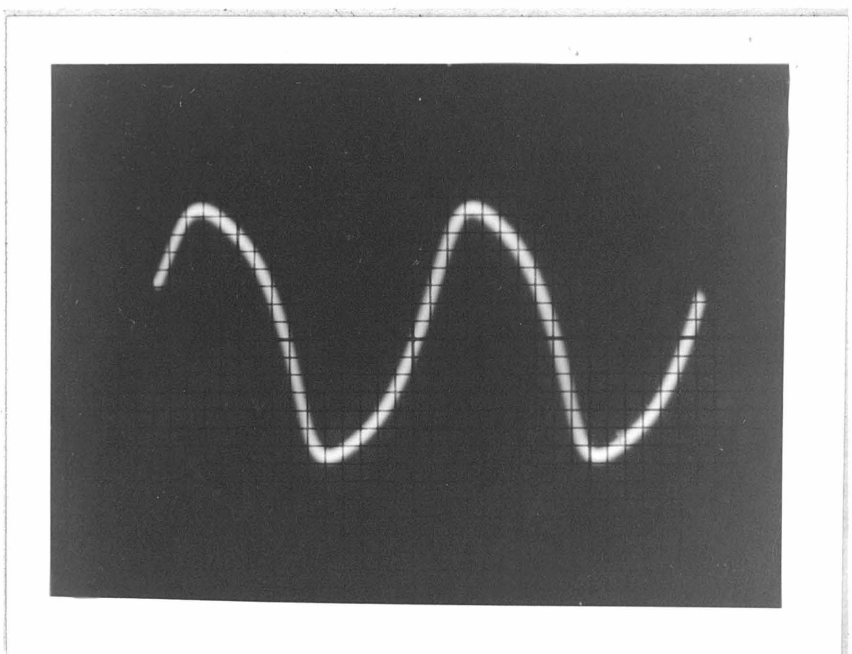

Fig 24

If observations are made on the wave form just before oscillation stops, $b \cong 1$, a sine wave results. From equation (49)

$$U = 2\sqrt{b-1}\,\sin\frac{\tau}{\sqrt{K}} \quad . \quad . \quad . \quad . \quad . \quad (50)$$

This shows that as long as there is some shunt capacity (K = finite), as $b \to 1$ the wave form will approach a sine wave whose limiting frequency is given by:*

$$f_0 = \frac{1}{2\pi\sqrt{K}} \quad . \quad . \quad . \quad . \quad . \quad (51)$$

The observed wave form is the following.

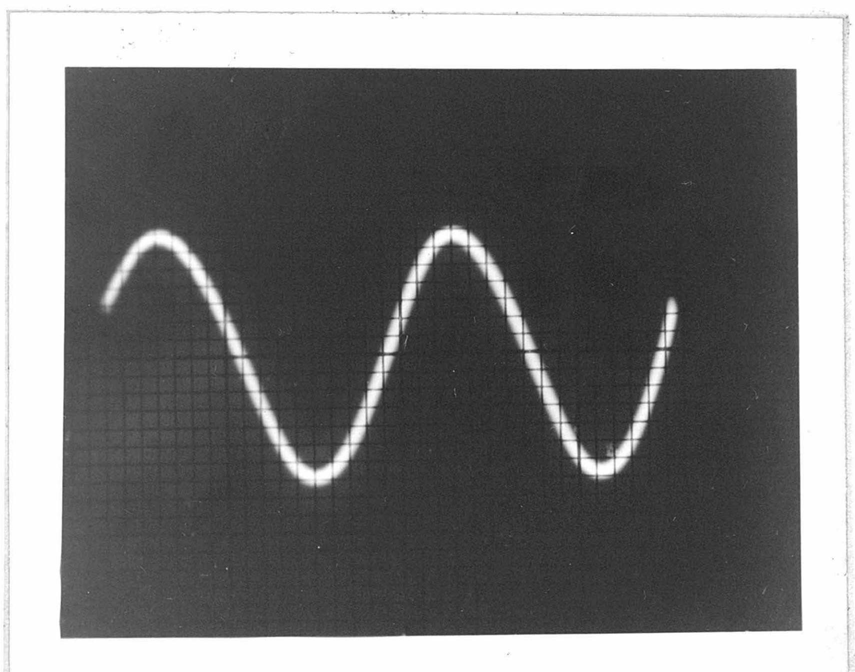

Fig 25

* f_0 in reduced time units (pt).

