Analysis in Linear Topological Spaces

Thesis by

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Introduction

During the past five years the study of linear topological spaces, those non-metrisable spaces intermediate between the pure spaces of ensemble topology and the normed spaces of General Analysis, has received much attention, particularly under the hands of the Russian and Polish schools headed by Kolmogoroff and Tychonoff. Interest in the slightly weaker spaces of topological group type has also been greatly stimulated by the search for strong purely topological foundations of group theory by Schreier, von Dantzig and others, and for an abstract formulation of continuous group theory as in the work of Michal and Elconin.

Because of the implied necessity of theories in the large for situations such as these, it was felt desirable to determine what analytic entities could be defined, while preserving as large a portion of the usual properties as possible, directly for the base spaces, without the rigid local intermediary of the norm.

I should like to express here my appreciation of the sustained assistance and advice of Professor A.D.Michal in the development of this thesis.

Résumé

Part 1 of this thesis is devoted to a general study of topological and geometrical properties of a linear space subjected to a topology of neighborhood type. Some new results on questions of boundedness and compactness are given.

Part 2 contains essentially work done in collaboration with Professor A.D.Michal during 1935 and 1936. It defines differentials of extended Fréchet and Gateaux type as well as derivatives for functions on the real numbers. Elementary properties are considered.

Parts 3 and 4 develop the theory of integrals of Riemann type, examine the relations to the derivative, and in particular verify the existence of the unique primitive solution of the differential equation $dy/d\mu = f(\mu)$, μ real.

Part 5 furnishes two types of existence theorem technique for the ordinary linear equation $dy/d\mu = f(\mu, y)$ in the linear topological space.

Various examples are considered in an appendix.

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Symbols : Those not enlarged upon in the text.

$x, y, p, q, \ldots$	abstract elements, points
$\alpha, \beta, \lambda, \mu, \ldots$	real numbers
$A, B, P, S, \ldots$	sets of points
$A \supset B$	the set A contains the set B
$B \subset A$	the set B is contained in the set A
$p \in P$	the element p is a member of the set P
$p \bar{\in} S$	p is not a member of the set S
0	the null class, zero element, zero
(p)	a set consisting of one element, p
$A \cap B$	intersection of A and B
Π ; $\mathfrak{P}(, ,)$	logical product
$\oplus$; $\mathfrak{S}(, ,)$	logical sum
$\{x\}$	a set of which x is a typical member
$\{x_m\}$, $\{P_m\}$	sequences
$. \supset .$	logical implication
$\Vert \ldots \Vert$	Banach norm
$\sim$	corresponds to ; is associated with

Numbers in square brackets refer to references listed in the bibliography.

Part 1

§ 1.1

Let L denote any ensemble of elements (points) of completely unspecified nature, forming under the undefined operations of 'addition' of elements, and 'multiplication' of elements by real numbers, a real linear space. The following postulates are to be fulfilled, with $f, g \in L$, α any real number implying αf, $f+g \in L$:

1.1.1 $f+g = g+f$

1.1.2. $(f+g)+h = f+(g+h)$

1.1.3. $1.f = f$

1.1.4. $\alpha(\beta f) = (\alpha\beta) f$

1.1.5. $(\alpha+\beta) f = \alpha f + \beta f$

1.1.6. $\alpha(f+g) = \alpha f + \alpha g$

1.1.7. $f+h = g+h$ implies $f = g$.

The rules of computation for o and $-f$ are readily deduced. In particular, $0.f$ is independent of f and hence may be named $o\ (\in L)$ uniquely. Cf. [2] pp 95–97.

Applying the same symbols of operation to sets $\subset L$, the following definitions are unambiguous ($f, g \in L$; $S, T \subset L$; α, β real) :

<u>Definition 1.1.1.</u> αS is the set of all αf, $f \in S$.

<u>Definition 1.1.2.</u> $S+T$ is the set of all $f+g$, $f \in S$, $g \in T$.

<u>Lemma 1.1.1.</u> $\alpha(S+T) = \alpha S + \alpha T$; $\alpha(\beta S) = (\alpha\beta) S$; $S+T = T+S$; $(S+T)+R \doteq S+(T+R)$; $\alpha S + \beta S \supset (\alpha+\beta) S$; $(S \pm T) \mp T \supset S$. The last two inclusions are, in general, proper.

<u>Proof.</u> Immediate by definition.

N.B. $S - S \neq o$, (o) .

§ 1.2

A topological ordering essentially of non-metric neighborhood type, is introduced in L through the following postulates wherein 'set' of the kind qualified below is the undefined notion. Because of the uniformity of the support space, only the situation at the origin need be considered. Let a set $\mathcal{U}$ of 'sets' $U \subset L$ be given such that :

1.2.1. $U \in \mathcal{U}$ implies there exists $V \in \mathcal{U}$, $V + V \subset U$.

1.2.2. $U, V \in \mathcal{U}$ imply there exists $W \in \mathcal{U}$, $W \subset U \cap V$.

1.2.3. $U \in \mathcal{U}$, $-1 \leq \alpha \leq +1$ imply there exists $V \in \mathcal{U}$, $\alpha V \subset U$.

1.2.4. There is a sequence $\{U_m\} \subset \mathcal{U}$ such that $\prod_m U_m = (o)$.

1.2.5. $f \in L$, $U \in \mathcal{U}$ imply there exists α, $f \in \alpha U$.

1.2.6. $\alpha \neq 0$, $V \in \mathcal{U}$ imply there exists $U \in \mathcal{U}$, $\alpha V = U$.

1.2.7. $U, V \in \mathcal{U}$ imply there exists $W \in \mathcal{U}$, $U + V = W$.

Remarks : By 1.2.4, $\mathcal{U}$ is non-vacuous. By 1.2.5, no $U (\in \mathcal{U}) = (o)$. By 1.2.3, $o.V \subset U$. Hence $U \in \mathcal{U}$ implies $o \in \mathcal{U}$.
The first five of these postulates are those given by v.Neumann, [1] p 4. His space convexity postulate $U + U \subset 2U$ is omitted. Various non-metric consistency and independence examples will be considered in the appendix.

We rephrase some fundamental set theoretic and topological definitions in terms of this topology.

Definition 1.2.1. The interior S_i of a set $S \subset L$ is the set of all $f \in S$ for which a $U \in \mathcal{U}$ exists with $f + U \subset S$.
Cf. [1] p 5.

<u>Definition 1.2.2.</u> Given $S \subset L$. A point p ($\bar{\in} S$ necessarily) is a limit point of S, if for every $U \in \mathcal{U}$, there exists a $q \in S$, $q \neq p$, such that $q \in p + U_i$.

The definitions of derived set, closure, and complement are quite as usual. Cf. e.g. [3].

<u>Definition 1.2.3.</u> The frontier set of $S \subset L$ is given by $F(S) = \overline{S} \cap \overline{C(S)}$, where $\overline{S}$ is the closure of S and $C(S)$ is the complement of S in L.

<u>Definition 1.2.4.</u> $S_i = D(\overline{S}, F(S))$ where D indicates the logical difference.

This is quite equivalent to Def.1.2.1.

§ 1.3

We indicate some topologies equivalent to $(\mathcal{U})$, in the sense of Hausdorff. Cf. [4] p 31.

Theorem 1.3.1. The following are equivalent topologies for L :

(α) $(\mathcal{U})$, original fundamental sets ;

(β) $(\mathcal{U}_i)$, the interiors of the U ;

(γ) $(\alpha\mathcal{U}_i)$, the interiors of the sets αU, $-1 \leq \alpha \leq +1$, $U \in \mathcal{U}$.

Proof. Ad $(\alpha) \equiv (\beta)$: $U_i \subset U$ ([1] p 5) ; there exists $V \subset U_i$ by Def.1.2.1 and $0 \in U_i$. Ad $(\beta) \equiv (\gamma)$: By Post.1.2.3, there exists V, $\alpha V \subset U$, $-1 \leq \alpha \leq +1$; $\alpha V_i \subset U_i$ ([1] p 6). Also $\alpha V_i \supset V_i$ by definition of αV .

$(\mathcal{U}_i)$ describes a regular Hausdorff topology ([1] p 6).

Remark : Topological properties remain invariant under a change to an equivalent neighborhood system.

Definition 1.3.1. $S \subset L$ is open if $S = S_i$; S is closed if $C(S)$ is open. $S_{cl} = C(\,C(S)_i\,)$.

Lemma 1.3.1. S_i is the greatest open set $\subset S$. S_{cl} is the smallest closed set $\supset S$. Cf. [1] p 6.

Theorem 1.3.2. Through Def.1.3.1, $(\mathcal{U})$ generates a topological space in the sense of Kuratowski, [9] p 15, using closed hulls as the primitive topological ordering.

Proof. Take $\overline{S} = S_{cl}$. Use Theorem 1.3.1 (β), v.Neumann's Theorem 6, [1] p 6, and the general equivalence theorem of Alexandroff and Hopf, [4] p 43.

§ 1.4

This section is concerned with boundedness, a notion permitted to L because of its linearity. The definitions given coincide with the usual ideas for a metric topology.

Definition 1.4.1.1. $S \subset L$ is bounded if for each $U \in \mathcal{U}$, there is determined a real $\alpha = \alpha(U, S)$, such that $S \subset \alpha U$.
(v.Neumann, [1] p 7)

Definition 1.4.1.2. $S \subset L$ is bounded if for every $x \in S$, there is determined a real $\lambda = \lambda(x, S)$ such that $\mu > \lambda$ implies $\mu x \,\bar{\in}\, S$.
(Michal and Paxson, [14])

Definition 1.4.1.3. $S \subset L$ is bounded if for every sequence $\{x_m\} \subset S$, and every real sequence $\{\alpha_m\}$ such that $\alpha_m \to 0$ with m^{-1}, $\alpha_m x_m \to 0$ with m^{-1}, under the assigned topology.
(Kolmogoroff, [5] p 30)

Definition 1.4.2. $\{x_m\} \subset L$ is convergent to x, i.e. $x_m \to x$, if for every $U \in \mathcal{U}$, $m \geq m_0(U)$ implies $x_m \in x + U$.

Theorem 1.4.1. If $x_m \to x$, then x is unique.
Proof. For suppose $x_m \to x^*$, $x^* \neq x$. Then U arbitrary $\in \mathcal{U}$, $m \geq m_1(U)$ implies $x_m - x \in U$, $m \geq m_2(U)$ implies $x_m - x^* \in U$. Put $m_0 = max(m_1, m_2)$; $x - x^* \in U - U \neq 0$ (Lemma 1.1.1). By Post.1.2.3, there exists U', $\pm U \subset V$, V now arbitrary. $x - x^* \in V + V \subset W$, W now arbitrary by Post.1.2.1. Put $W = U_m$ in Post.1.2.4, and $x - x^*$ can only be 0, for the contradiction.

<u>Theorem 1.4.2.</u> Definitions 1.4.1.1, 1.4.1.2, 1.4.1.3 are completely equivalent.

<u>Proof.</u> Ad (1.4.1.1).$\supset$.(1.4.1.2) : Let S , $o \in S$, be bounded (1.4.1.1). Suppose there exists $x_0 \in S$ such that $l.u.b.\ \lambda\ (\lambda x_0 \in S)$ does not exist finitely. Then $\lambda x_0 \in \alpha U$, unbounded λ , finite α , every $U \in \mathcal{U}$. Contradiction of Post.1.2.4.

Ad (1.4.1.2).$\supset$.(1.4.1.1) : Let S , $o \in S$ be bounded (1.4.1.2). Then for every $x_0 \in S$, there is determined $\delta(x_0, S)$ such that $\mu x_0\ (\mu > \delta)\ \bar{\in}\ S$. Take $\delta x_0 \in S$ (we may always take $\bar{S}$, for if $\bar{S}$ is bounded so also is S_i). Consider the set of such δ , $\Delta(S)$. By Post.1.2.5, $\delta x_0 \in \alpha U$. By Post.1.2.3, $\beta V \subset U$. Hence $\gamma V \subset \alpha U$, $-\alpha \leq \gamma \leq +\alpha$. That is, given any U , a centered convex set (cf. Def.1.6.2) as large as desired may be constructed for αU , by taking α as large as necessary. Consider the sheaf of line segments through the o-element. The class of δ's terminate these finitely in S by hypothesis. By isomorphic mapping, derive a set of α , denoted by A , so that $\alpha \sim \delta$, ? $A \sim \Delta$. Take $l.u.b.\ A = \alpha^* \in A$. Then $S \subset \alpha^* U$.

Ad (1.4.1.1).$\supset$.(1.4.1.3) : Let S , $o \in S \subset L$ be bounded (1.4.1.1). Let $\{\alpha_m\}$ be any real null sequence, and $\{x_m\}$ any sequence $\subset S$. Let U be arbitrary $\in \mathcal{U}$. Let $\lambda V \subset U$ for $-1 \leq \lambda \leq 1$ by Post.1.2.3. For all m , $x_m \in S \subset \alpha(S, V).V$, by hypothesis. This is a uniform condition, α not depending on m . Hence $\alpha_m x_m \in \alpha_m \alpha V$. Since α is finite and $\alpha_m \to o$, for some $m_0(\alpha)$, $m \geq m_0$ implies $|\alpha_m \alpha| \leq 1$. $m \geq m_0 :\supset: \alpha_m x_m \subset U$.

Ad (1.4.1.3).$\supset$.(1.4.1.1) : Let S, $o \in S \subset L$ be bounded (1.4.1.3). Then if x be arbitrary $\in S$, and $\{\alpha_n\}$ any real null sequence, $V \in \mathcal{U}$ such that $\lambda V \subset U$ (arbitrary) for $-1 \le \lambda \le 1$, we have $\alpha_n x \in V \subset \lambda V$. If βp be any line through the origin, there exists a finite $\overline{\beta}$ such that $\delta > \overline{\beta}$ implies $\delta p \,\overline{\in}\, S$. For choosing $p \in S$, $\overline{\in}$ some $W \in \mathcal{U}$, we would have, supposing the contrary, some sequence $\{\alpha_{n_i}^{-1} p\} \subset S$ such that the hypothesis would be contradicted, since $p \,\overline{\in}\, W$. $\{\alpha_{n_i}\}$ is some null subsequence of some null sequence $\{\alpha_n\}$ Hence (1.4.1.3).$\supset$.(1.4.1.2), completing the discussion.

D.H.Hyers [6] has shown independently that (1.4.1.3) and (1.4.1.1) are equivalent definitions.

<u>Lemma 1.4.1.</u> L is a Hausdorff group under 'addition'.
<u>Proof.</u> We use the equivalent topology $(\mathcal{U}_i)$ of Theorem 1.3.1, which is a regular Hausdorff topology after [1], Theorem 6. We must show [7] first that given $a + b + U_i$ there exist V_i and W_i such that $a + V_i + b + W_i \subset a + b + U_i$. This is clear after Post.1.2.1, with $V_i + V_i \subset U_i$. And second that given $a + U_i$ there exists $-a + V_i$ such that $-[-a + V_i] \subset a + U_i$. This follows from Post.1.2.3.