(b) Experimental Check of Limiting Frequency (f_o)

For reasons stated before the grid to ground capacitance, C_g, is kept as small as possible. In this case using equations (4) and (5), equation (51) becomes:

$$f_o = \frac{1}{2\pi\sqrt{R_g C + R_p(C+C_p)}} \cdot \sqrt{\frac{g_m}{C_p}} \quad . \; . \; . \; . \; (52)$$

where also

$$k = \frac{g_m}{\frac{1}{R_g}\left(1+\frac{C_p}{C}\right) + \frac{1}{R_p}} = 1 \quad . \; . \; . \; (53)$$

In order to check these equations it was found necessary to insert a 100,000 ohm series grid resistor r because grid current flow effectively lowered the measured value of the grid resistor R_g. This meant, however, that since the rC_g time constant corresponded to 1 megacycle, the only frequencies that could be checked were those up to 10,000 c.p.s. if accuracy within a few per cent was desired.

As before the resistances R_g and R_p represent the average of the grid and plate resistors respectively. The grid resistors and the plate resistors were repeatedly set equal to within 2 % or 3 %. Similarly the capacitance readings C and C_p are the average of condensers that are equal within 2% or 3%. The value of g_m recorded is the average mutual conductance of the tubes for the bias at which they were operated. No corrections to g_m were necessary here because the theoretical approximation is sufficiently accurate. Frequencies were measured as before by comparing the multivibrator wave with the Western Electric 13 A oscillator. The data is tabulated as follows:

Table 3. Limiting Frequencies

R_p(ohms)	R_g(ohms)	C(uf)	C_p(uf)	g_m(umhos)	(obs) f_o(cps)	(calc) f_o(cps)	(obs) b
1110	1275	.1233	.1210	2320	1070	1068	0.966
1110	1175	.0576	.0521	2350	2450	2465	0.932
1110	882	.1233	.0521	2350	1930	1940	0.932
1110	1944	.0576	.1210	2360	1270	1270	0.940
1110	1285	.545	.545	2325	240	232	0.944
1110	733	.545	.1210	2330	640	639	0.939
1110	3705	.0110	.0521	2330	3170	3225	0.955
1110	1209	.0110	.00945	2350	11800	13800	0.963

The experimental and theoretical values of f_o check within 3% up to approximately 10,000 cycles per second where it will be remembered that the grid capacity and series grid resistance time constant tends to make the observed frequency come out low. This effect is directly observable by adding, say, 20 u.u.f. to the grid capacity and noticing that the frequency drops 700 cycles.

The parameter b, which should equal 1 at this limiting frequency, comes out consistently low by 5 or 6%. Possibly further experimentation would clear up this point. However, as it stands the theory checks within 5 or 6% with the experimental values.

VI. Effect of Grid Current

While the introduction of series grid resistors to prevent grid current flow is a useful device at the lower frequencies (up to 10,000 c.p.s.), if higher frequencies are desired this resistance must be omitted in order to maintain a steep wave front. The original analysis cannot be carried through because the grid voltages are no longer equal and opposite in phase.

The effect of grid current flow in the grid circuit is to lower (R_g), the grid resistor, to about 500 ohms during the positive swing of the grid. In the plate circuit the flow of grid current changes the transfer characteristic so that it is not as symmetrical about the bias voltage. These factors increase the frequency of the multivibrator over that calculated in the previous theory. The grid wave forms are changed in a somewhat predictable manner. For instance, for large values of the parameter b ($b > 10$) the grid current cuts off the positive swing of the grid voltage. Thus:

Grid

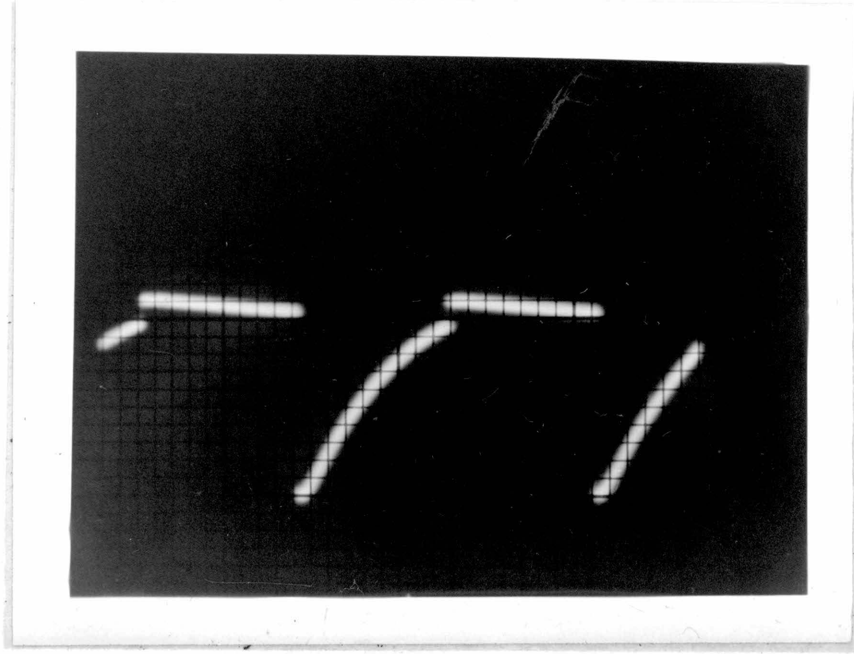

Fig 26.

Since the grid resistor effectively is much smaller than the plate resistor due to grid current flow, the exponential distortion is larger. Hence:

Plate

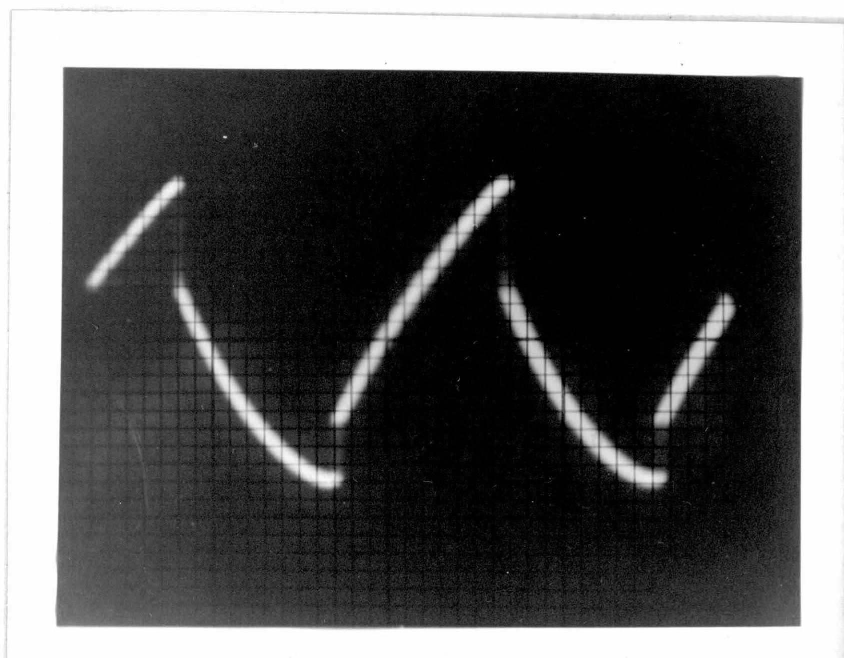

Fig 27

For small values of b, the flow of grid current does not affect the grid circuit as much as the plate circuit. The effect is to change the grid transfer characteristic from the one assumed here to that assumed by Van der Pol with the result that the grid wave forms are as follows:

Fig 28

This is seen to be of the form calculated by Van der Pol.(9)

The exponential distortion of the plate voltage for small b is reduced because the plate resistors are smaller. The plate current distortion, however, is just as large. Hence

(9) Relaxation Oscillations, Phil. Mag. (7) 2, p. 986 (1926).

Plate

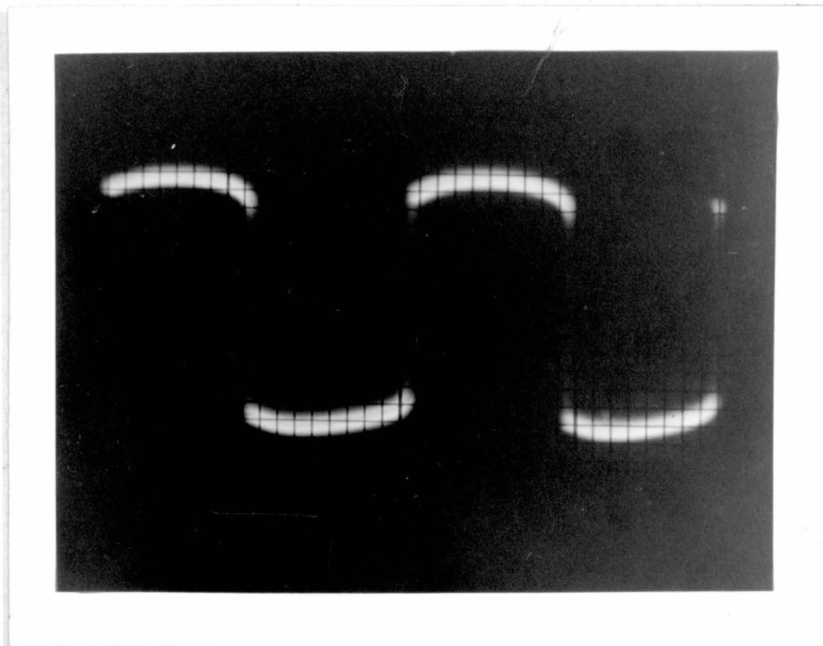

Fig 29

VII. Practical Results

(a) Distortions

Since the multivibrator is used to produce "square waves" it is important to examine the wave form on the plate of the tubes in order to determine what factors make the wave depart from a true rectangular shape. Aside from the ultimate limit to squareness caused by the shunt capacities and the lead inductances, there are two other sources of distortion. First, the flow of current through the condensers (C) results in an exponential beginning of the wave. Secondly, the flow of current through the vacuum tube just before "flip-over" occurs gives rise to the final curvature of the wave. These distortions can be minimized by a proper choice of circuit constants. In general it can be said that the sharper the transfer characteristic (large g_m) the more quickly will the "flip-over" take place. To be more specific, the parameter b must be much greater than 1. This means that the "flip-over" region is only a small part of the actual grid swing.

To correct for the exponential distortion the grid resistor (R_g) must be large compared with the plate resistor (R_p). If this is not feasible, the cathode current may be used by inserting a small resistor in the cathode circuit. Also, cathode followers may be inserted after each tube. Still another method is to introduce another tube in such a way that there is no condenser connection to the plate from which the output signal is taken. These methods will be illustrated below.

To eliminate the distortion due to plate current flow before "flip-over", it is necessary to increase the mutual conductance (g_m) of the transfer characteristic. One method of doing this is to make use of the discontinuous properties of trigger circuits. W. Nottingham[10] has suggested a circuit which has the following "transfer characteristic."

(10) Class Notes (1940).

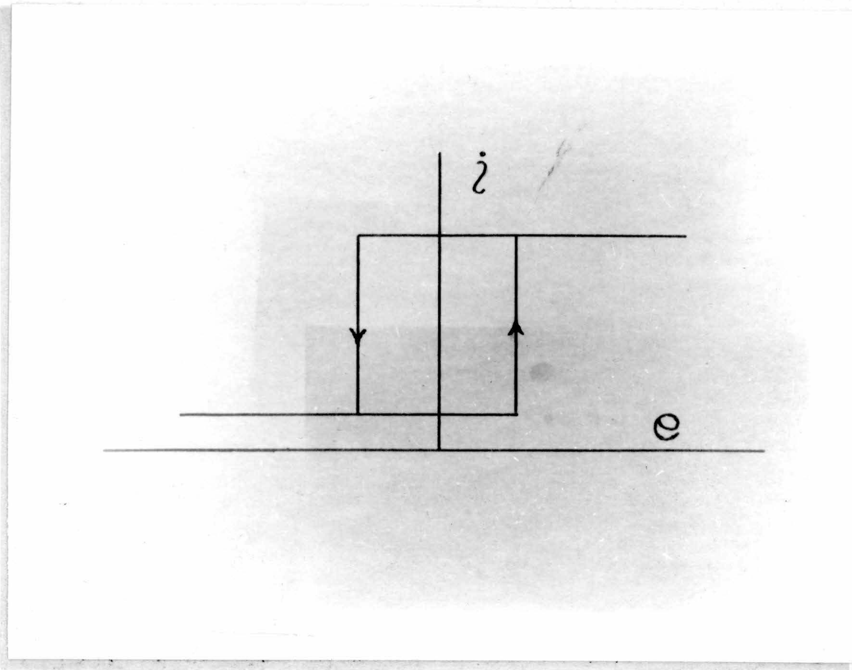

Fig. 30

This may be used to give an effective (g_m) which is essentially infinite (limited only by the tube capacities).