<u>Theorem 1.4.3.</u> If one $U \in \mathcal{U}$ is bounded, then L is homeomorphic with a metric space, i.e. it is metrisible.
<u>Proof.</u> Taking account of Lemma 1.4.1, after a theorem of Birkhoff [7], we have only to show that the first Hausdorff countability axiom, [2] p 227, is verified at the origin, L being linear. Let $U^* \in \mathcal{U}$ be the bounded set of the hypothesis. By Post.1.2.3, $\alpha V^* \subset U^*$, $-1 \le \alpha \le +1$. Then αV_i^* is clearly bounded also.

By Theorem 1.3.1, $(\alpha \mathcal{U}_i)$ is a topology equivalent to $(\mathcal{U})$. Hence we may verify the countability axiom for it. Let $\{\mu_m\}$ be a real sequence satisfying $\mu_m \to 0$ with m^{-1} and $\mu_m > \mu_{m+1} > 0$. Then the sequence of sets $\{\mu_m \alpha V_i^*\}$ is monotonic decreasing. Now we have only to show that given any open set S, $0 \in S \subset L$, for some m_0, $\mu_{m_0} \alpha V_i^* \subset S$. By Def.1.3.1, $U \in \mathcal{U}$ exists such that $U \subset S$. Let $W \subset V^* \cap U$ by Post.1.2.2. Let

$$\bar{\lambda} = g.l.b.\ \lambda \ (\lambda x \in W,\ x \in F(\alpha V^*))\ ;$$

$$\bar{\lambda}_m = g.l.b.\ \lambda_m (\lambda_m x \in \mu_m \alpha V_i^*,\ x \in F(\alpha V^*)).$$

By boundedness $\bar{\lambda}_m \to 0$ with m^{-1}. Hence for some m_0, $\bar{\lambda}_{m_0} < \bar{\lambda}$. Thus for some V, $\alpha V \subset W$, $-1 \leq \alpha \leq +1$;

$$\mu_{m_0} \alpha V_i^* \subset \alpha V \subset W \subset U \subset S,$$

completing the proof.

Remark : It will be shown in the appendix that Hilbert space weakly topologised, which is an instance of the postulates, possesses no bounded neighborhood.

Definition 1.4.3. A function to L is bounded if its set of values lies in a bounded set.

Theorem 1.4.4. If M_1, M_2 are bounded, and α is any real number and β is such that $-1 \leq \beta \leq 1$, then $\mathcal{P}(M_1, M_2)$; $\mathcal{S}(M_1, M_2)$; $\mathcal{D}(M_1, M_2)$; $M_1 \pm M_2$; αM_1; βM_1, are all bounded sets.

Proof. Clear. Contradicting Def.1.4.1.3 contradicts the boundedness of either M_1 or M_2.

§ 1.5

This section considers various aspects of continuity. Definitions may be phrased as desired in terms of open sets because of the v.Neumann result that $(\mathcal{U})$ describes a regular Hausdorff topology.

<u>Definition 1.5.1.</u> A function $f(x)$ on L to L will be called continuous at $x=x_0$ if any open set S, $0 \in S$, determines some open set T, $0 \in T$, such that $x \in x_0 + T$ implies $f(x) \in f(x_0) + S$.

Continuity of a function of several variables in the set is written in an obvious fashion.

<u>Definition 1.5.2.</u> A function $f(\alpha)$ on (α_0, α_1) to L will be called continuous at $\alpha = \alpha^*$ if any open set S, $0 \in S \subset L$, determines a real open interval $I(S)$, $0 \in I$, such that $\alpha \in \mathcal{P}((\alpha_0, \alpha_1), \alpha^* + I)$ implies $f(\alpha) \in f(\alpha^*) + S$.

<u>Definition 1.5.3.</u> A function $f(\alpha)$ on (α_0, α_1) to L will be called uniformly continuous on (α_0, α_1) if, given any open set S, $0 \in S \subset L$, there is determined an open real interval $I(S)$, $0 \in I$, the same for all $\alpha \in (\alpha_0, \alpha_1)$, such that $\mu \in I$ and $\alpha + \mu \in (\alpha_0, \alpha_1)$ imply $f(\alpha + \mu) \in f(\alpha) + S$.

<u>Definition 1.5.4.</u> If $f(\alpha)$ is continuous at every point of (α_0, α_1) it is said to be continuous on (α_0, α_1).

<u>Theorem 1.5.1.</u> If $f(\alpha)$ on the closed real interval $[\alpha_0, \alpha_1]$ to L is continuous on $[\alpha_0, \alpha_1]$, then $f(\alpha)$ is uniformly continuous on $[\alpha_0, \alpha_1]$.

<u>Proof.</u> A set of open intervals covering $[\alpha_0, \alpha_1]$ is the set of I's, $I(S, \alpha^*)$, of the continuity hypothesis.

Thus the Heine-Borel theorem may be applied giving a finite set G of closed intervals $J(\alpha_m^*)$ such that

$$J(\alpha^*) \subset \alpha^* + I(S, \alpha^*) .$$

That is, $\alpha \in J(\alpha_m^*)$ implies $f(\alpha) \in f(\alpha_m^*) + S$

Let $g.l.b.\ G = J_\delta$. Let $\lambda, \bar{\lambda} \in [\alpha_0, \alpha_1]$ be any two numbers satisfying $\lambda \in \bar{\lambda} + J_\delta$. Then $\lambda, \bar{\lambda}$ lie in (a) the same or (b) adjacent intervals. Suppose (b) : let λ_0 be the common end point. Then there are determined β_1, β_2, one in each interval such that

$$f(\lambda) \in f(\beta_1) + S$$

$$f(\bar{\lambda}) \in f(\beta_2) + S \qquad -f(\bar{\lambda}) \in -f(\beta_2) - S$$

$$f(\lambda_0) \in f(\beta_1) + S \qquad -f(\lambda_0) \in -f(\beta_1) - S$$

$$f(\lambda_0) \in f(\beta_2) + S$$

Hence

$$f(\lambda) - f(\bar{\lambda}) - f(\lambda_0) + f(\lambda_0) \in f(\beta_1) - f(\beta_2) - f(\beta_1) + f(\beta_2) + T,$$

where $T = S - S - S + S \neq 0$. Now T is open and arbitrary. Thus $f(\lambda) \in f(\bar{\lambda}) + T$ whenever $\lambda \in \bar{\lambda} + J_\delta$, and J_δ does not depend on $\bar{\lambda}$. The reasonong proceeds inquite the same fashion for $\lambda, \bar{\lambda}$ in the same interval.

<u>Theorem 1.5.2.</u> αx and $x + y$ $(x, y \in L)$ are continuous functions of α, x and x, y respectively.

<u>Proof.</u> [1], Theorem 7.

Theorem 1.5.3. Let $f(\alpha)$ be continuous on the closed interval $[\alpha_0, \alpha_1]$ to L. Then $f(\alpha)$ is bounded.

Proof. Suppose the contrary. Then there exists a sequence $\{f(\alpha_m)\}, \{\alpha_m\} \subset [\alpha_0, \alpha_1]$, such that if $\{\beta_m\}$ is any real null monotonic sequence, $\beta_m f(\alpha_m) \not\to 0$ with β_m. Since $[\alpha_0, \alpha_1]$ is compact, α_m has a limit point $\alpha \in [\alpha_0, \alpha_1]$. Hence by continuity $f(\alpha_m) \to f(\alpha)$ as $m \to \infty$. But $\beta_m f(\alpha) \to 0$ with m^{-1} by Theorem 1.5.2. Hence $\beta_m f(\alpha_m) \to 0$, using Theorem 1.4.1, giving the contradiction.

Remark : It is to be noted that open sets are, in general, unbounded. For they contain U's which are not necessarily bounded. Vide the remark following Theorem 1.4.3.

Theorem 1.5.4. Let $f(\mu, y)$ be on $[\alpha_0, \alpha_1] \times L_0$ to L continuously in the pair μ, y. Let $y = y(\mu)$ be continuously on $[\alpha_0, \alpha_1]$ to L_0, open. Then $z(\mu) = f(\mu, y(\mu))$ is continuous in μ and hence bounded.

Proof. By hypothesis any open E, $0 \in E \subset L$ determines $\delta > 0$, and an open S, $0 \in S \subset L$, such that $|\mu - \mu_0| < \delta$, $\mu + \delta \in [\alpha_0, \alpha_1]$, $y \in y_0 + S$ imply $f(\mu, y) \in f(\mu_0, y_0) + E$.
Also S determines $\rho > 0$ such that $|\mu - \mu_0| < \rho$, $\mu + \rho \in [\alpha_0, \alpha_1]$ imply $y \in y_0 + S$. Let $\lambda = \min(\rho, \delta)$. Then $|\mu - \mu_0| < \lambda$, $\mu + \lambda \in [\alpha_0, \alpha_1]$ imply $z(\mu) \in z_0 + E$. Boundedness by Theorem 1.5.3.

Theorem 1.5.5. Let $f(\mu, y(\mu))$ be continuous in the pair μ, y, $\mu \in [\alpha_0, \alpha_1]$. Let $g(f)$ be continuous in f. Then g is continuous in y.

Proof. Immediate from the general Sierpiński theorem [3], when the first place of $f(\mu, \gamma)$ keeps any value in $[\alpha_0, \alpha_1]$.

These last two theorems are taken in a convenient form for the sequel.

Theorem 1.5.6. If $f(x)$, $g(x)$ on L to L are continuous at $x = x_0$, and $A(\alpha)$ is continuous at $\alpha = \alpha_0$ on the real numbers to the real numbers, then $f(x) \pm g(x)$ are continuous at $x = x_0$, and $A(\alpha) f(x)$ is continuous at $x = x_0$, $\alpha = \alpha_0$.

Proof. Immediate from continuity of a continuous iteration and Theorem 1.5.2.

For purposes of reference we restate the definition of topological completeness given by v.Neumann in [1].

Definition 1.5.5. A sequence $\{x_m\} \subset L$ is fundamental, or is a Cauchy sequence, if for every $U \in \mathcal{U}$ there exists an $n_1 = n_1(U)$ such that $m, n \geq n_1$ imply $x_m - x_n \in U$.

L is a complete space if every fundamental sequence is convergent.

§ 1.6

Various aspects of convexity are now treated which have some intrinsic interest and subsequent importance.

Definition 1.6.1. $S \subset L$ is convex if $x_0, x_1 \in S$, $0 \leq \alpha \leq 1$ imply $\alpha x_0 + (1-\alpha) x_1 \in S$.

Definition 1.6.2. S, $0 \in S \subset L$ is centered convex if $x \in S$, $0 \leq \alpha \leq 1$ imply $\alpha x \in S$.

Definition 1.6.2 is of course the weaker of the two and has as a useful example the sets αY of Post.1.2.3.

Definition 1.6.3. The convex hull of a set $S \subset L$ is the set of all elements of the form $\sum_{i=1}^{n} \alpha_i x_i$, $x_i \in S$, $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$.

Theorem 1.6.1. The convex hull of a set is convex in the sense Of Def.1.6.1.

Proof. Let $\sum_{i=1}^{n} \alpha_i x_i$, $\sum_{i=1}^{s} \beta_i y_i \in E_{c.h.}$.

Consider the form $\mu \sum_{i=1}^{n} \alpha_i x_i + (1-\mu) \sum_{i=1}^{s} \beta_i y_i$ where $0 \leq \mu \leq 1$, $0 \leq \mu\alpha_i, (1-\mu)\beta_i \leq 1$ and $\sum_{i=1}^{n} \mu\alpha_i + \sum_{i=1}^{s} (1-\mu)\beta_i = 1$.

Theorem 1.6.2. A necessary and sufficient condition that a set $S \subset L$ be convex is that, α, $\bar{\alpha} \geq 0$ being any real numbers,

$$(\alpha + \bar{\alpha}) S = \alpha S + \bar{\alpha} S.$$

Cf. [8], where this theorem is stated but not proved.

Proof. Ad necessity : Let S be convex, and let $\sum_{i=1}^{s} \alpha_i x_i \in S$.

Then

$$(\alpha + \bar{\alpha}) \sum_{i=1}^{s} \alpha_i x_i = \sum_{i=1}^{s} \alpha \alpha_i x_i + \sum_{i=1}^{s} \bar{\alpha} \alpha_i x_i,$$

which is in $\alpha S + \bar{\alpha} S$. Conversely let

$$\sum_{i=1}^{s} \alpha_i x_i, \quad \sum_{i=1}^{n} \beta_i y_i \in S.$$

Form $\sum_{i=1}^{s} \alpha\alpha_i x_i + \sum_{k=s+1}^{t=n+s} \bar{\alpha}\alpha_k x_k$. Then $\sum_{i=1}^{s} \alpha\alpha_i + \sum_{i=1}^{n} \bar{\alpha}\beta_i = \alpha+\bar{\alpha}$.
We may therefore write for this $\sum_{\ell=1}^{t} \mu_\ell x_\ell$ or $(\alpha+\bar{\alpha})\sum_{\ell=1}^{t} \gamma_\ell x_\ell$,
where $\sum_{\ell=1}^{t} \gamma_\ell = 1$ and for $1 \leq \ell \leq s$, $1 \geq \gamma_\ell = \alpha\alpha_\ell[\alpha+\bar{\alpha}]^{-1} \geq 0$,
and for $s \leq k \leq t$, $1 \geq \gamma_k = \bar{\alpha}\alpha_k[\alpha+\bar{\alpha}]^{-1} \geq 0$.
Hence the set inclusion goes both ways and the condition subsists.
Ad sufficiency : Suppose for all $\alpha, \bar{\alpha} \geq 0$, $(\alpha+\bar{\alpha})S = \alpha S + \bar{\alpha} S$.

Thus $$S = \frac{\alpha}{\alpha+\bar{\alpha}} S + \frac{\bar{\alpha}}{\alpha+\bar{\alpha}} S.$$

Hence if $x_1, x_2 \in S$ so also is $\mu x_1 + (1-\mu)x_2$, $0 \leq \mu \leq 1$.
The construction for the general linear form is now obvious.

In the discussion of a non-effective existence theorem in Part 5, sets of the following type will be useful.

Definition 1.6.4. A set $M \subset L$ will be called suitably convex if it is convex and has $0 \in L$ in its interior.

Remark : The neighborhoods in the previously mentioned weakly topologised Hilbert space are of this type and are unbounded.

Theorem 1.6.3. Let M, M_1, M_2 be suitably convex, and let α be any real number. Then

(1) there exists $U \in \mathcal{U}$, $U \subset M_i$;

(2) $M \neq (0)$;

(3) M may not be a line segment through $0 \in L$;

(4) $M_1 \cap M_2$ is suitably convex;

(5) $M_1 + M_2$ is suitably convex;

(6) αM is suitably convex.

Proof. Ad (1) : M_i is open $0 \in M_i$, Def.1.3.1.

Ad (2) : No $U(\in \mathcal{U}) = (0)$.