(b) Practical Circuits

1. Multivibrator Utilizing Cathode Followers

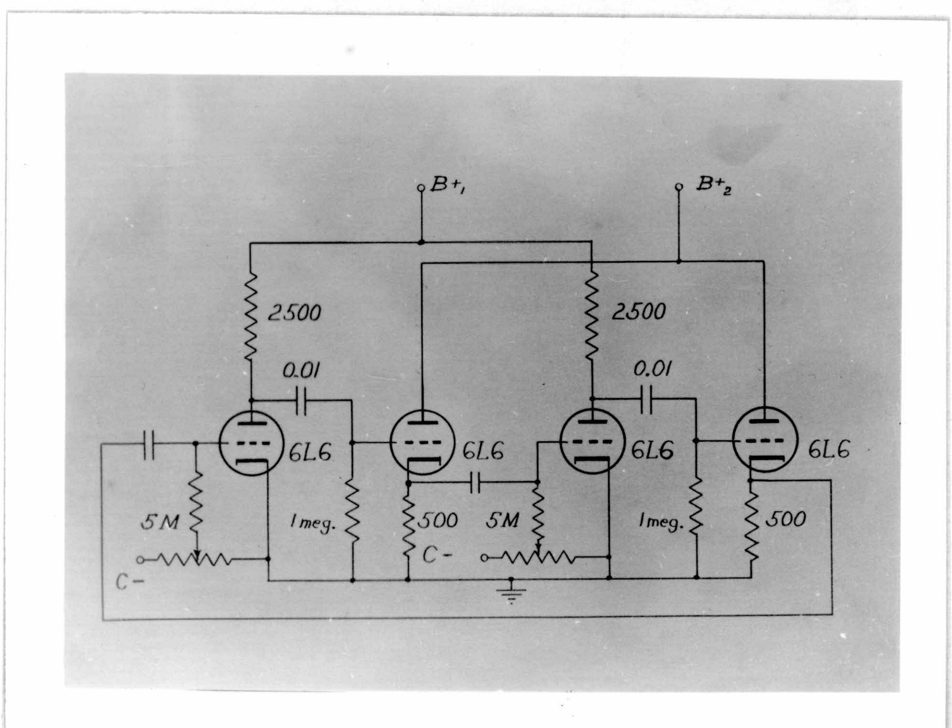

Fig 31

2. Modified Nottingham Trigger Circuit

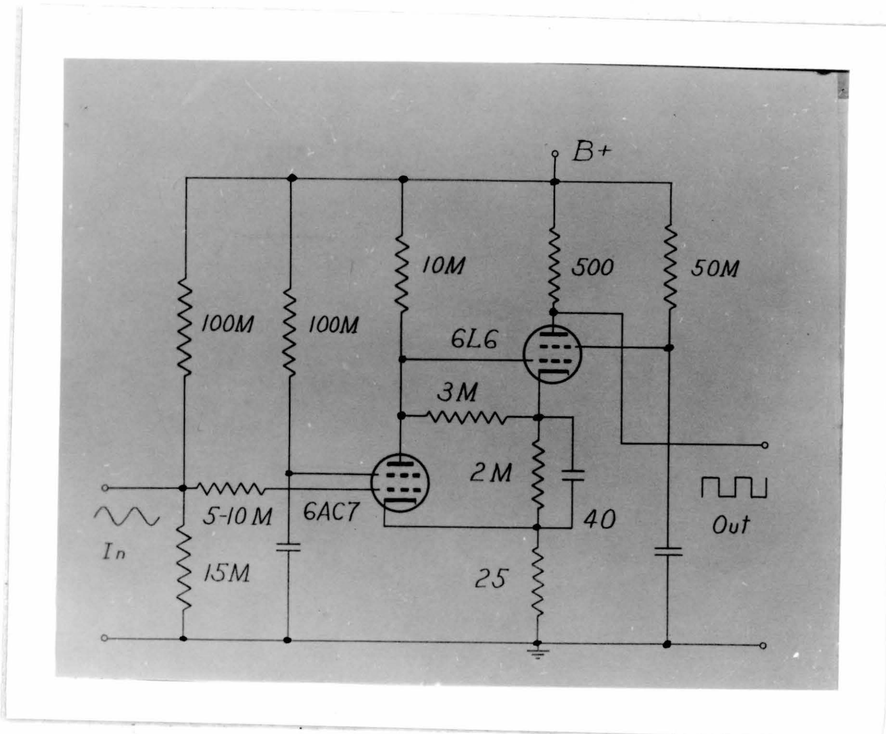

Fig 32

The operation is based on the regeneration introduced by the 25 ohm cathode resistor. When the voltage input (e) is sufficiently positive the bias on the 6AC7 will be positive and current will flow, thus biasing the 6L6 to cut-off. The voltage of the point (A) is then, say, 400 volts. As (e) is made less positive and finally negative there comes a point, namely the cut-off point of the 6AC7, when the 6AC7 conducts less. But in doing so the 6L6 conducts more and by virtue of the 25 ohm cathode resistor more bias appears on the 6AC7. Thus the regeneration introduced by this resistor caused an extremely sharp cut-off. The