Ad (3) : No $U(\in \mathcal{U})$ is a line segment by Post.1.2.5.

Ad (4) : That $0 \in M_1 \cap M_2$ is obvious. Let x , y be any two points of $R = (M_1 \cap M_2)_i$. Then all points $\lambda x + (1-\lambda) y$, $0 \leq \lambda \leq 1$, lie in R because they must lie in both M_1 and M_2.

Ad (5) : $0 \in (M_1 + M_2)_i$ by definition of sum. Convexity : If (a) $x, y \in M_1$ or $x, y \in M_2$, so also $\lambda x + (1-\lambda) y \in M_1, M_2$. Hence $\lambda x + (1-\lambda) y \in M_1 + M_2$ since $0 \in M_1, M_2$.
If (b) $x \in M_1 (\bar{\in} M_2)$, $y \in M_2 (\bar{\in} M_1)$; $\lambda x \in M_1$, $(1-\lambda) y \in M_2$, since λ and $1-\lambda$ are ≤ 1 , $0 \in M_1, M_2$. Thus the sum is in $M_1 + M_2$. If (c) $x, y \bar{\in} M_1, M_2$ but $\in M_1 + M_2$, then $x = x_1 + x_2$
$y = y_1 + y_2$; $x_1, y_1 \in M_1$, $x_2, y_2 \in M_2$; $\lambda x_1 + (1-\lambda) y_1 \in M_1$, $\lambda x_2 + (1-\lambda) y_2 \in M_2$.
Hence $\lambda x + (1-\lambda) y \in M_1 + M_2$, $0 \leq \lambda \leq 1$.

Ad (6) : $0 \in M_i$ implies $0 \in \alpha M_i$.
Let x, y , $\lambda x + (1-\lambda) y \in M$, $0 \leq \lambda \leq 1$.
Take $\bar{x}, \bar{y} \in \alpha M$. Then $\bar{x}/\alpha$, $\bar{y}/\alpha \in M$.
$\lambda \bar{x}/\alpha + (1-\lambda) \bar{y}/\alpha \in M$, $\lambda \bar{x} + (1-\lambda) \bar{y} \in \alpha M$, $0 \leq \lambda \leq 1$.

Remark : In general neither the logical sum nor the logical difference of two suitably convex sets will be suitably convex.

§ 1.7

Because of the fashion in which the definition of a differential will be phrased in Part 2, an intimate knowledge of the properties of frontier points is desirable. Theorems on representability in terms of frontier points will also be useful in the theory of the integral.

Lemma 1.7.1. In L, the line $\lambda \bar{x}$, $\bar{x} \in L$ and fixed, λ a real variable, is dense-in-itself.

Proof. By Theorem 1.5.2, $\lambda \bar{x}$ is a continuous function of λ. Thus given any open set S, $o \in S \subset L$, there is determined a real interval I, $0 \in I$, such that for any $\bar{\lambda}$, $\lambda \in \bar{\lambda} + I(S)$ implies $\lambda \bar{x} \in \bar{\lambda} \bar{x} + S$. But I determines a positive integer $m(I)$ such that $n \geq m$ implies $n^{-1} \bar{\lambda} \in I$. Replace λ by $[1 + n^{-1}] \bar{\lambda}$ on the line $\lambda \bar{x}$. $\bar{\lambda} \bar{x}$ is a limit point of a sequence on the line.

Lemma 1.7.2. Let $E \subset L$ be a bounded set. Then for every $x \in L$ there is determined $\lambda(x, E)$ such that $\lambda x \in F(E)$, the frontier set of E.

Proof. Apply Def.1.4.1.2 and Lemma 1.7.1.

Lemma 1.7.3. If E is bounded, $o \in E \subset L$, then every $z \in E$ is of the form $z = \lambda x$, $0 \leq \lambda \leq 1$, $x \in F(E)$.

Proof. Every $z \in E$ lies on some line λy, $y \in L$. Let $\lambda^* = l.u.b.\ \lambda\ (\mu > \lambda$ implying $\mu y \,\bar{\in}\, E)$. Then $x\,(\in F(E)) = \lambda^* y$. The result follows by continuity.

Lemma 1.7.4. $U \in \mathfrak{U}$, $\alpha \neq 0$, $p \in F(U)$ imply there exists $V \in \mathfrak{U}$, $\alpha U_i^{p} = V_i^{\alpha p}$. (*)

(*) If p is in the frontier set of a set U we write U^{p}.

Proof. By Post.1.2.6, $\alpha U = V$. Since $\alpha U_i = (\alpha U)_i$, $\alpha U_i = V_i$. For any $U^* \in \mathcal{U}$ there exists $r \in U_i$ such that $r \in p + U_i^*$. Hence $\alpha r \in \alpha p + \alpha U_i^* = \alpha p + W_i$, $W \in \mathcal{U}$. Now $r \in U_i$ implies $\alpha r \in V_i$. Thus αp is a limit point of V_i. Symmetrically, since $p \in F(U)$, it is clear that αp is also a limit point of $C(V_i)$, the complement of V_i in L.

Lemma 1.7.5. $U \in \mathcal{U}$, $p \in L$, $q \in F(U)$ imply
$$p + U_i^q = (p + U)_i^{q+p} = (p + U^q)_i .$$

Proof. Immediate.

§ 1.8

Some purely topological aspects of L are now considered, principally compactness and connectedness.

Definition 1.8.1. A topological space is called compact, after Fréchet, if every infinite set in it possesses at least one limit point under the assigned topology.

Remark : It is sufficient that every denumerable sequence have a limit point.

Theorem 1.8.1. A necessary and sufficient condition that be compact is that every monotone decreasing sequence of non-empty closed sets

$$F_1 \supset F_2 \supset \cdots \supset F_k \supset \cdots$$

have a non-empty intersection.

Proof. The necessity is demonstrated in [4] p 85. The sufficiency is mentioned but not shown in [4] p 86, and [10] p 259. We demonstrate the latter for our situation.

$(\mathcal{U}_i)$ describes a regular Hausdorff topology as remarked previously. Hence $p\,(\in L) \neq 0$ implies there exist $U, V \in \mathcal{U}$ such that $\underline{P}(U, p+V) = 0$.(Def.1.2.1). Also $p \in U_i . \supset : \exists\, V \in \mathcal{U} . p+V \subset U.$ Let E be any infinite set $\subset L$. Let $U \in \mathcal{U}$. By Post.1.2.3, U is infinite. Let $a_1 \in E$. Put $S_1 = E \cap (a_1 + U)$, $\overline{S_1} = F_1$, (the closure). Let $W_1, W_2 \in \mathcal{U}$ be such that $(a_2 \in S_1,\ a_2 \neq a_1)$, $(a_1 + W_1) \cap (a_2 + W_2) = 0$. Take $W_3 \in \mathcal{U}$ such that $a_2 + W_3 \subset a_1 + U$. By Post.1.2.2, there exists $W_4 \subset W_3 \cap W_2$.

Put $S_2 = (a_2 + W_4) \cap E$, $\overline{S}_2 = F_2$. Then F_1 and F_2 are closed and non-empty with $F_1 \supset F_2$ quite properly. Repeat this construction inductively giving a sequence

$$F_1 \supset F_2 \supset \dots \supset F_k \supset \dots$$

such as the hypothesis describes.

Let $b_k \in F_k \ominus F_{k+1}$, $k = 1, 2, \dots$. Since the F_k are distinct so also are the b_k . Let $\{\{ b_k \}\}$ denote the class of all such sequences. We are to show that one at least of these sequences has a limit point.

To that end, consider the class of frontier points of the non-empty ΠF_k , denoted by $F(\Pi F_k)$. We assert that for at least one $p \in F(\Pi F_k)$ and any open set S , $p \in S \subset L$, there is determined $n_0 = n_0(S, p)$ such that $S \supset$ some points in F_{n_0} but not in F_{n_0+1} . For there exist points $\subset S$, $\overline{\subset}\ F(\Pi F_k)$. Since the sequential inclusion of $\{ F_k \}$ is proper the above assertion follows.

Therefore p is a limit point of at least one of the sequences $\{\{ b_k \}\}$. But every such sequence is in E , completing the proof.

<u>Definition 1.8.2.</u> Cf. [10] p 264.

(1) If a point of a space is characterized by a property E, it is called an E-point of the space.

(2) A space R is said to be closed in relation to all of its E-points (E-closed) in case it is impossible to add to R a point ξ such that in the extended space $R \oplus \xi$:

(a) ξ is an E-point, and (b) ξ is not isolated.

(3) $x \in R$ is a $\varkappa$-point if there exists in R a denumerable sequence converging to x .

Remark : For example the complex plane, although locally compact, is not compact in the large (it has sequences such as $1, 2, 3, \ldots$). Addition of an ideal point, the point at infinity, with a stereographic remetrisation that changes distance but not convergence makes the complete plane compact. This may also be done for a locally compact Banach space. Cf. [11] p 221.

Alexandroff and Urysohn indicate but do not prove a theorem of the following type, [10] p 265.

Theorem 1.8.2. Let $\mathcal{R}$ be a regular topological space. Vide [4] p 68. A necessary and sufficient condition that $\mathcal{R}$ be compact is that it be $\mathcal{H}$-closed.

Proof. In a Hausdorff neighborhood space the neighborhoods are open sets,

Ad necessity : Let $\mathcal{R}$ be compact. Let $U_0(p)$ be any neighborhood of a point p . Then by regularity we have inductively

$$U_1(p) \subset U_0(p) \;, \ldots, \; U_k(p) \subset U_{k-1}(p) \,,$$

quite properly. Thus $\overline{U}_1 \supset \overline{U}_2 \supset \cdots \supset \overline{U}_k \supset \cdots$ properly, where the bar indicates set closure.

Consider the set of sequences $\{\{p_n\}\} \equiv \{U_n \ominus U_{n+1}\}$. Every $\{p_n\}$ has a limit point by compactness. p is the limit point of at least one such sequence. For let $V(p)$ be any neighborhhod of p . If $V(p) \overline{\subset} \prod_n \overline{U}_n$ the statement is obvious for then $V(p)$ overlaps the sets U_n for some determined n_0 . Let $V(p) \subset \prod_n \overline{U}_n$. Then the above process is repeated using $V(p)$ in place of $U_0(p)$ and a fresh denumerable sequence is added to $\{p_n\}$.

Hence the arbitrary $V(p)$ is penetrated and $p_n \to p$. Every $p \in R$ is a $\mathcal{K}$-point.

Ad $\mathcal{K}$-closure : Suppose a non-isolated point q could be added to R such that in $R \oplus q$, q is a $\mathcal{K}$-point. Then there is a distinct termed sequence $\{q_n\}$, $q_n \to q$. Deleting q , $\{q_n\}$ is an infinite set of R with no limit point, denying compactness.

Ad sufficiency : Suppose R is not compact. Then there exists in R at least one sequence $\{p_n\}$, $p_n \neq p_m$, with no limit point in R . Then we can add to R a non-isolated point q such that $p_n \to q$. To this end, define for the extended space $R \oplus q$,

$$U_n(q) = q \oplus \sum_{i=n+1}^{\infty} U(p_i)$$

where $\sum$ is a set theoretic sum and the $U(p_i)$ are chosen by the separation axiom such that $U(p_i) \wedge U(p_j) = 0$ for $i \neq j$.

Hence R is not $\mathcal{K}$-closed, completing the proof by contradiction.

Remark : Every point of L is a $\mathcal{K}$-point by Lemma 1.7.1, but L is not $\mathcal{K}$-closed since it contains sequences such as $\{np\}$ p fixed, n taking integral values. But if L possesses one compact neighborhood it may be prolonged into a compact space after the fashion indicated by Fréchet in [12] p 351.

Theorem 1.8.3. Every compact set in L is bounded.

Proof. Vide [6]. This is of course expected since compactness generalizes the Bolzano-Weierstrass property.

Corollary 1.8.3. Every bicompact set, Def.1.8.4, in L is bounded.

Proof. A bicompact set is compact.

Remark : No unbounded set in L is compact.

Definition 1.8.3. A system of sets of a topological space constitutes a covering of a set A of the space if each $p \in A$ is also in at least one of the sets of the system.

Cf. [4] p 47.

Definition 1.8.4. A set A in a topological space is said to be bicompact if every open covering of A contains a finite subset that also covers A .

Theorem 1.8.4. If a set $S \subset L$ is compact, then every Cauchy sequence that penetrates S infinitely is convergent; i.e. S as a relative space is complete.

Proof. Such a sequence is an infinite subset of S and hence must have a limit point.

Definition 1.8.5. A set $S \subset L$ is said to be connected if there do not exist two non-vacuous, disjoint, closed sets such that

$$S \subset M_1 \oplus M_2 \ .$$

Remark : Connectedness is a topological property by [4], Corollary p 53.

Theorem 1.8.5. If $S, T \subset L$ are connected, and α is any real number, then αS , $S + T$ are connected, and $R \subset L$, $-1 \leq \beta \leq +1$ imply that βR is connected.

<u>Proof.</u> αx on S to αS is a homeomorphism through Theorem 1.5.2, and connectedness is a topological property, giving the first result.

Let $p \in T$. Then as above, $p + S$ is connected. $T + S$ is obtained by a continuous displacement of this set.

The last result is obvious. $p, q \in R$ imply $\beta p, \beta q \in \beta R$, $-1 \leq \beta \leq +1$, and the segments βp, βq are self connected and joined to one another at the origin.

<u>Remarks</u> : R need not be connected. S need not intersect T. An example in the appendix will show that all of the fundamental sets of the assigned topology $(\mathcal{U})$ need not be connected.

Part 2.

§ 2.1

This part is concerned with definitions and elementary properties of topological differentials, of extended Fréchet [16] and Gateaux [17] types, and of derivatives, devised and developed in collaboration with Professor A.D.Michal. Cf. [13] and [14]. For further properties and discussion see these latter papers.

We are principally interested here in developing sufficient machinery for the sequel.

§ 2.2

<u>Definition 2.2.1.</u> Let $f(x)$ be a function defined on L to L, although not necessarily on or to the whole of L. If there exists a function $f(x;z)$ on $L \times L$ to L, which satisfies :
(1) $f(x;z)$ is linear in z, i.e. additive and continuous,
(2) given $\rho > 0$, there is determined a $V(\rho) \in \mathcal{U}$ such that for some $U \in \mathcal{U}$, which has ρz as one of its frontier points,

$$f(x_0+z) - f(x_0;z) \in f(x_0) + U_i^{\rho z},$$

whenever $z\ (\neq 0) \in V_i$,
then we shall say that $f(x)$ has a differential $f(x_0;z)$ at $x = x_0$.

<u>Remarks</u> : We notice that if L is specialised to a Banach space, i.e. subjected to a norm topology, the sets U being the spheres $\|x\| \leq \delta$, U_i the open spheres $\|x\| < \delta$, the usual Fréchet differential for a function on $\mathcal{B}$ to $\mathcal{B}$ arises.