voltage at the point (A) now drops to around 300 volts. As (e) is made more negative nothing happens. However, if (e) is now made less negative and then positive, when (e) reaches the former discontinuity nothing happens since the bias across the 25 ohm resistor is different. The return to the initial state does take place when (e) is equal to $\left[25 \begin{array}{c}\text{Main 6L6}\\ \text{Current}\end{array} - E_g \text{ (Cut off 6AC7)}\right]$. When This point is reached the action is again regenerative in that the bias across the 25 ohm resistor changes in the same direction that (e) is changing. The cycle is then complete. The hysteresis loop of the voltage at the point (A) is as follows:

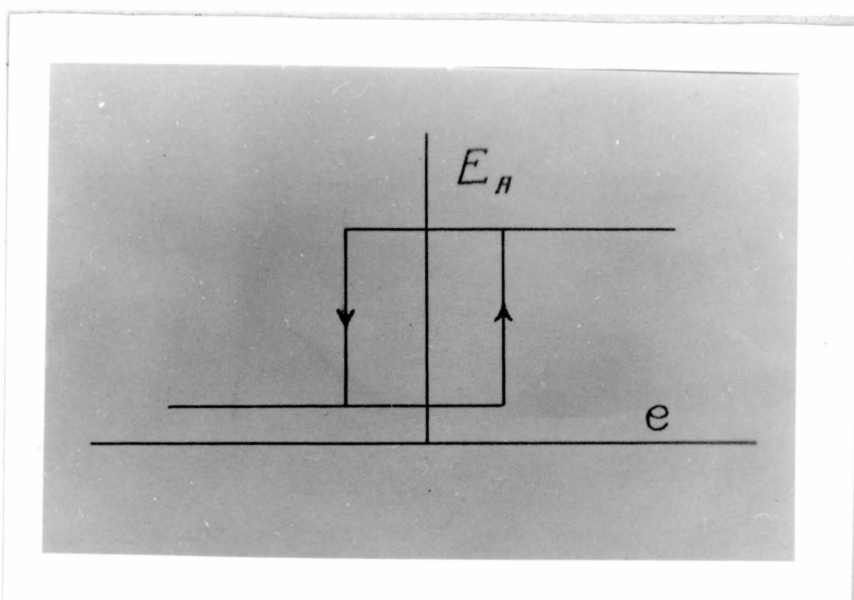

Fig 33

A 6AG7 may be used to advantage in place of the 6L6 if the resistors are increased.

3. Super-Regenerative Multivibrator.

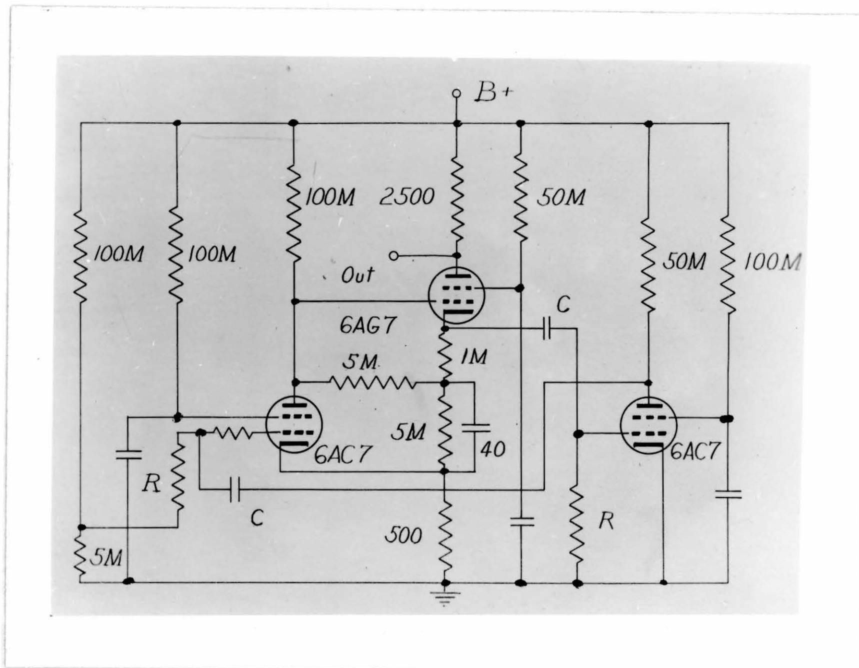

Fig 34

The advantages of this circuit are fourfold. First, because of the sharp cut-off trigger circuit (6AC7 - 6AG7), the pulse contains no distortions due to plate current flow before "flip-over". Secondly, there is no distortion of the output from the flow of current in the condensers (C). Thirdly, the size of the pulse on the grid of the second 6AC7 is independent of the frequency (value of R) within limits. This allows the circuit to be used at higher frequencies. Lastly, the frequency is not determined by the cut-off characteristic of the second 6AC7 since the circuit is regenerative without this tube. This tube merely serves to switch the equilibrium positions of the first regenerative unit. The result is a square wave where sharpness is solely determined by the shunt capacities of the tubes.

In conclusion, the author wishes to express his appreciation to Dr. W. H. Pickering for suggesting the problem, and for his advice and interest throughout the rest of the work.

Appendix I. Proof that $u + v \to 0$ as $\tau \to \infty$.

$$K\frac{d^2u}{d\tau^2} + \frac{du}{d\tau} + u + \frac{b}{1+v^2}\cdot\frac{dv}{d\tau} = 0 \quad \cdots\cdots (1)$$

$$K\frac{d^2v}{d\tau^2} + \frac{dv}{d\tau} + v + \frac{b}{1+u^2}\cdot\frac{du}{d\tau} = 0 \quad \cdots\cdots (2)$$

Let:

$$u+v=2R \quad ; \quad u-v=2S \quad \cdots\cdots (3)$$

Solve (3) for u and v and substitute into (1) and (2). This gives:

$$K\frac{d^2R}{d\tau^2} + K\frac{d^2S}{d\tau^2} + \frac{dR}{d\tau} + \frac{dS}{d\tau} + R + S + \frac{b}{1+(S-R)^2}\left[\frac{dR}{d\tau} - \frac{dS}{d\tau}\right] \quad (4)$$

$$K\frac{d^2R}{d\tau^2} - K\frac{d^2S}{d\tau^2} + \frac{dR}{d\tau} - \frac{dS}{d\tau} + R - S + \frac{b}{1+(S+R)^2}\left[\frac{dR}{d\tau} + \frac{dS}{d\tau}\right] \quad (5)$$

Experimentally there is no question of whether or not a unique periodic solution of (1) and (2) exists. However, mathematically it might be proved by reducing (1) and (2) to three first order equations and then applying some generalizations given by Lefshetz.[1] Here it will be assumed that u and v have unique single-valued

(1) Existence of Periodic Solutions for Certain Differential Equations S. Lefshetz, Proc. Nat'l Academy of Science 29, pp. 29-32 (1943).

solutions. Thus from equation (3) R and S are single-valued functions of τ.

Consider (1) and (2) solved for u and v, then define:

$$P(\tau) = 1 + \frac{b}{2}\left[\frac{1}{1+(S-R)^2} + \frac{1}{1+(S+R)^2}\right] \quad \cdots (6)$$

$$Q(\tau) = \frac{b}{2}\left[\frac{1}{1+(S-R)^2} - \frac{1}{1+(S+R)^2}\right] \quad \cdots (7)$$

$$M(\tau) = 1 - \frac{b}{2}\left[\frac{1}{1+(S-R)^2} + \frac{1}{1+(S+R)^2}\right] \quad \cdots (8)$$

Adding and subtracting equations (4) and (5), and making use of (6), (7) and (8), there results:

$$K\frac{d^2R}{d\tau^2} + P(\tau)\frac{dR}{d\tau} + R = Q(\tau)\frac{dS}{d\tau} \quad \cdots (9)$$

$$K\frac{d^2S}{d\tau^2} + M(\tau)\frac{dS}{d\tau} + S = -Q(\tau)\frac{dR}{d\tau} \quad \cdots (10)$$

The object now is to prove that R $\rightarrow$ 0. Notice that in equation (9) each term may be thought of as a force by using a dynamical analogy. The first term represents the inertial force, the second the resistance force, and the third the restoring force. On the other side there is an applied force.