The translation of Def.2.2.1(2) runs : given $\rho > 0$, there is determined $\epsilon(\rho) > 0$ such that

$$\| f(x_0 + z) - f(x_0) - f(x_0; z) \| < \| \rho z \| = \rho \| z \| ,$$

for all $\| z \| < \epsilon$, $z \neq 0$.

The general non-metric character of these spaces of L -type apparently inhibits, under this definition at least, the consideration of differentiability for transformations between two distinct spaces, as is common in the Banach situation. There the liaison induced by the two norms mapping on the same set, the real numbers, permits the notion. Here, however, points selected from one space could not, in general, be frontier points for the U lying in the value space.

<u>Theorem 2.2.1.</u> If $f(x)$ on L to L has a differential $f(x_0; z)$ at $x = x_0$, then $f(x)$ is continuous at $x = x_0$.
<u>Proof.</u> By definition $f(x_0 + z) \in f(x_0) + f(x_0; z) + U_i^{\rho z}$ for $z \in V_i(\rho)$. By Post.1.2.5, there exists a set of α such that $f(x_0; z) \in \alpha W$, where $W \in \mathcal{U}$ arbitrarily. By continuity of the differential at $z = 0$, any open set W_i determines an open set S , $0 \in S$, such that $f(x_0; z) \in \alpha W$ for $z \in V_i(\rho)$ and $f(x_0; z) \in W_i$ for $z \in S(W)$.
Consequently $f(x_0; z) \in W_i$ for $z \in \mathcal{P}(V_i, S) = I(\rho, W)$, I open, $0 \in I$. Put $W_i + U_i^{\rho z} = R(z, \rho, W)$, R open. But z depends on ρ and W , which are arbitrary. Hence let R be arbitrary, determining ρ and W and hence I . Thus $z \in I(R)$ implies $f(x_0 + z) \in f(x_0) + R$, I and R open.

<u>Theorem 2.2.2.</u> Let $f(x)$ have a differential $f(x_0; z)$ at $x = x_0$. Let α be any real number. Then $\alpha f(x)$ has a differential at $x = x_0$, given by $\alpha f(x_0; z)$.

<u>Proof.</u> We must show that for $\bar{\rho} > 0$ arbitrary

$$\alpha f(x_0 + z) - \alpha f(x_0; z) \in \alpha f(x_0) + \overline{U}_i^{\,\bar{\rho} z}$$

for $z \in \overline{V}_i(\bar{\rho}),\ z \neq 0$. In Def.2.2.1(2) take $\rho = \bar{\rho}/\alpha$. Since ρ and $\bar{\rho}$ are arbitrary and α is fixed $V_i(\bar{\rho}/\alpha) = \overline{V}_i(\bar{\rho})$. For $z \in \overline{V}_i(\bar{\rho})$, then, we also have

$$\alpha f(x_0 + z) - \alpha f(x_0; z) \in \alpha f(x_0) + \alpha U_i^{\,\bar{\rho} z/\alpha}$$

Apply now Lemma 1.7.4, giving $\alpha U_i^{\,\bar{\rho} z/\alpha} = \overline{U}_i^{\,\bar{\rho} z}$.

$\alpha f(x_0; z)$ is continuous in z by Theorem 1.5.6, and is obviously additive.

It is also possible to define in L an extended form of the Gateaux differential [17] usually defined for normed vector spaces.

<u>Definition 2.2.2.</u> Let $f(x)$ be on L to L. If there exists a function $f(x, y)$ on $L \times L$ to L, such that given any $U \in \mathcal{U}$ there is determined $\delta(U) > 0$ so that for all $|\lambda| < \delta$,

$$f(x_0 + \lambda y) - \lambda f(x_0, y) \in f(x_0) + \lambda U_i\ ,$$

then $f(x_0, y)$ will be called the extended Gateaux differential of $f(x)$ at $x = x_0$.

<u>Theorem 2.2.3.</u> If $f(x)$ has an extended Fréchet differential at $x = x_0$, then it has an extended Gateaux differential at $x = x_0$, and the two are equal.

<u>Proof.</u> In Def.2.2.1(2) put $z = \lambda y$. Then

$$f(x_0 + \lambda y) - f(x_0; \lambda y) \in f(x_0) + U_i^{\rho \lambda y}$$

for $\lambda y \in V_i(\rho)$, wherein y is held fixed. By continuity of λy, $|\lambda| < \delta(V_i(\rho))$ implies $\lambda y \in V_i(\rho)$ implies

$$f(x_0 + \lambda y) - \lambda f(x_0; y) \in f(x_0) + \lambda \overline{U}_i^{\rho y},$$

using the linearity of $f(x_0; z)$ in z and Lemma 1.7.4. Since y is fixed $\overline{U}^{\rho y} = \overline{U}(\rho)$. Conversely, selecting $\overline{U}$ determines ρ and hence V so that $V = V(\overline{U})$. Finally $|\lambda| < \delta = \delta(V) = \delta(\overline{U})$ implies

$$f(x_0 + \lambda y) - \lambda f(x_0, y) \in f(x_0) + \lambda \overline{U}_i$$

where $f(x_0, y) = f(x_0; y)$.

<u>Remark</u> : It is possible to phrase the extended Gateaux differential as a limit. First write

$$\lambda^{-1}[f(x_0 + \lambda y) - f(x_0)] - f(x_0, y) \in U_i$$

for $|\lambda| < \delta(U)$. Now form the sequence

$$\{f_m\} \equiv \{\overline{\lambda}.2^{1-m}[f(x_0 + \overline{\lambda}.2^{1-m} y) - f(x_0)]\}$$

where $\overline{\lambda}$ is any fixed number. By taking m sufficiently large $\lambda = \overline{\lambda}.2^{1-m}$ can be made as small as desired. Hence whatever δ any selected U determines, an m_1 is determined so that for $m \geq m_1(U)$, $|\lambda| < \delta(U)$. Thus $f(x_0, y)$ is the limit of $\{f_m\}$ as $m \to \infty$.

Theorem 2.2.4. If the extended Fréchet differential exists at $x = x_0$ for a function $f(x)$, then that differential is unique.

Proof. Apply Theorem 2.2.3. Then the uniqueness of $f(x_0, y)$ follows from the preceding remark and Theorem 1.4.1.

Let $f(\alpha)$ be a function defined on a real number interval to some set $\subset L$. Then one may define a derivative for $f(\alpha)$ as follows :

Definition 2.2.3. If there exists a function $f'(\alpha)$ on R to L, such that for any $U \in \mathcal{U}$ there is determined $\delta(U) > 0$ with $|\lambda| < \delta$ implying

$$\lambda^{-1} [f(\alpha_0 + \lambda) - f(\alpha_0)] \in f'(\alpha_0) + U_i ,$$

then $f'(\alpha_0)$ will be called the derivative of $f(\alpha)$ at $\alpha = \alpha_0$.

Remark : Properties of the usual type hold for this derivative.

Theorem 2.2.5. If $f'(\alpha_0)$ exists, then it is unique.

Proof. For Def.2.2.3 implies a sequential approach to $f'(\alpha_0)$ as in the remark following Theorem 2.2.3. Uniqueness by Theorem 1.4.1.

Theorem 2.2.6. If $f(x)$ on L to L has a differential in the sense of Def.2.2.1, at all points of a convex set $L_c \subset L$ (Def.1.6.1), then if $x_0, x_1 \in L_c$

$$\varphi(t) \equiv f(x_0 + t(x_1 - x_0))$$

has a derivative $\varphi'(t)$ for all $t \in [0, 1]$.

<u>Proof</u>. Since $f(x; \delta x)$ exists for all $x \in L_c$, $f(x, \delta x)$ exists for all $x \in L_c$. Hence phrasing $f(x, \delta x)$ as a limit we have, putting $\bar{\lambda} . 2^{1-m} = \lambda_m$,

$$\lim_{\lambda_m \to 0} \lambda_m^{-1} [f(x + \lambda_m \delta x) - f(x)] = f(x, \delta x).$$

Since L_c is convex, $\delta x = \delta t (x_1 - x_0)$. Thus

$$\lim_{\lambda_m \to 0} \lambda_m^{-1} [f(x_0 + t(x_1 - x_0) + \lambda_m \delta t (x_1 - x_0)) - f(x_0 + t(x_1 - x_0))]$$
$$= f(x_0 + t(x_1 - x_0), \delta t (x_1 - x_0))$$

$$\lim_{\mu \to 0} \mu^{-1} [\varphi(t + \mu) - \varphi(t)] = \varphi^*, \quad 0 \leq t \leq 1.$$

Note that x_0, x_1 are only abstract parameters.

Thus $\varphi^* = \varphi'(t)$ exists uniquely for all $t \in [0, 1]$.

<u>Remark</u> : A weaker theorem, $f(x, \delta x)$ implying the unique existence of $\varphi'(t)$, is tacit in the above proof.

For further properties and theorems see Part 4.

Part 3

§ 3.1

We develop here the technical facts necessary for a discussion of existence theorems for equations involving a derivative of the type of Definition 2.2.3, and for a consideration of further differential properties.

The concern is with an integral of Riemann type for functions on $\mathcal{R}$, the real number system, to the space $\mathcal{L}$ with topology described by $(\mathcal{U})$. It is frequently possible to abstract to some extent the classical Banach space technique of Kerner [18] and Graves [19].

§ 3.2

<u>Definition 3.2.1.</u> Let $f(\alpha)$ be topologically bounded (Def.1.4.1) on the real interval $[\alpha_0, \alpha_1]$ to L. Consider any subdivision π of $[\alpha_0, \alpha_1]$,

$$\alpha_0 = \bar{\alpha}_1 \leq \bar{\alpha}_2 \leq \ldots \leq \bar{\alpha}_\nu \leq \ldots \leq \bar{\alpha}_m = \alpha_1 .$$

Let β_ν be an arbitrary point in $(\bar{\alpha}_\nu, \bar{\alpha}_{\nu+1})$, $\bar{\alpha}_\nu \leq \beta_\nu \leq \bar{\alpha}_{\nu+1}$, $\nu = 1, \ldots, m-1$. Then, if given any $U \in \mathcal{U}$, there is determined a real $\delta(U) > 0$ such that for all subdivisions π with $\underset{\nu}{l.u.b.}(\bar{\alpha}_{\nu+1} - \bar{\alpha}_\nu) \leq \delta$, ($(\bar{\alpha}_\nu, \bar{\alpha}_{\nu+1})$ has the same sign as $[\alpha_0, \alpha_1]$), there exists an element $I \in L$ such that for every choice of $\beta_\nu \in (\bar{\alpha}_\nu, \bar{\alpha}_{\nu+1})$,

$$\sum_\pi \equiv \sum_\pi f(\beta_\nu)(\bar{\alpha}_{\nu+1} - \bar{\alpha}_\nu) \in I + U_i ,$$

I is called the definite integral of $f(\alpha)$ from α_0 to α_1 and is denoted as usual by $\int_{\alpha_0}^{\alpha_1} f(\alpha)\, d\alpha$.

If I exists, $f(\alpha)$ is said to be integrable on $[\alpha_0, \alpha_1]$.

It is also possible to phrase this definition in terms of partial sums.

<u>Definition 3.2.2.</u> By Def.3.2.1, a δ-subdivision π determines an m-subdivision, i.e. $[\alpha_0, \alpha_1]$ into m parts, $m \geq m_1(U)$. The limit of the sequence of partial sums $\{\sigma_m\}$ as $m \to \infty$,

$$\sigma_m = \sum_{\nu=1}^{m-1} \Delta\bar{\alpha}_\nu \, f(\beta_\nu), \quad \Delta\bar{\alpha}_\nu = \bar{\alpha}_{\nu+1} - \bar{\alpha}_\nu ,$$

if it exists, is also called the definite integral of $f(\alpha)$ from α_0 to α_1.

<u>Remark</u> : This might be phrased as a Cauchy condition and a complete space used.

Lemma 3.2.1. The I of Def.3.2.1 implies the existence of the $\sigma(\sigma_m \to \sigma)$ of Def.3.2.2 and the equality of the two.

Proof. Obvious.

Lemma 3.2.2. If the integral of Def.3.2.1 exists for a function $f(\alpha)$, then that integral is unique.

Proof. I implies σ which is unique by Theorem 1.4.1.

Hereafter when completeness is required for L it will be assumed explicitly.

The following theorem is obvious, using a direct abstraction of the usual proof. Cf. [19].

Theorem 3.2.1. Let $f(\alpha)$, $g(\alpha)$ be integrable on $[\alpha_0, \alpha_1]$ to L. Let $\bar{\alpha}$ be such that $\alpha_0 < \bar{\alpha} < \alpha_1$. Let μ be any real number. Then

(1) $$\int_{\alpha_0}^{\alpha_1} (f+g)\, d\alpha = \int_{\alpha_0}^{\alpha_1} f\, d\alpha + \int_{\alpha_0}^{\alpha_1} g\, d\alpha ;$$

(2) $$\int_{\alpha_0}^{\alpha_1} \mu f\, d\alpha = \mu \int_{\alpha_0}^{\alpha_1} f\, d\alpha ;$$

(3) $$\int_{\alpha_0}^{\bar{\alpha}} f\, d\alpha + \int_{\bar{\alpha}}^{\alpha_1} f\, d\alpha = \int_{\alpha_0}^{\alpha_1} f\, d\alpha .$$

For the third statement L must be complete (Def.1.5.5).

With the essential restriction that the range of a function $f(\alpha)$ be a closed, convex set $L_c^a \subset L$ (Def.1.6.1), the following mean value theorem may be stated.

<u>Theorem 3.2.2.</u> Let L be complete. Let $\lambda(\alpha)$ be on $[\alpha_0, \alpha_1]$ to the positive real numbers. Let $f(\alpha)$ be on $[\alpha_0, \alpha_1]$ to $L_c^{cl} \subset L$, where L_c^{cl} is a closed convex set. Let $\lambda(\alpha)$ and $\lambda(\alpha) f(\alpha)$ be integrable on $[\alpha_0, \alpha_1]$. Then there exists an element $x \in L_c^{cl}$ such that

$$\int_{\alpha_0}^{\alpha_1} \lambda(\alpha)\, f(\alpha)\, d\alpha = \int_{\alpha_0}^{\alpha_1} \lambda(\alpha)\, d\alpha \,.\, x \,.$$

<u>Proof.</u> Consider for any m-subdivision the element

$$\sigma_m^* = \Big[\sum_{\nu=1}^{m-1} \Delta\bar{\alpha}_\nu \,.\, \lambda(\beta_\nu) f(\beta_\nu)\Big]\Big[\sum_{\nu=1}^{m-1} \Delta\bar{\alpha}_\nu \,.\, \lambda(\beta_\nu)\Big]^{-1}.$$

Since L_c^{cl} is convex and $f(\beta_\nu) \in L_c^{cl}$, $\sigma_m^* \in L_c^{cl}$. As $m \to \infty$, σ_m^* tends to a unique element x by completeness, which is in L_c^{cl} since this set is closed. Hence

$$\int_{\alpha_0}^{\alpha_1} \lambda(\alpha)\, f(\alpha)\, d\alpha = \int_{\alpha_0}^{\alpha_1} \lambda(\alpha)\, d\alpha \,.\, x \,.$$

<u>Remark</u> : It would clearly be sufficient to require only that every fundamental sequence penetrating L_c^{cl} infinitely be convergent. Cf. in this connection Theorem 1.8.4.