To prove the theorem use the first integral of the system, namely the 'energy integral". Multiply (9) by dR and integrate. The result is:

$$W = \frac{1}{2}K\left(\frac{dR}{d\tau}\right)^2 + \frac{1}{2}R^2 = W_0 + \int_0^\tau \left[Q(\tau)\frac{dS}{d\tau}\frac{dR}{d\tau} - P(\tau)\left(\frac{dR}{d\tau}\right)^2\right] d\tau \quad (11)$$

W then is the 'energy' associated with the R coordinate of the system and is equal to the initial 'energy' plus that added by the $Q(\tau)$ integral minus that dissipated by the resistance $P(\tau)$.

In a similar fashion the 'energy' integral of (10) is:

$$U = \frac{1}{2}K\left(\frac{dS}{d\tau}\right)^2 + \frac{1}{2}S^2 = U_0 - \int_0^\tau \left[Q(\tau)\frac{dS}{d\tau}\frac{dR}{d\tau} + M(\tau)\left(\frac{dS}{d\tau}\right)^2\right] d\tau \quad (12)$$

U has a similar interpretation for the coordinate S. The total 'energy' of the system is:

$$H = U + W \quad . \; . \; . \; . \; . \; . \; . \quad (13)$$

$$H = H_0 - \int_0^\tau \left[M\left(\frac{dS}{d\tau}\right)^2 + P\left(\frac{dR}{d\tau}\right)^2\right] d\tau \quad . \; . \; . \quad (14)$$

The dissipation function for this system is then:

$$F = M\left(\frac{dS}{d\tau}\right)^2 + P\left(\frac{dR}{d\tau}\right)^2 \quad . \; . \; . \; . \quad (15)$$

Thus:

$$H = H_0 - \int_0^{\tau} F \, d\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

But for large τ, H does not approach a constant.

$$\lim_{\tau \to \infty} H \neq \text{Const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This would mean that the steady state solution of (9) and (10) was sinusoidal, and it is easily seen that a sinusoidal solution, in general, does not satisfy the equations. Furthermore, since the 'energy' fluctuations Δ H must be bounded, it follows that the 'energy' must oscillate about some value. This means that the dissipation function F must oscillate about zero for large τ. But from equation (6):

$$P(\tau) > 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and $\left(\frac{dR}{d\tau}\right)^2$ and $\left(\frac{dS}{d\tau}\right)^2$ are essentially positive; M(τ) changes sign depending on the magnitude of R and S. Also since R and S are single-valued functions of τ, M(τ) must change sign as τ increases. Thus F is made up of an oscillating function $M\left(\frac{dS}{d\tau}\right)^2$ and a non-oscillating function $P\left(\frac{dR}{d\tau}\right)^2$.

But the dissipation function must oscillate about zero for large τ. Hence for large τ:

$$P(\tau)\left(\frac{dR}{d\tau}\right)^2 \to 0 \quad \cdot \cdot \cdot \cdot \cdot \quad (19)$$

Using (18):

$$\frac{dR}{d\tau} \to 0 \quad \cdot \cdot \cdot \cdot \cdot \cdot \quad (20)$$

Thus:

$$\frac{d^2R}{d\tau^2} \longrightarrow 0 \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (21)$$

Using (7) equation (9) approaches:

$$R = \frac{2\,b\,RS}{[1+(S-R)^2][1+(S+R)^2]} \cdot \frac{dS}{d\tau} \quad \cdot \cdot \cdot \cdot \quad (22)$$

But this is not the solution for S since equation (10) approaches

$$K\frac{d^2S}{d\tau^2} + M\frac{dS}{d\tau} + S = 0 \quad \cdot \cdot \cdot \cdot \quad (23)$$

Hence:

$$R \to 0 \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (24)$$

since this solves (22) and satisfies (1) and (2). But R = u + v, hence for large values of r.

$$u = -v \qquad \ldots\ldots\ldots \quad (25)$$

Q.E.D.

Error

I have since found errors in my reasoning and am unable to prove the theorem. However I still believe it to be true and have already proved it for large values of u and v $(K=0)$.

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