<u>Corollary 3.2.2.</u> Let $f(\alpha)$ be integrable on $[\alpha_0, \alpha_1]$ to L_c^{cl}. Then there exists $\bar{x} \in L_c^{cl}$ such that

$$\int_{\alpha_0}^{\alpha_1} f(\alpha)\, d\alpha = (\alpha_1 - \alpha_0) \,.\, \bar{x} \,.$$

<u>Proof.</u> In the theorem put $\lambda(\alpha) = 1$.

Definition 3.2.3. Let $\{f_m(\alpha)\}$ be a sequence of functions on $[\alpha_0, \alpha_1]$ to L. $\{f_m(\alpha)\}$ will be said to converge uniformly to a limit function $f(\alpha)$ if given any $U \in \mathcal{U}$, the same for all $\alpha \in [\alpha_0, \alpha_1]$, there is determined a positive integer $m_0(U)$ such that $m \geq m_0$ implies $f_m(\alpha) \in f(\alpha) + U_i$ for each $\alpha \in [\alpha_0, \alpha_1]$.

Theorem 3.2.3. Let L be complete. Let each member of the sequence $\{f_m(\alpha)\}$ be integrable on $[\alpha_0, \alpha_1]$ to L. Let $\{f_m(\alpha)\}$ converge uniformly on $[\alpha_0, \alpha_1]$ to $f(\alpha)$. Then

(1) $f(\alpha)$ is integrable on $[\alpha_0, \alpha_1]$,

(2) $$\lim_{m \to \infty} \int_{\alpha_0}^{\alpha_1} f_m(\alpha)\, d\alpha = \int_{\alpha_0}^{\alpha_1} f(\alpha)\, d\alpha.$$

Proof. Letting π be any p-subdivision of $[\alpha_0, \alpha_1]$, writing $\bar{\alpha}_{\nu+1} - \bar{\alpha}_\nu = \Delta \bar{\alpha}_\nu$, using the uniform convergence hypothesis and Post.1.2.6, we have

$$\Delta \bar{\alpha}_\nu\, f_m(\beta_\nu) \in \Delta \bar{\alpha}_\nu\, f(\beta_\nu) + {}_\nu\bar{U}_i .$$

Using Post.1.2.7 inductively, $m \geq m_0$ implies

(1) $$\sum_{\nu=1}^{p-1} \Delta \bar{\alpha}_\nu\, f_m(\beta_\nu) \in \sum_{\nu=1}^{p-1} \Delta \bar{\alpha}_\nu\, f(\beta_\nu) + U_i^*,$$

where $U^*(\in \mathcal{U}) = \sum_{\nu=1}^{p-1} {}_\nu\bar{U}$.

Now effect a $p_k^{(k)}$- subdivision of $[\alpha_0, \alpha_1]$ such that

$$\alpha_0 = \bar{\alpha}_1^{(k)} \leq \beta_1^{(k)} \leq \bar{\alpha}_2^{(k)} \leq \dots \leq \bar{\alpha}_\nu^{(k)} \leq \beta_\nu^{(k)} \leq \bar{\alpha}_{\nu+1} \leq \dots$$
$$\leq \bar{\alpha}_{p_k - 1}^{(k)} \leq \bar{\alpha}_{p_k}^{(k)} = \alpha_1 .$$

Then the partial sum has as a typical member

(2) $$\sigma_k = \sum_{\nu=1}^{p_k - 1} \Delta \bar{\alpha}_\nu^{(k)}\, f_{m_0}(\beta_\nu^{(k)})$$

with a δ_k that $\to 0$.

Then for any selected k , $\{\sigma_k\}$ converges to $\int_{\alpha_0}^{\alpha_1} f_{m_0}(\alpha)\, d\alpha$ as $p_k \to \infty$.

Note that the sequence $\{\sigma_k\}$, has $\delta_k \to 0$. Hence the integral is being approached as a limit and one may find $m_0(U)$ such that $k, l \geq m_0$ implies

(3) $$\sigma_k \in \sigma_l + U_i .$$

In (1) take $m = m_0$, and take $\pi = \pi_k = \pi_l$. Then

(4) $$\sigma_k \in \sum_{\nu=1}^{p_k - 1} \Delta \bar{\alpha}_\nu^{(k)} f(\beta_\nu^{(k)}) + U_i^* \qquad \text{and}$$

(5) $$\sigma_l \in \sum_{\nu=1}^{p_l - 1} \Delta \bar{\alpha}_\nu^{(l)} f(\beta_\nu^{(l)}) + U_i^* .$$

By (3), (4), (5), $k, l \geq m_0$ implies

(6) $$\sum_{\nu=1}^{p_k - 1} \Delta \bar{\alpha}_\nu^{(k)} f(\beta_\nu^{(k)}) \in \sum_{\nu=1}^{p_l - 1} \Delta \bar{\alpha}_\nu^{(l)} f(\beta_\nu^{(l)}) + \tilde{U}_i .$$

This considered as a π_k sequence is therefore fundamental and by completeness

$$\sum_{\nu=1}^{p_k - 1} \Delta \bar{\alpha}_\nu^{(k)} f(\beta_\nu^{(k)}) \quad \text{is convergent.}$$

But since $\delta_k \to 0$ as k increases, it follows that $f(\alpha)$ is integrable on $[\alpha_0, \alpha_1]$.

Now in (1) let $\delta \to 0$, i.e. $p \to \infty$. Then, clearly, $m \geq m_0$ implies

$$\int_{\alpha_0}^{\alpha_1} f_m(\alpha)\, d\alpha \in \int_{\alpha_0}^{\alpha_1} f(\alpha)\, d\alpha + U_i^* .$$

The proof is completed by the observation that U^* and $\tilde{U}$ are arbitrary since their composing sets were such.

§ 3.3

This section develops the fundamental theorem of this integral calculus.

Theorem 3.3.1. Let $\zeta(\alpha)$ be continuous on $[\alpha_0, \alpha_1]$ to L. Let π decompose the closed interval $[\alpha_0, \alpha_1]$ in such a fashion that for any two points β_1, β_2 of any interval of the decomposition and for an arbitrary $U \in \mathcal{U}$, $\zeta(\beta_1) - \zeta(\beta_2) \in U_i$ (a uniformity condition). Further let $\overline{\pi}$ be obtained from π by the intercalation of arbitrarily more end points. Then there is determined a bounded set T, $0 \in T \subset L$, such that

$$\sum_{\overline{\pi}} \in \sum_{\pi} + T \qquad \left(\sum_{\pi} \text{ as in Def.3.2.1}\right).$$

Proof. $\sum_{\overline{\pi}} \equiv \sum_{\overline{\pi}} \Delta\alpha^*_\nu \, \zeta(\beta^*_\nu)$, $\nu = 1, \ldots, m \geq n$,
$\sum_{\pi} \equiv \sum_{\pi} \Delta\overline{\alpha}_\mu \, \zeta(\beta_\mu)$, $\mu = 1, \ldots, n-1$.
Consider any subinterval $(\alpha^*_\nu, \alpha^*_{\nu+1})$ of $\overline{\pi}$, $\beta^*_\nu \in (\alpha^*_\nu, \alpha^*_{\nu+1})$. This interval is in some $(\overline{\alpha}_\mu, \overline{\alpha}_{\mu+1})$ of π, properly or improperly. In general suppose that $(\alpha^*_\nu, \alpha^*_{\nu+1}), \ldots, (\alpha^*_{\nu+\lambda}, \alpha^*_{\nu+\lambda+1})$ are all in $(\overline{\alpha}_\mu, \overline{\alpha}_{\mu+1})$. Then by hypothesis we have the p relations, arising after suitable multiplication,

$$\Delta\alpha^*_\nu \, \zeta(\beta^*_\nu) \in \Delta\alpha^*_\nu \, \zeta(\beta_\mu) + \Delta\alpha^*_\nu \cdot U_i \,.$$

Hence, via Posts.1.2.6 and 1.2.7, remembering that $\alpha^*_\nu = \overline{\alpha}_\mu$, $\alpha^*_{\nu+\lambda+1} = \overline{\alpha}_{\mu+1}$,

$$(1) \quad \sum_{\rho=0}^{\lambda} \Delta\alpha^*_{\nu+\rho} \, \zeta(\beta^*_{\nu+\rho}) \in \Delta\overline{\alpha}_\mu \, \zeta(\beta_\mu) + W_i \,.$$

The W above depends not only on U but also on $\overline{\pi}$, $W \in \mathcal{U}$.
By the continuity of $\zeta(\alpha)$ and Theorem 1.5.3, $\zeta(\alpha)$ is to a bounded set $S \subset L$. Hence W may be replaced by a bounded set $T(U, \overline{\pi}) \subset W$ (Theorem 1.4.4).
This is quite equivalent to the extraction from U of a bounded set E, $0 \in E$, and writing for the set on the r.h.s. of (1)

$$\sum_{\rho=0}^{\lambda} \Delta \alpha^{*}_{\nu+\rho} \; E = \overline{E}, \tag{2}$$

where $\overline{E}$ is bounded by Theorem 1.4.4.
By Lemma 1.7.3, all elements of E are of the form λx, $x \in F(E)$, $0 \leq \lambda \leq 1$. Now by (2)

$$\overline{E} \subset -\alpha^{*}_{\nu} E + \alpha^{*}_{\nu+1} E + \dots - \alpha^{*}_{\nu+\lambda} E + \alpha^{*}_{\nu+\lambda+1} E. \tag{3}$$

Consider the set $a_1 E + a_2 E$, $0 < a_1 \leq a_2$.
The frontier of E is represented parametrically by $x \in F(E)$, the frontier of $a_2 E$ by $a_2 x$, as in Lemma 1.7.4.
Construct from $a_2 E$ the set $E^{*} = E^{*}(a_2, E)$, that contains all points of the form $\lambda a_2 x$, $x \in F(E)$, $0 \leq \lambda \leq 1$.
Now certainly $a_2 E \subset E^{*}$ and $a_1 E \subset E^{*}$. E^{*} is bounded.
Thus $a_1 E + a_2 E \subset E^{*} + E^{*} = \overline{E}^{*}(a_2, E)$.
Finally, by finite induction the r.h.s. of (3) is included in a set $G(\overline{\alpha}_{\mu+1}, U)$.

The idea of this process has been to obtain a set for the inclusion (1) that does not depend on all the end points of $\overline{\pi}$ but only on the extreme right hand one.

Now for each of the intervals of $\overline{\pi}$ we have a relation of the type, via (1)

$$\sum_{\rho=0}^{\lambda} \Delta \alpha^{*}_{\nu+\rho} \; \zeta(\beta^{*}_{\nu+\rho}) \in \Delta \overline{\alpha}_{\mu} \; \zeta(\beta_{\mu}) + G,$$

wherein G depends on $\overline{\alpha}_{\mu+1}$ and U.

By addition of all such relations for each interval, and application again of reasoning similar to the above, we have the desired result

$$\sum_{\overline{\pi}} \in \sum_{\pi} + T(\alpha_1, U), \quad 0 \in T,$$

where T is bounded.

<u>Theorem 3.3.2.</u> Let $f(\alpha)$ be continuous on $[\alpha_0, \alpha_1]$ to L. Let L be complete. Then $f(\alpha)$ is integrable on $[\alpha_0, \alpha_1]$.

<u>Proof.</u> Let $\{\pi_m\}$ be a quite arbitrary sequence of decompositions of the interval $[\alpha_0, \alpha_1]$. Let $\delta_m \to 0$, i.e.
$\underset{\nu}{l.u.b.}\, \Delta \overline{\alpha}_\nu^{(m)} \to 0$ with m^{-1}.

By Theorem 1.5.1, $f(\alpha)$ is uniformly continuous. Hence $\alpha \in \overline{\alpha} + J(P)$ implies $f(\alpha) \in f(\overline{\alpha}) + P$, P open, $0 \in P \subset L$. Let σ be such that $\pm \sigma \in J$. Then there is determined $m_0(\sigma)$ such that $m \geq m_0$ implies $\delta_m < \delta$. Let also $n \geq m_0$. Then for any two points α, $\overline{\alpha}$ of a subinterval of π_n, or π_m, $f(\alpha) \in f(\overline{\alpha}) + P$.

Form by the addition of the effects of π_n and π_m the decomposition $\overline{\pi}$. Then by Theorem 3.3.1, taking $P = U_i$, there is determined a bounded set T, $0 \in T \subset L$, such that

$$\sum_{\pi_m} \in \sum_{\overline{\pi}} + T(\alpha_1, U), \quad \sum_{\pi_n} \in \sum_{\overline{\pi}} + T(\alpha_1, U).$$

By subtraction, for $m, n \geq m_0$,

$$\sum_{\pi_m} \in \sum_{\pi_n} + S(\alpha_1, U'), \quad (S \text{ bounded}).$$

Since U is quite arbitrary and α_1 is fixed, S may be taken in its turn arbitrary. Thus S is imbeddable in an arbitrary open set. By hypothesis L is complete. Hence $\sum_{\pi_n}$ converges to a unique limit. But π_n is arbitrary. $f(\alpha)$ is integrable.

Part 4

§ 4.1

This chapter considers various theorems representing the interaction of the integral and derivative theories previously developed. It also establishes the fundamental existence theorem for the most primitive differential equation. The functions used are again those on a real number interval to a suitable set in the space L .

§ 4.2

Theorem 4.2.1. If $f(\alpha)$ on $[\alpha_0, \alpha_1]$ to L has a derivative on $[\alpha_0, \alpha_1]$ (Def.2.2.3) which is integrable (Def.3.2.1) on $[\alpha_0, \alpha_1]$, then
$$\int_{\alpha_0}^{\alpha_1} f'(\alpha)\, d\alpha = f(\alpha_1) - f(\alpha_0).$$

Proof. Take $\alpha_0 < \alpha_1$. Let $U \in \mathcal{U}$ arbitrary. By definition there is determined a real $\delta(U) > 0$ such that, for every subdivision π with $\underset{\nu}{l.u.b.}\, \Delta\bar{\alpha}_\nu \leq \delta$ and for every $\beta_\nu \in (\bar{\alpha}_\nu, \bar{\alpha}_{\nu+1})$ of π , we have

$$(1) \qquad \sum_{\pi} \Delta\bar{\alpha}_\nu f'(\beta_\nu) - \int_{\alpha_0}^{\alpha_1} f'(\alpha)\, d\alpha \in U_i .$$

Now for each $\beta \in [\alpha_0, \alpha_1]$ there exists a positive number $t_\beta \leq \delta(U)$ such that for each point β' of $[\alpha_0, \alpha_1]$ satisfying $|\beta' - \beta| \leq t_\beta$,

$$(2) \qquad f(\beta') - f(\beta) - (\beta' - \beta) f'(\beta) \in |\beta' - \beta|\, U_i .$$

The open intervals $I_\beta \equiv (\beta - t_\beta < \rho' < \beta + t_\beta)$ constitute an open covering of $[\alpha_0, \alpha_1]$. Applying the Heine-Borel theorem a finite set of them, with centers at $\rho_1 < \rho_2 < \cdots < \rho_m$ also cover $[\alpha_0, \alpha_1]$, with $\rho_1 = \alpha_0$, $\rho_m = \alpha_1$.

By (2), for each I_k, when $\beta \in I_k$ or is an end point of I_k,

(3) $$f(\beta) - f(\rho_k) - f'(\rho_k)(\beta - \rho_k) \in |\beta - \rho_k| U_i .$$

Then by a simple extension of the reasoning used by Graves, [19],

(4) $$f(\bar{\alpha}_{\nu+1}) - f(\bar{\alpha}_\nu) - f'(\rho_\nu)\Delta\bar{\alpha}_\nu \in \Delta\bar{\alpha}_\nu U_i$$

for every ν. Hence by (4) and Posts.1.2.6 and 1.2.7, with W now arbitrary

(5) $$\sum_\pi [f(\bar{\alpha}_{\nu+1}) - f(\bar{\alpha}_\nu) - \Delta\bar{\alpha}_\nu f'(\rho_\nu)] \in W_i ,$$

so that with (1)

$$\sum_\pi [f(\bar{\alpha}_{\nu+1}) - f(\bar{\alpha}_\nu)] - \int_{\alpha_0}^{\alpha_1} f'(\alpha)\,d\alpha \in U_i + W_i \subset W_i^* .$$

But for any π, the sum above is $f(\alpha_1) - f(\alpha_0)$.

Hence

$$g = f(\alpha_1) - f(\alpha_0) - \int_{\alpha_0}^{\alpha_1} f'(\alpha)\,d\alpha \in W_i^* .$$

But W^* is arbitrary. Therefore g must be in each member of the sequence of Post.1.2.4. Thus g can only be the zero element, completing the proof.

<u>Theorem 4.2.2.</u> Let $f(x)$ on L to L have a vanishing differential $f(x; z)$ (Def.2.2.1) at all points of a convex set $L_c \subset L$. Then $f(x)$ is abstractly constant on this set.

<u>Proof.</u> $\varphi'(t)$ exists by Theorem 2.2.6, and $\varphi'(t) = 0$ for $0 \le t \le 1$. Hence $\varphi'(t)$ is integrable and by Theorem 4.2.1, $\varphi(1) - \varphi(0) = 0$. Since $\varphi(t) = f(x_0 + t(x_1 - x_0))$, $f(x_1) = f(x_0)$. But x_1, x_0 are any two points of L_c. Hence $f(x)$ is constant over L_c.

The following theorem is similar but has a somewhat different proof.

Theorem 4.2.3. Let L be complete. Let $f(x)$ be on L to L. Let the differential $f(x;z)$ exist, continuous in x, for all points of a convex set $L_c \subset L$. Then

$$\int_0^1 f(x_1 + \alpha(x_2 - x_1); x_2 - x_1)\, d\alpha = f(x_2) - f(x_1), \; x_1, x_2 \in L_c.$$

Proof. If $x_1, x_2 \in L_c$, $x = x_1 + \alpha(x_2 - x_1) \in L_c$, $0 \leq \alpha \leq 1$. Then by linearity, $f(x;z) = f(x_1 + \alpha(x_2 - x_1); x_2 - x_1)\,\delta\alpha$.

But $$f(x_1 + \alpha(x_2 - x_1); x_2 - x_1) = \lim_{\delta\alpha \to 0} \frac{\varphi(\alpha + \delta\alpha) - \varphi(\alpha)}{\delta\alpha} = \frac{d\varphi}{d\alpha},$$

for all $\alpha \in [0, 1]$. Take $\delta\alpha = d\alpha$. Since $f(x;z)$ is continuous in x it is continuous in α, and hence by Theorem 3.3.2, since L is complete, $d\varphi/d\alpha$ is integrable on $[0, 1]$. The result follows from Theorem 4.2.1.

Definition 4.2.1. Let $f(\alpha)$ be on $[\alpha_0, \alpha_1]$ to L. If there exists a function $F(\mu)$ such that $F(\mu) = \int_{\alpha_0}^{\mu} f(\alpha)\, d\alpha$ for all $\mu \in [\alpha_0, \alpha_1]$, it will be called the indefinite integral of $f(\alpha)$.

Remark : If $f(\alpha)$ is continuous on $[\alpha_0, \alpha_1]$ it is integrable on every subinterval and $F(\mu)$ exists on $[\alpha_0, \alpha_1]$ to L.

Lemma 4.2.1. Let L be complete. If $f(\alpha)$ is continuous on $[\alpha_0, \alpha_1]$ to L, and if $\lambda > 0$, $\alpha_0 \leq \bar{\mu} < \bar{\mu} + \lambda \leq \alpha_1$, then

$$\lim_{\lambda \to 0} \lambda^{-1} \int_{\bar{\mu}}^{\bar{\mu}+\lambda} f(\alpha)\, d\alpha = f(\bar{\mu}).$$

<u>Proof</u>. By Theorem 3.3.2, $f(\alpha)$ is integrable on every sub-interval of $[\alpha_0, \alpha_1]$. Hence $\lambda^{-1} \int_{\bar{\mu}}^{\bar{\mu}+\lambda} f(\alpha)\, d\alpha$ always exists for suitable λ.

By Theorem 1.5.3, $f(\alpha)$ is on $(\bar{\mu}, \bar{\mu}+\lambda)$ to $f(\bar{\mu}) + E(\lambda)$ where E is a bounded set. Construct the closed convex hull of E, E_c^{cl}. $f(\alpha)$ is clearly to $f(\bar{\mu}) + E_c^{cl}$. By Corollary 3.2.2,

$$\lambda^{-1} \int_{\bar{\mu}}^{\bar{\mu}+\lambda} f(\alpha)\, d\alpha = \bar{x}(\lambda)$$

where $\bar{x} \in f(\bar{\mu}) + E_c^{cl}$. As $\lambda \to 0$, $E_c^{cl} \to (0)$. Hence in the limit, $\bar{x}(0) = f(\bar{\mu})$.

<u>Theorem 4.2.4.</u> Let L be complete. Let $f(\alpha)$ be continuous at $\alpha = \bar{\mu}$. Then $F'(\bar{\mu}) = f(\bar{\mu})$ ($F(\mu)$ as in Def.4.2.1).

<u>Proof</u>. By Theorem 3.2.1,

$$F(\bar{\mu}+\lambda) - F(\bar{\mu}) = \int_{\bar{\mu}}^{\bar{\mu}+\lambda} f(\alpha)\, d\alpha .$$

Also $\lambda = \int_{\bar{\mu}}^{\bar{\mu}+\lambda} d\alpha$. Hence

$$\lambda f(\bar{\mu}) = \int_{\bar{\mu}}^{\bar{\mu}+\lambda} f(\bar{\mu})\, d\alpha .$$

These relations are true for λ sufficiently small, i.e. for regions close enough to $\bar{\mu}$ so that in them the continuity conditions on $f(\alpha)$ apply.

Again by Theorem 3.2.1,

$$F(\bar{\mu}+\lambda) - F(\bar{\mu}) - \lambda f(\bar{\mu}) = \int_{\bar{\mu}}^{\bar{\mu}+\lambda} [f(\alpha) - f(\bar{\mu})]\, d\alpha .$$

By Lemma 4.2.1,

$$\lim_{\lambda \to 0} \lambda^{-1} [F(\bar{\mu}+\lambda) - F(\bar{\mu})] = f(\bar{\mu}),$$

giving the result.

Theorem 4.2.5. Let $F(\alpha)$ be on $[\alpha_0, \alpha_1]$ to L. Let $F'(\alpha)$ vanish on $[\alpha_0, \alpha_1]$ and let $F(\alpha_0)=0$. Then $F(\alpha)$ vanishes on $[\alpha_0, \alpha_1]$.

Proof. $F'(\alpha)$ exists on $[\alpha_0, \alpha_1]$ to the closed set (0) in L. By definition it is integrable from α_0 to $\mu \leq \alpha_1$. Hence

$$I(\mu) = \int_{\alpha_0}^{\mu} F'(\alpha)\, d\alpha = F(\mu) - F(\alpha_0)$$

by Theorem 4.2.1. But $I(\mu)=0$ and $F(\alpha_0)=0$. Hence $F(\mu)=0$.

Theorem 4.2.6. Let $f(\alpha)$ be continuous on $[\alpha_0, \alpha_1]$ to L. The indefinite integral $\int_{\alpha_0}^{\mu} f(\alpha)\, d\alpha$, which is zero for $\mu = \alpha_0$, is the unique solution of the equation $F'(\mu) = f(\mu)$.

Proof. If there were another distinct function $F^*(\mu)$ that satisfied these conditions, then $F^*(\alpha_0) = 0$, and $F^{*\prime}(\mu) = f(\mu)$. Then $F(\alpha_0) - F^*(\alpha_0) = g(\alpha_0) = 0$. Also $g'(\mu)$ vanishes identically in μ. By Theorem 4.2.5, $g(\mu)=0$, and F and F^* can not be distinct.

This assures the existence of a unique primitive solution for $F'(\mu) = f(\mu)$.

Part 5

§ 5.1

This part is devoted to a study along two essentially different lines of the existence of solutions of the differential equation $dy/d\mu = f(\mu, y)$ where $f(\mu, y)$ is on $R \times L$ to L and y on R to L.

One attack is through a Cauchy approximation technique for whose successful completion it is apparently necessary, for reasons mentioned below, to make rather severe demands on the form of the function $f(\mu, y)$, the space remaining locally unconditioned.

In the other direction an extension of the Brouwer fixed point theorem is applied to yield a non-effective, i.e. non-constructive, existence theorem. But this approach, in its turn, while demanding little of the form of $f(\mu, y)$ makes rather severe pseudo-local demands on the space, in the sense elucidated below, without affecting what is in general the non-metric character of the space, under the assigned topology.

§ 5.2

We proceed first to the non-effective theorem. For this discussion it is necessary to consider in some detail an abstract function space associated with the space L .

Let F denote the space of all continuous functions on the closed interval $[0, 1]$ (this is selected for definiteness; any closed real interval could be chosen ab initio) to the space L . By Theorem 1.5.3, F is a subspace of the space of all bounded functions on $[0,1]$ to L . If $y(\alpha) \in F$ then also by Theorem 1.5.1, $y(\alpha)$ is uniformly continuous. Obviously every point $\in L$, i.e. the constant functions, lies $\in F$.

<u>Theorem 5.2.1.</u> F is alinear space. Cf. 1.1.1 to 1.1.7, p 1.
<u>Proof.</u> Addition and multiplication by reals are well defined through the postulates on L . $x(\alpha) + y(\alpha) = z(\alpha)$ means that corresponding points in L are added, $x(\alpha_1) + y(\alpha_1) = z(\alpha_1)$. Similarly for $\lambda x(\alpha) = w(\alpha)$. That $z(\alpha)$ and $w(\alpha)$ are in F follows from Theorem 1.5.6. With the closures established the required postulates are verified through those on L .

We topologize F in the following way, with 'set' as conditioned below again the undefined notion. Let there be given a set $\mathcal{U}_F$ of 'sets' $U_F \subset F$ where U_F consists of all continuous functions on $[0, 1]$ to a suitably convex set $M \subset L$. Cf. Def.1.6.4. Then the discussions involving $\mathcal{U}_F$ are thrown back on a discussion of the set $\mathfrak{M}$ of suitable sets M , thanks to Theorem 5.2.1.

<u>Theorem 5.2.2.</u> F is of the same linear topological type as L .

<u>Proof</u>. Linearity by Theorem 5.2.1.

Let U_F , V_F be any two neighborhoods of the zero function.

Ad Post.1.2.1 : Let $U_F \ (\in \mathcal{U}_F) \sim M \ (\in \mathfrak{M})$. Then $M/2 + M/2 = M$, since M is convex. I.e. $x \in M$, $x/2 \in M/2$, $M \subset M/2 + M/2$; $x, y \in M$, $x/2 + y/2 \in M$, $M \supset M/2 + M/2$. $0 \in (M/2)_i$.
Hence $M/2$ is suitably convex. Take $W_F \sim M/2$.

Ad Post.1.2.2. : Theorem 1.6.3(4).

Ad Post.1.2.3 : If M is convex, so also is $N = M - M$ by Theorem 1.6.3 (5,6). N is such that if $x \in N$, then $-x \in N$.
To prove this statement : (1) let $x \in M$, then $-x \in -M$ $-x + 0 \in N$, $-x \in N$; (2) let $x \,\overline{\in}\, M$, $\overline{\in}\, -M$, but $x \in N$. Then $x = x_1 - x_2$, $x_1, x_2 \in M$. Thus $-x = x_2 - x_1$, $x_2 - x_1 \in N$.
Hence $\alpha N \subset N$ for $-1 \leq \alpha \leq 1$ by convexity since $0 \in N_i$.

Now we wish to show that there exists $\beta \neq 0$ such that $\beta N \subset M$.
For then $\beta N = N^* \subset M$.

Both N and M are origin centered convex (Def.1.6.2).

Let $x \in F(M)$, and suppose to avoid triviality that $\mathcal{P}(c(M), \mathcal{P}(M, N)) \neq 0$. Then for some $x \in F(M)$ and some $\lambda > 1$ we have $\lambda x \in F(N)$. Therefore $x \in \lambda^{-1} F(N)$.
Consider the set $\{\lambda^{-1}\}$ of such λ^{-1} . This set is clearly bounded from zero since M contains a U , $0 \,\overline{\in}\, F(U)$, and αx is continuous. Let $\beta = \text{g.l.b.} \{\lambda^{-1}\}$.

Then $\beta N \subset M$. $\beta \alpha N = \alpha N^* \subset M$, $-1 \leq \alpha \leq +1$.

Ad Post.1.2.4 : Let $U \in \mathcal{U}$. Denote by U_{conv} the convex hull of U . Then U_{conv} is suitably convex, $U \subset U_{conv}$. U contains a centered convex set αV , some $V \in \mathcal{U}$, $-1 \leq \alpha \leq +1$. Let $x \in F(\alpha V)$; $x \neq 0$. Take $\beta = g.l.b.\{\lambda^{-1}\}$ where $\lambda x \in F(U_{conv})$. β is bounded from zero. $\beta U_{conv} \subset \alpha V \subset U$. βU_{conv} is suitably convex by Theorem 1.6.3(6). Thus every $U \in \mathcal{U}$ contains a suitably convex kernel. Finally let U be successively the U_m of Post.1.2.4, giving the result.

Ad Post.1.2.5 : Take any $f(\alpha) \in F$. Then since $f(\alpha)$ is continuous, it is to a bounded set $S \subset L$. Hence given any $U \in \mathcal{U}$, there exists α with $S \subset \alpha U$. For any suitably convex M take $U \subset M$ since $0 \in M_i$. Then $f(\alpha) \subset S \subset \alpha U \subset \alpha M$.
αM is suitably convex by Theorem 1.6.3(6). $f \in U_F \sim \alpha M$.

Ad Posts.1.2.6 and 1.2.7 : Theorem 1.6.3(5,6).

This completes the proof.

<u>Corollary 5.2.2.1.</u> The topology $(\mathcal{U}_F)$ describes a regular Hausdorff topology for F .

<u>Corollary 5.2.2.2.</u> If $f, g \in F$ and α is any real number, then $f+g$, αf are continuous in f , g and α , f , respectively.

<u>Lemma 5.2.1.</u> F is spatially convex in the sense of v.Neumann. Vide p 1. F is locally convex.

<u>Proof</u>. For the first result we show that if $M \in JJJ$, then

$M + M = 2M = M^* \in JJJ$.

That $M+M \supset 2M$ is clear.

Let $x, y \in M$. Then $x+y \in M+M$ $(x \neq y)$, $(x+y)/2 \in (M+M)/2$

$(x+y)/2 \in M$ by convexity. $x+y \in 2M$, $M+M \subset 2M$.

To prove the second statement take $U_F \in \mathcal{U}_F$, $M \sim U_F$.

Let $f(\alpha), g(\alpha) \subset M$. Since F is linear $\lambda f(\alpha)+(1-\lambda) g(\alpha) \in F$,

for each λ, $0 \leq \lambda \leq 1$. But M is convex. Hence for each

$\alpha_0 \in [0,1]$, $\lambda f(\alpha_0) + (1-\lambda) g(\alpha_0) \subset M$.

Thus the local convexity.

Lemma 5.2.2. Let $y(\mu)$ be on $[0,1]$ to $L_0 \subset L$, $0 \in L_0$.

If $dy/d\mu$ exists for all $\mu \in [0,1]$, then $y(\mu)$ is continuous in μ.

Proof. By Def.2.2.3, $U \in \mathcal{U}$, $\mu \in [0,1]$, $|\lambda| < \delta(U)$, $\mu+\lambda \in [0,1]$

imply $\lambda^{-1}[y(\mu+\lambda) - y(\mu)] \in dy/d\mu + U_i$.

Then $y(\mu+\lambda) - y(\mu) \in \lambda[dy/d\mu + U_i]$. Here $|\lambda| < 1$.

Let S be that open set consisting of all points of the form

αx, $x \in F(dy/d\mu + U)$, $|\alpha| < 1$. Then $|\lambda| < \delta(S)$

implies $y(\mu+\lambda) - y(\mu) \in S$.

Lemma 5.2.3. Let E be an arbitrary open set, $0 \in E \subset L$.

Let M be any suitably convex set with $0 \in M \subset L$. Then if

$\{\alpha_m\}$ is a proper real null sequence there is determined

$m_0 = m_0(E, M)$ such that $m \geq m_0$ implies $\alpha_m M \subset E$.

Proof. Since E is open, $0 \in E$, there exists $U(\in \mathcal{U}) \subset E$.

Apply the proof of Theorem 5.2.2 (1.2.4).

Lemma 5.2.4. Let $I(\alpha, g) = \int_0^\alpha g(\mu)\, d\mu$, $0 \leq \alpha \leq 1$.

Then $I(\alpha, g)$ is a linear function of g on F to F, providing

$g(\mu)$ is to $M \in \mathfrak{M}$ and L is complete.

<u>Proof.</u> First, it is clearly additive and homogeneous of degree one by Theorem 3.2.1.

Second, it is defined on F since through Theorem 1.6.3 (5) $M + M \in \mathfrak{M}$ if $M \in \mathfrak{M}$. It is also to F by the mean value Theorem 3.2.2.

Ad continuity : This need only be shown at the zero function. Let $E \subset F$ be open, $0 \in E$. By Lemma 5.2.3 choose m_0 so that $\alpha_m M = N \subset E$. By Theorem 3.2.2, $g(\mu) \in N$ implies $\int_0^\alpha g(\mu)\,d\mu = \alpha g^*(\alpha)$ and $g^* \in N$. Since $\alpha \in [0,1]$ and N is convex, we also have $\alpha g^* \in N \subset E$ completing the proof.

<u>Lemma 5.2.5.</u> Let N be a bicompact set $\subset L$. Let $\mathcal{B}$ be that set $\subset F$ consisting of all continuous functions on $[0,1]$ to N. Then $\mathcal{B}$ is bicompact in F.

<u>Proof.</u> Let $\mathfrak{R}$ be any infinite open covering of $\mathcal{B}$. Then if $R \in \mathfrak{R}$, let W be that set on N to which R corresponds, i.e. which contains all the functions composing R. W is taken as the minimal such set. Then W is open. For let $r(\alpha) \in F(R)$. Then $r(\alpha)$ is the limit of a sequence of functions $\subset R$. $r(\alpha) \mathrel{\overline{\in}} R$. Hence for each value of $\alpha \in [0,1]$, $r(\alpha_0) \in F(W)$ by the definition of W. Thus $W = W_i$. Therefore corresponding to $\mathfrak{R}$ is an infinite open covering $\mathfrak{W}$ of N. This contains a finite covering since N is bicompact and hence $\mathfrak{R}$ also contains a finite covering, for the result.

<u>Remark</u> : By a similar proof the same result holds for compact sets replacing bicompact ones in the above lemma.

Tychonoff [20] has given the following extension of the Brouwer fixed point theorem. Cf. also [21], [23], and especially [22].

Theorem 5.2.3. Every continuous mapping of a convex bicompact set of a locally convex linear topological space into itself possesses at least one fixed point.

The abstract function space F is suitable for an application of this theorem by virtue of Theorems 5.2.2 and 5.2.3, Corollary 5.2.2.2, and Lemma 5.2.1. Tychonoff speaks of a linear space as being topological when a topological ordering has been introduced in some fashion under which the linear operations are continuous.

The following existence theorem may now be stated :

Theorem 5.2.4. Let $f(\mu, y)$ be continuous in the pair μ, y, and defined on $[0,1] \times L_0$ $(0 \in L_0 \subset L)$ to a bicompact and convex set $A \subset L$, $0 \in A$. Then within the class of continuous functions $y(\mu)$ on $[0,1]$ to L_0, the differential system

(1) $$dy/d\mu = f(\mu, y) \qquad y(0) = 0$$

admits at least one solution.

Proof. A is bicompact, hence compact, hence bounded and complete by Theorems 1.8.3 and 1.8.4. Therefore the integral theory applies for functions to A.

By the theorems established in Parts 3 and 4, the solution of the problem is completely equivalent to assuring ourselves that a solution of the integral equation

(2) $$y(\mu) = \int_0^{\mu} f(\lambda, y(\lambda))\,d\lambda, \quad 0 \le \mu \le 1,$$

exists.

Let A_F denote the set in the space F that contains all functions $g(\mu) \subset A$. First it follows that

$$\int_0^{\alpha} g(\mu)\,d\mu \in A .$$

For by Corollary 3.2.2, $\int_0^{\alpha} g(\mu)\,d\mu = \alpha . \bar{g}$, $\bar{g} \in A$. By convexity, since $0 \in A$, $\alpha \bar{g} = g^*(\alpha) \subset A$, $g^* \in A_F$. $L_0 \cap A \neq 0$. Let $\{\alpha_n\}$ be a proper real null sequence and as in Lemma 5.2.3 choose n_0 such that for some $n \ge n_0$, $\alpha_n A \subset L_0$. Put $\alpha_n A = A^*$ which is clearly convex and, as a subset of a bicompact set, also bicompact.

By Theorem 1.5.3, members of the class of continuous functions on $[0,1]$ to L_0 may be taken to a bounded set $A^* \subset L_0$. Since by hypothesis $f(\mu, y(\mu))$ is continuous in the pair of variables, it follows from Theorem 1.5.4 that it is continuous as a function of μ. Hence $\int_0^{\alpha} f(\mu, y(\mu))\,d\mu$ is well defined.

Since $f(\mu, y(\mu))$ is a given fixed function, the above remarks show that

(3) $$z(\alpha) = \int_0^{\alpha} f(\mu, y(\mu))\,d\mu$$

is an integral transformation taking A_F^* $(A_F^* \sim A^*, A_F^* \subset A_F)$ into itself.

By Lemma 5.2.4, (3) is a continuous transformation and by Lemma 5.2.5, A_F^* is bicompact, and clearly convex. Thus Theorem 5.2.3 applies and at least one solution of (2) and hence of (1) is assured.

<u>Remarks</u> : This technique apparently furnishes no information as to the uniqueness of the solution achieved.

It is to be noted that the conditioning of the space L by the requirement that a bicompact set be imbeddable in it does not inhibit the assigned topology. For example, any finite line segment in L is such a convex and bicompact set, and in this connection compare particularly § 5.5, where a solution in this pre-existing situation is explicitly exhibited.

That is, A is not a neighborhood nor will it in general contain a neighborhood, for these are in general unbounded. Thus the imposition is not local in the customary sense of that word. In particular the non-metrisability of the space is not affected. We feel that such pseudo-local conditions have an as yet unmarked importance in the theory of linear spaces such as these.

§ 5.3

When the Cauchy method of successive approximations is applied to the differential equation $dy/d\mu = f(\mu, y)$ to construct a solution in the small for spaces in which a norm is defined, a sphere is selected for a Lipschitz condition in which the differences of the function values of $f(\mu, y)$ are required to lie, uniformly independently of μ. That is, it is supposed that there exists a number $M > 0$, independent of μ and y such that

$$(1) \qquad \| f(\mu, y) - f(\mu, \bar{y}) \| \leq M \cdot \| y - \bar{y} \| .$$

The rigidity of the norm obviates any essential knowledge of the form of the function $f(\mu,y)$.

One natural way of abstracting this condition to our situation is along the lines of Part 2. One could demand that there exist a neighborhood U with frontier point $M(y - \bar{y})$ such that

$$(2) \qquad f(\mu, y) - f(\mu, \bar{y}) \in U^{M(y-\bar{y})},$$

analagously to (1). But since $M(y - \bar{y})$ is only one frontier point of a rather fluid set U, which may also be unbounded, $f(\mu, y) - f(\mu, \bar{y})$ is not in general expressible in terms of $M(y - \bar{y})$, for use in the integral, as is effectively the case, in norm, for (1).

Rather abortive attempts and results, assuming (2) show that a stronger inclusion idea is necessary. Something along the lines of the definition inclusion for the derivative and the integral rather than one pivoting on one frontier point is required, if $f(\mu, y)$ is to remain unconditioned in form.

We content ourselves here however with the demonstration of solutions for the differential system in which the function $f(\mu, y)$ is taken essentially linear. An extension to polynomial forms appears practicable.

As a prélude to this we consider briefly questions of uniform convergence of series of functions.

§ 5.4

<u>Definition 5.4.1.</u> The series $\sum_{n=0}^{\infty} f_n(\alpha)$ of functions $f_n(\alpha)$ on a real interval $[\lambda_1, \lambda_2]$ to $L_0 \subset L$, is said to be uniformly convergent on that interval, if given any open set S, $0 \in S \subset L$, there is determined $n_0 = n_0(S)$, which does not depend on α, such that for all $n \geq n_0$ and for all $\alpha \in [\lambda_1, \lambda_2]$,

$$\sum_{\nu=0}^{\infty} f_\nu(\alpha) - \sum_{\nu=0}^{n} f_\nu(\alpha) \in S.$$

The following useful test is the abstraction of a theorem of Weierstrass :

<u>Theorem 5.4.1.</u> If $\{f_n(\alpha)\}$ is a sequence of functions on $[\lambda_1, \lambda_2]$ to $L_0 \subset L$, then $\sum_{n=0}^{\infty} f_n(\alpha)$ is uniformly convergent on $[\lambda_1, \lambda_2]$ if there exists a convergent series of positive real numbers $\sum_{n=0}^{\infty} \beta_n$ say, and a bounded set T, $0 \in T \subset L$, such that

$$f_n(\alpha) \in \beta_n T$$

for all n and all $\alpha \in [\lambda_1, \lambda_2]$. L is supposed complete.

<u>Proof.</u> Since $\beta_n \to 0$ and T is bounded

$$\mathcal{I}(\beta_n T) = (0).$$

Hence for each value of α, $f_n(\alpha) \to 0 \in L$. Consider any fixed value of α, say $\bar{\alpha}$. We then have a series of constant abstract elements, $\sum \bar{f}_n$, with $\bar{f}_n \to 0$.

Since $\sum \beta_n$ is convergent, given $\delta > 0$, there is determined $n_0(\delta)$ such that $n > n_0$ implies

$$\sum_{\nu=0}^{\infty} \beta_\nu - \sum_{\nu=0}^{n} \beta_\nu \in (0, \delta).$$

Hence

$$\sum_{\nu=0}^{\infty} \bar{f}_\nu - \sum_{\nu=0}^{n} \bar{f}_\nu \in \delta . T$$

But δ is independent of α and hence n_0 is also.
Given any open set S, $0 \in S \subset L$, δ may be chosen so that $\delta . T \subset S$ since T is bounded. Cf. Lemma 5.2.3. Hence the uniformity when completeness is used on the partial sums.

The next theorem also abstracts readily :

<u>Theorem 5.4.2.</u> Let $\sum_{n=0}^{\infty} \beta_n(\alpha)$ be a uniformly convergent series of real functions on $[\lambda_1, \lambda_2]$. Then the series $\sum_{n=0}^{\infty} f_n(\alpha)$ of functions on $[\lambda_1, \lambda_2]$ to L converges uniformly on $[\lambda_1, \lambda_2]$ if there exists a bounded set T, $0 \in T \subset L$, such that

$$f_n(\alpha) \in \beta_n(\alpha) . T$$

for all n and all $\alpha \in [\lambda_1, \lambda_2]$.

<u>Theorem 5.4.3.</u> The sum of a uniformly convergent series of continuous functions on $[\lambda_1, \lambda_2]$ to L is a continuous function.
<u>Proof.</u> We use the notation of Theorem 5.4.1, with $s(\alpha) = \sum_{n=0}^{\infty} f_n(\alpha)$, $s_n(\alpha) = \sum_{\nu=0}^{n} f_\nu(\alpha)$. Then as usual $s(\alpha) = s_n(\alpha) + r_n(\alpha)$ where $r_n(\alpha)$ is the remainder after n terms.

Let α and $\alpha + \delta\alpha$ be in $[\lambda_1, \lambda_2]$. Then
$s(\alpha + \delta\alpha) - s(\alpha) = s_n(\alpha + \delta\alpha) - s_n(\alpha) + r_n(\alpha + \delta\alpha) - r_n(\alpha)$.
Given any open set S we can choose n_0 such that $n \geq n_0(S)$ implies $r_n(\alpha + \delta\alpha)$, $r_n(\alpha) \in S$ for all permitted α and $\delta\alpha$.

Fix $N > n_0$. Then $s_N(\alpha)$ is a continuous function of α since it is the sum of N continuous functions. Hence given an open $S \subset L$, $\rho > 0$ is determined such that $\delta\alpha \in (0, \rho)$ implies $s_N(\alpha + \delta\alpha) - s_N(\alpha) \in S$. Hence for $\delta\alpha \in (0, \rho)$, $s(\alpha + \delta\alpha) - s(\alpha) \in S + S - S$. The r.h.s. is clearly open and arbitrary.

§ 5.5

Definition 5.5.1. A function $f(\alpha)$ on the real numbers to a bounded set $E \subset L$ is said to be measurable if for every bounded set $S \subset L$ the set of real numbers for which $f(\alpha) \in S$, written $E[f \in S]$, is measurable.

Remark : The characteristic properties of measurable functions are readily carried over.

Theorem 5.5.1. Let (1) $dy/d\mu = f(\mu, y)$ be the first order differential equation for which a solution is desired. Let $y(\mu)$ be on R_0 to L_0; $f(\mu, y)$ on $R_0 \times L_0$ $(0 \in L_0 \subset L)$ to $S \subset L$, with S bounded, such that $f(\mu, 0)$ is measurable. Let $f(\mu, y)$ be additive in the set μ, y. Let $y(0) = 0$. And lastly take L complete.

Then there exists a continuous function $y(\mu)$ on R_0 to L_0 that satisfies (1) and also the initial condition that $y(0) = 0$.

Proof. As previously remarked in Theorem 5.2.4 the solution is equivalent to the determination of a continuous solution of

$$(2) \qquad y(\mu) = \int_0^\mu f(\lambda, y(\lambda))\, d\lambda .$$

By additivity, $f(\mu, y) = f(\mu, 0) + f(0, y)$ and $f(\mu_1 + \mu_2, 0) = f(\mu_1, 0) + f(\mu_2, 0)$.

Now as is readily verified the analyses of both Sierpiński [24] and Banach [25] translate with minor modifications to this range space. Hence we have

(3) $$\zeta(\mu, 0) = \mu\, \zeta(1, 0)$$

and $\zeta(\mu, 0)$ is a continuous function of μ. Hence from the additivity relations it is clear that $\zeta(\mu, y)$ is a continuous function of μ for each y. By Theorem 3.3.2, $\zeta(\mu, 0)$ is integrable and we may write

(4) $$y_1(\mu) = \int_0^\mu \zeta(\lambda, 0)\, d\lambda = \frac{\mu^2}{2}\zeta(1, 0) \in \frac{\mu^2}{2} \cdot S,$$

by virtue of (3).

Thus $y_1(\mu)$ is continuous and so $\zeta(\mu, y_1(\mu))$ is continuous. Write

(5) $$\begin{aligned} y_2(\mu) - y_1(\mu) &= \int_0^\mu [\zeta(\lambda, y_1(\lambda)) - \zeta(\lambda, 0)]\, d\lambda \\ &= \int_0^\mu \frac{\mu^2}{2} \zeta(0, \zeta(1, 0))\, d\lambda. \end{aligned}$$

The homogeneity of $\zeta(\lambda, y)$ in y is shown in the usual way. Hence

$$y_2(\mu) - y_1(\mu) = \frac{\mu^3}{3!}\zeta(0, \zeta(1, 0)) \in \frac{\mu^3}{3!} \cdot S.$$

One proceeds now as is usual with the method of successive approximations, defining recursively

(6) $$y_{m+1}(\mu) = \int_0^\mu \zeta(\lambda, y_m(\lambda))\, d\lambda.$$

There arises

(7) $$y_m(\mu) - y_{m-1}(\mu) \in \frac{\mu^{m+1}}{(m+1)!} \cdot S.$$

By Theorems 5.4.1 and 5.4.3 it is clear that the sequence

$$y_1 + (y_2 - y_1) + (y_3 - y_2) + \cdots + (y_m - y_{m-1}) + \cdots$$

is uniformly convergent to a continuous function $y(\mu)$.
Then, proceeding to the limit in (6) we have

$$y(\mu) = \int_0^{\mu} f(\lambda, y(\lambda))\, d\lambda .$$

Clearly $y(0) = 0$.

Remark : The same theorem holds if $f(\mu, y)$ is subjected to the following alternative conditions :

(1) $f(\mu, y)$ is to a bounded, closed and convex set $S \subset L$;

(2) for each pair $y, \bar{y} \in L_0$, there exists a biunivocal and bicontinuous transformation T, that is homogeneous, such that

$$f(\mu, y) - f(\mu, \bar{y}) = T(\beta(\mu).(y - \bar{y}))$$

where $\beta(\mu)$ is a real function and $x \in S$ implies $T(x) \in S$.

Appendix

§ A.1

Consistency example for Postulates 1.2.1 - 1.2.7.

It is readily verified that complex Hilbert space, cf. [26], as weakly topologised by v.Neumann [1], satisfies Posts. 1.2.1 - 1.2.5.

The definition of fundamental set is : for $n = 1, 2, \dots,$ and $\varphi_1, \varphi_2, \dots \in H$, $\delta > 0$ define $U_2(\varphi_1, \dots, \varphi_n; \delta)$ as the set of all points $\zeta \in H$ satisfying the n inequalities

$$|(\zeta, \varphi_\nu)| \leq \delta .$$

Ad Post.1.2.6 : For $|(\zeta, \varphi_\nu)| \leq \delta$ and a given $\alpha\ (\neq 0)$,

$$|(\alpha\zeta, \varphi_\nu)| \leq |\alpha| . \delta = \eta \quad , \quad \alpha\zeta \in V_2 .$$

Ad Post.1.2.7 : Take two neighborhoods $U_2(\varphi_1, \dots, \varphi_n; \delta)$ and $V_2(\bar{\varphi}_1, \dots, \bar{\varphi}_m; \bar{\delta})$ for which

$$|(\zeta, \varphi_\nu)| \leq \delta \ , \nu = 1, \dots, n \ ; \ |(\bar{\zeta}, \bar{\varphi}_\mu)| \leq \bar{\delta} \ , \mu = 1, \dots, m \ .$$

We must show that there exist $\bar{\bar{\varphi}}_\rho$ and an $\eta > 0$ such that

$$|(\zeta + \bar{\zeta} \ , \ \bar{\bar{\varphi}}_\rho)| \leq \eta .$$

Take $\bar{\bar{\varphi}}_\rho = \varphi_\nu + \bar{\varphi}_\mu$ so that $\rho \leq nm$.

Now $|(\zeta + \bar{\zeta}, \varphi_\nu + \bar{\varphi}_\mu)| \leq \delta + \bar{\delta} + |(\zeta, \bar{\varphi}_\mu)| + |(\bar{\zeta}, \varphi_\nu)|$.

By Schwartz' inequality, for any $\zeta \in H$, we have

$$|(\zeta, \bar{\varphi}_\mu)| \leq |(\zeta, \zeta)|^{\frac{1}{2}} . |(\bar{\varphi}_\mu, \bar{\varphi}_\mu)|^{\frac{1}{2}} \leq K_1 \, |(\zeta, \zeta)|^{\frac{1}{2}} .$$

Also

$$|(\zeta, \varphi_\nu)| \leq |(\zeta, \zeta)|^{\frac{1}{2}} . |(\varphi_\nu, \varphi_\nu)|^{\frac{1}{2}} \leq K_2 \, |(\zeta, \zeta)|^{\frac{1}{2}} .$$

Hence for any $f \in U_2$, $f \neq 0$,

$$|(f, \overline{\varphi_\mu})| \leq K_3 \, |(f, \varphi_\nu)| \leq K_3 \delta , \quad K_3 = K_1 / K_2 .$$

Similarly, $|(\overline{f}, \varphi_\nu)| \leq \overline{K_3} \cdot \overline{\delta}$.

Hence take $\eta = \delta + \overline{\delta} + K_3 \delta + \overline{K_3}\, \overline{\delta}$ and $U_2 + V_2 = W_2$.

Notice that each of the so defined fundamental sets are convex. For if $f, g \in U_2$, $0 \leq \alpha \leq 1$ then

$$|(\alpha f + \overline{1-\alpha}\, g , \varphi_\nu)| \leq |(\alpha f, \varphi_\nu)| + |(\overline{1-\alpha}\, g, \varphi_\nu)|$$
$$\leq \alpha \delta + \overline{1-\alpha}\, \delta = \delta .$$

Hence $\alpha f + \overline{1-\alpha}\, g \in U_2$.

However every neighborhood is unbounded. To show this consider $U_2(\varphi_1, \ldots, \varphi_n ; \delta)$.

Now $\varphi_1, \ldots, \varphi_n$ determine a linear manifold $\{\varphi_1, \ldots, \varphi_n\}$. Since n is finite there exists $f^* \in H$ orthogonal to this manifold. Hence if $\{\alpha_n\}$ is a real null sequence $\alpha_n^{-1} f^* \in U_2$ since $|(f^*, \varphi_\nu)| = 0$ for $\nu = 1, \ldots, n$. Further $f^* \neq 0$. Thus as $\alpha_n \to 0$, $\alpha_n \alpha_n^{-1} f^* \to f^* \neq 0$.

V.Neumann proves that $(\mathcal{U}_2)$ violates both of the Hausdorff countability axioms. Cf. [1] and [2]. Hence after Birkhoff [7] it follows that H under $(\mathcal{U}_2)$ is not homeomorphic with a metric space.

§ A.2

The space of all bounded linear operators in Hilbert space which form a linear space by [26] p 65, also weakly topologised by v.Neumann [1], verifies these postulates also. Denoting the space by $\mathcal{B}$, it is subjected to the topology $(\mathcal{U}_5)$ as follows: For any $n = 1, 2, \ldots, \varphi_1, \psi_1, \ldots, \varphi_n, \psi_n \in H, \delta > 0$ define $U_5(\varphi, \psi; n; \delta)$ as the set of all bounded linear operators $A \in \mathcal{B}$ satisfying the n inequalities

$$|(A\varphi_\nu, \psi_\nu)| \leq \delta .$$

We verify the postulates for this space.

Ad 1.2.1 : Clear from the convexity of the fundamental sets. In fact $U_5 + U_5 = 2U_5$.

Ad 1.2.2 : Consider the sets $U_5(\varphi, \psi; n; \delta)$ and $V_5(\bar{\varphi}, \bar{\psi}; \bar{n}; \bar{\delta})$. $\varphi_\nu, \psi_\nu, \bar{\varphi}_{\bar{\nu}}, \bar{\psi}_{\bar{\nu}}$ are fixed in H . Let $m = n + \bar{n}$, $\epsilon = \min(\delta, \bar{\delta})$. Then the set $\subset \mathcal{B}$ such that

$$|(A\varphi_\nu, \psi_\nu)| \leq \epsilon \quad , \nu = 1, \ldots, n + \bar{n}$$

is a subset of both U_5 and V_5 .

Ad 1.2.3 : Clear with $U_5 = V_5$ since $|(\alpha f, g)| = |\alpha| . |(f, g)|$.

Ad 1.2.4 : Select $n = 1, \varphi, \psi \in H$. Let $\{\delta_m\}$ be a real null sequence. Take the sequence $\{U_5(\varphi, \psi; 1; \delta_m)\}$ and suppose there exist~~s~~ed a non-null operator common to all members of the sequence. By continuity it would be bounded from the null operator in the inner product and $|(T\varphi, \psi)| \leq \delta_m$ would be contradicted.

Ad 1.2.5 : Take $U_5(\varphi, \psi; n; \delta)$ and a fixed $T \in \mathcal{B}$. Since T is bounded $|(T\varphi_\nu, \psi_\nu)| \leq C \|\varphi_\nu\| . \|\psi_\nu\|$,

Put $M = \max_{\nu} (\|\varphi_\nu\|, \|\psi_\nu\|)$. Then $|(T\varphi_\nu, \psi_\nu)| \leq \frac{cM^2\delta}{\delta}$.
Hence $|(\delta/cM^2 \cdot T\varphi_\nu, \psi_\nu)| \leq \delta$, $\nu = 1, 2, \ldots, n$.
Thus $\delta T/cM^2 = A \in U_5$. $\alpha = cM^2/\delta$, $T \in \alpha U_5$.
Ad 1.2.6 : $\alpha V_5(\varphi, \psi; n; \delta) = U_5(\varphi, \psi; n; \delta . |\alpha|)$.
Ad 1.2.7 : Take $U_5(\varphi, \psi; n; \delta)$, $V_5(\bar\varphi, \bar\psi; \bar m; \bar\delta)$.
Let $m = \max(n, \bar m)$, $c = l.u.b.$ of the bounds of the operators of U_5, $\bar c$ of the bounds of those of V_5.
Let $M = \max_{\nu, \bar\nu} (\|\varphi_\nu\|, \|\bar\varphi_{\bar\nu}\|, \|\psi_\nu\|, \|\bar\psi_{\bar\nu}\|)$.
Then $|([A+B](\varphi_\nu + \bar\varphi_{\bar\nu}), \psi_\nu + \bar\psi_\nu)| \leq \delta + \bar\delta + 3M^2(c+\bar c)$

Thus W_5 is the neighborhood $W_5(\Phi, \Psi; m; \rho)$
where for $\nu = 1, \ldots, n$, $\Phi_\nu = \varphi_\nu + \bar\varphi_\nu$, $\Psi_\nu = \psi_\nu + \bar\psi_\nu$;
and for $\bar\nu = n+1, \ldots, m$, $\Phi_{\bar\nu} = \bar\varphi_{\bar\nu}$, $\Psi_{\bar\nu} = \bar\psi_{\bar\nu}$;
with $\rho = \delta + \bar\delta + 3M^2(c + \bar c)$.
Clearly the set of operators of $U_5 + V_5$ is exhauted by the $A + B$ of W_5.

This space again violates, [1], both countability axioms.

It is obvious that all linear normed spaces are of topological type L.

For further examples of linear topological spaces see [6].

§ A.3

Various independence examples.

The numeration refers to the postulate denied.

1.2.7 : Consider the linear space of ordered number pairs in the plane, subjected to the customary vector addition and multiplication by real numbers. Define the fundamental sets about the origin as those sets bounded by closed, simple, origin centered symmetric curves whose minimal radius vectors occur on the axes and whose maximal ones occur on the lines $y = \pm x$,

1.2.6 : Take real Banach space. For the sets $\mathcal{U}$ take the set of all spheres of rational radius only about the origin.

1.2.5 : The linear continuum with all $U = (0)$.

1.2.4 : The linear continuum with all U = the whole space.

The complete intra-independence of this set of postulates has not been established. It is suggested that 1.2.2 and 1.2.3 may be dependent, as also may be 1.2.1 and 1.2.7.

§ A.4

The following example shows that all of the fundamental sets need not be connected. Take the linear continuum with sets $\mathcal{U}$ of three types :

(α) open segments about the origin, $|x| < M$;

(β) open sets about the origin, $|x| < M$, plus the points $|x| > N, N > M$;

(γ) the space itself.

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